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ASSIGNMENT - 1

Mathematical Methods

1) Given ODE : $x \cdot \left(\frac{d^2y}{dx^2}\right) + 2 \cdot \left(\frac{dy}{dx}\right) + \frac{(xy)}{2} = 0 \quad \ldots \text{(i)}$

Substitute : $z = y\sqrt{x} \quad \Rightarrow \quad y = \frac{z}{\sqrt{x}} = x^{-1/2} \cdot z \quad \ldots \text{(ii)}$

Then, $\frac{dy}{dx} = \frac{d}{dx}\left(x^{-1/2} \cdot z\right)$

$$\Rightarrow \frac{dy}{dx} = x^{-1/2} \cdot \frac{dz}{dx} - \frac{1}{2} \times x^{-3/2} \cdot z \quad \ldots \text{(iii)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(x^{-1/2} \cdot \frac{dz}{dx} - \frac{1}{2} \times x^{-3/2} \cdot z\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = x^{-1/2} \times \frac{d^2z}{dx^2} - x^{-3/2} \cdot \frac{dz}{dx} + \frac{3}{4} x^{-5/2} \cdot z \quad \ldots \text{(iv)}$$

Substituting (ii), (iii), (iv) in (i), we get:

$$x\left(x^{-1/2} \cdot \frac{d^2z}{dx^2} - x^{-3/2} \cdot \frac{dz}{dx} + \frac{3}{4} \cdot x^{-5/2} \cdot z\right)$$

$$+ 2 \times \left(x^{-1/2} \cdot \frac{dz}{dx} - \frac{1}{2} \times x^{-3/2} \cdot z\right) + \frac{x}{2} \times \frac{z}{\sqrt{x}} = 0$$

$$\Rightarrow x^{1/2} \cdot \frac{d^2z}{dx^2} - x^{-1/2} \cdot \frac{dz}{dx} + \frac{3}{4} \cdot x^{-3/2} \cdot z + 2x^{-1/2} \cdot \frac{dz}{dx} - x^{-3/2} \cdot z + \frac{1}{2} x^{1/2} \cdot z = 0$$

$$\Rightarrow x^{1/2} \cdot \frac{d^2z}{dx^2} + x^{-1/2} \cdot \frac{dz}{dx} - \frac{1}{4} x^{-3/2} \cdot z + \frac{1}{2} x^{1/2} \cdot z = 0 \quad \ldots \text{(v)}$$

Multiplying both sides of (v) by $x^{3/2}$, we get:

$$x^2 \cdot \left(\frac{d^2z}{dx^2}\right) + x \cdot \left(\frac{dz}{dx}\right) + \left(\frac{x^2}{2} - \frac{1}{4}\right) z = 0 \quad \ldots \text{(vi)}$$

Now, let $u = \frac{x}{\sqrt{2}}$. Then, $\frac{dz}{dx} = \frac{dz}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{2}} \cdot \frac{dz}{du}$

And, $\frac{d^2z}{dx^2} = \frac{d}{dx}\left(\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{1}{\sqrt{2}} \cdot \frac{dz}{du}\right)$

$$\Rightarrow \frac{d^2z}{dx^2} = \frac{d}{du}\left(\frac{1}{\sqrt{2}} \times \frac{dz}{du}\right) \times \frac{du}{dx} = \frac{1}{2} \times \frac{d^2z}{du^2}$$

Substituting these in (vi), we get:

$$(u\sqrt{2})^2 \times \frac{1}{2} \cdot \frac{d^2z}{du^2} + (u\sqrt{2}) \times \frac{1}{\sqrt{2}} \times \frac{dz}{du} + \left(\frac{(u\sqrt{2})^2}{2} - \frac{1}{4}\right) z = 0$$

$$\Rightarrow u^2 \cdot \frac{d^2z}{du^2} + u \cdot \frac{dz}{du} + \left(u^2 - \frac{1}{4}\right) z = 0 \quad \cdots\cdots \text{(vii)}$$

Now, (vii) is the Bessel's equation of order $\frac{1}{2}$.

Since $\frac{1}{2}$ is not an integer, the solution of (vii) is given by:

$$z = A \times J_{\frac{1}{2}}(u) + B \times J_{-\frac{1}{2}}(u)$$

As $y = \frac{z}{\sqrt{x}} \Rightarrow z = y\sqrt{x}$ and $u = \frac{x}{\sqrt{2}}$, we have:

$$y\sqrt{x} = A \times J_{\frac{1}{2}}(x/\sqrt{2}) + B \times J_{-\frac{1}{2}}(x/\sqrt{2})$$

$$\Rightarrow y = \frac{A}{\sqrt{x}} \times J_{\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right) + \frac{B}{\sqrt{x}} \times J_{-\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right)$$; where A, B are arbitrary constants

[Ans.]

This is the solution to eqn. (i).

2) To Prove : $\displaystyle\int_0^1 \frac{u\, J_0(xu)}{(1-u^2)^{1/2}}\, du = \frac{\sin x}{x}$

We know, $J_0(x) = 1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 \cdot 4^2} - \dfrac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

$$\Rightarrow J_0(xu) = 1 - \frac{x^2 u^2}{2^2} + \frac{x^4 u^4}{2^2 \cdot 4^2} - \frac{x^6 u^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$\therefore \int_0^1 \frac{u\, J_0(xu)}{(1-u^2)^{1/2}} \cdot du = \int_0^1 \frac{u}{\sqrt{1-u^2}} \cdot \left[1 - \frac{x^2 u^2}{2^2} + \frac{x^4 u^4}{2^2 \cdot 4^2} - \cdots\right] \cdot du$$

Substitute : $u = \sin\theta \Rightarrow du = \cos\theta \cdot d\theta$

So, $\displaystyle\int_0^1 \frac{u}{\sqrt{1-u^2}} \times \left[1 - \frac{x^2 u^2}{2^2} + \frac{x^4 u^4}{2^2 \cdot 4^2} - \cdots\right] \cdot du$

when $u = 0 \to \theta = 0$
when $u = 1 \to \theta = \frac{\pi}{2}$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \times \left[1 - \frac{x^2}{2^2} \cdot \sin^2\theta + \frac{x^4}{2^2 \cdot 4^2} \sin^4\theta - \cdots\right] \cdot \cos\theta \cdot d\theta$$

$$= \int_0^{\pi/2} \sin\theta \cdot d\theta - \frac{x^2}{2^2} \int_0^{\pi/2} \sin^3\theta \cdot d\theta + \frac{x^4}{2^2 \cdot 4^2} \int_0^{\pi/2} \sin^5\theta \cdot d\theta - \cdots \quad \cdots (i)$$

Now, using Wallis Formula, we know :

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{(n-1)(n-3) \cdots 4 \cdot 2}{n(n-2)(n-4) \cdots 5 \cdot 3} \quad ; \text{ for } n = 3, 5, 7, 9, \ldots$$

Hence, (i) reduces to :

$$\left[-\cos\theta\right]_0^{\pi/2} - \frac{x^2}{4} \times \frac{2}{3} + \frac{x^4}{2^2 \cdot 4^2} \times \frac{4 \times 2}{5 \times 3} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} \times \frac{6 \times 4 \times 2}{7 \times 5 \times 3}$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$\therefore \int_0^1 \frac{u \cdot J_0(xu)}{(1-u^2)^{1/2}} \cdot du = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$= \frac{1}{x} \times \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \frac{1}{x} \times (\sin x)$$

[From the Taylor series expansion of $\sin x$ about $x=0$]

$$= \frac{\sin x}{x}$$

[Proved.]

3) To Prove : $J_n(x) = \dfrac{x^n}{2^{n-1}\Gamma(n)} \displaystyle\int_0^{\pi/2} \sin\theta \cdot \cos^{2n-1}(\theta) \cdot J_0(x\sin\theta)\, d\theta$;

where $n > \frac{-1}{2}$.

We know, $J_0(x) = 1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2.4^2} - \dfrac{x^6}{2^2.4^2.6^2} + \cdots$

$$\Rightarrow J_0(x\sin\theta) = 1 - \frac{x^2\sin^2\theta}{2^2} + \frac{x^4 \cdot \sin^4\theta}{2^2 \cdot 4^2} - \frac{x^6\sin^6\theta}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

Then, we have :

$$\frac{x^n}{2^{n-1} \times \Gamma(n)} \int_0^{\pi/2} \sin\theta \cdot \cos^{2n-1}\theta \cdot J_0(x\sin\theta) \cdot d\theta$$

$$= \frac{x^n}{2^{n-1} \times \Gamma(n)} \times \int_0^{\pi/2} \sin\theta \cdot \cos^{(2n-1)}\theta \cdot \left[1 - \frac{x^2}{2^2}\sin^2\theta + \frac{x^4}{2^2 . 4^2}\sin^4\theta - \cdots\right] \cdot d\theta$$

$$= \frac{x^n}{2^{n-1} \times \Gamma(n)} \times \left[\int_0^{\pi/2} \sin\theta \cdot \cos^{2n-1}\theta \cdot d\theta - \frac{x^2}{2^2}\int_0^{\pi/2} \sin^3\theta \cdot \cos^{2n-1}\theta \cdot d\theta + \frac{x^4}{2^2 \cdot 4^2}\int_0^{\pi/2} \sin^5\theta \cdot \cos^{(2n-1)}\theta \cdot d\theta - \cdots\right]$$

Multiplying and dividing by 2, we get :

$$= \frac{x^n}{2^n \times \Gamma(n)} \times \left[2\int_0^{\pi/2} \sin\theta \cdot \cos^{2n-1}\theta \cdot d\theta - \frac{x^2}{2^2} \times 2\int_0^{\pi/2} \sin^3\theta \cdot \cos^{2n-1}\theta \cdot d\theta + \frac{x^4}{2^2 \cdot 4^2} \times 2\int_0^{\pi/2} \sin^5\theta \cdot \cos^{2n-1}\theta \cdot d\theta \; - \cdots \right]$$

$$= \frac{x^n}{2^n \times \Gamma(n)} \times \left[\beta(1,n) - \frac{x^2}{2^2} \times \beta(2,n) + \frac{x^4}{2^2 \times 4^2} \times \beta(3,n) - \cdots \right]$$

[We reached this step, since:
From an equivalent definition of Beta function,
$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta \;]$$

$$= \frac{x^n}{2^n \times \Gamma(n)} \times \left[\frac{\Gamma(n) \times \Gamma(1)}{\Gamma(n+1)} - \frac{x^2}{2^2} \times \frac{\Gamma(n) \times \Gamma(2)}{\Gamma(n+2)} + \frac{x^4}{2^2 \cdot 4^2} \times \frac{\Gamma(n) \times \Gamma(3)}{\Gamma(n+3)} \; - \cdots \right]$$

[$\because$ we know, $\beta(m,n) = \frac{\Gamma(m) \times \Gamma(n)}{\Gamma(m+n)}$]

$$= \left(\frac{x}{2}\right)^n \times \left[\frac{\Gamma(1)}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \times \frac{\Gamma(2)}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \times \frac{1}{1^2 \times 2^2} \times \frac{\Gamma(3)}{\Gamma(n+3)} - \cdots \right]$$

$$= \left(\frac{x}{2}\right)^n \times \left[(-1)^0 \times \frac{1}{0! \times \Gamma(n+0+1)} + (-1)^1 \times \left(\frac{x}{2}\right)^2 \times \frac{1}{1! \times \Gamma(n+1+1)} + (-1)^2 \times \left(\frac{x}{2}\right)^4 \times \frac{\Gamma(3)}{\{\Gamma(3)\}^2 \times \Gamma(n+2+1)} + (-1)^3 \times \left(\frac{x}{2}\right)^6 \times \frac{\Gamma(4)}{\{\Gamma(4)\}^2 \times \Gamma(n+3+1)} + \cdots \right]$$

$$= \sum_{r=0}^{\infty} (-1)^r \times \left(\frac{x}{2}\right)^{n+2r} \times \frac{1}{r! \times \Gamma(n+r+1)} = \underline{\underline{J_n(x)}}$$

Hence, $J_n(x) = \frac{x^n}{2^{n-1} \times \Gamma(n)} \times \int_0^{\pi/2} \sin\theta \cdot \cos^{2n-1}(\theta) \cdot J_0(x\sin\theta) \cdot d\theta \; ; \; n > -\frac{1}{2}$

[Proved]

4) To Prove : $\int_0^x t \cdot [J_n(t)]^2 \cdot dt = \frac{x^2}{2} \times [J_n^2(x) - J_{n-1}(x) \cdot J_{n+1}(x)]$ .... (i)

$$LHS = \int_0^x t \cdot [J_n(t)]^2 \cdot dt = \int_0^x [J_n(t)]^2 \cdot t \cdot dt$$

$$= \left[(J_n(t))^2 \times \frac{t^2}{2} \right]_0^x - \int_0^x \frac{t^2}{2} \times 2 J_n(t) \cdot J_n'(t) \cdot dt$$

[By using integration by parts. Taking $[J_n(t)]^2$ as first function]

$$= \frac{x^2}{2} \times J_n^2(x) - \int_0^x t^2 \times J_n(t) \cdot J_n'(t) \cdot dt \quad \text{.... (ii)}$$

Now, we solve for : $I = \int_0^x t^2 \cdot J_n(t) \cdot J_n'(t) \cdot dt$,

$$I = \int_0^x t^2 \cdot J_n(t) \cdot J_n'(t) \cdot dt$$

$$= \int_0^x t^2 \cdot J_n(t) \times \frac{1}{2} \times [J_{n-1}(t) - J_{n+1}(t)] \cdot dt$$

[$\because$ From Recurrence formula, $J_n'(x) = \frac{1}{2} \times [J_{n-1}(x) - J_{n+1}(x)]$]

$$= \frac{1}{2} \int_0^x \left(t^2 \cdot J_n(t) \cdot J_{n-1}(t) - t^2 \cdot J_n(t) \cdot J_{n+1}(t) \right) \cdot dt$$

$$= \frac{1}{2} \times \int_0^x \left[\left(t^{n+1} \times J_n(t)\right) \times \left(t^{-(n-1)} \times J_{n-1}(t)\right) - \left(t^{n+1} \times J_{n+1}(t)\right) \times \left(t^{-(n-1)} \times J_n(t)\right) \right] \cdot dt$$

[$\because t^2 = t^{n+1} \times t^{1-n} = t^{n+1} \times t^{-(n-1)}$]

Now, by Recurrence formula,

we know :

$$\frac{d}{dx}[x^n \cdot J_n(x)] = x^n \cdot J_{n-1}(x) \quad \text{and,} \quad \frac{d}{dx}[x^{-n} \cdot J_n(x)] = -x^{-n} \cdot J_{n+1}(x)$$

So, from here, we get :

$$t^{n+1} \times J_n(t) = \frac{d}{dt}[t^{n+1} \times J_{n+1}] \quad \text{and,} \quad -t^{-(n-1)} \times J_n(t) = \frac{d}{dt}[t^{-(n-1)} \times J_{n-1}(t)]$$

Using these in the expression of I, we get :

$$I = \frac{1}{2} \times \int_0^x \left[\frac{d}{dt}\left\{ t^{n+1} J_{n+1}(t) \right\} \times \left(t^{-(n-1)} \times J_{n-1}(t) \right) + \left(t^{n+1} J_{n+1}(t) \right) \times \frac{d}{dt}\left\{ t^{-(n-1)} \times J_{n-1}(t) \right\} \right] \cdot dt$$

$$\Rightarrow I = \frac{1}{2} \times \int_0^x \frac{d}{dt}\left\{ \left(t^{n+1} J_{n+1}(t) \right) \times \left(t^{-(n-1)} J_{n-1}(t) \right) \right\} \cdot dt$$

[$\because$ Using the u-v rule of derivative, $u'v + uv' = (uv)'$]

$$\Rightarrow I = \frac{1}{2} \times \int_0^x \frac{d}{dt}\left\{ t^2 \cdot J_{n+1}(t) \cdot J_{n-1}(t) \right\} \cdot dt$$

[$\because t^{n+1} \times t^{-(n-1)} = t^{n+1-n+1} = t^2$]

$$\Rightarrow I = \frac{1}{2} \times \int_0^x d\left[t^2 \cdot J_{n+1}(t) \cdot J_{n-1}(t) \right]$$

$$= \frac{1}{2} \times \left[t^2 \times J_{n+1}(t) \cdot J_{n-1}(t) \right]_0^x = \frac{1}{2} \times x^2 J_{n+1}(x) \times J_{n-1}(x) \quad \cdots \text{(iii)}$$

Putting the expression of I from (iii) in (ii), we get :-

$$LHS = \frac{x^2}{2} \times J_n^2(x) - I = \frac{x^2}{2} \times J_n^2(x) - \frac{x^2}{2} \times J_{n-1}(x) \times J_{n+1}(x)$$

$$= \frac{x^2}{2} \times \left[J_n^2(x) - J_{n-1}(x) \times J_{n+1}(x) \right] = RHS.$$

Hence, $\int_0^x t \cdot [J_n(t)]^2 \cdot dt = \frac{x^2}{2} \times \left[J_n^2(x) - J_{n-1}(x) \cdot J_{n+1}(x) \right]$ [Proved.]

5) By Recurrence formula, we know:

$$J_n(x) = \frac{x}{2n} \cdot [J_{n-1}(x) + J_{n+1}(x)]$$

$$\Rightarrow J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \quad \dots \text{(i)}$$

Putting $n = 1, 2, 3$ in (i) we get:

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \quad \dots \text{(ii)}$$

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x) \quad \dots \text{(iii)}$$

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x) \quad \dots \text{(iv)}$$

Substituting (ii) in (iii),

$$J_3(x) = \frac{4}{x} \times \left\{ \frac{2}{x} J_1(x) - J_0(x) \right\} - J_1(x)$$

$$\Rightarrow J_3(x) = \left(\frac{8}{x^2} - 1 \right) J_1(x) - \frac{4}{x} J_0(x) \quad \dots \text{(v)}$$

Substituting (v) and (ii) in (iv), we get :-

$$J_4(x) = \frac{6}{x} \times \left[\left(\frac{8}{x^2} - 1 \right) J_1(x) - \frac{4}{x} J_0(x) \right] - \left[\frac{2}{x} J_1(x) - J_0(x) \right]$$

$$\Rightarrow J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x} \right) J_1(x) + \left(1 - \frac{24}{x^2} \right) J_0(x) \qquad \text{[Ans.]}$$