§ Let f and g be differentiable on $[a,b]$. If Wronskian $W(f,g)(t_0)$ is non-zero for some t_0 in $[a,b]$, then f and g are L.I. on $[a,b]$. If f and g are L.D., then W is zero $\forall\, t$ in $[a,b]$.

Proof:

$$c_1 f(t) + c_2 g(t) = 0$$
$$c_1 f'(t) + c_2 g'(t) = 0$$

$$D = W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

If $W \neq 0$ at some t_0, only trivial solution exists. Hence they are L.I.

§ Let y_1 and y_2 be solutions of the diff. eqn.

$$y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on $[a,b]$. Then the Wronskian is given by

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)\, dt}$$

where c is a constant depending on y_1 and y_2, not on t. The Wronskian is either 0 $\forall\, t$ in $[a,b]$ or no t in $[a,b]$

Proof $W = y_1 y_2' - y_1' y_2$

$$W' = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$$
$$= y_1 y_2'' - y_1'' y_2.$$

$$y_1'' + p(t) y_1' + q(t) y_1 = 0 \quad —(1)$$
$$y_2'' + p(t) y_2' + q(t) y_2 = 0 \quad —(2)$$

$(1) \times (-y_2) + (2) \times y_1$

$$(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0$$
$$\Rightarrow W' + p(t) W = 0$$

$$\frac{dW}{W} = -p(t)\,dt$$

Result follows.

Ex $y_1(t) = 1 - t$ $y_2(t) = t^3$ cannot both be the solutions to a diff. eqn. $y'' + p(t) y + q(t) = 0$, $p(t), q(t)$ being continuous in $[-1, 5]$.

Sol^n. $y_1' = -1$ $y_2' = 3t^2$

$$W(y_1, y_2) = (1-t)3t^2 - t^3(-1) = 3t^2 - 2t^3$$

W is 0 at $t = 0$ but non-zero at $t = 1$. $\therefore$ y_1, y_2 cannot both be the sol^n..