ASSIGNMENT 2

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Mathematical Methods

1) (i) We know : $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+2r}$

Putting $n=0$, we get : $J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times \Gamma(r+1)} \times \left(\frac{x}{2}\right)^{2r}$

$$\Rightarrow J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times r!} \times \left(\frac{x}{2}\right)^{2r} \quad \left[\because \text{ when } n \in \mathbb{N}, \ \Gamma(n) = (n-1)!\right]$$

..... (i)

Now, we know that $J_n(x)$ series is uniformly convergent $\forall x \in \mathbb{R}$.

So we can do term by term differentiation in eqn. (i) to get :

$$J_0'(x) = \frac{d}{dx}\left\{\sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \left(\frac{x}{2}\right)^{2r}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \frac{d}{dx}\left(\frac{x}{2}\right)^{2r} = \sum_{r=1}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times 2r \times \left(\frac{x}{2}\right)^{2r-1} \times \frac{1}{2}$$

[$\because$ The $r=0$ term is a constant term in $J_0(x)$, so it's derivative $=0$, and hence, here the summation starts from $r=1$]

$$\Rightarrow J_0'(x) = \sum_{r=1}^{\infty} \frac{(-1)^r}{r!\,(r-1)!} \times \left(\frac{x}{2}\right)^{2r-1}$$

Changing the variable in summation from r to m, putting $m = r-1$, So $r = m+1$,

we get :

$$J_0'(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!\,m!} \times \left(\frac{x}{2}\right)^{2(m+1)-1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1) \times (-1)^m}{m! \times \Gamma(m+2)} \times \left(\frac{x}{2}\right)^{2m+1} \quad \left[\because \text{ For } m \in \mathbb{N} \cup \{0\}, \ \Gamma(m+2) = (m+1)!\right]$$

$$= -\sum_{m=0}^{\infty} \frac{(-1)^m}{m! \times \Gamma(1+m+1)} \times \left(\frac{x}{2}\right)^{1+2m} = -J_1(x)$$

Hence, $J_0'(x) = -J_1(x)$ [Proved.]

(ii) We know the recurrence relation:

$$J_n'(x) = \frac{1}{2} \times \left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Putting $n = 1$, $J_1'(x) = \frac{1}{2} \times \left[J_0(x) - J_2(x)\right]$ .... (i)

And, in the previous part, we proved: $J_0'(x) = -J_1(x)$

So, $J_1(x) = -J_0'(x)$

Differentiating both sides w.r.t x, we get:

$J_1'(x) = -J_0''(x)$ .... (ii)

From (i) and (ii), we get:

$$-J_0''(x) = \frac{1}{2} \times \left[J_0(x) - J_2(x)\right]$$

$\Rightarrow J_0''(x) = \frac{1}{2} \times \left[J_2(x) - J_0(x)\right]$ $\Rightarrow J_2(x) - J_0(x) = 2J_0''(x)$ [Proved.]

(iii) We know the recurrence relation:

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x)$$

Putting $n = 1$, $J_2(x) = \frac{1}{x} J_1(x) - J_1'(x)$ .... (i)

And, we also know: $J_0'(x) = -J_1(x)$ $\Rightarrow J_1(x) = -J_0'(x)$ .... (ii)

$\Rightarrow J_1'(x) = -J_0''(x)$ .... (iii)

Using eqn. (ii) and (iii) in eqn. (i), we get:

$$J_2(x) = \frac{-1}{x} J_0'(x) + J_0''(x)$$

$\therefore$ $J_2(x) = J_0''(x) - \left(\frac{1}{x}\right) \cdot J_0'(x)$ [Proved.]

2) <u>To Prove:</u> $J_{n+1}(x) = x\int_0^1 J_n(xy)\cdot y^{n+1}\cdot dy$ ..... (i)

We know, $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times \Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+2r}$

$$\Rightarrow J_n(xy) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times \Gamma(n+r+1)} \times \left(\frac{xy}{2}\right)^{n+2r}$$

$$\Rightarrow y^{n+1}\cdot J_n(xy) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times \Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+2r} \times y^{n+1+n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times \Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+2r} \times y^{2(n+r)+1} \quad \text{.... (ii)}$$

$\therefore$ RHS of (i) $= x\int_0^1 J_n(xy)\cdot y^{n+1}\cdot dy$

$$= x\int_0^1 \sum_{r=0}^{\infty} \left\{\frac{(-1)^r}{r!\,\Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+2r} \times y^{2(n+r)+1}\right\}\cdot dy \quad \text{[From (ii)]}$$

$$= x\sum_{r=0}^{\infty}\left\{\int_0^1 \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+2r}\cdot y^{2(n+r)+1}\cdot dy\right\}$$

[Interchanging the integral and summation]

$$= x\sum_{r=0}^{\infty}\left\{\frac{(-1)^r}{r!\,\Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+2r}\cdot \int_0^1 y^{2(n+r)+1}\cdot dy\right\}$$

$$= x\sum_{r=0}^{\infty}\left\{\frac{(-1)^r}{r! \times \Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+2r} \times \left[\frac{y^{2(n+r+1)}}{2(n+r+1)}\right]_0^1\right\}$$

$$= x \times \sum_{r=0}^{\infty}\left\{\frac{(-1)^r}{r! \times \Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+2r} \times \frac{1}{2(n+r+1)}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times (n+r+1) \times \Gamma(n+r+1)} \times \left(\frac{x}{2}\right)^{n+1+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times \Gamma(n+r+2)} \times \left(\frac{x}{2}\right)^{n+1+2r}$$

[$\because n \times \Gamma(n) = \Gamma(n+1)$
So, $(n+r+1) \times \Gamma(n+r+1) = \Gamma(n+r+2)$]

$$\therefore\ x\int_0^1 J_n(xy)\cdot y^{n+1}\cdot dy = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\times\Gamma(n+1+k+1)}\times\left(\frac{x}{2}\right)^{n+1+2k}$$

$$= J_{n+1}(x). \quad = \text{LHS of (i)}.$$

Hence, $J_{n+1}(x) = x\int_0^1 J_n(xy)\cdot y^{n+1}\cdot dy$ [Proved.]

3) We know the Recurrence relation:

$$\frac{d}{dx}\left[x^{-n}\cdot J_n(x)\right] = -x^{-n}\cdot J_{n+1}(x)$$

$$\Rightarrow \int x^{-n}\cdot J_{n+1}(x)\cdot dx = -x^{-n}\cdot J_n(x) \quad \cdots\cdots \text{(i)}$$

we will use (i) to find $\int J_3(x)\cdot dx$.

$$\int J_3(x)\cdot dx = \int x^2\cdot\left[x^{-2}\cdot J_3(x)\right]\cdot dx + c$$

$$= x^2\cdot\left[-x^{-2}\cdot J_2(x)\right] + \int 2x\cdot\left[x^{-2}\cdot J_2(x)\right]\cdot dx + c$$

[Integrating by parts, choosing x^2 as first function and $x^{-2}\cdot J_3(x)$ as second function.

Also, $\int x^{-2}\cdot J_3\, dx = -x^{-2}\cdot J_2$ from eqn. (i)]

$$= -J_2(x) + 2\int x^{-1}\cdot J_2(x)\cdot dx + c$$

$$= -J_2(x) + 2\times\left[-x^{-1}\times J_1(x)\right] + c$$

[From (i)]

$$\therefore \int J_3(x)\cdot dx = c - J_2(x) - \frac{2}{x}J_1(x) \quad \cdots\cdots \text{(ii)}$$

Now, we know the recurrence relation:

$$J_n(x) = \frac{x}{2n}\times\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$\Rightarrow J_{n+1}(x) = \frac{2n}{x}\cdot J_n(x) - J_{n-1}(x) \quad \cdots \text{(iii)}$$

Putting $n = 1$ in (iii), $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$ .... (iv)

Putting eqn. (iv) in eqn. (ii), we get:

$$\int J_3(x) \cdot dx = c - \left(\frac{2}{x} \cdot J_1(x) - J_0(x)\right) - \frac{2}{x} J_1(x)$$

$$= c + J_0(x) - \frac{4}{x} J_1(x) \quad \text{[Ans.]}$$

[where c = constant of integration]

4) (i) As proved in Q.1) part (i), we have:

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \left(\frac{x}{2}\right)^{2r}$$

$$\Rightarrow \frac{d}{dx} J_0(x) = \frac{d}{dx} \left\{ \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \left(\frac{x}{2}\right)^{2r} \right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \frac{d}{dx}\left(\frac{x}{2}\right)^{2r} = \underbrace{\frac{1}{1} \times \frac{d}{dx}(1)}_{\text{The } r=0 \text{ term}} + \sum_{r=1}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \frac{d}{dx}\left(\frac{x}{2}\right)^{2r}$$

$$\text{So, } \frac{d}{dx} J_0(x) = 0 + \sum_{r=1}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times 2r \times \left(\frac{x}{2}\right)^{2r-1} \times \frac{1}{2}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\, r!} \times \left(\frac{x}{2}\right)^{2r-1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{m!\,(m+1)!} \times \left(\frac{x}{2}\right)^{2(m+1)-1}$$

[Putting $m = r-1$, i.e. $r = m+1$, m as the new variable of summation]

$$= \sum_{m=0}^{\infty} \frac{(-1) \times (-1)^m}{m! \times \Gamma(m+2)} \times \left(\frac{x}{2}\right)^{2m+1}$$

[$\because$ for $m \in \mathbb{Z}^+ \cup \{0\}$, $\Gamma(m+2) = (m+1)!$]

$$= -\sum_{m=0}^{\infty} \frac{(-1)^m}{m! \times \Gamma(1+m+1)} \times \left(\frac{x}{2}\right)^{1+2m} = -J_1(x)$$

Hence, $\frac{d}{dx}[J_0(x)] = -J_1(x)$ [Proved.]

(ii) $$I = \int_a^b J_0(x) \cdot J_1(x) \cdot dx$$

$$= \int_a^b J_0(x) \times -\frac{d}{dx}[J_0(x)] \cdot dx$$

$\left[\because \text{From previous part, } \frac{d}{dx}[J_0(x)] = -J_1(x) \right]$

$$= -\int_a^b J_0(x) \cdot d[J_0(x)] \quad = \quad \int_b^a J_0(x) \cdot d[J_0(x)]$$

$$\therefore I = \left[\frac{\{J_0(x)\}^2}{2} \right]_b^a \quad = \quad \frac{J_0^2(a) - J_0^2(b)}{2}$$

Hence, $$\int_a^b J_0(x) \cdot J_1(x) \cdot dx = \frac{1}{2} \times \left[J_0^2(a) - J_0^2(b) \right]$$ [Proved.]

5) To Prove: $$\int_0^\infty e^{-ax} \cdot J_0(bx)\, dx = \frac{1}{\sqrt{a^2 + b^2}}$$; for $a > 0$

We know, $$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \left(\frac{x}{2}\right)^{2r}$$

$$\Rightarrow J_0(bx) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \left(\frac{bx}{2}\right)^{2r}$$

So, $$\int_0^\infty e^{-ax} \cdot J_0(bx) \cdot dx = \int_0^\infty e^{-ax} \cdot \left[\sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \left(\frac{bx}{2}\right)^{2r} \right] \cdot dx$$

$$= \sum_{r=0}^{\infty} \left[\frac{(-1)^r}{\{r!\}^2} \times \left(\frac{b}{2}\right)^{2r} \times \int_0^\infty e^{-ax} \cdot x^{2r} \cdot dx \right] \quad \ldots (i)$$

[Interchanging the integral and summation]

Substitute:

$z = ax \quad \Rightarrow x = \frac{z}{a} \quad \Rightarrow dx = \frac{1}{a} \cdot dz$

When $x = 0 \rightarrow z = 0$

when $x = \infty \rightarrow z = \infty$

So, ① becomes : (after the substitution)

$$\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{\{r!\}^2}\times\left(\frac{b}{2}\right)^{2r}\times\int_0^{\infty} e^{-z}\cdot\left(\frac{z}{a}\right)^{2r}\cdot\frac{1}{a}\cdot dz\right]$$

$$=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{\{r!\}^2}\times\left(\frac{b}{2}\right)^{2r}\times\frac{1}{a^{2r+1}}\int_0^{\infty} e^{-z}\cdot z^{2r}\cdot dz\right]$$

$$=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{\{r!\}^2}\times\frac{1}{2^{2r}}\times\left(\frac{b}{a}\right)^{2r}\times\frac{1}{a}\times\Gamma(2r+1)\right]$$ $$\left[\because \Gamma(n+1)=\int_0^{\infty} e^{-x}\cdot x^n\cdot dx\right]$$

$$=\frac{1}{a}\times\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{\{r!\}\{r!\}}\times\frac{1}{2^{2r}}\times\left(\frac{b}{a}\right)^{2r}\times(2r)!\right]$$ $$\left[\because \text{for } n\in\mathbb{N},\ \Gamma(n)=(n-1)!\right]$$

$$=\frac{1}{a}\times\sum_{r=0}^{\infty}\left[(-1)^r\times\frac{(2r)!}{2^r}\times\frac{1}{2^r\times r!\times r!}\times\left(\frac{b^2}{a^2}\right)^r\right]$$

$$=\frac{1}{a}\times\sum_{r=0}^{\infty}\left[(-1)^r\times\frac{[1\times3\times\cdots\times(2r-1)]\times[2\times4\times\cdots\times2r]}{2^r}\times\frac{1}{[2\times4\times\cdots\times2r]\times r!}\times\left(\frac{b^2}{a^2}\right)^r\right]$$

$$=\frac{1}{a}\times\sum_{r=0}^{\infty}\left[(-1)^r\times\frac{1\times3\times5\times\cdots\times(2r-1)}{2\times4\times\cdots\times2r}\times\left(\frac{b^2}{a^2}\right)^r\right]$$

$$=\frac{1}{a}\times\left(1+\frac{b^2}{a^2}\right)^{-1/2}$$ $$\left[\because \text{From Negative Binomial Expansion, we can see: } \left(1+\frac{b^2}{a^2}\right)^{-1/2}=1-\frac{1}{2}\times\left(\frac{b^2}{a^2}\right)+\frac{1\times3}{2\times4}\times\left(\frac{b^2}{a^2}\right)^2-\cdots\right]$$

$$=\frac{1}{a}\times\left(\frac{a^2+b^2}{a^2}\right)^{-1/2}$$

$$=\frac{1}{a}\times\left(\frac{a^2}{a^2+b^2}\right)^{1/2}=\frac{1}{a}\times\frac{|a|}{\sqrt{a^2+b^2}}=\frac{1}{\cancel{a}}\times\frac{\cancel{a}}{\sqrt{a^2+b^2}}$$ $$\left[\because a>0,\ \text{So, } |a|=a\right]$$

$$=\frac{1}{\sqrt{a^2+b^2}}$$

Hence, $\int_0^{\infty} e^{-ax}\cdot J_0(bx)\cdot dx=\dfrac{1}{\sqrt{a^2+b^2}}$; for $a>0$ [Proved.]