Strum-Liouville problem

Strum-Liouville problem is a second order homogeneous boundary value problem of the form

$$\frac{d}{dx}\{r(x)y'\} + \{q(x) + \lambda p(x)\}y = 0$$

or, $$[r(x)y']' + [q(x) + \lambda p(x)]y = 0 \quad \text{— (1)}$$

that satisfies the boundary conditions at the two end points a and b i.e. $a \le x \le b$,

$$a_1 y(a) + a_2 y'(a) = 0 \text{ and } b_1 y(b) + b_2 y'(b) = 0 \quad \text{— (2)}$$

where $p(x)$, $q(x)$, $r(x)$ and $r'(x)$ are real valued continuous functions on $[a,b]$, $p(x)$ and $r(x)$ are positive on $[a,b]$ and the constant λ is an arbitrary parameter. Also that a_1, a_2, b_1, b_2 are real constants such that a_1, a_2 are not both zero and so are b_1, b_2.

Clearly $y=0$ is always a solution of S-L problem for any value of λ. $y=0$ is called trivial solution of the problem. The non-zero solutions of the S-L problem given by (1) and (2) are called eigenfunctions of the problem and the value of λ for which such solutions exist, are called eigenvalues of the problem.

A special case:
Let $p=r=1$ and $q=0$ in (1). Let $a_1 = b_1 = 1$ and $a_2 = b_2 = 0$ in (2). The S-L problem reduces to

$$y'' + \lambda y = 0 \text{ with } y(a) = 0,\ y(b) = 0.$$

This is the simplest form of S-L problem.

Orthogonality of eigenfunctions

Theorem: Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the S-L problem that correspond to different eigenvalues λ_m and λ_n respectively. Then y_m, y_n are orthogonal on that interval w.r.t. the weight function $p(x)$.

Proof: Consider the S-L problem

$$[r(x)y']' + [q(x) + \lambda p(x)]y = 0 \quad -(1)$$

$$a_1 y(a) + a_2 y'(a) = 0 \quad -(2a)$$

$$b_1 y(b) + b_2 y'(b) = 0 \quad -(2b)$$

Let y_m and y_n be ~~be~~ eigenfunctions of the above S-L problem that correspond to different eigenvalues λ_m and λ_n. Then by definition of eigenfunctions y_m and y_n both satisfy (1).

Hence
$$(ry_m')' + (q + \lambda_m p)y_m = 0 \quad -(3)$$
$$(ry_n')' + (q + \lambda_n p)y_n = 0 \quad -(4)$$

Multiplying (3) by y_n and (4) by y_m and subtracting,

$$(ry_m')'y_n - (ry_n')'y_m + (\lambda_m - \lambda_n)p y_m y_n = 0$$

$$\Rightarrow (\lambda_m - \lambda_n)p y_m y_n = (ry_n')'y_m - (ry_m')'y_n$$

$$\Rightarrow (\lambda_m - \lambda_n)p y_m y_n = \frac{d}{dx}\left\{(ry_n')y_m - (ry_m')y_n\right\} \quad -(5)$$

Integrating both sides of (5) over x from a to b

$$(\lambda_m-\lambda_n)\int_a^b p\, y_m y_n\, dx = \left[r\,(y_n' y_m - y_m' y_n)\right]_a^b$$

$$= r(b)\left\{y_n'(b)y_m(b) - y_m'(b)y_n(b)\right\}$$

$$- r(a)\left\{y_n'(a)y_m(a) - y_m'(a)y_n(a)\right\} \quad \text{--- (6)}$$

Case I Let $r(a) = r(b) = 0$

Then (6) becomes $(\lambda_m-\lambda_n)\int_a^b p\, y_m y_n\, dx = 0$ — (7)

i.e. $\int_a^b p\, y_m y_n\, dx = 0$

Case II Let $r(b)=0$ but $r(a)\neq 0$

(6) will reduce to

$$(\lambda_m-\lambda_n)\int_a^b p\, y_m y_n\, dx = -r(a)\left\{y_n'(a)y_m(a) - y_m'(a)y_n(a)\right\} \quad \text{--- (8)}$$

Since y_m and y_n both satisfy (2a)

$$a_1 y_m(a) + a_2 y_m'(a) = 0 \quad \text{--- (9)}$$

$$a_1 y_n(a) + a_2 y_n'(a) = 0 \quad \text{--- (10)}$$

Let $a_2 \neq 0$. Multiplying (10) by $y_m(a)$ and (9) by $y_n(a)$ and then subtracting,

$$a_2\left\{y_n'(a)\,y_m(a) - y_m'(a)\,y_n(a)\right\} = 0$$

$$\Rightarrow y_n'(a)y_m(a) - y_m'(a)y_n(a) = 0 \quad \text{--- (11)}$$

$\because a_2 \neq 0$

If $a_2 = 0$, then let $a_1 \neq 0$
Now multiplying (9) by $y_n'(a)$ and (10) by $y_m'(a)$, subtracting

$$a_1 \{ y_n'(a)\, y_m(a) - y_m'(a)\, y_n(a) \} = 0$$

$$\Rightarrow \quad y_n'(a)\, y_m(a) - y_m'(a)\, y_n(a) = 0 \qquad \because a_1 \neq 0$$

$$\therefore \int_a^b p\, y_m y_n \, dx = 0$$

Case III Let $r(a) = 0$ $r(b) \neq 0$
Case IV Let $r(a) \neq 0$ $r(b) \neq 0$
Case V Let $r(a) = r(b)$
} similar proofs.

Reality of eigenvalues

Theorem

All eigenvalues of S-L problem are real.

Proof: S-L problem is

$$[r(x)\, y']' + [q(x) + \lambda p(x)]\, y = 0 \quad -(1)$$
$$a_1 y(a) + a_2 y'(a) = 0 \quad -(2a)$$
$$b_1 y(b) + b_2 y'(b) = 0 \quad -(2b)$$

Let $y(x)$ be an eigenfunction corresponding to an eigenvalue $\lambda = \alpha + i\beta$, where α, β are real. This eigenfunction $y(x)$ satisfies (1), (2a) and (2b) and may be a complex valued function. Taking complex conjugates of all the terms in (1), (2a) & (2b),