Séminaire Lotharingien de Combinatoire **XX** (2023)
Article #YY, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

The maximum likelihood degree of a matroid

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Abstract. We prove that the maximum likelihood degree of a matroid M equals its beta invariant. For an element e of M that is neither a loop nor a coloop, this is defined to be the degree of the intersection of the Bergman fan of (M,e) and the inverted Bergman fan of N=(M/e)*. Equivalently, for a generic weight vector w on E-e, this is the number of ways to find weights (0,x) on M and y on N with x+y=w such that on each circuit of M (resp. N), the minimum x-weight (resp. y-weight) occurs at least twice.

Keywords: matroid, Bergman fan, maximum likelihood degree

1 Introduction

During the Workshop on Nonlinear Algebra and Combinatorics from Physics at the Center for the Mathematical Sciences and Applications at Harvard University in April 2022, Sturmfels [15] posed one of those combinatorial problems that is deceivingly simple to state, but whose answer requires a deeper understanding of the objects at hand.

Conjecture 1.1. *[15]* (Combinatorial version) *Let M be a matroid on ground set E and let e be an element that is neither a loop nor a coloop. Let M/e be the contraction of M by e and let $N = (M/e)^{\perp}$ be its dual matroid. Given a vector $\mathsf{w} \in \mathbb{R}^{E-e}$, we wish to find weight vectors $(0, \mathsf{x}) \in \mathbb{R}^E$ on M (where e has weight 0) and $\mathsf{y} \in \mathbb{R}^{E-e}$ on N such that*

- *on each circuit of M, the minimum x-weight occurs at least twice,*
- *on each circuit of N, the minimum y-weight occurs at least twice, and*
- $\mathsf{w} = \mathsf{x} + \mathsf{y}$.

Prove that, for generic w, the number of solutions is the beta invariant $\beta(M)$.

The following is our main result.

Theorem 1.2. (Combinatorial version) *Conjecture 1.1 is true.*

We now restate Theorem 1.2 in tropical terms; see relevant definitions in Section 2.3.

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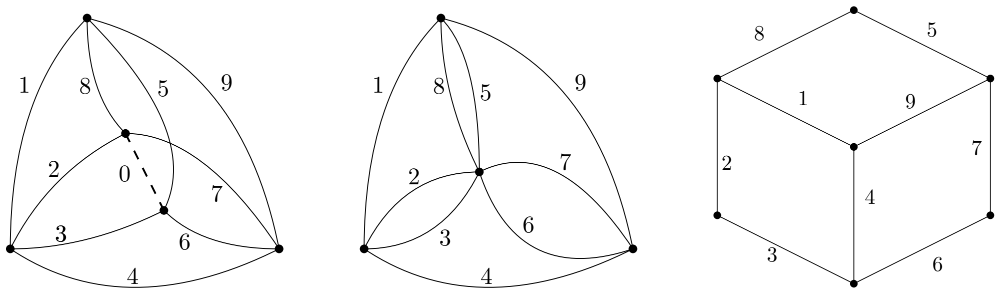

Figure 1: A graph G, its contraction $G/0$, and its dual $H = (G/0)^{\perp}$.

Theorem 1.3. (Geometric version) *Let M be a matroid on E, and let $e \in E$ be an element that is neither a loop nor a coloop. Let (M, e) be the affine matroid of M with respect to e, and let $N = (M/e)^{\perp}$. Then the degree of the intersection of the Bergman fan $\Sigma_{(M,e)}$ and the inverted Bergman fan $-\Sigma_N$ is*

$$\deg(\Sigma_{(M,e)} \cdot -\Sigma_{(M/e)^{\perp}}) = \beta(M)\,.$$

Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, and Telen [1] first encountered (a special case of) Problem 1.1 in their study of the maximum likelihood estimation for linear discrete models. Using algebro-geometric results of Huh and Sturmfels [10], which built on earlier work of Varchenko [16], they proved Theorem 1.2 and Theorem 1.3 for matroids realizable over the real numbers.

We prove the equivalent Theorem 1.2 and Theorem 1.3 for all matroids. Following the original motivation, we call the answer to Conjecture 1.1 the maximum likelihood degree of a matroid; our main result is that it equals the beta invariant.

We first prove Theorem 1.2 combinatorially, relying on the tropical geometric fact that the number of solutions is the same for all generic w. We show that when the entries of w are super-increasing with respect to some order $<$ on E, the solutions to Problem 1.1 are naturally in bijection with the β-nbc bases of the matroid with respect to $<$. It is known that the number of such bases is the beta invariant for any order $<$.

We then sketch a proof of Theorem 1.3 that relies on recent developments on the combinatorial algebraic geometry of matroids. It combines the theory of tautological classes of matroids of Berget, Eur, Spink, and Tseng [7] with a powerful observation that they made, based on Derksen and Fink's work on matroid valuations [9]: If two quantities $f(M)$ and $g(M)$ coincide for matroids realizable over $\mathbb{R}$, and if the functions $f(-)$ and $g(-)$ are valuative under matroid subdivisions, then $f(M)$ and $g(M)$ coincide in general. This proof is seemingly shorter, but it relies on the validity of the theorem when M is realizable over $\mathbb{R}$, which was established in the previous work [1], combining results of [10, 16].

This is an extended abstract of our results in [6].

2 Notation and preliminaries

2.1 The lattice of set partitions

A set partition $\boldsymbol{\lambda}$ of a set E is a collection of subsets, called blocks, of E, say $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_\ell\}$, whose union is E and whose pairwise intersections are empty. We write $\boldsymbol{\lambda} \models E$. We let $|\boldsymbol{\lambda}| = \ell$ be the number of blocks of $\boldsymbol{\lambda}$. If $e \in E$ and $\boldsymbol{\lambda} \models E$, we write $\boldsymbol{\lambda}(e)$ for the block of $\boldsymbol{\lambda}$ that contains e.

We define the *linear space of a set partition* $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_\ell\} \models E$ to be

$$\begin{aligned} \mathsf{L}(\boldsymbol{\lambda}) &:= \operatorname{span}\{\mathsf{e}_{\lambda_1}, \ldots, \mathsf{e}_{\lambda_\ell}\} \subseteq \mathbb{R}^E \\ &= \{\mathsf{x} \in \mathbb{R}^E \mid x_i = x_j \text{ whenever } i, j \text{ are in the same block of } \boldsymbol{\lambda}\}, \end{aligned}$$

where $\{\mathsf{e}_i : i \in E\}$ is the standard basis of $\mathbb{R}^E$, $\mathsf{e}_S = \sum_{s \in S} e_s$ for $S \subsetneq E$. Notice that $\dim \mathsf{L}(\boldsymbol{\lambda}) = |\boldsymbol{\lambda}|$. The map $\boldsymbol{\lambda} \mapsto \mathsf{L}(\boldsymbol{\lambda})$ is a bijection between the set partitions of E and the flats of the *braid arrangement*, which is the hyperplane arrangement in $\mathbb{R}^E$ given by the hyperplanes $x_i = x_j$ for $i \neq j$ in E.

If $\boldsymbol{\lambda} \models 0 \sqcup E$ then we write $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} = \{\mathsf{x} \in \mathbb{R}^E : (0, \mathsf{x}) \in \mathsf{L}(\boldsymbol{\lambda}) \subseteq \mathbb{R}^{0 \sqcup E}\}$.

2.2 The intersection graph of two set partitions

The following construction from [3] will play an important role.

Definition 2.1. *Let $\boldsymbol{\lambda} \models 0 \sqcup E$ and $\boldsymbol{\mu} \models E$ be set partitions. The* intersection graph $\Gamma = \Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *is the bipartite graph with vertex set $\boldsymbol{\lambda} \sqcup \boldsymbol{\mu}$ and edge set E, where the edge e connects the parts $\lambda(e)$ of $\boldsymbol{\lambda}$ and $\mu(e)$ of $\boldsymbol{\mu}$ containing e. On this graph, the vertex corresponding to $\boldsymbol{\lambda}(0)$ is marked with a hollow point.*

The intersection graph may have several parallel edges connecting the same pair of vertices. Notice that the label of a vertex in Γ is just the set of labels of the edges incident to it. Therefore we can remove the vertex labels, and simply think of Γ as a bipartite multigraph on edge set E. This is illustrated in Figure 2.

Lemma 2.2. *Let $\boldsymbol{\lambda} \models 0 \sqcup E$ and $\boldsymbol{\mu} \models E$ be set partitions and $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ be their intersection graph.*

1. *If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ has a cycle, then $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu})) = \emptyset$ for generic*[1] $\mathsf{w} \in \mathbb{R}^E$.
2. *If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is disconnected, then $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is not a point for any $\mathsf{w} \in \mathbb{R}^E$.*
3. *If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree, then $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is a point for any $\mathsf{w} \in \mathbb{R}^E$.*

[1] This means that this property holds for all w outside of a set of measure 0.

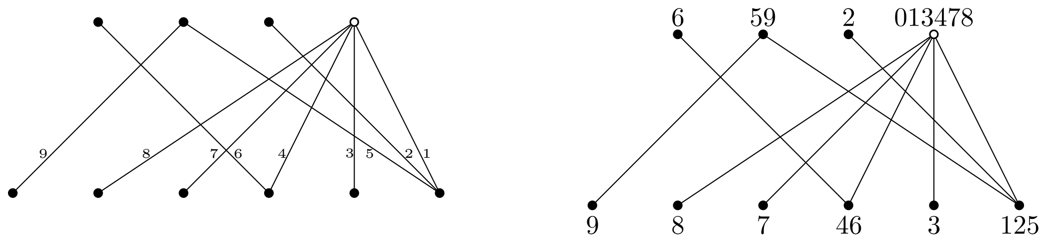

Figure 2: For $\boldsymbol{\lambda} = \{6, 59, 2, 013478\} \models [0,9]$ and $\boldsymbol{\mu} = \{9, 8, 7, 46, 3, 125\} \models [9]$, the intersection graph is a tree. We omit brackets for legibility. **Left:** The edges are labelled by the elements of $[9]$. **Right:** The vertices are labelled by the blocks of the set partitions.

Proof. Let $\mathsf{x} \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} \in \mathsf{L}(\mu)$ such that $\mathsf{x}+\mathsf{y} = \mathsf{w}$. Write $x_{\lambda(i)} := x_i$ and $y_{\mu(j)} := y_j$ for simplicity. The subspace $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is cut out by the equalities

$$\begin{aligned} x_{\lambda(i)} + y_{\mu(i)} &= w_i \qquad \text{for } i \in E, \\ x_{\lambda(0)} &= 0. \end{aligned}$$

This system has $|E|+1$ equations and $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$ unknowns. The linear dependences among these equations are controlled by the cycles of the graph $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. More precisely, the first $|E|$ linear functionals $\{x_{\lambda(i)} + y_{\mu(i)} \,:\, i \in E\}$ gives a realization of the graphical matroid of $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. The last equation is clearly linearly independent from the others.

If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ has a cycle with edges $i_1, i_2, \ldots, i_{2k}$ in that order, then the above equalities imply that $w_{i_1} - w_{i_2} + w_{i_3} - \cdots - w_{i_{2k}} = 0$. For a generic w, this equation does not hold, so $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu})) = \emptyset$.

If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is disconnected, let A be the set of edges in a connected component not containing the vertex $\lambda(0)$. If $\mathsf{x} \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} \in \mathsf{L}(\boldsymbol{\mu})$ satisfy $\mathsf{x}+\mathsf{y} = \mathsf{w}$ and $x_0 = 0$, then $\mathsf{x} + r\mathsf{e}_A \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} - r\mathsf{e}_A \in \mathsf{L}(\boldsymbol{\mu})$ also satisfy those equations for any real number r. Therefore $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is not a point.

Finally, if $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree, then its number of vertices is one more than the number of edges, that is, $|E| + 1 = |\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$, so the system of equations has equally many equations and unknowns. Also, these equations are linearly independent since $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree. It follows that the system has a unique solution. $\square$

When $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree, we call $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ an *arboreal pair*.

Lemma 2.3. *Let $\boldsymbol{\lambda} \models 0 \sqcup E$ and $\boldsymbol{\mu} \models E$ be an arboreal pair of set partitions and let $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ be their intersection tree. Let $\mathsf{w} \in \mathbb{R}^E$. The unique vectors $\mathsf{x} \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} \in \mathsf{L}(\boldsymbol{\mu})$ such that $\mathsf{x}+\mathsf{y} = \mathsf{w}$ and $x_0 = 0$ are given by*

$$\begin{aligned} x_{\lambda_i} &= w_{e_1} - w_{e_2} + \cdots \pm w_{e_k} \qquad \textit{where } e_1 e_2 \ldots e_k \textit{ is the unique path from } \lambda_i \textit{ to } \lambda(0) \\ y_{\mu_j} &= w_{f_1} - w_{f_2} + \cdots \pm w_{f_l} \qquad \textit{where } f_1 f_2 \ldots f_l \textit{ is the unique path from } \mu_j \textit{ to } \lambda(0) \end{aligned}$$

for any i and j.

Proof. This follows readily from the fact that, for each $1 \leq i \leq k$, the values of $x_{\lambda(e_i)}$ and $y_{\mu(e_i)}$ on the vertices incident to edge i have to add up to w_{e_i}. $\square$

Example 2.4. *Let $\boldsymbol{\lambda} = \{6, 59, 2, 013478\} \models [0,9]$ and $\boldsymbol{\mu} = \{9, 8, 7, 46, 3, 125\} \models [9]$. These set partitions form an arboreal pair, as evidenced by their intersection tree, shown in Figure 2. We have, for example, $y_9 = w_9 - w_5 + w_1$ because the path from $\mu(9) = \{9\}$ to $\lambda(0) = \{013478\}$ uses edges $9, 5, 1$ in that order. The remaining values are:*

$$x_6 = w_6 - w_4, \quad x_{59} = w_5 - w_1, \quad x_2 = w_2 - w_1, \quad x_{13478} = 0,$$
$$y_9 = w_9 - w_5 + w_1, \quad y_8 = w_8, \quad y_7 = w_7, \quad y_{46} = w_4, \quad y_3 = w_3, \quad y_{1235} = w_1.$$

Definition 2.5. *A vector* $\mathsf{w} \in \mathbb{R}^{n+1}$ *is* super-increasing *if $w_{i+1} > 3w_i > 0$ for $1 \leq i \leq n$.*

Lemma 2.6. *Let* w *be super-increasing. For any $1 \leq a < b \leq n+1$ and any choice of ϵ_is and δ_is in $\{-1, 0, 1\}$, we have $w_a + \sum_{i=1}^{a-1} \epsilon_i w_i < w_b + \sum_{j=1}^{b-1} \delta_j w_j$.*

Proof. Omitted. $\square$

Definition 2.7. *Given a super-increasing vector* $\mathsf{w} \in \mathbb{R}^{n+1}$ *and a real number x, we will say x* is near w_i *and write $x \approx w_i$ if $w_i - (w_1 + \cdots + w_{i-1}) \leq x \leq w_i + (w_1 + \cdots + w_{i-1})$. Note that if $x \approx w_i$ and $y \approx w_j$ for $i < j$ then $x < y$.*

2.3 Matroids, Bergman fans, and tropical geometry

We assume familiarity with basic notions in matroid theory; for definitions and proofs, see [13, 17]. We also state here some facts from tropical geometry that we will need; see [11, 12] for a thorough introduction.

Let M be a matroid on E of rank $r+1$. The *dual matroid* $M^\perp$ is the matroid on E whose set of bases is $\{B^\perp \mid B \text{ is a basis of } M\}$, where $B^\perp := E - B$. The following lemma is useful to how M and $M^\perp$ interact; see [2, Lemma 3.14] and [13, Proposition 2.1.11].

Lemma 2.8. *If F is a flat of M and G is a flat of $M^\perp$, then $|F \cup G| \neq |E| - 1$.*

Definition 2.9. *Fix a linear order $<$ on M. A* broken circuit *is a set of the form $C - \min_< C$ where C is a circuit of M. An* nbc-basis *of M is a basis of M that contains no broken circuits. A* βnbc-basis *of M is an nbc-basis B such that $B^\perp \cup 0 \setminus 1$ is an nbc-basis of $M^\perp$.*

Theorem 2.10. *[8] The number of βnbc-bases of M is the* beta invariant $\beta(M)$ *which is given by $\beta(M) := |\chi'_M(1)|$, where χ_M is the* characteristic polynomial *of M:*

$$\chi_M(t) := \sum_{X \subseteq E} (-1)^{|X|} t^{r(M) - r(X)}.$$

For each basis $B = \{b_1 > \cdots > b_r > b_{r+1}\}$ of the matroid M, we define the complete flag of flats

$$\mathcal{F}_M(B) := \{\emptyset \subsetneq \mathrm{cl}_M\{b_1\} \subsetneq \mathrm{cl}_M\{b_1, b_2\} \subsetneq \cdots \subsetneq \mathrm{cl}_M\{b_1, \ldots, b_r\} \subsetneq E\}.$$

The following characterization of nbc-bases will be useful.

Lemma 2.11. *Let M be a matroid of size $n+1$ and rank $r+1$, and B a basis of M. Then B is an nbc-basis of M if and only if $b_i = \min F_i$ for $i = 1, \ldots, r+1$.*

Proof. Omitted. □

An *affine matroid* (M, e) on E is a matroid M on E with a chosen element $e \in E$.

Definition 2.12. *[14] The* Bergman fan *of a matroid M on E is*

$$\Sigma_M = \{\mathrm{x} \in \mathbb{R}^E \mid \min_{c \in C} x_c \text{ is attained at least twice for any circuit } C \text{ of } M\}.$$

The Bergman fan of an affine matroid (M, e) *on E is*

$$\Sigma_{(M,e)} = \{\mathrm{x} \in \mathbb{R}^{E-e} \mid (0, \mathrm{x}) \in \Sigma_M\}.$$

The motivation for this definition comes from tropical geometry. A subspace $V \subset \mathbb{R}^E$ determines a matroid M_V on E, and the tropicalization of V is precisely the Bergman fan of M_V. Similarly, an affine subspace $W \subset \mathbb{R}^{E-e}$ determines an affine matroid (M_W, e) on E, where e represents the hyperplane at infinity. The tropicalization of W is the Bergman fan $\Sigma_{(M_W,e)}$.

Theorem 2.13. *[5] The Bergman fan of a matroid M is equal to the union of the cones*

$$\begin{aligned} \sigma_{\mathcal{F}} &= \mathit{cone}(\mathrm{e}_{F_1}, \ldots, \mathrm{e}_{F_{r+1}}) \\ &= \{\mathrm{x} \in \mathbb{R}^E \mid x_a \geq x_b \text{ whenever } a \in F_i \text{ and } b \in F_j \text{ for some } 1 \leq i \leq j \leq r+1\} \end{aligned}$$

for the complete flags $\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E\}$ of flats of M. It is a tropical fan with weights $w(\mathcal{F}) = 1$ for all $\mathcal{F}$.

If Σ_1 and Σ_2 are tropical fans of complementary dimensions, then Σ_1 and $v + \Sigma_2$ intersect transversally at a finite set of points for any generic vector $v \in \mathbb{R}^n$. Furthermore, each intersection point p is equipped with a multiplicity $w(p)$ that depends on the respective intersecting cones, in such a way that the quantity

$$\deg(\Sigma_1 \cdot \Sigma_2) := \sum_{p \in \Sigma_1 \cap (v + \Sigma_2)} w(p)$$

is constant for generic v [12, Proposition 4.3.3, 4.3.6]; this is the *degree* of the intersection.

In all the tropical intersections that arise in this paper, one can verify that the multiplicity $w(p)$ is equal to 1 for every intersection point p. Therefore the degree of the intersection will be simply the number of intersection points:

$$\deg(\Sigma_{(M,e)} \cdot -\Sigma_{(M/e)^\perp}) := |\Sigma_{(M,e)} \cap (v - \Sigma_{(M/e)^\perp})|$$

for generic $v \in \mathbb{R}^{E-e}$.

3 Combinatorial proof of the main theorem

Let M be a matroid on $[0,n]$ of rank $r+1$ such that 0 is not a loop nor a coloop. Then $M/0$ has rank r, and $N=(M/0)^\perp$ has rank $n-r$. For any basis B of M containing 0, $B^\perp=[0,n]-B$ is a basis of $N=(M/0)^\perp$. Conversely, every basis of N equals $B^\perp$ for a basis B of M containing 0.

Let us construct an intersection point in $\Sigma_{(M,0)}\cap(\mathsf{w}-\Sigma_N)$ for each β-nbc basis of M.

Lemma 3.1. *Let M be a matroid on $E=[0,n]$ of rank $r+1$ such that 0 is not a coloop, and let $N=(M/0)^\perp$. Let $\mathsf{w}\in\mathbb{R}^n$ be super-increasing. For any β-nbc basis B of M, there exist unique vectors $(0,\mathsf{x})\in\sigma_{\mathcal{F}_M(B)}$ and $\mathsf{y}\in\sigma_{\mathcal{F}_N(B^\perp)}$ such that $\mathsf{x}+\mathsf{y}=\mathsf{w}$.*

Proof. First we show that the set partitions $\boldsymbol{\pi}$ of $\mathcal{F}=\mathcal{F}_M(B)$ and $\boldsymbol{\pi}^\perp$ of $\mathcal{F}^\perp=\mathcal{F}_N(B^\perp)$ form an arboreal pair. Since they have sizes $|B|=r+1$ and $|B^\perp|=n-r$, respectively, their intersection graph has $n+1$ vertices and n edges. Therefore it is sufficient to prove that the intersection graph $\Gamma_{\boldsymbol{\pi},\boldsymbol{\pi}^\perp}$ is connected; this implies that it is a tree.

Assume contrariwise, and let A be a connected component not containing the edge 1. Let $a>1$ be the smallest edge in A. Then a is the smallest element of its part $\boldsymbol{\pi}(a)$ in $\boldsymbol{\pi}$, and since B is nbc in M, this implies $a\in B$. Similarly, since $B^\perp$ is nbc in N, this also implies $a\in B^\perp$. This is a contradiction.

It follows from Lemma 2.2 that there exist unique $(0,\mathsf{x})\in\mathsf{L}(\boldsymbol{\pi})$ and $\mathsf{y}\in\mathsf{L}(\boldsymbol{\pi}^\perp)$ such that $\mathsf{x}+\mathsf{y}=\mathsf{w}$. It remains to show that $(0,\mathsf{x})\in\sigma_{\mathcal{F}}$ and $\mathsf{y}\in\sigma_{\mathcal{F}^\perp}$.

Lemma 2.3 provides formulas for x and y in terms of the paths from the various vertices of the tree of $\Gamma_{\boldsymbol{\pi},\boldsymbol{\pi}^\perp}$ to $\pi(0)$. To understand those paths, let us give each edge e an orientation as follows:

$$\begin{aligned}\pi(e)\longrightarrow\pi^\perp(e)&\quad\text{if}\quad\min\pi(e)>\min\pi^\perp(e),\\ \pi(e)\longleftarrow\pi^\perp(e)&\quad\text{if}\quad\min\pi(e)<\min\pi^\perp(e).\end{aligned}$$

We never have $\min\pi(e)=\min\pi^\perp(e)$, because as above, that would imply $e\in B\cap B^\perp$.

We claim that every vertex other than $\pi(0)$ has an outgoing edge under this orientation. Consider a part $\pi_i\neq\pi(0)$ of $\boldsymbol{\pi}$; let $\min\pi_i=b$. Edge b connects $\pi_i=\pi(b)$ to $\pi^\perp(b)\ni b$, and we cannot have $\min\pi^\perp(b)>b=\min\pi(b)$, so we must have $\pi_i\to\pi^\perp(b)$. The same argument works for any part $\pi_j^\perp$ of $\boldsymbol{\pi}^\perp$.

Now, since B is an nbc basis of M, every element $b\in B$ is minimum in $\pi(b)$, so there is a directed path that starts at $\pi(b)$ and can only end at $\pi(0)$ whose first edge is b. Furthermore, by the definition of the orientation, the labels of the edges decrease along this path. Thus in the alternating sum $x_b=w_b\pm\cdots$ given by Lemma 2.3, the first term dominates, and $x_b\approx w_b$. Similarly, since $B^\perp$ is an nbc basis of N, $y_c\approx w_c$ for all $c\in B^\perp$.

Therefore, if we write $B=\{b_1>\cdots>b_r>b_{r+1}=0\}$, since w is super-increasing, it follows that $x_{b_1}>x_{b_2}>\cdots>x_{b_r}>x_{b_{r+1}}=0$, so indeed $(0,\mathsf{x})\in\sigma_{\mathcal{F}}$. Similarly, if we write $B^\perp = E - B = \{c_1 > \cdots > c_{n-r} > c_{n-r+1} = 1\}$, then $y_{c_1} > y_{c_2} > \cdots > y_{c_{n-r+1}}$, so $\mathsf{y} \in \sigma_{\mathcal{F}^\perp}$. The desired result follows. □

Example 3.2. *The graphical matroid M of the graph G in Figure 1 has six β-nbc bases: 0256, 0257, 0259, 0368, 0378, and 0379. Let us compute the intersection point in $\Sigma_{(M,0)} \cap (\mathsf{w} - \Sigma_N)$ associated to 0257 for the super-increasing vector $\mathsf{w} = (10^0, 10^1, \ldots, 10^8) \in \mathbb{R}^9$.*

For $B = 0257$, we have $B^\perp = 134689$. Then

$$\begin{aligned} \mathcal{F}_M(B) &= \{\emptyset \subsetneq 7 \subsetneq 57 \subsetneq 2457 \subsetneq 0123456789\} \\ \mathcal{F}_N(B^\perp) &= \{\emptyset \subsetneq 9 \subsetneq 89 \subsetneq 689 \subsetneq 46789 \subsetneq 346789 \subsetneq 123456789\}, \end{aligned}$$

give rise to the corresponding set compositions

$$\boldsymbol{\pi} = 7|5|24|013689, \qquad \boldsymbol{\pi}^\perp = 9|8|6|47|3|125.$$

This is indeed an arboreal pair, as evidenced by their intersection graph in Figure 3.

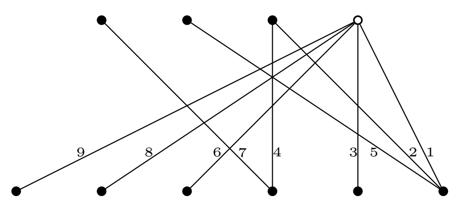

Figure 3: The intersection graph of $\boldsymbol{\pi} = 7|5|24|13689$ and $\boldsymbol{\pi}^\perp = 9|8|6|47|3|125$.

Lemma 3.1 gives us the unique points $(0, \mathsf{x}) \in \mathcal{F}_{\boldsymbol{\pi}}$ and $\mathsf{y} \in \mathcal{F}_{\boldsymbol{\tau}}$ such that $\mathsf{x} + \mathsf{y} = \mathsf{w}$; they are given by the paths to the special vertex $\pi(0)$ in the intersection tree $\Gamma_{\boldsymbol{\pi},\boldsymbol{\pi}^\perp}$. For example $x_7 = 10^6 - 10^3 + 10^1 - 10^0 = 999009$ and $y_4 = 10^3 - 10^1 + 10^0 = 991$ are given by the paths 7421 and 421 from $\pi(7) = \pi_1$ and $\pi^\perp(7) = \pi^\perp_4$ to $\pi(0)$, respectively. In this way we obtain:

$$\begin{array}{rrrrrrrrrr} \mathsf{x} = & 0 & 9 & 0 & 9 & 9999 & 0 & 999009 & 0 & 0 \\ \mathsf{y} = & 1 & 1 & 100 & 991 & 1 & 100000 & 991 & 10000000 & 100000000 \\ \mathsf{w} = & 1 & 10 & 100 & 1000 & 10000 & 100000 & 1000000 & 10000000 & 100000000 \end{array}$$

and $\mathsf{x} \in \Sigma_{(M,0)} \cap (\mathsf{w} - \Sigma_N)$. We invite the reader to record the weights $(0, \mathsf{x})$ and y in the graphs G and H of Figure 1, and verify that in each cycle the minimum weight appears at least twice.

Conversely, the following lemma shows that any intersection point between $\Sigma_{(E,e)}$ and $v - \Sigma_N$ is of the form constructed in Lemma 3.1; that is, it comes from a β-nbc basis.

Lemma 3.3. *Let M be a matroid on $E = [0, n]$ of rank $r + 1$, such that 0 is not a loop nor a coloop, and $N = (M/0)^\perp$. Let $\mathsf{w} \in \mathbb{R}^n$ be generic and super-increasing. Let*

$$\begin{aligned} \mathcal{F} &= \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E\} \\ \mathcal{G} &= \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{n-r-1} \subsetneq G_{n-r} = E - 0\} \end{aligned}$$

be complete flags of the matroids M and N, respectively, such that $\Sigma_{(M,0)}$ and $\mathsf{w} - \Sigma_N$ intersect at $\sigma_{\mathcal{F}}$ and $\mathsf{w} - \sigma_{\mathcal{G}}$. Then there exists a β-nbc basis B of M such that $\mathcal{F} = \mathcal{F}_M(B)$ and $\mathcal{G} = \mathcal{F}_N(B^\perp)$.

Proof. By Lemma 2.2, the set compositions $\boldsymbol{\pi}$ and $\boldsymbol{\tau}$ of $\mathcal{F}$ and $\mathcal{G}$ form an arboreal pair. In particular, $\pi_a \cap \tau_b = (F_a - F_{a-1}) \cap (G_b - G_{b-1})$ cannot have more than one element for any a and b. We proceed in several steps.

1. Our first step will be to show that in the intersection tree $\Gamma_{\boldsymbol{\pi},\boldsymbol{\tau}}$, the top right vertex π_{r+1} contains 0 and 1, the bottom right vertex τ_{n-r} contains 1, and thus the edge 1 connects these two rightmost vertices.

Each G_i is a flat of $N = M^\perp - 0$, so $G_i^\bullet := \mathrm{cl}_{M^\perp}(G_i) \in \{G_i, G_i \cup 0\}$ is a flat of $M^\perp$. Consider the flag of flats of $M^\perp$

$$\mathcal{G}^\bullet := \{\emptyset = G_0^\bullet \subsetneq G_1^\bullet \subsetneq \cdots \subsetneq G_{n-r-1}^\bullet \subsetneq G_{n-r}^\bullet = E\},$$

where $G_{n-r}^\bullet = E$ because 0 is not a coloop of $M^\perp$ and $G_0^\bullet = \emptyset$ because 0 is not a loop of $M^\perp$. Let m be the minimal index such that $0 \in G_m^\bullet$, so

$$\mathcal{G}^\bullet := \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{m-1} \subsetneq G_m \cup 0 \subsetneq \cdots \subsetneq G_{n-r-1} \cup 0 \subsetneq G_{n-r} \cup 0 = E\},$$

Consider the unions of the flat F_r with the coflats in $\mathcal{G}^\bullet$; let j be the index such that

$$F_r \cup G_{j-1}^\bullet \neq E, \qquad F_r \cup G_j^\bullet = E$$

The former cannot have size $|E| - 1$because it is the union of a flat and a coflat. Therefore $(F_r \cup G_j^\bullet) - (F_r \cup G_{j-1}^\bullet) = (E - F_r) \cap (G_j^\bullet - G_{j-1}^\bullet)$ has size at least 2. But $\mathcal{F}$ and $\mathcal{G}$ are arboreal so $\pi_{r+1} \cap \tau_j = (E - F_r) \cap (G_j - G_{j-1})$ has size at most 1. This has two consequences:

a) $G_j^\bullet = G_j \cup 0$ and $G_{j-1}^\bullet = G_{j-1}$, that is, $j = m$.

b) $0 \in E - F_r = \pi_{r+1}$.

Similarly, consider the unions of the coflat $G_{n-r-1}^\bullet$ with the flats in $\mathcal{F}$; let i be the index such that

$$F_{i-1} \cup G_{n-r-1}^\bullet \neq E, \qquad F_i \cup G_{n-r-1}^\bullet = E.$$

An analogous argument shows that $(F_i - F_{i-1}) \cap (E - G_{n-r-1}^\bullet)$ has size at least 2, whereas $\pi_i \cap \tau_{n-r} = (F_i - F_{i-1}) \cap (E - 0 - G_{n-r-1})$ has size at most 1. This has three consequences:

c) $G_{n-r-1}^\bullet = G_{n-r-1}$, that is, $m = n - r$.

d) $0 \in F_i - F_{i-1}$, which in light of b) implies that $i = r + 1$.

e) $(F_i - F_{i-1}) \cap (E - 0 - G_{n-r-1}) = \pi_{r+1} \cap \tau_{n-r} = \{e\}$ for some element $e \in E - 0$. But $e \in \pi_{r+1}$ means that $x_e = 0$ is minimum among all x_is for any $(0, \mathrm{x}) \in \sigma_{\mathcal{F}}$, and $e \in \tau_{n-r}$

means that y_e is minimum among all y_is for any $\mathsf{y} \in \sigma_{\mathcal{G}}$. Since $\mathsf{w} = \mathsf{x} + \mathsf{y}$ for some such x and y, $w_e = x_e + y_e$ is minimum among all w_is, and since w is super-increasing, $e = 1$.

It follows that in the intersection tree $\Gamma_{\pi,\tau}$, the top right vertex π_{r+1} contains 0 and 1 by d) and e), the bottom right vertex τ_{n-r} contains 1 by e), and thus 1 connects them.

2. Next we claim that for any path in the tree $\Gamma_{\pi,\tau}$ that ends with the edge 1, the first edge has the largest label.[2] Assume contrariwise, and consider a containment-minimal path P that does not satisfy this property; its edges must have labels satisfying $e < f > f_2 > \cdots > f_k$ sequentially. If edge e goes from $\pi(e)$ to $\tau(e)$, Lemma 2.3 gives $x_e = w_e - w_f \pm (\text{terms smaller than } w_f) \approx -w_f < 0 = x_1$, contradicting that $(0, \mathsf{x}) \in \sigma_{\mathcal{F}}$. If e goes from $\tau(e)$ to $\pi(e)$, we get $y_e = w_e - w_f \pm (\text{terms smaller than } w_f) \approx -w_f < w_1 = y_1$, contradicting that $\mathsf{y} \in \sigma_{\mathcal{G}}$.

3. Now define

$$\begin{aligned} b_i &:= \min(F_i - F_{i-1}) \quad \text{for} \quad i = 1, \dots, r+1, \\ c_j &:= \min(G_j - G_{j-1}) \quad \text{for} \quad j = 1, \dots, n-r. \end{aligned}$$

Then $B := \{b_1, \dots, b_{r+1}\}$ and $C := \{c_1, \dots, c_{n-r}\}$ are bases of M and N, and $\mathcal{F} = \mathcal{F}_M(B)$ and $\mathcal{G} = \mathcal{F}_N(C)$. We claim that B is β-nbc and $C = B^{\perp}$.

To do so, we first notice that the path from vertex $\pi_i = F_i - F_{i-1}$ (resp. $\tau_j = G_j - G_{j-1}$) to edge 1 must start with edge b_i (resp. c_j): if it started with some larger edge $b' \in F_i - F_{i-1}$, then the path from edge b_i to edge 1 would not start with the largest edge. This has two consequences:

f) The sets B and C are disjoint. If we had $b_i = c_j = e$, then edge e, which connects vertices $\pi_i = F_i - F_{i-1}$ and $\tau_j = G_j - G_{j-1}$, would have to be the first edge in the paths from both of these vertices to edge 1; this is impossible in a tree. We conclude that B and C are disjoint. Since $|B| = r+1$ and $|C| = n-r$, we have $C = B^{\perp}$.

g) For each i we have $x_{b_i} \approx w_{b_i}$, because the path from τ_i to vertex 0 – which is the path from τ_i to edge 1, with edge 1 possibly removed – starts with the largest edge b_i, so Lemma 2.3 gives $x_{b_i} = w_{b_i} \pm (\text{smaller terms}) \approx w_{b_i}$. Similarly $y_{c_i} \approx w_{c_i}$. Now, $(0, \mathsf{x}) \in \sigma_{\mathcal{F}}$ gives $x_{b_1} > \cdots > x_{b_{r+1}}$, which implies $w_{b_1} > \cdots > w_{b_{r+1}}$, which in turn gives

$$b_1 > \cdots > b_r > b_{r+1}; \qquad \text{and analogously,} \qquad c_1 > \cdots > c_{n-r-1} > c_{n-r} = 1.$$

The former implies that B is nbc in M by Lemma 2.11. The latter, combined with c), implies that $c_1 > \cdots > c_{n-r-1} > 0$ respectively are the minimum elements of the flats $G_1^{\bullet}, \dots, G_{n-r-1}^{\bullet}, G_{n-r}^{\bullet} = E$ that they sequentially generate, so $C \cup 0 \setminus 1 = B^{\perp} \cup 0 \setminus 1$ is nbc in $M^{\perp}$. It follows that B is β-nbc in M.

We conclude that B is β-nbc in M, $\mathcal{F} = \mathcal{F}_M(B)$, and $\mathcal{G} = \mathcal{F}_N(B^{\perp})$, as desired. $\square$

[2] This implies that the edge labels decrease along any such path, but we will not use this in the proof.

Combinatorial proof of Theorem 1.2. This follows by combining the previous two lemmas. □

4 Geometric proof of the main theorem

Sketch of geometric proof of Theorem 1.3. The *base polytope* of a matroid M on ground set E is the polytope

$$P(M) = \text{convex hull of } \{\mathrm{e}_B : B \text{ a basis of } M\} \subset \mathbb{R}^E.$$

Let $1_{P(M)} : \mathbb{R}^E \to \mathbb{Z}$ denote the function defined by $1_{P(M)}(x) = 1$ if $x \in P(M)$ and $1_{P(M)}(x) = 0$ if otherwise. Let f be a function on the set of matroids on E taking values in an abelian group. Such a function f is said to be *valuative* if $\sum_i a_i f(M_i) = 0$ for any finite collection of matroids $\{M_i\}$ and integers $\{a_i\}$ such that $\sum_i a_i 1_{P(M_i)} = 0$. We will use the following key fact about valuative functions [7, Lemma 5.9], which was deduced from [9].

Lemma 4.1. *If f and g are valuative functions on matroids such that $f(M) = g(M)$ for all matroids M realizable over $\mathbb{R}$, then in fact $f(M) = g(M)$ for all (not necessarily realizable) matroids.*

Our strategy then is to show that both sides of the equality in Theorem 1.3 are valuative functions on matroids. Since the equality was shown to hold for matroids realizable over $\mathbb{R}$ in [1], we can then conclude that it holds for all matroids by the lemma above.

That the right-hand-side, i.e., the beta invariant of the matroid, is a valuative function was shown by Ardila, Fink, and Rincón in [4]. For the left-hand-side, we use the framework of *tautological classes* of matroids of Berget, Eur, Spink, and Tseng. Let $\mathcal{S}_M$ and $\mathcal{Q}_M$ be the tautological classes of matroids as defined in [7], and let c_i denote their Chern classes. In this framework, using [7, Proposition 5.11 & Theorem 7.6], one can rewrite the left-hand-side as

$$\deg(\Sigma_{(M,e)} \cap -\Sigma_{(M/e)^\perp}) = c_{n+1-r}(\mathcal{Q}_M) \cdot c_{r-1}(\mathcal{S}^\vee_{M/\star \oplus U_{0,\star}}).$$

A minor variation of the proof of [7, Proposition 5.6] shows that this is also valuative.

For a complete proof, see [6]. □

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