HYPERBOLIC DISCOUNTING MODEL: GENERAL FUNCTIONAL FORMS

1. Elements of the model

1.1. **Consumer.** Imagine a representative consumer with quasi-linear preferences across two goods: a polluting good, x, and a numeraire, m. The consumer's instantaneous utility function is

$$U(x, m) = u(x) + m. \tag{1}$$

The function $u(x)$ must satisfy the Inada conditions in order for the inverse demand to be defined over all possible values of x.**[does this need a proof?]**

The consumer's problem is then

$$\max_x u(x) + m \tag{2}$$

$$\text{s.t. } px + m \leq I. \tag{3}$$

The FOC of this problem is

$$u'(x) = p \tag{4}$$

so inverse demand in this market is

$$p = u'(x). \tag{5}$$

1.2. **Monopolist.** The manufacturer of good x faces costs $C(x)$ and monopolises the market. His instantaneous profit function is

$$\pi = R(x) - C(x) \tag{6}$$

$$= u'(x).x - C(x). \tag{7}$$

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1.3. **Regulator.** The regulator recognises the pollution damage, $\phi(k_t)$, caused by the existence of the stock, k_t, of good x. The stock of the good evolves according to the rule

$$k_t = \theta k^{t-1} + x^{t-1} \qquad \theta \in [0,1]. \tag{8}$$

Instantaneous welfare is

$$w(x_t, k_t) = \pi(x_t) + CS(x_t) - \phi(k_t) \tag{9}$$

$$= u'(x_t).x_t - C(x_t) + \int_0^{x_t} u'(x)\, dx - u'(x_t).x_t - \phi(k_t) \tag{10}$$

$$= u(x_t) - C(x_t) - \phi(k_t) \tag{11}$$

The regulator suffers from time inconsistency and is modelled as having quasi-hyperbolic preferences with $\beta\delta$-discounting. As a result, the regulator's net present valuation of welfare is

$$W(x,k) = w(x_t, k_t) + \beta \sum_{i=1}^{\infty} \delta^i w(x_{t+i}, k_{t+i}). \tag{12}$$

2. Laissez Faire equilibrium

Suppose that the monopolist acted unregulated. He then solves a static problem in each period:

$$\max_x u'(x).x - C(x). \tag{13}$$

The FOC of this problem is

$$u''\left(x^\ell\right).x^\ell + u'\left(x^\ell\right) - C'\left(x^\ell\right) = 0 \tag{14}$$

where x^ℓ denotes the laissez-faire level of output chosen by the monopolist. This shows the point at which marginal profit is equal to zero.

3. Regulation with precommitment

Suppose that the regulator can directly choose $\{x_\tau\}_{\tau=t}^{\infty}$ at time t. The regulator must maximise

$$W_t(x,k) = \beta \sum_{i=1}^{\infty} \delta^i \left[u(x_{t+i}) - C(x_{t+i}) - \phi(k_{t+i})\right] + \left[u(x_t) - C(x_t) - \phi(k_t)\right] \tag{15}$$

where δ is the discount rate and β is the quasi-hyperbolic modifier on the future discount rate.

This can be notated more simply as

$$W_0 = w(x_0, k_0) + \beta\delta w(x_1, \theta k_0 + x_0) + \beta\delta^2 w(x_2, x_1 + \theta x_0 + \theta^2 k_0) + \dots \tag{16}$$

Maximising this requires taking first-order conditionss with respect to $x_t \forall t \in [0, \infty)$. Derivatives are notated as usual with the time superscript taken from the time subscript of the state variable.

$$\frac{\partial W_0}{\partial x_0} = w_1^0 + \beta\delta w_2^1 + \theta\beta\delta^2 w_2^2 + \dots = 0 \tag{17}$$

$$\frac{\partial W_0}{\partial x_1} = \beta\delta w_1^1 + \theta\beta\delta^2 w_2^2 + \theta\beta\delta^3 w_2^3 + \dots = 0 \tag{18}$$

$$\frac{\partial W_0}{\partial x_2} = \beta\delta^2 w_1^2 + \theta\beta\delta^3 w_2^3 + \theta\beta\delta^4 w_2^4 + \dots = 0 \tag{19}$$

$$\vdots \tag{20}$$

Compacting these conditions, (17)−(18)×θ gives

$$w_1^0 + \beta\delta \left(w_2^1 - \theta w_1^1 \right) = 0, \tag{21}$$

and ((19)−(21)×θ)/$\beta\delta$ gives

$$w_1^t + \delta \left(w_2^{t+1} - \theta w_1^{t+1} \right) = 0 \qquad \forall\, t \geq 2 \tag{22}$$

which expands to

$$u'(x_t) - C'(x_t) - \delta\theta \left(u'(x_{t+1}) - C'(x_{t+1}) \right) - \delta\phi'(k_{t+1}) = 0 \qquad \forall\, t \geq 2 \tag{23}$$

3.1. **Comparison to laissez-faire outcome.** At this point it is important to see how the first-best outcome compares to the laissez-faire outcome previously derived (equation (14)). The laissez faire equilibrium is characterised by equation (14):

$$u'\left(x^\ell\right) - C'\left(x^\ell\right) + u''\left(x^\ell\right) x^\ell = 0. \tag{24}$$

This equates the monopolist's marginal profit to zero. Since profit is concave, marginal profit is a decreasing function.

In order to tractably compare this equilibrium condition to the first best outcome it is necessary to focus upon the steady state of the dynamic game. Suppose that $x_1 = x_2 = \ldots = x_T = \bar{x}$, and similarly for the state variable, k_t. Now rearrange equation (23)

$$u'\left(\bar{x}\right) - C'\left(\bar{x}\right) = \frac{\delta\phi'\left(\bar{k}\right)}{1 - \delta\theta} \tag{25}$$

and add $u''(\bar{x})\,\bar{x}$ to both sides:

$$u'(\bar{x}) - C'(\bar{x}) + u''(\bar{x})\,\bar{x} = \frac{\delta\phi'\left(\bar{k}\right)}{1-\delta\theta} + u''(\bar{x})\,\bar{x}. \tag{26}$$

Remembering that marginal profit is a decreasing function, if $\frac{\delta\phi'\left(\bar{k}\right)}{1-\delta\theta} + u''(\bar{x})\,\bar{x} > 0$ then $\bar{x} < x^{\ell}$ and vice versa.

Signing the component parts gives

$$\delta\phi'(\bar{x}) > 0 \tag{27}$$

$$1 - \beta\delta > 0 \tag{28}$$

by assumption, and

$$u''(\bar{x})\,\bar{x} < 0 \tag{29}$$

because $u(\cdot)$ exhibits diminishing marginal utility. So if

$$\frac{\delta\phi'\left(\bar{k}\right)}{1-\delta\theta} > u''(\bar{x})\,\bar{x} \tag{30}$$

then $\bar{x} < x^{\ell}$. The left hand component of the inequality shows the cost of the externality, while the right hand component shows the deadweight loss due to the exercise of monopoly power. The externality effect causes the quantity produced to be too high, while the firm's market power depresses production. Regulation to diminish pollution is only worthwhile when the former outweighs the latter. Henceforth, we shall assume that equation (30) holds.

4. Regulation without precommitment

Now suppose that the regulator can still directly choose output but is no longer able to precommit to future output decisions. The problem must be formulated recursively in order to solve for a time consistent output path.

The problem will first be solved with reduced form notation and then functional forms can later be substituted in.

Let the MPE strategy of the regulator be $x_t = f(k_t)$. Then his current period value function is

$$U(k_t) = \max_{x_t}\left\{w(x_t, k_t) + \beta\delta V(\theta k_t + x_t)\right\}. \tag{31}$$

From period $t+1$ onward the value function is $V(k_t)$.

$$V(k_t) = w\left(f(k_t), k_t\right) + \delta V\left(\theta k_t + f(k_t)\right). \tag{32}$$

The current period's FOC is

$$w_1^t + \beta\delta V_1^{t+1} = 0 \tag{33}$$

$$\therefore V_1^{t+1} = -\frac{w_1^t}{\beta\delta}. \tag{34}$$

Differentiating (32) gives

$$V_1^t = w_1^t f_1^t + w_2^t + \delta V_1^{t+1}(\theta f_1^t) \tag{35}$$

and substituting in equation (34) gives the reduced form Euler-Lagrange equation:

$$w_1^t + \beta\delta(w_1^{t+1} f_1^{t+1} + w_2^{t+1}) - \delta(\theta + f_1^{t+1})w_1^{t+1} = 0. \tag{36}$$

This expands to give

$$\begin{aligned} u'(x_t) - C'(x_t) + \beta\delta\left(\left(u'(x_{t+1}) - C'(x_{t+1})\right) f_1^{t+1} - \phi'(k_t)\right) \\ - \delta(\theta + f_1^{t+1})\left(u'(x_{t+1}) - C'(x_{t+1})\right) = 0. \end{aligned} \tag{37}$$

It is unnecessary to compare this outcome directly to the first best **[I hope.]** Rather note that it is significantly different from the first best outcome's generalised Euler equation and it will thus produce a different price path. **[This is totally inadequate. Is this even true?!]**

5. Regulation with delegation

The delegation game involves the regulator setting a tax rate for pollution simultaneously with the monopolist's choice of output in each period. Both the tax and the choice of output are feedback strategies. We consider only a linear tax, however there are two possible ways to levy it. It can be levied on either emissions or upon the stock of pollution.

The case of a tax on emissions will be considered first and then the problem with a pollution tax will be solved.

5.1. **The welfare function.** The regulator's problem changes for two reasons: first, because he gains revenue from taxation and, secondly, because we introduce a cost to changing the tax rate. Economists are often criticised by policymakers for excluding the costs of implementation when they recommend taxes. Here, we explicitly include the costs of implementing and modifying tax schemes in the regulator's welfare function.

Suppose that the tax is levied on emissions, the value of the tax revenue to the regulator is γ, and the the cost of changing policies is $\kappa\rho(\tau_t, \tau_{t-1})$ [**would** $\rho\left(|\tau_t - \tau_{t-1}|\right)$ **be more appropriate and accurate?**], where τ_t is the period t tax rate. Then the welfare function becomes

$$w(x_t, k_t, \tau_t, \tau_{t-1}) = u(x_t) - C(x_t) - \phi(k_t) + (\gamma - 1)\tau_t x_t - \kappa\rho(\tau_t, \tau_{t-1}). \tag{38}$$

[**what conditions should I impose on** ρ] Note that, if $\gamma = 1$ then the tax is a simple transfer from the monopolist to the consumers and doesn't directly affect the welfare function. Similarly, if $\kappa = 0$ then there are no costs of policy adjustment.

5.2. **The profit function.** With taxation the monopolist's instantaneous profit becomes

$$u'(x).x - C(x) - \tau_t x_t. \tag{39}$$

5.3. **The game.** Let the MPE strategy of the monopolist be $x_t = h(\tau_{t-1}, k_t)$ and the MPE strategy of the regulator be $\tau_t = g(\tau_{t-1}, k_t)$.

5.3.1. *Regulator.* Current period value function:

$$U(\tau_{t-1}, k_t) = \max_{\tau_t}\left\{w\big(h(\tau_{t-1}, k_t), k_t, \tau_t, \tau_{t-1}\big) + \beta\delta V\big(\tau_t, \theta k_t + h(\tau_{t-1}, k_t)\big)\right\} \tag{40}$$

Continuation value function:

$$\begin{aligned} V(\tau_{t-1}, k_t) = w\big(h(\tau_{t-1}, k_t), k_t, g(\tau_{t-1}, k_t), \tau_{t-1}\big) \\ + \delta V\big(g(\tau_{t-1}, k_t), \theta k_t + h(\tau_{t-1}, k_t)\big) \end{aligned} \tag{41}$$

FOC:

$$w_3^t + \beta\delta V_1^{t+1} = 0 \tag{42}$$

$$\therefore V_1^{t+1} = -\frac{w_3^t}{\beta\delta} \tag{43}$$

Envelope conditions:

$$V_1^t = w_1^t h_1^t + w_3^t g_1^t + w_4^t + \delta\left[V_1^{t+1} g_1^t + V_2^{t+1} h_1^t\right] \tag{44}$$

$$V_2^t = w_1^t h_2^t + w_2^t + w_3^t g_2^t + \delta\left[V_1^{t+1} g_2^t + V_2^{t+1}\left(\theta + h_2^t\right)\right] \tag{45}$$

Now substituting (43) into (44) gives

$$w_1^t h_1^t + w_3^t g_1^t + w_4^t - \frac{w_3^t g_1^t}{\beta} + \delta V_2^{t+1} h_1^t + \frac{w_3^{t-1}}{\beta\delta} = 0 \tag{46}$$

$$\therefore V_2^{t+1} = \frac{w_3^t g_1^t}{\beta\delta h_1^t} - \frac{w_1^t h_1^t + w_3^t g_1^t + w_4^t}{\delta h_1^t} - \frac{w_3^{t-1}}{\beta\delta^2 h_1^t} \tag{47}$$

and (47), (44)→ (45):

$$\begin{aligned} &\frac{w_3^{t-1} g_1^{t-1} - \beta\left(w_1^{t-1} h_1^{t-1} + w_3^{t-1} g_1^{t-1} + w_4^{t-1}\right)}{\delta h_1^{t-1}} \\ &\qquad - \frac{w_3^{t-1-1}}{\delta^2 h_1^{t-1}} = \beta(w_1^t h_2^t + w_2^t + w_3^t g_2^t) - g_2^t w_3^t \\ &\qquad\qquad + \frac{(\theta + h_2^t)}{h_1^t}\left[w_3^t g_1^t - \beta\left(w_1^t h_1^t + w_3^t g_1^t + w_4^t\right) - \frac{w_3^{t-1}}{\delta}\right]. \end{aligned} \tag{48}$$

In the special case where $\gamma = 1, \kappa = 0$ this can be simplified by substituting in the partial derivatives of the welfare function.

5.3.2. *Monopolist.* Since the monopolist discounts exponentially, he has a stationary value function: