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ASSIGNMENT - 2

Mathematical Methods

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1) (i) Given ODE : $2y'' + 18y = 6\tan(3t)$

$\Rightarrow y'' + 9y = 3\tan(3t) \quad \ldots (i)$

[For applying the method of variation of parameters, we would make the coefficient of $y'' = 1$]

Now, corresponding homogenous ODE :

$y'' + 9y = 0$

→ characteristic equation : $\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$

$\Rightarrow \lambda = 0 \pm 3i$

So, $y_1 = e^{0 \cdot t} \cdot \sin(3t) = \sin(3t)$

$y_2 = e^{0 \cdot t} \cdot \cos(3t) = \cos(3t)$

These are ~~the~~ two L.I. solns. of $y'' + 9y = 0$.

So, general soln. of the homogenous part :

$y_c = c_1 y_1 + c_2 y_2 = c_1 \sin(3t) + c_2 \cos(3t)$

Now, let the particular integral of (i) be :

$y_p = u(t) \cdot y_1(t) + v(t) \cdot y_2(t)$

$\Rightarrow y_p = u(t) \cdot \sin(3t) + v(t) \cdot \cos(3t)$

By method of variation of parameters, we choose $u(t)$ and $v(t)$ such that :

$u' y_1 + v' y_2 = 0 \Rightarrow u' \sin(3t) + v' \cos(3t) = 0 \quad \ldots (ii)$

and, $u' y_1' + v' y_2' = 3\tan(3t) \Rightarrow 3u' \cos(3t) - 3v' \sin(3t) = 3\tan(3t)$

$\Rightarrow u' \cos(3t) - v' \sin(3t) = \tan(3t) \quad \ldots (iii)$

So, $u(t)$ and $v(t)$ are chosen such that they satisfy equations (ii) and (iii). From (ii) and (iii), we have :

$v' = -u' \tan(3t)$ [From (ii)]

Putting in (iii), $u' \cos(3t) + u' \tan(3t) \sin(3t) = \tan(3t)$

$\Rightarrow u' \times \left[\cos(3t) + \frac{\sin^2(3t)}{\cos(3t)} \right] = \frac{\sin(3t)}{\cos(3t)}$

$$\Rightarrow u' \left[\frac{\cos^2(3t) + \sin^2(3t)}{\cos(3t)} \right] = \frac{\sin(3t)}{\cos(3t)}$$

$$\Rightarrow u' = \sin(3t) \quad \Rightarrow u = \int \sin(3t)\,dt = \frac{-\cos(3t)}{3} \quad \dots \text{(iv)}$$

$$\text{So, } v' = \frac{-\sin^2(3t)}{\cos(3t)} = \frac{\cos^2(3t) - 1}{\cos(3t)} = \cos(3t) - \sec(3t)$$

$$\Rightarrow v = \int (\cos(3t) - \sec(3t)) \cdot dt$$

$$\Rightarrow v = \frac{\sin(3t)}{3} - \frac{1}{3} \ln|\sec(3t) + \tan(3t)| \quad \dots \text{(v)}$$

From (iv) and (v), we get the particular integral as:

$$y_p = u \cdot \sin(3t) + v \cdot \cos(3t)$$

$$\Rightarrow y_p = \frac{-1}{3} \sin(3t)\cos(3t) + \frac{1}{3} \sin(3t)\cos(3t) - \frac{1}{3}\cos(3t) \cdot \ln|\sec(3t) + \tan(3t)|$$

$$\Rightarrow y_p = \frac{-1}{3} \cos(3t) \cdot \ln|\sec(3t) + \tan(3t)| \quad \dots \text{(vi)}$$

Hence, the general solution of (i) is given by:

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 \sin(3t) + c_2 \cos(3t) - \frac{\cos(3t)}{3} \cdot \ln|\sec(3t) + \tan(3t)|$$

[Ans.]

(ii) Given ODE: $y'' - 2y' + y = \dfrac{e^t}{t^2+1} \quad \dots$ (i)

Corresponding homogenous ODE:

$$y'' - 2y' + y = 0$$

→ Characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0$

$\Rightarrow \lambda = 1, 1$

(Equal roots)

So, $y_1 = e^t$, $y_2 = t \cdot e^t$ } These are two L.I solns. of $y'' - 2y' + y = 0$

So, complementary function of (i) is:

$$y_c = c_1 e^t + c_2 t \cdot e^t \quad \dots \text{(ii)}$$

Now, let the particular integral of (i) be :

$$y_p = u(t)\cdot y_1(t) + v(t)\cdot y_2(t)$$

$$\Rightarrow y_p = u(t)\cdot e^t + v(t)\cdot te^t$$

By the method of variation of parameters, we choose $u(t)$ and $v(t)$ such that they satisfy the following eqns. :

$$u'y_1 + v'y_2 = 0 \quad \Rightarrow u'.e^t + v'.te^t = 0$$

$$\Rightarrow u' + t.v' = 0 \quad \cdots \text{(iii)} \qquad \left[\because e^t \neq 0 \text{ for any } t \in \mathbb{R}\right]$$

$$\text{And, } u'y_1' + v'y_2' = \frac{e^t}{t^2+1} \quad \Rightarrow u'e^t + v'e^t(t+1) = \frac{e^t}{t^2+1}$$

$$\Rightarrow u' + v'(t+1) = \frac{1}{t^2+1} \quad \cdots\cdots \text{(iv)}$$

From (iii), $u' = -t.v'$. Putting it in (iv) and solving for v',

we get : $-v't + v't + v' = \dfrac{1}{t^2+1}$

$$\Rightarrow v' = \frac{1}{t^2+1} \quad \Rightarrow v = \int \frac{dt}{t^2+1} \quad \Rightarrow v = \tan^{-1}(t) \quad \cdots \text{(v)}$$

$$\text{And, } u' = -t.v' = \frac{-t}{t^2+1} \quad \Rightarrow u = \int \frac{-t.dt}{t^2+1} = \frac{-1}{2}\ln(t^2+1) \quad \cdots \text{(vi)}$$

Hence we obtained u and v as in (v) and (vi). Putting these, we have :

$$y_p = u.e^t + v.te^t \quad \Rightarrow y_p = e^t \times \left[t \times \tan^{-1}(t) - \frac{1}{2}\ln(t^2+1)\right] \cdots \text{(vii)}$$

So, from (ii) and (vii), the general solution of (i) is :

$$y = y_c + y_p$$

$$\Rightarrow y = c_1e^t + c_2.te^t + e^t \times \left[t \times \tan^{-1}(t) - \frac{1}{2}\ln(t^2+1)\right]$$

[Ans.]

2) Given ODE: $t \cdot y'' - (t+1) \cdot y' + y = t^2$

$\Rightarrow y'' - \frac{(t+1)}{t} \cdot y' + \frac{1}{t} \cdot y = t$ ....(i) [Dividing by t, so as to make coefficient of $y'' = 1$]

Also given that the two L.I solutions of the corresponding homogenous equation of (i) are:

$y_1 = e^t$ and $y_2 = t+1$

Let the particular integral of (i) be:

$y_p = u(t) \cdot y_1(t) + v(t) \cdot y_2(t)$

$\Rightarrow y_p = u(t) \cdot e^t + v(t) \cdot (t+1)$

By the method of variation of parameters, $u(t)$ and $v(t)$ should be such that:

$u'y_1 + v'y_2 = 0 \Rightarrow u'e^t + v'(t+1) = 0$ ....(ii)

And, $u'y_1' + v'y_2' = t \Rightarrow u'e^t + v' = t$ ...(iii)

$u'(t)$ and $v'(t)$ should satisfy the system of eqns. (ii) & (iii).

from (ii) and (iii), $u'e^t = -v'(t+1) = t - v'$

$\Rightarrow -v't = t \Rightarrow v' = -1$

$\Rightarrow v = \int -1 \cdot dt = -t$ ..(iv)

And, $u'e^t = (t+1) \Rightarrow u' = e^{-t} \cdot (t+1)$

$\Rightarrow u = \int e^{-t} \cdot (t+1) \cdot dt = -(t+1) \cdot e^{-t} + \int e^{-t} \cdot dt$

$\Rightarrow u = -(t+1)e^{-t} - e^{-t} = -(t+2)e^{-t}$ .... (v)

So, from (iv) and (v), we get the particular integral as:

$y_p = -(t+2) - t(t+1) = -t^2 - 2t - 2$ ....(vi)

Hence, the general solution of (i) is:

$y = c_1 e^t + c_2(t+1) - t^2 - 2t - 2$ [Ans.]

3) Consider the 3rd order ODE :

$$y''' + a(x)\cdot y'' + b(x)\cdot y' + c(x)\cdot y = r(x) \quad \cdots (i)$$

Suppose the corresponding homogenous ODE of (i) has three L.I solutions $y_1(x)$, $y_2(x)$ and $y_3(x)$.

Let the particular integral of (i) be :

$$y_P = u(x).y_1(x) + v(x)\cdot y_2(x) + \omega(x)\cdot y_3(x)$$

or, $y_P = u\cdot y_1 + v\cdot y_2 + \omega\cdot y_3$ [in short hand notation, where $u, v, \omega, y_1, y_2, y_3$ are all functions of x]

Then, $y_P' = u\cdot y_1' + v\cdot y_2' + \omega\cdot y_3' + (u'y_1 + v'y_2 + \omega' y_3)$

We'll choose u, v, ω such that :

$$u'y_1 + v'y_2 + \omega' y_3 = 0 \quad \cdots (ii)$$

In that case, $y_P' = u\cdot y_1' + v y_2' + \omega\cdot y_3'$

$\Rightarrow y_P'' = u\cdot y_1'' + v\cdot y_2'' + \omega\cdot y_3'' + (u'\cdot y_1' + v'\cdot y_2' + \omega'\cdot y_3')$

Again, we set $u'\cdot y_1' + v'\cdot y_2' + \omega'\cdot y_3' = 0 \quad \cdots (iii)$

So then, $y_P'' = u\cdot y_1'' + v\cdot y_2'' + \omega\cdot y_3''$

$\Rightarrow y_P''' = u\cdot y_1''' + v\cdot y_2''' + \omega\cdot y_3''' + u'\cdot y_1'' + v'\cdot y_2'' + \omega'\cdot y_3''$

Putting the expressions of y_P, y_P', y_P'', y_P''' in (i), we get :

$$u\cdot(y_1''' + a(x)\cdot y_1'' + b(x)\cdot y_1' + c(x)\cdot y_1) + v\cdot(y_2''' + a(x)\cdot y_2'' + b(x)\cdot y_2' + c(x)\cdot y_2)$$
$$+ \omega\cdot(y_3''' + a(x)\cdot y_3'' + b(x)\cdot y_3' + c(x)\cdot y_3) + u'\cdot y_1'' + v'\cdot y_2'' + \omega'\cdot y_3'' = r(x) \quad \cdots (iv)$$

Now, as $y_1(x)$, $y_2(x)$, $y_3(x)$ are solutions of the homogenous ODE associated with (i), so :

$$y_1''' + a(x)\cdot y_1'' + b(x)\cdot y_1' + c(x)\cdot y_1 = 0$$
$$y_2''' + a(x)\cdot y_2'' + b(x)\cdot y_2' + c(x)\cdot y_2 = 0$$
$$y_3''' + a(x)\cdot y_3'' + b(x)\cdot y_3' + c(x)\cdot y_3 = 0$$

Using these equations, eqn. (iv) simplifies to :

$$u' y_1'' + v' \cdot y_2'' + w' \cdot y_3'' = r(x) \quad \cdots\cdots \text{(v)}$$

Hence, we obtain a system of equations given by (ii), (iii) and (v) for $u'(x)$, $v'(x)$ and $w'(x)$.

In matrix form, this system of equations is represented by :

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r(x) \end{bmatrix}$$

By Cramer's Rule, the solution of this system is given by :

$$u' = \frac{W_1}{W} \quad , \quad v' = \frac{W_2}{W} \quad , \quad w' = \frac{W_3}{W}$$

where, W = determinant of coefficient matrix
$= W(y_1, y_2, y_3)$ [i.e. Wronskian of y_1, y_2, y_3]

and W_i = determinant obtained from W by replacing the i^{th} column by $[0 \;\; 0 \;\; r(x)]^T$; for $i = 1, 2, 3$

Then, $u = \displaystyle\int \frac{W_1}{W(y_1, y_2, y_3)} dx$, $v = \displaystyle\int \frac{W_2}{W(y_1, y_2, y_3)} dx$

and $w = \displaystyle\int \frac{W_3}{W(y_1, y_2, y_3)} \cdot dx$ ..... (vi)

Hence, the particular integral of (i) is given by :

$y_p = u(x) \cdot y_1(x) + v(x) \cdot y_2(x) + w(x) \cdot y_3(x)$, where $u(x)$, $v(x)$ and $w(x)$ are given by eqn. (vi)

[Proved.]