INTEGRAL CALCULUS

1) Mean value theorem: If f is continuous on $[a\ b]$, then for some c in $[a\ b]$ we have

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

2) Weighted mean value theorem:

Assume f & g are continuous on $[a\ b]$. If g never changes sign in $[a\ b]$ then

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

3) Let m and M be the minimum and maximum values for $f(x)$ on $[a\ b]$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

4) Let $f(x)$ and $g(x)$ be integrable functions on $[a,b]$ and let $f(x) \le g(x)$, $a \le x \le b$. Then,

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

5) $$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

- Convergence of improper integrals:

  Lecture notes ✓

- Beta & Gamma functions:

  Lecture notes ✓

- ~~Integration~~ Differentiation under integral sign:

  Leibnitz rule:

$$\Phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\, dx$$

$$\frac{d\Phi}{d\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx + f(u_2(\alpha),\alpha)\frac{\partial u_2}{\partial \alpha} - f(u_1(\alpha),\alpha)\frac{du_1}{d\alpha}$$

  - Evaluation of integrals
  - Integral Calculus, Shanti Narayan, P.K. Mittal.

- Double integrals:

  Lecture notes (must)

Ex: Evaluate

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dx\,dy$$

R is bounded by $x=0,\ y=0,\ x+y=1$.

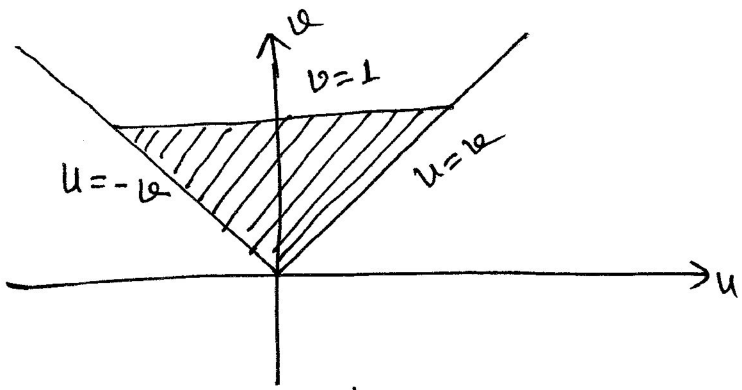
S. ·

S. Let
$$\left.\begin{aligned} x-y &= u \\ x+y &= v \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \frac{u+v}{2} \\ y &= \frac{v-u}{2} \end{aligned}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}.$$

<u>New domain</u>:
$$x=0 \Rightarrow u=-v$$
$$y=0 \Rightarrow u=v$$
$$x+y=1 \Rightarrow v=1.$$



$$\iint_R \cos\frac{x-y}{x+y}\,dx\,dy = \int_0^1 \int_{-v}^{v} \cos\left(\frac{u}{v}\right)\frac{1}{2}\cdot du\,dv$$

$$= \frac{1}{2}\int_0^1 \left[v\sin\left(\frac{u}{v}\right)\right]_{-v}^{v} dv$$

$$= \frac{1}{2}\int_0^1 \left[v(\sin(1)) - (v\sin(-1))\right] dv$$

$$= \sin 1 \int_0^1 v\,dv \Rightarrow \sin(1)\frac{1}{2}$$ Ans.

Application of double integral:

- Evaluation of surface area:

If $z = f(x,y)$ be the equation of the surface, then

$$S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

- Evaluation of volume.

$$V = \iint_D z\, dx\, dy.$$

NOTE: Volume can also be evaluated as $V = \iint_V dx\, dy\, dz$.

Example: Find the area of the paraboloid $2z = \frac{x^2}{a} + \frac{y^2}{b}$ inside the cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: The required area is

$$S = 4\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx dy$$

where D is the positive octant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$S = 4\iint_D \sqrt{1+\frac{x^2}{a^2}+\frac{y^2}{b^2}}\, dx dy$$

Subst. $\frac{x}{a} = r\cos\theta$ $\quad \frac{y}{b} = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

$$S = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \sqrt{1+r^2}\; ab\, r\, dr\, d\theta$$

$$= 4ab \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \sqrt{1+r^2}\; r\, dr\, d\theta$$

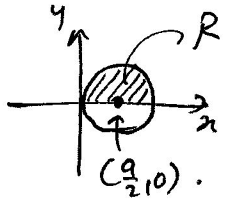
Let $1+r^2 = t \Rightarrow 2r\,dr = dt$

$$= 2ab \int_{\theta=0}^{\pi/2} \frac{2}{3}(1+r^2)^{3/2}\Big|_0^1 \, d\theta$$

$$= \frac{2}{3}\pi ab\left[2^{3/2} - 1\right] \quad \text{Ans.}$$

Q. Find the volume common to a sphere $x^2+y^2+z^2=a^2$ and a circular cylinder $x^2+y^2=ax$.

S. Required volume.



$$V = 4\iint_R z\, dx\, dy$$

$$= 4\iint_R \sqrt{a^2-x^2-y^2}\; dx\, dy$$

Subst. $x = r\cos\theta \quad y = r\sin\theta$.

$\Rightarrow \quad r^2 = ar\cos\theta \Rightarrow r(r - a\cos\theta) = 0$

$\Rightarrow$ R is bounded by $\theta = 0$ to $\pi/2$

& $r = 0$ to $a\cos\theta$.

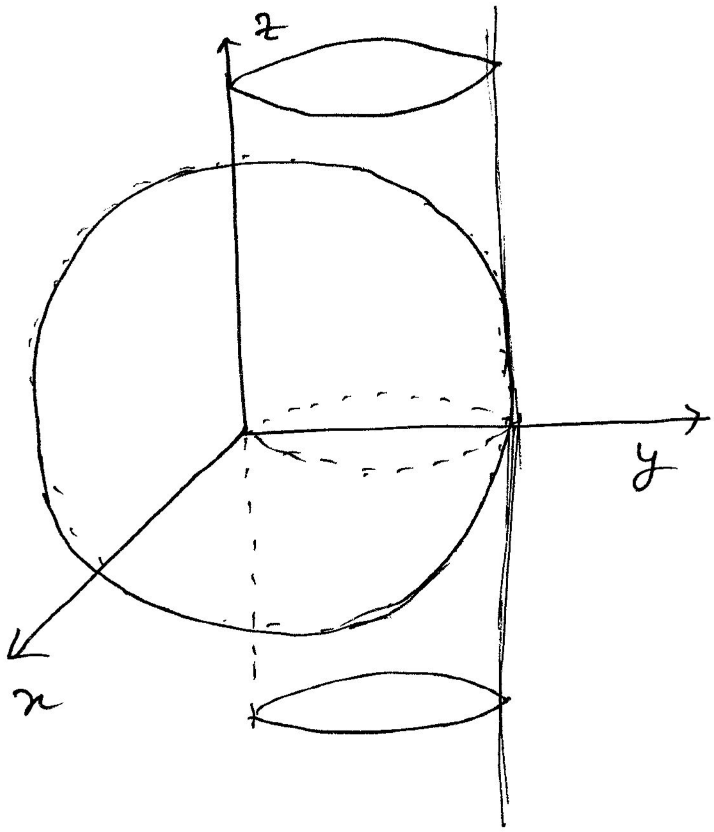
$$V = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} \sqrt{a^2 - r^2}\; r \cdot dr\, d\theta$$

$$= -2 \int_0^{\pi/2} \frac{2}{3} \left(a^2 - r^2\right)^{3/2} \Big|_0^{a\cos\theta} d\theta$$

$$= -\frac{4}{3} \cdot \int_0^{\pi/2} \left(a^3 \sin^3\theta - a^3\right) d\theta$$

$$= -\frac{4}{3} \cdot a^3 \left[2/3 - \pi/2\right] . = \frac{2}{9} a^3 (3\pi - 4) .$$

Q Using triple integral find the volume common to a sphere $x^2+y^2+z^2=a^2$ and a circular cylinder $x^2+y^2=ax$.



S: $V = \iiint_V dx\, dy\, dz$

$$= 4\int_{x=0}^{a} \int_{y=0}^{\sqrt{ax-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} dx\, dy\, dz$$

$$= 4\int_0^a \int_{y=0}^{\sqrt{ax-x^2}} \sqrt{a^2-x^2-y^2}\; dy\, dx.$$

⋮

as above.

$$= \frac{2}{9} a^3 \left(\pi - 4/3\right).$$

Ans

⑦

Q: Find the volume of the solid formed by two paraboloids:

$$z_1 = x^2+y^2 \quad \text{and} \quad z_2 = 1-x^2-y^2.$$

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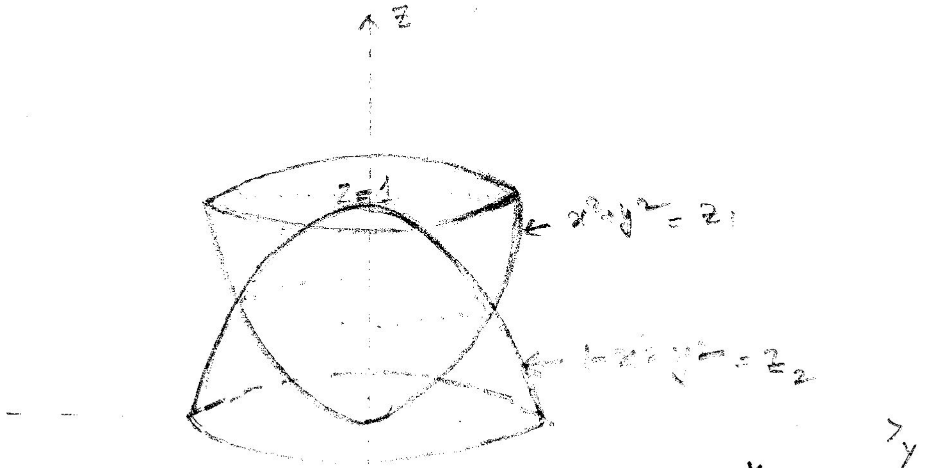
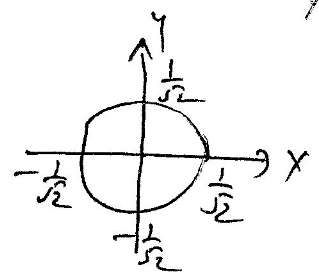


Intersection curve:

$$x^2+y^2 = 1-x^2-y^2$$

$$\Rightarrow 2(x^2+y^2) = 1$$

$$\Rightarrow x^2+y^2 = \frac{1}{2}$$



$$V = \iiint dx\,dy\,dz = \int_{x=-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{y=-\sqrt{\frac{1}{2}-x^2}}^{+\sqrt{\frac{1}{2}-y^2}} \int_{z=x^2+y^2}^{z=1-x^2-y^2} dz\,dy\,dx$$

In cylindrical coordinate: $x = r\cos\theta \quad y = r\sin\theta \quad z = z.$

$$V = 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{\frac{1}{\sqrt{2}}} \int_{z=r^2}^{1-r^2} r \; dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \int_0^{\frac{1}{\sqrt{2}}} r \cdot (1-r^2-r^2)\,dr\,d\theta$$

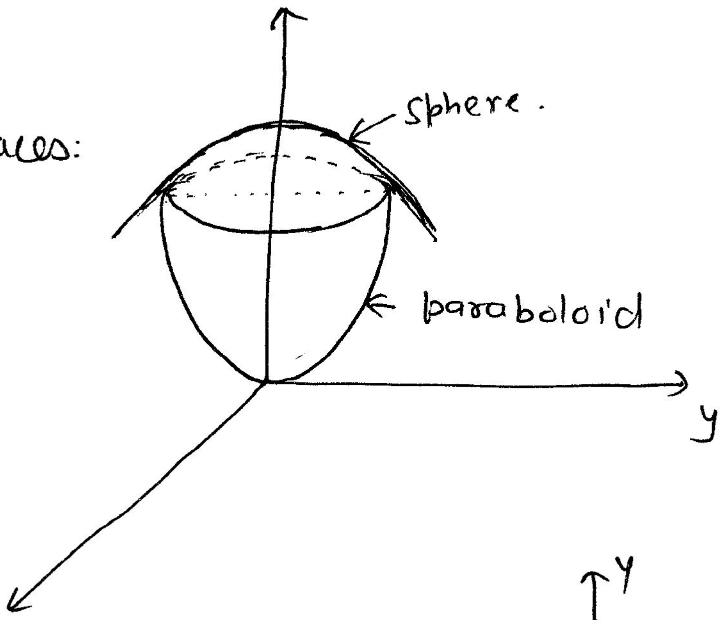
$$= 2\pi \cdot \int_0^{\frac{1}{\sqrt{2}}} r(1-2r^2)\,dr$$

$$= \frac{2\pi}{-4} \cdot \frac{1}{2} (1-2r^2)^2 \Big|_0^{\frac{1}{\sqrt{2}}} = -\frac{\pi}{2} \cdot \frac{1}{2} \cdot (-1) = \frac{\pi}{4}.$$

Ans.

Example: Find the volume of the solid bounded by the sphere $x^2+y^2+z^2=6$ and the paraboloid $x^2+y^2=z$.

Sol:



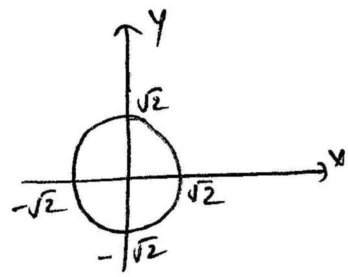
intersection of these surfaces:

$$z+z^2-6=0$$

$$\Rightarrow z = \frac{-1 \pm \sqrt{1+24}}{2}$$

$$= -\frac{1}{2} \pm \frac{5}{2}$$

$$= 2, -3$$



Intersection curve: $x^2+y^2=2, \quad z=2.$

$$V = \iiint_V dz\,dy\,dx = \int_{x=-\sqrt{2}}^{\sqrt{2}} \int_{y=-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{z=x^2+y^2}^{\sqrt{6-x^2-y^2}} dz\,dy\,dx.$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left[\sqrt{6-x^2-y^2} - (x^2+y^2)\right] dy\,dx.$$

Changing to polar coordinates:

$x = r\cos\theta \quad y = r\sin\theta. \quad |J| = r.$

$$V = 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{\sqrt{2}} r\cdot\left[\sqrt{6-r^2} - r^2\right] dr\,d\theta$$

$$= \frac{4\pi}{2}\left[-\frac{1}{2}\left\{\frac{2}{3}(6-r^2)^{3/2}\right\}_0^{\sqrt{2}} - \frac{1}{4}\cdot\{r^4\}_0^{\sqrt{2}}\right]$$

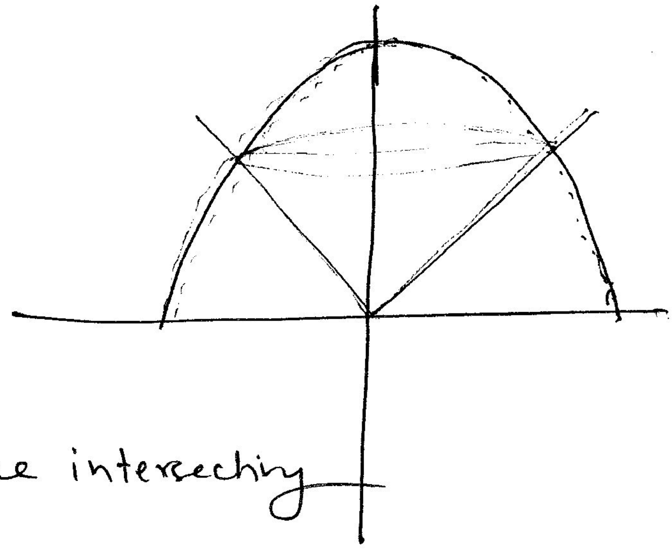
$$= 2\pi\cdot\left[-\frac{1}{2}\left\{\frac{2}{3}\cdot\left(4^{3/2}-6^{3/2}\right)\right\}-\frac{1}{4}\cdot(4)\right]$$

$$= 2\pi\left[\frac{1}{3}\left(6\sqrt{6}-8\right)-1\right]$$

$$= \frac{2\pi}{3}\cdot\left[6\sqrt{6}-11\right]$$

Ans.

Example: Calculate the volume of the solid bounded by the paraboloid $z = 2-x^2-y^2$ and the conic surface $z=\sqrt{x^2+y^2}$



S. Intersecting surface:

$$2-x^2-y^2 = \sqrt{x^2+y^2}$$

Clearly $x^2+y^2=1$ is the intersecting surface.

$$V = \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=\sqrt{x^2+y^2}}^{2-x^2-y^2} dz\,dy\,dx.$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(\left(2-x^2-y^2\right)-\sqrt{x^2+y^2}\right)\cdot dy\,dx.$$

Changing to polar coordinates

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left[(2-r^2) - r \right] r \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{r=0}^{1} (2r - r^2 - r^3) \, dr \, d\theta$$

$$= 2\pi \cdot \left[1 - \frac{1}{3} - \frac{1}{4} \right] = 2\pi \left(\frac{12-4-3}{12} \right)$$

$$= \frac{5\pi}{6} .$$

Ans

Limit, Continuity & differentiability:

Carry over the concept of real valued function.

let $\bar{f}(t) = x(t)\,\bar{i} + y(t)\,\bar{j}$

$$\lim_{t\to a} \bar{f}(t) = \lim_{t\to a} x(t)\,\bar{i} + \lim_{t\to a} y(t)\,\bar{j}$$

We say $\bar{f}(t)$ is continuous at $t=a$ if

$$\lim_{t\to a} \bar{f}(t) = \bar{f}(a).$$

$$\frac{d\bar{f}}{dt} = \bar{f}'(t) = \lim_{\Delta t\to 0} \frac{\bar{f}(t+\Delta t) - \bar{f}(t)}{\Delta t}$$

provided this limit exists.

$$\bar{f}'(t) = \frac{dx(t)}{dt}\,\bar{i} + \frac{dy(t)}{dt}\,\bar{j}\,.$$

Directional derivative: $\boxed{D_b f = \bar{\nabla} f \cdot \bar{b}}$

Q. Find the directional derivative of $\cancel{f} = x^2 y z^3$ along the curve $x = e^{-u}$, $y = 2\sin u + 1$, $z = u - \cos u$ at the point P where $u=0$.

S. tangent vector to the curve is

$$\frac{d\bar{r}}{du} = -e^{-u}\,\bar{i} + 2\cos u\,\bar{j} + (1+\sin u)\,\bar{k}$$

at P: $\frac{d\bar{r}}{du} = -\bar{i} + 2\bar{j} + \bar{k}$; Point P is $(1, 1, -1)$.

Unit tangent vector $T_0 = (-\bar{i} + 2\bar{j} + \bar{k})/\sqrt{6}$

$$D_T f = \left[2xyz^3\,\bar{i} + x^2z^3\,\bar{j} + (3x^2yz^2)\,\bar{k}\right] \cdot \left[-\bar{i} + 2\bar{j} + \bar{k}\right]/\sqrt{6} = \boxed{\frac{\cancel{\sqrt{6}}}{2}}$$

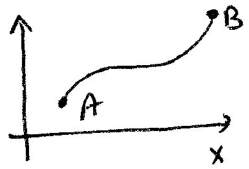
Conservative vector field:

if: $\bar{V} = \nabla f$ f is called potential function

$\bar{V}$ is called conservative vector field.

In this case ($\bar{v}$ is a conservative vector field):

$$\int_C \bar{V} \cdot d\bar{r} = f(B) - f(A)$$



Test for conservative field: $\bar{F} = F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\,\hat{j} + F_3(x,y,z)\,\hat{k}$

$\bar{F}$ is conservative $\Longleftrightarrow$ $\underbrace{\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}}_{\bar{\nabla} \times \bar{F} = 0.}$

Q. Determine if the vector field $\bar{F} = (yz, xz, xy)$ is conservative.

S
$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z) = 0.$$

Hence, the vector field is conservative.

Q Show that the integral $\int_{AB} (3x^2y + y)\,dx + (x^3+x)\,dy$ is path independent and calculate this integral. The points A & B are $(1,2)$ & $(4,5)$ respectively.

S.
$$\left.\begin{aligned} \frac{\partial F_1}{\partial y} &= 3x^2+1 \\ \frac{\partial F_2}{\partial x} &= 3x^2+1 \end{aligned}\right\} \Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$\Rightarrow \bar{F}$ is conservative and hence the line integral is path independent.

Let $F = \bar{\nabla}\varphi \Rightarrow F_1\hat{i} + F_2\hat{j} = \frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j}$

$\Rightarrow \frac{\partial \varphi}{\partial x} = 3x^2y + y \Rightarrow \varphi = x^3y + g(y)$

$\frac{\partial \varphi}{\partial y} = x^3 + x = x^3 + g'(y) \Rightarrow g'(y) = x$

$\Rightarrow g(y) = xy + c.$

$\Rightarrow \varphi = x^3y + xy + c.$

$$\int_{AB} F_1 dx + F_2 dy = \varphi(4,5) - \varphi(1,2)$$

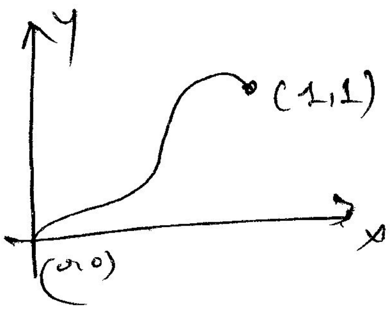
$$= 64 \times 5 + 20 - 2 - 2$$

$$= 320 + 20 - 4$$

$$= 336.$$

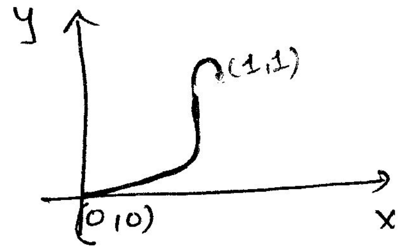
Q: Evaluate $\int_C \bar{F} \cdot d\bar{r}$ for $F = (y^2 + 2xy)\hat{i} + (x^2 + 2xy)\hat{j}$

$C: x^9 + (1-xy)^{1/5} = 1$ from $(0,0)$ to $(1,1)$.



Question: Evaluate $\int_C \bar{F}\cdot d\bar{r}$ for $\bar{F} = (y^2+2xy)\bar{i} + (x^2+2xy)\bar{j}$

$C: \quad (x^9) + (1-xy)^{1/5} = 1$ from $(0,0)$ to $(1,1)$



Solution: Note that $\frac{\partial}{\partial x}(x^2+2xy) - \frac{\partial}{\partial y}(y^2+2xy)$

$$= 2x+2y-2y-2x = 0$$

$\Rightarrow \bar{F}$ is conservative. $\quad (\bar{\nabla}\times\bar{F} = 0)$

$\Rightarrow$ Line integral is path independent.

WAY 1: $\quad \exists\, \varphi$ such that $\bar{F} = \bar{\nabla}\varphi$

$$\Rightarrow \frac{\partial\varphi}{\partial x} = y^2+2xy \Rightarrow \varphi = y^2x + x^2y + f(y)$$

$$\& \quad \frac{\partial\varphi}{\partial y} = x^2+2xy = 2yx + x^2 + f'(y) \Rightarrow f'(y) = 0$$

$$\Rightarrow f(y) = C.$$

$$\Rightarrow \varphi = xy^2 + yx^2 + C.$$

$$\int_C \bar{F}\cdot d\bar{r} = \varphi(1,1) - \varphi(0,0) = 2-0 = 2.$$

WAY 2: Take a path straight line from $(0,0)$ to $(1,1)$

parametric equation $\bar{r}(t) = t\,\hat{i} + t\,\hat{j}$; $0\le t\le 1$.

$$\int F\cdot dr = \int_{t=0}^{1} 2(t^2+2t^2)\,dt = 2\cdot\left[\tfrac{1}{3}+\tfrac{2}{3}\right] = 2.$$

Ans

Q. Determine if the vector field $\bar{F}(x,y) = (x+y,\ x-y)$ is conservative? If it is, find its potential.

S. $\left.\begin{aligned} \frac{\partial}{\partial y}(x+y) &= 1 \\ \frac{\partial}{\partial x}(x-y) &= 1 \end{aligned}\right\}$ given vector field is conservative.

$$\bar{F} = \nabla\varphi \Rightarrow \frac{\partial\varphi}{\partial x} = x+y \quad \& \quad \frac{\partial\varphi}{\partial y} = x-y.$$

$$\Rightarrow \varphi = \frac{x^2}{2} + xy + g(y)$$

$$\Rightarrow x + g'(y) = x - y \Rightarrow g'(y) = \cancel{x-1}\ -y$$

$$g(y) = -\frac{y^2}{2} + c$$

$$\boxed{\varphi = \frac{x^2}{2} + xy - \frac{y^2}{2} + c}$$

Q: Determine if the vector field ~~F(x,y) = (x+y,~~
$\bar{F}(x,y,z) = (yz,\ xz+2y,\ xy+1)$. If it is find its potential.

S. $$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz+2y & xy+1 \end{vmatrix} = \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z)$$
$$= 0.$$

$$\bar{F} = \nabla\varphi \Rightarrow \quad yz = \frac{\partial\varphi}{\partial x} \quad xz+2y = \frac{\partial\varphi}{\partial y} \quad xy+1 = \frac{\partial\varphi}{\partial z}$$

$$\varphi = xyz + g(y,z) \Rightarrow \frac{\partial\varphi}{\partial y} = xz + \frac{\partial g(y,z)}{\partial y} = xz+2y$$

$$\Rightarrow g(y,z) = y^2 + h(z)$$

$$\Rightarrow \varphi = xyz + h(z) + y^2 \Rightarrow \frac{\partial\varphi}{\partial z} = xy + h'(z) = xy + \phi \Rightarrow h'(z) = \phi \Rightarrow h(z) = c+z$$

$$\boxed{\varphi = xyz + y^2 + c + z}$$

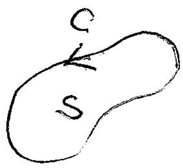
Green's theorem:

If D is a region in $\mathbb{R}^2$, ∂D is its boundary, oriented in the positive sence, and F_1 & F_2 are C^1 functions of x & y then

$$\oint_{\partial D} F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy.$$

Stokes theorem:

$$\oint_c \bar{F}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times \bar{F})\cdot \bar{n}\, d\sigma$$



Divergence theorem:

$$\iint_S \bar{F}\cdot\bar{n}\, d\sigma = \iiint_V \nabla\cdot F\, dv$$

region V is enclosed by S.

Ex. Evaluate the line integral $\oint_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz$ where C is the curve formed by intersection of the sphere $x^2+y^2+z^2=1$ with the plane $x+2y+2z=0$.

(Hint use stokes theorem)

S: normal to the surface cut by the sphere from the plane.

$$\bar{n} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+2z & x+2z & x+2y \end{vmatrix} = \bar{i}(2-2) - \bar{j}(1-2) + \bar{k}(1-1)$$

$$= \bar{j}$$

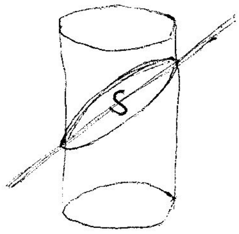
Using Stokes theorem.

$$\oint_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz$$

$$= \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, d\sigma = \frac{2}{3} \iint_S d\sigma$$

$$= \frac{2}{3}\pi .$$

<u>Example</u>: Use stokes theorem to calculate the line integral $\oint_C y^3 dx - x^3 dy + z^3 dz$. The curve is the intersection of the cylinder $x^2+y^2=a^2$ and the plane $x+y+z=b$.



<u>S</u>. Unit normal to the surface S

$$\bar{n} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}.$$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & -x^3 & z^3 \end{vmatrix} = \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(-3x^2-3y^2)$$

$$= -3\bar{k}\,(x^2+y^2)$$

$$\oint_C \cdots = \iint_S (\bar{\nabla}\times\bar{F}) \cdot n\, d\sigma = \iint_S -\frac{3}{\sqrt{3}} (x^2+y^2)\, d\sigma$$

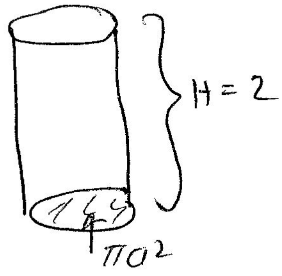
$$= -\sqrt{3} \iint_S (x^2+y^2)\, d\sigma$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA = \frac{\sqrt{3}}{1} \cdot dx\,dy$$

$$\oint_C \cdots = -\sqrt{3} \cdot \iint_{x^2+y^2 \le a^2} (x^2+y^2)\, \sqrt{3} \cdot dx\,dy = -3 \cdot \int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2\, r\, dr\, d\theta$$

$$= -3 \cdot \frac{1}{4} a^4 \cdot 2\pi = -\frac{3}{2} a^4 \pi$$

Q: Use the divergence theorem to evaluate the surface integral $\iint_S \bar{F}\cdot\bar{n}\,d\sigma$ of the vector field $\bar{F}=(x,y,z)$, where S is the surface of the solid bounded by the cylinder $x^2+y^2=a^2$ and the planes $z=-1$, $z=1$.



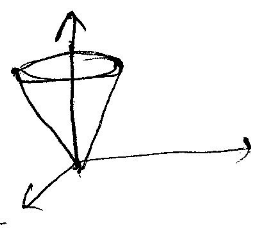
S.: $\iint_S \bar{F}\cdot\bar{n}\,d\sigma = \iiint_V \nabla\cdot\bar{F}\,dv$

$$= \iiint_V 3\,dx\,dy\,dz = 3\cdot 2\pi a^2$$

$$= 6\pi a^2$$

Q: Use divergence theorem to evaluate $\iint_S \bar{F}\cdot\bar{n}\,d\sigma$ of the $\bar{F}=(x^3,y^3,z^3)$, where S is the surface of a solid bounded by the cone $x^2+y^2-z^2=0$ and the plane $z=1$.

S.: $\iint_S \bar{F}\cdot\bar{n}\,d\sigma = \iiint_V (3x^2+3y^2+3z^2)\,dx\,dy\,dz$



Changing to cylindrical coordinates:

$$\iint_S \bar{F}\cdot\bar{n}\,d\sigma = 3\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=r}^{1}(r^2+z^2)\,r\,dz\,d\theta\,dr$$

$$= 3\int_0^1\int_0^{2\pi}\left[r^3(1-r)+\tfrac{1}{3}r(1-r^3)\right]d\theta\,dr$$

$$= 6\pi\cdot\left[\frac{1}{4}-\frac{1}{5}+\frac{1}{3\cdot 2}-\frac{1}{3\times 5}\right] = 6\pi\cdot\left(\frac{15-12+10-4}{60}\right)$$

$$= \cancel{6}\pi\cdot\frac{9}{\cancel{6}10} = \boxed{\frac{9\pi}{10}}$$

Q: Verify Divergence theorem for the following problem (8)

Let Σ be the piecewise smooth closed surface consisting of the surface Σ_1 of the cone $z=\sqrt{x^2+y^2}$ for $x^2+y^2 \le 1$, together with the flat cap Σ_2 consisting of the disk $x^2+y^2 \le 1$ in the $z=1$ plane.

~~This surf~~ Let $F = (x\hat{i}+y\hat{j}+z\hat{k})$.

S: $\iiint_V \nabla\cdot F \, dv = \iiint_V 3 \cdot dv$

$$= 3 \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1} r\,dz\,dr\,d\theta$$

$$= 3 \cdot 2\pi \cdot \int_0^1 r(1-r)\,dr$$

$$= 6\pi \left(\frac{1}{2}-\frac{1}{3}\right)$$

$$= 6\pi \cdot \frac{1}{6} = \underline{\underline{\pi}}. \quad \text{Ans.}$$

Using surface integral:

On Σ_1:

$$\hat{n} = \frac{1}{\sqrt{2}}\left(\frac{x}{z}\hat{i} + \frac{y}{z}\hat{j} - \hat{k}\right)$$

$$\hat{F}\cdot\hat{n} = \frac{1}{\sqrt{2}}\left(\frac{x^2}{z}+\frac{y^2}{z}-z\right) = 0.$$

$$\iint_{\Sigma_1} F\cdot\hat{n}\,d\sigma = 0. \quad //$$

On Σ_2: $\hat{F}\cdot\hat{n} = z$.

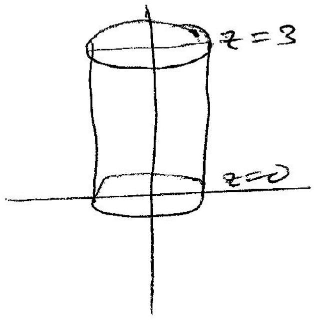
$$\iint_{\Sigma_2} \hat{F}\cdot\hat{n}\,d\sigma = \iint_{\Sigma_2} z\,d\sigma = \iint_{\Sigma_2} d\sigma \quad (\text{on } \Sigma_2, z=1)$$

$$= \underline{\underline{\pi}} \quad //$$

Q: Using Gauss divergence theorem, evaluate

$$\iint_S \bar{F}\cdot\hat{n}\cdot d\sigma \quad \text{where} \quad \bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$$

~~and~~ and S: surface of $x^2+y^2 \le 4$, $z \ge 0$ to $z \le 3$



S: $\iiint_V \operatorname{div}\bar{F}\cdot dV$

$$= \iiint_V (4-4y+2z)\, dx\,dy\,dz$$

Use cylindrical coordinates:

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{3} \left(4 - 4\cdot r\,\overset{\sin\theta}{\cancel{\cos\theta}} + 2z\right) r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \left[4\cdot 3 - 4\cdot r\cos\theta\cdot 3 + 9\right] dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \left(21 - 12\, r\cos\theta\right) dr\,d\theta$$

$$= \int_0^{2\pi} 21\cdot 2 - 12\cos\theta\cdot\frac{1}{2}\cdot 4^2\, d\theta$$

$$= \int_0^{2\pi} (42 - 24\cdot\cos\theta)\, d\theta$$

$$= 42\cdot[2\pi - 24\cdot 0] =$$

$$= 84\pi$$

Ans