By virtue of (I), we see that the system of equations (H) possess a unique solution in $b_1(t), b_2(t), \ldots, b_n(t)$. Already we have seen that unique values of $c_1(t), \ldots, c_n(t)$ exist. Hence (B) shows that $a_1(t), \ldots, a_n(t)$ are defined uniquely.

Thus we have established the existence and uniqueness of Green's function $G(x,t)$. We have also indicated a procedure for constructing the Green's function in the above proof of the theorem.

Solved example based on construction of Green's function

Ex-1 Find the Green's function of the BVP

$$y''(0), \quad y(0) = y(l) = 0$$

Sol^n: The G.S. of $y'' = 0$ —(1) with $y(0) = 0$ —(2) and $y(l) = 0$ —(3) is $y = Ax + B$ and putting B.C, the sol^n is $y(x) = 0$. $\therefore$ Only trivial sol^n exists. So Green's f^n will be unique.

By property (i), Greens f^n is assumed to have the form

$$G(x,t) = \begin{cases} a_1 x + a_2 & 0 \le x < t \\ b_1 x + b_2 & t < x \le l \end{cases}$$

The proposed Green's function must satisfy the following three properties:

(i) $G(x,t)$ is continuous at $x = t$ i.e.

$$b_1 t + b_2 = a_1 t + a_2$$

$$\Rightarrow (b_1 - a_1) t + (b_2 - a_2) = 0 \quad \text{—— (4)}$$

(ii) The derivative of G has a discontinuity of magnitude $-\frac{1}{p_0(t)}$ at the point $x=t$, where $p_0(x)$ = coefficient of the highest order derivative in Eq. (1) $= 1$

$$\therefore \left(\frac{\partial G}{\partial x}\right)_{x=t+0} - \left(\frac{\partial G}{\partial x}\right)_{x=t-0} = -1$$

$$\Rightarrow \quad b_1 - a_1 = -1 \quad \text{——} \quad (5)$$

(iii) $G(x,t)$ must satisfy the B.C. (2) and (3),

$$\left.\begin{array}{lll} & G(0,t)=0 & \therefore a_2 = 0 \\ \text{and} & G(l,t)=0 & \therefore b_1 l + b_2 = 0 \end{array}\right\} \text{——} (6)$$

Solving (4), (5) and (6),

$$a_2 = 0,\ b_2 = t,\ b_1 = -\frac{t}{l},\ a_1 = 1 - \frac{t}{l}$$

$$\therefore a_1 x + a_2 = \left(1 - \frac{t}{l}\right)x = \frac{x}{l}(l-t)$$

$$b_1 x + b_2 = -\frac{t}{l}x + t = \frac{t}{l}(l-x)$$

$$G(x,t) = \begin{cases} \frac{x}{l}(l-t) & 0 \le x < t \\ \frac{t}{l}(l-x) & t < x \le l \end{cases}$$