Date
09/09/2019

Lecture 3

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Note :- Equations reducible to Bessel's equation

We consider

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0 \quad \text{(16)}$$

Let $z = \lambda x$, so that

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \lambda\frac{dy}{dz}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\lambda\frac{dy}{dz}\right)$$

$$= \lambda\frac{d}{dx}\left(\frac{dy}{dz}\right) = \lambda\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \lambda^2\frac{d^2y}{dz^2}$$ (By chain Rule)

Then eq^n 16 becomes

$$z^2\left(\frac{d^2y}{dz^2}\right) + z\left(\frac{dy}{dz}\right) + (z^2 - n^2)y = 0 \quad \longrightarrow 17$$

which is Bessel's equation of order n. (n is an integer)

$\therefore$ the general solution of eq^n 17 is

$$y = A\,J_n(z) + B\,Y_n(z)$$

or, $y = A\,J_n(\lambda x) + B\,Y_n(\lambda x)$

Thus, $J_n(\lambda x)$ & $Y_n(\lambda x)$ are solutions of eq^n 16 which is called the modified Bessel's eq^n.

Note 2 :- Bessel's function of the second kind of order n

Def^n This is denoted by $Y_n(x)$ & is defined by

$$Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)},$$

$n \neq$ integer

& $$Y_n(x) = \lim_{\nu \to n} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)},$$

n is an integer.

④

HW

Q) Integration of Bessel's equation

$$xy'' + y' + xy = 0$$

in series form $\left[x^2y'' + xy' + (x^2 - 0^2)y = 0\right.$ $(\times x)$

$x = 0$.

(Bessel's function of zeroth order i.e, $J_0(x)$).

Solve it independently by Frobenius method.

Hint:- $xy'' + y' + xy = 0$.

$$\text{Let } y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0$$

$$y', \quad y'', \quad \ldots x^{k-1}$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots \infty$$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n},$$

$\Rightarrow$

$$\boxed{J_0(0) = 1.}$$

where $J_0(x)$ is known as Bessel's function of zeroth order.

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Ex/ Prove that

(i) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

(ii) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

(iii) $\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}.$

Soln:- By the defn of $J_n(x)$, we have

By def^n of $J_n(x)$, we have

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \cdots\right] \quad (1)$$

Replacing n by $(-\frac{1}{2})$ in eq^n (1) & simplifying, we get

$$J_{-\frac{1}{2}}(x) = \frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}}\underbrace{\Gamma(-\frac{1}{2}+1)}_{=\Gamma(\frac{1}{2}) = \sqrt{\pi}}}\left[1 - \frac{x^2}{2.2/\frac{1}{2}} + \frac{x^4}{1.2.3.4} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\underbrace{\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]}_{=\cos x}$$

$$\therefore J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x ,$$

(ii) Replacing n by $(1/2)$ in eqn (1)

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\Gamma(3/2)}\left[1 - \frac{x^2}{1\cdot2\cdot3} + \frac{x^4}{1\cdot2\cdot3\cdot4\cdot5} - \cdots\right]$$

$$= \sqrt{\frac{\cancel{x}}{2}} \frac{1}{\underbrace{1/2\,\Gamma(1/2)}_{(=\sqrt{\pi})}} \frac{1}{\cancel{x}\,\sqrt{x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}\right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x .$$

Squaring & adding (i) & (ii),

$$\left[J_{-1/2}(x)\right]^2 + \left[J_{1/2}(x)\right]^2 = \left(\frac{2}{\pi x}\right)\underbrace{\left(\sin^2 x + \cos^2 x\right)}_{=1}$$

$$= 2/\pi x.$$

HW

Q2) Prove that

$$\lim_{z \to 0} \frac{J_n(z)}{z^n} = \frac{1}{2^n \Gamma(n+1)}, \quad n > -1.$$

EX3 / Write the general solution of the following equations:

(i) $x^2\left(\frac{d^2y}{dx^2}\right)+x\left(\frac{dy}{dx}\right)+\left(x^2-25\right)y=0.$

(ii) $x^2\left(\frac{d^2y}{dx^2}\right)+x\left(\frac{dy}{dx}\right)+\left(x^2-9/16\right)y=0$

(iii) $\frac{d^2y}{dx^2}+\left(\frac{1}{x}\right)\times\left(\frac{dy}{dx}\right)+\left(1-\frac{1}{6.25x^2}\right)y=0.$

(iv) $x^2\left(\frac{d^2z}{dx^2}\right)+x\left(\frac{dz}{dx}\right)+\left(x^2-64\right)z=0.$

(v) $z\left(\frac{d^2y}{dz^2}\right)+\left(\frac{dy}{dz}\right)+zy=0.$

Hint :- $(\times z)$

$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+z^2y=0.$

sol^n:- (i) Given $x^2y'' + xy' + (x^2 - 5^2)y = 0$

which is Bessel's equation of order 5, which is an integer.

$\therefore$ its general solution is $y = A J_5(x) + B Y_5(x)$, where A & B are arbitrary constants.

(ii) Hint:-

$y = A J_{3/4}(x) + B J_{-3/4}(x)$.

— — A, B are arbitrary constants

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~~EXY~~ Solve the following d. eq^n :-

(i) $x^2\left(\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right) + \left(4x^4 - \frac{1}{4}\right)y = 0.$

H.W. (ii) $x\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} + \left(y/4\right) = 0,$

[Hint:- by using the substitution $z = \sqrt{x}$]

Sol^n (i) :- Suppose that $z = x^2$ $\rightarrow (1)$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \underbrace{\frac{dz}{dx}}_{(=2x)} = 2x \cdot \frac{dy}{dz} \rightarrow (2)$$

& $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x\frac{dy}{dz}\right)$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(2x\frac{dy}{dz}\right)$$

$$= 2\cdot\frac{dy}{dz} + 2x\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$(2x)$

$$= 2\frac{dy}{dz} + (2x)^2\frac{d^2y}{dz^2}.$$

Substituting the above values in (by ①) the given eq^n we get

$$x^2\left(2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2}\right) + 2x^2\frac{dy}{dz} + \left(4x^4 - \tfrac{1}{4}\right)y = 0$$

$$\Rightarrow 4z^2\frac{d^2y}{dz^2} + 4z\frac{dy}{dz} + 4\left\{z^2 - \left(\tfrac{1}{4}\right)^2\right\}y = 0$$

$$\Rightarrow z^2\left(\frac{d^2y}{dz^2}\right) + z\left(\frac{dy}{dz}\right) + \left(z^2 - \left(\tfrac{1}{4}\right)^2\right)y = 0 \quad \to ③$$

eq^n ③ is a Bessel's eq^n of order $(1/4)$. Since $(1/4)$ is a positive non-integral real number, hence sol^n of ③ is $y = AJ_{1/4}(z) + BJ_{-1/4}(z)$

$$\therefore y = AJ_{1/4}(x) + BJ_{-1/4}(x)$$

where A, B are arbitrary constants

—x—