Date
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Lecture 7

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EX/ Show that

(i) $x \sin x = 2\left(2^2 J_2 - 4^2 J_4 + 6^2 J_6 - \dots\right)$

HW (ii). $x \cos x = 2\left(1^2 J_1 - 3^2 J_3 + 5^2 J_5 - \dots\right)$

Sol^n:- We know that.

$$\cos(x\sin\phi) = J_0 + 2\cos(2\phi)J_2 + 2\cos(4\phi)J_4 + \dots \rightarrow (1)$$

Differentiating (1) w.r.t 'ϕ'

$$-\sin(x\sin\phi)\cdot x\cos\phi = 0 - 2\cdot 2 J_2 \sin(2\phi) - 2\cdot 4\sin(4\phi)J_4 + \dots \rightarrow (2)$$

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Differentiating (2) w.r.t ϕ

$$-\cos(x\sin\phi)\cdot(x\cos\phi)^2 + \sin(x\sin\phi)\cdot(x\sin\phi)$$

$$= -2\cdot 2^2 J_2 \cos(2\phi)$$
$$-2\cdot 4^2\cdot J_4 \cos(4\phi)$$
$$-2\cdot 6^2\cdot J_6 \cos(6\phi)$$
$$-\cdots \longrightarrow (3)$$

Replacing ϕ by $(\pi/2)$ in eq^n (3) we get

$$x\sin x = 2\left(2^2 J_2 - 4^2 J_4 + 6^2 J_6 - \cdots\right)$$

§/ Bessel's integrals

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Show that

(i) $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi$,

where n is a positive integer.

(ii) $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi$

where n is any integer

HW

(iii) $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\phi)\,d\phi$

$= \frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\,d\phi$

HW

(iv) Deduce that

$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{\left(2^m \cdot m!\right)^2}$$

<u>Sol^n</u> :- (i) We will use the following results:

$$\int_0^{\pi} \cos m\phi \cos n\phi \, d\phi = \int_0^{\pi} \sin(m\phi)\sin(n\phi)\, d\phi$$

$$= \begin{cases} \pi/2, & \text{when } m = n \\ 0, & \text{when } m \neq n. \end{cases} \longrightarrow (1)$$

$$\& \cos(x \sin\phi) = J_0 + 2J_2 \cos(2\phi) + 2J_4 \cos(4\phi) + \cdots \longrightarrow (2)$$

$$\& \sin(x \sin\phi) = 2J_1 \sin\phi + 2J_3 \sin 3\phi + 2J_5 \sin 5\phi + \cdots \longrightarrow (3)$$

Multiplying eqn 2 by $\cos(n\phi)$ & then integrate w.r.t 'ϕ' betn limits

Hard Ex 0 to π & using 1, we get

$$\int_{\phi=0}^{\pi} \cos(x\sin\phi).\cos n\phi \, d\phi = 0, \text{ if } n \text{ is odd} \rightarrow (4)$$

$$= \pi J_n, \text{ if } n \text{ is even} \rightarrow (5)$$

Again, Multiplying both sides 2 3 by $\sin(n\phi)$ & then integrating w.r.t 'ϕ' betn the limits 0 to π & using 1, we get –

Ex

$$\int_0^{\pi} \sin(x\sin\phi).\sin n\phi \, d\phi = \begin{cases} \pi J_n, \text{ if } n \text{ is odd} \rightarrow (6) \\ 0, \text{ if } n \text{ is even} \rightarrow (7) \end{cases}$$

Let n be odd.

Adding 4 & 6, we get

$$\int_0^{\pi} \left[\cos(x\sin\phi)\cos n\phi + \sin(x\sin\phi)\sin n\phi\right] d\phi$$

$$= \pi J_n$$

or $\displaystyle\int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi = \pi J_n$

$$\Rightarrow J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi \longrightarrow 8$$

Next, let n be even. Then adding 5 & 7, we get as before

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi$$

$\therefore$ eqn (8) holds for each positive integer (even as well as odd).

(ii) Let n be any integer. Then as in part (i), if n is a positive integer, we have

$$\boxed{J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\,d\phi} \quad \text{(9)}$$

Next, let n be a negative integer, so that $n = -m$ where m is a positive integer.

To prove $J_{-m}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(-m\phi - x\sin\phi)\,d\phi$ (10)

Let $\phi = \pi - \theta$

so that $d\phi = -d\theta$

Limits of

when $\phi = 0$, $\theta = \pi$

& $\phi = \pi$, $\theta = 0$

Then, we have

R.H.S of eqn (10)

$$= \frac{1}{\pi}\int_{\pi}^{0} \cos\left[\left(-m(\pi-\theta) - x\sin(\pi-\theta)\right)\right](-d\theta)$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos\left[(m\theta - x\sin\theta) - m\pi\right] d\theta$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left[\cos(m\theta - x\sin\theta)\underbrace{\cos(m\pi)}_{=(-1)^m} + \sin(m\theta - x\sin\theta)\cdot\underbrace{\sin(m\pi)}_{(=0)}\right] d\theta$$

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$$= \frac{1}{\pi}\int_0^{\pi} (-1)^m \cos(m\theta - x\sin\theta)\, d\theta$$

$$= \frac{(-1)^m}{\pi}\int_0^{\pi} \cos(m\phi - x\sin\phi)\, d\phi$$

(changing θ by ϕ)

$= (-1)^m J_m(x)$ [using eqn 9, as

$= J_{-m}(x)$ m is a positive integer]

$=$ L.H.S of eqn 10.

// proved.

Thus, eqn 9 & 10 shows that the reqd. result holds for any integer.

HW Q) Use the generating function to show that

$$J_n(-x) = (-1)^n J_n(x)$$

Sol^n:- We know that

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = \exp\left\{\frac{x}{2}\left(z - \frac{1}{z}\right)\right\} \rightarrow (*)$$

Hint:- Replace x by $(-x)$ in $(*)$.

Orthogonality of Bessel's function

If λ_i & λ_j are roots of the equation

$$J_n(\lambda a) = 0,$$ then

$$\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx$$

$$= \begin{cases} 0, & \text{if } i \neq j \text{ (different roots)} \\ \dfrac{a^2}{2} J_{n+1}^2(\lambda_i a), & \text{if } i = j \text{ (equal roots)} \end{cases}$$

i.e., $\int_0^a x J_n(\lambda_i x) \cdot J_n(\lambda_j x)\, dx$

$$= \frac{a^2}{2} J_{n+1}^2(\lambda_i a)\, \delta_{ij},$$

where δ_{ij} = kronecker delta

$$= \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

Proof :— **Case 1** :— Let $i \neq j$ & let λ_i & λ_j be unequal roots (distinct) of $J_n(\lambda a) = 0$.

$\therefore J_n(\lambda_i a) = 0$ & $J_n(\lambda_j a) = 0 \rightarrow (1)$

Let $u(x) = J_n(\lambda_i x)$ & $v(x) = J_n(\lambda_j x)$ $\rightarrow (2)$

then u & v are Bessel's function satisfying the modified Bessel's equation

$$x^2 y'' + x y' + (\lambda^2 x^2 - n^2) y = 0 \rightarrow (3)$$

$$\therefore x^2 u'' + x u' + (\lambda_i^2 x^2 - n^2) u = 0 \rightarrow (4)$$

$$\& \quad x^2 v'' + x v' + (\lambda_j^2 x^2 - n^2) v = 0 \rightarrow (5)$$

Multiplying (4) by v & eqn (5) by u & then subtracting, we get

$$x^2(vu'' - uv'') + x(u'v - v'u) + x^2(\lambda_i^2 - \lambda_j^2)uv = 0$$

$(\div x)$

or, $$x(vu'' - uv'') + (u'v - v'u) = x(\lambda_j^2 - \lambda_i^2)uv$$

or, $$x\frac{d}{dx}(vu' - uv') + 1.(u'v - v'u) = x(\lambda_j^2 - \lambda_i^2)uv$$

or, $$\frac{d}{dx}\left\{x(vu' - uv')\right\} = x(\lambda_j^2 - \lambda_i^2)uv \rightarrow (6)$$

Integrating (6) w.r.t x from 0 to a,

$$(\lambda_j^2-\lambda_i^2)\int_0^a x\,u\,v\,dx = \Big[x(vu'-uv')\Big]_{x=0}^{a} \longrightarrow (7)$$

Using (2), eqn (7) reduces to

$$(\lambda_j^2-\lambda_i^2)\int_0^a x\,J_n(\lambda_i x)\cdot J_n(\lambda_j x)\,dx$$

$$= \Big[x\big\{J_n(\lambda_j x)\cdot J_n'(\lambda_i x) - J_n(\lambda_i x)\cdot J_n'(\lambda_j x)\big\}\Big]_0^a$$

$$= a\big\{\underbrace{J_n(\lambda_j a)}_{=0} J_n'(\lambda_i a) - \underbrace{J_n(\lambda_i a)}_{=0}\cdot J_n'(\lambda_j a)\big\}$$

$= 0$ [by (1)]

Since, $\lambda_i \neq \lambda_j$, the above eq^n gives

$$\int_0^a x\, J_n(\lambda_i x)\cdot J_n(\lambda_j x)\, dx$$

$$= 0, \text{ when } i \neq j \quad \text{——} \quad (8)$$

Case II :- Let $i = j$ (equal roots)

Multiplying eq^n (4) by $2u'$, we have

$$\underbrace{2x^2 u'' u' + 2x u'^2 + 2(\lambda_i^2 x^2 - n^2)}\, uu' = 0$$

or, $$\frac{d}{dx}\left\{x^2 u'^2 - n^2 u^2 + \lambda_i^2 x^2 u^2\right\} - 2\lambda_i^2 x u^2 = 0$$

$\Rightarrow 2\lambda_i^2 x u^2 = \frac{d}{dx}\left(x^2 u'^2 - n^2 u^2 + \lambda_i^2 x^2 u^2\right)$ $\rightarrow$ (9)

Integrating (9) w.r.t 'x' from 0 to a, we get

$$2\lambda_i^2 \int_0^a x u^2 \, dx = \left[x^2 u'^2 - n^2 u^2 + \lambda_i^2 x^2 u^2\right]_0^a$$ $\rightarrow$ (10)

Using (1) & (2) & noting that $J_n(0) = 0$, we have

$$2\lambda_i^2 \int_0^a x J_n^2(\lambda_i x) \, dx = a^2\left[\{J_n'(\lambda_i x)\}^2\right]_{x=a}$$ $\rightarrow$ (11)

From recurrence relation IV, we have

$$\frac{d}{dx}\{J_n(x)\} = \frac{n}{x}J_n(x) - J_{n+1}(x) \rightarrow (12)$$

Replacing x by $\lambda_i x$ in (12), we get

$$\frac{d}{d(\lambda_i x)} J_n(\lambda_i x) = \frac{n}{(\lambda_i x)} J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$$

or, $$\frac{1}{\lambda_i}\frac{d}{dx}\{J_n(\lambda_i x)\} = \frac{n}{(\lambda_i x)} J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$$

or, $$J_n'(\lambda_i x) = \frac{n}{x} J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x)$$

$$\therefore \left[\left\{J_n'(\lambda_i x)\right\}^2\right]_{x=a}$$

$$= \left[\left\{\frac{n}{x} J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x)\right\}^2\right]_{x=a}$$

$$= \left\{0 - \lambda_i J_{n+1}(\lambda_i a)\right\}^2 \quad [\text{by } (1)]$$

$$= \lambda_i^2 J_{n+1}^2(\lambda_i a)$$

Putting this value in eq^n (11) & dividing both sides of the resulting eq^n by $(2\lambda_i^2)$, we get

$$\int_0^a x J_n^2(\lambda_i x)\,dx = \frac{a^2}{2} J_{n+1}^2(\lambda_i a).$$

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Combining (8) & (13), we have

$$\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\,dx$$

$$= \frac{a^2}{2} J_{n+1}^2(\lambda_i a)\,\delta_{ij}$$

(14)

Hypergeometric Function

§/ Pochhammer Symbol (Defⁿ) :-

Let n be a positive integer, then Pochhamer symbol is denoted and defined by

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) \rightarrow (1)$$

$$(\alpha)_0 = 1 \rightarrow (2)$$

Deduction :-

1) $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$

$$\therefore (\alpha)_n = \frac{1.2.3\ldots(\alpha-1)\alpha(\alpha+1)\ldots(\alpha+n-1)}{1.2.3\ldots(\alpha-1)}$$

$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad \left[\begin{array}{l}\text{as} \\ \Gamma(p) = (p-1)\Gamma(p-1)\end{array}\right.$$

Thus, $(\alpha)_n = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \rightarrow (3)$

2) $(\alpha)_{n+1} = \alpha(\alpha+1)(\alpha+2)\ldots[\alpha+n+1-1]$

$$= \alpha\left[(\alpha+1)(\alpha+2)\ldots(\alpha+1+n-1)\right]$$

$$= \alpha(\alpha+1)_n$$

Thus, $(\alpha)_{n+1} = \alpha(\alpha+1)_n \rightarrow (4)$

3) $(\alpha+n)(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1).(\alpha+n)$

$= \alpha(\alpha+1)\cdots(\alpha+n-1)(\alpha+n+1-1)$

$= (\alpha)_{n+1}$.

Thus, $(\alpha+n)(\alpha)_n = (\alpha)_{n+1} \rightarrow (5)$

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§/ General hypergeometric function (Def^n)

The general hypergeometric function is denoted and defined by

$${}_mF_n(\alpha_1, \alpha_2, \ldots \alpha_m; \beta_1, \beta_2, \ldots, \beta_n; x)$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r \cdots (\alpha_m)_r}{(\beta_1)_r (\beta_2)_r \cdots (\beta_n)_r} \cdot \frac{x^r}{r!} \longrightarrow (1)$$

It can also be written as

$${}_mF_n\left[\begin{matrix}\alpha_1, \alpha_2, \ldots \alpha_m; \\ \beta_1, \beta_2, \ldots, \beta_n;\end{matrix} x\right] \longrightarrow (2)$$

Note :- We shall consider only two special cases of (1) here.

They are given in the following two relations for $m = n = 1$ & $m=2, n=1$ resly.

£/ Confluent hypergeometric (or, Kummer) function (Defn)

Confluent hypergeometric function is denoted by

$_1F_1(\alpha; \beta; x)$ or,

$F(\alpha; \beta; x)$ or, $M(\alpha, \beta, x)$

& is defined as

$$F(\alpha;\beta;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \cdot \frac{x^n}{n!} \longrightarrow (1)$$

$$= 1 + \frac{\alpha}{1.\beta} x + \frac{\alpha(\alpha+1)}{1.2.\beta(\beta+1)} x^2 + \ldots \longrightarrow (2)$$

8/ Hypergeometric function (Def^n).

Hypergeometric function is denoted by

$_2F_1(\alpha;\beta;\gamma;x)$ or, simply $F(\alpha;\beta;\gamma;x)$ & is defined by

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!} \to (1)$$

$$= 1 + \frac{\alpha \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \gamma(\gamma+1)} x^2 + \cdots \to (2)$$

In particular, if $\alpha = 1$, & $\beta = \gamma$, then series (2) takes the form

$$1 + x + x^2 + x^3 + \cdots$$

which is a geometric series. Since (2) reduces the a geometric series as a particular case

(2) is called hyper-geometric series

$$= 1 + \frac{\alpha . \beta}{\gamma} \frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!}$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{x^3}{3!} + \cdots \quad (3)$$