

# The tropical critical points of an affine matroid

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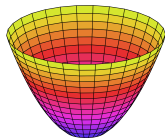
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Slides can be found at `raulpenaguiao.github.io/`  
Joint work with Federico Ardila and Chris Eur

# Optimization of a monomial

Fix some vector  $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$ .

Optimize  $f_{\mathbf{w}} : \mathbf{x} \mapsto x_1^{w_1} \cdots x_n^{w_n}$  on a variety  $X \subset (\mathbb{C}^*)^n$ .



**Figure:** A variety where we can optimize  $f_{\mathbf{w}}$

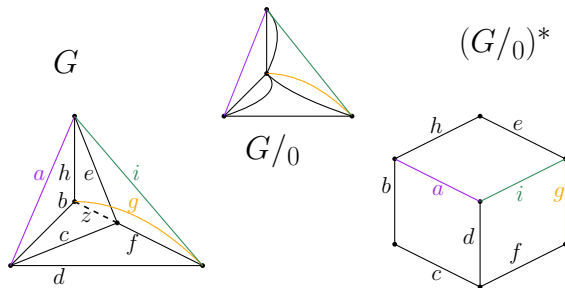
What is the number of critical points of  $f$ ? Does it depend on the choice of  $\mathbf{w}$ ? For generic  $\mathbf{w}$ , no!

This number is called the **maximum likelihood degree** of a model  $X$ .  
If  $X$  is a vector space,  $\text{MLDeg}(X) = \beta(M(V))$ .

# Edge weight problem

Given  $G = (V, E)$ , fix some vector  $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$ .

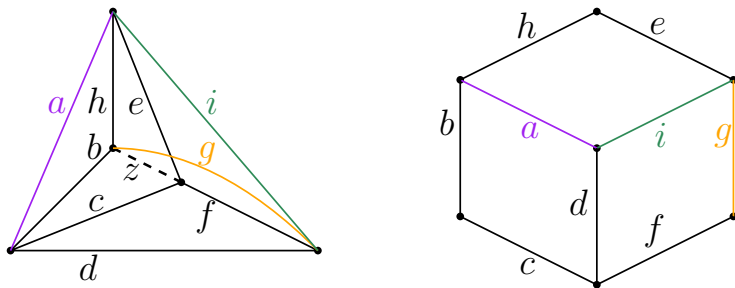
Find coefficients  $\mathbf{x} = (x_e)_{e \in E}$ ,  $\mathbf{y} = (y_e)_{e \in E}$  that are **compatible**?



**Figure:** Find  $\mathbf{x}$  and  $\mathbf{y}$  edge weights that are *compatible* with  $G$  and  $(G \setminus z)^*$ .

- The sum of the weights is  $\mathbf{w}$ .
- **(Compatible)** Every cycle has at least two minimal edges.

Fix  $\mathbf{w} = (0, 1, 1, 2, 2, 5, 3, 4, 7)$ .



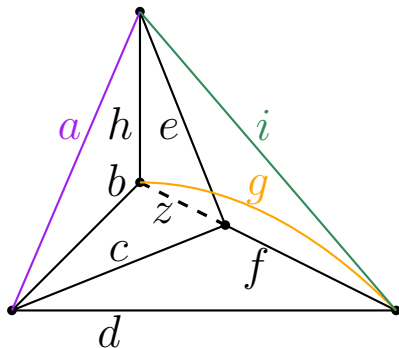
**Figure:** Find  $\mathbf{x}$  and  $\mathbf{y}$  edge weights that are *compatible* with  $G$  and  $(G \setminus z)^*$ .

	$a$	$b$	$c$	$d$	$e$	$f$	$g$	$h$	$i$
$\mathbf{x}$	00	01	00	00	02	03	00	00	02
$\mathbf{y}$	000	110	111	222	20	52	33	44	75
$\mathbf{w}$	0	1	1	2	2	5	3	4	7

- 1 Introduction
- 2 Matroids
- 3 The Bergman Fan
- 4 Degree of Bergman Fan

# Graphical matroid

Given a graph  $G = (V, E)$ , the collection of edges  $E$  forms a matroid.



Independent sets  $\mapsto$  forests

Basis  $\mapsto$  Spanning forests

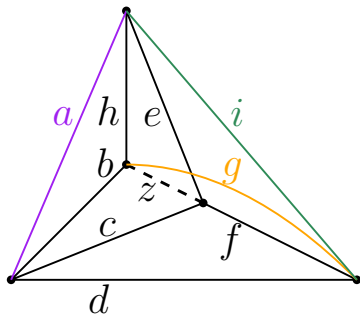
Circuits  $\mapsto$  Simple cycles

Rank of set  $A \subseteq E \mapsto$  size of largest spanning forest

# Flats

Maximal sets with a fixed rank.

That is,  $F$  is a flat if for any  $i \notin F$ ,  $r_M(F \cup i) > r_M(F)$ .

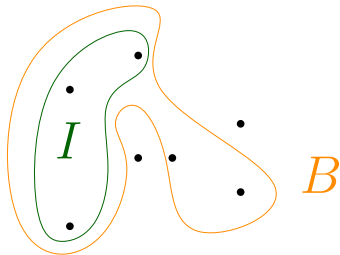


$\{\emptyset, \text{matchings}, \text{complete subgraphs}, \dots\}$

$\{\emptyset \subsetneq a \subsetneq za \subsetneq zafg \subsetneq zabcdefghi\}$

# The uniform matroid

Basis of the uniform matroid  $U_{n,k}$  = all sets of size  $k$  in  $[n]$ .  
Any set of size  $\leq k$  is independent.



**Figure:** Matroid  $U_{7,5}$  along with a basis  $B$  and independent set  $I$ .

Any set of size  $\leq k - 1$  is a flat.  
Any complete flag of flats is of the form

$$\{\emptyset \subsetneq \{v_1\} \subsetneq \{v_1, v_2\} \subsetneq \cdots \subsetneq \{v_1, \dots, v_{k-1}\} \subsetneq [n]\}$$



# The Bergman Fan

$$\Sigma_M := \{ \vec{x} \in \mathbb{R}^n \mid \forall \text{ circuits } C \text{ we have } \min_{c \in C} x_c \text{ is attained twice} \}.$$

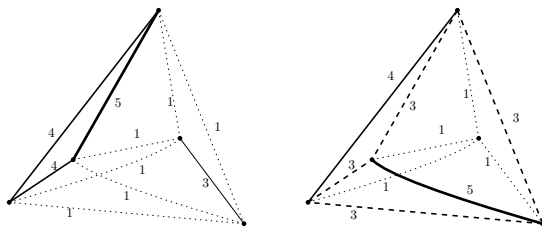
$$\Sigma_M := \{ \vec{x} \in \mathbb{R}^n / \mathbb{1}\mathbb{R} \mid \forall \text{ circuits } C \text{ we have } \min_{c \in C} x_c \text{ is attained twice} \}.$$

If  $M = U_{n,k}$ , any set of size  $k + 1$  is a circuit.

A point  $\vec{x} \in \mathbb{R}^n$  is on the Bergman fan if  $\#\{i \in E \mid x_i > \min \vec{x}\} \leq k - 1$ .

$$\Sigma_M = \bigcup_{|I|=n-k+1} \{ \vec{x} \in \mathbb{R}^n / \mathbb{1}\mathbb{R} \mid \arg \min \vec{x} \subseteq I \}.$$

$$\Sigma_{U_{3,2}} = \{(a, a, b) \mid a \leq b\} \cup \{(a, b, a) \mid a \leq b\} \cup \{(b, a, a) \mid a \leq b\}.$$



**Figure:** Two elements in the Bergman fan of the graphical matroid

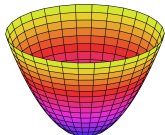
### Theorem (Sturmfels and Feichner, 2004)

*The Bergman Fan of a matroid decomposes into the following cones*

$$\Sigma_M = \bigcup_{\mathcal{F} \text{ flag of flats}} \mathcal{C}_{\mathcal{F}} = \bigcup_{\substack{F_1 \subset \dots \subset F_k \\ \text{flag of flats}}} \{x_i \geq x_j \text{ whenever } i \in F_k, j \notin F_k\}.$$

## Problem

*Can we compute the degree of the tropical variety  $\Sigma_M$ ?*



**Figure:** A variety  $X$ : Degree = # of intersections with a **line** in  $\mathbb{C}^n$ .

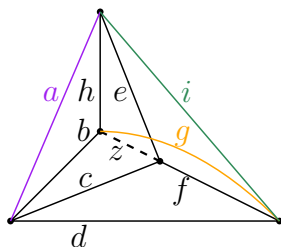
Just intersect it with a hyperplane with dimension  $= n - \dim X$

Note: Hyperplanes in the tropical world are Bergman fans of  $U_{n,k}$ .

# Example of degree computation

Consider  $M$  the graphical matroid of  $K_5$ , of rank 4 (so  $\dim \Sigma_M = 3$ ). One has  $\Sigma_M \subseteq \mathbb{R}^{10}/\mathbb{R}\mathbf{1}$ , we intersect it with a hyperplane of  $\dim = 6$ .

$$|\Sigma_{U_{10,7}} + \underbrace{(1, 10, 100, 1000, \dots, 10^9)}_{\vec{w}} \cap \Sigma_M| = ?$$



**Figure:** There is only one  $x \in \Sigma_M$  and  $y \in \Sigma_U$  such that  $y + w = x$ . Such vector  $x$  belongs to a **cone determined by the greedy basis of  $M$** .

# Activities

Fix total order in  $V$ , ground set of a matroid  $M$ .

- $i \in B$  is internal activity if  $i = \min C^\perp$ , where  $C^\perp \subseteq B^c \cup i$  is a cocircuit (a cut of the graph).
- $e \notin B$  is external activity if  $e = \min C$ , where  $C \subseteq B \cup i$  is a circuit.

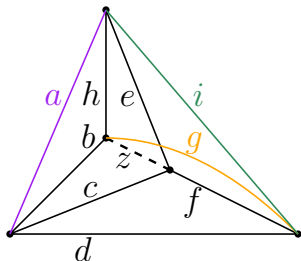


Figure: Fix order  $z < a < b < c < d < e < f < g < h < i$ .

$$i(befg) = 0, \quad e(befg) = 2, \quad i(zadf) = 2, \quad e(zadf) = 1$$

# Tutte polynomial

$$T_M(x, y) = \sum_{A \subseteq V} (x - 1)^{r_M(V) - r_M(A)} (y - 1)^{|A| - r_M(A)}$$

$$T_M(x, y) = \sum_{B \in \mathcal{B}} x^{i(B)} y^{e(B)} = \sum_{i,j} b_{i,j} x^i y^j$$

## Observation

# of bases with no external activities (called *nbc* bases):  
**independent of the order chosen.**

# of bases with no external activities and one internal activity (called  $\beta$ -*nbc* bases) **independent of the order chosen**

# Degree computations

Theorem (Greedy basis algorithm)

$$\deg(\Sigma_M) = \Sigma_M \cap (\mathbf{w} + \Sigma_{U_{n,n-k-1}}) = 1$$

Theorem (Adiprasito, Huh, Katz, 2018)

$$\deg(-\Sigma_M) = (-\Sigma_M) \cap (\mathbf{w} + \Sigma_{U_{n,n-k-1}}) = \#\{nbc \text{ bases}\}$$

Theorem (Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, Telen 2021 and Ardila, Eur, P 2022)

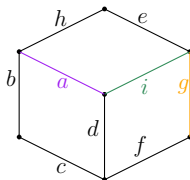
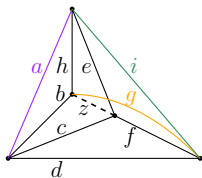
$$\deg(\Sigma_{(M/0)^\perp} \cdot -\Sigma_M) = (-\Sigma_M) \cap (\mathbf{w} + \Sigma_{(M/0)^\perp}) = \#\{\beta - nbc \text{ bases}\}$$

# Finding Nemø points form nbc bases

$$\deg(\Sigma_{(M/0)^\perp} \cdot -\Sigma_M) = ?$$

Fix generic  $\mathbf{w} = (10^{i-1})_i$ .

Find  $\mathbf{x} \in \Sigma_M, \mathbf{y} \in \Sigma_{(M/0)^\perp}$  such that  $\mathbf{x} + \mathbf{y} = \mathbf{w}$ .



Find a  $\beta$ -nbc basis  $B = zbeg \rightarrow g|e|bd|zacfhi$ ,

corresponding  $\beta$ -nbc cobasis  $B^\perp = acdfhi \rightarrow i|h|f|gd|c|eba$ .

	$a$	$b$	$c$	$d$	$e$	$f$	$g$	$h$	$i$
$\mathbf{x}$	0	9	0	9	$10^4 - 1$	0	$10^6 - 10^3 + 9$	0	0
$\mathbf{y}$	1	1	100	$10^3 - 9$	1	$10^5$	$10^3 - 9$	$10^7$	$10^8$
$\mathbf{w}$	1	10	100	$10^3$	$10^4$	$10^5$	$10^6$	$10^7$	$10^8$



# Biblio

- Agostini D., Brysiewicz T., Fevola C., Kühne L., Sturmfels B., Telen S. (2021). *Likelihood Degenerations*. [Motivation behind ML degree computations](#)
- Adiprasito K., Huh J., Katz E. (2018). *Hodge Theory for combinatorial geometries* [Degree computations in matroids](#)
- Ardila F., Eur C., RP (2022) *The maximum likelihood of a matroid*.

# Thank you

