Advanced Stochastic Processes

Sidney I. Resnick

Contents

Chapter 1

Preliminaries

1.1 Generating Functions

Definition 1.1 (Generating Function)**.** The generating function of a *non-negative integer valued* rv X is

$$\mathbb{P}(s) = \sum_{k=0}^{\infty} p_k s^k$$

or is equivalently

$$P(s) = \mathbb{E}s^X$$

Remark. Generating functions are useful for computing mass functions of sums of independant rvs, calculating moments, limits of distributions, and more.

1.1.1 Generating Functions of Common Distributions

Just factor the s^k out.

Poisson: $X \sim p(k; \lambda)$

$$P(s) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{(\lambda)^k}{k!} s^k = e^{\lambda(s-1)}$$

Binomial: $X \sim b(k; n, p)$

$$P(s) = (q + ps)^n$$

Geometric: $X \sim g(k; p)$

$$P(s) = \frac{p}{1 - qs}$$

1.1.2 Differentiation of Generating Functions

$$\frac{d^n}{ds^n}\mathbb{P}(s) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} p_k s^{k-n}$$

Thus, evaluating at $s = 0$:

$$\frac{d^n}{ds^n}\mathbb{P}(s)\Big|_{s=0} = n!p_n$$

Proposition 2. *A generating function uniquely dtermines its sequence*

Note that differentiating moments and evaluating at $s = 1$ gives you information about various moments i.e.

$$\lim_{s\uparrow 1}\frac{d^n}{ds^n}\mathbb{P}(s) = \mathbb{E}[X(X-1)...(X-2)(X-n+1)] \tag{2.1}$$

Proposition 3. *The gf of a convolution is the product of the gf's (true for any 2 sequences)*

$$P_{X_1+X_2}(s) = P_{X_1}(s)P_{X_2}(s)$$

Proof.

$$P_{X_1+X_2}(s) = \mathbb{E}s^{X_1}\mathbb{E}s^{X_2}$$

□

Example 3.1. A binomial $X \sim b(k; n, p)$ can be decomposed as a sum of Bernoullis $X_i \sim \mathcal{B}(k; p)$ as $X = X_1 + X_2 + ... + X_n$ thus,

$$P_{X_1} = (q + ps)$$

and

$$P_X(s) = (q + ps)^n$$

Example 3.2. A sum of Poissons is a poisson. $X_1 \sim p(k; \lambda), X_2 \sim p(k; \mu)$ and with $X \sim p(k; \lambda + \mu)$ we have,

$$P_{X_1+X_2}(s) = e^{\lambda(s-1)}e^{\mu(s-1)} = P_X(s)$$

3.0.1 Summing a random number of rvs

Let N be a random variable and let $S_N = X_1 + X_2 + ... + X_N$. The following result comes from expanding the formula and swithcing the order of summation.

$$P_{S_N} = P_N(P_X(s)) \tag{3.1}$$