Computing degrees of a Bergman fan in a funny way

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Slides can be found at `raulpenaguiao.github.io/`
Joint work with Federico Ardila and Chris Eur

Optimization of a monomial

Fix some vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n_{>0}$.
Consider $f_{\mathbf{t}} : \mathbf{x} \mapsto x_1^{w_1} \ldots x_n^{w_n}$ from a variety $X \subset (\mathbb{C}^*)^n$.

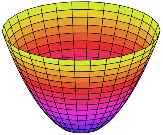


Figure: A variety where we can optimize $f_{\mathbf{w}}$

What is the number of critical points of f? Does it depend on the choice of $\mathbf{w}$? For generic $\mathbf{w}$, no! This number is called the **maximum likelihood degree** of a model X.

Edge weight problem

Fix some vector $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n_{>0}$.

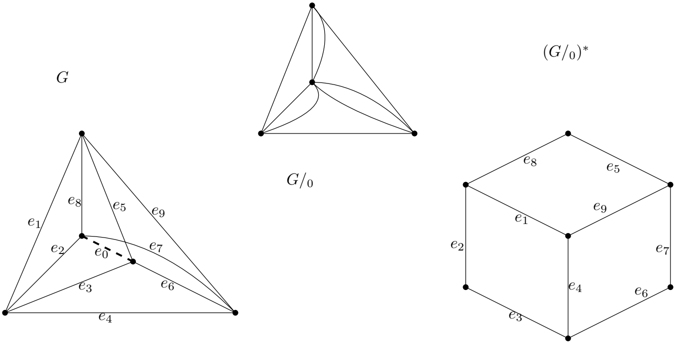


Figure: Find $\mathbf{x}$ and $\mathbf{y}$ edge weights that are *compatible*.

- The sum of the weights is $\mathbf{w}$.
- Every cycle has at least two minimal edges (edge 0 has weight 0).

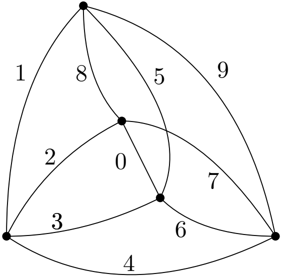
Fix $\mathbf{w} = (0, 1, 1, 2, 2, 5, 3, 4, 7)$.

Figure: Find $\mathbf{x}$ and $\mathbf{y}$ edge weights that are *compatible*.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
$\mathbf{x}$	0	1	0	0	2	3	0	0	2
$\mathbf{y}$	0	0	1	2	0	2	3	4	5
$\mathbf{w}$	0	1	1	2	2	5	3	4	7

Rank function

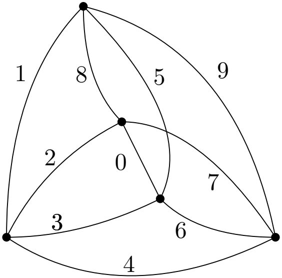
$$r_M(A) = \max_{I \text{ independent}} |A \cap I| \, .$$



$r(1238) = 3$ $\qquad r(57) = 2$ $\qquad r(\emptyset) = 0$

Flats

Maximal sets with a fixed rank.
That is, F is a flat if for any $i \notin F$, $r_M(F \cup i) > r_M(F)$.



$\{\emptyset,$ matchings , complete subgraphs , ...$\}$

$$\{\emptyset \subsetneq 1 \subsetneq 01 \subsetneq 0167 \subsetneq 0123456789\}$$

Flats: The uniform matroid

Basis of the uniform matroid $U(n,k)$ = all sets of size k in $[n]$.
Any set of size $\leq k$ is independent.
Any set of size $\leq k-1$ is a flat.
Any complete flag of flats is of the form

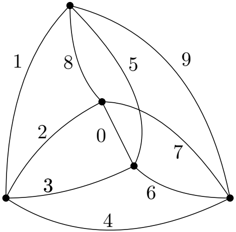
$$\{\emptyset \subsetneq \{v_1\} \subsetneq \{v_1, v_2\} \subsetneq \cdots \subsetneq \{v_1, \ldots, v_{k-1}\} \subsetneq [n]\}$$

$$v_1 \,|\, v_2 \,|\, \ldots \,|\, v_{k-1} \,|\, ([n] \setminus \{v_1, \ldots, v_{k-1}\})$$

Activities

Fix total order in V, ground set of a matroid M.

- $i \in B$ is internal activity if $i = \min C^{\perp}$, where $C^{\perp} \subseteq B^c \cup i$ is a cocircuit.
- $e \notin B$ is external activity if $e = \min C$, where $C \subseteq B \cup i$ is a circuit.



$$i(2567) = 0,\ \ e(2567) = 2,\ \ i(0146) = 2,\ \ e(0146) = 1$$

Tutte polynomial: Deletion-contraction

$$T_M(x,y) = \sum_{A \subseteq V} (x-1)^{r_M(V)-r_M(A)}(y-1)^{|A|-r_M(A)}$$

Deletion-contraction invariant if e is not loop nor coloop:

$$T_M(x,y) = T_{M\backslash e}(x,y) + T_{M/_e}(x,y)$$

$$T_M(x-1,y-1) = \sum_{B\in\mathcal{B}} x^{i(B)}y^{e(B)} = \sum_{i,j} b_{i,j}x^i y^j$$

Observation

*$\#$ of bases with no external activities (called nbc bases): **independent of the order chosen**.*
*$\#$ of bases with no external activities and one internal activity (called β-nbc bases) **independent of the order chosen***

The Bergman Fan

$$\Sigma_M = \{\vec{x} \in \mathbb{R}^n/_{\mathbb{R}\mathbb{1}} | \forall_{C \in \mathcal{C}} \text{ s.t. } \min_{c \in C} x_c \text{ attained twice } \} .$$

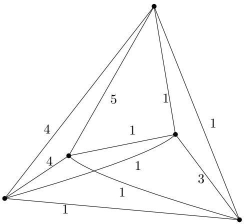
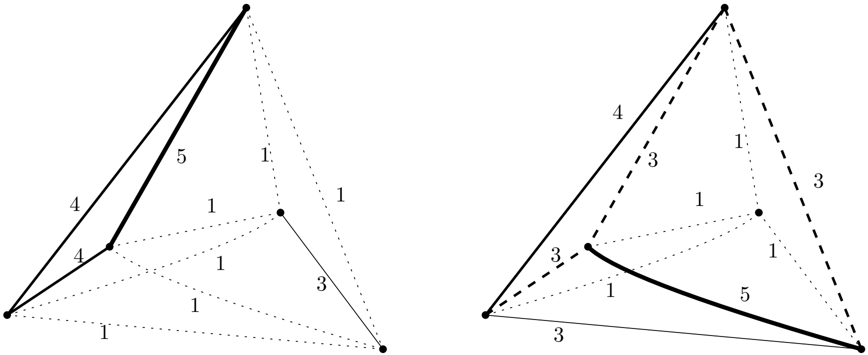


Figure: Two elements in the Bergman fan of the graphical matriod



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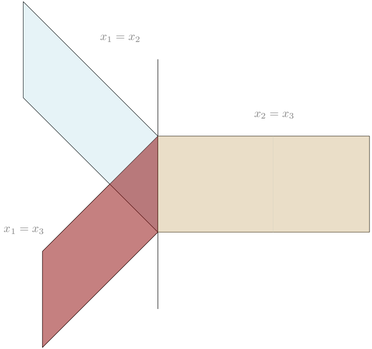


Figure: The Bergman Fan is a polyhedral fan

Theorem (Sturmfels and Feichner, 2004)

The Bergman Fan of a matroid decomposes into the following cones

$$\Sigma_M = \bigcup_{\mathcal{F} \text{ flag of flats}} \mathcal{C}_{\mathcal{F}} = \bigcup_{F_1 \subset \ldots \text{ flag of flats}} \{x_i \geq x_j \text{ whenever } i \in F_k, j \notin F_k\}\,.$$

Problem

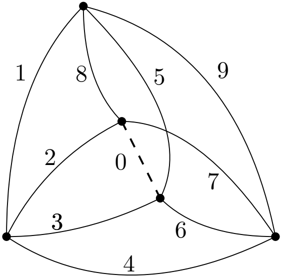
Can we compute the degree of the tropical variety Σ_M?

Just intersect it with a hyperplane!
Note: Hyperplanes in the tropical world are Bergman fans of $U_{n,k}$.

Example of degree computation

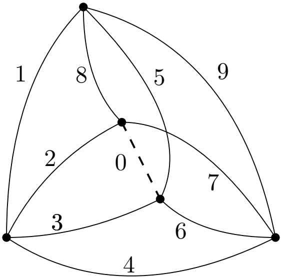
Consider M the graphical matroid of K_5, of rank 4. One has $\Sigma_M \subseteq \mathbb{R}^{10}/_{\mathbb{R}\mathbb{1}}$

$$|\Sigma_{U_{10,7}} + \underbrace{(1, 10, 100, 1000, ..., 10^9)}_{\vec{\omega}} \cap \Sigma_M| = ?$$



$$\vec{v} = (10^6, 10^4, 10^4, 10^4, 10^4, 10^6, 10^6, 10^8, 10^8, 10^9)\,.$$

$\vec{v}$ belongs to the cone $\mathcal{C}_{9|87|650|1234} \subseteq \Sigma_M$
and $\vec{v} - \vec{\omega}$ belongs to $\mathcal{C}_{7|0|5|1|2|3|4689} \subseteq \Sigma_{U_{10,7}}$



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and $\vec{v} - \vec{\omega}$ belongs to $\mathcal{C}_{7|0|5|1|2|3|4689} \subseteq \Sigma_{U_{10,7}}$
where $\vec{\omega} = (1, 10, \ldots, 10^9)$.

Degree computations

Theorem

$$\deg(\Sigma_M) = 1$$

Theorem (Adiprasito, Huh, Katz, 2018)

$$\deg(-\Sigma_M) = \#\{\textit{nbc bases}\}$$

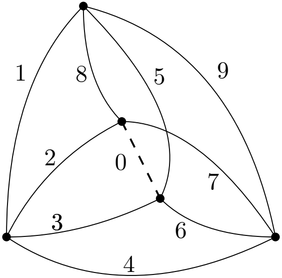
Theorem (Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, Telen 2021 and Ardila, Eur, P 2022)

$$|\Sigma_{(M\backslash 0)^\perp} \cap_{st} -\Sigma_M| = \#\{\beta - \textit{nbc bases}\}$$

Examples

If M is the graphical matroid

$$|\left(\Sigma_{U_{10,7}} + \vec{\omega}\right) \cap \Sigma_M|$$



Then the unique intersection point is

$$\vec{v} = (10^6, 10^4, 10^4, 10^4, 10^4, 10^6, 10^6, 10^8, 10^8, 10^9)\,.$$

Theorem

The unique intersection point of

$$\Sigma_{U_{n,n-r+1}} + \vec{\omega} \cap \Sigma_M$$

lies in $\mathcal{C}_{\mathcal{F}(B)}$, the cone corresponding to the greedy basis with respect to the order induced by $\vec{\omega}$.

Theorem

Each intersection point in

$$|\Sigma_M \cap (\omega - \Sigma_{U_{n,n-r+1}})$$

lies in $\mathcal{C}_{\mathcal{F}(B)}$, the cone of Σ_M corresponding to an nbc-basis with respect to the order induced by $\vec{\omega}$.

Theorem

Each intersection point in

$$|\Sigma_M \cap (\omega - \Sigma_{(M\backslash 0)^\perp})$$

lies in $\mathcal{C}_{\mathcal{F}(B)}$, the cone of Σ_M corresponding to a β-nbc-basis with respect to the order induced by $\vec{\omega}$.

Biblio

- Agostini D., Brysiewicz T., Fevola C., Kühne L., Sturmfels B., Telen S. (2021). *Likelihood Degenerations*. Motivation behind ML degree computations
- Adiprasito K., Huh J., Katz E. (2018). *Hodge Theory for combinatorial geometries* Degree computations in matroids
- Ardila F., Eur C., RP (2022) *The maximum likelihood of a matroid* All these computations are here! To appear

Thank you