Finally if we assert that $G(x,y)$ is continuous at $x=y$ while $G'(x,y)$ has discontinuity there by property 3, then (15) reduces to

$$u(y) = \int_0^l f(x)\, G(x,y)\, dx \quad —(16)$$

and thus we obtain the solution of the given BVP. $G(x,y)$ satisfies

$$G'' + k^2 G = 0 \qquad x \neq y$$
$$G(0,y) = G(l,y) = 0$$
$$G(x,y) \text{ is continuous at } x=y$$
$$G'(y,y^+) - G'(y,y^-) = -1$$
$$G(x,y) = G(y,x)$$

If we write $-u(y) = -\int_0^l f(x)\, G(x,y)\, dx$

$$= \int_0^l \left[u''(x) + k^2 u(x)\right] G(x,y)\, dx$$

and integrate by parts using the B.C. imposed on $u(x)$ and $G(x,y)$, we obtain

$$-u(y) = \int_0^l \left[G''(x,y) + k^2 G(x,y)\right] u(x)\, dx \quad —(17)$$

Now we have noted that $G'' + k^2 G$ is 0 everywhere except possibly at $x=y$ and yet (17) is a formal representation of the required function u.

Clearly $G'' + k^2 G$ cannot be a function in the usual sense. This is written as

$$G''(x,y) + k^2 G(x,y) = -\delta(x-y) \quad —(18)$$

where δ is the Dirac delta function.

[$\delta(x-a)=0 \quad \text{if } x\neq a$

$$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

$$\delta(-x) = \delta(x)$$]

Integral equation : Definition

An integral equation is an equation in which the unknown function appears under integral sign.

For example, for $a\le x\le b$, $a\le t\le b$, the equations

$$\int_a^b K(x,t)\,y(t)\,dt = f(x)$$

$$y(x) - \lambda\int_a^b K(x,t)\,y(t)\,dt = f(x)$$

where the function $y(x)$ is the unknown function while the function $f(x)$ and $K(x,t)$ are known functions and λ, a and b are constants, are all integral equations.

Fredholm integral equation:

$$y(x) = f(x) + \lambda\int_a^b K(x,t)\,y(t)\,dt$$

Volterra integral equation:

$$y(x) = f(x) + \lambda\int_a^x K(x,t)\,y(t)\,dt$$

Definition of Green's function

We consider a linear differential equation

$$Ly = -\,\phi(x) \qquad \text{—}\ (1)$$

where L is a linear differential operator, $\phi(x)$ is known and $y(x)$ is to be determined. $y(x)$ can be determined if we can determine the inverse operator L^{-1}. Since L is a differential operator, L^{-1} must be an integral operator. The kernel of this integral operator is known as the Green's function for the diff. eqn. (1). Symbolically, if $G(x,t)$ is the Green's function, then

$$L^{-1}[y] = \int G(x,t)\,y(t)\,dt$$

with $L\left[G(x,t)\right] = -\,\delta(x-t)$, $\delta(x-t)$ is the Dirac delta f^n.

Once the Green's f^n. for the diff. eqn. (1) is determined, its solution can be derived as

$$y(x) = \int G(x,t)\,\phi(t)\,dt$$

$$\Big[\, Ly = L\int G(x,t)\,\phi(t)\,dt = \int L\,G(x,t)\,\phi(t)\,dt$$

$$= \int -\,\delta(x-t)\,\phi(t)\,dt$$

$$= -\int +\,\delta(t-x)\,\phi(t)\,dt$$

$$= -\,\phi(x)$$

Properties of Green's function

We consider a linear homogeneous differential eqn. of order n

$$L[y] = 0 \qquad \text{---(1)}$$

where L is the differential operator

$$L \equiv p_0(x)\frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x) \qquad \text{---(2)}$$

where $p_0(x), p_1(x), \ldots p_n(x)$ are continuous on $[a,b]$, $p_0(x) \neq 0$ on $[a,b]$ and the boundary conditions are

$$V_k(y) = 0 \quad (k = 1, 2, 3, \ldots n) \qquad \text{---(3)}$$

where $V_k(y) = \alpha_k y(a) + \alpha_k^{(1)} y'(a) + \cdots + \alpha_k^{(n-1)} y^{n-1}(a)$

$$+ \beta_k y(b) + \beta_k^{(1)} y'(b) + \cdots + \beta_k^{(n-1)} y^{n-1}(b) \qquad \text{---(4)}$$

where the linear forms $V_1, V_2, \ldots V_n$ in $y(a), y'(a) \ldots y^{n-1}(a)$ $y(b), y'(b), \ldots y^{n-1}(b)$ are linearly independent.

Suppose that the homogeneous b.v.p. given by (1) to (4) has only a trivial solution $y(x) = 0$. Then the Green's function of the b.v.p. (1) to (4) is the function $G(x,t)$ constructed for any point t, $a < t < b$ and which has the following four properties:

(i) In each of the intervals $[a, t)$ and $(t, b]$ the function $G(x,t)$ considered as a function of x is a solution of (1) i.e. $L[G] = 0$ ---(5)

(ii) $G(x,t)$ is continuous and has continuous derivatives w.r.t. x upto order $(n-2)$ for $a \le x \le b$.

(iii) $(n-1)$th derivative of $G(x,t)$ w.r.t. x at the point $x=t$ has a jump discontinuity, the jump being equal to $-\frac{1}{p_0(t)}$ i.e.

$$\left(\frac{\partial^{n-1} G}{\partial x^{n-1}}\right)_{x=t+0} - \left(\frac{\partial^{n-1} G}{\partial x^{n-1}}\right)_{x=t-0} = -\frac{1}{p_0(t)} \quad \text{—(6)}$$

(iv) $G(x,t)$ satisfies the B.C. (3) i.e.

$$V_K(G)=0 \quad k=1,2,\ldots n \quad \text{—(7)}$$

Adjoint eqn. of a 2nd order linear diff. eqn.

We consider a 2nd order homogeneous linear diff. eqn.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad \text{—(1)}$$

where $a_0(x)$ has a continuous 2nd order derivative

$a_1(x)$ " " " 1st " "

$a_2(x)$ is continuous and $a_0(x)>0$ on $a \le x \le b$

Here L is a differential operator given by

$$L = a_0(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x) \quad \text{—— (2)}$$

Thus (1) can be written as $Ly(x)=0$ —— (3)

The adjoint operator M of L is defined as

$$My(x) = \frac{d^2}{dx^2}\{a_0(x)y(x)\} - \frac{d}{dx}\{a_1(x)y(x)\} + a_2(x)y(x) \quad \text{— (4)}$$

Adjoint of (1) is given by

$$My(x) = 0$$

i.e. $$\frac{d^2}{dx^2}\{a_0(x)y(x)\} - \frac{d}{dx}\{a_1(x)y(x)\} + a_2(x)y(x) = 0 \quad \text{— (5)}$$

Self adjoint equation

If the adjoint of any linear homogeneous equation is identical with the equation itself, then the given eqn. is known as self adjoint equation.

Necessary and sufficient condition for a second order homogeneous linear ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

to be self adjoint is that $a_0'(x) = a_1(x)$ on $a \le x \le b$

Proof: By definition, the adjoint eqn. of

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y(x) = 0 \quad \text{— (1)}$$

is $$\frac{d^2}{dx^2}(a_0(x)y) - \frac{d}{dx}\{a_1(x)y(x)\} + a_2(x)y(x) = 0$$

i.e. $$a_0(x)\frac{d^2y}{dx^2} + \{2a_0'(x) - a_1(x)\}\frac{dy}{dx} + \{a_0''(x) - a_1'(x) + a_2(x)\}y = 0 \quad \text{— (2)}$$

The condition is necessary

Let (1) be self adjoint equation. Then (2) must be identical with (1).

i.e. $$2a_0'(x) - a_1(x) = a_1(x) \quad —(3)$$

and $$a_0''(x) - a_1'(x) + a_2(x) = a_2(x) \quad —(4)$$

From (4), $$a_0''(x) = a_1'(x)$$

$$\therefore a_0'(x) = a_1(x) + c \quad —(5)$$

Substituting the value of $a_0'(x)$ from (5) in (3)

$$2[a_1(x) + c] - a_1(x) = a_1(x) \quad \therefore c = 0$$

$\therefore$ (5) yields $a_0'(x) = a_1(x)$

The condition is sufficient

Let for (1), $a_0'(x) = a_1(x) \quad —(6)$

$$\therefore 2a_0'(x) - a_1(x) = 2a_1(x) - a_1(x) = a_1(x) \quad —(7)$$

and $$a_0''(x) - a_1'(x) + a_2(x) = a_1'(x) - a_1'(x) + a_2(x) = a_2(x) \quad —(8)$$

By (7) and (8), (2) reduces to

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0$$

which is same as (1).

Result

If the BVP (1) to (4) is self-adjoint, then Green's f^n is symmetric i.e. $G(x,t) = G(t,x)$.