¡¡¡¡¡¡¡ .mine ======= ¿¿¿¿¿¿¿ .r23 ¡¡¡¡¡¡¡ .mine ======= ¿¿¿¿¿¿¿ .r23 ¡¡¡¡¡¡¡ .mine ======= ¿¿¿¿¿¿¿ .r23 ¡¡¡¡¡¡¡ .mine ======= ¿¿¿¿¿¿¿ .r23

HYPERBOLIC DISCOUNTING MODEL

1. Elements of the model

1.1. **Consumer.** Imagine a representative consumer with quasi-linear preferences across two goods: a polluting good, x, and a numeraire, y. The consumer's instantaneous utility function is

$$u = ax - \frac{b}{2}x^2 + y. \tag{1}$$

The consumer's budget constraint is

$$p_x x + y = W, \tag{2}$$

hence, demand for x is

$$x = \frac{a}{b} - \frac{p}{b} \tag{3}$$

and inverse demand is

$$p = a - bx. \tag{4}$$

1.2. **Monopolist.** The manufacturer of good x faces costs cx and monopolises the market. His instantaneous profit function is

$$\pi = ax - bx^2 - cx. \tag{5}$$

1.3. **Regulator.** The regulator recognises the pollution damage, $\phi(k_t)$, caused by the existence of the stock, k_t, of good x. The stock of the good evolves according to the rule

$$k_t = \theta k^{t-1} + x^{t-1} \qquad \theta \in [0, 1]. \tag{6}$$

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Instantaneous welfare is given by

$$w_t = \pi_t + CS_t - \phi(k_t) \tag{7}$$

$$= \left[ax - bx^2 - cx\right] + \left[(a - p_t^*)\,\frac{x_t^*}{2}\right] - \phi(k_t), \tag{8}$$

where the superscripted asterisk denotes the equilibrium value of the variable.

Let the pollution function be $\phi(k_t) = dk_t^2$. Now the instantaneous welfare function can be written

$$w_t = (a - c)x_t - \frac{b}{2}x_t^2 - dk_t^2. \tag{9}$$

The regulator suffers from time inconsistency and is modelled as having quasi-hyperbolic preferences with $\beta\delta$-discounting.

2. Laissez Faire equilibrium

Suppose that the monopolist acted unregulated. He then solves a static problem in each period:

$$MR = MC \tag{10}$$

$$a - 2bx = c \tag{11}$$

$$x^* = \frac{a - c}{2b} \tag{12}$$

$$p^* = \frac{a + c}{2}. \tag{13}$$

Thus

$$CS\,(x^*) = \frac{(a - c)^2}{8b} \tag{14}$$

and

$$\pi\,(x^*) = \frac{(a - c)^2}{4b}. \tag{15}$$

In the dynamic, infinite horizon case these profit and CS values will be summed to infinity with a discount rate of δ. They will simply be multiplied by $\frac{1}{1-\delta}$.Thus

$$CS\,(x^*) = \frac{(a - c)^2}{8b} \tag{16}$$

and

$$\pi\left(x^{*}\right)=\frac{(a-c)^{2}}{4b}. \tag{17}$$

3. Regulation with precommitment

Suppose that the regulator can directly choose $\{x_\tau\}_{\tau=t}^{\infty}$ at time t. The regulator must maximise

$$W_t=\beta\sum_{i=1}^{\infty}\delta^{i}\left[(a-c)x_{t+i}-\frac{b}{2}x_{t+i}^{2}-dk_{t+i}^{2}\right]+\left[(a-c)x_t-\frac{b}{2}x_t^{2}-dk_t^{2}\right] \tag{18}$$

where $k_t=\theta k_{t-1}+x_{t-1}$ describes the evolution of the state variable, δ is the discount rate and β is the quasi-hyperbolic modifier on the future discount rate.

Notating current instantaneous welfare in an abbreviaNotating current instantaneous welfare in an abbreviated fashion as $w(x_t,k_t)$ allows the NPV of total future welfare to be written as

$$W_0=w(x_0,k_0)+\beta\delta w(x_1,\theta k_0+x_0)+\beta\delta^{2}w(x_2,x_1+\theta x_0+\theta^{2}k_0)+\dots \tag{19}$$

Maximising this requires taking first-order conditionss with respect to $x_t\forall t\in[0,\infty)$. Derivatives are notated as usual with the time superscript taken from the time subscript of the state variable.

$$\frac{\partial W_0}{\partial x_0}=w_1^0+\beta\delta w_2^1+\theta\beta\delta^2 w_2^2+\ldots=0 \tag{20}$$

$$\frac{\partial W_0}{\partial x_1}=\beta\delta w_1^1+\theta\beta\delta^2 w_2^2+\theta\beta\delta^3 w_2^3+\ldots=0 \tag{21}$$

$$\frac{\partial W_0}{\partial x_2}=\beta\delta^2 w_1^2+\theta\beta\delta^3 w_2^3+\theta\beta\delta^4 w_2^4+\ldots=0 \tag{22}$$

$$\vdots \tag{23}$$

Compacting these conditions, (40)−(41)×θ gives

$$w_1^0+\beta\delta\left(w_2^1-\theta w_1^1\right)=0, \tag{24}$$

and ((43)−(45)×θ)/$\beta\delta$ gives

$$w_1^t+\delta\left(w_2^t-\theta w_1^{t+1}\right)=0\qquad\forall\ t\geq 2. \tag{25}$$

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$$W_0=w(x_0,k_0)+\beta\delta w(x_1,\theta k_0+x_0)+\beta\delta^{2}w(x_2,x_1+\theta x_0+\theta^{2}k_0)+\dots \tag{26}$$

Maximising this requires taking first-order conditionss with respect to $x_t \forall t \in [0, \infty)$. Derivatives are notated as usual with the time superscript taken from the time subscript of the state variable.

$$\frac{\partial W_0}{\partial x_0} = w_1^0 + \beta\delta w_2^1 + \theta\beta\delta^2 w_2^2 + \ldots = 0 \tag{27}$$

$$\frac{\partial W_0}{\partial x_1} = \beta\delta w_1^1 + \theta\beta\delta^2 w_2^2 + \theta\beta\delta^3 w_2^3 + \ldots = 0 \tag{28}$$

$$\frac{\partial W_0}{\partial x_2} = \beta\delta^2 w_1^2 + \theta\beta\delta^3 w_2^3 + \theta\beta\delta^4 w_2^4 + \ldots = 0 \tag{29}$$

$$\vdots \tag{30}$$

Compacting these conditions, (18)−(19)×θ gives

$$w_1^0 + \beta\delta\left(w_2^1 - \theta w_1^1\right) = 0, \tag{31}$$

and ((20)−(22)×θ)/$\beta\delta$ gives

$$w_1^t + \delta\left(w_2^t - \theta w_1^{t+1}\right) = 0 \qquad \forall\, t \geq 2. \tag{32}$$

4. Regulation without precommitment

Now suppose that the regulator can still directly choose output but is no longer able to precommit to future output decisions. The problem must be formulated recursively in order to solve for a time consistent output path.

The problem will first be solved with reduced form notation and then functional forms can later be substituted in.

Let the MPE strategy of the regulator be $x_t = f(k_t)$. Then his current period value function is

$$U(k_t) = \max_{x_t} \left\{w(x_t, k_t) + \beta\delta V(\theta k_t + x_t)\right\}. \tag{33}$$

From period $t+1$ onward the value function is $V(k_t)$.

$$V(k_t) = w\left(f(k_t), k_t\right) + \delta V\left(\theta k_t + f(k_t)\right). \tag{34}$$

The current period's FOC is

$$w_1^t + \beta\delta V_1^{t+1} = 0 \tag{35}$$

$$\therefore V_1^{t+1} = -\frac{w_1^t}{\beta\delta}. \tag{36}$$

Differentiating (25) gives

$$V_1^t = w_1^t f_1^t + w_2^t + \delta V_1^{t+1}(\theta f_1^t) \tag{37}$$

and substituting in equation (27) gives the reduced form Euler-Lagrange equation:

$$w_1^t + \beta\delta(w_1^{t+1} f_1^{t+1} + w_2^{t+1}) - \delta(\theta + f_1^{t+1}) w_1^{t+1} = 0. \tag{38}$$

5. Regulation with delegation

The delegation game involves the regulator setting a tax rate for pollution simultaneously with the monopolist's choice of output in each period. Both the tax and the choice of output are feedback strategies. We consider only a linear tax, however there are two possible ways to levy it. It can be levied on either emissions or upon the stock of pollution.

The case of a tax on emissions will be considered first and then the problem with a pollution tax will be solved.

5.1. **The welfare function.**

The regulator's problem changes for two reasons: first, because he gains revenue from taxation and, secondly, because we introduce a cost to changing the tax rate. Economists are often criticised by policymakers for excluding the costs of implementation when they recommend taxes. Here, we explicitly include the costs of implementing and modifying tax schemes in the regulator's welfare function.

Suppose that the tax is levied on emissions, the value of the tax revenue to the regulator is γ, and the the cost of changing policies is $\kappa(\tau_t - \tau_{t-1})^2$, where τ_t is the period t tax rate. Then the welfare function becomes

$$w_t = (a-c)x_t - bx_t^2 + (\gamma - 1)\tau_t x_t + (a - p^*)\frac{x_t^*}{2} - dk_t^2 - \kappa(\tau_t - \tau_{t-1})^2 \tag{39}$$

$$= \omega(x_t, k_t) + (\gamma - 1)\tau_t x_t - \kappa\rho(\tau_t, \tau_{t-1}). \tag{40}$$

Note that, if $\gamma = 1$ then the tax is a simple transfer from the monopolist to the consumers and doesn't directly affect the welfare function. Similarly, if $\kappa = 0$ then there are no costs of policy adjustment.

5.2. **The profit function.** With taxation the monopolist's instantaneous profit becomes

$$\pi_t = (a - c - \tau_t)x_t - bx_t^2. \tag{41}$$

5.3. **The game.** Let the MPE strategy of the monopolist be $x_t = h(\tau_{t-1}, k_t)$ and the MPE strategy of the regulator be $\tau_t = g(\tau_{t-1}, k_t)$.

5.3.1. *Regulator.* Current period value function:

$$U(\tau_{t-1}, k_t) = \max_{\tau_t} \Big\{ \omega\big(h(\tau_{t-1}, k_t), k_t\big) + (\gamma - 1)\tau_t x_t - \kappa\rho(\tau_t, \tau_{t-1}) + \beta\delta V\big(\tau_t, \theta k_t + h(\tau_{t-1}, k_t)\big) \Big\}. \tag{42}$$

Continuation value function:

$$V(\tau_{t-1}, k_t) = \omega\left(h(\tau_{t-1}, k_t), k_t\right) + (\gamma - 1)g(\tau_{t-1}, k_t)x_t - \kappa\rho(\tau_t, \tau_{t-1}) + \delta V\left(g(\tau_{t-1}, k_t), \theta k_t + h(\tau_{t-1}, k_t)\right). \tag{43}$$

FOC:

$$(\gamma - 1)x_t + \kappa\rho_1^t + \beta\delta V_1^{t+1} = 0 \tag{44}$$

$$V_1^{t+1} = -\frac{(\gamma - 1)x_t + \kappa\rho_1^t}{\beta\delta}. \tag{45}$$

Envelope conditions from (34):

$$V_1^t = \omega_1^t h_1^t + (\gamma - 1)x_t g_1^t - \kappa\rho_2^t + \delta V_1^{t+1} g_1^t + \delta V_2^{t+1} h_1^t \tag{46}$$

$$V_2^t = \omega_2^t + \omega_1^t h_2^t + (\gamma - 1)x_t g_2^t + \delta V_1^{t+1} g_2^t + \delta V_2^{t+1}(\theta + h_2^t). \tag{47}$$

Solving for the Euler-Lagrange equation: (36) $\rightarrow$ (37) gives

$$\omega_1^t h_1^t + (\gamma - 1)x_t g_1^t - \kappa\rho_2^t + \delta\left(V_2^{t+1} h_1^t - g_1^t \frac{(\gamma - 1)x_t + \kappa\rho_1^t}{\beta\delta}\right) + \frac{(\gamma - 1)x_{t-1} + \kappa\rho_1^{t-1}}{\beta\delta} = 0 \tag{48}$$

$$\therefore V_2^{t+1} = \frac{g_1^t(\kappa\rho_1^t + (\gamma - 1)x_t)}{\beta\delta h_1^t} - \frac{\kappa\rho_1^{t-1} + (\gamma - 1)x_{t-1}}{\beta\delta h_1^t} + \frac{\kappa\rho_2^t - (\gamma - 1)x_t g_1^t}{\delta h_1^t} - \frac{\omega_1^t}{\delta}. \tag{49}$$

Now (36), (40) $\to$ (38) gives the result

$$
\begin{aligned}
(50)\quad & h_1^t\kappa\rho_2^{t-1}\beta\delta - h_1^t\omega_1^{t-1}h_1^{t-1}\beta\delta - h_1^t g_1^{t-1}\delta x^{t-1} + h_1^t x^{t-2} \\
& - \delta h_1^{t-1}h_2^t\kappa\rho_1^{t-1} - \delta^2 h_1^{t-1}\kappa\rho_2^t\beta\theta - \delta^2 h_1^{t-1}h_2^t\kappa\rho_2^t\beta + \delta^2 h_1^{t-1}h_2^t g_1^t x^t \\
& + h_1^t g_1^{t-1}x^{t-1}\beta\delta - \delta^2 h_1^{t-1}h_2^t g_1^t x^t\beta + \delta^2 h_1^{t-1}g_1^t\kappa\rho_1^t\theta + \delta^2 h_1^{t-1}h_2^t g_1^t\kappa\rho_1^t + \delta^2 h_1^{t-1}g_1^t x^t\theta \\
& + \delta^2 h_1^{t-1}g_1^t x^t\beta\gamma h_2^t - \delta^2 h_1^{t-1}g_1^t x^t\gamma h_2^t - h_1^t g_1^{t-1}x^{t-1}\beta\delta\gamma + h_1^t\kappa\rho_1^{t-2} - h_1^t g_1^{t-1}\delta\kappa\rho_1^{t-1} \\
& + g_2^t h_1^t\delta^2 h_1^{t-1}x^t\gamma - x^t g_2^t\beta h_1^t\delta^2 h_1^{t-1}\gamma + \delta h_1^{t-1}x^{t-1}\gamma\theta + \delta h_1^{t-1}x^{t-1}\gamma h_2^t - \delta^2 h_1^{t-1}g_1^t x^t\gamma\theta \\
& + \delta^2 h_1^{t-1}g_1^t x^t\beta\gamma\theta + h_1^t g_1^{t-1}\delta x^{t-1}\gamma - g_2^t h_1^t\delta^2 h_1^{t-1}\kappa\rho_1^t - \omega_2^t\beta h_1^t\delta^2 h_1^{t-1} + x^t g_2^t\beta h_1^t\delta^2 h_1^{t-1} \\
& - \delta h_1^{t-1}\kappa\rho_1^{t-1}\theta - g_2^t h_1^t\delta^2 h_1^{t-1}x^t + \delta^2 h_1^{t-1}\omega_1^t h_1^t\beta\theta - \delta^2 h_1^{t-1}g_1^t x^t\beta\theta - \delta h_1^{t-1}h_2^t x^{t-1} \\
& - \delta h_1^{t-1}x^{t-1}\theta - h_1^t x^{t-2}\gamma\beta h_1^t\delta^2 h_1^{t-1} = 0
\end{aligned}
$$

Note that, if $\gamma = 1, \kappa = 0$ then this simplifies to

$$(51)\qquad \textit{something}$$

5.3.2. *Monopolist.* Since the monopolist discounts exponentially, he has a stationary value function:

$$(52)\qquad \Pi(\tau_{t-1}, k_t) = \max_{x_t}\left\{\pi(x_t, g(\tau_{t-1}, k_t)) + \delta\Pi\left(g(\tau_{t-1}, k_t), \theta k_t + x_t\right)\right\}.$$

FOC:

$$(53)\qquad \pi_1^t + \delta\Pi_2^{t+1} = 0$$

$$(54)\qquad \therefore \Pi_2^{t+1} = -\frac{\pi_1^t}{\delta}.$$

Envelope conditions:

$$(55)\qquad \Pi_1^t = \pi_2^t g_1^t + \delta\Pi_1^{t+1}g_1^t$$

$$(56)\qquad \Pi_2^t = \pi_2^t g_2^t + \delta\Pi_1^{t+1}g_2^t + \theta\delta\Pi_2^{t+1}.$$

Solving for the Euler equation: (45)$\to$(47) gives

$$(57)\qquad -\frac{\pi_1^{t-1}}{\delta} = \pi_2^t g_2^t - \theta\pi_1^t + \delta\Pi_1^{t+1}g_2^t$$

$$(58)\qquad \Pi_1^{t+1} = \frac{\theta\pi_1^t - \pi_2^t g_2^t - \frac{1}{\delta}\pi_1^{t-1}}{\delta g_2^t}.$$

Now, (49)→(46) yields the Euler-Lagrange equation:

$$\pi_2^t g_1^t + \frac{g_1^t}{g_2^t}\left(\theta\pi_1^t - \pi_2^t g_2^t - \frac{1}{\delta}\pi_1^{t-1}\right) - \frac{\theta\pi_1^{t-1} - \pi_2^{t-1} g_2^{t-1} - \frac{1}{\delta}\pi_1^{t-2}}{\delta g_2^{t-1}} = 0 \tag{59}$$

5.4. **Taxation of the pollution stock.**

5.4.1. *Regulator.* The instantaneous welfare function becomes

$$w_t = (a-c)x_t - bx_t^2 + (\gamma - 1)\tau_t k_t + (a - p^*)\frac{x_t^*}{2} - dk_t - \kappa(\tau_t - \tau_{t-1})^2 \tag{60}$$

$$= w(x_t, \tau_t, \tau_{t-1}, k_t). \tag{61}$$

Since the reduced form of the welfare function is the same, the reduced form Euler equation wil be the same as equation (41).

5.4.2. *Monopolist.* The instantaneous profit function becomes

$$\pi_t = (a-c)x_t - bx_t^2 - \tau_t k_t \tag{62}$$

$$= \pi(x_t, \tau_t, k_t). \tag{63}$$

The Bellman function is then

$$\Pi(\tau_{t-1}, k_t) = \max_{x_t}\left\{\pi\left(x_t, g(\tau_{t-1}, k_t), k_t\right) + \delta\Pi\left(g(\tau_{t-1}, k_t), \theta k_t + x_t\right)\right\}. \tag{64}$$

FOC:

$$\Pi_2^{t+1} = -\frac{\pi_1^t}{\delta}. \tag{65}$$

Envelope conditions:

$$\Pi_1^t = \pi_2^t g_1^t + \delta\Pi_1^{t+1} g_1^t \tag{66}$$

$$\Pi_2^t = \pi_2^t g_2^t + \pi_3^t + \delta\Pi_1^{t+1} g_2^t + \delta\theta\Pi_2^{t+1}. \tag{67}$$

Solving for the Euler equation: (56)→ (58) rearranges to

$$\Pi_1^{t+1} = \frac{\theta\pi_1^t - \pi_3^t - \pi_2^t g_2^t - \frac{1}{\delta}\pi_1^{t-1}}{\delta g_2^t} \tag{68}$$

and then substituting (59) into (57) gives the Euler-Lagrange equation:

$$\pi_2^t g_1^t + \frac{g_1^t}{g_2^t}\left(\theta\pi_1^t - \pi_2^t g_2^t - \pi_3^t - \frac{1}{\delta}\pi_1^{t-1}\right) - \frac{\theta\pi_1^{t-1} - \pi_2^{t-1} g_2^{t-1} - \pi_3^t - \frac{1}{\delta}\pi_1^{t-2}}{\delta g_2^{t-1}} = 0 \tag{69}$$