Solution of BVP using Green's function

Ex Using Green's function, solve the BVP

$$y'' + y = x, \quad y(0) = y\left(\frac{\pi}{2}\right) = 0$$

Sol^n: Consider the associated BVP $y'' + y = 0$

with $y(0) = y\left(\frac{\pi}{2}\right) = 0$

G.S. is $y = A\cos x + B\sin x$ & B.C.s give $A = B = 0$

It is having only trivial solution.

$$\text{Let } G(x,t) = \begin{cases} a_1\cos x + a_2\sin x & 0 \le x < t \\ b_1\cos x + b_2\sin x & t < x \le \frac{\pi}{2} \end{cases} \quad \text{— (3)}$$

Green's f^n must also satisfy

(i) $b_1\cos t + b_2\sin t = a_1\cos t + a_2\sin t$

$\Rightarrow (b_1 - a_1)\cos t + (b_2 - a_2)\sin t = 0$

(ii) $$\left(\frac{\partial G}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G}{\partial x}\right)_{x=t-0} = -1$$

$\Rightarrow -b_1\sin t + b_2\cos t - (-a_1\sin t + a_2\cos t) = -1$

$\Rightarrow -(b_1 - a_1)\sin t + (b_2 - a_2)\cos t = -1$

(iii) $G(0,t) = 0$ so that $a_1 = 0$

$G\left(\frac{\pi}{2}, t\right) = 0$ " " $b_2 = 0$

Let $b_1 - a_1 = c_1$ $\quad b_2 - a_2 = c_2$

$c_1\cos t + c_2\sin t = 0$

$-c_1\sin t + c_2\cos t + 1 = 0$

$$\therefore \frac{c_1}{\sin t} = \frac{c_2}{-\cos t} = \frac{1}{\cos^2 t + \sin^2 t}$$

$\therefore c_1 = \sin t \quad c_2 = -\cos t$

$b_1 - a_1 = \sin t \quad a_1 = 0 \quad b_1 = \sin t$

$b_2 - a_2 = -\cos t \quad b_2 = 0 \quad a_2 = \cos t$

$$\therefore G(x,t) = \begin{cases} \cos t \sin x & 0 \le x < t \\ \sin t \cos x & t < x \le \frac{\pi}{2} \end{cases}$$

Solⁿ. of the D.E is given as

$$y(x) = \int_0^{\pi/2} G(x,t)\,\phi(t)\,dt$$

$\phi(x) = -x$ so that $\phi(t) = -t$

Reqd. solⁿ. is given by

$$y(x) = -\int_0^{\pi/2} G(x,t)\,t\,dt$$

$$= -\left[\int_0^{x} t\,G(x,t)\,dt + \int_x^{\pi/2} t\,G(x,t)\,dt\right]$$

$$= -\left[\int_0^{x} t \sin t \cos x\,dt + \int_x^{\pi/2} t \cos t \sin x\,dt\right]$$

$$= -\cos x \int_0^{x} t \sin t\,dt - \sin x \int_x^{\pi/2} t \cos t\,dt$$

$$= -\cos x \left[(-t\cos t)_0^{x} - \int_0^{x}(-\cos t)\,dt\right]$$

$$- \sin x \left[(t \sin t)_x^{\pi/2} - \int_x^{\pi/2} \sin t\,dt\right]$$

$$= \cos x\left[-x\cos x + \sin x\right] - \sin x\left[\frac{\pi}{2} - x\sin x - \cos x\right]$$

$$= x - \frac{\pi}{2}\sin x$$

Orthogonal set of functions and Strum-Liouville problem

Orthogonality

Two functions $f(x)$ and $g(x)$ defined on some interval $a \le x \le b$ are said to be orthogonal on $a \le x \le b$ if

$$\int_a^b f(x)\, g(x)\, dx = 0$$

The norm of $f(x)$, denoted by $\|f(x)\|$ is defined by

$$\|f(x)\| = \left\{ \int_a^b f^2(x)\, dx \right\}^{1/2}$$

Orthogonal set of functions

Consider the set of functions $\{f_n(x)\}$ $n=1,2,3,\ldots$ defined in $a \le x \le b$. Then $\{f_n(x)\}$ is said to be an orthogonal set of functions in $a \le x \le b$ if

$$\int_a^b f_m(x)\, f_n(x)\, dx = 0, \qquad m \ne n$$

Orthonormal set of functions

$$\int_a^b f_m(x)\, f_n(x)\, dx = \begin{cases} 0 & \text{when } m \ne n \\ 1 & \text{when } m = n \end{cases}$$

[(i) Orthogonal (ii) norm 1]

Orthogonality w.r.t. a weight function

Let $p(x) > 0$. $f(x)$ & $g(x)$ defined in $[a,b]$.
Then $f(x)$ and $g(x)$ are orthogonal w.r.t. $p(x)$ if

$$\int_a^b p(x)\, f(x)\, g(x) = 0$$

$$\therefore \|f(x)\| = \left\{ \int_a^b p(x)\, f^2(x)\, dx \right\}^{1/2}$$

Orthogonal set of functions w.r.t. a weight function

$$\int_a^b p(x)\, f_m(x)\, f_n(x)\, dx = 0 \qquad m \neq n$$

Orthonormal set of functions w.r.t. a weight function

$$\int_a^b p(x)\, f_m(x)\, f_n(x) = \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n \end{cases}$$

Working rule for getting orthonormal set $\{f_n(x)\}$ of functions corresponding to a known orthogonal set $\{f_n(x)\}$ where $n = 1, 2, 3, \ldots$ and none of the functions $f_n(x)$ has norm zero.

Divide each f^n. $f_n(x)$ by its norm $\| f_n(x) \|$ and get a new f^n. $\varphi_n(x) = \dfrac{f_n(x)}{\| f_n(x) \|}$

$$\| \varphi_n(x) \| = \left\{ \int_a^b \varphi_n^2(x)\, dx \right\}^{1/2}$$

$$= \left[\int_a^b \left\{ \frac{f_n(x)}{\| f_n(x) \|} \right\}^2 dx \right]^{1/2}$$

$$= \frac{1}{\| f_n(x) \|} \left\{ \int_a^b f_n^2(x)\, dx \right\}^{1/2}$$

$$= \frac{1}{\| f_n(x) \|} \| f_n(x) \| = 1$$

$\therefore$ $\{ \varphi_n(x) \}$ is an orthonormal set of functions.

Gram-Schmidt process of orthonormalization

Let $\{f_n(x)\}$ where $n = 1, 2, \ldots n$ be a set of L.I. functions for each of which norm $\|f_n(x)\|$ exists and is non-zero. Then we wish to obtain an orthonormal set $\{\varphi_n(x)\}$ where $n = 1, 2, 3, \ldots$ such that

$$\int_a^b \varphi_m(x)\,\varphi_n(x)\,dx = \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n \end{cases} \quad \text{--- (1)}$$

We select $f_1(x)$ and obtain $\varphi_1(x) = \dfrac{f_1(x)}{\|f_1(x)\|}$ — (2)

We next choose $f_2(x)$ and let $F_2(x) = f_2 + c\varphi_1$ — (3)
where c is chosen in such a manner that F_2 is orthogonal to φ_1.

i.e. $\int_a^b F_2 \varphi_1\,dx = 0$

$$\Rightarrow \int_a^b (f_2 + c\varphi_1)\,\varphi_1\,dx = 0 \quad \text{by (3)}$$

$$\Rightarrow \int_a^b f_2\varphi_1\,dx + c\int_a^b \varphi_1^2\,dx = 0$$

$$\Rightarrow \int_a^b f_2\varphi_1\,dx + c = 0 \quad \text{by (1)}$$

$$\Rightarrow c = -\int_a^b f_2\,\varphi_1\,dx$$

$\therefore$ (3) gives $F_2 = f_2 - \varphi_1 \int_a^b f_2\,\varphi_1\,dx$ — (4)

Now we take $\varphi_2(x) = \dfrac{F_2}{\|F_2\|}$ — (5)

Next we choose f_3 and let $F_3(x) = f_3 + c_1\varphi_1 + c_2\varphi_2$ — (6)
where c_1 and c_2 are chosen in such a manner that F_3 is orthogonal to φ_1 and φ_2 i.e.

$$\int_a^b (f_3 + c_1\varphi_1 + c_2\varphi_2)\varphi_1 \, dx = 0 \quad \text{and}$$

$$\int_a^b (f_3 + c_1\varphi_1 + c_2\varphi_2)\varphi_2 \, dx = 0$$

$\therefore \int_a^b f_3\varphi_1 \, dx + c_1 = 0$ and $\int_a^b f_3\varphi_2 \, dx + c_2 = 0$ by (1)

$$\therefore c_1 = -\int_a^b f_3\varphi_1 \, dx \quad \text{and}$$

$$c_2 = -\int_a^b f_3\varphi_2 \, dx \qquad \text{— (7)}$$

Using (7), (6) gives

$$F_3 = f_3 - \varphi_1\int_a^b f_3\varphi_1 \, dx - \varphi_2\int_a^b f_3\varphi_2 \, dx \qquad \text{— (8)}$$

$$\therefore \varphi_3(x) = \frac{F_3}{\|F_3\|}$$

Proceeding in above manner, $\varphi_n = \dfrac{F_n}{\|F_n\|}$

Ex With the help of $1, x, x^2$ construct three functions φ_0, φ_1 and φ_2 which are orthogonal over $-1 \le x \le 1$.

Sol^n: We take $\varphi_0(x) = 1$

Next we choose $\varphi_1 = x + c\,\varphi_0(x) = x + c$

Let φ_1 be orthogonal to φ_0 so that

$$\int_{-1}^{1} \varphi_0 \varphi_1 \, dx = 0 \quad \text{or,} \quad \int_{-1}^{1} (x+c)\, dx = 0$$

$$\Rightarrow \left[\frac{x^2}{2} + cx\right]_{-1}^{1} = 0 \quad \text{giving } c = 0$$

$\therefore \varphi_1 = x$

Next we choose $\varphi_2 = x^2 + c_1 \varphi_0 + c_2 \varphi_1 = x^2 + c_1 + c_2 x$

Let φ_2 be orthogonal to both φ_0 and φ_1 so that

$$\int_{-1}^{1} \varphi_2 \varphi_0 \, dx = 0 \quad \& \quad \int_{-1}^{1} \varphi_2 \varphi_1 \, dx = 0$$

$$\Rightarrow \frac{2}{3} + 2c_1 = 0 \qquad c_2 \times \frac{2}{3} = 0$$

$$c_1 = -\frac{1}{3} \qquad c_2 = 0$$

$$\therefore \varphi_0 = 1 \qquad \varphi_1 = x \qquad \varphi_2 = x^2 - \frac{1}{3}$$

Ex Given the set of functions $1, x, x^2, x^3, \ldots$. Obtain from these a set of functions which are mutually orthonormal in $(-1,1)$.

Solⁿ: Let $\{\varphi_n(x)\}$ be the required orthonormal set of fⁿs. so that

$$\int_{-1}^{1} \varphi_n \varphi_m \, dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \quad \text{— (1)}$$

Step-1 Choose $f_1(x) = 1$ and take $\varphi_1(x) = \dfrac{f_1(x)}{\|f_1(x)\|}$

Now $\|f_1(x)\| = \left[\int_{-1}^{1} f_1^2(x)\,dx\right]^{1/2} = \left[\int_{-1}^{1} 1^2\,dx\right]^{1/2} = \sqrt{2}$

$$\therefore \varphi_1(x) = \frac{1}{\sqrt{2}} \quad \text{— (2)}$$

Step 2. Choose $f_2(x) = x$ and take a function

$$g_2(x) = f_2(x) + c\,\varphi_1(x) = x + \frac{c}{\sqrt{2}} \quad \text{— (3)}$$

Let $g_2(x)$ and $\varphi_1(x)$ be orthogonal in $(-1,1)$.

$\therefore \int_{-1}^{1} g_2(x)\varphi_1(x)\,dx = 0 \Rightarrow \int_{-1}^{1} \left(x + \frac{c}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\,dx = 0$

$\Rightarrow \left[\frac{x^2}{2} + \frac{cx}{\sqrt{2}}\right]_{-1}^{1} = 0$ so that $c = 0$

$\therefore g_2(x) = x$ by (3). Now $\|g_2(x)\| = \left[\int_{-1}^{1} g_2^2\,dx\right]^{1/2} = \left[\frac{2}{3}\right]^{1/2}$

$$\therefore \varphi_2(x) = \frac{g_2(x)}{\|g_2(x)\|} = \left(\frac{3}{2}\right)^{1/2} x \quad \text{— (4)}$$

Proceeding in this manner, $\varphi_3(x) = \frac{1}{2}\left(\frac{5}{2}\right)^{1/2}(3x^2 - 1)$

$\varphi_4(x) = \left(\frac{7}{2}\right)^{1/2}\left(\frac{5x^3 - 3x}{2}\right)$ etc.