Date
11/11/2019

Lecture 15

Assignment 2 on Tensor Calculus

submit by Thursday.
(14th Nov 2019)

End - Sem Examination
(50 marks) Time - 3 hours

Syllabus :- Bessel's function of first kind & Second kind their properties, generating f^n, orthogonality f^n:
Hyper - geometric f^n & series their properties.
Tensors kinds of tensor with examples, symmetric

Symmetric tensor, skew-symmetric tensor with examples, Operations on tensors, line element, length & metric tensor.

Christoffel symbols of first & second kind — their various properties.

(IV) $$\frac{\partial g^{mk}}{\partial x^l} = -g^{mi}\begin{Bmatrix} k \\ i\,l \end{Bmatrix} - g^{ki}\begin{Bmatrix} m \\ i\,l \end{Bmatrix}$$

Sol^n:- we know $$g^{ik}\, g_{ij} = \delta_j^k = 1 \text{ or } 0 \quad \longrightarrow (1)$$

Differentiating (1) w.r.t x^l, we obtain

$$\frac{\partial g^{ik}}{\partial x^l}\, g_{ij} + \frac{\partial g_{ij}}{\partial x^l}\, g^{ik} = 0 \quad \longrightarrow (2)$$

Inner multiplication of eq^n (2) by g^{jm}, we get

$$\frac{\partial g^{ik}}{\partial x^l}\, \underbrace{g_{ij}\, g^{jm}}_{=\delta_i^m} + \frac{\partial g_{ij}}{\partial x^l}\, g^{ik} . g^{jm} = 0$$

$$\Rightarrow \frac{\partial g^{ik}}{\partial x^l}\, \delta_i^m + g^{jm} . g^{ik} \frac{\partial g_{ij}}{\partial x^l} = 0$$

$$\therefore \frac{\partial g^{mk}}{\partial x^l} = -g^{jm} g^{ik} \Big[[il,j] + [jl,i] \Big]$$

$$= -g^{ik} g^{jm} [il,j] \quad [\text{by prop. III}]$$

$$\qquad - g^{jm} \underbrace{g^{ik} [jl,i]}$$

$$= -g^{ki} \begin{Bmatrix} m \\ il \end{Bmatrix} - g^{jm} \begin{Bmatrix} k \\ jl \end{Bmatrix} \quad [\text{by prop. II}]$$

$$= -g^{mi} \begin{Bmatrix} k \\ il \end{Bmatrix} - g^{ki} \begin{Bmatrix} m \\ il \end{Bmatrix}$$

$$\left[g^{ki} [jl,i] = \begin{Bmatrix} k \\ jl \end{Bmatrix} \right]$$

$$g^{jm} \begin{Bmatrix} k \\ jl \end{Bmatrix}$$

$$= g^{im} \begin{Bmatrix} k \\ il \end{Bmatrix} \quad \left[\begin{array}{c}\text{changing} \\ j \text{ by } i\end{array}\right]$$

$$= g^{mi} \begin{Bmatrix} k \\ il \end{Bmatrix}$$

Q1) Prove that the transformations of Christoffel symbols form a group.

$\left(\text{i.e., } x^i \to \bar{x}^i \to \bar{\bar{x}}^i \right)$

Q2) Deduce an expression for

$$\left\{ \begin{matrix} i \\ i\ j \end{matrix} \right\}.$$

or, Show that $\left\{ \begin{matrix} i \\ i\ j \end{matrix} \right\} = \frac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^j}$

$$= \frac{\partial}{\partial x^j} \left(\log \sqrt{g} \right)$$

where, $g = |g_{lm}|$, where g_{lm} is the fundamental tensor of rank 2.

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Q3) If the metric of a V_N is such that

$$g_{ij} = 0 \text{ for } i \neq j,$$ show that

$$\left[{i \atop j\,k} \right] = 0, \quad \left[{j \atop j\,j} \right] = \frac{1}{2g_{ii}} \frac{\partial g_{jj}}{\partial x^i},$$

$$\left[{i \atop i\,j} \right] = \frac{\partial}{\partial x^j} \left[\log \sqrt{g_{ii}} \right],$$

$$\left[{i \atop i\,i} \right] = \frac{\partial}{\partial x^i} \left[\log \sqrt{g_{ii}} \right],$$

where i, j & k are not equal & the summation convention does not apply.

ie., $i \neq j \neq k$, & $g_{ij} = 0$ for $i \neq j$,

$$g_{ij} = 0 = g_{ji}$$

Note :— since g is an invariant (or, scalar), $g = \bar{g}$, it does not follow that $\begin{bmatrix} i \\ i\,j \end{bmatrix}$ is a covariant tensor

If g is negative $(g < 0)$, the above eqn may be altered to

$$\begin{bmatrix} i \\ i\,j \end{bmatrix} = \frac{\partial}{\partial x^j} \left\{ \log \sqrt{-g} \right\},$$

$$(\text{as } -g > 0)$$

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Q 4) Calculate the Christoffel symbols corresponding to the metric

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1)^2 \sin^2 x^2 \, (dx^3)^2.$$

Hint :- we know

$$ds^2 = g_{ij} \, dx^i dx^j$$

(for $i, j = 1, 2, 3$)

Here, $g_{11} = 1$, $g_{22} = (x^1)^2$, $g_{33} = (x^1)^2 \sin^2 x^2$

& $g_{12} = g_{21} = 0$ i.e., $g_{ij} = 0$, when $i \neq j$.

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(V) Tensor laws of transformation of Christoffel symbols

Q) Show that Christoffel symbols, are not tensors.

proof :- The fundamental tensor g_{ij}, being a covariant tensor of rank 2, transforms according to the eq^n, for co-ordinates x^i to $\bar{x}^i$ as

$$\bar{g}_{lm} = \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^m} g_{ij} \longrightarrow (1)$$

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Diff. this eq^n (1) w.r.t $\bar{x}^n$, we get

$$\frac{\partial \bar{g}_{lm}}{\partial \bar{x}^n} = \frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^n}$$

$$+ \frac{\partial^2 x^i}{\partial \bar{x}^l \partial \bar{x}^n}\cdot\frac{\partial x^j}{\partial \bar{x}^m} g_{ij}$$

$$+ \frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial^2 x^j}{\partial \bar{x}^m \partial \bar{x}^n} g_{ij}$$

$$\left[\frac{d}{dx}(uvw) = uvw' + uv'w + u'vw \right]$$

$$= T_1 + T_2 + T_3 \rightarrow (2)$$

Slly, $$\bar{g}_{mn} = \frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial x^k}{\partial \bar{x}^n} g_{jk} \rightarrow (3)$$

& $$\bar{g}_{nl} = \frac{\partial x^k}{\partial \bar{x}^n}\frac{\partial x^i}{\partial \bar{x}^l} g_{ki} \rightarrow (4)$$

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On Diff. eq^n (3) w.r.t $\bar{x}^l$

& eq^n (4) w.r.t $\bar{x}^m$, we get

$$\frac{\partial \bar{g}_{mn}}{\partial \bar{x}^l} = T_4 + T_5 + T_6 \longrightarrow (5)$$

$$\frac{\partial \bar{g}_{nl}}{\partial \bar{x}^m} = T_7 + T_8 + T_9 \longrightarrow (6)$$

[changing of dummy suffixes k by i in eq^n (5) & k by j in (6),

$$\left[\text{as } \frac{\partial^2 x^i}{\partial \bar{x}^n \partial \bar{x}^m} = \frac{\partial^2 x^i}{\partial \bar{x}^m \partial \bar{x}^n}\right]$$

$T_2 = T_6$, $T_3 = T_8$

$$\therefore g_{ij} \frac{\partial^2 x^i}{\partial \bar{x}^l \partial \bar{x}^n} \frac{\partial x^j}{\partial \bar{x}^n} = g_{ij} \frac{\partial^2 x^j}{\partial \bar{x}^l \partial \bar{x}^n} \frac{\partial x^i}{\partial \bar{x}^m} \longrightarrow (7)$$

(11)

Now, $\frac{1}{2}\left[\frac{\partial}{\partial \bar{x}^n}(5) + \frac{\partial}{\partial \bar{x}^l}(6) - \frac{\partial}{\partial \bar{x}^n}(2)\right]$, gives

$$\overline{[l\,m, n]} = [i\,j, k]\,\frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial x^k}{\partial \bar{x}^n}$$

$$+ g_{ij}\,\frac{\partial x^i}{\partial \bar{x}^n}\,\frac{\partial^2 x^j}{\partial \bar{x}^l\,\partial \bar{x}^m} \longrightarrow (8)$$

slly,

$$\overline{\left\{ {p \atop l\,m} \right\}} = \left\{ {s \atop i\,j} \right\}\frac{\partial x^i}{\partial \bar{x}^l}\,\frac{\partial x^j}{\partial \bar{x}^m}\,\frac{\partial \bar{x}^p}{\partial x^s}$$

$$+ \frac{\partial \bar{x}^p}{\partial x^j}\,\frac{\partial^2 x^j}{\partial \bar{x}^l\,\partial \bar{x}^m} \longrightarrow (9)$$

The above eq^n 8 & 9 represent the transformation laws of the Christoffel symbols of first & second kinds resp'ly.

eq^n 8 & 9 prove that they do not follows the tensor law of transformation & hence these are <u>not</u> tensors.

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eg:, However, $x^i = a^i \bar{x}^l + b^i$

linear transforms _____ ...