ASSIGNMENT 6

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Mathematical Methods

1) Given ODE: $2x^2 \cdot \frac{d^2y}{dx^2} - x \cdot \frac{dy}{dx} + (1-x^2)y = 0$ .... (i)

Expressing it in the form of: $y'' + P(x) \cdot y' + Q(x) \cdot y = 0$,

we get:
$$\frac{d^2y}{dx^2} - \frac{1}{2x} \cdot \frac{dy}{dx} + \frac{(1-x^2)}{2x^2} y = 0$$

So, $P(x) = \frac{-1}{2x}$ and $Q(x) = \frac{(1-x^2)}{2x^2}$

Clearly, $P(x)$ and $Q(x)$ are not defined at $x=0$, and so are not analytic about $x=0$.

So, $x=0$ is not an ordinary point of (i).

Now, $xP(x) = \frac{-1}{2}$ and $x^2 Q(x) = \frac{(1-x^2)}{2}$ are both analytic about $x=0$. Hence, $x=0$ is ~~an ordi~~ a regular singular point of (i).

Now, let $y = x^k \sum_{n=0}^{\infty} a_n \cdot x^n = \sum_{n=0}^{\infty} a_n \cdot x^{(n+k)}$ be a soln. of (i).

$$\Rightarrow y' = \sum_{n=0}^{\infty} (n+k) \cdot a_n \cdot x^{n+k-1}, \quad \text{and}$$

$$y'' = \sum_{n=0}^{\infty} (n+k)(n+k-1) \cdot a_n \cdot x^{n+k-2}$$

Putting these in (i), we get:

$$2x^2 \cdot \sum_{n=0}^{\infty} (n+k) \cdot (n+k-1)\, a_n \cdot x^{n+k-2} - x \cdot \sum_{n=0}^{\infty} (n+k) \cdot a_n \cdot x^{n+k-1}$$
$$+ (1-x^2) \sum_{n=0}^{\infty} a_n \cdot x^{n+k} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \{2(n+k)(n+k-1) - (n+k)\} \cdot a_n \cdot x^{n+k} + \sum_{n=0}^{\infty} a_n \cdot x^{n+k} - \sum_{n=0}^{\infty} a_n \cdot x^{n+k+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n \times \{(n+k-1)(2n+2k-1)\} x^{n+k} - \sum_{n=2}^{\infty} a_{n-2} \times x^{n+k} = 0$$

$$\Rightarrow a_0 \times (k-1)(2k-1)\, x^k + a_1 \times k(2k+1)\, x^{k+1} + \sum_{n=2}^{\infty} \{a_n \times (n+k-1)(2n+2k-1) - a_{n-2}\} x^{n+k} = 0$$

Equating coefficients of powers of x on both sides :

Coeff. of x^k : $a_0 (k-1)(2k-1) = 0 \Rightarrow k = 1$ or $\frac{1}{2}$ $[\because a_0 \neq 0]$

$\downarrow$

Indicial equation : has distinct real roots not differing by an integer

Coeff. of x^{k+1} : $a_1 \times (k)(2k+1) = 0$. But, $k = 1$ or $\frac{1}{2}$ $\Rightarrow k(2k+1) \neq 0$.

So, $a_1 = 0$.

Coeff. of x^{n+k} (for $n \geq 2$) : $a_n \times (n+k-1)(2n+2k-1) - a_{n-2} = 0$

$$\Rightarrow a_n = \frac{a_{n-2}}{(n+k-1)(2n+2k-1)}$$

From here, as $a_1 = 0$, we get $a_1 = a_3 = a_5 = \cdots = a_{2m+1} = 0$.

Case 1: For $k = 1$: $a_n = \dfrac{a_{n-2}}{n(2n+1)} = \dfrac{a_0}{[2 \times 4 \times \cdots \times n] \times [5 \times 9 \times \cdots \times (2n+1)]}$

Then, $y_1 = x^1 \times \sum_{n=0}^{\infty} a_n \cdot x^n = a_0 x \times \left[1 + \frac{a_2}{a_0} x^2 + \frac{a_4}{a_0} x^4 + \cdots\right]$

$$\Rightarrow y_1 = a_0 x \times \left[1 + \frac{x^2}{2\times 5} + \frac{x^4}{2\times 4\times 5\times 9} + \cdots\right] \quad \cdots \text{(ii)}$$

<u>Case 2</u>: $k = \frac{1}{2}$: $a_n = \dfrac{a_{n-2}}{(n-\frac{1}{2})(2n)} = \dfrac{a_{n-2}}{n(2n-1)}$

$$\Rightarrow a_n = \frac{a_0}{[2\times 4\times\cdots\times n]\times[3\times 7\times\cdots(2n-1)]}$$

So, $y_2 = x^{\frac{1}{2}} \times \sum\limits_{n\geq 0}^{\infty} a_n \cdot x^n = a_0 x^{\frac{1}{2}} \times \left[1 + \frac{a_2}{a_0}x^2 + \frac{a_4}{a_0}x^4 + \cdots\right]$

$$\Rightarrow y_2 = a_0 x^{\frac{1}{2}} \times \left[1 + \frac{x^2}{2\times 3} + \frac{x^4}{2\times 4\times 3\times 7} + \cdots\right] \quad \cdots \text{(iii)}$$

The general solution of (i) is given by : $y = Ay_1 + By_2$.

From (ii) and (iii), we get:

$$y = Ax\left[1 + \frac{x^2}{2\times 5} + \frac{x^4}{2\times 4\times 5\times 9} + \cdots\right] + Bx^{\frac{1}{2}} \times \left[1 + \frac{x^2}{2\times 3} + \frac{x^4}{2\times 4\times 3\times 7} + \cdots\right]$$

<u>Ans.</u>

2) Given ODE : $\dfrac{d^2y}{dx^2} - y = 0 \quad \cdots \text{(i)}$

Substitution : $z = \dfrac{1}{x} \Rightarrow \dfrac{dz}{dx} = \dfrac{-1}{x^2}$

Then, $\dfrac{dy}{dx} = \dfrac{dy}{dz} \times \dfrac{dz}{dx} = \dfrac{-1}{x^2} \times \dfrac{dy}{dz}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dz} \times \frac{2}{x^3} - \frac{1}{x^2} \times \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{x^3} \times \frac{dy}{dz} - \frac{1}{x^2} \times \frac{d}{dz}\left(\frac{dy}{dz}\right) \times \frac{dz}{dx}$$

$$= \frac{2}{x^3} \times \frac{dy}{dz} + \frac{1}{x^4} \times \frac{d^2y}{dz^2} = 2z^3 \times \frac{dy}{dz} + z^4 \times \frac{d^2y}{dz^2}$$

$\therefore$ (i) transforms to : $z^4 \cdot \frac{d^2y}{dz^2} + 2z^3 \cdot \frac{dy}{dz} - y = 0$ ... (ii)

Now, let $y = z^k \sum_{n=0}^{\infty} a_n \cdot z^n = \sum_{n=0}^{\infty} a_n \cdot z^{(n+k)}$ be soln. to (ii)

$\Rightarrow y' = \sum_{n=0}^{\infty} (n+k) \cdot a_n \cdot z^{n+k-1}$, and

$y'' = \sum_{n=0}^{\infty} (n+k) \cdot (n+k-1) \cdot a_n \cdot z^{n+k-2}$

Putting these in (ii), we get :

$$\sum_{n=0}^{\infty} (n+k) \cdot (n+k-1) \cdot a_n \cdot z^{n+k+2} + \sum_{n=0}^{\infty} 2(n+k) a_n \cdot z^{n+k+2} - \sum_{n=0}^{\infty} a_n \cdot z^{n+k} = 0$$

$\Rightarrow$ Equating the coeff. of z^k and z^{k+1} from both sides :

Coeff. of z^k : $-a_0 = 0 \Rightarrow a_0 = 0$ [But we take $a_0 \neq 0$ for soln. to exist in this method]

Coeff. of z^{k+1} : $-a_1 = 0 \Rightarrow a_1 = 0$

Hence this method (Frobenius or power series method) fails here. This is because, $z = 0$ is not an ordinary or regular singular point of (ii).

3) We know, $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) \cdot t^n$

Putting $x = 0$: $(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0) \cdot t^n$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(0) \cdot t^n = 1 - \frac{1}{2} t^2 + \frac{\frac{1}{2} \times \frac{3}{2}}{2!} t^4 - \frac{\frac{1}{2} \times \frac{3}{2} \times \frac{5}{2}}{3!} t^6 + \cdots$$

Comparing coefficients of t^n from both sides :

(i) $P_n(0) = 0$ for $n =$ odd.

(ii) For $n =$ even, $P_n(0) = (-1)^{n/2} \times \dfrac{1 \times 3 \times 5 \times \cdots \times (n-1)}{2^{n/2} \times (n/2)!}$

$\Rightarrow P_n(0) = (-1)^{n/2} \times \dfrac{1 \times 2 \times 3 \times 4 \times \cdots \times (n-1) \times n}{2^{n/2} \times 2 \times 4 \times 6 \times \cdots \times n \times \left(\frac{n}{2}\right)!}$

$= (-1)^{n/2} \times \dfrac{n!}{2^{n/2} \times 2^{n/2} \times 1 \times 2 \times 3 \times \cdots \times \frac{n}{2} \times \left(\frac{n}{2}\right)!}$

$\Rightarrow P_n(0) = (-1)^{n/2} \times \dfrac{n!}{2^n \times \left\{\left(\frac{n}{2}\right)!\right\}^2}$ [Proved.]

4) Given: $P_n(x) = \dfrac{1}{2^n \times n!} \times \dfrac{d^n}{dx^n} (x^2-1)^n$

(i) $\int_{-1}^{1} P_n(x) \cdot dx = \dfrac{1}{2^n \times n!} \int_{-1}^{1} \dfrac{d^n}{dx^n} (x^2-1)^n \cdot dx = \dfrac{1}{2^n \times n!} \times \left[\dfrac{d^{(n-1)}}{dx^{(n-1)}} (x^2-1)^n\right]_{-1}^{1}$

$n \neq 0$

$\Rightarrow n \geq 1$

Now, as $(x^2-1)^n$ has $(x-1)$ and $(x+1)$ as factors with multiplicity n, so $(n-1)$th derivative of $(x^2-1)^n$ will have $(x-1)$ and $(x+1)$ both as factors.

So, $\left[\dfrac{d^{(n-1)}}{dx^{(n-1)}} (x^2-1)^n\right]_{-1}^{1} = 0 \Rightarrow \int_{-1}^{1} P_n(x) \cdot dx = 0$ for $n \neq 0$.

[Proved.]

(ii) $\int_{-1}^{1} P_0(x) \cdot dx = \int_{-1}^{1} 1 \cdot dx$ $\left[\because P_0(x) = \dfrac{1}{2^0 \times 0!} \times (x^2-1)^0 = 1\right]$

$= [x]_{-1}^{1} = 2$ [Proved.]