Recall $\sum_{k=1}^{\infty} a_k(x)$, $a_k(x) \geq 0$,

$a_k(x)$ are meas. Then

i) $$\int \sum_{k=1}^{\infty} a_k(x) = \sum_{k=1}^{\infty} \int a_k(x)$$

ii) $$\sum_{k=1}^{\infty} \int a_k(x)\,dx < \infty$$

$$\Rightarrow \int \sum a_k(x)\,dx < \infty \quad ✓$$

$$\Rightarrow \sum_{k=1}^{\infty} a_k(x) < \infty \quad \text{a.e. } x.$$

An alternative proof of Borel-Cantelli lemma :-

Recall B.C. Lemma :- $\{E_k\}$ be a collection of measurable sets, with $\sum_{k=1}^{\infty} m(E_k) < \infty$, then $m(E) = 0$ where

$$E = \limsup_{k \to \infty} E_k$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k \quad ✓$$

$$= \{x \in \mathbb{R}^d \mid x \in E_k \text{ for infinitely many } k\text{'s}\}$$

Proof :- Let $a_k(x) = \chi_{E_k}(x)$

$x \in E_k$ for infinitely many k, iff

$$\sum_{k=1}^{\infty} a_k(x) = \infty$$

T.S.

$$m\underbrace{\left\{x : \sum_{k=1}^{\infty} a_k(x) = \infty\right\}}_{E} = 0$$

$m(E_k)$

$= \int \chi_{E_k}$

$= \int a_k(x)$

$\sum m(E_k)$

$= \sum \int a_k(x) < \infty$

$\Downarrow$

$\sum \int a_k(x) < \infty$

$\Rightarrow \int \sum a_k(x) < \infty$

$\Rightarrow \sum a_k(x) < \infty$ a.e.

$\Rightarrow m\left\{x : \sum_{k=1}^{\infty} a_k(x) = \infty\right\} = 0$

$\Rightarrow m(E) = 0$

Lebesgue integration for general case

$\rightsquigarrow$ $f : \mathbb{R}^d \to \mathbb{R}$, f is meas.

We say f is "Lebesgue integrable" or simply "integrable", if the non-negative meas. function $|f|$ is integrable.

Define :-

$\rightsquigarrow$ $f^+(x) = \max\{f(x), 0\}$

$f^-(x) = \max\{-f(x), 0\}$

Clear that:

$f^+ \geq 0, \quad f^- \geq 0$

Claim

$f = f^+ - f^-$ ✓

$f^+, f^- \leq |f|$

Whenever, f is integrable, then f^+ & f^- are both integrable.

Define, Lebesgue integral of f as,

$$\boxed{\int f = \int f^+ - \int f^-}$$

$f = f_1 - f_2$ ✓ one particular decomposition.

$\int f = \int f_1 - \int f_2$, $f_1, f_2 \geq 0$

$f = g_1 - g_2$ ✓ is another decomposition , $g_1, g_2 \geq 0$

$$f_1 - f_2 = g_1 - g_2$$

$$\underset{✓}{f_1} + \underset{✓}{g_2} = g_1 + f_2$$

$$\int f_1 + g_2 = \int g_1 + f_2$$

$$\int f_1 \overset{\shortparallel}{+} \int g_2 = \int g_1 + \int f_2$$

$$\Rightarrow \int f_1 - \int f_2 = \int g_1 - \int g_2$$ ✓

$\Longrightarrow$ The integral of Lebesgue integrable functions is linear, additive, monotonic & satisfies the triangle inequality.

Proposition :- Suppose f is integrable on $\mathbb{R}^d$. Then for every $\varepsilon > 0$

i) $\exists$ a set of finite measure B such that
$$\int_{B^c} |f| < \varepsilon.$$

ii) There is a $\delta > 0$ such that
$$\int_E |f| < \varepsilon \quad \text{whenever } m(E) < \delta$$

[Absolute Continuity of measure]

Proof :- replacing, f with $|f|$, we may consider w.l.o.g. that $f \geq 0$

i) B_N = ball of radius N & center at the origin.

Consider $f_N(x) = f(x)\, \chi_{B_N}(x)$, then

$f_N \geq 0$, is measurable, ✓

$f_N(x) \leq f_{N+1}(x)$ ✓

$\lim_{N \to \infty} f_N(x) = f(x)$ ✓

By MCT,

$\lim_{N \to \infty} \int f_N = \int f.$

$\Rightarrow$ for large N,

$$0 \leq \int f - \int f \chi_{B_N} < \varepsilon.$$

$$\Rightarrow 0 \leq \int (1 - \chi_{B_N}) f < \varepsilon$$

$$\Rightarrow 0 \leq \int \chi_{B_N^c} f < \varepsilon$$

$$\Rightarrow \quad 0 \leq \int_{B_N^c} f < \varepsilon .$$

<u>2nd Part</u> :- $\quad f \geq 0$

Let $f_N(x) = f(x) \chi_{E_N}$

where $E_N = \{ x : f(x) \leq N \}$

$$f_N \geq 0 , \quad f_N(x) \leq f_{N+1}(x) .$$

$$f_N \longrightarrow f \text{ as } N \to \infty$$

<u>Apply MCT</u>, $\lim_{n\to\infty} \int f_N = \int f$

<u>Given</u>, $\varepsilon > 0$, $\exists$ $\underline{\underline{N}} > 0$ such that

$$\int (f - f_N) < \varepsilon/2 \quad \text{—} \quad (*)$$

We choose δ s.t. $N\delta < \varepsilon/2$

If $m(E) < \delta$, then

$$\int_E f$$

$$= \int_E f - f_N + \int_E f_N$$

$$\leq \int_{\mathbb{R}^d} f - f_N + \int_E f_N$$

$$\leq \int_{\mathbb{R}^d} f - f_N + \int_E f \cdot \chi_{E_N}$$

$$\leq \int f - f_N + N \, m(E_N \cap E)$$

$$\leq \varepsilon/2 + N \, \underline{m(E)}$$

$$= \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon$$

8. $f : [a,b] \to \mathbb{R}$
$g : [a,b] \to \mathbb{R}$ $\Big\}$ Cont.

$\int_a^b f = \int_a^b g$. P.T. $\exists$ a pt

$c \in (a,b)$ s.t. $f(c) = g(c)$.

Proof: $h = f - g$

$\int_a^b h = 0$, h is cont. on $[a,b]$

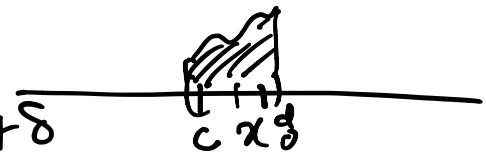
Let, $F(x) = \int_a^x h(t)\,dt$

$F(a) = 0$ F is diff., since h is cont

$F(b) = 0$,   Rolle's thm,

$F'(c) = 0, \quad a < c < b$

$\Rightarrow h(c) = 0$ ✓



$$\int_{x_0-\delta}^{x_0+\delta} f \quad , \qquad f > 0 \text{ on } (x_0-\delta,\ x_0+\delta)$$

$$[c,d] \subseteq (x_0-\delta,\ x_0+\delta)$$

x_0

$$\geq \int_c^d f\,dx$$

$$> m \int_c^d dx$$

$$= m\,(d-c)$$

$$> 0$$

$f > 0$, f is cont. on $[c,d]$

$\Rightarrow$ f has a minimum $= m$

$\exists\ x_0 \in [c,d]$

$f(x_0) = m > 0$

$f \geq m$ on $[c,d]$

9. $f : [0,1] \to \mathbb{R}$ cont.

$$\int_0^x f(t)\,dt = \int_x^1 f(t) \quad \forall x \in [0,1]$$

P.T. $f(x) = 0 \quad \forall\ x \in [0,1]$

$$G'(x) = f(x) \checkmark$$

$$F(x) = \int_0^x f(t)\,dt$$

$$F(1) = \int_0^1 f(t)\,dt = 0$$

$$F(0) = 0$$

$$\Rightarrow \int_0^c f + \int_c^1 f = 0 \qquad 0 < c < 1$$

$$\Rightarrow 2\int_0^c f = 0$$

$$\Rightarrow F(c) = 0 \quad \forall\, c \in (0,1)$$

$$\Rightarrow f(c) = 0 \quad \forall\, c \in (0,1)$$

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$\{f_n\}$ ← seq. of R-int.

$\downarrow$ pt. wise

f ← ? R integrable

Ans: Not

$Q = \{r_1, r_2, \dots\}$

$$f_n(x) = \begin{cases} 1, & x \in \{r_1, r_2, \dots, r_n\} \\ 0, & \text{otherwise} \end{cases}$$

$\downarrow$

$$f(x) = \begin{cases} 1, & x \in Q \\ 0, & x \in Q^c \end{cases}$$

↑ not Riemann integrable.

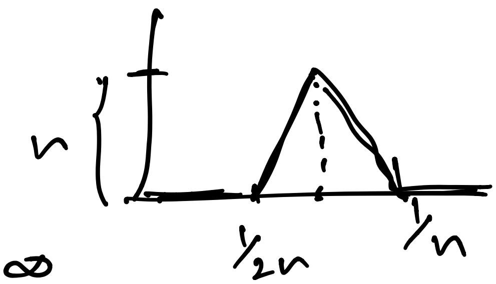
2nd Part

$f_n : [0,1] \to R$

$$f_n(x) = \begin{cases} 0, & x \le \frac{1}{2n} \text{ or } x \ge \frac{1}{n} \\ & \end{cases}$$

← like in the graph.

$f_n \longrightarrow 0$ as $n \to \infty$
$\qquad\ \ \, \parallel$
$\qquad\ \ \, f$



$\int f = 0.$

$$\lim_{n\to\infty} \int_0^1 f_n(x) = \frac{1}{4} \neq 0.$$