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Lecture 12

Ex/ A symmetric tensor of second order has at most $\frac{1}{2}N(N+1)$ different components. Prove this. (independent)

Sol^n:- Suppose A_{ij} is a second order covariant tensor which has N^2 — component in the space V_N. Then the components are as follows:

$$\begin{matrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1N} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2N} \\ - & - & - & - & - \\ A_{N1} & A_{N2} & A_{N3} & \cdots & A_{NN} \end{matrix}$$

$A \to A_{ij}$ is symmetric in ij

$A_{ij} = A_{ji}$ ie, $A_{12} = A_{21}$, $A_{32} = A_{23}$ etc.

The no. of independent components corresponding to a repeated suffix (ie, $A_{11}, A_{22}, A_{33}, \ldots, A_{NN}$) is N.

The no. of ~~independent~~ components corresponding to different suffixes is $(N^2 - N)$. However, due to symmetry property (ie, $A_{12} = A_{21}$ etc.). This number reduces to half ie, $(N^2 - N)/2$.

Thus, the total no. of independent components in the tensor A_{ij} is

$$N + (N^2 - N)/2 = \frac{1}{2} N (2 + N - 1)$$

$$= \boxed{\frac{1}{2} N (N+1)} .$$

The same can be proved for a symmetric contra-variant tensor A^{ij} of rank 2 (ie, $A^{ij} = A^{ji}$).

H.W.
EX/ A skew-symmetric (or, anti-symmetric) tensor of second order has atmost $\frac{1}{2}N(N-1)$ independent (different) components.

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EX/ If $\bar{A}_{ij}$ is a skew-symmetric tensor, prove that

$$\left(\delta_j^i \delta_l^k + \delta_l^i \delta_j^k\right) A_{ik} = 0.$$

Sol^n:- As A_{ij} is a skew-symmetric tensor in the indices i, j, we have

$$A_{ij} = -A_{ji} \quad \text{or,} \quad A_{ij} + A_{ji} = 0$$

So, $\delta^{i}_{j}\,\delta^{k}_{l}\,A_{ik} + \delta^{i}_{l}\,\delta^{k}_{j}\,A_{ik}$

$$= \delta^{i}_{j}\,A_{il} + \delta^{i}_{l}\,A_{ij}$$

$$= A_{jl} + A_{lj}$$

$$= 0 \;\; // \quad [\text{as } A_{jl} = -A_{lj}$$

EX/ Show that a tensor of the type (2,0) can be expressed as a sum of a symmetric tensor & a skew-symmetric tensor.

Hint: We know that any tensor A^{ij} is known as a tensor of the type $(2,0)$ (i.e., a contravariant tensor of order 2).

Clearly, $$A^{ij} = \frac{1}{2}\left(A^{ij} + A^{ji}\right) + \frac{1}{2}\left(A^{ij} - A^{ji}\right)$$

$$= B^{ij} + C^{ij} \rightarrow (1)$$

where, $$B^{ij} = \frac{1}{2}\left(A^{ij} + A^{ji}\right) \quad 2$$

symmetric

$$C^{ij} = \frac{1}{2}\left(A^{ij} - A^{ji}\right).$$

i.e., $C^{ij} = -C^{ji}$

skew-symmetric

$$B^{ij} = \frac{1}{2}\left(A^{ij} + A^{ji}\right)$$

$$\Rightarrow B^{ji} = \frac{1}{2}\left(A^{ji} + A^{ij}\right)$$

$$= \frac{1}{2}\left(A^{ij} + A^{ji}\right)$$

$$= B^{ij}$$

$\therefore B^{ij}$ is symm.

HW

EX) Show that a second rank covariant tensor (or, a tensor of type (0,2)) is expressible as a sum of two tensors one of which is symmetric & the other is skew-symmetric.

8/ Fundamental Operations with Tensors –

It includes (a) Addition, (b) Subtraction, (c) Outer multiplication, (d) Contraction, (e) ~~inne~~ inner multiplication. (f) Quotient law : (Quotient law)

(a) **Addition** ! — (8)

The sum of two or more tensors of the same rank (or, order) & type (i.e., having the same no. of contravariant indices & the same no. of covariant indices) is also a tensor of the same rank & type.

Thus, if A^{ij}_{k} & B^{ij}_{k} are mixed tensors of rank (2,1) or, 3 then their sum

$C^{ij}_{k} = A^{ij}_{k} + B^{ij}_{k}$ is also a mixed tensor of the same rank (2,1) or 3.

Note 1 – Addition of tensors is commutative (ie, $A^{ij}_{k} + B^{ij}_{k} = B^{ij}_{k} + A^{ij}_{k}$)

& associative (ie, $\left(A^{ij}_{k} + B^{ij}_{k}\right) + D^{ij}_{k} = A^{ij}_{k} + \left(B^{ij}_{k} + D^{ij}_{k}\right)$)

(b) Subtraction :– The difference of two tensors of the same rank (ie, order) & type (ie, having the same no. of contravariant indices & the same no. of covariant indices) is also a tensor of the same rank & type (or, character)

ie, if A^{ij}_{k} & B^{ij}_{k} are mixed tensors of rank (2,1) or 3 then their difference $D^{ij}_{k} = A^{ij}_{k} - B^{ij}_{k}$ is also a mixed tensor of the same rank (2,1) or 3.

(d) **Outer multiplication:-**

The product of two tensors is a tensor whose rank (or, order) is the sum of the ranks of the given tensors, this product which involves (contains) ordinary multiplication of the components of the tensor, is called the 'outer product' of tensors.

Thus, $C^{ijk}_{lm} = A^{ij}_{l} B^{k}_{m}$ is the 'outer product' or, 'open product' or, outer multiplication of mixed tensors A^{ij}_{l} & B^{k}_{m}.

Note :- However, not every tensor

can be expressed as a product of two tensor of lower rank.

Note :- The 'Division' in the usual sense, one tensor by another tensor is not defined.

d) Contraction :-

eg, In the tensor (mixed) of rank 5 or, (3, 2)

A^{ijk}_{lm}, if we set (fix) $k = m$.

then we obtain

$$A^{ijk}_{lm} = A^{ij}_{l},$$ a mixed

tensor of rank $(5-2)=3$

Further, by setting $j=l$,

we obtain $A^{ij}_{l} = A^{i}$,

which is a tensor of rank 1

(or, a contravariant vector of rank 1).

If one contravariant & one covariant index (or, suffix) are set (made) equal in a tensor, then

The result indicates that a summation over equal indices is to be taken according to the summation convention. This resulting sum is a tensor of rank two less than that of the original tensor. This process is called 'contraction'