Using intersection of fans to construct more fans?

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Raúl Penaguião

SFSU

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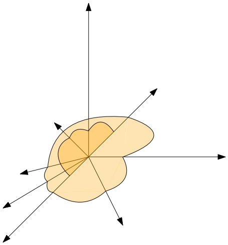
Slides can be found at `raulpenaguiao.github.io/`

Outline of the talk

1. Introduction
   - Convex fans and Matroids
   - The Bergman fan
2. Intersection of perturbations
   - The simple case
   - The derivative fan

What is a convex fan

A coherent collection of cones. Like the Braid fan or:



Questions that we can ask:

- Is it a pure fan?
- Is it complete?
- What is the number of **maximal cones**?
- What is its f-vector?

What is a matroid

- A matroid is described by its collection of independent sets.
- A matroid is described by its collection of circuits.
- A matroid is described by its collection of bases.
- A matroid is described by its rank function.
- A matroid is described by its collection of flats.

Matroids can come from graphs (called graphical matroids)

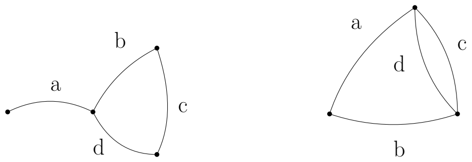
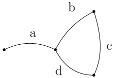
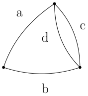


Figure: To two graphs correspond two matroids.

What is a matroid





Bases

$$\mathcal{B}(M_1) = \{abc, abd, acd\}\,.$$

$$\mathcal{B}(M_2) = \{ab, ac, ad, bc, bd\}\,.$$

Circuits

$$\mathcal{C}(M_1) = \{bcd\}\,.$$
$$\mathcal{C}(M_2) = \{abc, abd, cd\}\,.$$

Flats

$$\mathcal{F}(M_1) = \{\text{any edge}, bcd, ab, ac, ad, abcd\}\,.$$
$$\mathcal{F}(M_2) = \{a, b, cd, abcd\}$$

The Bergman fan

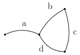
$$\Sigma_M = \{(x_i)_i \in \mathbb{R}^n | \ \forall_{C \in \mathcal{C}(M)} \min_{c \in C} x_c \text{ is attained twice}\} .$$

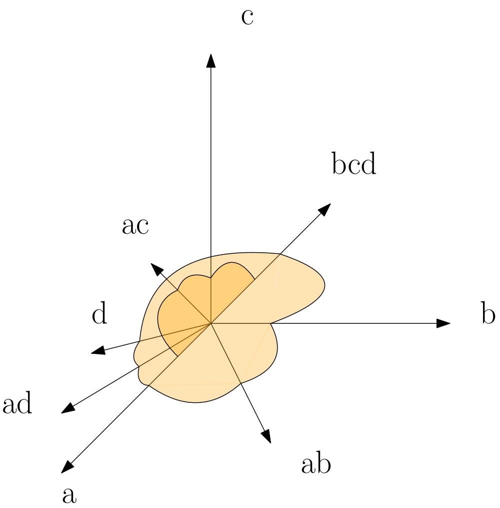
$$\Sigma_M = \biguplus_{\substack{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \\ \text{each } F_i \text{ is a flat}}} \operatorname{cone}(\vec{e}_{F_1}, \ldots, \vec{e}_{F_r}) .$$

Questions that we can ask:

- Why are these the same thing?
- (Geometric questions) Is this a complete fan? What is the dimension of this fan? How many maximal cones are there?
- Can I see some pictures?

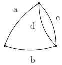
Pictures

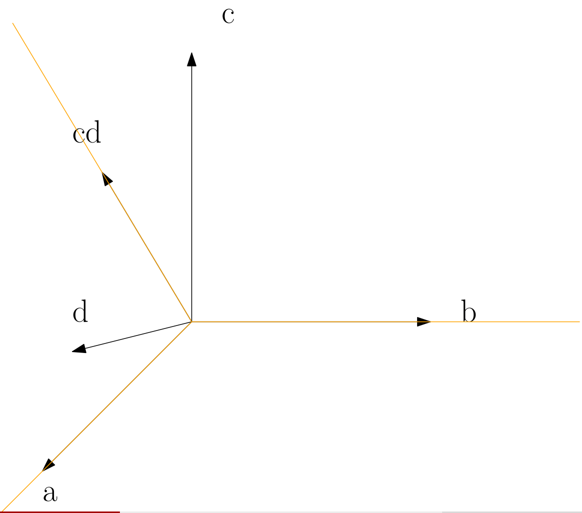
In $\mathbb{R}^4/\mathbb{1}\mathbb{R}$. 



Pictures

In $\mathbb{R}^4/\mathbb{1}\mathbb{R}$.





Pictures

Observations:

- Rank of matroid = dimension of Bergman fan.
- A generic perturbation of these fans seems to intersect at a unique point.

The simple case

Theorem (Rau and Mikhalkin)

If M is a matroid of rank r on n elements, and $U = U_{n,n-r+1}$ is the uniform matroid of rank $n-r+1$, then for a generic $\vec{v}$ we have that

$$(\Sigma_M + \vec{v}) \cap \Sigma_U$$

intersects at a unique point.

A derivative construction fan

For each generic vector $\vec{v}$, find the cones

$$\mathcal{C}_{\mathcal{F}} = \mathcal{C}_{\emptyset \subsetneq F_1 \subsetneq \dots} = \operatorname{cone}(\vec{e}_{F_1}, \dots, \vec{e}_{F_r})$$

in the Bergman fan of M, and

$$\mathcal{C}_{\vec{t}} = \mathcal{C}_{(t_1, \dots, t_{n-r})} = \operatorname{cone}(\vec{e}_{t_1}, \dots)$$

in the Bergman fan of U, such that

$$(\mathcal{C}_{\mathcal{F}} + \vec{v}) \cap \mathcal{C}_{\vec{t}} \neq \emptyset$$

The flags $\mathcal{F} = \mathcal{F}(\vec{v})$ and $\vec{t} = \vec{t}(\vec{v})$ depend on $\vec{v} \in \mathbb{R}^n$.This splits $\mathbb{R}^n$ into cones that form a complete fan Ω_M.

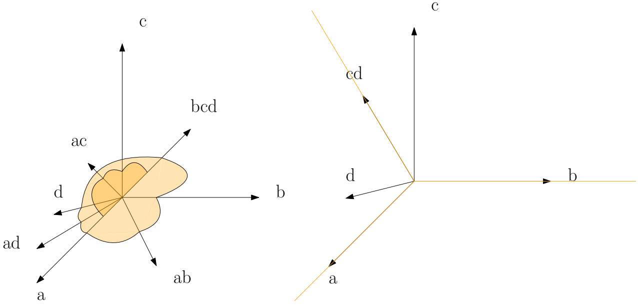
Enumeration results for the derivative fan

How many cones does Ω_M have, for a given matroid M?
If M is the graphical matroid of the complete graph K_v, then it is a matroid in $n = \binom{v}{2}$ elements and the number of cones c satisfies a catalan style recurrence relation:

v	2	3	4	5	6	7	8
n	1	3	6	10	15	21	28
c	1	6	576	9.1×10^6	9.9×10^{14}	5.4×10^{28}	1.6×10^{50}

Other fans?

What happens if we study the generic intersection theory on two different matroids?
Explore the two matroids from the beginning!



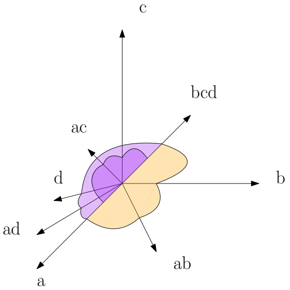
Other fans?

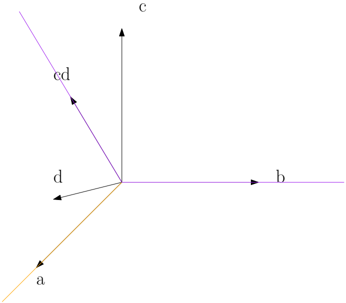
Compatibility condition? Which cones from Σ_{M_1} and Σ_{M_2} **can** intersect in a unique point?
By the picture, it is clear that not all cones can intersect eachother in a point.
Compatibility condition:

$$\lambda(\mathcal{F}_1) \wedge \lambda(\mathcal{F}_2) = \{\{1\}, \dots, \{n\}\}$$

There are 12 pairs of flags of flats that are compatible between M_1 and M_2.
Does it mean that we have a fan Ω_{M_1,M_2}? Rau & Mikhalkin does not guarantee us that!

Other fans?





Other fans?

Holy smokes Raul waitaminute there's a lot of things that I don't understand there!
Me too!
Question does this refine the Braid fan?
More questions?

Thank you