Theorem

If the BVP given by (1) to (4) has only the trivial solution $y(x) \equiv 0$, then the operator L has a unique Green's f^n. $G(x,t)$.

Proof: Suppose that $y_1(x), y_2(x), \cdots y_n(x)$ be L.I. sol^{n}s. of the eqn. (1). Then by virtue of property (i) of Green's f^n, the unknown Green's f^n. $G(x,t)$ must have the following representation on the intervals $[a,t)$ and $(t,b]$.

$$G(x,t) = \begin{cases} a_1 y_1(x) + a_2 y_2(x) + \cdots + a_n y_n(x) & a \le x < t \\ b_1 y_1(x) + b_2 y_2(x) + \cdots + b_n y_n(x) & t < x \le b \end{cases} \quad \text{---}(A)$$

where $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are some f^ns. of t.

Again by virtue of property (ii), the continuity of $G(x,t)$ and of its first $(n-2)$ derivatives w.r.t. x at the point $x=t$ gives rise to the following relations:

$$\left[b_1 y_1(t) + b_2 y_2(t) + \cdots + b_n y_n(t)\right] - \left[a_1 y_1(t) + \cdots + a_n y_n(t)\right] = 0 \quad \text{---}(A_1)$$

$$\left[b_1 y_1'(t) + \cdots + b_n y_n'(t)\right] - \left[a_1 y_1'(t) + \cdots + a_n y_n'(t)\right] = 0 \quad \text{---}(A_2)$$

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$$\left[b_1 y_1^{(n-2)}(t) + \cdots + b_n y_n^{(n-2)}(t)\right] - \left[a_1 y_1^{(n-2)}(t) + \cdots + a_n y_n^{(n-2)}(t)\right] = 0 \quad \text{---}(A_{n-1})$$

Finally by property (iii),

$$\left[b_1 y_1^{(n-1)}(t) + \cdots + b_n y_n^{(n-1)}(t)\right] - \left[a_1 y_1^{(n-1)}(t) + \cdots + a_n y_n^{(n-1)}(t)\right] = -\frac{1}{p_0(t)} \quad \text{---}(A_n)$$

Let $c_k(t) = b_k(t) - a_k(t)$ where $k = 1, 2, \ldots n$ — (B)

Rewriting $(A_1), (A_2), \cdots (A_n)$ with the help of (B), we obtain a system of linear equations in $c_k(t)$:

$$c_1 y_1(t) + \cdots + c_n y_n(t) = 0 \quad \cdots (B_1)$$

$$c_1 y_1'(t) + \cdots + c_n y_n'(t) = 0 \quad \cdots (B_2)$$

$$\cdots$$

$$c_1 y_1^{(n-2)}(t) + \cdots + c_n y_n^{(n-2)}(t) = 0 \quad \cdots (B_{n-1})$$

and $$c_1 y_1^{(n-1)}(t) + \cdots + c_n y_n^{(n-1)}(t) = -\frac{1}{p_0(t)} \quad \cdots (B_n)$$

The determinant D of the system $(B_1), \cdots (B_n)$ is

$$D = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

$$= W(y_1, y_2, \cdots, y_n) \quad \text{— (C)}$$

where $W(y_1, y_2, \cdots, y_n)$ is the Wronskian of the functions $y_1, y_2, \cdots y_n$. Since $y_1, y_2, \cdots y_n$ are taken as L.I., it follows that $D = W(y_1, y_2, \cdots, y_n) \neq 0$ — (D)

Sine $D \neq 0$, it follows that the system of equations $(B_1), \cdots (B_n)$ possess a unique solution for c_k, $k = 1, 2, \cdots n$.

Now to find $a_k(t)$ and $b_k(t)$, we use the property (4) of Green's function. First we write $V_k(y)$ in the form

$$V_k(y) = P_k(y) + q_k(y) \quad \text{-- (E)}$$

where $P_k(y) = \alpha_k y(a) + \alpha_k^{(1)} y'(a) + \cdots + \alpha_k^{(n-1)} y^{(n-1)}(a)$ -- (E_1)

and $q_k(y) = \beta_k y'(a) + \beta_k^{(1)} y'(b) + \cdots + \beta_k^{(n-1)} y^{(n-1)}(b)$ -- (E_2)

Now by virtue of eqn. (7) of the property (iv), we have

$$\begin{aligned}
V_k(G) &= P_k(G) + q_k(G) \\
&= \alpha_k G(a,t) + \alpha_k^{(1)} G'(a,t) + \cdots + \alpha_k^{(n-1)} G^{(n-1)}(a,t) \\
&\quad + \beta_k G(b,t) + \beta_k^{(1)} G'(b,t) + \cdots + \beta_k^{(n-1)} G^{(n-1)}(b,t) \\
&= \alpha_k \left[a_1 y_1(a) + a_2 y_2(a) + \cdots + a_n y_n(a)\right] \\
&\quad + \alpha_k^{(1)} \left[a_1 y_1'(a) + a_2 y_2'(a) + \cdots + a_n y_n'(a)\right] \\
&\quad + \cdots \\
&\quad + \beta_k \left[b_1 y_1(b) + b_2 y_2(b) + \cdots + b_n y_n(b)\right] \\
&\quad + \beta_k^{(1)} \left[b_1 y_1'(b) + b_2 y_2'(b) + \cdots + b_n y_n'(b)\right] \\
&\quad + \cdots \\
&= a_1 \left[\alpha_k y_1(a) + \alpha_k^{(1)} y_1'(a) + \cdots\right] \\
&\quad + a_2 \left[\alpha_k y_2(a) + \alpha_k^{(1)} y_2'(a) + \cdots\right] + \cdots \\
&\quad + b_1 \left[\beta_k y_1(b) + \beta_k^{(1)} y_1'(b) + \cdots\right] \\
&\quad + b_2 \left[\beta_k y_2(b) + \beta_k^{(1)} y_2'(b) + \cdots\right] \\
&= a_1 P_k(y_1) + \cdots + a_n P_k(y_n) + b_1 q_k(y_1) + \cdots + b_n q_k(y_n) \\
&= 0, \quad k = 1, 2, \ldots, n \quad \text{— (F)}
\end{aligned}$$

From eqn. (B), $a_k = b_k - c_k$, $k = 1, 2, \dots n$ — (G)

Using (G), (F) becomes

$$(b_1 - c_1) p_k(\nu_1) + \dots + (b_n - c_n) p_k(\nu_n) + b_1 q_k(\nu_1) + \dots$$
$$+ \dots + b_n q_k(\nu_n) = 0$$

$$\Rightarrow b_1 \left[p_k(\nu_1) + q_k(\nu_1) \right] + \dots + b_n \left[p_k(\nu_n) + q_k(\nu_n) \right]$$
$$= c_1 p_k(\nu_1) + \dots + c_n p_k(\nu_n)$$

$$\Rightarrow b_1 V_k(\nu_1) + \dots + b_n V_k(\nu_n) = c_1 p_k(\nu_1) + \dots + c_n p_k(\nu_n)$$
$$k = 1, 2, \dots, n \quad \text{— (H)}$$

Here (H) is a system of n linear equations for determination of $b_1, b_2, \dots, b_n$. Since we have assumed the linear independence of the forms $V_1, V_2, \dots, V_n$, it follows that the determinant of the system (H) is non-zero i.e.

$$\begin{vmatrix} V_1(\nu_1) & V_1(\nu_2) & \dots & V_1(\nu_n) \\ V_2(\nu_1) & V_2(\nu_2) & & V_2(\nu_n) \\ - & - & - & - \\ V_n(\nu_1) & V_n(\nu_2) & \dots & V_n(\nu_n) \end{vmatrix} \neq 0 \quad \text{--- (I)}$$