~~Balz~~

Analysis —

Balzano — Weierstrass Theorem:—

A bounded sequence of real numbers has a convergent subsequence.

We give below another proof. In fact, if we can prove the following result, then Balzano — Weierstrass theorem follows easily from Monotone convergence theorem.

Monotone ~~Sequence~~ Sequence Theorem:

If $X=(x_n)$ is a sequence of real numbers, then there is a subsequence of X that is monotone.

We first define the following term:—

Peak:— For a sequence (x_n), we say that x_m is a peak if $x_m \geq x_n$ if $n \geq m$.

Now, we shall consider the following two cases—

Case:— I X has a finite number (possibly 0) peaks. Let these peaks be listed by increasing subscripts: $x_{m_1}, x_{m_2}, --, x_{m_r}$. Let $s_1 := m_r + 1$.

Since x_{s_1} is not a peak, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Again, x_{s_2} is not a peak, so there exists $s_3 > s_2$ such that $x_{s_3} > x_{s_2}$. Continuing this way, we obtain an increasing subsequence of X.

Case:— II X has infinitely many peaks. We list the peaks by increasing subscripts: $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$,

By definition of a peak,

$$x_{m_1} \geq x_{m_2} \geq \cdots \geq x_{m_k} \geq \cdots$$

Thus we get a decreasing subsequence of X.

Example:-

A) Let's consider the sequence (x_n) where $x_n := \sin\frac{n\pi}{2}$. Then (x_n) is a bounded sequence as $|x_n| \leq 1$ $\forall n \in \mathbb{N}$. Then

(i) The subsequence $(x_{4n-3}) := \left(\sin\frac{(4n-3)\pi}{2}\right)$

$= (1)$.

So, this subsequence converges to 1.

(ii) The subsequence $(x_{2n}) := (\sin n\pi) = (0)$, converges to 0.

(iii) The subsequence (x_{2n-1}) is a divergent sequence.

B) We know that the set of all rational numbers is countable and therefore can be written as a sequence (r_n). So, by Density Theorem, every real number in $[0,1]$ is the limit of a subsequence of (r_n).

Defn. Let (x_n) be a sequence.

(A) A real number that is the limit of a subsequence of (x_n) is called a subsequential limit of (x_n).

(B) Let (x_n) be a bounded sequence. Let S be the set of subsequential limit of (x_n). Then the ~~lea~~ supremum of S is said to be the upper limit or the limit superior of (x_n) and is denoted by $\overline{\lim}\, x_n$ or $\lim\sup x_n$.
Similarly, the infimum of S is said to be the lower limit or the limit inferior of (x_n) is denoted by $\underline{\lim}\, x_n$ or $\lim\inf x_n$.

Remark 1 -

(A) Let (x_n) be a sequence which is bounded. ~~above~~ Let x^* be the supremum of ~~terms of~~ (x_n) S, the set of all subsequential limits of (x_n).
Then, (x_n) has a subsequential limit $l_1 > x^* - 1$. So, there is a natural number r_1 such that $x_{r_1} > x^* - 1$ and $x^* + 1 \geq x_{r_1}$.
Similarly, there is a natural number $r_2 > r_1$ such that
$x^* + 1/2 \geq x_{r_2} > x^* - \frac{1}{2}$. ~~and~~
Proceeding similarly, we get a strickly increasing sequence of natural numbers $(r_1, r_2, r_3, ——)$ such that $x_{r_n} > x^* - \frac{1}{n}$ and $x^* + 1/n \geq x_{r_n}$.

So, $x^* + 1/n > x_{r_n} > x^* - 1/n \quad \forall n \in \mathbb{N}$.

So, x^* is a subsequential limit of (x_n).
So, x^* is the greatest element of S.

(B) Let (x_n) be a sequence which unbounded above. ~~Let~~ Then we define $\overline{\lim}\, x_n = \infty$.

(C) Let (x_n) be unbounded below. Then $\underline{\lim}\, x_n = -\infty$.

(D) Let (x_n) be unbounded above but bounded below. If there is at least ~~one~~ one subsequential limit, then $\underline{\lim}\, x_n$ is defined to be the infimum of the set of all subsequential limit. If there is no subsequential limit, then $\underline{\lim}\, x_n$ is defined to be ∞

(E) Similarly, we define $\overline{\lim}\, x_n$ when (x_n) is unbounded below but bounded above.

There are equivalent definitions of limit superior for a bounded sequence, which is given as the following result:-

Theorem:- If (x_n) is a bounded sequence, then the following statement for a real number x^* are equivalent—

(a) $x^* = \lim \sup (x_n)$.

(b) x^* is the infimum of the set V of $v \in \mathbb{R}$ such that $x_n > v$ for at most a finite number of $n \in \mathbb{N}$.

(c) if ~~...~~ $u_m = \sup \{x_n ; n \geq m\}$, then

$$x^* = \inf \{u_m : m \in \mathbb{N}\} = \lim (u_m).$$

(d) if $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$, but an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$.

Proof: (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (d) Fix $\varepsilon > 0$. Since (x_n) is a bounded sequence, we know that x^* is a sequential limit and there are infinitely many n such that

$$x_n \in (x^* - \varepsilon, x^* + \varepsilon).$$

Also, if there are infinitely many n such that $x_n > x^* + \varepsilon$, then $x_n \in [x^* + \varepsilon, B]$ for infinitely many n where B is an upper bound of (x_n) (here we have assumed that $x^* < B$). So, there is a subsequential limit which is $> x^*$, contradicting the fact that x^* is the limit superior.

Therefore, if $v \in V$ be such that $v < x^*$, then $v < x_n$ for infinitely many n, So, $v \geq x^*$. Let $x^* + \varepsilon_1 > x^*$. Take $x^* + \varepsilon_{1/2}$. Then, $x^* + \varepsilon_{1/2} < x^* + \varepsilon$ and $x^* + \varepsilon_{1/2} \in V$. So, x^* is the infimum of V.

(b) $\Rightarrow$ (c) Let x^* be the infimum of V.

Then for $\varepsilon > 0$, $\exists$ $v \in V$ s.t. $x^* \leq v < x^* + \varepsilon$.

Now, there are at most finitely many n (let they be $n_1, n_2, -, n_r$ in increasing subscripts) such that $v < x_n$.

Let $m = \max\{n_1, -, n_r\} + 1$

Then, $x^* + \varepsilon \geq x_n \; \forall n \geq m$.

$\Rightarrow x^* + \varepsilon \geq u_n \; \forall n \geq m$

$\Rightarrow x^* + \varepsilon \geq \inf\{u_n : n \in \mathbb{N}\}$.

$\Rightarrow x^* \geq \inf\{u_n : n \in \mathbb{N}\}$.

Let us denote $\inf\{u_n ; n \in \mathbb{N}\}$ by y

Then for $\varepsilon > 0$, $\exists$ $u_m \geq y$ and $u_m < y+\varepsilon$.

$\Rightarrow$ $x_n \leq u_m < y+\varepsilon$ $\forall n \geq m$.

$\Rightarrow y+\varepsilon \in V$.

$\Rightarrow x^* \leq y+\varepsilon$

$\Rightarrow x^* \leq y \Rightarrow x^* = y$.

(c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a) can be proved similarly.

By studying lim superior and lim inferior of a sequence, we can conclude about the convergence of that sequence.

<u>Theorem:</u> A bounded sequence (x_n) is convergent if and only if $\overline{\lim}\, x_n = \underline{\lim}\, x_n$.

<u>Proof:</u> Every subsequence of a convergent sequence is convergent. So, $\overline{\lim}\, x_n = \underline{\lim}\, x_n$.

Conversely, let $\underline{\lim}\, x_n = \underline{\lim}\, x_n = l$ (say)

So, for a pre-assigned $\varepsilon > 0$, there are natural numbers k_1 and k_2 such that

$l+\varepsilon > x_n$ $\forall n \geq k_1$ (as $l = \overline{\lim}\, x_n$)

and $l-\varepsilon < x_n$ $\forall n \geq k_2$ (as $l = \underline{\lim}\, x_n$)

Let $k = \max\{k_1, k_2\}$. Then,

$l-\varepsilon < x_n < l+\varepsilon$ $\forall n \geq k$

So, (x_n) is convergent and $\lim x_n = l$.

Cor: Let $X = (x_n)$ be a bounded sequence such that every convergent subsequence of X converges to x. Then (x_n) converges to x.

The Cauchy Criterion

Apart from knowing the value of the limit in advance and ~~the~~ restricting the sequence to be monotone, we need a powerful condition to ~~prove~~ study convergence of a sequence. The Cauchy Criterion is such a condition. We start with examples.

Example:

1) Let (x_n) be a sequence defined by $x_1 = 0$, $x_2 = 1$ and

$$x_{n+2} := \frac{1}{2}(x_{n+1} + x_n) \ \forall n \geq 1.$$

Note that,
$$x_{n+2} - x_{n+1} = \tfrac{1}{2}(x_{n+1} + x_n) - x_{n+1}$$
$$= \tfrac{1}{2}(-x_{n+1} + x_n)$$

$$\therefore |x_{n+2} - x_{n+1}| = \tfrac{1}{2}|x_{n+1} - x_n|$$
$$= \frac{1}{2^2}|x_n - x_{n-1}|$$
$$= \frac{1}{2^n}|x_2 - x_1|$$
$$= \frac{1}{2^n}$$

Let $m > n$, then $|x_m - x_n|$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$= \left(\frac{1}{2}\right)^{m-2} + \cdots + \left(\tfrac{1}{2}\right)^{n-1} = \frac{4}{2^n}\left[1 - \left(\frac{1}{2}\right)^{m-n}\right] < \frac{4}{2^n}.$$

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2) Let (x_n) be a sequence defined by

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

If $m > n$, then $x_m - x_n = \frac{1}{n+1} + \cdots + \frac{1}{m}$

$$> \frac{(m-n)}{m}$$

Let $m = 2n$, then $x_{2n} - x_n > \frac{n}{2n} = 1/2$.

By the following results, we can show, with the above observations, that the sequence in Example-1) is convergent whereas the sequence in Example-2) is divergent.

Cauchy sequence:-

A sequence $X = (x_n)$ of real numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists a natural number $H(\varepsilon)$ such that for all natural numbers $m, n \geq H(\varepsilon)$, $|x_n - x_m| < \varepsilon$.

Examples of Cauchy sequence:

1) In the Example 1), for a given $\varepsilon > 0$, there exists a natural number k such that $\frac{1}{2^n} < \frac{\varepsilon}{4}$ $\forall n \geq k$. $\therefore |x_m - x_n| < \varepsilon$ $\forall m, n \geq k$.

2) In the Example 2), for $\varepsilon < 1/2$, such a natural number k doesn't exist.

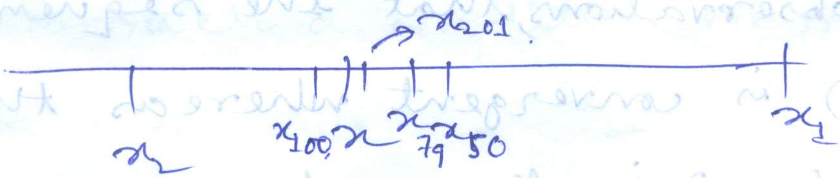
3) Let (x_n) be such that $x_n = \frac{1}{n}$.

Then, $|x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} < \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\forall m, n \geq k$ where

$\frac{1}{k} < \varepsilon/2$.

Now, our goal is to show that the Cauchy sequences are precisely the convergent sequences. We first prove that a convergent sequence is a Cauchy sequence.



We see that the distance between two terms are becoming smaller and smaller as ~~n → ∞~~ n gets larger.

Lemma: If $X = (x_n)$ is a convergent sequence of real numbers, then X is a Cauchy sequence.

Proof: Let $x = \lim x_n$. Then for every $\varepsilon > 0$, there is a natural number k such that $|x_n - x| < \varepsilon/2$ $\forall n \geq k$.

Then, $|x_n - x_m| \leq |x_n - x| + |x - x_m|$

$< \varepsilon/2 + \varepsilon/2 \quad \forall n, m \geq k$

$= \varepsilon$

$\therefore$ (x_n) is a Cauchy sequence.

Cauchy Convergence Criterion

A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof: The previous lemma proves one direction.

Converse part:—

Claim: A Cauchy sequence is bounded.

Proof: Let $X = (x_n)$ be a Cauchy sequence. Then we have a natural number k_1 such that

$$|x_n - x_m| < 1 \quad \forall m, n \geq k_1$$

$$\therefore |x_n| \leq |x_n - x_{k_1}| + |x_{k_1}| \leq |x_{k_1}| + 1$$

Let $M := \sup\{|x_1|, \ldots, |x_{k_1 - 1}|, |x_{k_1}| + 1\}$.

Then, $|x_n| \leq M \quad \forall n \in \mathbb{N}$.

By Bolzano-Weierstrass ~~Theore~~ Theorem, (x_n) has a convergent subsequence $X' := (x_{n_r})$.

Let X' converge to x.

Claim: X is convergent and converges to x.

Proof: As X is a Cauchy sequence, so there exists a natural number k_2 such that

$$|x_n - x_m| < \varepsilon/2 \quad \forall m, n \geq k_2$$

As, X' converges to x, there exists a natural number k_3 such that

$$|x_{n_r} - x| < \varepsilon/2 \quad \forall r \geq k_2$$

Let $k := \max\{k_1, n_{k_2}\}$. Then $n_k \geq n_{k_2}$ and $n_k \geq k_1$.

Then $\forall n \geq k$, $|x_n - x|$

$$\leq |x_n - x_{n_k}| + |x_{n_k} - x|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\therefore$ (x_n) is convergent and $\lim x_n = x$.

Remarks:

(A) To prove that a Cauchy sequence is convergent, we have applied Balzano–Weierstrass theorem, which uses mainly the completeness property of $\mathbb{R}$.

(B) Assume that $S \subset \mathbb{R}$ be a non-empty ~~bounded~~ set, so which is bounded above by M. Let $m < M$ be another real number such that

~~$[m, M] \cap \{x_n \mid n \in \mathbb{N}\} \neq \Phi$~~ $[m, M] \cap S \neq \Phi$.

~~If $[m, M]$ contains only finitely many terms of elements of $\{x_n \mid n \in \mathbb{N}\}$ then then S, then the maximum element is the supremum of S.~~

~~Case-1 If S is a singleton set, then that element is the supremum of S.~~

Case-2 If

$$[m, M] \cap S \neq \Phi$$

If $[m, M]$ contains finitely many elements of S, then the maximum among them is the supremum of S.

If $[m, M] \cap S$ contains infinitely many elements, then split $[m, M]$ into

$I_1 := [m, \frac{m+M}{2}]$ and $I_2 := [\frac{m+M}{2}, M]$,

If $I_2 \cap S = \Phi$, then work with I_1 (i.e. again split I_1 in two equal length subintervals). If $I_2 \cap S$ is a finite ~~sum~~ set, then the maximum is the supremum of S. If $I_2 \cap S$ is an infinite set, then again split I_2 into two equal length subintervals.

Continuing this way and taking one element ~~from each~~ in each step, we form a Cauchy sequence (z_n). If we assume that each Cauchy sequence is convergent, then (z_n) converges to z, say. We can prove that z is the supremum of S.

Thus we have established that the ~~Compleme~~ Completeness property of $\mathbb{R}$ is equivalent to the Cauchy Convergence Criterion.

Defn A subset S of $\mathbb{R}$ is said to be complete if every Cauchy sequence in S converges to a point in S.

Example

1) A subset $S \subset \mathbb{R}$ is complete ~~in $\mathbb{R}$~~ if and only if S is closed in $\mathbb{R}$.

Proof Let S be complete and let x be a limit point of S. Then we shall form a sequence (x_n)

in S such that $\lim x_n = x$.

Take the nbhd $N_{1/n}(x)$, then $(N_{1/n}(x) - \{x\}) \cap S \neq \phi$.

Take one elt from $(N_{1/n}(x) - \{x\}) \cap S$, call this x_n. Then one can show that $\lim x_n = x$.

So, (x_n) is a Cauchy sequence in S and so is convergent in S and due to the uniqueness of limit of a sequence, $x \in S$.

Conversely, let S be a closed set and (x_n) be a Cauchy sequence in S. So, (x_n) is a convergent sequence. Let $x = \lim x_n$. ~~and~~

If $x = x_n$ for some n, then $x \in S$, so (x_n) is convergent in S.

If $x \neq x_n$ for all n, then x is a limit point of S, so $x \in S$. $\therefore (x_n)$ is convergent in S.

So, S is complete.

So, each closed interval is complete.

2) $\mathbb{Q}$ is not complete.

Application of Cauchy Convergence Criterion:-

The polynomial $x^3 - 5x + 1 = 0$ has a root r with $0 < r < 1$.

Verification:- We choose means of iteration.

Choose x_1 such that $0 < x_1 < 1/2$ and write $x_{n+1} := \frac{1}{5}(x_n^3 + 1)$

Then, $0 < x_n < \frac{1}{2}, \forall n \in \mathbb{N}$,

$\therefore |x_{n+1} - x_n| = \left|\frac{1}{5}(x_n^3+1) - 1/5\,(x_{n-1}^3+1)\right|$

$= \frac{1}{5}\left|x_n^3 - x_{n-1}^3\right|$

$= 1/5\,|x_n - x_{n-1}|\,|x_n^2 + x_n x_{n-1} + x_{n-1}^2|$.

$\leq \frac{3}{5}|x_n - x_{n-1}|$.

$\therefore |x_{n+1} - x_n| \leq \left(\frac{3}{5}\right)^{n-1}|x_2 - x_1|$.

Let $n > m$. $\therefore |x_n - x_m|$

$\leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m|$.

$\leq \left(\left(\frac{3}{5}\right)^{n-2} + \cdots + \left(\frac{3}{5}\right)^{m-1}\right)|x_2 - x_1|$.

$\leq \left(\frac{3}{5}\right)^{m-1} \cdot \frac{5}{2} \cdot |x_2 - x_1|$.

Choose a natural number M such that

$\left(\frac{3}{5}\right)^{M-1} \cdot 5/2\,|x_2 - x_1| < \varepsilon$,

$\Leftrightarrow M > \dfrac{\ln\left(\dfrac{6\varepsilon}{25|x_2 - x_1|}\right)}{\ln(3/5)}$.

$\therefore \forall n, m \geq M$, $|x_n - x_m| < \varepsilon$

$\therefore (x_n)$ is a Cauchy sequence, so is convergent.

Let $x := \lim x_n$.

$\therefore$ Taking limit on both sides of

$x_{n+1} = 1/5\,(x_n^3 + 1)$, we get

$x^3 - 5x + 1 = 0$

As, $0 \le x_n \le 1/2 \Rightarrow 0 \le x \le 1/2 \Rightarrow 0 < x < 1$.

Here we provide a general situation—

<u>Defⁿ</u> A seq $X := (x_n)$ of real numbers is ~~con~~ said to be contractive if there exists a constant C, $0 < C < 1$ such that

$$|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n| \text{ for all } n \in \mathbb{N}.$$

C is the ~~co~~ constant of the contractive sequence.

Similar to the previous application, we can prove the following result—

<u>Theorem</u> Every contractive sequence is a Cauchy sequence, and therefore is convergent.

For the cases of contractive sequences, it is often important and also possible to have an estimate of the error at the nth stage of calculating the limit.

<u>Cor</u> If (x_n) is a contractive sequence with constant C, $0 < C < 1$, and if $x = \lim x_n$, then

(i) $|x - x_n| \le \dfrac{C^{n-1}}{1-C} |x_2 - x_1|$,

(ii) $|x - x_n| \le \dfrac{C}{1-C} |x_n - x_{n-1}|$.

<u>Proof</u> (i) $|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|$

$$\le C^n |x_2 - x_1|.$$

$$\therefore |x_{n+1} - x_n| \le C^{n-1} |x_2 - x_1|.$$

~~Taking $n \to \infty$~~ Let $m > n$, then

$$|x_m - x_n| \le |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n|$$

$\leq c^{n-2}|x_2-x_1| + \cdots + c^{n-1}|x_2-x_1|$

$$= c^{n-1}\frac{1-c^{m-n}}{1-c}|x_2-x_1| \leq \frac{c^{n-1}}{1-c}|x_2-x_1|.$$

Let $m\to\infty$, then we get $|x-x_n| \leq \dfrac{c^{n-1}}{1-c}|x_2-x_1|$.

(ii) We have observed that

~~$|x_m - x_n| \leq c^{n-2}$~~

$$|x_{n+k}-x_{n+k-1}| \leq c^k|x_n-x_{n-1}|.$$

$\therefore |x_m - x_n|$

$\leq |x_m - x_{m-1}| + \cdots + |x_{n+1}-x_n|$

$\leq (c^{m-n} + \cdots + c)|x_n - x_{n-1}|.$

$\leq \dfrac{c}{1-c}|x_n-x_{n-1}|.$

Let $m\to\infty$ $\therefore |x-x_n| \leq \dfrac{c}{1-c}|x_n-x_{n-1}|.$

Back to Application of Cauchy Convergence Criterion:

Let $x_1 = 1/2$, $\therefore x_2 = \frac{1}{5}((.5)^3+1) = .225.$

$\therefore |x_2-x_1| = .5 - .225 = .275.$

Here $c = 3/5.$

$\therefore |x-x_n| \leq \dfrac{(3/5)^{n-1}}{1-3/5}|x_2-x_1| = \dfrac{(.6)^{n-1}}{.4}\times .225$

$= (.6)^{n-1}\times .5625.$

$\therefore |x-x_6| \leq .04374$. Here, $x_6 = .20164$.

So, we can find x with error $< .04374$.

Series

An infinite series is 'defined' to be 'an expression of the form'

$$x_1 + x_2 + \dots + x_n + \dots$$

Defⁿ Let $X := (x_n)$ be a sequence in $\mathbb{R}$. The infinite series (or simply the series) generated by X is the sequence $S := (s_k)$ defined by

$$s_1 := x_1,$$
$$s_2 := x_1 + x_2 = s_1 + x_2$$
$$\dots$$
$$s_k := x_1 + \dots + x_k = s_{k-1} + x_k.$$
$$\dots$$

The numbers s_k are called the partial sums of the series. If $\lim_n S$ exists, then the series is said to be convergent and this limit is said to be the sum or the value of the series. If this limit does not exist, we say that the series S is divergent.

Example:-

1) Consider the geometric series $\sum_{n=0}^{\infty} r^n$

$$= 1 + r + r^2 + \dots + r^n + \dots$$

Here $s_n := 1 + r + r^2 + \dots + r^n$

$$= \frac{1 - r^{n+1}}{1 - r}$$

$$\therefore \left| s_n - \frac{1}{1-r} \right| \le \frac{|r|^{n+1}}{1-r}.$$

Here $|r|^{n+1} \to 0$ as $n \to \infty$ for $|r| < 1$.

$\therefore s_n \to \frac{1}{1-r}$ if $|r| < 1$.

$\therefore$ The geometric series converges to $\frac{1}{1-r}$ when $|r| < 1$.

Note that here the sequence (x_n), where $x_n := r^n$) does not converge if $|r| \geq 1$. Then the following results implies that $\sum_{n=0}^{\infty} r^n$ ~~diverges~~ diverges if $|r| \geq 1$.

The nth term Test:

If the series $\sum x_n$ converges, then $\lim x_n = 0$

Proof: $x_n = s_n - s_{n-1}$.

$\therefore \lim x_n = \lim (s_n) - \lim (s_{n-1})$

$= 0$.

The following Cauchy Criterion is a reformulation of Cauchy Criterion for sequences.

Cauchy Criterion for Series:

The series $\sum x_n$ converges if and only if for every $\varepsilon > 0$ there exists a natural number $M(\varepsilon)$ such that if $m > n \geq M(\varepsilon)$, then

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon.$$

Example: We saw before the sequence (x_n), defined by $x_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is divergent. So, the series $\sum_{n \geq 1} \frac{1}{n}$ is divergent.

Limits

"Mathematical Analysis" is generally understood to refer to that area of mathematics where various limiting concepts are applied or studied. In the preceding chapter we studied the limit of a sequence of real numbers. In this chapter we will encounter the notion of a limit.

The introductory notion of a limiting process first emerged in the work of Isaac Newton and Gottfried Leibniz in 1680s. Both of them realized the need to formulate a notion of function and the idea of quantities being closed to one another. Each of them didn't know ~~about~~ about other's work initially and their creative insights were quite different. Later Leonhard Euler and Cauchy ~~tried~~ among many other mathematicians tried to formulate the concept of limit with rigor. Finally, Karl Weierstrass ~~in~~ succeeded in ~~for~~ formulating a precise definition of limit and his definition of limit is the one we use today.

__Def:__ ~~The~~ The intuitive idea of the function having a limit L at c is that the values f(x) are close to L when x is closed to (but different from) c. It is necessary to have a technical way of working with the idea of "close to" and this is accomplished in the $\varepsilon-\delta$ definition given below.

To have the limit of a function f at c, it is necessary that f be defined at points near c. Though f is not needed to be defined at c. For that, we assume that c is a limit point of D, the domain of f.

Defⁿ Let $A \subseteq \mathbb{R}$, and let c be the cluster point of A. For a function $f: A \to \mathbb{R}$, a real number L is said to be a limit of f at c, if given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x-c| < \delta$, then $|f(x) - L| < \varepsilon$.

Equivalently, corresponding to any neighbourhood V of L, there exists a neighbourhood W of c such that $f(x) \in V$ for all $x \in (W - \{c\}) \cap D$.

This is expressed by the symbol $\lim_{x \to c} f(x) = L$.

Examples:—

a) Let $f(x) := b \; \forall x \in \mathbb{R}$.

Let $\varepsilon > 0$ be arbitary. Choose $\delta = 1$.

Then if $0 < |x-c| < 1$, then we have $|f(x) - b| = |b-b| = 0 < \varepsilon$.

$\therefore \lim_{x \to c} f(x) = b$.

b) Let $f(x) := x \; \forall x \in \mathbb{R}$.

If $\varepsilon > 0$, we choose δ to be ε. Then if $0 < |x-c| < \delta$, then $|f(x) - c| = |x-c| < \delta = \varepsilon$.

$\therefore \lim_{x \to c} x = c.$

c) Let $f(x) := \cancel{x^2 \; \forall x \in \mathbb{R}} \; \frac{1}{x} \; \forall x \neq 0$

Here the domain is $\mathbb{R} - \{0\}$ and $c > 0$ is a limit pt. of it.

Let $c > 0$ Take x in such a way that

$$|x - c| < \frac{c}{2} \therefore \; |c| \leq |x - c| + |x|$$

$$\therefore |x| \geq |c| - |x - c|$$

$$> c - \frac{c}{2}$$

$$= \frac{c}{2}.$$

Also, if $x < 0$, then $x < -\frac{c}{2}$, $\therefore |x - c| = \frac{3c}{2} > \frac{c}{2}$

$\therefore |x - c| < c/2 \Rightarrow x > 0$

Let $\varepsilon > 0$, $\therefore \left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x - c|}{|x||c|}$

Let $\delta := \min\left\{ \frac{c}{2}, \frac{c^2}{2} \varepsilon \right\}$, Let $|x - c| < \delta$,

$$\therefore \left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x - c|}{|x||c|} < \frac{\frac{c^2}{2} \varepsilon}{c/2 \cdot c} = \varepsilon.$$

Then $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$.

d)

Let $f(x) := \frac{x^2 - 4}{x - 2}$, $x \neq 2$

$= 10$, $x = 2$.

Here the domain of the function is $\mathbb{R}$ and 2 is a limit point of it..

When $x \neq 2$, $|f(x) - 4| = \left| \frac{x^2 - 4}{x - 2} - 4 \right| = |x - 2|$

Let us choose $\varepsilon > 0$.

Then for $0 < |x - 2| < \varepsilon$, $|f(x) - 4| < \varepsilon$. ~~So lim~~

So, $\lim_{x\to 2} f(x) = 4$.

e) $\lim_{x\to 0}\left(\frac{1}{x}\right)$ doesnot exist.

Here $f(x) := \frac{1}{x}$, $x \neq 0$. $\mathbb{R}-\{0\}$ is the domain of the function. Though 0 is not in the domain, 0 is a limit point of it.

Choose the sequence $\left(\frac{1}{n}\right)$. This is convergent and $\lim_{n\to\infty} \frac{1}{n} = 0$. ~~But~~ $\cancel{\lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} n}$

But $f\left(\frac{1}{n}\right) = n$, so $\left(f\left(\frac{1}{n}\right)\right)$ is a sequence which is not convergent as it is not bounded.

From the above observation and fallowing results, we can conclude that $\lim_{x\to 0}\left(\frac{1}{x}\right)$ doesnot exist.

Sequential Criterion for limits:-

The fallowing important formulation of limit of a function is in terms of limits of sequences.

Theorem (Sequential Criterion):-

Let $f: A \to \mathbb{R}$ and let c be a limit point of A. Then the fallowing are equivalent-

(i) $\lim_{x\to c} f(x) = L$;

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

Proof:- (i) $\Rightarrow$ (ii) Let (x_n) be a sequence in A with

$\lim (x_n) = c$ and $x_n \neq c$ for all n, let $\varepsilon > 0$, then there exists a ~~natural number k such that~~ positive numbers δ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

And as $\lim\limits_{n \to \infty} x_n = c$, so there exists a ~~natur~~ natural number k such that for all $n \geq k$,

$$0 < |x_n - c| < \delta.$$

Then, for all $n \geq k$, $|f(x_n) - L| < \varepsilon$.

$\therefore$ $(f(x_n))$ converges to L.

(ii) $\Rightarrow$ (i) If (i) is not true, then there exists a $\varepsilon_0 > 0$ such that for any $\delta > 0$, there exists a real number x_δ such that

$$0 < |x_\delta - c| < \delta \text{ but } |f(x_\delta) - L| \geq \varepsilon_0.$$

Choose $\delta = 1/n$, there exists x_n such that

$$0 < |x_n - c| < 1/n \text{ but } |f(x_n) - L| \geq \varepsilon_0$$

$\therefore$ $(x_n) \in A - \{c\}$ is a sequence which converges to c, but the sequence $(f(x_n))$ doesnot converge to L, which is a contradiction to (ii).

Divergence Criteria:—

This is a corallary to the above theorem, which is often needed to show (i) that a certain number is not the limit, or (ii) that the function doesnot have a limit at a point.

Cor: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of A.

(a) If $L \in \mathbb{R}$, then f doesn't have limit L at c if and only if there exists a sequence (x_n) in A

with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but $(f(x_n))$ does not converge to L.

(b) The function f does not have a limit at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge in $\mathbb{R}$.

Proof:- Proof is obvious from the previous theorem.

Example:-

Let $f(x) := \sin\left(\frac{1}{x}\right), x \neq 0$. Here the domain D of f is $\mathbb{R} - \{0\}$. 0 is a limit point of D. We shall show that f does not have a limit point at 0. To establish that we shall find two sequences (x_n) and (y_n) in D with $\lim(x_n) = 0$, $\lim(y_n) = 0$, but $\lim(f(x_n)) \neq \lim(f(y_n))$.

So, we can construct a new sequence (z_n) with

$$z_{2n} = x_n$$

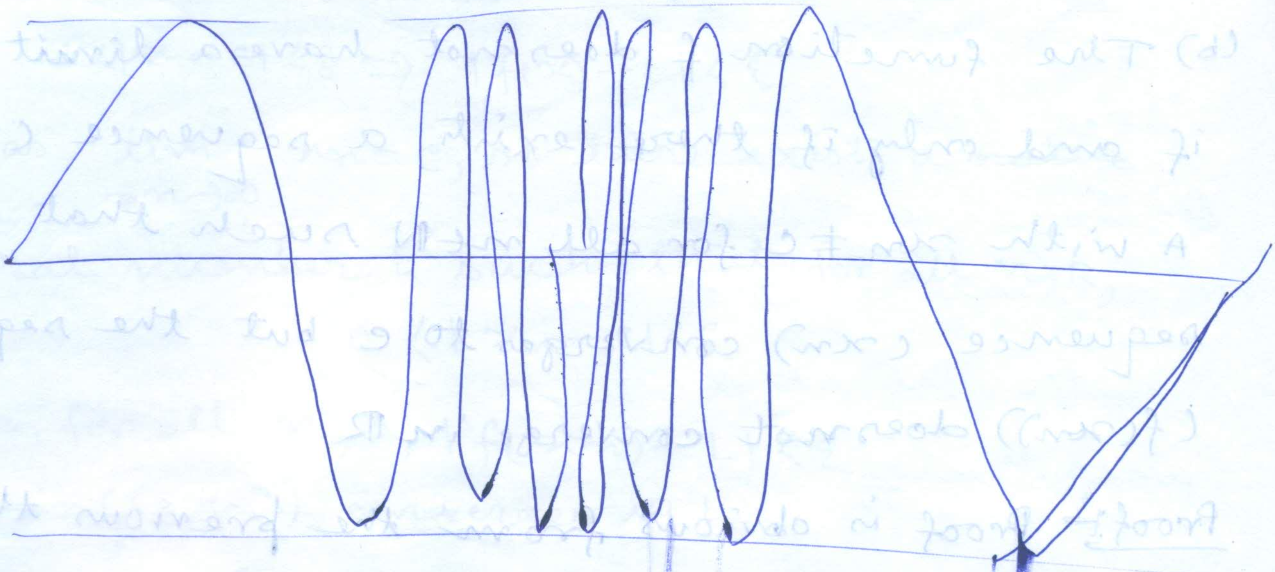
$$z_{2n-1} = y_n$$

and then $\lim(z_n) = 0$, but $\lim(f(z_n))$ does not exist. Then by Divergence Criteria, $\lim\limits_{x \to 0} f(x)$ does not exist.

Let $x_n := \frac{1}{n\pi}$ and $y_n := \frac{1}{(2n+1)\pi}$ $\frac{1}{(4n+1)\pi/2}$.

Then $\lim x_n = \lim y_n = 0$, but $f(x_n) = 0 \ \forall n$.

and $f(y_n) = 1$. So, $\lim_{n\to\infty}(f(x_n)) = 0$ and $\lim_{n\to\infty}(f(y_n)) = 1$



Properties of limit

We shall now obtain results which are important in calculating limits of functions.

Defn: Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$. Let c be a limit point of A. We say that f is bounded on a neighbourhood of c if there exists a δ-neighbourhood $V_\delta(c)$ of c and a constant $M > 0$ such that $|f(x)| \leq M$ $\forall x \in (V_\delta(c) - \{c\}) \cap A$.

Thm: Let $A \subseteq \mathbb{R}$ and c be a limit point of A. Let $f: A \to \mathbb{R}$ have a limit at c. Then f is bounded on some neighbourhood of c.

Proof: Let $L := \lim_{x\to c} f(x)$. Then for $\varepsilon = 1$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in A$, then

$|f(x) - L| < 1$

Then $|f(x)| \le |f(x) - L| + |L| < |L| + 1$ when $0 < |x - c| < \delta$ and $x \in A$.

Here $M = |L| + 1$

The next result is similar to the result on the limit of sum, difference, product and quotient of convergent sequences.

Theorem:- Let $A \subseteq \mathbb{R}$ and let $f, g : A \to \mathbb{R}$. Let c be a limit point of A and $b \in \mathbb{R}$

(a) If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} g(x) = \cancel{b_2 L_2} \; ~~then~~ \; L_2$

then

$$\lim_{x \to c} (f+g) = L_1 + L_2, \quad \lim_{x \to c} (f-g) = L_1 - L_2,$$

$$\lim_{x \to c} (fg) = L_1 L_2, \quad \lim_{x \to c} (bf) = bL.$$

(b) If $h : A \to \mathbb{R}$, $h(x) \ne 0$ for all $x \in \mathbb{R}$ and $\lim_{x \to c} h(x) = H \ne 0$, then

$$\lim_{x \to c} \left(\frac{f}{h}\right) = \frac{L_1}{H}.$$

Theorem Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and c be a limit point of A. If

$a \le f(x) \le b \quad \forall x \in A$, then if $\lim_{x \to c} f$ exists, then

$$a \le \lim_{x \to c} f \le b$$

Squeeze Theorem:- Let $A \subseteq \mathbb{R}$, $f, g, h : A \to \mathbb{R}$ and let c be a limit point of A. If

$f(x) \le g(x) \le h(x) \quad \forall x \in A$ and if

Then $|f(x)| \le |f(x)-L| + |L| < |L|+1$ when $0<|x-c|<\delta$ and $x \in A$.

Here $M = |L|+1$

The next result is similar to the result on the limit of sum, difference, product and quotient of convergent sequences.

Theorem:- Let $A \subseteq \mathbb{R}$ and let $f, g : A \to \mathbb{R}$. Let c be a limit point of A and $b \in \mathbb{R}$

(a) If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} g(x) = L_2$ then

$$\lim_{x\to c}(f+g) = L_1 + L_2, \quad \lim_{x \to c}(f-g) = L_1 - L_2,$$

$$\lim_{x \to c}(fg) = L_1 L_2, \quad \lim_{x\to c}(bf) = bL.$$

(b) If $h : A \to \mathbb{R}$, $h(x) \neq 0$ for all $x \in \mathbb{R}$ and $\lim_{x\to c} h(x) = H \neq 0$, then

$$\lim_{x \to c}\left(\frac{f}{h}\right) = \frac{L_1}{H}.$$

Theorem:- Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and c be a limit point of A. If

$a \le f(x) \le b \;\; \forall x \in A$, then if $\lim_{x \to c} f$ exist, then

$$a \le \lim_{x\to c} f \le b$$

Squeeze Theorem:- Let $A \subseteq \mathbb{R}$, $f, g, h : A \to \mathbb{R}$ and let c be a limit point of A. If

$f(x) \le g(x) \le h(x) \;\; \forall x \in A$ and if