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Pattern Hopf algebras, antipode and reciprocity

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Abstract. The permutation pattern Hopf algebra is a commutative filtered and connected Hopf algebra. Its product structure stems from counting patterns of a permutation. The Hopf algebra was shown to be a free commutative algebra and to fit into a general framework of pattern Hopf algebras, via species with restrictions. In this work we introduce the cancellation-free and grouping-free formula for the antipode of the permutation pattern Hopf algebra, via the sign-reversing involution method. This formula has applications on polynomial invariants on permutations, in particular for obtaining reciprocity theorems. In fact, we study a brand new polynomial invariant on permutations amenable to this method. On our way, we also introduce the packed word patterns Hopf algebra and present a formula for its antipode. Species in parking functions are also introduced.

Keywords: permutations, presheaves, species with restrictions, species, Hopf algebras, free algebras, antipode, cancellation-free, chromatic, reciprocity.

1 Introduction

In his now celebrated "Lemma 14", Takeuchi (see [17], Lemma 14) obtained a quite general formula for the antipode of a Hopf algebra. This is an antipode formula for any filtered Hopf algebra, which can be applied in much generality and fits the framework of **pattern Hopf algebras**, introduced in [12]. However, it has been observed that it is not the most economical formula, as it leaves some cancellations to be made.

Economical formulas for antipodes in Hopf algebras in combinatorics have played an important role in extracting old and new combinatorial equations, see [2, 5, 9, 15, 19]. In particular, with such a formula, Humpert and Martin were able to explain, in [9], an elusive *reciprocity relation* on graphs first presented by Stanley in [16]. A method for obtaining a cancellation-free formula that seem to work with a large family of Hopf algebras was brought forth by Sagan and Benedetti in [5], called *sign-reversing involution method*, which we leverage here.

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Parallel to this development is the study of permutation patterns. This is a study with roots in computer science, pioneered by Knuth in [10], where a description for the *stack sortable* permutations was presented via permutation patterns. In the meantime, permutation patterns established as an area of expertise in combinatorics, see [11].

The second author, in [18], using the classical notion of restriction $\tau|_I$ of a permutation, introduced yet another tool to study permutation patterns, by building the permutation pattern Hopf algebra. Specifically, if we consider finite sums of functions of the form

$$\binom{\tau}{\pi} := \mathbf{pat}_\pi(\tau) := \#\{ \text{ sets } I \text{ such that } \pi = \tau|_I \} ,$$

the central functions in the study of permutation patterns, we span a vector space that is closed for pointwise product:

$$\mathbf{pat}_{\pi_1} \mathbf{pat}_{\pi_2} = \sum_\sigma \binom{\sigma}{\pi_1, \pi_2} \mathbf{pat}_\sigma \ . \tag{1.1}$$

The coefficients $\binom{\sigma}{\pi_1, \pi_2}$ that arise in this product formula count the so called **quasi-shuffle signatures**, or **QSS**, of σ from π_1, π_2. We precise this definition in Definition 1.1

The algebra corresponding to this pointwise product is the **permutation pattern Hopf algebra** $\mathcal{A}(\texttt{Per})$, and is shown to be free in [18]. The construction was generalized to other combinatorial objects, in [12], as long as there is a notion of restrictions, for instance graphs or marked permutations. The resulting structures are called **pattern Hopf algebras**. Some known Hopf algebras can be constructed as the pattern algebra of a species with restrictions. An example is Sym, the Hopf algebra of *symmetric functions*, which is free. It is conjectured that all pattern Hopf algebras are free.

In this paper, which is an extended abstract of [13] and [14], we provide a cancellation-free and grouping-free antipode formula for the pattern Hopf algebras on permutations and on packed words. This is an application of the sign-reversing involution method. We also present an original species with restrictions structure on parking functions. This is part of a project to interpret common combinatorial objects as species with restrictions.

1.1 The permutation pattern Hopf algebra and the main result

Definition 1.1 (QSS on permutations)**.** A QSS of σ from $\pi_1, \dots, \pi_n$ is a tuple $\vec{\mathbf{I}} = (I_1, \dots, I_n)$ of sets on the ground set of the permutation σ, that cover this ground set and that the restricted permutation $\sigma|_{I_i}$ is exactly a pattern of π_i.

It was shown in [12] that $\binom{\sigma}{\pi_1, \dots, \pi_n}$, the coefficient that arises in the iterated product of n elements in $\mathcal{A}(\texttt{Per})$, counts the number of ways of covering σ with n permutations, each fitting the patterns $\pi_1, \dots, \pi_n$. This algebra $\mathcal{A}(\texttt{Per})$ can be endowed with a Hopf algebra structure with the help of the diagonal sum of permutations, $\oplus$, also called *shifted* *concatenation of permutations*. If one lets $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ be the decomposition of π into $\oplus$-indecomposable permutations under the $\oplus$ product, we define the coproduct:

$$\Delta\, \mathbf{pat}_\pi = \sum_{k=0}^{n} \mathbf{pat}_{\pi_1 \oplus \cdots \oplus \pi_k} \otimes \mathbf{pat}_{\pi_{k+1} \oplus \cdots \oplus \pi_n} \in \mathcal{A}(\mathtt{Per}) \otimes \mathcal{A}(\mathtt{Per})\,.$$

To present the antipode formula we need to refine the notion of QSS.

Definition 1.2 (Interlacing QSS on permutations)**.** A QSS $\vec{\mathbf{I}} = (I_1, \ldots, I_n)$ of σ is said to be **non-interlacing** if there is some $i = 1, \ldots, n-1$ such that $I_i < I_{i+1}$ and $\sigma(I_i) < \sigma(I_{i+1})$. Otherwise, we say that the QSS is **interlacing**.

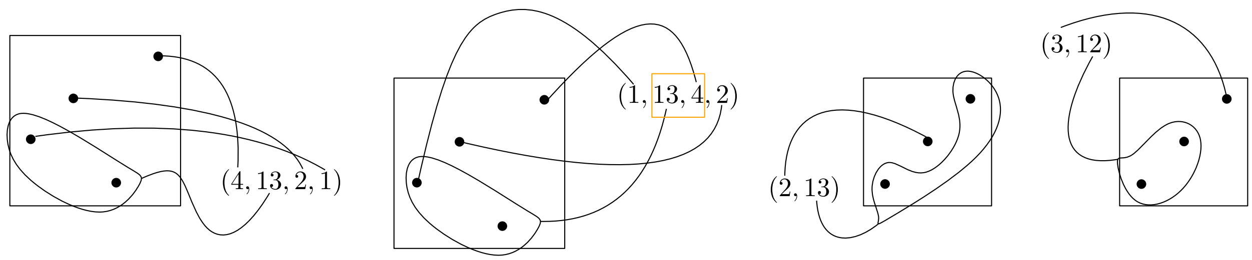


Figure 1: From left to right: the permutation 2314, along with a QSS from 1, 21, 1, 1. Another QSS, this one is not interlacing: in orange the two sets that do not interlace. The permutation 123, along with its two interlacing QSS from 1, 12.

Example 1.3 (Interlacing QSS)**.** The permutation 2314 has several QSS from 1, 21, 1, 1, for instance $(4, 13, 2, 1)$ and $(1, 13, 4, 2)$, but from these two, only the first is interlacing. In Figure 1, one can see these two QSS. Further computations can show that

$$\binom{2314}{1,21,1,1} = 36 \quad \text{and} \quad \begin{bmatrix} 2314 \\ 1,21,1,1 \end{bmatrix} = 8.$$

One can observe that there are three QSS of 123 from 1, 12, but one of them is non-interlacing (the QSS $(1, 23)$), so $\left[\begin{smallmatrix} 123 \\ 1,21 \end{smallmatrix}\right] = 2$. In Figure 1, one can see these two QSS.

Theorem 1.4 (Cancellation-free and groupping-free antipode formula for Permutation pattern Hopf algebra)**.** Let $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ be the decomposition of a permutation π into $\oplus$-indecomposable permutations. Then, we have the following formula for the antipode of $\mathbf{pat}_\pi$:

$$S(\mathbf{pat}_\pi) = (-1)^n \sum_\sigma \begin{bmatrix} \sigma \\ \pi_1, \ldots, \pi_n \end{bmatrix} \mathbf{pat}_\sigma\,,$$

where the sum runs over all permutations σ, and the coefficients count the number of **interlacing QSS** of σ from $\pi_1, \ldots, \pi_n$.

Example 1.5. Consider $\pi = 132 = 1 \oplus 21$. It comes from the antipode axioms that the antipode on $\mathbf{pat}_{132}$ is:

$$S(\mathbf{pat}_{132}) = \mathbf{pat}_{132} + \mathbf{pat}_1 \, \mathbf{pat}_{21} = 3 \, \mathbf{pat}_{321} + 2 \, \mathbf{pat}_{231} + 2 \, \mathbf{pat}_{312} + \mathbf{pat}_{213} + 2 \, \mathbf{pat}_{21} \, .$$

These coefficients can be seen as enumerating quasi-shuffle signatures of 132 from 1 and 12 that are **interlacing**, according to Theorem 1.4.

Similarly to the case of graphs, the antipode formula in (1.4) begets reciprocity results, see Section 5. Also, the notion of *bialgebra in cointeraction* (c.f. [8]) can also be used to obtain antipode formulas for the pattern Hopf algebra. This is discussed in [14].

2 The pattern Hopf algebra

2.1 Species with restriction

The general setting for our approach to patterns is given by the notion of *species with restrictions*, a terminology due to Schmitt (see [15]) and used by the first author in[1] [12].

Let $\mathbb{K}$ be a field of arbitrary characteristic. We denote by $\mathsf{Set}^{\times}$, $\mathsf{Vect}_{\mathbb{K}}$ and $\mathsf{Set}^{\times}_{\hookrightarrow}$ the category of finite sets and bijections between finite sets, the category of $\mathbb{K}$-vector spaces and linear maps between vector spaces and the category of finite sets with injections as morphisms, respectively. **Set species**, **vector species** and **species with restriction** are defined respectively as contravariant functors $\mathrm{P} : \mathsf{Set}^{\times} \to \mathsf{Set}$, $\mathbf{p} : \mathsf{Set}^{\times} \to \mathsf{Vect}_{\mathbb{K}}$ and $\mathtt{R} : \mathsf{Set}^{\times}_{\hookrightarrow} \to \mathsf{Set}$. Given a species with restrictions $\mathtt{R}$ and two finite sets I, J such that $J \subseteq I$, the **restriction map** $\mathrm{res}_{I,J} : \mathtt{R}[I] \to \mathtt{R}[J]$ is the image under the functor $\mathtt{R}$ of the inclusion $J \hookrightarrow I$, so $\mathrm{res}_{I,J} := \mathtt{R}[\hookrightarrow]$. By functoriality, these maps satisfy the contravariant axioms

$$\mathrm{res}_{J,K} \circ \mathrm{res}_{I,J} = \mathrm{res}_{I,K} \qquad , \qquad \mathrm{res}_{I,I} = \mathrm{id}_{\mathtt{R}[I]}, \tag{2.1}$$

for any finite sets $I \supseteq J \supseteq K$. Set species, vector species and species with restrictions form categories Sp, $\mathsf{Sp}_{\mathbb{K}}$ and Spr, respectively. In all those categories, the Cauchy product is denoted by $\cdot$. This allows us to define **associative species with restrictions** in the usual way see [12].

Given a species with restrictions $\mathtt{R}$, we can construct a linearized comonoid in $(\mathsf{Sp}, \cdot)$ as follows: Let $\mathbf{r} = \mathbb{K}\mathtt{R}$ be the linearization of $\mathtt{R}$. Given a decomposition $I = S \sqcup T$, consider the linear map $\Delta_{S,T} : \mathbf{r}[I] \to \mathbf{r}[S] \otimes \mathbf{r}[T]$ given by

$$\Delta_{S,T}(x) := \mathrm{res}_{I,S}(x) \otimes \mathrm{res}_{I,T}(x), \tag{2.2}$$

for any $x \in \mathtt{R}[I]$. Let $\epsilon_{\emptyset} : \mathbf{r}[\emptyset] \to \mathbb{K}$ the linear extension of the map sending every element of $\mathtt{R}[\emptyset]$ to 1. Hence, $(\mathbf{r}, \Delta, \epsilon)$ is a linearized comonoid.

[1] In this work, species with restrictions are called *combinatorial presheaves*.

Consider now a linearized comonoid $\mathbf{p} = \mathbb{K}\mathrm{P}$ in $(\mathsf{Sp}, \cdot)$. That is a comonoid in set species that has no notion of restriction. In this case, the coproduct gives a pure tensor

$$\Delta_{S.T}(x) = x|_S \otimes x/_S,$$

for each $x \in \mathrm{P}[I]$ and for each decomposition $I = S \sqcup T$. We may then define restriction maps on $\mathbf{p}$ either by the rule $x \mapsto x|_J$ or $x \mapsto x/_{I\setminus J}$ for $x \in \mathbf{p}[I]$. Each of this two restriction maps turns $\mathbf{p}$ into a species with restriction. When $\mathbf{p}$ is cocommutative, then both restriction maps coincide. We have then the following characterization of monoids of species with restrictions (see [3], Proposition 8.29, for other characterizations).

Theorem 2.1 (Schmitt, Aguiar-Mahajan)**.** There is an equivalence between the category of monoids in $(\mathsf{Spr}, \cdot)$ and the category of linearized cocommutative bimonoids.

2.2 Pattern functions and the pattern Hopf algebra

Given a species with restrictions $\mathtt{R}$ and two finite sets I and J, two objects $a \in \mathtt{R}[I], b \in \mathtt{R}[J]$ are said to be **isomorphic objects**, or $a \sim b$, if there is a bijection $\sigma : I \to J$ such that $\mathtt{R}[\sigma](b) = a$. The collection of equivalence classes of a species with restrictions $\mathtt{R}$ is denoted by $\mathcal{G}(\mathtt{R}) := \bigcup_{n\geq 0} \mathtt{R}[n]_{\mathfrak{S}_n}$. In this way, the set $\mathcal{G}(\mathtt{R})$ is the collection of all the $\mathtt{R}$-objects up to isomorphism. Recall that for every two finite set I, J such that $J \subseteq I$, there is a restriction map $\mathrm{res}_{I,J} : \mathtt{R}[I] \to \mathtt{R}[J]$, defined as the image by $\mathtt{R}$ of the map $J \hookrightarrow I$. If $b \in \mathtt{R}[I]$, we write $b|_J := \mathrm{res}_{I,J}(b)$.

Definition 2.2 (Patterns coefficients)**.** Let $\mathtt{R}$ be a species with restrictions. Given two finite sets I, J such that $J \subseteq I$ and two objects $b \in \mathtt{R}[I], a \in \mathtt{R}[K]$, we say that the subset $J \subseteq I$ is a **pattern** of a in b if $b|_J \sim a$. More precisely, $J \subseteq I$ is a pattern of a in b if there exists a bijection $\sigma : J \to K$ such that $\mathtt{R}[\sigma](a) = \mathrm{res}_{I,J}(b)$. The **pattern coefficient** of a in b is

$$\binom{b}{a}_{\mathtt{R}} := \mathbf{pat}_a(b) := \left|\{J \subseteq I \,:\, b|_J \sim a\}\right| . \tag{2.3}$$

From [12], this number only depends on the isomorphism classes of $a \in \mathtt{R}[J]$ and $b \in \mathtt{R}[I]$.

Definition 2.3 (Patterns functions and patterns spaces)**.** Let $\mathtt{R}$ be a species with restrictions. Given a finite set I, we define the **pattern function** associated to $a \in \mathtt{R}[I]$ as the function $\mathbf{pat}_a : \mathcal{G}(\mathtt{R}) \to \mathbb{K}$ given by $b \mapsto \mathbf{pat}_a(b)$, for all $b \in \mathcal{G}(\mathtt{R})$. The linear span of the pattern functions is denoted by $\mathcal{A}(\mathtt{R}) := \mathbb{K}\{\mathbf{pat}_a \,:\, a \in \mathcal{G}(\mathtt{R})\}$.

Theorem 2.4 ([12])**.** Let $(\mathtt{R}, \square, 1)$ be an associative species with restrictions. The space $\mathcal{A}(\mathtt{R})$ is closed under pointwise multiplication and has a unit. It forms an algebra, called the **pattern algebra associated to** $\mathtt{R}$. More precisely, if $a, b \in \mathcal{G}(\mathtt{R})$,

$$\mathbf{pat}_a\, \mathbf{pat}_b = \sum_c \binom{c}{a, b}_{\mathtt{R}} \mathbf{pat}_c\,, \tag{2.4}$$

where the coefficients $\binom{c}{a,b}_{\mathtt{R}}$ are the number of "quasi-shuffles" of a, b that result in c, specifically, if we take $c \in \mathtt{R}[C]$ to be a representative of the equivalence class c, then:

$$\binom{c}{a,b}_{\mathtt{R}} = \left|\{(I,J) \text{ such that } I \cup J = C\,,\ c|_I \sim a,\, c|_J \sim b\}\right| .$$

If we also consider the following coproduct in the pattern algebra $\mathcal{A}(\mathtt{R})$:

$$\Delta_{\square}\, \mathbf{pat}_a = \sum_{\substack{b,c\,\in\,\mathcal{G}(\mathtt{R}) \\ a=b\,\square\, c}} \mathbf{pat}_b \otimes \mathbf{pat}_c\,, \tag{2.5}$$

then the pattern algebra of $\mathtt{R}$ together with the coproduct $\Delta_{\square}$, and a natural choice of counit, forms a bialgebra. If additionally $|\mathtt{R}[\emptyset]| = 1$, it forms a filtered Hopf algebra.

2.3 Species on linear orders, permutations and packed words

The set species of linear orders, $\mathtt{L}$ is defined has $\mathtt{L}[I] = \{$ total orders in I $\}$. This can be endowed with a notion of restriction and product quite canonically.

We will generalize the framework used in [4], where a rather unusual definition of permutations was introduced, in order to defines our species with restriction on permutations and packed words. First, we call a linear partial order $\leq$ on a set I the pullback of a surjective map $I \to [m]$. We call this function the rank of $\leq$, or $rk_{\leq}$. We abuse notation and also call the integer m the rank of the order $\leq$.

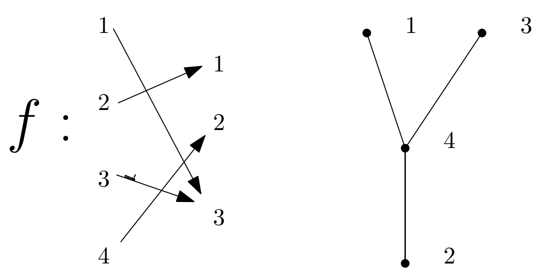


Figure 2: Left: If $f = \{a \mapsto 3, b \mapsto 1, c \mapsto 3, d \mapsto 2\}$ is a surjective map $\{a,b,c,d\} \to [3]$, the inherited order is $\{b < d < \{a,c\}\}$ and has rank three. **Right:** The Hasse diagram of the linear partial order inherited by f.

In this way, a permutation π on I is a pair of total orders $(\leq_P, \leq_V)$, and a packed word ω on I is a pair of orders $(\leq_P, \leq_V)$ where $\leq_P$ is a total order in I, and $\leq_V$ is a linear partial order on I. In particular, note that any permutation on I can be seen as a packed word on I, as any total order is a linear partial order. This relates to the usual notion of packed words as a word in $[m]$ in the following way: if we order the elements of $I = \{a_1 \leq_P \cdots \leq_P a_k\}$, then the corresponding packed word is

$rk_{\leq_V}(a_1) rk_{\leq_V}(a_2) \dots rk_{\leq_V}(a_k)$. Conversely, for any packed word $\omega = p_1 \dots p_k$, there are several pairs of isomorphic orders $(\leq_P, \leq_V)$ corresponding to the word ω. If $I \hookrightarrow J$, by restricting the orders on I to orders on J, we obtain a restriction to a packed word on J, which results on species with restrictions structures denoted by `Per` and `PWo`.

It will be useful to represent packed words ω in I as rectangle diagrams labeled by I. This is done in the following way: let $1 \leq m \leq |I|$ be the rank of ω, we place the elements of I in an $m \times |I|$ grid so that the elements are placed horizontally according to the $\leq_P$ order, and vertically according to the $\leq_V$ order.

$$\begin{array}{|c|c|c|}\hline & & 3 \\ \hline 1 & & \\ \hline & 2 & \\ \hline\end{array} \quad , \quad \begin{array}{|c|c|c|c|c|}\hline & e & & & b \\ \hline & & & a & \\ \hline d & & c & & \\ \hline\end{array} \; .$$

Figure 3: Left: permutation $\pi = \{1 <_P 2 <_P 3, 2 <_V 1 <_V 3\}$ in $\{1,2,3\}$. **Right**: packed word $\omega = 13123 = \{d > e > c > a > b, \{b,e\} > \{a\} > \{d,c\}\}$ in $\{a,b,c,d,e\}$.

If $f : J \to I$ is an injective map, the preimage of each order $\leq_P, \leq_V$ is well defined. Furthermore, the preimage of $\leq_P$ is a total order on J, whereas the preimage of $\leq_V$ is a total order on J. Thus, this defines the packed word $\texttt{PWo}[f](\omega)$. The $*$ operation on partial orders can be extended to a **diagonal sum** operation $\oplus$ on packed words. This is the shifted concatenation of packed words. This endows both `Per` and `PWo` with an associative species with restrictions structure.

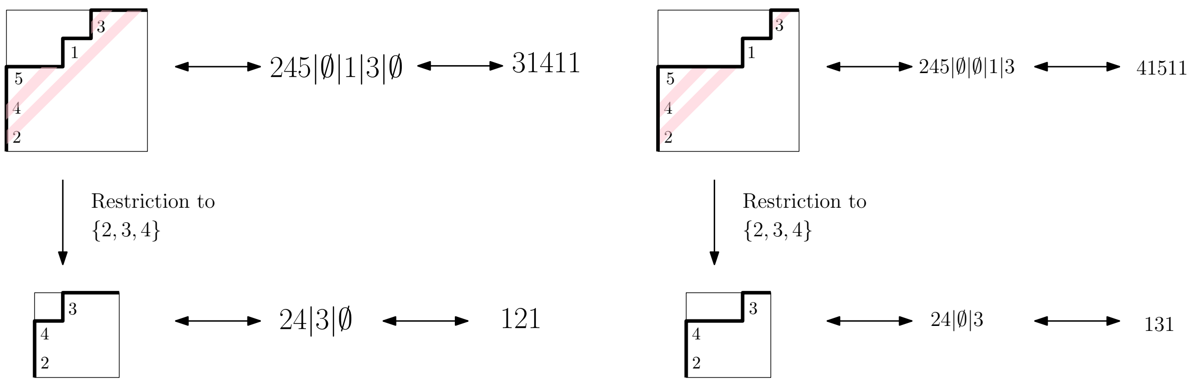
2.4 Species on parking functions

A **parking function** $\mathfrak{p} = a_1 \dots a_n$ is a sequence of integers in $[n]$ such that, after reordering $a^{(1)} \leq a^{(2)} \leq \dots \leq a^{(n)}$, we have $a^{(i)} \leq i$ for all i. Given an $n \times n$ square grid, a **Dyck path** on this square is an edge path, from the lower left corner to the upper right corner, that is always above the main diagonal. It is a classical result that the number of Dyck paths of size n is enumerated by Catalan numbers.

In [13] we describe a structure of species with restrictions on parking functions `PF`, by leveraging a recent construction of parking functions via labelled Dyck paths, see [6]. This is done via a notion of **tunnels**, introduced by Deutsch and Elizalde in [7], and is described in Figure 4. In particular, it recovers a notion of patterns in parking functions studied in [1]. One further defines a shifted concatenation $\oplus$ on parking functions.

The following can be established by the same methods presented in [12].

Proposition 2.5 (Species with restrictions on parking functions)**.** The species `PF` forms a species with restrictions structure. Furthermore, $\oplus$ endows `PF` with a monoid structure, and the resulting Hopf algebra $\mathcal{A}(\texttt{PF})$ is NCF, therefore it is free (see Lemma 3.2).



(a) The parking function 31411 and its restriction

(b) The parking function 41511 and its restriction

Figure 4

3 $\mathtt{L} \times -$ species with restrictions

In [12, Corollary 3.4.] it was shown that in species with restrictions $\mathtt{h}$, any factorization into $*$-indecomposibles is unique up to order. In some species with restrictions, we can also drop the "up to order" adjective, as there is exactly one factorization into $*$-indecomposibles full stop. We precise this in the *non-commuting factorization* definition.

Definition 3.1 (Non-Commuting Factorization on pattern Hopf algebras)**.** A monoidal species with restrictions is called a **non-commuting factorization** species, or simply an NCF species, if any element x has a unique factorization into $*$-indecomposibles elements $x = x_1 * \cdots * x_n$.

Lemma 3.2 (Linear species with restrictions have NCF, [13], Lemma 6.2)**.** Let $\mathtt{R}$ be a connected species with restrictions. Then $\mathtt{L} \times \mathtt{R}$ has NCF. Furthermore, $\mathcal{A}(\mathtt{R})$ is free.

The last part of the above lemma is deduced from [18], after we recognize that $\mathtt{L} \times \mathtt{R}'$ has NCF. In the following, we will refer to associative and connected species of the form $\mathtt{L} \times \mathtt{R}$ as **ordered species with restrictions**. The pattern Hopf algebra on permutations, on packed words and on parking functions all satisfy the NCF property. In fact, permutations, packed words and parking functions are of the form $\mathtt{L} \times \mathtt{R}$. This property allows for an easy manipulation of the coproduct, and results in a more tractable approach to the antipode formula.

Definition 3.3 (Composition poset and cumulative sum)**.** Recall that we write $\mathcal{C}_n$ for the set of compositions of size n. We will define the bijection $\mathbf{CS} : \mathcal{C}_n \to 2^{[n-1]}$, as follows. If $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathcal{C}_n$, define $f_i = \sum_{j=1}^{i} \alpha_j$ and $\mathbf{CS}(\alpha) := \{f_i \,|\, i = 1, \dots, \ell - 1\}$.

This bijection allows us to define an order $\leq$ in $\mathcal{C}_n$, via the pullback from the boolean poset in $2^{[n-1]}$. This order can be defined as well as follows: we say that $\alpha \leq \beta$ if α arises from β after merging and adding consecutive entries.

Definition 3.4 (Compositions and QSS)**.** Consider again $\mathtt{R}$ an ordered species with restrictions. Let $x \in \mathtt{R}[J], y \in \mathtt{R}[I]$, and $x = x_1 * \cdots * x_n$ be the unique factorization of x into indecomposables. Further say that $y = (\leq_y, \iota)$, where $\leq_y$ is a total order in I. Let $\vec{\mathbf{I}} = (I_1, \ldots, I_n)$ be a QSS of y from $x_1, \ldots, x_n$ and consider a composition $\alpha \models n$.

Suppose that $\mathbf{CS}(\alpha) = \{f_1, \ldots, f_{\ell(\alpha)-1}\}$ and use the convention that $f_0 = 0$ and $f_{\ell(\alpha)} = n$. Then we define for $i = 1, \ldots, \ell(\alpha)$:

$$I_i^\alpha := I_{f_{i-1}+1} \cup \cdots \cup I_{f_i} \quad , \quad x_i^\alpha := x_{f_{i-1}+1} * \cdots * x_{f_i}.$$

For a partial order $\leq$ on a set I, and two sets $A, B \subseteq I$, we say that $A \lneq B$ if A, B are disjoint and $a \leq b$ for any $a \in A$ and $b \in B$. Two indices $i < j$ are said to be **merged** by α if there is some k in $\{1, \ldots, n\}$ such that $f_{k-1} < i < j \leq f_k$. We say that a QSS $\vec{\mathbf{I}}$ of y from $x_1, \ldots, x_n$ is α-stable if $(I_i^\alpha)_i$ is a QSS of y from $(x_i^\alpha)_i$ and, whenever $x_i \sim x_j$ and $i < j$ are merged with α, then $I_i \lneq_y I_j$. Finally, we define

$$\mathcal{I}_{\vec{\mathbf{I}}}^{x,y} := \left\{ \alpha \models n \,\middle|\, \vec{\mathbf{I}} \text{ is an } \alpha\text{-stable QSS of } y \text{ from } (x_i)_i \right\} . \tag{3.1}$$

Example 3.5 (α-stable QSS on $\mathtt{PWo}$)**.** Consider the packed word $\rho = 21 \oplus 111 = 21333 = (1 < 2 < 3 < 4 < 5, 2 < 1 < \{3,4,5\})$ on the set $[5]$. This packed word has three QSS from $21, 1, 11$, precisely $\vec{\mathbf{I}}_1 = (12, 3, 45)$, $\vec{\mathbf{I}}_2 = (12, 4, 35)$ and $\vec{\mathbf{I}}_3 = (12, 5, 34)$. All three are $(1,1,1)$-stable. We now observe that $\vec{\mathbf{I}}_1, \vec{\mathbf{I}}_2$ and $\vec{\mathbf{I}}_3$ are $(2,1)$-stable, but neither $(1,2)$-stable nor (3)-stable. Indeed, $\vec{\mathbf{I}}_1^{(1,2)} = \vec{\mathbf{I}}_2^{(1,2)} = \vec{\mathbf{I}}_3^{(1,2)} = (12, 345)$, and $\rho|_{345} = 111$ is not $\rho|_{I_2} \oplus \rho|_{I_3}$ for any of the QSS. On the other hand, $\vec{\mathbf{I}}_1^{(2,1)} = (123, 45)$, $\vec{\mathbf{I}}_2^{(2,1)} = (124, 35)$ and $\vec{\mathbf{I}}_3^{(2,1)} = (125, 34)$, in each case is easy to note that $\rho|_{I_1^{(2,1)}} = 21 \oplus 1$ and $\rho|_{I_2^{(2,1)}} = 11$ for each of the QSS. Therefore, in each case we get $\mathcal{I}_{\vec{\mathbf{I}}}^{21\oplus1\oplus11, 21333} = \{(1,1,1), (2,1)\}$.

Define the composition $\mu_i := (\underbrace{1, \ldots, 1}_{i-1 \text{ times}}, 2, 1, \ldots, 1)$. If $\vec{\mathbf{I}} = (I_1, \ldots, I_n)$ is a μ_i-stable QSS, then $y|_{I_1 \cup I_{i+1}} = y|_{I_i} * y|_{I_{i+1}}$. This motivates the following lemma.

Lemma 3.6 ([13], Lemma 6.2)**.** Let $\mathtt{R}$ be an ordered species with restriction. Consider x, y objects in $\mathtt{R}$, such that $y = (\leq_y, m)$ and $x = x_1 * \cdots * x_n$ a factorization into indecomposibles. If $\vec{\mathbf{I}} = (I_1, \ldots, I_n)$ is a QSS of y from $x_1, \ldots x_n$ that is μ_i-stable, then $I_i \lneq_y I_{i+1}$.

In the case of packed words, a stronger claim can be used to compute α-stability (see Lemma 4.2). Also, if $\mathtt{R}$ is a species with restrictions, $y, x_1, \ldots, x_n \in \mathcal{G}(\mathtt{R})$ and $\vec{\mathbf{I}}$ a QSS of y from $x_1 \ldots, x_n$, then $\mathcal{I}_{\vec{\mathbf{I}}}^{x_1 * \cdots * x_n, y}$ has a unique maximal element, $\mathbb{1} = (1, \ldots, 1)$.

Theorem 3.7 ([13], Theorem 6.10)**.** For an ordered species with restrictions $\mathtt{R}$, and an element x, along with its factorization $x = x_1 * \cdots * x_n$, the antipode of $\mathcal{A}(\mathtt{R})$ is given by:

$$S(\mathbf{pat}_x) = \sum_y \mathbf{pat}_y \sum_{\substack{\vec{\mathbf{I}} \text{ QSS of } y \\ \text{from } x_1,\ldots,x_n}} \sum_{\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}} (-1)^{\ell(\alpha)} .$$

Proposition 3.8 (Filtered structure of $\mathcal{I}$, [13], Proposition 6.11)**.** For an ordered species with restrictions $\mathtt{R}$, and an elements $x \in \mathtt{R}[J], y \in \mathtt{R}[I]$, along with a factorization into $*$-indecomposibles $x = x_1 * \cdots * x_n$ and a QSS $\vec{\mathbf{I}}$ of y from $x_1, \ldots, x_n$, we have that $\mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ is a filter. That is, if $\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ and $\beta \geq \alpha$ then $\beta \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$. Furthermore, if $\mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ has a unique minimal element distinct from $\mathbb{1}$, then $\sum_{\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}} (-1)^{\ell(\alpha)} = 0$.

The consequence is, for pattern algebras, that computing the antipode corresponds to find which $\mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ have a unique minimal element. In this case, the antipode formula reduces to counting how many of these minimal elements are the composition $\mathbb{1}$. These will contribute with $(-1)^n$ to the total coefficient of $\mathbf{pat}_y$.

4 The antipode formula for the pattern algebra on packed words

For a partial order $\leq$ on a set I, and two sets $A, B \subseteq I$, recall that we say that $A \lneq B$ if $A \cap B = \emptyset$ and $a \leq b$ for any $a \in A$ and $b \in B$. We recall the definition of non-interlacing QSS on packed words.

Definition 4.1 (Interlacing QSS on packed words)**.** Let $\rho, \omega_1, \ldots, \omega_n$ be packed words, where $\rho = (\leq_P, \leq_V)$ is a packed word on I. Let $\vec{\mathbf{I}} = (I_1, \ldots, I_n)$ be a QSS of ρ from $\omega_1, \ldots, \omega_n$. We say that this QSS is **non-interlacing** if there exists some $i = 1, \ldots, n-1$ such that $I_i \lneq_P I_{i+1}$ and $I_i \lneq_V I_{i+1}$. If no such i exists, we say that the QSS is **interlacing**.

Additionally, let $\left[{\rho \atop \omega_1, \ldots, \omega_n}\right]$ be the number of interlacing QSS of ρ from $\omega_1, \ldots, \omega_n$.

Recall that $\mu_i = (\underbrace{1, \ldots, 1}_{i-1 \text{ times}}, 2, 1, \ldots, 1)$. The next lemma was proved in [13, Lemma 7.2 and Lemma 7.3].

Lemma 4.2. Let ω and $\rho = (\leq_P, \leq_V)$ be packed words, such that $\omega = \omega_1 \oplus \cdots \oplus \omega_n$ is its factorization into $\oplus$-indecomposables. Let ρ be another packed word, and $\vec{\mathbf{I}}$ a QSS of ρ from $\omega_1, \ldots, \omega_n$. Then $\vec{\mathbf{I}}$ is μ_i-stable if and only if $I_i \lneq_P I_{i+1}$ and $I_i \lneq_V I_{i+1}$. Moreover, $\mathcal{I}_{\vec{\mathbf{I}}}^{\omega,\rho}$ has a unique minimal element and, therefore, it is an interval in $\mathcal{C}_n$.

Theorem 4.3. Let ω be a packed word, and $\omega = \omega_1 \oplus \cdots \oplus \omega_n$ be its decomposition into $\oplus$-indecomposible packed words. Then, on the pattern Hopf algebra of packed words, we have the following cancellation free and grouping free formula:

$$S(\mathbf{pat}_\omega) = (-1)^n \sum_\rho \begin{bmatrix} \rho \\ \omega_1, \ldots, \omega_n \end{bmatrix} \mathbf{pat}_\rho .$$

Proof. From Theorem 3.7, we only need to establish that

$$\sum_{\alpha \in \mathcal{I}^{\rho,\omega}_{\vec{\mathbf{I}}}} (-1)^{\ell(\alpha)} = (-1)^n \mathbb{1}[\vec{\mathbf{I}} \text{ is interlacing QSS of } \rho \text{ from } \omega_1, \dots, \omega_n] \,. \tag{4.1}$$

Further, from Proposition 3.8 we know that the sum $\sum_{\alpha \in \mathcal{I}^{\rho,\omega}_{\vec{\mathbf{I}}}} (-1)^{\ell(\alpha)}$ vanishes whenever $\mathcal{I}^{\rho,\omega}_{\vec{\mathbf{I}}}$ is an interval with more than one element. From Lemma 4.2, we know that $\mathcal{I}^{\rho,\omega}_{\vec{\mathbf{I}}}$ is indeed an interval. The minimal interval is $\mathbb{1}$ if and only if $\vec{\mathbf{I}}$ is an interlacing QSS from Lemma 4.2. This concludes the proof. □

Notice that this proof hides a sign-reversing involution in it. Specifically, it was used in establishing Proposition 3.8.

5 Polynomial invariant on pattern Hopf algebras

Polynomial invariants are pervasive in the study of combinatorial objects. Central figure is the chromatic polynomial on graphs. The invariant that we introduce in this paper is the following polynomial, general to any species with restrictions:

Definition 5.1 (Polynomial invariant on a species with restrictions $\mathtt{R}$)**.** Let x be an object in the connected associative species with restrictions $(\mathtt{R}, *, 1)$. Then we define the invariant χ^{x} on $\mathtt{R}$: fix y an object, then

$$\chi^{\mathrm{x}}_{\mathrm{y}}(n) \coloneqq \chi^{\mathrm{x}}(\mathbf{pat}_{\mathrm{y}})(n) \coloneqq \mathbf{pat}_{\mathrm{y}}(\mathrm{x}^{\star n}) \,.$$

Theorem 5.2 (Polynomial invariants in pattern algebras)**.** If $\mathtt{R}$ is a connected associative species with restrictions and x on of its objects, then χ^{x} is a Hopf algebra morphism $\mathcal{A}(\mathtt{R}) \to \mathbb{K}[x]$.

Theorem 5.3. Let ρ be a packed word, and let $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ be the decomposition of a packed word π into $\oplus$-indecomposibles, then

$$\chi^{\rho}_{\pi}(-x) = (-1)^n \sum_{\sigma} \begin{bmatrix} \sigma \\ \pi_1, \dots, \pi_n \end{bmatrix} \chi^{\rho}_{\omega}(x).$$

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