

~~Part~~
07/11/2019
Defn:-

Lecture 14

(1)

Length or Magnitude of a vector

The magnitude (or length) \overrightarrow{A} of a

contravariant vector A^i is

defined by

$$(A)^2 = e_{(A)} g^{ij} A^i A^j$$

where $e_{(A)}$ is the indicator $\begin{cases} +1 & \text{if } A \text{ real} \\ -1 & \text{if } A \text{ not real} \end{cases}$ (1)

The magnitude A is an invariant (ie, $\overline{A} = A$) : (how? EX)

Similarly, the magnitude B of a covariant vector B_i is defined by the "g" $(B)^2 = e_{(B)} g^{ij} B_i B_j$ (2)

where $\epsilon(B)$ is the indicator

of the vector B_i ; & it is

clear that B is an invariant
 (\equiv) (i.e., $\bar{B} = B$)

Also, a vector whose magnitude
is unity, is called a Unit

vector & a vector whose

magnitude is zero, is called
a null vector.

. Unit tangent vector:

We have,

$$ds^2 = g_{ij} dx^i dx^j$$

where from $g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = e = \pm 1$

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This shows that

$\frac{dx^i}{ds}$ is a contravariant vector of magnitude one,
(ie, unit contravariant vector)

& this unit vector is denoted
as unit tangent vector
to some curve C in a
Riemannian space V_N .

g/ Associate Tensor

The inner product of the fundamental tensor g_{ij} &
the contravariant vector A^j

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is the covariant vector $g_{ij} A^j$

which is said to associate to A_i . We define

$$A_j = g_{ij} A^j \rightarrow (1)$$

Similarly, we define $A^i = g^{ij} A_j$

form covariant

vector A^i .

The vectors A^i & A_j are associated to each other.

The relation bet' n a vector & its

associate is

reciprocal. form

the vector associate to A^i is

$$\begin{aligned} & \sum_{j=1}^N g^{ij} A_j \\ &= g^{i1} A_1 + g^{i2} A_2 \\ &+ \dots + g^{ii} A_i \\ &+ \dots + g^{iN} A_N \\ &= \boxed{\textcircled{A}^i} A^i \end{aligned}$$

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$$g^{ij} A_j = \underbrace{g^{ij} g_{jk}}_{= S_k^i} A^k \quad [Eq(1)]$$

$$= S_k^i A^k = A^i$$

This process of association
is often referred to as
'lowering the superscript'
or 'raising the subscript' nicely

$$\text{Also, } e_{(A)} g^{ij} \cancel{A^i} A^j = e_{(A)} g^{ij} g^{ik} A_k.$$

$$= e_{(A)} \cancel{\int^{kl} A_k A_l} \left[g^{ij} g^{ik} \right] = g_{kj}$$

which shows that the magnitude of associate vectors $\left[g_{ki} g_{il} \right] = \cancel{g^{kl}}$

"Angle bet'n vectors":

Defn:- The angle bet'n two
unit vectors $A^i \& B^i$
 is defined by

$$\begin{aligned}\cos \theta &= g_{ij} A^i B^j = A_j B^j \\ &= g^{jk} \underbrace{A_j B_k}_{\rightarrow (1)} = A^k B_k.\end{aligned}$$

It follows from (1) that
 the angle bet'n two vectors
 $A^i \& B^i$ which are not
 necessarily unit vectors,
 is given by

$$\therefore \cos \theta = \frac{g_{ij} A^i B^j}{\sqrt{e_{(A)} e_{(B)}} \lambda_m^{1/2} \lambda_n^{1/2} g_{mn}}$$

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\rightarrow (2)

• Orthogonality :-

Two vectors are said to be orthogonal to one another if the angle bet'n them is a right angle.

[i.e., $\theta = 90^\circ$, $\therefore \cos \theta = 0$.]

The N/S condition

for the orthogonality of two vectors

$A^i \otimes B^j$ is

$$\underbrace{g_{ij}}_{A^i B^j = 0} \rightarrow \textcircled{3}$$

* ~~Yours~~

8) Christoffel symbols
or, Brackets

We first investigate the
two functions formed

from the fundamental

tensor g_{ij} . These are

called the Christoffel
symbols or brackets of the
first & second kinds, defined

respectively by

$$[ijj,k] = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{jj}}{\partial x^k} \right) \quad \rightarrow ①$$

$$\epsilon [ij] = g^{lk} [ij,k] \quad \rightarrow ②$$

Other symbols used for

$$[ij] \text{ are } [ij,l] \text{ or, } [ij]$$

The christoffel symbols of
1st & 2nd kind $[ij,k]$ &
 $[ij]$ are not tensors.

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Properties :-

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I) Christoffel symbols of both kinds are symmetric w.r.t the indices $i \& j$,

i.e., $[i^j; k] = [j^i, k]$

$$\text{e.g. } [i^j; k] = [j^l; i].$$

Soln :- By defn

$$[i^j; k] = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)$$

As g_{ij} is symmetric $\Rightarrow g_{ij} = g_{ji}$

$$[i^j; k] = \frac{1}{2} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k} \right)$$

$$= [j^i; k] \rightarrow (3)$$

$$= [j^i; k] = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^k} \right)$$

$$\text{Again, } \left[\begin{smallmatrix} 1 \\ i; j \end{smallmatrix} \right] = g^{lk} \left[\begin{smallmatrix} i; j, k \end{smallmatrix} \right] \quad (3)$$

$$= g^{lk} \left[\begin{smallmatrix} j; i, k \end{smallmatrix} \right]$$

$$\therefore \left[\begin{smallmatrix} 1 \\ i; j \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 \\ j; i \end{smallmatrix} \right]. \quad \checkmark \rightarrow (4)$$

$$\text{II) } \left[\begin{smallmatrix} 1 \\ i; j, m \end{smallmatrix} \right] = g_{lm} \left[\begin{smallmatrix} 1 \\ i; j \end{smallmatrix} \right]$$

Sol:- We have, $\left[\begin{smallmatrix} 1 \\ i; j \end{smallmatrix} \right] = g^{lk} \left[\begin{smallmatrix} i; j, k \end{smallmatrix} \right] \rightarrow (1)$

Inner multiplication of (1) by g_{lm}

$$\begin{aligned} g_{lm} \left[\begin{smallmatrix} 1 \\ i; j \end{smallmatrix} \right] &= g_{lm} g^{lk} \left[\begin{smallmatrix} i; j, k \end{smallmatrix} \right] \quad \text{gives} \\ &= \delta_m^k \left[\begin{smallmatrix} i; j, k \end{smallmatrix} \right] \quad \left[\because g_{lm} g^{lk} = \delta_m^k \right] \end{aligned}$$

$$\therefore f_{lm} \begin{bmatrix} 1 \\ i, j \end{bmatrix} = f_m^k \begin{bmatrix} 1 \\ i, j, k \end{bmatrix}$$

↑

$$= \begin{bmatrix} 1 \\ i, j, m \end{bmatrix} \left[\sum_{k=1}^n f_m^k \begin{bmatrix} 1 \\ i, j, k \end{bmatrix} \right] = \begin{bmatrix} 1 \\ i, j, m \end{bmatrix}$$

Thus, $\begin{bmatrix} 1 \\ i, j, m \end{bmatrix} = f_{lm} \begin{bmatrix} 1 \\ i, j \end{bmatrix}$,

~~HW~~ show that

(iii) $\frac{\partial g_{ik}}{\partial x^j} = \underbrace{\begin{bmatrix} 1 \\ i, j, k \end{bmatrix}}_{\text{red}} + \underbrace{\begin{bmatrix} k \\ k, j, i \end{bmatrix}}_{\text{red}}$

(iv) $\frac{\partial g^{mk}}{\partial x^l} = -g^{mi} \begin{bmatrix} k \\ i, l \end{bmatrix} - g^{ki} \begin{bmatrix} m \\ i, l \end{bmatrix}$

Hint: - we know $g_{ik} g_{ij} = S_j^k = \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases}$

Diff. w.r.t x^l .