ASSIGNMENT 5

Name: Raushan Sharma

Mathematical Methods

Roll : 18MA20058

1) Consider a second order contravariant tensor of type (2,0) : A^{ij}.

Suppose that it is a skew-symmetric tensor w.r.t the pair of indices i and j in x^i coordinate system, i.e. $A^{ji} = -A^{ij}$.

We'll show that in some other coordinate system $\bar{x}^i$ also, this tensor is skew-symmetric w.r.t these two indices.

By transformation law, we have :

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial \bar{x}^j}{\partial x^l} \cdot A^{kl}$$

$$= \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial \bar{x}^j}{\partial x^l} \cdot (-A^{lk}) \qquad [\because A^{kl} = -A^{lk}]$$

$$= - \frac{\partial \bar{x}^j}{\partial x^l} \cdot \frac{\partial \bar{x}^i}{\partial x^k} \cdot A^{lk}$$

$$= -\bar{A}^{ji} \qquad \text{[from the transformation law of tensors]}$$

$\therefore \bar{A}^{ij} = -\bar{A}^{ji}$. So, the components A^{ij} of a tensor which is skew-symmetric w.r.t indices i and j in x^i coordinate system is again skew-symmetric w.r.t indices i and j in any other $\bar{x}^i$ coordinate system, as $\bar{A}^{ij} = -\bar{A}^{ji}$.

Now, consider a covariant tensor of second order A_{ij} which is skew-symmetric in x^i coordinate system, i.e. $A_{ji} = -A_{ij}$

We'll show that on transformation to $\bar{x}^i$ coordinate system also, it will remain skew-symmetric.

By transformation law,

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \times \frac{\partial x^l}{\partial \bar{x}^j} \times A_{kl}$$

$$= \frac{\partial x^k}{\partial \bar{x}^i} \times \frac{\partial x^l}{\partial \bar{x}^j} \times (-A_{lk}) \qquad [\because A_{kl} = -A_{lk}]$$

$$= -\frac{\partial x^l}{\partial \bar{x}^j} \times \frac{\partial x^k}{\partial \bar{x}^i} \times A_{lk}$$

$$= -\bar{A}_{ji} \qquad \therefore \bar{A}_{ij} = -\bar{A}_{ji}$$

So we have proved the statement for second order tensors skew-symmetric w.r.t two contravariant or two covariant indices. For higher order tensors also it can be proved in a similar way, and hence we conclude : If a tensor is skew-symmetric w.r.t a pair of indices in one coordinate system, it is so in every coordinate system. [Proved.]

2) Let the components of a tensor be $A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$. The tensor is an (r, s) type tensor. Suppose its components vanish in x^i coordinate system, i.e. $A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = 0$.

Then, by law of transformation of tensors,

$$\bar{A}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \times \frac{\partial \bar{x}^{i_2}}{\partial x^{k_2}} \times \cdots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{k_r}} \times \frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}} \times \frac{\partial x^{l_2}}{\partial \bar{x}^{j_2}} \times \cdots \times \frac{\partial x^{l_s}}{\partial \bar{x}^{j_s}} \times A^{k_1 k_2 \cdots k_r}_{l_1 l_2 \cdots l_s}$$

$$= 0 \qquad \left[\because A^{k_1 k_2 \cdots k_r}_{l_1 l_2 \cdots l_s} = 0\right]$$

So, components of the tensor vanish identically in any other coordinate system $\bar{x}^i$ also, as $\bar{A}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = 0$.

Hence, if the components of a tensor vanish in one-coordinate system, they vanish identically in all coordinate systems. [Proved.]

3) Contraction of a tensor: The process of getting a tensor of lower rank / order (reduced by 2) by putting a covariant index equal to a contravariant index, and performing the summation according to summation convention, is known as contraction.

For example, consider a mixed tensor A^{ijk}_{lm} of order 5, of type (3,2). Then by law of transformation,

$$\bar{A}^{ijk}_{lm} = \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^k}{\partial x^r} \times \frac{\partial x^s}{\partial \bar{x}^l} \times \frac{\partial x^t}{\partial \bar{x}^m} \times A^{pqr}_{st}$$

Put the covariant index l = contravariant index i.

$$\bar{A}^{ijk}_{im} = \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^k}{\partial x^r} \times \frac{\partial x^s}{\partial \bar{x}^i} \times \frac{\partial x^t}{\partial \bar{x}^m} \times A^{pqr}_{st}$$

$$= \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^k}{\partial x^r} \times \frac{\partial x^t}{\partial \bar{x}^m} \times \frac{\partial x^s}{\partial x^p} \times A^{pqr}_{st}$$

$$= \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^k}{\partial x^r} \times \frac{\partial x^t}{\partial \bar{x}^m} \times \delta^s_p \times A^{pqr}_{st} \qquad \left[\because \frac{\partial x^s}{\partial x^p} = \delta^s_p\right]$$

$$= \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^k}{\partial x^r} \times \frac{\partial x^t}{\partial \bar{x}^m} \times A^{pqr}_{pt}$$

This is the law of transformation of a tensor of rank 3.

So, A^{ijk}_{im} is a tensor of rank 3 and type (2,1) whereas A^{ijk}_{lm} is of order 5 and type (3,2).

This shows that contraction reduces rank of a tensor by 2.

Now, to prove : $a_{ij} \times a^{ij} = \delta^j_j$

Let $d = |a_{ij}|$ be the determinant with elements a_{ij} and $d \neq 0$

Then, the reciprocal tensor of a_{ij} is defined as:

$$a^{ij} = \frac{\text{cofactor of } a_{ij} \text{ in the determinant } |a_{ij}|}{d} = \frac{B_{ij}}{d} \quad (\text{let.})$$

Then, from the properties of determinant, we know that the sum of an element multiplied by its cofactor over any row/column ~~(or, all elements)~~ gives the determinant.

So, $a_{ij} \times B_{ij} = d \Rightarrow a_{ij} \cdot \frac{B_{ij}}{d} = 1 \Rightarrow a_{ij} \times a^{ij} = 1$

And, we know, $\delta^j_j = 1$. Hence, $a_{ij} \times a^{ij} = \delta^j_j$ [Proved.]

In general, we have the result : $a_{ij} \times a^{kj} = \delta^k_i$

So, $a_{ij} \times a^{ij} = \delta^i_i = \delta^j_j$

4) Given : $c_{ij} A^i A^j$ is an invariant, for A^i being an arbitrary contravariant vector.

So, $\bar{c}_{ij} \bar{A}^i \bar{A}^j = c_{kl} A^k A^l$

$$\Rightarrow \bar{c}_{ij} \times \frac{\partial \bar{x}^i}{\partial x^k} A^k \times \frac{\partial \bar{x}^j}{\partial x^l} A^l = c_{kl} A^k A^l \qquad \left[\because A^i \text{ is a contravariant vector}\right]$$

$$\Rightarrow \underbrace{\left(\bar{c}_{ij} \times \frac{\partial \bar{x}^i}{\partial x^k} \times \frac{\partial \bar{x}^j}{\partial x^l} - c_{kl}\right)}_{\text{Let this be } B_{kl}} A^k A^l = 0 \qquad \ldots\ldots (i)$$

Let this be B_{kl}.

Then, $B_{kl} A^k A^l = 0$. As k, l are dummy indices, we now interchange them to get : $B_{lk} A^l A^k = 0$. So, $(B_{kl} + B_{lk}) A^k A^l = 0$ ....(ii)
As A^i is an arbitrary contravariant vector, so we must have: $B_{kl} + B_{lk} = 0$

$$\Rightarrow \bar{c}_{ij} \times \frac{\partial \bar{x}^i}{\partial x^k} \times \frac{\partial \bar{x}^j}{\partial x^l} - c_{kl} + \bar{c}_{ji} \times \frac{\partial \bar{x}^j}{\partial x^l} \times \frac{\partial \bar{x}^i}{\partial x^k} - c_{lk} = 0$$

$$\Rightarrow \left(\bar{c}_{ij} + \bar{c}_{ji}\right) \times \frac{\partial \bar{x}^i}{\partial x^k} \times \frac{\partial \bar{x}^j}{\partial x^l} = (c_{kl} + c_{lk})$$

$$\Rightarrow \left(\bar{c}_{ij} + \bar{c}_{ji}\right) = \frac{\partial x^k}{\partial \bar{x}^i} \times \frac{\partial x^l}{\partial \bar{x}^j} \cdot (c_{kl} + c_{lk}) \qquad \ldots (iii)$$

Hence, from (iii), we conclude that $(c_{ij} + c_{ji})$ is a covariant tensor of order 2. [Proved.]

5) To Prove : $\left\{ {i \atop i\ j} \right\} = \dfrac{\partial \log(\sqrt{g})}{\partial x^j}$

Here, $g = |g_{ij}|$ is the determinant

We know, $g_{ik} * g^{il} = \delta^l_k \Rightarrow g_{ik} * g^{ik} = \delta^k_k = 1$

Also, by definition, $g^{ik} = \dfrac{G_{ik}}{g}$; where G_{ik} = cofactor of g_{ik} in the determinant $|g_{ik}|$

$\Rightarrow g * g^{ik} = G_{ik} \quad \Rightarrow g * g_{ik} * g^{ik} = g_{ik} * G_{ik}$

$\Rightarrow g = g_{ik} * G_{ik} \quad [\because g_{ik} * g^{ik} = 1]$

..... (i)

Differentiating (i) partially w.r.t g_{ik}, we get :

$$\frac{\partial g}{\partial g_{ik}} = G_{ik}$$

Now, $\dfrac{\partial g}{\partial x^j} = \dfrac{\partial g}{\partial g_{ik}} * \dfrac{\partial g_{ik}}{\partial x^j} = G_{ik} * \dfrac{\partial g_{ik}}{\partial x^j}$

But, we know, $G_{ik} = g * g^{ik}$ [shown earlier]

So, $\dfrac{\partial g}{\partial x^j} = g * g^{ik} * \dfrac{\partial g_{ik}}{\partial x^j}$

$$\Rightarrow \frac{1}{g} * \frac{\partial g}{\partial x^j} = g^{ik} * \frac{\partial g_{ik}}{\partial x^j}$$

$$\Rightarrow \frac{1}{g} * \frac{\partial g}{\partial x^j} = g^{ik} * \left\{ [jk, i] + [ij, k] \right\} \qquad \left[\because \frac{\partial g_{ik}}{\partial x^j} = [jk, i] + [ij, k]\right]$$

$$= g^{ik} [jk, i] + g^{ik} [ij, k]$$

$$= \left\{ {k \atop j\ k} \right\} + \left\{ {i \atop i\ j} \right\}$$

$$\Rightarrow \frac{1}{g} \times \frac{\partial g}{\partial x^j} = \left\{ {i \atop j\ i} \right\} + \left\{ {i \atop j\ i} \right\} \qquad [\because k \text{ is a dummy index}]$$

$$\Rightarrow \frac{1}{g} \times \frac{\partial g}{\partial x^j} = 2 \times \left\{ {i \atop j\ i} \right\}$$

$$\Rightarrow \frac{1}{2g} \times \frac{\partial g}{\partial x^j} = \left\{ {i \atop j\ i} \right\} \qquad \Rightarrow \frac{\partial \log(\sqrt{g})}{\partial x^j} = \left\{ {i \atop j\ i} \right\} \qquad \left[\because \partial \log(\sqrt{g}) = \frac{1}{2g} \times \partial g\right]$$

Hence, $\left\{ {i \atop i\ j} \right\} = \frac{\partial \log(\sqrt{g})}{\partial x^j}$ [Proved.] $\left[\because \left\{ {i \atop j\ i} \right\} = \left\{ {i \atop i\ j} \right\}\right]$

6) For each pair (i, j), or for each independent g_{ij}, there are n distinct Christoffel symbols of each kind due to another free index k in each Christoffel symbol.

And, since g_{ij} is a symmetric tensor of rank 2, so it has $\frac{n(n+1)}{2}$ independent components at max.

Hence, the number of independent components of Christoffel's symbol of each kind is $= n \times \frac{n(n+1)}{2}$

$$= \frac{n^2(n+1)}{2}$$

Now we prove that the transformation of Christoffel's symbols form a group i.e. possess the transitive property.

Let the symbols in x^i coordinate system be transformed to the coordinate system $\bar{x}^i$ and that be again transformed to $\bar{\bar{x}}^i$.

When coordinates x^i are transformed to $\bar{x}^i$, the law of transformation of Christoffel's symbols of second kind is:

$$\overline{\begin{Bmatrix} k \\ i\ j \end{Bmatrix}} = \frac{\partial \bar{x}^k}{\partial x^s} \times \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} + \frac{\partial \bar{x}^k}{\partial x^s} \times \frac{\partial x^p}{\partial \bar{x}^i} \times \frac{\partial x^q}{\partial \bar{x}^j} \times \begin{Bmatrix} s \\ p\ q \end{Bmatrix} \quad \ldots \text{(i)}$$

When coordinates $\bar{x}^i$ are transformed to $\bar{\bar{x}}^i$, then:

$$\overline{\overline{\begin{Bmatrix} \mu \\ u\ v \end{Bmatrix}}} = \overline{\begin{Bmatrix} k \\ i\ j \end{Bmatrix}} \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^\mu}{\partial \bar{x}^k} + \frac{\partial^2 \bar{x}^k}{\partial \bar{\bar{x}}^u . \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^\mu}{\partial \bar{x}^k}$$

$$= \begin{Bmatrix} s \\ p\ q \end{Bmatrix} \times \frac{\partial \bar{x}^k}{\partial x^s} \times \frac{\partial x^p}{\partial \bar{x}^i} \times \frac{\partial x^q}{\partial \bar{x}^j} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^\mu}{\partial \bar{x}^k}$$

[From (i)]

$$+ \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} \times \frac{\partial \bar{x}^k}{\partial x^s} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} \cdot \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^\mu}{\partial \bar{x}^k} + \frac{\partial^2 \bar{x}^k}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^\mu}{\partial \bar{x}^k}$$

$$\Rightarrow \overline{\overline{\begin{Bmatrix} \mu \\ u\ v \end{Bmatrix}}} = \begin{Bmatrix} s \\ p\ q \end{Bmatrix} \times \frac{\partial x^p}{\partial \bar{\bar{x}}^u} \times \frac{\partial x^q}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^\mu}{\partial x^s} + \frac{\partial^2 \bar{x}^k}{\partial \bar{\bar{x}}^u \cdot \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^\mu}{\partial \bar{x}^k}$$

$$+ \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} \times \frac{\partial \bar{\bar{x}}^\mu}{\partial x^s} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \quad \ldots \text{(ii)}$$

Now, we know that: $$\frac{\partial x^s}{\partial \bar{x}^i} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} = \frac{\partial x^s}{\partial \bar{\bar{x}}^u} \quad \ldots \text{(iii)}$$

Differentiating partially eqn. (iii) w.r.t $\bar{\bar{x}}^v$, we get:

$$\frac{\partial}{\partial \bar{\bar{x}}^v}\left(\frac{\partial x^s}{\partial \bar{x}^i}\right) \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} + \frac{\partial x^s}{\partial \bar{x}^i} \times \frac{\partial}{\partial \bar{\bar{x}}^v}\left(\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u}\right) = \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v}$$

$$\Rightarrow \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} + \frac{\partial x^s}{\partial \bar{x}^i} \times \frac{\partial^2 \bar{x}^i}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} = \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} \quad \ldots \text{(iv)}$$

Multiplying (iv) by $\dfrac{\partial \bar{\bar{x}}^k}{\partial x^s}$, we get :

$$\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} \times \frac{\partial \bar{\bar{x}}^k}{\partial x^s} + \frac{\partial \bar{\bar{x}}^k}{\partial \bar{x}^i} \times \frac{\partial^2 \bar{x}^i}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} = \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^k}{\partial x^s}$$

Replacing i with k in 2nd term on LHS [$\because i$ is a dummy index there],

$$\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} \times \frac{\partial \bar{\bar{x}}^k}{\partial x^s} + \frac{\partial \bar{\bar{x}}^k}{\partial \bar{x}^k} \times \frac{\partial^2 \bar{x}^k}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} = \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^k}{\partial x^s} \quad \ldots \text{(v)}$$

Using (v) in eqn. (ii), we get :

$$\overline{\overline{\begin{Bmatrix} k \\ u \; v \end{Bmatrix}}} = \begin{Bmatrix} s \\ p \; q \end{Bmatrix} \times \frac{\partial x^p}{\partial \bar{\bar{x}}^u} \times \frac{\partial x^q}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^k}{\partial x^s} + \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^k}{\partial x^s} \quad \ldots \text{(vi)}$$

And, eqn. (vi) is similar to eqn. (i), which shows that if we made direct transformation from x^i to $\bar{\bar{x}}^i$ coordinate system, we get the same law of transformation.

Hence Christoffel symbol of second kind possess group property (i.e. transitive property) upon transformation.

Now, we know that $[ij, m] = g_{km} \begin{Bmatrix} k \\ i \; j \end{Bmatrix}$; where g_{km} is the fundamental tensor.

and, as $\begin{Bmatrix} k \\ i \; j \end{Bmatrix}$ possess group property upon transformation and $[ij, m]$, i.e. christoffel symbol of 1st kind, is a product of g_{km} and $\begin{Bmatrix} k \\ i \; j \end{Bmatrix}$, and both follow transitive property, so $[ij, m]$ also follows transitive property.

Note that, g_{km} follows transitive property as we had earlier proved that all types of tensors [in general, any (r, s) type mixed tensor] possesses group property in transformation.

Hence, we conclude that the laws of transformations of Christoffel symbols possess group properties. [Proved.]