Dated 28/10/2019

Lecture 10

§ Co-ordinate Transformations

Let $(x^1, x^2, \dots, x^N)$ & $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^N)$ be the co-ordinates of a point (in the space V_N) in two different frames of reference.

Suppose there exists N independent relations betn the co-ordinates of the two systems in the following form:

$$\left.\begin{aligned} \bar{x}^1 &= \bar{x}^1(x^1, x^2, \dots, x^N) \\ \bar{x}^2 &= \bar{x}^2(x^1, x^2, \dots, x^N) \\ &\;\;\vdots \\ \bar{x}^N &= \bar{x}^N(x^1, x^2, \dots, x^N) \end{aligned}\right\} \quad \text{--- (1)}$$

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which is written, in kreiyen

$$\bar{x}^i = \bar{x}^i (x^1, x^2, \ldots, x^N)$$

$$(i = 1, 2, \ldots, N)$$

or, $\bar{x}^i = \bar{x}^i$, $(i = 1, 2, \ldots, N)$

or, $(\bar{x}^i = \bar{x}^i)$ $\longrightarrow$ 2

It is assumed that the above functions involved are single-valued, continuous & have continuous derivatives (partial)

Under this cond^n, (on solving eq^n(1)), conversely, to each set of co-ordinates $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$, we obtain a unique set

3'

$(x^1, x^2, \ldots, x^N)$ or, x^i $(i = 1, 2, \ldots, N)$ given by

$$x^i = x^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$$

$$(i = 1, 2, \ldots, N) \longrightarrow (3)$$

Thus, the relations (2) or (3) define a transformation of co-ordinates from one frame of reference to another.

Note:– On differentiating equations (1) or (2), partially w.r.t x^r, we get

$$d\bar{x}^i = \sum_{r=1}^{N} \frac{\partial \bar{x}^i}{\partial x^r} dx^r$$

$$\Rightarrow d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^r} dx^r \quad \text{(using summation convention)} \quad (i = 1, 2, \ldots, N)$$

Sǁly, partial differentiation of eqn (3) w.r.t $\bar{x}^r$ & using the summation convention, we obtain

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^r} d\bar{x}^r$$

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(2) $\left(\sum_{i=1}^{N} a_i x^i\right)^2$ will be written as $(a_i a_j x^i x^j)$, instead of $(a_i x^i a_i x^i)$

(3) The rank or order of a tensor is defined as the total no. of indices per component.

x^2, $(x^N)^2$

x_2, $(x_N)_2$

Defn of Tensor :-

A tensor is the mathematical representation of a physical entity (or, object) that may be characterized by magnitude & multiple directions.

A tensor is the generalized form of vectors. It is specified by its rank or order, which includes the total no. of indices per component. It may be noted that scalars require no subscript (x) vectors require a single subscript (x_i) & tensors require two (x_{ij}) or more (x_{ijk}) subscripts etc.

Thus, tensors of rank 0 are scalars & of rank 1 are vectors.

i.e, the general form of tensor includes both scalars & vectors.

eg, tensor — inertia

relates angular velocity (ω) of a rotating body with its angular momentum

A tensor of rank m in N-dimensions has N^m components

As such in 3-dimensional (dim) space, a second-rank tensor (A_{ij}, A^{ij}, A^i_j) $(i,j = 1,2,3)$ is represented by $3^2 = 9$ numbers.

eg., in 3-D space, it is seen

<u>scalar</u> (x)

<u>vector</u> (x_1, x_2, x_3) or, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

<u>Tensor (2 rank)</u>

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

3×3

8 / Types of Tensors

Covariant, contravariant & mixed Tensors.

(Tensors of first order or rank)

1. Covariant vectors.

A set of N- functions

A_i or, $A_1, A_2, \ldots, A_N$ of N-co-ordinates

x^i or, $x^1, x^2, \ldots, x^N$

(i.e., $A_i = A_i(x^i)$, $i = 1 - N$)

are said to be the components of a 'covariant' vector or, 'covariant' tensor

$$(x^1, \ldots, x^N) \to (\bar{x}^1, \ldots, \bar{x}^N)$$

of the first rank or first order, if they transform from x^j to $\bar{x}^i$ or, $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N$ co-ordinate system, according to the eqn

$$\bar{A}_i = \sum_{j=1}^{N} \frac{\partial x^j}{\partial \bar{x}^i} A_j, \quad i = 1, 2, \ldots, N$$

or, by $\bar{A}_i = \dfrac{\partial x^j}{\partial \bar{x}^i} A_j \to (1)$

Kronecker Delta

The Kronecker Delta, denoted by δ^i_j, $i = 1, \ldots, N$ is defined as follows:

$$\delta^i_j = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}$$

so that $\delta^1_1 = \delta^2_2 = \dots = \delta^N_N = 1$

whereas,

$\delta^1_2 = 0 = \delta^3_2$ etc.

As its notation indicates

δ^i_j (i, j real or free suffixes)

is a mixed tensor of second order (or, rank 2). It is also denoted by $\delta_{ij} = \delta^i_j = \delta^{ij}$

clearly, $\delta_{ij} = \delta_{ji}, \forall i, j$, hence it is called a symmetric tensor of rank 2. It has the following properties.