7.1.19

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Boundary value problems (BVP)

$$y'' = F(x, y, y')$$

2nd order (y'')

Ordinary ($y = f(x)$) ↳ one variable.

Linear (in y)

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)\,y = C(x) \qquad a < x < b$$

A, B, C can be any function of x or constant not all zero.

Numerical solution corresponds to a well posed problem.

well posed: unique solution; varies continuously with condition.
No singularity / multiple solution etc.

BVP if $y(x)$ is given at the boundary points.

ie $y(a) = y_a$ and $y(b) = y_b$ are prescribed.

Task to get $y(x)$ for $a < x < b$

Initial Value Problem (IVP) if

$y(a) = y_a$ $y_a'(x) = y_a'$ are prescribed.

Find $y(x) = ?$ $x > a$

In IVP all the conditions are prescribed at initial point.

IVP $\frac{d^2y}{dx^2} = F(x, y, y')$

$$y(a) = y_a \qquad y'(a) = y_a'$$

Let $z(x) = \frac{dy}{dx}$

$$\left[\begin{array}{ll} \frac{dz}{dx} = F(x, y, z) & \frac{dy}{dx} = z \\ y(a) = y_a & z(a) = y_a' \end{array}\right]$$

Scanned by CamScanner

We can solve by P-C* or R-K* methods.

n^{th} order IVP

$$\frac{d^n y}{dx^n} = F(x, y, y', y'', \ldots, y^{(n-1)})$$

Initial conditions $y(a) = y_a$ $y'(a) = y'_a \cdots y^{(n-1)}_a = y^{(n-1)}_a$

$$z_1 = y'$$
$$z_2 = y''$$
$$\vdots$$
$$z_{n-1} = y^{(n-1)}$$

$$\frac{dz_{n-1}}{dx} = F(x, y, z_1, z_2, \ldots, z_{n-1})$$

$$\frac{dy}{dx} = z_1$$

$$\frac{dz_1}{dx} = z_2$$

$$\vdots$$

$$\frac{dz_{n-2}}{dx} = z_{n-1}$$

A system of n first order ODE with conditions

$y(a) = y_a$ $z_1(a) = y'_a$ $\ldots$ $z_{n-1}(a) = y_a^{n-1}$

$\therefore$ Any n^{th} order IVP reduced to system of n 1st order ODEs.

Not true for BVP ✗

Can use shooting methods* to reduce to IVP and solved iteratively.

Summary: Solve as $y(x; \alpha)$ ← missing iv condition in bvp

Correct $\alpha \Rightarrow y(b, \alpha) = y_b$

Finite difference methods.

$$y'' = F(x, y, y')$$

$$y'' + A(x)\, y' + B(x)\, y = C(x) \qquad a < x < b.$$

$$y_i'' + A_i y_i' + B_i y_i' = C_i \quad \text{at} \quad x = x_i$$

$$x_i = a + ih$$

$$\boxed{\text{EOC}}$$

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Finite Difference

$$y'' = F(x, y, y')$$

$$y'' + A(x)y' + B(x)y = C(x), \qquad a < x < b$$

$$y_i'' + A_i y_i' + B_i y_i = C_i \quad \text{at } x = x_i$$

$$x_i = a + ih.$$

$$i = 0, 1, \ldots, n.$$

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

$a = x_0$ $\qquad$ $b = x_n$

To find y_i for $i = 1$ to $n-1$ (unknowns)

To satisfy:

$$(*) \quad y_i'' + A_i y_i' + B_i y_i = C_i \qquad i = 1 \ldots n-1$$

Relation $(*)$ forms $n-1$ equations involving y_i'', y_i', y_i

We want to express y_i'' and y_i' in terms of y_i's by finite difference methods.

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \ldots \qquad (\mathrm{I})$$

$$= y_i + hy_i' + \underbrace{O(h^2)}_{\text{very small}} \approx y_i + hy_i'$$

$$\therefore \quad y_i' = \frac{y_{i+1} - y_i}{h} - \left[\frac{h}{2!} y_i'' + \frac{h^2}{3!} y_i''' + \ldots\right]$$

$$\simeq \frac{y_{i+1} - y_i}{h} + O(h)$$

First Order forward difference method.

Truncation :- when ommiting infinite terms from a infinite series leaving only finite terms.

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' + \quad \ldots (\text{II})$$

$$y_i' = \frac{y_i - y_{i-1}}{h} + O(h) \quad \text{First order \underline{backward} difference}$$

(I) − (II)

$$y_{i+1} - y_i = 2h y_i' + \frac{2h^3}{3!} y_i''' + \ldots$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2) \quad \text{Second } \cancel{\text{First}} \text{ order \underline{Central} difference.}$$

(I) + (II)

$$y_{i+1} + y_{i-1} = 2y_i + h^2 y_i'' + \frac{2h^4}{4!} y_{i+1}'''' + \ldots$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2)$$

Second order Central difference (also)

$\boxed{\text{HW}}$

$$y_i' = Ay_i + By_{i-1} + Cy_{i-2} + O(h^2)$$

$$y_i'' = \bar{A}y_i + \bar{B}y_{i+1} + \bar{C}y_{i+2} + O(h^2)$$

$x \longrightarrow$

Substitute y_i' and y_i'' by finite difference formulae in (*).

We set.
We use second order difference formula to replace derivative.

Discretized equation

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + A_i \frac{y_{i+1} - y_{i-1}}{2h} + B_i y_i = C_i \qquad i = 1, 2, \ldots, n-1$$

Procedure = discretization

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i$$

$$a_i = \frac{1}{h^2} - \frac{A_i}{2h}, \quad b_i = B_i - \frac{2}{h^2}, \quad c_i = \frac{1}{h^2} + \frac{A_i}{2h}, \quad d_i = C_i$$

$i = 1, \ldots, n-1$

$$a_1 y_0 + b_1 y_1 + c_1 y_2 = d_1$$

$$\text{or } b_1 y_1 + c_1 y_2 = d_1 - a_1 y_a \quad \dots (1)$$

$$a_2 y_1 + b_2 y_2 + c_2 y_3 = d_2 \quad \dots (2)$$

$$a_3 y_2 + b_3 y_3 + c_3 y_4 = d_3 \quad \dots (3)$$

$$\vdots$$

$$a_{n-1} y_{n-2} + b_{n-1} y_{n-1} = d_{n-1} - c_{n-1} y_n \quad \dots (n-1)$$

Which are n-1 linear algebraic equations involving $y_1, y_2, \dots, y_{n-1}$ i.e. (n-1) unknowns. ⇒ compact/closed systems.

$$Ax = d$$

$$x = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_{n-1} \end{bmatrix} \quad A = \begin{bmatrix} b_1 & c_1 & 0 & 0 & & \\ a_2 & b_2 & c_2 & 0 & & 0 \\ 0 & a_3 & b_3 & c_3 & & \\ 0 & 0 & & \ddots & \ddots & \\ & 0 & & a_{n-2} & b_{n-2} & c_{n-2} \\ & & & & a_{n-1} & b_{n-1} \end{bmatrix} \quad d = \begin{bmatrix} d_1 - a_1 y_a \\ d_2 \\ d_3 \\ \vdots \\ d_{n-2} \\ d_{n-1} - c_{n-1} y_b \end{bmatrix}$$

which is a (n-1) eqns in (n-1) unknowns.

Tridiagonal System of Linear Equations

$x^T = [y_1 \; y_2 \; \dots \; y_{n-1}]$ → vector of unknowns

$d^T = [d_1 \; d_2 \; \dots \; d_{n-1}]$ → vector of knowns

Eliminate a_i's the reduced matrix is.

$$\begin{bmatrix} b_1' & c_1' & 0 & 0 & 0 & \dots & 0 \\ 0 & b_2' & c_2' & 0 & 0 & \dots & 0 \\ 0 & 0 & b_3' & c_3' & & & \\ \vdots & \vdots & & & \ddots & \ddots & \\ \vdots & \vdots & & & & & c_{n-2}' \\ 0 & 0 & & & & & b_{n-1}' \end{bmatrix} \rightarrow \begin{bmatrix} 1 & c_1' & & & & \\ & 1 & c_2' & & 0 & \\ & & 1 & c_3' & & \\ & & & 1 & & \\ & 0 & & & 1 & c_{n-2}' \\ & & & & & 1 \end{bmatrix}$$

Thomas Algorithm.

to solve a tridiagonal system.

Analytical algorithm, not iterative $\left.\begin{array}{l}\text{Gauss Jacobi} \\ \text{Gauss Seidel}\end{array}\right\}$ iterative

$\boxed{\text{Ex. Derive Thomas Algorithm}}$

$$Ax = d \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad A, d \to \text{same.}$$

$$x_n = d_n' \qquad x_i = d_i' - c_i' x_{i+1} \qquad i = n-1, n-2, \dots 1$$

where $c_1' = \dfrac{c_1}{b_1}$ $\quad d_1' = \dfrac{d_1}{b_1}$

and $c_i' = \dfrac{c_i}{b_i - a_i c_{i-1}'}$; $d_i' = \dfrac{d_i - a_i d_{i-1}'}{b_i - a_i c_{i-1}'}$ $\quad i = 2, 3, \dots, n$

$\boxed{\text{Lab task based on Thomas Algorithm}}$

Ex $x^2 y'' + x y' = 1$

$y(1) = 0 \quad y(1.4) = 0.0566 \qquad h = 0.1$

~~$y_0 = 0 \quad y_1 = 0$~~.

$y_0 = 0 \quad y_1, y_2, y_3 \to$ unknown $\quad y_4 = 0.0566$.

$x_0 = 1 \quad x_1 = 1.1 \quad x_2 = 1.2 \quad x_3 = 1.3 \quad x_4 = 1.4$.

$A_i = 1/x_i \quad B_i = 0 \quad C_i = 1/x_i^2$

~~$a_1 = \frac{1}{h^2} - A_1$~~

$A_0 = \frac{1}{1}$ $\quad A_1 = \frac{1}{1.1} \quad A_2 = \frac{1}{1.2} \quad A_3 = \frac{1}{1.3} \quad A_4 = \frac{1}{1.4}$

$B_i = 0 \quad C_0 = \frac{1}{1^2} \quad C_1 = \frac{1}{1.1^2} \quad C_2 = \frac{1}{1.2^2} \quad C_3 = \frac{1}{1.3^2} \quad C_4 = \frac{1}{1.4^2}$

~~$a_0 = \frac{1}{0.1^2} - \frac{1}{2\times 0.1} = 100 - 5 = 95$~~ $\quad a_1 = 100 - \frac{1/1.1}{2\times 0.1} = 95.4545$

$a_2 = 100 - \frac{1/1.2}{2\times 0.1} = 95.833 \quad a_3 = 96.15385 \quad a_4 = 96.42857$

$b_i = -\frac{2}{0.1^2} = -200$ $\quad c_0 = \frac{1}{h^2} + \frac{A_0}{2h} = 100 + \frac{1}{2 \times 0.1} = 105$

$c_1 = 104.5454 \quad c_2 = 104.16667 \quad c_3 = 103.84615 \quad c_4 = 103.57143$

$$\begin{bmatrix} -200 & 104.5454 & 0 \\ 95.4545 & -200 & 104.1667 \\ 0 & 95.833 & -200 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.8264 \\ 0.69444 \\ -5.285976 \end{bmatrix}$$

$y = -\frac{1}{2}(\log x)^2$ Compute for

$h = 0.1, 0.05, 0.01$

and compare with exact.

$\boxed{\text{EOC}}$

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$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)\,y = C(x) \qquad a < x < b \quad \ldots (1)$$

$$x_i = a + i\delta x.$$

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i \quad \ldots (2)$$

$$i = 1, 2, \ldots, n-1$$

Compatible discrete equation.

(1) is the differential equation

(2) is the difference equation.

The difference equation is said to be compatible to the differential equation if the truncation error tends to zero as the step sizes tends to zero. i.e. $\delta x \to 0$.

$$TE \to 0 \quad \text{as} \quad \delta x \to 0$$

Truncation Error (T.E) is the residue by which the exact solution of the differential equation fails to satisfy the difference equation

Let $Y(x)$ be the exact solⁿ

$$TE = a_i Y_{i-1} + b_i Y_i + c_i Y_{i+1} - d_i$$

Compatible $\Rightarrow$ $TE \to 0$ as $\delta x \to 0$

Expand all variables about x_i

$$\left(\frac{1}{\delta x^2} + \frac{A_i}{2\delta x}\right) Y_{i-1} + \left(-\frac{2}{\delta x^2} , B_i\right) Y_i + \left(\frac{1}{\delta x^2} + \frac{A_i}{2\delta x}\right) Y_{i+1} - C_i = T - t$$

$P(a < x < a+b)$

$a \in (0,1) =$

$(\infty\ a(1) =$

$\int_0^a [0,7]\, dx$

0 0.7

$$T.E. = \left(\frac{1}{\delta x^2} - \frac{A_i}{2\delta x}\right)\cdot\left(Y_i - \delta_x Y_i' + \frac{\delta x^2}{2} Y_i'' - \dots\right)$$

$$+\left(-\frac{2}{x^2} \cancel{} + B_i\right) Y_i + \left(\frac{1}{\delta x^2} + \frac{A_i}{2\delta x}\right)\left(Y_i + \delta_x Y_i' + \dots\right)$$

$$= \left[\frac{1}{2} Y_i'' + \frac{1}{2} Y_i'' + \frac{A_i}{2} Y_i' + \frac{A_i}{2} Y_i' + B_i Y_i - C_i\right] + \dots$$

$$= \left[Y_i'' + A_i Y_i' + B_i Y_i - C_i\right] + \delta x^2\left[\frac{A_i}{3!} Y_i'''\right] + \dots$$

$$= \delta x^2\left[\frac{A_i}{3!} Y_i'''\right] + \dots$$

The order T.E. is $O(\delta x^2)$

T.E. $\to 0$ as $\delta x \to 0$ since TE is expressed as power of δx.

$\Rightarrow$ compatible or consistent

X ——————— X.

Boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$

$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2$$

We have $(n-1) + 2$ variable $\{y(a), y(b)\}$

and $(n-1)$ discretised equation $+ 2$ B.C. $= n$

$\therefore$ $n+1$ both $\neq$ compatible.

We need second order forward and backward difference.

$$y_0' = A y_0 + B y_1 + C y_2 + O(h^2)$$
$$y_n' = \bar{A} y_n + \bar{B} y_{n-1} + \bar{C} y_{n-2} + O(h^2)$$
} HW.

$\boxed{\text{EoC}}$.

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① $y_n' = A y_n + B y_{n-1} + C y_{n-2} + O(h^2)$

② $y_n' = \bar{A} y_n + \bar{B} y_{n+1} + \bar{C} y_{n+2} + O(h^2)$

① $y_n' = (\ldots) y_n + (\ldots) y_n' + (\ldots) y_n'' + \ldots$

$$A + B + C = 0$$

$$B + 2C = \frac{-1}{h}$$

$$\frac{Bh^2}{2} + \frac{C(2h)^2}{2} = 0$$

$$y_n' = \frac{1}{2\delta x}\left[3y_n - 4y_{n-1} + y_{n-2}\right] + O(h^2)$$

→ Backward difference.

②

$$\bar{A} + \bar{B} + \bar{C} = 0$$

$$\bar{B}\,\delta x + 2\bar{C}\,\delta x = 1$$

$$\bar{B}\frac{\delta x^2}{2} + \frac{(2\delta x)^2}{2}\bar{C} = 0.$$

$$y_n' = \frac{1}{2\delta x}\left[-3y_n + 4y_{n+1} - y_{n+2}\right] + O(h^2)$$

→ forward difference.

Using these, the b.c.s can be discretized as

$$\alpha_1 y_0 + \frac{\beta_1}{2\delta x}\left[-3y_0 + 4y_1 - y_2\right] = \gamma_1$$

$$y_0 = f(y_1, y_2)$$

$$y_n = g(y_{n-1}, y_{n-2})$$

Rough

$k_1 e^{x\sqrt{\lambda}} + k_2 e^{-x\sqrt{\lambda}}$

$k_1 e^{x\sqrt{\lambda}} + k_2 e^{-x\sqrt{\lambda}}$

$k_1 + k_2 = 1$

$k_1\sqrt{\lambda} e^{\sqrt{\lambda}} = k_2\sqrt{\lambda} e^{-\sqrt{\lambda}}$

$\frac{k_1}{k_2} = e^{-2\sqrt{\lambda}}$

$k_1 = e^{-2\sqrt{2}} k_2$

$k_2 = \frac{1}{1+e^{-2\sqrt{2}}}$

$k_1 = \frac{e^{-2\sqrt{2}}}{1+e^{-2\sqrt{2}}}$

k_2

$\theta'' = \lambda\theta \quad , \quad \theta(0) = 1 \quad \theta'(1) = 0$

$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{h^2} = \lambda\theta_i \Big\} \quad \theta_0 \ldots \theta_n.$ $n-1$

$\theta_0 = 1$ ✓

$\cancel{\theta_n = 1}$

$\frac{1}{2\delta x}\left[3\theta_n - 4\theta_{n-1} + \theta_{n-2}\right] = 0 \Big\}$ ✓ $n+1$

$\frac{1}{h^2}\theta_{i-1} - \left(\frac{2}{h^2} + \lambda\right)\theta_i + \frac{1}{h^2}\theta_{i+1}$

$\sqrt{\lambda} e^{\sqrt{\lambda}x} = 0$

$\theta_n = \frac{4\theta_{n-1} - \theta_{n-2}}{3}$

$\theta = e^{\sqrt{\lambda}x}$

$\theta' = \sqrt{\lambda} e^{\sqrt{\lambda}x}$

$\theta'' = \lambda e^{\sqrt{\lambda}x} = \lambda\theta.$

$ae^{\sqrt{\lambda}x} +$

$a\sqrt{\lambda} e^{\sqrt{\lambda}x}$

$a\lambda e^{\sqrt{\lambda}x}$

$= \lambda\theta$

$ae^{x\sqrt{2}}$

$e^{-x\sqrt{2}} = 1$

$ae^{\sqrt{2}x} + be^{-\sqrt{2}x}$

$\cancel{a+b=1}$

$\cancel{\sqrt{2}ae^{\sqrt{2}} - \sqrt{2}be^{-\sqrt{2}} = 0}$

$a - b = 0$

$\frac{1}{2}\left(e^{\sqrt{2}x} + e^{-\sqrt{2}x}\right)$

(1) $\theta_0 = 1$

(2) $16\theta_0 - 34\theta_1 + 16\theta_2 = 0$

(3) $16\theta_1 - 34\theta_2 + 16\theta_3 = 0$

(4) $16\theta_2 - 34\theta_3 + 16\theta_4 = 0$

(5) $3\theta_4 - 4\theta_3 + \theta_2 = 0$

$$\begin{bmatrix} -34 & 16 & 0 & 0 \\ 16 & -34 & 16 & 0 \\ 0 & 16 & -34 & 16 \\ 0 & 1 & -4 & 3 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} -16 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\theta_0 = 0 \quad \theta_1 = 0.7432 \quad \theta_2 = 0.5793 \quad \theta_3 = 0.4878.$

Q $y'' - 2xy' - 2y = -4x \qquad 0 < x < 1$

$y(0) - y'(0) = 0$

$2y(1) - y'(1) = 1$

(Lab Task)

Show for $h = 0.1$

$y_n \sim 3.72$

Also solve by fictitious pts.

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \cancel{2}x\left(\frac{y_{i+1} - y_{i-1}}{\cancel{2}h}\right) - 2y_i = -4x_i$$

$$\left(\frac{1}{h^2} + \frac{x_i}{h}\right) y_{i-1} - \left(\frac{2}{h^2} + 2\right) y_i + \left(\frac{1}{h^2} - \frac{x_i}{h}\right) y_{i+1} = 4x_i$$

$$y_0 - \frac{1}{2h}\left[-3y_0 + 4y_1 - y_2\right] = 0$$

$$2y_n - \frac{1}{2h}\left[3y_n - 4y_{n-1} + y_{n-2}\right] = 1$$

$$y_0 = \frac{4y_1 - y_2}{2h\left(1 + \frac{3}{2h}\right)} = \frac{4y_1 - y_2}{2h + 3} \qquad y = \frac{1}{x}$$

$$\cancel{y_n = \frac{1 - 4y_{n-1} + y_{n-2}}{2h\left(2 - \frac{3}{2h}\right)}} \qquad -\frac{1}{x^2} \quad -\frac{2}{x^3}$$

$y = x$

xv

$y = x$ $\qquad y = kx$

~~Show~~

$y_1\left[\cancel{\frac{1}{0.1^2} + \frac{0.2}{0.1}}\right] \quad \left(\frac{1}{0.1^2} +\right. \qquad \cancel{x} - x^2 \qquad \cancel{2}xy' = -\cancel{2}y$

$\qquad \frac{y'}{y} = -\frac{1}{x}$

$\| \qquad \ln y = \ln x$

$10\cancel{2}y_1 \quad \cancel{y_1}\left(\frac{1}{h}\right.$

$x - k \qquad y = kx$

$y_0\left(\frac{1}{h^2} + x_1\right.$

HW

$y'' + y = 0 \quad , \quad 0 < x < 1$

$y(0) = 0 , y(1) = 1$

$h = 0.25$

Exact $y = \frac{\sin x}{\sin 1}.$

Lab Task

Check for grid independency

~~X~~

Higher order BVP

$y''' + A(x) y'' + B(x) y' + C(x) y = D(x)$

$a < x < b$

$y(a) = y_a \quad y'(a) = y_a' , \quad y'(b) = y_b'$

$y_i''' = (y_i'')' = \frac{y_{i+1}'' - y_{i-1}''}{2h} = \frac{1}{2h^3}[y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}]$

Assume fictitious pts y_{-1}, y_{n+1}

Calculate BCs at 0, n using central diff & fictitious pt.

But unknowns :- $y_{-1}, y_1, \ldots, y_n, y_{n+1}$ $(n+2)$

Eqn. $\underbrace{(n-1)}_{\text{discret.}} + \underbrace{2}_{BC}$

Not a compact system

So

$z = y'$

$z'' + Az' + Bz + Cy = 0$

$y(a) = y_a \quad z(a) = y_a' \quad z(b) = y_b'$

(1) $y_i' = z_i$

$$\frac{y_{i+1} - y_{i-1}}{2h} = z_i$$

(2) $z_i'' + A_i z_i' + B_i z_i + C_i y_i = D_i$

$2(n-1) = 2n-2$ row

$2n-1$ rows

yet not closed X.

$\boxed{\text{EoC}}$

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$$y''' + A(x)\,y'' + B(x)\,y' + C(x)\,y = D(x) \qquad a < x < b$$

$$y(a) = y_a \quad y'(a) = y'_a \quad y'(b) = y'_b$$

$$\begin{cases} \dfrac{dy}{dx} = z \\ z'' + Az' + Bz + Cy = D \end{cases}$$

$$y(a) = y_a \quad z(a) = y'_a \quad z(b) = y'_b$$

$x_i = a + i\delta x \qquad \delta x = h$, step size

x_i = grid pts / node points.

$$y_i' = z_i$$

$$z_i'' + A_i z_i' + B_i z_i + C_i y_i = D_i \qquad i = 1, 2, \dots, n-1$$

$$y_i - y_{i-1} = \int_{x_{i-1}}^{x_i} z\,dx \longrightarrow \text{trapezoidal.}$$

← since y_a prescribed

If y_b was prescribed use $\int_{x_i}^{x_{i+1}} z\,dx$.

$$= \frac{h}{2}(z_i + z_{i-1}) + O(h^3) \qquad \frac{1}{2}(z(x_i) + z(x_{i-1}))$$

$$i = 1, 2, \dots n-1$$

$$y_i - y_{i-1} - \frac{h}{2} z_i - \frac{h}{2} z_{i-1} = 0 \quad \dots (1)$$

Applying central diff scheme,

$$\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + A_i \frac{z_{i+1} - z_{i-1}}{2h} + B_i z_i + C_i y_i = D_i \quad (2)$$

$$i = 1, 2, \dots n-1.$$

Variable: $y_1, y_2, \dots, y_{n-1}$ & $z_1, z_2 \dots z_{n-1}$ = $2(n-1)$ involved in $2(n-1)$ equations which is a closed system.

(1) and (2) are coupled systems.

Let $X_i = \begin{pmatrix} y_i \\ z_i \end{pmatrix}$

$$A^i x_{i-1} + B^i x_i + C^i x_{i+1} = D^i \qquad i = 1, 2, \ldots, n-1$$

$$A^i = \begin{pmatrix} -1 & -\frac{h}{2} \\ 0 & \frac{1}{h^2} - \frac{A_i}{2h} \end{pmatrix}$$

$$B^i = \begin{pmatrix} 1 & -\frac{h}{2} \\ C_i & -\frac{2}{h^2} + B_i \end{pmatrix}$$

$$C^i = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{A_i}{2h} \end{pmatrix}$$

$$D^i = \begin{pmatrix} 0 \\ D_i \end{pmatrix}$$

$$D^1 = \begin{pmatrix} y_a + \frac{h}{2} y_a' \\ D_1 - \left(\frac{1}{h^2} - \frac{A_i}{2h}\right) y_a' \end{pmatrix}$$

$$D^{n-1} = \begin{pmatrix} 0 \\ D_{n-1} - \left(\frac{1}{h^2} + \frac{A_{n-1}}{2h}\right) y_b' \end{pmatrix}$$

Let $X^T = [x_1 \; x_2 \; x_3 \ldots x_{n-1}]$

$AX = D$

$$\begin{bmatrix} B^1 & C^1 & 0 & \cdots & 0 \\ A^2 & B^2 & C^2 & \cdots & 0 \\ 0 & A^3 & \cdots & & 0 \\ & 0 & & A^{n-1} & B^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} D^1 \\ D^2 \\ \vdots \\ D^{n-1} \end{bmatrix}$$

A is a block matrix whose coeff. are 2×2 matrices

A is block-tridiagonal matrix

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Q $y''' + 4y'' + y' - 6y = 1$

$y(0) = 0, \; y'(0) = 0, \; y'(1) = 1$

$AX = D$ $\qquad h = \frac{1}{3}$

3

$2(n-1)$

4

$z = \frac{dy}{dx} = 0$

$z'' + 4z' + z - 6y = 1$

$y(0) = 0 \quad z(0) = 0 \quad z(1) = 1$

$y_i - y_{i-1} - \frac{h}{2}(z_i + z_{i-1}) = 0$

$z_{i-1}\left(\frac{1}{h^2} - \frac{2}{h}\right) + z_i\left(-\frac{2}{h^2} + 1\right) + z_{i+1}\left(\frac{1}{h^2} + \frac{2}{h}\right) - 6y_i = 1 \;\;(2)$

$A_i = \begin{pmatrix} -1 & -\frac{h}{2} \\ 0 & \frac{1}{h^2} - \frac{2}{h} \end{pmatrix} \qquad C_i = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{2}{h} \end{pmatrix}$

$B_i = \begin{pmatrix} 1 & -\frac{h}{2} \\ -6 & 1 - \frac{2}{h^2} \end{pmatrix} \qquad D_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$D_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - A_1 X_0$

$D_{n-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - C_{n-1} X_n$

$AX = D$

$$A = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & c_3 & & \vdots \\ \vdots & 0 & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix} \qquad D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

a_i, b_i, c_i are $m \times m$ matrices.

y(1

Then $x_n = d_n'$ $x_i = d_i' - c_i' x_{i+1}$

where $c_1' = (b_1)^{-1} c_1$, $d_1' = (b_1)^{-1} d_1$

$b_i' = b_i - a_i c_{i-1}'$

$c_i' = (b_i')^{-1} c_i$

$d_i' = (b_i')^{-1} (d_i - a_i d_{i-1}')$

Look UP

Lab Task

Implement Block Tridiagonal.

\# $y^{iv} + 81y = 81x^2$ $0 < x < 1$

$y(0) = y(1) = y''(0) = y''(1) = 0$

$z = y''$

$z'' + 81y = 81x^2$

$y(0) = y(1) = 0$ $z(0) = z(1) = 0$.

Discretise by central diff.

$z_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$

or, $\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) - z_i = 0$

$\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + 81 y_i = 81 x_i^2$

$A_i = \begin{pmatrix} \frac{1}{h^2} & 0 \\ 0 & \frac{1}{h^2} \end{pmatrix}$

$B_i = \begin{pmatrix} -\frac{2}{h^2} & -1 \\ 81 & -\frac{2}{h^2} \end{pmatrix}$ $C_i = \begin{pmatrix} \frac{1}{h^2} & 0 \\ 0 & \frac{1}{h^2} \end{pmatrix}$

$$D_i = \begin{pmatrix} 0 \\ 81 x_i^2 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 0 \\ 81 x_1^2 \end{pmatrix} - \begin{pmatrix} y_0''/h^2 \\ z_0/h^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 81 x_1^2 \end{pmatrix} \qquad D_{n-1} = \begin{pmatrix} 0 \\ 81 x_{n-1}^2 \end{pmatrix}$$

$y_0'' = 0$, $z_0 = 0$.

$\cancel{D_{n-1}} =$

$y'''' + 81y = 81x^2$

$$y(0) = y(1) = y''(0) = y''(1) = 0$$

$$y_i^{(iv)} = (y_i'')'' = \frac{y_{i+1}'' - 2y_i'' + y_{i-1}''}{h^2}$$

$$= \frac{1}{h^4}\left[y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} \right] + 81 y_i = 81 x_i^2$$

$i = 1, 2, \dots, n-1$

Introduce $i = 1, 2, \dots n-1$

b.c.s $\qquad a[-1]$

$y_0 = 0$, $y_n = 0$, $y_0'' = 0$, $y_n'' = 0$

$y_0'' = 0 \Rightarrow y_{-1} = -y_1$

$y_n'' = 0 \Rightarrow y_{n+1} = -y_{n-1}$

Thus $n-1$ eq^n in $n-1$ var. $\quad h = 1/3$

$$y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} + y_i = x_i^2$$

x_{-1}	x_0	x_1	x_2	x_3	x_4
$-1/3$	0	$1/3$	$2/3$	1	$4/3$

$$\cancel{y_3} - 4y_2 + 6y_1 - \cancel{4y_0} + \cancel{y_{-1}} + \cancel{y_1} = 1/9$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 + y_2 = x_2^2$$

$$-5\cancel{y_3} + 7y_2 - 4y_1 + \cancel{y_0} = 4/9$$

$$\cancel{\begin{bmatrix} -4 & 1 \\ 7 & -5 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 4/9 \end{bmatrix}}$$

$$\begin{bmatrix} 6 & -4 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 4/9 \end{bmatrix}$$

$$6y_1 - 4y_2 = \frac{1}{9}$$

$$-4y_1 + 6y_2 = \frac{4}{9}$$

$\frac{11}{90}$ $\frac{7}{45}$

66 −

y

$$Q: y^{iv} - y''' + y = x^2$$

HW $\quad y(0) = y'(0) = 0$

$y(1) = 2,\ y'(1) = 0$

$\frac{14}{\times 4}{56}$

$-44 + \cancel{98}$ 84

12

$6\times$ 42

98

-58

49 ×2 98 44 58 90

\# $$y^{iv} + 81y = 729x^2$$

HW $\quad y(0) = y'(0) = y''(1) = y'''(1) = 0$

$h = 1/3$

\# $$y^{iv} + 2y = \frac{x^2}{9} + \frac{2}{3}x + 4.$$

$$y(0) = y'(0) = y(3) = y'(3) = 0$$

Nonlinear BVP

$$F(y'', y', y, x) = 0 \quad a < x < b$$

$y(a) = y_a \quad y(b) = y_b$

F is any arbit. function.

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Eg $y_i'' + y_i^2 + \sin y_i = x_i^2$

Discretized equation is

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \left(\frac{y_{i+1} - y_{i-1}}{2h}\right)^2 + \sin y_i = x_i^2$$

$\boxed{Ax=b}$. X

$\phi_i(y_{i-1}, y_i, y_{i+1}) = 0 \qquad i = 1, 2, \ldots, n-1$

Newton Raphson method.

$\phi(x) = 0$ root is α., $\phi(\alpha) = 0$.

$x_{i+1} = x_i - h\frac{f'}{f}$

$x_{i+1} = x_i - h\frac{f'}{f}$

Let x_0 be app. of α.

$\alpha = x_0 + \text{Error} = x_0 + \Delta x_0$

$$\phi(x_0 + \Delta x) = \phi(x_0) + \Delta x\, \phi'(x_0) + \frac{\Delta x_0^2}{2}\phi''(x_0) \ldots = 0.$$

$$\Delta x_0 = -\frac{\phi(x_0)}{\phi'(x_0)}$$

$$x_{i+1} = x_i - \frac{\phi_i}{\phi_i'}$$

$|x_{k+1} - x_k| < \varepsilon \Rightarrow$ stop.

ε

n

$M, N > n$

$|x_m - x_n| < \varepsilon$

$(x_m - f) < \varepsilon$.

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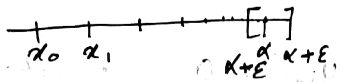
$$\{x_k \mid k \geq 0\} \text{ with } \lim_{k\to\infty} x_k = \alpha$$

For any $N \geq 0$ $\exists$ a ε such that

$$|x_{k+N} - x_k| < \epsilon \qquad N \geq 0$$



$$|x_k - \alpha| < \epsilon \qquad k \geq 0$$

$$\alpha - \epsilon < x_k < \alpha + \epsilon$$

$$|x_{k+1} - x_k| < \epsilon \qquad k \geq 0$$

$$|x_{k+1} - \alpha| \leq |x_k - \alpha|^2$$

Non linear BVP

$$f(y'', y', y, x) = 0 \qquad a < x < b$$

$$y(a) = y_a \qquad y(b) = y_b.$$

At grid pt. x_i, discretized eqn is

$$F_i(y_1, y_2, \ldots, y_{n-1}) = 0 \qquad i = 1, 2, \ldots, n-1$$

which is a system $(n-1)$ non linear algebraic equations involving $(n-1)$ variable $y_1, y_2, \ldots, y_{n-1}$

We solve (*) by taking Newtons linearization technique.

Let at k^{th} iteration, $y_i^{(k)}$ is the approximate solution of y_i with error Δy_i

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i \qquad i = 1, 2, \dots, n-1$$

~~whit~~ where y_i's are the exact solution of $(*)$

So, $F_i(y_1^{(k)} + \Delta y_1, y_2^{(k)} + \Delta y_2, \dots, y_{n-1}^{(k)} + \Delta y_{n-1}) = 0$

Expand by Taylor series. about $y_1^{(k)} \dots y_2^{(k)} \dots y_{n-1}^{(k)}$

We get

$$F(y_1^{(k)}, y_2^{(k)}, \dots, y_{n-1}^{(k)}) + \Delta y_1 \frac{\partial F_i}{\partial y_1}\Big|^{(k)} + \Delta y_2 \frac{\partial F_i}{\partial y_2}\Big|^{(k)} + \dots + \Delta y_{n-1} \frac{\partial F_i}{\partial y_{n-1}}\Big|^{(k)} + \frac{1}{2}(\Delta y_1)^2 \frac{\partial^2 F_i}{\partial y_1^2}\Big|^{(k)} + \dots + \frac{1}{2}(\Delta y_{n-1})^2 \frac{\partial^2 F_i}{\partial y_{n-1}^2}\Big|^{(k)} = 0$$

Assume Δy_i's are small enough, to drop the ~~quad~~ quadratic and higher order terms, then we get reduced form as.

$$F_i(y_1^{(k)}, \dots, y_{n-1}^{(k)}) + \Delta y_1 \frac{\partial F_i}{\partial y_1}\Big|^{(k)} + \dots + \Delta y_{n-1} \frac{\partial F_i}{\partial y_{n-1}}\Big|^{(k)} = 0$$

which leads to $(n-1)$ linear algebraic system of equations involving $n-1$ unknowns. $\Delta y_1, \dots, \Delta y_{n-1}$, which can be solved.

The next approximation for y_i is obtained as

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i \qquad i = 1, 2, \dots, n-1$$

Repeat till ~~$\|y^{(k+1)} - y^{(k)}\|$~~ $\max\limits_i |y_i^{(k+1)} - y_i^{(k)}| < \varepsilon$

Iteration starts with initial approximation for $y_i^{(0)}\ \forall i$

Working rule for Newtons linearization ~~that~~ technique.

1. Discretize the non linear BVP
2. Assume $y_i^{(0)}$ for all i
3. obtain Δy_i by solving the linear system (**
4. Find $y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$, $k \geq 0$
5. ~~Repeat~~ till

$$\max_i |y_i^{(k+1)} - y_i^{(k)}| < \epsilon$$

$$Ax = b \Rightarrow, \quad x^T = (\Delta y_1, \Delta y_2, \ldots \Delta y_{n-1})$$

Q. $y'' - (y')^2 - y^2 + y + 1 = 0$

$y(0) = 1/2$, $y(\pi) = -1/2$ (L.T.)

$$\cancel{\frac{y_{i+1}^{(0)} - 2y_i^{(0)} + y_{i-1}^{(0)}}{h^2}}$$

$$\cancel{\frac{(y_{i+1}^{(0)} + \Delta y_{i+1}^{(0)}) - 2(y_i^{(0)} + \Delta y_i^{(0)}) + (y_{i-1}^{(0)} + \Delta y_{i-1}^{(0)})}{h^2}}$$

$$\cancel{(y_{i+1}^{(0)} + \Delta y_{i+1}^{(0)}) - (}$$

$$\frac{y_{i+1}^{(k+1)} - 2y_i^{(k+1)} + y_{i-1}^{(k+1)}}{h^2} - \frac{1}{4h^2}\left(y_{i+1}^{(k+1)} - y_{i-1}^{(k+1)}\right)^2 = y_i^{(k+1)^2} + y_i^{(k+1)} + 1$$

$i = 1, 2, \ldots, n-1.$

$$y_i^{(k+1)} = \cancel{x}\, y_i^{(k)} + \Delta y_i$$

$d \; \frac{h}{h^2}$

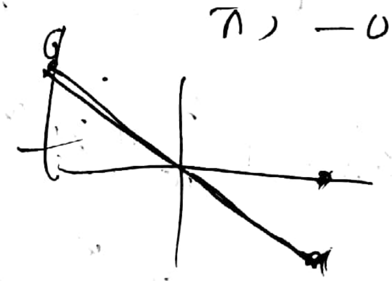
$$\Rightarrow \quad a_i \Delta y_{i-1} + b_i \Delta y_i + a_i \Delta y_{i+1} = d_i \qquad i = 1, 2, \ldots, n-1$$

$$\left[\frac{1}{h^2} + \frac{1}{4h^2} 2\left(y_{i+1}^{(k)} - y_{i-1}^{(k)}\right)\right] \Delta y_{i-1}$$
$$+ \Delta y_i \left[-\frac{2}{h^2} - 2y_i^{(k)} + 1\right] + \Delta y_{i+1}\left[\frac{1}{h^2} + \frac{1}{4h^2} 2(\;\ldots\right]$$

$\boxed{\text{Foc}}$

$0, 0.5$

$\pi, -0.5$



$\frac{1}{4h^2}\left(y_{i+1}^2 + \right.$

$$y = -\frac{1}{\pi}x + 0.5$$

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$\Delta y_0 = \Delta y_n = 0$ for all k

$\therefore$ no iteration at endpoints.

Q $y'' = 2 + y^2 \quad y(0) = y(1) = 0$. (L.T.)

$\Delta y_0 = \Delta y_n = 0$.

$$\left(\frac{y_{i+1}^{(k+1)} - 2y_i^{(k+1)} + y_{i-1}^{(k+1)}}{h^2} \right) = 2 + \left(y_i^{(k+1)}\right)^2$$

$$\Rightarrow \left\{ \frac{y_{i+1}^{(k)} - 2y_i^{(k)} + y_{i-1}^{(k)}}{h^2} = 2 + \left(y_i^{(k)} + \Delta y_i\right)^2 \right.$$

$$\left. + \left(\frac{\Delta y_{i+1} - 2\Delta y_i + \Delta y_{i-1}}{h^2} \right) \right\}$$

$$-\frac{1}{h^2}\Delta y_{i+1} + \left(\frac{2}{h^2} + 2y_i^{(k)}\right)\Delta y_i - \frac{1}{h^2}\Delta y_{i-1}$$

$$= \frac{1}{h^2}\left(y_{i+1}^{(k)} - 2y_i^{(k)} + y_{i-1}^{(k)}\right) - 2 - y_i^{2(k)}$$

$$\frac{1}{h^2}\Delta y_{i-1} + \left(2y_i^{(k)} - \frac{2}{h^2}\right)\Delta y_i + \frac{1}{h^2}\Delta y_{i+1}$$

$$= \frac{1}{h^2}\left(y_{i+1}^{(k)} - 2y_i^{(k)} + y_{i-1}^{(k)}\right) - 2 - y_i^{2(k)}$$

Choose $\Delta y_i^{(0)}$ s.t. b.c. satisfied.

Say. if $f(0) = f(1) = 0$

$\Rightarrow f(x) = x(1-x)$ can be used.

$y_0 = 0 \quad y_1^{(0)} = \frac{2}{9} \quad y_2^{(0)} = \frac{2}{9} \quad y_3 = 0 \qquad \frac{2}{3}$

~~$9\Delta y_1$~~

$9(\Delta y_0) - \left(\frac{4}{9} + 18\right)\Delta y_1 + 9\Delta y_2 = 2 + \frac{4}{81} - 9\left(\frac{2}{9} - \frac{4}{9} + 0\right)$

$9\Delta y_0 - \frac{166}{9}\Delta y_1 + 9y_2 = \frac{328}{81}$ $\qquad 162$

$2 + \frac{4}{81} \qquad \frac{328}{81}$

$0 - \frac{166}{9}\Delta y_1 + 9y_2 = \frac{328}{81}$

$\frac{166}{81} + 2 \qquad 18 \quad 9 \quad 162$

$9\Delta y_1 - \left(18 + \frac{4}{9}\right)\Delta y_2 + 9\Delta y_3 = 2 + \frac{4}{9} - 9\left(0 - 2\cdot\frac{2}{9} + \frac{2}{9}\right)$ $\qquad \frac{22}{9} + 2$

$9\Delta y_1 - \frac{166}{9}\Delta y_2 = \frac{328}{81}$

$166 \quad 81 \quad 247$

$-166\Delta y_1 + 81\Delta y_2 = \frac{328}{9}$

$81\Delta y_1 - 166\Delta y_2 = \frac{328}{9}$

$-85\Delta y_1 - 85\Delta y_2 = \frac{656}{9}$ $\qquad -\frac{656}{9\times 85}$

$\Delta y_1 + \Delta y_2 = -\frac{80}{85} = -\frac{16}{17}$ $\qquad \boxed{-\frac{328}{9\times 85}}$

$-247\Delta y_1 + 247\Delta y_2 = 0$

$\Delta y_1 = \Delta y_2 \doteq -0.4287$

$\Delta y_0 = 0 \quad y_1 = -0.20654 \quad y_2 = -0.20654 \quad y_3 = 0$

$-18.4131\,\Delta y_1 + 9\Delta y_2 = \cancel{9(0.206540)}\ 0.183736$

$9\Delta y_1 - 18.4131\Delta y_2 = 0.183736$

$f''' + ff'' + 1 - (f')^2 = 0$ (L.T)

at $\eta = 0$, $f = 0$, $f' = 0$

~~Let~~ $\eta = 10$, $f' = 1$.

$F = f'$

$F'' + fF' + 1 - F^2 = 0$

$$\frac{F_{i-1} - 2F_i + F_{i+1}}{h^2} + f_i \frac{F_{i+1} - F_{i-1}}{2h} + 1 - F_i^2 = 0$$

$$f_i - f_{i-1} = \frac{h}{2}(F_i + F_{i-1})$$

$-(F_i + \Delta F_i$

$$\left(\frac{1}{h^2} - \frac{f_i}{2h}\right)\Delta F_{i-1} + \left(-\frac{2}{h^2} - 2F_i\right)\Delta F_i + \left(\frac{1}{h^2} + \frac{f_i}{2h}\right)\Delta F_{i+1}$$
$$+ \Delta f_i \left(\frac{F_{i+1} - F_{i-1}}{2h}\right)$$
$$+ \frac{F_{i-1} - 2F_i + F_{i+1}}{h^2} + \frac{f_i(F_{i+1} - F_{i-1})}{2h} + 1 - F_i^2 = 0$$

$$\frac{h}{2}\Delta F_{i-1} + \frac{h}{2}\Delta F_i - \Delta f_i + \Delta f_{i-1}$$
$$+ \frac{h}{2}(F_i + F_{i-1}) + f_{i-1} - f_i = 0$$

$$A_i = \begin{pmatrix} 1 & \frac{h}{2} \\ 0 & \frac{1}{h^2} - \frac{f_i}{2h} \end{pmatrix} \qquad B_i = \begin{pmatrix} -1 & \frac{h}{2} \\ \frac{F_{i+1} - F_{i-1}}{2h} & -\frac{2}{h^2} - 2F_i \end{pmatrix}$$

$$C_i = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{f_i}{2h} \end{pmatrix}$$

$D_i =$

D

D_n

$f''' +$

$\eta =$

$\eta =$

$$D_i = \begin{pmatrix} f_i - f_{i-1} - \frac{h}{2}(F_i + F_{i-1}) \\ F_i^2 - 1 - \frac{f_i(F_{i+1} - F_{i-1})}{2h} - \frac{F_{i-1} - 2F_i + F_{i+1}}{h^2} \end{pmatrix}$$

$i = 1, 2, \ldots, n-1.$

$$D_1^* = D_1^* - A_1 \cancel{X_0}^{\,0} = D_1$$

$$D_{n-1} = D_{n-1} - C_{n-1}\cancel{X_n}^{\,0} = D_{n-1}$$

$f''' + ff'' + f'^2 = 0$

$\eta = 0: \quad f = 0, f'' = 0$

$\eta = 10 \quad f' = 0.$

$F'(0) = 0 \quad F_0' = \frac{-3F_0 + 4F_1 - F_2}{2h} = 0.$

$F_0 = \frac{1}{3}(4F_1 - F_2)$

5.2.19

Quasilinearization Technique.

The ~~x~~ non linear ODE / PDE is linearized iteratively

Non linear BVP is $F(y'', y', y, x) = 0$

$$y(a) = y_a \quad y(b) = y_b . \qquad a < x < b$$

Treat F as a function of y'', y', y at any x

At every iteration F is reduced to a linear form.

At $(k+1)^{th}$ Iteration, expand F about the known form of y'', y', y evaluated at k^{th} iteration

ie $\cancel{y''(x), y'(x)}$, $y''^{(k)}$, $y'^{(k)}$, $y^{(k)}$

Expand $F(y'', y', y, x)$ about $\cancel{y''(k), y'(k), y}$ $y''^{(k)}$, $y'^{(k)}$, $y^{(k)}$ by the taylor series expansion:-

$$F(y'', y', y, x) = F(y''^{(k)}, y'^{(k)}, y^{(k)}) + (y'' - y''^{(k)}) \frac{\partial F}{\partial y''}\Big|^{(k)} + (y' - y'^{(k)}) \frac{\partial F}{\partial y'}\Big|^{(k)} + \dots$$

$\cancel{h\frac{\partial}{\partial x} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}}$

Variables with superscript k is the known evaluated at the previous iteration $k (\geq 0)$

Retain only upto linear terms of $(y'' - y''^{(k)})$, $(y' - y'^{(k)})$, $(y - y^{(k)})$ and denote the variables with superscript $k+1$, we get.

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$$F(y''^{(k)}, y'^{(k)}, y^{(k)}, x) + (y''^{(k+1)} - y''^{(k)})\frac{\partial F}{\partial y''}\Big|^{(k)} + (y'^{(k+1)} - y'^{(k)})\frac{\partial F}{\partial y'}\Big|^{(k)} + (y^{(k+1)} - y^{(k)})\frac{\partial F}{\partial y}\Big|^{(k)} = 0 \quad \ldots (1)$$

In this expression $y^{(k+1)}$ and its derivatives $y''^{(k+1)}$, $y'^{(k+1)}$ appears linearly

And since $y^{(k+1)}$ is an approximation of $y(x)$, thus $y^{(k+1)}$ satisfy the b.c.s.

$$y^{(k+1)}(a) = y_a \qquad y^{(k+1)}(b) = y_b \quad \ldots (2)$$

Thus y^{k+1} satisfy a linear BVP.

Solve by finite diference.

~~Note: May~~

For numerical computation

Solve (1) with b.c.s (2) to get $y_i^{(k+1)}$ for $i = 1, 2, \ldots, n-1$

Repeat the process till

$$\max_{1 \le i \le n} |y_i^{(k+1)} - y_i^{(k)}| < \epsilon$$

Iteration process starts with assumption for $y^{(0)}$, $y'(0)$, $y''(0)$ $\forall x$.

The condition required for $y^{(0)}(x)$ is that it has to satisfy the b.c.s $y^{(0)}(a) = y_a$ $y^{(0)}(b) = y_b$

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Q $y'' - (y')^2 - y^2 + y + 1 = 0$ (L.T.)

$y(0) = \frac{1}{2} \quad y(\pi) = -\frac{1}{2}$

$F(y'', y', y) = 0 = y'' - (y')^2 - y^2 + y + 1$

$y(x)^{(0)} = \frac{1}{2}\cos(x)$

$$F''(y''^{(k)}, y'^{(k)}, y^{(k)}) + (y''^{(k+1)} - y''^{(k)}) \frac{\partial F}{\partial y''}\Big|^{(k)} + (y'^{(k+1)} - y'^{(k)}) \frac{\partial F}{\partial y'}\Big|^{(k)} + (y^{(k+1)} - y^{(k)}$$

$$y''^{(k)} - (y'^{(k)})^2 - (y^{(k)})^2 + y^{(k)} + 1 + (y''^{(k+1)} - y''^{(k)})1 + (y'^{(k+1)} - y'^{(k)})(-2y'^{(k)})$$
$$+ (y^{(k+1)} - y^{(k)})(-2y^{(k)} + 1) = 0$$

$$y''^{(k+1)} - 2y'^{(k)} y'^{(k+1)} + (1 - 2y^{(k)}) y^{(k+1)}$$
$$= \cancel{-y''^{(k)}} (y'^{(k)})^2 - 2(y'^{(k)})^2 + (y^{(k)})^2 - 2(y^{(k)})^2 - 1$$

$$\frac{y_{i+1}^{(k+1)} - y_{i-1}^{(k+1)}}{2h}$$

$$y''^{(k+1)} - 2y'^{(k)} y'^{(k+1)} + (1 - 2y^{(k)}) y^{(k+1)} = -(y'^{(k)})^2 - (y^{(k)})^2 - 1$$

$$\cancel{= (y'^{(k)})^2 - 2}$$

$$\frac{y_{i-1}^{(k+1)} - 2y_i^{(k+1)} + y_{i+1}^{(k+1)}}{h^2} \qquad y^{(k+1)}(0) = \frac{1}{2} \qquad y^{(k+1)}\pi = -\frac{1}{2}$$

At every iteration $(k+1)$ we need to solve the above linear BVP

Solve $y_i^{(k+1)}$ by FDM

0	$\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{3\pi}{5}$	$\frac{4\pi}{5}$	$\frac{5\pi}{5}$
0	1	2	3	4	5
0	1	2	3	4	

$$\frac{y_{i-1} - 2 y_i + y_{i+1}}{h^2} - 2y'^{(}$$

$$y_{i-1}^{(k+1)}\left(\frac{1}{h^2} + \frac{2y'^{(k)}}{2h}\right) + y_i^{(k+1)}\left(-\frac{2}{h^2} + (1 - 2y^{(k)})\right)$$
$$+ y_{i+1}^{(k+1)}\left(\frac{1}{h^2} - \frac{2y'^{(k)}}{2h}\right) = -$$

$y'' + 2y = 0$ (LT)

Q Solve by the quasilinearization technique.

L.T.

$$f''' + ff'' + (f')^2 = 0$$

$$f(0) = 0, \quad f''(0) = 0, \quad f'(10) = 0$$

$$F(f''', f'', f', f) = 0$$

$$\left(f''' + ff'' + (f')^2\right)^{(k)} + \left(f'''^{(k+1)} - f'''^{(k)}\right) 1 + \left(f''^{(k+1)} - f''^{(k)}\right) f^{(k)} + \left(f'^{(k+1)} - f'^{(k)}\right) 2f'^{(k)} + \left(f^{(k+1)} - f^{(k)}\right) f''^{(k)} = 0$$

$$f'''^{(k+1)} + f^{(k)} f''^{(k+1)} + 2f'^{(k)} f'^{(k+1)} + f''^{(k)} f^{(k+1)} = -\cancel{f^{(k)} f''^{(k)}} - \cancel{(f'^{(k)})^2} + \cancel{f^{(k)} f''^{(k)}} + 2(f'^{(k)})^2 + f^{(k)} f''^{(k)}$$

$$f'''^{(k+1)} + f^{(k)} f''^{(k+1)} + 2 f'^{(k)} f'^{(k+1)} + f''^{(k)} f^{(k+1)} = (f'^{(k)})^2 + f^{(k)} f''^{(k)}$$

$$f^{(k+1)}(0) = 0 \quad f''^{(k+1)}(0) = 0 \quad f'^{(k+1)}(0) = 0.$$

Leading to Block Tridiagonal.

-2.857

0.5 -1.829

-2.357 × 5

$\cancel{2 \times 3.14}$

$\cancel{6.28}$

9.15

$0.5 - \sin\left(\frac{1}{2}\right)$

$-\frac{k\cos}{2}\left(\frac{\pi}{2}\right)$

$y_0^{(k+1)}$

$F^{(k+1)} = f'^{(k+1)} 2$

$F''^{(k+1)} + f^{(k)} F'^{(k+1)} + 2f'^{(k)} F^{(k+1)} + f''^{(k)} f^{(k+1)}$

Spline Interpolation

$$\begin{cases} y(x) \approx p_n(x) \\ p_n(x_i) = y_i \end{cases} \qquad i = 0, 1, \dots, n$$

$x_i \rightarrow$ node pts.

$y(\alpha) \simeq p_n(\alpha)$ α is not a node pt In $[x_0, \dots, x_n]$

Huge diff b/w $y'(\alpha)$, & $p_n'(\alpha)$

We go for piecewise polynomial interpolation.

Set $S(x) = p_k(x)$; for $x_k \leq x \leq x_{k+1}$

$p_k(x)$ is the piecewise interpolation ~~technique~~ polynomial that interpolates $y(x)$ in $[x_k, x_{k+1}]$

Now, $p_k(x_k) = y_k$
$p_k(x_{k+1}) = y_{k+1}$

$y(x) \sim p_k(x)$ for $x_k \leq x \leq x_{k+1}$

$p_{k+1}(x_{k+1}) = y_{k+1}$

$y(x) \sim p_{k+1}(x)$ in $[x_{k+1}, x_{k+2}]$

$\Rightarrow p_k(x_{k+1}) = p_{k+1}(x_{k+1})$ $k = 1, 2, \dots n-1$

$x_1, x_2, \dots, x_{n-1} \longrightarrow$ Knot pts.

$p_k(x)$ is a cubic polynomial.

~~$p_k(x)$ have same value~~

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p_k have same value, same slope and same curvature at the knot points.

In other words $S(x)$ has the continuity of the first and second derivatives in $[x_0, x_n]$

Then $p_k'(x_k) = p_{k-1}'(x_k)$

$p_k''(x_k) = p_{k-1}''(x_k)$

In addition

$p_k(x) = y_k$

$p_k(x_k) = p_{k-1}(x_k)$

p_k is a cubic polynomial.

$ax^3 + bx^2 + cx + d$

p_k'' is a linear fn.

Let $p_k''(x_k) = S''(x_k) = M_k$

and $p_k''(x_{k+1}) = S''(x_{k+1}) = M_{k+1}$

$3(-1)2(-1)$

$x -$

M_k, M_{k+1} are unknown

$$p_k''(x) = \frac{x - x_{k+1}}{x_k - x_{k+1}} M_k + \frac{x - x_k}{x_{k+1} - x_k} M_{k+1}$$

Integrate twice wrt. x

$h = x_{k+1} - x_k$

ie equispaced points.

$$p_k(x) = \frac{M_k}{h} \frac{(x - x_{k+1})^3}{6} + \frac{M_{k+1}}{h} \frac{(x - x_k)^3}{6} + C_k(x - x_k) + D_k(x_{k+1} - x)$$

where C_k, D_k are arbit constants.

$p_k(x_k) = y_k$ $\qquad$ $p_k(x_{k+1}) = y_{k+1}$

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$$D_k = \frac{y_k - \frac{M_k h^2}{6}}{h} = \frac{y_k}{h} - \frac{M_k h}{6} = \frac{1}{h}\left(y_k - M_k\frac{h^2}{6}\right)$$

$$C_k = \frac{1}{h}\left(y_{k+1} - M_{k+1}\frac{h^2}{6}\right)$$

$$p_k(x) = \frac{M_k}{6}\left[\frac{(x_{k+1}-x)^3}{h} - h(x_{k+1}-x)\right] + \frac{y_k}{h}(x_{k+1}-x)$$

$$+ \frac{M_{k+1}}{6}\left[\frac{(x-x_k)^3}{h} - h(x-x_k)\right] + \frac{y_{k+1}}{h}(x-x_k)$$

$k = 0, 1, 2, \ldots, n-1$ in $[x_k, x_{k+1}]$

Also $p_k'(x_k) = p_{k-1}'(x_k)$

$$\cancel{\frac{M_k}{6}\left[h^2 - h^2\right.}$$

$$LHS = \frac{M_k}{6}\left[\frac{3(-1)h^2}{h} + h\right] - \frac{y_k}{h} + \frac{M_{k+1}}{6}\left[3\times 0 - h\right] + \frac{y_{k+1}}{h}\times 0$$

$$= \frac{M_k}{6}\left[-2h\right] - \frac{y_k}{h} - \frac{h M_{k+1}}{6} + \frac{y_{k+1}}{h}$$

$$\cancel{= \frac{y_{k+1}-y_k}{h} - \frac{M_k h}{2}}$$

$$= \frac{y_{k+1}-y_k}{h} - \frac{h}{6}\left(M_{k+1} + 2M_k\right)$$

RHS

Cx + D

$$\cancel{\frac{M_{k-1}}{6}}\Big[$$

$$p_{k-1}(x) = \frac{M_{k-1}}{6}\left[\frac{(x_k - x)^3}{h} - h(x_k - x)\right] + \frac{y_{k-1}}{h}(x_k - x) + \frac{M_k}{6}\left[\frac{(x - x_{k-1})^3}{h} - h(x - x_{k-1})\right] + \frac{y_k}{h}(x - x_{k-1})$$

$$RHS = \frac{M_{k-1}}{6}\left[0 + h\right] - \frac{y_{k-1}}{h} + \frac{M_k}{6}\left[\frac{3h^2}{h} - h\right] + \frac{y_k}{h}$$

$$= \frac{y_k - y_{k-1}}{h} + \frac{h}{6}(2M_k + M_{k-1})$$

$$\frac{y_{k+1} - y_k}{h} - \frac{h}{6}(M_{k+1} + 2M_k) = \frac{y_k - y_{k-1}}{h} + \frac{h}{6}(2M_k + M_{k-1})$$

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h} = \frac{h}{6}(M_{k+1} + 4M_k + M_{k-1})$$

$$M_{k+1} + 4M_k + M_{k-1} = \frac{6}{h^2}(y_{k-1} - 2y_k + y_{k+1})$$

$$= \frac{6}{h^2}\Delta^2 y_{k-1}$$

$$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$$

$$y_{k+1} - y_k$$

$$\Delta^2 y_{k-1} = \Delta y_k - \Delta y_{k-1}$$

We need to know M_0 & M_n.

Either of

1. free end → $M_0 = M_n = 0$
2. $M_0 = M_1$ & $M_n = M_{n-1}$ → periodic.
3. $M_0 = \sigma_0$ $M_n = \sigma_n$ given.

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* Solve the above using tridiagonal system.
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Eg

	x_0	x_1	x_2	x_3
x	1	2	3	4
y	1.5	2.2	3.1	4.3

Cubic spline interpolation with free ends.

$$S(x) = \{p_k(x) \mid x \in [x_k, x_{k+1}], k = 0, 1, \ldots, n-1$$

$$y_k = p_k(x_k) = p_{k-1}(x_k)$$

Interpolation polynomial

$\Rightarrow$ polynomial is equal to function value at the interpolation pts / node pts

$$M_0 = M_3 = 0$$

$$M_{k-1} + 4M_k + M_{k+1} = \frac{6\,\Delta^2 y_{k-1}}{h^2}$$

$h = 1$

$$\cancel{M_0}^{\,0} + 4M_1 + M_2 = 6\underbrace{(y_0 - 2y_1 + y_2)}_{= 0.2} = 1.2$$

$$M_1 + 4M_2 + \cancel{M_3}^{\,0} = 6(0.3) = 1.8$$

~~$M_2 + 4M_3 + M_4 = 6$~~

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.8 \end{bmatrix}$$

$$\frac{-0.6}{3}$$

$$M_1 + M_2 = 0.6$$

$$M_1 - M_2 = -0.2$$

$$M_1 = 0.2, \; M_2 = 0.4$$

$$p_k(x) = \frac{M_k}{6}\left[\frac{(x_{k+1}-x)^3}{h} - h(x_{k+1}-x)\right] + \frac{y_k}{h}(x_{k+1}-x)$$

$$+ \frac{M_{k+1}}{6}\left[\frac{(x-x_k)^3}{h} - h(x-x_k)\right] + \frac{y_{k+1}}{h}(x-x_k)$$

$$p_0(x) = \frac{M_1}{6}\left[(x-1)^3 - (x-1)\right] + y_0(2-x) + y_1(x-1)$$

$$= \frac{0.2}{6}\left[(x-1)^3 - (x-1)\right] + \overset{1.5}{\cancel{y_0}}(2-x) + \overset{2.2}{\cancel{y_1}}(x-1)$$

$$= \frac{0.2}{6}\left[0.2^3 - 0.2\right] + 1.5 \times \overset{12}{0.8} + 2.2 \times 0.2$$

$$= \frac{0.2}{\cancel{6}}\left[-\overset{0.032}{\cancel{0.192}}\right] + \overset{1.2}{\cancel{\;}} + 0.44$$

~~$= 3.14 - 0.0064$~~

$$= 1.64 - 0.0064$$

$$= 1.6336$$

$$\begin{array}{r} 0.008 \\ 0.2 \\ \hline \end{array}$$

$$\frac{0.384}{6}$$

$$0.64$$

$$y'(1) = p_0'(1) = 0.66$$

\# $y_k = p_m(x_k)$; $k = 0, 1, 2, \ldots, n$

Special case $y_k = 0 \quad \forall k$

$= y_l \quad k = l$

$p_m(x_k) = 0$, $k = 0, 1, \ldots n$, $k \neq l$.

$= y_l$. $k = l$.

$$p_m(x) = A_m(x-x_0)(x-x_1)\ldots(x-x_{l-1})(x-x_{l+1})\ldots(x-x_n)$$

$$A_n = \frac{y_l}{(x_l-x_0)(x_l-x_1)\ldots(x_l-x_n)}$$

no $(x_l - x_l)$

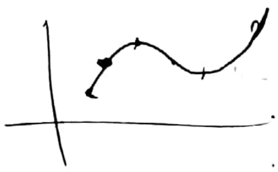
$$p_n(x) = y_l \frac{(x-x_0)\cdots(x-x_{l-1})(x-x_{l+1})\cdots(x-x_n)}{(x_l-x_0)\cdots(x_l-x_{l-1})(x_l-x_{l+1})\cdots(x_l-x_n)}$$

Let $l(x) = \prod_{0}^{n}(x-x_k)$

$$l'(x_l) = (x_l-x_0)\cdots(x_l-x_{l-1})(x_l-x_{l+1})\cdots(x_l-x_n)$$

$$p_n(x) = y_l \frac{l(x)}{(x-x_l)\,l'(x_l)}$$

Then $p_n(x) = y_0 \frac{l(x)}{(x-x_0)\,l'(x_0)} + y_1 \frac{l(x)}{(x-x_1)\,l'(x_1)} + \cdots + y_n \frac{l(x)}{(x-x_n)\,l'(x_n)}$



$x^2(x_0-x)$

$$p_m(x) \equiv a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0.$$

$$p_m(x_k) = y_k$$

$$a_m x_0^m + \cdots + a_0 = y_0$$

$$a_m x_1^m + \cdots + a_0 = y_1$$

$$\vdots$$

$$x_m x_n^m + \cdots + a_0 = y_n$$

Solvable for $m=n$

& unique for distinct x_i

§ Solve linear BVP using spline interpolation

$$y'' + a(x)\,y' + b(x)\,y = c(x) \qquad a < x < b.$$

$$y(a) = y_a \quad y(b) = y_b$$

Special case $a(x) = 0$

$$y'' + b(x)\,y = c(x)$$

$$M_k + b_k y_k = c_k \qquad k = 0, 1, \ldots, n$$

If $y(x)$ is determined through cubic spline then

$$M_{k-1} + 4M_k + M_{k+1} = \frac{6}{h^2}\Delta^2 y_{k-1}$$

$k = 1, 2, \ldots, n-1$

Substitute M_k in terms of y_k using ②

$$c_{k-1} - b_{k-1} y_{k-1} + 4c_k - 4b_k y_k + c_{k+1} - b_{k+1} y_{k+1} = \frac{6}{h^2}\left(y_{k-1} - 2y_k + y_{k+1}\right)$$

$$\left(\frac{6}{h^2} + b_{k-1}\right) y_{k-1} + \left(4b_k - \frac{12}{h^2}\right) y_k + \left(b_{k+1} + \frac{6}{h^2}\right) y_{k+1}$$

$$= c_{k-1} + 4c_k + c_{k+1} \qquad k = 1, 2, \ldots n-1$$

Solve by thomas.

$y'' - y = 0 \quad y(0) = 0 \quad y(1) = 1$

$h = \frac{1}{2}$

use spline interpolation to find $y(\frac{1}{2})$

$$M_0 + 4M_1 + M_2 = \frac{6}{h^2}(y_0 - 2y_1 + y_2)$$

$$= 24(1 - 2y_1)$$

$$M_k = y_k$$

$\cancel{y_k + 4y}$

$$\cancel{y_0}^{0} + 4y_1 + \cancel{y_2}^{1} = 24 - 48y_1$$

$$y_1 = \frac{23}{52}$$

If $a(x) \neq 0$

$$y'' + a(x)y'(x) + b(x)y = c(x)$$

$$a(x)y' = c(x) - y'' - b(x)y$$

$$y'(x) \sim p_k'(x)$$

$$p_{k-1}'(x) = y_k'$$

$$\frac{h}{6}a_k M_{k-1} + a_k \frac{M_k h}{3} + a_k \frac{\Delta y_{k-1}}{h}$$
$$= c_k - M_k - b_k y_k$$

$1, 2, \ldots n$

$$M_{k-1} + 4M_k + M_{k+1} = \frac{6}{h^2}(y_{k+1} - 2y_k + y_{k-1})$$

$1, 2, \ldots, n-1$

Sweet are the uses of adversity

Eg $y'' + 2y' + y = 30x$

$y(0) = 0$, $y(1) = 0$

$h = \frac{1}{2}$

(a) $p'_{k-1}(x_k) = y_k$ (b) $p_k'(x_k) = y_k'$

$k = 1, 2, \ldots n$ $k = 0, 1, \ldots n-1$

$$\frac{h}{6} a_k M_{k-1} + \left(a_k \frac{h}{3} + 1\right) M_k = c_k - b_k y_k + \frac{a_k}{h}(y_k - y_{k-1})$$

$k = 1, 2, \ldots n.$

$a_k p_k'(x_k) = a_k y_k' = \ldots$

$$-a_k h \frac{M_k}{3} - a_k \frac{M_{k+1} h}{6} + a_k \frac{\Delta y_k}{h} = c_k - b_k y_k - M_k$$

$k = 0, \ldots n-1$

$y'' + 2y' + y = 30x$ H.T.

$y(0) = 0$ $y(1) = 0$ $h = 1/2$

~~a = 1 b = 2~~ $a = 2$ $b = 1$ $c = 30x$

$$-2 \cdot \frac{1}{2} \frac{M_0}{3} - \frac{2}{6} M_1 \frac{1}{2} + 2 \frac{y_1 - y_0}{1/2} = 0 - 1 y_0 - M_0.$$

$$\frac{2M_0}{3} - \frac{M_1}{6} + 4y_1 \cancel{- 3y_0} = 0.$$

$$\frac{1}{12} 2 M_0 + \left(2 \frac{1/2}{3} + 1\right) M_1 = 15 - y_1 + \frac{2}{1/2}(y_1 - y_0)$$

$$\frac{M_0}{6} + \frac{4M_1}{3} - 3y_1 \cancel{+ 4y_0} = 15$$