Date
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Lecture 2

Note :-

Since n is not an integer & r is always integral, the factor $\Gamma(-n+r+1)$ in (10) is always finite & non-zero (why?)

For $2r < n$, eqn (10) shows that $J_{-n}(x)$ contains negative powers of x. On the other hand, eqn (9) shows that $J_n(x)$ is not containing any negative powers of x at all.

$\therefore$ we find that at $x=0$

$J_n(x)$ is finite, while

$J_{-n}(x)$ is infinite, so that

one cannot be expressed

as a constant multiple

of the other.

From these arguments

we conclude that $J_n(x)$

& $J_{-n}(x)$ are two independent

solutions of (1), when

n is not an integer.

Thus, the general solution of Bessel eq^n (1) when n is not an integer is $y = AJ_n(x) + BJ_{-n}(x)$ where A & B are arbitrary constants.

Bessel's f^n of first kind & of order n is denoted by $J_n(x)$ & is defined as

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n},$$

$\longrightarrow$ (1)

where n is any non-negative constant.

Remark 1 :- When n is an integer

$\Gamma(n+r+1) = (n+r)!$ & so (1) may be rewritten as

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot 1}{m! \cdot (n+m)!} \left(\frac{x}{2}\right)^{2m+n}$$

(4)

$\longrightarrow$ (2)

Replacing n by 0 & 1 in turn in eq^n (2), we get Bessel's functions of orders 0 & 1, & are given by

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$J_0(x) = 1 - \frac{x^2}{2^2 \cdot (1!)^2} + \frac{x^4}{2^4 \cdot (2!)^2} - \cdots \longrightarrow (2^r)$$

$$|J_0(x)| \leq 1$$

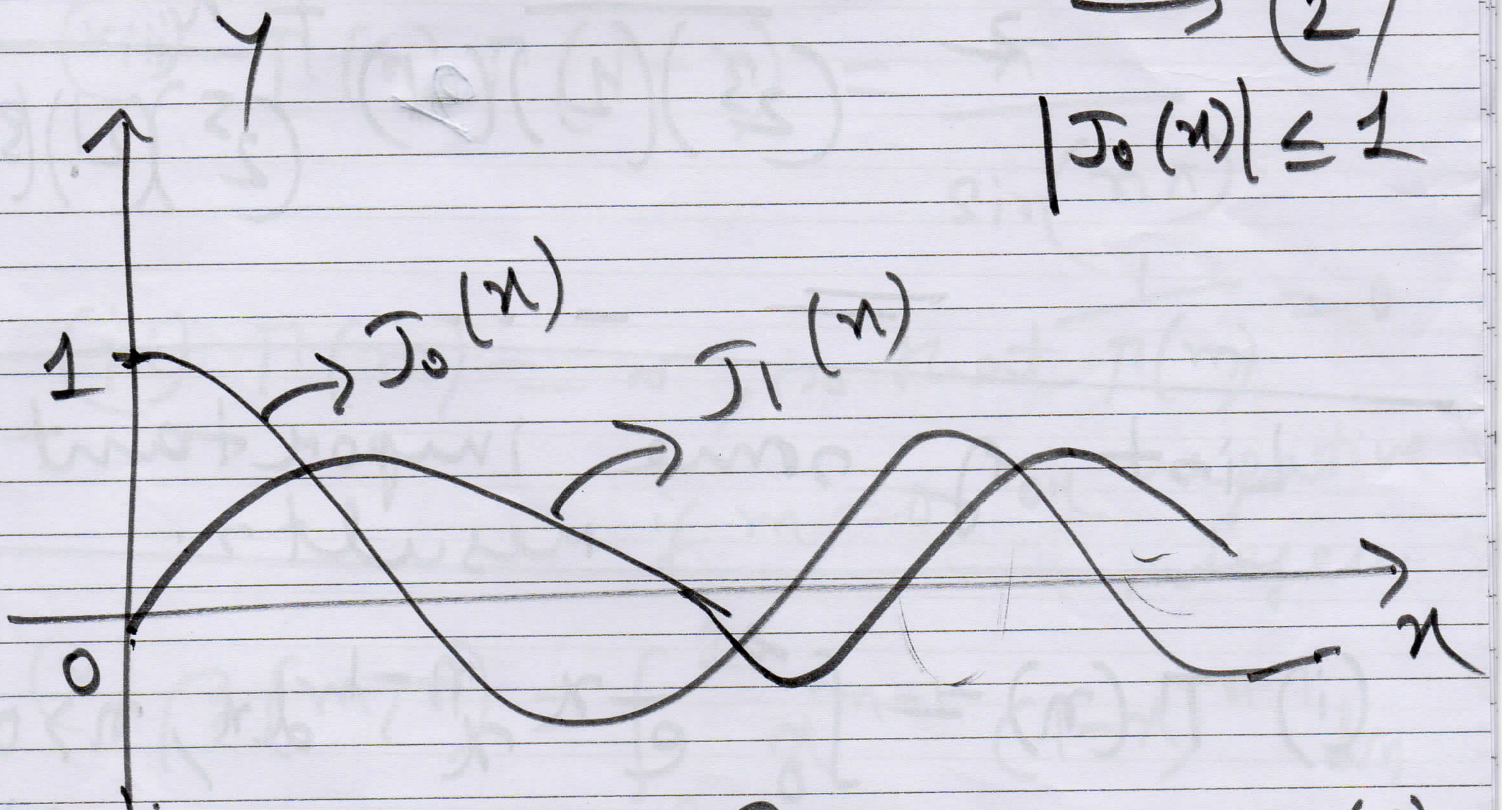

Fig 1:- Graphs of $J_0(x)$ & $J_1(x)$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \cdots$$

$$= \frac{x}{2} - \frac{x^3}{(2^3)(1!)(2!)} + \frac{x^5}{(2^5)(2!)(3!)} - \cdots$$

List of some important results.

(i) $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx, \; n > 0$

$(x = t^2)$

(ii) $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\,dx.$

(iii) $\Gamma(1) = 1$, (iv) $\Gamma(1/2) = \sqrt{\pi}$

(v) $\Gamma(n+1) = n\Gamma(n), \; n > 0$

(vi) $\Gamma(n+1) = n!$, if n is a positive integer

(vii) $\beta(m,n) = \beta(n,m)$

(viii) $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)}$

(ix) $\Gamma(m) = \infty$, so that $\dfrac{1}{\Gamma(m)} = 0$ if $m = 0$ or negative integer.

(x) $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$, $m>0, n>0$

(xi) $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

(xii) $$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma(n)\Gamma(n+\tfrac{1}{2})$$

Relation betn $J_n(x)$ & $J_{-n}(x)$

n being an integer

Th-1 Show that when n is

(i) positive integer,

$$J_{-n}(x) = (-1)^n J_n(x)$$

(ii) ~~any~~ negative integer,

$$J_{-n}(x) = (-1)^n J_n(x).$$

prob :- (i) Let n be a positive integer,

we know that

$$J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m \cdot \frac{1}{m!\,\Gamma(-n+m+1)} \left(\frac{x}{2}\right)^{2m-n} \longrightarrow (1)$$

Since $n>0$, so $\Gamma(-n+m+1) = \Gamma(m-n+1)$

$= (m-n)!$

$$\left[\begin{array}{l} \Gamma(-n+m+1) \\ = \Gamma(m-n+1) \\ = (m-n)! \end{array}\right.$$

is infinite (& so $\dfrac{1}{\Gamma(-n+m+1)}$ is zero)

for $m = 0, 1, 2, \ldots, (n-2), (n-1)$.

keeping this in mind we see that the sum over m in (1) must be taken from n to infinity. Thus

$$J_{-n}(x) = \sum_{M=n}^{\infty} (-1)^M \frac{1}{M!\,\Gamma(-n+M+1)} \left(\frac{x}{2}\right)^{2M-n}$$

$$= \sum_{m=0}^{\infty} (-1)^{m+n} \cdot \frac{1}{(m+n)!\,\Gamma(m+1)} \cdot \left(\frac{x}{2}\right)^{2m+n} \qquad (2)$$

Let $m = M-n \Rightarrow M = m+n$

when [redacted]

$M=n,\ m=0$

$M=\infty,\ m=\infty$

$$\therefore J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m \cdot (-1)^n \frac{1}{\Gamma(m+n+1)\,m!} \left(\frac{x}{2}\right)^{2m+n}$$

$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+n+1)\,m!} \left(\frac{x}{2}\right)^{2m+n}$$

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$$\Rightarrow J_{-n}(x) = (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+n+1)\, m!} \left(\frac{x}{2}\right)^{2m+n}$$

$$= (-1)^n J_n(x) \quad \left[\text{by def}^n \text{ of } J_n(x).\right] \longrightarrow (3)$$

(ii) Let $n < 0$

Let p be a positive integer such that $n = -p$

Since $p > 0$, from part (i) above we have

$$J_{-p}(x) = (-1)^p J_p(x)$$

so that $J_p(x) = (-1)^{-p} J_{-p}(x)$

But $p = -n$, hence the above result becomes

$$J_{-n}(x) = (-1)^n J_n(x).$$

which is of the same form as eqn (3).

Hence, the reqd. result holds for any integer.

Note 1.— When n is an integer, $J_{-n}(x)$ is not independent of $J_n(x)$ (why?). Hence, $y = A J_n(x) + B J_{-n}(x)$ is not the general soln of Bessel's equation.

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Th-2/ The two independent solutions of Bessel's eqn may be taken to be $J_n(x)$ &

$$Y_n(x) = \frac{\cos(n\pi) J_n(x) - J_{-n}(x)}{\sin(n\pi)},$$

for all values of n : $\longrightarrow$ (5)

Proof :- Case I :- Let n be not an integer.

Since n is not an integer, $\sin(n\pi) \neq 0$. Hence (5) shows that $Y_n(x)$ is a linear combination of $J_n(x)$ & $J_{-n}(x)$. But we know that $J_n(x)$ & $J_{-n}(x)$ are independent solⁿ. Thus, we find that $J_n(x)$ & $Y_n(x)$ are two independent solⁿs of Bessel's equation.

Case II :- Let n be an integer

Then we have

$$\cos(n\pi) = (-1)^n, \quad \sin(n\pi) = 0$$

$$\& \quad J_{-n}(x) = (-1)^n J_n(x).$$

Using these values in eqn (5), we find that $Y_n(x)$ has the form $(0/0)$ & so $Y_n(x)$ is undefined.

$$\text{or } Y_n(x) = \frac{\cancel{(-1)^n J_n(x) - (-1)^n J_n(x)}}{0}$$

$$= \frac{0}{0} \text{ (form)}$$

$\nu \to n u$

To make $Y_n(x)$ meaningful we define it as

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

$$= \lim_{\nu \to n} \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} \to \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{\left[\frac{\partial}{\partial\nu}\left[\{\cos(\nu\pi) J_\nu(x)\} - J_{-\nu}(x)\right]\right]_{\nu=n}}{\left[\left(\frac{\partial}{\partial\nu}\right)\sin(\nu\pi)\right]_{\nu=n}}$$

$$= \frac{\left[-\pi\sin(\nu\pi) J_\nu(x) + \cos(\nu\pi)\frac{\partial J_\nu(x)}{\partial\nu} - \frac{\partial}{\partial\nu} J_{-\nu}(x)\right]_{\nu=n}}{\left[\pi\cos(\nu\pi)\right]_{\nu=n}}$$

$$= \frac{\cos n\pi \left[\frac{\partial}{\partial v} J_v(x)\right]_{v=n} - \left[\frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{\pi \cos(n\pi)}$$

$$= \frac{(-1)^n \left[\frac{\partial}{\partial v} J_v(x)\right]_{v=n} - (-1)^{2n} \left[\frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{\pi (-1)^n}$$

$$= \frac{\cancel{(-1)^n} \left[\left[\frac{\partial}{\partial v} J_v(x)\right]_{v=n} - (-1)^n \left[\frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n} \right]}{\pi \cancel{(-1)^n}}$$

$$= \frac{1}{\pi} \left[\frac{\partial}{\partial v} J_v(x) - (-1)^n \frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}$$

$\longrightarrow$ 7

We now establish the following two results about $Y_n(x)$ as given by (6):

(i) $Y_n(x)$ is a solution of Bessel's eq^n

(ii) $Y_n(x)$ is a solution independent of $J_n(x)$.

Proof (i):- Since $J_\nu(x)$ & $J_{-\nu}(x)$ are solutions of Bessel's of order ν, we must have

$$x^2 \frac{d^2 J_\nu}{dx^2} + x \frac{dJ_\nu}{dx} + (x^2 - \nu^2) J_\nu = 0 \quad \rightarrow (8)$$

$$\& \ x^2 \frac{d^2 J_{-\nu}}{dx^2} + x \frac{dJ_{-\nu}}{dx} + (x^2 - \nu^2) J_{-\nu} = 0 \quad \rightarrow (9)$$

Differentiating (8) & (9) w.r.t v, we obtain —

$$x^2 \frac{d^2}{dx^2}\left(\frac{2}{2v} J_v\right) + x\frac{d}{dx}\left(\frac{2J_v}{2v}\right) + (x^2-v^2)\cdot\frac{2J_v}{2v} - 2vJ_v = 0 \quad \rightarrow (10)$$

$$x^2 \frac{d^2}{dx^2}\left(\frac{2J_{-v}}{2v}\right) + x\frac{d}{dx}\left(\frac{2J_{-v}}{2v}\right) + (x^2-v^2)\frac{2J_{-v}}{2v} - 2vJ_{-v} = 0 \quad \rightarrow (11)$$

Multiplying (11) by $(-1)^v$ & subtracting from (10) gives

$$x^2\frac{d^2}{dx^2}\left\{\frac{2}{2v}J_v - (-1)^v\frac{2}{2v}J_{-v}\right\} + x\frac{d}{dx}\left(\frac{2J_v}{2v} - (-1)^v\frac{2J_{-v}}{2v}\right)$$

$$+(x^2-v^2)\left\{\frac{2}{2v}J_v - (-1)^v\frac{2}{2v}J_{-v}\right\} - 2v\left\{J_v - (-1)^v J_{-v}\right\} = 0 \quad \rightarrow (12)$$

Taking $\nu = n$ in eqn (12), & using (in the limit)
(7), we have

$$x^2 \frac{d^2}{dx^2}\left\{\pi Y_n(x)\right\} + x\frac{d}{dx}\left\{\pi Y_n(x)\right\}$$

$$+ (x^2 - n^2)\,\pi Y_n(x) - 2n\,\underbrace{\left[J_n(x) - (-1)^n J_{-n}(x)\right]}_{(=0)} = 0$$

Since, n is an integer,

$$J_{-n}(x) = (-1)^n J_n(x)$$

[by Th-1]

∴ the above eqn reduces

to $x^2 Y_n'' + x Y_n' + (x^2 - n^2) Y_n = 0$

This shows that $Y_n(x)$ is also a soln of Bessel's eqn of order n. (13)

(ii) Ex/ We know that an explicit expression of $Y_n(x)$ for n integral is given by

$$Y_n(x) = \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma - \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r}\right)\right] J_n(x)$$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\,(n+m)!}\left(\frac{x}{2}\right)^{n+2m} \times \sum_{r=1}^{m}\left[\frac{1}{r}+\frac{1}{(r+n)}\right]$$

$$-\frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{-n+2m} \rightarrow (14)$$

where γ is Euler's constant.

From eq^n (4), we find that $Y_n(x)$ is infinite when $x=0$, whereas $J_n(x)$ is finite when $x=0$.

So, $Y_n(x)$ as given by eq^n (6) & $J_n(x)$ are two independent solutions of Bessel's eq^n of order n.

Note :- General sol^n of Bessel's eq^n when n is an integer is

$$y = A J_n(x) + B Y_n(x),$$

A & B being arbitrary constants.

$Y_n(x)$ is known as Bessel's function of order n of the second kind. $Y_n(x)$ is called the Neumann function of order n & is denoted by $N_n(x)$.

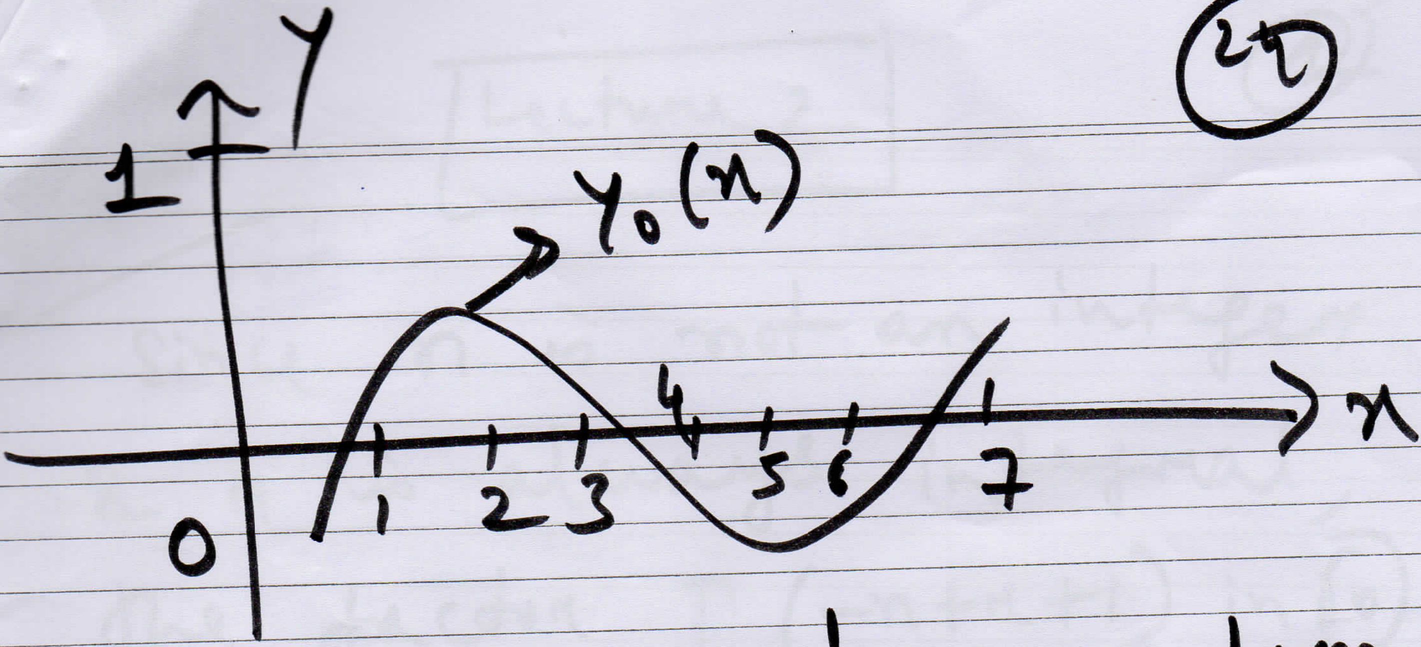

Fig 2 :- Bessel function of the second kind $Y_0(x)$.

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