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Assignment - 1

Mathematical Methods

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1) Given: $y = 2x$, $y_1 = x$ and $y_2 = x^2$.

we can write: $2x = (2) * x + (0) * x^2$

$\Rightarrow y = 2 * y_1 + 0 * y_2$ → Hence, y is a linear combination of y_1 and y_2.

2) Given: $y_1 = \sin x$, $y_2 = \cos x$ and $y_3 = \sin(x+1)$

Then, $\sin(x+1) = \sin x . \cos 1 + \cos x . \sin 1$

$= (\cos 1). \sin x + (\sin 1). \cos x$

$\Rightarrow y_3 = (\cos 1) \cdot y_1 + (\sin 1) \cdot y_2$ [cos1 and sin1 are constants]

→ Hence, y_3 is a linear combination of y_1 and y_2.

3) (i) By using definition of linear dependence / independence of functions, we need to identify if $f(x)$, $g(x)$ in each subpart are linearly dependent or independent.

(a) $f(x) = 9\cos 2x$ $\quad g(x) = 2\cos^2 x - 2\sin^2 x$

$= 2(\cos^2 x - \sin^2 x) = 2\cos 2x$

Now, consider:

$c_1 f + c_2 g = 0 \quad \forall x \in \mathbb{R}$ [i.e. identically equal to 0]

$\Rightarrow c_1 * (9\cos 2x) + c_2 * (2\cos 2x) = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow (9c_1 + 2c_2) \cos 2x = 0 \quad \forall x \in \mathbb{R}$

we can see that $c_1 = -2$ and $c_2 = 9$ are satisfying this eqn. [and there are even more such solutions],

and here, $c_1 \neq 0$, $c_2 \neq 0$.

So, since the equation: $c_1 f + c_2 g = 0$ has non-trivial solution, so f and g are L.D.

(b) $f(t) = 2t^2$, $g(t) = t^4$

Consider the equation: $c_1 f + c_2 g = 0$

$\Rightarrow c_1 \times 2t^2 + c_2 \times t^4 = 0 \quad \forall t \in \mathbb{R}$ [i.e. identically equal to 0]

$\Rightarrow c_2 t^4 + 2c_1 t^2 = 0 \quad \forall t \in \mathbb{R}$

Since it is identically equal to 0, we equate the corresponding coefficients of t^4 and t^2 on both sides :

So, $2c_1 = 0 \Rightarrow c_1 = 0$ and $c_2 = 0$

So, $c_1 f + c_2 g = 0$ has only the trivial solution $c_1 = c_2 = 0$ here. Hence, $f(t)$ and $g(t)$ are L.I.

(ii) Now, for the same two examples in part (i), we verify the result by using Wronskian.

(a) $f(x) = 9\cos 2x$, $g(x) = 2\cos 2x$

$\Rightarrow f'(x) = -18\sin 2x$, $g'(x) = -4\sin 2x$

So, $W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 9\cos 2x & 2\cos 2x \\ -18\sin 2x & -4\sin 2x \end{vmatrix}$

$\Rightarrow W(f,g) = -36\cos(2x)\sin(2x) + 36\cos(2x).\sin(2x)$

$= 0$

And by the definition of L.D, we had already seen that $f(x)$ and $g(x)$ are L.D.

So, f and g are L.D $\Rightarrow W(f,g) = 0$ [Verified]

[Note that, we can't guarantee the other way round that if $W(f,g) = 0$ then f, g are L.D]

(b) $f(t) = 2t^2$, $g(t) = t^4$

$\Rightarrow f'(t) = 4t$, $g'(t) = 4t^3$

Then, $w(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix}$

$\Rightarrow w(f,g) = 8t^5 - 4t^5 = 4t^5$

As $w(f,g) \neq 0$, [i.e. we are having atleast one point t where $w(f,g) \neq 0$]

so f and g are L.I.

And, by using definition of L.I also, we had earlier found the f and g are indeed L.I.

Hence, $w(f,g) \neq 0 \Rightarrow f(t)$ and $g(t)$ are L.I. [verified.]

4) $f(x) = 6^x$, $g(x) = 6^{x+2}$

$\Rightarrow f'(x) = 6^x . \ln 6$, $g'(x) = 6^{x+2} . \ln 6$

Then, $w(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 6^x & 6^{x+2} \\ 6^x . \ln 6 & 6^{x+2} . \ln 6 \end{vmatrix}$

$= 6^{2x+2} . \ln 6 - 6^{2x+2} . \ln 6 = 0$

However, $w(f,g) = 0$ doesn't guarantee whether f, g are L.D or L.I. [A standard example to show this is for $f = x^2$, $g = x|x|$; $x \in \mathbb{R}$]

Justification:

consider $f(x) = 9\cos 2x$, $g(x) = 2\cos 2x$

→ then $w(f,g) = 0$ [already found in Q. 3) b)]

and we found that $f(x) = 9\cos 2x$, $g(x) = 2\cos 2x$ are L.D.

Now, consider $f(x) = x^2$, $g(x) = x|x| = \begin{cases} x^2 ; & x \geq 0 \\ -x^2 ; & x < 0 \end{cases}$

$\Rightarrow f'(x) = 2x$, $g'(x) = \begin{cases} 2x ; & x \geq 0 \\ -2x ; & x < 0 \end{cases}$

Then, for $x \geq 0$, $w(f,g) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0$

And for $x < 0$, $W(f,g) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0$

So, $W(f,g) = 0 \quad \forall x \in \mathbb{R}$

However, in this case $f(x) = x^2$ and $g(x) = x|x|$ are L.I.

This shows that from $W(f,g) = 0$, we can't guarantee whether $f(x)$ and $g(x)$ are L.D or L.I.

So, in such a case, for being sure whether $f(x) = 6^x$ and $g(x) = 6^{x+2}$ are L.D or L.I, we will use the basic definition of L.D and L.I.

Consider the equation:

$$c_1 f + c_2 g = 0 \quad \forall x \in \mathbb{R} \quad \text{[i.e. identically equal to 0]}$$

$$\Rightarrow c_1 \times 6^x + c_2 \times 6^{x+2} = 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow 6^x \times (c_1 + 36c_2) = 0$$

$$\Rightarrow c_1 + 36c_2 = 0 \quad [\because 6^x \text{ can never be equal to } 0,\ 6^x > 0]$$

→ This eqn. has non-trivial soln., one of them being $c_1 = -36$, $c_2 = 1$.

Hence, $f(x)$ and $g(x)$ are L.D.

5) Given IVP: $\dfrac{dy}{dx} = \dfrac{4x^2 - 7x}{3y^2 + 2}$; $y(1) = 1$

Then, $(3y^2+2)\,dy = (4x^2 - 7x)\,dx$

Integrating both sides:

$$\int_1^y (3y^2+2)\,dy = \int_1^x (4x^2 - 7x)\,dx \quad [\because \text{when } x=1,\ y=1]$$

$$\Rightarrow \left[y^3 + 2y\right]_1^y = \left[\frac{4x^3}{3}\right]_1^x - \left[\frac{7x^2}{2}\right]_1^x$$

$$\Rightarrow y^3 + 2y - 3 = \frac{4x^3}{3} - \frac{4}{3} - \frac{7x^2}{2} + \frac{7}{2}$$

$$\Rightarrow 6y^3 + 12y - 18 = 8x^3 - 8 - 21x^2 + 21$$

$$\Rightarrow 6y^3 + 12y = 8x^3 - 21x^2 + 31 \quad \text{[Ans.]}$$

6) Given IVP : $y'' = 2$; $y'(0) = 6$, $y(0) = 0$

Corresponding homogenous ODE : $y'' = 0$

→ Characteristic eqn : $\lambda^2 = 0 \Rightarrow \lambda = 0, 0$ [Two equal roots]

So, $y_1 = e^{0.x} = 1$ is a soln. of this homogenous ODE.

$y_2 = x \cdot e^{0.x} = x$ is also a soln. of this homogenous ODE.

So, complementary function, $y_c = c_1 y_1 + c_2 y_2$

$\Rightarrow y_c = c_1 + c_2 x$ ..... (i)

Now, Particular Integral :

By using Method of undetermined coefficients,

let the particular integral be : $y_p = ax^2 + bx + c$

Then, $y_p' = 2ax + b$, $y_p'' = 2a$

Substituting in the original ODE,

$y_p'' = 2 \Rightarrow 2a = 2 \Rightarrow a = 1$ [b, c can be anything. For simplicity, take $b = c = 0$]

So, $y_p = x^2$ .... (ii)

Hence, from (i) and (ii), general soln. of the ODE is :

$y = y_c + y_p \Rightarrow y = c_1 + c_2 x + x^2$

Now, $y(0) = 0 \Rightarrow c_1 = 0$

And, $y'(0) = 6 \Rightarrow c_2 = 6$

$\therefore y = x^2 + 6x$ [Ans.]

7) Given BVP : $y'' + 4y = 0$

→ characteristic eqn. : $\lambda^2 + 4 = 0 \Rightarrow \lambda = 0 + 2i, 0 - 2i$

So, $y_1 = e^{0.x} \cdot \sin(2x) = \sin 2x$ is a soln. of this ODE.

$y_2 = e^{0.x} \cdot \cos(2x) = \cos 2x$ is also a soln. of this ODE

So, general solution of the ODE :

$y = c_1 y_1 + c_2 y_2 = c_1 \sin(2x) + c_2 \cos(2x)$

Now, given BCs:

(i) $y(0) = -2 \Rightarrow c_2 = -2$ $\quad [\because y = c_1 \sin(2x) + c_2 \cos(2x)]$

$y(\frac{\pi}{4}) = 10 \Rightarrow c_1 * \sin(\frac{\pi}{2}) + c_2 * \cos(\frac{\pi}{2}) = 10$

$\Rightarrow c_1 = 10$

So in this case, we have a unique soln. to the BVP:

$y = 10 \sin(2x) - 2\cos(2x)$ [Ans.]

(ii) $y(0) = -2 \Rightarrow c_2 = -2$

$y(2\pi) = -2 \Rightarrow c_1 * 0 + c_2 * 1 = -2 \Rightarrow c_2 = -2.$

and c_1 can take any value

So in this case, we have infinitely many solutions to the BVP, where the general soln. is given by:

$y = c_1 * \sin(2x) - 2\cos(2x)$ [Ans.]

(iii) $y(0) = -2 \Rightarrow c_2 = -2$

$y(2\pi) = 3 \Rightarrow c_1 * 0 + c_2 * 1 = 3 \Rightarrow c_2 = 3$

But $c_2 = -2$ and $c_2 = 3$ are both conflicting and both can't happen simultaneously.

Hence, this case has no solution to the BVP. [Ans.]