Uniform Continuity

Let's study the continuity of the function $f:[0,\infty)\to\mathbb{R}$ defined by $f(x)=x^2$ at various points.

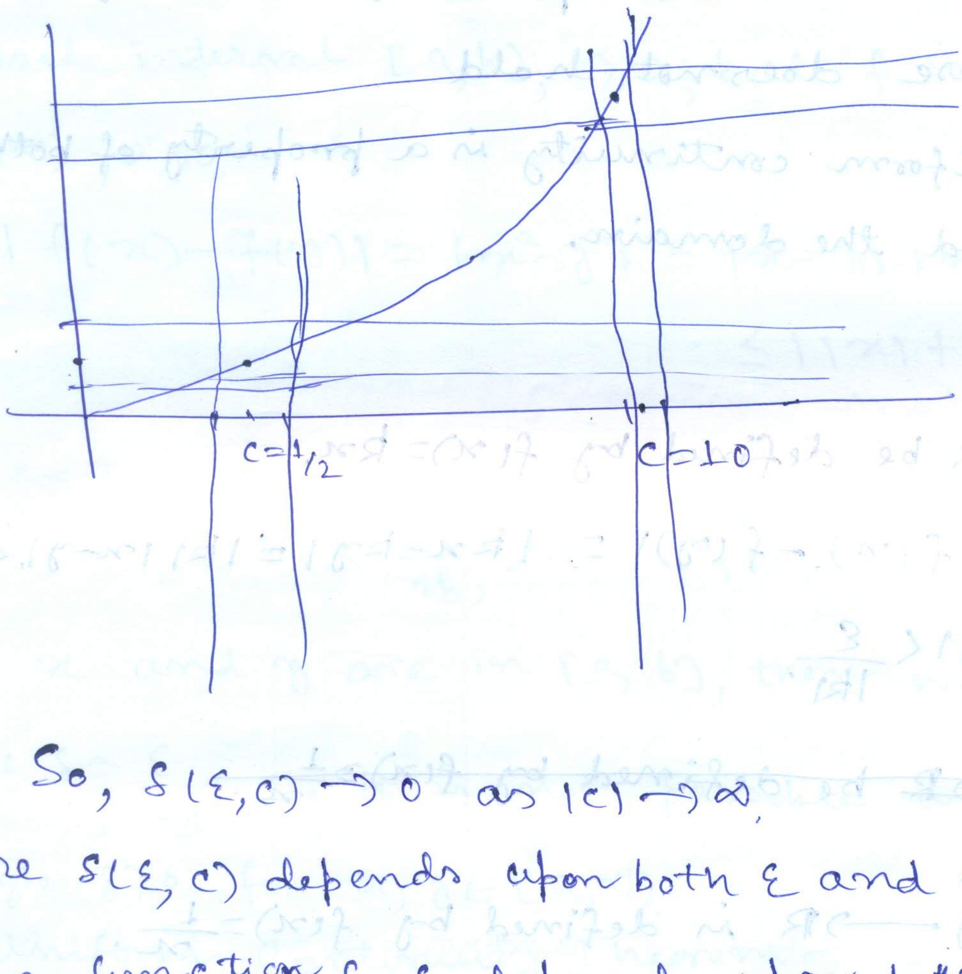
$$|x^2-c^2| = |x+c|\,|x-c|$$

$$\le (|x|+|c|)\,|x-c|$$

$$< 3|c|\,|x-c| \quad \text{if } |x-c|<|c|, \text{ then}$$

$$|x| \le |x-c|+|c| \le |c|+|c| = 2|c|$$

$$\Rightarrow |x^2-c^2|<\varepsilon \text{ if } |x-c|<\frac{\varepsilon}{3|c|}=\delta(\varepsilon,c)$$



So, $\delta(\varepsilon,c)\to 0$ as $|c|\to\infty$.

Here $\delta(\varepsilon,c)$ depends upon both ε and c.

For a function f, δ depends upon both $\varepsilon>0$ and $c\in A$ reflect the fact that f may change its value ~~rapi~~ rapidly near certain points and slowly near other points.

If we can choose $\delta(\varepsilon,c)$ independently of c or uniformly ~~ofc~~ on A, then f is uniformly continuous.

Def:- Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. We say that f is uniformly continuous on A if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, y \in A$ are any numbers satisfying $|x-y| < \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$.

Note:-

1) If f is uniformly continuous on ~~B~~ A, then for $B \subset A$, f is ~~uni~~ uniformly continuous on B.

2) If f is ~~uni~~ uniformly continuous on A, then f is continuous at each point of A. Though the ~~conve~~ converse does not hold.

3) So, the uniform continuity is a property of both the function and the domain.

Example:-

1) $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = kx$.

Then, $|f(x) - f(y)| = |kx - ky| = |k|\,|x-y| < \varepsilon$

if $|x-y| < \dfrac{\varepsilon}{|k|}$.

~~2) $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$.~~

2) $f: [1, \infty) \longrightarrow \mathbb{R}$ is defined by $f(x) = \frac{1}{x}$.

$\therefore |f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \dfrac{|x-c|}{|xc|} \leq |x-c|$

as $|xc| \geq 1$.

$\therefore |f(x) - f(c)| < \varepsilon$ if $|x-c| < \varepsilon$.

3) $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$.

$\therefore |f(x) - f(y)| = |\sin x - \sin y|$

$$= 2 \left|\sin \frac{x-y}{2}\right| \left|\cos \frac{x+y}{2}\right|$$

$$\leq 2 \left|\sin \frac{x-y}{2}\right|$$

$$\leq 2 \cdot \frac{|x-y|}{2} \quad \text{since } |\sin x| \leq |x|$$

$$= |x-y|.$$

$\therefore$ if $|x-y| < \varepsilon$ implies $|f(x) - f(y)| < \varepsilon$.

4) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.

Then f is uniformly ~~cont~~ continuous on any closed interval $[a,b]$, $a \geq 0$, but f is not so on $[a, \infty)$, $a \geq 0$.

$$|f(x) - f(y)| = |x^2 - y^2| = |x-y|\,|x+y|$$

$$\leq \big||x| + |y|\big|\,|x-y|$$

$$< 2b\,|x-y|$$

$\therefore$ If $|x-y| < \frac{\varepsilon}{2b}$, then $|f(x) - f(y)| < \varepsilon$.

If x and y are in $[a,b]$, then we need to take $\delta = \frac{\varepsilon}{2b}$. So, it is not possible to take a single $\delta > 0$ for $x, y \in [a, \infty)$.

5) <u>Uniform Continuity Theorem:-</u>

Let A be a ~~closed an~~ closed and bounded set and let $f: A \to \mathbb{R}$ be continuous on A. Then f is uniformly continuous on A.

To prove this we need the following result where we get equivalent criteria for a function to be uniformly continuous.

Nonuniform Continuity Criteria:-

Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. Then the following statements are equivalent –

(i) f is not uniformly continuous on A,

(ii) there exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.

(iii) there exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Proof of Uniform Continuity Theorem:-

If f is not uniformly continuous on A, then there exists $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $|x_n - u_n| < \frac{1}{n}$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Here (x_n) and ~~(x_n)~~ (u_n) are bounded, by Balzano–Weierstrass Theorem, there is a subsequence (x_{n_r}) of (x_n) that converges to an element x.

$\therefore$ $|u_{n_r} - x| \leq |u_{n_r} - x_{n_r}| + |x_{n_r} - x|$

$< \varepsilon + \varepsilon$ for all $r \geq k$ for some k.

So, (u_{n_r}) converges to x.

Since f is continuous at x, so $(f(x_{n_r}))$ and $(f(u_{n_r}))$ converge to $f(x)$ which contradicts the fact that

$$|f(x_{n_r}) - f(u_{n_r})| \geq \varepsilon_0.$$

So, f is uniformly ~~contin~~ continuous on A.

6) Def^n Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. If there exits a constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x-y|$$

for all $x, y \in A$, then f is said to be a Lipschitz function on A.

or, $\left|\frac{f(x)-f(y)}{x-y}\right| \leq K$, $x, y \in A$, $x \neq y$,

i.e. the slopes of all lines joining two points $(x, f(x))$ and $(y, f(y))$ on the graph $y = f(x)$ over A are bounded by some number K.

Theorem:- If $f: A \to \mathbb{R}$ is a Lipschitz ~~funct~~ function, then f is uniformly continuous on A.

Proof. Let $\delta := \frac{\varepsilon}{K}$. Then $x, y \in A$ and $|x-y| < \delta$ implies $|f(x) - f(y)| < K \cdot \frac{\varepsilon}{K} = \varepsilon$.

$\therefore$ f is uniformly continuous on A.

Examples:-

(a) Let $f : [0, M] \to \mathbb{R}$ be defined by $f(x) = x^n$.

Then, $|f(x) - f(c)| = |x^n - c^n| = |x-c| \cdot |x^{n-1} + x^{n-2}c + \cdots + c^{n-1}|$

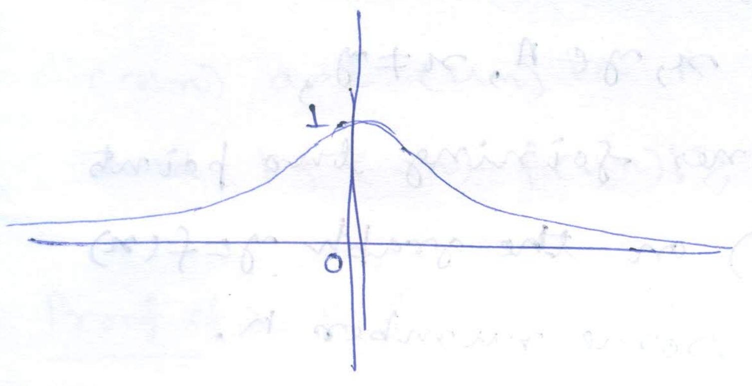
$$\leq |x-c| \left(|x|^{n-1} + |x|^{n-2}|c| + \cdots + |c|^{n-1} \right)$$

$$\leq nM^{n-1} |x-c|.$$

$\therefore f$ is a Lipschitz function. So, f is uniformly continuous on $[0, M]$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{1+x^2}$.

Then $|f(x) - f(c)| = \left| \dfrac{1}{1+x^2} - \dfrac{1}{1+c^2} \right| = \dfrac{|c^2 - x^2|}{(1+x^2)(1+c^2)}$



$$\leq \frac{|x-c|\,(|x| + |c|)}{(1+x^2)(1+c^2)}.$$

$$\leq |x-c| \left(\frac{|x|}{1+x^2} + \frac{|c|}{1+c^2} \right)$$

$$\leq |x-c| \left(\tfrac{1}{2} + \tfrac{1}{2} \right)$$

$$= |x-c|.$$

$\therefore f$ is uniformly continous on $\mathbb{R}$.

(c) Let $g(x) := \sqrt{x}$ on the set $A := [0, \infty)$.

As g is continuous on the closed and bounded set $I := [0, 2]$, so g is uniformly continuous. So, for $\varepsilon > 0$, $\exists\ \delta_I$ s.t. $|x-y| < \delta_I \Rightarrow |g(x) - g(y)| < \varepsilon$.

Let $J := [1, \infty)$

$$|g(x) - g(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{|\sqrt{x} + \sqrt{y}|} \leq \frac{|x-y|}{2}.$$

$\therefore$ By Lipschitz criterion, g is uniformly continuous on J.

So, for $\epsilon > 0$, $\exists\ \delta_J > 0$ s.t. $|x-y| < \delta_J \Rightarrow |g(x)-g(y)| < \epsilon$ and $x, y \in I \cup J = A$

Let $\delta := \min\{1, \delta_I, \delta_J\}$. Then $|x-y| < \delta$ implies

$$|g(x)-g(y)| < \epsilon.$$

So, g is uniformly continuous on A.

Continuous Extension Theorem

Consider the two examples:-

a) $f(x) = \frac{1}{x}$ and b) $f(x) = \sin\frac{1}{x}$, $\forall x \in (0,1]$.

The function $f = \frac{1}{x}$ can't be extended on $[0,1]$ as

$\lim_{x\to 0} \frac{1}{x}$ doesn't exist.

Also the function $f(x) = \sin\frac{1}{x}$ can't be extended on $[0,1]$ as

$\lim_{x\to 0} \sin\frac{1}{x}$ does not exist as if we take

$x_n = \frac{1}{n\pi}$, $\lim_{n\to\infty} \sin\frac{1}{x_n} = \lim_{n\to\infty} \sin n\pi = 0$ and

$y_n = \frac{1}{(2n+\frac{1}{2})\pi}$, then $\lim_{n\to\infty} \sin\frac{1}{y_n} = \lim_{n\to\infty} \sin(2n+\frac{1}{2})\pi$

$$= \lim_{n\to\infty} \sin \pi/2 = 1.$$

Also note that f is ~~not uniformly~~ continuous on $(0,1)$ and can be extended on $[0,1]$, ~~then~~ f is uniformly continuous on $[0,1]$, so f is uniformly continuous on $(0,1)$

So, we can ask — under what conditions is a function uniformly continuous on $(0,1)$?

Continuous Extension Theorem:-

A function f is uniformly continuous on the interval (a,b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a,b]$.

To prove this, we need the following theorem —

Theorem:- If $f: A \to \mathbb{R}$ is uniformly continuous on $A(\subseteq \mathbb{R})$ and if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

Proof:- Let (x_n) be a Cauchy sequence in A. Then for $\varepsilon > 0$, we get $\delta > 0$ such that $|x-y| < \delta$ implies

$$|f(x) - f(y)| < \varepsilon. \quad \text{—— (a)}$$

For $\delta > 0$, there is a natural no. N s.t. for $n, m \geq N$,

$$\cancel{|x-y|}\ |x_n - x_m| < \delta. \quad \text{—— (b)}$$

$\therefore$ (a) and (b) imply

$$|f(x) - f(y)| < \varepsilon \quad \forall n, m \geq N.$$

$\therefore (f(x_n))$ is a Cauchy sequence.

Application:-

a) $f: (0,1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$.

Then take $(x_n) = \left(\frac{1}{n}\right)$ is a Cauchy sequence, but $(f(x_n)) = (n)$ is not a Cauchy sequence.

$\therefore f$ is not uniformly continuous on $(0,1)$

b) $f: (0,1) \to \mathbb{R}$ be defined by $f(x) = \sin\frac{1}{x}$.

Then again take $(x_{2n}) = \left(\frac{1}{n\pi}\right)$ and $(x_{2n-1}) =$

$\frac{1}{(2n+\frac{1}{2})\pi}$. Then (x_n) is a convergent sequence, so is Cauchy. But $(f(x_n)) = (1, 0, 1, 0, \ldots)$ is not Cauchy. $\therefore \sin\frac{1}{x}$ is not uniformly convergent on $(0,1)$.

c) x^2 is not uniformly convergent on $\mathbb{R}$ as take $x_n = n$ and $c_n = n + \frac{1}{n}$. $\therefore |x_n - c_n| \to 0$

But, $|x_n^2 - c_n^2| = |n^2 - (n+\frac{1}{n})^2| = |2 + \frac{1}{n^2}| > 2$.

So, by 'nonuniform continuity criteria', $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof of Continuous Extension ~~Theore~~ Theorem:-

~~It~~ is obvious that ~~if~~ if we can extend at the end points a and b, then f is uniformly continuous on $[a,b]$, so again f is uniformly continuous on (a,b).

Conversely, suppose f is uniformly continuous on (a,b). Now, we shall extend f to a. The extension of f to b is similar.

We shall show that $\lim_{x \to a} f(x) = L$ exists.

Let (x_n) be a sequence in (a,b) with $\lim(x_n) = a$.

Then (x_n) is a Cauchy sequence. So, by the previous theorem, $(f(x_n))$ is a Cauchy sequence, so is a convergent sequence. Let $\lim_{n\to\infty} (f(x_n)) =: L$.

Let (y_n) be another sequence in (a,b) with $\lim y_n = a$.

Then, $\lim_{n\to\infty} (x_n - y_n) = 0$

So, ~~$(x_n - y_n)$ is a~~ by uniform continuity of f,

$$\lim_{n\to\infty} (f(x_n) - f(y_n)) = 0, \text{ so}$$

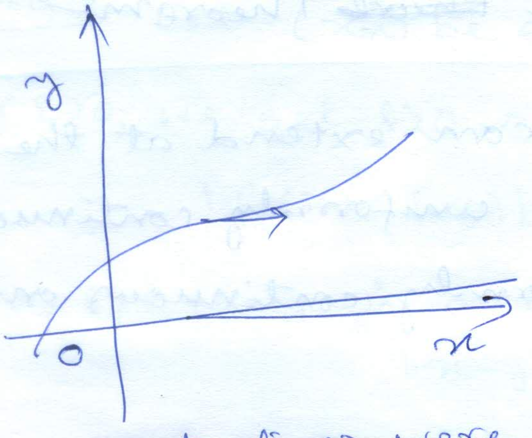
$$\lim_{n\to\infty} (f(y_n)) = \lim_{n\to\infty} (f(y_n) - f(x_n)) + \lim_{n\to\infty} f(x_n)$$

$$= 0 + L = L.$$

$\therefore$ By sequential criteria of limit, $\lim_{x\to a} f(x) = L$.

$\therefore$ If we define $f(a) := L$, then f is continuous at a. Similarly we can extend f to b.

Differentiation



Previously, a curve was generally described as a locus of points satisfying some geometric properties and tangent lines were obtained through geometric description.

For example:- Circle is $p(x,y) := (x-a)^2 + (y-b)^2 = r^2$

Tangent line at (x,y) is the line which is perpendicular to the line passing through (a,b) and (x,y).

Descartes and Fermat described curves and surfaces using algebraic expressions and tangent lines were obtained by differentiation. The problem of finding tangent lines and the problem of finding maximum and minimum values were ~~fo~~ first connected by Fermat. Though it's mainly work of

Newton and Leibnitz (separately) that the area under curves can be calculated by reversing the differentiation (integration). This drew attension of many mathematicians and led to the theory of differential and integral calculus.

Def: Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$, and let $c \in I$. A real number L is said to be the derivative of f at c if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in I$ and $0 < |x-c| < \delta(\varepsilon)$,

then
$$\left| \frac{f(x)-f(c)}{x-c} - L \right| < \varepsilon, \text{ i.e.}$$

$$L = \lim_{x \to c} \frac{f(x)-f(c)}{x-c},$$ provided the limit exists.

We say that f is differentiable at c and we write $f'(c)$ for L.

We now show that continuity of f at a point c is necessary (but not sufficient) condition for the existence of the derivative at c.

Theorem: If $f: I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

Proof: For $x \in I$, $x \neq c$, we have

$$f(x) - f(c) = \left(\frac{f(x)-f(c)}{x-c} \right) \cdot (x-c)$$

Since $f'(c)$ exists, $\lim_{x \to c} (f(x)-f(c))$

$$= \lim_{x \to c} \left(\frac{f(x)-f(c)}{x-c} \right) \cdot \lim_{x \to c} (x-c)$$

$$= f'(c) \cdot 0 = 0.$$

Note:- The continuity of f at c doesn't ensure differentiability at c. For example:-
$f(x) = |x|$ is continuous, but is not differentiable at $x = 0$.

Theorem:- Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f, g : I \to \mathbb{R}$ be functions that are differentiable at c. Then:

(a) If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c, and
$$(\alpha f)'(c) = \alpha f'(c).$$

(b) The function $f+g$ is differentiable at c and
$$(f+g)'(c) = f'(c) + g'(c),$$

(c) (Product rule) The function fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(d) (Quotient rule) If $g(c) \neq 0$, then the function $\frac{f}{g}$ is differentiable at c and
$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

The Chain Rule:-

The "Chain Rule" provides a formula for finding the derivative of a composite function $g \circ f$ in terms of the derivatives of g and f. We first prove the following theorem—

Carathéodory's Theorem:—

Let $f: I \to \mathbb{R}$, let $c \in I$. Then f is differentiable at c if and only if there exists a function ϕ on I that is continuous at c and satisfies

$$f(x) - f(c) = \phi(x)(x-c), \text{ for } x \in I.$$

In this case, we have $\phi(c) = f'(c)$.

Proof:— $(\Rightarrow)$ If $f'(c)$ exists, then we define ϕ by

$$\phi(x) := \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{for } x \neq c, x \in I \\ f'(c) & \text{, for } x = c. \end{cases}$$

Here $\displaystyle\lim_{x \to c} \phi(x) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c) = \phi(c)$.

$\therefore$ ϕ is continuous at c and $f(x) - f(c) = \phi(x)(x-c)$.

$(\Leftarrow)$ $\displaystyle\phi(c) = \lim_{x \to c} \phi(x)$ (as ϕ is continuous at c)

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ exists.}$$

$\therefore$ f is differentiable at c and $f'(c) = \phi(c)$.

Chain rule:— Let I, J be intervals in $\mathbb{R}$ and let $f: J \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be defined by $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Proof:- Since $f'(c)$ exists, by Carathéodory's Theorem there exists a function ϕ on J such that ϕ is continuous at c and $f(x)-f(c)=\phi(x)(x-c)$ for $x\in J$ and ~~$\phi'(c)=$~~ $\phi(c)=f'(c)$.

Since $g'(f(c))$ exists, there exists a function ψ on I such that ψ is continuous at $\underbrace{f(c)}_{=d}$ and

$$g(y)-g(d)=\psi(y)(y-d) \text{ for } y\in I \text{ and } \psi(d)=g'(d).$$

Put $y=f(x)$ and $d=f(c)$, so we obtain

$$g(f(x))-g(f(c))=\psi(f(x))(f(x)-f(c))$$
$$=[\psi\circ f(x)\,.\,\phi(x)](x-c)$$

for all $x\in J$.

Since $\psi\circ f(x).\phi(x)$ is continuous at c and $\psi\circ f(c).\phi(c)=g'(f(c)).f'(c)$, by Carathéodory's theorem, ~~$(g\circ f)'(c)=$~~ $g\circ f$ is differentiable at c and

$$(g\circ f)'(c)=g'(f(c)).f'(c).$$

We will now relate the derivative of a function to the derivative of its inverse function.

Theorem:- Let I be an interval in $\mathbb{R}$ and let $f: I\to\mathbb{R}$ be strictly monotone and continuous on I. Let $J:=f(I)$ and let $g: J\to\mathbb{R}$ be the strictly monotone and continuous function inverse to f. If f is differentiable at c and $f'(c)\neq 0$, then g is differentiable at $d:=f(c)$ and

$$g'(d)=\frac{1}{f'(c)}=\frac{1}{f'(g(d))}.$$

Before proving this theorem, we first say when a function is called strictly monotone and why g is continuous.

Def^n: A function $f: A \to \mathbb{R}$ is said to be strictly increasing if whenever $x_1, x_2 \in A$ and $x_1 < x_2$, then $f(x_1) < f(x_2)$.

Similarly, a function $f: A \to \mathbb{R}$ is said to be strictly decreasing if whenever $x_1, x_2 \in A$ and $x_1 < x_2$, then $f(x_1) > f(x_2)$.

If f is either strictly increasing or strictly decreasing, we say that f is strictly monotone on A.

As f is strictly monotone function, so it is injective and so it has an inverse. In fact we can say more—

Continuous Inverse Theorem:

Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be strictly monotone and continuous on I. Then the function g inverse to f is strictly monotone and continuous on $J := f(I)$.

We are not proving continuous inverse theorem and directly prove the previous theorem.

Proof: From Carathéodory's theorem, there exists a function ϕ, continuous at c, such that

$$f(x)-f(c)=\phi(x)(x-c), \text{ and } \phi(c)=f'(c).$$

Since $\phi(c)\neq 0$, there exists a $\delta>0$ such that

$$x\in(c-\delta, c+\delta) \Rightarrow \phi(x)\neq 0.$$

Let $U:=f(I\cap(c-\delta, c+\delta))$. ~~Then~~ Then, for $y\in U$,

$$\begin{aligned} y-d &= f(g(y))-f(c) \\ &= \phi(g(y)).(g(y)-c) \\ &= \phi(g(y)).(g(y)-g(d)). \end{aligned}$$

Since, $\phi(g(y))\neq 0$ for $y\in U$, we get

$$g(y)-g(d)=\frac{1}{\phi(g(y))}(y-d).$$

Since $\frac{1}{\phi\circ g}$ is continuous at d, so by

Carathéodory's theorem, we ~~conculd~~ conclude that

g is differentiable at d and

$$g'(d)=\frac{1}{\phi(g(d))}=\frac{1}{\phi(c)}=\frac{1}{f'(c)}.$$

Examples:-

a) Let $f:I\to\mathbb{R}$ be defferentiable on I and that $f(x)\neq 0$ $\forall x\in I$. Let $g(y)=1/y$ for $y\in\mathbb{R}-\{0\}$. Then,

$g'(y)=-\frac{1}{y^2}$ for $y\in\mathbb{R}-\{0\}$

$\therefore$ We have

$$\left(\frac{1}{f}\right)'(x)=(g\circ f)'(x)=g'(f(x)).f'(x)$$

$$= -\frac{f'(x)}{(f(x))^2} \quad \forall x \in I$$

b) Let $g(x) := |x| \; \forall x \in \mathbb{R}$.

Then, $g(x) = x \quad \forall x \geq 0$

$\quad\quad\quad\;\; = -x \quad \forall x < 0.$

Therefore, $g'(x) = 1 \quad \forall x > 0$

$\quad\quad\quad\quad\quad\;\; = -1 \quad \forall x < 0.$

and g is not differentiable at $x=0$ as if g is differentiable at $x=0$, then there exists a function φ continuous at 0 such that

$$g(x) - g(0) = \varphi(x)(x-0)$$

$$\Rightarrow |x| = \varphi(x).x$$

$$\Rightarrow \varphi(x) = 1 \text{ for } x > 0$$

$$-1 \text{ for } x < 0, \text{ which}$$

contradicts that φ is continuous at $x=0$.

Let $f: I \longrightarrow \mathbb{R}$ be a differentiable function. Then by the chain rule, the composite function $g \circ f = |f|$ is differentiable at all points x where $f(x) \neq 0$. For $f(x) \neq 0$,

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

$$= \begin{cases} f'(x) & \text{if } f(x) > 0 \\ -f'(x) & \text{if } f(x) < 0 \end{cases}$$

In fact, for $f(c)=0$, we can say the following:—

$|f|$ is differentiable at c if and only if $f'(c)=0$.

c) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0. \end{cases}$$

f is differentiable at $x\neq 0$ and

$$f'(x) = \sin\frac{1}{x} + x\left(-\tfrac{1}{x^2}\right)\cos\frac{1}{x}, \quad x\neq 0$$ (using Chain rule and product rule)

$$= \sin\frac{1}{x} - \tfrac{1}{x}\cos\frac{1}{x}, \quad x\neq 0.$$

$$\lim_{x\to 0}\frac{x\sin\frac{1}{x}-0}{x} = \lim_{x\to 0}\sin\frac{1}{x} \text{ doesn't exist.}$$

$\therefore$ f is not differentiable at $x=0$.

d) Let $f(x) := x^3+2x+1 \ \forall x\in\mathbb{R}$.

Then f is strictly increasing on $\mathbb{R}$ and

$f'(x) = 3x^2+2 \neq 0$.

$\therefore$ $g=f^{-1}$ is differentiable at every point

and $\cancel{g'(x) = g'(f(x)) = (f^{-1})'(f(x))}$

$$\cancel{= \frac{1}{3(f(x))^2+2}}$$

$$g'(f(x)) = (f^{-1})'(f(x))$$

$$= \frac{1}{f'(x)} = \frac{1}{3x^2+2}.$$

e) Let $n \in \mathbb{N}$ be even and $f : I \to \mathbb{R}$ $(I = [0, \infty))$ be defined by $f(x) = x^n$. Then f is strictly increasing and continuous on I. So, the inverse function $g(y) = y^{1/n}$ for $y \in J := [0, \infty)$ is also strictly increasing and continuous on J. We have

$f'(x) = n x^{n-1}$. $\therefore f'(x) \neq 0$ for $x > 0$.

$$\therefore g'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x) \Rightarrow y = x^n$$
$$\Rightarrow x = y^{1/n}$$

$$= \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{n} y^{1/n - 1}.$$

$$g'(0) = \lim_{y \to 0} \frac{y^{1/n} - 0}{y} = \lim_{y \to 0} \frac{1}{y^{1 - 1/n}}$$

$1 - 1/n > 0 \Rightarrow y^{1 - 1/n} > 0$ and $\to 0$ as $y \to 0$

$\frac{1}{y^{1-1/n}}$ is unbounded, so

$\lim_{y \to 0} \frac{1}{y^{1-1/n}}$ does not exist.

f) For $n \in \mathbb{N}$ and n odd, $f(x) = x^n$, $x \in \mathbb{R}$ is strictly increasing and continuous. So, its inverse $g(y) = y^{1/n} : \mathbb{R} \to \mathbb{R}$ is continuous and is differentiable at $y \neq 0$ and

$$g'(y) = \frac{1}{n} y^{1/n - 1}.$$

g) Let $f(x) = \sin x$. Then f is strictly increasing on the interval $I := [-\pi/2, \pi/2]$. So, $\sin x$ has a continuous inverse on I, which we denote by Arcsin. Arcsin exists on $J := [-1, 1]$ and as $(\sin x)' = \cos x \neq 0$ on $(-\pi/2, \pi/2)$, so Arcsin is also differentiable on ~~$(-\pi/2, \pi/2)$ and~~ $(-1, 1)$ and for $y \in (-1, 1)$,

$$(\text{Arcsin})'(y) = \frac{1}{(\sin)'(x)} = \frac{1}{\cos x}$$

$$= \frac{1}{\sqrt{1-(\sin x)^2}}$$

$$= \frac{1}{\sqrt{1-y^2}}.$$

h) Let $f(x) = \sin^{-1} \frac{2x}{1+x^2}$. Here $\sin^{-1}$ is same as $\arcsin x$.

Let $g(x) = \frac{2x}{1+x^2}$, $h(x) = \sin^{-1} x$, $|x| \le 1$.

Here $|g(x)| \le 1$, so $h \circ g$ is defined and

~~$h'(c) =$~~ $f'(c) = h'(g(c)) \cdot g'(c)$.

Now, h is differentiable if

The Mean Value Theorem

The Mean Value Theorem relates the values of a function to values of its derivatives and so is one of the most important results in real analysis.

The problem of finding tangent lines and the seemingly unrelated problem of finding maximum or minimum values of a function were first seen to have a connection by Fermat in the 1630s.

Maxima and Minima:-

a) A function $f: I \to \mathbb{R}$ is said to have a global maximum (or an absalute maximum) on I if there exists a point $c \in I$ such that $f(c) \geq f(x)$ for all $x \in I$. c is said to be a global maximum point for f on I.

b) f is said to have a global minimum (or an absalute minimum) on I if there exists a point $c \in I$ such that $f(c) \leq f(x)$ for all x in I. c is said to be a global minimum point for f on I.

c) f is said to have a local maximum (or relative maximum) at a point $c \in I$ if there exists a neighbourhood $N(c, \delta)$ of c such that

$f(c) \geq f(x)$ for all $x \in N(c,\delta) \cap I$

d) f is said to have a <u>local minimum</u> (or relative minimum) at a point $c \in I$ if there ~~exists~~ exist a neighbourhood $N(c,\delta)$ of c such that $f(c) \leq f(x)$ for all $x \in N(c,\delta) \cap I$.

e) f is said to have a <u>local extremum</u> (or relative extremum) at a point $c \in I$ if f has a local maximum or a local minimum at c.

Note:- A global maximum (or minimum) is always a local maximum (or minimum).

The next result provides the theoretical justification for finding point at which f has local extrema by examining the zeros of the derivative.

<u>Interior Extremum Theorem:-</u>

Let c be an interior pt. of the interval I at which $f: I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$.

<u>Proof:-</u> Without loss of generality, we assume that f has a relative maximum. The proof for the case of a relative minimum is similar.

Since $f'(c)$ exists, so $f'(c) > 0$ or < 0 or $= 0$.

If $f'(c) > 0$, then there exists a ~~neighbourhood~~ ~~continuous~~ function $\varphi(x)$ s.t. φ is continuous at c &

$f(x) - f(c) = \varphi(x)(x-c)$ and $\varphi(c) = f'(c)$.

Since $f'(c) > 0$ and $\varphi(x)$ is continuous at c, so $\exists$ $\delta > 0$ and for $x \in (c-\delta, c+\delta) \cap I$, $\varphi(x) > 0$,

$\therefore \dfrac{f(x) - f(c)}{x-c} > 0$ for $x \in (c-\delta, c+\delta) \cap I$ and $x \neq c$.

So, $x > c \Rightarrow f(x) - f(c) > 0 \Rightarrow f(x) > f(c)$ — (a)

If $x < c$, then $f(x) - f(c) < 0 \Rightarrow f(x) < f(c)$ — (b)

Here (a) contradicts that f has a relative maximum at c. Similarly, we can't have $f'(c) < 0$, $\therefore f'(c) = 0$.

Note:

(1) During the course of the proof of Interior Extremum Theorem, we have actually proved much more. Let's define the following first.

Defn

Let $I \subset \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ be a function. Let c be an interior pt. of I.

a) f is said to be increasing at c if there exists a $\delta > 0$ such that

$$f(x) < f(c) \text{ for all } x \in (c-\delta, c) \cap I$$

and $f(x) > f(c)$ for all $x \in (c, c+\delta) \cap I$.

b) f is said to be decreasing at c if there exists a $\delta > 0$ such that

$$f(x) > f(c) \text{ for all } x \in (c-\delta, c) \cap I$$

and $f(x) < f(c)$ for all $x \in (c, c+\delta) \cap I$

We have already proved the following while proving Interior Extremum Theorem—

Theorem:- Let $I \subset \mathbb{R}$ be an interval and a function $f: I \to \mathbb{R}$ be differentiable at $c \in I$.

(i) If $f'(c) > 0$ then f is increasing at c.

(ii) If $f'(c) < 0$ then f is decreasing at c.

(2) We have an immediate corallary to Interior Extremum Theorem—

Cor:- Let $f: I \to \mathbb{R}$ be continuous on an interval I and suppose that f has a relative ~~maxi~~ extremum at an interior pt. c of I. Then either the derivative of f at c doesn't exist, or it is equal to 0.

Example:- $f(x) = |x|$ has ~~an~~ relative minimum at $x=0$, but f is not differentiable at $x=0$.

(3) Going through the proof, one can immediately see that c has to be an interior pt. of I, otherwise the result is not true. For example:- $f: I \to \mathbb{R}$ is defined as $f(x) = x$. Then $x = 0, 1$ yield the global minimum and maximum respectively (and hence relative ~~maxim~~ minimum and maximum respectively). But at neither pt, f has derivative 0.

We now obtain a sufficient condition for a function to have ~~an~~ relative extremum at an interior pt of an interval.

First Derivative Test for Extrema:

Let f be continuous on the interval $I := [a,b]$ and let c be an interior point of I. Assume that f is differentiable on (a,c) and (c,b). Then:

(a) If there is a neighbourhood $(c-\delta, c+\delta) \subseteq I$ such that $f'(x) \geq 0$ for $c-\delta < x < c$ and $f'(x) \leq 0$ for $c < x < c+\delta$, then f has a relative maximum at c.

(b) If there is a neighbourhood $(c-\delta, c+\delta) \subseteq I$ such that $f'(x) \leq 0$ for $c-\delta < x < c$ and $f'(x) \geq 0$ for $c < x < c+\delta$, ~~then f~~ then f has a relative minimum at c.

Note: The converse of the theorem is not true.

Example: let $f(x) = 2x^2 + x^2 \sin 1/x, \; x \neq 0$

$$= 0, \quad x = 0.$$

Then f has a local minimum at $x=0$.

$$f'(x) = 4x + 2x \sin 1/x - \cos \frac{1}{x}, \quad x \neq 0$$

$$= 0, \quad x = 0$$

Here f' takes both positive and negative values on both sides of 0 (in the immediate neighbourhood)

To prove this, we need one of the most important theorems in Real Analysis, which is as follows:

Rolle's Theorem: Suppose that f is continuous on a closed interval $I := [a,b]$, that the derivative f' exists at every point of the open interval (a,b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a,b) such that $f'(c) = 0$.