The feasible region of consecutive occurrences of permutations is a cycle polytope

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Part 1: Feasible regions and overlap graphs

Permutation patterns

A permutation of size n is an arrangement of points in an $n \times n$ grid, in such a way that each column and each row has exactly one point.

Permutation pattern

By selecing a set of columns I on a permutation τ, and disregarding resulting empty rows, we obtain a new permutation $\pi = \tau|_I$, which is called a **pattern** of τ. The set I is called an **occurrence** of π in τ, and **interval occurrence** if it is a set of consecutive columns.

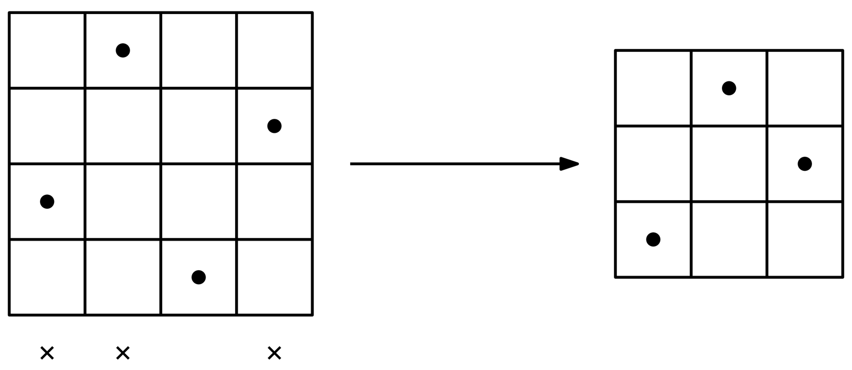

Figure 1: An occurrence of 231 in 2413.

$$\mathrm{occ}(\pi, \tau) = \#\{\text{ occurrences of } \pi \text{ in } \tau\,\}$$

$$\mathrm{coc}(\pi, \tau) = \#\{\text{ interval occurrences of } \pi \text{ in } \tau\,\}$$

The **proportion of (consecutive) occurrences** is defined as

$$\widetilde{\mathrm{occ}}(\pi, \tau) = \frac{\mathrm{occ}(\pi, \tau)}{|\tau|^{|\pi|}}, \quad \widetilde{\mathrm{coc}}(\pi, \tau) = \frac{\mathrm{coc}(\pi, \tau)}{|\tau|}.$$

Feasible region

For a fixed k, a given sequence of permutations $\{\sigma^{(n)}\}_{n\geq 0}$ such that $|\sigma^{(n)}| \to \infty$ gives rise to a **feasible point** $\vec{x} = (x_\pi)_{\pi \in \mathcal{S}_k}$ in $\mathbb{R}^{\mathcal{S}_k}$ defined by

$$x_\pi := \lim_{n\to\infty} \widetilde{\mathrm{coc}}(\pi, \sigma^{(n)})\,.$$

The feasible region of consecutive occurrences

The **feasible region** P_k is the set of feasible points in $\mathbb{R}^{\mathcal{S}_k}$, that is:

$$\left\{\vec{x} \in \mathbb{R}^{\mathcal{S}_k} | \exists \sigma^{(n)} \text{ s.t. } \vec{x} = (\lim_{n\to\infty} \widetilde{\mathrm{coc}}(\pi, \sigma^{(n)}))_{\pi \in \mathcal{S}_k}\right\}$$

Example: for $k = 2$, the sequence $\sigma^{(n)} = 21^{\oplus n}$ gives rise to the limit point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

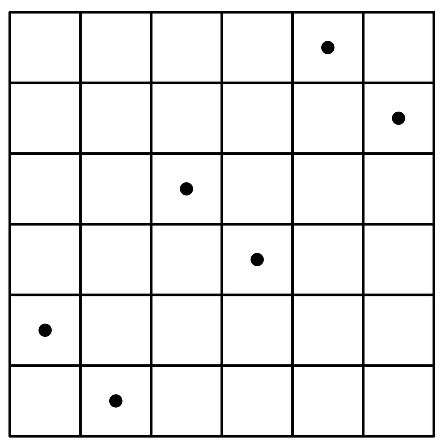

Figure 2: The permutation $\sigma^{(3)}$ from the example above, with $\mathrm{coc}(12, \sigma^{(3)}) = 2$ and $\mathrm{coc}(21, \sigma^{(3)}) = 3$.

The overlap graph

The overlap graph is the central tool that will allow us to decode the structure of P_k.

The overlap graph

The **overlap graph** $\mathcal{O}v(k)$ is an oriented graph on the vertex set $\mathcal{S}_{k-1}$ and such that for each $\pi \in \mathcal{S}_k$ there is an edge from $\pi|_{\{1,\dots,k-1\}}$ to $\pi|_{\{2,\dots,k\}}$.

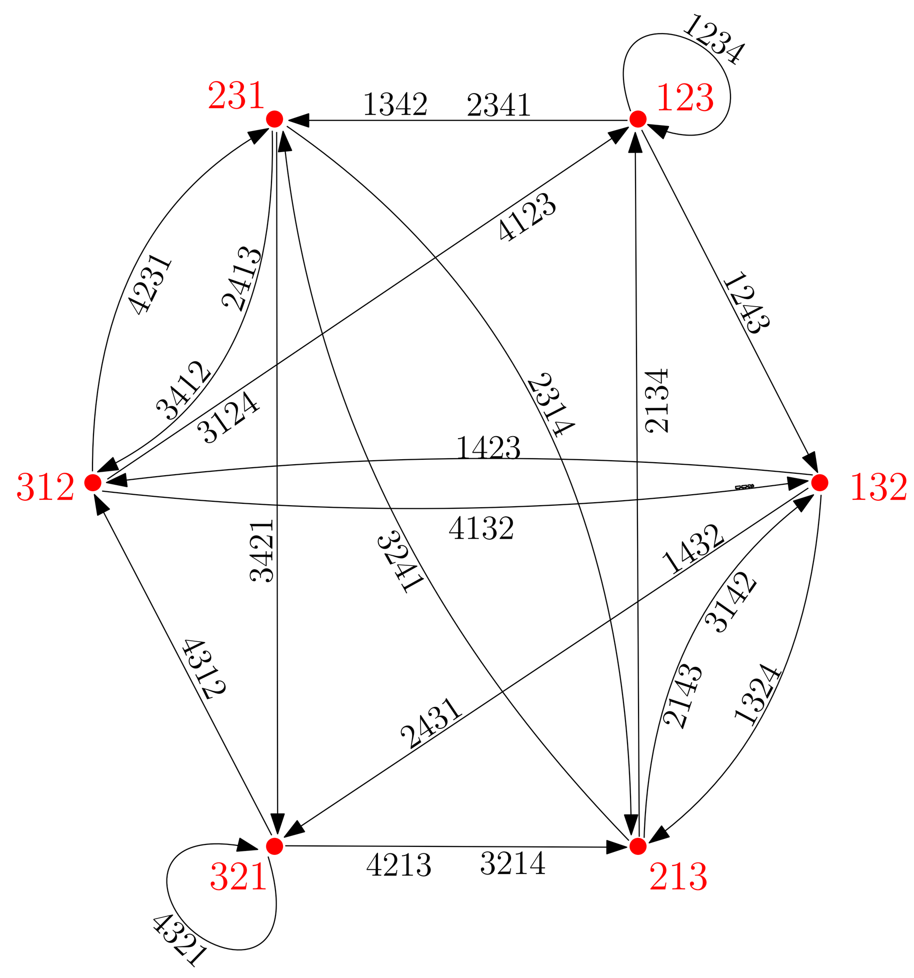

Figure 3: The overlap graph $\mathcal{O}v(4)$.

Part 2: Cycle polytopes and face structure

Cycle polytopes

A simple cycle in an oriented graph is a cycle with no (non-trivial) repeated vertices.

The cycle polytope of a graph

Let G be an oriented graph and $\mathcal{C}$ one of its simple cycles. We define the vector $\vec{a}_\mathcal{C} \in \mathbb{R}^{E(G)}$ as $(\vec{a}_\mathcal{C})_e = |\mathcal{C}|^{-1}$ if $e \in \mathcal{C}$, and 0 otherwise.
The **cycle polytope** of G is defined as

$$P(G) := \mathrm{conv}\{\vec{a}_\mathcal{C} |\ \mathcal{C} \text{ is a simple cycle of } G\}\,.$$

We have combinatorial characterisations of the geometric properties of $P(G)$.

Theorem (see [1])

If G is strongly connected, then $P(G)$ is a polytope of dimension $\#E(G) - \#V(G)$, given by the equations

$$\sum_{e \in E(G)} x_e = 1\,,$$

$$\sum_{be(e)=v} x_e = \sum_{en(e)=v} x_e \text{ for all } v \in V(G)\,,$$

where we denote by $be(e)$ and $en(e)$ the beginning vertex (resp. ending) vertex of an edge e. Furthermore, to each face it corresponds a unique subgraph of G.

Characterization of feasible regions

In the following theorem, we find that the feasible region for consecutive patterns, unlike classical patterns, is a convex polytope.

Theorem (see [1])

The feasible region P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$. From this, it results that $\dim P_k = k! - (k-1)!$, and the vertices of P_k correspond to simple cycles in $\mathcal{O}v(k)$.

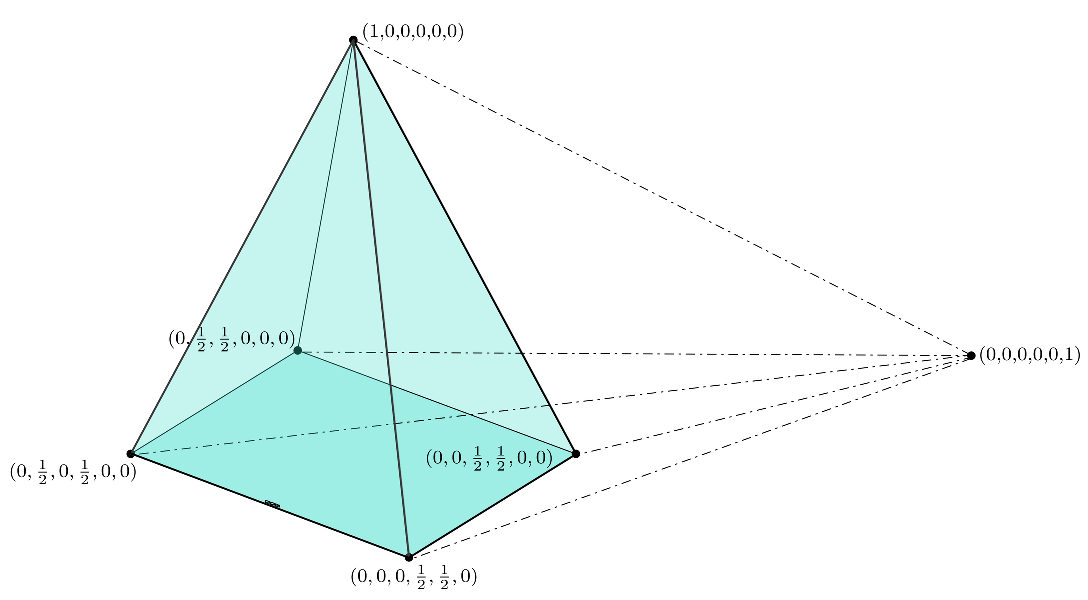

Figure 4: The feasible region for $k = 3$ drawn in a 4-dimensional space.

Pattern avoiding permutations

Given a set of permutations $\mathcal{P}$, we define the family $\mathrm{Av}(\mathcal{P})$ of permutations τ that have no occurrences of any permutation $\pi \in \mathcal{P}$. Such families are called **permutation classes**.

The restricted feasible region

Fix k, and let $\mathcal{A}$ be a *permutation class*. A sequence of permutations $\{\sigma^{(n)}\}_{n\geq 0} \subseteq \mathcal{A}$ such that $|\sigma^{(n)}| \to \infty$ gives rise to an $\mathcal{A}$-feasible point in $\mathbb{R}^{\mathcal{S}_k}$. We define $P_k^\mathcal{A}$ as the set of $\mathcal{A}$-feasible points in $\mathbb{R}^{\mathcal{S}_k}$.

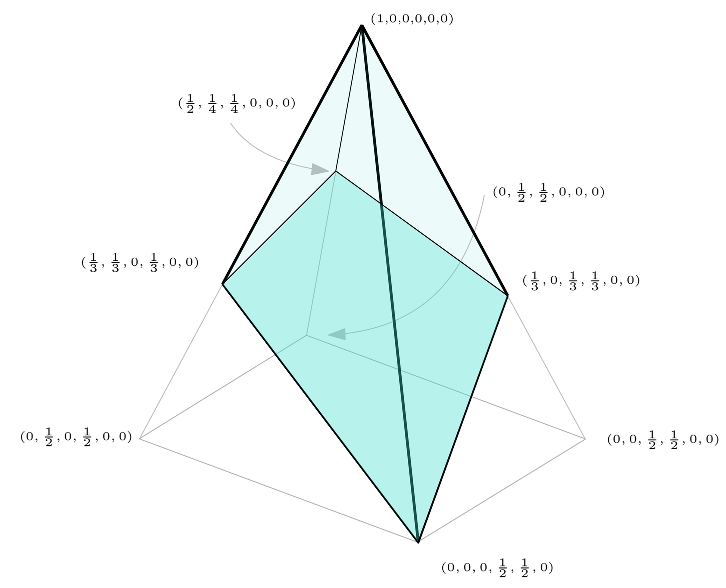

Figure 5: The $\mathrm{Av}(321)$-feasible region for $k = 3$.

Partial results for: $\mathrm{Av}(231), \mathrm{Av}(n \dots 1)$ for any $n \geq 2$, see [2].

References

[1] Borga J. and Penaguiao R., The feasible region for consecutive patterns of permutations is a cycle polytope, ArXiv preprint: 1910.02233 (2019).

[2] Borga J. and Penaguiao R., The feasible region for consecutive patterns of some permutation classes is a cycle polytope, in preparation.