$$\Rightarrow -\frac{1}{k}\int_{c_2}^{x} f(y)\cos ky\, dy = -\frac{1}{k}\int_0^{x} f(y)\cos ky\, dy - \frac{1}{k\sin kl}\int_0^{l} f(y)\sin k(y-l)\, dy$$

Putting this value in (7),

$$u(x) = \frac{\cos kx}{k}\int_0^{x} f(y)\sin ky\, dy - \frac{\sin kx}{k}\int_0^{x} f(y)\cos ky\, dy - \frac{\sin kx}{k\sin kl}\int_0^{l} f(y)\sin k(y-l)\, dy$$

$$= \frac{1}{k}\int_0^{x} f(y)\sin k(y-x)\, dy - \frac{\sin kx}{k\sin kl}\int_0^{l} f(y)\sin k(y-l)\, dy \quad \text{——(10)}$$

$$= \frac{1}{k}\int_0^{x} f(y)\sin k(y-x)\, dy - \frac{\sin kx}{k\sin kl}\int_0^{x} f(y)\sin k(y-l)\, dy - \frac{\sin kx}{k\sin kl}\int_x^{l} f(y)\sin k(y-l)\, dy$$

$$= \int_0^{x} \frac{f(y)\sin ky \sin k(l-x)}{k\sin kl}\, dy + \int_x^{l} \frac{f(y)\sin kx \sin k(l-y)}{k\sin kl}\, dy$$

$$= \int_0^{l} f(y)\, G(x,y)\, dy \quad \text{—— (11)}$$

where
$$G(x,y) = \begin{cases} \dfrac{\sin ky \sin k(l-x)}{k\sin kl}, & 0 \le y \le x \\[2ex] \dfrac{\sin kx \sin k(l-y)}{k\sin kl}, & x \le y \le l \end{cases} \quad \text{——(12)}$$

This function $G(x,y)$ is known as Green's function for the Eq. (1) and B.C. (2). It's existence for this particular problem is assured provided $\sin kl \neq 0$. Therefore when $G(x,y)$ exists and it is known explicitely, then we can immediately write down the solution to the BVP (1) and (2) in the simple form (11).

We notice that $G(x,y)$ defined in (12) has the following properties:

1. It satisfies the homogeneous form of the given differential equation i.e. $G'' + k^2 G = 0$ in each of the intervals $0 \le y < x$, $x < y \le l$. The behaviour of G at $y = x$ is at this moment, uncertain.

2. The function G is continuous at $y = x$ since

$$\lim_{y \to x-} G(x,y) = \frac{\sin kx \sin k(l-x)}{k \sin kl} = \lim_{y \to x+} G(x,y)$$

3. The derivative of G w.r.t. y is discontinuous at $y = x$. This can be seen as follows;

$$G'(x, x^-) = \lim_{y \to x^-} G'(x,y) = \frac{\cos kx \sin k(l-x)}{\sin kl}$$

$$G'(x, x^+) = \lim_{y \to x^+} G'(x,y) = \frac{-\sin kx \cos k(l-x)}{\sin kl}$$

$$\therefore G'(x, x^+) - G'(x, x^-) = -1$$

4. $G(x,y)$ satisfies $G(x,0) = G(x,l) = 0$ and thus $G(x,y)$ satisfies the B.C. of the problem.

5. $G(x,y) = G(y,x)$

With these properties of the Green's function in mind, we now try to solve BVP (1) & (2) assuming that the Green's function $G(x,y)$ exists.

By multiplying both sides of (1) by a f^n. $G(x,y)$ and integrating w.r.t. x over $0 \le x \le l$, we obtain

$$\int_0^l (u'' + k^2 u)\, G(x,y)\, dx = -\int_0^l f(x)\, G(x,y)\, dx \quad \text{—(13)}$$

We exclude the point $x = y$ from the range of integration and write,

$$\int_0^l (u'' + k^2 u)\, G(x,y)\, dx = \lim_{\xi \to y-} \int_0^{\xi} (u'' + k^2 u)\, G(x,y)\, dx$$

$$+ \lim_{\eta \to y+} \int_{\eta}^{l} (u'' + k^2 u)\, G(x,y)\, dx \quad \text{—(14)}$$

Treating each integral on the RHS separately, we integrate twice by parts

$$\int_0^{\xi} (u'' + k^2 u)\, G(x,y)\, dx = \int_0^{\xi} u'' G\, dx + \int_0^{\xi} k^2 u G\, dx$$

$$= G u' \Big|_0^{\xi} - \int_0^{\xi} G' u'\, dx + \int_0^{\xi} k^2 u\, G\, dx$$

$$= G u' \Big|_0^{\xi} - G' u \Big|_0^{\xi} + \int_0^{\xi} G'' u\, dx + \int_0^{\xi} k^2 u\, G\, dx$$

$$= \Big[G u' - G' u \Big]_0^{\xi} + \int_0^{\xi} u\, (G'' + k^2 G)\, dx$$

Similarly

$$\int_\eta^l (u''+k^2u)\,G(x,y)\,dx = \Big[Gu'-G'u\Big]_\eta^l + \int_\eta^l u\,(G''+k^2G)\,dx$$

If we choose $G(x,y)$ to satisfy $G''+k^2G=0$ in $0\le x\le \xi$ and $\eta\le x\le l$, then the integrals on the RHS vanish. Inserting the remaining integrated terms into (14) and taking appropriate limits,

$$\begin{aligned}-\int_0^l f(x)\,G(x,y)\,dx = \Big\{ & G(y,y^-)\,u'(y^-) - G'(y,y^-)\,u(y^-) \\ & - \cancel{G(0,y)}\,u'(0) + G'(0,y)\,\cancel{u(0)}^{B.C.} \\ & + \cancel{G(l,y)}\,u'(l) - G'(l)\,\cancel{u(l)}^{B.C.} \\ & - G(y,y^+)\,u'(y^+) + G'(y,y^+)\,u(y^+)\Big\}\end{aligned}$$

provided $G(x,y) = G(y,x)$

If now we assume that $G(x,y)$ satisfies the B.C. (2), then we can obtain

$$\begin{aligned}\int_0^l f(x)\,G(x,y)\,dx = & -u(y)\Big\{G'(y,y^+) - G'(y,y^-)\Big\} \\ & + u'(y)\Big\{G(y,y^+) - G(y,y^-)\Big\} \qquad -(15)\end{aligned}$$

by assuming u & u' to be continuous and hence

$$u(y^+) = u(y^-) = u(y)$$
$$u'(y^+) = u'(y^-) = u'(y)$$