Mathematical Methods

Koeli Ghoshal
Dept. of Mathematics
IIT, KGP

2020

Basic theory of linear differential equation

Definition

A linear ODE of order n in the dependent variable y and the independent variable x is an equation of the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}(y)}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y$$
$$= F(x) \quad —(1)$$

where $a_0 \neq 0$. Here $a_0, a_1, \ldots, a_n$ and F are continuous real functions on a real interval $a \le x \le b$ and $a_0 \neq 0$ for any x on $a \le x \le b$. The right hand member $F(x)$ is called the non-homogeneous term. If $F=0$, then the equation reduces to

$$a_0(x)\frac{d^n y}{dx^n} + \cdots + a_n(x)y = 0. \quad —(2)$$

and is called homogeneous.

For $n=2$, Eq. (1) reduces to the second order non-homogeneous linear ODE

$$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x) \quad —(3)$$

and the corresponding homogeneous equation

$$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad —(4)$$

Theorem (on Initial value problem or IVP)

Consider the nth-order linear diff. eqn.

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \quad -(A)$$

where $a_0, a_1, \cdots a_n$ and F are continuous real functions on a real interval $a \le x \le b$ and $a_0(x) \ne 0$ for any x on $a \le x \le b$. Let x_0 be any point of the interval $a \le x \le b$ and let $c_0, c_1, \cdots, c_{n-1}$ be n arbitrary real constants. Then $\exists$ a unique solution of (A) where

$$y(x_0) = c_0,\ y'(x_0) = c_1,\ \cdots,\ y^{n-1}(x_0) = c_{n-1}$$

and the solution is defined over the entire interval $a \le x \le b$.

Example

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + x^3y = e^x$$

$$y(1) = 2, \quad y'(1) = -5$$

There exists unique solution of the problem by the above theorem.

Corollary

Let f $[y = f(x)]$ be a solution of the n-th order homogeneous linear ODE

$$a_0(x)\frac{d^ny}{dx^n} + \cdots + a_n(x)y = 0$$

such that $f(x_0) = 0,\ f'(x_0) = 0,\ f''(x_0) = 0,\ \cdots,\ f^{n-1}(x_0) = 0$ where $x_0 \in [a,b]$. Then $f(x) = 0\ \forall x$ in $a \le x \le b$.

Example

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + x^2y = 0$$

$$f(2) = f'(2) = f''(2) = 0$$

$\exists$ only trivial solution for this ODE.

Basic theorem on linear homogeneous ODE

$$a_0(x)\frac{d^n y}{dx^n} + \cdots + a_n(x)y = 0 \quad \text{—(A)}$$

Let $f_1, f_2, \cdots, f_m$ be any m solutions of (A). Then $c_1f_1 + c_2f_2 + \cdots + c_mf_m$ is also a sol^n. of (A) where $c_1, c_2, \cdots c_m$ are m arbitrary constants.

Linear dependence and linear independence of functions

The n functions $f_1, f_2, \cdots f_n$ are called L.D. on $a \le x \le b$ if $\exists$ constants $c_1, c_2, \cdots c_n$, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$.

For linear independence $\rightarrow c_1 = c_2 = \cdots = c_n = 0$

Example: $\sin x$, $3\sin x$ and $-\sin x$ are L.D. in $-1 \le x \le 2$ for $\exists$ c_1, c_2, c_3 not all zero such that

$$c_1 \sin x + c_2(3\sin x) + c_3(-\sin x) = 0.$$

For example, $c_1 = 1$, $c_2 = 1$, $c_3 = 4$

Theorem

The nth order homogeneous linear ODE

$$a_0(x)\frac{d^n y}{dx^n} + \cdots + a_n(x)y = 0 \quad \text{—(1)}$$

always possesses n L.I. solutions. Linear combination of these n L.I. sol^n. gives the G.S. of the ODE-(1). These n L.I. sol^{n}s. are called fundamental set of solutions.

Example

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

Given 3 sol^{n}s. as e^x, e^{-x} & e^{2x}. These are L.I. So they constitute a fundamental set of solutions. G.S. can be written as $y = c_1e^x + c_2e^{-x} + c_3e^{2x}$

Wronskian

Wronskian of n f^ns. $f_1, f_2, \ldots f_n$ is given by

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & & f_n' \\ - & - & - & - \\ - & - & - & \vdots \\ f_1^{n-1} & f_2^{n-1} & & f_n^{n-1} \end{vmatrix}$$

Theorem

The n solutions $f_1, f_2, \ldots, f_n$ of the nth order homogeneous linear ODE are L.I. if and only if the Wronskian of $f_1, f_2, \ldots f_n$ is not zero.

Example

The solutions $\sin x$ and $\cos x$ of $\frac{d^2y}{dx^2} + y = 0$ are L.I.

Because $W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x$
$= -1 \neq 0$

Boundary value problem (BVP)

A BVP for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions.

Types of B.C. $\longrightarrow$ $y'' + p(x)y' + q(x)y = r(x)$, $a<x<b$

(i) Dirichlet or First kind : $y(a) = \eta_1$, $y(b) = \eta_2$

(ii) Neumann or Second kind : $y'(a) = \eta_1$, $y'(b) = \eta_2$

(iii) Robin or Third kind : $\alpha_1 y(a) + \alpha_2 y'(a) = \eta_1$

$\beta_1 y(b) + \beta_2 y'(b) = \eta_2$

BVPs do not behave as nicely as IVPs. As such existence of sol^n. for BVP is not easy to determine.

Example

Consider $y'' + y = 0$ — (A)

(i) The BVP (A) with B.C. $y(0) = 1$, $y(\frac{\pi}{2}) = 1$ has a unique solution.

(ii) The BVP (A) with B.C. $y(0) = 1$, $y(\pi) = 1$ has no solⁿ.

(iii) The BVP (A) with B.C. $y(0) = 1$, $y(2\pi) = 1$ has many solⁿs.

Two point BVP

We consider $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = f(x)$

or $L[y] = f(x)$

Here f, q be continuous fⁿ. in $[a,b]$, $p\ (>0)$ be continuously differentiable fⁿ. and does not vanish on $[a,b]$. Two B.C.s are given by

$$a_1 y(a) + a_2 y'(a) = \eta_1 \Rightarrow u_1[y] = \eta_1$$
$$b_1 y(b) + b_2 y'(b) = \eta_2 \Rightarrow u_2[y] = \eta_2$$

where both a_1, a_2 or b_1, b_2 are not zero.

Non-homogeneous form

$$\left.\begin{array}{l} L[y] = f(x) \\ u_1[y] = \eta_1, \quad u_2[y] = \eta_2 \end{array}\right\} \text{— (A)}$$

Homogeneous form

$$\left.\begin{array}{l} L[y] = 0 \\ u_1[y] = 0 \quad u_2[y] = 0 \end{array}\right\} \text{— (B)}$$

Lemma

(i) A linear combination of solutions of the homogeneous BVP (B) is also a solution of the homogeneous BVP (B).

(ii) If u, v are two solutions of the non-homogeneous BVP, then their difference $u-v$ is a solution of the homogeneous BVP.

(iii) If y solves the non-homogeneous BVP (A)
and z " " homogeneous BVP (B),
then $y+z$ " " non-homogeneous BVP (A).

Lemma

Let φ_1, φ_2 be a fundamental pair of solutions to the ODE $L[y]=0$. Then the following are equivalent

(i) The non-homogeneous BVP has a unique solution for any given constants η_1 and η_2 and a given continuous function f on the interval $[a,b]$.

(ii) The associated homogeneous BVP has only trivial solution.

Try to make connection with system of linear equations $Ax=b$ and $Ax=0$.