ASSIGNMENT - 3

Name: Raushan Sharma

Roll : 18MA20058

Advanced Numerical Techniques

1) Solve : $u_t = u_{xx}$; where $u(x,0) = \sin \pi x$ for $0 < x < 1$

$u(0,t) = u(1,t) = 0$ for $t > 0$

for $\delta x = \frac{1}{4}$ and $r = \frac{1}{2}$

by (a) Explicit method (b) Implicit method

(c) Crank Nicolson method

Soln.: $r = \frac{1}{2} \Rightarrow \frac{\delta t}{(\delta x)^2} = \frac{1}{2} \Rightarrow \delta t = \frac{1}{2} \times \frac{1}{16} \Rightarrow \delta t = \frac{1}{32}$.

Then, $x_i = i\delta x = \frac{i}{4}$. Let $u(x_i, t_n) \simeq u_i^n$

So, $u_i^0 = \sin(\pi x_i) = \sin\left(\frac{i\pi}{4}\right) \Rightarrow u_0^0 = 0$, $u_1^0 = \frac{1}{\sqrt{2}}$, $u_2^0 = 1$, $u_3^0 = \frac{1}{\sqrt{2}}$, $u_4^0 = 0$

and, $u_0^n = 0$, $u_N^n = 0$

(a) Explicit method :

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\delta x)^2}$$

$$\Rightarrow 32 \times \left[u_i^{n+1} - u_i^n\right] = 16 \times \left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]$$

$$\Rightarrow 2u_i^{n+1} - 2u_i^n = u_{i+1}^n - 2u_i^n + u_{i-1}^n$$

$$\Rightarrow 2u_i^{n+1} = u_{i+1}^n + u_{i-1}^n \Rightarrow u_i^{n+1} = \frac{u_{i-1}^n + u_{i+1}^n}{2} \quad \dots \text{(i)}$$

Then, we have : $u_0^1 = 0$, $u_1^1 = \frac{u_0^0 + u_2^0}{2} = 0.5$,

$u_2^1 = \frac{u_1^0 + u_3^0}{2} = \frac{1}{\sqrt{2}} = 0.707107$, $u_3^1 = \frac{u_2^0 + u_4^0}{2} = 0.5$, $u_4^1 = 0$

[Ans.]

(b) Implicit method : $\dfrac{u_i^{n+1} - u_i^n}{\delta t} = \dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\delta x)^2}$

$\Rightarrow 2u_i^{n+1} - 2u_i^n = u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}$

$\Rightarrow u_{i-1}^{n+1} - 4u_i^{n+1} + u_{i+1}^{n+1} = -2u_i^n$

So, for $n=0$, we get the equations :

$u_0' - 4u_1' + u_2' = -2u_1^0 = -\sqrt{2}$ .... (i)

$u_1' - 4u_2' + u_3' = -2u_2^0 = -2$ .... (ii)

$u_2' - 4u_3' + u_4' = -2u_3^0 = -\sqrt{2}$ .... (iii)

So, $\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ -2 \\ -\sqrt{2} \end{bmatrix}$

$\therefore u_1' = 0.54692$

$u_2' = 0.77346$ [Ans]

$u_3' = 0.54692$

(c) Crank Nicolson method :

$\dfrac{u_i^{n+1} - u_i^n}{\delta t} = \dfrac{1}{2} \times \left[\dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\delta x)^2} + \dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\delta x)^2} \right]$

$\Rightarrow \cancel{32}^{4} \times [u_i^{n+1} - u_i^n] = \dfrac{1}{\cancel{2}} \times \cancel{16}^{8} \times \left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n \right]$

$\Rightarrow 4u_i^{n+1} - 4u_i^n = u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n$

$\Rightarrow u_{i-1}^{n+1} - 6u_i^{n+1} + u_{i+1}^{n+1} = -u_{i-1}^n - 2u_i^n - u_{i+1}^n$

So, for $n=0$, we get the equations :

$u_0' - 6u_1' + u_2' = -u_0^0 - 2u_1^0 - u_2^0 = -\sqrt{2} - 1$ ... (i)

$u_1' - 6u_2' + u_3' = -u_1^0 - 2u_2^0 - u_3^0 = -\sqrt{2} - 2$ .... (ii)

$u_2' - 6u_3' + u_4' = -u_2^0 - 2u_3^0 - u_4^0 = -\sqrt{2} - 1$ ... (iii)

So,
$$\begin{bmatrix} -6 & 1 & 0 \\ 1 & -6 & 1 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} -\sqrt{2}-1 \\ -\sqrt{2}-2 \\ -\sqrt{2}-1 \end{bmatrix}$$

$\therefore\ u_1' = 0.526456$, $u_2' = 0.744521$, $u_3' = 0.526456$ [Ans.]

2) Consider the following finite difference scheme :

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\delta t} - \frac{u_{i+1}^{n} - 2\{\theta u_i^{n+1} + (1-\theta) u_i^{n-1}\} + u_{i-1}^{n}}{(\delta x)^2} = 0$$

for $0 < \theta < 1$. Find whether this scheme is consistent ~~with~~

when : (a) $\delta t = r\delta x$ (b) $\delta t = r(\delta x)^2$.

Find θ for consistency and also the PDE with which it will be consistent.

Soln: By Taylor's expansions about (x_i, t_n), we get the following :

$$u_i^{n+1} = u(x_i, t_{n+1}) = u(x_i, t_n + \delta t)$$
$$= u(x_i, t_n) + \delta t \times \frac{\partial u}{\partial t}\Big|_i^n + \frac{(\delta t)^2}{2!} \times \frac{\partial^2 u}{\partial t^2}\Big|_i^n + \frac{(\delta t)^3}{3!} \times \frac{\partial^3 u}{\partial t^3}\Big|_i^n$$
$$+ \frac{(\delta t)^4}{4!} \times \frac{\partial^4 u}{\partial t^4}\Big|_i^n + O((\delta t)^5) \quad \ldots\ (i)$$

$$u_i^{n-1} = u(x_i, t_{n-1}) = u(x_i, t_n - \delta t)$$
$$= u(x_i, t_n) - \delta t \times \frac{\partial u}{\partial t}\Big|_i^n + \frac{(\delta t)^2}{2!} \times \frac{\partial^2 u}{\partial t^2}\Big|_i^n - \frac{(\delta t)^3}{3!} \times \frac{\partial^3 u}{\partial t^3}\Big|_i^n$$
$$+ \frac{(\delta t)^4}{4!} \times \frac{\partial^4 u}{\partial t^4}\Big|_i^n + O((\delta t)^5) \quad \ldots\ (ii)$$

Using (i) - (ii), we get :

$$u_i^{n+1} - u_i^{n-1} = 2\delta t \times \frac{\partial u}{\partial t}\Big|_i^n + 2 \times \frac{(\delta t)^3}{3!} \times \frac{\partial^3 u}{\partial t^3} + O((\delta t)^5)$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^{n-1}}{2\delta t} = \frac{\partial u}{\partial t}\Big|_i^n + \frac{(\delta t)^2}{6} \times \frac{\partial^3 u}{\partial t^3} + O((\delta t)^4) \quad \ldots\ (iii)$$

Now, $u_{i+1}^n + u_{i-1}^n = u(x_i,t_n) + \delta x \times \frac{\partial u}{\partial x}\Big|_i^n + \frac{(\delta x)^2}{2!}\times\frac{\partial^2 u}{\partial x^2}\Big|_i^n + \frac{(\delta x)^3}{3!}\times\frac{\partial^3 u}{\partial x^3}\Big|_i^n + \frac{(\delta x)^4}{4!}\times\frac{\partial^4 u}{\partial x^4}\Big|_i^n + \dots$

$+ u(x_i,t_n) - \delta x \times \frac{\partial u}{\partial x}\Big|_i^n + \frac{(\delta x)^2}{2!}\times\frac{\partial^2 u}{\partial x^2}\Big|_i^n - \frac{(\delta x)^3}{3!}\times\frac{\partial^3 u}{\partial x^3}\Big|_i^n + \frac{(\delta x)^4}{4!}\times\frac{\partial^4 u}{\partial x^4}\Big|_i^n + \dots$

$\Rightarrow u_{i+1}^n + u_{i-1}^n = 2u(x_i,t_n) + (\delta x)^2\times\frac{\partial^2 u}{\partial x^2}\Big|_i^n + \frac{(\delta x)^4}{12}\times\frac{\partial^4 u}{\partial x^4}\Big|_i^n + O((\delta x)^6)$ [$\because$ odd powers of (δx) terms get cancelled out]

$\Rightarrow \frac{u_{i+1}^n + u_{i-1}^n}{(\delta x)^2} = \frac{2u(x_i,t_n)}{(\delta x)^2} + \frac{\partial^2 u}{\partial x^2}\Big|_i^n + \frac{(\delta x)^2}{12}\times\frac{\partial^4 u}{\partial x^4}\Big|_i^n + O((\delta x)^4)$ .... (iv)

And, in (i) and (ii), we have found expansions of u_i^{n+1} and u_i^{n-1}.

Using (i) and (ii),

$\theta u_i^{n+1} + (1-\theta)u_i^{n-1} = u(x_i,t_n) + (2\theta-1)\times\delta t\times\frac{\partial u}{\partial t}\Big|_i^n + \frac{(\delta t)^2}{2!}\times\frac{\partial^2 u}{\partial t^2}\Big|_i^n + O((\delta t)^3)$

$\Rightarrow \frac{2}{(\delta x)^2}\left\{\theta u_i^{n+1} + (1-\theta)u_i^{n-1}\right\} = \frac{2u(x_i,t_n)}{(\delta x)^2} + \frac{2(2\theta-1)\times\delta t}{(\delta x)^2}\times\frac{\partial u}{\partial t}\Big|_i^n + \frac{(\delta t)^2}{(\delta x)^2}\times\frac{\partial^2 u}{\partial t^2}\Big|_i^n$
$+ O\left(\frac{(\delta t)^3}{(\delta x)^2}\right)$ .... (v)

From (iii), (iv) and (v), we get:

Truncation error for this scheme is: [at the point (x_i,t_n)]

$$T.E = u_t - u_{xx} + \frac{(\delta t)^2}{6}u_{ttt} - \frac{(\delta x)^2}{12}u_{xxxx} + (2\theta-1)\times\frac{2\delta t}{(\delta x)^2}\times u_t$$
$$+ \frac{(\delta t)^2}{(\delta x)^2}\times u_{tt} + O\left(\frac{(\delta t)^3}{(\delta x)^2}, (\delta t)^4, (\delta x)^4\right)$$

(a) When $\delta t = r\delta x$, we get:

$$T.E = u_t - u_{xx} + r^2 u_{tt} + (2\theta-1)\times\frac{2r}{\delta x}\times u_t + \frac{(\delta t)^2}{6}u_{ttt} - \frac{(\delta x)^2}{12}u_{xxxx}$$
$$+ O\left(\frac{(\delta t)^3}{(\delta x)^2}, (\delta t)^4, (\delta x)^4\right)$$

If $\theta \neq \frac{1}{2}$, then the term $(2\theta-1)\times\frac{2r}{\delta x}\times u_t \to \infty$ as $\delta x \to 0$.

So, $T.E \to \infty$ as $\delta x \to 0$. Hence, when $\delta t = r\delta x$ and $\theta \neq \frac{1}{2}$ the scheme is inconsistent.

And for $\theta = \frac{1}{2}$,

$$T.E. = u_t - u_{xx} + r^2 u_{tt} + \frac{(\delta t)^2}{6} u_{ttt} - \frac{(\delta x)^2}{12} u_{xxxx} + O\left(\frac{(\delta t)^3}{(\delta x)^2}, (\delta t)^4, (\delta x)^4\right)$$

When $\delta t \to 0$ and $\delta x \to 0$, and the PDE is : $u_t - u_{xx} + r^2 u_{tt} = 0$,

then $T.E \to 0$.

Hence, when $\delta t = r\delta x$ and $\theta = \frac{1}{2}$, the scheme is consistent with

the PDE : $u_t - u_{xx} + r^2 u_{tt} = 0$ [Ans.]

(b) when $\delta t = r(\delta x)^2$, we get :

$$T.E = u_t - u_{xx} + 2r(2\theta - 1) u_t + r\cdot\delta t \times u_{tt} + \frac{(\delta t)^2}{6} u_{ttt} - \frac{(\delta x)^2}{12} \times u_{xxxx}$$
$$+ O\left(\frac{(\delta t)^3}{(\delta x)^2}, (\delta t)^4, (\delta x)^4\right)$$

When $\delta t \to 0$ and $\delta x \to 0$ and the PDE is : $u_{xx} = [1 + 2r(2\theta - 1)] u_t$

then $T.E \to 0$.

Hence, when $\delta t = r(\delta x)^2$ ~~and~~ the scheme is consistent with the PDE :

$u_{xx} - [1 + 2r(2\theta - 1)] u_t = 0$. [Ans.]

3) Expand by Taylor's series and show that the Truncation Error of Crank Nicolson Scheme is $O(\delta t^2, \delta x^2)$ and the Crank Nicolson scheme is consistent.

Soln: The PDE : $u_t = \gamma u_{xx}$, when discretized using the Crank Nicolson scheme, gives :

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{\gamma}{2} \times \left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\delta x)^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\delta x)^2}\right]$$

Let $\delta t = k$ and $\delta x = h$. Then the Truncation error is given by :

$$T.E = \frac{u_i^{n+1} - u_i^n}{k} - \frac{r}{2} \times \left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \right]$$

Now, by Taylor series expansions, we have : (supposing, u_i^n is the actual soln. to the PDE)

[For simplicity, we have used notations u_i^n and $u(x_i, t_n)$ interchangeably]

$$u_i^{n+1} = u(x_i, t_{n+1}) = u(x_i, t_{n+1/2} + \tfrac{k}{2})$$

$$\Rightarrow u_i^{n+1} = u(x_i, t_{n+1/2}) + \left[\frac{k}{2} \times u_t + \frac{(\frac{k}{2})^2}{2!} \times u_{tt} + \frac{(\frac{k}{2})^3}{3!} \times u_{ttt} \right]_{(x_i, t_{n+1/2})} + O(k^4) \quad \dots (i)$$

And, $u_i^n = u(x_i, t_n) = u(x_i, t_{n+1/2} - \frac{k}{2})$

$$\Rightarrow u_i^n = u(x_i, t_{n+1/2}) + \left[-\frac{k}{2} \times u_t + \frac{(\frac{k}{2})^2}{2!} \times u_{tt} - \frac{(\frac{k}{2})^3}{3!} \times u_{ttt} \right]_{(x_i, t_{n+1/2})} + O(k^4) \quad \dots (ii)$$

Then, using (i) - (ii), we get :

$$u_i^{n+1} - u_i^n = \left[k \times u_t + \frac{k^3}{12} \times u_{ttt} \right]_{(x_i, t_{n+1/2})} + O(k^5)$$

[$\because$ Even power terms of k get cancelled out]

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{k} = u_t(x_i, t_{n+1/2}) + \frac{k^2}{12} u_{ttt}(x_i, t_{n+1/2}) + O(k^4) \quad \dots (iii)$$

Now, $u_{i+1}^n = u(x_{i+1}, t_n) = u(x_i + h, t_n)$

$$\Rightarrow u_{i+1}^n = u(x_i, t_n) + \left[h \times u_x + \frac{h^2}{2!} \times u_{xx} + \frac{h^3}{3!} \times u_{xxx} \right]_{(x_i, t_n)} + O(h^4) \quad \dots (iv)$$

And, $u_{i-1}^n = u(x_{i-1}, t_n) = u(x_i - h, t_n)$

$$\Rightarrow u_{i-1}^n = u(x_i, t_n) + \left[-h \times u_x + \frac{h^2}{2!} \times u_{xx} - \frac{h^3}{3!} \times u_{xxx} \right]_{(x_i, t_n)} + O(h^4) \quad \dots (v)$$

From (iv) and (v), we'll get:

$$u_{i+1}^n + u_{i-1}^n = 2u(x_i, t_n) + h^2 u_{xx}(x_i, t_n) + O(h^4)$$

$$\Rightarrow u_{i+1}^n - 2u_i^n + u_{i-1}^n = h^2 u_{xx}(x_i, t_n) + O(h^4) \quad \dots (vi)$$

Similarly, if we put t_{n+1} in place of t_n in (iv), (v) and (vi), and do Taylor's expansions about the point (x_i, t_{n+1}), we'll get :

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$$u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} = h^2 u_{xx}(x_i, t_{n+1}) + O(h^4) \quad \ldots\ldots \text{(vii)}$$

Now, putting (iii), (vi) and (vii) in the expression of Truncation error, we get :

$$T.E = \frac{u_i^{n+1} - u_i^n}{k} - \frac{\nu}{2} \times \left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \right]$$

$$= u_t(x_i, t_{n+1/2}) + \frac{k^2}{12} u_{ttt}(x_i, t_{n+1/2}) + O(k^4)$$

$$- \frac{\nu}{2} \times \left[u_{xx}(x_i, t_{n+1}) + \frac{h^2}{12} u_{xxxx}(x_i, t_{n+1}) + O(h^4) + u_{xx}(x_i, t_n) + \frac{h^2}{12} u_{xxxx}(x_i, t_n) + O(h^4) \right]$$

$$\Rightarrow T.E = \left[u_t(x_i, t_{n+1/2}) - \frac{\nu}{2} \times \left\{ u_{xx}(x_i, t_n) + u_{xx}(x_i, t_{n+1}) \right\} \right]$$

$$+ \frac{k^2}{12} u_{ttt}(x_i, t_{n+1/2}) - \frac{\nu}{2} \times \frac{h^2}{12} \times \left[u_{xxxx}(x_i, t_n) + u_{xxxx}(x_i, t_{n+1}) \right] + O(k^4) + O(h^4)$$

$$= \left[u_t(x_i, t_{n+1/2}) - \frac{\nu}{2} \times \left\{ 2u_{xx}(x_i, t_{n+1/2}) + \frac{k^2}{4} \times \left.\frac{\partial^2 u_{xx}}{\partial t^2}\right|_{(x_i, t_{n+1/2})} + O(k^4) \right\} \right]$$

By Taylor's expansion about $(x_i, t_{n+1/2})$

$$+ \frac{k^2}{12} u_{ttt}(x_i, t_{n+1/2}) - \nu \times \frac{h^2}{12} \times u_{xxxx}(x_i, t_{n+1/2}) + O(h^4 + k^4)$$

$$= \left[u_t - \nu \times u_{xx} \right]_{(x_i, t_{n+1/2})} + \left[\frac{k^2}{12} \times u_{ttt} - \nu \times \frac{h^2}{12} \times u_{xxxx} - \nu \times \frac{k^2}{8} \times \frac{\partial^4 u}{\partial t^2 \partial x^2} \right]_{(x_i, t_{n+1/2})} + O(h^4 + k^4)$$

Now, as the given PDE is $u_t = \nu u_{xx}$, and $u(x_i, t_{n+1/2})$ satisfies it, so $\left[u_t - \nu u_{xx} \right]_{(x_i, t_{n+1/2})} = 0$.

$$\therefore T.E = \left[\frac{k^2}{12} u_{ttt} - \nu \times \frac{h^2}{12} \times u_{xxxx} - \nu \times \frac{k^2}{8} \times \frac{\partial^4 u}{\partial t^2 \partial x^2} \right]_{(x_i, t_{n+1/2})} + O(h^4 + k^4)$$

$$= O(k^2, h^2) = O(\delta t^2, \delta x^2)$$

And, as $k \to 0$, $h \to 0$, $T.E \to 0$

$\Rightarrow$ The scheme is consistent with the given PDE.

$\therefore$ Truncation error of Crank Nicolson Scheme is $O(\delta t^2, \delta x^2)$ [Proved.]

4) Perform the Von Neumann Stability analysis for the :

a) Implicit scheme

Discretized equation for implicit scheme :

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \gamma \times \left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\delta x)^2} \right]$$

$$\Rightarrow u_i^{n+1} - u_i^n = r \times \left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} \right] \quad \text{Taking } r = \frac{\gamma \times \delta t}{(\delta x)^2}$$

So, $\xi_j^{n+1} - \xi_j^n = r \times \left[\xi_{j+1}^{n+1} - 2\xi_j^{n+1} + \xi_{j-1}^{n+1} \right]$; Put $\xi_j^n = A^n e^{i\theta j}$

$$\Rightarrow A^{n+1} e^{i\theta j} - A^n e^{i\theta j} = r \times \left[A^{n+1} e^{i\theta(j+1)} - 2A^{n+1} e^{i\theta j} + A^{n+1} e^{i\theta(j-1)} \right]$$

$$\Rightarrow \xi - 1 = r \left[\xi e^{i\theta} - 2\xi + \xi e^{-i\theta} \right]$$ [Dividing by $A^n e^{i\theta j}$ and putting $\frac{A^{n+1}}{A^n} = \xi$]

$$\Rightarrow \xi - 1 = r\xi \times \left[2\cos\theta - 2 \right]$$

$$\Rightarrow \xi \times \left[1 + 2r(1-\cos\theta) \right] = 1 \quad \Rightarrow \xi = \frac{1}{1 + 2r(1-\cos\theta)}$$

$$\Rightarrow \xi = \frac{1}{1 + 2r \times 2\sin^2\theta/2} = \frac{1}{1 + 4r\sin^2\theta/2}$$. For stability, $|\xi| \leq 1$

Then, for any $r \geq 0$, we always have $1 + 4r\sin^2\theta/2 \geq 1$

So, $|\xi| \leq 1$ is satisfied unconditionally.

Hence, the Implicit scheme is unconditionally stable .

b) Crank Nicolson scheme

Discretized equation for Crank Nicolson scheme :

$$u_i^{n+1} - u_i^n = \frac{r}{2} \times \left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n \right]$$

$$\Rightarrow \xi_j^{n+1} - \xi_j^n = \frac{r}{2} \times \left[\xi_{j+1}^{n+1} - 2\xi_j^{n+1} + \xi_{j-1}^{n+1} + \xi_{j+1}^n - 2\xi_j^n + \xi_{j-1}^n \right]$$

$$\Rightarrow A^{n+1} e^{i\theta j} - A^n e^{i\theta j} = \frac{r}{2} \times \left[A^{n+1} \left(e^{i\theta(j+1)} - 2e^{i\theta j} + e^{i\theta(j-1)} \right) + A^n \left(e^{i\theta(j+1)} - 2e^{i\theta j} + e^{i\theta(j-1)} \right) \right]$$

$$\Rightarrow (A^{n+1} - A^n) e^{i\theta j} = \frac{r}{2} \times (A^{n+1} + A^n)\left(e^{i\theta(j+1)} - 2e^{i\theta j} + e^{i\theta(j-1)}\right)$$

$$\Rightarrow \xi - 1 = \frac{r}{2} \times (\xi + 1)\left(e^{i\theta} - 2 + e^{-i\theta}\right)$$

$$\Rightarrow \frac{\xi - 1}{\xi + 1} = \frac{r}{2} \times (2\cos\theta - 2) = r(\cos\theta - 1) = \frac{r\cos\theta - r}{1}$$

$$\Rightarrow \frac{\xi}{-1} = \frac{r(\cos\theta - 1) + 1}{r(\cos\theta - 1) - 1} = \frac{1 - 2r\sin^2\frac{\theta}{2}}{-1 - 2r\sin^2\frac{\theta}{2}}$$

$$\Rightarrow \xi = \frac{1 - 2r\sin^2\frac{\theta}{2}}{1 + 2r\sin^2\frac{\theta}{2}}$$

Now, for any $r \geq 0$ $\left[\text{Note that, } r = \frac{\nu \times \delta t}{(\delta x)^2} \text{ is } \geq 0\right]$

$1 - 2r\sin^2\frac{\theta}{2} \geq -1 - 2r\sin^2\frac{\theta}{2}$ and $1 - 2r\sin^2\frac{\theta}{2} \leq 1 + 2r\sin^2\frac{\theta}{2}$

$\Rightarrow \frac{1 - 2r\sin^2\theta/2}{1 + 2r\sin^2\theta/2} \geq -1$ and $\frac{1 - 2r\sin^2\theta/2}{1 + 2r\sin^2\theta/2} \leq 1$

$\therefore -1 \leq \xi \leq 1 \Rightarrow |\xi| \leq 1$ is satisfied unconditionally.

Hence, the Crank Nicolson scheme is unconditionally stable.

5) Prove that the Leap Frog scheme (central time, central space) is unconditionally unstable.

Soln: Discretized equation for Leap Frog scheme:

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\delta x)^2} \qquad [\text{For the PDE: } u_t = \nu u_{xx}]$$

$$\Rightarrow u_i^{n+1} - u_i^{n-1} = \nu \times \frac{2\delta t}{(\delta x)^2} \cdot (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$\Rightarrow u_i^{n+1} - u_i^{n-1} = 2r \cdot (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \qquad \left[\text{where, } r = \frac{\nu\,\delta t}{(\delta x)^2}\right]$$

So, $\xi_j^{n+1} - \xi_j^{n-1} = 2r \times (\xi_{j+1}^n - 2\xi_j^n + \xi_{j-1}^n)$

$$\Rightarrow A^{n+1}e^{i\theta j} - A^{n-1}e^{i\theta j} = 2r \times \left(A^n e^{i\theta(j+1)} - 2A^n e^{i\theta j} + A^n e^{i\theta(j-1)}\right)$$

>

$$\Rightarrow \xi - \frac{1}{\xi} = 2r \times \left[e^{i\theta} - 2 + e^{-i\theta}\right]$$

$$= 2r \times \left[2\cos\theta - 2\right] = -8r\sin^2\theta/_2$$

$$\Rightarrow \xi^2 - 1 = -8r\sin^2\theta/_2\,\xi \qquad \Rightarrow \xi^2 + 8r\sin^2\tfrac{\theta}{2}\,\xi - 1 = 0$$

Then, $$\xi = \frac{-8r\sin^2\theta/_2 \pm \sqrt{64r^2\sin^4\theta/_2 + 4}}{2}$$

$$\Rightarrow \xi = -4r\sin^2\theta/_2 \pm \sqrt{16r^2\sin^4\theta/_2 + 1}$$

Then $\xi = -4r\sin^2\theta/_2 + \sqrt{16r^2\sin^4\theta/_2 + 1}$ or, $\xi = -4r\sin^2\theta/_2 - \sqrt{16r^2\sin^4\theta/_2 + 1}$

Among these two, the one with greater magnitude is ↗

So, for stability, we need $|\xi| \leq 1$. However,

$$\left| -4r\sin^2\theta/_2 - \sqrt{16r^2\sin^4\theta/_2 + 1} \right| = \left| 4r\sin^2\theta/_2 + \sqrt{16r^2\sin^4\theta/_2 + 1} \right|$$

$$\geq 1 \quad \forall \theta \in R, \ \forall r \geq 0.$$

Hence, we get $|\xi| \geq 1$ $\forall \theta \in R, \forall r \geq 0$ for the root of the quadratic with greater magnitude.

Thus, the Leap Frog Scheme is Unconditionally unstable. [Proved.]

6) Consider the non linear PDE : $\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = \gamma \times \frac{\partial^2 u}{\partial x^2}$; $t > 0$, $0 < x < a$

I.C : $u(x,0) = f(x)$ for $0 < x < a$

B.C : $u(0,t) = U_0$ and $u(a,t) = U_a$; for $t > 0$.

Discretize by the Crank Nicolson Scheme, then applying Newton's linearization technique, find the ensuing tri-diagonal system that needs to be solved at every iteration.

Soln: Discretizing using the Crank Nicolson scheme :

$$\frac{u_j^{n+1}-u_j^n}{\delta t} + \frac{1}{2}\times\left[u_j^{n+1}\times\left(\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\delta x}\right)\right] - \frac{\nu}{2}\times\left[\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\delta x)^2}\right]$$

$$= \underbrace{\frac{\nu}{2}\times\left[\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{(\delta x)^2}\right] - \frac{1}{2}\times\left[u_j^{n}\times\left(\frac{u_{j+1}^{n}-u_{j-1}^{n}}{2\delta x}\right)\right]}_{\text{Let's call this } RHS_1} \quad \ldots\ldots (*)$$

Now, we apply Newton's linearization technique to solve (*) iteratively :

Let $(u_j^{n+1})^{(k+1)} = (u_j^{n+1})^{(k)} + \Delta u_j^{n+1}$ at any $(k+1)^{th}$ iteration.

Then,
$$\frac{(u_j^{n+1})^{(k)}+\Delta u_j^{n+1}-u_j^n}{\delta t} + \frac{1}{2}\times\left\{(u_j^{n+1})^{(k)}+\Delta u_j^{n+1}\right\}\times\left[\frac{(u_{j+1}^{n+1})^{(k)}-(u_{j-1}^{n+1})^{(k)}+\Delta u_{j+1}^{n+1}-\Delta u_{j-1}^{n+1}}{2\delta x}\right]$$

$$-\frac{\nu}{2}\times\left[\frac{(u_{j+1}^{n+1})^{(k)}-2\times(u_j^{n+1})^{(k)}+(u_{j-1}^{n+1})^{(k)}+\Delta u_{j+1}^{n+1}-2\Delta u_j^{n+1}+\Delta u_{j-1}^{n+1}}{(\delta x)^2}\right] = RHS_1$$

$$\Rightarrow \frac{1}{\delta t}\Delta u_j^{n+1} + \frac{1}{2}\times\left\{\frac{(u_{j+1}^{n+1})^{(k)}-(u_{j-1}^{n+1})^{(k)}}{2\delta x}\right\}\Delta u_j^{n+1} + \frac{1}{2}\times\left\{\frac{(u_j^{n+1})^{(k)}}{2\delta x}\right\}\times\Delta u_{j+1}^{n+1}$$

$$-\frac{1}{2}\times\frac{(u_j^{n+1})^{(k)}}{2\delta x}\times\Delta u_{j-1}^{n+1} - \frac{\nu}{2\times(\delta x)^2}\times\Delta u_{j+1}^{n+1} + \frac{\nu}{(\delta x)^2}\times\Delta u_j^{n} - \frac{\nu}{2\times(\delta x)^2}\times\Delta u_{j-1}^{n+1}$$

$$= RHS_1 - \frac{(u_j^{n+1})^{(k)}-u_j^n}{\delta t} - \frac{(u_j^{n+1})^{(k)}}{2}\times\left[\frac{(u_{j+1}^{n+1})^{(k)}-(u_{j-1}^{n+1})^{(k)}}{2\delta x}\right] + \frac{\nu}{2}\times\left[\frac{(u_{j+1}^{n+1})^{(k)}-2\times(u_j^{n+1})^{(k)}+(u_{j-1}^{n+1})^{(k)}}{(\delta x)^2}\right] \quad \ldots (**)$$

Let's denote this RHS by RHS_2 .

Then, writing (**) in the tri-diagonal form : $a_j\Delta u_{j-1}^{n+1}+b_j\Delta u_j^{n+1}+c_j\Delta u_{j+1}^{n+1}=d_j$,

we get : $a_j = -\frac{(u_j^{n+1})^{(k)}}{4\delta x} - \frac{\nu}{2\times(\delta x)^2}$; $b_j = \frac{1}{\delta t} + \frac{(u_{j+1}^{n+1})^{(k)}-(u_{j-1}^{n+1})^{(k)}}{4\delta x} + \frac{\nu}{(\delta x)^2}$

$c_j = \frac{(u_j^{n+1})^{(k)}}{4\delta x} - \frac{\nu}{2\times(\delta x)^2}$; $d_j = RHS_2$ [Ans.]

7) Prove that the ADI scheme is $O(\delta t^2, \delta x^2, \delta y^2)$. Let $\delta t = k$, $\delta x = h_1$, $\delta y = h_2$

Soln: Step 1 :

$$T.E_1 = \frac{u^{n+1/2}_{i,j} - u^n_{i,j}}{k/2} - \gamma \times \left[\frac{u^{n+1/2}_{i+1,j} - 2u^{n+1/2}_{i,j} + u^{n+1/2}_{i-1,j}}{h_1^2} + \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{h_2^2} \right] \quad \ldots (i)$$

[$u^n_{i,j}$ be the actual soln. of the PDE, and let $u^n_{i,j} = u(x_i, y_j, t_n)$]

Step 2 :

$$T.E_2 = \frac{u^{n+1}_{i,j} - u^{n+1/2}_{i,j}}{k/2} - \gamma \times \left[\frac{u^{n+1/2}_{i-1,j} - 2u^{n+1/2}_{i,j} + u^{n+1/2}_{i+1,j}}{h_1^2} + \frac{u^{n+1}_{i,j-1} - 2u^{n+1}_{i,j} + u^{n+1}_{i,j+1}}{h_2^2} \right] \quad \ldots (ii)$$

Suppose the Truncation error for step 1 is $T.E_1$ and that for step 2 is $T.E_2$. Then, the T.E for ADI scheme can be obtained as: $T.E = T.E_1 + T.E_2$

(i) + (ii) gives : ~~T.E =~~

$$T.E = \frac{u^{n+1}_{i,j} - u^n_{i,j}}{k/2} - \gamma \times \left[2 \times \left\{ \frac{u^{n+1/2}_{i-1,j} - 2u^{n+1/2}_{i,j} + u^{n+1/2}_{i+1,j}}{h_1^2} \right\} + \left\{ \frac{u^n_{i,j-1} - 2u^n_{i,j} + u^n_{i,j+1}}{h_2^2} \right\} + \left\{ \frac{u^{n+1}_{i,j+1} - 2u^{n+1}_{i,j} + u^{n+1}_{i,j-1}}{h_2^2} \right\} \right] \quad \ldots (iii)$$

Now, as solved in T.E for Crank Nicolson scheme, in Q.3, we get :

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{k/2} = 2u_t(x_i, y_j, t_{n+1/2}) + \frac{k^2}{6} u_{ttt}(x_i, y_j, t_{n+1/2}) + O(k^4) \quad \ldots (iv)$$

And, $$\frac{u^{n+1/2}_{i-1,j} - 2u^{n+1/2}_{i,j} + u^{n+1/2}_{i+1,j}}{h_1^2} = u_{xx}(x_i, y_j, t_{n+1/2}) + \frac{h_1^2}{12} \times u_{xxxx}(x_i, y_j, t_{n+1/2}) + O(h_1^4) \quad \ldots (v)$$

And, $$\frac{u^n_{i,j-1} - 2u^n_{i,j} + u^n_{i,j+1}}{h_2^2} + \frac{u^{n+1}_{i,j-1} - 2u^{n+1}_{i,j} + u^{n+1}_{i,j+1}}{h_2^2} = \frac{\{u^n_{i,j-1} + u^{n+1}_{i,j-1}\} - 2 \times \{u^n_{i,j} + u^{n+1}_{i,j}\} + \{u^n_{i,j+1} + u^{n+1}_{i,j+1}\}}{h_2^2}$$

$$= \frac{2u^{n+1/2}_{i,j-1} + \cancel{\frac{k^2}{4} \times u_{tt}(x_i, y_{j-1}, t_{n+1/2})} - 4u^{n+1/2}_{i,j} - \cancel{\frac{k^2}{2} \times u_{tt}(x_i, y_j, t_{n+1/2})} + 2u^{n+1/2}_{i,j+1} + \cancel{\frac{k^2}{4} \times u_{tt}(x_i, y_{j+1}, t_{n+1/2})} + O(k^4)}{h_2^2}$$

$$= \frac{2 \times [u^{n+1/2}_{i,j-1} - 2u^{n+1/2}_{i,j} + u^{n+1/2}_{i,j+1}]}{h_2^2} = \frac{2h_2^2 u_{yy}(x_i, y_j, t_{n+1/2}) + \frac{h_2^4}{6} u_{yyyy}(x_i, y_j, t_{n+1/2}) + O(h_2^6)}{h_2^2}$$

$$= 2u_{yy}(x_i, y_j, t_{n+1/2}) + \frac{h_2^2}{6} u_{yyyy}(x_i, y_j, t_{n+1/2}) + O(h_2^4) \quad \ldots (vi)$$

Using (iv), (v) and (vi), and the fact that $u(x_i, y_j, t_{n+1/2})$ will satisfy the PDE : $u_t = \gamma[u_{xx} + u_{yy}]$ as u is the exact soln, so :

$$T.E = \frac{k^2}{6} \times u_{ttt}(x_i, y_j, t_{n+1/2}) - \gamma \times \frac{h_1^2}{6} \times u_{xxxx}(x_i, y_j, t_{n+1/2})$$

$$- \gamma \times \frac{h_2^2}{6} \times u_{yyyy}(x_i, y_j, t_{n+1/2}) + O(k^4, h_1^4, h_2^4)$$

$$\left[\because 2u_t - 2\gamma \times (u_{xx} + u_{yy}) = 0\right]$$

Thus, $T.E = O(k^2, h_1^2, h_2^2) = O(\delta t^2, \delta x^2, \delta y^2)$

and $T.E \to 0$ as $\delta t \to 0$, $\delta x \to 0$, $\delta y \to 0$. Thus, ADI scheme is also consistent with the PDE : $u_t = \gamma \times (u_{xx} + u_{yy})$. [Proved.]

Q4 Prove that the ADI scheme is unconditionally stable.

Soln: Step 1 of ADI scheme gives : $\left[\text{Taking } \delta x = \delta y \text{ and } \mu = \frac{\gamma \times \delta t}{(\delta x)^2}\right]$

$$\xi_{ij}^{n+1/2} - \xi_{i,j}^{n} = \frac{\mu}{2} \times \left(\xi_{i+1,j}^{n+1/2} - 2\xi_{i,j}^{n+1/2} + \xi_{i-1,j}^{n+1/2} + \xi_{i,j+1}^{n} - 2\xi_{i,j}^{n} + \xi_{i,j-1}^{n}\right) \quad \ldots (i)$$

Step 2 of ADI scheme gives :

$$\xi_{i,j}^{n+1} - \xi_{i,j}^{n+1/2} = \frac{\mu}{2} \times \left(\xi_{i+1,j}^{n+1/2} - 2\xi_{i,j}^{n+1/2} + \xi_{i-1,j}^{n+1/2} + \xi_{i,j+1}^{n+1} - 2\xi_{i,j}^{n+1} + \xi_{i,j-1}^{n+1}\right) \quad \ldots (ii)$$

Subtracting (i) from (ii) :-

$$\xi_{i,j}^{n+1} - 2\xi_{i,j}^{n+1/2} + \xi_{i,j}^{n} = \frac{\mu}{2} \times \left[\xi_{i,j+1}^{n+1} - 2\xi_{i,j}^{n+1} + \xi_{i,j-1}^{n+1} - \xi_{i,j+1}^{n} + 2\xi_{i,j}^{n} - \xi_{i,j-1}^{n}\right]$$

$$\Rightarrow A^{n+1} - 2A^{n+1/2} + A^{n} = \frac{\mu}{2} \times \left[A^{n+1}e^{i\theta} - 2A^{n+1} + A^{n+1}e^{-i\theta} - A^{n}e^{i\theta} + 2A^{n} - A^{n}e^{-i\theta}\right]$$

$$\Rightarrow \xi - 2\sqrt{\xi} + 1 = \frac{\mu}{2} \times \left[(\xi - 1)(e^{i\theta} - 2 + e^{-i\theta})\right]$$

$$\Rightarrow \xi - 2\sqrt{\xi} + 1 = (\xi - 1) \times \frac{\mu}{2} \times 2(\cos\theta - 1) = \mu\xi(\cos\theta - 1) - \mu(\cos\theta - 1)$$

$$\Rightarrow (1 + \mu - \mu\cos\theta)\xi - 2\sqrt{\xi} + (1 - \mu + \mu\cos\theta) = 0$$

Take $\sqrt{\xi} = m$. Then we get a quadratic in m:

$$(1 + r - r\cos\theta) m^2 - 2m + (1 - r + r\cos\theta) = 0$$

$$\Rightarrow m = \frac{2 \pm \sqrt{4 - 4\,(1 + (r - r\cos\theta))(1 - (r - r\cos\theta))}}{2(1 + r - r\cos\theta)}$$

$$= \frac{2 \pm 2\sqrt{1 - 1 + (r - r\cos\theta)^2}}{2(1 + r - r\cos\theta)} = \frac{1 \pm r(1-\cos\theta)}{1 + r - r\cos\theta}$$

$$\therefore m = \frac{1 + r(1-\cos\theta)}{1 + r(1-\cos\theta)} \quad \text{or} \quad m = \frac{1 - r(1-\cos\theta)}{1 + r(1-\cos\theta)}$$

$$\Rightarrow m = 1 \quad \text{or} \quad m = \frac{1 - 2r\sin^2\frac{\theta}{2}}{1 + 2r\sin^2\frac{\theta}{2}}$$

In both the cases, $|m| \leq 1 \Rightarrow |m^2| \leq 1 \Rightarrow \underline{|\xi| \leq 1}$

Hence, the ADI scheme is unconditionally stable. [Proved.]

9) $\frac{\partial u}{\partial t} = \nabla^2 u$; $-1 < x, y < 1$, $t > 0$ where

$u(x,y,0) = \cos\frac{\pi x}{2} \cdot \cos\frac{\pi y}{2}$ and $u = 0$ on $x = \pm 1$, $y = \pm 1$.

Solve this using ADI scheme for $\delta x = \delta y = \frac{1}{2}$ and $r = \frac{1}{6}$.

Soln: $r = \frac{\delta t}{(\delta x)^2} \Rightarrow \frac{1}{6} = \frac{\delta t}{1/4} \Rightarrow \delta t = \frac{1}{24}$.

step 1: $$\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{1/48} = \frac{u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2}}{1/4} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{1/4}$$

$$\Rightarrow 12u_{i,j}^{n+1/2} - 12u_{i,j}^{n} = u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2} + u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}$$

$$\Rightarrow -u_{i-1,j}^{n+1/2} + 14u_{i,j}^{n+1/2} - u_{i+1,j}^{n+1/2} = u_{i,j-1}^{n} + 10u_{i,j}^{n} + u_{i,j+1}^{n} \longrightarrow \text{Tri diagonal system.}$$

Put $n=0$: Then, for $j=1$: we get the system:

$$\begin{bmatrix} 14 & -1 & 0 \\ -1 & 14 & -1 \\ 0 & -1 & 14 \end{bmatrix} \begin{bmatrix} u_{1,1}^{1/2} \\ u_{2,1}^{1/2} \\ u_{3,1}^{1/2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_{1,1}^{1/2} = 0,\ u_{2,1}^{1/2} = 0,\ u_{3,1}^{1/2} = 0.$$

For $j=2$: We get the system :

$$\begin{bmatrix} 14 & -1 & 0 \\ -1 & 14 & -1 \\ 0 & -1 & 14 \end{bmatrix} \begin{bmatrix} u_{1,2}^{1/2} \\ u_{2,2}^{1/2} \\ u_{3,2}^{1/2} \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} u_{1,2}^{1/2} &= 0.0515464, \\ u_{2,2}^{1/2} &= 0.7216495, \\ u_{3,2}^{1/2} &= 0.0515464 \end{aligned}$$

For $j=3$: Again we get same system as for $j=1$, and so we get :

$u_{1,3}^{1/2} = 0$, $u_{2,3}^{1/2} = 0$, $u_{3,3}^{1/2} = 0$.

So, at $t = \frac{\delta t}{2}$ i.e. $t_{1/2}$, we get the values : (i.e. $t = \frac{1}{48} = 0.020833$)

$x \backslash y$	-1	-0.5	0	0.5	1
-1	0	0	0	0	0
-0.5	0	0	0.0515464	0	0
0	0	0	0.7216495	0	0
0.5	0	0	0.0515464	0	0
1	0	0	0	0	0

Now, Step 2 : $12\, u_{i,j}^{n+1} - 12 u_{i,j}^{n+1/2} = u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2} + u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}$

$\Rightarrow -u_{i,j-1}^{n+1} + 14\, u_{i,j}^{n+1} - u_{i,j+1}^{n+1} = u_{i-1,j}^{n+1/2} + 10\, u_{i,j}^{n+1/2} + u_{i+1,j}^{n+1/2}$

Put $n=0$: Then, for $i=1$: We get the system :

$$\begin{bmatrix} 14 & -1 & 0 \\ -1 & 14 & -1 \\ 0 & -1 & 14 \end{bmatrix} \begin{bmatrix} u_{1,1}^{1} \\ u_{1,2}^{1} \\ u_{1,3}^{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1.2371135 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} u_{1,1}^{1} &= 0.0063769 \\ u_{1,2}^{1} &= 0.089276232 \\ u_{1,3}^{1} &= 0.0063769 \end{aligned}$$

For $i=2$: We get the system :

$$\begin{bmatrix} 14 & -1 & 0 \\ -1 & 14 & -1 \\ 0 & -1 & 14 \end{bmatrix} \begin{bmatrix} u_{2,1}^{1} \\ u_{2,2}^{1} \\ u_{2,3}^{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 7.3195878 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} u_{2,1}^{1} &= 0.03773 \\ u_{2,2}^{1} &= 0.52822 \\ u_{2,3}^{1} &= 0.03773 \end{aligned}$$

For $i=3$: Again we get the same system as for $i=1$, and so :

$u_{3,1}^{1} = 0.0063769$, $u_{3,2}^{1} = 0.089276232$, $u_{3,3}^{1} = 0.0063769$

So, at $t = \delta t = \frac{1}{24}$, i.e. at t_1, we get the values :

$x \backslash y$	-1	-0.5	0	0.5	1
-1	0	0	0	0	0
-0.5	0	0.0063769	0.089276232	0.0063769	0
0	0	0.03773	0.52822	0.03773	0
0.5	0	0.0063769	0.089276232	0.0063769	0
1	0	0	0	0	0

10) $u_t + u u_x = \nu \times u_{xx}$; $\nu = 1$ $\rightarrow$ So, $u_t + u u_x = u_{xx}$

And, $u(x,0) = \sin \pi x$; $0 < x < 1$ and $u(0,t) = u(1,t) = 0$; $t > 0$

Find the discretized equation that needs to be solved.

Soln: Discretizing using Crank Nicolson scheme, we get :

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2} \times \left[u_j^{n+1} \times \left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\delta x} \right) \right] - \frac{1}{2} \times \left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\delta x)^2} \right]$$

$$= \frac{1}{2} \times \left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2} \right] - \frac{1}{2} \times \left[u_j^n \times \left(\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} \right) \right] \quad \ldots\ldots \circledast$$

Then, we apply Newton's linearization technique to solve $\circledast$ iteratively.

Substitute : $(u_j^{n+1})^{(k+1)} = (u_j^{n+1})^{(k)} + \Delta u_j^{n+1}$ at any $(k+1)^{th}$ iteration.

Then, simplifying, we get a tri-diagonal system as we found in Q.6.

Tri-diagonal system : $a_j \Delta u_{j-1}^{n+1} + b_j \Delta u_j^{n+1} + c_j \Delta u_{j+1}^{n+1} = d_j$

where, $a_j = -\frac{(u_j^{n+1})^{(k)}}{4\delta x} - \frac{1}{2(\delta x)^2}$, $b_j = \frac{1}{\delta t} + \frac{(u_{j+1}^{n+1})^{(k)} - (u_{j-1}^{n+1})^{(k)}}{4\delta x} + \frac{1}{(\delta x)^2}$,

and $c_j = \frac{(u_j^{n+1})^{(k)}}{4\delta x} - \frac{1}{2(\delta x)^2}$ and d_j has a long expression, which we have found in Q.6.

Solving this tri diagonal system iteratively, will yield the solution.

11) $u_t = \nabla^2 u \;;\; 0 \le x, y \le 1$ and $t > 0$

$u(x,y,0) = \sin \pi x \sin \pi y$ and $u = 0$ on the boundary.

Derive the ensuing tri-diagonal system at each step, discretized by the ADI scheme.

Soln: Discretization by ADI scheme gives:

Step 1: $$\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{\delta t} = \frac{\nu}{2} \times \left[\frac{u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2}}{(\delta x)^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\delta y)^2} \right]$$

$$\Rightarrow \frac{-\nu}{2(\delta x)^2} u_{i+1,j}^{n+1/2} + \left(\frac{1}{\delta t} + \frac{\nu}{(\delta x)^2} \right) u_{i,j}^{n+1/2} - \frac{\nu}{2(\delta x)^2} u_{i-1,j}^{n+1/2}$$

$$= \frac{1}{\delta t} u_{i,j}^{n} + \frac{\nu}{2} \times \left[\frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\delta y)^2} \right] \quad \text{.....} \; (i)$$

So, (i) gives a tri diagonal system for step 1 of ADI scheme. Now,

Step 2: $$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+1/2}}{\delta t} = \frac{\nu}{2} \times \left[\frac{u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2}}{(\delta x)^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{(\delta y)^2} \right]$$

$$\Rightarrow \frac{-\nu}{2(\delta y)^2} \times u_{i,j+1}^{n+1} + \left(\frac{1}{\delta t} + \frac{\nu}{(\delta y)^2} \right) u_{i,j}^{n+1} - \frac{\nu}{2(\delta y)^2} \times u_{i,j-1}^{n+1}$$

$$= \frac{1}{\delta t} \times u_{i,j}^{n+1/2} + \frac{\nu}{2} \times \left[\frac{u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2}}{(\delta x)^2} \right] \quad \text{....} \; (ii)$$

So, (ii) gives a tri diagonal system for step 2 of ADI scheme.

And then, we can use the boundary conditions:

$u(x,y,0) = \sin \pi x \sin \pi y \Rightarrow u_{i,j}^{0} = \sin(\pi i) \sin(\pi j)$

And, ~~u=0~~ $u = 0$ on boundary.

12) Consider the PDE : $u_t = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ → i.e. 3 space variables.

Think about how can we extend the ADI scheme to a 3-step process and solve this?

Soln: We can:

1) Divide each time step into 3 fractional steps → at $t_{n+1/3}$ and $t_{n+2/3}$

2) Then, apply schemes which are implicit in x-, y-, and z- space and explicit in the other two space variables, respectively.

Step 1:
$$\frac{u_{i,j,k}^{n+1/3} - u_{i,j,k}^{n}}{\delta t} = \frac{1}{3} \times \left[\frac{u_{i+1,j,k}^{n+1/3} - 2u_{i,j,k}^{n+1/3} + u_{i-1,j,k}^{n+1/3}}{(\delta x)^2} + \frac{u_{i,j+1,k}^{n} - 2u_{i,j,k}^{n} + u_{i,j-1,k}^{n}}{(\delta y)^2} + \frac{u_{i,j,k+1}^{n} - 2u_{i,j,k}^{n} + u_{i,j,k-1}^{n}}{(\delta z)^2} \right]$$

Step 2:
$$\frac{u_{i,j,k}^{n+2/3} - u_{i,j,k}^{n+1/3}}{\delta t} = \frac{1}{3} \times \left[\frac{u_{i+1,j,k}^{n+1/3} - 2u_{i,j,k}^{n+1/3} + u_{i-1,j,k}^{n+1/3}}{(\delta x)^2} + \frac{u_{i,j+1,k}^{n+2/3} - 2u_{i,j,k}^{n+2/3} + u_{i,j-1,k}^{n+2/3}}{(\delta y)^2} + \frac{u_{i,j,k+1}^{n+1/3} - 2u_{i,j,k}^{n+1/3} + u_{i,j,k-1}^{n+1/3}}{(\delta z)^2} \right]$$

Step 3:
$$\frac{u_{i,j,k}^{n+1} - u_{i,j,k}^{n+2/3}}{\delta t} = \frac{1}{3} \times \left[\frac{u_{i+1,j,k}^{n+2/3} - u_{i,j,k}^{n+2/3} + u_{i-1,j,k}^{n+2/3}}{(\delta x)^2} + \frac{u_{i,j+1,k}^{n+2/3} - 2u_{i,j,k}^{n+2/3} + u_{i,j-1,k}^{n+2/3}}{(\delta y)^2} + \frac{u_{i,j,k+1}^{n+1} - u_{i,j,k}^{n+1} + u_{i,j,k-1}^{n+1}}{(\delta z)^2} \right]$$

13) $\nabla^2 u = -10(x^2+y^2+1)$; $0 < x, y < 3$

$u=0$ on the boundary and $\delta x = \delta y = 1$

Soln: Discretize using central difference scheme: $x_i = 0 + i \times \delta x = i$, $y_j = j$ $u_{0j} = u_{i0} = 0$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{1} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{1} = -10\,(i^2 + j^2 + 10)$$

$i, j = 1, 2$

$i=1, j=1$: $u_{2,1} - 2u_{1,1} + u_{1,2} - 2u_{1,1} = -10 \times (12) = -120$ .... (i)

$i=2, j=1$: $u_{3,1} - 2u_{2,1} + u_{1,1} + u_{2,2} - 2u_{2,1} = -150$ .... (ii)

$i=1, j=2$: $u_{2,2} - 2u_{1,2} + u_{1,3} - 2u_{1,2} + u_{1,1} = -10 \times (15) = -150$ .... (iii)

$i=2, j=2$: $u_{3,2} - 2u_{2,2} + u_{1,2} + u_{2,3} - 2u_{2,2} + u_{2,1} = -10 \times (18) = -180$ .... (iv)

Using BCs $u=0$ on the boundary, $u_{0j} = u_{i0} = 0$ and

$u_{3j} = u_{i3} = 0$.

So, the equations further reduce to :

$u_{2,1} - 4u_{1,1} + u_{1,2} = -120$, $-4u_{2,2} + u_{1,1} + u_{2,2} = -150$,

$u_{2,2} - 4u_{1,2} + u_{1,1} = -150$, $u_{1,2} - 4u_{2,2} + u_{2,1} = -180$

Thus,
$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & 0 & -4 & 1 \\ 1 & -4 & 0 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} -120 \\ -150 \\ -150 \\ -180 \end{bmatrix} \Rightarrow \left.\begin{array}{l} u_{1,1} = 67.5, \\ u_{1,2} = 75, \\ u_{2,1} = 75, \\ u_{2,2} = 82.5 \end{array}\right\} \text{[Ans.]}$$

14) Solve : $\nabla^2 u - 2 \times \frac{\partial u}{\partial x} = -2$ in R where $R : 0 < x < 1,\ 0 < y < 1$

and $u = 0$ on ∂R and $h = 1/3$

Soln: On discretizing, we get :

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{1/9} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{1/9} - 2 \times \frac{u_{i+1,j} - u_{i-1,j}}{2 \times 1/3} = -2$$

$$\Rightarrow 9u_{i-1,j} - 18u_{i,j} + 9u_{i+1,j} + 9u_{i,j-1} - 18u_{i,j} + 9u_{i,j+1} - 3u_{i+1,j} + 3u_{i-1,j} = -2$$

$$\Rightarrow 12u_{i-1,j} - 36u_{i,j} + 6u_{i+1,j} + 9u_{i,j-1} + 9u_{i,j+1} = -2$$

Using Boundary conditions, $u_{i,0} = u_{0,j} = 0$ and $u_{i,3} = u_{3,j} = 0$.

$i=1, j=1$: $-36u_{1,1} + 6u_{2,1} + 9u_{1,2} = -2$ .... (i)

$i=1, j=2$: $-36u_{1,2} + 6u_{2,2} + 9u_{1,1} = -2$ .... (ii)

$i=2, j=1$: $12u_{1,1} - 36u_{2,1} + 9u_{2,2} = -2$ .... (iii)

$i=2, j=2$: $12u_{1,2} - 36u_{2,2} + 9u_{2,1} = -2$ ... (iv)

On solving this system of equations, we get :

$u_{1,1} = \frac{22}{219}$, $u_{1,2} = \frac{22}{219}$, $u_{2,1} = \frac{26}{219}$, $u_{2,2} = \frac{26}{219}$ [Ans.]

15) Show that $u_t + c\,u_x = 0$ is stable for :

a) FTBS when $c > 0$

b) FTFS when $c < 0$.

Soln: a) Taking $\gamma = \frac{c\,\delta t}{\delta x}$, the discretized eqn. by FTBS is :

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c \times \frac{u_j^n - u_{j-1}^n}{\delta x} = 0 \quad \Rightarrow u_j^{n+1} - u_j^n + \gamma \times (u_j^n - u_{j-1}^n) = 0$$

So, $\xi_j^{n+1} - \xi_j^n + \gamma \times (\xi_j^n - \xi_{j-1}^n) = 0$

$$\Rightarrow A^{n+1} e^{i\theta j} - A^n e^{i\theta j} + \gamma \times \left(A^n e^{i\theta j} - A^n e^{i\theta(j-1)}\right) = 0$$

$$\Rightarrow \xi - 1 + \gamma \times (1 - e^{-i\theta}) = 0 \quad \Rightarrow \xi = 1 - \gamma \times (1 - \cos\theta + i\sin\theta)$$

$$\Rightarrow \xi = (1 + \gamma\cos\theta - \gamma) - i\gamma\sin\theta$$

$|\xi|^2 = (1 + \nu\cos\theta - \nu)^2 + \nu^2\sin^2\theta = [1 - \nu(1-\cos\theta)]^2 + \nu^2\sin^2\theta$

$\Rightarrow |\xi|^2 = 1 + 2\nu^2 - 2\nu^2\cos\theta - 2\nu + 2\nu\cos\theta$

$= 1 + 2\nu(\nu-1)(1-\cos\theta)$

Then, for $|\xi|^2 \le 1$ for stability, we need : $|\xi|^2 \le 1$

$2\nu(\nu-1)(1-\cos\theta) \le 0$

$\Rightarrow \nu(\nu-1) \le 0 \quad \Rightarrow \quad 0 \le \nu \le 1 \quad \rightarrow$ But $\nu \ne 0$

So, $\underline{0 < \nu \le 1}$

And as $\nu > 0$, So $c > 0$

Hence, FTBS is stable for $c > 0$, and for its stability, $\nu \in (0, 1]$.

b) Discretized eqn. by FTFS is :

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c \times \frac{u_{j+1}^n - u_j^n}{\delta x} = 0 \quad \Rightarrow u_j^{n+1} - u_j^n + \nu \times (u_{j+1}^n - u_j^n) = 0$$

So, $\xi_j^{n+1} - \xi_j^n + \nu \times (\xi_{j+1}^n - \xi_j^n) = 0 \quad \Rightarrow \xi - 1 + \nu \times [e^{i\theta} - 1] = 0$

$\Rightarrow \xi = 1 - \nu \times [\cos\theta - 1 + i\sin\theta] = [1 + \nu(1-\cos\theta)] - i\nu\sin\theta$

$\Rightarrow |\xi|^2 = [1 + \nu(1-\cos\theta)]^2 + \nu^2\sin^2\theta$

$= 1 + 2\nu^2 - 2\nu^2\cos\theta + 2\nu - 2\nu\cos\theta$

$= 1 + 2\nu(\nu+1)(1-\cos\theta)$

Then, for stability, we need : $|\xi|^2 \le 1$

$\Rightarrow 2\nu(\nu+1)(1-\cos\theta) \le 0 \quad \Rightarrow \nu(\nu+1) \le 0$

$\Rightarrow -1 \le \nu \le 0 \quad \rightarrow$ But $\nu \ne 0$

So, $-1 \le \nu < 0$

And, as $\nu < 0$, So $c < 0$.

Hence, FTFS is stable for $c < 0$ and for its stability, $\nu \in [-1, 0)$.

[Proved.]