$$[r(x)\bar{y}']' + [q(x) + \bar{\lambda}\, p(x)]\bar{y} = 0 \qquad \text{--- (3)}$$

$$a_1 \bar{y}(a) + a_2 \bar{y}'(a) = 0 \qquad \text{--- (4a)}$$
$$b_1 \bar{y}(b) + b_2 \bar{y}'(b) = 0 \qquad \text{--- (4b)}$$

The above equations (3), (4a) and (4b) show that $\bar{y}(x)$ is the eigenfunction corresponding to the eigenvalue $\bar{\lambda} = \alpha - i\beta$.

Multiplying (1) by $\bar{y}$ and (3) by y and subtracting

$$(ry')'\bar{y} - (r\bar{y}')'y + (\lambda - \bar{\lambda})\, p y \bar{y} = 0$$

$$\Rightarrow (\lambda - \bar{\lambda})\, p y \bar{y} = (r\bar{y}')'y - (ry')'\bar{y}$$
$$= \frac{d}{dx}\left\{ (r\bar{y}')y - (ry')\bar{y} \right\}$$

$$\therefore (\lambda - \bar{\lambda}) \int_a^b p y \bar{y}\, dx = \left[(r\bar{y}')y - (ry')\bar{y} \right]_a^b$$

$$= r(b)\left\{ \bar{y}'(b) y(b) - y'(b)\bar{y}(b) \right\}$$
$$- r(a)\left\{ \bar{y}'(a) y(a) - y'(a)\bar{y}(a) \right\}$$
$$= 0 \quad \text{[with the help of previous derivations]}$$

In S-L problem $p(x) > 0$ and $y\bar{y} = |y|^2$

$$\therefore (\lambda - \bar{\lambda}) \int_a^b p(x)\, |y|^2\, dx = 0$$

$\lambda = \alpha + i\beta \quad \bar{\lambda} = \alpha - i\beta \quad \therefore \lambda - \bar{\lambda} = 2i\beta$

$$\rightarrow 2i\beta \int_a^b p|y|^2 dx = 0 \quad \therefore \beta \int_a^b p\, |y|^2\, dx = 0$$

$\int_a^b p|y|^2 dx \neq 0 \quad \therefore \beta = 0$

$\therefore \lambda$ is real.

Ex Find the eigenvalues and the corresponding eigenfunctions of $X'' + \lambda X = 0$, $X(0) = 0$, $X'(L) = 0$.

Solⁿ: Case I Let $\lambda = 0$ Then $X(x) = Ax + B$

Putting B.C. $A = B = 0$. So only trivial solution.

Case-II Let λ be -ve or $\lambda = -\mu^2$, $\mu \neq 0$

$$X'' - \mu^2 X = 0$$

$$X(x) = Ae^{\mu x} + Be^{-\mu x}$$

Putting B.C. $A = B = 0$. So only trivial solution.

Case III Let λ be +ve or $\lambda = \mu^2$, $\mu \neq 0$

$$X'' + \mu^2 X = 0$$

$$X(x) = A\cos\mu x + B\sin\mu x$$

$$X'(x) = -A\mu\sin\mu x + B\mu\cos\mu x$$

$X(0) = 0$ gives $0 = A$

$X'(L) = 0$ $\quad B\mu\cos\mu L = 0$

$B \neq 0$ $\because$ it will again lead to trivial solⁿ.

$\therefore \cos\mu L = 0$

$\therefore \mu L = (2n-1)\frac{\pi}{2}$ $\quad n = 1, 2, 3, \ldots$

$\mu = (2n-1)\frac{\pi}{2L}$

$\therefore X(x) = B\sin\left\{(2n-1)\frac{\pi}{2L}x\right\}$ $\quad n = 1, 2, 3, \ldots$

$\lambda = \mu^2 = (2n-1)^2\frac{\pi^2}{4L^2}$ $\quad n = 1, 2, 3, \ldots$

$\therefore$ $X_n(x) = B_n\sin\left\{(2n-1)\frac{\pi}{2L}x\right\}$

$\lambda_n = (2n-1)^2\frac{\pi^2}{4L^2}$ $\quad n = 1, 2, 3, \ldots$

Ex Find all the eigenvalues and eigenfunctions of

$$4(e^{-x}y')' + (1+\lambda)e^{-x}y = 0$$

$$y(0)=0, \quad y(l)=0$$

Sol^n: $4(e^{-x}y'' - e^{-x}y') + (1+\lambda)e^{-x}y = 0$

$\Rightarrow \quad 4y'' - 4y' + (1+\lambda)y = 0$

Case I Let $\lambda = 0$; $4y'' - 4y' + y = 0$

A.E. $4m^2 - 4m + 1 = 0$

$\Rightarrow (2m-1)^2 = 0 \qquad m = \frac{1}{2}, \frac{1}{2}$

$y(x) = (A + Bx)e^{\frac{x}{2}}$

Putting B.C. only trivial solution is obtained.

Case II Let $\lambda = -\mu^2$

$4y'' - 4y' + (1-\mu^2)y = 0$

A.E. $4m^2 - 4m + (1-\mu^2) = 0$

$$m = \frac{4 \pm \sqrt{16 - 16(1-\mu^2)}}{8} = \frac{1}{2} \pm \frac{\mu}{2}$$

$\therefore y = Ae^{(\frac{1}{2}+\frac{\mu}{2})x} + Be^{(\frac{1}{2}-\frac{\mu}{2})x}$

Putting B.C. $A = B = 0$

$\therefore$ Only trivial solution.

Case III $\quad \lambda = \mu^2$

$$4y'' - 4y' + (1+\mu^2)y = 0$$

$$4m^2 - 4m + (1+\mu^2) = 0$$

$$m = \frac{4 \pm \sqrt{16 - 16(1+\mu^2)}}{8} = \frac{1}{2} \pm i\frac{\mu}{2}$$

$$y(x) = e^{\frac{x}{2}}\left\{A\cos\frac{\mu x}{2} + B\sin\frac{\mu x}{2}\right\}$$

$y(0) = 0$ gives $A = 0$

$y(1) = 0 \quad 0 = B\sin\frac{\mu}{2}$

$\sin\frac{\mu}{2} = 0$ i.e. $\frac{\mu}{2} = n\pi$

i.e. $\mu = 2n\pi \quad n = 1, 2, 3, \ldots$

$$\therefore y(x) = Be^{\frac{x}{2}}\sin n\pi x$$

$\lambda = \mu^2 = 4n^2\pi^2 \quad n = 1, 2, 3, \ldots$

$y_n(x) = e^{\frac{x}{2}}\sin n\pi x \quad$ Taking $B = 1$

$\lambda_n = 4n^2\pi^2 \quad n = 1, 2, 3, \ldots$

Power series solution of second order ODE

Some basic definitions

Power series: An infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots \quad (1)$$

is called a power series in $(x-x_0)$. In particular, a power series in x is an infinite series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

For example, the exponential function e^x has the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots \quad (2)$$

The power series (1) converges (absolutely) for $|x| < R$, where

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

provided the limit exists. R is said to be the radius of convergence of P.S. (1). The interval $(-R, R)$ is said to be the interval of convergence.

The interval of convergence of P.S. (2) is $(-\infty, \infty)$.

Analytic function

A f^n. $f(x)$ defined on an interval containing the point $x = x_0$ is called analytic at x_0 if its Taylor series

$$\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} (x-x_0)^n$$

exists and converges to $f(x)$ for all x in the interval of convergence.

Ordinary and singular points

A point $x = x_0$ is called an ordinary point of the eqn.

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{—}(1)$$

if both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$. If the point $x = x_0$ is not an ordinary point of (1), then it is called a singular point of (1). There are two types of singular points; (i) regular singular point (ii) irregular singular point. A singular point $x = x_0$ of (1) is called a regular singular pt. of (1) if both $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are analytic at $x = x_0$. A singular pt. which is not regular is called an irregular singular point.

Ex Determine whether $x=0$ is an ordinary point or a regular singular point of $2x^2 \frac{d^2y}{dx^2} + 7x(x+1)\frac{dy}{dx} - 3y = 0$

Solⁿ: Comparing with standard eqn. $y'' + P(x)y' + Q(x)y = 0$,

here $P(x) = \frac{7(x+1)}{2x}$ & $Q(x) = -\frac{3}{2x^2}$

$\therefore$ Both $P(x)$ and $Q(x)$ are undefined at $x=0$; so $P(x)$ and $Q(x)$ are not analytic at $x=0$. Thus $x=0$ is not an ordinary point and so $x=0$ is a singular pt.

Also $(x-0)P(x) = \frac{7(x+1)}{2}$ $\quad (x-0)^2 Q(x) = -\frac{3}{2}$

$\therefore$ Both are analytic at $x=0$. $\therefore$ $x=0$ is a regular singular point.

Power series solution about an ordinary point

Ex Find the P.S. solution of the equation

$$(x^2+1)y'' + xy' - xy = 0$$ in powers of x (i.e. about $x=0$)

Sol^n: Here $x=0$ is an ordinary point. To solve the eqn. we take the P.S. $y = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n$

$$\therefore y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substituting in the diff. eqn.

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

$$\Rightarrow 2c_2 + (6c_3 + c_1 - c_0)x + \sum_{n=2}^{\infty}\left[n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_{n-1}\right]x^n = 0$$

$\therefore 2c_2 = 0 \qquad \therefore c_2 = 0$

$6c_3 + c_1 - c_0 = 0 \qquad \therefore c_3 = \dfrac{c_0 - c_1}{6}$

$$n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_{n-1} = 0 \quad \forall\ n \geq 2$$

$$\therefore c_{n+2} = \frac{c_{n-1} - n^2 c_n}{(n+1)(n+2)} \quad \forall\ n \geq 2$$

This is called recurrence relation.

Putting $n=2$ in recurrence relation

$$c_4 = \frac{c_1 - 4c_2}{12} = \frac{c_1}{12} \quad \text{as } c_2 = 0$$

Putting $n=3$, $c_5 = -\frac{9c_3}{20} = -\frac{9}{20}\left(\frac{c_0-c_1}{6}\right) = -\frac{3}{40}(c_0-c_1)$

$$\therefore y = c_0 + c_1 x + \left(\frac{c_0-c_1}{6}\right)x^3 + \frac{1}{12}c_1 x^4 - \frac{3}{40}(c_0-c_1)x^5 + \cdots$$

$$= c_0\left(1+\frac{1}{6}x^3 - \frac{3}{40}x^5 + \cdots\right) + c_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 \cdots\right)$$

which is the reqd. solution near $x=0$ where c_0 & c_1 are arbitrary constants.

Series solution about a regular singular point

Consider the eqn. $x^2\frac{d^2y}{dx^2} + y = 0$ for which $x=0$ is a singular point. Putting $y = \sum_{n=0}^{\infty} a_n x^n$ in the eqn., we get

$$\sum_{2}^{\infty} a_n n(n-1)x^n + \sum_{0}^{\infty} a_n x^n = 0$$

which shows that for values of $n=0,1,2,\cdots$, $a_n = 0$. Hence there is no series of the form $y = \sum_{n=0}^{\infty} a_n x^n$ which may be the solution of the above equation.

With a slight modification, solutions near the singularities can be obtained. This method is known as Frobenius method.

Consider the eqn.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{——— (1)}$$

This is written in the form

$$x^2 \frac{d^2y}{dx^2} + xP(x)\, x\frac{dy}{dx} + x^2 Q(x) y = 0 \qquad -(2)$$

Now $xP(x)$ and $x^2Q(x)$ are regular at $x=0$. So we can put

$$xP(x) = p_0 + p_1 x + \cdots + p_s x^s + \cdots$$
$$x^2 Q(x) = q_0 + q_1 x + \cdots + q_s x^s + \cdots \qquad (3)$$

For the solution of (2), we put

$$y = x^k(c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots), \quad c_0 \neq 0$$
$$= c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \cdots + c_n x^{k+n} + \cdots \qquad (4)$$

where $c_0, c_1, \ldots$ are constants and $c_0 \neq 0$.

From (4),

$$\frac{dy}{dx} = c_0 k x^{k-1} + c_1(k+1)x^k + \cdots + c_n(k+n)x^{k+n-1} + \cdots$$

$$\frac{d^2y}{dx^2} = c_0 k(k-1)x^{k-2} + c_1(k+1)k x^{k-1} + \cdots + c_n(k+n)(k+n-1)x^{k+n-2} + \cdots$$

Substituting these and rearranging terms, L.H.S. of (2) becomes

$$c_0\left\{k(k-1) + kp_0 + q_0\right\} x^{k-2}$$
$$+ \left[c_1\left\{(k+1)k + (k+1)p_0 + q_0\right\} + c_0(kp_1 + q_1)\right] x^{k-1} + \cdots = 0 \quad -(6)$$

Thus (4) will be a sol^n. of eqn. (1) if the coefficients of each term containing similar powers of x on the RHS of (6) be zero.

Equating the coefficients of x^{k-1} to zero,

$$c_1\{(k+1)k+(k+1)p_0+q_0\} + c_0(kp_1+q_1) = 0$$

This gives $\frac{c_1}{c_0}$ = a f^n of $k = f_1(k)$, say

Similarly, equating the coefficients of x^{k} to zero, we shall get c_2 in terms of c_1 and c_0 and hence in terms of c_0 only so that

$$\frac{c_2}{c_0} = f_2(k) \text{ say and so on}$$

Thus equating the coefficients of all powers of x on the ~~RHS~~ LHS of (6) other than that of x^{k-2} to zero, we get

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = c_0\{k(k-1)+kp_0+q_0\}x^{k-2} \quad \text{— (7)}$$

Then y is obtained as

$$y = c_0 x^k\left(1+\frac{c_1}{c_0}x+\frac{c_2}{c_0}x^2+\cdots+\frac{c_n}{c_0}x^n+\cdots\right)$$
$$= c_0x^k\left(1+xf_1(k)+x^2f_2(k)+\cdots+x^nf_n(k)+\cdots\right) \quad \text{— (8)}$$

From (7), $k^2+(p_0-1)k+q_0=0$ since $c_0 \neq 0$ — (9)
This eqn. is obtained by equating the coefficient of the lowest degree term of x on the RHS of (6) to zero and is called the indicial eqn. This eqn. gives the possible values of k in (4). Roots of the indicial eqn. will give the exponent of x with which the series for y will commence in (4).

Three important cases may arise !

The roots of the indicial eqn.

(i) may have distinct roots, not differing by an integer

(ii) may be equal

(iii) are distinct but differ by an integer

Subcase of (iii)

(a) Some coefficient becomes infinity

(b) " " " indeterminate.

Details can not be discussed here. We move directly to solution of Legendre differential eqn.