Unit I
Electromagnetic theory

What do we got here??????????????????

- Scalar and vectors fields, concept of gradient, divergence and curl, dielectric constant, Gauss theorem and Stokes theorem (qualitative).
- Poisson and Laplace equations, continuity equation.
- Maxwell electromagnetic equations (differential and integral forms)
- Physical significance of Maxwell equations.
- Ampere Circuital Law, Maxwell displacement current and correction in Ampere Circuital Law.

1. Scalar and vectors fields, concept of gradient, divergence and curl, dielectric constant, Gauss theorem and Stokes theorem (qualitative).

Outline:

- ❑ Scalar and Vectors
- ❑ Dot product and cross product
- ❑ Unit vectors
- ❑ Scalar field
- ❑ Vector field
- ❑ Ordinary derivative
- ❑ Introduction to the ‘del’ operator or ‘nabla’ operator

Introduction to Scalar and Vectors

Scalar: Scalar is a quantity which can be expressed by a single number representing its **magnitude.**
Examples: mass, density and temperature.

Vector: Vector is a quantity which is specified by **both magnitude and direction**.
Examples: Force, Velocity and Displacement.

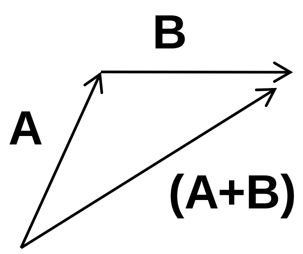
Basics about vectors

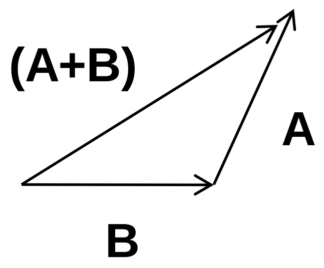
Four vector operations

(a)Addition of two vectors
(b) Multiplication by a scalar
(c)Dot product of two vectors
(d)Cross product of two vectors

Addition of two vectors

(a)This rule generalizes the obvious procedure for combining two displacements.





e.g: **three miles east followed by 4 miles north gets you to the same place as 4 miles north followed by 3 miles east.**

Addition is *commutative*. i.e. **A+B = B+A**
Addition is also *associative i.e.* **(A+B)+C= A+(B+C)**

Multiplication by a scalar

Multiplication of a vector by a positive scalar '*m*' multiplies the magnitude but leaves the direction unchanged.
(If 'm' is negative, the direction is reversed).

Scalar multiplication is distributive : m (**A**+**B**)= m**A** + m**B**

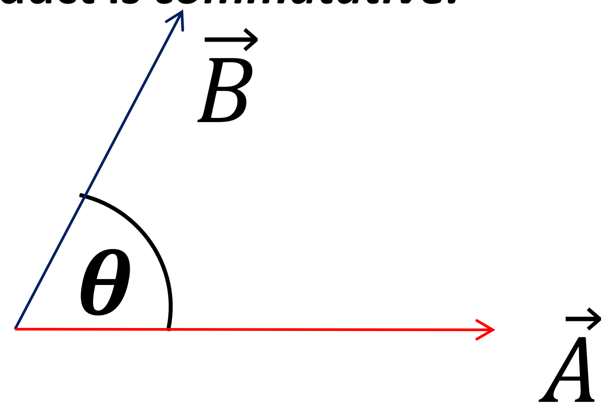
Dot product of two vectors (Scalar Product)

The dot product of two vectors is defined by

$\vec{A}\,.\,\vec{B}=|\vec{A}||\vec{B}|\cos\theta$**, where θ is the angle they form when placed tail-to-tail.**

Note that $|\vec{A}|$ *and* $|\vec{B}|$ **are scalar quantities. The dot product is *commutative.***

$\vec{A}\,.\,\vec{B}=\vec{B}\,.\,\vec{A}$



As well as *distributive*

$\vec{A}.(\vec{B}+\vec{C})=\vec{A}\,.\,\vec{B}+\vec{A}\,.\,\vec{C}$

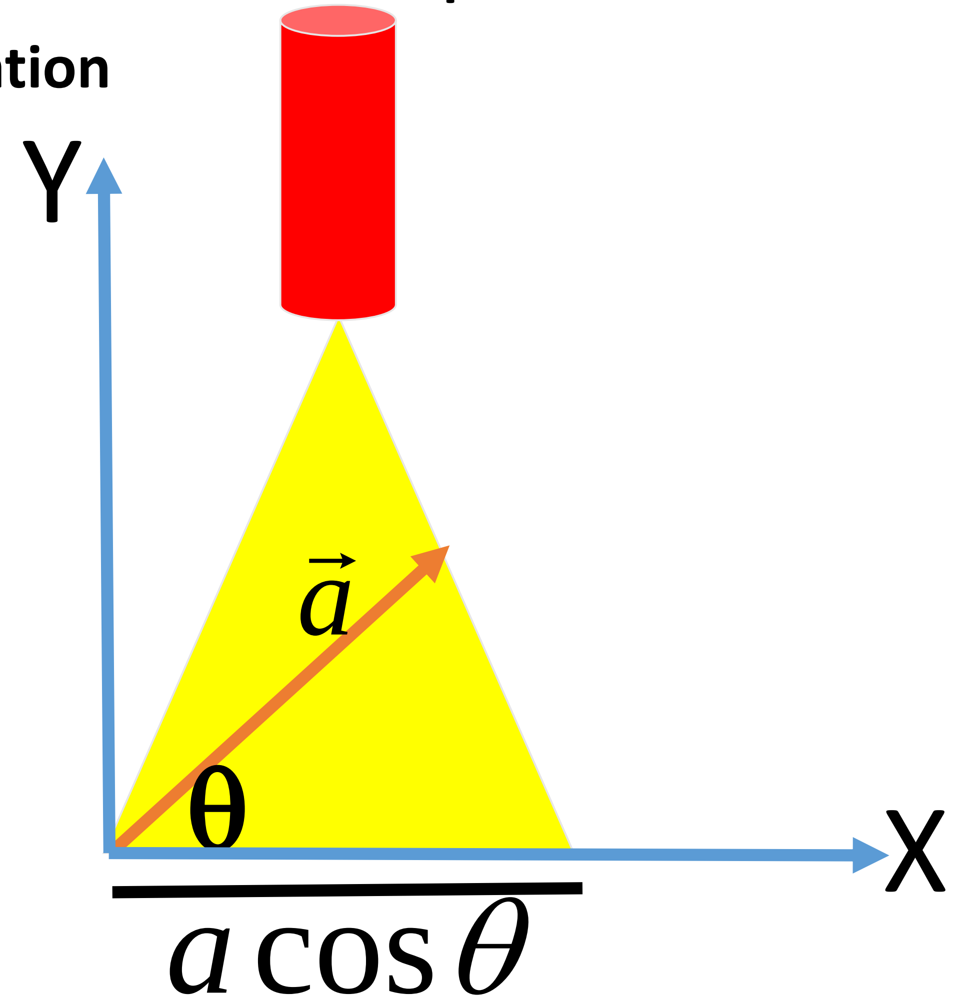
Geometrically $\vec{A}\,.\,\vec{B}$ **is the product of** $\vec{A}$ **times the projection of** $\vec{B}$ **along** $\vec{A}$ **(or the product of** $\vec{B}$ **times the projection of** $\vec{A}$ **along** $\vec{B}$**)**

Projection of a Vector and Vector Components

- When we want a component of a vector along a particular direction, it is useful to think of it as a projection.
- The projection always has length *a* cos q , where *a* is the length of the vector and *q* is the angle between the vector and the direction along which you want the component.
- You should know how to write a vector in unit vector notation

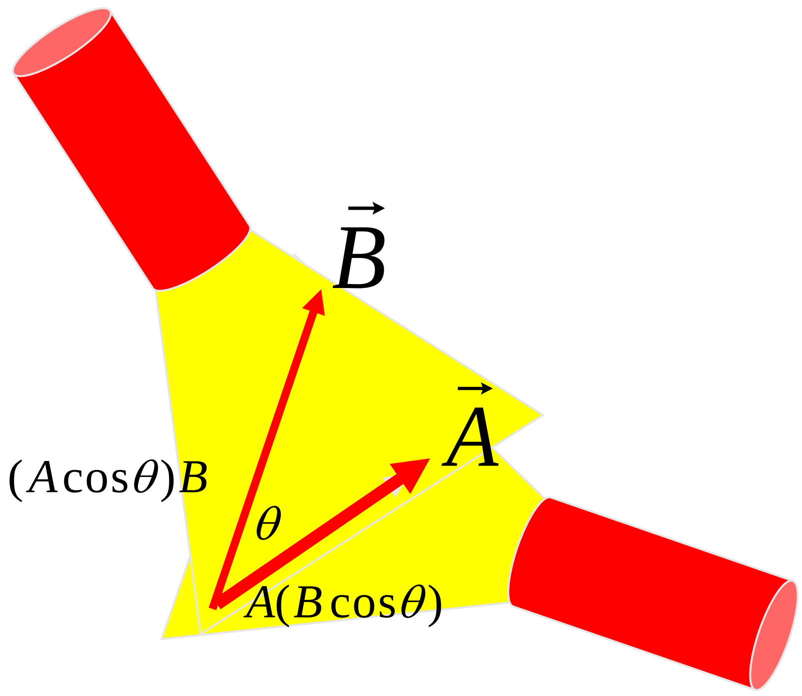
$$\vec{a} = a_x\hat{i} + a_y\hat{j} \quad \text{or} \quad \vec{a} = (a_x, a_y)$$

Just imagine a light falls on the vector a, then the length of shadow of the vector a on to the desired axis, is it's projection on that axis.



Projection of a Vector: Dot Product

- **The dot product says something about how parallel two vectors are.**
- **The dot product (scalar product) of two vectors can be thought of as the projection of one onto the direction of the other.**



- **Components**

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\vec{A}\cdot\hat{i} = A\cos\theta = A_x$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Projection of a Vector: Dot Product

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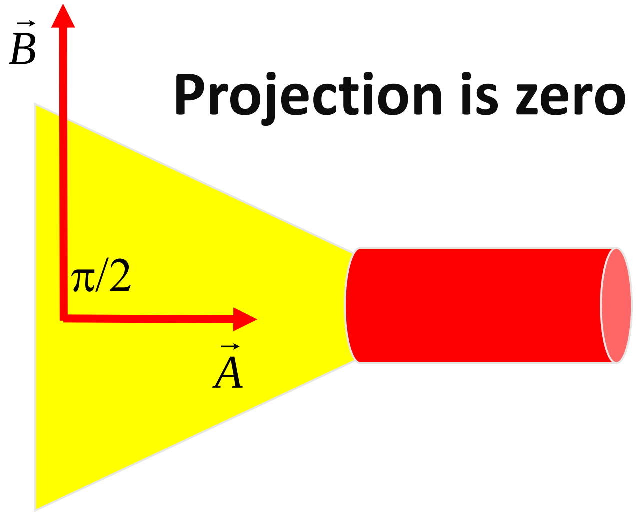


- Components

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

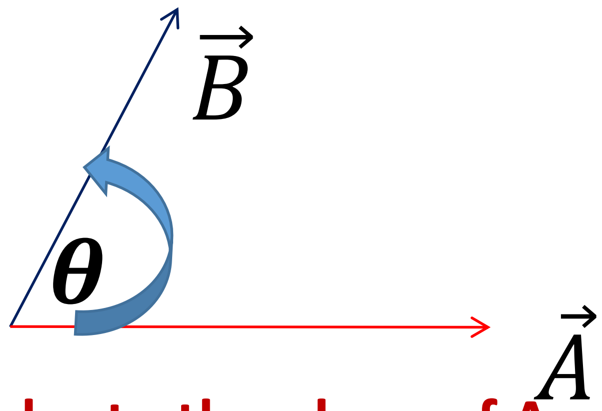
$$\vec{A}\cdot\hat{i} = A\cos\theta = A_x$$

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Cross product of two vectors

The Cross product of two vectors is defined by

$$\vec{A}\times\vec{B}\equiv AB\sin\theta\hat{n}$$



Where $\hat{n}$ is a unit vector (vector of length 1) pointing perpendicular to the plane of A and B. θ is the angle between vector A and vector B, in the direction from vector A to vector B. Please note, it is a vector product and the end result is also a vector. What about the direction? We will discuss in subsequent slide…….
In or Out of the plane ? ……….Follow Right hand rule

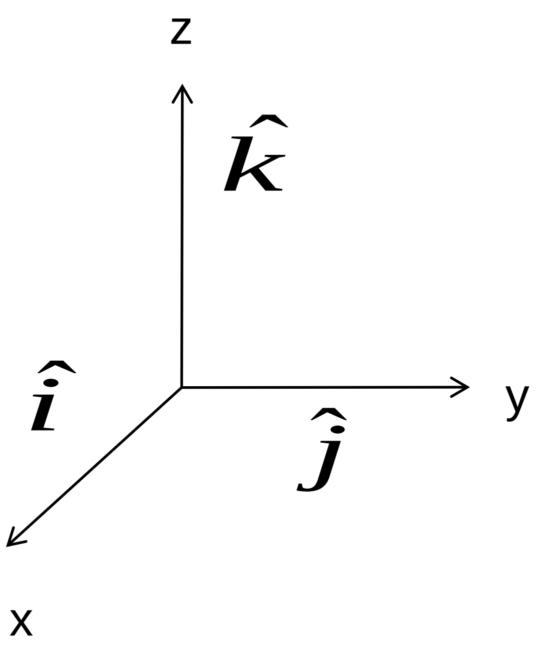
Cross product is distributive :

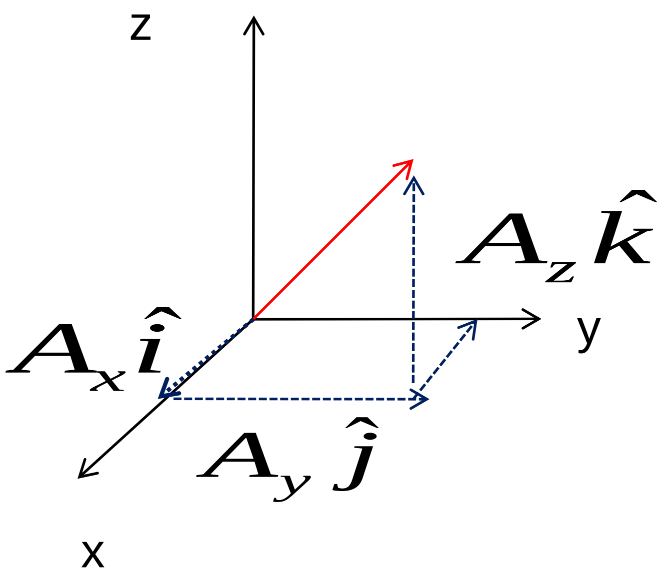
$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=(\vec{A}\times\vec{B})+(\vec{A}\times\vec{C})$$

But not Commutative: In fact

$$\left(\vec{B}\times\vec{A}\right)=-(\vec{A}\times\vec{B})$$

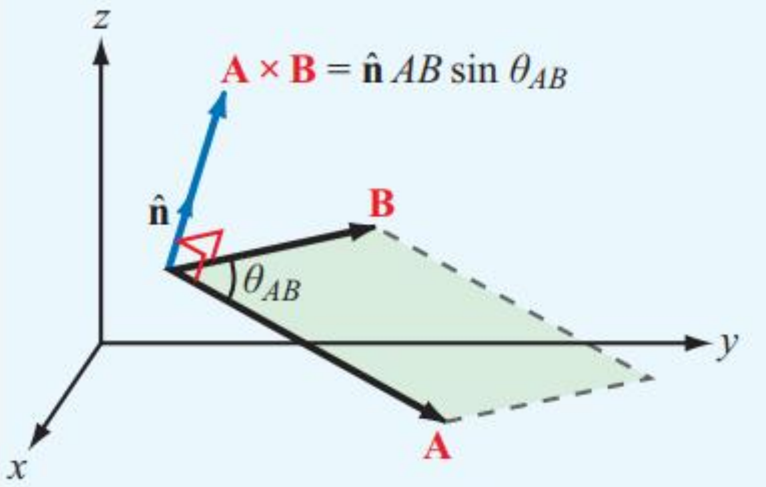
Vector algebra: component form



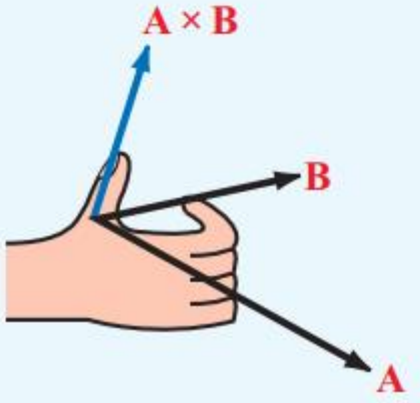


$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Ax, Ay and Az are called components of A, geometrically these are the projections of A vector along the x , y and z axis.



(a) Cross product



(b) Right-hand rule

- The cross product is *distributive*

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

- To calculate use the determinant formula

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) \\ &\quad + \hat{\mathbf{z}}(A_x B_y - A_y B_x)\end{aligned}$$

Triple Products

scalar triple product:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \quad \textit{cyclic}$$

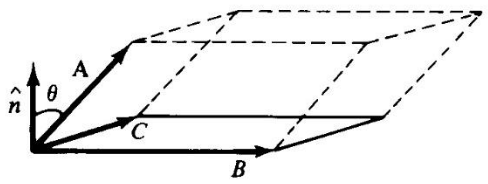
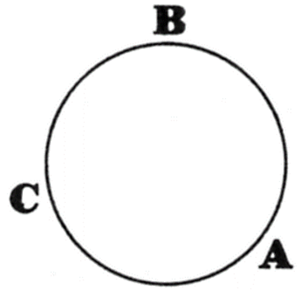


Figure 1.12



So, it does not matter where the dot and cross are.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) \equiv (\vec{A}\vec{B}\vec{C})$$

volume

$$|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})| = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

- An interchange of rows changes just the sign of a determinant.

$$\begin{aligned}\vec{A}\cdot(\vec{B}\times\vec{C}) &= (\vec{A}\times\vec{B})\cdot\vec{C} \\ &= \vec{C}\cdot(\vec{A}\times\vec{B}) \\ &= -(\vec{A}\times\vec{C})\cdot\vec{B}\end{aligned}$$

vector triple product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

bac - cab

rule

3. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

4. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$

The product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is sometimes called the *scalar triple product* or *box product* and may be denoted by $[\mathbf{ABC}]$. The product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is called the *vector triple product*.

Dot product and cross product of unit vectors

$\hat{i}\ \hat{j}\ \hat{k}$ are mutually perpendicular unit vectors along the x, y , z axes respectively

$$\Rightarrow \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$$

$$\Rightarrow \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$$

$$\Rightarrow \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\Rightarrow \hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}$$

$$\Rightarrow \hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$$

$$\Rightarrow \hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$$

Some vector operations rules

Rule 1: To add vectors , add like components.

$$\vec{A}+\vec{B}=(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})+(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})$$

$$=(A_x+B_x)\hat{i}+(A_y+B_y)\hat{j}+(A_z+B_z)\hat{k}$$

Rule 2: To multiply by a scalar, multiply each component

$$a\vec{A}=(aA_x)\hat{i}+(aA_y)\hat{j}+(aA_z)\hat{k}$$

Rule 3: To calculate the dot product, multiply like components and add.

$$\vec{A}.\vec{B}=(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}).(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})$$

$$=A_xB_x+A_yB_y+A_zB_z$$

Some vector operations rules-Continued

Rule 4: To calculate the cross product, form the determinant whose first row is i^,j^,k^ whose second row is **A**(in component form) and whose third row is **B**.

$$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

This expression can be written more easily as a determinant.

Normalize a vector

- v is represented by n-tuple ($v_1, v_2, \ldots v_n$)
- Magnitude (length): the distance from the tail to the head.

$$|v| = \sqrt{v_1^{\ 2} + v_2^{\ 2} + \ldots + v_n^{\ 2}}$$

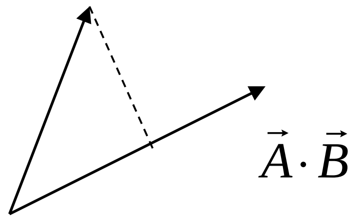
- Normalization: Scale a vector to have a unity length, unit vector,

$$\hat{\boldsymbol{v}} = \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}$$

Application of vector multiplication

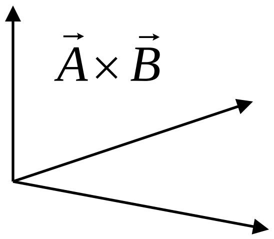
1) Dot product

$$\vec{A}\cdot\vec{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$$



2) Cross product

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}, \quad \left|\vec{A}\times\vec{B}\right| = AB\sin\theta$$



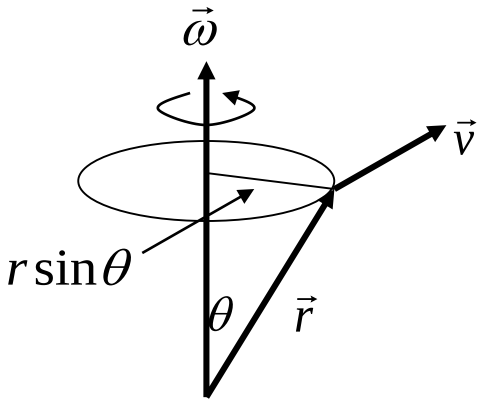
- Example

a) Work $\quad W = Fd\cos\theta = \vec{F}\cdot\vec{d}$

$$dW = \vec{F}\cdot d\vec{r}$$

b) Torque $\quad \vec{\tau} = \vec{r}\times\vec{F}$

c) Angular velocity $\quad \vec{v} = \vec{\omega}\times\vec{r}$



Vector relations in the three common coordinate systems

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\quad |\mathbf{A}| =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\quad \overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\quad \mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\quad \mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $\quad d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $\quad dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

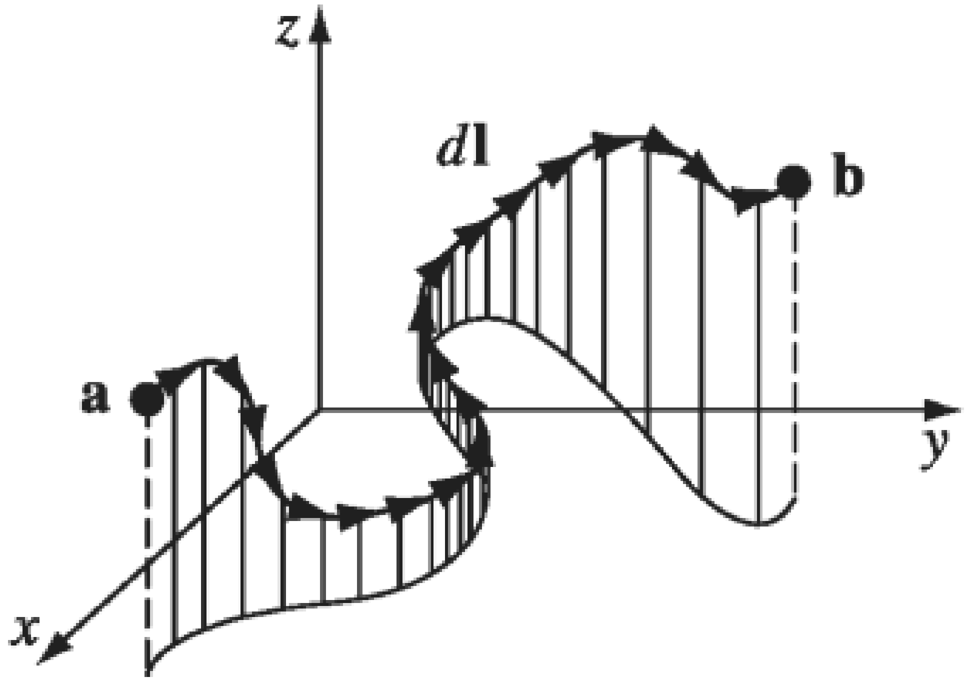
Line, Surface and volume integrals

Line Integral/Linear Integral: It is an integral expression of the form

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l},$$

Where V is a vector function and dl is the infinitesimal displacement vector and the integral is to be carried out along a prescribed path P from point A to point B. If the path in the question is a closed loop, we would put a circle in the integral sign.

$$\oint \mathbf{v} \cdot d\mathbf{l}.$$



Ordinarily, the value of a line integral depends critically on the path taken from a to b, but there is an important special class of vector functions for which the line integral is *independent* of path and is determined entirely by the end points. It will be our business in due course to characterize this special class of vectors. (A *force* that has this property is called conservative.)

Surface Integral

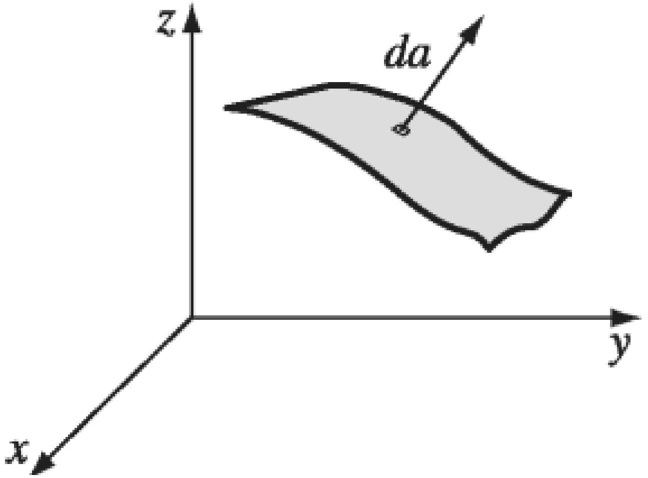
A surface integral is an expression of the form

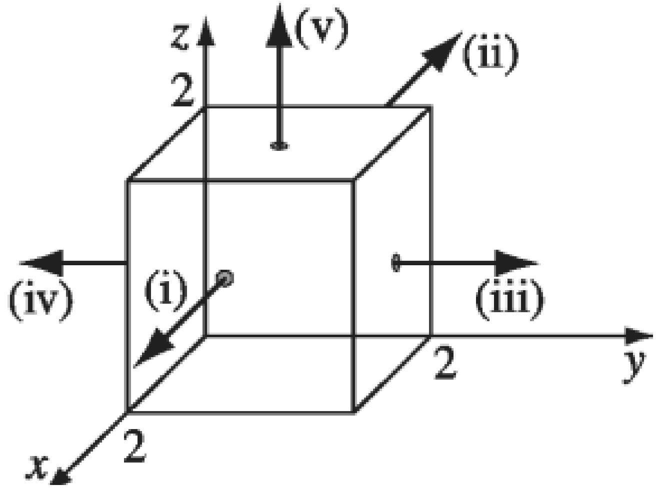
$$\int_S \mathbf{v} \cdot d\mathbf{a},$$

where v is again some vector function, and the integral is over a specified surface *S.* Here da is an infinitesimal patch of area, with direction perpendicular to the surface. There are, of course, *two* directions perpendicular to any surface, so the *sign* of a surface integral is intrinsically ambiguous. If the surface is *closed* (forming a "balloon"), in which case we would again put a circle on the integral sign,

$$\oint \mathbf{v} \cdot d\mathbf{a},$$

Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is *independent* of the surface and is determined entirely by the boundary line





Volume Integral

A volume integral is an expression of the form

$$\int_{\mathcal{V}} T\, d\tau,$$

where T is a scalar function and *dr* is an infinitesimal volume element. In Cartesian coordinates,

$$d\tau = dx\, dy\, dz.$$

For example, if *T* is the density of a substance (which might vary from point to point), then the volume integral would give the total mass. Occasionally we shall encounter volume integrals of *vector* functions:

$$\int \mathbf{v}\, d\tau = \int (v_x\, \hat{\mathbf{x}} + v_y\, \hat{\mathbf{y}} + v_z\, \hat{\mathbf{z}}) d\tau = \hat{\mathbf{x}} \int v_x d\tau + \hat{\mathbf{y}} \int v_y d\tau + \hat{\mathbf{z}} \int v_z d\tau$$

because the unit vectors (i, y, and z) are constants, they come outside the integral.

Derivation

- How do we show that $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$
- Start with
$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
- Then
$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$
$$= A_x\hat{i}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$
- But
$$\hat{i}\cdot\hat{j} = 0;\ \ \hat{i}\cdot\hat{k} = 0;\ \ \hat{j}\cdot\hat{k} = 0$$
$$\hat{i}\cdot\hat{i} = 1;\ \ \hat{j}\cdot\hat{j} = 1;\ \ \hat{k}\cdot\hat{k} = 1$$
- So
$$\vec{A}\cdot\vec{B} = A_x\hat{i}\cdot B_x\hat{i} + A_y\hat{j}\cdot B_y\hat{j} + A_z\hat{k}\cdot B_z\hat{k}$$
$$= A_x B_x + A_y B_y + A_z B_z$$

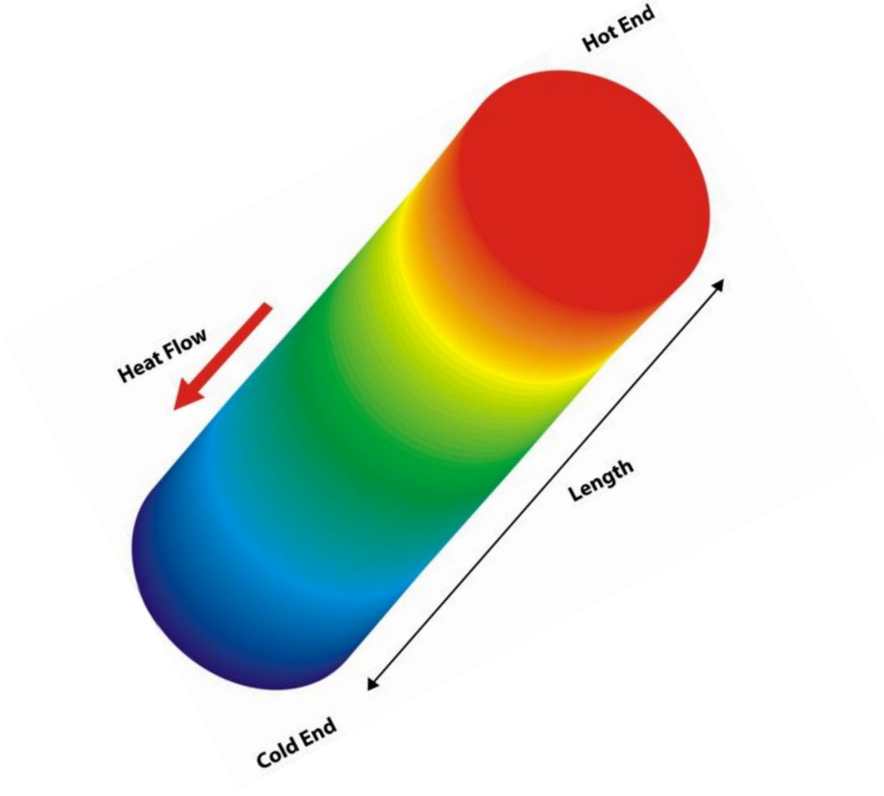
Derivation

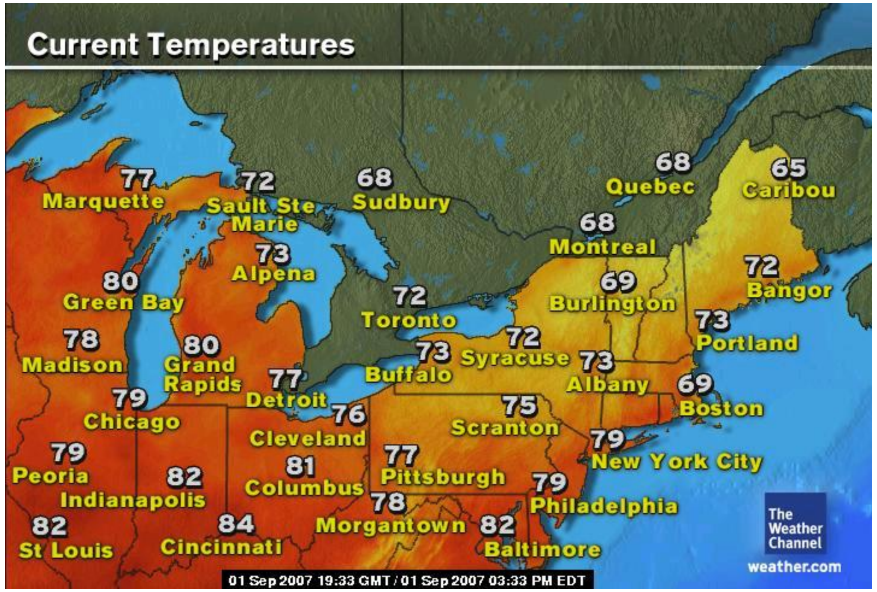
- How do we show th at $\vec{A}\times\vec{B}=(A_yB_z-A_zB_y)\hat{i}+(A_zB_x-A_xB_z)\hat{j}+(A_xB_y-A_yB_x)\hat{k}$
- Start with
$$\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$$
$$\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$$
- Then
$$\begin{aligned}\vec{A}\times\vec{B}&=(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})\\&=A_x\hat{i}\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})+A_y\hat{j}\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})+A_z\hat{k}\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})\end{aligned}$$
- But
$$\hat{i}\times\hat{j}=\hat{k};\;\;\hat{i}\times\hat{k}=-\hat{j};\;\;\hat{j}\times\hat{k}=\hat{i}$$
$$\hat{i}\times\hat{i}=0;\;\;\hat{j}\times\hat{j}=0;\;\;\hat{k}\times\hat{k}=0$$
- So
$$\begin{aligned}\vec{A}\times\vec{B}&=A_x\hat{i}\times B_y\hat{j}+A_x\hat{i}\times B_z\hat{k}+A_y\hat{j}\times B_x\hat{i}+A_y\hat{j}\times B_z\hat{k}\\&\quad+A_z\hat{k}\times B_x\hat{i}+A_z\hat{k}\times B_y\hat{j}\end{aligned}$$

Scalar Field

If at every point in a region, a scalar function has a defined value, the region is called a **scalar field**.

Example: Temperature distribution in a rod.





Or such as this temperature map

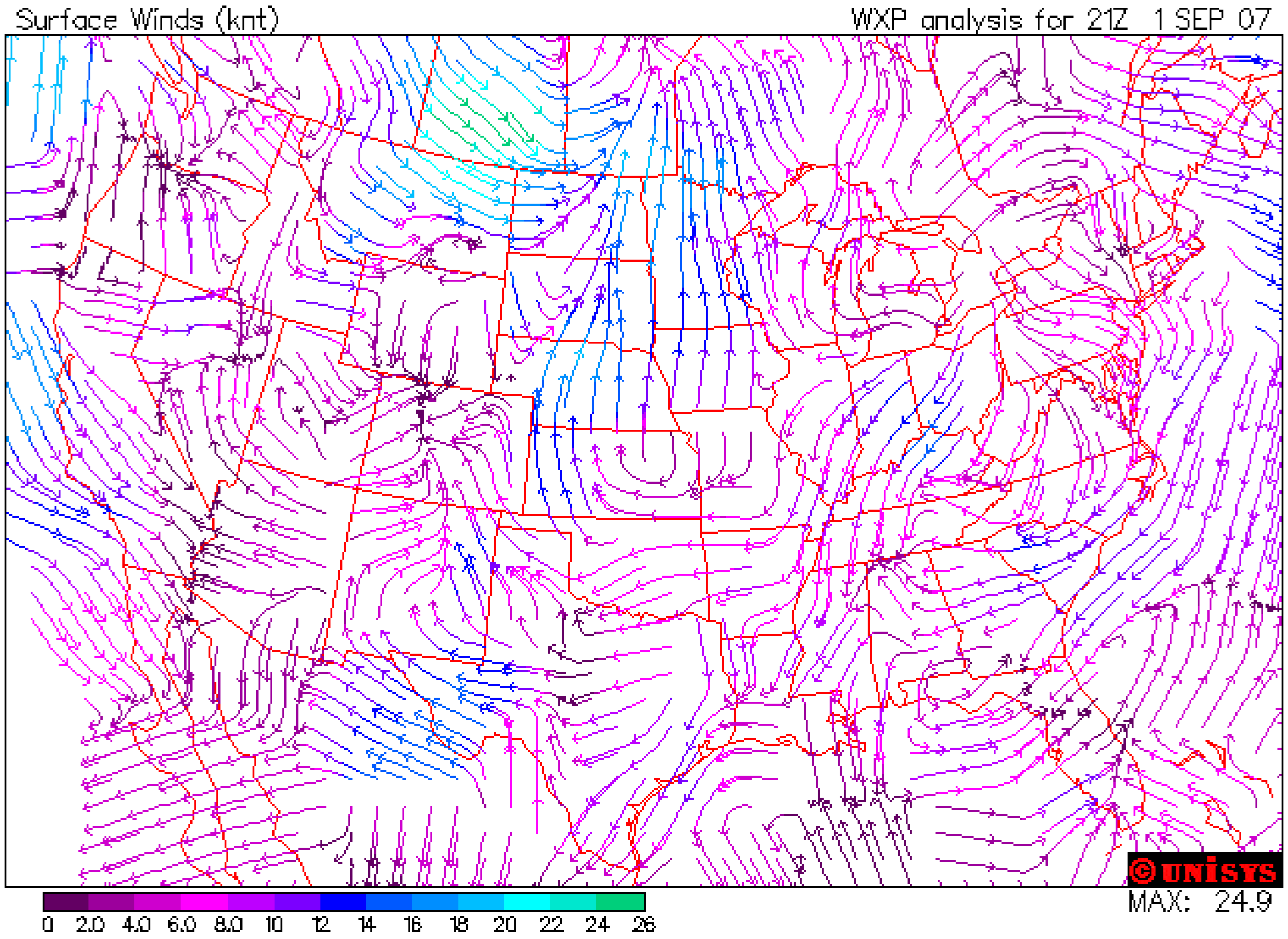
Ref: SIMUION

Vector Field

If at every point in a region, a vector function has a defined value, the region is called a **vector field**.

For example, the velocity v of the wind if varying from place to place.
(Hurricane)
Img. Ref: NASA

Another Example—(surface winds)



Ordinary derivatives

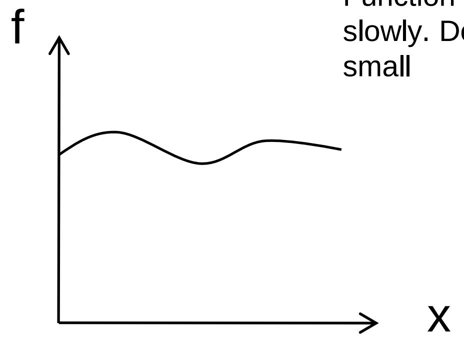
If f(x) is a function of one or more variable, then what does the derivative df/dx do for us?

Ans: It tells us how rapidly the function f(x) varies when we change the x by a tiny amount dx.

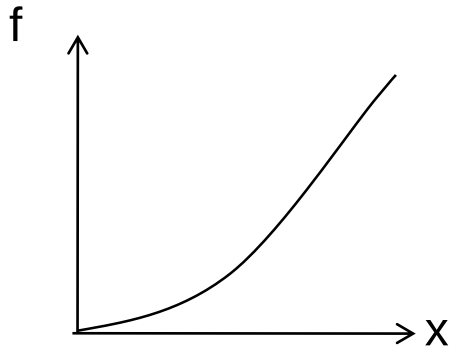
$$df = \left(\frac{\partial f}{\partial x}\right) dx$$

If we change x by an amount dx, then *f* changes by an amount df; the derivative is the proportionality factor.

Geometrical interpretation: The derivative $\left(\frac{\partial f}{\partial x}\right)$ is the slope of the graph of f versus x



Function varies slowly. Derivative is small



Function varies rapidly. Derivative is not small

Now suppose we have a function of three variable….. Say temperature T(x,y,z) in a room.

How fast T vary?

We want to generalize the notion of "derivative" to functions like T which depends not on one but on three variables.

The knowledge of partial derivatives tells us how T changes when we alter three variables by infinitesimal amounts dx, dy ,dz.

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

This equation is reminiscent to a dot product….

?

$$dT = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k} \right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \left(\vec{\nabla} T\right).\left(d\vec{l}\right)$$

where

$$\vec{\nabla} T = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k} \right)$$

is the gradient of T. It is a vector quantity with three components, it is the generalized derivative (3D version).

Q. If $\phi(x, y, z) = xy^2 + 4yz^2$ **Calculate** $\vec{\nabla}\phi$

at the position (2,1,0)

Del Operator, Gradient, Divergence and Curl---- Vector operations

THE VECTOR DIFFERENTIAL OPERATOR DEL, written ∇, is defined by

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

This vector operator possesses properties analogous to those of ordinary vectors. It is also called 'nabla'.
'Nabla' is neither a scalar or a vector, it is an operator.

Gradient or Gradient operation

Let $\varphi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e. φ defines a differentiable scalar field). Then the gradient of φ , written $\nabla\varphi$ or grad φ, is defined by,

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$ **vector**

Please note that the end product is a vector field. It tells us the rate of change of a scalar quantity in a particular direction, lets say temperature.

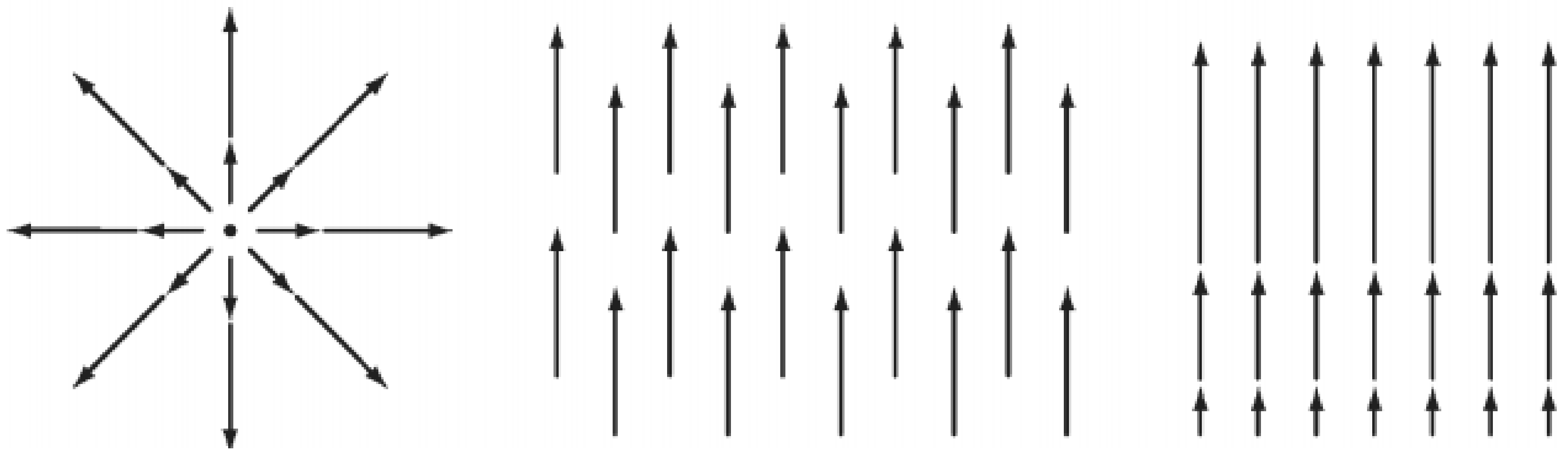
Divergence or Divergence operation

It is a dot product between del operator and a vector.

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}})$$
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

And you observed, it is a scalar: Right???

The name divergence is well chosen and it is a measure of how much the vector v spreads out (diverges) or converges in from the point or out of a point in question.



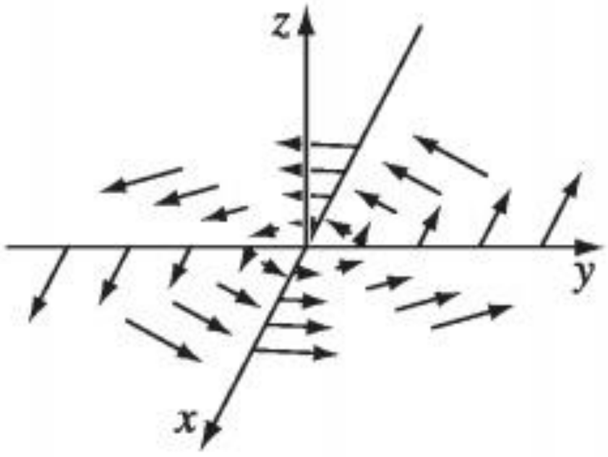
Curl or Rotation operation

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

It is the cross product between del operator and any vector and the end product is also a vector.

The name curl is also well chosen, it is a measure of how much the vector v swirls/rotates around the point. Thus the three vectors in previous slide, all have zero curl (as you can easily check for yourself), whereas the vector in bottom figure have a non-zero curl, pointing in the *z* direction.



The “del” operator

The gradient has the formal appearance of a vector, multiplying a scalar (say T)

$$\vec{\nabla} T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) T$$

The term in the parenthesis is called “del”

$$\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right)$$

“del” is a **vector operator** that acts upon T not a vector that multiplies T.

It doesn't multiply T rather *is an instruction to differentiate*.

1. When it acts on a scalar function T : The gradient $\vec{\nabla} T$
2. When it acts on a vector function v, via dot product: The divergence $\vec{\nabla} . \vec{v}$
3. When it acts on a vector function , via the cross product: The curl $\vec{\nabla} \times \vec{v}$

Divergence Theorem

$$\iiint_{\text{volume}\,\tau} \nabla \cdot \mathbf{V}\, d\tau = \iint_{\substack{\text{surface}\\ \text{inclosing}\,\tau}} \mathbf{V} \cdot \mathbf{n}\, d\sigma$$

It converts volume integral to surface integral. The divergence of any vector in a closed volume is numerically equal to the surface integral of the same vector, taken around the same volume (surface which bounds the volume).

Stokes' Theorem

It converts a surface integral to a line integral. The curl of any vector taken over a surface is numerically equal to the line integral of the same vector surrounding the same surface.

$$\boxed{\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}.}$$

These two theorems are fundamental theorem of divergence and curl

Laplacian Operator

$$\nabla^2\phi = \nabla\cdot\nabla\phi = \mathbf{div}\,\mathrm{grad}\,\phi$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

It always acts on a scalar quantity.

$\nabla^2\phi = 0$ is Laplace' equation.

$\nabla^2\phi = \dfrac{1}{a^2}\dfrac{\partial^2\phi}{\partial t^2}$ is the wave equation.

$\nabla^2\phi = \dfrac{1}{a^2}\dfrac{\partial\phi}{\partial t}$ is the diffusion equation or equation of heat conduction

Cylindrical and spherical polar coordinate

Cylindrical:

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Spherical:

$$\nabla = \hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}$$

Some Useful Rules

f, g: Scalar quantity. $\vec{A}$ **and** $\vec{B}$ **are vector quantities.**

$$\Rightarrow \vec{\nabla}(f+g) = \vec{\nabla}f + \vec{\nabla}g$$

$$\Rightarrow \vec{\nabla}\bullet\left(\vec{A}+\vec{B}\right) = \left(\vec{\nabla}\bullet\vec{A}\right)+\left(\vec{\nabla}\bullet\vec{B}\right)$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{A}+\vec{B}\right) = \left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{\nabla}\times\vec{B}\right)$$

$$\Rightarrow \vec{\nabla}(fg) = f\left(\vec{\nabla}g\right)+g\left(\vec{\nabla}f\right)$$

$$\Rightarrow \vec{\nabla}\bullet\left(f\vec{A}\right) = f\left(\vec{\nabla}\bullet\vec{A}\right)+\vec{A}\bullet\left(\vec{\nabla}f\right)$$

$$\Rightarrow \vec{\nabla}\bullet\left(\vec{A}\times\vec{B}\right) = \vec{B}\bullet\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\bullet\left(\vec{\nabla}\times\vec{B}\right)$$

$$\Rightarrow \vec{\nabla}\times\left(f\vec{A}\right) = f\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\times\left(\vec{\nabla}f\right)$$

Some Useful Rules

f, g: Scalar quantity. $\vec{A}$ and $\vec{B}$ are vector quantities.

$$\nabla \frac{f}{g} = \frac{g\nabla f - f\nabla g}{g^2}$$

$$\nabla . \frac{\vec{A}}{g} = \frac{g\left(\nabla.\vec{A}\right) - \vec{A}.(\nabla g)}{g^2}$$

$$\nabla\ X \frac{\vec{A}}{g} = \frac{g\left(\nabla\ X\ \vec{A}\right) - \vec{A}\ X(\nabla g)}{g^2}$$

$\nabla\ X\ (\nabla\ (g)) = 0$ **Very Important**

$\nabla\ X\ \nabla\ X\ (\vec{A}\ ) = \nabla\ (\nabla\ .\ \vec{A}\ ) - \nabla^2\ \vec{A}$ **Very Important**

$\nabla\ .\ \nabla\ X\ (\vec{A}\ ) = 0$ **Very Important**

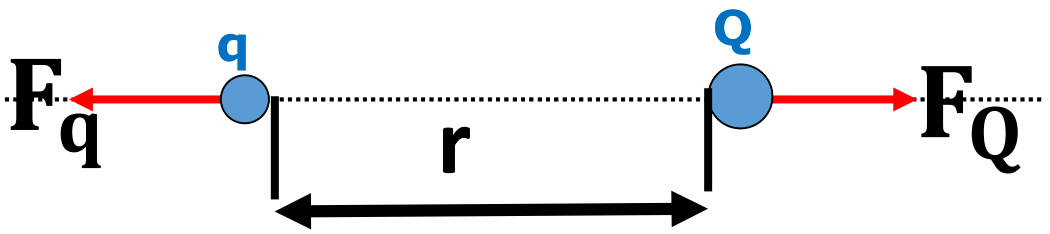
2. Electrostatics, Gauss’s Law, Poisson’s and Laplace’s equations and continuity equation.

Electrostatics: The Physics which deals with static charge(s).

Coulomb's Law

$$F_E = k\frac{q \cdot Q}{r^2}$$

$$\vec{F}_q \neq \vec{F}_Q$$



More than two charges? **SUPERPOSE them!!!**

$$K = \frac{1}{4\pi\varepsilon_0}$$

ε_0=8.85 X 10^{-12} $\frac{c^2}{N \cdot m^2}$ Permittivity of free space

Electric Field

$$\vec{E} = \frac{\vec{F}_E}{q}$$

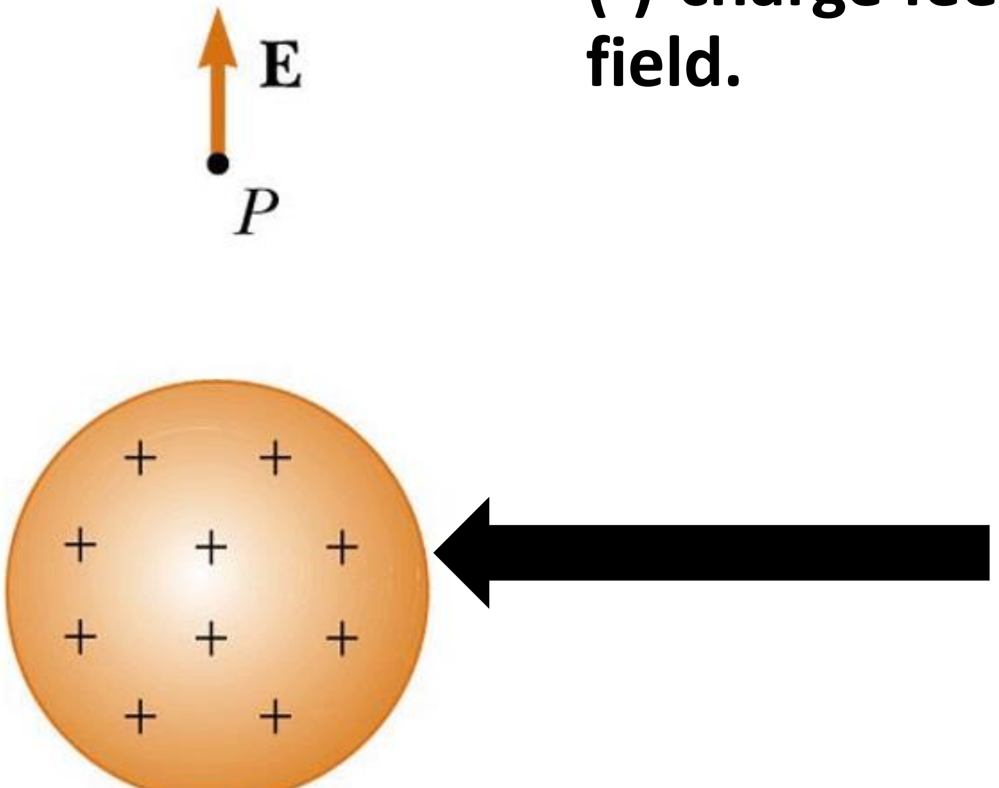
Force on a positive unit charge (N/C).

Therefore, E-field superposable!!!

If there is E-field, there is force!!!

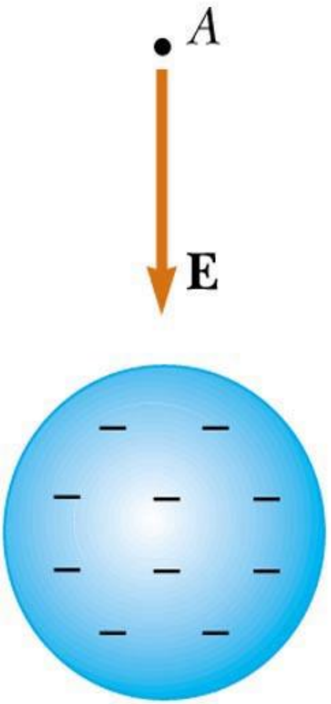
(+)-charge feels the force in the same direction of the E-field

(-)-charge feels the force in the opposite direction of the E-field.

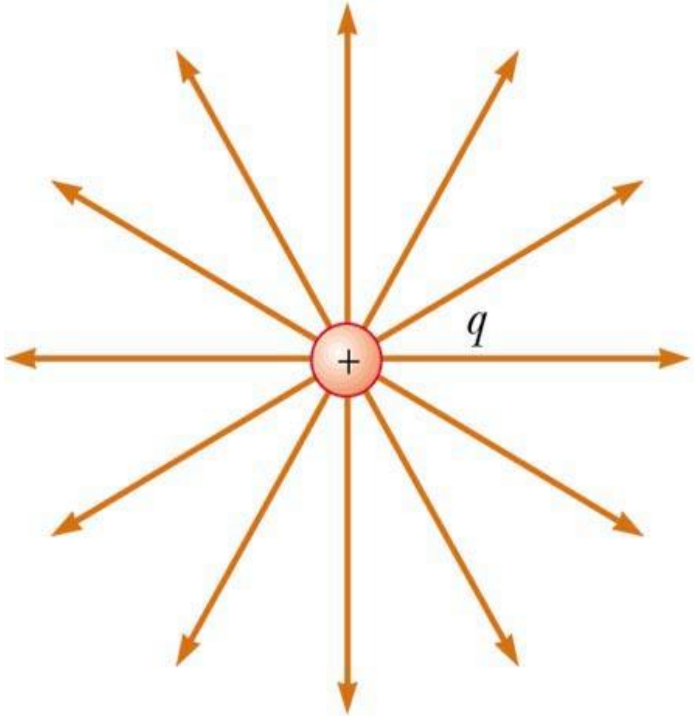


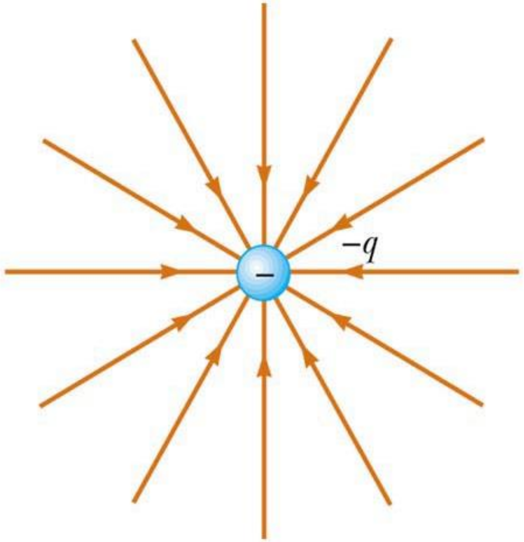
- **The electric field produced by a positive charge is directed away from the charge**
  - **A positive unit charge would be repelled from the positive source charge**

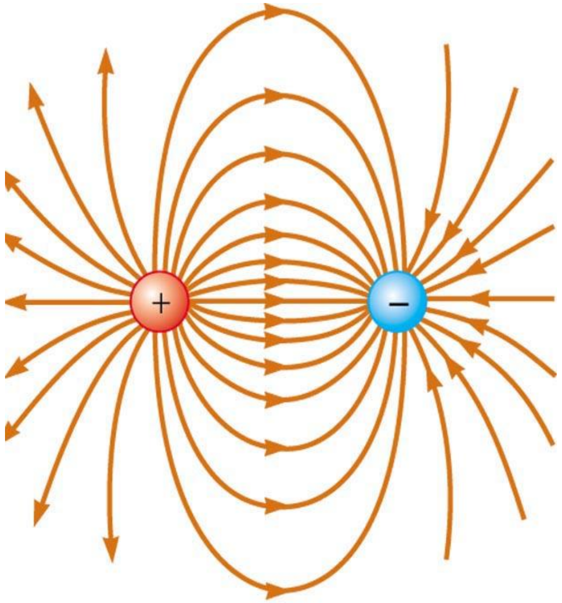
Electric Field

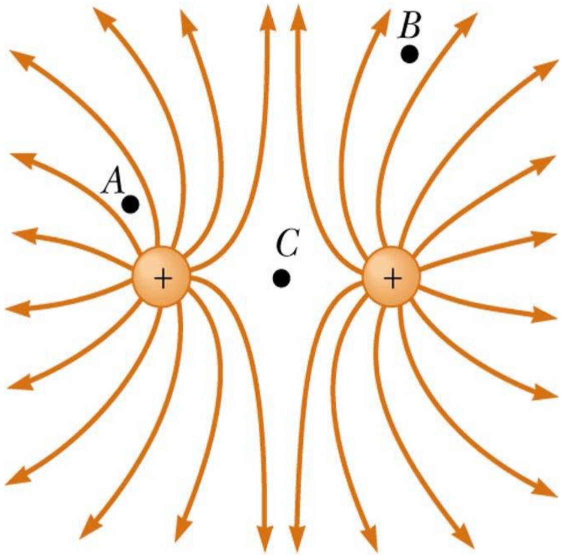


- **The electric field produced by a negative charge is directed toward the charge**
  - **A positive unit charge would be attracted to the negative source charge**









Electrost-atics Field-lines pattern

Electrostatic Potential or electrostatic potential energy

It is the work done by an electric field to move an unit charge from infinity to a particular point in space or the work done by an electric field to move an unit charge from a particular point in space to infinity. So, the potential difference between two points in space in presence of an electric field is the work done to move that unit charge between those two points.

$$V(p) = -\int_{\vartheta}^{p} \vec{E} \cdot d\vec{l}$$

Where ϑ is some standard reference point ; *V* depends only on the point *P. V* is called the *electric potential*.

$$V(b) - V(a) = -\int_{\vartheta}^{b} \vec{E} \cdot d\vec{l} - \left(-\int_{\vartheta}^{a} \vec{E} \cdot d\vec{l} \right) = -\int_{a}^{b} \vec{E} \cdot d\vec{l}$$

The fundamental theorem for gradients $\Rightarrow$ $V(b) - V(a) = \int_{a}^{b} (\nabla V) \cdot d\vec{l}$

$$\int_{a}^{b} (\nabla V) \cdot d\vec{l} = -\int_{a}^{b} \vec{E} \cdot d\vec{l} \Longrightarrow \vec{E} = -\nabla V$$

Electrostatic Potential or electrostatic potential energy

We just noticed that Electric field (E) = - Gradient (Electrostatic Potential)

$$\vec{E} = -\nabla V$$

Theorem: **If the curl of a vector field vanishes, then the vector can be expressed as the negative of the gradient of a scalar potential. This is a fundamental theorem of vector calculus and it holds true for any vector field.**

Electric field travels in straight line, so it is an non-rotational vector. So, the curl of an electric field is always zero.

$$\nabla \, X \, \vec{E} = 0$$

Therefore,

$$\vec{E} = -\nabla V$$

$\vec{E}$: Curl less or Irrotational or non-rotational vector

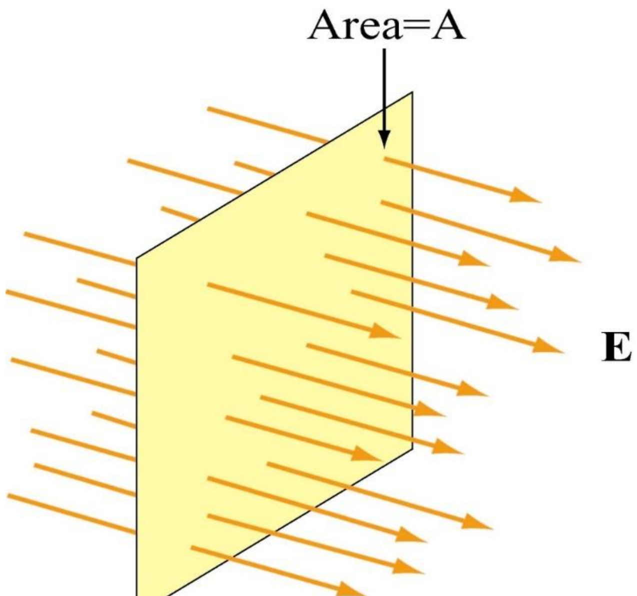
$\nabla \times \vec{E} = 0$ Therefore, $\int_a^b \vec{E} \cdot d\vec{l}$ → Independent of the path

$\oint \vec{E} \cdot \vec{dl} = 0$ → For any closed loop

Gauss's Law in electrostatics

Electric Flux Φ_E

Case I: E is constant vector field perpendicular to planar surface S of area A



$$\Phi_E = \iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$\Phi_E = +EA$$

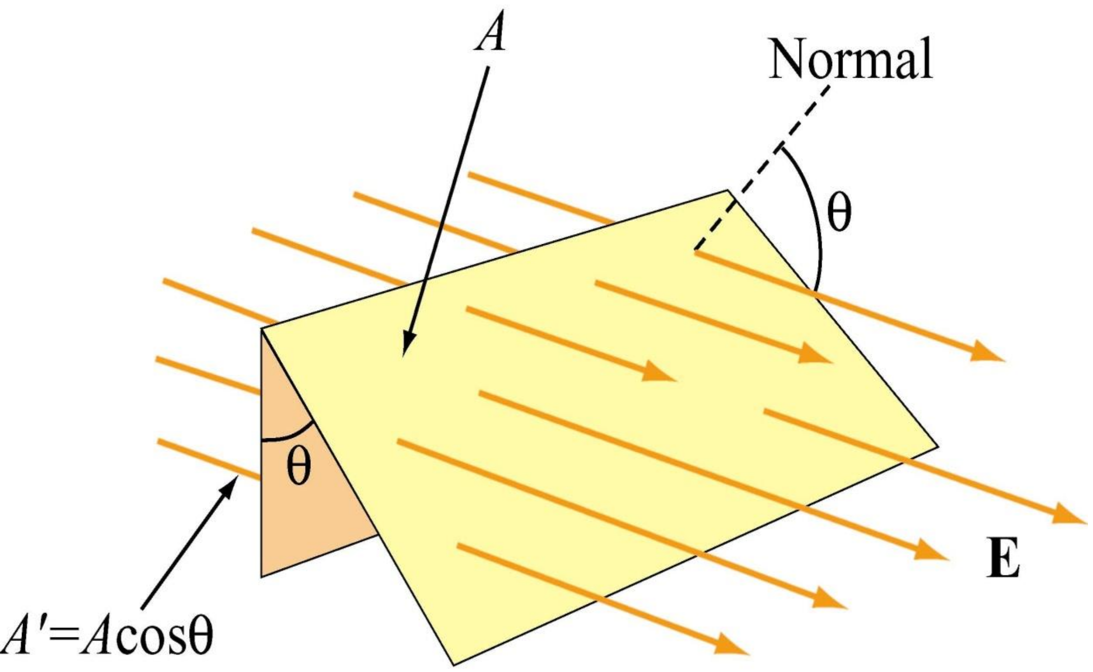
Gauss’s Law in electrostatics

Electric Field Lines: The number of field lines, also known as lines of force, are related to strength of the electric field. More appropriately it is the number of field lines crossing through a given surface that is related to the electric field

Flux: How much of something (Electric field lines) passes through some surface.

Case II: E is constant vector field directed at angle θ to planar surface S of area A

Gauss's Law in electrostatics



$$\Phi_E = \iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$d\vec{\mathbf{A}} = dA\,\hat{\mathbf{n}}$$

$$\Phi_E = EA\cos\theta$$

$$\boxed{\Phi_E = \underset{\text{closed surface S}}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}}$$

Electric flux Φ_E (the surface integral of E over closed surface *S*) is proportional to the total charge inside the volume enclosed by *S*.

$$\Phi_E = \oint\!\!\!\oint_{\substack{\text{closed}\\ \text{surface}\,S}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enclosed}}{\varepsilon_0}$$

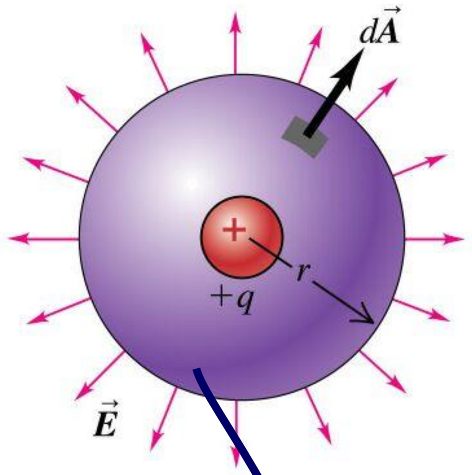
Note: Integral must be over closed surface

This is the mathematical expression of Gauss's law in electrostatics.

Statement: ***The net flux through any closed surface is numerically equal to the net (total) charge*** **inside** ***that surface (enclosed by the surface) divided by*** ε_0.

Applications of Gauss's Law

Electric field at a distance r from a point change????????????



What is electric flux that comes from a point charge?

We start from $$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

The problem has spherical symmetry, we therefore use a sphere as the Gaussian surface

The integral over the surface area of the sphere yields

$$A = 4\pi r^2$$

Applications of Gauss's Law

$$\Phi_E = \int \vec{E} \cdot d\vec{A} = E \times 4\pi r^2$$

According to Gauss's Law,

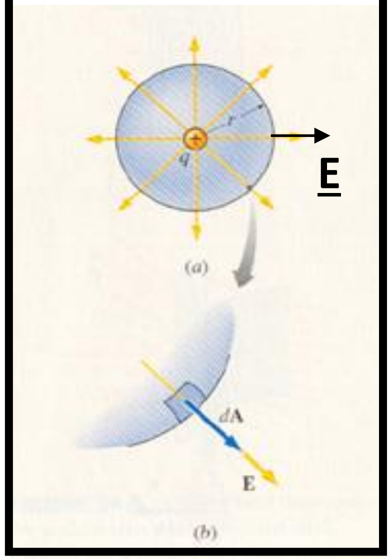
$$\Phi_E = \int \vec{E} \cdot d\vec{A} = E \times 4\pi r^2 = \frac{q_{enclosed}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$$

Therefore,

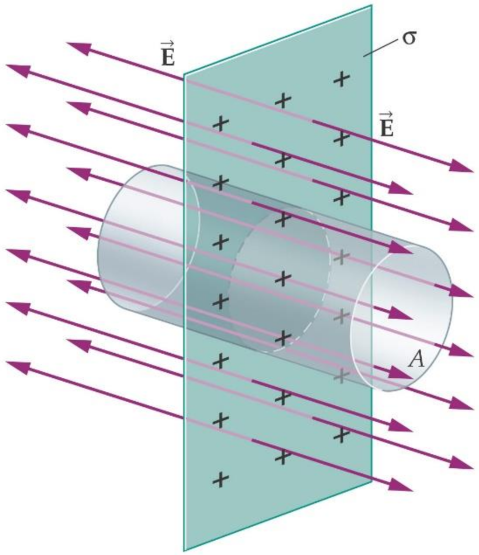
$$\mathbf{E} = \frac{\mathbf{q}}{4\pi\varepsilon_0 \mathbf{r}^2}$$

Gauss's Law is always true, but is only useful for certain very simple problems with great symmetry.

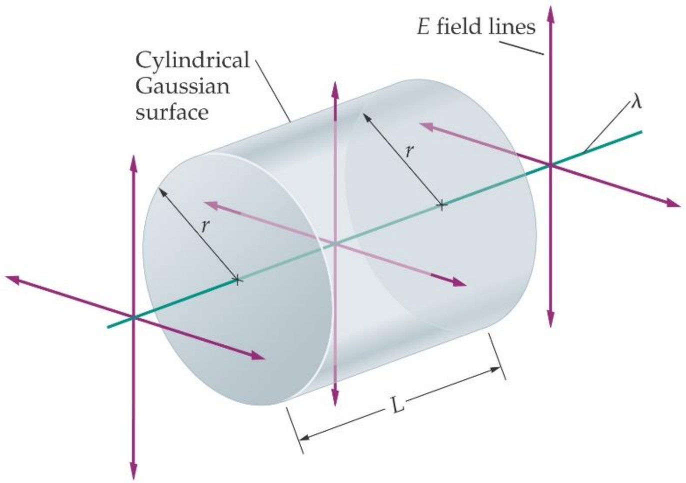
Spherical geometry



Planar geometry

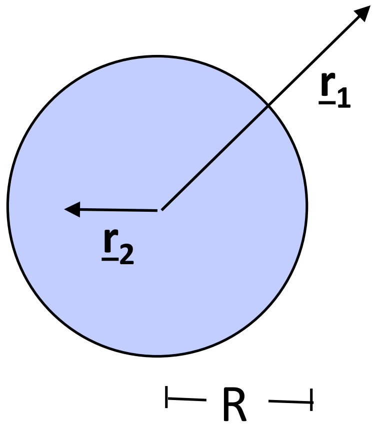


Cylindrical geometry



Problem: Sphere of Charge Q

A charge Q is uniformly distributed through a solid non-conducting sphere of radius R. What is the electric field as a function of $\underline{r}$?. Find $\underline{E}$ at $\underline{r}_1$ and $\underline{r}_2$.



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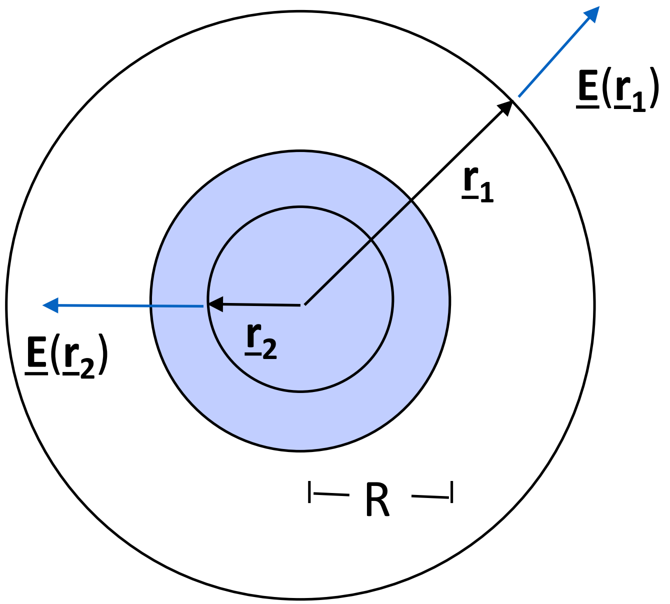
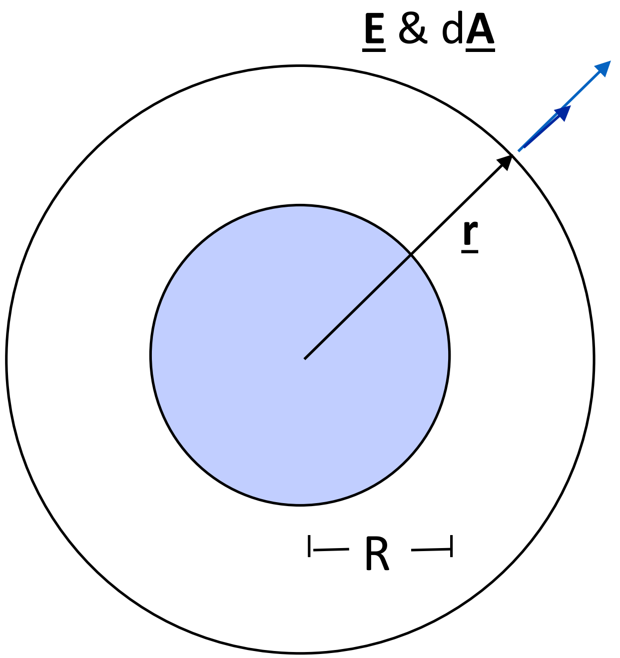


Use symmetry!

This is spherically symmetric. That means that E(r) is radially outward, and that all points, at a given radius (|r|=r), have the same magnitude of field.

Problem: Sphere of Charge Q

First find E(r) at a point outside the charged sphere. Apply Gauss's law, using as the Gaussian surface the sphere of radius r pictured.



What is the enclosed charge?

Problem: Sphere of Charge Q

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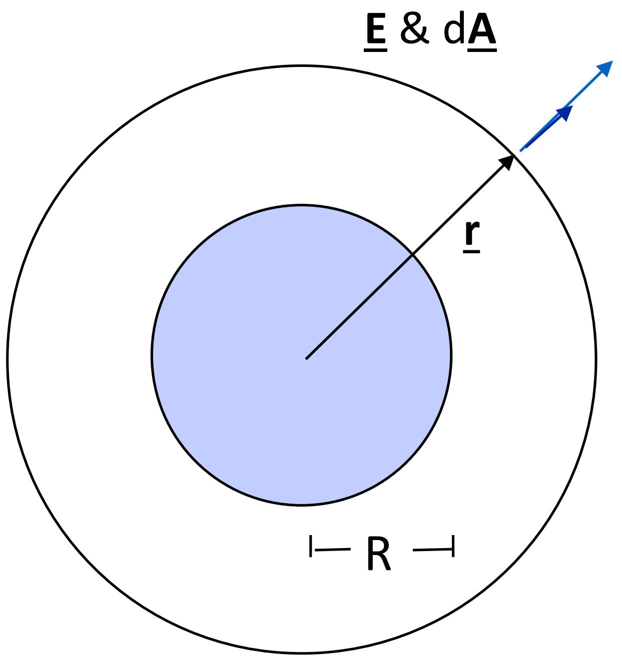

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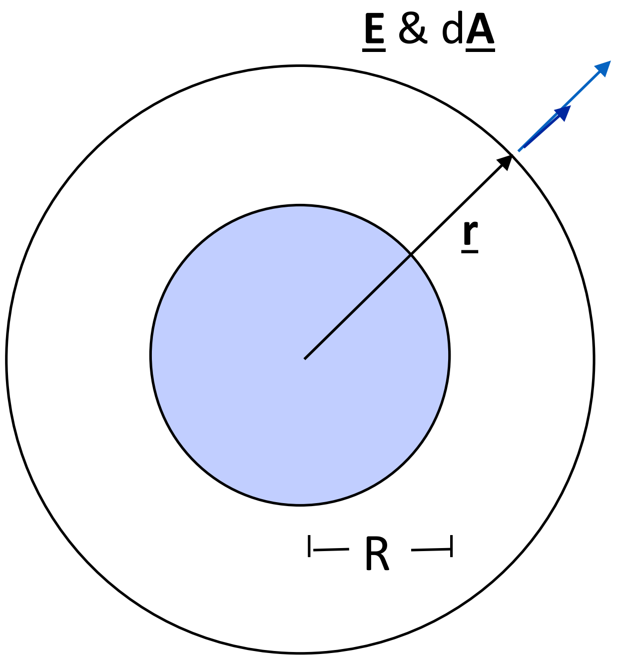
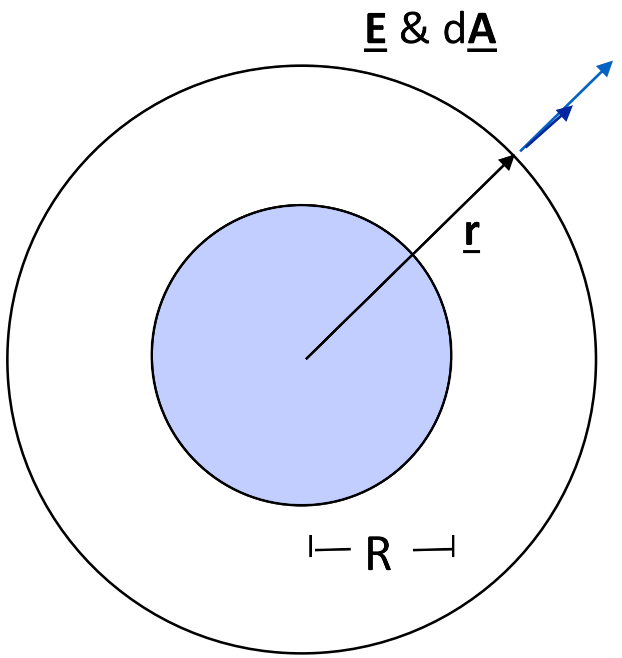


What is the enclosed charge? Q

What is the flux through this surface?

Problem: Sphere of Charge Q

First find <u>E</u>(<u>r</u>) at a point outside the charged sphere. Apply Gauss's law, using as the Gaussian surface the sphere of radius r pictured.



What is the enclosed charge? **Q**

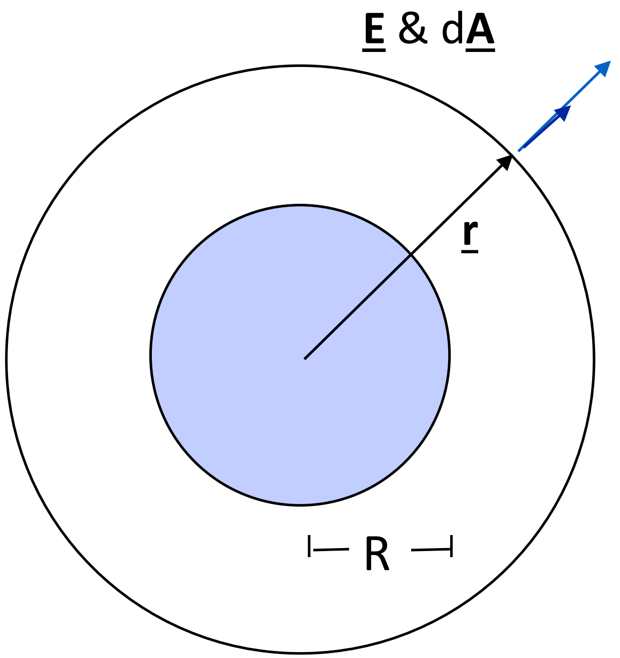
What is the flux through this surface?

$$\Phi = \oint \vec{\mathbf{E}} \bullet d\vec{\mathbf{A}} = \oint E\,dA$$

$$= E\oint dA = EA = E(4\pi r^2)$$

Problem: Sphere of Charge Q

First find $\underline{E}(\underline{r})$ at a point outside the charged sphere. Apply Gauss's law, using as the Gaussian surface the sphere of radius r pictured.



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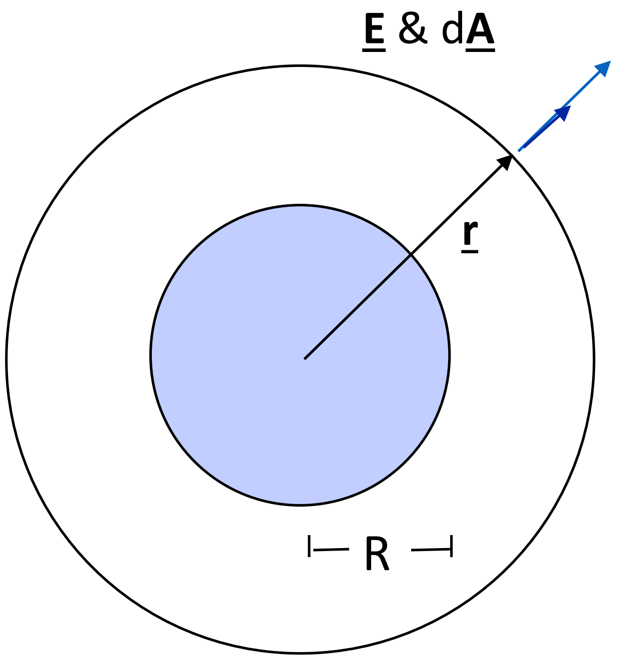
$$= E \oint dA = EA = E(4\pi r^2)$$

Gauss $\Rightarrow$ $\Phi = Q/\varepsilon_o$

$$Q/\varepsilon_0 = \Phi = E(4\pi r^2)$$

Problem: Sphere of Charge Q

First find E(r) at a point outside the charged sphere. Apply Gauss's law, using as the Gaussian surface the sphere of radius r pictured.



What is the enclosed charge? Q

What is the flux through this surface?

$$\Phi = \oint \vec{\mathbf{E}} \bullet d\vec{\mathbf{A}} = \oint E\, dA$$

$$= E \oint dA = EA = E(4\pi r^2)$$

Gauss: $$\Phi = Q / \varepsilon_o$$

$$Q/\varepsilon_0 = \Phi = E(4\pi r^2)$$

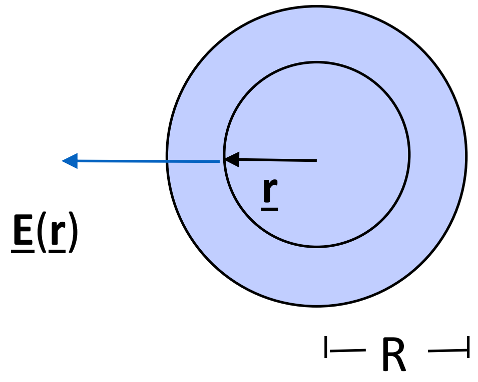
So $$\boxed{\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_o} \frac{Q}{r^2} \hat{\mathbf{r}}}$$

Exactly as though all the charge were at the origin!
(for r>R)

Problem: Sphere of Charge Q

Next find <u>E</u>(<u>r</u>) at a point inside the sphere. Apply Gauss's law, using a little sphere of radius r as a Gaussian surface.

What is the enclosed charge?
That takes a little effort. The little sphere has some fraction of the total charge. What fraction?



That's given by volume ratio: $$Q_{enc} = \frac{r^3}{R^3} Q$$

Again the flux is: $$\Phi = EA = E(4\pi r^2)$$

Setting $\Phi = Q_{enc} / \varepsilon_o$ gives $$E = \frac{(r^3/R^3)Q}{4\pi\varepsilon_o r^2}$$

For r<R

$$\boxed{\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_o R^3} r\hat{\mathbf{r}}}$$

Poisson's equation, Laplace's equation and continuity equation.

Let us re-write the Gauss's Law, we learned before,

$$\underset{\text{closed surface}}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enclosed}}{\varepsilon_0}$$

This expression known as integral form of Gauss's Law

If we apply divergence theorem, the above equation can be written as,

$$\oiiint \left(\nabla \cdot \vec{E}\right) d\tau = \frac{1}{\varepsilon_0} \oiiint \rho \, d\tau$$

Expressed both sides as volume integration. In the right side the total charge is expressed in terms of volume charge density (ρ), integrated over total volume.

As the integration is performed inside the same material, the volume integral on both sides gets cancelled.

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ This expression known as differential form of Gauss's Law

Let's plug in the expression of electric field as we learnt before in terms of electric potential $(\vec{E} = -\nabla\ V)$. The left side of the expression gets the form as mentioned bellow.

$$\nabla \cdot \vec{E} = \nabla \cdot \left(-\nabla V\right) = -\nabla^2 V$$

$$\left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

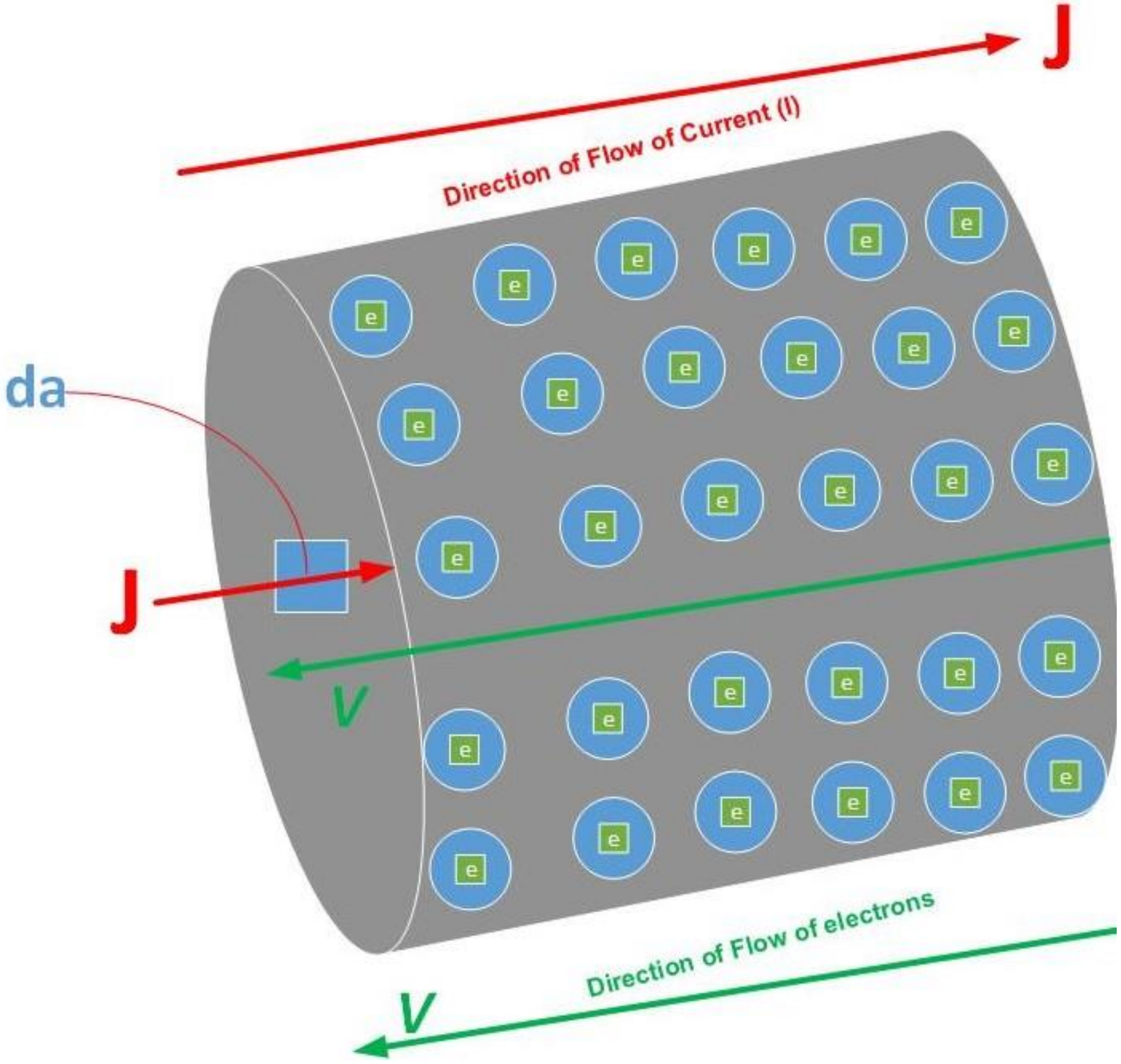
Therefore,

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ → **Poisson's Equation**

In a charge free region (ρ = 0)

$$\nabla^2 V = 0$$ → **Laplace's Equation**

Continuity Equation



Current Density (J): When the flow of charge is distributed throughout a 3 D region, one can describe it by current density.

$\vec{J}$: Current per unit area perpendicular to the flow.

$\overrightarrow{da}$: Cross-sectional area perpendicular to the flow of the current.

$$\vec{J} = \frac{dI}{\overrightarrow{da}}$$

ρ: Volume charge density.

$\vec{V}$: Velocity of electron.

$$\vec{J} = \rho \vec{V}$$

$$I = \oiint \vec{J} \cdot \overrightarrow{da}$$

$$I = \oiint \vec{J} . \overrightarrow{da}$$

Now, we will convert this expression into volume integral using divergence theorem.

$$\boxed{I = \oiint \vec{J} . \overrightarrow{da} = \iiint (\nabla . \vec{J})\, d\tau}$$

$-\frac{dq}{dt}$: Rate of flow of charge with time at a particular cross-section.

Again, we can write

$$I = \oiint \vec{J} . \overrightarrow{da} = -\frac{dq}{dt}$$

q: Total charge in a particular volume $= \iiint \rho .\, d\tau$

$$\frac{dq}{dt} = \frac{d}{dt} \iiint \rho \,.\, d\tau$$

Now, if we keep only time as variable and keep the surface as constant (Charges flows in a particular direction only)…we can easily express this as partial derivative.

$$\frac{dq}{dt} = \frac{d}{dt}\iiint \rho \, . \, d\tau = \frac{\partial}{\partial t}\iiint \rho \, . \, d\tau$$

Now,

$$I = \oiint \vec{J} \, . \, \overrightarrow{da} = \oiiint (\nabla \, . \, \vec{J}) \, d\tau = -\frac{\partial}{\partial t}\iiint \rho \, . \, d\tau$$

Therefore,

$$\oiiint (\nabla \, . \, \vec{J}) \, d\tau = -\frac{\partial}{\partial t}\iiint \rho \, . \, d\tau$$

Since, the integration is computed inside the same material (for both sides of the above equation), the volume integral will be cancelled.

$$\boxed{\nabla . \vec{J} = -\frac{\partial \rho}{\partial t}} \Rightarrow$$ **Continuity equation.**

Conservation of Charge:- "The Principle of conservation of charge states that Charge can neither be created nor be destroyed, Although equal amounts of positive and negative charge(s) may be simultaneously created, obtained by separation, destroyed or lost by recombination.

Read carefully, these two statements are similar. So, continuity equation proves the conservation of charge theorem.

Equation of Continuity:- "The total current flowing out of some volume is equal to the rate of decrease of charge within that volume".

Continuity equation indicates that the current, or charge per second, diverging from a small volume or per unit volume is equal to the rate of decrease of charge per unit volume at every point.

Electrostatic in Dielectric materials

Conductors and Insulators

Conductor: Charges are free to move Electrons weakly bound to atoms.

Example: metals

Insulator: Charges are NOT free to move Electrons strongly bound to atoms. Dielectric materials are insulators.

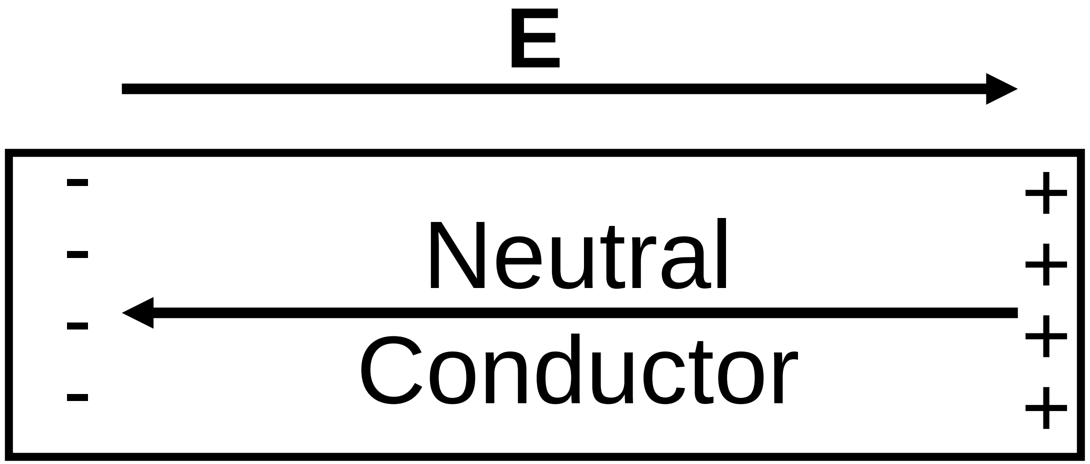
Examples: plastic, paper, wood

Conductors

Conductors have free charges

→ E must be zero inside the conductor

→ Conductors are equipotential objects

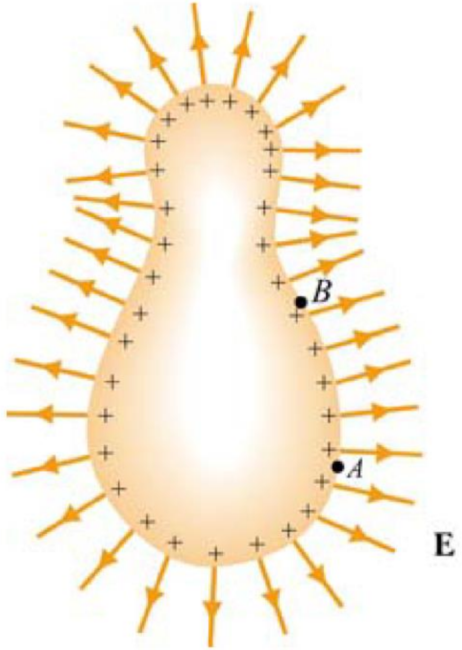


Conductors are equipotential objects:

1) E = 0 inside.

2) E perpendicular to surface.

3) Net charge inside is 0.



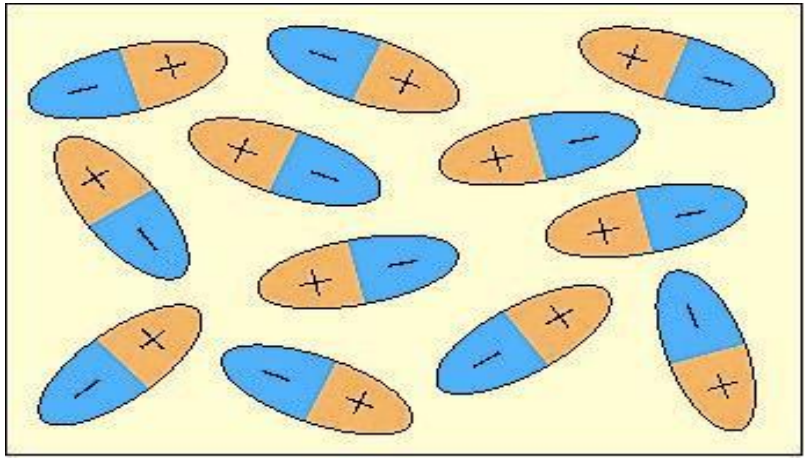
When a dielectric is placed in a charged capacitor, it reduces the potential difference between the two plates.

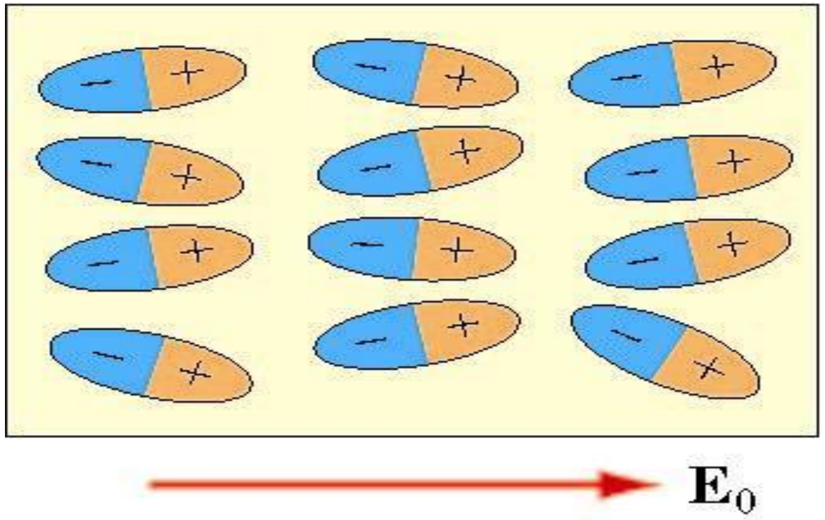
HOW???

We will observe Gauss's law gets modified inside any dielectric material.

Molecular View of Dielectrics

Polar Dielectrics : Dielectrics with permanent electric dipole moments. Example: Water



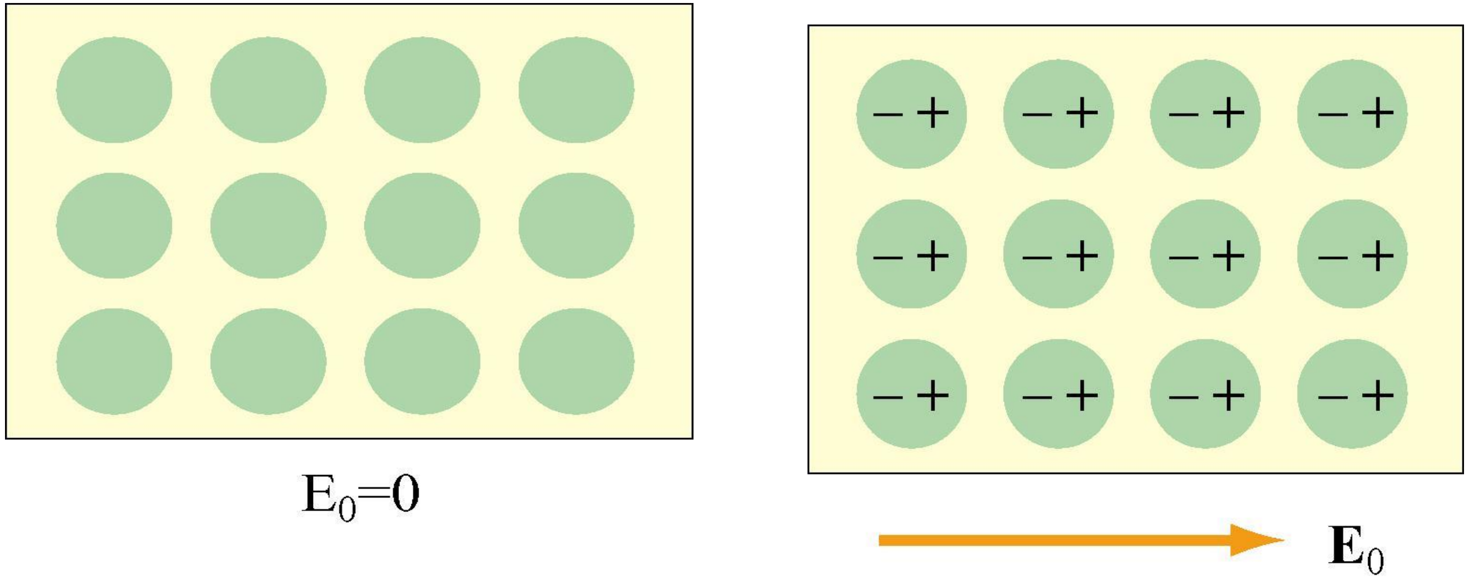


Molecular View of Dielectrics

Non-Polar Dielectrics

Dielectrics with induced electric dipole moments

Example: CH_4



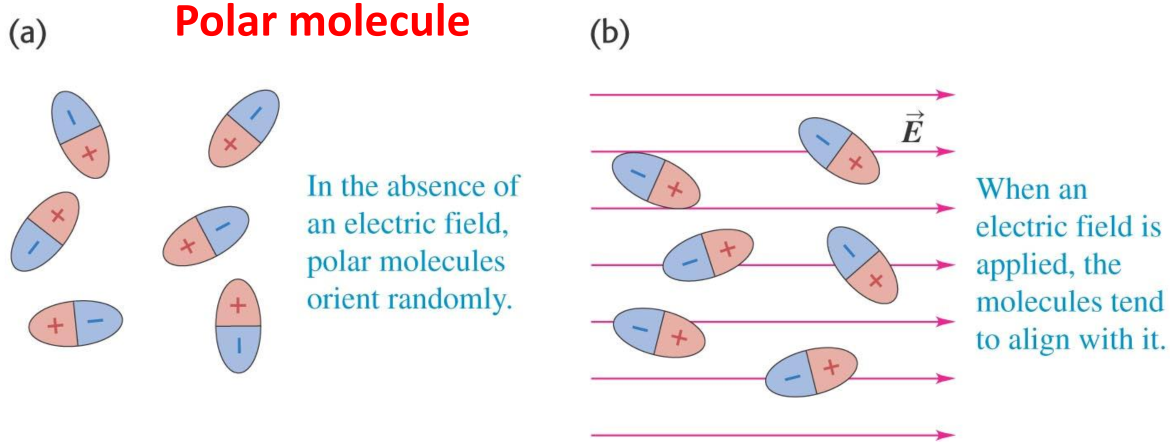
Why does dielectric behave different from conductor?

Where does a dielectric get it's dipole moments from? **We will explore these questions shortly……**

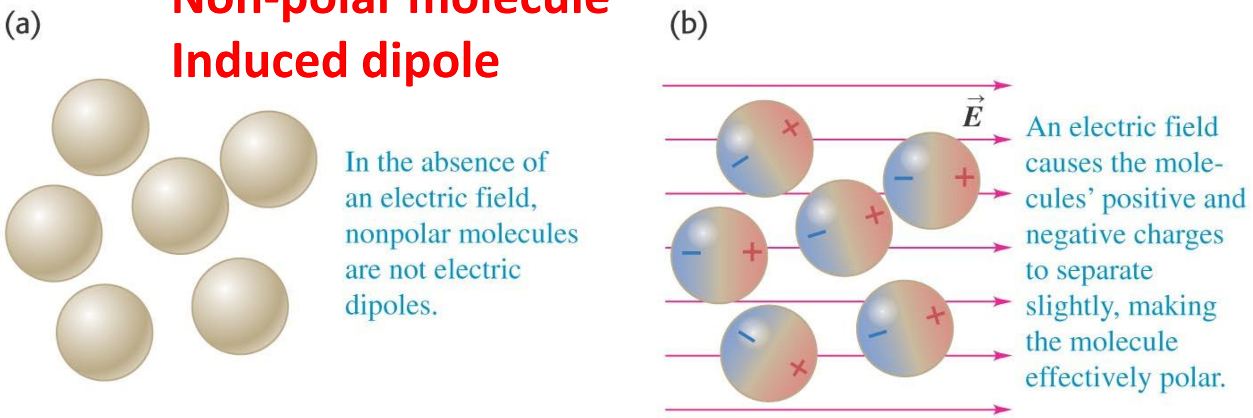
The very basic difference between an insulator and a conductor is the difference between their energy band gap (Energy difference between conduction band and valence band). Conductors have very less band gap and in most of the cases the conduction band and valence bands are almost overlapped, so electrons can easily travels from valence band to conduction band even in room temperature. And there are huge number of electrons in the conduction band. So, electrons do not stay closure to the atoms and they are free to roam around the conductor.

On the contrary, due to huge band gap, the electrons inside the insulator stay near to their respective atoms and they are also known as bound charges. In dielectric, most of the electrons are bound to their respective atoms and there are very few electrons in the conduction band. So, the population of electrons are very high in the valence band. When the electrons stay near to the atoms and as atoms are highly packed (very close to each other), these electrons are also subjected to face various electrostatics forces due to other nearby electrons. Because of these forces, the electrons displaces a bit from their original orientations/position, creating a positive and negative charge separations and thus a dipole.

Polar molecule

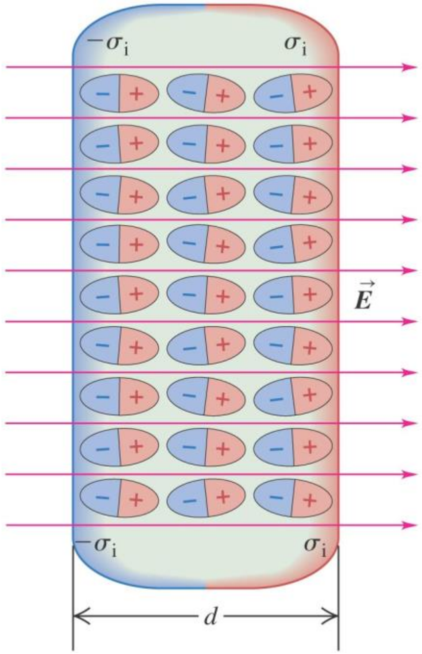


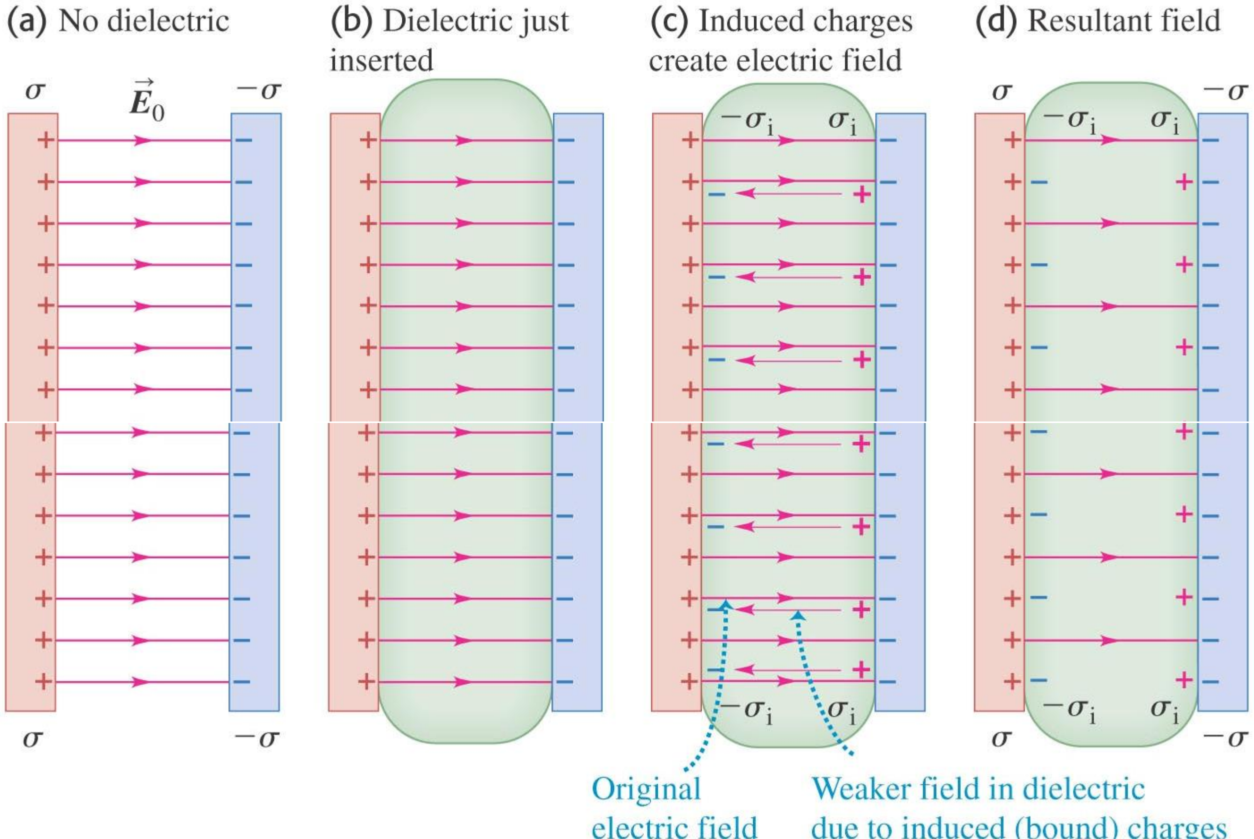
Non-polar molecule
Induced dipole



Polarization and Electric Field Lines

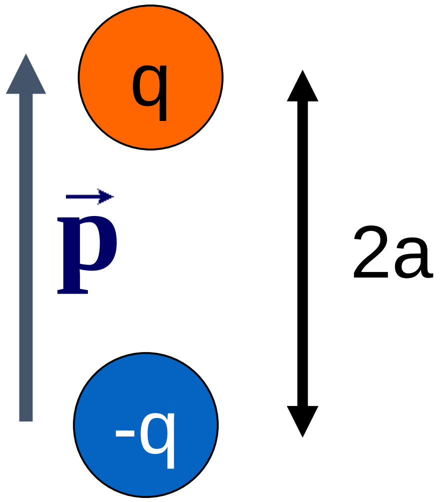
Polarization of a dielectric in electric field gives rise to bound charges on the surfaces, creating σi, - σi.





Electric Dipole

Two equal but opposite charges *+q* and *–q*, separated by a distance 2a



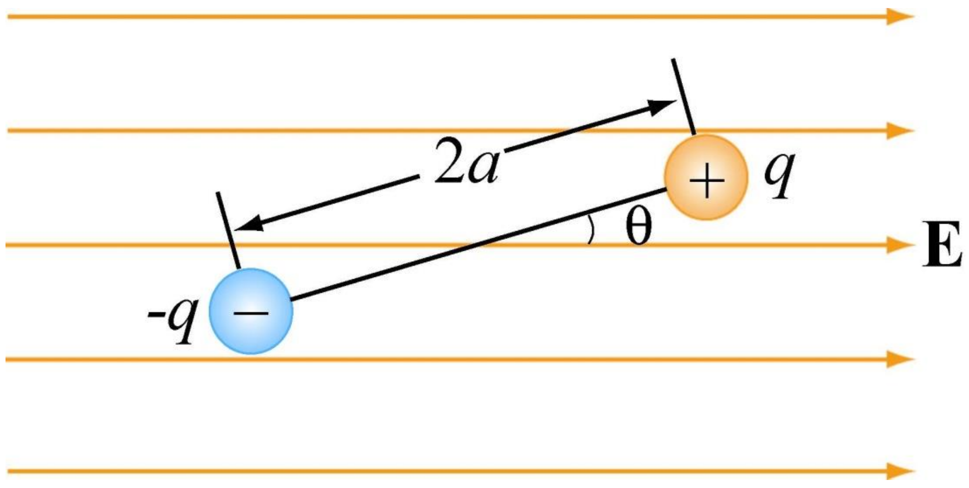
Dipole Moment

$$\vec{\mathbf{p}} \equiv \text{charge} \times \text{displacement}$$

$$= q \times 2a\hat{\mathbf{j}} = 2qa\hat{\mathbf{j}}$$

$\vec{\mathbf{p}}$ **Points from negative to positive charge**

Dipole in Uniform Field



$$\vec{\mathbf{E}} = E\hat{\mathbf{i}}$$

$$\vec{\mathbf{p}} = 2qa(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})$$

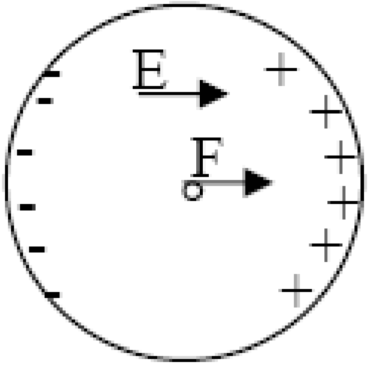
Total Net Force: $\vec{\mathbf{F}}_{net} = \vec{\mathbf{F}}_{+} + \vec{\mathbf{F}}_{-} = q\vec{\mathbf{E}} + (-q)\vec{\mathbf{E}} = 0$

Torque on Dipole: $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$

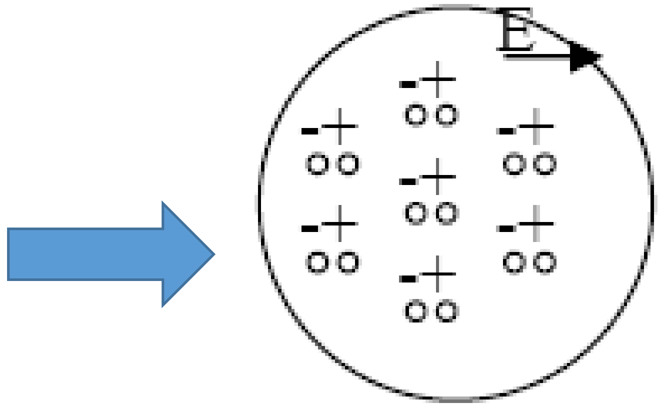
$$\tau = rF_{+}\sin(\theta) = (2a)(qE)\sin(\theta) = pE\sin(\theta)$$

$\vec{\mathbf{p}}$ **tends to align with the electric field**

In conductors charges will be pushed to the boundary by external field.



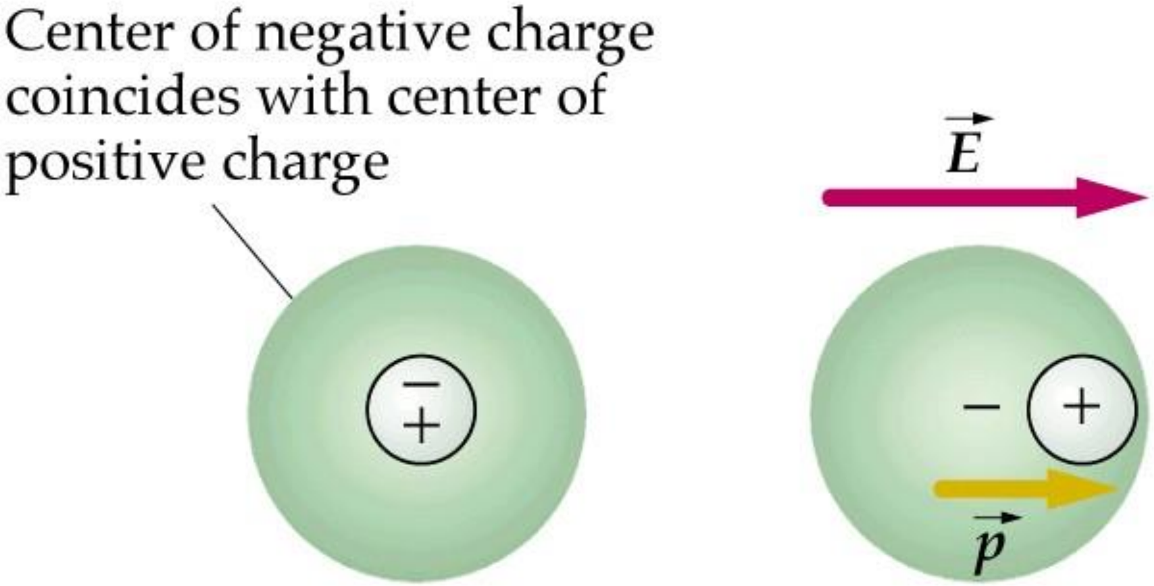
In dielectrics charges are attached to atoms or molecules.



The electric field can distort the charge distribution of a dielectric atom or molecule by two principal mechanisms: stretching and rotating.

Normally, the dipole moment is zero on large scales since atomic dipoles are oriented in random directions.

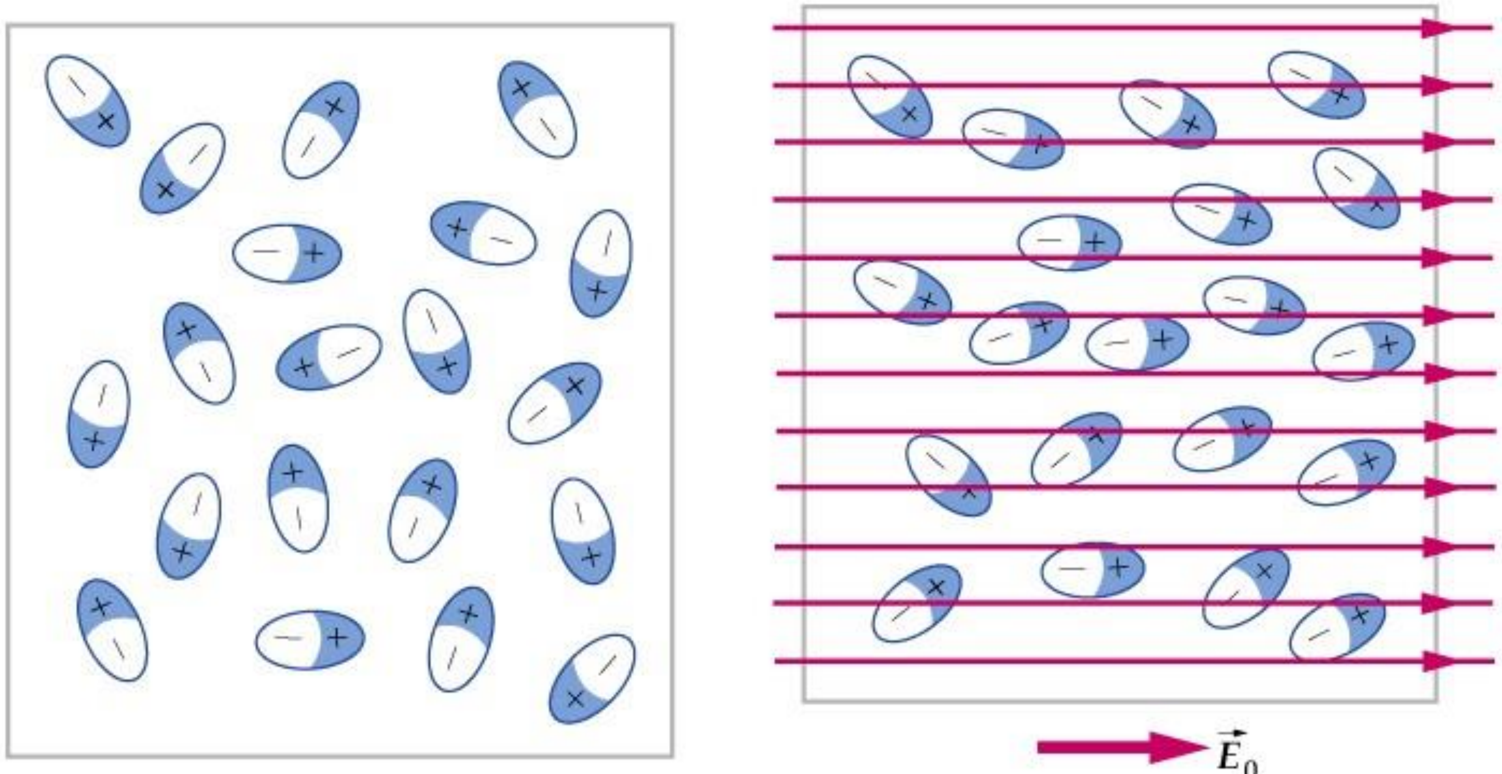
Immersion of a dielectric in an electric field polarizes atoms and tends to align the atomic dipoles.



$$\mathbf{p} = \alpha \mathbf{E}$$

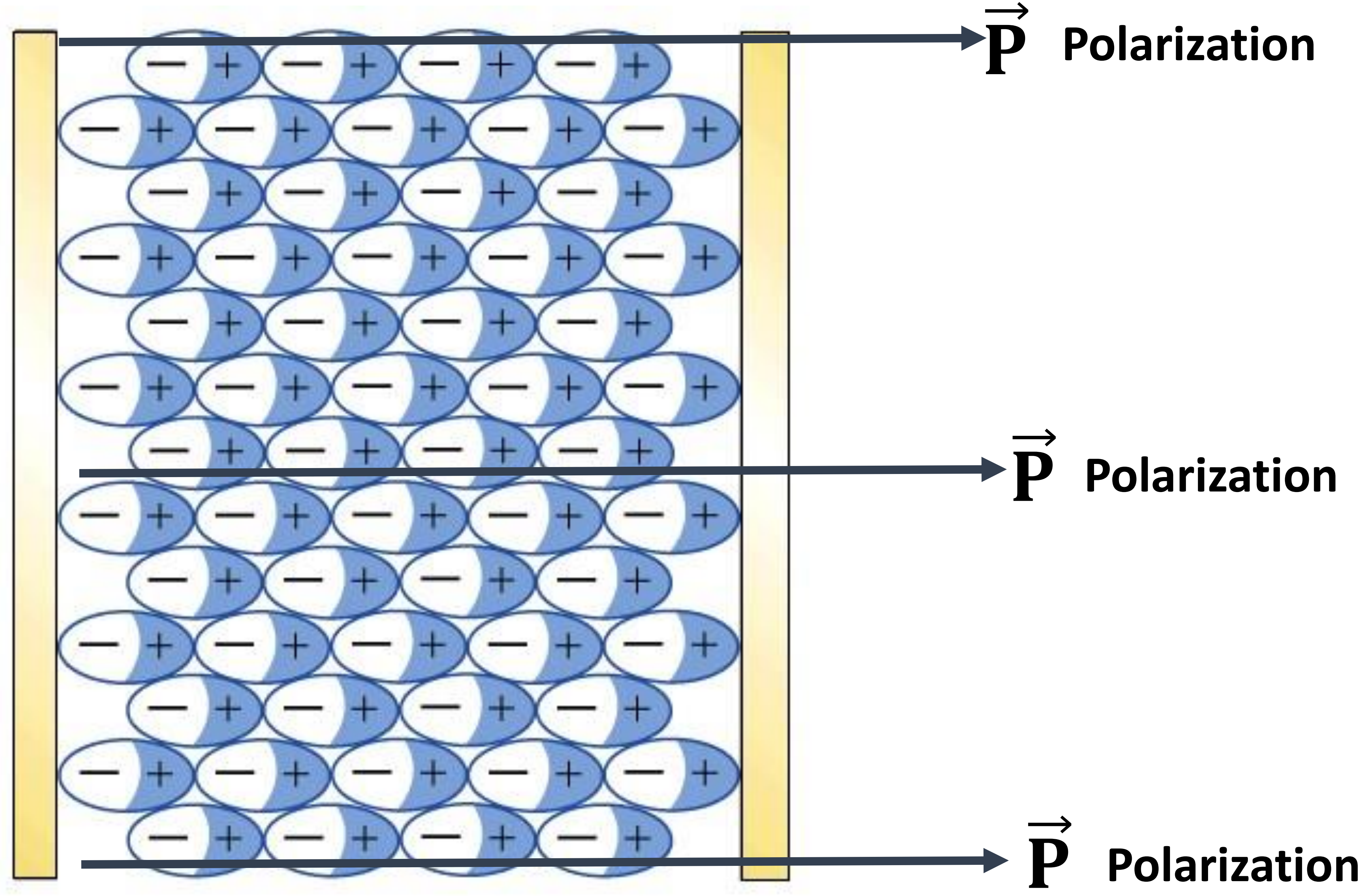
$\mathbf{p}$ **Dipole moment of an atom**

α **Atomic polarizability**



Considering the bulk of atoms together. If we sum up all the dipole moments of the atoms and then divide by it's total volume we get, dipole moment per unit volume.

$\overrightarrow{\mathbf{P}}$ **Polarization/Dipole moment per unit Volume**


$\vec{P}$ Polarization
$\vec{P}$ Polarization
$\vec{P}$ Polarization

We already introduced the concept of bound charges inside the dielectric in presence of an electric field. There are two types of bound charges, Volume bound charge (ρ_b), which stays inside the material and surface bound charge (σ_b), which stays on the surface of the material.

volume bound charge density $\rho_b = -\nabla \cdot \vec{\mathbf{P}}$

surface bound charge density $\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}},$

where $\hat{\mathbf{n}}$ is the normal to the surface.

Irrespective of these bound charges there are also some free charges inside a dielectric material. These free charges are not bound to any atoms and they are free to move inside the conduction band but it's number is significantly less than the number of bound charges.

$\boldsymbol{\rho_f}$ **Volume free charge density.**

Total charge density inside a dielectric $\boldsymbol{\rho = \rho_b + \rho_f}$

Let us apply Gauss's law inside a dielectric material. From the differential form of Gauss's law, one can write:

$$\nabla\,.\,\vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_b + \rho_f}{\varepsilon_0}$$

From the relationships we learnt in previous slides, we can express this as,

$$\nabla\,.\,\vec{E} = \frac{-\nabla.\vec{P} + \rho_f}{\varepsilon_0} \longrightarrow \nabla\,.\,(\varepsilon_0\vec{E} + \vec{P}) = \rho_f$$

Let us introduce a new vector, $\vec{D} = (\varepsilon_0\vec{E} + \vec{P})$ → Electric Displacement vector. This is equivalent to electric field inside a dielectric media.

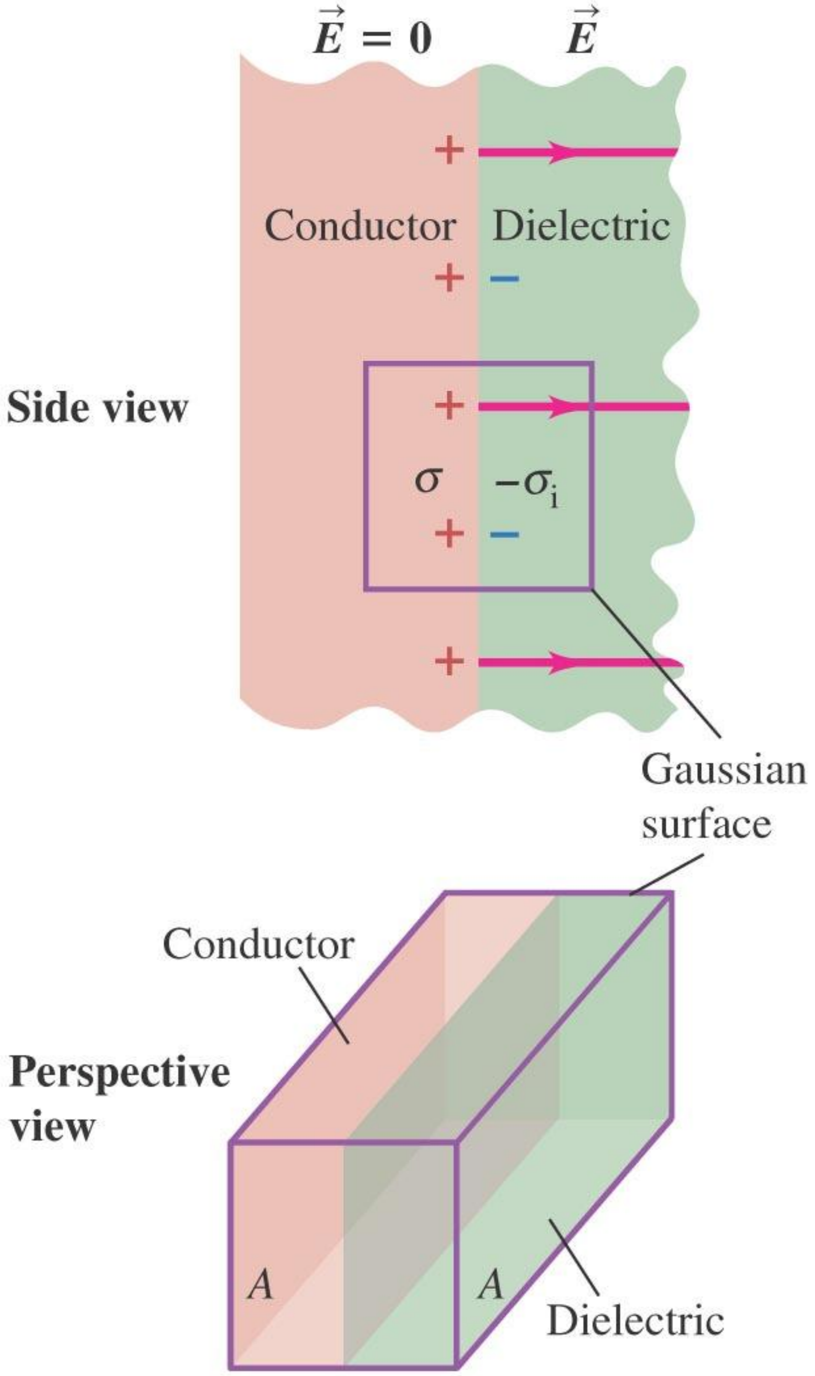
Therefore, the above expression can be expressed as: $\nabla\,.\,\vec{D} = \rho_f$

$\boxed{\nabla\,.\,\vec{D} = \rho_f}$ → Differential form of Gauss's law inside a dielectric medium

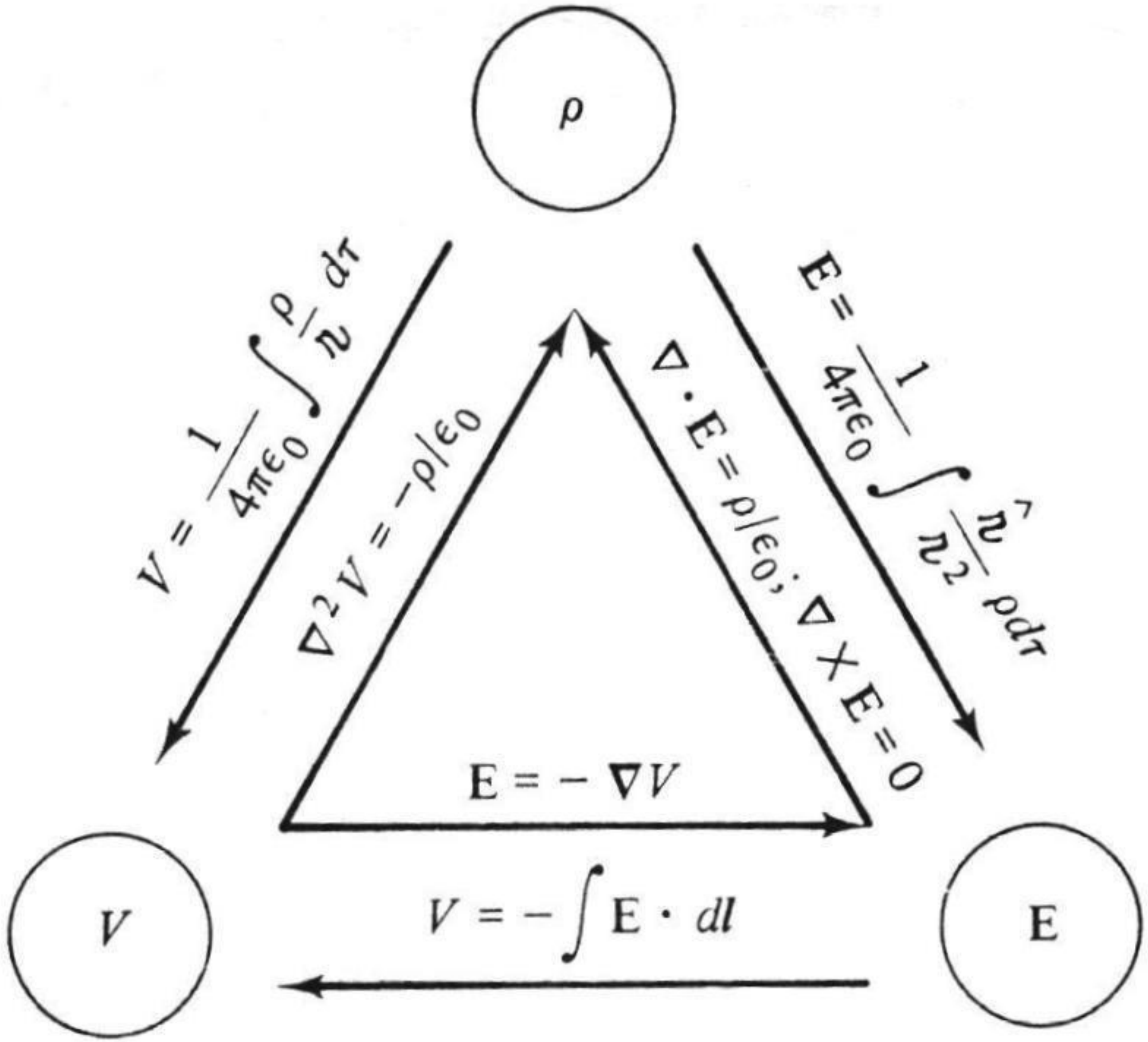
$$\oiint \overrightarrow{D}.\overrightarrow{da} = Q_{f,Enclosed}$$

Integral form of Gauss's Law inside a dielectric media.

$Q_{f,Enclosed}$: **Free charge enclosed by the Gaussian surface**



Electrostatics in one picture



3. Magneto-statics and electromagnetism

- Ampere Circuital Law, Maxwell displacement current and correction in Ampere Circuital Law.
- Maxwell electromagnetic equations (differential and integral forms)
- Physical significance of Maxwell equations.

Magneto-statics is the study of magnetic fields in systems where the currents are steady (not changing with time). It is the magnetic analogue of electrostatics, where the charges are stationary. Here, electrons are moving with a constant velocity.

Source of Magnetic field $\Rightarrow$ Current

Source of Current $\Rightarrow$ Moving Electrons

But how do we measure/calculate magnetic field? For simplicity, we will consider current and not moving electrons to calculate magnetic field.

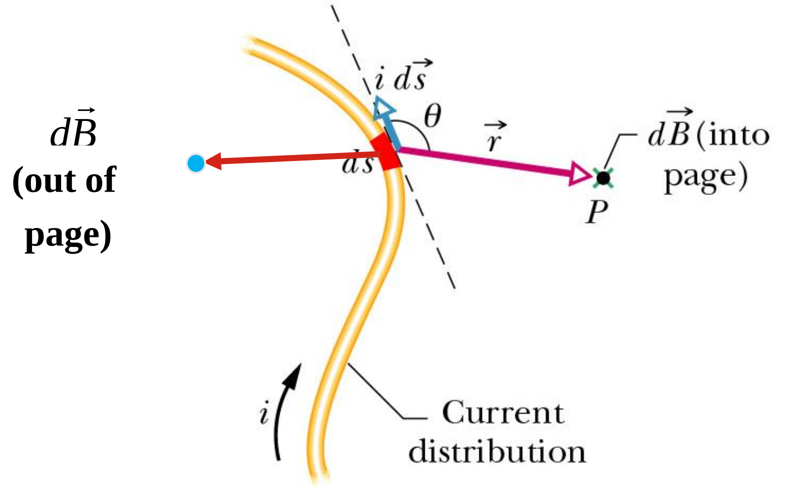
Biot-Savart Law

- **The magnetic field due to an element of current is**

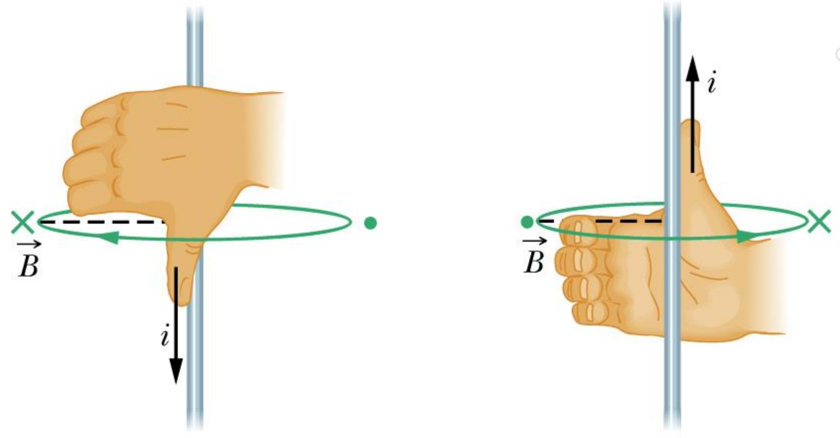
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\vec{r}}{r^3}$$

$$\frac{1}{\sqrt{\varepsilon_0\mu_0}} = c = \text{speed of light}$$

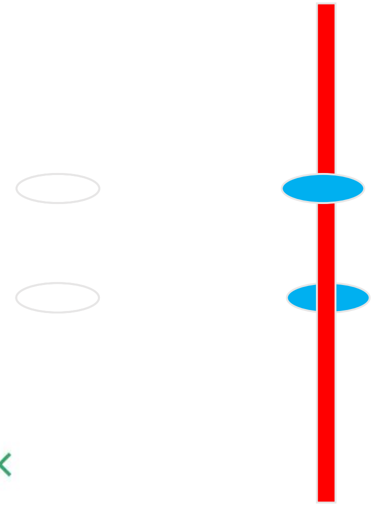
μ_0 = permeability constant
exactly $4\pi\times10^{-7}\ \mathrm{T\cdot m/A}$



- **The magnetic field wraps in circles around a wire. The direction of the magnetic field is easy to find using the right-hand rule.**
- **Put the thumb of your right hand in the direction of the current, and your fingers curl in the direction of B.**



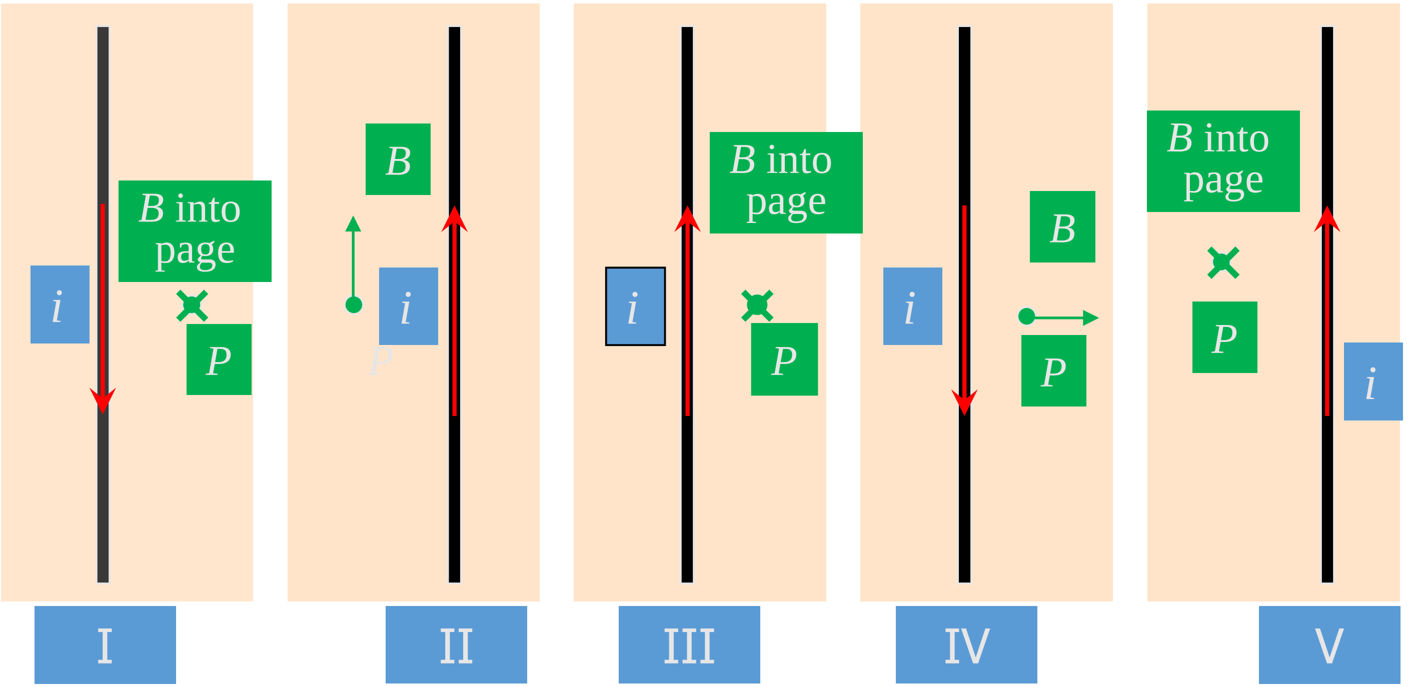
(a) (b)



Direction of Magnetic Field

1. **Which drawing below shows the correct direction of the magnetic field, *B*, at the point *P*?**

A. I.
B. II.
C. III.
D. IV.
E. V



B due to a Long Straight Wire

$$dB = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\vec{r}}{r^3}$$

Just add up all of the contributions *ds* to the current, keeping track of distance *r*.

$$B = 2\int_0^\infty dB = \frac{\mu_0 i}{2\pi}\int_0^\infty \frac{r\sin\theta\,ds}{r^3}$$

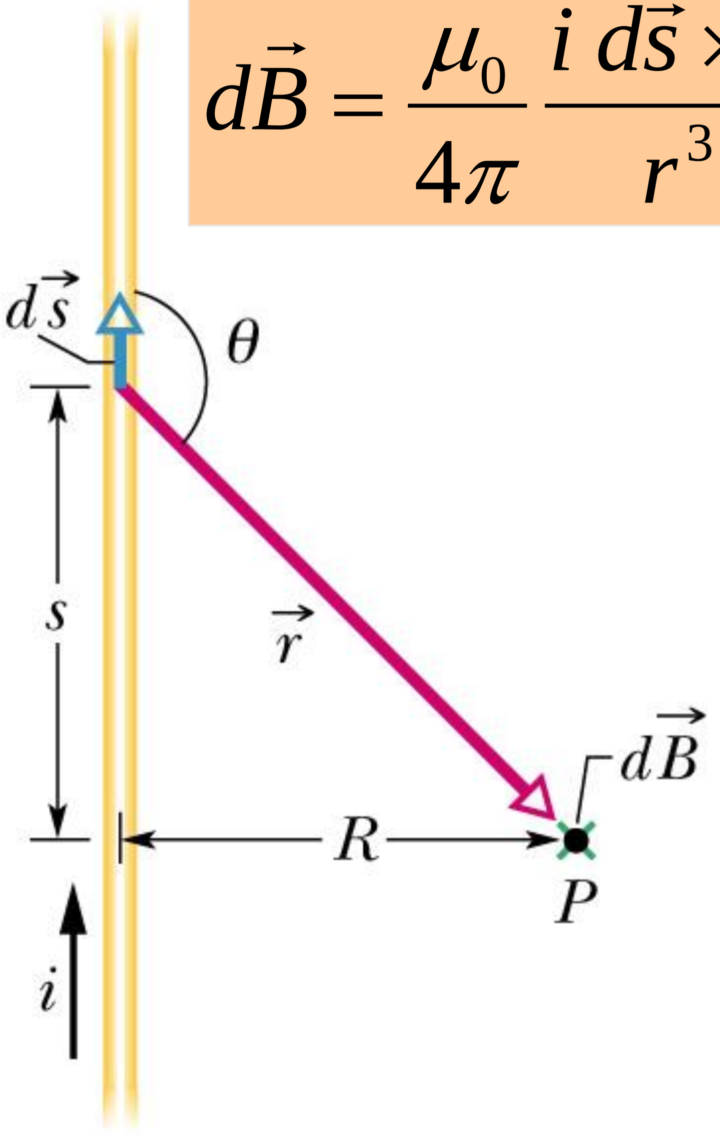
Notice that $r = \sqrt{R^2 + s^2}$

And *r* $\sin\theta$ **= *R*, So the integral becomes**

$$B = \frac{\mu_0 i}{2\pi}\int_0^\infty \frac{R\,ds}{(R^2+s^2)^{3/2}}$$

The integral is a little tricky, and finally

$$B = \frac{\mu_0 i}{2\pi R}\left[\frac{s}{\sqrt{R^2+s^2}}\right]_0^\infty = \frac{\mu_0 i}{2\pi R}$$



$$B = \frac{\mu_0 i}{2\pi R}$$

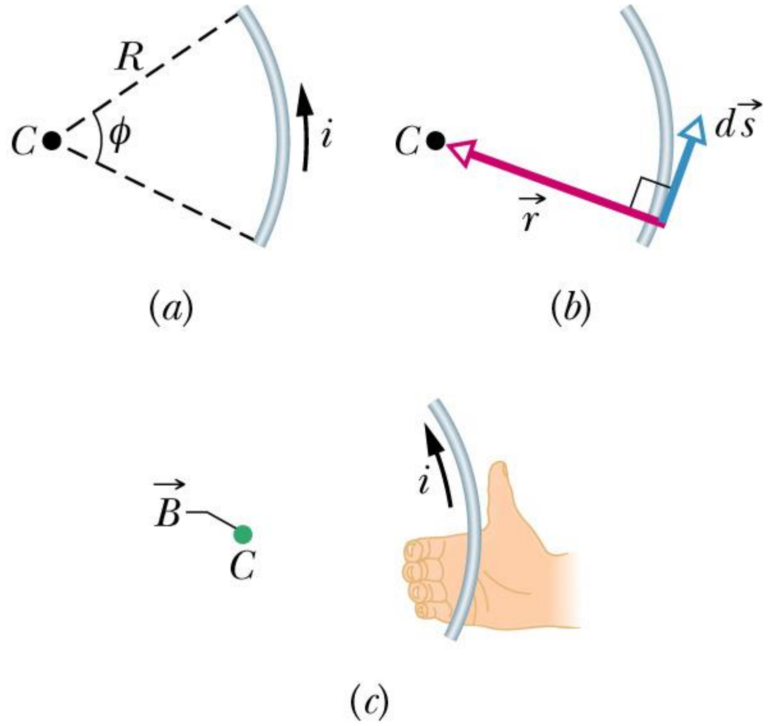
B due to current in a long straight wire

B at Center of a Circular Arc of Wire

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\vec{r}}{r^3}$$

- Just add up all of the contributions ds to the current, but now distance $r=R$ is constant, and $\vec{r}\perp d\vec{s}$.

$$B = \int_0^\phi dB = \frac{\mu_0 i}{4\pi R^2}\int_0^\phi ds$$



- Notice that $ds = Rd\phi$. So the integral becomes

$$B = \frac{\mu_0 i}{4\pi R^2}\int_0^\phi R d\phi = \frac{\mu_0 i\phi}{4\pi R}$$

$$B = \frac{\mu_0 i\phi}{4\pi R}$$

B due to current in circular arc

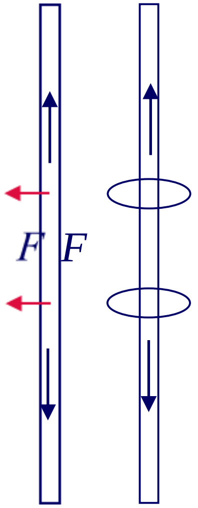
- For a complete loop, $\phi = 2\pi$, so $B = \frac{\mu_0 i}{2R}$

B at center of a full circle

Force Between Two Parallel Currents

$$\vec{F}_B = i\vec{L} \times \vec{B}$$

- Recall that a wire carrying a current in a magnetic field feels a force.
- When there are two parallel wires carrying current, the magnetic field from one causes a force on the other.
- When the currents are parallel, the two wires are pulled together.
- When the currents are anti-parallel, the two wires are forced apart.



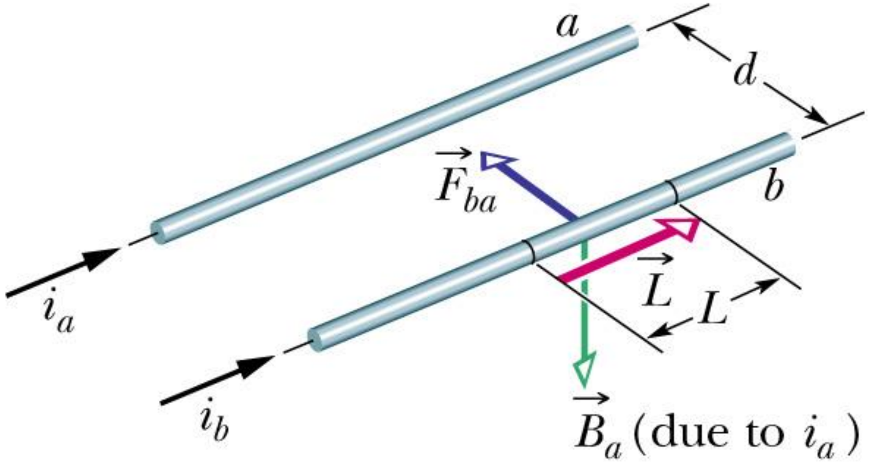
$$\vec{F}_{ba} = i_b\vec{L} \times \vec{B}_a$$

To calculate the force on *b* due to *a*,

$$B = \frac{\mu_0 i}{2\pi R} = \frac{\mu_0 i_a}{2\pi\, d}$$

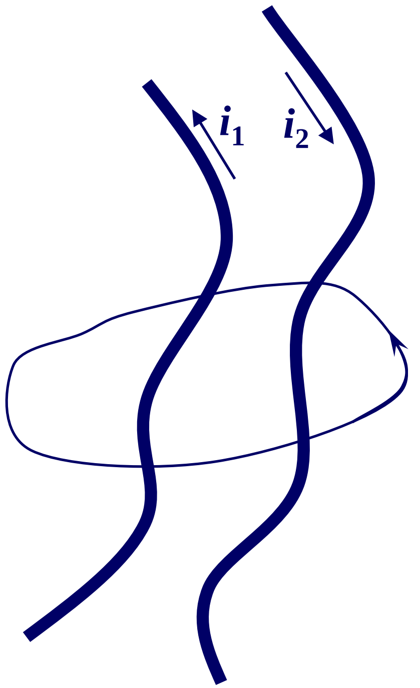
$$F_{ba} = \frac{\mu_0 i_a i_b L}{2\pi d}$$

Force between two parallel currents



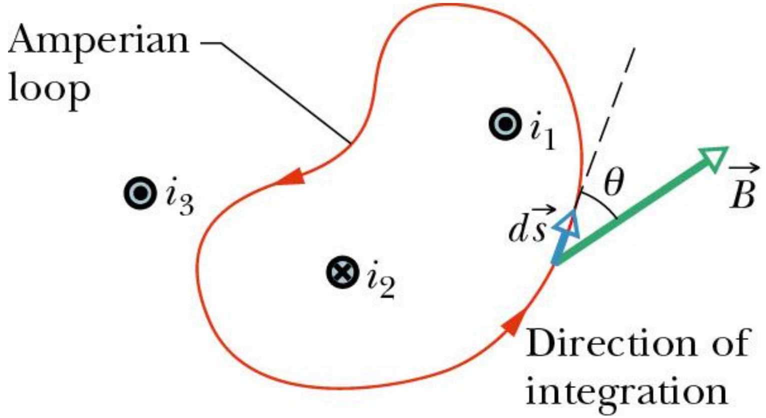
Ampere's Law/ Ampere's Circuital theorem

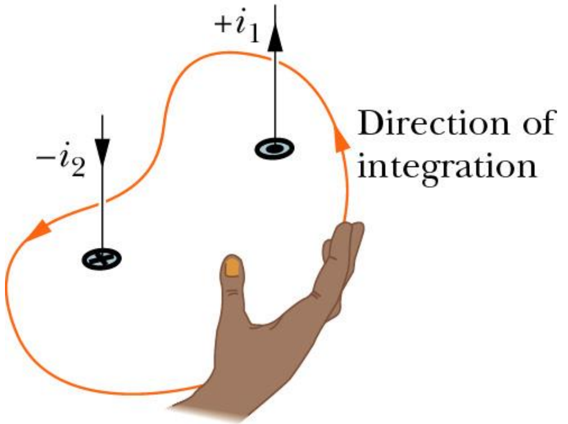
- Ampere's Law for magnetic fields is analogous to Gauss' Law for electric fields.
- Draw an "amperian loop" around a system of currents (like the two wires at right). The loop can be any shape, but it must be *closed*.
- Add up the component of $\vec{B}$ along the loop, for each element of length *ds* around this closed loop.
- The value of this integral is proportional to the current enclosed:



$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$$

Ampere's Law in integral form





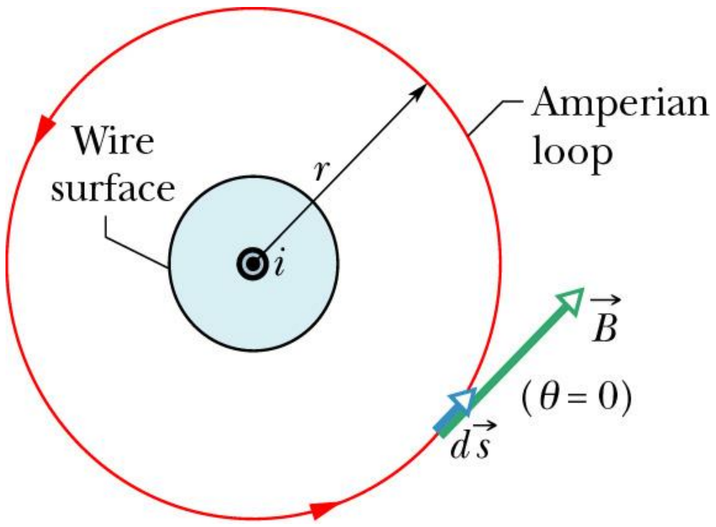
Magnetic Field Outside a Long Straight Wire with Current

- We already used the Biot-Savart Law to show that, for this case, $$B=\frac{\mu_0 i}{2\pi r}$$
- Let's show it again, using Ampere's Law:
- First, we are free to draw an Amperian loop of any shape, but since we know that the magnetic field goes in circles around a wire, let's choose a circular loop (of radius *r*).
- Then *B* and *ds* are parallel, and *B* is constant on the loop, so

$$\oint \vec{B}\cdot d\vec{s} = B2\pi r = \mu_0 i_{enc}$$

- And solving for B gives our earlier expression.

$$B=\frac{\mu_0 i}{2\pi r}$$



$$\oint \vec{B}\cdot d\vec{s} = \mu_0 i_{enc}$$

Ampere's Law

Let us start from Ampere's Law in integral form arrive at the Ampere's law in differential form.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$\mathbf{I_{enc}} = \oiint \vec{\mathbf{J}} \cdot \overrightarrow{\mathbf{da}}$$

Line Integral **Surface Integral**

Therefore, $\oint \vec{B} \cdot \overrightarrow{ds} = \mu_0 \oiint \vec{\mathbf{J}} \cdot \overrightarrow{\mathbf{da}}$

Using Stokes' theorem on the left hand side, the above expression can written as,

$$\oiint (\nabla \times \vec{B}) \cdot \overrightarrow{da} = \mu_0 \oiint \vec{\mathbf{J}} \cdot \overrightarrow{\mathbf{da}}$$

Since, the integration is computed inside a same material, the integral both sides cancels out.

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Ampere's Law in differential form

Unlike electric field, magnetic field is a rotational vector. $\nabla \times \vec{B} \neq 0$

Any vector which has zero divergence is known as divergence less or solenoidal vector. $\nabla \cdot \vec{B} = 0$, and therefore Magnetic field ($\vec{B}$) is also known as solenoidal vector.

Theorem: If the divergence of a vector field vanishes, then the vector can be expressed as the curl of another vector potential.

Since, $\nabla \cdot \vec{\mathrm{B}} = 0$ and therefore, $\vec{\mathrm{B}} = \nabla \times \vec{\mathrm{A}}$. $\vec{\mathrm{A}}$, is known as magnetic vector potential. It is similar to electric potential in electro-static.

When $\nabla \cdot \vec{\mathrm{B}} = 0$, $\iint \vec{\mathrm{B}} \cdot \overrightarrow{\mathrm{da}}$ is independent of the surface and $\oiint \vec{\mathrm{B}} \cdot \overrightarrow{\mathrm{da}} = 0$ for any closed surface.

You remember, the Poisson's equation in electrostatics! If not, I will repost that equation again here.

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

A similar equation exists in magneto statics also, we will get there shortly:

Let's start from differential form of Ampere's Law:

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

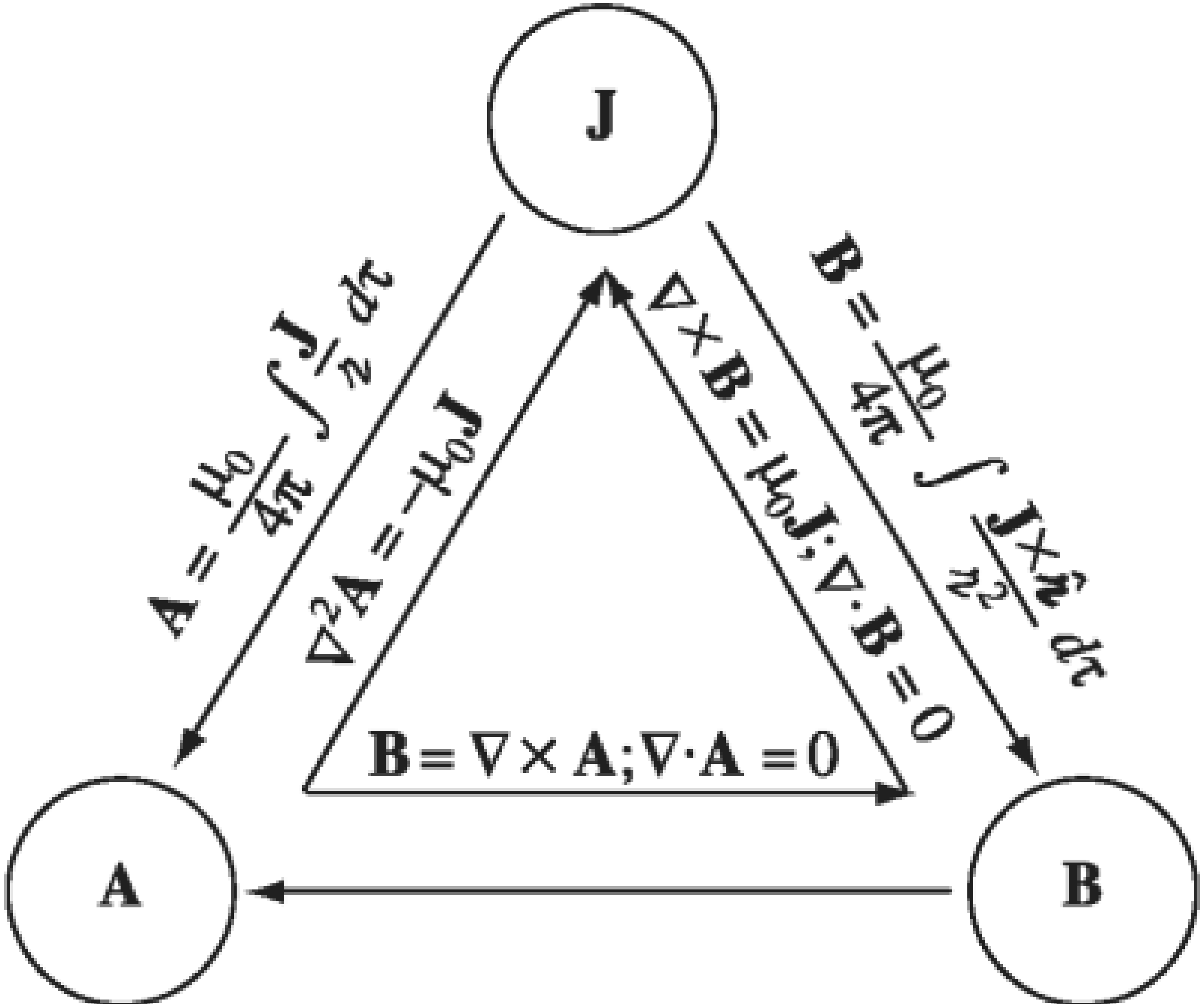
Plugging in the expression of $\vec{B}$ in terms of $\vec{A}$, the above expression becomes:

$$\nabla \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$$

$$\nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

As, $\nabla \cdot \vec{A} = 0$, then the above expression becomes, $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

Magneto-statics in one picture



Electromagnetism: Maxwell's Equations

“Electromagnetism is a branch of physics involving the study of the electromagnetic force, a type of physical interaction that occurs between electrically charged particles. The electromagnetic force is carried by electromagnetic fields composed of electric fields and magnetic fields, and it is responsible for electromagnetic radiation such as light.”- Wikipedia

Comparison between magneto-statics and electrostatics

$\nabla . \vec{E} = \frac{\rho}{\varepsilon_0}$ $\longrightarrow$ **Gauss’s Law**

$\nabla \times \vec{E} = 0$ $\longrightarrow$ **No Name**

Maxwell’s equations for electrostatics

$\nabla . \vec{B} = 0$ $\longrightarrow$ **No Name**

$\nabla \times \vec{B} = \mu_0 \vec{J}$ $\longrightarrow$ **Ampere’s Law**

Maxwell’s equations for magneto-statics

Maxwell's equations in electromagnetism before Maxwell's corrections

1. $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$	Gauss's Law
2. $\nabla \cdot \vec{B} = 0$	No Name (Non-existence of Magnetic Monopole)
3. $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$	Faraday's Law
4. $\nabla \times \vec{B} = \mu_0 \vec{J}$	Ampere's Law

These were the law's already invented by scientists before Maxwell's started working on electrodynamics theories. Maxwell's made correction to the last one (Ampere's Law) and re-derived Ampere's law (we will discuss this later) before his derivations of equations of electromagnetic waves. Because of his immense contributions in electrodynamics, all the above equations are combinedly known as Maxwell's numbered as first, second, third and forth as mentioned above chronologically. These are all in differential form and we will discover their integral form also.

Maxwell's First equation

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Differential Form, Also known as Gauss's Law.

By applying divergence theorem, it's integral form can be arrived. Already done in electrostatics.

$$\oiint_{\text{closed surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enclosed}}{\varepsilon_0}$$

Integral form of Maxwell's first equation

Maxwell's Second equation

$$\nabla \cdot \vec{B} = 0$$

Maxwell's Second equation, differential form. This equation conforms that there is no magnetic monopole.

Let us apply divergence theorem on both sides:

$$\boxed{\oiint \vec{B} \cdot \vec{da} = 0}$$

Maxwell's Second equation, integral form

Maxwell's Third equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Differential form

Also known as Faraday's law of induction

Statement: Whenever there is a rotational electric field (Curl of electric field), there has to a time variation of magnetic field but in opposite direction. Let's dig into this a bit more. Any rotational electric field produces a time variation of magnetic field. That means, we can produce magnetic field from electric field and *vice e versa*. How do you think electric generator work?

If we apply surface integral on the above expression it becomes,

$$\oiint (\nabla \times \vec{E}).\overrightarrow{da} = - \oiint \frac{\partial}{\partial t}(\vec{B}.\overrightarrow{da})$$ Applying Stokes's theorem on to the left

$$\boxed{\oint(\vec{E}\,.\,\overrightarrow{dl}) = -\frac{\partial}{\partial t}\left(\oiint \vec{B}\,.\,\overrightarrow{da}\right)}$$ Integral form

Derivation of Displacement Current

For a capacitor, $q = \varepsilon_0 EA$ **and** $I = \dfrac{dq}{dt} = \varepsilon_0 \dfrac{d(EA)}{dt}$ **.**

Now, the electric flux is given by EA, so: $I = \varepsilon_0 \dfrac{d(\Phi_E)}{dt}$**, where this current , not being associated with charges, is called the "Displacement current", I_d.**

Hence: $$\boxed{I_d = \varepsilon_0 \frac{d\Phi_E}{dt}}$$

Derivation of Displacement Current

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where this current, not being associated with charges, is called the "Displacement Current", I_d**.**

Hence:

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

and:

$$\oint \underline{B} \bullet \underline{dl} = \mu_0 (I + I_d)$$

Ampere's Law after Maxwell's Correction

$$\Rightarrow \oint \underline{B} \bullet \underline{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Let's take the surface integral on both sides

$$\oiint (\nabla \times \vec{B}) \cdot \overrightarrow{da} = \oiint \left(\mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \cdot \overrightarrow{da}$$

$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \oiint \frac{\partial \vec{E}}{\partial t} \overrightarrow{da}$$

Integral form of Maxwell's forth equation.

Differential form of Maxwell's Equations after Corrections

First	$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$	Gauss's Law
Second	$\nabla \cdot \vec{B} = 0$	No Name (Non-existence of Magnetic Monopole)
Third	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Faraday's Law of electromagnetic induction
Forth	$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$	Ampere's Law with Maxwell's Correction

Integral form of Maxwell's Equations after Corrections

First

$$\underset{\substack{\text{closed}\\\text{surface}}}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enclosed}}{\varepsilon_0}$$

Gauss's Law

Second

$$\oiint \overrightarrow{\mathbf{B}} \,.\, \overrightarrow{\mathbf{da}} = 0$$

No Name

Third

$$\oint (\vec{\mathbf{E}} \,.\, \overrightarrow{\mathbf{dl}}) = -\frac{\partial}{\partial t} (\oiint \overrightarrow{\mathbf{B}} \,.\, \overrightarrow{\mathbf{da}})$$

Faraday's Law of electromagnetic induction

Forth

$$\oint \overrightarrow{\mathbf{B}} . \overrightarrow{\mathbf{dl}} = \mu_0 \mathbf{I}_{enc} + \mu_0 \, \varepsilon_0 \oiint \frac{\partial \vec{\mathbf{E}}}{\partial t} \overrightarrow{\mathbf{da}}$$

Ampere's Law with Maxwell's Correction