ASSIGNMENT - 3

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Mathematical Methods

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1) To Prove: $\int_0^t J_0\left(\sqrt{x(t-x)}\right) dx = 2\sin\left(\frac{t}{2}\right)$

We know, $J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \left(\frac{x}{2}\right)^{2r}$

$$\Rightarrow J_0\left(\sqrt{x(t-x)}\right) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \frac{[x(t-x)]^r}{2^{2r}}$$

So, $$\int_0^t J_0\left(\sqrt{x(t-x)}\right).dx = \int_0^t \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \frac{[x(t-x)]^r}{2^{2r}} . dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \frac{1}{2^{2r}} \times \int_0^t x^r (t-x)^r . dx$$

[$\because$ The series is uniformly convergent, so we can interchange the summation and integral signs]

Now, substitute: $x = tz \Rightarrow dx = t.dz$

when $x=0 \rightarrow z=0$

when $x=t \rightarrow z=1$

Then we have:

$$\int_0^t J_0\left(\sqrt{x(t-x)}\right).dx = \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \frac{1}{2^{2r}} \times \int_0^1 t^r.z^r.t^r.(1-z)^r.t\,dz$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \frac{1}{2^{2r}} \times t^{2r+1} \int_0^1 z^r.(1-z)^r.dz$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2} \times \frac{t^{2r+1}}{2^{2r}} \times B(r+1, r+1)$$

[$\because B(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}.dx$]

Then, $\int_0^t J_0\left(\sqrt{x(t-x)}\right) \cdot dx = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times r!} \times \frac{t^{2r+1}}{2^{2r}} \times \frac{\Gamma(r+1) \times \Gamma(r+1)}{\Gamma(2r+2)}$

$\left[\because \beta(m,n) = \frac{\Gamma(m) \times \Gamma(n)}{\Gamma(m+n)}\right]$

$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \times r!} \times \frac{t^{2r+1}}{2^{2r}} \times \frac{r! \times r!}{(2r+1)!}$ $\left[\because \Gamma(n) = (n-1)! \text{ for } n \in \mathbb{N}\right]$

$= \sum_{r=0}^{\infty} \frac{(-1)^r}{2^{2r}} \times \frac{t^{2r+1}}{(2r+1)!} = 2 \times \sum_{r=0}^{\infty} (-1)^r \times \frac{\left(\frac{t}{2}\right)^{2r+1}}{(2r+1)!}$

$= 2\sin\left(\frac{t}{2}\right)$ $\left[\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$ $\forall x \in \mathbb{R}$

Hence, $\int_0^t J_0\left(\sqrt{x(t-x)}\right) \cdot dx = 2\sin\left(\frac{t}{2}\right)$ [Proved.]

2) To Prove: $J_n(x+y) = \sum_{r=-\infty}^{\infty} J_r(x) \cdot J_{n-r}(y)$

From the Generating function for $J_n(x)$, we know:

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) \cdot t^n \quad \ldots\ldots (i)$$

So, $e^{\frac{(x+y)}{2} \times \left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x+y)\, t^n \quad \ldots (ii)$ [By replacing x with $(x+y)$]

So, $J_n(x+y)$ is the coefficient of t^n in the expansion of

$$e^{\frac{(x+y)}{2} \times \left(t - \frac{1}{t}\right)}$$

But again, $e^{\frac{(x+y)}{2}\times\left(t-\frac{1}{t}\right)} = e^{\frac{x}{2}\times\left(t-\frac{1}{t}\right)} \times e^{\frac{y}{2}\times\left(t-\frac{1}{t}\right)}$

$$= \sum_{r=-\infty}^{\infty} J_r(x)\cdot t^r \times \sum_{s=-\infty}^{\infty} J_s(y)\cdot t^s \qquad \left[\text{Using (i)}\right]$$

$$\Rightarrow e^{\frac{(x+y)}{2}\times\left(t-\frac{1}{t}\right)} = \sum_{r,s=-\infty}^{\infty} J_r(x)\cdot J_s(y)\cdot t^{r+s} \quad \ldots\ldots \text{(iii)}$$

Now, for a particular value of r here, we obtain t^n and its coefficient by putting $s = n-r$.

So, for a fixed r, the contribution to the coefficient of t^n is:

$$J_r(x)\cdot J_{n-r}(y).$$

Hence, the total coefficient of t^n in (iii) is: $\sum_{r=-\infty}^{\infty} J_r(x)\cdot J_{n-r}(y)$.

So, equating the coefficients of t^n from (ii) and (iii), we get:

$$J_n(x+y) = \sum_{r=-\infty}^{\infty} J_r(x)\cdot J_{n-r}(y) \qquad \left[\text{Proved.}\right]$$

3) We have, ${}_2F_1(a-1, b-1; c; x) - {}_2F_1(a, b-1; c; x)$

$$= \sum_{r=0}^{\infty} \frac{(a-1)_r \times (b-1)_r}{(c)_r} \times \frac{x^r}{r!} - \sum_{r=0}^{\infty} \frac{(a)_r \times (b-1)_r}{(c)_r} \times \frac{x^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{\left[(a-1)_r - (a)_r\right] \times (b-1)_r}{(c)_r} \times \frac{x^r}{r!}$$

$$= \sum_{r=1}^{\infty} \frac{\left[(a-1)_r - (a)_r\right] \times (b-1)_r}{(c)_r} \times \frac{x^r}{r!} \quad \ldots \text{(i)}$$

$\left[\because \text{The } r=0 \text{ term will be } 0, \text{ as } (\alpha)_0 = 1.\ \text{So, } (a-1)_0 = (a)_0 = 1\right]$

Now, we know : $(\alpha)_{n+1} = \alpha \times (\alpha+1)_n$ where $(\alpha)_n$ is the Pochhammer symbol.

Putting $\alpha = a-1$, we have : $(a-1)_r = (a-1) \times (a)_{r-1}$ .... (ii)

And, $(a)_r = a \times (a+1) \cdots (a+r-1) = (a+r-1) \times [a \times (a+1) \times \cdots (a+r-2)]$

$\Rightarrow (a)_r = (a+r-1) \times [a \times (a+1) \times \cdots (a+(r-1)-1)]$

$\Rightarrow (a)_r = (a+r-1) \times (a)_{r-1}$ ..... (iii)

From (ii) and (iii), $(a-1)_r - (a)_r = [a-1-a-r+1] \times (a)_{r-1}$

$\Rightarrow (a-1)_r - (a)_r = -r \times (a)_{r-1}$ ..... (iv)

Also, we have : $(b-1)_r = (b-1) \times (b)_{r-1}$ ... (v)

And, $(c)_r = c \times (c+1)_{r-1}$ .... (vi)

Using (iv), (v) and (vi) and putting them in (i), we get :

$${}_2F_1(a-1, b-1; c; x) - {}_2F_1(a, b-1; c; x)$$

$$= \sum_{r=1}^{\infty} \frac{-r \times (a)_{r-1} \times (b-1) \times (b)_{r-1}}{c \times (c+1)_{r-1}} \times \frac{x^r}{r!}$$

$$= \sum_{m=0}^{\infty} \frac{(1-b)}{c} \times (m+1) \times \frac{(a)_m \times (b)_m}{(c+1)_m} \times \frac{x^{m+1}}{(m+1)!}$$

[Substituting $m = r-1$ $\Rightarrow r = m+1$]

$$= \left(\frac{x}{c}\right) \times (1-b) \times \sum_{m=0}^{\infty} \frac{(a)_m \times (b)_m}{(c+1)_m} \times \frac{x^m}{m!} = \left(\frac{x}{c}\right) \times (1-b) \times {}_2F_1(a, b; c+1; x)$$

[Proved.]

4) For $\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \neq 0, -1, -2, \dots$ and $|x| < 1$, the Hypergeometric equation is given by:

$$x(1-x) \cdot \frac{d^2y}{dx^2} + [\gamma - (\alpha+\beta+1)x] \cdot \frac{dy}{dx} - (\alpha\beta) y = 0 \quad \dots\dots (i)$$

When $\gamma \notin \mathbb{Z}$, the general solution of (i) is given by: (about $x = 0$)

$$y = A \times {}_2F_1(\alpha, \beta; \gamma; x) + B \times x^{1-\gamma} \times {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x) \quad \dots\dots (ii)$$

where A and B are constants.

We'll use these facts to solve the next problems.

(i) $x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0 \quad \dots\dots (iii)$

Comparing this eqn. with (i), we have:

$\gamma = \frac{3}{2}$, $\alpha + \beta + 1 = 2$ and $\alpha\beta = -2$

$\Rightarrow \alpha + \beta = 1$ and $\alpha\beta = -2$

Then, $t^2 - (1)t + (-2) = 0$ $\rightarrow$ has roots $t = \alpha, \beta$

$\Rightarrow t^2 - t - 2 = 0 \Rightarrow (t+1)(t-2) = 0 \Rightarrow t = 2, -1$

So, $\alpha = 2$, $\beta = -1$, $\gamma = \frac{3}{2}$.

As $\gamma \notin \mathbb{Z}$, the solution of (iii), using (ii), is given by:

$$y = A \times {}_2F_1\left(2, -1; \frac{3}{2}; x\right) + B \times x^{-1/2} \cdot {}_2F_1\left(\frac{3}{2}, \frac{-3}{2}; \frac{1}{2}; x\right) \quad \text{[Ans.]} \quad \dots (iv)$$

Further, as $\beta = -1$ is a negative integer, ${}_2F_1\left(2, -1; \frac{3}{2}; x\right)$ will be a finite polynomial.

So, ${}_2F_1\left(2,-1;\frac{3}{2};x\right) = 1 + \frac{2\times(-1)}{\frac{3}{2}} \times \frac{x}{1!} + \frac{[2\times 3]\times[(-1)\times 0]}{\left[\frac{3}{2}\times\frac{5}{2}\right]} \times \frac{x^2}{2!} + \cdots$

$= 1 - \frac{4}{3}x$ [Rest all higher degree terms will have coeff. = 0]

Hence, solution of (iii) is :

$y = A \times \left(1 - \frac{4}{3}x\right) + B \times \frac{1}{\sqrt{x}} \times {}_2F_1\left(\frac{3}{2}, \frac{-3}{2}; \frac{1}{2}; x\right)$ [Ans.]

[where A, B are constants]

(ii) $(x - x^2)y'' + \left(\frac{3}{2} - 2x\right)y' - \left(\frac{y}{4}\right) = 0$

$\Rightarrow x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' - \left(\frac{1}{4}\right)y = 0$ .... (v)

Comparing (v) with (i), we get :

$\gamma = \frac{3}{2}$, $\alpha + \beta + 1 = 2$ and $\alpha\beta = \frac{1}{4}$

$\Rightarrow \alpha + \beta = 1$ and $\alpha\beta = \frac{1}{4}$

Then, $t^2 - (1)t + \frac{1}{4} = 0$ → has roots $t = \alpha, \beta$

$\Rightarrow 4t^2 - 4t + 1 = 0 \Rightarrow (2t-1)^2 = 0 \Rightarrow t = \frac{1}{2}, \frac{1}{2}$

So, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{3}{2}$.

As $\gamma \notin \mathbb{Z}$, the solution of (v), using (ii), is given by :

$y = A \times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x\right) + B \times x^{-1/2} \times {}_2F_1\left(0, 0; \frac{1}{2}; x\right)$ .... (vi) [Ans.]

Further, we have : ${}_2F_1\left(0, 0; \frac{1}{2}; x\right) = 1 + \frac{0\times 0}{\frac{1}{2}} \times \frac{x}{1!} + \cdots = 1 + 0 + 0 + \cdots$

$= 1$

So, $y = A \times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x\right) + \frac{B}{\sqrt{x}}$; where A, B are constants

Also, we know : $x\, F\left(\frac{1}{2}, \frac{1}{2} ; \frac{3}{2} ; x^2\right) = \sin^{-1} x$

Hence, $\sqrt{x} \times {}_2F_1\left(\frac{1}{2}, \frac{1}{2} ; \frac{3}{2} ; x\right) = \sin^{-1}(\sqrt{x})$

$$\Rightarrow {}_2F_1\left(\frac{1}{2}, \frac{1}{2} ; \frac{3}{2} ; x\right) = \frac{\sin^{-1}(\sqrt{x})}{\sqrt{x}}$$

Hence, solution of (v) becomes :

$$y = \frac{A \sin^{-1}(\sqrt{x}) + B}{\sqrt{x}}$$ [Ans.] where A, B are constants

5) Legendre equation :

$$(1-x^2)\frac{d^2y}{dx^2} - 2x \cdot \frac{dy}{dx} + n(n+1)\, y = 0 \quad \cdots\cdots \text{(i)}$$

Substitute : $x^2 = t$ $\rightarrow$ Then $\frac{dt}{dx} = 2x$

Now, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2x \cdot \frac{dy}{dt} \quad \cdots\cdots \text{(ii)}$ [By Chain Rule of differentiation]

And, $\frac{d^2y}{dx^2} = (2) \times \frac{dy}{dt} + 2x \times \frac{d}{dx}\left(\frac{dy}{dt}\right)$ [Product rule : $(uv)' = u'v + uv'$]

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \frac{dy}{dt} + 2x \times \frac{d}{dt}\left(\frac{dy}{dt}\right) \times \frac{dt}{dx} = 2 \times \frac{dy}{dt} + 4x^2 \times \frac{d^2y}{dt^2} \quad \cdots\cdots \text{(iii)}$$

Putting (ii) and (iii) in (i) and using the fact that $x^2 = t$, we get :

$$(1-t) \times \left[2 \times \frac{dy}{dt} + 4t \times \frac{d^2y}{dt^2}\right] - 2x \times 2x \frac{dy}{dt} + n(n+1)\, y = 0$$

$$\Rightarrow 4t(1-t)\cdot\frac{d^2y}{dt^2} + 2(1-t)\times\frac{dy}{dt} - 4t\times\frac{dy}{dt} + n(n+1)y = 0$$

$$\Rightarrow t(1-t)\cdot\frac{d^2y}{dt^2} + \frac{(1-3t)}{2}\cdot\frac{dy}{dt} + \frac{n(n+1)}{4}y = 0$$

$$\Rightarrow t(1-t)\cdot\frac{d^2y}{dt^2} + \left(\frac{1}{2}-\frac{3}{2}t\right)\cdot\frac{dy}{dt} + \frac{n(n+1)}{4}y = 0 \quad \ldots\ldots \text{(iv)}$$

Eqn. (iv) is now a hypergeometric differential equation. Comparing with the standard form, we have:

$$\gamma = \frac{1}{2}, \quad \alpha+\beta = \frac{1}{2} \quad \text{and} \quad \alpha\beta = \frac{-n(n+1)}{4}$$

Then, $k^2 - \left(\frac{1}{2}\right)k + \frac{(-n)(n+1)}{4} = 0$ $\rightarrow$ has roots $k = \alpha, \beta$

$$\Rightarrow 4k^2 - 2k - n(n+1) = 0 \quad \Rightarrow 4k^2 - n^2 - 2k - n = 0$$

$$\Rightarrow (2k)^2 - (n)^2 - (2k+n) = 0 \quad \Rightarrow (2k+n)(2k-n) - (2k+n) = 0$$

$$\Rightarrow (2k+n)(2k-n-1) = 0 \quad \rightarrow k = \frac{-n}{2}, \frac{n+1}{2}$$

So, $\alpha = \frac{n+1}{2}$, $\beta = \frac{-n}{2}$, $\gamma = \frac{1}{2}$. As $\gamma \notin \mathbb{Z}$, the soln. of (iv) is:

$$y = A\times F\left(\frac{n+1}{2}, \frac{-n}{2}; \frac{1}{2}; t\right) + B\times t^{1/2}\times F\left(\frac{n}{2}+1, \frac{1-n}{2}; \frac{3}{2}; t\right)$$

Hence, soln. of (i) is given by: (Put $t = x^2$)

$$y = A\times F\left(\frac{n+1}{2}, \frac{-n}{2}; \frac{1}{2}; x^2\right) + B\times x\times F\left(\frac{n}{2}+1, \frac{1-n}{2}; \frac{3}{2}; x^2\right) \quad \text{[Ans.]}$$

[where A, B are constants]

We can see, when n = even natural no., $F\left(\frac{n+1}{2}, \frac{-n}{2}; \frac{1}{2}; x^2\right)$ will be a finite polynomial as $\frac{-n}{2}$ will be a negative integer. Similarly, when n = odd natural no., $x\times F\left(\frac{n}{2}+1, \frac{1-n}{2}; \frac{3}{2}; x^2\right)$ will be polynomial. This polynomial soln. in both cases ~~are~~ give us the Legendre polynomials.