The discrete signature Veronese variety

Probabilistic methods, Signatures, Cubature and Geometry

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Slides can be found at `raulpenaguiao.github.io/`

Joint work with Carlo Belingeri and Bernd Sturmfels.

Path signatures

Given a ($\mathcal{C}^1$) path $\mathbf{X} : [0,1] \to \mathbb{R}^d$, we can define its (continuous) signature $\sigma^{(k)} \in \mathcal{T}^k(\mathbb{R})$:

$$\sigma_{\omega_1 \dots \omega_k}(\mathbf{X}) = \int_{0<t_1<\cdots<t_k<1} X'_{\omega_1}(t_1) \cdots X'_{\omega_k}(t_k) d\mathbf{t}\,,$$

defined for $\omega_i \in \{1, \dots, d\}$.
These satisfy the **shuffle relations**:

$$\sigma_\omega(\mathbf{X})\sigma_\tau(\mathbf{X}) = \sum_{\alpha \in \omega ⧢ \tau} \sigma_\alpha(\mathbf{X})\,.$$

Example: $\sigma_1(\mathbf{X})^2 = 2\sigma_{11}(\mathbf{X})$.

Discrete path signatures

Given a sequence of vectors $\mathbf{X} = (\mathbf{X}^0, \dots, \mathbf{X}^N) \in (\mathbb{R}^d)^{N+1}$, we can define its (discrete) signatures:

$$\boldsymbol{\mathcal{S}}_{p_1,\dots,p_k}(\mathbf{X}) = \sum_{1 \leq t_1 < \cdots < t_k \leq N} p_1(\mathbf{X}^{t_1} - \mathbf{X}^{t_1 - 1}) \cdots p_k(\mathbf{X}^{t_k} - \mathbf{X}^{t_k - 1}),$$

defined for p_i **non-constant monomials** in $\{x_1, \dots, x_d\}$.

Discrete path signatures

$\mathbf{X} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix}$, $N = 3$, $d = 2$.
$p_1 = x_1 x_2$ and $p_2 = x_1$ so $k = 2$.

$$\boldsymbol{\mathcal{S}}_{p_1,\dots,p_k}(\mathbf{X}) = \sum_{1 \leq t_1 < \cdots < t_k \leq N} p_1(\mathbf{X}^{t_1} - \mathbf{X}^{t_1 - 1}) \cdots p_k(\mathbf{X}^{t_k} - \mathbf{X}^{t_k - 1})\,,$$

$$\begin{aligned} \boldsymbol{\mathcal{S}}_{p_1,p_2}(\mathbf{X}) =& p_1(\mathbf{X}^1 - \mathbf{X}^0) p_2(\mathbf{X}^2 - \mathbf{X}^1) + p_1(\mathbf{X}^1 - \mathbf{X}^0) p_2(\mathbf{X}^3 - \mathbf{X}^2) \\ &+ p_1(\mathbf{X}^2 - \mathbf{X}^1) p_2(\mathbf{X}^3 - \mathbf{X}^2) \\ =& 2 \times (-1) \times 0 + 2 \times (-1) \times 1 + 0 \times 0 \times 1 = -2\,. \end{aligned}$$

$\mathbf{X} \in (\mathbb{R}^d)^{N+1} \to \mathbf{X} \in (\mathbb{R}^d)^{N+1}$ $\qquad$ $(p_1, p_2) \to 12|1$

Recovering paths

Tree-like excursions $\rightarrow$ Time warping
Recovering paths from signature was done by Pfeffer, Seigal and Sturmfels

- Inverse problem: given a signature of a polynomial path, what is the specific path that we started with?
- Optimal path with a given signature (minimal distance covered, etc.)

This should all be possible in the discrete world.

Where do discrete signatures live

For $d = 2$ and $h \leq 3$, we have the following $\mathcal{I}_{d,h}$

$$\mathcal{I}_{2,1} = \{1, 2\}$$

$$\mathcal{I}_{2,2} = \{11, 12, 22, 1|1, 1|2, 2|1, 2|2\}$$

$$\begin{aligned}\mathcal{I}_{2,3} = &\{111, 112, 122, 222, 1|11, 1|12, 1|22, 2|11, 2|12, 2|22, 11|1, 11|2, 12|1, \\ &12|2, 22|1, 22|2, 1|1|1, 1|1|2, 1|2|1, 1|2|2, 2|1|1, 2|1|2, 2|2|1, 2|2|2\}\end{aligned}$$

$\#\mathcal{W}_{d,h}$	$h=1$	$h=2$	$h=3$
$d=2$	2	7	24
$d=3$	3	15	73
$d=4$	4	26	164

Shuffle relations $\to$ quasi-shuffle relations.

$$\boldsymbol{\mathcal{S}}_\omega(\mathbf{X})\boldsymbol{\mathcal{S}}_\tau(\mathbf{X}) = \sum_{\alpha \in \omega \overline{\sqcup\!\sqcup} \tau} \sigma_\alpha(\mathbf{X})\,.$$

Example: $\boldsymbol{\mathcal{S}}_{x_1}(\mathbf{X})\boldsymbol{\mathcal{S}}_{x_2}(\mathbf{X}) = \boldsymbol{\mathcal{S}}_{x_1,x_2}(\mathbf{X}) + \boldsymbol{\mathcal{S}}_{x_2,x_1}(\mathbf{X}) + \boldsymbol{\mathcal{S}}_{x_1x_2}(\mathbf{X})$.

Quasi-shuffle

$$
\begin{aligned}
\boldsymbol{\mathcal{S}}_{x_1}(\mathbf{X})\boldsymbol{\mathcal{S}}_{x_2}(\mathbf{X}) &= \left(\sum_{1\leq i\leq N} x_1(\mathbf{X}^i)\right)\left(\sum_{1\leq i\leq N} x_2(\mathbf{X}^i)\right) \\
&= \sum_{1\leq i,j\leq N} \mathbf{X}_1^i \mathbf{X}_2^j \\
&= \sum_{1\leq i<j\leq N} \mathbf{X}_1^i \mathbf{X}_2^j + \sum_{1\leq j<i\leq N} \mathbf{X}_1^i \mathbf{X}_2^j + \sum_{1\leq i\leq N} \mathbf{X}_1^i \mathbf{X}_2^i \\
&= \boldsymbol{\mathcal{S}}_{x_1,x_2}(\mathbf{X}) + \boldsymbol{\mathcal{S}}_{x_2,x_1}(\mathbf{X}) + \boldsymbol{\mathcal{S}}_{x_1x_2}(\mathbf{X})
\end{aligned}
$$

Extra term comes from **measure zero diagonals.**

The varieties - continuous case

$\mathcal{W}_d = \{1, \dots, d\}$.
$\mathcal{W}_{d,k}$ the set of words of length k on the characters $\mathcal{W}_d$.
The variety $\mathcal{U}_{d,k,N}$ is the closure of the image of $\sigma^{(k)}$ on **polynomial** paths of degree N. In this way, $\mathcal{U}_{d,k,N} \subset \mathbb{R}^{\mathcal{W}_{d,k}} = \mathcal{T}^{(k)}(\mathbb{R}^d)$.

$$\mathcal{U}_{d,k,1} \subset \mathcal{U}_{d,k,2} \subset \cdots .$$

Let $\mathcal{U}_{d,k}$ be the limit of this chain.

Theorem (Améndola, Friz and Sturmfels 2020)

$\mathcal{U}_{d,n}$ is precisely the image of the closure of $\sigma^{(k)}$.

The varieties

$\mathcal{M}_d$ the set of non-constant monomials on $\{x_1, \dots, x_d\}$.
$\mathcal{I}_d$ the set of words in $\mathcal{M}_d$.
Height of a word $\vec{p} = (p_1, \dots, p_k)$ in $\mathcal{M}_d$

$$h(\vec{p}) = \sum_i \deg p_i \,.$$

Do not mistake **height** of a word with its **lenght**, generally smaller.
$\mathcal{I}_{d,h}$ the set of words in $\mathcal{M}_d$ of height h.
The variety $\mathcal{V}_{d,h,N}$ is the closure of the image of $\boldsymbol{\mathcal{S}} : (\mathbb{R}^d)^N \to \mathbb{R}^{\mathcal{I}_{d,h}}$. In this way, $\mathcal{V}_{d,h,N} \subset \mathbb{R}^{\mathcal{I}_{d,h}}$.

$$\mathcal{V}_{d,h,1} \subset \mathcal{V}_{d,h,2} \subset \cdots \,.$$

Let $\mathcal{V}_{d,n}$ be the limit of this chain.

Some dimension considerations

Theorem (Hoffman 2000)

Shuffle algebra and quasi-shuffle algebra are isomorphic via an exponential map.

Theorem (Améndola, Friz and Sturmfels 2019)

The dimension of $U_{d,n}$ is $\lambda_{d,n} - 1$, where $\lambda_{d,n}$ is the number of Lyndon words on d characters of length at most n.

Theorem (Belingeri, P. and Sturmfels 2023)

The dimension of $V_{d,h}$ is $\mu_{d,n} - 1$, where $\mu_{d,n}$ is the number of Lyndon words on $\mathcal{M}$ of height at most h.

$\#\mathcal{W}_{d,h}$	$h=1$	$h=2$	$h=3$
$d=2$	2	7	24
$d=3$	3	15	73
$d=4$	4	26	164

$\mu_{d,h}$	$h=2$	$h=3$	$h=4$
$d=2$	2	4	12
$d=3$	3	9	36
$d=4$	4	16	80

Some degree considerations

The degree of $\mathcal{V}_{d,h}$ is related with the **number of solutions** of an inverse problem.

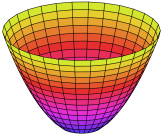

Figure: A paraboloid, degree two and dimension three variety. Credit to Krishnavedala - Wikipedia

Theorem (Bounds on the degree of this variety)

Coming soon!

Small cases

We can compute **dimension and degree** of these varieties using **Macaulay2**.

dim	$V_{2,1}$	$V_{2,2}$	$V_{2,3}$	$V_{3,1}$	$V_{3,2}$	$V_{3,3}$	$V_{4,1}$	$V_{4,2}$
dim	2	7	12	3	9	36	4	16
deg	(...)	(...)	(...)	(...)	(...)	(...)	(...)	(...)

Biblio

- Améndola, Carlos, Peter Friz, and Bernd Sturmfels. ”Varieties of signature tensors.” Forum of Mathematics, Sigma. Vol. 7. Cambridge University Press, 2019.
- Galuppi, Francesco. “The rough Veronese variety.” Linear algebra and its applications 583 (2019): 282-299.
- Diehl, Joscha, Ebrahimi-Fard, Kurusch, and Tapia, Nikolas. ”Time-warping invariants of multidimensional time series.” Acta Applicandae Mathematicae 170.1 (2020): 265-290.

Thank you