Chapter-3

Hypergeometric Functions

3.1 Introduction

Let us list some special types of second order linear differential equations with variable coefficients which find their wide applications in many physical and engineering problems:

$$x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0 \qquad \ldots(1)$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \ldots(2)$$

$$x^2y'' + xy' + (x^2-n^2)y = 0 \qquad \ldots(3)$$

$$y'' - 2xy' + 2ny = 0 \qquad \ldots(4)$$

$$x\,y'' + (1-x)y' + ny = 0 \qquad \ldots(5)$$

$$(x^2-1)y'' + xy' - n^2y = 0 \qquad \ldots(6)$$

The above differential equations (1), (2), (3), (4), (5) and (6) are respectively known as **Gauss's hypergeometric differential equation, Legendre's differential equation, Bessel's differential equation Hermite's differential equation, Laguerre's differential equation and Chebyshev differential equation.** These differential equations may be solved about their ordinary and regular singular points in terms of convergent series which are some functions of x. These functions are known as **special functions**, e.g. the solution of (1) in a special case, is called **Gauss's hypergeometric function**, the solution of (2) in a special case, is called **Legendre's polynomial**; the solution of (3) in a special case, gives **Bessel's function**, the solution of (4) in a special case, gives **Hermite's polynomial**, the solution of (5) in a special case, gives **Laguerre's polynomial** and the solution of (6) gives **Chebyshev's polynomial.** Here, we are concerned only with hypergeometric function and its various properties. Gauss's hypergeometric function has a wide applications in many problems arising in physical and engineering sciences.

3.2 Gauss's Hypergeometric Differential Equation

Differential equation $x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0$ …(1) where $a,\ b$ and c are constants and it is assumed that c is not an integer, is known as **hypergeometric differential equation** or **Gauss's hypergeometric differential equation,** named after **Carl Friedrich Gauss** (1777-1855), a great German mathematician, who contributed to algebra, number theory, mechanics, complex analysis, differential geometry, non-Euclidean geometry, astronomy including differential equation and numerical analysis.

A particular solution of Gauss's hypergeometric differential equation (1) is known as **Gauss's hypergeometric function** or simply **hypergeometric function.**

The singular points of Gauss's hypergeometric differential equation (1) are only $x = 0$, $x = 1$ and $x = \infty$. More precisely, the singular points $x = 0$, $x = 1$ and $x = \infty$ of (1) are all regular and, therefore, it can be stated that the Gauss's hypergeometric differential equation is a **Fuchsian differential equation**.

3.3 Exponents of Gauss's Hypergeometric Differential Equation and Papperitz Symbol

Consider the Gauss's hypergeometric differential equation

$$x(x-1)y'' + \{c-(a+b+1)x\}y' - aby = 0 \qquad \ldots(1)$$

Since, all the three singular points $x = 0$, $x = 1$ and $x = \infty$ of the Gauss's hypergeometric differential equation (1) are regular, therefore, we can find the series solution of (1) by Frobenius method about all the three regular singular points $x = 0$, $x = 1$ and $x = \infty$ separately.

While using Frobenius method, we find the exponents of the hypergeometric differential equation (1) as follows:

(i) When we solve (1) about the regular singular point $x = 0$, we get the exponents 0 and (1–c).
(ii) When we solve (1) about the regular singular point $x = 1$, we get the exponents 0 and (c – a – b).
(iii) When we solve (1) about the regular singular point $x = \infty$, we get the exponents a and b.

All these informations can be symbolically represented by

$$y = P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{matrix}\right\} \qquad \ldots(2)$$

The symbol on the right side of (2) is called the **Riemann- P function** or the **Papperitz symbol** or simply **P-symbol**. The main use of this symbol is that it is capable of representing all solutions of the Gauss's hypergeometric differential equation.

3.4 Pochhammer Symbol or Barne's Symbol

Let n be a positive integer. Then, **Pochhammer symbol** is denoted and defined by

$$(a)_n = a(a+1)(a+2)\ldots\ldots(a+n-1) \qquad \ldots(1)$$

with $$(a)_0 = 1 \qquad \ldots(2)$$

The Pochhammer symbol is also known as **Barne's symbol.**

The following deductions may be obtained by using Pochhammer symbol :

(i) $(a)_n = a(a+1)(a+2)\dots(a+n-1) = \frac{1.2.3\ldots\ldots(a-1)a(a+1)(a+2)\ldots\ldots(a+n-1)}{1.2.3\ldots(a-1)}$

$= \frac{\Gamma(a+n)}{\Gamma(a)}$, by using the formula $\Gamma(p) = (p-1)\Gamma(p-1)$

Thus, we have $\quad (a)_n = \frac{\boldsymbol{\Gamma(a+n)}}{\boldsymbol{\Gamma(a)}} \quad$ …(3)

(ii) $(a)_{n+1} = a(a+1)(a+2)\dots(a+n-1)(a+n)$

$= a[(a+1)(a+2)\dots(a+n-1)(a+n)]$

$= a[(a+1)(a+1+1)+\cdots(a+1+n-1)] = a(a+1)_n$

Thus, we have $\quad \boldsymbol{(a)_{n+1} = a(a+1)_n} \quad$ …(4)

(iii) $(a+n)(a)_n = a(a+1)(a+2)\ldots\ldots(a+n-1)(a+n) = (a)_{n+1}$

Thus, we have $\quad \boldsymbol{(a+n)(a)_n = (a)_{n+1}} \quad$ …(5)

(iv) $a(a+1)_n = a[(a+1)(a+1+1)+\cdots+(a+1+n-2)(a+1+n-1)]$

$= [a(a+1)(a+2)+\cdots+(a+n-1)](a+n)$

$= (a)_n(a+n) = (a+n)(a)_n$

Thus, we have $\quad \boldsymbol{(a+n)(a)_n = a(a+1)_n} \quad$ …(6)

3.5 Generalized Hypergeometric Function and Confluent Hypergeometric Function

The generalized hypergeometric function or the **general hypergeometric function** is denoted and defined by

$${}_mF_n\begin{bmatrix} a_1, & a_2,\ldots, & a_m; x \\ b_1, & b_2,\ldots, & b_m; \end{bmatrix} = {}_mF_n(a_1, a_2 \ldots, a_m; b_1, b_2, \ldots, b_n\,; x)$$

$$= \sum_{r=1}^{\infty} \frac{(a_1)_r(a_2)_r \dots (a_m)_r}{(b_1)_r(b_2)_r \dots (b_n)_r} \cdot \frac{x^r}{r!} \qquad \dots(1)$$

In the notation ${}_mF_n(a_1, a_2 \dots \dots, a_m; b_1, b_2, \dots \dots, b_n\,; x)$, m denotes the number of numerator parameters and n, the denominator parameters.

The Confluent hypergeometric function is denoted by ${}_1F_1(a; b; x)$ or $F(a; b; x)$ or $M(a; b; x)$ and is defined by

$$F(a; b; x) = \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r} \cdot \frac{x^r}{r!} \qquad \dots(2)$$

In the notation ${}_1F_1(a; b; x)$, there is one numerator parameter a and one denominator parameter b. If there is no confusion between hypergeometric and confluent functions, then $F(a; b; x)$ may be used in place of ${}_1F_1(a; b; x)$. Sometimes, we use the following modified definition of the confluent hypergeometric function:

$$F(a; b; x) = 1 + \frac{a}{1.b} x + \frac{a(a+1)}{1.2.b(b+1)} x^2 + \cdots \qquad \dots(3)$$

3.6 Gauss's Hypergeometric Function

Gauss's hypergeometric function is denoted by ${}_2F_1(a, b; c; x)$. In this notation ${}_2F_1(a, b; c; x)$, there are two numerator parameters a and b and one denominator parameter c. It is also denoted by $F\,(a, b; c; x)$ and is defined by

$$F\,(a, b; c; x) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r} \frac{x^r}{r!} \qquad \dots(1)$$

or $$F\,(a, b; c; x) = 1 + \frac{a.b}{1.c} x + \frac{a(a+1).b(b+1)}{1.2.c(c+1)} x^2 + \cdots \qquad \dots(2)$$

In particular, if $a = 1$, $b = 1$ and $c = 1$, then (2) takes the form

$$F(1, 1; 1; x) = 1 + x + x^2 + \dots..$$

which is a geometric series. This is the reason why (2) is called a hypergeometric series or hypergeometric function.

Sometimes, we use the following modified definition of the hypergeometric series:

$$F(a,b;c;x) = 1 + \frac{a.b}{c}\frac{x}{1!} + \frac{a(a+1).b(b+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2).b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots \quad \ldots(3)$$

The hypergeometric function $F(a,b;c;x)$ can also be put in the following different forms:

(i) $F(a,b;c;x) = (1-x)^{c-a-b}F(c-a, c-b; c; x)$ ...(4)

(ii) $F(a,b;c;x) = (1-x)^{-a}F\left(a, c-a; c; \frac{x}{x-1}\right)$ ...(5)

(iii) $F(a,b;c;x) = (1-x)^{-b}F\left(b, c-a; c; \frac{x}{x-1}\right)$ ...(6)

3.7 Solution of Gauss's Hypergeometric Differential Equation

Consider the Gauss's hypergeometric differential equation

$$x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0 \quad \ldots(1)$$

where a, b and c are constants and it is assumed that c is not an integer.

Dividing this equation by $x\,(1-x)$, we get

$$y'' + \frac{\{c-(a+b+1)x\}}{x(1-x)}y' - \frac{a\,b}{x(1-x)}y = 0 \quad \ldots(2)$$

Comparing this with $y'' + P(x)y' + Q(x)y = 0$, we have

$$P(x) = \frac{\{c-(a+b+1)x\}}{x(1-x)} \quad \text{and} \quad Q(x) = \frac{-a\,b}{x(1-x)}$$

Here, we see that $P(x) \to \infty$ at the points $x = 0$, $x = 1$ and $x = \infty$ and $Q(x) \to \infty$ at the points $x = 0$ and $x = 1$. Thus, the point $x = 0$, $x = 1$

and $x = \infty$ are the singular points of the hypergeometric equation (1). Now, let us find whether these singular points are regular or irregular.

If the singular points are regular, then we find the solution of Gauss's hypergeometric differential equation (1) by using **Frobenius method.**

Again, if there is no singular point of (1) or (2), i.e., if there is an ordinary point of (1) or (2), then, we can find the solution of Gauss's hypergeometric differential equation (1) by using **power series method**.

Now, let us try to find the series solution of (1) about the points x = 0, $x = 1$ and $x = \infty$ separately.

3.7.1 Series Solution of Gauss's Hypergeometric Differential Equation about $x = 0$:

It can be seen that $x = 0$ is a singular point of Gauss's hypergeometric differential equation (1). Now, let us decide whether, it is regular or irregular. For this, we find .

$$(x-0)P(x) = \frac{\{c-(a+b+1)x\}}{1-x} \quad \text{and} \quad (x-0)^2Q(x) = \frac{-abx}{(1-x)}$$

from which, we note that $(x - 0)$ P(x) and $(x - 0)^2$ Q(x) are analytic and, therefore, the point $x = 0$ is *a* regular singular point.

Now, to find the series solution of (1), we proceed as follows:

Let $\quad y = \sum_{m=0}^{\infty} C_m\, x^{k+m}, C_0 \neq 0$ ...(3)

be the series solution of the hypergeometric equation (1) about its singular point $x = 0$.

$$y' = \sum_{m=0}^{\infty} C_m\,(k+m)x^{k+m-1}, \; y'' = \sum_{m=0}^{\infty} C_m\,(k+m)(k+m-1)x^{k+m-2} \quad \text{...(4)}$$

Putting the values of y, y′ and y″ from (3) and (4) in (1), we get

$$(x - x^2)\ \textstyle\sum_{m=0}^{\infty} C_m\,(k+m)(k+m-1)x^{k+m-2}$$

$$+\{c-(a+b+1)x\}\textstyle\sum_{m=0}^{\infty} C_m\,(k+m)x^{k+m-1} - ab\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\textstyle\sum_{m=0}^{\infty} C_m\,\{(k+m)(k+m-1)+c(k+m)\}x^{k+m-1}$$

$$-\textstyle\sum_{m=0}^{\infty} C_m\,\{(k+m)(k+m-1)+(a+b+1)(k+m)+ab\}x^{k+m} = 0$$

or $$\textstyle\sum_{m=0}^{\infty} C_m\,(k+m)(k+m-1+c)x^{k+m-1}$$

$$-\textstyle\sum_{m=0}^{\infty} C_m\,\{(k+m)^2+(a+b)(k+m)+ab\}x^{k+m} = 0$$

or $$\textstyle\sum_{m=0}^{\infty} C_m\,(k+m)(k+m-1+c)x^{k+m-1}$$

$$-\textstyle\sum_{m=0}^{\infty} C_m\,(k+m+a)(k+m+b)x^{k+m} = 0 \quad \ldots(5)$$

which is an identity and, therefore, the coefficients of various powers of *x* in it should be zero.

Thus, equating to zero, the coefficient of smallest power of *x*, namely x^{k-1} from (5), we get

$$C_0 k(k-1+c) = 0$$

Since $C_0 \neq 0$, therefore, we must have $k\,(k-1+c) = 0$ …(6)

which is known as the indicial equation or governing equation.

The roots of the indicial equation i.e., the exponents are $k = k_1 = 1$ and $k = k_2 = (1 - c)$, which are unequal and not differing by an integer because by assumption, c is not an integer.

Now, we obtain recurrence relation by equating to zero, the coefficient of general lowest power of *x* in the identity i.e., by equating to zero, the coefficient of x^{k+m-1} from (5), we have

$$C_m(k+m)(k+m-1+c) - C_{m-1}(k+m-1+a)(k+m-1+b) = 0$$

or $$C_m = \frac{(k+m-1+a)(k+m-1+b)}{(k+m)(k+m-1+c)} C_{m-1} \qquad \ldots(7)$$

Now, we find the two linearly independent solutions corresponding to the exponents k = 0 and k = (1 – c), as follows:

Case I. When k = 0, then from (7), we have

$$C_m = \frac{(m-1+a)(m-1+b)}{m(m-1+c)} C_{m-1} \qquad \ldots(8)$$

Putting m = 1, 2, 3,…. in (8), we get

$$C_1 = \frac{a.b}{1.c} C_0, C_2 = \frac{(a+1)(b+1)}{2.(c+1)} C_1 = \frac{a(a+1).b(b+1)}{1.2.c(c+1)} C_0,$$

$$C_3 = \frac{(a+2).(b+2)}{3.(c+2)} C_2 = \frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3c(c+1)(c+2)} C_0 \text{ and so on.}$$

Now, substituting these values and k = 0 and replacing C_0 by A in (3), we get

$$y = A\left[1 + \frac{a.b}{1.c}x + \frac{a(a+1).b(b+1)}{1.2c(c+1)}x^2 + \frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\ c(c+1)(c+2)}x^3 + \cdots\right]$$

We can also write this solution as ;

or $$y = A\sum_{r=1}^{\infty} \frac{(a)_r(b)_r}{(c)_r}\frac{x^r}{r!} \qquad \ldots(9)$$

If we take A = 1 in (9), then the series in R.H.S. of is called hypergeometric series and is represented or denoted by $F(a, b; c; x)$ and is known as **hypergeometric function**.

Thus, we say that $F(a, b; c; x)$ is a solution of the hypergeometric differential equation (1).

Case II. When k = (1 – c), then from (7), we have

$$C_m = \frac{(1-c+m-1+a)(1-c+m-1+b)}{(1-c+m)(1-c+m-1+c)} C_{m-1}$$

or $$C_m = \frac{(1-c+a+m-1)(1-c+b+m-1)}{m(2-c+m-1)} C_{m-1}$$

or $$C_m = \frac{(a'+m-1)(b'+m-1)}{m(c'+m-1)} C_{m-1} \qquad \ldots(10)$$

where $a' = 1 - c + a, b' = 1 - c + b \; and \; c' = 2 - c.$

Now, putting m = 1, 2, 3,……. in (10), we get

$$C_1 = \frac{a'.b'}{1.c'} C_0, C_2 = \frac{(a'+1).(b'+1)}{2.(c'+1)} C_1 = \frac{a'(a'+1).b'(b'+1)}{1.2c'(c'+1)} C_0,$$

$$C_3 = \frac{(a'+2).(b'+2)}{3.(c'+2)} C_2 = \frac{a'(a'+1)(a'+2).b'(b'+1)(b'+2)}{1.2.3c'(c'+1)(c'+2)} C_0 \text{ and so on.}$$

Substituting these values and k = (1 – c) and replacing C_0 by B in (3), we get

$$y = B\ x^{1-c}\left[1 + \frac{a'.b'}{1.c'} x + \frac{a'(a'+1).b'(b'+1)}{1.2c'(c'+1)} x^2 + \cdots\right]$$

We can also write this solution as :

$$y = Bx^{1-c} \sum_{r=0}^{\infty} \frac{(a')_r (b')_r}{(c')_r} \frac{x^r}{r!} \qquad \ldots(11)$$

If we take B = 1 in (11), then the series in R.H.S. of (11) would become $x^{c-1}F(a', b'; c'; x) i.e., x^{c-1}F(1 - c + a, 1 - c + b; 2 - c; x)$ which is another independent solution of the hypergeometric differential equation (1).

Hence, the general series solution of the hypergeometric differential equation (1) is given by

$$y = AF(a, b; c; x) + Bx^{c-1}F(1 - c + a, 1 - c + b; 2 - c; x) \qquad \ldots(12)$$

where A and B are arbitrary constants.

3.7.2 Series Solution of Gauss's Hypergeometric Differential Equation about *x* = 1:

It can be seen that $x = 1$ is a regular singular point of Gauss's hypergeometric differential equation

$$(x-1)xy'' + \{(a+b+1)x - c\}y' + aby = 0 \qquad \ldots(1)$$

Therefore, the solution of (1) can be obtained in series of powers of (x – 1) as follows :

To find the solution of (1), we change the variable from x to t by taking $x = 1 - t$. This transfers the point $x = 1$ to $t = 0$ and, therefore, we obtain the series solution of the following transformed differential equation (2) in terms of the series of powers of t:

$$-t(1-t)\frac{d^2y}{dt^2} - \{(a+b+1)(1-t) - c\}\frac{dy}{dt} + aby = 0$$

or $$t(t-1)\frac{d^2y}{dt^2} - \{(-c+a+b+1) - (a+b+1)t\}\frac{dy}{dt} + aby = 0$$

or $$t(1-t)\frac{d^2y}{dt^2} + \{(-c+a+b+1) - (a+b+1)t\}\frac{dy}{dt} - aby = 0 \qquad \ldots(2)$$

From (2), we observe that when $(-c + a + b + 1) = c'$, then this differential equation is similar to Gauss's hypergeometric differential equation

$$x(1-x)\frac{d^2y}{dx^2} + \{c - (a+b+1)x\}\frac{dy}{dx} - aby = 0.$$

So, equation (2) can be solved in a singular way as we have done earlier. The **roots of indicial equation** i.e. the **exponents** in this case, will be $k = k_1 = 0$ and $k = k_2 = 1 - c'$, i.e. $k = k_1 = 0$ and $k = k_2 = 1 - (-c + a + b + 1) = c - a - b$.

The two linearly independent solutions of (2) corresponding to $k = k_1 = 0$ and $k = k_2 = c - a - b$ are given by

$y = AF(a, b; c'; t)$, where $c' = -c + a + b + 1$

or $\quad y = AF(a, b; -c + a + b + 1; t)$ …(3)

and $\quad y = Bt^{1-c'}F(1 - c' + a, 1 - c' + b; 2 - c'; t)$, where $1 - c' = c - a - b$

or $\quad y = Bt^{c-a-b}F(c - b, c - a; c - a - b + 1; t)$ …(4)

Now, replacing t by (1 – x) in (3) and (4), we get

$$y = AF(a, b; -c + a + b + 1; 1 - x) \qquad \ldots(5)$$

$$y = B(1 - x)^{c-a-b}F(c - b, c - a; c - a - b + 1; 1 - x) \qquad \ldots(6)$$

Thus, the general solution of (1) is given by

$$y = AF(a, b; -c + a + b + 1; 1 - x) + B(1 - x)^{c-a-b}F(c - b, c - a; c - a - b + 1; 1 - x) \qquad \ldots(7)$$

where A and B are arbitrary constants.

3.7.3 Series Solution of Gauss's Hypergeometric Differential Equation about $x = \infty$:

It can be seen that $x = \infty$ is a regular singular point of Gauss's hypergeometric differential equation

$$x(1 - x)\frac{d^2y}{dt^2} + \{c - (a + b + 1)x\}\frac{dy}{dt} - aby = 0 \qquad \ldots(1)$$

Thus, the solution of (1) can be obtained in series about $x = \infty$ by taking $x = \frac{1}{t}$ in (1), so that (1) is transformed to the following differential equation:

$$t^2(1 - t)\frac{d^2y}{dt^2} + \{2t(1 - t) - (a + b + 1)t + ct^2\}\frac{dy}{dt} + aby = 0 \qquad \ldots(2)$$

Now, to find its solution, we proceed as follows:

Let $\quad y = \sum_{m=0}^{\infty} C_m t^{k+m}, C_0 \neq 0$ …(3)

be the series solution of (2).

$\therefore \ \frac{dy}{dt} = \sum_{m=0}^{\infty} C_m \,(k+m)t^{k+m-1}, \frac{d^2y}{dt^2} = \sum_{m=0}^{\infty} C_m \,(k+m)(k+m-1)t^{k+m-2}$ ...(4)

Substituting the values of y, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ from (3) and (4) in (2), we get

$$t^2(1-t)\sum_{m=0}^{\infty} C_m \,(k+m)(k+m-1)t^{k+m-2}$$

$$+\{2t(1-t)-(a+b+1)t+ct^2\}\sum_{m=0}^{\infty} C_m \,(k+m)t^{k+m-1}$$

$$+ab\sum_{m=0}^{\infty} C_m \,t^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1)+(k+m)(1-a-b)+ab\}t^{k+m}$$

$$-\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1)+(k+m)(2-c)\}t^{k+m+1} = 0 \qquad ...(5)$$

which is an identity and, therefore, the coefficients of various powers of *t* in it should be zero.

Thus, equating to zero, the coefficient of the smallest power of t, namely t^k from the identity (5), we get

$$C_0\{k(k-1)+k(1-a-b)+ab\} = 0$$

or $$C_0\{k^2-k(a+b)+ab\} = 0$$

Since $C_0 \neq 0$, therefore, we must have

$$k^2 - k\,(a+b) + ab = 0 \qquad ...(6)$$

which is the indicial equation, (quadratic in k), which gives the two indicial roots k_1 and k_2 as: $k = k_1 = a$ and $k = k_2 = b$.

$\therefore$ The exponents are $k = k_1 = a$ and $k = k_2 = b$.

Again, equating to zero, the coefficient of the general lowest degree, namely t^{k+m} from (5), we get

$$C_m\{(k+m)(k+m-1)+(k+m)(1-a-b)+ab)\}$$

$$-C_{m-1}\{(k+m-1)(k+m-2)+(k+m-1)(2-c)\}=0$$

or$C_m=\{(k+m)^2-(k+m)(a+b)+ab\}-C_{m-1}\{(k+m-1)(k+m-c)\}=0$

or $$C_m=\frac{(k+m-1)(k+m-c)}{(k+m)\{(k+m)-a+b\}+ab}C_{m-1} \quad \ldots(7)$$

which is the recurrence relation which connects C_m and C_{m-1}.

Now, we proceed as follows:

Putting k = *a* in the recurrence relation (7), we have

$$C_m=\frac{(m-1+a)(m-c+a)}{m(m+a+b)}C_{m-1} \quad \ldots(8)$$

Again, putting m = 1, 2, 3,………. in (8), we get

$$C_1=\frac{a.(1-c+a)}{1.(a+b+1)}C_0$$

$$C_2=\frac{(a+1)(2-c+a)}{2(a+b+2)}C_1=\frac{a(a+b).(1-c+a)(1-c+a+1)}{2.(a+b+1)(a+b+2)}C_0$$

and so on.

Putting these values and k = *a* in (3) and replacing C_0 by A, we get

$$y=At^a\left[1+\frac{a.(1-c+a)}{1.(a+b+1)}t+\frac{a(a+1).(1+c+a)(1-c+a+1)}{1.2(a+b+1)(a+b+1)}t^2+\cdots\right]$$

or $$y=A\,t^a\sum_{m=a}^{\infty}\frac{(a)_r(1-c+a)_r}{(a+b+1)_r}\frac{t^r}{r!}=Ax^{-a}\sum_{r=0}^{\infty}\frac{(a_r)(1-c+a)_r}{(a+b+1)_r}.\frac{r^{-r}}{r!} \quad \ldots(9)$$

Next, replacing *t* by 1/*x* in (9), we have

$$y=Ax^{-a}\left[a,1-c+a;a+b+1;\frac{1}{x}\right] \quad \ldots(10)$$

Similarly, the another solution corresponding to k = b is given by

$$y = Bx^{-b}\left[b, 1-c+a; a+b+1; \frac{1}{x}\right] \quad \ldots(11)$$

Hence, the general solution of hypergeometric differential equation (1) is given by

$$y = Ax^{-a}\left[a, 1-c+a; a+b+1; \frac{1}{x}\right] + Bx^{-b}\left[b, 1-c+a; a+b+1; \frac{1}{x}\right] \quad \ldots(12)$$

where A and B are arbitrary constants.

3.7.4 Radius of Convergence of Hypergeometric Series

Let us consider the following hypergeometric series:

$$F(a,b;c;x) = \sum_{r=1}^{\infty} \frac{(a)_r(b)_r}{(c)_r} \cdot \frac{x^r}{r!} \quad \ldots(1)$$

We apply D'Alembert's ratio test to find the radius convergence of the above hypergeometric series as follows:

The n^{th} and $(n+1)^{th}$ terms y_n and y_{n+1} of (1) are respectively given by $y_n = \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{x^n}{n!}$ and $y_{n+1} = \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}} \cdot \frac{x^{n+1}}{(n+1)!}$.

$$\therefore \lim_{n\to\infty}\left|\frac{y_{n+1}}{y_n}\right| = \lim_{n\to\infty}\left|\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}} \cdot \frac{x^{n+1}}{(n+1)!} \Big/ \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{x^n}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(a)_{n+1}(b)_{n+1}(c)_n}{(a)_n(b)_n(c)_{n+1}} \cdot \frac{n!x}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(a+n)(b+n)}{(c+n)(n+1)}x\right|$$

$$= \lim_{n\to\infty}\left|\frac{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}{\left(1+\frac{c}{n}\right)\left(1+\frac{1}{n}\right)}x\right| = |x| \quad \ldots(2)$$

Therefore, by ratio test, the series will be convergent only when |x| < 1. Thus, the radius of convergence of the hypergeometric series about x = 0 is unity. It is also obvious from the fact that the nearest singular point to x = 0 of the hypergeometric equation is x = 1.

It can be seen by using Raabe's test that about $x = 1$, the series is convergent if **Re** $[c - a - b] > 0$.

Many familiar functions are special cases of the hypergeometric function. For example,

(i) $(1-x)^n = F(-n, 1; 1; x)$ …(3)

(ii) $(1+x)^2 + (1-x)^n = 2F\left(-\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}; \frac{1}{2}; x^2\right)$ …(4)

(iii) $\ln(1+x) = xF(1,1; 2; -x)$ …(5)

(iv) $ln\left(\frac{1+x}{1-x}\right) = 2xF(\frac{1}{2}, 1; \frac{3}{2}; -x^2$ …(6)

(v) $sin^{-1}x = xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)$ …(7)

(vi) $tan^{-1}x = xF\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2\right)$ …(8)

If **Re** [c] > **Re** [b] > 0, then F $(a, b; c; x)$ may be represented as

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt \quad \text{…(9)}$$

This representation (9) is called as the **integral representation for the hypergeometric function** and may be obtained by expanding binomial as $(1-xt)^{-a} = \sum_{r=0}^{\infty}(a)_r \frac{(xt)^r}{r!}, |xt| < 1$,and then, integrating term by term and finally using the result $\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Now, by using (9), we find the following theorems:

1.Gauss's Theorem: If **Re** $[c - a - b] > 0$, then F $(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$.

2. Vandermonde's Theorem: F $(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}$.

In particular, from **Vandermonde's theorem,** we have

$$F(-n, n+1; 1; 1) = \frac{(1-n-1)_n}{(1)_n} = \frac{(-n)_n}{(1)_n} = \frac{(-n)(-n+1)(-n+2)...(-n+n-1)}{1(1+1)(1+2)(1+3)...(1+n-1)}$$

$$= \frac{(-1^n)[n(n-1)(n-2)...3.2.1]}{1.2.3....n} = (-1)^n$$

3.8 Symmetric Property of Hypergeometric Function

Gauss's hypergeometric function remains unaltered (i.e. does not change) if the numerator parameters a and b are interchanged, keeping the denominator parameterc fixed. Thus, we have

$$F(a, b; c; x) = F(b, a; c; x)$$

This is called **symmetric property of the Gauss's hypergeometric function.**

Proof : By definition, we have

$$F(a, b; c; x) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \cdot \frac{x^r}{r!} \qquad ...(1)$$

and $$F(b, a; c; x) = \sum_{r=0}^{\infty} \frac{(b)_r (a)_r}{(c)_r} \cdot \frac{x^r}{r!} \qquad ...(2)$$

Now, from (1) and (2), we get

$$F(a, b; c; x) = F(b, a; c; x) \qquad ...(3)$$

3.9 Differentiation of Hypergeometric Functions

The differentiation of the hypergeometric function $F(a, b; c; x)$ w.r.to x is given by

$$\frac{d}{dx} F(a, b; c; x) = \frac{ab}{c} F(a+1, b+1; c+1; x)$$

Proof: By definition, we have

$$F(a,b;c;x) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r} . \frac{x^r}{r!} \qquad \ldots(1)$$

Differentiating both sides of (1) w.rto x, we have

$$\frac{d}{dx}F(a,b;c;x) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r} . \frac{rx^r}{r!}$$

$$= \sum_{r=1}^{\infty} \frac{(a)_r(b)_r}{(c)_r} . \frac{x^{r-1}}{(r-1)!}$$, since the term with r = 0 vanishes.

$$= \sum_{s=0}^{\infty} \frac{(a)_{s+1}(b)_{s+1}}{(c)_{s+1}} . \frac{x^x}{s!}$$, by putting r – 1= s

$$= \sum_{s=0}^{\infty} \frac{a(a+1)_s . b(b+1)_s}{c.(c+1)_s} . \frac{x^s}{s!}.$$

$$= \frac{ab}{c} \sum_{s=0}^{\infty} \frac{(a+1)_s . (b+1)_s}{(c+1)_s} . \frac{x^s}{s!}.$$

or $$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x) \qquad \ldots(2)$$

3.10 Contiguous Hypergeometric Functions

According to Gauss, a function is said to be contiguous to F(*a*; *b*; *c;* *x*) when it is increased or decreased by one and only one of the parameters *a, b, c* by unity.

Thus, there exist six hypergeometric functions contiguous to F (*a, b; c; x*). These are denoted and defined as follows:

(i) $F(a^+) = F_{a^+} = F(a+1,b;c;x)$ …(1)
(ii) $F(a^-) = F_{a^-} = F(a-1,b;c;x)$ …(2)
(iii) $F(b^+) = F_{b^+} = F(a,b+1;c;x)$ …(3)
(iv) $F(b^-) = F_{b^-} = F(a,b-1;c;x)$ …(4)
(v) $F(c^+) = F_{c^+} = F(a,b;c+1;x)$ …(5)
(vi) $F(c^-) = F_{c^-} = F(a,b;c-1;x)$ …(6)

The following relation, which is called the **contiguity relationship,** may be used as and when required:

$$(a-b)F(a,b;c;x) = aF(a+1,b;c;x) - bF(a,b+1;c;x)$$

Illustrative Examples

Example 1. Prove that $\frac{d^n}{dx^n}F(a,b;c;x) = \frac{(a)_n(b)_n}{(c)_n}F(a+n,b+n;c+n;x)$ and hence deduce that $\left[\frac{d^n}{dx^n}F(a,b;c;x)\right]_{x=0} = \frac{(a)_n(b)_n}{(c)_n}$.

Proof: By definition, the function F(a, b; c; x) is given by

$$F(a,b;c;x) = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}.\frac{x^r}{r!} \qquad \ldots(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x) \qquad \ldots(2)$$

Now, for each positive integer, we have to show that

$$\frac{d}{dx}F(a,b;c;x) = \frac{(a)_n(b)_n}{(c)_n}F(a+1,b+1;c+1;x) \qquad \ldots(3)$$

We do this job by the method of mathematical induction and, therefore, we proceed as follows:

Since $a=(a)_1, b=(b)_1$ and $c=(c)_1$, then from (2), we note that

$$\frac{d}{dx}F(a,b;c;x) = \frac{(a)_1(b)_1}{(c)_1}F(a+1,b+1;c+1;x) \qquad \ldots(4)$$

Now, let us assume that (3) is true for a particular value of n, say m, so that we have

$$\frac{d^m}{dx^m}F(a,b;c;x) = \frac{(a)_m(b)_m}{(c)_m}F(a+m,b+m;c+m;x) \qquad \ldots(5)$$

Now, we shall show that (3) is true for n = m + 1.

Thus, differentiating (5) w.r.t. x, we get

$$\frac{d^{m+1}}{dx^{m+1}} F\big(a, b; c; x\big) = \frac{(a)_m (b)_m}{(c)_m} \frac{d}{dx} F(a+m, b+m; c+m; x)$$

$$= \frac{(a)_m (b)_m}{(c)_m} \frac{(a+m)(b+m)}{(c+m)} F(a+m+1, b+m+1; c+m+1; x)$$

$$= \frac{(a+m)(a)_m . (b+m)(b)_m}{(c+m)(c)_m} F(a+m+1, b+m+1; c+m+1; x)$$

$$= \frac{(a)_{m+1} (b)_{m+1}}{(c)_{m+1}} F(a+m+1, b+m+1; c+m+1; x) \qquad \text{...(6)}$$

This shows that (3) is true for n = m + 1. Thus, we see that if (3) is true for n = m, then (3) is also true for each positive integer. Thus, it is proved that

$$\frac{d^n}{dx^n} F(a, b; c; x) = \frac{(a)_n (b)_n}{(c)_n} F(a+n, b+n; c+n; x) \qquad \text{...(7)}$$

Putting $x = 0$ in the above relation (7), we get

$$\left[\frac{d^n}{dx^n} F(a, b; c; x)\right]_{x=0} = \frac{(a)_n (b)_n}{(c)_n} F(a+n, b+n; c+n; 0)$$

$$= \frac{(a)_n (b)_n}{(c)_n} \left[\sum_{r=0}^{\infty} \frac{(a+n)_r (b+n)_r}{(c+n)_r} \frac{x^r}{r!}\right]_{x=0}$$

$= \frac{(a)_n (b)_n}{(c)_n}$, since the term in bracket is unity at x = 0.

Thus, we have $\left[\frac{d^n}{dx^n} F(a, b; c; x)\right]_{x=0} = \frac{(a)_n (b)_n}{(c)_n}$. ...(8)

Example 2. Prove the following :

(i) $e^x = F(a; a; x)$ (ii) $(1-x)^{-a} = F(a, b; b; x)$

(iii) $\ln(1+x) = x\, F(1,1; 2; -x)$ (iv) $sin^{-1}x = x\, F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)$.

and (v) $tan^{-1}x = x\, F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -x^2\right)$

Proof: **(i)** By definition of $F(a; b; x)$, we have

$$F(a; b; x) = \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r} . \frac{x^r}{r!}$$

or $$F(a; b; x) = 1 + \frac{a}{b} . \frac{x}{1!} + \frac{a(a+1)}{b(b+1)} . \frac{x^2}{2!} + \cdots \qquad \ldots(1)$$

Replacing b by$a$ in (1), we have

$$F(a; a; x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x$$

Thus, we have $\boldsymbol{e^x = F(a; a; x)}$. $\qquad \ldots(2)$

(ii) By definition of $F(a, b; c; x)$, we have

$$F(a, b; c; x) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} . \frac{x^r}{r!}$$

or $$F(a, b; r; x) = 1 + \frac{a.b}{a} \frac{x}{1!} + \frac{a(a+1).b(b+1)}{c(c+1)} . \frac{x^2}{2!} + \cdots \qquad \ldots(3)$$

Replacing c by b in (3), we have

$$F(a, b; b; x) = 1 + \frac{a}{1!} x + \frac{a(a+1)}{2!} x^2 + \cdots$$

$$= 1 + \frac{(-a)}{1!}(-x) + \frac{(-a)(-a-1)}{2!}(-x)^2 + \cdots$$

$= (1 - x)^{-a}$, by using binomial theorem.

Thus, we have $\boldsymbol{(1 - x)^{-a} = F(a, b; b; x)}$ $\qquad \ldots(4)$

(iii) By definition of $F(a, b; c; x)$, we have

$$F(a, b; c; x) = 1 + \frac{a.b}{c} \frac{x}{1!} + \frac{a(a+1).b(b+1)}{c(c+1)} . \frac{x^2}{2!} + \cdots \qquad \ldots(5)$$

Replacing a, b, c and x respectively by 1, 1, 2 and – x in (5), we get

$$F(1,1; 2; -x) = 1 + \frac{1.1}{2} \frac{(-x)}{1!} + \frac{1.2.1.2}{2.3} . \frac{(-x)^2}{2!} + \frac{1.2.3.1.2.3}{2.3.4} . \frac{(-x)^3}{2!}$$

Multiplying both sides of it by x, we get

$$xF(1,1;2;-x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = ln(1+x)$$

Thus, we have $\boldsymbol{ln(1+x) = F(1,1;2;-x)}$ …(6)

(iv) Replacing a, b, c and x respectively by $\frac{1}{2}, 1, \frac{3}{2}$ and $-x^2$ in (5), we get

$$F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right) = 1 + \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{3}{2}}\frac{x^2}{1!} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{\frac{3}{2}\cdot\frac{5}{2}}\frac{(x^2)^2}{2!} + \cdots$$

$$= 1 + 1^2\frac{x^2}{3!} + 1^2.3^2.\frac{x^5}{5!} + 1^2.3^2.5^2.\frac{x^6}{7!} + \cdots$$

Multiplying both sides of it by x, we get

$$F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right) = 1 + 1^2\frac{x^2}{3!} + 1^2.3^2.\frac{x^5}{5!} + 1^2.3^2.5^2.\frac{x^6}{7!} + \cdots = sin^{-1}x$$

Thus, we have $\boldsymbol{xF\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right) = sin^{-1}x}$ …(7)

(v) Replacing a, b, c and x respectively by $\frac{1}{2}, 1, \frac{3}{2}$ and $-x^2$ in (5), we get

$$F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-x^2\right) = 1 + \frac{\frac{1}{2}.1}{\frac{3}{2}}\frac{(-x)^2}{1!} + \frac{\frac{1}{2}\cdot\frac{3}{2}.1.2}{\frac{3}{2}\cdot\frac{5}{2}}\frac{(-x^2)^2}{2!} + \cdots$$

$$= 1\frac{x^2}{3} + \frac{x^5}{5} - \cdots$$

Multiplying both sides of it by x, we get

$$xF\left(\frac{1}{2},1;\frac{3}{2};-x^2\right) = x - \frac{x^2}{3} + \frac{x^4}{5} - \cdots = tan^{-1}x$$

Thus, we have $\boldsymbol{tan^{-1}x = xF\left(\frac{1}{2},1;\frac{3}{2};-x^2\right)}$ …(8)

Example 3. Prove that $\lim_{a\to\infty}\ {}_2F_1\left(1,a;1;\frac{x}{a}\right) = e^x$.

Solution. By definition, we have

$${}_2F_1\left(1,a;1;\frac{x}{a}\right) = 1 + \frac{1.a}{1.1}\left(\frac{x}{a}\right) + \frac{1.2.a(a+1)}{1.2.1.2}\left(\frac{x}{a}\right)^2 +\ldots$$

or $${}_2F_1\left(1,a;1;\frac{x}{a}\right) = 1 + \frac{x}{1!} + \left(1+\frac{1}{a}\right)\frac{x^2}{2!} + \left(1+\frac{1}{a}\right)\left(1+\frac{2}{a}\right)\frac{x^3}{3!} +\ldots$$

$$\therefore \quad \lim_{a\to\infty}\ {}_2F_1\left(1,a;1;\frac{x}{a}\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} +\ldots = e^x.$$

Example 4. Prove that ${}_2F_1(a;1;a;x) = (1-x)^{-1}$.

Solution. By definition, we have

$$F(a;1;a;x) = 1 + \frac{a.1}{1.a}x + \frac{a(a+1).1.2}{1.2.a(a+1)}x^2 + \frac{a(a+1)(a+2)1.2.3.}{1.2.3.a(a+1)(a+2)}x^2 + \ldots$$

$$= 1 + x + x^2 + x^3 + \ldots$$

or $$F(a;1;a;x) = (1-x)^{-1}$$

EXERCISE 3

1. Show that

(i) $(1-x)^{-1} = F(1,1;1;x)$ (ii) $(1+x)^n = F(-n,1;1;-x)$

(iii) $\ln x = -x\,F(1,1;2;x)$ (iv) $\ln(1-x) = -x\,F(1,1;2;x)$.

2. It is given that $F(a,b;c;x) = 1 + \frac{a.b}{c}\frac{x}{1!} + \frac{a(a+1).b(b+1)}{c(c+1)}.\frac{x^2}{2!} + \cdots$ is the solution of the differential equation $x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0$.
Then, show that $\left[\frac{d}{dx}F(a,b;c;x)\right]_{x=0} = \frac{ab}{c}$.

3. Show that $\lim_{b\to\infty} F(a,b;c;x/b) = F(a;c;x)$.

4. Prove that $b\,F(a;b;x) = bF(a-1;b;x) + x\,F(a;b+1;x)$, where symbols have their usual meaning.

5. Prove that $(a-b)F(a;b;c;x) = a\,F(a+1;b;c;x) - bF(a,b+1;c;x)$.

6. Show that $(1-x)^n = 1 - n\,x\,F(1-n,1;2;x)$, where n is a natural number.

7. Show that $(1-x)^{-1} = F(a,1;a;x)$, where the symbols have their usual meaning.

8. Find the value of $\frac{d^2}{dx^2}F(a,b;b;x)$.

9. Write the general solution of $x(x-1)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0$ about x = 0 by using the solution of Gauss's hypergeometric equation. Before giving general solution, write the numerator parameters and denominator parameter used in hypergeometric function.

10. Write the general solution of differential equation $(x-x^2)y'' + \left(\frac{3}{2} - 2x\right)y' - \frac{1}{4}y = 0$ about x = 0 by using the solution of Gauss's hypergeometric differential equation.

11. Solve the following Legendre's differential equation by changing it to a hypergeometric differential equation:

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0.$$

12. Find the value of $F\left(2,-1;\frac{3}{2};x\right)$.

13. Prove that $aF(a+1;b;x)-(b-1)F(a;b-1;x)=(a-b+1)F(a;b;x)$.

14. Show that $F\left(0,0;\frac{1}{2};x\right)=1$.

15. Show that $(-1)^n=F(-n,n+1;1;1)$.

ANSWERS

8. $\frac{d^2}{dx^2}F(a;b;b;x)=\frac{(a)_2(b)_2}{(c)_2}F(a+2,b+2;b+2;x)$.

9. Numerator parameters are: $a=2$, $b=-1$, denominator parameter is: $c=\frac{3}{2}$ and general solution is: $y=A\left(1-\frac{4}{3}x\right)+B\,x^{-1/2}F\left(\frac{3}{2},-\frac{3}{2};\frac{1}{2};x\right)$.

10. $y=A\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x\right)+B\,x^{-1/2}F\left(0,0;\frac{1}{2};x\right)$.

11. On putting $x^2=t$, the equation is transformed to $t(1-t)\frac{d^2y}{dt^2}+\left(\frac{1}{2}-\frac{3}{2}t\right)\frac{dy}{dt}+\frac{n(n+1)}{4}y=0$, Numerator parameters are: $a=\frac{n+1}{2}, b=\frac{-n}{2}$, denominator parameter is: $c=\frac{1}{2}$ and general solution is: $y=A\left(\frac{n+1}{2},\frac{-n}{2};\frac{1}{2};x^2\right)+B\,x\,F\left(\frac{n+1}{2},\frac{1-n}{2};\frac{3}{2};x^2\right)$.

12. $F\left(2,-1;\frac{3}{21};x\right)=1-\frac{4}{3}x$.

OBJECTIVE TYPE QUESTIONS

Choose the correct alternative in the following questions:

1. Which of the following symbol is capable for representing all solutions of the Gauss's hypergeometric differential equation:

(A) Papperitz symbol (B) Pochhammer symbol

(C) Barne's symbol (D) Factorial symbol.

2. How many singular points are there in the Gauss's hypergeometric differential equation

$$x(x-1)y'' + \{c-(a+b+1)x\}y' - aby = 0.$$

(A) Only one | (B) Only two

(C) Only three | (D) Infinite

3. The point $x = 0$ is ………… point of the Gauss's hypergeometric differential equation.

(A) Ordinary | (B) Regular singular

(C) Irregular singular | (D) Equilibrium

4. In the notation ${}_2F_1(a,b;c;x)$ for Gauss's hypergeometric function, 2 represents the number of………

(A) Numerator parameters | (B) Denominator parameters

(C) Exponents | (D) Variables

5. The radius of convergence of the Gauss's hypergeometric series F(a, b; c; x) is :

(A) Zero | (B) Unity

(C) Infinite | (D) $a + b + c$

6. $F(-n, n+1; 1,1)$ is the expansion of

(A) 1 | (B) $(-1)^n$

(C) ln n | (D) e^{-n}

7. $\frac{d}{dx}F(a,b;c;x) = \cdots$

(A) $\frac{ab}{c}F(a+1,b+1;c+1;x)$ (B) $\frac{ab}{c}F(a+1,b;c;x)$

(C) $\frac{ab}{c}F(a,b+1;c;x)$ (D) $\frac{ab}{c}F(a,b;c+1;x)$

8. How many ordinary points are there for the Gauss's hypergeometric differential equation

$$x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0.$$

(A) Only one (B) Only two

(C) Only three (D) Infinite

9. If **Re** $[c-a-b] > 0$, then $F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ is known as:

(A) Gauss's theorem (B) Vandermonde's theorem

(C) Kummer's theorem (D) Hadamard's theorem

10. $F(a,b;c;x) = \cdots$

(A) $F(a,c;b;x)$ (B) $F(a,b;x;c)$

(C) $F(b,a;c;x)$ (D) $F(b,c;a;x)$

11. Gauss's hypergeometric differential equation is a Fuchsian class of differential equation because it has all its singular points

(A) Regular (B) Irregular

(C) Both (A) and (B) (D) None of these

12. Gauss's hypergeometric differential equation has the singular points

(A) $0,1,-1$ (B) $0,1,\infty$

(C) $0,-1,\infty$ (D) $0,n,-n$

13. What is the nature of the point x = 1 for the Gauss's hypergeometric differential equation $x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0$.

(A) Regular singular point (B) Irregular singular point

(C) Ordinary point (D) None of these

14. $F(a,b;b;x) = \cdots$

(A) $(1-x)^a$ (B) $(1+x)^{-1}$

(C) $(1-x)^{-a}$ (D) $(1+x)^a$

15. $F(1,1;2;-x) = \cdots$

(A) $\ln\ (1-x)$ (B) $(1+x)^{-1}$

(C) $(1-x)^{-1}$ (D) $\ln\ (1+x)$

ANSWERS

1. (A) **2.** (C) **3.** (B) **4.** (A) **5.** (B) **6.** (B) **7.** (A) **8.** (D) **9.** (A)

10. (C) **11.** (A) **12.** (B) **13.** (A) **14.** (C) **15.** (D)