# ASSIGNMENT - 4

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## Mathematical Methods

1) To show : Equations of transformation of a mixed tensor possess the group property (transitive property).

Let $A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ be a general mixed tensor of rank $(r+s)$ of the type $(r,s)$ in a coordinate system $x^i$ $(i = 1, 2, \ldots, n)$.

Let the coordinates $x^i$ be transformed to the coordinate system $\bar{x}^i$ and then $\bar{x}^i$ is transformed to $\bar{\bar{x}}^i$.

When, coordinates $x^i$ are transformed to $\bar{x}^i$, by the transformation of a mixed tensor, we have :

$$\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{m_1}} \times \frac{\partial \bar{x}^{p_2}}{\partial x^{m_2}} \times \cdots \times \frac{\partial \bar{x}^{p_r}}{\partial x^{m_r}} \times \frac{\partial x^{n_1}}{\partial \bar{x}^{q_1}} \times \frac{\partial x^{n_2}}{\partial \bar{x}^{q_2}} \times \cdots \times \frac{\partial x^{n_s}}{\partial \bar{x}^{q_s}} \times A^{m_1 m_2 \cdots m_r}_{n_1 n_2 \cdots n_s} \quad \cdots\cdots (i)$$

And, when coordinates $\bar{x}^i$ are transformed to $\bar{\bar{x}}^i$, by the transformation law of a mixed tensor, we have :

$$\bar{\bar{A}}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = \frac{\partial \bar{\bar{x}}^{i_1}}{\partial \bar{x}^{p_1}} \times \frac{\partial \bar{\bar{x}}^{i_2}}{\partial \bar{x}^{p_2}} \times \cdots \times \frac{\partial \bar{\bar{x}}^{i_r}}{\partial \bar{x}^{p_r}} \times \frac{\partial \bar{x}^{q_1}}{\partial \bar{\bar{x}}^{j_1}} \times \cdots \times \frac{\partial \bar{x}^{q_s}}{\partial \bar{\bar{x}}^{j_s}} \times \bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} \quad \cdots\cdots (ii)$$

$$= \frac{\partial \bar{\bar{x}}^{i_1}}{\partial \bar{x}^{p_1}} \times \frac{\partial \bar{\bar{x}}^{i_2}}{\partial \bar{x}^{p_2}} \times \cdots \times \frac{\partial \bar{\bar{x}}^{i_r}}{\partial \bar{x}^{p_r}} \cdot \frac{\partial \bar{x}^{q_1}}{\partial \bar{\bar{x}}^{j_1}} \times \cdots \times \frac{\partial \bar{x}^{q_s}}{\partial \bar{\bar{x}}^{j_s}} \times \left( \frac{\partial \bar{x}^{p_1}}{\partial x^{m_1}} \times \cdots \times \frac{\partial \bar{x}^{p_r}}{\partial x^{m_r}} \times \frac{\partial x^{n_1}}{\partial \bar{x}^{q_1}} \times \cdots \times \frac{\partial x^{n_s}}{\partial \bar{x}^{q_s}} \times A^{m_1 m_2 \cdots m_r}_{n_1 n_2 \cdots n_s} \right)$$

[From (i)]

$$\Rightarrow \bar{A}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \times \frac{\partial \bar{x}^{i_2}}{\partial x^{m_2}} \times \cdots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{m_r}} \times \frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \times \cdots \times \frac{\partial x^{n_s}}{\partial \bar{x}^{j_s}} \times A^{m_1 m_2 \cdots m_r}_{n_1 n_2 \cdots n_s} \quad \cdots (iii)$$

Now, eqn. (iii) is nothing but the law of transformation of a mixed tensor from $x^i$ coordinate system to $\bar{x}^i$.

Hence we conclude that, equations of transformation of a mixed tensor possess the group property (transitive property).

[Proved.]

2) Consider two tensors: $A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ of type $(r, s)$ and $B^{k_1 k_2 \cdots k_{r'}}_{l_1 l_2 \cdots l_{s'}}$ of type $(r', s')$.

Then, according to the law of transformation:

$$\bar{A}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \times \frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \times \cdots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \times \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \times \frac{\partial x^{q_2}}{\partial \bar{x}^{j_2}} \times \cdots \times \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} \times A^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} \quad \cdots (i)$$

And,

$$\bar{B}^{k_1 k_2 \cdots k_{r'}}_{l_1 l_2 \cdots l_{s'}} = \frac{\partial \bar{x}^{k_1}}{\partial x^{m_1}} \times \frac{\partial \bar{x}^{k_2}}{\partial x^{m_2}} \times \cdots \times \frac{\partial \bar{x}^{k_{r'}}}{\partial x^{m_{r'}}} \times \frac{\partial x^{n_1}}{\partial \bar{x}^{l_1}} \times \frac{\partial x^{n_2}}{\partial \bar{x}^{l_2}} \times \cdots \times \frac{\partial x^{n_{s'}}}{\partial \bar{x}^{l_{s'}}} \times B^{m_1 m_2 \cdots m_{r'}}_{n_1 n_2 \cdots n_{s'}} \quad \cdots (ii)$$

Then, from (i) and (ii), outer product of these two tensors follows:

$$\bar{A}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} \times \bar{B}^{k_1 k_2 \cdots k_{r'}}_{l_1 l_2 \cdots l_{s'}} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \times \cdots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \times \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \times \cdots \times \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} \times \frac{\partial \bar{x}^{k_1}}{\partial x^{m_1}} \times \cdots \times \frac{\partial \bar{x}^{k_{r'}}}{\partial x^{m_{r'}}} \times$$

$$\frac{\partial x^{n_1}}{\partial \bar{x}^{l_1}} \times \cdots \times \frac{\partial x^{n_{s'}}}{\partial \bar{x}^{l_{s'}}} \times A^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} \times B^{m_1 m_2 \cdots m_{r'}}_{n_1 n_2 \cdots n_{s'}} \quad \cdots (iii)$$

Now, if we take product of $A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ and $B^{k_1 k_2 \cdots k_{r'}}_{l_1 l_2 \cdots l_{s'}}$ as :

$$\bar{C}^{i_1 \cdots i_r k_1 \cdots k_{r'}}_{j_1 \cdots j_s l_1 \cdots l_{s'}} = \bar{A}^{i_1 \cdots i_r}_{j_1 \cdots j_s} \times \bar{B}^{k_1 \cdots k_{r'}}_{l_1 \cdots l_{s'}} ,$$ and let :

$$C^{p_1 \cdots p_r m_1 \cdots m_{r'}}_{q_1 \cdots q_s n_1 \cdots n_{s'}} = A^{p_1 \cdots p_r}_{q_1 \cdots q_s} \times B^{m_1 \cdots m_{r'}}_{n_1 \cdots n_{s'}} ,$$ then eqn. (iii) can be

written as :

$$\bar{C}^{i_1 \cdots i_r k_1 \cdots k_{r'}}_{j_1 \cdots j_s l_1 \cdots l_{s'}} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \times \cdots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \times \frac{\partial \bar{x}^{k_1}}{\partial x^{m_1}} \times \cdots \times \frac{\partial \bar{x}^{k_{r'}}}{\partial x^{m_{r'}}} \times \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \times \cdots \times \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} \times$$

$$\frac{\partial x^{n_1}}{\partial \bar{x}^{l_1}} \times \cdots \times \frac{\partial x^{n_{s'}}}{\partial \bar{x}^{l_{s'}}} \times C^{p_1 p_2 \cdots p_r m_1 \cdots m_{r'}}_{q_1 \cdots q_s n_1 \cdots n_{s'}} \quad \cdots \text{(iv)}$$

Eqn. (iv) is the law of transformation of a mixed tensor of rank $r+r'+s+s'$, and of type $(r+r', s+s')$.

Hence it follows that the tensor product, or outer product of two tensors of type $(r, s)$ and $(r', s')$ is a tensor of rank $r+s+r'+s'$ (i.e. sum of ranks of the two tensors) and of type $(r+r', s+s')$.

order/ rank = $r+s$     order/ rank = $r'+s'$

[Proved.]

3) We have already proved in the previous question that the open product (or, outer product) of two tensors of type $(r, s)$ and $(r', s')$ is a tensor of type $(r+r', s+s')$ and is of rank $(r+r'+s+s')$.

Also, a vector is a tensor of rank 1. So, a vector is either of type $(1,0)$ or of type $(0,1)$ according as the vector is contravariant or covariant.

Case 1: Consider two vectors, both of which are contravariant

So both are of type $(1,0)$.

So their open product is a tensor of type $(1+1, 0+0)$
$= (2,0)$

$\Rightarrow$ Their open product is a tensor of rank $2+0 = 2$

Case 2: Consider two vectors, both of which are covariant.

So both are of type $(0,1)$

So their open product is a tensor of type $(0+0, 1+1)$
$= (0,2)$

$\Rightarrow$ Their open product is a tensor of rank $0+2 = 2$.

Case 3: Consider two vectors, one contravariant and one covariant.

So they are of type $(1,0)$ and $(0,1)$

So their open product is a tensor of type $(1+0, 0+1)$
$= (1,1)$

$\Rightarrow$ Their open product is a tensor of rank $1+1 = 2$

Hence, from cases 1, 2 and 3 we can say that the open product of two vectors is a tensor of order (or, rank) = 2.

[Proved.]

However, the converse is not true. Not every tensor can be expressed as a product (open product) of two tensors of lower rank.

For instance, consider the standard basis of $\mathbb{R}^2$, which is $\{e_1, e_2\}$ where : $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now consider the tensor : $e_1 \otimes e_1 + e_2 \otimes e_2$. [This is of rank 2, as proved in previous part]

We would show that there does not exist any vectors in $\mathbb{R}^2$ whose tensor product will give $e_1 \otimes e_1 + e_2 \otimes e_2$.

To the contrary, suppose there are vectors : $(a_1 e_1 + a_2 e_2)$ and $(b_1 e_1 + b_2 e_2)$, i.e. vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ such that :

$$(a_1 e_1 + a_2 e_2) \otimes (b_1 e_1 + b_2 e_2) = e_1 \otimes e_1 + e_2 \otimes e_2$$

$$\Rightarrow a_1 b_1 (e_1 \otimes e_1) + a_2 b_2 (e_2 \otimes e_2) + a_1 b_2 (e_1 \otimes e_2) + a_2 b_1 (e_2 \otimes e_1)$$

$$= e_1 \otimes e_1 + e_2 \otimes e_2$$

On comparing coefficients, we must have : $a_1 b_1 = a_2 b_2 = 1$

$\Rightarrow a_1, b_1, a_2, b_2 \neq 0$

But then, $a_1 b_2 \neq 0$ and $a_2 b_1 \neq 0$.

→ Then LHS and RHS can't be same. So, $e_1 \otimes e_1 + e_2 \otimes e_2$ can't be expressed as open product of two vectors. Hence, the converse doesn't hold true. [Proved.]

4) Consider two tensors : $A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ of rank $(r+s)$ and $B^{k_1 k_2 \cdots k_m}_{l_1 l_2 \cdots l_n}$ of rank $(m+n)$. Then, by law of transformation,

$$\bar{A}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \times \cdots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \times \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \times \cdots \times \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} \times A^{p_1 \cdots p_r}_{q_1 \cdots q_s} \quad \cdots (i)$$

, and,

$$\bar{B}^{k_1 \cdots k_m}_{l_1 \cdots l_n} = \frac{\partial \bar{x}^{k_1}}{\partial x^{\alpha_1}} \times \cdots \times \frac{\partial \bar{x}^{k_m}}{\partial x^{\alpha_m}} \times \frac{\partial x^{\beta_1}}{\partial \bar{x}^{l_1}} \times \cdots \times \frac{\partial x^{\beta_n}}{\partial \bar{x}^{l_n}} \times B^{\alpha_1 \cdots \alpha_m}_{\beta_1 \cdots \beta_n} \quad \cdots (ii)$$

Also, define : $\bar{C}^{i_1 \cdots i_r k_1 \cdots k_m}_{j_1 \cdots j_s l_1 \cdots l_n} = \bar{A}^{i_1 \cdots i_r}_{j_1 \cdots j_s} \times \bar{B}^{k_1 \cdots k_m}_{l_1 \cdots l_n}$ and,

$$C^{p_1 p_2 \cdots p_r \alpha_1 \cdots \alpha_m}_{q_1 \cdots q_s \beta_1 \cdots \beta_n} = A^{p_1 \cdots p_r}_{q_1 \cdots q_s} \times B^{\alpha_1 \cdots \alpha_m}_{\beta_1 \cdots \beta_n}$$

Then, multiplying (outer product) (i) and (ii), we get :

$$\bar{C}^{i_1 \cdots i_r k_1 \cdots k_m}_{j_1 \cdots j_s l_1 \cdots l_n} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \times \cdots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \times \frac{\partial \bar{x}^{k_1}}{\partial x^{\alpha_1}} \times \cdots \times \frac{\partial \bar{x}^{k_m}}{\partial x^{\alpha_m}} \times \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \times \cdots \times \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} \times$$

$$\frac{\partial x^{\beta_1}}{\partial \bar{x}^{l_1}} \times \cdots \times \frac{\partial x^{\beta_n}}{\partial \bar{x}^{l_n}} \times C^{p_1 p_2 \cdots p_r \alpha_1 \cdots \alpha_m}_{q_1 \cdots q_s \beta_1 \cdots \beta_n} \quad \cdots (iii)$$

Eqn. (iii) is the law of transformation of a mixed tensor of rank $r+s+m+n$, which is the sum of ranks of the tensors whose outer product has been taken.

Hence, the outer product of two tensors is a tensor whose order (or, rank) is the sum of the orders of the two tensors.

[Proved.]

5) As $A^p_r$ is a mixed tensor of rank 2, of type (1, 1) and $B^{rs}_t$ is a mixed tensor of rank 3, of type (2, 1), then by the law of transformation for mixed tensors:

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial x^r}{\partial \bar{x}^j} \times A^p_r \quad \ldots \text{(i)}$$ and,

$$\bar{B}^{kl}_m = \frac{\partial \bar{x}^k}{\partial x^q} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times B^{qs}_t \quad \ldots \text{(ii)}$$

In (ii), putting $k=j$, we get:

$$\bar{B}^{jl}_m = \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times B^{qs}_t \quad \ldots \text{(iii)}$$

Multiplying (i) and (iii), we get:

$$\bar{A}^i_j \bar{B}^{jl}_m = \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial x^r}{\partial \bar{x}^j} \times \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times A^p_r \times B^{qs}_t$$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times \frac{\partial x^r}{\partial x^q} \times A^p_r B^{qs}_t$$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times \delta^r_q \times A^p_r \times B^{qs}_t \qquad \left[\because \frac{\partial x^r}{\partial x^q} = \delta^r_q\right]$$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times A^p_r \times B^{rs}_t \qquad \left[\because \delta^r_q B^{qs}_t = B^{rs}_t\right]$$

$$\ldots \text{(iv)}$$

Now, eqn. (iv) is the law of transformation of a mixed tensor of rank 3. Hence, $A^p_r \cdot B^{rs}_t$ (inner product) is a tensor of rank 3. [Proved.]

6) We'll show that $g_{ij}$ is symmetric tensor.

We know, $g_{ij}$ can be written as:

$$g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) + \frac{1}{2}(g_{ij} - g_{ji})$$

$\Rightarrow g_{ij} = A_{ij} + B_{ij}$ ; where $A_{ij} = \frac{1}{2}(g_{ij} + g_{ji})$ is symmetric

and $B_{ij} = \frac{1}{2}(g_{ij} - g_{ji})$ is skew-symmetric

Then, $g_{ij}\, dx^i dx^j = (A_{ij} + B_{ij})\, dx^i dx^j$

$\Rightarrow (g_{ij} - A_{ij})\, dx^i dx^j = B_{ij}\, dx^i dx^j$ .... (i)

Interchanging the dummy indices in $B_{ij}\, dx^i dx^j$, we have:

$B_{ij}\, dx^i dx^j = B_{ji}\, dx^i dx^j$

$\Rightarrow B_{ij}\, dx^i dx^j = -B_{ij}\, dx^i dx^j$ [$\because B_{ij}$ is skew-symmetric. So, $B_{ji} = -B_{ij}$]

$\Rightarrow 2 \times B_{ij}\, dx^i dx^j = 0$

$\Rightarrow B_{ij}\, dx^i dx^j = 0$ .... (ii)

So, from (i) and (ii), $(g_{ij} - A_{ij})\, dx^i dx^j = 0$

As $dx^i$, $dx^j$ are arbitrary, we must have: $g_{ij} = A_{ij}$

So, $g_{ij}$ is symmetric [$\because A_{ij}$ is symmetric]

Hence, $g_{ij}$ is a covariant symmetric tensor of rank 2, so it will have $n^2$ components. Out of these $n^2$ components, $n$ components $g_{11}, g_{22}, \dots, g_{nn}$ are independent at max.

And, due to the symmetry of $g_{ij}$, out of the remaining $(n^2-n)$ components, only half of them are independent at max,

because: $g_{12} = g_{21}$, $g_{23} = g_{32}$, ... etc.

Hence, total no. of independent components $g_{ij}$ of the metric tensor cannot exceed $n + \frac{n^2-n}{2} = \frac{n^2+n}{2}$

$$= \frac{n(n+1)}{2}.$$ [Proved]