ANT (Contd.)

23/3/21

(HT) Expand by Taylor series and show that ADI scheme is $O(\delta t^2, \delta x^2, \delta y^2)$ and is consistent.

* Von Neumann analysis reveals that the ADI scheme is unconditionally stable.

(HT) Derive the tridiagonal system:

$$u_t = \nabla^2 u \quad ; \quad 0 \le x, y \le 1 \quad , t > 0$$

$$u(x, y, 0) = \sin \pi x \sin \pi y$$

$u = 0$ on the boundary

Derive the ensuing tri-diagonal system at each step discretized by ADI scheme.

(HT) Think!!

Q. $u_t = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$ → How to extend ADI scheme to a 3-step process and solve this?

Laplace or Poisson equation: (steady state situation, i.e. independent of time)

$$\left.\begin{aligned} \nabla^2 \phi = 0 &\rightarrow \text{Laplace eqn.} \\ \nabla^2 \phi = f(x,y) &\rightarrow \text{Poisson eqn.} \end{aligned}\right\} \begin{aligned} &\text{Elliptic PDE} \\ &b^2 - 4ac < 0 \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow \text{Hyperbolic PDE} \quad (b^2 - 4ac > 0)$$

- Elliptic PDE: $\nabla^2 \phi = 0$ or $f(x,y)$

all conditions on ϕ are prescribed on the boundary, for this it also refers as boundary value problem.

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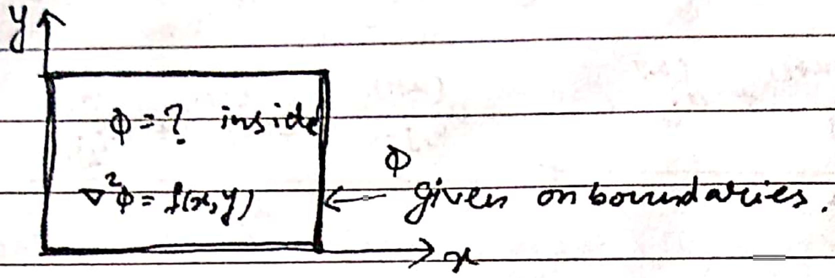
$\nabla^2\phi = f(x,y) \qquad x, y \in R$

ϕ is prescribed on the boundary ∂R of R.

$R: 0<x<a, \quad 0<y<b \qquad \partial R: x=0, a; \; y=0, b$



$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f(x,y) \quad ; \; x_i = i\,\delta x, \; i = 0, 1, \ldots, N$$

$$y_j = j\,\delta y, \; j = 0, 1, \ldots, M$$

Discretize through central diff. scheme:

$$\frac{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}}{\delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}}{\delta y^2} = f_{ij}$$

$$i = 1, 2, \ldots, N-1 \qquad j = 1, 2, \ldots, M-1$$

which are $(N-1)\times(M-1)$ equations involving $(N-1)\times(M-1)$ variables ϕ_{ij}.

Q. $\nabla^2 u = -10(x^2 + y^2 + 10) \quad ; \; 0 < x, y < 3$

HT

$u = 0$ on the boundary $\qquad \delta x = \delta y = 1$

$i, j = 1, 2$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{1} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{1} = -10(i^2 + j^2 + 10)$$

$i=1, j=1$: $\quad u_{21} - 2u_{11} + u_{12} - 2u_{11} = -10(12)$

$\Rightarrow u_{21} + u_{12} - 4u_{11} = -120 \quad \ldots (i)$

$i=2, j=1$: $\quad u_{31} - 2u_{21} + u_{11} + u_{22} - 2u_{21} = -10(15)$

$\Rightarrow u_{31} + u_{11} + u_{22} - 4u_{21} = -150 \quad \ldots (ii)$

Then find eqns. for
$i=1, j=2$
$i=2, j=2$

Gauss Seidel Method to solve the system of equations iteratively :

u_{ij} is the unknown at ij^{th} eqn. in $(k+1)^{th}$ iteration.

So,
$$\frac{u_{i-1,j}^{(k+1)} - 2u_{ij}^{(k+1)} + u_{i+1,j}^{(k)}}{\delta x^2} + \frac{u_{i,j-1}^{(k+1)} - 2u_{ij}^{(k+1)} + u_{i,j+1}^{(k)}}{\delta y^2} = f_{ij}$$

$u_{i-1,j}^{(k+1)}$ and $u_{i,j-1}^{(k+1)}$ have already been updated in the $(k+1)^{th}$ iteration. So they are also known.

Take $\beta = \frac{\delta x}{\delta y}$. Then, by simplifying,

$$\left(\frac{2}{\delta x^2}+\frac{2}{\delta y^2}\right) u_{ij}^{(k+1)} = \frac{u_{i-1,j}^{(k+1)} + u_{i+1,j}^{(k)}}{\delta x^2} + \frac{u_{i,j-1}^{(k+1)} + u_{i,j+1}^{(k)}}{\delta y^2} - f_{ij}$$

$$\Rightarrow (2+2\beta^2)\, u_{ij}^{(k+1)} = u_{i-1,j}^{(k+1)} + u_{i+1,j}^{(k)} + \beta^2\left[u_{i,j-1}^{(k+1)} + u_{i,j+1}^{(k)}\right] - \delta x^2 f_{ij}$$

$$\Rightarrow u_{ij}^{(k+1)} = \frac{u_{i-1,j}^{(k+1)} + u_{i+1,j}^{(k)} + \beta^2\left[u_{i,j-1}^{(k+1)} + u_{i,j+1}^{(k)}\right]}{2(1+\beta^2)} - \frac{\delta x^2 f_{ij}}{2(1+\beta^2)}$$

→ To start the iteration, guess $u_{ij}^{(0)}$ at all grid points.

This procedure is repeated till :

$$\text{Max}\left|u_{ij}^{(k+1)} - u_{ij}^{(k)}\right| < \epsilon$$

[The problem with Gauss Seidel is that its convergence is slow]

Diagonally dominance is a sufficient condition for convergence.

GS - ~~iterati~~ iteration converges at a slow rate.

Successive over-relaxation : [To have faster convergence]

$$\bar{u}_{ij}^{(k+1)} = \bar{u}_{ij}^{(k)} + \omega \left(u_{ij}^{(k+1)} - \bar{u}_{ij}^{(k)} \right)$$

(HT) $\bar{u}_{ij}^{(k+1)}$ is the modified value at $(k+1)^{th}$ iteration

ω is the relaxation parameter $1 \le \omega \le 2$

$\omega = 1 \rightarrow$ is the normal Gauss Seidel iteration.

For $0 < \omega < 1 \rightarrow$ Successive Under - relaxation.

Write the coefficients of successive over and under relaxation.

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Q. $\frac{\partial T}{\partial t} + u \cdot \frac{\partial T}{\partial x} = \alpha \cdot \frac{\partial^2 T}{\partial x^2}$; $0 < x < 1$, $t > 0$

(Lab) $T(0,t) = 0$, $T(1,t) = 100$, $t > 0$

$T(x,0) = 100x$

Q. $\frac{\partial T}{\partial t} + T \cdot \frac{\partial T}{\partial x} = \gamma \cdot \frac{\partial^2 T}{\partial x^2}$; $0 < x < 1$, $t \ge 0$ [Non linear]

(HT) $T(x,0) = f(x)$, $T(0,t) = T(1,t) = 0$

Discretize by the Crank Nicolson scheme and use Newton's linearization technique and determine the ensuing tri-diagonal system which needs to be solved.

Hint : $t_n \rightarrow t_{n+1}$ $\left(T_i^{n+1}\right)^{(k+1)} = \left(T_i^{n}\right)^{(k)} + \Delta T_i$

$\delta x = 0.25$, $r = 1$

(HT.) Q) $\nabla^2 u - 2\frac{\partial u}{\partial x} = -2$ in R , $u = 0$ on ∂R

$R : 0 < x < 1$, $0 < y < 1$; $h = \frac{1}{3}$

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Q. $-\nabla^2 u + 0.1 u = 1$; $0 < x < 1$

(HT) $u = 0$ on $x = 0, y = 0$

$\frac{\partial u}{\partial n} = 0$ on $x = 1, y = 1$; $n \to$ unit normal

$\delta x = \delta y = 0.5$

Hyperbolic PDE :

$$\frac{\partial^2 u}{\partial t^2} = c^2 \times \frac{\partial^2 u}{\partial x^2} \; ; \; u(x,0) = f(x) \; ; \; \frac{\partial u}{\partial t}(x,0) = g(x) \; ; \quad a \le x \le b$$

and $u(a,t)$ and $u(b,t)$ are given.

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\delta t^2} = c^2 \times \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2} \; ; \; \gamma = \frac{c\,\delta t}{\delta x}$$

$$\Rightarrow u_j^{n+1} = \gamma^2 \times (u_{j+1}^n + u_{j-1}^n) + 2(1-\gamma^2) u_j^n - u_j^{n-1} \; ; \quad j = 1, 2, \dots, N-1$$

[Explicit scheme]

$\underline{n=0}$: $\frac{\partial u}{\partial t}\Big|_j^0 = g_j$, $\frac{u_j^1 - u_j^{-1}}{2\delta t} = g_j$; $u_j^{-1} = u_j^1 - 2\delta t \cdot g_j$

$$u_j^1 = \gamma^2(u_{j+1}^0 + u_{j-1}^0) + 2(1-\gamma^2)u_j^0 - \{u_j^1 - 2\delta t\, g_j\}$$

$$u_j^1 = \frac{\gamma^2}{2}\left[(f_{j+1} + f_{j-1}) + (1-\gamma^2) f_j\right] + \delta t\, g_j \; ; \quad j = 1, 2, \dots, N-1$$

$\Rightarrow$ $\gamma < 1$ for stability

(HT) Solve this hyperbolic PDE for $c = 1$, $\delta x = \frac{1}{3}$, $\gamma = 0.5$

$u(0,t) = u(1,t) = 0$

$u(x,0) = \sin \pi x$; $\frac{\partial u}{\partial t}(x,0) = 0$; $0 \le x \le 1$

Find for $n = 1, 2, 3$.

* This scheme is also called ~~leapfrog~~ Leapfrog scheme. (CTCS)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$$

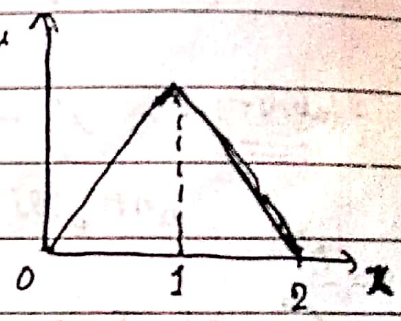
These two have similar characters

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} &= z \\ \frac{\partial z}{\partial t} - c\frac{\partial z}{\partial x} &= 0 \end{aligned}\right\} \text{Hyperbolic PDE.}$$

Both are wave equations.

$$\boxed{\frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} = 0} \rightarrow \text{1st order Hyperbolic PDE.}$$

$$u(x,0) = \begin{cases} x & ; \ 0 < x \le 1 \\ 2-x & ; \ 1 < x \le 2 \\ 0 & ; \ \text{otherwise} \end{cases}$$



$t=0$

$$\frac{dx}{c} = \frac{dt}{1} = \frac{du}{0} \qquad u = c_1 \ , \ x - ct = c_2$$

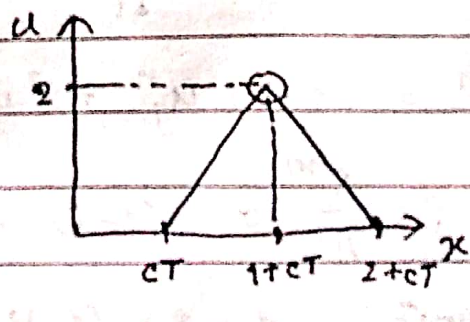
$$f(c_1, c_2) = 0$$

So, $u = f(x - ct)$; f is any arbitrary function.

$$u(x,0) = f(x) = \begin{cases} x & ; \ 0 < x \le 1 \\ 2-x & ; \ 1 \le x \le 2 \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$f(x - ct) = \begin{cases} x - ct & ; \ ct \le x \le 1 + ct \\ 2 - x + ct & ; \ 1 + ct \le x \le 2 + ct \end{cases}$$

$$u(x,T) = f(x - cT)$$



$t = T$

We can do the same thing for $\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0$. Just in that case, movement of wave will be in opposite direction. $u = g(x + ct)$

$u_t + c u_x = 0$; c is either $c = c(x,t)$ or a constant

$u(x,0) = u_0(x) \rightarrow$ I.C and B.C over x is prescribed.

$t_n \rightarrow t_{n+1}$; $n \geq 0$

FTCS

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c \times \left(\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} \right) = 0$$

$$\Rightarrow u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

Stability: $u_j^n \rightarrow A^n e^{i\theta j}$ $\cancel{A^{n+1} e^{i\theta(j+1)}}$

$$A^{n+1} e^{i\theta j} = A^n e^{i\theta j} - \frac{\nu}{2}\left(A^n e^{i\theta(j+1)} - A^n e^{i\theta(j-1)}\right)$$

$\xi = \dfrac{A^{n+1}}{A^n}$

$$\Rightarrow \xi = 1 - \frac{\nu}{2} \times \left(e^{i\theta} - e^{-i\theta}\right)$$

$\Rightarrow \xi = 1 - i\nu \sin\theta$. Now, for stability, $|\xi| \leq 1$

$$\Rightarrow |\xi|^2 = 1 + \nu^2 \sin^2\theta > 1$$

$\rightarrow$ Unconditionally unstable.

Hence FTCS can't be used.

FTBS (Euler scheme)

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c \times \frac{u_j^n - u_{j-1}^n}{\delta t} = 0$$

$$\Rightarrow u_j^{n+1} = (1-\nu)\, u_j^n + \nu\, u_{j-1}^n$$

Then, $\xi = \dfrac{A^{n+1}}{A^n} = (1-\nu) + \nu(\cos\theta - i\sin\theta)$

$$\Rightarrow |\xi|^2 = 1 - 2\nu(1-\nu)(1-\cos\theta)$$

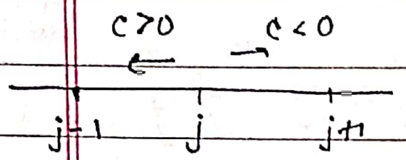
So, for stability: $0 < \nu \leq 1$; $\nu = \dfrac{c\,\delta t}{\delta x} > 0 \Rightarrow c > 0$

(ii) If $c<0$, FTFS scheme is used.

Q. Show that $u_t + c u_x = 0$, stable for

(HT) a) FTBS when $c>0$ b) FTFS when $c<0$

Find the value of ν and check for consistency.



This is also called Upwind Scheme.

5th Apr $u_t + c u_x = 0$ FTCS $\rightarrow$ Unconditionally unstable.

So we use Upwind Scheme: if $c>0$, FTBS i.e. $\frac{u_j^{n+1} - u_j^n}{\delta t} + c \times \frac{u_j^n - u_{j-1}^n}{\delta x} = 0$

if $c<0$, FTFS i.e. $\frac{u_j^{n+1} - u_j^n}{\delta t} + c \times \frac{u_{j+1}^n - u_j^n}{\delta x} = 0$

* Upwind implies one sided differencing for the space derivatives i.e. u_x.

This scheme is stable for $\nu \rightarrow$ CFL number

$|\nu| = \frac{|c|\,\delta t}{\delta x} \leq 1$ Courant Fedrick Lewis Number

Q) $u_t + u_x = 0$ $\delta x = 1/4$, $\nu = 1/2$

(HT) $u(x,0) = \begin{cases} x^2 ; & 0 \leq x \leq 1 \\ 0 ; & x > 1 \end{cases}$, $u(0,t) = 0$

Solve for $n \geq 0$, 2×0.0156, 2×0.0937, 2×0.2812, 2×0.5937

$u_j^{n+1} = (1-\nu) u_j^n + \nu u_{j-1}^n$

(HT) Q) $\frac{\partial T}{\partial t} + u \cdot \frac{\partial T}{\partial x} = 0$; $u = 0.1$; $T(x,0) = 200x$; $0 < x < 0.5$

(Lab) $= 200(1-x)$; $0.5 < x < 1$

$\nu = \frac{u\,\delta t}{\delta x} = 0.1$, $\delta x = 0.05$

Compare with exact.

FTBS $\rightarrow$ $O(\delta t, \delta x)$

classmate

Date

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Modified equation:

Initial eqn: $\frac{u_j^{n+1} - u_j^n}{\delta t} + c \cdot \frac{u_j^n - u_{j-1}^n}{\delta x} = 0$

Expand by Taylor series:

$$\frac{1}{\delta t} \times \left[\left\{ u_j^n + \delta t \, u_t \Big|_j^n + \frac{\delta t^2}{2!} \times u_{tt} \Big|_j^n + \frac{\delta t^3}{3!} \times u_{ttt} \Big|_j^n + \cdots \right\} - u_j^n \right]$$

$$+ \frac{c}{\delta x} \times \left[u_j^n - \left\{ u_j^n - \delta x \, u_x \Big|_j^n + \frac{\delta x^2}{2!} \cdot u_{xx} \Big|_j^n - \frac{\delta x^3}{3!} \times u_{xxx} \Big|_j^n + \cdots \right\} \right] = 0$$

at (x_j, t_n), we get:

$$u_t + c\, u_x = \frac{-\delta t}{2} u_{tt} + \frac{c\,\delta x}{2} u_{xx} - \frac{\delta t^2}{3!} u_{ttt} - \frac{c\,\delta x^2}{3!} u_{xxx} + \cdots \quad \cdots (*)$$

So, T.E $\sim O(\delta t, \delta x)$ consistent.

Replace the time derivatives by space derivatives in RHS

Differentiate $(*)$ w.r.t t:

$$u_{tt} + c\, u_{xt} = \frac{-\delta t}{2} u_{ttt} + \frac{c\,\delta x}{2} u_{xxt} - \frac{\delta t^2}{6} u_{tttt} - \frac{c\,\delta x^2}{6} u_{xxxt} + \cdots \quad \text{—(i)}$$

Differentiate $(*)$ w.r.t x:

$$u_{xt} + c\, u_{xx} = \frac{-\delta t}{2} \times u_{ttx} + \frac{c\,\delta x}{2} u_{xxx} - \frac{\delta t^2}{6} u_{tttx} - \frac{c\,\delta x^2}{6} u_{xxxx} \quad \text{—(ii)}$$

(i) $- c \times$ (ii) :-

$$u_{tt} - c^2 u_{xx} = \delta t \times \left[\frac{-u_{ttt}}{2} + \frac{c}{2} \times u_{ttx} \right] + O(\delta t^2) \quad \cdots (a)$$

$$+ \delta x \times \left[\frac{c}{2} u_{xxt} - \frac{c^2}{2} u_{xxx} \right] + O(\delta x^2)$$

Similarly, $u_{ttt} = -c^3 u_{xxx} + O(\delta t, \delta x) \quad \cdots (b)$

Substitute ⓐ, ⓑ in ⊛, we get :-

$$u_t + cu_x = -c^2 \frac{\delta t}{2} u_{xx} + c\frac{\delta x}{2} u_{xx} + \frac{\delta t^2}{6} c^3 u_{xxx} - \frac{c\delta x^2}{6} u_{xxx} + \cdots$$

$$\boxed{\text{i.e. } u_t + cu_x = c\frac{\delta x}{2}[1-\nu] u_{xx} + O(\delta x^2, \delta t^2)} \; ; \; \nu = \frac{c\,\delta t}{\delta x} \quad \text{---} (**)$$

If u is the soln. of the PDE, then LHS $u_t + cu_x = 0$

This eqn. is called the modified equation i.e., the PDE which the difference scheme (FTBS) is actually solving.

The least order term in δx is $\frac{c}{2}(1-\nu)u_{xx}$

which is called the dissipation error.

$\frac{c}{2}(1-\nu)$ is the numerical diffusion.

Note: Presence of this 2nd order space derivatives tends to spread out the sharp discontinuity in the u-profile.

Hyperbolic PDEs:
- Dissipative scheme: ~~whose least order~~ term in T.E. involves even order derivative w.r.t x. (occurs when we use one sided schemes like FTBS, FTFS)
- Dispersive scheme: T.E involves odd-order derivatives w.r.t x (occurs when we use central diff. like FTCS)

Lax scheme: FTCS can be made stable by replacing : (HT) check stability

$$u_j^n = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$$

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\delta t} + c \times \frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} = 0 \longrightarrow \text{Lax Scheme}$$

Find the modified eqn. (H.T)

$$u_t + cu_x = \frac{c}{2}\delta x\left(\frac{1}{\nu} - \nu\right)u_{xx} + \frac{c(\delta x)^2}{3}(1-\nu^2)u_{xxx} + \cdots$$

stable $|\nu| = |c| \times \frac{\delta t}{\delta x} \le 1$

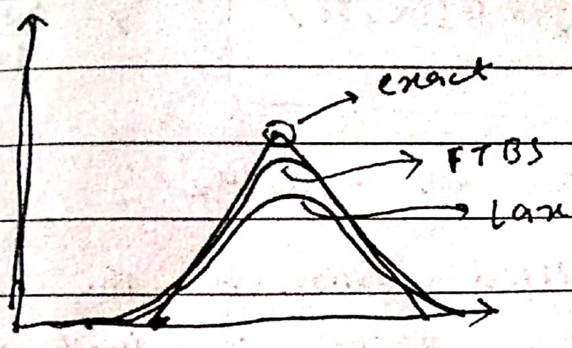
If $\nu \ne 1$, it is a dissipative scheme.

$$T.E = O\left(\delta t, \frac{\delta x^2}{\delta t}\right) \to \text{H.T}$$

The Lax scheme involves large dissipation

Thus, Lax scheme:

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right) \quad ; n \geq 0$$



Dissipative scheme

- Lax-Wendroff scheme:

This scheme can be derived from the Taylor series ~~expa~~ expansion:

$$u_j^{n+1} = u_j^n + \delta t\, u_t\Big|_j^n + \frac{\delta t^2}{2} u_{tt}\Big|_j^n + \frac{\delta t^3}{6} \times u_{ttt}\Big|_j^n + \cdots$$

Replace u_t, u_{tt} etc. by space derivatives i.e. $u_x, u_{xx}, \ldots$ by using the PDE

$$u_t = -c u_x \quad \text{So, } \frac{\partial u}{\partial t} = -c \times \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial t} \equiv -c \times \frac{\partial}{\partial x}$$

$$\frac{\partial u_t}{\partial t} = -\frac{\partial}{\partial t}\left(c \times \frac{\partial u}{\partial x}\right) = c \times \frac{\partial}{\partial x}\left(c \frac{\partial u}{\partial x}\right) = c^2 u_{xx}$$

$$\text{So, } u_{tt} = c^2 u_{xx}$$

$$\text{so, } u_j^{n+1} = u_j^n - \delta t \times c u_x\Big|_j^n + \frac{\delta t^2}{2} \times c^2 u_{xx}\Big|_j^n + O(\delta t^3)$$

Discretize the space derivatives by central diff. scheme:

$$u_j^{n+1} = u_j^n - \frac{c}{2}\frac{\delta t}{\delta x} \times \left(u_{j+1}^n - u_{j-1}^n\right) + \underbrace{\frac{1}{2} \times \nu^2}_{\frac{\nu^2}{2}} \times \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Lax wendroff scheme *

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\nu^2}{2} \times \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

which is an explicit scheme and T.E is $O(\delta t^2, \delta x^2)$ and stable if $|\nu| \le 1$ → (H.T)

Also find the modified eqn. → (H.T)

$$u_t + c u_x = -c \times \frac{(\delta x)^2}{6}(1-\nu^2) u_{xxx} - \frac{c(\delta x)^3}{8} \nu(1-\nu^2) u_{xxxx} + \dots$$

T.E = $O(\delta t^2, \delta x^2)$, consistent.

Lowest order term involves 3rd order derivative, which produces a dispersion error.

This ~~term~~, dispersion error is associated with wiggles formation near a sharpness in the solution.

(HT) Solve the two 1st order PDEs by all methods: FTBS, Lax, Lax Wendroff

(Lab) → Plot u vs x at $t = t^*$, along with the exact soln.

- Two step Lax-Wendroff scheme:

Step 1: Apply Lax method at the mid point $(j+\frac{1}{2})$ with half time step $(n+\frac{1}{2})$. i.e. $t_n \to t_{n+\frac{1}{2}}$

$$\frac{u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \frac{1}{2}\left[u_{j+1}^n + u_j^n\right]}{\delta t/2} + c \times \frac{u_{j+1}^n - u_j^n}{2 \times \delta x/2} = 0$$

$$\Rightarrow u_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \times (u_{j+1}^n + u_j^n) - \frac{\nu}{2} \times (u_{j+1}^n - u_j^n) \longrightarrow \boxed{\text{Predictor step}}$$

Step 2: $t_{n+\frac{1}{2}} \to t_{n+1}$; ~~central diff. at $(j, n+\frac{1}{2})$~~ with half space and time steps.

$$\frac{u_j^{n+1} - u_j^n}{2 \times \delta t/2} + c \times \left[\frac{u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{2 \times \delta x/2}\right] = 0$$

$$\Rightarrow u_j^{n+1} = u_j^n - \nu \times \left[u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right] \longrightarrow \boxed{\text{Corrector step}}$$

~~Step 1: Predictor step~~

Step 2: (corrector step): central diff. approx. at $(n+\frac{1}{2})$ time step at the mid-pt. $(j+\frac{1}{2})$

Replace $u_{j+1/2}^{n+1/2}$, $u_{j-1/2}^{n+1/2}$ from Step 1 to Step 2, we get the single step Lax-Wendroff scheme.

T.E $\sim O(\delta t^2, \delta x^2)$, stable for $|\nu| \le 1$

Q) $u_t + u_x = 0$; $u(x,0) = \exp(-20(x-2)^2) + \exp(-(x-5)^2)$

Solve upto $t = 17$

$0 \le x \le 25$, $\delta x = 0.05$, $\nu = 0.8$

Q) $u_t + 0.1\, u_x = 0$

$$u(x,0) = \begin{cases} 20x \;; & 0 \le x \le 0.05 \\ 20(0.1 - x) \;; & 0.05 \le x \le 0.1 \\ 0 \;; & 0.1 \le x \le 1 \end{cases}$$

$u(0,t) = 0$, $\nu = 0.8$, $\delta x = \frac{1}{4}$

Solve by all methods. $u_t + c u_x = 0$

- MacCormack Scheme:

$$u_j^{n+1} = u_j^n + u_t|_j^n \delta t + u_{tt}|_j^n \frac{\delta t^2}{2} + O(\delta t^3)$$

Instead of replacing u_{tt} by using the PDE, we consider a first-order expansion of $u_{tt}|_j^n$ as:

$$u_{tt}|_j^n = \left(u_t|_j^{n+1} - u_t|_j^n\right)/\delta t \quad \text{—} \circledast$$

$$u_j^{n+1} = u_j^n + u_t|_j^n \delta t + \frac{\delta t^2}{2} \times \frac{1}{\delta t} \cdot \left[u_t|_j^{n+1} - u_t|_j^n\right] + O(\delta t^3)$$

$$= u_j^n + \frac{\delta t}{2} \cdot \left[u_t|_j^{n+1} + u_t|_j^n\right] + O(\delta t^3)$$

Since $u_t = -c u_x$ from the PDE,

$$u_j^{n+1} = u_j^n - \frac{c\delta t}{2} \times \left[u_x|_j^n + u_x|_j^{n+1} \right] + O(\delta t^3)$$

The implicit derivative $u_x|_j^{n+1}$ creates problem

~~If we replace u_x by central differences, then it will be an implicit $O(\delta t^2, \delta x^2)$ scheme which is difficult to handle due to the implicit nature.~~

MacCormac proposed a Predictor step to obtain u_j^{n+1} by FTFS,

i.e. $\bar{u}_j^{n+1} = u_j^n - \nu(u_{j+1}^n - u_j^n)$; $\nu = \frac{c\delta t}{\delta x}$.

Use $\bar{u}_j^{n+1}$. Evaluate $u_x|_j^{n+1}$ by first order backward scheme.

$$u_j^{n+1} = u_j^n - \frac{c\delta t}{2} \times \left[\frac{u_{j+1}^n - u_j^n}{\delta x} + \underbrace{\frac{\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}}{\delta x}}_{\text{Using predicted values.}} \right] \quad \text{(Corrector step)}$$

$$\Rightarrow u_j^{n+1} = u_j^n - \frac{\nu}{2} \times \left[u_{j+1}^n - u_j^n \right] - \frac{\nu}{2} \times \left[\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1} \right]$$

Using predictor step, $\bar{u}_j^{n+1} = u_j^n - \nu \cdot (u_{j+1}^n - u_j^n) \Rightarrow -\frac{\nu}{2}(u_{j+1}^n - u_j^n) = \frac{1}{2} \times [\bar{u}_j^{n+1} - u_j^n]$

So, $u_j^{n+1} = u_j^n + \frac{1}{2} \times [\bar{u}_j^{n+1} - u_j^n] - \frac{\nu}{2} \times [\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}]$

$$\Rightarrow u_j^{n+1} = \frac{1}{2} \times [\bar{u}_j^{n+1} + u_j^n] - \frac{\nu}{2} \times [\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}] \quad \ldots (**) \quad \text{(Corrector step)}$$

So, the two-step MacCormack method is :

Predictor: $\bar{u}_j^{n+1} = u_j^n - \nu(u_{j+1}^n - u_j^n)$

Corrector: $u_j^{n+1} = \frac{1}{2} \times [u_j^n + \bar{u}_j^{n+1}] - \frac{\nu}{2} \times [\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}]$

(HT) For a linear PDE, this scheme is identical to the Lax-Wendroff single step method. T.E $= O(\delta t^2, \delta x^2)$ stable $|\nu| \leq 1$

Now, we'll discuss a non-linear PDE : $\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 0$

$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{1}{2}u^2) = 0 \rightarrow$ can be treated as $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$

By chain rule, $\frac{\partial f}{\partial x} = \left(\frac{\partial F}{\partial u}\right) \times \frac{\partial u}{\partial x} = A \frac{\partial u}{\partial x}$; $A = \frac{\partial F}{\partial u} \rightarrow$ Jacobian

So, $\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0$, $A = \frac{\partial F}{\partial u}$

If u is a vector, $A = \frac{\partial F}{\partial u}$ is a matrix, referred to as Jacobian.

$$\frac{\partial u}{\partial t} = -A \frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 f}{\partial t \partial x} = \frac{-\partial}{\partial x}\left(\frac{\partial F}{\partial t}\right)$$

$$= \frac{-\partial}{\partial x}\left(\frac{\partial F}{\partial u} \times \frac{\partial u}{\partial t}\right) = \frac{-\partial}{\partial x}\left(-A \cdot \frac{\partial F}{\partial x}\right)$$

So, $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(A \cdot \frac{\partial F}{\partial x}\right)$$

Now, as we do in Lax Wendroff scheme, just replacing time derivatives on RHS, using space derivatives, we get:

$$u_j^{n+1} = u_j^n + \delta t \times \frac{\partial u}{\partial t}\Big|_j^n + \frac{\delta t^2}{2!} \times \frac{\partial^2 u}{\partial t^2}\Big|_j^n + O(\delta t^3)$$

$$= u_j^n - \delta t \times \frac{\partial F}{\partial x}\Big|_j^n + \frac{\delta t^2}{2} \times \frac{\partial}{\partial x}\left(A \frac{\partial F}{\partial x}\right)\Big|_j^n + O(\delta t^3)$$

Now, space derivatives are discretized by central differences.

$$\frac{\partial}{\partial x}\left(A \frac{\partial F}{\partial x}\right)\Big|_j^n = \frac{1}{\delta x} \times \left[A \frac{\partial F}{\partial x}\Big|_{j+1/2}^n - A \frac{\partial F}{\partial x}\Big|_{j-1/2}^n \right]$$

$$= \frac{1}{\delta x^2} \times \left[A\Big|_{j+1/2}^n \times (F_{j+1}^n - F_j^n) - A\Big|_{j-1/2}^n \times (F_j^n - F_{j-1}^n) \right]$$

And, $A\Big|_{j+1/2}^n = A \times \left(\frac{u_j^n + u_{j+1}^n}{2}\right)$ and $A_{j-1/2}^n = A \times \left(\frac{u_j^n + u_{j-1}^n}{2}\right)$

Now, single step Lax-Wendroff scheme is:

$$u_j^{n+1} = u_j^n - \frac{\delta t}{\delta x} \times \left(\frac{F_{j+1}^n - F_{j-1}^n}{2}\right) + \frac{1}{2} \times \left(\frac{\delta t}{\delta x}\right)^2 \times \left[A_{j+1/2}^n (F_{j+1}^n - F_j^n) - A_{j-1/2}^n (F_j^n - F_{j-1}^n) \right]$$

$\rightarrow$ T.E $= O(\delta t^2, \delta x^2)$

Which ~~becomes~~ involves the evaluation of the Jacobian $A_{j+1/2}^n$ and $A_{j-1/2}^n$, which becomes complicated when a system of ~~eq~~ equations is considered, i.e. u is a vector.

Thus, we propose a two-step Lax-Wendroff scheme, proposed by Richtmyer.

Two step Lax-Wendroff scheme; (proposed by Richtmyer)

Predictor: $\bar{u}_j^{n+1} = \frac{1}{2} \times (u_j^n + u_{j+1}^n) - \frac{\delta t}{2\delta x}(F_{j+1}^n - F_j^n)$

Corrector: $u_j^{n+1} = u_j^n - \frac{\delta t}{\delta x} \times \left[F(\bar{u}_j) - F(\bar{u}_{j-1}) \right]$

Q) $\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 0$; $x>0$, $t>0$

(HT) $u(x,0) = x$; $x>0$ $\quad u(0,t) = 0$; $t>0$

Solve by: (1) Single step Lax Wendroff

(2) Two step Lax Wendroff scheme

$\delta x = 0.2$, $r = 0.05$

(HT) Q) $u_t + u u_x = 0$; $u(x,0) = 1$; $0 \le x \le 2$ → Using both methods mentioned above.

$u(x,0) = 0$; $2 \le x \le 4$

Q) $u_t + c u_x = \nu (u_{xx} + u_{yy})$; $0 \le x,y \le 1$, $t>0$

$u(0,x,y) = f(x,y)$ $\quad 0 \le x,y \le 1$

u is zero on the boundary.

Discretize by ADI scheme and derive the ensuing linear system of equations.