A new cancellation free formula for permutation pattern Hopf algebras

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Slides can be found at `https://raulpenaguiao.github.io/`

Permutations - square diagrams

Permutation π on a set $S = \{s_1, \dots, s_n\}$ is a pair of orders on S.

$$132 = \begin{array}{|c|c|c|}\hline & s_1 & \\ \hline & & s_3 \\ \hline s_2 & & \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline & \cdot & \\ \hline & & \cdot \\ \hline \cdot & & \\ \hline \end{array}$$

Counting occurrences of a pattern

Permutation π on a set $S = \{s_1, \dots, s_n\}$ and $I \subseteq S$.
The **restriction to** I can be defined $\pi|_I$ and is a permutation in I.
We can count occurrences! How many times does a particular diagram occur? If $\pi = 132$ as above,

$$\pi\big|_{\{s_2,s_3\}} = \left.\begin{array}{|c|c|c|}\hline & s_1 & \\ \hline & & s_3 \\ \hline s_2 & & \\ \hline\end{array}\right|_{\{s_2,s_3\}} = \begin{array}{|c|c|}\hline & s_3 \\ \hline s_2 & \\ \hline\end{array}$$

We write

$$\mathbf{p}_{12}(132) = 2,\ \mathbf{p}_{123}(123456) = 20,\ \mathbf{p}_{2413}(762341895) = 0\,.$$

Permutation pattern algebra

Proposition (Linear independence)

The set $\{\mathbf{p}_\pi \mid \pi \in \uplus_{n\geq 0} S_n\}$ *is linearly independent* - *Triangularity argument*

Proposition (Product formula)

There are coefficients $\binom{\sigma}{\pi, \tau}$ *that count the number of quasi-shuffles of* σ *with permutations* π*,* τ *such that:*

$$\mathbf{p}_\pi \cdot \mathbf{p}_\tau = \sum_\sigma \binom{\sigma}{\pi, \tau} \mathbf{p}_\sigma \,,$$

where σ *runs over equivalence classes of pairs of orders.*

Permutation pattern algebra

Example try it with $\tau = 231$!

$$\mathbf{p}_{12}\,\mathbf{p}_{1} = 3\,\mathbf{p}_{123} + \mathbf{p}_{312} + \mathbf{p}_{231} + 2\,\mathbf{p}_{213} + 2\,\mathbf{p}_{132} + 2\,\mathbf{p}_{12}\ .$$

$$\mathbf{p}_{12}(231)\,\mathbf{p}_{1}(231) = 1 \times 3$$

$$\begin{aligned} &3\,\mathbf{p}_{123}(231) + \mathbf{p}_{312}(231) + \mathbf{p}_{231}(231) + 2\,\mathbf{p}_{213}(231) \\ +&2\,\mathbf{p}_{132}(231) + 2\,\mathbf{p}_{12} \\ &= 0 + 0 + 1 + 0 + 0 + 2 \times 1\,. \end{aligned}$$

Permutation pattern algebra - adding another ingredient

$$\pi \oplus \tau = \begin{array}{|c|c|}\hline & \tau \\ \hline \pi & \\ \hline\end{array} \qquad \pi \ominus \tau = \begin{array}{|c|c|}\hline \pi & \\ \hline & \tau \\ \hline\end{array}$$

By *magic properties* of dualization, these give coproducts on $\mathcal{A}(\texttt{Per})$, for instance:

$$\Delta \, \mathbf{p}_\pi = \sum_{\pi = \tau_1 \oplus \tau_2} \mathbf{p}_{\tau_1} \otimes \mathbf{p}_{\tau_2} \, ,$$

so that we have a Hopf algebra

$$\mathbf{p}_\pi(\sigma_1 \oplus \sigma_2) = \Delta \, \mathbf{p}_\pi(\sigma_1 \otimes \sigma_2) \, .$$

$$\Delta \, \mathbf{p}_{21354} = \mathbf{p}_\emptyset \otimes \mathbf{p}_{21354} + \mathbf{p}_{21} \otimes \mathbf{p}_{132} + \mathbf{p}_{213} \otimes \mathbf{p}_{21} + \mathbf{p}_{21354} \otimes \mathbf{p}_\emptyset \ .$$

Permutation pattern algebra

Free pattern algebras: as simple as polynomial algebras.

Theorem (Vargas, 2014)

The linear span of pattern functions $\mathcal{A}(\texttt{Per})$ form a Hopf algebra.
The Hopf algebra $\mathcal{A}(\texttt{Per})$ is free commutative.
A free generator family is given by a family of Lyndon permutations.

Outline of the talk

1. Introduction
   - Permutations
   - Species with restrictions
2. The freeness question
   - The commutative case
   - Permutations
   - Marked permutations
3. Antipodes

Pattern algebra

What do we need to have a pattern Hopf algebra?

- Assignment $S \mapsto h[S] = \{$structures over $S\}$ + notion of *relabelling*.
- For any inclusion $V \hookrightarrow W$, a restriction map $h[W] \to h[V]$.
- An associative *monoid* operation $*$ with unit that is compatible with restrictions.
- A unique element of size zero.

- $\mathtt{G}[V] = \{$ graphs on the vertex set V $\}$.
- Induced subgraphs $\to$ restrictions.
- The disjoint union of graphs.
- The empty graph fortunately exists!

Simple example - binomial identities

Define $\texttt{Set}[S] = \{*_{\#S}\}$ to have a unique element.

$$\mathbf{p}_{*_n}(*_m) = \binom{m}{n} \qquad \binom{*_d}{*_a, *_b} = \binom{d}{a}\binom{a}{a+b-d}.$$

So we obtain the following binomial identity:

$$\mathbf{p}_{*_a}(*_c)\,\mathbf{p}_{*_b}(*_c) = \sum_{d\geq 0} \binom{d}{a}\binom{a}{a+b-d}\mathbf{p}_{*_d}(*_c)$$

Monoidal structure - Disjoint union: $*_n *_m = *_{n+m}$.

$$\Delta\,\mathbf{p}_{*_a} = \sum_{k=0}^{a} \mathbf{p}_{*_k} \otimes \mathbf{p}_{*_{a-k}}, \qquad \mathcal{A}(\texttt{Set}) = k[\mathbf{p}_{*_1}]$$

Another example: words

$\mathtt{Word}_{\mathcal{K}}[n] = \{$ words of size n using an alphabet $\mathcal{K}$ characters $\}$

The resulting algebra is isomorphic to the **shuffle algebra** on the alphabet $\mathcal{K}$, and is also free.

Other known Hopf algebras that arise as a pattern algebra

$$\mathtt{SPart}[I] = \{\text{set partitions of } I\}$$

This has a species with restrictions structure,
$\{A_1, \dots\}|_J = \{A_1 \cap J, \dots\}$.
This also has a product structure (disjoint union of partitions).

Proposition

$\mathcal{A}(\mathtt{SPart})$ *is isomorphic to the Hopf algebra of symmetric functions* Sym*.*

Conjecture

$\mathcal{A}(\mathtt{SComp})$ *is isomorphic to the Hopf algebra of quasi-symmetric functions.*

Freeness question - commutative casae

When is the pattern algebra free? Vargas showed in 2014 that for permutations, we get a free algebra.

Theorem (P - 2020)

If $\mathtt{h}$ *is a commutative, then* $\mathcal{A}(\mathtt{h})$ *is free. The free generators are the indecomposable objects with respect to the commutative product.*

To show freeness: generator set $\mathcal{I}$ such that all products of elements in $\mathcal{I}$ are l.i.
These are l.i. because they correspond to the product of the usual basis $\{\mathbf{p}_a\}$ with an upper triangular matrix (for a suitable choice of order).
In any commutative species with restrictions, any object a has a **unique** factorization up to order of factors:

$$a = a_1 \star \cdots \star a_k \, .$$

Furthermore, $\mathbf{p}_a$ arises with a non-zero coefficient in $\prod_i \mathbf{p}_{a_i}$.

Unique factorisation theorem on permutations

Back to permutations, which are non-commutative (so the theorem does not apply).
Vargas used the $\oplus$ product on permutations to obtain a unique factorisation theorem on permutations.

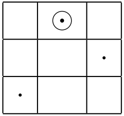
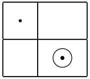
$$\pi = \tau_1 \oplus \cdots \oplus \tau_k = \begin{array}{|c|c|c|}\hline & & \tau_k \\ \hline & \iddots & \\ \hline \tau_1 & & \\ \hline\end{array}$$

This is not unique **up to reordering of factors**, so a new factorization needs to be found.
A factorization into **Lyndon permutations** was cooked up such that any permutation π has a unique factorization into **Lyndon permutations** $\ell_1 \geq \cdots \geq \ell_k$ such that

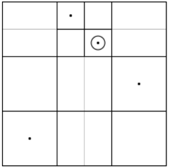
$$\pi = \ell_1 \oplus \cdots \oplus \ell_k \,.$$

Marked permutations

Species of marked permutations.

$\pi =$  $= 1\bar{3}2\,,\ \sigma =$  $= 2\bar{1}$

Inflation of $\pi * \sigma = 14\bar{3}2$ is



Freeness on marked permutations

Marked permutations have an *exotic* unique factorization theorem.

Theorem (P - 2020)

The Hopf algebra $\mathcal{A}(\mathtt{MPer})$ is free generated by the Lyndon marked permutations.

Conjecture

Any pattern algebra is free.

Antipodes on Hopf algebras

The antipode is to a Hopf algebra as the inverse map is to a group.
An antipode $S : H \to H$ on a Hopf algebra is a map that satisfies

$$\begin{tikzcd}
 & H \otimes H \arrow[r, "id_H \otimes S"] & H \otimes H \arrow[dr, "\mu"] & \\
\mathrm{H} \arrow[ur, "\Delta"] \arrow[r, "\epsilon"] \arrow[dr, "\Delta"'] & k \arrow[rr, "\iota"] & & H \\
 & H \otimes H \arrow[r, "S \otimes id_H"] & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Obs: $\iota \circ \epsilon(x)$ is the coefficient of the identity element of x.

Antipode formula

Example: the polynomial algebra $k[x]$ has the antipode $S(x^n) = (-x)^n$

$$
\begin{aligned}
&\mu \circ (id \otimes S_{k[x]}) \circ \Delta(x^n) = \\
&\mu \circ (id \otimes S_{k[x]}) \left(\sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k} \right) = \\
&\mu \left(\sum_{k=0}^{n} \binom{n}{k} x^k \otimes (-x)^{n-k} \right) = \\
&\sum_{k=0}^{n} \binom{n}{k} (-1)^k x^n = 0 = \iota \circ \epsilon(x^n)
\end{aligned}
$$

Permutation patterns - what to do with antipodes

A chromatic polynomial is a Hopf algebra morphism $\chi : H \to k[x]$.
Because it is a Hopf algebra morphism, it satisfies

$$\chi_{S(a)}(x) = S(\chi_a(x)) = \chi_a(-x)$$

Theorem (Reciprocity results - Stanley, 1969)

For a graph G

$$\chi_G(-1) = \#\{ \textit{ acyclic orientations of } G\}$$

Takeuchi formula

Proposition (Takeuchi's formula)

If $H = (H, \mu, \iota, \Delta, \epsilon, S)$ is a filtered Hopf algebra, then

$$S = \sum_{k \geq 0} (-1)^k \mu^{\circ(k-1)} \circ (\mathrm{id}_H - \iota \circ \epsilon)^{\otimes k} \circ \Delta^{\circ(k-1)} .$$

In the polynomial algebra $k[x]$ the formula gives us

$$S(x^n) = \sum_{\vec{\boldsymbol{\pi}} \models [n]} (-1)^{\ell(\vec{\boldsymbol{\pi}})} x^n .$$

Problem (Cancellation-free and grouping-free antipode formulas)

Can we find more economic formulas for our favourite Hopf algebras?

Sign-reversing involution formulas

$$S(x^n) = \sum_{\vec{\boldsymbol{\pi}} \models [n]} (-1)^{\ell(\vec{\boldsymbol{\pi}})} x^n \, .$$

Benedetti and Sagan define a sign-reversing involution

$$\beta : \{\text{set compositions }\} \to \{\text{set compositions}\} \, ,$$

such that either $(-1)^{\ell(\vec{\boldsymbol{\pi}})} = -(-1)^{\ell(\beta(\vec{\boldsymbol{\pi}}))}$ or $\beta(\vec{\boldsymbol{\pi}}) = \vec{\boldsymbol{\pi}}$.

$$S(x^n) = \sum_{\vec{\boldsymbol{\pi}} \text{ fixed point of } \beta} (-1)^{\ell(\vec{\boldsymbol{\pi}})} x^n = (-1)^{\ell((\{1\}, \dots, \{n\}))} x^n = (-1)^n x^n \, .$$

Permutation patterns

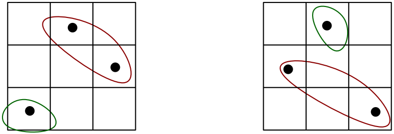
What does the Takeuchi formula give us for permutations? Let π and σ be permutations, and $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ a decomposition into **irreducibles**.

$$S(\mathbf{p}_\pi) = \sum_\sigma \mathbf{p}_\sigma \sum_{\substack{\vec{\mathbf{I}} \text{ QSS of } \sigma \\ \text{from } \pi_1, \dots, \pi_n}} \sum_{\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{\pi,\sigma}} (-1)^{\ell(\alpha)} .$$

The cover $(I_1, \dots, I_n)$ is a QSS of σ from π_i if $\sigma|_{I_i} = \pi_i$.

Permutation patterns

Let $\left[{\sigma \atop \pi_1,\dots,\pi_n}\right]$ be the number of interlacing QSS of σ from $\pi_1, \dots, \pi_n$



$$\left[{312 \atop 1,21}\right] = 2$$

$$\binom{132}{1, 21} = 1$$

$$\left[{132 \atop 1,21}\right] = 0$$

Permutation patterns

Theorem (P., Vargas, 2022+)

Let π be a permutation, and $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ be its decomposition into irreducible permutations. Then, on the pattern Hopf algebra of permutations, we have the following cancellation-free and grouping-free formula:

$$S(\mathbf{p}_\pi) = (-1)^n \sum_{\sigma} \mathbf{p}_\sigma \left[{}^{\sigma}_{\pi_1, \ldots, \pi_n} \right] .$$

$$S(\mathbf{p}_\pi) = \sum_{\sigma} \mathbf{p}_\sigma \sum_{\substack{\vec{\mathbf{I}} \text{ QSS of } \sigma \\ \text{from } \pi_1, \ldots, \pi_n}} \sum_{\alpha \in \mathcal{I}^{\pi, \sigma}_{\vec{\mathbf{I}}}} (-1)^{\ell(\alpha)} .$$

Permutation patterns

Theorem

There are no Hopf algebra morphisms $\mathcal{A}(\texttt{Per}) \to k[x]$.

Biblio

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Thank you