The discrete signature Veronese variety

Probabilistic methods, Signatures, Cubature and Geometry

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Slides can be found at `raulpenaguiao.github.io/`
Joint work with Carlo Belingeri and Bernd Sturmfels.

Path signatures

Given a ($\mathcal{C}^1$) path $\mathbf{X} : [0,1] \to \mathbb{R}^d$, we can define its (continuous) signature $\sigma^{(k)} \in \mathcal{T}^k(\mathbb{R})$:

$$\sigma_{\omega_1 \dots \omega_k}(\mathbf{X}) = \int_{0<t_1<\cdots<t_k<1} X'_{\omega_1}(t_1) \cdots X'_{\omega_k}(t_k) d\mathbf{t} \,,$$

defined for $\omega_i \in \{1, \dots, d\}$.
These satisfy the **shuffle relations**:

$$\sigma_\omega(\mathbf{X})\sigma_\tau(\mathbf{X}) = \sum_{\alpha \in \omega ⧢ \tau} \sigma_\alpha(\mathbf{X}) \,.$$

Example: $\sigma_1(\mathbf{X})^2 = 2\sigma_{11}(\mathbf{X})$.

Discrete path signatures

Given a sequence of vectors $\mathbf{X} = (\mathbf{X}^0, \dots, \mathbf{X}^N) \in (\mathbb{R}^d)^{N+1}$, we can define its (discrete) signatures:

$$\boldsymbol{\mathcal{S}}_{p_1,\dots,p_k}(\mathbf{X}) = \sum_{0 \leq t_1 < \dots < t_k \leq N} p_1(\mathbf{X}^{t_1} - \mathbf{X}^{t_1 - 1}) \cdots p_k(\mathbf{X}^{t_k} - \mathbf{X}^{t_k - 1}) \,,$$

defined for p_i **non-constant monomials** in $\{x_1, \dots, x_d\}$.

Discrete path signatures

$\mathbf{X} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix}$, $N = 3$, $d = 2$.
$p_1 = x_1 x_2$ and $p_2 = x_1$ so $k = 2$.

$$\boldsymbol{\mathcal{S}}_{p_1,\dots,p_k}(\mathbf{X}) = \sum_{0 \le t_1 < \cdots < t_k \le N} p_1(\mathbf{X}^{t_1} - \mathbf{X}^{t_1 - 1}) \cdots p_k(\mathbf{X}^{t_k} - \mathbf{X}^{t_k - 1}) \, ,$$

$$\begin{aligned} \boldsymbol{\mathcal{S}}_{p_1,p_2}(\mathbf{X}) =& p_1(\mathbf{X}^1 - \mathbf{X}^0) p_2(\mathbf{X}^2 - \mathbf{X}^1) + p_1(\mathbf{X}^1 - \mathbf{X}^0) p_2(\mathbf{X}^3 - \mathbf{X}^2) \\ &+ p_1(\mathbf{X}^2 - \mathbf{X}^1) p_2(\mathbf{X}^3 - \mathbf{X}^2) \\ =& 2 \times (-1) \times 0 + 2 \times (-1) \times 1 + 0 \times 0 \times 1 \, . \end{aligned}$$

Tree-like excursions $\rightarrow$ Time warping

Where do discrete signatures live

For $d = 2$ and $h \leq 3$, we have the following $\mathcal{I}_{d,h}$

$$\mathcal{I}_{2,1} = \{1, 2\}$$

$$\mathcal{I}_{2,2} = \{11, 12, 22, 1|1, 1|2, 2|1, 2|2\}$$

$$\begin{aligned}\mathcal{I}_{2,3} =& \{111, 112, 122, 222, 1|11, 1|12, 1|22, 2|11, 2|12, 2|22, 11|1, 11|2, 12|1,\\ & 12|2, 22|1, 22|2, 1|1|1, 1|1|2, 1|2|1, 1|2|2, 2|1|1, 2|1|2, 2|2|1, 2|2|2\}\end{aligned}$$

dim	$h=1$	$h=2$	$h=3$
$d=2$	2	7	24
$d=3$	3	15	73
$d=4$	4	26	164

Shuffle relations $\to$ quasi-shuffle relations.

$$\boldsymbol{\mathcal{S}}_\omega(\mathbf{X})\boldsymbol{\mathcal{S}}_\tau(\mathbf{X}) = \sum_{\alpha\in\omega\overline{\sqcup\!\sqcup}\tau} \sigma_\alpha(\mathbf{X})\,.$$

Example: $\boldsymbol{\mathcal{S}}_{x_1}(\mathbf{X})\boldsymbol{\mathcal{S}}_{x_2}(\mathbf{X}) = \boldsymbol{\mathcal{S}}_{x_1,x_2}(\mathbf{X}) + \boldsymbol{\mathcal{S}}_{x_2,x_1}(\mathbf{X}) + \boldsymbol{\mathcal{S}}_{x_1x_2}(\mathbf{X})$.

Quasi-shuffle

$$\begin{aligned}\boldsymbol{\mathcal{S}}_{x_1}(\mathbf{X})\boldsymbol{\mathcal{S}}_{x_2}(\mathbf{X}) &= \left(\sum_{1\leq i\leq N} x_1(\mathbf{X}^i)\right)\left(\sum_{1\leq i\leq N} x_2(\mathbf{X}^i)\right)\\ &= \sum_{1\leq i,j\leq N} \mathbf{X}_1^i\mathbf{X}_2^j\\ &= \sum_{1\leq i<j\leq N} \mathbf{X}_1^i\mathbf{X}_2^j + \sum_{1\leq j<i\leq N} \mathbf{X}_1^i\mathbf{X}_2^j + \sum_{1\leq i\leq N} \mathbf{X}_1^i\mathbf{X}_2^i\\ &= \boldsymbol{\mathcal{S}}_{x_1,x_2}(\mathbf{X}) + \boldsymbol{\mathcal{S}}_{x_2,x_1}(\mathbf{X}) + \boldsymbol{\mathcal{S}}_{x_1x_2}(\mathbf{X})\end{aligned}$$

Extra term comes from **measure zero diagonals.**

The varieties

Define $\mathcal{W}_{d,n}$ as the set of words of length n on the characters $\mathcal{W}_d = \{1, \dots, d\}$.
Define $\mathcal{M}_d$ as the set of non-constant monomials on $\{x_1, \dots, x_d\}$.
Height of a word $\vec{p} = (p_1, \dots, p_k)$ in $\mathcal{M}_d$ is $h(\vec{p}) = \sum_i \deg p_i$.

We define $\mathcal{I}_d$ to be the set of words in $\mathcal{M}_d$, and $\mathcal{I}_{d,h}$ the ones of height h. Do not mistake **height** of a word with its **lenght**, generally smaller.
The variety $\mathcal{V}_{d,h,N}$ is the closure of the image of $\boldsymbol{\mathcal{S}} : (\mathbb{R}^d)^{N+1} \to \mathbb{R}^{\mathcal{I}_{d,h}}$.
In this way, $\mathcal{V}_{d,h,N} \subset \mathbb{R}^{\mathcal{I}_{d,h}}$.

$$\mathcal{V}_{d,h,1} \subset \mathcal{V}_{d,h,2} \subset \cdots .$$

Let $\mathcal{V}_{d,n}$ be the limit of this chain.

The universal discrete signature variety

Some degree considerations

Theorem (Hoffman 2000)

Shuffle algebra and quasi-shuffle algebra are isomorphic via an exponential map

Theorem (Améndola, Friz and Sturmfels 2019)

The dimension of $U_{n,d}$ is $\lambda_{n,d} - 1$, where $\lambda_{n,d}$ is the number of Lyndon words on d characters of length at most n.

Theorem (Belingeri, P. and Sturmfels 2023)

The dimension of $V_{h,d}$ is $\mu_{n,d} - 1$, where $\mu_{n,d}$ is the number of Lyndon words on $\mathcal{M}$ of height at most h.

Some small cases

Let's see what we can

dim	$h=1$	$h=2$	$h=3$
$d=2$	2	7	24
$d=3$	3	15	73
$d=4$	4	26	164

$\mu_{d,h}$	$h=2$	$h=3$	$h=4$
$d=2$	2	4	12
$d=3$	3	9	36
$d=4$	4	16	80

Small cases

Biblio

- AmÃ©ndola, Carlos, Peter Friz, and Bernd Sturmfels. ”Varieties of signature tensors.” Forum of Mathematics, Sigma. Vol. 7. Cambridge University Press, 2019.
- Galuppi, Francesco. “The rough Veronese variety.” Linear algebra and its applications 583 (2019): 282-299.
- Diehl, Joscha, Ebrahimi-Fard, Kurusch, and Tapia, Nikolas. ”Time-warping invariants of multidimensional time series.” Acta Applicandae Mathematicae 170.1 (2020): 265-290.

Thank you