The concept of Green's function

Boundary value problems ~~or~~ often arises in Mathematics and their solution is a major concern to the mathematicians. Here we explain one particular method to solve a BVP that needs construction of a function called Green's function. First we will discuss Method of variation of parameter to solve a 2nd order linear ODE.

Method of variation of parameters

Let us consider $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = R(x)$ — (1)

Let the C.F. of Eq. (1) be known i.e. the G.S. of

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = 0 \qquad \text{— (2)}$$

is known. Let the G.S. of (2) be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are constants, y_1 and y_2 are two L.I. sol^{n}s. of (2). We replace c_1 and c_2 by unknown functions $v_1(x)$ and $v_2(x)$. Our problem is to determine v_1 and v_2 so that

$$y_p = v_1 y_1 + v_2 y_2 \qquad \text{— (3)}$$

satisfies (1). This will be a particular solution of (1). Eq. (3) will satisfy Eq. (1).

We differentiate Eq. (3) and obtain

$$Dy_p = (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2) \qquad \text{— (4)}$$

We choose v_1 and v_2 in such a way that

$$v_1' y_1 + v_2' y_2 = 0 \qquad \text{— (5)}$$

This will reduce Eq. (4) to

$$Dy_p = v_1 y_1' + v_2 y_2' \quad \text{— (6)}$$

$$\therefore D^2 y_p = (v_1 y_1'' + v_2 y_2'') + (v_1' y_1' + v_2' y_2') \quad \text{— (7)}$$

On substituting (3), (6), (7) into (1) and rearranging,

$$v_1(y_1'' + Py_1' + Qy_1) + v_2(y_2'' + Py_2' + Qy_2) + v_1' y_1' + v_2' y_2' = R(x) \quad \text{— (8)}$$

Since y_1 and y_2 are solutions of Eq. (2),

$$y_1'' + Py_1' + Qy_1 = 0$$

$$y_2'' + Py_2' + Qy_2 = 0$$

and Eq. (8) becomes $\quad v_1' y_1' + v_2' y_2' = R(x) \quad$ — (9)

We have thus two equations (5) and (9) from which we can solve for v_1' and v_2'. In fact,

$$v_1' = \frac{-y_2 R(x)}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \text{and} \quad v_2' = \frac{y_1 R(x)}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Denominators are Wronskians $W(y_1, y_2)$ and it is not zero as y_1 and y_2 are L.I.

$$\therefore v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)}\, dx, \quad v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)}\, dx$$

$\therefore$ We obtain a particular solⁿ. of Eq. (1) namely

$$y_p = v_1(x) y_1(x) + v_2(x) y_2(x)$$

Now we will come back to our original point of discussion (Green's function).

Let us consider the ODE

$$\frac{d^2u}{dx^2} + k^2u = -f(x) \qquad 0<x<l \quad -(1)$$

with $u(0) = u(l) = 0 \qquad -(2)$

To solve the BVP (1) with (2), we employ method of variation of parameter.

Let the solution be of the form

$$u(x) = A(x)\cos kx + B(x)\sin kx \quad -(3)$$

$$Du = -kA(x)\sin kx + kB(x)\cos kx + A'(x)\cos kx + B'(x)\sin kx$$

$$A'(x)\cos kx + B'(x)\sin kx = 0 \quad -(4)$$

$$\therefore D^2u = -k^2A(x)\cos kx - k^2B(x)\sin kx - kA'(x)\sin kx + kB'(x)\cos kx$$

Substituting in (1),

$$-k^2A(x)\cos kx - k^2B(x)\sin kx - kA'(x)\sin kx + kB'(x)\cos kx + k^2A(x)\cos kx + k^2B(x)\sin kx = -f(x)$$

$$\Rightarrow A(x)\left[-k^2\cos kx + k^2\cos kx\right] + B(x)\left[-k^2\sin kx + k^2\sin kx\right] - kA'(x)\sin kx + kB'(x)\cos kx = -f(x)$$

$$\Rightarrow -kA'(x)\sin kx + kB'(x)\cos kx = -f(x) \quad -(5)$$

So solving Eq. (4) and (5), we get

$$A'(x) = \frac{f(x)\sin kx}{k}, \qquad B'(x) = -\frac{f(x)\cos kx}{k}$$

∴ Solution of (1) can be written as

$$u(x) = \frac{\cos kx}{k}\int_{c_1}^{x} f(y)\sin ky\,dy - \frac{\sin kx}{k}\int_{c_2}^{x} f(y)\cos ky\,dy \quad \text{--- (7)}$$

where c_1 and c_2 are constants which must be so chosen as to ensure that the B.C. (2) are satisfied.

Inserting the condition $u(0)=0$ in (7), we find that we must choose c_1 such that

$$\int_{c_1}^{0} f(y)\sin ky\,dy = 0 \quad \text{--- (8)}$$

Since $f(y)$ is assumed arbitrary, we must choose $c_1 = 0$.

The condition $u(l)=0$ when inserted in (7) will require that

$$u(l) = \frac{\cos kl}{k}\int_{0}^{l} f(y)\sin ky\,dy - \frac{\sin kl}{k}\int_{c_2}^{l} f(y)\cos ky\,dy = 0$$

$$\Rightarrow \frac{-\sin kl}{k}\int_{c_2}^{0} f(y)\cos ky\,dy - \frac{\sin kl}{k}\int_{0}^{l} f(y)\cos ky\,dy$$
$$+ \frac{\cos kl}{k}\int_{0}^{l} f(y)\sin ky\,dy = 0$$

$$\Rightarrow -\frac{\sin kl}{k}\int_{c_2}^{0} f(y)\cos ky\,dy + \frac{1}{k}\int_{0}^{l} f(y)\sin k(y-l)\,dy = 0 \quad \text{--- (9)}$$

$$\Rightarrow -\frac{\sin kl}{k}\int_{c_2}^{x} f(y)\cos ky\,dy - \frac{\sin kl}{k}\int_{x}^{0} f(y)\cos ky\,dy$$
$$= -\frac{1}{k}\int_{0}^{l} f(y)\sin k(y-l)\,dy$$

$$\Rightarrow -\frac{\sin kl}{k}\int_{c_2}^{x} f(y)\cos ky\,dy = -\frac{\sin kl}{k}\int_{0}^{x} f(y)\cos ky\,dy$$
$$-\frac{1}{k}\int_{0}^{l} f(y)\sin k(y-l)\,dy$$