The tropical critical points of an affine matroid

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Raul Penaguiao

MPI MiS Leipzig

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Slides can be found at `raulpenaguiao.github.io/`
Joint work with Federico Ardila and Chris Eur

Optimization of a monomial

Fix some vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$.
Optimize $f_{\mathbf{w}} : \mathbf{x} \mapsto x_1^{w_1} \cdots x_n^{w_n}$ on a variety $X \subset (\mathbb{C}^*)^n$.

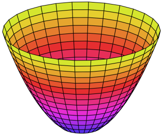


Figure: A variety where we can optimize $f_{\mathbf{w}}$

What is the number of critical points of f? Does it depend on the choice of $\mathbf{w}$? For generic $\mathbf{w}$, no!
This number is called the **maximum likelihood degree** of a model X.
If X is a vector space, MLDeg$(X) = \beta(M(V))$.

Edge weight problem

Given $G = (V, E)$, fix some vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$.

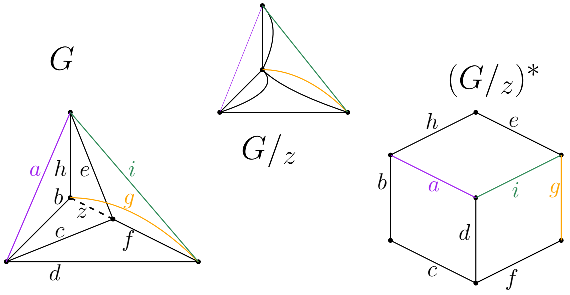


Figure: Find $\mathbf{x}$ and $\mathbf{y}$ edge weights that are *compatible* with G and $(G \setminus z)^*$.

- The sum of the weights is $\mathbf{w}$.
- **(Compatible)** Every cycle has at least two minimal edges.

Fix $\mathbf{w} = (0, 1, 1, 2, 2, 5, 3, 4, 7)$.

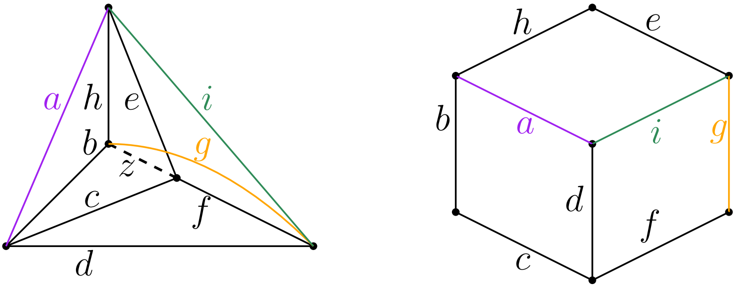


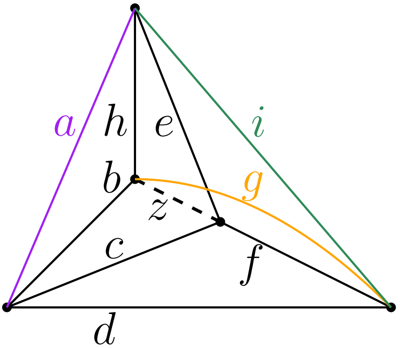
Figure: Find $\mathbf{x}$ and $\mathbf{y}$ edge weights that are *compatible* with G and $(G \setminus z)^*$.

	a	b	c	d	e	f	g	h	i
$\mathbf{x}$	00	01	00	00	02	03	00	00	02
$\mathbf{y}$	000	110	111	222	20	52	33	44	75
$\mathbf{w}$	0	1	1	2	2	5	3	4	7

1. Introduction
2. Matroids
3. The Bergman Fan
4. Degree of Bergman Fan

Graphical matroid

Given a graph $G = (V, E)$, the collection of edges E forms a matroid.



Independent sets $\mapsto$ forests

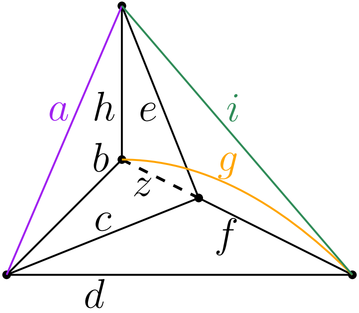
Basis $\mapsto$ Spanning forests

Circuits $\mapsto$ Simple cycles

Rank of set $A \subseteq E$ $\mapsto$ size of largest spanning forest

Flats

Maximal sets with a fixed rank.
That is, F is a flat if for any $i \notin F$, $r_M(F \cup i) > r_M(F)$.



$\emptyset$, matchings , complete subgraphs , . . .

$$\{\emptyset \subsetneq a \subsetneq za \subsetneq zafg \subsetneq zabcdefghi\}$$

The uniform matroid

Basis of the uniform matroid $U_{n,k}$ = all sets of size k in $[n]$.
Any set of size $\leq k$ is independent.

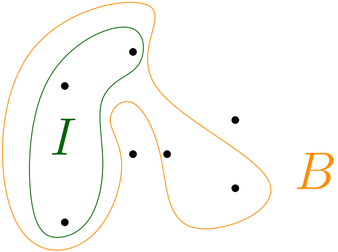


Figure: Matroid $U_{7,5}$ along with a basis B and independent set I.

Any set of size $\leq k-1$ is a flat.
Any complete flag of flats is of the form

$$\{\emptyset \subsetneq \{v_1\} \subsetneq \{v_1, v_2\} \subsetneq \cdots \subsetneq \{v_1, \ldots, v_{k-1}\} \subsetneq [n]\}$$

The Bergman Fan

$$\Sigma_M := \{\vec{x} \in \mathbb{R}^n | \forall \text{ circuits } C \text{ we have } \min_{c \in C} x_c \text{ is attained twice } \}.$$

$$\Sigma_M := \{\vec{x} \in \mathbb{R}^n /_{\mathbb{1}\mathbb{R}} | \forall \text{ circuits } C \text{ we have } \min_{c \in C} x_c \text{ is attained twice } \}.$$

If $M = U_{n,k}$, any set of size $k+1$ is a circuit.
A point $\vec{x} \in \mathbb{R}^n$ is on the Bergman fan if $\#\{i \in E | x_i > \min \vec{x}\} \leq k-1$.

$$\Sigma_M = \bigcup_{|I|=n-k+1} \{\vec{x} \in \mathbb{R}^n /_{\mathbb{R}\mathbb{1}} | \arg\min \vec{x} \subseteq I\}.$$

$$\Sigma_{U_{3,2}} = \{(a,a,b) | a \leq b\} \cup \{(a,b,a) | a \leq b\} \cup \{(b,a,a) | a \leq b\}.$$

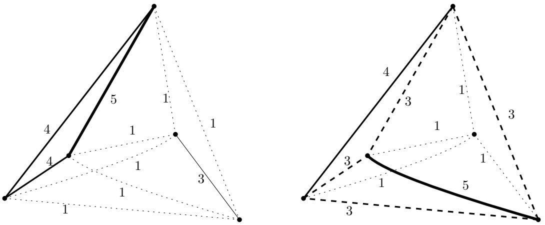


Figure: Two elements in the Bergman fan of the graphical matroid

Theorem (Sturmfels and Feichner, 2004)

The Bergman Fan of a matroid decomposes into the following cones

$$\Sigma_M = \bigcup_{\mathcal{F} \text{ flag of flats}} \mathcal{C}_{\mathcal{F}} = \bigcup_{\substack{F_1 \subset \cdots \subset F_k \\ \text{flag of flats}}} \{x_i \geq x_j \text{ whenever } i \in F_k, j \notin F_k\} .$$

Problem

Can we compute the degree of the tropical variety Σ_M?

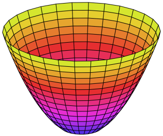


Figure: A variety X: Degree $= \#$ of intersections with a **line in** $\mathbb{C}^n$.

Just intersect it with a hyperplane with dimension $= n - \dim X$
Note: Hyperplanes in the tropical world are Bergman fans of $U_{n,k}$.

Example of degree computation

Consider M the graphical matroid of K_5, of rank 4 (so $\dim \Sigma_M = 3$). One has $\Sigma_M \subseteq \mathbb{R}^{10}/_{\mathbb{R}\mathbb{1}}$, we intersect it with a hyperplane of $dim = 6$.

$$|\Sigma_{U_{10,7}} + \underbrace{(1, 10, 100, 1000, ..., 10^9)}_{\vec{\omega}} \cap \Sigma_M| = ?$$

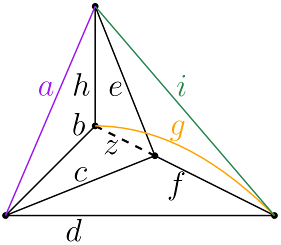


Figure: There is only one $\mathbf{x} \in \Sigma_M$ and $\mathbf{y} \in \Sigma_U$ such that $\mathbf{y} + \mathbf{w} = \mathbf{x}$. Such vector $\mathbf{x}$ belongs to a **cone determined by the greedy basis of** M.

Activities

Fix total order in V, ground set of a matroid M, basis B.

- $e \in B$ is internal activity if $e = \min C^{\perp}$, where $C^{\perp} \subseteq B^c \cup e$ is a cocircuit (a cut of the graph).
- $e \notin B$ is external activity if $e = \min C$, where $C \subseteq B \cup i$ is a circuit.

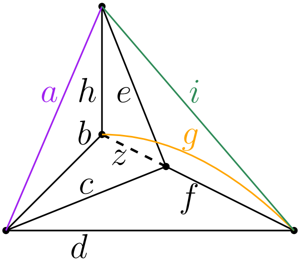


Figure: Fix order $z < a < b < c < d < e < f < g < h < i$.

$i(befg) = 0, \ e(befg) = 2, \ i(zadf) = 2, \ e(zadf) = 1$

$i(B) = 0$ nbc basis. $i(B) = 0$ and $e(B) = 1$ β-nbc basis

Degree computations

Theorem (Greedy basis algorithm)

$$\deg(\Sigma_M) = \Sigma_M \cap (\mathbf{w} + \Sigma_{U_{n,n-k-1}}) = 1$$

Theorem (Adiprasito, Huh, Katz, 2018)

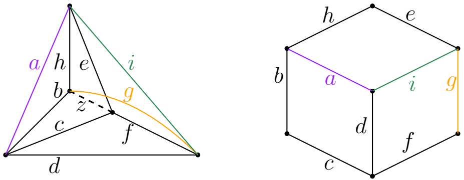
$$\deg(-\Sigma_M) = (-\Sigma_M) \cap (\mathbf{w} + \Sigma_{U_{n,n-k-1}}) = \#\{\textit{nbc bases}\ \}$$

Theorem (Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, Telen 2021 and Ardila, Eur, P 2022)

$$\deg(\Sigma_{(M/_0)^{\perp}} \cdot -\Sigma_M) = (-\Sigma_M) \cap (\mathbf{w} + \Sigma_{(M/_0)^{\perp}}) = \#\{\beta - \textit{nbc bases}\ \}$$

Finding ~~Nemo~~ points form nbc bases

$\deg(\Sigma_{(M/_0)^\perp} \cdot -\Sigma_M) =?$ Fix generic $\mathbf{w} = (10^{i-1})_i$.
Find $\mathbf{x} \in \Sigma_M$, $\mathbf{y} \in \Sigma_{(M/_0)^\perp}$ such that $\mathbf{x} + \mathbf{y} = \mathbf{w}$.



Find a β-nbc basis $B = zbeg \to g|e|bd|zacfhi$,

corresponding β-nbc cobasis $B^\perp = acdfhi \to i|h|f|gd|c|eba$.

	a	b	c	d	e	f	g	h	i
$\mathbf{x}$	0	9	0	9	$10^4 - 1$	0	$10^6 - 10^3 + 9$	0	0
$\mathbf{y}$	1	1	100	$10^3 - 9$	1	10^5	$10^3 - 9$	10^7	10^8
$\mathbf{w}$	1	10	100	10^3	10^4	10^5	10^6	10^7	10^8

Biblio

- Agostini D., Brysiewicz T., Fevola C., Kühne L., Sturmfels B., Telen S. (2021). *Likelihood Degenerations*. Motivation behind ML degree computations
- Adiprasito K., Huh J., Katz E. (2018). *Hodge Theory for combinatorial geometries* Degree computations in matroids
- Ardila F., Eur C., RP (2022) *The maximum likelihood of a matroid*.

Thank you