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Mathematical Methods Roll: 18MA20058

1) From the Generating function of $P_n(x)$, we have:

$$\sum_{n=0}^{\infty} P_n(x) \cdot t^n = (1 - 2xt + t^2)^{-1/2} = [1 - t(2x - t)]^{-1/2}$$

On expanding RHS, we get:

$$\sum_{n=0}^{\infty} P_n(x) \cdot t^n = 1 + \frac{1}{2} \times t(2x-t) + \frac{\frac{1}{2} \times \frac{3}{2}}{2!} \times t^2 (2x-t)^2$$

$$+ \frac{\frac{1}{2} \times \frac{3}{2} \times \frac{5}{2}}{3!} \times t^3 (2x-t)^3 + \frac{\frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \times \frac{7}{2}}{4!} \times t^4 (2x-t)^4$$

$+ \ldots$ (terms with powers of $t \geq 5$)

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x) \cdot t^n = 1 + \frac{1}{2} t(2x-t) + \frac{3}{8} t^2 (2x-t)^2 + \frac{5}{16} t^3 (2x-t)^3$$

$$+ \frac{35}{128} t^4 (2x-t)^4 + \ldots$$ (terms with powers of $t \geq 5$)

Comparing like powers of t from LHS and RHS :

Coeff. of t^0 : $P_0(x) = 1$

Coeff. of t^1 : $P_1(x) = \frac{1}{2} \times (2x) = x$

Coeff. of t^2 : $P_2(x) = \frac{1}{2} \times (-1) + \frac{3}{8} \times (2x)^2 = \frac{3}{2} x^2 - \frac{1}{2}$

$$\Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Coeff. of t^3 : $P_3(x) = \frac{3}{8} \times {}^2C_1 \times (2x) \times (-1) + \frac{5}{16} \times (2x)^3$

$$\Rightarrow P_3(x) = \frac{-3}{2} x + \frac{5}{2} x^3 = \frac{1}{2}(5x^3 - 3x)$$

Coeff. of t^4 : $P_4(x) = \frac{3}{8} \times (-1)^2 + \frac{5}{16} \times {}^3C_1 \times (2x)^2 \times (-1) + \frac{35}{128} \times (2x)^4$

$$\Rightarrow P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Hence, we got : $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2} \times (5x^3 - 3x)$ and $P_4(x) = \frac{1}{8} \times (35x^4 - 30x^2 + 3)$

[Proved.]

2) We have, $1 = P_0(x)$ ... (i) , $x = P_1(x)$ .... (ii)

$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow \frac{2}{3}P_2(x) = x^2 - \frac{1}{3}$

$$\Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x) \quad \text{.... (iii)}$$

$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow \frac{2}{5}P_3(x) = x^3 - \frac{3}{5}x$

$$\Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) \quad \text{.... (iv)}$$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \Rightarrow \frac{8}{35}P_4(x) = x^4 - \frac{30}{35}x^2 + \frac{3}{35}$

$$\Rightarrow x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}P_0(x)$$

$$= \frac{8}{35}P_4(x) + \frac{6}{7} \times \left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right) - \frac{3}{35}P_0(x) \quad \text{[From (iii)]}$$

$$\Rightarrow x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{2}{7}P_0(x) - \frac{3}{35}P_0(x) \quad \text{.... (v)}$$

$\therefore P = x^4 + 2x^3 + 2x^2 - x - 3$

$$= \left(\frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{2}{7}P_0 - \frac{3}{35}P_0\right) + 2 \times \left(\frac{2}{5}P_3 + \frac{3}{5}P_1\right)$$

$$+ 2 \times \left(\frac{2}{3}P_2 + \frac{1}{3}P_0\right) - (P_1) - 3 \times (P_0) \quad \text{[From (i) - (v)]}$$

$$\Rightarrow P = \frac{8}{35}P_4 + \frac{4}{5}P_3 + \left(\frac{4}{7}+\frac{4}{3}\right)P_2 + \left(\frac{6}{5}-1\right)P_1 + \left(\frac{2}{7}-\frac{3}{35}+\frac{2}{3}-3\right)P_0$$

$$\Rightarrow P = \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{224}{105}P_0(x)$$

[Proved.]

3) We know, $P_0 = P_1'$ $\quad [\because P_0 = 1,\ P_1 = x]$

And, in the recurrence relation $(2n+1)P_n = P'_{n+1} - P'_{n-1}$,

putting $n = 1, 2, 3, \ldots$ we get :

$n=1$: $\quad 3P_1 = P_2' - P_0'$

$n=2$: $\quad 5P_2 = P_3' - P_1'$

$n=3$: $\quad 7P_3 = P_4' - P_2'$

$\vdots$

$(n-2)$: $\quad (2n-3)P_{n-2} = P'_{n-1} - P'_{n-3}$

$(n-1)$: $\quad (2n-1)P_{n-1} = P_n' - P'_{n-2}$

n : $\quad (2n+1)P_n = P'_{n+1} - P'_{n-1}$

Adding all these (the first one $P_0 = P_1'$ also), we get :

$$P_0 + 3P_1 + 5P_2 + \cdots + (2n+1)P_n = P'_{n+1} + P_n' - P_0'$$

$$= P'_{n+1} + P_n' \quad [\because P_0 = 1 \Rightarrow P_0' = 0]$$

Hence, $P'_{n+1} + P_n' = P_0 + 3P_1 + 5P_2 + \cdots + (2n+1)P_n$ [Proved.]

4) Integrating by parts, we get :

$$\int_{-1}^{1} (1-x^2) \cdot P_m' \cdot P_n' \cdot dx = \left[(1-x^2) \cdot P_m' \cdot P_n \right]_{-1}^{1} - \int_{-1}^{1} \frac{d}{dx} \{ (1-x^2) \cdot P_m' \} P_n \cdot dx$$

$$= 0 - \int_{-1}^{1} \{ (1-x^2) P_m'' - 2x \cdot P_m' \} \cdot P_n \cdot dx \quad \cdots \text{(i)}$$

Now, as $P_m(x)$ is a solution of Legendre's equation, it satisfies : $(1-x^2) \cdot P_m'' - 2x \cdot P_m' + m(m+1) P_m = 0$

$$\Rightarrow (1-x^2) \cdot P_m'' - 2x \cdot P_m' = -m(m+1) P_m \quad \cdots \text{(ii)}$$

Putting (ii) in (i), we get :

$$\int_{-1}^{1} (1-x^2) \cdot P_m' \cdot P_n' \cdot dx = \int_{-1}^{1} m(m+1) \cdot P_m \cdot P_n \cdot dx$$

$$= m(m+1) \int_{-1}^{1} P_m(x) \cdot P_n(x) \cdot dx$$

$$= \underline{\underline{0}}$$ [$\because$ $m \neq n$ is given . So P_m and P_n are orthogonal on $-1 \leq x \leq 1$]

[Proved.]

5) From the recurrence relation : $(2n+1) x P_n = (n+1) P_{n+1} + n P_{n-1}$,

we get : $$x P_{n-1} = \frac{1}{(2n-1)} \times \left[n P_n + (n-1) P_{n-2} \right] \quad \cdots \text{(i)}$$

And, $$x P_{n+1} = \frac{1}{(2n+3)} \times \left[(n+2) P_{n+2} + (n+1) P_n \right] \quad \cdots \text{(ii)}$$

Multiplying (i) and (ii), we get :

$$x^2 P_{n+1} P_{n-1} = \frac{1}{(2n-1)(2n+3)} \times \Big[n(n+2) P_n P_{n+2} + n(n+1) P_n^2 + (n-1)(n+2) P_{n-2} P_{n+2} + (n^2-1) P_n P_{n-2} \Big]$$

Integrating both sides from -1 to 1 and using the property that $P_m(x)$ and $P_n(x)$ are orthogonal when $m \neq n$, we get :

$$\int_{-1}^{1} x^2 P_{n-1} P_{n+1} \cdot dx = \frac{n(n+1)}{(2n-1)(2n+3)} \int_{-1}^{1} P_n^2 \cdot dx$$

$$\Rightarrow \int_{-1}^{1} x^2 \cdot P_{n+1} \cdot P_{n-1} \cdot dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)} \qquad \left[\because \int_{-1}^{1} P_n^2 \cdot dx = \frac{2}{2n+1}\right]$$

[Proved.]

6) We know, $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) \cdot t^n$

Differentiating partially w.r.t t, we get :

$$\frac{-1}{2} \times (1-2xt+t^2)^{-3/2} \times (2t-2x) = \sum_{n=1}^{\infty} n \cdot P_n(x) \cdot t^{n-1}$$

Multiplying by t on both sides :

$$t(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} n \cdot P_n(x) \cdot t^n$$

$$\Rightarrow (tx-t^2)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n \cdot P_n(x) \cdot t^n \quad \cdots\cdots (i) \qquad \left[\because \text{At } n=0, \text{ it is } 0\right]$$

Again, we know, $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) \cdot t^n$

Multiplying by t on both sides :

$$t(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) \cdot t^{n+1}$$

Now, differentiating partially w.r.t t, we get :

$$(1-2xt+t^2)^{-1/2} - \frac{1}{2} \times t(1-2xt+t^2)^{-3/2} \times (2t-2x) = \sum_{n=0}^{\infty} P_n(x) \cdot (n+1) \cdot t^n$$

$$\cdots\cdots (ii)$$

Adding (i) and (ii),

$$\sum_{n=0}^{\infty} (2n+1)\cdot P_n(x)\cdot t^n = (1-2xt+t^2)^{-1/2} + 2x(tx-t^2)(1-2xt+t^2)^{-3/2}$$

$$= (1-2xt+t^2)^{-3/2} \times \left[1-2xt+t^2+2xt-2t^2\right]$$

$$= (1-2xt+t^2)^{-3/2} \times (1-t^2)$$

Hence, $$\frac{1-h^2}{(1-2xh+h^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1)\cdot P_n(x)\cdot h^n$$ [Proved.]

7) We know, $$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\cdot t^n$$

Putting $x=1$: $$(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1)\cdot t^n$$

So, $$\sum_{n=0}^{\infty} P_n(1)\cdot t^n = \left[(1-t)^2\right]^{-1/2} = (1-t)^{-1}$$

$$= 1+t+t^2+t^3+\dots$$

Equating the coefficients of t^n from both sides, we get :

$P_n(1) = 1 \quad \forall\, n = 0, 1, 2, \dots$ [Proved.]