Assignment 4

Mathematical Methods

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1) Given equation $Ly = 0$ is :

$$x^2 \cdot \frac{d^2y}{dx^2} + (2x^3+1)\cdot\frac{dy}{dx} + y = 0 \quad \cdots\cdots \text{(i)}$$

$\therefore$ Adjoint of (i) is given by :

$$My(x) \equiv \frac{d^2}{dx^2}(x^2 \cdot y) - \frac{d}{dx}\left((2x^3+1)\cdot y\right) + y$$

$$= \frac{d}{dx}(2xy + x^2 \cdot y') - 6x^2y - (2x^3+1)y' + y$$

$$= 2y + 2xy' + 2xy' + x^2y'' - 6x^2y - (2x^3+1)y' + y$$

$$= x^2 \cdot y'' - (2x^3 - 4x + 1)y' - 3(2x^2-1)y$$

Then, adjoint equation of (i) is : $My(x) = 0$

$$\Rightarrow x^2 \cdot y'' - (2x^3 - 4x + 1)y' - 3(2x^2-1)y = 0 \qquad \text{[Ans.]}$$

2) Given ODE : $x^2 \cdot \frac{d^2y}{dx^2} - 2x \cdot \frac{dy}{dx} + 2y = 0 \quad \cdots\cdots \text{(i)}$

We know, that the ODE $a_0(x)\cdot y'' + a_1(x)\cdot y' + a_2(x)y = 0$ is self adjoint if and only if (i.e. the condition is both necessary and sufficient) $a_0'(x) = a_1(x)$.

Here, $a_0(x) = x^2 \Rightarrow a_0'(x) = 2x \neq a_1(x)$ as $a_1(x) = -2x$

Hence, (i) is not a self adjoint equation. [Ans.]

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3) Let $\{f_n\}_{n=0}^{\infty}$ be a set of functions where:

$$f_n(x) = \cos\left(\frac{n\pi x}{L}\right) \quad ; \quad -L \leq x \leq L$$

<u>To Prove:</u> $\{f_n\}_{n=0}^{\infty}$ is a set of mutually orthogonal functions on $-L \leq x \leq L$

<u>Proof:</u> We have, for $m \neq n$,

$$\int_{-L}^{L} f_m(x) \cdot f_n(x) \cdot dx$$

$$= \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot dx = \frac{1}{2} \int_{-L}^{L} 2\cos\left(\frac{m\pi x}{L}\right) \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot dx$$

$$= \frac{1}{2} \times \int_{-L}^{L} \left\{ \cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right) \right\} \cdot dx$$

$$= \frac{1}{2} \times \left[\frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{\frac{(m+n)\pi}{L}} \right]_{-L}^{L} + \frac{1}{2} \times \left[\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{\frac{(m-n)\pi}{L}} \right]_{-L}^{L}$$

$$= \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = \underline{0} \qquad \left[\because \sin k\pi = 0 \quad \forall k \in \mathbb{Z}\right]$$

Hence, $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ is mutually orthogonal on $-L \leq x \leq L$.

[Proved.]

4) Let $\{f_n\}_{n=1}^{\infty}$ be a set of functions where:

$$f_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad ; \quad -L \leq x \leq L$$

<u>To Prove:</u> $\{f_n\}_{n=1}^{\infty}$ is a set of mutually orthogonal functions on $-L \leq x \leq L$

<u>Proof:</u> We have, for $m \neq n$:

$$\int_{-L}^{L} f_m(x) \cdot f_n(x) \cdot dx$$

$$= \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \cdot dx = \frac{1}{2} \int_{-L}^{L} 2\sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \cdot dx$$

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$$\Rightarrow \int_{-L}^{L} f_m(x)\cdot f_n(x)\cdot dx = \frac{1}{2}\cdot\int_{-L}^{L}\left\{\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right\}\cdot dx$$

$$= \frac{1}{2}\times\left[\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{\frac{(m-n)\pi}{L}}\right]_{-L}^{L} - \frac{1}{2}\times\left[\frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{\frac{(m+n)\pi}{L}}\right]_{-L}^{L}$$

$$= 0 - 0 = \underline{\underline{0}} \qquad [\because \sin k\pi = 0 \quad \forall k\in\mathbb{Z}]$$

Hence, $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is mutually orthogonal on $-L\le x\le L$.

[Proved.]

5) Given BVP: $y'' + \lambda y = 0 \quad \ldots\ (i)$

with BCs: $y(0) = 0 = y(L) \quad \ldots\ (ii)$

Case 1: when $\lambda = 0$

→ General soln. of (i) in this case: $y(x) = Ax + B$

Using BC's in (ii), $y(0) = 0 \Rightarrow B = 0$

$y(L) = 0 \Rightarrow AL = 0 \Rightarrow A = 0$

→ Only trivial soln. $y(x) \equiv 0$ in this case. So $\lambda = 0$ is not an eigenvalue of given BVP.

Case 2: when $\lambda < 0$ → take $\lambda = -\mu^2$; $\mu \neq 0$.

Then, (i) becomes: $y'' - \mu^2 y = 0$.

Auxiliary equation: $m^2 - \mu^2 = 0 \Rightarrow m = \pm\mu$

So, general solution of (i) in this case: $y(x) = Ae^{\mu x} + Be^{-\mu x}$

Using BCs in (ii), $y(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$

And, $y(L) = 0 \Rightarrow Ae^{\mu L} + Be^{-\mu L} = 0$

$\Rightarrow A\left(e^{\mu L} - e^{-\mu L}\right) = 0 \quad [\because B = -A]$

$\Rightarrow A = 0 \quad [\because \mu \neq 0 \Rightarrow e^{\mu L} - e^{-\mu L} \neq 0]$

$\Rightarrow B = 0$ → Only trivial soln. $y \equiv 0$

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Case 3: When $\lambda > 0$ → take $\lambda = \mu^2$; $\mu \neq 0$

Then, (i) becomes: $y'' + \mu^2 y = 0$

→ General soln. of (i) in this case : $y(x) = A\cos(\mu x) + B\sin(\mu x)$

Using the BCs in (ii), $y(0) = 0 \Rightarrow A = 0$

And, $y(L) = 0 \Rightarrow B\sin(\mu L) = 0$

Now, for non trivial solutions, $B \neq 0$. Then $\sin(\mu L) = 0$

$\Rightarrow \mu L = n\pi$; $n = 1, 2, 3, \ldots$

$\Rightarrow \mu = \frac{n\pi}{L}$; $n = 1, 2, 3, \ldots$

Then, $\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2}$ are the eigenvalues of the BVP, i.e. values where non-zero solutions to the BVP exist. [Ans.]

and the corresponding solutions are: $y_n(x) = B\sin(\mu x)$
$= B\sin\left(\frac{n\pi x}{L}\right)$

6) Given BVP: $y'' + \lambda y = 0$ ... (i)

with BCs: $y'(0) = 0$ & $y'(L) = 0$ .... (ii)

Case 1: $\lambda = 0$

Then, (i) becomes: $y'' = 0$

→ General solution of (i) in this case : $y = Ax + B$

$\Rightarrow y'(x) = A$

Using BCs in (ii), $y'(0) = 0 \Rightarrow A = 0$

And $y'(L) = 0 \Rightarrow A = 0$, so B can be anything

So in this case, we have $y(x) = B$; where $B \neq 0$ as non-zero solutions to the BVP.

So, $\lambda = 0$ is an eigenvalue with eigenfunction : $y_0(x) = B$

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Case 2: When $\lambda < 0$ $\rightarrow$ take $\lambda = -\mu^2$; $\mu \neq 0$

Then, (i) becomes : $y'' - \mu^2 y = 0$

General solution of (i) in this case : $y(x) = Ae^{\mu x} + Be^{-\mu x}$

$\Rightarrow y'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$

Using BCs in (ii), $y'(0) = 0 \Rightarrow A\mu - B\mu = 0 \Rightarrow A = B$ [$\because \mu \neq 0$]

And, $y'(L) = 0 \Rightarrow A\mu e^{\mu L} - B\mu e^{-\mu L} = 0$

$\Rightarrow A\mu \left[e^{\mu L} - e^{-\mu L}\right] = 0$ [$\because A = B$]

$\Rightarrow A = 0$ [$\because \mu \neq 0 \Rightarrow e^{\mu L} - e^{-\mu L} \neq 0$]

$\Rightarrow B = 0$

$\rightarrow$ Only trivial solution $y(x) \equiv 0$ in this case. So $\lambda < 0$ is not an eigenvalue for this BVP for any $\lambda < 0$.

Case 3: When $\lambda > 0$ $\rightarrow$ take $\lambda = \mu^2$; $\mu \neq 0$

Then (i) becomes : $y'' + \mu^2 y = 0$

General solu. of (i) in this case : $y(x) = A\cos(\mu x) + B\sin(\mu x)$

$\Rightarrow y'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x)$

Using BCs in (ii), $y'(0) = 0 \Rightarrow B\mu = 0 \Rightarrow B = 0$ [$\because \mu \neq 0$]

And, $y'(L) = 0 \Rightarrow -A\mu\sin(\mu L) = 0$

As $\mu \neq 0$, for non-trivial solutions, $A \neq 0$, then $\sin(\mu L) = 0$

$\Rightarrow \mu L = n\pi$; $n = 1, 2, 3, \ldots$

Then, $\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2}$; $n = 1, 2, 3, \ldots$ are also eigenvalues of this BVP with corresponding eigenfunctions : $y_n(x) = A\cos\left(\frac{n\pi x}{L}\right)$

$\therefore$ Combining the three cases, $\lambda = \frac{n^2\pi^2}{L^2}$; $n = 0, 1, 2, 3, \ldots$ are the eigenvalues of this BVP [$\because \lambda = 0$ is also an eigenvalue] with corresponding eigenfunctions : $y_n(x) = A\cos\left(\frac{n\pi x}{L}\right)$; $n = 0, 1, 2, \ldots$

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7) Given BVP: $y'' + \lambda y = 0$ .... (i)

with BCs: $y(0) = 0$ and $y(1) + y'(1) = 0$ .... (ii)

Case 1: when $\lambda = 0$

Then, (i) becomes: $y'' = 0 \Rightarrow y = Ax + B \Rightarrow y'(x) = A$

Using BCs in (ii), $y(0) = 0 \Rightarrow B = 0$

And, $y(1) + y'(1) = 0 \Rightarrow A + A = 0 \Rightarrow A = 0$

→ Only trivial solution $y \equiv 0$ in this case. So $\lambda = 0$ is not an eigenvalue of this BVP.

Case 2: $\lambda < 0$ → take $\lambda = -\mu^2$; $\mu \neq 0$

Then (i) becomes: $y'' - \mu^2 y = 0$

General soln. of (i) in this case: $y = Ae^{\mu x} + Be^{-\mu x}$

Using BCs in (ii), $y(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$

And, $y'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$

So, $y(1) + y'(1) = 0 \Rightarrow Ae^{\mu} + Be^{-\mu} + A\mu e^{\mu} - B\mu e^{-\mu} = 0$

$\Rightarrow Ae^{\mu}(1+\mu) + Be^{-\mu}(1-\mu) = 0$

$\Rightarrow A \times [e^{\mu}(1+\mu) - e^{-\mu}(1-\mu)] = 0$ $[\because B = -A]$

$\Rightarrow A \times [2\sinh(\mu) + 2\mu\cosh(\mu)] = 0$

$\Rightarrow A \times 2\cosh(\mu) \times [\tanh(\mu) + \mu] = 0$

Now, for $\mu \neq 0$, $\cosh(\mu) \neq 0$ and $\tanh(\mu) \neq -\mu$

So, $A = 0 \Rightarrow B = 0$

→ Only trivial solution $y \equiv 0$ in this case. So $\lambda < 0$ is not an eigenvalue for this BVP for any $\lambda < 0$.

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Case 3: $\lambda > 0$ $\rightarrow$ take $\lambda = \mu^2$; $\mu \neq 0$

Then (i) becomes: $y''(x) + \mu^2 y(x) = 0$

General solution of (i) in this case: $y = A\cos(\mu x) + B\sin(\mu x)$

Using BCs in (ii), $y(0) = 0 \Rightarrow A = 0$

Then, $y = B\sin(\mu x) \Rightarrow y' = B\mu\cos(\mu x)$

So, $y(1) + y'(1) = 0 \Rightarrow B\sin\mu + B\mu\cos\mu = 0$

For non-trivial solutions to the BVP, $B \neq 0$

Hence, $\sin\mu + \mu\cos\mu = 0 \Rightarrow \tan\mu = -\mu$ [$\because \cos\mu \neq 0$ as when $\cos\mu = 0$, it is not satisfied]

So, $\lambda = \mu^2$ where μ satisfies: $\tan\mu = -\mu$ $(\mu \neq 0)$ are the eigenvalues to this BVP and the corresponding eigenfunctions are: $y = B\sin(\mu x)$ [Ans.]

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