Pattern Hopf algebras on marked permutations and enriched set species

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In this talk we introduce pattern Hopf algebras in combinatorial structures. We start by considering the functions that count patterns. These pattern functions satisfy a product relation, and we are able to endow the linear span of pattern functions with a compatible coproduct. In this way, several combinatorial objects generate a Hopf algebra. For example, the Hopf algebra on permutations studied by Vargas in [6] is a particular case of this construction.

Questions of algebraic nature arise when dealing with these Hopf algebras: freeness, the character group, and so on. We discuss here the freeness of these Hopf algebras.

A particular case of such a Hopf algebra structure, defined on marked permutations, is of interest to us. These objects have an inherent multiplication structure that stems from the inflation operation on permutations, and this product operation is central in establishing that this Hopf algebra is a free algebra.

Introduction

The notion of substructures is important in mathematics, and particularly in combinatorics. In graph theory, minors and induced subgraphs are the main examples of studied substructures. Other objects also get attention in this topic: set partitions, trees, paths and, to a bigger extent, permutations, where the study of substructures leads us to the concept of a pattern in permutations.

A priori unrelated, Hopf algebras are a natural tool in algebraic combinatorics to study graphs, set compositions and permutations. For instance, the celebrated Hopf algebra on permutations named after Malvenuto and Reutenauer sheds some light on the structure of shuffles in permutations. Other examples of Hopf algebras in combinatorics are the word quasisymmetric functions with a basis indexed by set compositions, and the permutation pattern Hopf algebra introduced by Vargas in [6]. A seminal work on the interactions between combinatorics and Hopf algebras is [4].

With that in mind, we build upon the notion of species, as presented in [1] by Aguiar and Mahajan, in order to connect these two areas of algebraic combinatorics, introducing in [5] the notion of presheaf.

In the 1950's, Chen, Fox and Lyndon introduced to us, in [3], the concept of a Lyndon word, and used it to establish that the shuffle algebra on words is free. This gave our comunity a readily availabe tool to establish the freeness of a Hopf algebra, with a surprising amount of flexibility. Indeed, Vargas used this same method in [6] to establish the freeness of the pattern algebra on permutations.

The particular case of a pattern Hopf algebra that interests us is the one on marked permutations. We exploit the remarkable product structure that arises from the inflation of two marked permutations. The study of the freeness of the pattern Hopf algebra on marked permutations reduces to a unique factorisation theorem, presented below, and a careful tunning of the application of the theory of Lyndon words to establish the freeness of an algebra.

Pattern functions, cover numbers and the inflation of marked permutations

For us, a permutation in a set S is a pair of orders in S, as described in [2]. A marked permutation π^* on a set I is a pair of orders in $I \sqcup \{*\}$, that is a permutation on $I \sqcup \{*\}$, and its size is $|\pi^*| = \#I$. If $J \subseteq I$, then $\pi^*|_J$ is the restriction of the underlying permutation to $J \sqcup \{*\}$, and so is a marked permutation on the set J.

An occurence of a marked permutation π^* in another marked permutation τ^* is an occurence of the underlying permutation where the marked element on both permutations match. Equivalently, is a set $J \subseteq I$ such that $\pi^*|_J$ and τ^* are isomorphic marked permutations.

Consider the following marked permutations: $\pi_1^* = 3\bar{2}1, \pi_2^* = \bar{2}1, \pi_3^* = 14\bar{2}3$. Then π_2^* is a pattern in π_1^*, but not in π_3^*.

Let us write $\mathbf{p}_{\pi^*}(\tau^*)$ for the number of occurences of π as a pattern in τ. Then, in the previous examples, we have that $\mathbf{p}_{\pi_2^*}(\pi_1^*) = 1$ and $\mathbf{p}_{\pi_2^*}(\pi_3^*) = 0$.

Our first observation, is that these pattern functions have a product formula.

Proposition 1. *Let π_1, π_2^*, τ^* be marked permutations, then*

$$\mathbf{p}_{\pi_1^*}(\tau^*)\,\mathbf{p}_{\pi_2^*}(\tau^*) = \sum_{\sigma^*} \binom{\sigma^*}{\pi_1^*, \pi_2^*} \mathbf{p}_{\sigma^*}(\tau^*)\,, \tag{1}$$

where the coefficients $\binom{\sigma^*}{\pi_1^*,\pi_2^*}$ have an explicit combinatorial interpretation, and where the sum runs over marked permutations σ^* up to isomorphism, with size at most $|\pi_1^*| + |\pi_2^*|$. In particular, span$\{\mathbf{p}_{\pi^*}\,|\pi^*$ marked permutation$\} = \mathcal{A}(\texttt{MPer})$ is a graded algebra.

If π_1^*, π_2^* are two marked permutations, we can define the inflation product $\pi_1^* * \pi_2^*$ of these marked permutations by inflating the marked element of π_1^* with the marked permutation π_2^*. In fig. 1, we have a graphical expression of $1\bar{3}2 * \bar{2}1$.

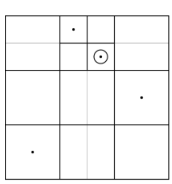

Figure 1: The inflation product of the marked permutations $1\bar{3}2$ and $\bar{2}1$ is $14\bar{3}2$.

A marked permutation π^* is called simple when any factorisation $\pi^* = \tau_1^* \cdot \tau_2^*$ has $\tau_1^* = \bar{1}$ or $\tau_2^* = \bar{1}$. For example $\bar{1}423$ is a simple marked permutation, although it has a decomposition as an $\oplus$ product.

We remark that the inflation product in marked permutations is not commutative. However, factorisations into simples are not unique. For instance, both $\bar{1}32$ and $21\bar{3}$ are simple, but $\bar{1}32 * 21\bar{3} = 21\bar{3} * \bar{1}32$. This is an example of a *two factor transposition*.

Define the coproduct $\Delta\, \mathbf{p}_{\pi^*} = \sum_{\pi^* = \tau_1^* \cdot \tau_2^*} \mathbf{p}_{\tau_1^*} \otimes \mathbf{p}_{\tau_2^*}$ in $\mathcal{A}(\texttt{MPer})$. This coproduct is compatibe with the product of pattern functions. With this, $\mathcal{A}(\texttt{MPer})$ is a Hopf algebra with a basis indexed by marked permutations.

Proposition 2. *Let α^* be a marked permutation with two factorisations $a_1^* * \cdots * a_k^*$ and $b_1^* * \cdots * b_j^*$ into simple marked permutations. Then these factorisations can be obtained from one to the other by explicit two factor transpositions. In particular, $k = j$ and the simple factors are the same up to order.*

The freeness follows via a careful application of the theory of Lyndon words.

Theorem 3. *The pattern algebra on marked permutations is free commutative.*

Other pattern algebras

The process of building a Hopf algebra out of combinatorial structures with a restriction and a product, that we went through above, is quite general.

For instance, the family of graphs with the disjoint union is a monoidal presheaf, and the family of permutations with the $\oplus$ product is one as well. It turns out that having a commutative product is enough to guarantee a unique factorisation theorem, hence:

Theorem 4. *Any pattern algebra with a commutative monoidal structure is a free commutative algebra.*

The surprising feature of this result is that the coalgebra structure, which is the one that pertains the monoidal strucutre, dictates whether the algebra structure is free. As a consequence, pattern algebras in graphs, posets and matroids are free.

References

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