Name: Raushan Sharma

Roll : 18MA20058

ASSIGNMENT 3

Mathematical Methods

1) (i) Given BVP : $\frac{d^4y}{dx^4} = 0$ .... (i)

with BCs : $y(0) = y'(0) = y''(1) = y'''(1) = 0$ .... (ii)

The general solution of (i) is given by :

$$y = A + Bx + Cx^2 + Dx^3$$

Using the given BCs, we have :

$y(0) = 0 \Rightarrow A = 0$, $y'(0) = 0 \Rightarrow B = 0$

$y''(1) = 0 \Rightarrow 2C + 6D = 0$, $y'''(1) = 0 \Rightarrow 6D = 0$

$\Rightarrow C = D = 0$.

Hence, the homogenous BVP given by (i) and (ii) has only the trivial solution $y(x) \equiv 0$.

$\therefore$ The Green's function for this BVP will be unique, and is given by :

$$G(x,t) = \begin{cases} a_1 + a_2x + a_3x^2 + a_4x^3 \ ; & 0 \le x < t \\ b_1 + b_2x + b_3x^2 + b_4x^3 \ ; & t < x \le 1 \end{cases}$$

$G(x,t)$ will satisfy the following properties :

(1) Continuity of G, $\frac{\partial G}{\partial x}$, $\frac{\partial^2 G}{\partial x^2}$ at $x = t$:

$$a_1 + a_2t + a_3t^2 + a_4t^3 = b_1 + b_2t + b_3t^2 + b_4t^3$$

$$\Rightarrow (b_1 - a_1) + (b_2 - a_2)t + (b_3 - a_3)t^2 + (b_4 - a_4)t^3 = 0$$

Define : $c_i = b_i - a_i$ $(i = 1, 2, 3, 4)$

Then we get : $c_1 + c_2t + c_3t^2 + c_4t^3 = 0$ .... (iii)

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And, from the continuity of $\frac{\partial G}{\partial x}$ and $\frac{\partial^2 G}{\partial x^2}$ at $x = t$,

we get:

$$c_2 + 2c_3 t + 3c_4 t^2 = 0 \quad \cdots\cdots \text{(iv)}$$

And, $$2c_3 + 6c_4 t = 0 \quad \cdots\cdots \text{(v)}$$

(2) $\frac{\partial^3 G}{\partial x^3}$ has a jump discontinuity at $x = t$:

$$\left(\frac{\partial^3 G}{\partial x^3}\right)_{x=t^+} - \left(\frac{\partial^3 G}{\partial x^3}\right)_{x=t^-} = \frac{-1}{P_0(t)}$$

[$P_0(x) =$ coeff. of y'''' in (i) $\Rightarrow P_0(x) = 1$ $\therefore P_0(t) = 1$]

$$\Rightarrow 6b_4 - 6a_4 = \frac{-1}{1} \Rightarrow 6c_4 = -1 \quad \cdots\cdots \text{(vi)}$$

So we have the system of equations given by (iii) to (vi)

as:
$$c_1 + c_2 t + c_3 t^2 + c_4 t^3 = 0$$
$$c_2 + 2c_3 t + 3c_4 t^2 = 0$$
$$2c_3 + 6c_4 t = 0$$
$$6c_4 = -1$$

Solving this system, we get:

$$c_1 = \frac{t^3}{6}, \quad c_2 = \frac{-t^2}{2}, \quad c_3 = \frac{t}{2}, \quad c_4 = \frac{-1}{6}$$

(3) Now, $G(x,t)$ also satisfies the BCs given by (ii)

So, $G(0,t) = 0 \Rightarrow a_1 = 0 \quad \cdots\cdots$ (vii)

$G'(0,t) = 0 \Rightarrow a_2 = 0 \quad \cdots\cdots$ (viii)

$G''(1,t) = 0 \Rightarrow 2b_3 + 6b_4 = 0 \Rightarrow b_3 + 3b_4 = 0 \quad \cdots\cdots$ (ix)

$G'''(1,t) = 0 \Rightarrow 6b_4 = 0 \Rightarrow b_4 = 0 \quad \cdots$ (x)

$b_4 = 0$ and $b_3 + 3b_4 = 0 \Rightarrow b_3 = 0$.

So, $c_1 = \frac{t^3}{6}$ and $a_1 = 0 \Rightarrow b_1 = c_1 + a_1 = \frac{t^3}{6}$

$c_2 = \frac{-t^2}{2}$ and $a_2 = 0 \Rightarrow b_2 = c_2 + a_2 = \frac{-t^2}{2}$

$c_3 = \frac{t}{2}$ and $b_3 = 0 \Rightarrow a_3 = b_3 - c_3 = \frac{-t}{2}$

$c_4 = \frac{-1}{6}$ and $b_4 = 0 \Rightarrow a_4 = b_4 - c_4 = \frac{1}{6}$

Hence, $$G(x,t) = \begin{cases} -\frac{t}{2}x^2 + \frac{1}{6}x^3 & ; \ 0 \le x < t \\ \frac{t^3}{6} - \frac{t^2}{2}x & ; \ t < x \le 1 \end{cases}$$ [Ans]

(ii) Given BVP : $\frac{d^3y}{dx^3} = 0 \quad \cdots$ (i)

with BCs : $y(0) = 0$, $y'(1) = 0$, $y'(0) = y(1) \quad \cdots$ (ii)

The General soln. of (i) is given by :

$y = A + Bx + Cx^2$

Using the BCs , we have : $y(0) = 0 \Rightarrow A = 0$

$y'(1) = 0 \Rightarrow B + 2C = 0$

And, $y'(0) = y(1) \Rightarrow B = A + B + C \Rightarrow B = B + C$
$\Rightarrow C = 0$

So, $B = -2C = 0$

So, $y(x) \equiv 0$ is the only soln. of BVP. Hence the Green's function for the BVP given by (i) and (ii) is unique, and will be of the form :

$$G(x,t) = \begin{cases} a_1 + a_2 x + a_3 x^2 & ; \ 0 \le x < t \\ b_1 + b_2 x + b_3 x^2 & ; \ t < x \le 1 \end{cases}$$

① Continuity of G, $\frac{\partial G}{\partial x}$ at $x = t$:

Define : $c_i = b_i - a_i$; $i = 1, 2, 3$

$$c_1 + c_2 t + c_3 t^2 = 0 \quad \ldots\ldots \text{(iii)}$$

And, $c_2 + 2c_3 t = 0 \quad \ldots\ldots \text{(iv)}$

② Discontinuity of $\frac{\partial^2 G}{\partial x^2}$ at $x = t$:

$$2c_3 = \frac{-1}{1} \Rightarrow 2c_3 = -1 \quad \ldots \text{(v)}$$

From the system of equations (iii), (iv) and (v) :

$$c_1 = \frac{-t^2}{2}, \quad c_2 = t, \quad c_3 = \frac{-1}{2}$$

③ $G(x,t)$ satisfies the BCs given by (ii) :

$G(0,t) = 0 \Rightarrow a_1 = 0$ $\qquad$ $G'(1,t) = 0 \Rightarrow b_2 + 2b_3 = 0$

$$G'(0,t) = G(1,t) \Rightarrow a_2 = b_1 + b_2 + b_3$$

$$\Rightarrow b_2 - c_2 = b_1 + b_2 + b_3$$

$$\Rightarrow b_1 + b_3 = -t$$

Now, $c_1 = \frac{-t^2}{2}$ and $a_1 = 0 \Rightarrow b_1 = \frac{-t^2}{2}$

Then, $b_1 + b_3 = -t \Rightarrow b_3 = \frac{t^2}{2} - t$

$$\Rightarrow a_3 = b_3 - c_3 = \frac{t^2}{2} - t + \frac{1}{2}$$

Now, $b_2 + 2b_3 = 0 \Rightarrow b_2 = 2t - t^2$

$$\Rightarrow a_2 = b_2 - c_2 = t - t^2.$$

Hence, $$G(x,t) = \begin{cases} (t - t^2)x + \left(\frac{t^2}{2} - t + \frac{1}{2}\right)x^2 ; & 0 \le x < t \\ (2x - x^2)t + \left(\frac{x^2}{2} - x - \frac{1}{2}\right)t^2 ; & t < x \le 1 \end{cases}$$ [Ans.]

(iii) $y''' = 0$; .... (i)

Given BCs: $y(0) = 0$, $y(1) = 0$, $y'(0) = y'(1)$ ... (ii)

The General solution of (i) is given by:

$$y = A + Bx + Cx^2$$

Using the BCs in (ii), $y(0) = 0 \Rightarrow A = 0$

$y(1) = 0 \Rightarrow A + B + C = 0 \Rightarrow B + C = 0$

$y'(0) = y'(1) \Rightarrow B = B + 2C \Rightarrow C = 0 \Rightarrow B = 0$

So the given BVP has only the trivial soln. $y \equiv 0$.

Hence the Green's function of the BVP will be unique and is given by:

$$G(x,t) = \begin{cases} a_1 + a_2 x + a_3 x^2 \ ; & 0 \le x < t \\ b_1 + b_2 x + b_3 x^2 \ ; & t < x \le 1 \end{cases}$$

Now, define : $c_i = b_i - a_i$ (for $i = 1, 2, 3$)

From the continuity of G and $\frac{\partial G}{\partial x}$ at $x = t$ and the jump discontinuity of $\frac{\partial^2 G}{\partial x^2}$ at $x = t$, we get : $c_1 = \frac{-t^2}{2}$, $c_2 = t$, $c_3 = \frac{-1}{2}$

[This we have found in the previous question]

Now, $G(x,t)$ will satisfy the BCs given by (ii) :

$G(0,t) = 0 \Rightarrow a_1 = 0$; $G(1,t) = 0$

$\Rightarrow b_1 + b_2 + b_3 = 0$

$G'(0,t) = G'(1,t) \Rightarrow a_2 = b_2 + 2b_3$

$\Rightarrow b_2 - c_2 = 2b_3 + b_2 \Rightarrow b_3 = \frac{-t}{2}$

$c_1 = \frac{-t^2}{2}$ and $a_1 = 0 \Rightarrow b_1 = \frac{-t^2}{2}$

$b_2 = -b_1 - b_3 \Rightarrow b_2 = \frac{t^2}{2} + \frac{t}{2}$ and $c_2 = t$

So, $a_2 = b_2 - c_2 = \frac{t^2}{2} - \frac{t}{2}$

Hence, $$G(x,t) = \begin{cases} \left(\frac{t^2}{2} - \frac{t}{2}\right)x + \left(\frac{-t}{2} + \frac{1}{2}\right)x^2 \; ; & 0 \le x < t \\ \left(\frac{-x^2}{2} + \frac{x}{2}\right)t + \left(\frac{x}{2} - \frac{1}{2}\right)t^2 \; ; & t < x \le 1 \end{cases}$$ [Ans.]

(iv) Given BVP : $y'' + y = 0$ .... (i)

with BCs : $y(0) = y(1)$; $y'(0) = y'(1)$ .... (ii)

The general soln. of (i) is given by : $y = A\cos x + B\sin x$

Using the given BCs : $y(0) = y(1) \Rightarrow A = A\cos(1) + B\cos(1)$

And, $y'(0) = y'(1) \Rightarrow B = -A\sin(1) + B\cos(1)$

From here, $A = B = 0 \Rightarrow y(x) \equiv 0$ is the only soln. of the given BVP. So, the Green's function for this BVP will be unique, and is given by :

$$G(x,t) = \begin{cases} a_1\cos x + a_2\sin x \; ; & 0 \le x < t \\ b_1\cos x + b_2\sin x \; ; & t < x \le 1 \end{cases}$$

(1) Continuity of G at $x = t$:

Let $c_i = b_i - a_i$ $(i = 1, 2)$

Then, $c_1\cos t + c_2\sin t = 0$ .... (iii)

(2) Jump discontinuity of $\frac{\partial G}{\partial x}$ at $x = t$:

$$\left(\frac{\partial G}{\partial x}\right)_{x=t^+} - \left(\frac{\partial G}{\partial x}\right)_{x=t^-} = \frac{-1}{1}$$

$$\Rightarrow (-b_1 \sin t + b_2 \cos t) - (-a_1 \sin t + a_2 \cos t) = -1$$

$$\Rightarrow -c_1 \sin t + c_2 \cos t = -1 \quad \cdots \text{(iv)}$$

Solving (iii) and (iv), we get:

$$c_1 = \sin t \quad , \quad c_2 = -\cos t$$

(3) $G(x,t)$ satisfies the BCs given by (ii):

$$G(0,t) = G(1,t) \Rightarrow a_1 = b_1 \cos 1 + b_2 \sin 1$$

$$G'(0,t) = G'(1,t) \Rightarrow a_2 = -b_1 \sin 1 + b_2 \cos 1$$

So, $\quad b_1 (1-\cos 1) - b_2 \sin 1 = \sin t \quad \cdots \text{(v)}$

And, $\quad b_1 \sin 1 + b_2 (1-\cos 1) = -\cos t \quad \cdots \text{(vi)}$

Solving (v) and (vi):

$$b_1 = \frac{-1}{2\sin(\frac{1}{2})} \times \cos(t+\tfrac{1}{2}) \quad , \quad b_2 = \frac{-1}{2\sin(\frac{1}{2})} \times \sin(t+\tfrac{1}{2})$$

Then, $a_1 = b_1 - c_1 = \dfrac{-1}{2\sin(\frac{1}{2})} \cos(t+\tfrac{1}{2}) - \sin t$

$$= \frac{-\cos(t+\frac{1}{2}) - 2\sin(\frac{1}{2})\sin t}{2\sin(\frac{1}{2})} = \frac{-\cos(t+\frac{1}{2}) - \cos(t-\frac{1}{2}) + \cos(t+\frac{1}{2})}{2\sin(\frac{1}{2})}$$

$$\Rightarrow a_1 = \frac{-1}{2\sin(\frac{1}{2})} \times \cos(t-\tfrac{1}{2})$$

And, $a_2 = b_2 - c_2 = \dfrac{-1}{2\sin(\frac{1}{2})} \sin(t+\tfrac{1}{2}) + \cos t$

$$\Rightarrow a_2 = \frac{-\sin(t+\frac{1}{2}) + 2\sin(\frac{1}{2})\cos t}{2\sin(\frac{1}{2})}$$

$$\Rightarrow a_2 = \frac{-\sin(t+\frac{1}{2}) + \sin(t+\frac{1}{2}) - \sin(t-\frac{1}{2})}{2\sin(\frac{1}{2})} = \frac{-1}{2\sin(\frac{1}{2})} \times \sin(t-\frac{1}{2})$$

Hence, $$G(x,t) = \begin{cases} \frac{-1}{2\sin(\frac{1}{2})} \times \left[\cos(t-\frac{1}{2})\cdot\cos x + \sin(t-\frac{1}{2})\sin x\right] ; & 0 \le x < t \\ \frac{-1}{2\sin(\frac{1}{2})} \times \left[\cos(t+\frac{1}{2})\cos x + \sin(t+\frac{1}{2})\cdot\sin x\right] ; & t < x \le 1 \end{cases}$$

$$\Rightarrow G(x,t) = \begin{cases} \frac{-1}{2\sin(\frac{1}{2})} \times \cos(x-t+\frac{1}{2}) ; & 0 \le x < t \\ \frac{-1}{2\sin(\frac{1}{2})} \times \cos(t-x+\frac{1}{2}) ; & t < x \le 1 \end{cases}$$

[Ans.]

2) (i) Given BVP : $y'' + \pi^2 y = \cos(\pi x)$ ... (i)

with BCs : $y(0) = y(1)$; $y'(0) = y'(1)$ .... (ii)

Consider the corresponding homogenous ODE of (i) :-

$$y'' + \pi^2 y = 0$$

Its general soln. is of the form : $y(x) = A\cos(\pi x) + B\sin(\pi x)$

Using the given BCs in (ii), we get $A = B = 0$ just as in Q. 1 part (iv).

So, the corresponding homogenous ODE of (i) has only the trivial soln.. Hence the Green's function for this BVP will ~~won't~~ be unique, and is given by:

$$G(x,t) = \begin{cases} a_1\cos(\pi x) + a_2\sin(\pi x) ; & 0 \le x < t \\ b_1\cos(\pi x) + b_2\sin(\pi x) ; & t < x \le 1 \end{cases}$$

Also, define $c_i = b_i - a_i$ (for $i = 1, 2$)

① Continuity of G at $x=t$:

From here, we get :

$$c_1 \cos(\pi t) + c_2 \sin(\pi t) = 0 \quad \cdots\cdots \text{(iii)}$$

② Jump discontinuity of $\frac{\partial G}{\partial x}$ at $x=t$:

$$\left(\frac{\partial G}{\partial x}\right)_{x=t^+} - \left(\frac{\partial G}{\partial x}\right)_{x=t^-} = \frac{-1}{P_0(t)} \qquad \left[\begin{array}{l}\text{Here, } p_0(x) = 1 \\ \Rightarrow p_0(t) = 1\end{array}\right]$$

$$\Rightarrow \left[-b_1\pi \sin(\pi t) + b_2\pi \cos(\pi t)\right] - \left[-a_1\pi \sin(\pi t) + a_2\pi \cos(\pi t)\right] = -1$$

$$\Rightarrow -c_1 \sin(\pi t) + c_2 \cos(\pi t) = \frac{-1}{\pi} \quad \cdots\cdots \text{(iv)}$$

From (iii) and (iv), we get :

$$c_1 = \frac{\sin(\pi t)}{\pi} \quad \text{and} \quad c_2 = \frac{-\cos(\pi t)}{\pi}$$

③ Now, $G(x,t)$ satisfies the BCs given by (ii) :

$$G(0,t) = G(1,t) \Rightarrow a_1 = -b_1$$

$$\Rightarrow a_1 = -(c_1 + a_1) \Rightarrow 2a_1 = -c_1$$

$$\Rightarrow a_1 = \frac{-1}{2\pi} \sin(\pi t)$$

And, $b_1 = -a_1 = \frac{1}{2\pi} \sin(\pi t)$

$$G'(0,t) = G'(1,t) \Rightarrow a_2\pi = -b_2\pi$$

$$\Rightarrow a_2 = -b_2 \Rightarrow 2a_2 = -c_2$$

$$\Rightarrow a_2 = \frac{1}{2\pi} \cos(\pi t)$$

And, $b_2 = -a_2$

$$\Rightarrow b_2 = \frac{-1}{2\pi} \cos(\pi t)$$

$$\text{So, } G(x,t) = \begin{cases} \frac{-1}{2\pi} \sin(\pi t) \cos(\pi x) + \frac{1}{2\pi} \cos(\pi t) \sin(\pi x) \ ; & 0 \le x < t \\ \frac{1}{2\pi} \sin(\pi t) \cos(\pi x) - \frac{1}{2\pi} \cos(\pi t) \sin(\pi x) \ ; & t < x \le 1 \end{cases}$$

So, $G(x,t) = \begin{cases} \frac{1}{2\pi} \sin(\pi(x-t)) & ; \; 0 \le x < t \\ \frac{1}{2\pi} \sin(\pi(t-x)) & ; \; t < x \le 1 \end{cases}$

Now, the solution to the non-homogenous BVP given by (i) and (ii) is calculated as:

$$y = \int_0^1 G(x,t) \cdot \phi(t) \cdot dt \qquad [\text{Here, } \phi(t) = -\cos(\pi t)]$$

$$\Rightarrow y = -\int_0^x G(x,t) \cdot \cos(\pi t) \cdot dt - \int_x^1 G(x,t) \cdot \cos(\pi t) \cdot dt$$

$$\Rightarrow y = -\int_0^x \frac{1}{2\pi} \sin(\pi(t-x)) \cdot \cos(\pi t) \cdot dt - \int_x^1 \frac{1}{2\pi} \sin(\pi(x-t)) \cdot \cos(\pi t) \cdot dt$$

$$\Rightarrow y = \frac{x \sin(\pi x)}{4\pi} + \frac{(x-1)\sin(\pi x)}{4\pi}$$

$$\Rightarrow y = \frac{(2x-1)\sin(\pi x)}{4\pi} \qquad [\text{Ans.}]$$

(ii) Given BVP: $y'' + y = x^2$ .... (i)

with BCs: $y(0) = 0$; $y(\frac{\pi}{2}) = 0$ .... (ii)

Consider the corresponding homogenous ODE of (i):

$$y'' + y = 0$$

Its general solution is given by: $y = A\cos(x) + B\sin(x)$

Using the BCs in (ii), $y(0) = 0 \Rightarrow A = 0$ and

$y(\frac{\pi}{2}) = 0 \Rightarrow B = 0$

So this has only the trivial solution $y \equiv 0$. Hence, the Green's function for the given BVP is unique, and of the form:

$$G(x,t) = \begin{cases} a_1 \cos(x) + a_2 \sin(x) \ ; & 0 \le x < t \\ b_1 \cos(x) + b_2 \sin(x) \ ; & t < x \le \frac{\pi}{2} \end{cases}$$

Define: $c_i = b_i - a_i \quad (i = 1, 2)$

Now, from the continuity of G at $x = t$ and the jump discontinuity of $\frac{\partial G}{\partial x}$ at $x = t$, we get:

$c_1 = \sin t \ , \quad c_2 = -\cos t$ [This we have found in Q.1) part (iv) for similar BVP]

Now, $G(x,t)$ satisfies BCs given by (ii):

$G(0,t) = 0 \Rightarrow a_1 = 0 \quad \Rightarrow b_1 = c_1 = \sin t$

$G(\frac{\pi}{2}, t) = 0 \Rightarrow b_2 = 0 \quad \Rightarrow a_2 = -c_2 = \cos t$

$$\therefore \quad G(x,t) = \begin{cases} \cos t \sin x \ ; & 0 \le x < t \\ \sin t \cdot \cos x \ ; & t < x \le \frac{\pi}{2} \end{cases}$$

Then, the solution to the non-homogenous BVP given by (i) and (ii) is calculated as:

$$y = -\int_0^{\pi/2} G(x,t) \cdot t^2 \cdot dt$$ [$\because$ Here, $\phi(x) = -x^2 \Rightarrow \phi(t) = -t^2$]

$$\Rightarrow y = -\int_0^{x} \sin t \cdot \cos x \cdot t^2 \cdot dt - \int_x^{\pi/2} \cos t \cdot \sin x \cdot t^2 \cdot dt$$

$$= -\cos x \cdot \int_0^x \sin t \cdot t^2 dt - \sin x \int_x^{\pi/2} \cos t \cdot t^2 \cdot dt$$

$$= \left[-x \sin(2x) + (x^2 - 2) \cos^2 x + 2\cos x \right]$$

$$+ \left[(x^2 - 2) \sin^2 x + x \sin(2x) - \frac{(\pi^2 - 8)}{4} \sin x \right]$$

$$\Rightarrow y = (x^2 - 2) + 2\cos x - \frac{(\pi^2 - 8)}{4} \sin x$$ [Ans.]

(iii) $y'' - y = 2\sinh(1)$ .... (i)

with BCs : $y(0) = 0$; $y(1) = 0$ .... (ii)

Consider the corresponding homogenous ODE of (i) :

$$y'' - y = 0$$

The general solution of this is : $y = Ae^{x} + Be^{-x}$

Using the BCs given by (ii) , we get $A = 0$ and $B = 0$

So this has only the trivial solution $y \equiv 0$. Hence, the Green's function for this BVP will be unique, and is of the form :

$$G(x,t) = \begin{cases} a_1 e^{x} + a_2 e^{-x} \; ; & 0 \leq x < t \\ b_1 e^{x} + b_2 e^{-x} \; ; & t < x \leq 1 \end{cases}$$

Define : $c_i = b_i - a_i$ (for $i = 1, 2$) . Then we have :

(1) Continuity of G at $x = t$:

we get : $c_1 e^{t} + c_2 e^{-t} = 0$ .... (iii)

(2) Jump discontinuity of $\frac{\partial G}{\partial x}$ at $x = t$:

$$[b_1 e^{t} - b_2 e^{-t}] - [a_1 e^{t} - a_2 e^{-t}] = -1$$

$\Rightarrow c_1 e^{t} - c_2 e^{-t} = -1$ .... (iv)

From (iii) and (iv), $c_1 = \frac{-1}{2} e^{-t}$ and $c_2 = \frac{1}{2} e^{t}$

(3) $G(x,t)$ will satisfy the BCs given by (ii) :

$G(0,t) = 0 \Rightarrow a_1 + a_2 = 0 \Rightarrow (b_1 - c_1) + (b_2 - c_2) = 0$

$\Rightarrow b_1 + b_2 = \frac{e^{t} - e^{-t}}{2}$

$\Rightarrow b_1 + b_2 = \sinh(t)$

And, $G(1,t)=0 \Rightarrow b_1 e + b_2 e^{-1} = 0 \Rightarrow b_1 e^2 + b_2 = 0$

On solving these two equations, $b_1 = \dfrac{\frac{-1}{e} \times \sinh(t)}{2\sinh(1)}$

and $b_2 = \dfrac{e\sinh(t)}{2\sinh(1)}$

Then, $a_1 = b_1 - c_1 = \dfrac{-\frac{1}{e}\sinh(t)}{2\sinh(1)} + \frac{1}{2}e^{-t}$

$a_2 = b_2 - c_2 = \dfrac{e\sinh(t)}{2\sinh(1)} - \dfrac{1}{2}e^{t}$

Then, $a_1 e^x + a_2 e^{-x} = \dfrac{-e^{x-1}\cdot\sinh(t)}{2\sinh(1)} + \frac{1}{2}e^{(x-t)} + \dfrac{e^{(1-x)}\sinh(t)}{2\sinh(1)}$

$- \frac{1}{2}e^{(t-x)}$

$= \dfrac{\sinh(1-x)\cdot\sinh(t)}{\sinh(1)} + \sinh(x-t)$

$\Rightarrow a_1 e^x + a_2 e^{-x} = \dfrac{2\sinh(1-x)\sinh(t) + 2\sinh(1)\sinh(x-t)}{2\sinh(1)}$

$= \dfrac{\cosh(1+t-x) - \cosh(x+t-1) + \cosh(x-t+1) - \cosh(1+t-x)}{2\sinh(1)}$

$= \dfrac{\cosh(x-t+1) - \cosh(x+t-1)}{2\sinh(1)} = \dfrac{\sinh(x)\cdot\sinh(1-t)}{\sinh(1)}$

Similarly, $b_1 e^x + b_2 e^{-x} = \dfrac{-e^{x-1}\cdot\sinh(t)}{2\sinh(1)} + \dfrac{e^{1-x}\cdot\sinh(t)}{2\sinh(1)}$

$= \dfrac{\sinh(t)\cdot\sinh(1-x)}{\sinh(1)}$

$$\therefore\ G(x,t) = \begin{cases} \dfrac{\sinh(x)\cdot\sinh(1-t)}{\sinh(1)} & ; 0 \le x < t \\[2ex] \dfrac{\sinh(t)\cdot\sinh(1-x)}{\sinh(1)} & ; t < x \le 1 \end{cases}$$

Then, the solution to the non-homogenous BVP given by (i) and (ii) is calculated as :

$$y = -\int_0^1 2\sinh(1) \times G(x,t)\cdot dt$$

$$\Rightarrow y = -\int_0^x 2\sinh(1) \times \frac{\sinh(t)\cdot\sinh(1-x)}{\sinh(1)}\cdot dt - \int_x^1 2\sinh(1) \times \frac{\sinh(x)\sinh(1-t)}{\sinh(1)}\cdot dt$$

$$\Rightarrow y = -2\sinh(1-x)\cdot\int_0^x \sinh(t)\cdot dt - 2\sinh(x)\cdot\int_x^1 \sinh(1-t)\cdot dt$$

$$= -2\sinh(1-x)\cdot\left[\cosh(x)-1\right] - 2\sinh(x)\times\left[\cosh(1-x)-1\right]$$

$$\Rightarrow y = -2\times\left[\sinh(x)\cdot\cosh(1-x) + \cosh(x)\cdot\sinh(1-x)\right] + 2\times\left[\sinh(x)+\sinh(1-x)\right]$$

$$\Rightarrow y = -2\sinh(x+1-x) + 2\times 2\sinh\left(\frac{x+1-x}{2}\right)\cosh\left(\frac{x+x-1}{2}\right)$$

$$\Rightarrow y = -2\sinh(1) + 4\sinh\left(\tfrac{1}{2}\right)\cosh\left(x-\tfrac{1}{2}\right) \qquad \text{[Ans.]}$$

(iv) $y'' - y = -2e^x$ .... (i)

with BCs : $y(0) = y'(0)$ and $y(1) + y'(1) = 0$ .... (ii)

Consider the corresponding homogenous ODE of (i) :

$$y'' - y = 0$$

Its general solution is given by : $y = Ae^x + Be^{-x}$

Using the given BCs in (ii), we get $A=0$, $B=0$. So this has only the trivial soln. $y \equiv 0$. Hence the Green's function for this BVP will be unique and is of the form :

$$G(x,t) = \begin{cases} a_1 e^{x} + a_2 e^{-x} ; & 0 \le x < t \\ b_1 e^{x} + b_2 e^{-x} ; & t < x \le l \end{cases}$$

Define : $c_i = b_i - a_i$ for $i = 1, 2$

Now, from the continuity of G at $x = t$ and jump discontinuity of $\frac{\partial G}{\partial x}$ at $x = t$, we get :

$c_1 = \frac{-1}{2} e^{-t}$ and $c_2 = \frac{1}{2} e^{t}$ [This we have already found in the previous question]

Now, $G(x,t)$ will satisfy the BCs given by (ii) :-

$G(0,t) = G'(0,t) \Rightarrow a_1 + a_2 = a_1 - a_2 \Rightarrow a_2 = 0$

$\Rightarrow b_2 = c_2 + a_2 = \frac{1}{2} e^{t}$

And, $G(l,t) + G'(l,t) = 0 \Rightarrow [b_1 e^{l} + b_2 e^{-l}] + [b_1 e^{l} - b_2 e^{-l}] = 0$

$\Rightarrow 2 b_1 e^{l} = 0 \Rightarrow b_1 = 0$

So, $a_1 = b_1 - c_1 = \frac{1}{2} e^{-t}$

$$\therefore G(x,t) = \begin{cases} \frac{1}{2} e^{(x-t)} ; & 0 \le x < t \\ \frac{1}{2} e^{(t-x)} ; & t < x \le l \end{cases}$$

Then, the solution to the non-homogenous BVP given by (i) and (ii) is calculated as :

$$y = \int_0^l 2e^t \cdot G(x,t) \cdot dt = \int_0^x 2e^t \cdot \frac{1}{2} e^{(t-x)} \cdot dt + \int_x^l 2e^t \cdot \frac{1}{2} e^{(x-t)} \cdot dt$$

$$\Rightarrow y = e^{-x} \int_0^x e^{2t} \cdot dt + \int_x^l e^{x} \cdot dt = \frac{e^{-x}}{2} \times [e^{2x} - 1] + (l-x) e^{x}$$

$$\Rightarrow y = \sinh(x) + (l-x) e^{x}$$ [Ans.]