

HISTORICAL BEHAVIOR OF SKEW PRODUCTS AND ARCSINE LAWS

PABLO G. BARRIENTOS AND RAUL R. CHAVEZ

ABSTRACT. We study the occurrence of historical behavior for almost every point in the setting of skew products with one-dimensional fiber dynamics. Under suitable ergodic conditions, we establish that a weak form of the arcsine law leads to the non-convergence of Birkhoff averages along almost every orbit. As an application, we show that this phenomenon occurs for one-step skew product maps over a Bernoulli shift, where the stochastic process induced by the iterates of the fiber maps is conjugate to a random walk.

Furthermore, we revisit known examples of skew products that exhibit historical behavior almost everywhere, verifying that they fulfill the required ergodic and probabilistic conditions. Consequently, our results provide a unified and generalized framework that connects such behaviors to the arcsine distribution of the orbits.

1. INTRODUCTION

The study of the dynamics of a function f on a compact metric space X often focuses on the long-term statistical behavior of its orbits. Given a continuous observable $\phi: X \rightarrow \mathbb{R}$, the orbit of $x \in X$ is said to exhibit *historical behavior* if its *time average* (or *Birkhoff average*)

$$\frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$$

does not converge. Otherwise, the orbit exhibits *predictable behavior*, a terminology introduced by Ruelle [Rue01] and further developed by Takens [Tak08]. A particularly interesting case of predictable behavior occurs when there is an ergodic invariant probability measure μ whose basin of attraction (the set of points for which the time average converges to the spatial average $\int \phi d\mu$ for every continuous function ϕ) has positive measure with respect to the reference measure. Such probabilities are called *physical measures*. It turns out that, by Birkhoff's ergodic theorem, the set of *irregular points* (i.e., the orbits exhibiting historical behaviors) has zero measure with respect to every invariant probability measure. Nevertheless, this set may still be metrically large. Numerous classes of examples are known in which the set of irregular points is residual [Tak08, BKN⁺20, CV21]. Additionally, there exist examples where the set of irregular points has full topological entropy and Hausdorff dimension [BS00, FFW01, BNR⁺22], although in most of these examples, this set has zero Lebesgue measure. A paradigmatic dynamical configuration that leads to historical behavior for Lebesgue almost every point is the so-called *Bowen Eye* in [Tak94]. This model had a significant impact on the study of the statistical behavior of dynamical systems. Motivated by this example, Takens proposed in [Tak08] the following problem:

Takens' last problem: Are there persistent classes of smooth dynamical systems that have a set of irregular points with positive Lebesgue measure?

The first class of smooth dynamical systems where it is not possible to eliminate historical behavior by discarding negligible sets of surface C^2 diffeomorphisms and initial conditions was given by Kiriki–Soma [KS17]. Their construction, based on the existence of wandering domains, was later extended to higher (analytic) regularity [BB23], to higher dimensions ($d \geq 3$) and lower (C^1) regularity [Bar22], and to flows [LR17]. More recently, examples with historical behavior for Lebesgue almost every point have also been obtained for families of

rational maps of degree $d \geq 2$ on the Riemann sphere [Tal22], for reparameterizations of linear flows on the two torus [AG22], a class of transitive partially hyperbolic diffeomorphisms on \mathbb{T}^3 [CYZ20], and for certain one-dimensional maps [ATZ05, CL24, CMT24]. Such *non-statistical* systems (i.e., with historical behavior almost everywhere) are particularly interesting because they fundamentally lack a physical measure. The absence of a physical measure implies that long-term statistical predictions are impossible for a significant set of initial conditions.

In this paper, we approach this phenomenon from a different perspective than the classical mechanisms. We establish an abstract framework in which the prevalence of historical behavior almost everywhere is linked directly to underlying ergodic and probabilistic properties. In particular, we relate this behavior to probabilistic mechanisms such as the *arcsine law*. Our contribution is to place all the known constructions for skew product systems within a unified abstract setting, which both clarifies the underlying mechanism and extends it to new classes of examples. In this way, we position the arcsine law as a mechanism complementary to the classical ones (based on wandering domains or Bowen-type dynamics) and suggest that probabilistic ideas may provide a new pathway toward a positive resolution of Takens' last problem.

1.1. Presentation of results and context. We consider the class of *skew products*

$$F: \Omega \times M \rightarrow \Omega \times M, \quad F(\omega, x) \stackrel{\text{def}}{=} (\tau(\omega), f_\omega(x)) \quad (1.1)$$

where the base function $\tau: \Omega \rightarrow \Omega$ is an ergodic measure-preserving map on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the fiber maps $f_\omega: M \rightarrow M$ are measurable functions on a compact one-dimensional manifold M endowed with the Lebesgue measure Leb . We denote the compositions of the fiber maps by

$$f_\omega^0 \stackrel{\text{def}}{=} \text{id} \quad \text{and} \quad f_\omega^n \stackrel{\text{def}}{=} f_{\tau^{n-1}(\omega)} \circ \cdots \circ f_{\tau(\omega)} \circ f_\omega \quad n \geq 1. \quad (1.2)$$

Note that (1.1) includes the class of one-dimensional dynamics by taking the base space Ω as a singleton. We study conditions under which F exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -almost every point. For brevity, we write $(\mathbb{P} \times \text{Leb})$ -a.e. (or *almost everywhere*). Notably, such systems with historical behavior almost everywhere do not admit physical measures with respect to the reference measure $\mathbb{P} \times \text{Leb}$. See Section 2.1 for a more precise definition of historical behavior in this context.

We analyze the connection between historical behavior in Dynamical Systems and the arcsine law in Probability Theory. Recall that a random variable Y on $[0, 1]$ is *arcsine distributed* if,

$$\mathbb{P}(Y \leq \alpha) = \frac{2}{\pi} \arcsin \sqrt{\alpha}, \quad \text{for every } \alpha \in [0, 1].$$

Lévy [L39] introduced this distribution by demonstrating that the proportion of time during which a one-dimensional Wiener process is positive follows the arcsine distribution. Later, Erdős and Kac [EK47] formalized an asymptotic version of this result for a sequence of independent and identically distributed (i.i.d.) random variables. That is, $\{Y_n\}_{n \geq 1}$ is *asymptotically*

arcsine distributed if,

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \leq \alpha) = \frac{2}{\pi} \arcsin \sqrt{\alpha} \quad \text{for every } \alpha \in [0, 1].$$

In particular, a key consequence is that for any $\alpha \in (0, 1)$,

$$\liminf_{n \rightarrow \infty} \mathbb{P}(Y_n \leq \alpha) < 1 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \mathbb{P}(Y_n \geq \alpha) < 1.$$

This pair of conditions, which ensures the probability mass does not accumulate at either endpoint, can be seen as a *fluctuation law*. We identify that this weak form of the arcsine law together with a kind of ergodic condition implies historical behavior almost everywhere for skew products as in (1.1). While these conditions differ slightly between *one-step* and *mild* skew products, they consistently form the structural basis for the observed phenomena. A *mild* skew product is a map as in (1.1) where the fiber functions in (1.2) depend on the entire $\omega \in \Omega$. In contrast, by *one-step*, we understand a skew product where the base dynamics is a Bernoulli shift and the fiber dynamics depends only on the zero-coordinate of $\omega \in \Omega$.

In the one-step setting, Molinek [Mol94] utilized the arcsine law of Erdős and Kac to construct examples of skew products exhibiting historical behavior almost everywhere for generalized (T, T^{-1}) -transformations, where T is a north-south diffeomorphism. Also, Bonifant and Milnor [BM08] sketched a proof using again the arcsine law to show historical behavior almost everywhere for one-step maps with fiber functions having zero Schwarzian derivative. Our first result, Theorem A, establishes historical behavior almost everywhere for one-step skew products satisfying a weak form of the arcsine law. In this result, the ergodic condition arises from the triviality of the tail σ -algebra generated by the random process induced by iterating the fiber functions. A consequence of this result is that, if this random process is conjugate to a one-dimensional random walk with zero mean and positive finite variance, then F exhibits historical behavior almost everywhere. Using this result, we revisit the examples of Bonifant–Milnor and Molinek, showing that they are indeed conjugate to random walks. We also introduce new families of examples to which our theorem applies, such as the T^Ψ -transformations, which generalize the (T, T^{-1}) -transformations, and a novel construction involving a non-trivial coupling of random walks.

Our second result, Theorem B, concerns historical behavior almost everywhere for mild skew products F , under the ergodicity assumption of the non-invariant reference measure $\mathbb{P} \times \text{Leb}$ and a weak version of the arcsine law. In this context, the earliest examples showing historical behavior almost everywhere are the class of *skew-flows* introduced by Ji and Molinek [JM00]. A *skew-flow* is a skew product as in (1.1), where $f_\omega(x) = \varphi(\phi(\omega), x)$ with $\varphi : \mathbb{R} \times M \rightarrow M$ a flow and $\phi : \Omega \rightarrow \mathbb{R}$ a roof function. It is noteworthy that a recent example of a partially hyperbolic diffeomorphism on \mathbb{T}^3 by Crovisier et al. [CYZ20] also belongs to this class. These examples are conjugate to skew-translations, and through this conjugacy, we establish the ergodicity of the non-invariant reference measure $\mathbb{P} \times \text{Leb}$ and the arcsine law for the skew product F . As an application, our result yields new examples of non-statistical

diffeomorphisms on \mathbb{T}^{n+1} for $n \geq 2$ and proves a conjecture of Bonifant–Milnor [BM08] for skew-product endomorphisms of the cylinder $\mathbb{T} \times [0, 1]$.

The scope of Theorem B also includes the one-dimensional Thaler functions [Tha80, Tha83, Tha02], which are full branch maps topologically conjugate to uniformly expanding doubling functions. Aaronson et al. [ATZ05] first observed that these functions exhibit historical behavior almost everywhere. This has been more recently extended to generalized Thaler functions by Coates and Luzzatto [CL24] and Coates et al. [CMT24].

Therefore, as a consequence of our main results, we achieve historical behavior almost everywhere, revisiting and unifying all known examples in the literature, as far as we know, of skew product type with one-dimensional fiber dynamics. We also generalize all these examples and construct new classes of systems exhibiting such non-statistical behavior.

1.2. Historical behavior from random walks. We study skew product maps as in (1.1), where M is the interval $I = [0, 1]$, and at the base we consider the Bernoulli shift $\tau: \Omega \rightarrow \Omega$ on $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{A}^{\mathbb{N}}, \mathcal{F}^{\mathbb{N}}, p^{\mathbb{N}})$. Here, \mathcal{A} denotes an at most countable alphabet, \mathcal{F} is its discrete σ -algebra, and p is a probability measure having full support, i.e., $p(a) > 0$ for every $a \in \mathcal{A}$. Additionally, we assume that F is *one-step* (or *locally constant*), that is,

$$F(\omega, x) = (\tau(\omega), f_{\omega_0}(x)), \quad \text{for every } \omega = (\omega_i)_{i \geq 0} \in \Omega. \quad (1.3)$$

In this specific instance, the compositions of the fiber maps in (1.2) can be written as follows:

$$f_{\omega}^0 \stackrel{\text{def}}{=} \text{id} \quad \text{and} \quad f_{\omega}^n \stackrel{\text{def}}{=} f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0} \quad \text{for } \omega \in \Omega \text{ and } n \geq 1. \quad (1.4)$$

We also consider the following conditions on the fiber maps:

- (H0) For every $x \in (0, 1)$, the sequence $\{X_n^x\}_{n \geq 1}$ of random variables $X_n^x(\omega) = f_{\omega}^n(x)$ has a trivial tail σ -algebra $\mathcal{T}(\{X_n^x\}_{n \geq 1})$. That is, $\mathbb{P}(A) \in \{0, 1\}$ for every $A \in \mathcal{T}(\{X_n^x\}_{n \geq 1})$.
- (H1) For every $\omega \in \Omega$, $f_{\omega}: (0, 1) \rightarrow (0, 1)$ is a monotonically increasing measurable map.
- (H2) For every $x \in (0, 1)$, there exist $\alpha, \beta \in \Omega$ such that $f_{\alpha}(x) < x < f_{\beta}(x)$.

Condition (H0) is a strong ergodic assumption (see the definition of tail σ -algebra in Definition 3.1). As we will show, it can be ensured when $\{X_n^x\}_{n \geq 1}$ is conjugate to a random walk (see Definition 1.2). The monotonically increasing condition in (H1) means that if $x, y \in (0, 1)$ with $x \leq y$, then $f_{\omega}(x) \leq f_{\omega}(y)$. Continuity is not required, and the endpoints 0 and 1 are not necessarily fixed. Condition (H2) is a natural and simple hypothesis that forces the interaction between the dynamics at the endpoints of I , playing a role analogous to heteroclinic connections between equilibrium points in the classical Bowen eye.

Definition 1.1. *The system F satisfies the pointwise-fiber fluctuation law if there are constants $\gamma_0, \gamma_1 \in (0, 1)$ and points $x_0, x_1 \in (0, 1)$, such that*

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_{\omega}^j(x_i)) \leq \alpha\right) < 1, \quad \text{for every } \alpha \in (0, 1) \text{ and } i = 0, 1,$$

where $I_0(\gamma) \stackrel{\text{def}}{=} [0, \gamma]$ and $I_1(\gamma) \stackrel{\text{def}}{=} [\gamma, 1]$ for any $\gamma \in I$.

The following theorem establishes historical behavior almost everywhere for one-step skew products that satisfy the pointwise-fiber fluctuation law.

Theorem A. *Let F be a one-step skew product as in (1.3), satisfying conditions (H0)–(H2) and the pointwise-fiber fluctuation law with constants $\gamma_0 < \gamma_1$. Then, for every $x \in (0, 1)$,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x) \leq \gamma_0 < \gamma_1 \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (1.5)$$

In particular, F exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -almost every point.

To connect the condition (H0) and the pointwise-fiber fluctuation law with random walks, let $\{Y_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables taking values in the additive group G , where G is either \mathbb{R} or \mathbb{Z} . A *random walk* starting at $t \in G$ is defined as the sequence $\{S_n\}_{n \geq 0}$ of random variables given by

$$S_0 = t \quad \text{and} \quad S_n = S_{n-1} + Y_n, \quad n \geq 1. \quad (1.6)$$

The random variables Y_n are called steps, and their mean and variance are denoted by

$$\mu = \mathbb{E}[Y_1] \stackrel{\text{def}}{=} \int Y_1 d\mathbb{P} \quad \text{and} \quad \sigma^2 = \mathbb{E}[(Y_1 - \mu)^2].$$

For each $x \in I$, consider the sequence $\{X_n^x\}_{n \geq 0}$, where $X_n^x(\omega) = f_\omega^n(x)$. We denote by $\mathcal{O}(x) = \{X_n^x(\omega) : \omega \in \Omega, n \geq 0\}$ the orbit of x by the semigroup generated by $\{f_{\omega_0}\}_{\omega_0 \in \mathcal{A}}$.

Definition 1.2. *The sequence $\{X_n^x\}_{n \geq 0}$ is conjugate to a G -valued random walk if there exists a strictly monotonic injection $h: \mathcal{O}(x) \rightarrow G$ such that the step random variables*

$$Y_n^t(\omega) = S_n^t(\omega) - S_{n-1}^t(\omega) \quad \text{for } n \geq 1 \text{ and } \omega \in \Omega,$$

are independent and identically distributed, where $t = h(x)$ and $S_n^t(\omega) = (h \circ f_\omega^n \circ h^{-1})(t)$. In other words, there exists a random walk $\{S_n^t\}_{n \geq 0}$ on G starting at $t = h(x)$ such that $S_n^t = h(X_n^x)$ for $n \geq 1$.

As a consequence of the Hewitt-Savage zero-one law [HS55], we show that condition (H0) is satisfied when $\{X_n^x\}_{n \geq 0}$ is conjugate to a random walk for any $x \in (0, 1)$. Moreover, by a result of Erdős and Kac [EK47], such random walks satisfy the arcsine law, which, through the conjugation, implies that the pointwise-fiber fluctuation law holds for the skew product. This observation leads to the following consequence of the Theorem A, which we prove in detail in Section 8.

Proposition I. *Let F be a one-step skew product as in (1.3). Assume that for every $x \in (0, 1)$, the sequence $\{X_n^x\}_{n \geq 0}$, where $X_n^x(\omega) = f_\omega^n(x)$, is conjugate to a random walk on \mathbb{Z} or \mathbb{R} with mean zero and positive finite variance. Then, F satisfies (H0), and for every $\gamma, x \in (0, 1)$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(f_\omega^j(x)) \leq \alpha\right) = \frac{2}{\pi} \arcsin \sqrt{\alpha}, \quad \text{for every } \alpha \in (0, 1) \text{ and } i = 0, 1. \quad (1.7)$$

Moreover, if F also satisfies conditions (H1) and (H2), then (1.5) holds and F exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -almost every point.

1.3. Historical behavior from the ergodicity of the reference measure. In this subsection, we analyze the historical behavior of skew product functions

$$F: \Omega \times I \rightarrow \Omega \times I, \quad F(\omega, x) = (\tau(\omega), f_\omega(x)), \quad (1.8)$$

where $\tau: \Omega \rightarrow \Omega$ is an ergodic, measure-preserving transformation of a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $f_\omega: I \rightarrow I$ are measurable functions on the interval $I = [0, 1]$. In a form similar to Definition 1.1, we introduce a corresponding fluctuation law.

Definition 1.3. *The system F satisfies the skew-product fluctuation law if there exist constants $\gamma_0, \gamma_1 \in (0, 1)$ such that*

$$\liminf_{n \rightarrow \infty} (\mathbb{P} \times \text{Leb}) \left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x)) \leq \alpha \right) < 1, \quad \text{for every } \alpha \in (0, 1) \text{ and } i = 0, 1. \quad (1.9)$$

On the other hand, to relax condition (H0), we consider a generalization of ergodicity for the reference measure $\mu = \mathbb{P} \times \text{Leb}$ with respect to F . We say that μ is *ergodic* with respect to F if $\mu(A) \in \{0, 1\}$ for every measurable set A such that $F(A) \subset A$. Unlike in the classical setting where the probability measure μ is assumed to be F -invariant, this notion of ergodicity does not require μ to be F -invariant. In most applications, μ serves as a quasi-invariant measure for the \mathbb{Z} -action of F , which aligns this definition with the classical notion of ergodicity in this setting [GS00, Buf14].

Note that if $\mu = \mathbb{P} \times \text{Leb}$ is an F -invariant measure, by Birkhoff ergodic theorem the time average converges μ -a.e. point. Thus, F cannot exhibit historical behavior almost everywhere. The following result establishes historical behavior almost everywhere for mild skew products satisfying the fluctuation law distribution and assuming that μ is ergodic with respect to F . As a consequence, μ must be a *non-invariant* measure of F .

Theorem B. *Let F be a skew product as in (1.8). Assume that*

- (i) *the measure $\mathbb{P} \times \text{Leb}$ is ergodic with respect to F , and*
- (ii) *F satisfies the skew-product fluctuation law with $\gamma_0 < \gamma_1$.*

Then,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x) \leq \gamma_0 < \gamma_1 \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x) \quad \text{for } (\mathbb{P} \times \text{Leb})\text{-a.e. } (\omega, x) \in \Omega \times I.$$

In particular, F exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -almost every point.

1.4. Limit points of the sequence of empirical measures. In this section, we investigate the asymptotic behavior of the sequence of empirical measures

$$e_n(\omega, x) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_\omega^j(x)}.$$

We define the set

$$\mathcal{L}(\omega, x) \stackrel{\text{def}}{=} \left\{ \nu : \nu \text{ is an accumulation point in the weak* topology of } e_n(\omega, x) \right\}. \quad (1.10)$$

Both Theorem A and Theorem B show that these sequences of empirical measures fail to converge in the weak* topology for $(\mathbb{P} \times \text{Leb})$ -almost every point. Thus, $\mathcal{L}(\omega, x)$ is not a trivial set. Moreover, in the one-step case, when the fiber maps are continuous functions, these accumulation points are *stationary measures*, i.e., measures ν on I that induce F -invariant measures of the form $\mathbb{P} \times \nu$. However, for mild skew products, the product measure $\mathbb{P} \times \nu$ is not generally F -invariant for a given $\nu \in \mathcal{L}(\omega, x)$.

A distinctive feature of our setting is that no additional continuity assumptions on the fiber maps f_ω are required. Furthermore, the maps are not assumed to fix the endpoints of the interval $I = [0, 1]$. As a result of this flexibility, F -invariant measures may not exist in general. However, under the additional assumption that the fiber maps fix the endpoints of the interval, specifically,

$$f_\omega(0) = 0 \quad \text{and} \quad f_\omega(1) = 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \quad (1.11)$$

we can ensure the existence of trivial F -invariant measures $\mu_0 = \mathbb{P} \times \delta_0$ and $\mu_1 = \mathbb{P} \times \delta_1$, where δ_0 and δ_1 are the Dirac measures at the fixed points 0 and 1, respectively.

For many systems exhibiting historical behavior, such as those considered in this article, orbits spend an insignificant fraction of their time inside the interior of the interval. The following result describes the limit set of empirical measures under this vanishing occupation time of the interior of I .

Proposition II. *Let F be a skew product satisfying the assumptions of Theorem A or Theorem B, along with (1.11). Assume that*

(i) *for any $0 < \epsilon < 1/2$ it holds*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[\epsilon, 1-\epsilon]}(f_\omega^j(x)) = 0 \quad (\mathbb{P} \times \text{Leb})\text{-a.e. } (\omega, x) \in \Omega \times I; \quad (1.12)$$

(ii) *the constants γ_0 and γ_1 in the definition of the pointwise-fiber and skew-product fluctuation laws can be chosen arbitrarily close to 0 and 1, respectively.*

Then,

$$\mathcal{L}(\omega, x) = \left\{ \lambda \delta_0 + (1 - \lambda) \delta_1 : \lambda \in [0, 1] \right\} \quad \text{for } (\mathbb{P} \times \text{Leb})\text{-a.e. } (\omega, x) \in \Omega \times I. \quad (1.13)$$

Remark 1.4. In the one-step case, we get that (1.13) holds for every $x \in (0, 1)$ and \mathbb{P} -a.e. $\omega \in \Omega$ whenever, for every fixed $x \in (0, 1)$ and $0 < \epsilon < 1/2$, the limit of the occupational time in (1.12) holds \mathbb{P} -almost surely.

It is a classical fact that one-dimensional random walks with non-degenerate increments (i.e., non-frozen at a single value) spend asymptotically zero proportion of time in any fixed compact subset of the interior of their state space (see Proposition 9.7). We shall return to this phenomenon in Theorem 9.1, where we prove a vanishing interior occupation time law even with non-i.i.d. steps. Hence, in the setting of Proposition I, the assumption (1.12) follows directly whenever the fiber process is conjugate to such a random walk, and from Proposition II and Remark 1.4, we have the following:

Corollary III. *Let F be a one-step skew product satisfying the assumptions of Proposition I and (1.11). Then, for every $x \in (0, 1)$,*

$$\mathcal{L}(\omega, x) = \left\{ \lambda \delta_0 + (1 - \lambda) \delta_1 : \lambda \in [0, 1] \right\} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

1.5. Examples. This section shows how our results unify and extend known examples of skew products that exhibit historical behavior almost everywhere. As previously mentioned, these examples share structural features: the weak form of the arcsine law and a kind of ergodicity. Thus, we provide a cohesive framework to analyze them by applying Theorems A and B. We also construct, as an application of these results, new non-trivial examples.

1.5.1. T^Ψ -transformations

Consider the product space $(\Omega, \mathcal{F}, \mathbb{P}) \stackrel{\text{def}}{=} (\mathcal{A}^{\mathbb{N}}, \mathcal{F}^{\mathbb{N}}, p^{\mathbb{N}})$, where \mathcal{A} is an at most countable alphabet and p is a probability measure with full support. Let M be either $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ or $I = [0, 1]$, and let $T : M \rightarrow M$ be a *Morse-Smale homeomorphism of period one*. That is, T has a non-empty set of periodic points consisting solely of sinks and sources of period one. Consider a one-step random variable $\Psi : \Omega \rightarrow \mathbb{Z}$, i.e., $\Psi(\omega) = \Psi(\omega_0)$ for $\omega = (\omega_i)_{i \geq 0} \in \Omega$. The associated one-step skew product $F_{T^\Psi} : \Omega \times M \rightarrow \Omega \times M$, is defined as

$$F_{T^\Psi}(\omega, x) \stackrel{\text{def}}{=} (\tau(\omega), f_{\omega_0}(x)) \quad \text{where} \quad f_{\omega_0}(x) \stackrel{\text{def}}{=} T^{\Psi(\omega_0)}(x). \quad (1.14)$$

As we will see in Section 8.2, the sequence $\{X_n^x\}_{n \geq 0}$ of random variables $X_n^x(\omega) = f_\omega^n(x)$ is conjugated to the \mathbb{Z} -valued random walk $\{S_n\}_{n \geq 0}$ by $S_0(\omega) = 0$ and $S_n(\omega) = \Psi(\omega_0) + \dots + \Psi(\omega_{n-1})$ for $n \geq 1$. Hence, as a consequence of Proposition I and Corollary III, we obtain the following result:

Proposition IV. *If $\mathbb{E}[\Psi] = 0$ and $0 < \mathbb{E}[\Psi^2] < \infty$, the skew product F_{T^Ψ} in (1.14) satisfies (1.7) for Leb-a.e. $x \in M$ and exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point. Moreover, for every pair of consecutive fixed points p and q of T ,*

$$\mathcal{L}(\omega, x) = \left\{ \lambda \delta_p + (1 - \lambda) \delta_q : \lambda \in [0, 1] \right\} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } x \in (p, q).$$

Taking in our general framework the alphabet $\mathcal{A} = \{-1, 1\}$, the symmetric Bernoulli measure $\mathbb{P}(\omega_0 = -1) = \mathbb{P}(\omega_0 = 1) = \frac{1}{2}$, and $\Psi(\omega) = \omega_0$, the skew-product (1.14) becomes

$$F_T(\omega, x) \stackrel{\text{def}}{=} (\tau(\omega), T^{\omega_0}(x)), \quad \omega = (\omega_i)_{i \geq 0} \in \{-1, 1\}^{\mathbb{N}}, \quad x \in M.$$

This map is a generalization to the classical (T, T^{-1}) -transformation [Kal82]. Since in this case, we have that Ψ has mean zero and positive finite variance, Proposition IV applies to F_T . Molinek [Mol94, Theorem 8] established that if T is a north-south diffeomorphism of M , the natural extension of this one-step skew product F_T to $\{-1, 1\}^{\mathbb{Z}} \times M$, exhibits historical behavior almost everywhere. Since $(\bar{\Omega}, \bar{\mathbb{P}}, \tau) = (\mathcal{A}^{\mathbb{Z}}, p^{\mathbb{Z}}, \tau)$ is a measurable theoretic extension of $(\Omega, \mathbb{P}, \tau)$, Proposition 2.5 allows us to immediately generalize our result to the bi-sequence base space. This generalization includes and extends Molinek's result, as stated in the following corollary:

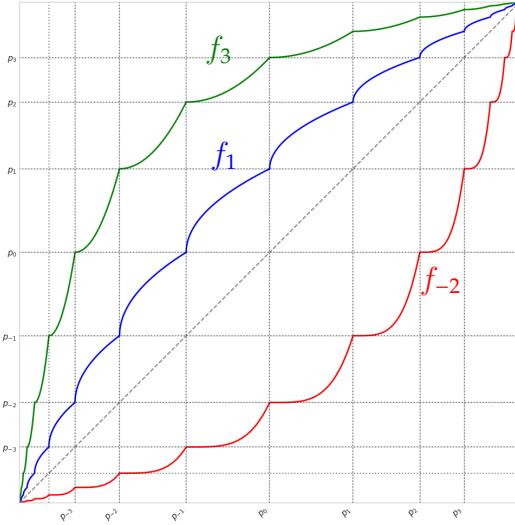


Figure 1. Fiber maps of a coupling random walk skew product with $\mathcal{A} = \{-2, 1, 3\}$ and probability vector $p = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. The one-step random variables are $Z(\omega_0) = \omega_0$, $\Psi(-2) = 2$, $\Psi(1) = -1$, and $\Psi(3) = 1$, using a north-south map $T(u) = u^2$ with partition points $p_k = \frac{2^k}{1+2^k}$ for $k \in \mathbb{Z}$.

Corollary V. Let T be a Morse-Smale diffeomorphism of period one on a one-dimensional compact manifold M , and consider the one-step skew product

$$\bar{F}_{T^\Psi} : \bar{\Omega} \times M \rightarrow \bar{\Omega} \times M, \quad \bar{F}_{T^\Psi}(\omega, x) = (\tau(\omega), T^{\Psi(\omega_0)}(x)).$$

If $\mathbb{E}[\Psi] = 0$ and $0 < \mathbb{E}[\Psi^2] < \infty$, then \bar{F}_{T^Ψ} exhibits historical behavior for $(\bar{\mathbb{P}} \times \text{Leb})$ -a.e. point, and for every pair of consecutive fixed points p and q of T ,

$$\mathcal{L}(\omega, x) = \left\{ \lambda \delta_p + (1 - \lambda) \delta_q : \lambda \in [0, 1] \right\} \quad \text{for } \bar{\mathbb{P}}\text{-a.e. } \omega \in \bar{\Omega}_2 \text{ and } x \in (p, q).$$

1.5.2. Coupling random walks

Consider $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{A}^{\mathbb{N}}, \mathcal{F}^{\mathbb{N}}, p^{\mathbb{N}})$ as in the previous subsection. Let $T : I \rightarrow I$ be a north-south homeomorphism, i.e., a homeomorphism of the interval $I = [0, 1]$ that fixes only the endpoints. Let $\{p_k\}_{k \in \mathbb{Z}} \subset (0, 1)$ be strictly increasing ladder points with $\lim_{k \rightarrow -\infty} p_k = 0$ and $\lim_{k \rightarrow +\infty} p_k = 1$. Denote $I_k = [p_k, p_{k+1})$ and $d_k = p_{k+1} - p_k$. Consider one-step random variables $Z : \Omega \rightarrow \mathbb{Z}$ and $\Psi : \Omega \rightarrow \mathbb{Z}$ and define the one-step skew product

$$F_{T^\Psi, Z} : \Omega \times I \rightarrow \Omega \times I, \quad F_{T^\Psi, Z}(\omega, x) = (\tau(\omega), f_{\omega_0}(x)), \quad (1.15)$$

where the fiber maps are given, for $x \in I_k$, by

$$f_{\omega_0}(x) = p_{k+Z(\omega_0)} + d_{k+Z(\omega_0)} \cdot T^{\Psi(\omega_0)}(u) \in I_{k+Z(\omega_0)}, \quad u = \frac{x - p_k}{d_k}, \quad (1.16)$$

with boundary conditions $f_{\omega_0}(0) = 0$, $f_{\omega_0}(1) = 1$ (see Figure 1).

The sequence $X_n^x(\omega) = f_\omega^n(x)$, $n \geq 0$, is not necessarily conjugate to a random walk. However, it can be expressed as

$$X_n^x = p_{Z_n} + d_{Z_n} \cdot u_n, \quad n \geq 0, \quad (1.17)$$

where

$$\begin{aligned} Z_0(\omega) &= k, & Z_n(\omega) &= Z_{n-1}(\omega) + Z(\omega_{n-1}), & n \geq 1, \\ S_0(\omega) &= 0, & S_n(\omega) &= S_{n-1}(\omega) + \Psi(\omega_{n-1}), & n \geq 1, \\ u_0(\omega) &= u, & u_n(\omega) &= T^{S_n(\omega)}(u_0), & n \geq 1. \end{aligned} \quad (1.18)$$

Since $\{u_n\}_{n \geq 1}$ is conjugate to the random walk $\{S_n\}_{n \geq 0}$, the process $\{X_n^x\}_{n \geq 0}$ can be regarded as a coupling of two random walks: a *macro* walk $\{Z_n\}_{n \geq 0}$ and a *micro* walk $\{S_n\}_{n \geq 0}$, both evolving on the integer lattice \mathbb{Z} .

As a consequence of Theorem A and Proposition II, we obtain:

Proposition VI. *Let Ψ and Z be one-step random variables with $\mathbb{E}[Z] = 0$ and $0 < \mathbb{E}[Z^2] < \infty$. Then, $F_{T^\Psi, Z}$ as in (1.15) has historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point. Moreover, for every $x \in (0, 1)$,*

$$\mathcal{L}(\omega, x) = \{\lambda \delta_0 + (1 - \lambda) \delta_1 : \lambda \in [0, 1]\} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

1.5.3. Skew-flow transformations

Let $(\Omega, \mathbb{P}, \tau)$ be a measure-preserving dynamical system, where $\tau : \Omega \rightarrow \Omega$ is either

- a topologically mixing (one-sided or two-sided) subshift of finite type, or
- the restriction of a C^1 -diffeomorphism to a topologically mixing hyperbolic basic set.

The probability measure \mathbb{P} is an equilibrium state associated with a Hölder continuous potential, also known as a *Hölder Gibbs measure* [Bow75].

Consider a one-dimensional compact manifold M , which is either $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ or $I = [0, 1]$. Let $\varphi : \mathbb{R} \times M \rightarrow M$ be a C^1 flow on M , and let $\phi : \Omega \rightarrow \mathbb{R}$ be a measurable function. Define the skew product

$$F_{\varphi, \phi}(\omega, x) = (\tau(\omega), f_\omega(x)) \quad \text{where} \quad f_\omega(x) = \varphi(\phi(\omega), x). \quad (1.19)$$

We study $F_{\varphi, \phi}$ via conjugation with the skew translation

$$T_\phi : \Omega \times \mathbb{R} \rightarrow \Omega \times \mathbb{R}, \quad T_\phi(\omega, x) \stackrel{\text{def}}{=} (\tau(\omega), x + \phi(\omega)).$$

By Theorem 9.1, $F_{\varphi, \phi}$ satisfies the vanishing occupational time (1.12) provided ϕ satisfies the following cohomological condition:

(C1) the equation $\phi = u \circ \tau - u$ has no solution in $L^2(\mathbb{P})$.

When such a solution u exists, ϕ is called *(additive) coboundary*. Ji and Molinek [JM00] analyzed the case where φ is a north-south flow on $M = \mathbb{T}$, showing that $F_{\varphi, \phi}$ exhibits historical behavior almost everywhere whenever ϕ has zero expectation and is a Hölder continuous and is not a coboundary. Under these same assumptions for ϕ , we show as a consequence of Theorem 9.15 that $F_{\varphi, \phi}$ satisfies an arcsine law. For a direct application

of Theorem B, ergodicity of reference measure is also required. However, this condition is not readily implied by the additive non-coboundary assumption (C1). Fortunately, the conjugacy between skew-flows and skew-translations provides the key to analyzing their long-term behavior. The decisive feature of the skew-translation is its additive, random-walk-like structure in the fiber, which allows the asymptotic behavior of an orbit to be studied independently of its initial point. This property effectively decouples the random process from the specific starting condition, thereby allowing us to derive the system's asymptotic properties directly from the ergodicity of the base dynamics together with the fluctuation law and the vanishing interior occupational time. This refined approach yields the following proposition:

Proposition VII. *Let φ and ϕ be, respectively, a Morse-Smale flow on M and a Hölder continuous function satisfying (C1) and $\mathbb{E}[\phi] = 0$. Then, $F_{\varphi,\phi}$ exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point. Moreover, given $x \in (p, q)$ where p and q are two consecutive equilibrium points of φ , we have that*

$$\mathcal{L}(\omega, x) = \left\{ \lambda \delta_p + (1 - \lambda) \delta_q : \lambda \in [0, 1] \right\} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Now, consider the skew product $\tilde{F}_{\varphi,\phi} : \Omega \times M \rightarrow \Omega \times M$, defined by

$$\tilde{F}_{\varphi,\phi}(\omega, x) = (\tau(\omega), f_\omega(x)), \quad \text{where } f_\omega(x) = -\varphi(\phi(\omega), x) \pmod{1}.$$

Here, the fiber functions f_ω are orientation-reversing, in contrast to (1.19), where they are orientation-preserving.

Crovisier et al. [CYZ20] studied the case where the base dynamics τ is a C^2 volume-preserving Anosov diffeomorphism on $\Omega = \mathbb{T}^2$, and φ is a north-south flow on $M = \mathbb{T}$ induced by a 1-periodic vector field X on \mathbb{R} , sufficiently close to zero, such that $X(-x) = -X(x)$ and $X(0) = X(1/2) = 0$. In particular, φ is a *symmetric Morse-Smale* flow. That is, φ is a Morse-Smale flow on M and satisfies the symmetric relation

$$\varphi(t, -x \pmod{1}) = -\varphi(t, x) \pmod{1} \quad \text{for every } (t, x) \in \mathbb{R} \times M.$$

This symmetry implies that $x = 1/2$ is necessarily an equilibrium point of φ . They showed that $\tilde{F}_{\varphi,\phi}$ exhibits historical behavior almost everywhere, provided $\phi : \Omega \rightarrow \mathbb{R}$ is Hölder, has zero expectation, and satisfies the following cohomological condition:

(C2) there do not exist $\lambda \in \mathbb{R} \setminus \{0\}$ and $\psi : \Omega \rightarrow \mathbb{S}^1$ in $L^2(\mathbb{P})$ such that

$$e^{i\lambda\phi(\omega)} = \frac{\psi(\tau(\omega))}{\psi(\omega)} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (1.20)$$

When $\lambda \neq 0$ and ψ satisfy (1.20), the function ϕ (or more precisely the character $e^{i\lambda\phi}$) is said to be *multiplicative coboundary*. It is not difficult to see that if ϕ is an additive coboundary, then ϕ is a multiplicative coboundary. Thus, condition (C2) implies (C1). But the converse is not necessarily true. We will prove in Theorem 9.9 that for the skew-translation T_ϕ , the conditions (C2) and $\mathbb{E}[\phi] = 0$ are equivalent to ergodicity with respect to $\mathbb{P} \times \text{Leb}$. Thus, under condition (C2), we would obtain the requirements for applying Theorem B.

Remark 1.5. When the function ϕ is Hölder continuous, Livšic's theorem [Liv72, Theorem 9] (and subsequent work by Parry-Pollicott [PP90, Propositions 4.2 and 4.12] and [PP97, Theorem 1 and 2]) ensures that if the cohomological equation has a measurable solution (almost everywhere), it must also have a continuous one (everywhere). Hence, if ϕ is an additive (resp. multiplicative) coboundary, then $\phi(\alpha) = 0$ (resp. $\lambda\phi(\alpha) \in 2\pi\mathbb{Z}$) for every fixed point $\alpha \in \Omega$ of τ . Thus, (C1) holds whenever $\phi(\alpha) \neq 0$ for some fixed point α . Furthermore, a sufficient condition to satisfy (C2) is the existence of fixed points $\alpha, \beta \in \Omega$ such that $\phi(\alpha)$ and $\phi(\beta)$ are rationally independent.

For symmetric Morse-Smale flows φ , the symmetry implies $\pi(\varphi(t, -x \bmod 1)) = \varphi(t, \pi(x))$ for all $(t, x) \in \mathbb{R} \times M$, where

$$\pi(x) = \begin{cases} x, & x \in [0, 1/2], \\ -x \bmod 1, & x \in [1/2, 1]. \end{cases}$$

Hence, we have

$$\Pi \circ \tilde{F}_{\varphi, \phi} = F_{\varphi, \phi} \circ \Pi, \quad \text{where } \Pi(\omega, x) = (\omega, \pi(x)).$$

By setting $I = [0, 1/2]$, since $\Pi: \Omega \times M \rightarrow \Omega \times I$ is a continuous surjection, $\tilde{F}_{\varphi, \phi}$ is an extension of the restriction of $F_{\varphi, \phi}$ to $\Omega \times I$ such that $\Pi_*(\mathbb{P} \times \text{Leb}) = \mathbb{P} \times (2 \cdot \text{Leb})$. Thus, as a consequence of Proposition VII and Proposition 2.5, we extend the result of [CYZ20] as follows:

Corollary VIII. *Let φ and ϕ be, respectively, a symmetric Morse-Smale flow and a Hölder continuous function satisfying (C1) and $\mathbb{E}[\phi] = 0$. Then, $\tilde{F}_{\varphi, \phi}$ exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point. Moreover, given $x \in (p, q)$ where p and q are two consecutive equilibrium points of φ in $[1/2, 1]$,*

$$\mathcal{L}(\omega, x) = \left\{ \lambda \frac{\delta_p + \delta_{\pi(p)}}{2} + (1 - \lambda) \frac{\delta_q + \delta_{\pi(q)}}{2} : \lambda \in [0, 1] \right\} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Remark 1.6. When $\Omega = \mathbb{T}^n$, $n \geq 2$, and $\tau: \mathbb{T}^n \rightarrow \mathbb{T}^n$ is a C^2 volume-preserving Anosov diffeomorphism, the skew products $F_{\varphi, \phi}$ and $\tilde{F}_{\varphi, \phi}$ can be realized as partially hyperbolic diffeomorphisms of \mathbb{T}^{n+1} , arbitrarily close to $\tau \times \text{Id}$ and $\tau \times (-\text{Id})$ respectively. In this setting, while $F_{\varphi, \phi}$ is non-transitive, $\tilde{F}_{\varphi, \phi}$ is transitive if and only if φ is a north-south flow.

1.5.4. Zero Schwarzian derivative

Recall that the Schwarzian derivative of every C^3 interval diffeomorphism f is defined by

$$Sf(x) \stackrel{\text{def}}{=} \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2.$$

Here, we consider a skew product as defined in (1.8), where $\tau: \Omega \rightarrow \Omega$ is a Bernoulli shift on a product space $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{A}^\mathbb{N}, \mathcal{F}^\mathbb{N}, p^\mathbb{N})$ where \mathcal{A} is at most countable alphabet, p is a probability measure with full support and the fiber maps f_ω are C^3 interval diffeomorphisms satisfying the following conditions:

- (S1) $f_\omega(0) = 0$, $f_\omega(1) = 1$, and $f_\omega \neq \text{id}$ for \mathbb{P} -a.e. $\omega \in \Omega$,
- (S2) $Sf_\omega(x) = 0$, for $(\mathbb{P} \times \text{Leb})$ -a.e. $(\omega, x) \in \Omega \times I$,

(S3) The Lyapunov exponents $\lambda(\delta_i) \stackrel{\text{def}}{=} \int \log(f'_\omega(i)) d\mathbb{P} = 0$ for $i = 0, 1$.

As indicated in [BM08], under conditions (S1)–(S3), the skew product can be written as

$$F_a: \Omega \times I \rightarrow \Omega \times I, \quad F_a(\omega, x) = (\tau(\omega), f_\omega(x)) \quad \text{with} \quad f_\omega(x) = \frac{a(\omega)x}{1 + (a(\omega) - 1)x} \quad (1.21)$$

where $a: \Omega \rightarrow (0, \infty)$ is a measurable function such that

$$\int \log a(\omega) d\mathbb{P} = 0 \quad \text{and} \quad a(\omega) \neq 1 \quad \text{for every } \omega \in \Omega. \quad (1.22)$$

We also consider the following additional assumptions when required:

- (E0) the function $a(\omega)$ depends only on the zero-coordinate of $\omega = (\omega_i)_{i \geq 0} \in \Omega$,
- (E1) $\int (\log a(\omega))^2 d\mathbb{P} < \infty$.

Under condition (E0), the skew product F_a is one-step.

Bonifant and Milnor claimed in [BM08, Theorem 6.2] that one-step skew products satisfying (S1)–(S3) exhibit historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point. However, they only provided a rough sketch of the proof with several incomplete steps. Here, we provide a complete proof of this result as a consequence of Proposition I, including the missing pieces (the finite moment condition (E1) and the at most countability of the alphabet \mathcal{A}).

Proposition IX. *Let F_a be a skew product as in (1.21) where $a: \Omega \rightarrow (0, \infty)$ satisfies (1.22), (E0), and (E1). Then F_a exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point. Moreover,*

$$\mathcal{L}(\omega, x) = \{\lambda\delta_0 + (1 - \lambda)\delta_1 : \lambda \in [0, 1]\} \quad \text{for any } x \in (0, 1) \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

In the above result, F_a is a one-step skew product. We can extend our results to mild skew products by establishing a connection between F_a and a skew-flow. Namely, let $\phi(\omega) = \log a(\omega)$ and define

$$\varphi(t, x) = \frac{e^t x}{1 + (e^t - 1)x}, \quad (t, x) \in \mathbb{R} \times I.$$

The function φ is the solution to the differential equation $\dot{x} = x(1 - x)$, which defines a north-south flow on $I = [0, 1]$. Then, the fiber maps of F_a in (1.21) can be written as

$$f_\omega(x) = \varphi(\phi(\omega), x) \quad \text{for } (\omega, x) \in \Omega \times I.$$

Notice that when the alphabet \mathcal{A} is finite, the Bernoulli probability $\mathbb{P} = p^\mathbb{N}$ is an equilibrium state for the Hölder continuous potential $\psi: \Omega \rightarrow \mathbb{R}$ defined by $\psi(\omega) = \log p(\omega_0)$. Thus, \mathbb{P} is a Hölder Gibbs measure for the full shift $\tau: \Omega \rightarrow \Omega$. Assuming additionally that ϕ is Hölder continuous, satisfies (C1), and $\mathbb{E}[\phi] = 0$, the skew product in (1.21) transforms into a skew-flow for which we can apply Proposition VII. This leads to the following result, which solves a conjecture for mild skew products posed by Bonifant and Milnor [BM08, conjecture before Hypothesis 6.1].

Corollary X. Let F_a be a skew product as in (1.21), where the alphabet \mathcal{A} is finite, $\phi(\omega) = \log a(\omega)$ is Hölder continuous and satisfies (1.22) and (C1). Then F_a exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point. Moreover,

$$\mathcal{L}(\omega, x) = \{\lambda\delta_0 + (1 - \lambda)\delta_1 : \lambda \in [0, 1]\} \quad \text{for } (\mathbb{P} \times \text{Leb})\text{-a.e. } (\omega, x) \in \Omega \times I.$$

Remark 1.7 (Solution to the Bonifant–Milnor conjecture). Bonifant and Milnor [BM08] studied skew products on the cylinder $\mathbb{T} \times I$ of the form $F(x, y) = (\ell x \bmod 1, f_x(y))$, $\ell \geq 2$, with fiber maps f_x of zero Schwarzian derivative. To prove the existence of historical behavior, they replaced the base by a Bernoulli shift and assumed F is one-step, conjecturing that the same holds for the expanding map. Since the map $x \mapsto \ell x \bmod 1$ is measure-theoretically isomorphic to a Bernoulli shift, F is isomorphic to a system F_a satisfying the hypotheses of Corollary X, provided the dependence $x \mapsto f_x$ is Hölder (in this setting, this Hölder continuity is equivalent to that of $\phi = \log a$). Thus, in view of Proposition 2.5, Corollary X confirms that the original Bonifant–Milnor skew product over the expanding map exhibits historical behavior for $(\text{Leb} \times \text{Leb})$ -a.e. point.

1.5.5. Interval functions

If Ω is a singleton, the skew product (1.8) can be interpreted as a measurable function $f: I \rightarrow I$. In this setting, Aaronson et al. [ATZ05, see comments after Theorem 2] observe that the so-called *Thaler functions* exhibit historical behavior almost everywhere. A function $f: I \rightarrow I$ is said to be a *Thaler functions* if the following conditions are satisfied: there exist $c \in (0, 1)$ and $p > 1$ such that

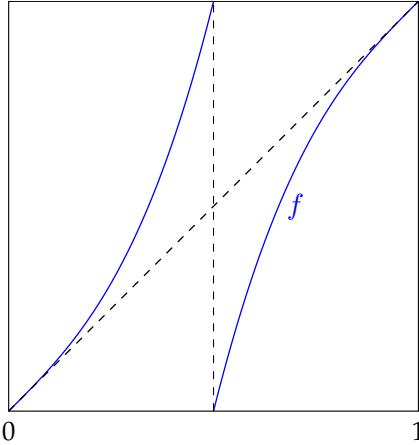
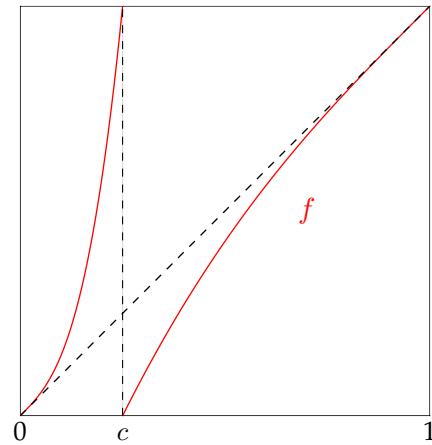
- (T1) f is full branch: the restrictions $f|_{(0,c)}: (0, c) \rightarrow (0, 1)$ and $f|_{(c,1)}: (c, 1) \rightarrow (0, 1)$ are increasing, onto, and C^2 , and admit C^2 -extensions to the closed intervals $[0, c]$ and $[c, 1]$, respectively;
- (T2) f is almost expanding: $f'(x) > 1$ for every $x \in (0, c^-] \cup [c^+, 1)$, $f'(0) = f'(1) = 1$, and f is convex and concave in neighborhoods of 0 and 1, respectively;
- (T3) $f(x) - x = h(x)x^{p+1}$ for $x \in (0, c)$, where $h(kx) \sim h(x)$ as $x \rightarrow 0$ for every $k \geq 0$;
- (T4) There exists $a \in (0, \infty)$ such that $(1 - x) - f(1 - x) \sim a^p(f(x) - x)$ as $x \rightarrow 0$.

One of the most characteristic and explicit example of a Thaler function is the symmetric Manneville-Pomeau function given by

$$f(x) = \begin{cases} x + 2^p x^{p+1} & \text{if } x \in [0, \frac{1}{2}), \\ x - 2^p(1 - x)^{p+1} & \text{if } x \in [\frac{1}{2}, 1], \end{cases} \quad p > 1. \quad (1.23)$$

See Figure 3. In the case $p = 2$ in (1.23), it was noted in [Ino00, BB03, Kel04] that these maps do not admit physical measures and satisfy the condition of occupational times (see condition (OT2) in Section 4).

This class of functions was introduced by Thaler [Tha80, Tha83], who proved that any such function is conservative and exact with respect to Leb and admits a σ -finite ergodic measure μ equivalent to the Lebesgue measure Leb such that, for every $\epsilon > 0$, we have

**Figure 2.** Symmetric Manneville-Pomeau**Figure 3.** Thaler functions

$\mu((\epsilon, 1 - \epsilon)) < \infty$. A direct consequence is that μ is ergodic with respect to f , and applying the ergodic theorem for infinite measure spaces (cf. [Aar97, Exercise 2.2.1]), we also have the condition of vanishing occupation time on $(\epsilon, 1 - \epsilon)$ as in (1.12) in Proposition II. Subsequently, Thaler [Tha02] established that every such map satisfies

$$\lim_{n \rightarrow \infty} \text{Leb}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(f^j(x)) \leq \alpha\right) = G_{\alpha, \beta}(x) \quad \text{for every } \gamma \in (0, 1) \text{ and } i \in \{0, 1\},$$

where $G_{\alpha, \beta}$ is the distribution function on $(0, 1)$, given by

$$G_{\alpha, \beta}(x) = \frac{1}{\pi \alpha} \left(\text{arccot}\left(\frac{\beta(1-x)^\alpha}{x^\alpha(1-\beta)\sin(\pi\alpha)}\right) + \cot(\pi\alpha) \right)$$

The parameters α and β are determined by the properties (T1)–(T4), namely

$$\alpha = \frac{1}{p} \quad \text{and} \quad \beta = \frac{f''(c^-)}{f'(c^-) + f'(c^+)/a}.$$

In particular, f satisfies the fluctuation law. Therefore, the following result is a direct consequence of Theorem B and Proposition II, revisiting the result observed in [ATZ05]. In this setting, (1.10) corresponds to the set $\mathcal{L}(x)$ of accumulation points in the weak* topology of the sequence of empirical measures $\frac{1}{n}(\delta_x + \dots + \delta_{f^{n-1}(x)})$.

Corollary XI. *Every Thaler function f has historical behavior almost everywhere. Moreover,*

$$\mathcal{L}(x) = \{\lambda \delta_0 + (1 - \lambda) \delta_1 : \lambda \in [0, 1]\}, \quad \text{for Leb-a.e. } x \in (0, 1).$$

Coates et al. [CLM23] consider interval functions that are similar to Manneville–Pomeau but may have zero or infinite derivatives at points of discontinuity. They proved that these functions admit a σ -finite ergodic measure equivalent to Leb. Subsequently, Coates and Luzzatto [CL24] demonstrated that such functions exhibit historical behavior almost everywhere by proving a condition on the occupational times which is a consequence of the fluctuation law (see condition (OT2) in Section 4 and Proposition 4.8). In Theorem 6.1,

we show that actually, this condition on the occupational times and the ergodicity of the reference measure is enough to get historical behavior almost everywhere.

More recently, Coates et al. [CMT24] proved historical behavior for almost every point for *generalized Thaler* functions (i.e. maps with k branches and k neutral fixed points, for $k \geq 2$). It follows from the Thaler result [Tha80, Tha83] that Leb is still ergodic with respect to f . Moreover, Sera and Yano [SY19] and Sera [Ser20] concluded that these generalized Thaler functions also satisfy a generalization of the arcsine law (c.f. [CMT24, Theorem 2.7]). Therefore, by adapting the fluctuation law to this generalization with several neutral fixed points, one can extend the same ideas of the proof of Theorem B to achieve historical behavior almost everywhere for this family of generalized Thaler maps.

In [CMT24], it is also observed that the generalized Thaler functions have a unique *natural measure*. A measure ν is called *natural* for a dynamical system f if there exists an absolutely continuous probability measure λ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \lambda = \nu$$

in the weak* topology. Note that every physical measure is a natural measure, but the converse does not hold. The existence of this measure in the generalized Thaler functions raises a question regarding systems with historical behavior almost everywhere. In particular:

Question 1. Let F be a skew product as in (1.1) that exhibits historical behavior almost everywhere, as studied here. Does a natural measure for F exist?

The results in [Mol94] provide a partial (positive) answer to Question 1 for one-step skew products of the generalized (T, T^{-1}) -transformation type, as in (1.14). However, this question remains open for mild skew products. In particular, in the context of Remark 1.6, Question 1 is closely related to a problem posed by Misiurewicz in [Has07, Question 9.4], which addresses the existence of natural measures that are not physical measures in smooth dynamical systems.

1.6. Organization of the paper: In Section 2, we introduce the definition of historical behavior for skew product maps. In Section 3, we summarize the preliminary concepts of probability theory that are useful throughout the paper. In Section 4, we explore the connections between the fluctuation laws introduced earlier. Furthermore, we show that these distributions govern the asymptotic occupation times of orbits, which are linked to historical behavior. Theorems A and B are proven, respectively, in Sections 5 and 6. In Section 7, we obtain Proposition II. Section 8 establishes the historical behavior of examples of one-step skew products, proving Propositions I, IV, VI and IX. In Section 9, the ergodicity (Theorem 9.9), vanishing interior occupational time (Theorem 9.1) and fluctuation law (Theorem 9.15) of skew-translations. We also provide a more detailed proof of Corollary III. Finally, in Section 10, we establish Proposition VII and elaborate some details of the proof of Corollary VIII.

2. HISTORICAL BEHAVIOR ON SKEW PRODUCT MAPS

In this section, we define historical behavior for skew products F as in (1.1) and show that this behavior also holds for appropriate extensions of F .

2.1. Definition of historical behavior. Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard probability space and that M is a compact manifold. Since Ω may be non-compact, $\Omega \times M$ is not necessarily compact. Therefore, it is important to proceed carefully when introducing the correct notion of historical behavior in this context. Considering that the skew products under consideration are the deterministic representation of the random dynamics given by the iteration of the fiber maps, we adopt the following perspective from [Nak17, Def. 1]:

Definition 2.1. Let $F: \Omega \times M \rightarrow \Omega \times M$ be a skew product as in (1.1). We say that F exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point (or almost everywhere) if, for $(\mathbb{P} \times \text{Leb})$ -almost every (ω, x) , there exists a continuous function $\varphi: M \rightarrow \mathbb{R}$ such that the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x))$$

does not exist.

Remark 2.2. Definition 2.1 implies the non-convergence of Birkhoff averages for the skew product F . This follows by considering a continuous map $\phi: \Omega \times M \rightarrow \mathbb{R}$, defined as $\phi(\omega, x) = \varphi(x)$ for $(\omega, x) \in \Omega \times M$. However, the non-convergence of Birkhoff averages for F is equivalent to the non-convergence, in the weak* topology, of the sequence of *empirical probability measures* given by

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{F^j(\omega, x)} \quad \text{for } n \geq 1,$$

only when Ω is compact.

Remark 2.3. The non-convergence of empirical measures is another common way to characterize historical behavior. Since the fiber space is the compact manifold M , Definition 2.1 ensures this non-convergence of empirical measures in the context of fiber dynamics. In other words, Definition 2.1 is equivalent to the non-convergence, in the weak* topology, of the sequence of measures

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_\omega^j(x)} \quad \text{for } n \geq 1,$$

for $(\mathbb{P} \times \text{Leb})$ -a.e. (ω, x) .

Remark 2.4. In Theorem A and B, as well as in other results on historical behavior for almost every point in this paper, we obtain a more uniform version of this concept than what is presented in Definition 2.1. Specifically, we demonstrate that Definition 2.1 applies with respect to the same function φ for $(\mathbb{P} \times \text{Leb})$ -a.e. point.

2.2. Historical behavior for extension maps. A canonical example is the extension of a non-invertible map on the base Ω (e.g., a one-sided shift) to its natural extension, which is invertible (e.g., a two-sided shift). In such cases, the preservation of historical behavior is an immediate consequence of our definition. By Remark 2.3, historical behavior is determined by the non-convergence of averages along the forward fiber orbits $\{f_\omega^j(x)\}_{j \geq 0}$. Since the forward fiber dynamics of an extended point $(\bar{\omega}, x)$ are identical to the dynamics of the original point (ω, x) , the existence of the limit is unaffected. However, the following proposition, which follows from [BNR⁺22, Lemma 5.1], addresses a more general scenario. It provides a framework for cases where the extension might involve a change in the fiber space itself and is not necessarily a simple extension of the base dynamics. It establishes that if a system F exhibits historical behavior uniformly for almost all points (as described in Remark 2.4), then any suitable extension \bar{F} will also exhibit historical behavior.

Given a continuous function $\varphi : M \rightarrow \mathbb{R}$, we define the set $I(F, \varphi)$ of φ -irregular points of the skew product F as follows

$$I(F, \varphi) \stackrel{\text{def}}{=} \left\{ (\omega, x) \in \Omega \times M : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x)) \text{ does not exist} \right\}.$$

Proposition 2.5. Let $\bar{F} : \bar{\Omega} \times \bar{M} \rightarrow \bar{\Omega} \times \bar{M}$ and $F : \Omega \times M \rightarrow \Omega \times M$ be skew products. Assume that there exists a continuous function $\Pi : \bar{\Omega} \times \bar{M} \rightarrow \Omega \times M$ of form

$$\Pi(\omega, x) = (\theta(\omega, x), \pi(x)) \in \Omega \times M, \quad (\omega, x) \in \bar{\Omega} \times \bar{M}.$$

such that

$$F \circ \Pi = \Pi \circ \bar{F} \quad \text{and} \quad \Pi_*(\bar{\mathbb{P}} \times \overline{\text{Leb}}) = \mathbb{P} \times \text{Leb}$$

where $\bar{\mathbb{P}}$, \mathbb{P} and $\overline{\text{Leb}}$, Leb are reference measures on $\bar{\Omega}$, Ω and the normalized Lebesgue measures on \bar{M} , M respectively. Then, for any continuous maps $\varphi : M \rightarrow \mathbb{R}$ it holds that

$$(\mathbb{P} \times \text{Leb})(I(F, \varphi)) \leq (\bar{\mathbb{P}} \times \overline{\text{Leb}})(I(\bar{F}, \varphi \circ \pi)).$$

In particular, if F exhibits historical behavior almost everywhere as in Definition 2.1 with respect to the same function $\varphi : M \rightarrow \mathbb{R}$ for $(\mathbb{P} \times \text{Leb})$ -a.e. point, then \bar{F} also exhibits historical behavior almost everywhere with respect to $\varphi \circ \pi$ for $(\bar{\mathbb{P}} \times \overline{\text{Leb}})$ -a.e. point.

3. PRELIMINARIES OF RANDOM VARIABLES

In this section, we introduce some notation and definitions from probability theory that are useful for our work. Thereafter, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space, and $\{X_n\}_{n \geq 0}$ is a sequence of real-valued random variables.

Definition 3.1. The tail σ -algebra generated by $\{X_n\}_{n \geq 0}$ is the σ -algebra on Ω given by

$$\mathcal{T}(\{X_n\}_{n \geq 0}) \stackrel{\text{def}}{=} \bigcap_{m=1}^{\infty} \mathcal{F}_m^{\infty}$$

where $\mathcal{F}_m^{\infty} = \sigma(X_m, X_{m+1}, \dots)$ denotes the σ -algebra generated by X_m, X_{m+1}, \dots

Lemma 3.2 ([Shi19, §4]). *For any constant $c > 0$, it holds that*

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} X_n(\omega) \geq c\right) \geq \limsup_{n \rightarrow \infty} \mathbb{P}(X_n(\omega) \geq c).$$

Lemma 3.3 ([Shi19, §4]). *For any constant $b \in \mathbb{R}$ and measurable function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$, the sets*

$$\left\{\limsup_{j \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(X_j(\omega)) \leq b\right\} \quad \text{and} \quad \left\{\limsup_{j \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(X_j(\omega)) \geq b\right\}$$

belong to the tail σ -algebra $\mathcal{T}(\{X_n\}_{n \geq 0})$.

The following result is a direct consequence of Hewitt-Savage Zero-One Law [HS55]. Further insights can be found in [Loe78, §26, Theorem B] or [BdH89].

Proposition 3.4. *Let $\{S_n\}_{n \geq 1}$ be the random variables defined by $S_n = X_0 + \dots + X_{n-1}$, $n \geq 1$. If X_i are i.i.d., then the tail σ -algebra $\mathcal{T}(\{S_n\}_{n \geq 1})$ is trivial.*

As a consequence of the above theorem, every random walk on a group has a trivial tail σ -algebra. The following result due to Erdős and Kac [EK47] concludes that the occupation times for random walks are asymptotically arcsine distributed.

Theorem 3.5 (Erdős-Kac arcsine law). *Let $\{S_n\}_{n \geq 1}$ be the random variables defined by $S_n = X_0 + \dots + X_{n-1}$. If $\{X_n\}_{n \geq 0}$ are i.i.d. random variables having mean zero and positive finite variance, then*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=1}^n \mathbb{1}_{(0,\infty)}(S_j) \leq \alpha\right) = \frac{2}{\pi} \arcsin \sqrt{\alpha} \quad 0 \leq \alpha \leq 1.$$

Now, we introduce the definition and some properties of Brownian motion. In this process, the arcsine distribution law also appears.

Definition 3.6. *We say that a real-valued stochastic process $\{B_t: t \geq 0\}$ is a Brownian motion or Wiener process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if the following conditions hold:*

- (i) *The process starts at 0: $\mathbb{P}(B_0 = 0) = 1$.*
- (ii) *The increments are independent, i.e., for all times $0 \leq t_1 \leq \dots \leq t_k$, the increments*

$$B_{t_k} - B_{t_{k-1}}, \dots, B_{t_2} - B_{t_1},$$

are independent random variables.

- (iii) *For $0 \leq s < t$, the increment $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$:*

$$\mathbb{P}(B_t - B_s < x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^x e^{-\frac{x^2}{2(t-s)}} dx, \quad x \in \mathbb{R}.$$

- (iv) *Almost surely, the function $t \mapsto B_t$ is continuous.*

Lévy [L39][L65] (see also [MP10, Sec. 5.4]) proved the following result:

Theorem 3.7 (Lévy arcsine law). *The occupation time above zero of a Brownian motion $\{B_t : t \geq 0\}$,*

$$\chi(B) \stackrel{\text{def}}{=} \int_0^1 \mathbb{1}_{(0,\infty)}(B_t) dt,$$

is arcsine distributed. That is, $\mathbb{P}(\chi(B) \leq \alpha) = \frac{2}{\pi} \arcsin \sqrt{\alpha}$ for any $0 \leq \alpha \leq 1$.

The following results due to Rudolff [Rud88, Proposition 2] can be seen as an asymptotically Brownian invariance principle:

Theorem 3.8 (Rudolph's invariance principle). *Let $(\Omega, \mathcal{F}, \mathbb{P}, \tau)$ be a measure-preserving invertible system where τ is either a subshift of finite type or a C^1 diffeomorphism restricted to a hyperbolic basic set, and \mathbb{P} is a Hölder Gibbs measure. Let $\phi: \Omega \rightarrow \mathbb{R}$ be a Hölder continuous function with $\mathbb{E}[\phi] = 0$ that is not an additive coboundary, i.e., satisfying (C1). Then*

$$\sigma^2 \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[(S_n)^2] > 0 \quad \text{where} \quad S_n \stackrel{\text{def}}{=} \sum_{j=0}^{n-1} \phi \circ \tau^j,$$

and there exists a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ joining $(\Omega, \mathcal{F}, \mathbb{P})$ with a space supporting a standard Brownian motion $\{B_t\}_{t \geq 0}$ such that

$$\lim_{t \rightarrow \infty} \frac{|S_{\lfloor t \rfloor} - \sigma \cdot B_t|}{t^{1/2-\beta}} = 0 \quad \overline{\mathbb{P}}\text{-almost surely, for some } 0 < \beta < 1/2.$$

We also recall the notion of convergence in distribution:

Definition 3.9. *Suppose M is a metric space and \mathcal{A} the Borel σ -algebra on M . Let $\{X_n\}_{n \geq 0}$ and X be M -valued random variables. We say that X_n converge in distribution to a limit X if*

$$\lim_{n \rightarrow \infty} \mathbb{E}(g(X_n)) = \mathbb{E}(g(X)) \quad \text{for every bounded continuous function } g: M \rightarrow \mathbb{R}.$$

The following theorem presents some equivalent conditions for this type of convergence.

Theorem 3.10 (Portmanteau Theorem). *Let $\{X_n\}_{n \geq 0}$ and X be M -valued random variables. The following statements are equivalent:*

- (a) X_n converges in distribution to X ;
- (b) for every bounded measurable function $\chi: M \rightarrow \mathbb{R}$ such that $\mathbb{P}(\chi \text{ is discontinuous at } X) = 0$,

$$\lim_{n \rightarrow \infty} \mathbb{E}[\chi(X_n)] = \mathbb{E}[\chi(X)].$$

Moreover, if $M = \mathbb{R}$, then also the above statements are equivalent to

- (c) $\lim_{n \rightarrow \infty} \mathbb{P}(X_n \leq \alpha) = \mathbb{P}(X \leq \alpha)$ for any continuity point $\alpha \in \mathbb{R}$ of the map $\alpha \mapsto \mathbb{P}(X \leq \alpha)$.

As a consequence, we have the following:

Corollary 3.11. *Let $\{X_n\}_{n \geq 0}$ be a sequence of M -valued random variables converging in distribution to a random variable X . Let $\chi: M \rightarrow \mathbb{R}$ be a bounded measurable function such that χ is continuous*

at X almost surely. Then, the sequence of real-valued random variables $\{\chi(X_n)\}_{n \geq 0}$ converges in distribution to $\chi(X)$. In particular,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\chi(X_n) \leq \alpha) = \mathbb{P}(\chi(X) \leq \alpha)$$

for every continuity point α of the accumulative distribution of X .

Proof. For any continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, $g \circ \chi$ is a bounded measurable function from M to \mathbb{R} such that $g \circ \chi$ is continuous at X almost surely. Then by item (b) in Theorem 3.10, $\mathbb{E}[g(\chi(X_n))] \rightarrow \mathbb{E}[g(\chi(X))]$. This means that $\chi(X_n)$ converges in distribution to $\chi(X)$. \square

The following lemma provides a condition similar to convergence in probability, which is sufficient to demonstrate that two sequences converge to the same distribution (c.f. [MP10, proof of the Donsker invariance principle], or [Liu21, Lemma 6.2]).

Lemma 3.12. *Let $\{X_n\}_{n \geq 0}$ and $\{Y_n\}_{n \geq 0}$ be sequences of random variables taking values in a Polish normed space $(M, \|\cdot\|)$, and let Y be a M -valued random variable. Suppose that $\{Y_n\}_{n \geq 0}$ and Y are identically distributed, and for every $\epsilon > 0$*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\|X_n - Y_n\| > \epsilon) = 0.$$

Then X_n converges in distribution to Y .

We will also use the following lemma:

Lemma 3.13. *Let $\{Y_n\}_{n \geq 1}$ be a sequence of \mathbb{R} -valued random variables. Assume there exists a constant C such that $\limsup_{n \rightarrow \infty} Y_n(\omega) = C$ for \mathbb{P} -a.e. $\omega \in \Omega$. If $\alpha \in \mathbb{R}$ satisfies $C < \alpha$, then*

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \geq \alpha) = 0.$$

Proof. For \mathbb{P} -a.e. $\omega \in \Omega$, by definition of limitsup, $Y_n(\omega) \geq \alpha > C$ occurs finitely many times. Let $E_n = \{Y_n \geq \alpha\}$. Then $\mathbb{1}_{E_n} \rightarrow 0$ pointwise and $|\mathbb{1}_{E_n}| \leq 1$, so by dominated convergence $\mathbb{P}(E_n) \rightarrow 0$. \square

The following result is motivated by condition (1.12) considered as a random process on the real line. This will be useful later to study random walks and skew-translations.

Lemma 3.14. *Let $\{S_n\}_{n \geq 0}$ be a sequence of \mathbb{R} -valued random variables such that for every compact set $K \subset \mathbb{R}$, it holds that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{L_i(0)}(S_j(\omega)) = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Then, for every $\kappa, y \in \mathbb{R}$, it holds

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{\bar{L}_i(\kappa)}(y + S_j(\omega)) - \mathbb{1}_{\bar{L}_i(0)}(S_j(\omega)) \right| = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } i = 0, 1$$

where $\bar{L}_0(s)$ and $\bar{L}_1(s)$ denote, respectively, either

$$\bar{I}_0(s) = (-\infty, s] \text{ and } \bar{I}_1(s) = [s, \infty) \text{ or } \bar{J}_0(s) = (-\infty, s) \text{ and } \bar{J}_1(s) = (s, \infty).$$

Proof. Fix $i \in \{0, 1\}$. Given $s \in \mathbb{R}$, define

$$A_n^s(\omega) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{\bar{L}_i(s)}(S_j(\omega)).$$

Let $K_s = [\min\{0, s\}, \max\{0, s\}]$ be the compact interval between 0 and s . We have that

$$|\mathbb{1}_{L_i(0)}(t) - \mathbb{1}_{L_i(s)}(t)| \leq \mathbb{1}_{K_s}(t) \quad \text{for every } t \in \mathbb{R}.$$

Then, for every $n \geq 1$,

$$|A_n^0(\omega) - A_n^s(\omega)| \leq \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{K_s}(S_j(\omega)).$$

By assumption, the occupational time in any compact set tends to zero almost surely and thus

$$\lim_{n \rightarrow \infty} |A_n^0(\omega) - A_n^s(\omega)| = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (3.1)$$

Finally, for arbitrary fixed $\kappa, y \in \mathbb{R}$, we have that $\mathbb{1}_{L_i(\kappa)}(y + S_j(\omega)) = \mathbb{1}_{L_i(\kappa-y)}(S_j(\omega))$ and hence

$$\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{\bar{L}_i(\kappa)}(y + S_j(\omega)) = A_n^{\kappa-y}(\omega).$$

Therefore, the proof of the corollary is completed by taking $s = \kappa - y$ in (3.1). \square

4. FLUCTUATION LAWS

Let F be a skew product as in (1.8) with fiber dynamics f_ω^n given by (1.2). Recall the *pointwise-fiber* and *skew-product fluctuation laws* from Definitions 1.1 and 1.3, respectively. For $\gamma \in (0, 1)$, define

$$I_0(\gamma) \stackrel{\text{def}}{=} [0, \gamma], \quad J_0(\gamma) \stackrel{\text{def}}{=} (0, \gamma), \quad I_1(\gamma) \stackrel{\text{def}}{=} [\gamma, 1], \quad J_1(\gamma) \stackrel{\text{def}}{=} (\gamma, 1).$$

We also use the unified notation $L_i(\gamma)$ to denote either $I_i(\gamma)$ or $J_i(\gamma)$ for $i = 0, 1$. The following result examines the connection between these two weak arcsine laws.

Proposition 4.1. *Given $\gamma_0, \gamma_1, \alpha \in (0, 1)$ and $i \in \{0, 1\}$, we have that*

$$\liminf_{n \rightarrow \infty} (\mathbb{P} \times \text{Leb}) \left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x)) \leq \alpha \right) < 1 \quad (4.1)$$

implies that there exist a set $B \subset I$ with positive Leb-measure such that

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x_i)) \leq \alpha \right) < 1 \quad \text{for every } x \in B. \quad (4.2)$$

Proof. Define

$$g_n(x) = \mathbb{P} \left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x)) \leq \alpha \right) \quad \text{and} \quad g(x) = \liminf_{n \rightarrow \infty} g_n(x).$$

By Fubini's theorem, (4.1) is equivalent to $\liminf_{n \rightarrow \infty} \int g_n d\text{Leb} < 1$ and (4.2) is equivalent to the set $B = \{x \in I : g(x) < 1\}$ having positive Leb-measure.

Since $\{g_n\}_{n \geq 0}$ are non-negative measurable functions, Fatou's lemma gives

$$\int g d\text{Leb} \leq \liminf_{n \rightarrow \infty} \int g_n d\text{Leb} < 1.$$

Suppose by contradiction that (4.2) is false. Then $\text{Leb}(B) = 0$. Since $0 \leq g_n \leq 1$, we have $0 \leq g \leq 1$. Therefore, $g(x) = 1$ for Leb-a.e. $x \in I$, which implies $\int g d\text{Leb} = 1$. This contradicts the previous inequality. Hence, the set B must have positive measure (i.e., (4.2) holds). \square

Definition 4.2. *The system F satisfies the fiberwise fluctuation law if there are $\gamma_0, \gamma_1 \in (0, 1)$ and subsets B_0, B_1 in I with positive Leb-measure, such that*

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x)) \leq \alpha\right) < 1, \quad \text{for every } \alpha \in (0, 1), x \in B_i \text{ and } i = 0, 1.$$

Remark 4.3. As a consequence of Proposition 4.1, one has the following relation between the three notions of fluctuation laws:

skew-product fluctuation \implies fiberwise fluctuation \implies pointwise-fiber fluctuation.

Now, we show sufficient conditions to get the fiberwise fluctuation law.

Proposition 4.4. *Let $\xi : (0, 1) \rightarrow (0, 1)$ be a measurable function such that there exist sets $B_0, B_1 \subset I$ with positive Leb-measure, and constants $\gamma_0, \gamma_1 \in (0, 1)$ satisfying that*

$$\xi(x_0) \leq \gamma_0 \quad \text{and} \quad \gamma_1 \leq \xi(x_1) \quad \text{for every } x_0 \in B_0 \text{ and } x_1 \in B_1$$

and

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{L_i(\xi(x))}(f_\omega^j(x)) \leq \alpha\right) < 1, \quad \text{for every } \alpha \in (0, 1), x \in B_i \text{ and } i = 0, 1,$$

Then F satisfies the fiberwise fluctuation law with constants γ_0 and γ_1 .

Proof. Since $\xi(x_0) \leq \gamma_0$ and $\gamma_1 \leq \xi(x_1)$, we have $\mathbb{1}_{L_i(\xi(x_i))} \leq \mathbb{1}_{I_i(\gamma_i)}$ for $i = 0, 1$. Then,

$$\mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x_i)) \leq \alpha\right) \leq \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{L_i(\xi(x_i))}(f_\omega^j(x_i)) \leq \alpha\right) \quad i = 0, 1.$$

Taking \liminf and by the assumption, we get that F satisfies the fiberwise fluctuation law. \square

Remark 4.5. In the above proposition, if we take the sets $B_0 = \{x_0\}$ and $B_1 = \{x_1\}$, i.e., containing only one single point, then we get that F satisfies the pointwise-fiber fluctuation law with constant $\gamma_0 = \xi(x_0)$ and $\gamma_1 = \xi(x_1)$.

As a consequence of the previous result, we get the following:

Corollary 4.6. *Let $\xi : (0, 1) \rightarrow (0, 1)$ be a continuous non-constant function such*

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{L_i(\xi(x))}(f_\omega^j(x)) \leq \alpha\right) < 1, \text{ for every } \alpha \in (0, 1), x \in (0, 1) \text{ and } i = 0, 1.$$

Then, F satisfies the fiberwise fluctuation law with constants $0 < \gamma_0 < \gamma_1 < 1$.

Proof. Since ξ is non-constant, there is $\tilde{x}_0, \tilde{x}_1 \in (0, 1)$ such that $\xi(\tilde{x}_0) < \xi(\tilde{x}_1)$. Moreover, by the continuity of ξ , we find closed neighborhoods $B_0, B_1 \subset I$ of \tilde{x}_0, \tilde{x}_1 respectively such that

$$\gamma_0 \stackrel{\text{def}}{=} \max\{\xi(x) : x \in B_0\} < \min\{\xi(x) : x \in B_1\} \stackrel{\text{def}}{=} \gamma_1.$$

Since the sets B_0, B_1 have positive Leb-measure, by Proposition 4.4, we conclude that F satisfies the fiberwise fluctuation law. \square

The next result will be useful for concluding the fiberwise fluctuation law by conjugation. For $s \in \mathbb{R}$, set

$$\bar{I}_0(s) \stackrel{\text{def}}{=} (-\infty, s], \quad \bar{I}_1(s) \stackrel{\text{def}}{=} [s, \infty) \quad \bar{J}_0(s) \stackrel{\text{def}}{=} (-\infty, s), \quad \bar{J}_1(s) \stackrel{\text{def}}{=} (s, \infty).$$

As before, we use the unified notation $\bar{L}_i(s)$ to denote either $\bar{I}_i(s)$ or $\bar{J}_i(s)$ for $i = 0, 1$.

Proposition 4.7. *Let $\mathcal{O} \subset I$ be forward f_ω -invariant for \mathbb{P} -a.e. $\omega \in \Omega$ and let $h : \mathcal{O} \rightarrow h(\mathcal{O}) \subset \mathbb{R}$ be a strictly monotone injection. Define $g_\omega \stackrel{\text{def}}{=} h \circ f_\omega \circ h^{-1}$ on $h(\mathcal{O})$. Fix $x \in \mathcal{O}, \gamma \in \mathcal{O}$ and put $t \stackrel{\text{def}}{=} h(x)$. Then for every $j \geq 0$ and \mathbb{P} -a.e. $\omega \in \Omega$, we have*

$$\mathbb{1}_{L_i(\gamma)}(f_\omega^j(x)) = \begin{cases} \mathbb{1}_{\bar{L}_i(h(\gamma))}(g_\omega^j(t)), & \text{if } h \text{ is increasing,} \\ \mathbb{1}_{\bar{L}_{1-i}(h(\gamma))}(g_\omega^j(t)), & \text{if } h \text{ is decreasing,} \end{cases} \quad i \in \{0, 1\}.$$

Proof. Since \mathcal{O} is forward invariant we have for \mathbb{P} -a.e. $\omega \in \Omega$ and every $j \geq 0$, the conjugacy relation $g_\omega^j(t) = h(f_\omega^j(x))$. If h is increasing then for each j and each $i \in \{0, 1\}$, we have that $f_\omega^j(x) \in L_i(\gamma)$ if and only if $g_\omega^j(t) = h(f_\omega^j(x)) \in \bar{L}_i(h(\gamma))$. So $\mathbb{1}_{L_i(\gamma)}(f_\omega^j(x)) = \mathbb{1}_{\bar{L}_i(h(\gamma))}(g_\omega^j(t))$. If h is decreasing, the inequalities reverse, $f_\omega^j(x) \in L_i(\gamma)$ if and only if $g_\omega^j(t) = h(f_\omega^j(x)) \in \bar{L}_{1-i}(h(\gamma))$. So $\mathbb{1}_{L_i(\gamma)}(f_\omega^j(x)) = \mathbb{1}_{\bar{L}_{1-i}(h(\gamma))}(g_\omega^j(t))$. \square

4.1. Asymptotic occupational times. Consider the following conditions for F :

(OT1) There are $x_0, x_1, \gamma_0, \gamma_1 \in (0, 1)$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x_i)) = 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \quad i = 0, 1.$$

(OT2) There exist sets $B_0, B_1 \subset I$ of positive Leb-measure and constants $\gamma_0, \gamma_1 \in (0, 1)$ such that for every $x_0 \in B_0$ and $x_1 \in B_1$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x_i)) = 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \quad i = 0, 1.$$

The following proposition shows that these asymptotic occupational times can be obtained when the skew products satisfy any previously introduced fluctuation laws.

Proposition 4.8. *Let F be a skew product $F : \Omega \times I \rightarrow \Omega \times I$.*

- (i) *If $\mathbb{P} \times \text{Leb}$ is ergodic and F satisfies the fiberwise fluctuation law, then (OT2) holds.*
- (ii) *If (H0) holds and F satisfies the pointwise-fiber fluctuation law, then (OT1) holds.*

Proof. To prove the first item, assume that F satisfies the fiberwise fluctuation law. Hence, there are sets $B_0, B_1 \subset I$ with positive Leb-measure and constant $\gamma_0, \gamma_1 \in (0, 1)$ such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x_i)) \leq \alpha\right) < 1 \quad \text{for every } \alpha \in (0, 1), x_i \in B_i \text{ and } i = 0, 1. \quad (4.3)$$

Now, fix $\alpha \in (0, 1)$ and define the sets

$$\begin{aligned} A_i(\alpha) &\stackrel{\text{def}}{=} \left\{(\omega, x) \in \Omega \times I : \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x)) > \alpha\right\} \\ B_i^n(\alpha) &\stackrel{\text{def}}{=} \left\{(\omega, x) \in \Omega \times B_i : \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x)) > \alpha\right\} = \left\{(\omega, x) : x \in B_i \text{ and } \omega \in B_i^n(\alpha, x)\right\} \end{aligned}$$

where

$$B_i^n(\alpha, x) \stackrel{\text{def}}{=} \left\{\omega \in \Omega : \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x)) > \alpha\right\} \quad \text{for } x \in B_i \text{ and } i = 0, 1.$$

Note that $A_i(\alpha)$ is F -invariant. By Lemma 3.2, Reverse Fatou's lemma and (4.3), we have

$$\begin{aligned} (\mathbb{P} \times \text{Leb})(A_i(\alpha)) &\geq \limsup_{n \rightarrow \infty} (\mathbb{P} \times \text{Leb})(B_i^n(\alpha)) = \limsup_{n \rightarrow \infty} \int_{B_i} \mathbb{P}(B_i^n(\alpha, x)) d\text{Leb} \\ &\geq \int_{B_i} \limsup_{n \rightarrow \infty} \mathbb{P}(B_i^n(\alpha, x)) d\text{Leb} > 0. \end{aligned}$$

By ergodicity of $\mathbb{P} \times \text{Leb}$, $(\mathbb{P} \times \text{Leb})(A_i(\alpha)) = 1$. Taking $\alpha \rightarrow 1$, we conclude that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x)) = 1 \quad \text{for } (\mathbb{P} \times \text{Leb})\text{-a.e. } (\omega, x) \in \Omega \times I, \quad i = 0, 1.$$

This proves (OT2).

Now, we prove the second item. Assume that F satisfies the pointwise-fiber fluctuation law and condition (H0). By Lemma 3.3, for each $\alpha \in (0, 1)$, the set

$$A_i(\alpha, x_i) \stackrel{\text{def}}{=} \left\{\omega \in \Omega : \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x_i)) \geq \alpha\right\}$$

belongs to the tail σ -algebra $\mathcal{T}(\{f_\omega^n(x_i)\}_{n \geq 0})$ for $i = 0, 1$. Hence, since by (H0) this tail is trivial, we get $\mathbb{P}(A_i(\alpha, x_i)) \in \{0, 1\}$. On the other hand, again by Lemma 3.2 and since F satisfies the

pointwise-fiber fluctuation law,

$$\mathbb{P}(A_i(\alpha, x_i)) \geq \limsup_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x_i)) > \alpha\right) > 0.$$

Therefore, $\mathbb{P}(A_i(\alpha, x_i)) = 1$, and taking $\alpha \rightarrow 1$, we obtain (OT1). \square

5. PROOF OF THEOREM A

Given a continuous function $\varphi: I \rightarrow \mathbb{R}$, we define the functions

$$U_\varphi(\omega, x) \stackrel{\text{def}}{=} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x)) \quad \text{and} \quad L_\varphi(\omega, x) \stackrel{\text{def}}{=} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x)).$$

It can be readily observed that U_φ and L_φ are invariant along the (forward) F -orbit of (ω, x) . Functions possessing this property are typically referred to as *first integral functions* of F . The following results show some properties of U_φ and L_φ under our settings.

5.1. First integral functions for one-step skew products. To prove Theorem A, we first show that the first integral functions are constant, and then evaluate these constants. In what follows, F denotes a skew product as in (1.3), being f_ω^n in (1.4) its fiber dynamics.

Proposition 5.1. *Assume that F satisfies (H0). Then, there exist two real-valued functions u, ℓ on I such that for every $x \in (0, 1)$*

$$U_\varphi(\omega, x) = u(x) \quad \text{and} \quad L_\varphi(\omega, x) = \ell(x) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Proof. Fix some constant $b \in \mathbb{R}$ and define the set $A(b, x) \stackrel{\text{def}}{=} \{\omega \in \Omega: U_\varphi(\omega, x) < b\}$. By Lemma 3.3, $A(b, x)$ belongs to the tail algebra. Consequently, according to (H0), the tail σ -algebra $\mathcal{T}(\{f_\omega^n(x)\}_{n \geq 0})$ is trivial. Then, the probability of $A(b, x)$ is either zero or one. Let

$$u(x) \stackrel{\text{def}}{=} \inf\{b: \mathbb{P}(A(b, x)) = 1\}.$$

Claim 5.1.1. $U_\varphi(\omega, x) = u(x)$ for \mathbb{P} -a.e. $\omega \in \Omega$.

Proof. By the definition of $u(x)$ as an infimum, for any integer $n \geq 1$, we have that the set $\mathcal{C}_n = \{\omega \in \Omega: U_\varphi(\omega, x) < u(x) + 1/n\}$ has full \mathbb{P} -measure for every $n \geq 1$. Since the countable intersection of the sets \mathcal{C}_n also has full measure, it holds that for \mathbb{P} -a.e. $\omega \in \Omega$, $U_\varphi(\omega, x) < u(x) + 1/n$ for all n . This implies that $U_\varphi(\omega, x) \leq u(x)$ for \mathbb{P} -a.e. $\omega \in \Omega$.

Conversely, again by the definition of $u(x)$, for any $n \geq 1$, $u(x) - 1/n$ is not an upper bound for the set of real numbers b satisfying $\mathbb{P}(A(b, x)) = 1$. As the tail σ -algebra is trivial, this necessarily implies that $\mathcal{N}_n = \{\omega \in \Omega: U_\varphi(\omega, x) < u(x) - 1/n\}$ has zero \mathbb{P} -measure. Since the union of \mathcal{N}_n is also a null set, we have that the complementary event, $U_\varphi(\omega, x) \geq u(x) - \frac{1}{n}$ for every $n \geq 1$, has full \mathbb{P} -measure. This implies that $U_\varphi(\omega, x) \geq u(x)$ for \mathbb{P} -a.e. $\omega \in \Omega$.

Combining these two observations, we conclude that $U_\varphi(\omega, x) = u(x)$ for \mathbb{P} -a.e. $\omega \in \Omega$. \square

The above claim provides the function u in the statement of the proposition. Considering the sets $\bar{A}(b, x) = \{\omega \in \Omega : L_\varphi(\omega, x) > b\}$ and arguing similarly, we also get that $L_\varphi(\omega, x) = \ell(x)$ for \mathbb{P} -a.e. $\omega \in \Omega$ where $\ell(x) = \sup\{b : \mathbb{P}(\bar{A}(b, x)) = 1\}$. This concludes the proof. \square

Proposition 5.2. *Let u, ℓ be the functions given in Proposition 5.1. Then, for every $x \in (0, 1)$,*

$$u(x) = u(f_\omega^k(x)) \quad \text{and} \quad \ell(x) = \ell(f_\omega^k(x)) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } k \geq 0.$$

Proof. We only prove the proposition for the function u . The proof for ℓ is similar and hence omitted. By Proposition 5.1, given any $x \in I$, there exists a set Ω_x with $\mathbb{P}(\Omega_x) = 1$, such that $U_\varphi(\omega, x) = u(x)$ for every $\omega \in \Omega_x$. Consider the set \mathcal{A}^n of all words of size $n \geq 1$. Now, define

$$A_n \stackrel{\text{def}}{=} \bigcap_{\omega \in \Omega} \tau^{-n}(\Omega_{f_\omega^n(x)}) = \bigcap_{\bar{w}=w_0 \dots w_{n-1} \in \mathcal{A}^n} \tau^{-n}(\Omega_{f_{w_{n-1}} \circ \dots \circ f_{w_0}(x)}). \quad (5.1)$$

Since \mathcal{A} is an alphabet at the most countable, the set \mathcal{A}^n is also countable. Since $\mathbb{P}(\Omega_{f_\omega^n(x)}) = 1$ and \mathbb{P} is τ -invariant, then $\mathbb{P}(\tau^{-n}(\Omega_{f_\omega^n(x)})) = 1$ for all $n \geq 1$. Now, as the intersection in (5.1) is countable, we have $\mathbb{P}(A_n) = 1$ for every $n \geq 1$. Define the set

$$\Lambda \stackrel{\text{def}}{=} \bigcap_{n \geq 0} A_n \quad \text{where } A_0 = \Omega_x.$$

As Λ is a countable intersection of sets with probability one, it follows that $\mathbb{P}(\Lambda) = 1$. This implies that for every word $\bar{w} = w_0 \dots w_{k-1}$ of size k , the cylinder

$$[\![\bar{w}]\!] \stackrel{\text{def}}{=} \{\omega = (\omega_i)_{i \geq 0} \in \Omega : \omega_i = w_i, i = 0, \dots, k-1\}$$

satisfies $\Lambda \cap [\![\bar{w}]\!] \neq \emptyset$. Now, choosing any $\omega \in \Lambda \cap [\![\bar{w}]\!]$, it holds

$$u(f_\omega^k(x)) = U_\varphi(\tau^k(\omega), f_\omega^k(x)) = U_\varphi(\omega, x) = u(x).$$

Since \bar{w} is an arbitrary word, we conclude that u is constant along the random orbit of x for every $\omega \in \Lambda$ and hence for \mathbb{P} -a.e. $\omega \in \Omega$. \square

Remark 5.3. In Propositions 5.1 and 5.2, one can consider one-step skew products F on $\Omega \times M$, where the fiber space M is any measurable space instead of I . This substitution is possible because only the measurability of the fiber maps is used.

Remark 5.4. The countability assumption for \mathcal{A} is not necessary for Proposition 5.1. This result holds even if $(\mathcal{A}, \mathcal{F}, p)$ is any probability space. However, in Proposition 5.2, the countability assumption of \mathcal{A} is crucial for the existence of the set A_n in (5.1).

Proposition 5.5. *Assume that F satisfies conditions (H0)–(H2) and consider a monotonically increasing function $\varphi : I \rightarrow \mathbb{R}$. Let u, ℓ be the functions given in Proposition 5.1. Then there exist constants $\bar{u}, \bar{\ell} \in \mathbb{R}$ such that*

$$u(x) = \bar{u} \quad \text{and} \quad \ell(x) = \bar{\ell}, \quad \text{for every } x \in (0, 1).$$

Proof. Again, we only prove the proposition for the function u and omitted the details for ℓ . By (H2), given any $x \in (0, 1)$, there are $\alpha, \beta \in \Omega$ such that $f_\alpha(x) < x < f_\beta(x)$.

Claim 5.5.1. *The function u is monotone increasing.*

Proof. Consider $x_1, x_2 \in (0, 1)$ with $x_1 < x_2$. Take $\omega \in \Omega$, since f_ω is monotonically increasing, i.e., (H1) holds, we have that $f_\omega^n(x_1) \leq f_\omega^n(x_2)$ for every $n \geq 1$. As the map φ also is increasing, we have that $\varphi(f_\omega^n(x_1)) \leq \varphi(f_\omega^n(x_2))$ for every $n \geq 0$. Therefore, the average satisfies

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x_1)) \leq \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x_2)).$$

Taking upper limits we have

$$U_\varphi(\omega, x_1) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x_1)) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x_2)) = U_\varphi(\omega, x_2).$$

By Proposition 5.1, we have $U_\varphi(\omega, x_1) = u(x_1)$ and $U_\varphi(\omega, x_2) = u(x_2)$, then $u(x_1) \leq u(x_2)$. This proves the claim. \square

By Proposition 5.2 we have that $u(f_\alpha(x)) = u(x) = u(f_\beta(x))$. Then, the claim implies that u is constant on the interval $(f_\alpha(x), f_\beta(x))$. Since x is arbitrary and belongs to this interval, we conclude that u is locally constant on $(0, 1)$. As u is monotone increasing and $(0, 1)$ is connected, there exists a constant $\bar{u} \in \mathbb{R}$ such that $u(x) = \bar{u}$ for every $x \in (0, 1)$. \square

Corollary 5.6. *Let F be a one-step skew product as in (1.3) satisfying conditions (H0)–(H2) and consider a monotonically increasing function $\varphi: I \rightarrow \mathbb{R}$. Then, there exist constants $\bar{u}, \bar{\ell} \in \mathbb{R}$ such that for every $x \in (0, 1)$ and \mathbb{P} -a.e. $\omega \in \Omega$, it holds $U_\varphi(\omega, x) = \bar{u}$ and $L_\varphi(\omega, x) = \bar{\ell}$.*

Proof. By Proposition 5.1, we have that $U_\varphi(\omega, x) = u(x)$ and $L_\varphi(\omega, x) = \ell(x)$ for \mathbb{P} -a.e. $\omega \in \Omega$. Now, by Proposition 5.5, $u(x) = \bar{u}$ and $\ell(x) = \bar{\ell}$ for every $x \in (0, 1)$, proving the corollary. \square

5.2. Historical behavior from one-step skew products. Recall the condition (OT1) defined in §4.1 and denote by id the identity function on I .

Theorem 5.7. *Let F be a one-step skew product as in (1.3) satisfying conditions (H0)–(H2). We also assume that F satisfies (OT1) with constant $\gamma_0, \gamma_1 \in (0, 1)$ where $\gamma_0 < \gamma_1$. Then*

$$L_{\text{id}}(\omega, x) \leq \gamma_0 \quad \text{and} \quad \gamma_1 \leq U_{\text{id}}(\omega, x) \quad \text{for every } x \in (0, 1) \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

In particular, F has historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point.

Proof. By Corollary 5.6, the functions U_{id} and L_{id} are constant. We now evaluate these constants. We first prove the statement for the function U_{id} .

Lemma 5.8. *For every $x \in (0, 1)$ it holds $U_{\text{id}}(\omega, x) \geq \gamma_1$ for \mathbb{P} -a.e. $\omega \in \Omega$.*

Proof. By (OT1), there exist $x_1 \in (0, 1)$ and $\Omega^+ \subset \Omega$ with $\mathbb{P}(\Omega^+) = 1$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_1(\gamma_1)}(f_\omega^j(x_1)) = 1 \quad \text{for every } \omega \in \Omega^+. \tag{5.2}$$

Since $\mathbb{1}_{I_1(\gamma_1)}(f_\omega^j(x_1)) \cdot \gamma_1 \leq f_\omega^j(x_1)$ for every $\omega \in \Omega^+$, applying the upper limit obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_1(\gamma_1)}(f_\omega^j(x_1)) \cdot \gamma_1 \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x_1) = U_{\text{id}}(\omega, x_1).$$

By (5.2) it follows that $\gamma_1 \leq U_\varphi(\omega, x_1)$ for every $\omega \in \Omega^+$.

On the other hand, by Corollary 5.6, there exists a constant \bar{u} satisfying that for every $x \in (0, 1)$ there is a set $\Omega_x \subset \Omega$ with $\mathbb{P}(\Omega_x) = 1$ such that $U_{\text{id}}(\omega, x) = \bar{u}$ for every $\omega \in \Omega_x$. Consider $\Omega_x^+ = \Omega_x \cap \Omega^+$. Since Ω_x^+ is an intersection of two sets with probability one, its follows that $\mathbb{P}(\Omega_x^+) = 1$. Moreover, $\gamma_1 \leq U_{\text{id}}(\omega, x_1) = \bar{u}$ for any $\omega \in \Omega_{x_1}^+$. This implies that $\gamma_1 \leq U_{\text{id}}(\omega, x)$ for every $\omega \in \Omega_x^+$ proving the lemma. \square

The proof of the statement for the function L_{id} is a variation of the proof of Lemma 5.8.

Lemma 5.9. *For Leb-a.e. $x \in (0, 1)$ it holds $L_{\text{id}}(\omega, x) \leq \gamma_0$ for \mathbb{P} -a.e. $\omega \in \Omega$.*

Proof. By (OT1), there exist $x_0 \in (0, 1)$ and $\Omega^- \subset \Omega$ with $\mathbb{P}(\Omega^-) = 1$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_0(\gamma_0)}(f_\omega^j(x_0)) = 1 \quad \text{for every } \omega \in \Omega^-.$$
 (5.3)

Hence, since $1 = \mathbb{1}_{I_0(\gamma_0)} + \mathbb{1}_{(\gamma_0, 1]}$, then

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(\gamma_0, 1]}(f_\omega^j(x_0)) = 0 \quad \text{for every } \omega \in \Omega^-.$$
 (5.4)

Moreover, since $f_\omega^j(x_0) \leq \mathbb{1}_{I_0(\gamma_0)}(f_\omega^j(x_0)) \cdot \gamma_0 + \mathbb{1}_{(\gamma_0, 1]}(f_\omega^j(x_0))$ for every $\omega \in \Omega^-$, applying the lower limit and having into account that $\liminf_{n \rightarrow \infty} (a_n + b_n) \leq \liminf_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$ for any pair of bounded sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$, (5.3) and (5.4), we obtain

$$L_{\text{id}}(\omega, x_0) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x_0) \leq \gamma_0 \quad \text{for every } \omega \in \Omega^-.$$

On the other hand, by Corollary 5.6, there exists a constant $\bar{\ell}$ satisfying that for every $x \in (0, 1)$ there is a set $\Omega_x \subset \Omega$ with $\mathbb{P}(\Omega_x) = 1$ such that $L_{\text{id}}(\omega, x) = \bar{\ell}$ for every $\omega \in \Omega_x$. Hence the set $\Omega_x^- = \Omega_x \cap \Omega^-$ has probability one and since $\bar{\ell} = L_{\text{id}}(\omega, x_0) \leq \gamma_0$ for any $\omega \in \Omega_{x_0}^-$, we conclude the proof. \square

Lemmas 5.8 and 5.9 show the first part of theorem. In particular, since the lower and the upper Lyapunov function are different, F has historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point. \square

5.3. Proof of Theorem A. As F satisfies the pointwise-fiber fluctuation law, by Proposition 4.8, the fiber maps of F satisfy condition (OT1). Then, by Theorem 5.7, F has historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point.

6. PROOF OF THEOREM B

Theorem 6.1. *Let F be a skew product as in (1.8) satisfying that $\mathbb{P} \times \text{Leb}$ is ergodic and (OT2) with constants $\gamma_0, \gamma_1 \in (0, 1)$ where $\gamma_0 < \gamma_1$. Then*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x) \leq \gamma_0 \quad \text{and} \quad \gamma_1 \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x)$$

for $(\mathbb{P} \times \text{Leb})$ -a.e. $(\omega, x) \in \Omega \times I$. In particular, F has historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point.

Proof. First, observe that by (OT2), for each $i = 0, 1$, the set of $(\omega, x) \in \Omega \times I$ for which

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma_i)}(f_\omega^j(x)) = 1 \tag{6.1}$$

has positive $(\mathbb{P} \times \text{Leb})$ -measure. Since these sets are F -invariant and $\mathbb{P} \times \text{Leb}$ is ergodic, (6.1) holds for $i = 0, 1$ and for every (ω, x) in a set $E \subset \Omega \times I$ with $(\mathbb{P} \times \text{Leb})(E) = 1$. Moreover, since $1 = \mathbb{1}_{I_0(\gamma_0)} + \mathbb{1}_{(\gamma_0, 1]}$, we also have that for every $(\omega, x) \in E$,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(\gamma_0, 1]}(f_\omega^j(x)) = 0. \tag{6.2}$$

Since $\mathbb{1}_{I_1(\gamma_1)}(f_\omega^j(x)) \cdot \gamma_1 \leq f_\omega^j(x)$, taking average and applying the upper limit, we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_1(\gamma_1)}(f_\omega^j(x)) \cdot \gamma_1 \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x) \stackrel{\text{def}}{=} U_{\text{id}}(\omega, x).$$

By (6.1), it follows that $\gamma_1 \leq U_{\text{id}}(\omega, x)$ for every $(\omega, x) \in E$.

Similarly, since $f_\omega^j(x) \leq \mathbb{1}_{I_0(\gamma_0)}(f_\omega^j(x)) \cdot \gamma_0 + \mathbb{1}_{(\gamma_0, 1]}(f_\omega^j(x))$, taking average and applying the lower limit, (6.1) and (6.2), we obtain $L_{\text{id}}(\omega, x) \leq \gamma_0$ for every $(\omega, x) \in E$. This concludes the proof. \square

6.1. Proof of Theorems B. Since F satisfies the skew-product fluctuation law, by Remark 4.3 and Proposition 4.8, condition (OT2) holds. Therefore, by Theorem 6.1, the skew product F has historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point.

7. PROOF OF PROPOSITION II

The first observation is that we can swap the order of the quantifiers in (1.12) as follows:

Lemma 7.1. *Condition (1.12) is equivalent to the following: for $(\mathbb{P} \times \text{Leb})$ -a.e. $(\omega, x) \in \Omega \times I$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[\epsilon, 1-\epsilon]}(f_\omega^j(x)) = 0 \quad \text{for every } 0 < \epsilon < 1/2. \tag{7.1}$$

Proof. It clear that (7.1) implies (1.12). Let us show the converse. For each rational number $\epsilon \in \mathbb{Q} \cap (0, 1/2)$, let S_ϵ be the full-measure set where the limit (1.12) is zero. The intersection $S_0 = \bigcap_{\epsilon \in \mathbb{Q} \cap (0, 1/2)} S_\epsilon$, being a countable intersection of full-measure sets, also has $(\mathbb{P} \times \text{Leb})$ -measure. Let $(\omega, x) \in S$. We must show that the limit condition (1.12) holds for all (not just rational) $\epsilon \in (0, 1/2)$. To do this, fix $\epsilon \in (0, 1/2)$. Since rationals are dense in reals, we can choose a rational number q such that $0 < q < \epsilon$. This implies $[\epsilon, 1 - \epsilon] \subset [q, 1 - q]$, and thus $\mathbb{1}_{[\epsilon, 1-\epsilon]} \leq \mathbb{1}_{[q, 1-q]}$. Then

$$0 \leq \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[\epsilon, 1-\epsilon]}(f_\omega^j(x)) \leq \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[q, 1-q]}(f_\omega^j(x)).$$

Since $(\omega, x) \in S_0 \subset S_q$, taking the limit as $n \rightarrow \infty$ shows that the right-hand side converges to 0 and therefore the limit of the left-hand side is also 0. As ϵ was arbitrary, condition (7.1). \square

Proposition 7.2. *Let F be a skew product as in (1.8) and consider a point $(\omega, x) \in \Omega \times I$ for which (7.1) holds. Then $\mathcal{L}(\omega, x) \subseteq \{\lambda\delta_1 + (1 - \lambda)\delta_0 : \lambda \in [0, 1]\}$.*

Proof. Let ν be an accumulation point in $\mathcal{L}(\omega, x)$, so that for some subsequence $\{n_k\}$, the empirical measures $e_{n_k}(\omega, x)$ converge to ν in the weak* topology. We aim to show that the support of ν is contained in $\{0, 1\}$, which is equivalent to showing $\nu((0, 1)) = 0$.

Let K be an arbitrary compact subset of the open interval $(0, 1)$. A probability measure on \mathbb{R} has at most a countable number of atoms. This implies that the set of $0 < \epsilon < 1/2$ for which the interval $C_\epsilon = [\epsilon, 1 - \epsilon]$ is not a ν -continuity set (i.e., $\nu(\{\epsilon\}) + \nu(\{1 - \epsilon\}) > 0$) is at most countable. Thus, since K is a compact in $(0, 1)$, we can choose an $0 < \epsilon < 1/2$ such that C_ϵ is a ν -continuity set and $K \subseteq C_\epsilon$. By the Portmanteau Theorem, the weak* convergence implies $\nu(C_\epsilon) = \lim_{k \rightarrow \infty} e_{n_k}(\omega, x)(C_\epsilon)$. From (7.1), this limit is zero, i.e.,

$$\nu(C_\epsilon) = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \mathbb{1}_{[\epsilon, 1-\epsilon]}(f_\omega^j(x)) = 0.$$

By the monotonicity of measure, since $K \subseteq C_\epsilon$, we have $\nu(K) = 0$. As K was an arbitrary compact subset of $(0, 1)$, and since the open interval $(0, 1)$ is a countable union of such compact sets, it follows by σ -additivity that $\nu((0, 1)) = 0$. This shows that ν is a convex combination of the Dirac measures δ_0 and δ_1 and the proposition is proven. \square

Proposition 7.3. *Let F be a skew product satisfying the assumptions of Theorem A or Theorem B, along with (1.11). Assume that the constants γ_0 and γ_1 in the definition of the fluctuation laws can be chosen arbitrarily close to 0 and 1, respectively. Then, for $(\mathbb{P} \times \text{Leb})$ -a.e. (ω, x) ,*

$$L_{\text{id}}(\omega, x) \stackrel{\text{def}}{=} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x) = 0 \quad \text{and} \quad U_{\text{id}}(\omega, x) \stackrel{\text{def}}{=} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_\omega^j(x) = 1 \quad (7.2)$$

Moreover, in the one-step case, (7.2) holds for every $x \in (0, 1)$ and \mathbb{P} -a.e. $\omega \in \Omega$.

Proof. Under the assumption of Theorem A, according to Corollary 5.6, there exists constants $\bar{u}, \bar{l} \in \mathbb{R}$ such that $L_{\text{id}}(\omega, x) = \bar{l}$ and $U_{\text{id}}(\omega, x) = \bar{u}$ for \mathbb{P} -a.e. $\omega \in \Omega$. Moreover, from Theorem 5.7

we get that $\bar{\ell} \leq \gamma_0 < \gamma_1 \leq \bar{u}$. Since by assumption we can take $\gamma_0 \rightarrow 0$ and $\gamma_1 \rightarrow 1$, we get that for every $x \in (0, 1)$, (7.2) holds for $(\mathbb{P} \times \text{Leb})$ -a.e. $\omega \in \Omega$.

We can arrive to similar conclusion under the assumption of Theorem B. To see this, we need the following lemma.

Lemma 7.4. *If $\mathbb{P} \times \text{Leb}$ is ergodic with respect to F , then, there are constants $\bar{u}, \bar{\ell}$ such that*

$$U_{\text{id}}(\omega, x) = \bar{u} \quad \text{and} \quad L_{\text{id}}(\omega, x) = \bar{\ell} \quad \text{for } (\mathbb{P} \times \text{Leb})\text{-a.e. } (\omega, x) \in \Omega \times I.$$

Proof. Given a constant $u \in \mathbb{R}$ define the set $A(u) \stackrel{\text{def}}{=} \{(\omega, x) \in \Omega \times I : U_{\text{id}}(\omega, x) < u\}$. Since $A(u)$ is an F -invariant set, by ergodicity of $\mathbb{P} \times \text{Leb}$, we have $(\mathbb{P} \times \text{Leb})(A(u)) \in \{0, 1\}$. Let

$$\bar{u} \stackrel{\text{def}}{=} \inf\{u : (\mathbb{P} \times \text{Leb})(A(u)) = 1\}.$$

Claim 7.4.1. $U_{\text{id}}(\omega, x) = \bar{u}$ for $(\mathbb{P} \times \text{Leb})$ -a.e. $(\omega, x) \in \Omega \times I$.

Proof. If $(\mathbb{P} \times \text{Leb})(A(\bar{u})) = 0$, then $U_{\text{id}}(\omega, x) \geq \bar{u}$ for $(\mathbb{P} \times \text{Leb})$ -a.e. (ω, x) . Moreover, by definition of \bar{u} , we have that $U_{\text{id}}(\omega, x) < \bar{u} + 1/n$ for every $n \geq 1$ and $(\mathbb{P} \times \text{Leb})$ -a.e. (ω, x) . Hence, by taking $n \rightarrow \infty$, we find that $U_{\text{id}}(\omega, x) = \bar{u}$ for $(\mathbb{P} \times \text{Leb})$ -a.e. (ω, x) proving the claim in this case.

To conclude the proof, we need to analyze also the case when $(\mathbb{P} \times \text{Leb})(A(\bar{u})) = 1$. Suppose, by contradiction, that there exists a set $B \subset \Omega \times I$ with $(\mathbb{P} \times \text{Leb})(B) > 0$ such that $U_{\text{id}}(\omega, x) \neq \bar{u}$ for every $(\omega, x) \in B$. By definition of \bar{u} , we have that $(\mathbb{P} \times \text{Leb})(A(\bar{u} - \frac{1}{n})) = 0$ for all $n \geq 1$. Consider the monotonically increasing sequence of sets

$$B_n \stackrel{\text{def}}{=} A\left(\bar{u} - \frac{1}{n}\right) \cap B, \quad \text{for } n \geq 1,$$

and note that $(\mathbb{P} \times \text{Leb})(B_n) = 0$ and therefore $\lim_{n \rightarrow \infty} (\mathbb{P} \times \text{Leb})(B_n) = 0$. However, by the monotonicity and since $(\mathbb{P} \times \text{Leb})(A(\bar{u})) = 1$,

$$\lim_{n \rightarrow \infty} (\mathbb{P} \times \text{Leb})(B_n) = (\mathbb{P} \times \text{Leb})(A(\bar{u}) \cap B) = (\mathbb{P} \times \text{Leb})(B) > 0,$$

which leads to a contradiction. Therefore, $U_{\text{id}}(\omega, x) = \bar{u}$ for $(\mathbb{P} \times \text{Leb})$ -a.e. point, proving the claim. \square

The above claim provides \bar{u} as in the statement of the proposition. Considering the sets $\bar{A}(\ell) = \{\omega \in \Omega : L_{\text{id}}(\omega, x) > \ell\}$ and arguing similarly, we also get that $L_{\text{id}}(\omega, x) = \bar{\ell}$ for $(\mathbb{P} \times \text{Leb})$ -a.e. (ω, x) where $\bar{\ell} = \sup\{\ell : (\mathbb{P} \times \text{Leb})(\bar{A}(\ell)) = 1\}$. This concludes the proof. \square

Now, Lemma 7.4, Theorem 6.1 and again the assumption that γ_0 and γ_1 can be taken arbitrarily close to 0 and 1 respectively, we get (7.2) for $(\mathbb{P} \times \text{Leb})$ -a.e. point. \square

Proposition 7.5. *Let (ω, x) be in $\Omega \times I$ satisfying (7.2). Then, $\mathcal{L}(\omega, x) \supseteq \{\lambda\delta_0 + (1-\lambda)\delta_1 : \lambda \in [0, 1]\}$.*

Proof. Given $(\omega, x) \in \Omega \times I$ for which (7.2) holds, let $\{n_k\}_{k \geq 1}$ be such that

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} f_{\omega}^j(x) = 0.$$

Let μ be an accumulation point in the weak* topology of the subsequence of empirical measure $e_{n_k}(\omega, x)$. For notational simplicity, we assume that $e_{n_k} \rightarrow \mu$ as $k \rightarrow \infty$. Then,

$$0 = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} f_\omega^j(x) = \lim_{k \rightarrow \infty} \int \text{id} \, de_{n_k} \rightarrow \int \text{id} \, d\mu.$$

This implies that μ is the Dirac measure δ_0 . By an analogous argument, we have that δ_1 is an accumulation point of $e_n(\omega, x)$. \square

7.1. Proof of Proposition II. Let S_0 and S_1 be the full-measure sets where, respectively, (7.1) and (7.2) hold. Hence, $S = S_0 \cap S_1$ has full $(\mathbb{P} \times \text{Leb})$ -measure and, for any $(\omega, x) \in S$ we can apply Proposition 7.2 to conclude that

$$\mathcal{L}(\omega, x) \subseteq L := \{\lambda\delta_0 + (1 - \lambda)\delta_1 : \lambda \in [0, 1]\}.$$

On the other hand, since $(\omega, x) \in S_1$, it satisfies condition (7.2) and, by Proposition 7.5, $\mathcal{L}(\omega, x) \supseteq L$. Since both inclusions hold for any point in the full-measure set S , the proof of Proposition II is complete.

Remark 7.6. Consider the one-step case assuming instead (1.12) that for every fixed $x \in (0, 1)$ and $0 < \epsilon < 1/2$, the limit in (1.12) holds \mathbb{P} -almost surely. Thus, the set S_ϵ in the proof of Lemma 7.1 is a bundle over $(0, 1)$ with fiber sets of full \mathbb{P} -measure and, consequently, so is S_0 . Similarly, in this one-step case, from Proposition 7.3, we have that S_1 is a similar bundle over $(0, 1)$. Thus, $S = S_0 \cap S_1$ is also a bundle over $(0, 1)$ with fiber sets of full \mathbb{P} -measure and the observation in Remark 1.4 follows.

8. PROOF OF PROPOSITIONS I, IV, VI AND IX

Let F be a one-step skew product as in (1.3) and fix $x \in (0, 1)$. In this section, we study the historical behavior of F when the Markov chain $\{X_n^x\}_{n \geq 0}$ with $X_n^x(\omega) = f_\omega^n(x)$ is conjugate to a random walk on the additive group $G = \mathbb{Z}$ or \mathbb{R} . Recall that according to Definition 1.2, this means that there is a strict monotonic injection $h: \mathcal{O}(x) \rightarrow G$ such that the step random variables $Y_n^t = S_n^t - S_{n-1}^t$, $n \geq 1$, are i.i.d., where $t = h(x)$ and $S_n^t(\omega) = h \circ f_\omega^n \circ h^{-1}(t)$. Here, the set $\mathcal{O}(x)$ denotes the orbit $\{X_n^x(\omega) : \omega \in \Omega, n \geq 0\}$. Now, define the random walk

$$S_0 = 0, \quad S_n(\omega) \stackrel{\text{def}}{=} \sum_{j=1}^n Y_j^t(\omega) = S_n^t(\omega) - t \quad \text{for } n \geq 1. \quad (8.1)$$

Notice that

$$S_n^t(\omega) = t + S_n(\omega) = g_\omega^n(t) \quad \text{where } g_\omega = h \circ f_\omega \circ h^{-1}. \quad (8.2)$$

Lemma 8.1. *Let $\{S_n\}_{n \geq 0}$ be a G -valued random walk starting $S_0 = 0$ with mean zero and positive finite variance. Then, for every $\kappa, t \in \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\kappa)}(t + S_j(\omega)) \leq \alpha\right) = \frac{2}{\pi} \arcsin \sqrt{\alpha} \quad \text{for every } \alpha \in (0, 1) \text{ and } i = 0, 1, \quad (8.3)$$

where $I_0(\kappa) = (-\infty, \kappa]$ and $I_1(\kappa) = [\kappa, \infty)$.

Proof. We prove (8.3) for $i = 1$; the case $i = 0$ follows analogously. Define

$$N_n(\omega) \stackrel{\text{def}}{=} \#\{j \in \{0, \dots, n-1\} : S_j(\omega) > 0\} = \sum_{j=0}^{n-1} \mathbb{1}_{(0,\infty)}(S_j(\omega)). \quad (8.4)$$

By assumption, the steps variables $Y_n = S_n - S_{n-1}$, $n \geq 1$, are i.i.d. with zero mean and positive finite variance. Applying the arcsine law (Theorem 3.5) to the sequence $\{Y_n\}_{n \geq 1}$ and using (8.4), we obtain (8.3) for the random process starting at $t = 0$ with target set $\bar{J}_1(0) = (0, \infty)$ instead $\bar{I}_1(0) = [0, \infty)$. The difference between the time averages for the closed and open intervals is the occupation time of the point $\{0\}$. Since the random walk has mean zero, it is recurrent. A classical result for one-dimensional recurrent random walks is that the occupational time spent at any compact set vanishes in the limit. Proposition 9.7 provides a new direct proof of this classic fact. Hence, since the time average over the open interval converges in distribution to the arcsine law, and the difference between the two averages converges almost surely (and thus in probability) to zero, Slutsky's theorem implies that the time average over the closed interval converges to the same arcsine distribution. This establishes that A_n^0 converges in distribution to the arcsine law, where

$$A_n^s = \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[s,\infty)}(S_j) \quad \text{for } s \in \mathbb{R} \text{ and } n \geq 1.$$

Once again, using the vanishing occupational time in compact sets for non-frozen random walks (see Proposition 9.7), Lemma 3.14 applies to get that $A_n^0 - A_n^{\kappa-t} \rightarrow 0$ almost surely (in particular in probability) and thus $A_n^{\kappa-t}$ also converges in distribution to the arcsine law. This proves (8.3) and completes the proof. \square

In view of (8.2), as an immediate consequence of the previous lemma and Proposition 4.7, we have the following:

Corollary 8.2. *Let F be a one-step skew product as in (1.3) such that the sequence of random variables $\{X_n^x\}_{n \geq 0}$ is conjugate to a G -valued random walk with mean zero and positive finite variance. Then, for every $\gamma \in (0, 1)$*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(f_\omega^j(x)) \leq \alpha\right) = \frac{2}{\pi} \arcsin \sqrt{\alpha} \quad \text{for every } \alpha \in (0, 1) \text{ and } i = 0, 1.$$

In the next proposition, we prove that the fiber maps satisfy (H0).

Proposition 8.3. *Let F be a one-step skew product as in (1.3) such that the sequence of random variables $\{X_n^x\}_{n \geq 0}$ is conjugate to a G -valued random walk. Then the tail σ -algebra $\mathcal{T}(\{X_n^x\}_{n \geq 1})$ is trivial. In particular, if the conjugation is established for all $x \in (0, 1)$, then F satisfies (H0).*

Proof. Consider again the random walk $\{S_n\}_{n \geq 0}$ in (8.1). By Proposition 3.4, the tail σ -algebra $\mathcal{T}(\{S_n\}_{n \geq 1})$ is trivial. Since tail σ -algebras are invariant under translations, we have

$$\mathcal{T}(\{S_n\}_{n \geq 1}) = \mathcal{T}(\{S_n^t - t\}_{n \geq 1}) = \mathcal{T}(\{S_n^t\}_{n \geq 1}).$$

Thus, $\mathcal{T}(\{S_n^t\}_{n \geq 1})$ is trivial. Finally, as the conjugation h is a measurable bijection, the σ -algebra generated by $\{X_n^x\}_{n \geq 1}$ equals that generated by $\{S_n^t\}_{n \geq 1}$ where $t = h(x)$. Consequently, $\mathcal{T}(\{X_n^x\}_{n \geq 1})$ is trivial, which completes the proof. \square

8.1. Proof of Proposition I. By assumption, $\{X_n^x\}_{n \geq 1}$ is conjugate to a G -valued random walk for all $x \in (0, 1)$. Hence, by Corollary 8.2, it follows that F satisfies the arcsine law. Also Proposition 8.3 shows that condition (H0) is satisfied. Consequently, assuming additionally that F also satisfies conditions (H1) and (H2), Theorem A implies that F has historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point.

8.2. Proof of Proposition IV. Recall that Ψ is a \mathbb{Z} -valued one-step random variable and T is a Morse-Smale homeomorphism of period one. Hence, it follows that the maps $f_\omega = T^{\Psi(\omega)}$ are monotone and increasing on each open interval J between consecutive fixed points p and q . Consequently, condition (H1) is established. On the other hand, since by assumption $\mathbb{E}[\Psi] = 0$ and $\Psi \neq 0$ because of $0 < \mathbb{E}[\Psi^2] < \infty$, we have $\alpha, \beta \in \Omega$ such that $\Psi(\alpha)\Psi(\beta) < 0$. Hence, again, since T is a Morse-Smale of period one, we have $T^{\Psi(\alpha)} < \text{id} < T^{\Psi(\beta)}$ or viceversa on J . This implies the verification of condition (H2).

For any $x \in J$, we define the monotonically increasing measurable bijection

$$h: \mathcal{O}_T(x) \rightarrow \mathbb{Z}, \quad h(T^t(x)) \stackrel{\text{def}}{=} t, \quad \text{for every } t \in \mathbb{Z},$$

where $\mathcal{O}_T(x) = \{T^t(x) : t \in \mathbb{Z}\}$ is the full orbit of x . Observe that $h^{-1}(t) = T^t(x)$. Then,

$$g_\omega(t) = h \circ f_\omega \circ h^{-1}(t) = h(f_\omega(T^t(x))) = h(T^{t+\Psi(\omega_0)}(x)) = t + \Psi(\omega_0),$$

which defines a \mathbb{Z} -valued random walk driven by Ψ . In particular, we have that the sequence of random variables $X_n^x(\omega) = f_\omega^n(x) = T^{\Psi(\omega)}(x)$ is conjugate to a random walk on \mathbb{Z} with mean zero and positive finite variance. Therefore, by Proposition I, $F_{T^\Psi}|_{\Omega \times J}$ exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -almost every point in $\Omega \times J$. Since M is the union of finitely many such intervals J , it follows that F_{T^Ψ} exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -almost every point. Finally, by Corollary III, for any $x \in J = (p, q)$, the limit set is given by $\mathcal{L}(\omega, x) = \{\lambda\delta_p + (1 - \lambda)\delta_q : \lambda \in [0, 1]\}$. This concludes the proof of Proposition IV.

8.3. Proof of Proposition VI. In this subsection, we study the skew product $F_{T^\Psi, Z}$.

Proposition 8.4. *The skew product $F_{T^\Psi, Z}$ in (1.15) satisfies conditions (H1) and (H2).*

Proof. To establish condition (H1), we must show that f_{ω_0} is monotone increasing on $(0, 1)$. Let $x, y \in (0, 1)$ with $x < y$. We consider two cases.

Case 1: $x, y \in I_k$ for some $k \in \mathbb{Z}$. On the interval $I_k = [p_k, p_{k+1})$, according to (1.16), f_{ω_0} is the composition $f_{\omega_0}(z) = v \circ T^{\Psi(\omega_0)} \circ u(z)$, where $u(z) = (z - p_k)/d_k$ and $v(z) = p_{k+Z(\omega_0)} + d_{k+Z(\omega_0)}z$. The maps u and v are strictly increasing affine functions (since $d_k, d_{k+Z(\omega_0)} > 0$), and the north-south homeomorphism $T^{\Psi(\omega_0)}$ is strictly increasing by definition. As a composition of strictly increasing functions, f_{ω_0} is strictly increasing on I_k . Thus, $f_{\omega_0}(x) < f_{\omega_0}(y)$.

Case 2: $x \in I_k$ and $y \in I_j$ for $k < j$. By definition of the partition, this implies $x < p_{k+1} \leq p_j < y$. The map f_{ω_0} sends the entire interval I_k to the interval $I_{k'}$ where $k' = k + Z(\omega_0)$, and similarly sends I_j to $I_{j'}$ where $j' = j + Z(\omega_0)$. Since $k < j$, we have $k' < j'$. The partition is ordered, so the interval $I_{k'}$ lies entirely to the left of $I_{j'}$, i.e., $\sup(I_{k'}) = p_{k'+1} \leq p_{j'} = \inf(I_{j'})$. As $f_{\omega_0}(x) \in I_{k'}$ and $f_{\omega_0}(y) \in I_{j'}$, it follows that $f_{\omega_0}(x) < p_{k'+1} \leq p_{j'} \leq f_{\omega_0}(y)$. Thus, $f_{\omega_0}(x) < f_{\omega_0}(y)$.

In both cases the map f_{ω_0} is strictly increasing proving (H1). We now verify condition (H2). By assumption, $\mathbb{E}[Z] = 0$ with $0 < \mathbb{E}[Z^2] < \infty$, so Z is non-degenerate and takes both positive and negative values. Then, for every $k \in \mathbb{Z}$ and $x \in I_k$, there exist $\alpha \in \Omega$ with $Z(\alpha) > 0$ such that f_α maps x to $I_{k+Z(\alpha)}$, moving toward 1 and therefore $f_\alpha(x) > x$; and $\beta \in \Omega$ with $Z(\beta) < 0$ such that f_β maps x to $I_{k+Z(\beta)}$, moving toward 0 and therefore $f_\beta(x) < x$. This completes the verification of (H2). \square

Recall that $F_{T^\Psi, Z}$ can be written according to (1.17) as a coupling $X_n^x = p_{Z_n} + d_{Z_n} \cdot u_n$ where $u_n = T^{S_n}(u_0)$, of a macro random walk $\{Z_n\}_{n \geq 0}$ and a micro walk $\{S_n\}_{n \geq 0}$ given in (1.18).

Proposition 8.5. *The skew product $F_{T^\Psi, Z}$ satisfies condition (H0), i.e., for every $x \in (0, 1)$, the sequence of random variables $\{X_n^x\}_{n \geq 1}$ has a trivial tail σ -algebra.*

Proof. First, note that if $x = p_k$ for some $k \in \mathbb{Z}$, then $\{X_n^x\}_{n \geq 0}$ is conjugate to the random walk $\{Z_n\}_{n \geq 0}$. Therefore, Proposition 8.3 implies that $\{X_n^x\}_{n \geq 0}$ has a trivial tail σ -algebra. Now consider $x \in (0, 1) \setminus \{p_k : k \in \mathbb{Z}\}$. Define $\phi: (0, 1) \rightarrow (0, 1)$ and $\kappa: (0, 1) \rightarrow \mathbb{Z}$ by

$$\kappa(t) \stackrel{\text{def}}{=} \text{the unique integer } k \text{ such that } t \in I_k, \quad \text{and} \quad \phi(t) \stackrel{\text{def}}{=} \frac{t - p_{\kappa(t)}}{d_{\kappa(t)}}.$$

These functions characterize the σ -algebra of $\{X_n^x\}_{n \geq 0}$ in terms of the tail σ -algebras of $\{Z_n\}_{n \geq 0}$ and $\{u_n\}_{n \geq 0}$. Recall that for a sequence of random variables $\{X_n^x\}_{n \geq 0}$, the generated σ -algebra is defined by

$$\sigma(X_n^x : n \geq 0) \stackrel{\text{def}}{=} \sigma\left(\bigcup_{n \geq 0} \sigma(X_n^x)\right).$$

Lemma 8.6. *Fix $k \in \mathbb{Z}$ and $x \in I_k \setminus \{p_k\}$. Then the following hold:*

- (i) *For every $n \geq 0$, $u_n \in (0, 1)$, $X_n^x \in I_{Z_n}$, and $X_n^x \notin \{p_k : k \in \mathbb{Z}\}$. Moreover, the functions κ and ϕ are Borel measurable and satisfy*

$$Z_n = \kappa(X_n^x) \quad \text{and} \quad u_n = \phi(X_n^x) \quad \text{for every } n \geq 0.$$

- (ii) *For every $m \geq 1$, $\sigma(X_n^x : n \geq m) = \sigma((Z_n, u_n) : n \geq m)$, and consequently,*

$$\mathcal{T}(\{X_n^x\}_{n \geq 1}) = \mathcal{T}(\{(Z_n, u_n)\}_{n \geq 1}). \tag{8.5}$$

Proof. To prove the first item, note that $x \in I_k \setminus \{p_k\}$ implies $u_0 = (x - p_k)/d_k \in (0, 1)$. Since T is a north-south homeomorphism, $T^n((0, 1)) \subset (0, 1)$ for all $n \in \mathbb{Z}$, and thus $u_n = T^{S_n}(u_0) \in (0, 1)$ for every $n \geq 0$. Therefore, for each $n \geq 0$, $X_n^x = p_{Z_n} + d_{Z_n} \cdot u_n \in (p_{Z_n}, p_{Z_n+1}) = I_{Z_n}$. The

uniqueness of the interval containing X_n^x ensures that κ is well-defined at $t = X_n^x$ and gives $Z_n = \kappa(X_n^x)$. It follows that

$$u_n = \frac{X_n^x - p_{Z_n}}{d_{Z_n}} = \frac{X_n^x - p_{\kappa(X_n^x)}}{d_{\kappa(X_n^x)}} = \phi(X_n^x).$$

The Borel measurability of κ and ϕ follows from the fact that κ is piecewise constant on the measurable partition $\{I_k\}_{k \in \mathbb{Z}}$, and ϕ is piecewise affine on the same partition.

Now, we prove the second item. Fix $m \geq 1$ and note that $(Z_n, u_n) = (\kappa(X_n^x), \phi(X_n^x))$ for every $n \geq m$. Since κ and ϕ are Borel measurable, the pair (Z_n, u_n) is a measurable function of X_n^x . This implies $\sigma(Z_n, u_n) \subset \sigma(X_n^x)$. Conversely, for every $n \geq m$, the identity $X_n^x = p_{Z_n} + d_{Z_n} \cdot u_n$ shows that X_n^x is a Borel function of (Z_n, u_n) , and hence $\sigma(X_n^x) \subset \sigma(Z_n, u_n)$. Therefore, $\sigma(X_n^x) = \sigma(Z_n, u_n)$ for every $n \geq m$. Taking the σ -algebra generated by the family $\{X_n^x : n \geq m\}$, we obtain

$$\sigma(X_n^x : n \geq m) = \sigma\left(\bigcup_{n \geq m} \sigma(X_n^x)\right) = \sigma\left(\bigcup_{n \geq m} \sigma(Z_n, u_n)\right) = \sigma((Z_n, u_n) : n \geq m),$$

where the second equality follows from the identity $\sigma(X_n^x) = \sigma(Z_n, u_n)$ for each n . Taking the intersection over all $m \geq 1$ yields $\mathcal{T}(\{X_n^x\}_{n \geq 1}) = \mathcal{T}(\{(Z_n, u_n)\}_{n \geq 1})$, which completes the proof of the lemma. \square

Now, as mentioned, $\{u_n\}_{n \geq 0}$ is conjugate to a random walk $\{S_n\}_{n \geq 0}$. That is, there is a monotonic bijection $h: \mathcal{O}_T(u_0) \rightarrow \mathbb{Z}$ such that $S_n = h(u_n)$ where $\mathcal{O}_T(u_0) = \{T^t(u_0) : t \in \mathbb{Z}\}$. See §8.2 for more details. Define the homeomorphism $H \stackrel{\text{def}}{=} \text{id} \times h$ from $\mathbb{Z} \times \mathcal{O}_T(u_0)$ to \mathbb{Z}^2 . This map conjugates (Z_n, u_n) to (Z_n, S_n) , i.e., $(Z_n, S_n) = H(Z_n, u_n)$. By (1.18), (Z_n, S_n) is driven by the one-step random variables Z and Ψ . This means that the step increments are

$$\Delta_n(\omega) = (Z_n(\omega) - Z_{n-1}(\omega), S_n(\omega) - S_{n-1}(\omega)) = (Z(\omega_{n-1}), \Psi(\omega_{n-1})).$$

Notice that $\Delta_n(\omega) = \chi(\omega_{n-1})$ where $\chi(a) = (Z(a), \Psi(a))$ is a measurable function on \mathcal{A} . Since the background process $\{\omega_n\}_{n \geq 0}$ is i.d.d., we also get that Δ_n is i.d.d. and thus, the joint process (Z_n, S_n) is a random walk on the abelian group \mathbb{Z}^2 . Hence, by Proposition 3.4, the tail σ -algebra $\mathcal{T}(\{(Z_n, S_n)\}_{n \geq 0})$ is trivial. Therefore, by Lemma 8.6 and since H is a homeomorphism, it follows that $\mathcal{T}(\{X_n^x\}_{n \geq 0}) = \mathcal{T}(\{(Z_n, u_n)\}_{n \geq 0}) = \mathcal{T}(\{(Z_n, S_n)\}_{n \geq 0})$ is trivial, which completes the proof of the proposition. \square

Lemma 8.7. *If the random variable Z has mean zero and finite positive variance, then the skew product $F_{T^\Psi, Z}$ satisfies the pointwise fiber fluctuation law with constants $\gamma_0 < \gamma_1$.*

Proof. For $i \in \{0, 1\}$, set $x_i = p_i$. Since the chain $\{X_n^{x_i}\}_{n \geq 0}$ is conjugate to the \mathbb{Z} -valued random walk $\{Z_n\}_{n \geq 0}$, which has step increments $Z_n(\omega) - Z_{n-1}(\omega) = Z(\omega_{n-1})$ with mean zero and positive finite variance, Corollary 8.2 implies that

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(f_\omega^j(x_i)) \leq \alpha\right) = \frac{2}{\pi} \arcsin \sqrt{\alpha} \quad \text{for every } \alpha, \gamma \in (0, 1) \text{ and } i = 0, 1$$

concluding the proposition. \square

By Lemmas 8.5, 8.4 and 8.7, $F_{T^\Psi, Z}$ satisfies (H0)–(H2) and pointwise-fiber fluctuation law. Therefore, by Theorem A, the skew product $F_{T^\Psi, Z}$ exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -almost every point. This completes the proof of Proposition VI.

8.4. Proof of Proposition IX. Let F_a be the skew product given in (1.21) where $a : \Omega \rightarrow (0, \infty)$ satisfies (1.22), (E0) and (E1), i.e., $a(\omega) = a(\omega_0)$ for every $\omega = (\omega_i)_{i \geq 0} \in \Omega$, and

$$\int \log a(\omega) d\mathbb{P} = 0, \quad \int (\log a(\omega))^2 d\mathbb{P} < \infty \quad \text{and} \quad a(\omega) \neq 1 \quad \text{for every } \omega \in \Omega.$$

We consider the preserving-orientation homeomorphism

$$h : (0, 1) \rightarrow \mathbb{R}, \quad h(x) = \log\left(\frac{x}{1-x}\right).$$

Then, since for any $x \in (0, 1)$

$$h \circ f_\omega(x) = \log\left(\frac{a(\omega_0)x}{1-x}\right) = h(x) + \log(a(\omega_0))$$

it follows that

$$g_\omega(t) \stackrel{\text{def}}{=} h \circ f_\omega \circ h^{-1}(t) = t + \log(a(\omega_0)) \quad \text{where } t = h(x).$$

From this and taking into account that

$$S_n^t(\omega) \stackrel{\text{def}}{=} g_{\omega_{n-1}} \circ \cdots \circ g_{\omega_0}(t) = t + \log(a(\omega_0)) + \cdots + \log(a(\omega_{n-1})), \quad n \geq 1,$$

we obtain that the sequence of step random variables

$$Y_n^t(\omega) \stackrel{\text{def}}{=} S_n^t(\omega) - S_{n-1}^t(\omega) = \log(a(\omega_{n-1})), \quad n \geq 1,$$

is independent and identically distributed. We also have that

$$\mu = \mathbb{E}[Y_1^t] = \int \log(a(\omega)) d\mathbb{P} = 0 \quad \text{and} \quad \sigma^2 = \mathbb{E}[(Y_1^t - \mu)^2] = \int (\log a(\omega))^2 d\mathbb{P} \in (0, \infty).$$

Thus, we conclude that for every $x \in (0, 1)$, the sequence $\{X_n^x\}_{n \geq 1}$ of random variables $X_n^x(\omega) = f_\omega^n(x)$ is conjugate to a random walk on \mathbb{R} with mean zero and finite variance.

To prove Proposition IX remains to show that F_a satisfies conditions (H1) and (H2). Clearly (H1) holds since the maps f_ω are interval diffeomorphisms with $f(0) = 0$ and $f(1) = 1$.

We begin by showing that F satisfies the condition (H2). Define

$$\Omega_- \stackrel{\text{def}}{=} \{\omega = (\omega_i)_{i \geq 0} \in \Omega : a(\omega_0) < 1\} \quad \text{and} \quad \Omega_+ \stackrel{\text{def}}{=} \{\omega = (\omega_i)_{i \geq 0} \in \Omega : a(\omega_0) > 1\}.$$

Note that since $a(\omega_0) \neq 1$ for every $\omega = (\omega_i)_{i \geq 0} \in \Omega$, we have $\mathbb{P}(\Omega_-) + \mathbb{P}(\Omega_+) = 1$.

Lemma 8.8. $\mathbb{P}(\Omega_-) > 0$ and $\mathbb{P}(\Omega_+) > 0$.

Proof. By contradiction, suppose that $\mathbb{P}(\Omega_1) = 1$. Hence,

$$\lambda(\delta_0) \stackrel{\text{def}}{=} \int \log(f'_\omega(0)) d\mathbb{P} = \int_{\Omega_-} \log(f'_\omega(0)) d\mathbb{P} = \int_{\Omega_-} \log(a(\omega_0)) d\mathbb{P} < 0.$$

Since, as mentioned in the introduction, the representation of the fiber maps f_ω of F_a is equivalent to conditions (S1)–(S3) we arrive to a contradiction with $\lambda(\delta_0) = 0$. A similar contradiction arises assuming that $\mathbb{P}(\Omega_+) = 1$. This proves the lemma. \square

Note that $f_\omega(x) < x$ for every $x \in (0, 1)$ and $\omega \in \Omega_-$. Similarly, $f_\omega(x) > x$ for every $x \in (0, 1)$ and $\omega \in \Omega_+$. Hence, for every $x \in (0, 1)$, we have that $\Omega_- \subset \{\omega \in \Omega : f_\omega(x) < x\}$ and $\Omega_+ \subset \{\omega \in \Omega : f_\omega(x) < x\}$. Now, Lemma 8.8 implies both sets have positive probability, proving the first part of the proposition. From this, it immediately follows that the skew product F_a satisfies the condition (H2).

Consequently, by Proposition I and Corollary III, F_a exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -a.e. point. Moreover, for every $x \in (0, 1)$, $\mathcal{L}(\omega, x) = \{\lambda\delta_0 + (1 - \lambda)\delta_1 : \lambda \in [0, 1]\}$ for \mathbb{P} -a.e. point, completing the proof of Proposition IX.

9. SKEW-TRANSLATIONS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space and $\tau : \Omega \rightarrow \Omega$ an ergodic, measure-preserving transformation. Let $\phi : \Omega \rightarrow \mathbb{R}$ be a measurable function and consider the skew-translation

$$T_\phi : \Omega \times \mathbb{R} \rightarrow \Omega \times \mathbb{R}, \quad T_\phi(\omega, y) = (\tau(\omega), y + \phi(\omega)). \quad (9.1)$$

Observe that the infinite product measure $\mu = \mathbb{P} \times \text{Leb}$ on $\Omega \times \mathbb{R}$ is T_ϕ -invariant.

9.1. Vanishing interior occupational time for skew-translations.

Theorem 9.1. *Let T_ϕ be a skew-translation as in (9.1). Assume one of the following conditions:*

- (1) $\phi : \Omega \rightarrow \mathbb{R}$ is not a multiplicative coboundary, i.e., (C2) holds;
- (2) (Ω, τ) is a (one-sided or two-sided) subshift of finite type or a hyperbolic basic set of a C^1 diffeomorphism and \mathbb{P} is a τ -invariant Hölder-Gibb measure. Moreover, $\phi : \Omega \rightarrow \mathbb{R}$ is Hölder and is not an additive coboundary, i.e., (C1) holds;
- (3) $(\Omega, \mathbb{P}, \tau)$ is a (one-sided or two-sided) Bernoulli shift and $\phi : \Omega \rightarrow \mathbb{R}$ is a non-zero one-step map.

Then for every fixed initial point $y \in \mathbb{R}$ and every compact set $K \subset \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_K(y + S_j(\omega)) = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

where $S_0 = 0$ and $S_j(\omega) \stackrel{\text{def}}{=} \sum_{i=0}^{j-1} \phi(\tau^i(\omega))$ for $j \geq 1$.

Proof. The proof proceeds by projecting the dynamics to a compact space. Fix $L > 0$ and denote by $\mathbb{T}_L = \mathbb{R}/(L\mathbb{Z})$ the circle of length L , equipped with Lebesgue measure Leb_L . Define

$$F_L : \Omega \times \mathbb{T}_L \rightarrow \Omega \times \mathbb{T}_L, \quad F_L(\omega, y) \stackrel{\text{def}}{=} (\tau\omega, y + \phi(\omega) \bmod L),$$

which preserves $\mu_L \stackrel{\text{def}}{=} \mathbb{P} \times \text{Leb}_L$. Our proof relies on the following proposition, which is of independent interest.

Proposition 9.2. *Under the assumption of Theorem 9.1, the set of lengths L for which F_L is not ergodic with respect to μ_L is at most countable. In particular, there exists a sequence $\{L_m\}_{m \geq 1}$ with $L_m \rightarrow \infty$ such that F_{L_m} is ergodic with respect to μ_{L_m} .*

Before proving the above proposition, let us conclude the proof of the theorem. If F_L is ergodic with respect to μ_L , Birkhoff Ergodic Theorem implies that the orbits are uniformly distributed for μ_L -a.e. (ω, y) . This means that for any continuous function $g : \mathbb{T}_L \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(y + S_j(\omega) \bmod L) = \int g d\text{Leb}_L \quad (9.2)$$

for μ_L -a.e. (ω, y) . Actually, we can strengthen this consequence:

Claim 9.2.1. *For any starting point $y \in \mathbb{R}$, there is $\Omega_{L,y} \subset \Omega$ with $\mathbb{P}(\Omega_{L,y}) = 1$ such that (9.2) holds for every $\omega \in \Omega_{L,y}$.*

Proof. Fix a continuous function $g : \mathbb{T}_L \rightarrow \mathbb{R}$ and $\omega \in \Omega$. Define

$$A_n^\omega(y) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{j=0}^{n-1} g(y + S_j(\omega) \bmod L) \quad y \in \mathbb{T}_L.$$

Since g is continuous on the compact space \mathbb{T}_L , it is uniformly continuous. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $d_{\mathbb{T}_L}(u, v) < \delta$ implies that $|g(u) - g(v)| < \varepsilon/2$. If $y, y' \in \mathbb{T}_L$ satisfy $d_{\mathbb{T}_L}(y, y') < \delta$, then for every $j \geq 0$, $d_{\mathbb{T}_L}(y + S_j(\omega), y' + S_j(\omega)) = d_{\mathbb{T}_L}(y, y') < \delta$, hence $|g(y + S_j(\omega) \bmod L) - g(y' + S_j(\omega) \bmod L)| < \varepsilon/2$. Consequently, for every $n \geq 1$,

$$|A_n^\omega(y) - A_n^\omega(y')| \leq \frac{1}{n} \sum_{j=0}^{n-1} |g(y + S_j(\omega) \bmod L) - g(y' + S_j(\omega) \bmod L)| < \frac{\varepsilon}{2}.$$

Thus the family $(A_n^\omega)_{n \geq 1}$ is equicontinuous.

Now, using Fubini, from (9.2), we have that for \mathbb{P} -a.e. $\omega \in \Omega$, there exists $\mathcal{T}_\omega \subset \mathbb{T}_L$ with $\text{Leb}_L(\mathcal{T}_\omega) = 1$ such that $A_n^\omega(y) \rightarrow \ell \stackrel{\text{def}}{=} \int g d\text{Leb}_L$ for all $y \in \mathcal{T}_\omega$ as $n \rightarrow \infty$. Fix arbitrary $y_0 \in \mathbb{T}_L$. Since \mathcal{T}_ω has full measure, it is dense on \mathbb{T}_L , and thus we can pick $y \in \mathcal{T}_\omega$ with $d_{\mathbb{T}_L}(y_0, y) < \delta$. By hypothesis $A_n^\omega(y) \rightarrow \ell$, so there exists $N \geq 1$ with $|A_n^\omega(y) - \ell| < \varepsilon/2$ for all $n \geq N$. For such n we obtain

$$|A_n^\omega(y_0) - \ell| \leq |A_n^\omega(y_0) - A_n^\omega(y)| + |A_n^\omega(y) - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where the first term is $< \varepsilon/2$ by equicontinuity and the second is $< \varepsilon/2$ by choice of N . As $\varepsilon > 0$ was arbitrary, we get that $A_n^\omega(y_0) \rightarrow \ell$ as $n \rightarrow \infty$. Since y_0 was arbitrary, the convergence holds for every $y_0 \in \mathbb{T}_L$. \square

Fix $y \in \mathbb{R}$ and a compact set $K \subset \mathbb{R}$. Let $\{L_m\}_{m \geq 1}$ be the sequence of lengths given by Proposition 9.2 that can be assumed to be $\text{Leb}(K) < L_m$ for all $m \geq 1$. Let $K_m \subset \mathbb{T}_{L_m}$ be the projection of K . For each $\varepsilon > 0$, choose a continuous $g_{\varepsilon,m} : \mathbb{T}_{L_m} \rightarrow [0, 1]$ such that

$$\mathbf{1}_{K_m} \leq g_{\varepsilon,m} \quad \text{and} \quad \int g_{\varepsilon,m} d\text{Leb}_{L_m} \leq \frac{\text{Leb}(K)}{L_m} + \varepsilon.$$

By the claim, for each m there exists a full \mathbb{P} -measure set $\Omega_{L_m,y}$ such that for every $\omega \in \Omega_{L_m,y}$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_K(y + S_j(\omega)) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} g_{\varepsilon,m}(y + S_j(\omega)) = \int g_{\varepsilon,m} d\text{Leb}_{L_m} \leq \frac{\text{Leb}(K)}{L_m} + \varepsilon.$$

Let $\Omega_y \stackrel{\text{def}}{=} \bigcap_{m \geq 1} \Omega_{L_m,y}$, which still has full \mathbb{P} -measure. Since $L_m \rightarrow \infty$, it follows

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_K(y + S_j(\omega)) = 0 \quad \text{for every } \omega \in \Omega_y.$$

This concludes the proof. \square

9.1.1. Proof of Proposition 9.2

We show that under the assumptions of Theorem 9.1, the map F_L is ergodic (for an appropriate sequence of lengths $L \rightarrow \infty$). Let $f \in L^2(\mu_L)$ be a F_L -invariant function, i.e., $f \circ F_L = f$. Expanding in a Fourier series in the fiber variable,

$$f(\omega, y) = \sum_{k \in \mathbb{Z}} c_k(\omega) e^{2\pi i k y / L}.$$

By the F_L -invariance of f , for each $k \in \mathbb{Z}$, it holds that

$$c_k \circ \tau = e^{-i\lambda\phi} c_k \quad \mathbb{P}\text{-a.e.}, \quad \text{where } \lambda = \frac{2\pi k}{L}. \quad (9.3)$$

For $k = 0$, we have $c_0 \circ \tau = c_0$, hence c_0 is constant almost everywhere by ergodicity of τ . For $k \neq 0$ we must show $c_k \equiv 0$ (provided L is appropriately chosen).

Assume $c_k \not\equiv 0$ for some $k \neq 0$, and set $B = \{\omega : c_k(\omega) \neq 0\}$. From (9.3), $\tau^{-1}(B) \subseteq B$ and, hence $\mathbb{P}(B) \in \{0, 1\}$. If $\mathbb{P}(B) = 0$ we are done; otherwise $\mathbb{P}(B) = 1$. Taking moduli in (9.3) gives $|c_k| \circ \tau = |c_k|$ almost everywhere, so again by ergodicity there is $r > 0$ with $|c_k(\omega)| = r$ for \mathbb{P} -a.e. $\omega \in B$. Define

$$\psi : \Omega \rightarrow \mathbb{S}^1, \quad \psi(\omega) = \frac{c_k(\omega)}{r} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in B \text{ and arbitrarily on the null complementary.}$$

Then dividing (9.3) by r , we get

$$e^{i\lambda\phi} = \frac{\psi \circ \tau}{\psi} \quad \mathbb{P}\text{-a.e.} \quad \text{with } \lambda = \frac{2\pi k}{L}. \quad (9.4)$$

Case (1): $\phi : \Omega \rightarrow \mathbb{R}$ is not a multiplicative coboundary, i.e., (C2) holds. Equation (9.4) yields for $t = -\lambda$, a nonzero solution ψ of the cohomological equation, which contradicts the hypothesis that ϕ satisfies (C2). Thus, $c_k \equiv 0$ for all $k \neq 0$. This holds for any choice of $L > 0$ and thus, in this case, F_L is ergodic with respect to μ_L for all $L > 0$.

Case (2): $(\Omega, \mathbb{P}, \tau)$ is a subshift of finite type or a hyperbolic basic set preserving a Hölder Gibbs measure. Moreover, $\phi : \Omega \rightarrow \mathbb{R}$ is Hölder and is not an additive coboundary. Since ϕ is a Hölder continuous function, then $e^{i\lambda\phi}$ is also Hölder. According to [PP97, Theorems 1 and 2],

this regularity, the base assumptions, and equation (9.4), imply that there exists a Hölder continuous function $\varphi : \Omega \rightarrow \mathbb{S}^1$ such that

$$\psi = \varphi \quad \text{and} \quad e^{i\lambda\phi} = \frac{\varphi \circ \tau}{\varphi} \quad \mathbb{P}\text{-a.e.}$$

By choosing any continuous lift of the circle-valued function, we write $\varphi(\omega) = e^{i\chi(\omega)}$ with $\chi : \Omega \rightarrow \mathbb{R}$ Hölder. Then $e^{i(\lambda\phi(\omega) - (\chi(\tau\omega) - \chi(\omega)))} = 1$, so there exists a function $n : \Omega \rightarrow \mathbb{Z}$ with

$$\lambda\phi(\omega) = \chi(\tau\omega) - \chi(\omega) + 2\pi n(\omega) \quad \text{for all } \omega \in \Omega. \quad (9.5)$$

Since ϕ and χ are continuous, rearranging gives that $n : \Omega \rightarrow \mathbb{Z}$ is also a continuous function. Thus, it is locally constant. The base systems are topologically transitive on each basic (or irreducible) component, hence a continuous \mathbb{Z} -valued function on such a component must be constant. Therefore there exists an integer $p \in \mathbb{Z}$ with $n(\omega) \equiv p$ on the transitive component supporting \mathbb{P} . Thus (9.5) simplifies to

$$\lambda\phi(\omega) = \chi(\tau\omega) - \chi(\omega) + 2\pi p. \quad (9.6)$$

Integrating (9.6) against the τ -invariant probability \mathbb{P} yields $\lambda\mu = 2\pi p$ where $\mu \stackrel{\text{def}}{=} \mathbb{E}[\phi]$.

If $\mu = 0$, then $0 = 2\pi p$, hence $p = 0$, and (9.6) becomes $\lambda\phi = \chi \circ \tau - \chi$. This says that ϕ is an additive coboundary, contradicting the hypothesis in this case (2). Therefore no nonzero Fourier coefficient c_k can exist and F_L is ergodic for every $L > 0$ in this subcase.

If $\mu \neq 0$, recall $\lambda = 2\pi k/L$ with $k \in \mathbb{Z} \setminus \{0\}$, the integrated identity becomes

$$L = \frac{k}{p}\mu \quad (\text{with } p \in \mathbb{Z}, k \in \mathbb{Z} \setminus \{0\}). \quad (9.7)$$

Equation (9.7) shows that any L which allows a nontrivial solution must lie in the countable set $\mathcal{E} \stackrel{\text{def}}{=} \{\frac{k}{p}\mu : k \in \mathbb{Z} \setminus \{0\}, p \in \mathbb{Z}\}$. Hence, for every $L \notin \mathcal{E}$ no nonzero k can produce a measurable (hence Hölder) solution, so all $c_k \equiv 0$ for $k \neq 0$ and F_L is ergodic.

Case (3): $(\Omega, \mathbb{P}, \tau)$ is a Bernoulli shift and $\phi : \Omega \rightarrow \mathbb{R}$ is a one-step function. The following lemma shows that it is enough to treat the skew-product over the one-sided Bernoulli shift.

Lemma 9.3. *The ergodicity of the one-sided skew product is equivalent to that of its two-sided natural extension.*

Proof. Denote by F_L^+ and F_L the skew-product over the one-sided and two-sided Bernoulli shifts $(\Omega_+, \mathbb{P}_+, \tau)$ and $(\Omega, \mathbb{P}, \tau)$ respectively. The ergodicity of the factor F_L^+ from the extension F_L is a well-known general fact. Conversely, assume F_L^+ is ergodic (with respect to $\mu_L^+ \stackrel{\text{def}}{=} \mathbb{P}_+ \times \text{Leb}_L$) and let $f \in L^2(\mu_L)$ be F_L -invariant. As before, we can expand f in the circle variable as follows:

$$f(\omega, \theta) = \sum_{k \in \mathbb{Z}} C_k(\omega) e^{2\pi i k \theta / L}, \quad C_k \in L^2(\mathbb{P}).$$

Invariance yields for each k the multiplicative cocycle

$$C_k(\tau\omega) = C_k(\omega) e^{-i\lambda\phi(\omega)}, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega \quad \text{where } \lambda = \frac{2\pi k}{L}. \quad (9.8)$$

Since C_0 is τ -invariant, the it is constant \mathbb{P} -a.e. from the ergodicity of \mathbb{P} . So, it suffices to show $C_k \equiv 0$ for every $k \neq 0$. Let \mathcal{F}_+ denote the future σ -algebra (generated by coordinates ω_j , $j \geq 0$) and set $C_k^+ = \mathbb{E}[C_k | \mathcal{F}_+]$. Since ϕ is \mathcal{F}_+ -measurable, taking conditional expectation in (9.8) yields

$$C_k^+(\tau\omega) = C_k^+(\omega) e^{-i\lambda\phi(\omega_0)} \quad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Identifying C_k^+ with the corresponding one-sided function, the Fourier series built from the C_k^+ is invariant under F_L^+ ; ergodicity of F_L^+ therefore implies $C_k^+ \equiv 0$ for all $k \neq 0$.

To lift $C_k^+ \equiv 0$ to $C_k \equiv 0$ fix $k \neq 0$. Since $C_k^+ = 0$ we have $\mathbb{E}[C_k H] = \mathbb{E}[C_k^+ H] = 0$ for every bounded \mathcal{F}_+ -measurable H . Let $R = A_- \times A_+$ be any finite cylinder rectangle (past-cylinder A_- , future-cylinder A_+). Choose $n \geq 1$ large so that $\tau^{-n}(R)$ depends only on nonnegative coordinates. Using (9.8) iterated n times and invariance of \mathbb{P} ,

$$\mathbb{E}[C_k \mathbf{1}_R] = \mathbb{E}[C_k \circ \tau^n \mathbf{1}_R \circ \tau^n] = \mathbb{E}[C_k e^{-i\lambda S_n} \mathbf{1}_{\tau^{-n}(R)}]$$

where $S_n(\omega) = \sum_{j=0}^{n-1} \phi(\omega_j)$. The factor $H = e^{-i\lambda S_n \phi} \mathbf{1}_{\tau^{-n}(R)}$ is \mathcal{F}_+ -measurable, hence the last integral vanishes. Since indicators of such rectangles span a dense subspace of $L^2(\mathbb{P})$, C_k is orthogonal to a dense set and thus $C_k \equiv 0$. Thus, all nonzero Fourier coefficients vanish, and therefore F_L is ergodic (with respect to $\mu_L \stackrel{\text{def}}{=} \mathbb{P} \times \text{Leb}$), completing the proof. \square

The following essential lemma, which connects the existence of a multiplicative coboundary to the characteristic function of the random step.

Lemma 9.4. *Let $(\Omega, \mathbb{P}, \tau)$ be a one-sided Bernoulli shift and consider one-step function $\phi : \Omega \rightarrow \mathbb{R}$. If there exist $\lambda \in \mathbb{R}$ and a measurable function $\psi \in L^2(\mathbb{P})$, not almost everywhere zero, satisfying the multiplicative cohomological equation*

$$\psi(\tau(\omega)) = e^{i\lambda\phi(\omega)} \psi(\omega) \quad \text{for } \mathbb{P}\text{-a.e. } \omega,$$

then

$$\Phi_\phi(\lambda) \stackrel{\text{def}}{=} \mathbb{E}[e^{i\lambda\phi}] = 1.$$

Proof. Let $\mathcal{H} = L^2(\mathbb{P})$. We write U_τ for the Koopman operator $U_\tau g = g \circ \tau$ and M_f for the multiplication $M_f g = fg$ by a complex-valued function f . The cohomological equation is equivalent to $T\psi = \psi$, where $T \stackrel{\text{def}}{=} M_{e^{-i\lambda\phi}} U_\tau$. Thus, 1 is an eigenvalue of T corresponding to the eigenfunction $\psi \neq 0$.

The adjoint operator is $T^* = U_\tau^* M_{e^{i\lambda\phi}} = P M_{e^{i\lambda\phi}}$, where $P = U_\tau^*$ is the transfer operator which coincides with the conditional expectation to the tail σ -algebra $\mathcal{F}_{\geq 1} = \sigma(\omega_1, \omega_2, \dots)$. Its action on a function $g \in \mathcal{H}$ is

$$(T^* g)(\omega) = \int e^{i\lambda\phi(a)} g(a, \tau(\omega)) dp(a) \quad \text{where } \mathbb{P} = p^{\mathbb{N}}.$$

That is, T^* is a complex Ruelle operator. The range of T^* is contained in the subspace $\mathcal{H}_{\text{tail}} = L^2(\Omega, \mathcal{F}_{\geq 1}, \mathbb{P})$ of functions that depend only on the tail coordinates $(\omega_1, \omega_2, \dots)$. If

$\mu \neq 0$ is an eigenvalue of T^* with eigenfunction h , then $h = \mu^{-1}T^*h$ must belong to $\mathcal{H}_{\text{tail}}$. For such a $\mathcal{F}_{\geq 1}$ -measurable function h , the action of T^* simplifies to multiplication:

$$(T^*h)(\omega) = \int e^{i\lambda\phi(a)}h(\omega) dp(a) = \eta \cdot h(\omega) \quad \text{where } \eta = \Phi_\phi(\lambda). \quad (9.9)$$

Thus, any nonzero eigenvalue of T^* is equal to η .

Let $u \stackrel{\text{def}}{=} \mathbb{E}[\psi | \mathcal{F}_{\geq 1}]$ be the conditional expectation of ψ with respect to the tail σ -algebra $\mathcal{F}_{\geq 1}$. From a geometric perspective, u is the orthogonal projection of ψ onto $\mathcal{H}_{\text{tail}}$, i.e., $\psi = u + (\psi - u)$ with $u \in \mathcal{H}_{\text{tail}}$ and $\langle \psi - u, h \rangle = 0$ for all $h \in \mathcal{H}_{\text{tail}}$. If $u \equiv 0$, then ψ would be orthogonal to $\mathcal{H}_{\text{tail}}$. Since $\text{Range}(T^*) \subseteq \mathcal{H}_{\text{tail}}$, this would imply $\langle \psi, T^*g \rangle = 0$ for all $g \in \mathcal{H}$. By duality, this means $\langle T\psi, g \rangle = 0$ for all g , so $T\psi = 0$. This contradicts $T\psi = \psi \neq 0$. Therefore, u is not almost everywhere zero.

Since $u \in \mathcal{H}_{\text{tail}}$, according to (9.9), $T^*u = \eta \cdot u$. Using the duality, $\langle T\psi, u \rangle = \langle \psi, T^*u \rangle$. Substituting $T\psi = \psi$ and $T^*u = \eta \cdot u$, we get

$$\langle \psi, u \rangle = \langle \psi, \eta \cdot u \rangle = \eta \cdot \langle \psi, u \rangle.$$

Hence, using that u is the orthogonal projection on $\mathcal{H}_{\text{tail}}$,

$$\langle \psi, u \rangle = \langle u + (\psi - u), u \rangle = \langle u, u \rangle + \langle \psi - u, u \rangle = \langle u, u \rangle = \|u\|^2.$$

Since u is nonzero almost everywhere, $\|u\|^2 > 0$. We can therefore divide by the nonzero quantity $\langle \psi, u \rangle$ to obtain $\eta = 1$ and conclude the proof. \square

Lemma 9.5. *Let X be a real random variable with characteristic function $\Phi_X(t) = \mathbb{E}[e^{itX}]$. If there exists $t_0 \neq 0$ with $|\Phi_X(t_0)| = 1$, then there are $a \in \mathbb{R}$ and $\delta > 0$ such that*

$$X(\omega) \in a + \delta\mathbb{Z} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Proof. Put $Z \stackrel{\text{def}}{=} e^{it_0 X}$. Then $|Z| = 1$ and by hypothesis $|\mathbb{E}[Z]| = 1$. Since $|\mathbb{E}[Z]| \leq \mathbb{E}[|Z|] = 1$, equality holds in the triangle inequality. The equality in the triangular inequality holds if and only if all realizations of Z lie on the same ray in the complex plane, that is, when Z has constant argument almost surely. Hence, there exists $\theta \in \mathbb{R}$ such that

$$e^{it_0 X(\omega)} = e^{i\theta} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Therefore there exists an integer-valued measurable function $K(\omega)$ with

$$t_0 X(\omega) = \theta + 2\pi K(\omega) \quad \mathbb{P}\text{-a.e. } \omega \in \Omega$$

Setting $a \stackrel{\text{def}}{=} \theta/t_0$ and $\delta \stackrel{\text{def}}{=} 2\pi/t_0$ (or $\delta \stackrel{\text{def}}{=} 2\pi/|t_0|$ to make it positive) we obtain $X(\omega) = a + \delta K(\omega) \in a + \delta\mathbb{Z}$ for \mathbb{P} -a.e. $\omega \in \Omega$ as required. \square

Lemma 9.6. *Let Z be an integer random variable with law $\mathbb{P}(Z = n) = p_n$ and let*

$$\Phi_Z(t) \stackrel{\text{def}}{=} \mathbb{E}[e^{itZ}] = \sum_{n \in \mathbb{Z}} p_n e^{itn}, \quad t \in \mathbb{R}.$$

If the law of Z is non-degenerate (i.e. not a Dirac mass at a single integer) then

$$|\Phi_Z(t)| < 1 \quad \text{for every } t \notin 2\pi\mathbb{Z}.$$

Proof. By the triangle inequality we always have $|\Phi_Z(t)| \leq \sum_n p_n |e^{itn}| = 1$. If $|\Phi_Z(t)| = 1$ then equality holds in the triangle inequality for the convex combination $\sum_n p_n e^{itn}$. For complex numbers of modulus one, equality in the triangle inequality for a convex combination occurs if all the summands are equal (or the sum degenerate to just one term). Otherwise, the convex combination lies strictly inside the convex hull of the unit circle. Hence e^{itn} must be equal (the same complex number) for every n in the support of Z . If the support contains at least two distinct integers $m \neq n$ then $e^{it(m-n)} = 1$, so $t(m-n) \in 2\pi\mathbb{Z}$, and therefore $t \in 2\pi\mathbb{Z}$ (since $m-n \in \mathbb{Z} \setminus \{0\}$). This proves the lemma. \square

Now we are ready to complete the proof of Proposition 9.2. Recall that from (9.4), there exist $\lambda \neq 0$ and $\psi : \Omega \rightarrow \mathbb{S}^1$ in $L^2(\mathbb{P})$ such that $\psi e^{i\lambda\phi} = \psi \circ \tau$ holds \mathbb{P} -a.e. Then, by Lemma 9.4, we conclude

$$\Phi_\phi(-\lambda) = 1, \quad \text{where } \Phi_\phi(t) \stackrel{\text{def}}{=} \mathbb{E}[e^{it\phi}].$$

Now, from Lemma 9.5 and since ϕ is one-step, there exist real numbers $a \in \mathbb{R}$ and $\delta > 0$ and an integer-valued one-step random variable Z such that

$$\phi(\omega) = a + \delta \cdot Z(\omega_0) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Using this lattice decomposition, we compute

$$\Phi_\phi(t) = e^{ita} \mathbb{E}[e^{it\delta Z}] = e^{ita} \Phi_Z(t\delta),$$

so $|\Phi_\phi(-\lambda)| = |\Phi_Z(-\lambda\delta)|$.

If the integer law of Z is non-degenerate, Lemma 9.6, $|\Phi_Z(t)| < 1$ for every $t \notin 2\pi\mathbb{Z}$, and equality $\Phi_Z(t) = 1$ can only occur when $t \in 2\pi\mathbb{Z}$. Consequently, for $\lambda = 2\pi k/L$ (with $k \neq 0$) the necessary condition $|\Phi_\phi(-\lambda)| = 1$ becomes $-\lambda\delta \in 2\pi\mathbb{Z}$ or equivalently, $k\delta/L \in \mathbb{Z} \setminus \{0\}$. Hence, for any choice of $L > 0$ with $\delta/L \notin \mathbb{Q}$, the only integer solution of $k\delta/L \in \mathbb{Z}$ with $k \in \mathbb{Z}$ is $k = 0$, arriving at a contradiction. Therefore, we must have $c_k \equiv 0$ for all $k \neq 0$. This shows the ergodicity of F_L whenever Z is non-degenerate for any L outside of a countable set of lengths.

If the integer law of Z is degenerate, say $Z = n_0$ almost surely, then the cocycle ϕ is almost everywhere constant. Then F_L is a direct product of shift map τ and the rigid rotation $R_\phi(y) = x + \phi \pmod L$. Again, as ϕ is non-zero, when L is irrational, R_ϕ is ergodic with respect to $\text{Leb}_{\mathbb{T}_L}$ and since τ is also \mathbb{P} -ergodic, we conclude that F_L is ergodic.

9.1.2. Consequences

Proposition 9.7. *Let $\{S_n\}_{n \geq 0}$ be an \mathbb{R} -valued stochastic process with fixed initial value $S_0 = t$. Assume that the increments $Y_n \stackrel{\text{def}}{=} S_n - S_{n-1}$ for $n \geq 1$ are i.i.d. with a common law $\mu \neq \delta_0$ (i.e., it is not frozen). Then for every compact set $K \subset \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_K(S_j) = 0 \quad \text{almost surely.}$$

Proof. Because S_n is measurable with respect to $\sigma(Y_1, \dots, Y_n)$ while Y_{n+1} is independent of $\sigma(Y_1, \dots, Y_n)$, it follows that Y_{n+1} is independent of S_n for each n . Hence the conditional distribution of the next increment given the present state coincides (almost surely) with the marginal law μ ; in particular, the law of the increment does not depend on the current value of the process. We now construct the canonical Bernoulli model that realizes the same marginal law. To do this, let \mathcal{A} be the topological support of μ endowed with its Borel σ -algebra, and consider the one-sided Bernoulli shift $(\Omega, \mathbb{P}, \tau)$ with $\Omega = \mathcal{A}^{\mathbb{N}}$ and $\mathbb{P} = \mu^{\mathbb{N}}$. Let $\phi : \Omega \rightarrow \mathbb{R}$ be the coordinate projection $\phi(\omega) = \omega_0$ (so $\phi_* \mathbb{P} = \mu$). Consider the skew-translation $T_\phi(\omega, x) = (\tau(\omega), x + \phi(\omega))$ and the corresponding process given by the fiber iteration

$$S'_n(\omega) = t + \sum_{j=0}^{n-1} \phi(\tau^j(\omega)), \quad n \geq 0.$$

Write $Y'_n \stackrel{\text{def}}{=} S'_n - S'_{n-1} = \phi(\tau^{n-1}(\omega))$ and note that, by construction, $\{Y'_n\}_{n \geq 1}$ is an i.i.d. sequence with common law μ .

On the other hand, the finite-dimensional distributions of the original process $\{S_n\}_{n \geq 0}$ are determined by the finite-dimensional distributions of its increments (Y_1, \dots, Y_N) for each N . Since both (Y_1, \dots, Y_N) and (Y'_1, \dots, Y'_N) have the product law $\mu^{\otimes N}$, the finite-dimensional distributions of (S_0, \dots, S_N) and (S'_0, \dots, S'_N) coincide for every N . The collection of these finite-dimensional laws is consistent and therefore determines a unique probability measure on the path space $\mathbb{R}^{\mathbb{N}}$; hence the path-space law of $\{S_n\}_{n \geq 0}$ equals that of $\{S'_n\}_{n \geq 0}$.

Since $\mu \neq \delta_0$, ϕ is a non-zero one-step map and thus Theorem 9.1 (alternative (3)) applies to the skew-translation T_ϕ . Hence, for every compact $K \subset \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_K(S'_j) = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Since the path-space measures of $\{S_n\}_{n \geq 0}$ and $\{S'_n\}_{n \geq 0}$ coincide, the measurable set of full measure on which the latter convergence holds is also a set of full measure for the original process. Therefore, the same almost-sure vanishing of the occupation time holds for $\{S_n\}_{n \geq 0}$, as required. \square

Corollary 9.8. *Let F be a one-step skew product as in (1.3). Assume that $\{X_n^x\}_{n \geq 0}$ is conjugate to a G -valued non-frozen random walk where $X_n^x(\omega) = f_\omega^n(x)$ and G is either \mathbb{Z} or \mathbb{R} . Then for every compact set $K \subset (0, 1)$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_K(f_\omega^j(x)) = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Proof. Let $\mathcal{O}(x)$ be the set $\{X_n^x(\omega) : \omega \in \Omega, n \geq 0\}$. By hypothesis, there is a strictly monotonic injection $h : \mathcal{O}(x) \rightarrow G$ such that the step increments $Y_n^t \stackrel{\text{def}}{=} S_n^t - S_{n-1}^t \in G$, $n \geq 1$, are i.i.d. non-degenerate random variables (i.e., with law $\mu \neq \delta_0$) where $S_n^t(\omega) \stackrel{\text{def}}{=} (h \circ f_\omega \circ h^{-1})(t)$ and $t = h(x)$. Thus, the random walk $\{S_n^t\}_{n \geq 0}$ satisfies the assumption of Proposition 9.7. Now,

fix a compact set K of $(0, 1)$ and let $K' = h(K)$. By Proposition 9.7,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{K'}(S_j^t(\omega)) = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

From here, as in Proposition 4.7, since $\mathbb{1}_{K'}(S_j^t(\omega)) = \mathbb{1}_K(f_\omega^j(x))$, we follow the vanishing occupational time for the sequence of iterated $f_\omega^j(x)$. \square

9.1.3. Proof of Corollary III

By Proposition I, F satisfy the arcsine law (1.7); in particular, the fluctuation parameters γ_0 and γ_1 can be chosen arbitrarily close to 0 and 1. By Corollary 9.8, the occupation time vanishes in the interior of I , i.e. (1.12) holds for every fixed x . Hence, the two hypotheses (i) and (ii) of Proposition II are satisfied. From that proposition and by Remark 1.4 follows that $\mathcal{L}(\omega, x) = \{\lambda\delta_0 + (1 - \lambda)\delta_1 : \lambda \in [0, 1]\}$ for every $x \in (0, 1)$, and \mathbb{P} -a.e. $\omega \in \Omega$ a required.

9.2. Ergodicity. Guivarc'h's [Gui89, Corollaire 3] treats essentially the same family of skew-extentions T_ϕ but under strong regularity assumptions and statistical hypotheses on the base map τ and the function ϕ . Under these hypotheses, Guivarc'h proves that if ϕ is *strictly aperiodic* (i.e., if for every constant $c \in \mathbb{R}$, $\phi - c$ is not a multiplicative coboundary) and $\mathbb{E}[\phi] = 0$, then the skew-translation T_ϕ is ergodic with respect to $\mu = \mathbb{P} \times \text{Leb}$. The following result characterizes the ergodicity of T_ϕ under a weaker cohomological condition for any probability-preserving ergodic base. See [Aar97, Corollary 8.2.5] for another different characterization of the ergodicity in terms of the essential values of ϕ .

Theorem 9.9. *The skew-translation T_ϕ given in (9.1) is ergodic with respect to μ if and only if $\mathbb{E}[\phi] = 0$ and ϕ is not a multiplicative coboundary, that is, condition (C2) holds.*

We divide the proof into several propositions. We first prove the following necessary conditions to ergodicity:

Proposition 9.10. *If T_ϕ is ergodic, then ϕ satisfies condition (C2).*

Proof. We argue by contraposition. Suppose that ϕ is a multiplicative coboundary. Then there exist $\lambda_0 \in \mathbb{R} \setminus \{0\}$ and a measurable function $\psi : \Omega \rightarrow \mathbb{S}^1$ satisfying

$$e^{i\lambda_0\phi(\omega)} = \frac{\psi(\tau(\omega))}{\psi(\omega)} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Define $g(\omega, y) \stackrel{\text{def}}{=} \psi(\omega)e^{-i\lambda_0 y} \in L^\infty(\mu)$. Then g is non-constant in the y -variable (since $\lambda_0 \neq 0$), and a direct computation using the coboundary identity yields

$$g(T_\phi(\omega, y)) = \psi(\tau(\omega))e^{-i\lambda_0(y+\phi(\omega))} = \psi(\omega)e^{-i\lambda_0 y} = g(\omega, y)$$

for μ -a.e. $(\omega, y) \in \Omega \times \mathbb{R}$. Hence, g is a non-constant T_ϕ -invariant function, contradicting the ergodicity of μ for T_ϕ . Therefore, ϕ cannot be a multiplicative coboundary. \square

Proposition 9.11. *If T_ϕ is ergodic, then $\mathbb{E}[\phi] = 0$.*

Remark 9.12. According to [Aar97, Corollary 8.1.5], the condition $\mathbb{E}[\phi] = 0$ is equivalent to T_ϕ being *conservative*; that is, for any measurable set A with $\mu(A) > 0$, there exists some integer $n \geq 1$ such that $\mu(A \cap T_\phi^{-n}(A)) > 0$. Thus, the previous proposition reads as follows:

$$\text{if } T_\phi \text{ is ergodic, then it is conservative.}$$

When the base map τ is invertible, this follows from [Aar97, Proposition 1.2.1]. In the case where τ is not invertible, this implication is new.

Proof of Proposition 9.11. We argue by contradiction. Suppose $\mathbb{E}[\phi] > 0$. The case $\mathbb{E}[\phi] < 0$ is analogous. By Birkhoff's ergodic theorem for $(\Omega, \mathbb{P}, \tau)$ gives $S_n(\omega)/n \rightarrow \mathbb{E}[\phi]$ for \mathbb{P} -a.e. $\omega \in \Omega$, where $S_n = \sum_{j=0}^{n-1} \phi \circ \tau^j$ for $n > 0$ and $S_0 = 0$. Hence $S_n \rightarrow \infty$ for \mathbb{P} -almost surely.

Fix a nonnegative, compactly supported, nonzero function $h \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $M > 0$ be such that $\text{supp } h \subset [-M, M]$, and set $I(h) = \int h d\text{Leb} > 0$. For $\varepsilon > 0$ define

$$f_\varepsilon(\omega, y) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} e^{-\varepsilon n} h(y - S_n(\omega)). \quad (9.10)$$

Since h has compact support and $S_n(\omega) \rightarrow \infty$ for \mathbb{P} -a.e. $\omega \in \Omega$, the sum in (9.10) is finite for μ -a.e. $(\omega, y) \in \Omega \times \mathbb{R}$ and f_ε is measurable. Define

$$g_\varepsilon \stackrel{\text{def}}{=} (e^\varepsilon - 1) f_\varepsilon.$$

Note that for every (ω, y) ,

$$0 \leq g_\varepsilon(\omega, y) \leq (e^\varepsilon - 1) \sum_{n \geq 0} e^{-\varepsilon n} \|h\|_\infty = e^\varepsilon \|h\|_\infty,$$

so $\|g_\varepsilon\|_\infty \leq 2\|h\|_\infty$ uniformly in $\varepsilon > 0$ small enough. Since $S_n(\tau(\omega)) = S_{n+1}(\omega) - \phi(\omega)$, we have

$$f_\varepsilon \circ T_\phi(\omega, y) = \sum_{n \geq 0} e^{-\varepsilon n} h(y + \phi(\omega) - S_n(\tau(\omega))) = \sum_{k \geq 1} e^{-\varepsilon(k-1)} h(y - S_k(\omega)) = e^\varepsilon (f_\varepsilon(\omega, y) - h(y)).$$

Consequently,

$$g_\varepsilon \circ T_\phi = (e^\varepsilon - 1)e^\varepsilon (f_\varepsilon - h) = e^\varepsilon g_\varepsilon - (e^\varepsilon - 1)e^\varepsilon h.$$

Thus,

$$\|g_\varepsilon \circ T_\phi - g_\varepsilon\|_{L^1(\mu)} = \|(e^\varepsilon - 1)((e^\varepsilon - 1)f_\varepsilon - e^\varepsilon h)\|_{L^1(\mu)} \leq (e^\varepsilon - 1)^2 \|f_\varepsilon\|_{L^1(\mu)} + (e^\varepsilon - 1)e^\varepsilon \|h\|_{L^1(\mu)}.$$

Moreover, since

$$\|f_\varepsilon\|_{L^1(\mu)} = \sum_{n \geq 0} e^{-\varepsilon n} \int h(y - S_n(\omega)) dy d\mathbb{P}(\omega) = \sum_{n \geq 0} e^{-\varepsilon n} \int h(y) dy = \frac{I(h)}{1 - e^{-\varepsilon}},$$

it follows that

$$\|g_\varepsilon \circ T_\phi - g_\varepsilon\|_{L^1(\mu)} \leq (e^\varepsilon - 1)^2 \frac{I(h)}{1 - e^{-\varepsilon}} + (e^\varepsilon - 1)e^\varepsilon I(h).$$

Consequently, we get

$$\lim_{\varepsilon \rightarrow 0^+} \|g_\varepsilon \circ T_\phi - g_\varepsilon\|_{L^1(\mu)} = 0. \quad (9.11)$$

On the other hand, since the family $\{g_\varepsilon\}_{\varepsilon > 0}$ is uniformly bounded in $L^\infty(\mu)$, by the Banach-Alaoglu theorem, there a weak* limit point in $L^\infty(\mu)$. Choose a sequence $\varepsilon_k \rightarrow 0^+$ such

that $g_k = g_{\varepsilon_k}$ converges weak* to some $g \in L^\infty(\mu)$. We will show that g is T_ϕ -invariant. To do this, fix $f \in L^1(\mu) \cap L^\infty(\mu)$. Then

$$\left| \int (g_k \circ T_\phi) f \, d\mu - \int g_k f \, d\mu \right| \leq \|f\|_{L^\infty} \cdot \|g_k \circ T_\phi - g_k\|_{L^1},$$

and by (9.11), the right-hand side tends to 0 as $k \rightarrow \infty$. By weak* convergence of g_k to g in $L^\infty(\mu)$, we also have $\lim_{k \rightarrow \infty} \int g_k f \, d\mu = \int g f \, d\mu$. Therefore

$$\lim_{k \rightarrow \infty} \int (g_k \circ T_\phi) f \, d\mu = \int g f \, d\mu.$$

Moreover, for each k ,

$$\int (g_k \circ T_\phi) f \, d\mu = \int g_k (Pf) \, d\mu,$$

where $P : L^1(\mu) \rightarrow L^1(\mu)$ is the pre-dual operator (Perron–Frobenius) of the Koopman operator $U : G \mapsto G \circ T_\phi$. Since $Pf \in L^1(\mu)$ and g_k converges to g in the weak* topology, $\lim_{k \rightarrow \infty} \int g_k (Pf) \, d\mu = \int g (Pf) \, d\mu = \int (g \circ T_\phi) f \, d\mu$. Comparing these limits gives

$$\int (g \circ T_\phi) f \, d\mu = \int g f \, d\mu \quad \text{for every } f \in L^1(\mu) \cap L^\infty(\mu).$$

By density this identity holds for all $f \in L^1(\mu)$, so g is T_ϕ -invariant.

Let us show the nontriviality and integrability of g . As shown earlier, $0 \leq g_k \leq 2\|h\|_\infty$ pointwise. Moreover,

$$\int g_k \, d\mu = (e^{\varepsilon_k} - 1) \sum_{n \geq 0} e^{-\varepsilon_k n} \int h(y) \, dy = e^{\varepsilon_k} I(h).$$

Thus, the family $\{g_k\}_{k \geq 0}$ is *uniformly bounded* in $L^1(\mu)$ by, say, $2I(h)$. The family also has *uniformly absolutely continuous integrals* in the sense that, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\int_A g_k \, d\mu < \epsilon$ for all $k \geq 0$ provided $\mu(A) < \delta$. To apply the Dunford-Pettis theorem, see [Bog07, Theorem 4.7.20], we also need the following *tightness* condition:

Claim 9.12.1. *For every $\epsilon > 0$, there is $L > 0$ such that $\int_{\Omega \times (\mathbb{R} \setminus [-L, L])} g_k \, d\mu < \epsilon$ for all $k \geq 0$.*

Proof. Recall $g_k = (e^{\varepsilon_k} - 1) \sum_{n \geq 0} e^{-\varepsilon_k n} h(y - S_n(\omega))$ and that $\text{supp } h \subset [-M, M]$. For every ω , $n \geq 0$ and $L > 0$,

$$\int_{|y| > L} h(y - S_n(\omega)) \, dy \leq I(h) \quad \text{and} \quad \int_{|y| > L} h(y - S_n(\omega)) \, dy \leq I(h) \mathbb{1}_{\{|S_n| > L - M\}}.$$

Hence

$$\int_{\Omega \times (|y| > L)} g_k \, d\mu \leq I(h) (e^{\varepsilon_k} - 1) \sum_{n \geq 0} e^{-\varepsilon_k n} \mathbb{P}(|S_n| > L - M). \quad (9.12)$$

Fix $\epsilon > 0$. Since $\varepsilon_k \rightarrow 0^+$, we may (after discarding finitely many indices) assume $\varepsilon_k \in (0, 1]$ for all $k \geq 0$. Split the sum in (9.12) at some $N \in \mathbb{N}$ as follows:

$$\int_{\Omega \times (|y| > L)} g_k \, d\mu \leq I(h) (e^{\varepsilon_k} - 1) \sum_{n=0}^{N-1} e^{-\varepsilon_k n} \mathbb{P}(|S_n| > L - M) + I(h) (e^{\varepsilon_k} - 1) \sum_{n=N}^{\infty} e^{-\varepsilon_k n}.$$

The tail is uniform in k (for $\varepsilon_k \in (0, 1]$),

$$I(h)(e^{\varepsilon_k} - 1) \sum_{n=N}^{\infty} e^{-\varepsilon_k n} \leq I(h)e^{\varepsilon_k(1-N)} \leq I(h)e^{1-N}.$$

Choose N large enough that this tail $< \epsilon/2$. For the other term, for each fixed $n \in \{0, \dots, N-1\}$, we have $\mathbb{P}(|S_n| > L - M) \rightarrow 0$ as $L \rightarrow \infty$. Also

$$(e^{\varepsilon_k} - 1) \sum_{n=0}^{N-1} e^{-\varepsilon_k n} \leq (e-1)N.$$

Hence

$$I(h)(e^{\varepsilon_k} - 1) \sum_{n=0}^{N-1} e^{-\varepsilon_k n} \mathbb{P}(|S_n| > L - M) \leq I(h)(e-1)N \max_{0 \leq n \leq N-1} \mathbb{P}(|S_n| > L - M).$$

Choose L large enough so that the right-hand side is $< \epsilon/2$. Combining both estimates,

$$\int_{\Omega \times (|y| > L)} g_k d\mu < \epsilon \quad \text{for all } k \geq 0.$$

which proves the claim. \square

Therefore $\{g_k\}_{k \geq 0}$ is *uniformly integrable* in the sense required by [Bog07, Theorem 4.7.20 (iv)], i.e., uniformly bounded in L^1 , uniformly absolutely continuous integrals and tightness. Consequently, it is relatively weakly compact in $L^1(\mu)$. Thus, passing to a further subsequence if necessary, we may assume that g_k converges weakly in $L^1(\mu)$ to some $\tilde{g} \in L^1(\mu)$. But weak* convergence in $L^\infty(\mu)$ and weak convergence in $L^1(\mu)$ determine the same limit as an element of the space of measurable functions (they give the same values on all test functions $f \in L^1(\mu)$), so \tilde{g} and g coincide almost everywhere. Hence $g \in L^1(\mu)$ and

$$\int g d\mu = \lim_{k \rightarrow \infty} \int g_k d\mu = \lim_{k \rightarrow \infty} e^{\varepsilon_k} I(h) = I(h) > 0.$$

This proves that the weak* limit g is nontrivial (indeed integrable with strictly positive integral). However, this yields a contradiction due to the ergodicity of T_ϕ . Since $g \in L^1(\mu) \cap L^\infty(\mu)$ is T_ϕ -invariant, it must be constant μ -almost everywhere. Moreover, because $g \in L^1(\mu)$ and $\mu(\Omega \times \mathbb{R}) = \infty$, the only constant function in $L^1(\mu)$ is zero. This contradicts the fact that $\int g d\mu = I(h) > 0$. Therefore, our initial assumption that $\mathbb{E}[\phi] \neq 0$ is false, and the proof is complete. \square

Conversely, we show that (C2) and $\mathbb{E}[\phi] = 0$ are sufficient conditions to guarantee ergodicity of T_ϕ . Below we prove a technical lemma for which we need to introduce the following notation: given a bounded interval I of \mathbb{R} , $U \subset \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $s \in \mathbb{R}$,

$$I_s(U) \stackrel{\text{def}}{=} \{y \in I : \bar{y} = (y + s) \bmod 1 \in U\}.$$

Lemma 9.13. *If $\text{Leb}_\mathbb{T}(U) = 1$, then $\text{Leb}(I_s(U)) = \text{Leb}(I)$ for every bounded interval I and $s \in \mathbb{R}$.*

Proof. Identify \mathbb{T} with the interval $[0, 1)$ and let I be a bounded interval. For $t \in \mathbb{T}$ set

$$n_I(t) \stackrel{\text{def}}{=} \#\{k \in \mathbb{Z} : t + k \in I\} = \sum_{k \in \mathbb{Z}} \mathbf{1}_I(t + k).$$

This function is measurable, and $0 \leq n_I(t) \leq \lceil \text{Leb}(I) \rceil$ for every $t \in \mathbb{T}$. Moreover,

$$\int_{\mathbb{T}} n_I(t) dt = \int_0^1 \sum_{k \in \mathbb{Z}} \mathbf{1}_I(t + k) dt = \sum_{k \in \mathbb{Z}} \int_k^{k+1} \mathbf{1}_I(u) du = \int_{\mathbb{R}} \mathbf{1}_I(u) du = \text{Leb}(I).$$

Write $N = \mathbb{T} \setminus U$. By hypothesis $\text{Leb}_{\mathbb{T}}(N) = 0$. Note that, for any $s \in \mathbb{R}$,

$$\text{Leb}(I_s(U)) = \int_I \mathbf{1}_U((y + s) \bmod 1) dy.$$

Change variables $t = (y + s) \bmod 1$ and use the periodic-counting description to obtain

$$\text{Leb}(I_s(U)) = \int_{\mathbb{T}} \mathbf{1}_U(t) n_I(t - s) dt = \int_{\mathbb{T}} n_I(t - s) dt - \int_N n_I(t - s) dt.$$

The first term equals $\int_{\mathbb{T}} n_I(t) dt = \text{Leb}(I)$ by translation invariance of Lebesgue measure on \mathbb{T} . The second term is an integral of the bounded measurable function $n_I(t - s)$ over the null set N , hence it equals 0. Therefore $\text{Leb}(I_s(U)) = \text{Leb}(I)$, proving the lemma. \square

Proposition 9.14. *If ϕ satisfies (C2) and $\mathbb{E}[\phi] = 0$, then T_{ϕ} is ergodic.*

Proof. Let $A \subset \Omega \times \mathbb{R}$ be a T_{ϕ} -invariant measurable set and let $B = \pi(A) \subset \Omega \times \mathbb{T}$ be its projection modulo 1. Denote by F the quotient skew-translation on $\Omega \times \mathbb{T}$ and write $\mu_{\mathbb{T}} = \mathbb{P} \times \text{Leb}_{\mathbb{T}}$. According to Proposition 9.2, under assumption (C2), F is ergodic with respect to $\mu_{\mathbb{T}}$ (see the proof of case (1), which holds for all $L > 0$). Note that since B is F -invariant, ergodicity implies that $\mu_{\mathbb{T}}(B) \in \{0, 1\}$.

Claim 9.14.1. *If $\mu_{\mathbb{T}}(B) = 0$, then $\mu(A) = 0$.*

Proof. By applying Fubini's theorem, we can write the measure of A as

$$\mu(A) = \int_B n_A d\mu_{\mathbb{T}}, \quad \text{where } n_A(\omega, t) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} \mathbf{1}_A(\omega, t + k).$$

The T_{ϕ} -invariance of A implies that n_A is F -invariant and, hence, n_A must be a constant almost everywhere. Thus, as $\mu_{\mathbb{T}}(B) = 0$ by hypothesis, it follows that $\mu(A) = 0$. \square

Claim 9.14.2. *If $\mu_{\mathbb{T}}(B) = 1$, then $\mu((\Omega \times \mathbb{R}) \setminus A) = 0$.*

Proof. Since $\mathbb{E}[\phi] = 0$, according to [Aar97, Corollary 8.1.5 and Proposition 8.1.2], there is a subset $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that $\liminf_{n \rightarrow \infty} |S_n(\omega)| = 0$ for every $\omega \in \Omega_0$ where we recall that $S_n = \sum_{j=0}^{n-1} \phi \circ \tau^j$. On the other hand, since B has full probability in $\Omega \times \mathbb{T}$, by Fubini, there is a full-measure set $\Omega_1 \subset \Omega$ such that for every $\omega \in \Omega_1$ the fiber $B_{\omega} \subset \mathbb{T}$ satisfies $\text{Leb}_{\mathbb{T}}(B_{\omega}) = 1$.

Fix $\omega \in \Omega_0 \cap \Omega_1 \stackrel{\text{def}}{=} \bar{\Omega}$ and let $I \subset \mathbb{R}$ be an arbitrary bounded interval. For each integer n , the invariance of A implies $A_{\tau^n(\omega)} \supset A_{\omega} + S_n(\omega)$, hence

$$A_{\omega} \cap (I - S_n(\omega)) \supset I_{-S_n(\omega)}(B_{\tau^n(\omega)}).$$

Since $\text{Leb}_{\mathbb{T}}(B_{\tau^n(\omega)}) = 1$, by Lemma 9.13, the right-hand side has Lebesgue measure $\text{Leb}(I) = \text{Leb}(I - S_n(\omega))$, and therefore

$$\text{Leb}((I - S_n(\omega)) \setminus A_\omega) = 0.$$

Consequently

$$\text{Leb}(I \setminus A_\omega) \leq \text{Leb}(I \setminus (I - S_n(\omega))) + \text{Leb}((I - S_n(\omega)) \setminus A_\omega) \leq |S_n(\omega)| + 0,$$

where the last inequality follows since the translation of an interval by t changes it by at most $|t|$ in Lebesgue measure. Since $\omega \in \bar{\Omega} \subset \Omega_0$, taking \liminf yields $\text{Leb}(I \setminus A_\omega) = 0$. Because I was an arbitrary bounded interval, this implies $\text{Leb}(\mathbb{R} \setminus A_\omega) = 0$. The conclusion holds for every ω in the full \mathbb{P} -measure subset $\bar{\Omega}$, so A is conull in $\Omega \times \mathbb{R}$. \square

Both claims above prove that A is null or conull in $\Omega \times \mathbb{R}$, showing the ergodicity of T_ϕ . \square

Proof of Theorem 9.9. Propositions 9.10, 9.11 and 9.14 conclude the main Theorem 9.9. \square

9.3. Fluctuation Law.

Theorem 9.15. *Let T_ϕ be a skew product as in (9.1) where (Ω, τ) is a (one-sided or two-sided) subshift of finite type or a hyperbolic basic set of a C^1 diffeomorphism, \mathbb{P} is a τ -invariant Hölder Gibbs measure and $\phi: \Omega \rightarrow \mathbb{R}$ is a Hölder continuous function with $\mathbb{E}[\phi] = 0$ and satisfying (C1). Then, for every $y \in \mathbb{R}$,*

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_{\bar{J}_i(y)}(y + S_j(\omega)) \leq \alpha\right) < 1 \quad \text{for every } \alpha \in (0, 1) \text{ and } i = 0, 1,$$

where $\bar{J}_0(y) = (-\infty, y)$, $\bar{J}_1(y) = (y, \infty)$ and, $S_0 = 0$ and $S_j = \phi + \phi \circ \tau + \cdots + \phi \circ \tau^{j-1}$, $j \geq 1$.

Proof. We first note that it is sufficient to prove the theorem for the invertible case, as the result for a one-sided subshift $(\Omega_+, \tau_+, \mathbb{P}_+)$ follows from its natural extension. Let $(\Omega, \tau, \mathbb{P})$ be this extension, where the measure satisfies $\mathbb{P}_+ = \pi_* \mathbb{P}$ for the canonical projection $\pi: \Omega \rightarrow \Omega_+$. It is a known result that if \mathbb{P}_+ is a Hölder Gibbs measure, then so is its lift \mathbb{P} . By defining the function on the invertible space as $\phi \circ \pi$, the Birkhoff sums are preserved since $S_j(\omega) = S_j(\pi(\omega))$ for any $\omega \in \Omega$. This implies that the probability of the set defined by the inequality in the theorem is identical in both systems for all n . Thus, we may assume henceforth that the base dynamics are invertible.

To prove the statement, it suffices to show it for $i = 1$; the argument for $i = 0$ is analogous. Let us interpolate $\{S_n\}_{n \geq 0}$ by

$$S_t = (\lfloor t \rfloor + 1 - t)S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)S_{\lfloor t \rfloor + 1}, \quad t \geq 0.$$

Since $S_t = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor) \cdot \phi \circ \tau^{\lfloor t \rfloor}$ and ϕ is bounded and τ invertible, by Theorem 3.8, we have $\sigma > 0$, $0 < \beta < 1/2$ and a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ joining $(\Omega, \mathcal{F}, \mathbb{P})$ with a standard Brownian motion $\{B_t: t \geq 0\}$ such that

$$0 \leq \frac{|S_t - \sigma B_t|}{t^{1/2-\beta}} \leq \frac{|S_{\lfloor t \rfloor} - \sigma B_{\lfloor t \rfloor}|}{t^{1/2-\beta}} + \frac{|\phi \circ \tau^{\lfloor t \rfloor}|}{t^{1/2-\beta}} \xrightarrow[t \rightarrow \infty]{} 0 \quad \bar{\mathbb{P}}\text{-almost surely.}$$

This concludes that

$$\lim_{t \rightarrow \infty} \frac{|S_t - \sigma B_t|}{t^{1/2-\beta}} = 0 \quad \bar{\mathbb{P}}\text{-almost surely.} \quad (9.13)$$

Let W_n and S_n^* be, respectively, the Brownian motion and the random function defined by rescaling $\{B_t : t \geq 0\}$ and $\{S_t : t \geq 0\}$ according to

$$W_n(t) \stackrel{\text{def}}{=} \frac{B_{nt}}{\sqrt{n}} \quad \text{and} \quad S_n^*(t) \stackrel{\text{def}}{=} \frac{S_{nt}}{\sqrt{n}} \quad \text{for } t \in [0, 1] \quad \text{and } n \geq 1.$$

Since $t \mapsto S_t$ and $t \mapsto B_t$ are $\bar{\mathbb{P}}$ -almost surely continuous function, from (9.13), we have that

$$\sup_{0 \leq t \leq 1} |S_t - \sigma B_t| < \infty \quad \text{and} \quad \sup_{t \geq 1} \frac{|S_t - \sigma B_t|}{t^{1/2-\beta}} < \infty \quad \bar{\mathbb{P}}\text{-almost surely.}$$

Consequently,

$$\begin{aligned} \|S_n^* - \sigma W_n\| &\stackrel{\text{def}}{=} \sup_{0 \leq t \leq 1} |S_n^*(t) - \sigma W_n(t)| = \frac{1}{\sqrt{n}} \sup_{0 \leq t \leq n} |S_t - \sigma B_t| \\ &= \frac{1}{\sqrt{n}} \sup_{0 \leq t \leq 1} |S_t - \sigma B_t| + \frac{1}{\sqrt{n}} \sup_{1 \leq t \leq n} t^{1/2-\beta} \frac{|S_t - \sigma B_t|}{t^{1/2-\beta}} \\ &\leq \frac{1}{\sqrt{n}} \sup_{0 \leq t \leq 1} |S_t - \sigma B_t| + \frac{n^{1/2-\beta}}{\sqrt{n}} \sup_{1 \leq t \leq n} \frac{|S_t - \sigma B_t|}{t^{1/2-\beta}} \\ &= \frac{1}{\sqrt{n}} \sup_{0 \leq t \leq 1} |S_t - \sigma B_t| + \frac{1}{n^\beta} \sup_{1 \leq t \leq n} \frac{|S_t - \sigma B_t|}{t^{1/2-\beta}} \xrightarrow[n \rightarrow \infty]{} 0 \quad \bar{\mathbb{P}}\text{-almost surely.} \end{aligned}$$

Thus, $\lim_{n \rightarrow 0} \|S_n^* - \sigma W_n\| = 0$ $\bar{\mathbb{P}}$ -almost surely where $\|\cdot\|$ is the sup-norm on the space $C^0([0, 1])$ of real-valued continuous function on $[0, 1]$. This implies that

$$\lim_{n \rightarrow \infty} \bar{\mathbb{P}}(\|S_n^* - \sigma W_n\| > \epsilon) = 0$$

for any $\epsilon > 0$ and consequently, by Lemma 3.12, S_n^* converges in distribution to σB where B is a standard Brownian motion.

Consider now the function $\chi: C^0([0, 1]) \rightarrow [0, 1]$ defined by

$$\chi(f) \stackrel{\text{def}}{=} \int_0^1 \mathbb{1}_{(0, \infty)}(f(t)) dt.$$

It is not hard to see that the function χ is continuous in every $f \in C^0([0, 1])$ with the property that $\text{Leb}(\{t \in [0, 1]: f(t) = 0\}) = 0$. Since the Brownian motion σB has this property $\bar{\mathbb{P}}$ -almost surely, we get from Corollary 3.11 that $\chi(S_n^*)$ converges in distribution to $\chi(\sigma B)$.

On the other hand,

$$\chi(\sigma B) \stackrel{\text{def}}{=} \int_0^1 \mathbb{1}_{(0, \infty)}(\sigma B_t) dt = \int_0^1 \mathbb{1}_{(0, \infty)}(B_t) dt \stackrel{\text{def}}{=} \chi(B)$$

and

$$\chi(S_n^*) = \int_0^1 \mathbb{1}_{(0, \infty)}(S_{nt}) dt = \sum_{j=0}^{n-1} \int_{\frac{j}{n-1}}^{\frac{j+1}{n}} \mathbb{1}_{(0, \infty)}(S_{nt}) dt = \frac{1}{n} \sum_{j=0}^{n-1} \int_j^{j+1} \mathbb{1}_{(0, \infty)}(S_\theta) d\theta.$$

For $\theta \in (j, j+1)$, we have that

- $\mathbb{1}_{(0,\infty)}(S_\theta) = \mathbb{1}_{(0,\infty)}(S_j)$ if $S_j \cdot S_{j+1} > 0$;
- $\mathbb{1}_{(0,\infty)}(S_\theta) \leq \mathbb{1}_{(0,\infty)}(S_j)$ if $S_j > 0$ and $S_{j+1} \leq 0$;
- $\mathbb{1}_{(0,\infty)}(S_\theta) \leq \mathbb{1}_{(0,\infty)}(S_{j+1})$ if $S_j \leq 0$ and $S_{j+1} \geq 0$.

Hence, we get that

$$\chi(S_n^*) \leq \frac{2}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(0,\infty)}(S_j).$$

Thus, by the arcsine law for Brownian motion (see Theorem 3.7), for any $\alpha \in (0, 1)$,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \overline{\mathbb{P}}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(0,\infty)}(S_j) \leq \alpha\right) &\leq \lim_{n \rightarrow \infty} \overline{\mathbb{P}}\left(\chi(S_n^*) \leq \frac{\alpha}{2}\right) = \\ &= \overline{\mathbb{P}}\left(\chi(\sigma B) \leq \frac{\alpha}{2}\right) \\ &= \overline{\mathbb{P}}\left(\chi(B) \leq \frac{\alpha}{2}\right) = \frac{2}{\pi} \arcsin \sqrt{\frac{\alpha}{2}} < 1. \end{aligned} \tag{9.14}$$

Finally, since $\mathbb{1}_{(0,+\infty)}(S_j(\omega)) = \mathbb{1}_{(y,\infty)}(y + S_j(\omega))$ for every $y \in \mathbb{R}$, the distributional result of equation (9.14) holds for the original process, as the law of the sums is invariant by the joining. Therefore, for every $y \in \mathbb{R}$,

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(y,\infty)}(y + S_j(\omega)) \leq \alpha\right) < 1 \quad \text{for every } \alpha \in (0, 1).$$

This completes the proof of the proposition. \square

We now establish the almost-sure counterpart to the fluctuation law. A direct consequence of the vanishing occupation time property is that the asymptotic behavior of the additive fiber process governed by the process $\{S_n\}_{n \geq 0}$ is independent of the initial fiber coordinate. This crucial decoupling from the fiber allows the fluctuation law, in conjunction with the ergodicity of the base map, to establish the following almost-sure occupation time property.

Corollary 9.16. *Under the assumptions of Theorem 9.15, for any fixed $\kappa, y \in \mathbb{R}$ it holds that*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{J_i(\kappa)}(y + S_j(\omega)) = 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \text{ and } i = 0, 1.$$

Proof. Let us prove the corollary for $i = 1$; for $i = 0$ is similar. Set

$$A_n(\omega) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(0,\infty)}(S_j(\omega))$$

and define $C(\omega) \stackrel{\text{def}}{=} \limsup_{n \rightarrow \infty} A_n(\omega)$. Since $S_j(\tau(\omega)) = S_{j+1}(\omega) - \phi(\omega)$, it follows that

$$A_n(\tau(\omega)) = \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(0,\infty)}(S_{j+1}(\omega) - \phi(\omega)) = \frac{1}{n} \sum_{j=1}^n \mathbb{1}_{(0,\infty)}(S_j(\omega) - \phi(\omega)).$$

Because ϕ is Hölder on the compact base, it is bounded; set

$$M \stackrel{\text{def}}{=} \sup_{\omega \in \Omega} |\phi(\omega)| < \infty, \quad K \stackrel{\text{def}}{=} [-M, M].$$

Using the elementary fact $|\mathbb{1}_{(0,\infty)}(t - \alpha) - \mathbb{1}_{(0,\infty)}(t)| \leq \mathbb{1}_K(t)$ for all $|\alpha| \leq M$, $t \in \mathbb{R}$, we obtain the uniform bound

$$|A_n(\tau(\omega)) - A_n(\omega)| \leq \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_K(S_j(\omega)). \quad (9.15)$$

By Theorem 9.1 (case (2)), we have that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mathbb{1}_K(S_j(\omega)) = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Combining this with (9.15) yields $\lim_{n \rightarrow \infty} |A_n(\tau(\omega)) - A_n(\omega)| = 0$ for \mathbb{P} -a.e. $\omega \in \Omega$. Hence $C(\tau(\omega)) = C(\omega)$ for \mathbb{P} -a.e. $\omega \in \Omega$. Ergodicity of $(\Omega, \mathcal{F}, \mathbb{P}, \tau)$ then implies that $C(\omega)$ is almost surely constant. Write $C \equiv C(\omega) \leq 1$ and assume that $C < 1$. Choose α with $C < \alpha < 1$. By Lemma 3.13 we have that $\lim_{n \rightarrow \infty} \mathbb{P}(A_n \geq \alpha) = 0$. On the other hand, by Theorem 9.15, the sequence A_n satisfies the fluctuation law, $\liminf_{n \rightarrow \infty} \mathbb{P}(A_n \leq \alpha) < 1$ for every $\alpha \in (0, 1)$. This implies that $\limsup_{n \rightarrow \infty} \mathbb{P}(A_n > \alpha) > 0$ which contradicts the previous null limit. Therefore the assumption $C < 1$ is false, and we must have $\limsup_{n \rightarrow \infty} A_n(\omega) = 1$ for \mathbb{P} -a.e. $\omega \in \Omega$. Now Lemma 3.14 implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(\kappa, \infty)}(y + S_j(\omega)) = \limsup_{n \rightarrow \infty} A_n(\omega) = 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

This completes the proof of the corollary. \square

10. PROOF OF PROPOSITION VII AND COROLLARY VIII

10.1. Proof of Proposition VII. Consider a skew flow $F_{\varphi, \phi}$ as in (1.19), where $\varphi: \mathbb{R} \times M \rightarrow M$ is a Morse-Smale flow on the one-dimensional compact manifold M , and $\phi: \Omega \rightarrow \mathbb{R}$ is a Hölder continuous function satisfying $\mathbb{E}[\phi] = 0$. Let $J = (p, q)$ be an open arc connecting two consecutive equilibrium points of φ , and fix $\theta \in J$. Define

$$\hat{H}: \Omega \times \mathbb{R} \rightarrow \Omega \times J, \quad \hat{H}(\omega, t) = (\omega, \varphi(t, \theta)).$$

It is straightforward to verify that \hat{H} is a homeomorphism satisfying $\hat{H} \circ T_\phi = F_{\varphi, \phi} \circ \hat{H}$, where T_ϕ is the skew-translation defined in (9.1). Additionally, the map $\hat{h}: \mathbb{R} \rightarrow J$ given by $\hat{h}(t) = \varphi(t, \theta)$ is also a homeomorphism, and $g_\omega = \hat{h}^{-1} \circ f_\omega \circ \hat{h}$, where $g_\omega(y) = y + \phi(\omega)$ and f_ω is given by (1.19). Letting $h = \hat{h}^{-1}$, we meet the conditions of Proposition 4.7.

Although assumption (C2) implies (C1), we will first prove the proposition under the hypothesis (C2) to illustrate how Theorem B can be used to obtain the result.

10.1.1. Proof under assumption (C2)

Since ϕ satisfies the assumptions of Theorems 9.9 and 9.15, we deduce the following:

- (i) $\hat{H}_*(\mathbb{P} \times \text{Leb}_{\mathbb{R}})$ is an ergodic $F_{\varphi,\phi}$ -invariant measure,
- (ii) for every $x \in J$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{J_i(x)}(f_\omega^j(x)) \leq \alpha\right) < 1 \quad \text{for every } \alpha \in (0, 1) \text{ and } i = 0, 1.$$

By Corollary 4.6, (ii) implies that $F_{\varphi,\phi}$ restricted to $\Omega \times I$, where I is the closure of J , satisfies the fiberwise fluctuation law with constants $\gamma_0 < \gamma_1$. Moreover, since $d\hat{H}_*(\mathbb{P} \times \text{Leb}_{\mathbb{R}}) = \hat{h}_* \text{Leb}_{\mathbb{R}} d\mathbb{P}$ and $\hat{h}_* \text{Leb}_{\mathbb{R}}$ is equivalent to Leb_I because \hat{h} is smooth, we conclude that $\hat{H}_*(\mathbb{P} \times \text{Leb}_{\mathbb{R}})$ is equivalent to $\mathbb{P} \times \text{Leb}_I$, where Leb_I is the normalized Lebesgue measure on I . Consequently, by (i), $\mathbb{P} \times \text{Leb}_I$ is ergodic with respect to $F_{\varphi,\phi}$.

Thus, in view of Proposition 4.8, the restriction of $F_{\varphi,\phi}$ to $\Omega \times I$, satisfies the assumptions of Theorem 6.1 (from which Theorem B follows) and exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -almost every $(\omega, x) \in \Omega \times I$. Since M is the union of finitely many intervals I , it follows that $F_{\varphi,\phi}$ exhibits historical behavior for $(\mathbb{P} \times \text{Leb})$ -almost every point in $\Omega \times M$.

Moreover, by Proposition II, the limit set is given by

$$\mathcal{L}(\omega, x) = \{\lambda\delta_p + (1 - \lambda)\delta_q : \lambda \in [0, 1]\}, \quad \text{for } (\mathbb{P} \times \text{Leb})\text{-almost every } (\omega, x),$$

where x lies between consecutive equilibrium points p and q of φ . This completes the proof of Proposition VII under the assumption (C2).

10.1.2. Proof under assumption (C1)

Since ϕ satisfies the assumptions of Theorem 9.1 (case (2)) and Corollary 9.16, we deduce the following:

- (i) for every $x \in J$ and compact set $K \subset J$ compact,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_K(f_\omega^j(x)) = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \text{ and } i = 0, 1.$$

- (ii) for every $\gamma, x \in J$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{J_i(\gamma)}(f_\omega^j(x)) = 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \text{ and } i = 0, 1.$$

Fix $\gamma, x \in J$. By (i), condition (1.12) (and thus (7.1)) holds, and hence Proposition 7.2 implies that $\mathcal{L}(\omega, x) \subset \{\lambda\delta_p + (1 - \lambda)\delta_q : \lambda \in [0, 1]\}$. On the other hand, by (ii), for \mathbb{P} -a.e. $\omega \in \Omega$, there exists a subsequence of integers $\{n_k\}_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} A_{n_k}(\omega) = 1$ where

$$A_n(\omega) = \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{J_i(\gamma)}(f_\omega^j(x)).$$

Consider the corresponding subsequence of empirical measures $\{e_{n_k}^\omega\}_{k \geq 1}$. This sequence lies in the space of probability measures on $I = [0, 1]$, which is compact in the weak* topology. Therefore, there must exist a convergent subsequence, which we re-index again by $\{n_k\}_{k \geq 1}$, converging to some limit measure $\nu \in \mathcal{L}(\omega, x)$. Hence $\nu = \lambda\delta_p + (1 - \lambda)\delta_q$ for some $\lambda \in [0, 1]$.

Note that $A_n(\omega) = e_n^\omega(J_i(\gamma))$. If $i = 0$, since $J_0(\gamma) = (p, \gamma) \subset [p, \gamma]$, we have that

$$1 = \lim_{k \rightarrow \infty} A_{n_k}(\omega, x) \leq \lim_{k \rightarrow \infty} e_{n_k}^\omega([p, \gamma]) \leq \nu([p, \gamma]) = \nu(\{p\}) = \lambda.$$

Hence $\lambda = 1$. This shows that the limit measure is $\nu = \delta_p$. We have thus found a subsequence of empirical measures that converges to δ_p , proving that $\delta_p \in \mathcal{L}(\omega, x)$. Otherwise, if $i = 1$, $J_1(\gamma) = (\gamma, q) \subset [\gamma, q]$ and thus,

$$1 = \lim_{k \rightarrow \infty} A_{n_k}(\omega) = \lim_{k \rightarrow \infty} e_{n_k}^\omega([\gamma, q]) \leq \nu([\gamma, q]) = \nu(\{q\}) = 1 - \lambda.$$

This implies that $\lambda = 0$, so $\nu = \delta_q$. Thus, $\delta_q \in \mathcal{L}(\omega, x)$.

We have shown that for every $x \in J$, for \mathbb{P} -a.e. $\omega \in \Omega$, the set of accumulation points $\mathcal{L}(\omega, x)$ contains both δ_p and δ_q . Since $\mathcal{L}(\omega, x)$ is a connected set, it must contain the entire segment of convex combinations connecting these two points. This completes the proof of the second inclusion and the proof of Proposition VII under the assumption (C1).

10.2. Proof of Corollary VIII. As mentioned in the introduction, this result follows directly from Proposition VII and 2.5. The only point that may require clarification is the description of the set of accumulation points of the sequence of empirical measures. To avoid confusion, we write $\tilde{\mathcal{L}}(\omega, x)$ and $\mathcal{L}(\omega, x)$ for limit set for $\tilde{F} = \tilde{F}_{\varphi, \phi}$ and $F = F_{\varphi, \phi}$ respectively.

To establish this, note first that if $p \in [1/2, 1]$ is an equilibrium point of φ , then $(f_\omega)_*\delta_p = \delta_{\pi(p)}$ and $(f_\omega)_*\delta_{\pi(p)} = \delta_p$ for \mathbb{P} -a.e. $\omega \in \Omega$. Consequently, the measure $\nu_p = \mathbb{P} \times \mu_p$, where $\mu_p = (\delta_p + \delta_{\pi(p)})/2$, is the unique F -invariant measure that is a convex combination of $\mathbb{P} \times \delta_p$ and $\mathbb{P} \times \delta_{\pi(p)}$. Moreover, $\Pi_*^{-1}(\{\mathbb{P} \times \delta_p\}) \cap \{F\text{-invariant measures}\} = \{\nu_p\}$. Now, since $\mathbb{P} \times \mu$ is \tilde{F} -invariant provided $\mu \in \tilde{\mathcal{L}}(\omega, x)$, we have

$$\Pi_*^{-1}(\{\mathbb{P} \times \mu : \mu \in \mathcal{L}(\omega, x)\}) \cap \{\tilde{F}\text{-invariant measures}\} = \{\lambda\nu_p + (1 - \lambda)\nu_q : \lambda \in [0, 1]\},$$

for \mathbb{P} -a.e. $\omega \in \Omega$ and x lying between consecutive equilibrium points p and q of φ in $[1/2, 1]$. Thus, it follows that $\tilde{\mathcal{L}}(\omega, x) = \{\lambda\mu_p + (1 - \lambda)\mu_q : \lambda \in [0, 1]\}$, as desired.

ACKNOWLEDGEMENT

This work is based on the master's thesis of the second author at the Pontifical Catholic University of Rio de Janeiro. We thank the second author's advisor, L. J. Díaz, for his guidance and support. We are also grateful to the members of the examination committee, M. Anderson, S. Luzzatto, R. Ruggiero, and A. Tahzibi, for their insightful comments and suggestions. We thank C. González-Tokman and D. Coates for their comments and questions, which enriched the discussions. The first author acknowledges support from grant PID2020-113052GB-I00 funded by MCIN, PQ 305352/2020-2 (CNPq), and JCNE E-26/201.305/2022

(FAPERJ). The second author was supported [in part] by CAPES – Finance Code 001 and FAPERJ Nota-10.

REFERENCES

- [Aar97] J. Aaronson. *An introduction to infinite ergodic theory*, volume 50 of *Math. Surv. Monogr.* Providence, RI: American Mathematical Society, 1997.
- [ATZ05] J. Aaronson, M. Thaler, and R. Zweimüller. Occupation times of sets of infinite measure for ergodic transformations. *Ergodic Theory Dynam. Systems*, 25(4):959–976, 2005.
- [AG22] M. Andersson and P.-A. Guihénéuf. Historic behaviour vs. physical measures for irrational flows with multiple stopping points. *Adv. Math.*, 409:Paper No. 108626, 71, 2022.
- [BS00] L. Barreira and J. Schmeling. Sets of “non-typical” points have full topological entropy and full Hausdorff dimension. *Israel J. Math.*, 116:29–70, 2000.
- [Bar22] P. G. Barrientos. Historic wandering domains near cycles. *Nonlinearity*, 35(6):3191, 2022.
- [BKN⁺20] P. G. Barrientos, S. Kiriki, Y. Nakano, A. Raibekas, and T. Soma. Historic behavior in nonhyperbolic homoclinic classes. *Proc. Amer. Math. Soc.*, 148(3):1195–1206, 2020.
- [BNR⁺22] P. G. Barrientos, Y. Nakano, A. Raibekas, and M. Roldan. Topological entropy and Hausdorff dimension of irregular sets for non-hyperbolic dynamical systems. *Dyn. Syst.*, 37(2):181–210, 2022.
- [BdH89] H. C. P. Berbee and W. T. F. den Hollander. Tail triviality for sums of stationary random variables. *Ann. Probab.*, 17(4):1635–1645, 1989.
- [BB23] P. Berger and S. Biebler. Emergence of wandering stable components. *Journal of the American Mathematical Society*, 36(2):397–482, 2023.
- [BB03] M. Blank and L. Bunimovich. Multicomponent dynamical systems: SRB measures and phase transitions. *Nonlinearity*, 16(1):387–401, 2003.
- [Bog07] V. I. Bogachev. *Measure theory*, volume 1. Springer-Verlag, Berlin, 2007.
- [BM08] A. Bonifant and J. Milnor. Schwarzian derivatives and cylinder maps. In *Holomorphic dynamics and renormalization*, volume 53 of *Fields Inst. Commun.*, pages 1–21. Amer. Math. Soc., Providence, RI, 2008.
- [Bow75] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume Vol. 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1975.
- [Buf14] A. I. Bufetov. Ergodic decomposition for measures quasi-invariant under Borel actions of inductively compact groups. *Mat. Sb.*, 205(2):39–70, 2014.
- [CV21] M. Carvalho and P. Varandas. Genericity of historic behavior for maps and flows. *Nonlinearity*, 34(10):7030–7044, 2021.
- [CL24] D. Coates and S. Luzzatto. Persistent non-statistical dynamics in one-dimensional maps. *Commun. Math. Phys.*, 405(4):34, 2024. Id/No 102.
- [CLM23] D. Coates, S. Luzzatto, and M. Muhammad. Doubly intermittent full branch maps with critical points and singularities. *Comm. Math. Phys.*, 402(2):1845–1878, 2023.
- [CMT24] D. Coates, I. Melbourne, and A. Talebi. Natural measures and statistical properties of non-statistical maps with multiple neutral fixed points. Preprint, arXiv:2407.07286 [math.DS] (2024), 2024.
- [CYZ20] S. Crovisier, D. Yang, and J. Zhang. Empirical measures of partially hyperbolic attractors. *Comm. Math. Phys.*, 375(1):725–764, 2020.
- [EK47] P. Erdős and M. Kac. On the number of positive sums of independent random variables. *Bull. Amer. Math. Soc.*, 53:1011–1020, 1947.
- [FFW01] A.-H. Fan, D.-J. Feng, and J. Wu. Recurrence, dimension and entropy. *J. London Math. Soc.* (2), 64(1):229–244, 2001.
- [GS00] G. Greschonig and K. Schmidt. Ergodic decomposition of quasi-invariant probability measures. *Colloq. Math.*, 84-85:495–514, 2000.
- [Gui89] Y. Guivarc'h. Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés. *Ergodic Theory Dynam. Systems*, 9(3):433–453, 1989.

- [Has07] B. Hasselblatt. Problems in dynamical systems and related topics. In *Dynamics, ergodic theory and geometry. Dedicated to Anatole Katok. Based on the workshop on recent progress in dynamics, Berkeley, CA, USA, from late September to early October, 2004*, pages 273–324. Cambridge: Cambridge University Press, 2007.
- [HS55] E. Hewitt and L. J. Savage. Symmetric measures on Cartesian products. *Trans. Amer. Math. Soc.*, 80:470–501, 1955.
- [Ino00] T. Inoue. Sojourn times in small neighborhoods of indifferent fixed points of one-dimensional dynamical systems. *Ergodic Theory Dyn. Syst.*, 20(1):241–257, 2000.
- [JM00] C. Ji and D. Molinek. Asymptotic physical measures for nearly Brownian skew products. *Ergodic Theory Dynam. Systems*, 20(2):473–481, 2000.
- [Kal82] S. A. Kalikow. T , T^{-1} transformation is not loosely Bernoulli. *Ann. of Math.* (2), 115(2):393–409, 1982.
- [Kel04] G. Keller. Completely mixing maps without limit measure. *Colloq. Math.*, 100(1):73–76, 2004.
- [KS17] S. Kiriki and T. Soma. Takens’ last problem and existence of non-trivial wandering domains. *Advances in Mathematics*, 306:524–588, 2017.
- [LR17] I. S. Labouriau and A. A. P. Rodrigues. On Takens’ last problem: tangencies and time averages near heteroclinic networks. *Nonlinearity*, 30(5):1876–1910, 2017.
- [Lé39] P. Lévy. Sur certains processus stochastiques homogènes. *Compositio Math.*, 7:283–339, 1939.
- [Lé65] P. Lévy. *Processus stochastiques et mouvement brownien*. Gauthier-Villars & Cie, Paris, 1965. Suivi d’une note de M. Loève, Deuxième édition revue et augmentée.
- [Liu21] C. Liu. Brownian motion as the limiting distribution of random walks, August 2021. University of Chicago REU paper.
- [Liv72] A. N. Livšic. Cohomology of dynamical systems. *Mathematics of the USSR-Izvestiya*, 6(6):1278, dec 1972.
- [Loe78] M. Loeve. *Probability theory. II. 4th ed*, volume 46 of *Grad. Texts Math.* Springer, Cham, 1978.
- [Mol94] D. K. Molinek. Asymptotic measures for skew products of Bernoulli shifts with generalized north pole–south pole diffeomorphisms. *Trans. Amer. Math. Soc.*, 345(1):263–291, 1994.
- [MP10] P. Mörters and Y. Peres. *Brownian motion*, volume 30 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.
- [Nak17] Y. Nakano. Historic behaviour for random expanding maps on the circle. *Tokyo J. Math.*, 40(1):165–184, 2017.
- [PP90] W. Parry and M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque*, page 268, 1990.
- [PP97] W. Parry and M. Pollicott. The Livšic cocycle equation for compact Lie group extensions of hyperbolic systems. *Journal of the London Mathematical Society*, 56(2):405–416, 1997.
- [Rud88] D. J. Rudolph. Asymptotically Brownian skew products give non-loosely Bernoulli K-automorphisms. *Invent. Math.*, 91(1):105–128, 1988.
- [Rue01] D. Ruelle. Historical behaviour in smooth dynamical systems. In *Global analysis of dynamical systems*, pages 63–66. Inst. Phys., Bristol, 2001.
- [Ser20] T. Sera. Functional limit theorem for occupation time processes of intermittent maps. *Nonlinearity*, 33(3):1183–1217, 2020.
- [SY19] T. Sera and K. Yano. Multiray generalization of the arcsine laws for occupation times of infinite ergodic transformations. *Trans. Amer. Math. Soc.*, 372(5):3191–3209, 2019.
- [Shi19] A. N. Shiryaev. *Probability. 2*, volume 95 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2019.
- [Tak94] F. Takens. Heteroclinic attractors: time averages and moduli of topological conjugacy. *Bol. Soc. Brasil. Mat. (N.S.)*, 25(1):107–120, 1994.
- [Tak08] F. Takens. Orbits with historic behaviour, or non-existence of averages. *Nonlinearity*, 21(3):T33–T36, 2008.

- [Tal22] A. Talebi. Non-statistical rational maps. *Mathematische Zeitschrift*, 302(1):589–608, 2022.
- [Tha80] M. Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.*, 37(4):303–314, 1980.
- [Tha83] M. Thaler. Transformations on [0, 1] with infinite invariant measures. *Israel J. Math.*, 46(1-2):67–96, 1983.
- [Tha02] M. Thaler. A limit theorem for sojourns near indifferent fixed points of one-dimensional maps. *Ergodic Theory Dynam. Systems*, 22(4):1289–1312, 2002.

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UFF
 RUA MÁRIO SANTOS BRAGA S/N - CAMPUS VALONGUINHOS, NITERÓI, BRAZIL

Email address: pgbarrientos@id.uff.br

DEPARTAMENTO DE MATEMÁTICA PUC-RIO
 MARQUÊS DE SÃO VICENTE 225, GÁVEA, RIO DE JANEIRO 22451-900, BRAZIL

Email address: r.rodriguezchav@aluno.puc-rio.br