Legendre differential equation and Legendre function

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{---} (1)$$

is called Legendre diff. eqn. where n is a non-ve integer. This eqn. is also written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

Let us assume $\quad y = \sum_{m=0}^{\infty} c_m x^{k-m}, \quad c_0 \neq 0 \quad \text{---} (2)$

$$y' = \sum_{m=0}^{\infty} c_m (k-m) x^{k-m-1}$$

$$y'' = \sum_{m=0}^{\infty} c_m (k-m)(k-m-1) x^{k-m-2}$$

Putting these values in Eq. (1),

$$(1-x^2)\sum_{m=0}^{\infty} c_m (k-m)(k-m-1)x^{k-m-2} - 2x\sum_{m=0}^{\infty} c_m(k-m)x^{k-m-1} + n(n+1)\sum_{m=0}^{\infty} c_m x^{k-m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m\left\{(k-m)(k-m-1) + 2(k-m) - n(n+1)\right\}x^{k-m} = 0 \quad \text{---} (3)$$

Now $(k-m)(k-m-1) + 2(k-m) - n(n+1)$
$= (k-m-n)(k-m+n+1)$

$$\Rightarrow \sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m(k-m-n)(k-m+n+1)x^{k-m} = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} c_{m-2}(k-m+2)(k-m+1)x^{k-m} - \sum_{m=0}^{\infty} c_m(k-m-n)(k-m+n+1)x^{k-m} = 0$$

$$\text{---} (4)$$

Eq. (4) is an identity. To get the indicial eqn., we equate to zero the coeff. of highest power of x i.e. x^k in (4) and obtain

$$c_0(k-n)(k+n+1) = 0$$

$$\Rightarrow (k-n)(k+n+1) = 0 \quad \text{as} \quad c_0 \neq 0 \qquad -(5)$$

$$k = n, -(n+1)$$

Next we equate to zero the coefficient of x^{k-1} in (4), and obtain $c_1(k-1-n)(k+n) = 0 \quad -(6)$

For $k=n$ and $-(n+1)$, neither $(k-1-n)$ nor $(k+n)$ is zero. So from (6), $c_1 = 0$. Finally equating to zero the coefficient of x^{k-m} in (4), we have

$$c_{m-2}(k-m+2)(k-m+1) - c_m(k-m-n)(k-m+n+1) = 0$$

$$\Rightarrow c_m = \frac{(k-m+2)(k-m+1)}{(k-m-n)(k-m+n+1)} c_{m-2} \quad -(7) \quad m \geq 2$$

Putting $m = 3, 5, 7, \ldots$ in (7) and noting that $c_1 = 0$

$$c_1 = c_3 = c_5 = c_7 = \ldots = 0 \quad -(8)$$

which holds for both $k = n$ and $k = -(n+1)$

Case - I

When $k=n$, (7) becomes

$$c_m = -\frac{(n-m+2)(n-m+1)}{m(2n-m+1)} c_{m-2} \quad -(9)$$

Putting $m = 2, 4, 6, \ldots$

$$c_2 = -\frac{n(n-1)}{2(2n-1)} c_0$$

$$c_4 = -\frac{(n-2)(n-3)}{4(2n-3)} c_2 = \frac{n(n-1)(n-2)(n-3)}{2.4\ldots(2n-1)(2n-3)} c_0$$

Re-writing Eq. (2) for $k = n$

$$y = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + c_3 x^{n-3} + \cdots \quad (10)$$

Using (8) and the above values of $c_2, c_4, \ldots$,
Eq. (10) becomes [c_0 is replaced by a]

$$y = a\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4\ldots(2n-1)(2n-3)} x^{n-4} + \cdots\right] \quad (11)$$

Case II When $k = -(n+1)$

Eq. (7) becomes $$c_m = \frac{(n+m-1)(n+m)}{m(2n+m+1)} c_{m-2} \quad (12)$$

Putting $m = 2, 4, 6, \ldots$

$$c_2 = \frac{(n+1)(n+2)}{2(2n+3)} c_0$$

$$c_4 = \frac{(n+3)(n+4)}{4(2n+5)} c_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4\,(2n+3)(2n+5)} c_0$$

and so on.

In (2) putting $k = -(n+1)$,

$$y = c_0 x^{-n-1} + c_1 x^{-n-2} + c_2 x^{-n-3} + c_3 x^{-n-4} \cdots \quad (13)$$

Using (8) and the values of $c_2, c_4, c_6, \cdots$ etc (13) becomes [replacing c_0 by b]

$$y = b\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \cdots\right] \quad \text{— (14)}$$

Eq. (11) and (14) are two L.I. solutions of Eq. (1). If we take $a = [1.3.5. \cdots (2n-1)]/n!$, the solⁿ. (11) is denoted by $P_n(x)$ and is called Legendre fⁿ. of the 1st kind or Legendre polynomial of degree n. Again if we take $b = n!/[1.3.5. \cdots (2n+1)]$, the solⁿ. (14) is denoted by $Q_n(x)$ and is called Legendre function of ~~1st~~ 2nd kind. ~~or Legendre polynomial of degree n.~~ Hence $P_n(x)$ and $Q_n(x)$ are two L.I. solution of Eq. (1).

Generating function of $P_n(x)$

$P_n(x)$ is the coefficient of h^n in the expansion in ascending powers of $(1-2hx+h^2)^{-1/2}$, $|x|<1$, $|h|<1$

Proof $\quad (1-2hx+h^2)^{-1/2} = \left\{1 - h(2x-h)\right\}^{-1/2}$

$$= 1 + \frac{1}{2} h(2x-h) + \frac{1.3}{2.4} h^2 (2x-h)^2 + \cdots$$

$$+ \frac{1.3. \cdots (2n-3)}{2.4. \cdots (2n-2)} h^{n-1} (2x-h)^{n-1}$$

$$+ \frac{1.3. \cdots (2n-1)}{2.4. \cdots 2n} h^n (2x-h)^n + \cdots$$

$\therefore$ Coefficient of h^n

$$= \frac{1.3.\cdots(2n-1)}{2.4.\cdots 2n}(2x)^n - \frac{1.3.\cdots(2n-3)}{2.4.\cdots 2n}\,{}^{(n-1)}C_1\,(2x)^{n-2} + \cdots\cdots$$

$$= \frac{1.3.\cdots(2n-1)}{n!}\left[x^n - \frac{2n}{2n-1}(n-1)\frac{x^{n-2}}{2^2} + \cdots\right]$$

$$= \frac{1.3.\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \cdots\right]$$

$$= P_n(x)$$

$\therefore$ We can say that $\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$

Orthogonal properties of Legendre polynomial

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0 \qquad \text{if } m \neq n$$

Proof: Legendre diff. eqn.

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

If $P_n(x)$ and $P_m(x)$ are two solutions, then

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0 \quad \text{--- (1)}$$

and $$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)P_m = 0 \quad \text{--- (2)}$$

Multiplying (1) by P_m and (2) by P_n and then subtracting,

$$P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} - P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}$$
$$+ \left\{n(n+1) - m(m+1)\right\} P_n P_m = 0$$

Integrating between the limits -1 to $+1$,

$$\int_{-1}^{+1} P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx - \int_{-1}^{+1} P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx$$
$$+\left\{n(n+1) - m(m+1)\right\}\int_{-1}^{+1} P_m P_n\, dx = 0$$

Integrating by parts,

$$\left[P_m(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1} \frac{dP_m}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx$$
$$- \left[P_n(1-x^2)\frac{dP_m}{dx}\right]_{-1}^{+1} + \int_{-1}^{+1}\frac{dP_n}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx$$
$$+ \left\{n(n+1) - m(m+1)\right\}\int_{-1}^{+1} P_m P_n\, dx = 0$$

$$\Rightarrow \left\{n(n+1) - m(m+1)\right\}\int_{-1}^{+1} P_m P_n\, dx = 0$$

$$\therefore \int_{-1}^{+1} P_m P_n\, dx = 0 \quad \text{if} \quad m \neq n.$$

<u>Aliter</u> $\left[(1-x^2)y'\right]' + \lambda y = 0$ where $\lambda = n(n+1)$ — (1)

So it is of the form of S-L eqn.

$$[r(x)y']' + [q(x) + \lambda p(x)]y = 0 \quad \text{— (2)}$$

Comparing (1) & (2), $r(x) = 1-x^2$, $q(x) = 0$, $p(x) = 1$

$\because r(1) = r(-1) = 0$, we need no B.C. to form a S-L problem.

$\therefore P_n(x)$, $n = 0, 1, 2, \ldots$ are orthogonal in $-1 \le x \le 1$

w.r.t. $p(x) = 1$ i.e. $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad m \neq n.$

Orthogonal property (2nd part)

Prove that $\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}$

Sol^n: We have $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Squaring both sides

$$(1-2xh+h^2)^{-1} = \sum_{n=0}^{\infty} h^{2n} \{P_n(x)\}^2 + 2 \sum_{\substack{m,n=0 \\ m \neq n}}^{\infty} h^{m+n} P_m(x) P_n(x)$$

Integrating between limits -1 to $+1$,

$$\sum_{n=0}^{\infty} \int_{-1}^{1} h^{2n} [P_n(x)]^2 dx + 2 \sum_{\substack{m,n=0 \\ m \neq n}}^{\infty} \int_{-1}^{1} h^{m+n} P_m(x) P_n(x) \, dx$$

$$= \int_{-1}^{+1} \frac{dx}{(1-2xh+h^2)}$$

or, $\sum_{n=0}^{\infty} \int_{-1}^{+1} h^{2n} [P_n(x)]^2 dx = \int_{-1}^{+1} \frac{dx}{(1-2xh+h^2)}$

[Other integrals in LHS are zero as $m \neq n$]

$$= -\frac{1}{2h} \{\ln(1-2xh+h^2)\}_{-1}^{+1}$$

$$= -\frac{1}{2h} \{\ln(1-h)^2 - \ln(1+h)^2\} = \frac{1}{2h} \left[\ln\left(\frac{1+h}{1-h}\right)^2\right]$$

$$= \frac{1}{h} \ln \left\{\frac{1+h}{1-h}\right\} = \frac{2}{h} \left\{h + \frac{h^3}{3} + \frac{h^5}{5} + \cdots\right\}$$

$$= 2\left\{1 + \frac{h^2}{3} + \frac{h^4}{5} + \cdots + \frac{h^{2n}}{2n+1} + \cdots\right\}$$

$$= \sum_{n=0}^{\infty} \frac{2h^{2n}}{2n+1}$$

Equating coeff. of h^{2n}, $\int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1}$

Recurrence formulae

I $\quad (2n+1)\, x P_n = (n+1) P_{n+1} + n P_{n-1}$

Proof $\quad (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Differentiating both sides w.r.t. 'h',

$$-\frac{1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h) = \sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-1/2} = (1-2xh+h^2) \sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

$$\Rightarrow (x-h) \sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2) \sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

$$\Rightarrow (x-h)\left[P_0(x) + h P_1(x) + \cdots + h^{n-1} P_{n-1}(x) + h^n P_n(x) + \cdots \right]$$
$$= (1-2xh+h^2)\left[P_1(x) + 2h P_2(x) + \cdots + (n-1) h^{n-2} P_{n-1}(x) + n h^{n-1} P_n(x) + (n+1) h^n P_{n+1}(x) + \cdots \right] \quad (1)$$

Equating the coefficients of h^n from two sides,

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2xn P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow (2n+1)\, x\, P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

In short, $(2n+1)\, x P_n = (n+1) P_{n+1} + n P_{n-1}$

<u>Note</u> Equating the coeff. of h^{n-1} from the two sides in (1)

$$x P_{n-1}(x) - P_{n-2}(x) = n P_n(x) - 2x(n-1) P_{n-1}(x) + (n-2) P_{n-2}(x)$$

$$\Rightarrow n P_n = (2n-1)\, x P_{n-1} - (n-1) P_{n-2}$$

II $\quad nP_n = xP_n' - P_{n-1}'$ dash denotes diff. w.r.t. x

Proof $\quad (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad — (1)$

Differentiating (1) w.r.t. 'h',

$$(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} n h^{n-1} P_n(x) \quad — (2)$$

Again differentiating (1) w.r.t. x,

$$h(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x)$$

$$\Rightarrow h(x-h)(1-2hx+h^2)^{-3/2} = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) \quad — (3)$$

From (2) and (3),

$$h\sum_{n=0}^{\infty} n h^{n-1} P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x)$$

$$\Rightarrow h\left[h^0 P_1(x) + 2h\, P_2(x) + \cdots + n h^{n-1} P_n(x) + \cdots\right]$$
$$= (x-h)\left[P_0'(x) + h\, P_1'(x) + \cdots + h^{n-1} P_{n-1}'(x) + h^n P_n'(x)\right]$$

Equating the coefficient of h^n on both the sides,

$$nP_n(x) = x\,P_n'(x) - P_{n-1}'(x)$$

i.e. $\quad nP_n = xP_n' - P_{n-1}'$

III $\quad (2n+1) P_n = P'_{n+1} - P'_{n-1}$

Proof: From rec. formula I

$$(2n+1) x P_n = (n+1) P_{n+1} + n P_{n-1}$$

Differentiating w.r.t. x

$$(2n+1) x P_n' + (2n+1) P_n = (n+1) P'_{n+1} + n P'_{n-1} \quad \text{— (1)}$$

From rec. formula II

$$x P_n' = n P_n + P'_{n-1} \quad \cancel{\text{— (2)}}$$

i.e. $(2n+1) x P_n' = (2n+1)(n P_n + P'_{n-1})$ — (2)

Eliminating xP_n' from (1) and (2),

$$(2n+1)(n P_n + P'_{n-1}) + (2n+1) P_n = (n+1) P'_{n+1} + n P'_{n-1}$$

$$\Rightarrow (2n+1)(n+1) P_n = (n+1) P'_{n+1} + n P'_{n-1} - (2n+1) P'_{n-1}$$

$$\Rightarrow (2n+1)(n+1) P_n = (n+1) P'_{n+1} - (n+1) P'_{n-1}$$

$$\Rightarrow (2n+1) P_n = P'_{n+1} - P'_{n-1}$$

IV $\quad (n+1) P_n = P'_{n+1} - x P_n'$

Proof: From rec. formula II and III

$$n P_n = x P_n' - P'_{n-1} \quad \text{— (1)}$$

$$(2n+1) P_n = P'_{n+1} - P'_{n-1} \quad \text{— (2)}$$

Subtracting (1) from (2),

$$(n+1) P_n = P'_{n+1} - x P_n'$$