STAT200 HW1

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Problem 1

Two dice are rolled, and the sum of the face values is six. What is the probability that at least one of the dice came up a three?

When rolling two dices (D_1, D_2) , the possible ways that the sum of two dice can equal six are as follows:

1. $D_1 = 1, D_2 = 2$
2. $D_1 = 2, D_2 = 4$
3. $D_1 = 3, D_2 = 3$
4. $D_1 = 4, D_2 = 2$
5. $D_1 = 5, D_2 = 1$

Therefore, there are five total ways in which two dice can sum to six. Of these five ways, there is just one way $(D_1 = 3, D_2 = 3)$ in which at least one of the dice comes up with a three. Therefore, the probability that at least one of the dice came up with a three is 1/5.
This can also be done using Bayes' theroem where $P(A)$ is probability of at least one dice being a three, and $P(B)$ being the probability that that the sum of the dice is six.

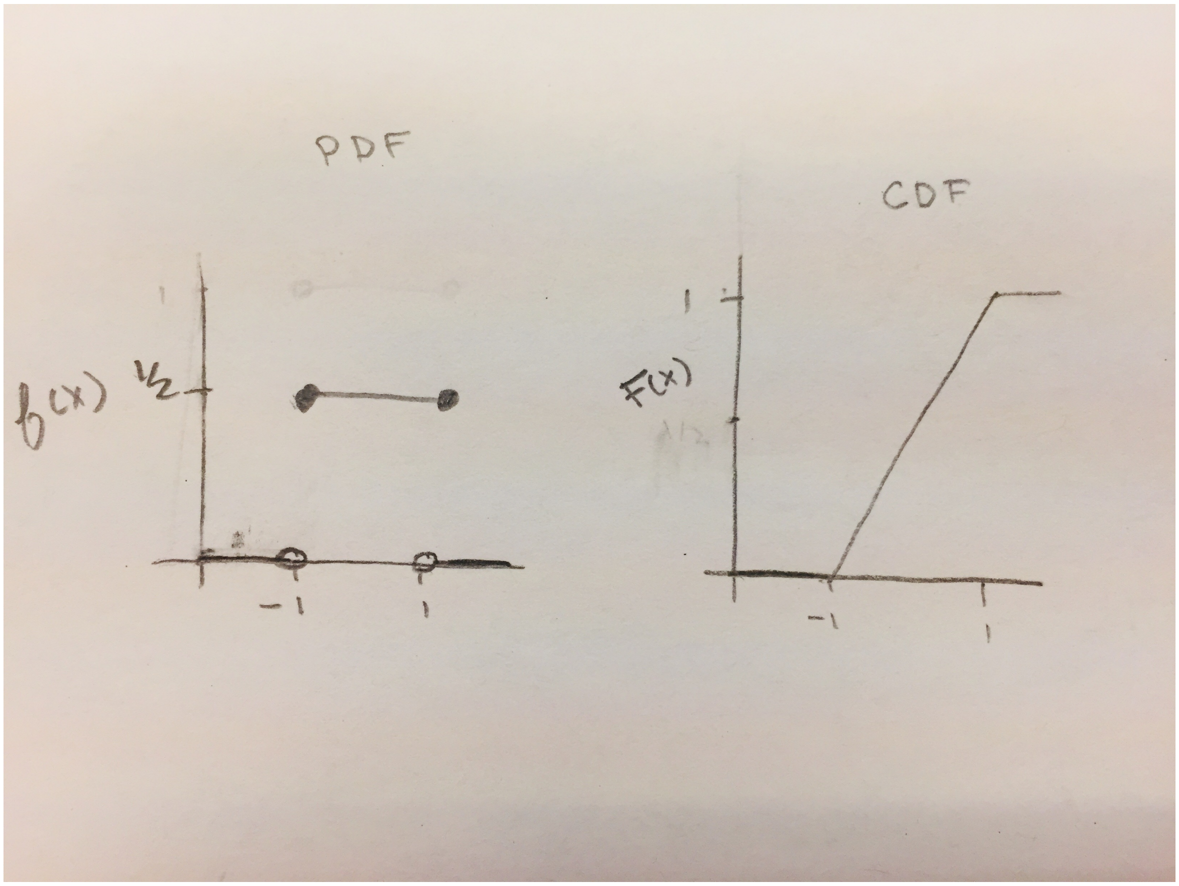
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(B) = 1, P(A) = \frac{1}{5}, P(B|A) = 1$$

$$P(A|B) = 1 * \frac{1}{5}/1 = 1/5$$

Problem 2

Sketch the PDF and CDF of a random variable that is uniform on $[-1, 1]$



Problem 3

Find the upper 75% and lower 25% quartiles of the exponential distribution with mean parameter $\lambda = 1$

The CDF when $\lambda = 1$ of the exponential distribution is as follows

$$F(x) = \int_{-\infty}^{\infty} f(u)du = \begin{cases} 1 - e^{-x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

Therefore, at the lower quartile:

$$1 - e^{-x} = 1/4$$
$$ln(3/4 = e^{-x}$$
$$x = -ln(3/4) \approx 0.288$$

And at the upper quartile:

$$1 - e^{-x} = 3/4$$
$$ln(1/4 = e^{-x}$$
$$x = -ln(1/4) \approx 1.386$$

Problem 4

If $X \sim N(\mu, \sigma^2)$, what is the distribution of $Z = (X = \mu)/\sigma$?

$$\begin{aligned}
F_Z(z) = P(Z \leq z) = P(\frac{X-\mu}{\sigma} \leq z) \\
= P(X \leq z\sigma + \mu) \\
= F_X(z\sigma + \mu) \\
\frac{d}{dz}F_Z(z) = \frac{d}{dz}F_X(z\sigma + \mu) \\
= \sigma f_X(z\sigma + \mu) \\
= \sigma \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{z\sigma+\mu-\mu}{\sigma})^2} \\
= \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}z^2} \\
Z \sim N(0,1)
\end{aligned}$$

What is the distribution of Z^2?

$$\begin{gathered}
U = Z^2 = \frac{(X-\mu)^2}{\sigma^2} \\
F_U(u) = P(U \leq u) = P(\frac{(X-\mu)^2}{\sigma^2} \leq u) \\
= P(\sqrt{(X-\mu)^2} \leq \sqrt{u\sigma^2}) \\
= P(-\sqrt{u}\sigma \leq X - \mu \leq \sqrt{u}\sigma) \\
= P(-\sqrt{u}\sigma + \mu \leq X \leq \sqrt{u}\sigma + \mu) \\
\text{Give this is normally distributed} \\
= 2F_X(\sqrt{u}\sigma + \mu) - 1 \\
\frac{d}{du}F_U(u) = \frac{d}{du}(2F_X(\sqrt{u}\sigma + \mu) - 1) \\
= \frac{1}{2\sqrt{u}}\sigma 2 f_X(\sqrt{u}\sigma + \mu) \\
= \frac{\sigma}{\sqrt{u}} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{\sqrt{u}\sigma+\mu-\mu}{\sigma})^2} \\
= \frac{1}{\sqrt{2u\pi}} e^{\frac{-u}{2}}
\end{gathered}$$

Problem 5

Let T_1 and T_2 be two independent exponentials with parameters λ_1 and λ_2 respectively. Find the density of $T_1 + T_2$

$$S = T_1 + T_2 \text{ therefore } T_2 = S - T_1$$

$$F_S(s) = \int_0^s \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 (s - t_1)} dt_1$$

$$= \lambda_1 \lambda_2 \int_0^s e^{-\lambda_1 t_1 + \lambda_2 t_1 - \lambda_2 s}$$

$$= \lambda_1 \lambda_2 \int_0^s e^{(\lambda_2 - \lambda_1) t_1 - \lambda_2 s}$$

$$= \lambda_1 \lambda_2 \frac{1}{\lambda_2 - \lambda_1} [e^{(\lambda_2 - \lambda_1) t_1 - \lambda_2 S}]_0^s$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} (-e^{-\lambda_2 s} + e^{-\lambda_1 s + \lambda_2 s - \lambda_2 s})$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} (-e^{-\lambda_2 s} + e^{-\lambda_1 s})$$

Problem 6

Let $X_1, ..., X_n$ be iid exponential random variables with rate $\lambda = 1$, what is the distribution of $U = min\{X_1, ..., X_n\}$?

As per Rice Ch3.7, the minimum of U is found by

$$f_U(u) = nf(u)[1 - F(u)]^{n-1}$$

therefore for an exponential function

$$f_U(u) = \lambda e^{-\lambda x} \text{ and } F_U(u) = 1 - \lambda e^{-\lambda x}$$

therefore

$$f_U(u) = n(\lambda e^{-\lambda x})(1 - (1 - \lambda e^{-\lambda x}))^{n-1}$$

we are given $\lambda = 1$ so

$$f_U(u) = n(e^{-x})(1 - (1 - e^{-x}))^{n-1}$$
$$= n(e^{-x-xn+x})$$
$$= ne^{-xn}$$

Problem 7

If X and Y are independent random variables with equal variances, find $Cov(X+Y, X-Y)$

$$
\begin{gathered}
Cov(X+Y, X-Y) = Cov(X,X) + Cov(Y,X) - Cov(X,Y) - Cov(Y,Y) \\
\text{because } Cov(Y,X) = Cov(X,Y) \\
= Cov(X,X) - Cov(Y,Y) \\
= Var(X) - Var(Y) \\
\text{Since we know these variances are equal where } Var(X) = Var(Y) \\
= Var(X) - Var(X) \\
= 0
\end{gathered}
$$