Ex Find the Green's function for the BVP

$$\frac{d^2y}{dx^2} + \mu^2 y = 0 \qquad y(0) = y(1) = 0$$

Sol^n:

$$y'' + \mu^2 y = 0 \quad \text{--- (1)}$$
$$y(0) = 0 \quad \text{--- (2)}$$
$$y(1) = 0 \quad \text{--- (3)}$$

G.S. of (1) $\rightarrow$ $y(x) = A\cos\mu x + B\sin\mu x$

Using B.C. $A = 0$, $B = 0$

$\therefore$ The BVP has only trivial sol^n.

$\therefore$ Unique Green's f^n. exists and is given by

$$G(x,t) = \begin{cases} a_1\cos\mu x + a_2\sin\mu x & 0 \le x < t \\ b_1\cos\mu x + b_2\sin\mu x & t < x \le 1 \end{cases} \quad \text{--- (4)}$$

Green's f^n. must also satisfy the following 3 properties

(i) $G(x,t)$ is continuous at $x = t$ i.e.

$$b_1\cos\mu t + b_2\sin\mu t = a_1\cos\mu t + a_2\sin\mu t$$
$$\Rightarrow (b_1 - a_1)\cos\mu t + (b_2 - a_2)\sin\mu t = 0 \quad \text{--- (5)}$$

(ii) The derivative of G has a discontinuity of magnitude $-\frac{1}{p_0(t)}$ at the point $x = t$, where $p_0(x) = 1$

$$\therefore \left(\frac{\partial G}{\partial x}\right)_{x = t+0} - \left(\frac{\partial G}{\partial x}\right)_{x = t-0} = -1$$

$$\Rightarrow \mu(-b_1\sin\mu t + b_2\cos\mu t) - \mu(-a_1\sin\mu t + a_2\cos\mu t) = -1$$

$$\Rightarrow -(b_1 - a_1)\sin\mu t + (b_2 - a_2)\cos\mu t = -\frac{1}{\mu} \quad \text{--- (6)}$$

(iii) $G(x,t)$ must satisfy the B.C. (2) and (3) i.e.

$$G(0,t)=0 \quad \text{so that} \quad a_1 = 0$$

and $\quad G(1,t)=0 \quad$ " " $\quad b_1\cos\mu + b_2\sin\mu = 0$

Let $\quad b_1 - a_1 = c_1$

$\quad\quad b_2 - a_2 = c_2$

Rewriting (5) & (6),

$$c_1\cos\mu t + c_2\sin\mu t = 0 \qquad \text{--- (7)}$$

$$-c_1\sin\mu t + c_2\cos\mu t + \frac{1}{\mu} = 0 \qquad \text{--- (8)}$$

$$\therefore \frac{c_1}{\frac{1}{\mu}\sin\mu t} = \frac{c_2}{-\frac{1}{\mu}\cos\mu t} = \frac{1}{\cos^2\mu t + \sin^2\mu t}$$

$$\therefore c_1 = \frac{1}{\mu}\sin\mu t \qquad c_2 = -\frac{1}{\mu}\cos\mu t$$

$$\therefore b_1 - a_1 = \frac{1}{\mu}\sin\mu t \qquad \text{--- (9)}$$

$$b_2 - a_2 = -\frac{1}{\mu}\cos\mu t \qquad \text{--- (10)}$$

Solving $\; a_1 = 0,\; b_1 = \frac{1}{\mu}\sin\mu t,\; b_2 = -\dfrac{\sin\mu t\cos\mu}{\mu\sin\mu}$

$$a_2 = -\frac{\sin\mu t\cos\mu}{\mu\sin\mu} + \frac{\cos\mu t}{\mu} = -\frac{\sin\mu(t-1)}{\mu\sin\mu}$$

$$a_1\cos\mu x + a_2\sin\mu x = -\frac{\sin\mu(t-1)\sin\mu x}{\mu\sin\mu}$$

$$b_1\cos\mu x + b_2\sin\mu x = -\frac{\sin\mu t\sin\mu(x-1)}{\mu\sin\mu}$$

$$G(x,t) = \begin{cases} -\dfrac{\sin\mu(t-1)\sin\mu x}{\mu\sin\mu} & 0 \le x < t \\[2ex] -\dfrac{\sin\mu t\sin\mu(x-1)}{\mu\sin\mu} & t < x \le 1 \end{cases}$$

Ex- Construct Green's function for the homogeneous BVP $\frac{d^4y}{dx^4} = 0$

$$y(0) = y'(0) = y(1) = y'(1) = 0$$

Sol^n: $y(x) = A + Bx + Cx^2 + Dx^3$

Applying B.C. $A = B = C = D = 0$

$\therefore$ Only trivial sol^n exists.

So Green's f^n is unique.

Let $G(x,t) = \begin{cases} a_1 + a_2 x + a_3 x^2 + a_4 x^3 & 0 \le x < t \\ b_1 + b_2 x + b_3 x^2 + b_4 x^3 & t < x \le 1 \end{cases}$ — (1)

(i) $G(x,t)$, $\frac{\partial G}{\partial x}$, $\frac{\partial^2 G}{\partial x^2}$ are continuous at $x = t$

$\therefore b_1 + b_2 t + b_3 t^2 + b_4 t^3 = a_1 + a_2 t + a_3 t^2 + a_4 t^3$ — (2)

$b_2 + 2b_3 t + 3b_4 t^2 = a_2 + 2a_3 t + 3a_4 t^2$ — (3)

$2b_3 + 6b_4 t = 2a_3 + 6a_4 t$ — (4)

(ii) $\frac{\partial^3 G}{\partial x^3}$ has a discontinuity of magnitude $-\frac{1}{p_0(t)}$

i.e. $\left(\frac{\partial^3 G}{\partial x^3}\right)_{x = t+0} - \left(\frac{\partial^3 G}{\partial x^3}\right)_{x = t-0} = -1$

$\Rightarrow 6b_4 - 6a_4 = -1$ — (5)

(iii) $G(x,t)$ satisfies the B.C.

i.e. $G(0,t) = 0$ so that $a_1 = 0$ — (6)

$G'(0,t) = 0$ " " $a_2 = 0$ — (7)

$G(1,t) = 0$ " " $b_1 + b_2 + b_3 + b_4 = 0$ — (8)

$G'(1,t) = 0$ " " $b_1 + 2b_3 + 3b_4 = 0$ — (9)

Let $c_k = b_k - a_k \qquad k = 1,2,3,4$

From (2), (3), (4) & (5) become

$$c_1 + c_2 t + c_3 t^2 + c_4 t^3 = 0 \qquad -(10)$$
$$c_2 + 2c_3 t + 3c_4 t^2 = 0 \qquad -(11)$$
$$2c_3 + 6c_4 t = 0 \qquad -(12)$$
$$6c_4 = -1 \qquad -(13)$$

Solving $c_4 = -\frac{1}{6}, \; c_3 = \frac{t}{2}, \; c_2 = -\frac{t^2}{2}, \; c_1 = \frac{t^3}{6} \quad -(14)$

$$\therefore \quad b_1 - a_1 = \frac{t^3}{6} \qquad -(15)$$
$$b_2 - a_2 = -\frac{t^2}{2} \qquad -(16)$$
$$b_3 - a_3 = \frac{t}{2} \qquad -(17)$$
$$b_4 - a_4 = -\frac{1}{6} \qquad -(18)$$

Solving (6), (7), (8), (9), (15), (16), (17), (18)

$a_1 = 0, \; a_2 = 0, \; a_3 = -\frac{t}{2} + t^2 - \frac{t^3}{2}, \; a_4 = \frac{1}{6} - \frac{t^2}{2} + \frac{t^3}{3}$

$b_1 = \frac{t^3}{6}, \; b_2 = -\frac{t^2}{2}, \; b_3 = -\frac{t^3}{2} + t^2, \; b_4 = -\frac{t^2}{2} + \frac{t^3}{3}$

$$\therefore G(x,t) = \begin{cases} x^2\left(t^2 - \frac{t}{2} - \frac{t^3}{2}\right) + x^3\left(\frac{1}{6} - \frac{t^2}{2} + \frac{t^3}{3}\right) & 0 \le x < t \\ t^2\left(x^2 - \frac{x}{2} - \frac{x^3}{2}\right) + t^3\left(\frac{1}{6} - \frac{x^2}{2} + \frac{x^3}{3}\right) & t < x \le 1 \end{cases}$$