Date
30/09/2019

Lecture 4

Q1) **Solve:-**

$$x\left(\frac{d^2y}{dx^2}\right)+2\left(\frac{dy}{dx}\right)+(xy)/2=0,$$

in terms of Bessel's function.

HW
Ex Q2) $$x\left(\frac{d^2z}{dx^2}\right)-2\left(\frac{dz}{dx}\right)+xz=0,$$

by using the substitution

$$y=\frac{z}{x^{3/2}}.$$

Sol^n 1) Given $x\left(\frac{d^2y}{dx^2}\right)+2\left(\frac{dy}{dx}\right)+(xy)/2=0$

We assume that $z=y\sqrt{x}$.

so that $y=z/\sqrt{x}$.

$$y = \frac{z}{\sqrt{x}} \rightarrow (2)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^{-1/2} z\right)$$

$$= x^{-1/2} \frac{dz}{dx} - \frac{x^{-3/2}}{2} z$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(x^{-1/2} \frac{dz}{dx}\right)$$

$$- \frac{1}{2}\frac{d}{dx}\left(x^{-3/2} z\right)$$

$$= x^{-1/2} \cdot \frac{d^2z}{dx^2} - x^{-3/2} \frac{dz}{dx} + \frac{3}{4} x^{-5/2} z$$

Substituting the above values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (1), we get

$$x\left(x^{-1/2}\frac{d^2z}{dx^2} - x^{-3/2}\frac{dz}{dx} + \frac{3}{4}z\,x^{-5/2}\right)$$

$$+\,2\left(x^{-1/2}\frac{dz}{dx} - \frac{x^{-3/2}}{2}z\right)$$

$$+\,\frac{x}{2}\cdot\frac{z}{\sqrt{x}} = 0$$

or, $$x^{1/2}\frac{d^2z}{dx^2} + x^{-1/2}\frac{dz}{dx} - \frac{z}{4}x^{-3/2} + \frac{1}{2}x^{1/2}z = 0 \longrightarrow (3)$$

Multiplying both sides of (3) by $x^{3/2}$ we get

$$x^2\left(\frac{d^2z}{dx^2}\right) + x\left(\frac{dz}{dx}\right) + \left(\frac{x^2}{2} - \frac{1}{4}\right)z = 0 \longrightarrow (4)$$

Let $$u = \frac{x}{\sqrt{2}} \longrightarrow (5)$$

$$\therefore \frac{dz}{dx} = \frac{dz}{du}\cdot\frac{du}{dx} = \frac{1}{\sqrt{2}}\cdot\frac{dz}{du}, \longrightarrow (6)$$

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$$\therefore \frac{d}{dx} = \frac{1}{\sqrt{2}} \frac{d}{du} \rightarrow (7)$$

$$\therefore \frac{d^2 z}{dx^2} = \frac{d}{dx}\left(\frac{dz}{dx}\right)$$

$$= \frac{1}{\sqrt{2}} \frac{d}{du}\left(\frac{1}{\sqrt{2}} \frac{dz}{du}\right)$$

$$\therefore \frac{d^2 z}{dx^2} = \frac{1}{2} \frac{d^2 z}{du^2} \rightarrow (8)$$

Substituting the above values in (4), we get

$$\cancel{2} u^2 \times \frac{1}{\cancel{2}} \frac{d^2 z}{du^2} + u\cancel{\sqrt{2}} \times \frac{1}{\cancel{\sqrt{2}}} \frac{dz}{du} + \left(u^2 - \tfrac{1}{4}\right) z = 0$$

which is a Bessel equation of order $\left(\tfrac{1}{2}\right)$. Since

$\left(\frac{1}{2}\right)$ is a ~~positive~~ non-integer, hence the reqd. solution is given by

$$z = A J_{1/2}(u) + B J_{-1/2}(u)$$

$$\boxed{\text{or } y\sqrt{x} = A J_{1/2}\left(x/\sqrt{2}\right) + B J_{-1/2}\left(x/\sqrt{2}\right), \quad [\text{by } (2) \ \& \ (5)] \\ \text{where } A \ \& \ B \text{ are arbitrary constants.}}$$

Ex 3 / Verify that the Bessel function $J_{1/2}(x) = \sin x \times \left(\frac{2}{\pi x}\right)^{1/2}$ satisfies the Bessel eqn of order $\left(\frac{1}{2}\right)$.

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$$x^2\left(\frac{d^2y}{dx^2}\right)+x\left(\frac{dy}{dx}\right)+\left(x^2-\tfrac{1}{4}\right)y=0$$

Let $y=J_{1/2}(x)=\sqrt{\frac{2}{\pi}}\left(x^{-1/2}\sin x\right)$

$$\therefore \frac{dy}{dx}=$$

$$\frac{d^2y}{dx^2}=$$

$$\boxed{0=0}$$

✓

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Ex 3/ Show that

$$d\left(x^n J_n(ax)\right)/dx = a x^n J_{n-1}(ax)$$

& hence deduce that

$$d\left(x J_1(x)\right)/dx = x J_0(x).$$

Sol^n:- We know that

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n} \longrightarrow (1)$$

$$\therefore x^n J_n(ax) = x^n \cdot \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{ax}{2}\right)^{2m+n}$$

$$\text{or, } x^n J_n(ax) = \sum_{m=0}^{\infty} \frac{(-1)^m a^{2m+n} \cdot x^{2m+2n}}{2^{2m+n} m!\,\Gamma(n+m+1)}$$

$$\therefore \frac{d}{dx}\left\{x^n J_n(ax)\right\} = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot a^{2m+n}\, 2(m+n) \cdot x^{2m+2n-1}}{2^{2m+n} \cdot m!\,\Gamma(n+m+1)} \longrightarrow (2)$$

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From 1

$$J_{n-1}(ax) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m)} \left(\frac{ax}{2}\right)^{2m+n-1}$$

$$a x^n J_{n-1}(ax) = \sum_{m=0}^{\infty} \frac{(-1)^m a^{2m+n} x^{2m+2n-1} \times 2(m+n)}{2^{2m+n-1} m!\,\Gamma(n+m) \times 2(m+n)}$$

$$\therefore a x^n J_{n-1}(ax) = \sum_{m=0}^{\infty} \frac{(-1)^m a^{2m+n} \cdot \times 2(m+n)\, x^{2m+2n-1}}{2^{2m+n} \cdot m!\,\Gamma(n+m+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m a^{2m+n} \times 2(m+n)\, x^{2m+2n-1}}{2^{2m+n} m!\,\Gamma(n+m+1)} \rightarrow 3$$

$\therefore$ from eq^n 2 & 3, we get

$$\frac{d}{dx}\{x^n J_n(ax)\} = a x^n J_{n-1}(x) \rightarrow 4$$

Putting $n=1$, $a=1$ in eq^n (4), we get

$$\boxed{\frac{d}{dx}\left(x J_1(x)\right) = x J_0(x)}$$

** Ex 5/ Show that

$$\int_0^1 \frac{u J_0(xu)}{(1-u^2)^{1/2}}\,du = \frac{\sin x}{x}$$

H.W
Ex 6/ Show that

$$\int_0^{\pi/2} J_1(z\cos\theta)\,d\theta = \frac{1-\cos z}{z}$$

Ex 7/ Prove that

$$J_n(x) = \frac{x^n}{2^{n-1}\Gamma(n)} \int_0^{\pi/2} \sin\theta \cos^{(2n-1)}\theta\, J_0(x\sin\theta)\,d\theta$$

where $n > -1/2$

5) Hint: we have

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2}$$

$$- \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$\therefore \int_0^1 \frac{u J_0(xu)}{(1-u^2)^{1/2}}\, du$$

$$= \int_0^1 \frac{u}{(1-u^2)^{1/2}} \left[1 - \frac{x^2}{4} u^2 + \frac{x^4}{4 \cdot 16} u^4 - \cdots \right] du$$

$$= \int_{\theta=0}^{\pi/2} \frac{\sin\theta}{\cancel{\cos\theta}}\, d\theta - \frac{x^2}{4} \int_0^{\pi/2} \sin^3\theta\, d\theta$$

$$+ \frac{x^4}{64} \int_0^{\pi/2} \sin^5\theta\, d\theta - \cdots$$

On putting $u = \sin\theta$

$\therefore du = \cos\theta\, d\theta$

Limits

when $u = 0,\ \theta = 0$

$u = 1,\ \theta = \pi/2$

$$= \left[-\cos\theta\right]_0^{\pi/2} - \frac{x^2}{\cancel{4}_2} \times \frac{\cancel{2}}{3} + \frac{x^4}{\cancel{64}_{\cancel{32}\,8}} \cdot \frac{\cancel{4}}{5} \cdot \frac{\cancel{2}}{3} - \cdots$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

[Using Wallis's formula (?) of integral calculus]

$$= \frac{1}{x}\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}$$

$$= \frac{\sin x}{x}.$$

Recurrence Relations

for $J_n(x)$

Prove that

I) $\dfrac{d}{dx}\left\{x^n J_n(x)\right\} = x^n J_{n-1}(x)$

II) $\dfrac{d}{dx}\left\{x^{-n} J_n(x)\right\} = -x^{-n} J_{n+1}(x)$

III) $J_n'(x) = J_{n-1}(x) - (n/x) J_n(x)$

or, $x J_n' = -n J_n + x J_{n-1}$

IV) $J_n'(x) = (n/x) J_n(x) - J_{n+1}(x)$

or, $x J_n' = n J_n - x J_{n+1}$

V) $J_n'(x) = \frac{1}{2}\left\{J_{n-1}(x) - J_{n+1}(x)\right\}$

or, $J_{n-1} - J_{n+1} = 2 J_n'$

VI) $J_{n-1}(x) + J_{n+1}(x) = \left(\frac{2n}{x}\right) J_n(x)$

or, $2n J_n = x\left(J_{n-1} + J_{n+1}\right)$

—x—