The tropical critical points of an affine matroid

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Slides can be found at `raulpenaguiao.github.io/`

Joint work with Federico Ardila and Chris Eur

Optimization of a monomial

Fix some vector $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n_{>0}$.
Consider $f_{\mathbf{t}} : \mathbf{x} \mapsto x_1^{w_1} \dots x_n^{w_n}$ from a variety $X \subset (\mathbb{C}^*)^n$.

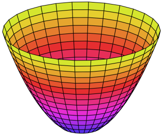


Figure: A variety where we can optimize $f_{\mathbf{w}}$

What is the number of critical points of f? Does it depend on the choice of $\mathbf{w}$? For generic $\mathbf{w}$, no! This number is called the **maximum likelihood degree** of a model X.
If X is a vector space, MLDeg$(X) = \beta(M)$.

Edge weight problem

Fix some vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n_{\geq 0}$. Can we find **compatible** weights $\mathbf{x}$ of the edges of G, $\mathbf{y}$ of the edges of $(G/_0)^*$?

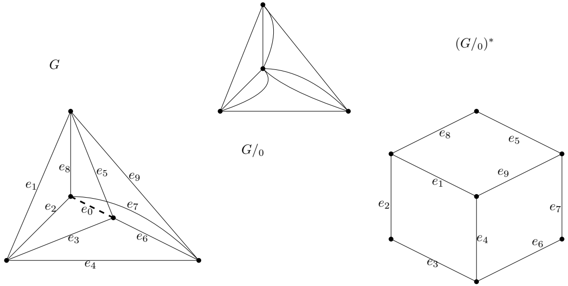


Figure: Find $\mathbf{x}$ and $\mathbf{y}$ edge weights that are *compatible*.

- The sum of the weights is $\mathbf{w}$.
- Every cycle has at least two minimal edges (edge 0 has weight 0).

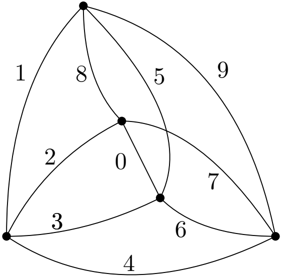
Fix $\mathbf{w} = (0, 1, 1, 2, 2, 5, 3, 4, 7)$.

Figure: Find $\mathbf{x}$ and $\mathbf{y}$ edge weights that are *compatible*.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
$\mathbf{x}$	00	01	00	00	02	03	00	00	02
$\mathbf{y}$	000	110	111	222	20	52	33	44	75
$\mathbf{w}$	0	1	1	2	2	5	3	4	7

Flats

Maximal sets with a fixed rank.
That is, F is a flat if for any $i \notin F$, $r_M(F \cup i) > r_M(F)$.



$\{\emptyset,$ matchings , complete subgraphs , ...$\}$

$$\{\emptyset \subsetneq 1 \subsetneq 01 \subsetneq 0167 \subsetneq 0123456789\}$$

Flats: The uniform matroid

Basis of the uniform matroid $U(n,k)$ = all sets of size k in $[n]$.
Any set of size $\leq k$ is independent.
Any set of size $\leq k-1$ is a flat.
Any complete flag of flats is of the form

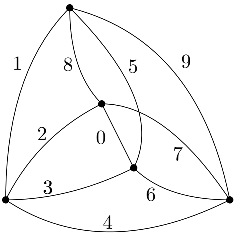
$$\{\emptyset \subsetneq \{v_1\} \subsetneq \{v_1, v_2\} \subsetneq \cdots \subsetneq \{v_1, \dots, v_{k-1}\} \subsetneq [n]\}$$

$$v_1 \,|\, v_2 \,|\, \dots \,|\, v_{k-1} \,|\, ([n] \setminus \{v_1, \dots, v_{k-1}\})$$

Activities

Fix total order in V, ground set of a matroid M.

- $i \in B$ is internal activity if $i = \min C^{\perp}$, where $C^{\perp} \subseteq B^c \cup i$ is a cocircuit (a cut of the graph).
- $e \notin B$ is external activity if $e = \min C$, where $C \subseteq B \cup i$ is a circuit.



$$i(2567) = 0,\ \ e(2567) = 2,\ \ i(0146) = 2,\ \ e(0146) = 1$$

Tutte polynomial

$$T_M(x, y) = \sum_{A \subseteq V} (x-1)^{r_M(V) - r_M(A)} (y-1)^{|A| - r_M(A)}$$

$$T_M(x, y) = \sum_{B \in \mathcal{B}} x^{i(B)} y^{e(B)} = \sum_{i,j} b_{i,j} x^i y^j$$

Observation

$\#$ of bases with no external activities (called nbc bases):
independent of the order chosen*.*
$\#$ of bases with no external activities and one internal activity (called β-nbc bases) ***independent of the order chosen***

The Bergman Fan

$$\Sigma_M = \{\vec{x} \in \mathbb{R}^n/_{\mathbb{R}\mathbb{1}} | \forall_{C \in \mathcal{C}} \text{ s.t. } \min_{c \in C} x_c \text{ attained twice } \}.$$

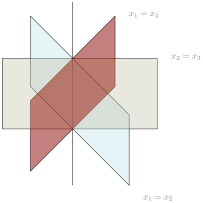
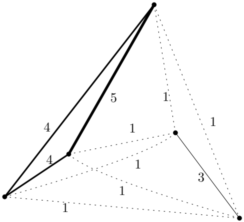


Figure: The Bergman Fan is a polyhedral fan in $\mathbb{R}^n/_{\mathbb{R}\mathbb{1}}$.



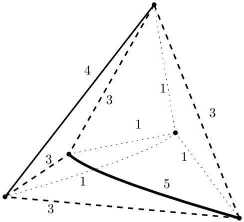


Figure: Two elements in the Bergman fan of the graphical matroid

Theorem (Sturmfels and Feichner, 2004)

The Bergman Fan of a matroid decomposes into the following cones

$$\Sigma_M = \bigcup_{\mathcal{F} \text{ flag of flats}} \mathcal{C}_{\mathcal{F}} = \bigcup_{F_1 \subset \dots \text{ flag of flats}} \{x_i \geq x_j \text{ whenever } i \in F_k, j \notin F_k\}\,.$$

Problem

Can we compute the degree of the tropical variety Σ_M?

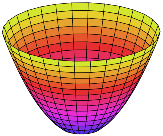


Figure: A variety: its degree is the $\#$ of intersections with a line **in the complex plane**.

Just intersect it with a hyperplane!
Note: Hyperplanes in the tropical world are Bergman fans of $U_{n,k}$.

Example of degree computation

Consider M the graphical matroid of K_5, of rank 4 (so $\dim \Sigma_M = 3$). One has $\Sigma_M \subseteq \mathbb{R}^{10}/_{\mathbb{R}\mathbb{1}}$, we intersect it with a hyperplane of $dim = 6$.

$$|\Sigma_{U_{10,7}} + \underbrace{(1, 10, 100, 1000, ..., 10^9)}_{\vec{\omega}} \cap \Sigma_M| = ?$$

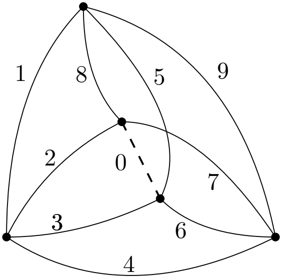


Figure: There is only one $\mathbf{x} \in \Sigma_M$ and $\mathbf{y} \in \Sigma_U$ such that $\mathbf{y} + \mathbf{w} = \mathbf{x}$. Such vector $\mathbf{x}$ belongs to the **greedy flag of flats**.

Degree computations

Theorem (Greedy basis algorithm)

$$\deg(\Sigma_M) = 1$$

Theorem (Adiprasito, Huh, Katz, 2018)

$$\deg(-\Sigma_M) = \#\{\textit{nbc bases}\}$$

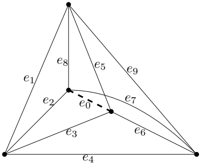
Theorem (Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, Telen 2021 and Ardila, Eur, P 2022)

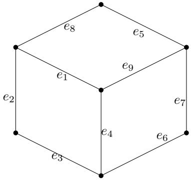
$$\deg(\Sigma_{(M/_0)^\perp} \cdot -\Sigma_M) = \#\{\beta - \textit{nbc bases}\} = \beta(M)$$

Finding ~~Nemo~~ points form nbc bases

$\deg(\Sigma_{(M/0)^\perp} \cdot -\Sigma_M) = ?$ Fix generic $\mathbf{w} = (10^{i-1})_i$.

Find $\mathbf{x} \in \Sigma_M$, $\mathbf{y} \in \Sigma_{(M/0)^\perp}$ such that $\mathbf{x} + \mathbf{y} = \mathbf{w}$.





Find a β-nbc basis $B = 0257 \to 7|5|24|013689$,

corresponding β-nbc cobasis $B^\perp = 134689 \to 9|8|6|74|3|521$.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
$\mathbf{x}$	0	9	0	9	$10^4 - 1$	0	$10^6 - 10^3 + 9$	0	0
$\mathbf{y}$	1	1	100	$10^3 - 9$	1	10^5	$10^3 - 9$	10^7	10^8
$\mathbf{w}$	1	10	100	10^3	10^4	10^5	10^6	10^7	10^8

Biblio

- Agostini D., Brysiewicz T., Fevola C., Kühne L., Sturmfels B., Telen S. (2021). *Likelihood Degenerations*. Motivation behind ML degree computations
- Adiprasito K., Huh J., Katz E. (2018). *Hodge Theory for combinatorial geometries* Degree computations in matroids
- Ardila F., Eur C., RP (2022) *The maximum likelihood of a matroid*.

Thank you