Math 300
Notes

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1 Overview

Below is a (hopefully comprehensive) list of all the topics that have been covered:

- 1 Complex Numbers
  - 1.1 The Algebra of Complex Numbers
  - 1.2 Point Representations of Complex Numbers
  - 1.3 Vectors and Polar Forms
  - 1.4 The Complex Exponential
  - 1.5 Powers and Roots
  - 1.6 Planar Sets
- 2 Analytic Functions
  - 2.1 Functions of a Complex Variable
  - 2.2 Limits and Continuity
  - 2.3 Analyticity
  - 2.4 The Cauchy-Riemann Equations
  - 2.5 Harmonic Functions
- 3 Elementary Functions
  - 3.1 Polynomials and Rational Functions
  - 3.2 The Exponential, Trigonometric, and Hyperbolic Functions
  - 3.3 The Logarithmic Function
  - 3.5 Complex Powers and Inverse Trigonometric Functions
- 4 Complex Integration
  - 4.1 Contours
  - 4.2 Contour Integrals

2 Definitions and Theorems

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Definition 1 (Cauchy-Riemann). $\longrightarrow$ *A complex function, $f : \mathbb{R} \to \mathbb{R}$ is considered to satisfy Cauchy-Riemann iff:*

$$\Big(u_x = v_y\Big) \wedge \Big(u_y = -v_x\Big) \tag{1}$$

where $f := u + iv$ and $u, v : \mathbb{R}^2 \to \mathbb{R}$.

Definition 2 (Differentiability and C-R). $\longrightarrow$ *Consider $f : \mathbb{C} \to \mathbb{C}$ and some $z_0 : \mathbb{C}$.*

$$f \text{ is differentiable at } z_0 \implies f \text{ satisfies C-R at } z_0 \tag{2}$$

Definition 3 (C-R and Differentiability). $\longrightarrow$ *Similarly, consider $f : \mathbb{C} \to \mathbb{C}$ and some $z_0 : \mathbb{C}$.*

$$\Big(f_x, f_y \text{ exist and are continuous at } z_0\Big) \wedge \Big(f \text{ satisfies C-R at } z_0\Big) \implies f \text{ is differentiable at } z_0 \tag{3}$$

2.5

Definition 4 (Laplace's Operator and Equation). $\longrightarrow$ *Let $\phi : \mathcal{C}^2(\mathbb{R}^n \to \mathbb{R})$ be a real-valued function in which at least all of its second partial derivatives exist. ϕ is said to solve Laplace's equation iff:*

$$\Delta(\phi) = 0 \tag{4}$$

where Δ, the Laplace operator, is defined as:

$$\Delta = \left(\phi \mapsto \sum_{j=2}^{n} f_{x_j x_j}\right) \tag{5}$$

Definition 5 (Harmonic Functions)**.** $\longrightarrow$ *A real-valued function, $f : \mathcal{C}(X \to Y)$ (i.e., an at least twice differentiable function), is considered harmonic on some domain D iff:*

$$\Delta(f) = \sum_{j=2}^{n} f_{x_j x_j} = 0 \tag{6}$$

for every point in D.

Theorem 1 (Analyticity and Harmonicity)**.** $\longrightarrow$ *Let $f : \mathbb{C} \to \mathbb{C}$ be some complex-valued function, where:*

$$f := u + iv \tag{7}$$

Thus:

$$f \textit{ is analytic on } D \implies u, v \textit{ are harmonic on } D \tag{8}$$

where D is some complex domain.

Theorem 2 (Existence of a Harmonic Conjugate)**.** $\longrightarrow$ *Consider some function, $u : \mathcal{C}^2(\mathbb{R}^2 \to \mathbb{R})$. Thus:*

$$u \textit{ is harmonic on } D \implies \Big(\big(\exists v : \mathcal{C}^2(\mathbb{R}^2 \to \mathbb{R}), v \textit{ is harmonic on } D\big) \wedge \big(f := u + iv \textit{ is analytic on } D\big)\Big) \tag{9}$$

Here, v is considered the harmonic conjugate of u.