and $\overline{\left\{ {p \atop lm} \right\}} = \left\{ {s \atop ij} \right\} \frac{\partial \bar{x}^p}{\partial x^s}\frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m}$ —(12)

which show that Christoffel symbols (eqs 11–12) are tensors relative to linear transformation of the type (11).

- Inner multiplication of eq (10) by $\frac{\partial x^r}{\partial \bar{x}^p}$, there results in

$$\frac{\partial^2 x^r}{\partial \bar{x}^l \partial \bar{x}^m} + \left\{ {r \atop ij} \right\} \frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m} = \overline{\left\{ {p \atop lm} \right\}} \frac{\partial x^r}{\partial \bar{x}^p}$$

ie, $$\frac{\partial^2 x^r}{\partial \bar{x}^l \partial \bar{x}^m} = \overline{\left\{ {p \atop lm} \right\}} \frac{\partial x^r}{\partial \bar{x}^p} - \left\{ {r \atop ij} \right\} \frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m} \quad \text{—(13)}$$

g_{ik} which (eq 13) ~~shows~~ expresses the second-order partial derivatives of x^r w.r.t. $\bar{x}^s$ in terms of the first derivatives and the Christoffel symbols of the second kind.

$$T_1 = \left\{ {s \atop ij} \right\} \frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m}\underbrace{\frac{\partial \bar{x}^p}{\partial x^s}\frac{\partial x^r}{\partial \bar{x}^p}}_{=\frac{\partial x^r}{\partial x^s}=\delta^r_s}$$

$$= \left\{ {r \atop ij} \right\} \frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m} \qquad \left[\because \delta^r_s \left\{ {s \atop ij} \right\} = \left\{ {r \atop ij} \right\}\right]$$

$$T_2 = \underbrace{\frac{\partial x^r}{\partial \bar{x}^p}\frac{\partial \bar{x}^p}{\partial x^j}}_{=\frac{\partial x^r}{\partial x^j}=\delta^r_j}\frac{\partial^2 x^j}{\partial \bar{x}^l \partial \bar{x}^m}$$

$$= \delta^r_j \frac{\partial^2 x^j}{\partial \bar{x}^l \partial \bar{x}^m} = \frac{\partial^2 x^r}{\partial \bar{x}^l \partial \bar{x}^m}$$

Ex 1 Prove that the transformations of Christoffel symbols form a group. ($x^i \to \bar{x}^i \to \bar{\bar{x}}^i$)

Note Derivative of a Determinant.

Let $a = \begin{vmatrix} a^1_1 & a^1_2 & \cdots & a^1_n \\ a^2_1 & a^2_2 & \cdots & a^2_n \\ \cdots & \cdots & \cdots & \cdots \\ a^n_1 & a^n_2 & \cdots & a^n_n \end{vmatrix}$. Suppose the elements a^i_j be $(i, j = 1, 2, \dots, n)$ functions of independent variables $x, y, z \dots$ etc.

Then

$$\frac{\partial a}{\partial x} = \begin{vmatrix} \frac{\partial a^1_1}{\partial x} & \frac{\partial a^1_2}{\partial x} & \cdots & \frac{\partial a^1_n}{\partial x} \\ a^2_1 & a^2_2 & \cdots & a^2_n \\ \cdots & \cdots & \cdots & \cdots \\ a^n_1 & a^n_2 & \cdots & a^n_n \end{vmatrix} + \begin{vmatrix} a^1_1 & a^1_2 & \cdots & a^1_n \\ \frac{\partial a^2_1}{\partial x} & \frac{\partial a^2_2}{\partial x} & \cdots & \frac{\partial a^2_n}{\partial x} \\ \cdots & \cdots & \cdots & \cdots \\ a^n_1 & a^n_2 & \cdots & a^n_n \end{vmatrix} + \cdots + \begin{vmatrix} a^1_1 & a^1_2 & \cdots & a^1_n \\ a^2_1 & a^2_2 & \cdots & a^2_n \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial a^n_1}{\partial x} & \frac{\partial a^n_2}{\partial x} & \cdots & \frac{\partial a^n_n}{\partial x} \end{vmatrix}$$

$$= D_1 + D_2 + \cdots + D_n$$

Now $D_1 = \frac{\partial a^1_1}{\partial x} A^1_1 + \frac{\partial a^1_2}{\partial x} A^2_1 + \cdots + \frac{\partial a^1_n}{\partial x} A^n_1 = \frac{\partial a^1_i}{\partial x} A^i_1$, where A^i_1 is the cofactor of a^1_i in the det $|a^i_j|$.

Similarly $D_2 = \frac{\partial a^2_i}{\partial x} A^i_2, \dots, D_n = \frac{\partial a^n_i}{\partial x} A^i_n$ (Combining)

Thus, $\frac{\partial a}{\partial x} = A^i_j \frac{\partial a^j_i}{\partial x}$, $\frac{\partial a}{\partial y} = A^i_j \frac{\partial a^j_i}{\partial y}$ etc.

In view of above, $\frac{\partial g}{\partial x^j} = G_{ik} \frac{\partial g_{ik}}{\partial x^j}$, G_{ik} = cofactor of g_{ik}

$\therefore \frac{\partial g}{\partial x^j} = g\, g^{ik} \frac{\partial g_{ik}}{\partial x^j}$

Also, $g^{ik} = \frac{(\text{cofactor of } g_{ik})}{g} = \frac{G_{ik}}{g}$

– Show that $\left\{ {i \atop ij} \right\} = \frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x^j} = \frac{\partial}{\partial x^j}(\log\sqrt{g})$.

We know $g = |g_{lm}|$, g_{lm} is fundamental tensor of rank 2

on differentiating g w.r.t. x^j, we get

$$\cancel{\frac{\partial g}{\partial x^j} = \frac{\partial g_{lm}}{\partial x^j} G(l,m)}$$

$$\frac{\partial g}{\partial x^j} = G(l,m)\cdot\frac{\partial g_{lm}}{\partial x^j}$$

$$\text{or } \frac{\partial g}{\partial x^j} = g g^{lm}\frac{\partial g_{lm}}{\partial x^j} \quad \text{—(1)}$$

$g = |g_{ij}|$ or, $g = |g$

$G(l,m) =$ cofactor of g

As $g^{lm} = \frac{\text{Cofactor of } g_{lm}}{g} = G(l,m)/g$

or, $G(l,m) = g g^{lm}$

Now $\left\{ {i \atop ij} \right\} = g^{im}[ij, m]$

$$= \frac{1}{2} g^{im}\left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right)$$

But $[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial}{\partial x^i}\right.$

$\left\{ {l \atop ij} \right\} = g^{lk}[ij, k]$

As g_{ij} is symmetric, $g_{ij} = g_{ji}$

$$g^{im}\frac{\partial g_{jm}}{\partial x^i} - g^{im}\frac{\partial g_{ij}}{\partial x^m} = \frac{\partial g_{ji}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^i} = 0,$$

$$= \frac{1}{2} g^{im}\frac{\partial g_{im}}{\partial x^j}$$

[by (1) $g^{lm}\frac{\partial g_{lm}}{\partial x^j} = \frac{1}{g}\frac{\partial g}{\partial x^j}$]

$\Rightarrow g^{im}\frac{\partial g_{im}}{\partial x^j} = \frac{1}{g}\frac{\partial g}{\partial x^j}$

$$= \frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x^j}$$

$$= \frac{1}{2}\frac{\partial}{\partial x^j}(\log g) = \frac{\partial}{\partial x^j}(\log g^{\frac{1}{2}})$$

$$\text{or, } \left\{ {i \atop ij} \right\} = \frac{\partial}{\partial x^j}\{\log\sqrt{g}\} \quad \text{—(2)}$$

Note: Since g is an invariant (or, scalar), $g = \bar{g}$, it does not follow that $\left\{ {i \atop ij} \right\}$ is a covariant tensor.

If g is negative $(g < 0)$, eq (2) above may be altered to

$$\left\{ {i \atop ij} \right\} = \frac{\partial}{\partial x^j}\{\log\sqrt{(-g)}\}$$

[as $(-g) > 0$ (+v

Ex 3 If the metric of a V_N is such that $g_{ij} = 0$ for $i \neq j$, show that $\left\{ {i \atop jk} \right\} = 0$, $\left\{ {i \atop jj} \right\} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i}$,

$\left\{ {i \atop ij} \right\} = \frac{\partial}{\partial x^j}\{\log\sqrt{g_{ii}}\}$, $\left\{ {i \atop ii} \right\} = \frac{\partial}{\partial x^i}\{\log\sqrt{g_{ii}}\}$,

where i, j and k are not equal and the summation convention does not apply.

– As per the question, $i \neq j \neq k$ and $g_{ij} = 0$ for $i \neq j$, $g_{ij} = 0 = g_{ji}$

(i) $\left\{ {i \atop jk} \right\} = g^{ii}[jk, i] = \frac{1}{2}g^{ii}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)$

$= \cancel{\frac{1}{2}g\left(g^{ii}\,0 + \frac{\partial g}{\partial}\right)}$

$= \frac{1}{2}g^{ii}(0 + 0 + 0) = 0$

$\left\{ \frac{\partial g_{ki}}{\partial x^j} = \right.$ $\frac{\partial g_{jk}}{}$

$\therefore \left\{ {i \atop jk} \right\} = 0$

$$\left\{ {i \atop j\,j} \right\} = g^{ii}[jj, i] = \frac{1}{2} g^{ii} \left(\frac{\partial g_{ji}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i} \right)$$

$$= \frac{1}{2} g^{ii} \left(0 + 0 - \frac{\partial g_{jj}}{\partial x^i} \right)$$

$$\therefore \left\{ {i \atop j\,j} \right\} = -\frac{1}{2 g_{ii}} \frac{\partial g_{jj}}{\partial x^i}$$

[as $g_{ij} = 0 = g_{ji}$ for $i \neq j$]

(iii) $$\left\{ {i \atop i\,j} \right\} = g^{ii}[ij, i] = \frac{1}{2} g^{ii} \left(\frac{\partial g_{ii}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^i} \right)$$

[$g^{ii} \cdot g_{ii} = \delta^i_i = 1 \Rightarrow g^{ii} = \frac{1}{g_{ii}}$]

$$= \frac{1}{2} \frac{1}{g_{ii}} \frac{\partial g_{ii}}{\partial x^j} = \frac{1}{2} \frac{\partial}{\partial x^j} (\log g_{ii})$$

[$\frac{\partial g_{ji}}{\partial x^i} = \frac{\partial g_{ij}}{\partial x^i}$ as $g_{ij} = g_{ji}$]

$$\therefore \left\{ {i \atop i\,j} \right\} = \frac{\partial}{\partial x^j} \left\{ \log \sqrt{g_{ii}} \right\}$$

(iv) $$\left\{ {i \atop i\,i} \right\} = g^{ii}[ii, i]$$

$$= \frac{1}{2} g^{ii} \left(\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i} \right)$$

[$\frac{\partial}{\partial x^j}(\log g_{ii}) = \frac{1}{g_{ii}} \frac{\partial g_{ii}}{\partial x^j}$]

$$= \frac{1}{2} \frac{1}{g_{ii}} \frac{\partial g_{ii}}{\partial x^i} = \frac{1}{2} \frac{\partial}{\partial x^i} (\log g_{ii})$$

$$\therefore \left\{ {i \atop i\,i} \right\} = \frac{\partial}{\partial x^i} \left(\log \sqrt{g_{ii}} \right)$$

Ex 1 Calculate the Christoffel symbols corresponding to the metric $ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1)^2 \sin^2 x^2 (dx^3)^2$.

We know $ds^2 = g_{ij} dx^i dx^j$ (for $i, j = 1, 2, 3$)

Here $g_{11} = 1$, $g_{22} = (x^1)^2$, $g_{33} = (x^1)^2 \sin^2 x^2$,

[$x_1 = x^1$, $x_2 = x^2$, $x_3 = x^3$]

and $g_{12} = g_{21} = 0$ etc. i.e. $g_{ij} = 0$ when $i \neq j$

The only non-zero Christoffel symbols of the second kind are as follows:

$$\left\{ {1 \atop 2\,2} \right\} = -\frac{1}{2 \times 1} \frac{\partial (x^1)^2}{\partial x^1} = -\frac{2x^1}{2} = -x^1 \quad (g_{11} = 1)$$

[$\left\{ {i \atop j\,j} \right\} = -\frac{1}{2 g_{ii}} \frac{\partial g_{jj}}{\partial x^i}$]

$$\left\{ {1 \atop 3\,3} \right\} = -\frac{1}{2 \times 1} \frac{\partial}{\partial x^1} (x^1)^2 \sin^2 x^2 = -x^1 \sin^2 x^2$$

$$\left\{ {2 \atop 1\,2} \right\} = + \frac{\partial}{\partial x^1} \left(\log \sqrt{(x^1)^2} \right) = \frac{\partial}{\partial x^1} \log x^1 = \frac{1}{x^1}$$

[$\left\{ {i \atop i\,j} \right\} = \frac{\partial}{\partial x^j} \left(\log \sqrt{g_{ii}} \right)$]

$$\left\{ {2 \atop 3\,3} \right\} = -\frac{1}{g_{22}} \frac{\partial g_{33}}{\partial x_2} = -\frac{1}{2(x^1)^2} \frac{\partial}{\partial x^2} (x^1)^2 \sin^2 x^2$$

$$= -\frac{(x^1)^2}{2(x^1)^2} 2 \sin x^2 \cdot \cos x^2 = -\sin x^2 \cos x^2$$

[$\left\{ {i \atop j\,j} \right\} = -\frac{1}{2 g_{ii}} \frac{\partial g_{jj}}{\partial x^i}$]

$$\left\{ {3 \atop 1\,3} \right\} = \frac{\partial}{\partial x^1} \left(\log \sqrt{g_{33}} \right) = \frac{\partial}{\partial x^1} \log \sqrt{(x^1)^2 \sin^2 x^2} = \frac{\sin^2 x^2}{x^1}$$

[$\left\{ {i \atop j\,k} \right\} = 0$]

$$\left\{ {3 \atop 2\,3} \right\} = \frac{\partial}{\partial x_2} \left(\log \sqrt{g_{33}} \right) = \frac{\partial}{\partial x^2} \left(\log \sqrt{\sin^2 x^2} \right) = \frac{\partial}{\partial x^2} \left(\log \sin x^2 \right) = \frac{1}{\sin x^2} \cos x^2 = \cot x^2$$

$$\begin{Bmatrix} i \\ j\,j \end{Bmatrix} = g^{ii}[jj, i] = \frac{1}{2} g^{ii}\left(\frac{\partial g_{ji}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i}\right)$$

$$= \frac{1}{2} g^{ii}\left(0 + 0 - \frac{\partial g_{jj}}{\partial x^i}\right) \qquad \left[\text{as } g_{ij} = 0 = g_{ji} \text{ for } i \neq j\right.$$

$$\therefore \begin{Bmatrix} i \\ j\,j \end{Bmatrix} = -\frac{1}{2 g_{ii}} \frac{\partial g_{jj}}{\partial x^i}$$

(III) $$\begin{Bmatrix} i \\ i\,j \end{Bmatrix} = g^{ii}[ij, i] = \frac{1}{2} g^{ii}\left(\frac{\partial g_{ii}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^i}\right) \qquad \left[g^{ii} \cdot g_{ii} = \delta^i_i = 1 \Rightarrow g^{ii} = \frac{1}{g_{ii}}\right.$$

$$= \frac{1}{2} \frac{1}{g_{ii}} \frac{\partial g_{ii}}{\partial x^j} = \frac{1}{2} \frac{\partial}{\partial x^j}(\log g_{ii}) \qquad \left[\frac{\partial g_{ji}}{\partial x^i} = \frac{\partial g_{ij}}{\partial x^i} \text{ as } g_{ij} = g_{ji}\right.$$

$$\therefore \begin{Bmatrix} i \\ i\,j \end{Bmatrix} = \frac{\partial}{\partial x^j}\left\{\log \sqrt{g_{ii}}\right\}$$

(IV) $$\begin{Bmatrix} i \\ i\,i \end{Bmatrix} = g^{ii}[ii, i] \qquad \left[\frac{\partial}{\partial x^j}(\log g_{ii}) = \frac{1}{g_{ii}} \frac{\partial g_{ii}}{\partial x^j}\right.$$

$$= \frac{1}{2} g^{ii}\left(\frac{\partial g_{ii}}{\partial x^i} + \underline{\frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i}}\right)$$

$$= \frac{1}{2} \frac{1}{g_{ii}} \frac{\partial g_{ii}}{\partial x^i} = \frac{1}{2} \frac{\partial}{\partial x^i}(\log g_{ii})$$

$$\therefore \begin{Bmatrix} i \\ i\,i \end{Bmatrix} = \frac{\partial}{\partial x^i}(\log \sqrt{g_{ii}})$$

Ex.1 Calculate the Christoffel symbols corresponding to the metric $ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1)^2 \sin^2 x^2 (dx^3)^2$.

We know $ds^2 = g_{ij}\, dx^i dx^j$ (for $i, j = 1, 2, 3$)

Here $g_{11} = 1$, $g_{22} = (x^1)^2$, $g_{33} = (x^1)^2 \sin^2 x^2$, $\qquad \left[x_1 = x^1,\ x_2 = x^2,\ x_3 = x^3\right.$

and $g_{12} = g_{21} = 0$ etc. i.e. $g_{ij} = 0$ when $i \neq j$

The only non-zero Christoffel symbols of the second kind are as follows:

$$\begin{Bmatrix} 1 \\ 2\,2 \end{Bmatrix} = -\frac{1}{2 \times 1} \frac{\partial (x^1)^2}{\partial x^1} = -\frac{2x^1}{2} = -x^1 \quad (g_{11} = 1) \qquad \left[\begin{Bmatrix} i \\ j\,j \end{Bmatrix} = -\frac{1}{2 g_{ii}} \frac{\partial g_{jj}}{\partial x^i}\right.$$

$$\begin{Bmatrix} 1 \\ 3\,3 \end{Bmatrix} = -\frac{1}{2 \times 1} \frac{\partial}{\partial x^1} (x^1)^2 \sin^2 x^2 = -x^1 \sin^2 x^2$$

$$\begin{Bmatrix} 2 \\ 1\,2 \end{Bmatrix} = + \frac{\partial}{\partial x^1}\left(\log \sqrt{(x^1)^2}\right) = \frac{\partial}{\partial x^1} \log x^1 = \frac{1}{x^1} \qquad \left[\begin{Bmatrix} i \\ i\,j \end{Bmatrix} = \frac{\partial}{\partial x^j}(\log \sqrt{g_{ii}})\right.$$

$$\begin{Bmatrix} 2 \\ 3\,3 \end{Bmatrix} = -\frac{1}{g_{22}} \frac{\partial g_{33}}{\partial x_2} = -\frac{1}{2(x^1)^2} \frac{\partial}{\partial x^2} (x^1)^2 \sin^2 x^2 \qquad \left[\begin{Bmatrix} i \\ j\,j \end{Bmatrix} = -\frac{1}{2 g_{ii}} \frac{\partial g_{jj}}{\partial x^i}\right.$$

$$= -\frac{(x^1)^2}{2(x^1)^2}\, 2 \sin x^2 \cdot \cos x^2 = -\sin x^2 \cos x^2$$

$$\begin{Bmatrix} 3 \\ 1\,3 \end{Bmatrix} = \frac{\partial}{\partial x^1}(\log \sqrt{g_{33}}) = \frac{\partial}{\partial x^1} \log \sqrt{(x^1)^2 \sin^2 x^2} = \frac{\sin^2 x^2}{x^1} \qquad \left[\begin{Bmatrix} i \\ j\,k \end{Bmatrix} = 0\right.$$

$$\begin{Bmatrix} 3 \\ 2\,3 \end{Bmatrix} = \frac{\partial}{\partial x^2}(\log \sqrt{g_{33}}) = \frac{\partial}{\partial x^2}\left(\log \sqrt{\sin^2 x^2}\right) = \frac{\partial}{\partial x^2}(\log \sin x^2) = \frac{1}{\sin x^2} \cos x^2 = \cot x^2$$

§24 The covariant derivative (differentiation)

I • The covariant derivative of a tensor or, vector A_p (covariant) w.r.t. x^q is denoted by $A_{p,q}$ and is defined by

$$A_{p,q} = \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop p\,q} \right\} A_s \, , \qquad (1)$$

which is a covariant tensor of rank (or order) two.

II • The covariant derivative of a tensor or vector A^p (contravariant) w.r.t. x^q, is denoted by $A^p_{,q}$ and is defined by

$$A^p_{,q} = \frac{\partial A^p}{\partial x^q} + \left\{ {p \atop q\,s} \right\} A^s , \qquad (2)$$

which is a mixed tensor of rank two.

III • The covariant derivative of a mixed tensor A^p_r w.r.t. x^q, is denoted by $A^p_{r,q}$ and is defined by

$$A^p_{r,q} = \frac{\partial A^p_r}{\partial x^q} - \left\{ {s \atop r\,q} \right\} A^p_s + \left\{ {p \atop q\,t} \right\} A^t_r \, , \qquad (3)$$

where $A^p_{r,q}$ is a tensor (mixed) of rank three of the type indicated by its

§25 ⊡ Tensor form of Gradient, Divergence and curl.

• **Gradient** : If ϕ is a scalar (function) or invariant then the gradient of ϕ is denoted by $\nabla\phi$ or grad ϕ and is defined by

$$\nabla\phi = \text{grad}\,\phi = \phi_{,p} = \frac{\partial\phi}{\partial x^p} \qquad (1)$$

where $\phi_{,p}$ is the covariant derivative of ϕ with respect to (w.r.t.)

Here ϕ is a scalar function of coordinates $x^1, x^2, \ldots, x^N$ or, x^p.

(∇ nebla)

• **Divergence** : The divergence of A^p is the contraction of its covariant derivative w.r.t. x^q, i.e., the contraction of $A^p_{,q}$.

Then $\text{div}\, A^p = A^p_{,p} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g}\, A^k \right)$. $\qquad (2)$

$$A_{p,q} - A_{q,p} = \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p}, \quad \text{—(3)}$$

which is a tensor of rank two.

The curl is also defined as $-\epsilon^{pqr} A_{p,q}$.

+

• Laplacian : The Laplacian of ϕ (a scalar function) is the divergence of grad ϕ (or, $\phi_{,p}$) or,

$$\nabla^2 \phi = \text{div}\, \phi_{,p}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{jk} \frac{\partial \phi}{\partial x^k}\right)$$

[grad $\phi = \phi_{,p} = \frac{\partial \phi}{\partial x^p}$; div $\phi_{,p} = \frac{1}{\sqrt{g}}$]

In case $g<0$, $\sqrt{g}$ must be replaced by $\sqrt{-g}$. Both cases $g>0$ and $g<0$ can be included by $\sqrt{|g|}$ in place of $\sqrt{g}$.

§25 Covariant differentiation of Vectors

I. By transformation law of coordinates x^i to $\bar{x}^i$, the covariant tensor A_j transforms to $\bar{A}_i$, so that

$$\bar{A}_i = A_j \frac{\partial x^j}{\partial \bar{x}^i}, \quad \text{—(1)}$$

On differentiating (partially) eq (1) w.r.t. $\bar{x}^l$, there results in

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^l} = \frac{\partial A_j}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^i} + A_j \frac{\partial^2 x^j}{\partial \bar{x}^l \partial \bar{x}^i} \quad \text{—(2)}$$

But $$\frac{\partial^2 x^j}{\partial \bar{x}^l \partial \bar{x}^i} = \overline{\left\{ {p \atop l\,i} \right\}} \frac{\partial x^j}{\partial \bar{x}^p} - \left\{ {j \atop m\,n} \right\} \frac{\partial x^m}{\partial \bar{x}^l} \frac{\partial x^n}{\partial \bar{x}^i} \quad \text{—(3)}$$

So that Inserting (3) in (2) and changing dummy indices,

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^l} = \overline{\left\{ {p \atop i\,l} \right\}} \underbrace{\left(A_j \frac{\partial x^j}{\partial \bar{x}^p}\right)}_{=\bar{A}_p \text{ by (1)}} + \left[\frac{\partial A_j}{\partial x^n} - \left\{ {r \atop j\,n} \right\} A_r\right] \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^l}$$

(Changing k by n, m by j)

or, $$\frac{\partial \bar{A}_i}{\partial \bar{x}^l} - \overline{\left\{ {m \atop i\,l} \right\}} \bar{A}_m = \left[\frac{\partial A_j}{\partial x^n} - \left\{ {r \atop j\,n} \right\} A_r\right] \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^l} \quad \text{—(4)}$$

(changing index p by m)

• Using the comma (,) notation for derivative, as

$$A_{j,n} \equiv \frac{\partial A_j}{\partial x^n} - \left\{ {r \atop j\,n} \right\} A_r \quad \text{—(5)}$$

So that $$\bar{A}_{i,l} \equiv \frac{\partial \bar{A}_i}{\partial \bar{x}^l} - \overline{\left\{ {m \atop i\,l} \right\}} \bar{A}_m \text{ —(6)},$$

and

$$A_{p,q} - A_{q,p} = \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p}, \quad \text{—(3)}$$

which is a tensor of rank two.

The curl is also defined as $-\epsilon^{pqr} A_{p,q}$.

• Laplacian: The Laplacian of ϕ (a scalar function) is the divergence of grad ϕ (or, $\phi_{,p}$) or,

$$\nabla^2 \phi = \text{div } \phi_{,p}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{jk} \frac{\partial \phi}{\partial x^k}\right)$$

[grad $\phi = \phi_{,p} = \frac{\partial \phi}{\partial x^p}$; div $\phi_{,p} = \frac{1}{\sqrt{g}}$]

In case $g<0$, $\sqrt{g}$ must be replaced by $\sqrt{-g}$. Both cases $g>0$ and $g<0$ can be included by $\sqrt{|g|}$ in place of $\sqrt{g}$.

§25 Covariant differentiation of Vectors

I. By transformation law of coordinates x^i to $\bar{x}^i$, the covariant tensor A_j transforms to $\bar{A}_i$, so that

$$\bar{A}_i = A_j \frac{\partial x^j}{\partial \bar{x}^i}, \quad \text{—(1)}$$

On differentiating (partially) eq (1) w.r.t. $\bar{x}^l$, there results in

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^l} = \frac{\partial A_j}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^i} + A_j \frac{\partial^2 x^j}{\partial \bar{x}^l \partial \bar{x}^i} \quad \text{—(2)}$$

But $\frac{\partial^2 x^j}{\partial \bar{x}^l \partial \bar{x}^i} = \overline{\left\{ {p \atop l\,i} \right\}} \frac{\partial x^j}{\partial \bar{x}^p} - \left\{ {j \atop m\,n} \right\} \frac{\partial x^m}{\partial \bar{x}^l} \frac{\partial x^n}{\partial \bar{x}^i} \quad$ —(3)

So that, Inserting (3) in (2) and changing dummy indices,

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^l} = \overline{\left\{ {p \atop i\,l} \right\}} \underbrace{\left(A_j \frac{\partial x^j}{\partial \bar{x}^p}\right)}_{=\bar{A}_p \text{ by(1)}} + \left[\frac{\partial A_j}{\partial x^n} - \left\{ {r \atop j\,n} \right\} A_r\right] \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^l}$$

(Changing k by n, m by j)

or $\frac{\partial \bar{A}_i}{\partial \bar{x}^l} - \overline{\left\{ {m \atop i\,l} \right\}} \bar{A}_m = \left[\frac{\partial A_j}{\partial x^n} - \left\{ {r \atop j\,n} \right\} A_r\right] \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^l} \quad$ —(4)

(Changing index p by m)

. Using the comma (,) notation for derivative, as

$$A_{j,n} \equiv \frac{\partial A_j}{\partial x^n} - \left\{ {r \atop j\,n} \right\} A_r \quad \text{—(5)}$$

So that $\bar{A}_{i,l} \equiv \frac{\partial \bar{A}_i}{\partial \bar{x}^l} - \overline{\left\{ {m \atop i\,l} \right\}} \bar{A}_m$ —(6),

and eq (4) is expressed as

which shows that $A_{j,n}$ is a covariant tensor of the second order, called the covariant derivative of A_j w.r.t. x^n.

II. By transformation law of coordinates x^i to $\bar{x}^i$, the contravariant vector (or tensor) A^k transforms to $\bar{A}^i$ so that

$$A^k = \bar{A}^i \frac{\partial x^k}{\partial \bar{x}^i}, \quad \text{——(1)}$$

or differentiating (1) partially w.r.t. x^j, we have

$$\frac{\partial A^k}{\partial x^j} = \frac{\partial \bar{A}^i}{\partial \bar{x}^n}\frac{\partial \bar{x}^n}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^i} + \bar{A}^i \frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^n}\frac{\partial \bar{x}^n}{\partial x^j}. \quad \text{——(2)}$$

The presence of last term on the right-hand side of eq (2) shows that the partial derivative $\frac{\partial A^k}{\partial x^j}$ do not form a tensor.

To obtain a tensor form, we use the expression

$$\frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^n} = \overline{\left\{ {p \atop i\,n} \right\}} \frac{\partial x^k}{\partial \bar{x}^p} - \left\{ {k \atop r\,s} \right\} \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^n} \quad \text{——(3)}$$

Inserting (3) in (2) and after changing the dummy indices

$$\frac{\partial A^k}{\partial x^j} = \frac{\partial \bar{A}^i}{\partial \bar{x}^m}\frac{\partial \bar{x}^m}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^i} + \bar{A}^i \frac{\partial \bar{x}^n}{\partial x^j}\Big[\underbrace{\overline{\left\{ {p \atop i\,n} \right\}} \frac{\partial x^k}{\partial \bar{x}^p}}_{=T_1} - \underbrace{\left\{ {k \atop r\,s} \right\} \frac{\partial x^r}{\partial \bar{x}^i}\cdots}_{=T_2}$$

or
$$\frac{\partial A^k}{\partial x^j} + \left\{ {k \atop r\,j} \right\} A^r = \left[\frac{\partial \bar{A}^i}{\partial \bar{x}^n} + \overline{\left\{ {i \atop r\,n} \right\}}\bar{A}^r\right]\frac{\partial \bar{x}^n}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^i} \quad \text{——(4)}$$

In $T_2 = \left(\bar{A}^i \frac{\partial x^r}{\partial \bar{x}^i}\right) = A^r$

$\bar{A}^i \frac{\partial \bar{x}^n}{\partial x^j} \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^n}$

$= \left(\bar{A}^i \frac{\partial x^r}{\partial \bar{x}^i}\right) \cdot \underbrace{\frac{\partial x^s}{\partial x^j}}_{=\delta^s_j}$

$= A^r \left\{ {k \atop r\,j} \right\}$

In $T_1 = \left[\frac{\partial \bar{A}^i}{\partial \bar{x}^n} + \bar{A}^r \overline{\left\{ {i \atop r\,n} \right\}}\right]\frac{\partial \bar{x}^n}{\partial x^j}\cdots$

$\bar{A}^i \frac{\partial \bar{x}^n}{\partial x^j} \overline{\left\{ {p \atop i\,n} \right\}} \frac{\partial x^k}{\partial \bar{x}^p}$

$= \bar{A}^r \frac{\partial \bar{x}^n}{\partial x^j} \overline{\left\{ {i \atop r\,n} \right\}} \frac{\partial x^k}{\partial \bar{x}^i}$

(changing i by r & p by i)

On introducing the comma (,) notation in (4),

$$A^k_{,j} = \frac{\partial A^k}{\partial x^j} + \left\{ {k \atop r\,j} \right\} A^r, \quad \text{——(5)}$$

and
$$\bar{A}^i_{,n} = \frac{\partial \bar{A}^i}{\partial \bar{x}^n} + \overline{\left\{ {i \atop r\,n} \right\}} \bar{A}^r, \quad \text{——(6)},$$

we have

$$A^k_{,j} = \bar{A}^i_{,n} \frac{\partial \bar{x}^n}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^i}, \quad \text{——(7)}$$

which shows that $A^k_{,j}$ is a mixed tensor of second order and it is called the covariant derivative of A^k w.r.t. x^j.

$$A^a_{b,c} = \frac{\partial A^a_b}{\partial x^c} + A^\alpha_b \left\{ {a \atop \alpha c} \right\} - A^a_\alpha \left\{ {\alpha \atop b c} \right\}, \quad -(6)$$

and
$$\bar{A}^i_{j,k} = \frac{\partial \bar{A}^i_j}{\partial \bar{x}^k} + \bar{A}^h_j \left\{ {i \atop h k} \right\} - \bar{A}^i_h \left\{ {h \atop j k} \right\}, \quad -(7)$$

Eq (5) yields to the form
$$\bar{A}^i_{j,k} \frac{\partial x^a}{\partial \bar{x}^i} = A^a_{bc} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} \quad -(8)$$

Inner multiplication of Eq (8) by $\frac{\partial \bar{x}^i}{\partial x^a}$, there results
$$\bar{A}^i_{j,k} = A^a_{bc} \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k}, \quad -(9)$$

as $\frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^a}{\partial \bar{x}^i} = \frac{\partial x^a}{\partial x^a} = \delta^a_a = 1$.

which is the covariant derivative of A^i_j.

The above eq (9) conforms (follows) the tensor law of transformation and it is a mixed tensor of order three.

$$\frac{\partial \bar{x}^i}{\partial \bar{x}^r}$$

$$= \bar{A}^i_j \frac{\partial x^a}{\partial \bar{x}^i} \cdot \left\{ {a \atop a s} \right\}$$

chang (r by a)

$$= \underbrace{A^a_b \frac{\partial x^b}{\partial \bar{x}^j}}_{\text{by (2)}} \left\{ {a \atop a s} \right\}$$

$$= A^\alpha_b \left\{ {a \atop \alpha c} \right\} \frac{\partial x^b}{\partial \bar{x}}$$

changing (a by α

In $T_4 = \left\{ {b \atop r s} \right\} \frac{\partial x^r}{\partial \bar{x}^j}$

$$= A^a_b \left\{ {b \atop b c} \right\} \frac{\partial x^b}{\partial \bar{x}^j}$$

changing (

$$= A^a_\alpha \left\{ {\alpha \atop b c} \right\} \frac{\partial x^b}{\partial \bar{x}^j}$$

changing

Ex 1 Find the covariant derivative of

(a) A^{ij}, (b) A_{ij}, (c) A^i_{jk}.

Ex 2 If $A_{ij} = B_{i,j} - B_{j,i}$, prove that
$$A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$$

Sol. $A_{ij} = B_{i,j} - B_{j,i} = \frac{\partial B_i}{\partial x^j} - \frac{\partial B_j}{\partial x^i}, \quad -(1)$

Similarly A_{jk} & A_{ki}.

then ~~$A_{ij,k} = B_{ij,k} + B_{jk,i} =$~~

Now
$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - A_{aj} \left\{ {a \atop i k} \right\} - A_{ia} \left\{ {a \atop j k} \right\} \quad -(2)$$

$$A_{jk,i} = \frac{\partial A_{jk}}{\partial x^i} - A_{ak} \left\{ {a \atop j i} \right\} - A_{ja} \left\{ {a \atop k i} \right\} \quad -(3)$$

$$A_{ki,j} = \frac{\partial A_{ki}}{\partial x^j} - A_{ai} \left\{ {a \atop j k} \right\} - A_{ka} \left\{ {a \atop i j} \right\} \quad -(4)$$

Therefore From (1) & (2), we get
$$A_{ij,k} = \left(\frac{\partial^2 B_i}{\partial x^j \partial x^k} - \frac{\partial^2 B_j}{\partial x^i \partial x^k} \right) - \left\{ {a \atop i k} \right\} \left(\frac{\partial B_a}{\partial x^j} - \frac{\partial B_j}{\partial x^a} \right) - \left\{ {a \atop j k} \right\} \left(\frac{\partial B_i}{\partial x^a} \right.$$

Similarly $A_{jk,i}$ & $A_{ki,j}$

$L.H.S = 0 = R.H.S.$

Here $A_{p,q} - A_{q,p} = \left(\frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\} A_s\right) - \left(\frac{\partial A_q}{\partial x^p} - \left\{ {s \atop qp} \right\} A_s\right)$

$$= \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p}$$

But $\left\{ {s \atop pq} \right\} = \left\{ {s \atop qp} \right\}$ as symmetric in lower indices

§26 Laws of Covariant Differentiation

Covariant derivatives obey the following laws:—

(I) The covariant derivative of the sum (+) (or difference (−)) of two tensors is the sum (or difference) of their covariant derivatives.

e.g., $(A_i \pm B_j)_{,k} = A_{i,k} \pm B_{j,k}$

(II) The covariant derivative of an outer (or inner) product of two tensors is equal to the sum of the two terms obtained by outer (or inner) multiplication of each tensor with the covariant derivative of the other tensor.

eg, (i) $(A_{ij} B^l)_{,m} = A_{ij,m} B^l + A_{ij} B^l_{,m}$.

(ii) $(A_{ij} B^j)_{,m} = A_{ij,m} B^j + A_{ij} B^j_{,m}$.

Ex 4 Show that $\operatorname{div} A^i = \operatorname{div} A_i$.

Ex 5 If A^{ijk} is a skew-symmetric tensor, show that
$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g}\, A^{ijk}\right) \text{ is a tensor.}$$

Ex 6 Prove that $\operatorname{div} A_j = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left(\sqrt{g}\, g^{rk} A_k\right) = \operatorname{div} A^j$.

is used for contraction of tensors (i.e, reducing the order of a tensor). Tensors have wide applications in diverse fields of Science & Engineering. There are cases of interest to the Theoretical computer scientists for applications of tensors methods which include the following: Signal processing, Quantum Information Theory, Machine Learning, Algebraic Statistics etc.

- Tensor e.g., stress, strain, angular momentum (a tensor of rank 2).
  Time is not a tensor (It is a coordinate).
  Christoffel symbol is not a tensor (as it does not transform as a tensor but rather as an object in jet bundle).
  Torque is defined as the rate of change of angular momentum (an. mo.), an. mo. is not a Vector in SR (Special Relativity) as there is no vector cross product in 4-dimensions (x, y, z, t). an. mo. is a tensor of rank 2.
- Tensors are used in the problems of mechanics (stress, elasticity, fluid mechanics, moment of inertia (m.i.), relativity, electrodynamics, electromagnetism etc. Tensors provide a concise mathematical framework for formulating & solving problems in Physics, Mechanics, Elasticity etc.
- Tensors may have an arbitrary number of indices. It is a type of Data structure, used in Linear Algeb

tensors for deformations (stress tensors) & Strain tensors for strain in Continuum Mech. Elasticity etc. (ii) Electromagnetic tensors (or Faraday's tensor) in electromagnetism.

- Application of Tensors: (1) It may be worthwhile mentioning that in switching over from Special Theory of Relativity (STR) (i.e, unaccelerated motion — motion with uniform or, constant velocity) to General Theory of Relativity (GTR) (i.e, accelerated motion — motion with non-uniform or, variable velocity), (STR → GTR) Great Albert Einstein (1879–1955) was searching for a tool for formulating his new theory (i.e, GTR). German mathematician Bernhard Riemann (1826–18..) suggested him to apply 'Tensors' in formulating the various concepts of General Relativity (GTR). Einstein has widely used tensors in general relativity (GTR).

(2) Applications of Tensors: Tensor calculus is a very powerful mathematical tool and hence tensors play a vital role in science and engineering where they are used to represent physical and synthetic objects and ideas in mathematical invariant forms. Tensor notation and techniques are used in many branches of mathematics, science and engineering such as Differential Geometry, Fluid Mechanics, Continuum Mechanics, General Relativity and Structural Engineering.

(1) Riemann-christoffel tensors of First & Second Kinds

(2) Curvature tensor

(3) Ricci tensor

(4) Bianchi's Identity

(5) Riemannian Curvature

In Electrodynamics:

- Maxwell's equations (in tensor form) are

$$\operatorname{div} E_i = 4\pi\rho$$

$$\operatorname{div} H_i = 0$$

$$\operatorname{curl} E_i + \frac{1}{c}\frac{\partial H_i}{\partial t} = 0$$

$$\operatorname{curl} H_i - \frac{1}{c}\frac{\partial E_i}{\partial t} = \frac{4\pi}{c} j_i$$

where j_i is the current density, ρ is the charge density, E_i is the electric field strength vector, H_i is the magnetic field strength vector, c is the velocity of a ray of light and t is the time.

- Kronecker delta or, Unit tensor δ is a rank-2 tensor in all dimensions. It is defined as

$$\delta_{ij}\ (\text{or } \delta^i_j) = \begin{cases} 1; & i=j \\ 0; & i\neq j \end{cases} \quad (i,j=1,2,\cdots,n)$$

where n is the space dimension and hence it can be considered as the 'identity matrix'.

e.g., in 3-D space, Kronecker (delta) δ tensor is given by

$$[\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (i,j=1,2,3)$$

The components of the covariant, contravariant and mixed types of this tensor are the same. i.e.,

$$\delta_{ij} = \delta^{ij} = \delta^i_j.$$

The kronecker delta δ tensor is 'symmetric', i.e.,

$$\delta_{ij} = \delta_{ji}, \quad \delta^{ij} = \delta^{ji} \text{ where } i,j=1,2,\cdots,n.$$

Moreover, it is 'conserved' under all proper and improper