Pattern Hopf algebras

Rough Algebra Day 2022, Berlin

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31st March, 2022

Slides can be found at
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Permutations - square diagrams

Permutation π on a set $S = \{s_1, \ldots, s_n\}$ is a pair of orders on S.

$$132 = \begin{array}{|c|c|c|}\hline & s_1 & \\ \hline & & s_3 \\ \hline s_2 & & \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline & \cdot & \\ \hline & & \cdot \\ \hline \cdot & & \\ \hline\end{array}$$

Counting occurrences of a pattern

Permutation π on a set $S = \{s_1, \dots, s_n\}$ and $I \subseteq S$.
The **restriction to** I can be defined $\pi|_I$ and is a permutation in I.
We can count occurrences! How many times does a particular diagram occur? If $\pi = 132$ as above,

$$\pi|_{\{1,3\}} = \left.\begin{array}{|c|c|c|}\hline & \cdot & \\ \hline & & \cdot \\ \hline \cdot & & \\ \hline\end{array}\right|_{\{1,3\}} = \begin{array}{|c|c|}\hline & \cdot \\ \hline \cdot & \\ \hline\end{array}$$

We write

$$\mathbf{p}_{12}(132) = 2,\ \mathbf{p}_{123}(123456) = 20,\ \mathbf{p}_{2413}(762341895) = 0\,.$$

Permutation pattern algebra

Proposition (Linear independence)

The set $\{\mathbf{p}_\pi \mid \pi \in \uplus_{n\geq 0} S_n\}$ *is linearly independent - Triangularity argument*

Proposition (Product formula)

There are coefficients $\binom{\sigma}{\pi, \tau}$ *that count the number of quasi-shuffles of* σ *with permutations* π*,* τ *such that:*

$$\mathbf{p}_\pi \cdot \mathbf{p}_\tau = \sum_\sigma \binom{\sigma}{\pi, \tau} \mathbf{p}_\sigma \,,$$

where σ *runs over equivalence classes of pairs of orders.*

Example try it!

$$\mathbf{p}_{12}\,\mathbf{p}_1 = 3\,\mathbf{p}_{123} + \mathbf{p}_{312} + \mathbf{p}_{231} + 2\,\mathbf{p}_{213} + 2\,\mathbf{p}_{132} \;.$$

Permutation pattern algebra

Theorem (Vargas, 2014)

The linear span of pattern functions $\mathcal{A}(\mathtt{Per})$ form a Hopf algebra.
The Hopf algebra $\mathcal{A}(\mathtt{Per})$ is free commutative. *what is free?*
A free generator family is given by a family of Lyndon permutations.

Permutation pattern algebra - adding another ingredient

$$\pi \oplus \tau = \begin{array}{|c|c|} \hline & \tau \\ \hline \pi & \\ \hline \end{array} \qquad \pi \ominus \tau = \begin{array}{|c|c|} \hline \pi & \\ \hline & \tau \\ \hline \end{array}$$

By *magic properties* of dualization, these give coproducts on $\mathcal{A}(\mathtt{Per})$, for instance:

$$\Delta \, \mathbf{p}_\pi = \sum_{\pi = \tau_1 \oplus \tau_2} \mathbf{p}_{\tau_1} \otimes \mathbf{p}_{\tau_2} \, ,$$

so that we have a Hopf algebra

$$\mathbf{p}_\pi(\sigma_1 \oplus \sigma_2) = \Delta \, \mathbf{p}_\pi(\sigma_1 \otimes \sigma_2) \, .$$

Outline of the talk

1. Introduction
   - Permutations
   - Combinatorial presheaves
2. Free pattern Hopf algebras
3. Non-cocommutative examples
   - Permutations
   - Marked permutations

Pattern algebra

What do we need to have a pattern Hopf algebra?

1. Assignment $S \mapsto h[S] = \{\text{structures over } S\}$ + notion of *relabelling*.
2. For any inclusion $V \hookrightarrow W$, a restriction map $h[W] \to h[V]$.
3. An associative *monoid* operation $*$ with unit that is compatible with restrictions.
4. A unique element of size zero.

$1 + 2 =$ *combinatorial presheaf* $\to$ Algebra.
$1 + 2 + 3 =$ *monoid in combinatorial presheaves* $\to$ bialgebra.
$1 + 2 + 4 =$ *connected presheaf* $\to$ Filtered (read graded) connected algebra.
$1 + 2 + 3 + 4 \to$ Hopf algebra.

A presheaf on graphs

- $\mathtt{G}[V] = \{$ graphs on the vertex set V $\}$.
- Induced subgraphs $\rightarrow$ restrictions.
- The disjoint union of graphs.
- The empty graph fortunately exists!

Theorem (P - 2019+)

If $\mathtt{h}$ is a connected commutative presheaf, then $\mathcal{A}(\mathtt{h})$ is free. The free generators are the indecomposable objects with respect to the commutative product.

Simple example - binomial identities

Define $\texttt{Set}[n] = \{*_n\}$ to have a unique element.

$$\mathbf{p}_{*_n}(*_m) = \binom{m}{n} \qquad \binom{*_d}{*_a, *_b} = \binom{d}{a}\binom{a}{a+b-d}.$$

So we obtain the following binomial identity:

$$\mathbf{p}_{*_a}(*_c)\,\mathbf{p}_{*_b}(*_c) = \sum_{d \geq 0} \binom{d}{a}\binom{a}{a+b-d} \mathbf{p}_{*_d}(*_c)$$

Monoidal structure - Disjoint union: $*_n *_m = *_{n+m}$.

$$\Delta\, \mathbf{p}_{*_a} = \sum_{k=0}^{a} \mathbf{p}_{*_k} \otimes \mathbf{p}_{*_{a-k}}, \qquad \mathcal{A}(\texttt{Set}) = \mathbb{R}\langle \mathbf{p}_{*_1} \rangle$$

Hey, can you tell me what happens on words?

The question of analysing patterns on words is a very natural one.
$\mathtt{Word}_{\mathcal{K}}[n] = \{$ words of size n using an alphabet $\mathcal{K}$ characters $\}$

The resulting algebra is very similar to the **shuffle algebra** on the alphabet $\mathcal{K}$, and is also free.

Unique factorisation theorem on permutations

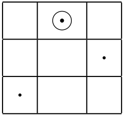
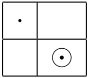
Vargas used the $\oplus$ product on permutations to obtain a unique factorisation theorem on permutations.

$$\pi = \tau_1 \oplus \cdots \oplus \tau_k = \begin{array}{|c|c|c|} \hline & & \tau_k \\ \hline & \iddots & \\ \hline \tau_1 & & \\ \hline \end{array}$$

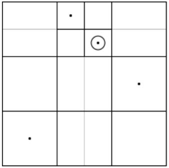
The factorisation is **not** unique up to order of factors.

Marked permutations

Presheaf of marked permutations.

$\pi =$ , $\sigma =$ 

Inflation of $\pi * \sigma$ is



Freeness on marked permutations

Theorem (P - 2020)

The Hopf algebra $\mathcal{A}(\mathtt{MPer})$ is free generated by the Lyndon marked permutations.

Conjecture

Any pattern algebra is free.

Hopfy questions about pattern algebras - the applications slide

Hey, can you tell us more about the antipode of these Hopf algebras?
Yes! Existence follows from the so called *Takeuchi formula*, but cancellation-free formulas have been found [P. Vargas 2022+].

What are the characters of this Hopf algebra?
Are there any characters? Evaluation characters! We can recover the inversion polynomial on permutations.

Biblio

- Aguiar, M., & Mahajan, S. A. (2010). Monoidal functors, species and Hopf algebras (Vol. 29). *Providence, RI: American Mathematical Society*. [All the category theory behind pattern algebras]
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- Kenyon, R., Kral, D., Radin, C., & Winkler, P. (2015). Permutations with fixed pattern densities. *arXiv preprint* arXiv:1506.02340. [For an unexpected application of Vargas' theorem!]

Thank you