Mathematics 300
Notes

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Basic Definitions

Base Definitions for Analyticity and Integrability

In order to start working with complex functions, we will need to start analyzing their inputs. We classify domains to complex functions using the following definitions.

Definition 1 (Interior Point)**.** *A point, $x \in \mathbb{C}$, is considered an interior point of some set, $D \subseteq \mathbb{C}$, iff every point in its neighbourhood is an element of D.*

$$x \text{ is an interior point of } D \iff \exists \epsilon \in \mathbb{R}^+, \forall x' \in \mathcal{N}(x, \epsilon), x' \in D \tag{1}$$

Definition 2 (Open Domain)**.** *An open domain, $D \subseteq \mathbb{C}$, is a set in which every point in D is an interior point.*

$$D \text{ is an open domain} \iff \forall x \in D, x \text{ is an interior point} \tag{2}$$

Definition 3 (Function Limit)**.** *A function, $f : D \to Y$, is said to have a limit of $L \in Y$ at the point $x_0 \in D$ iff f approaches L from **all** directions in the neighbourhood of x_0.*

$$f \text{ has a limit of } L \text{ at point } x_0 \tag{3}$$
$$\iff \forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \big(0 < |x - x_0| < \delta\big) \implies \big(|f(x) - f(x_0)| < \epsilon\big) \tag{4}$$

Definition 4 (Function Continuity)**.** *A function, $f : D \to Y$, is said to be continuous at the point $x \in D$ iff f has a limit of $L \in Y$ at the point x, **and** $f(x) = L$.*

$$f \text{ is continuous at } x \tag{5}$$
$$\iff \Big(f \text{ has a limit of } L \text{ at } x\Big) \wedge \Big(f(x) = L\Big) \tag{6}$$

Definition 5 (Function Differentiability)**.** *A function, $f : D \to Y$, is said to be differentiable at a point $x \in D$ iff the rate of change of f is continuous at x.*

$$f \text{ is differentiable at } x_0 \tag{7}$$
$$\iff \left(x \mapsto \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right) \text{ is continuous at } x_0 \tag{8}$$

Definition 6 (Function Analyticity)**.** *A function, $f : D \to Y$, is said to be analytical in the domain $D \subseteq \mathbb{C}$ iff it is differentiable at every point in D.*

$$f \text{ is analytical in } D \tag{9}$$
$$\iff \forall x \in D,\ f \text{ is differentiable at } x \tag{10}$$