**INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR**
**CLASS TEST / LABORATORY TEST**

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Signature of the Invigilator

| EXAMINATION ( Mid Semester / End Semester ) | SEMESTER ( Autumn / Spring ) |
| --- | --- |
| Roll Number | Section | Name: Koeli Ghoshal |
| Subject Number: MA20103 | Subject Name: PDE |

Lecture - 11 (continued) 7.8.2017

## Example on 1st type

**Ex** Form a pde by eliminating $a, b$ from

$$z = (x+a)(y+b)$$

**Sol^n**

$$z = (x+a)(y+b) \quad \text{---} (1)$$

$$\left.\begin{aligned} \frac{\partial z}{\partial x} = p = y+b \\ \frac{\partial z}{\partial y} = q = x+a \end{aligned}\right\} \begin{aligned} \text{---}(2) \\ \text{---}(3) \end{aligned}$$

Substituting (2) & (3) in (1), we get

$z = pq$ which is the reqd. pde.

**Ex** Form a pde by eliminating $a, b, c$ from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{---}(1)$$

**Sol^n** Differentiating (1) partially w.r.t. $x$ & $y$,

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \Rightarrow c^2 x + a^2 z\frac{\partial z}{\partial x} = 0 \quad \text{---}(2)$$

and $$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \Rightarrow c^2 y + b^2 z\frac{\partial z}{\partial y} = 0 \quad \text{---}(3)$$

Again differentiating (2) partially w.r.t. $x$,

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow \frac{c^2}{a^2} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

Substituting $\frac{c^2}{a^2} = -\frac{z}{x}\frac{\partial z}{\partial x}$ from (2),

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow -z\frac{\partial z}{\partial x} + x\left(\frac{\partial z}{\partial x}\right)^2 + xz\frac{\partial^2 z}{\partial x^2} = 0$$

Similarly differentiating (3) partially w.r.t. $y$ and substituting the value of $\frac{c^2}{b^2}$ from (3), we will get

$$-z\frac{\partial z}{\partial y} + y\left(\frac{\partial z}{\partial y}\right)^2 + yz\frac{\partial^2 z}{\partial y^2} = 0$$

<u>Example on Type 2</u>

<u>Ex</u> Form a pde by eliminating the $f^n$ $f$ from

$$z = y^2 + 2f\left(\frac{1}{x} + \ln y\right) \quad -(1)$$

Sol$^n$: $\frac{\partial z}{\partial x} = p = 2f'\left(\frac{1}{x} + \ln y\right)\cdot\left(-\frac{1}{x^2}\right)$

$$\Rightarrow -px^2 = 2f'\left(\frac{1}{x} + \ln y\right) \quad -(2)$$

$$\frac{\partial z}{\partial y} = q = 2y + 2f'\left(\frac{1}{x} + \ln y\right)\cdot\frac{1}{y}$$

$$\Rightarrow qy - 2y^2 = 2f'\left(\frac{1}{x} + \ln y\right) \quad -(3)$$

From (2) & (3), $-px^2 = qy - 2y^2$

$$\Rightarrow x^2 p + yq = 2y^2$$

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Lecture - 12 & 13 9.8.2017

Ex Form a pde by eliminating the function $\phi$ from $lx+my+nz = \phi(x^2+y^2+z^2)$.

Sol$^n$ $$l + n\frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2)\left\{2x + 2z\frac{\partial z}{\partial x}\right\} \quad -(1)$$

$$m + n\frac{\partial z}{\partial y} = \phi'(x^2+y^2+z^2)\left\{2y + 2z\frac{\partial z}{\partial y}\right\} \quad -(2)$$

Dividing (1) by (2),

$$\frac{l + n\frac{\partial z}{\partial x}}{m + n\frac{\partial z}{\partial y}} = \frac{2\left\{x + z\frac{\partial z}{\partial x}\right\}}{2\left\{y + z\frac{\partial z}{\partial y}\right\}}$$

$$\Rightarrow (ny - mz)\frac{\partial z}{\partial x} + (lz - nx)\frac{\partial z}{\partial y} = mx - ly$$

Ex Form a pde by eliminating $f$ from
$$z = e^{ax+by} f(ax-by)$$

Sol$^n$: $$\frac{\partial z}{\partial x} = e^{ax+by}\, a f'(ax-by) + a e^{ax+by} f(ax-by)$$

$$\frac{\partial z}{\partial y} = e^{ax+by}\{-bf'(ax-by)\} + b\, e^{ax+by} f(ax-by)$$

Multiplying $\frac{\partial z}{\partial x}$ by $b$ & $\frac{\partial z}{\partial y}$ by $a$ & adding

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\, e^{ax+by} f(ax-by)$$

$$\Rightarrow b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$$

# PDE of order one

$$f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$$

## Classification of first order pde

1. **Linear eqn.**

linear in $p$, $q$ & $z$ i.e. of the form

$$P(x,y)\, p + Q(x,y)\, q = R(x,y)\, z + S(x,y)$$

Ex $\quad y p - x q = xyz + x$

2. **Semi-linear eqn.**

linear in $p$ & $q$ and the coeff. of $p$ & $q$ are $f^n$s of $x$ & $y$ only

$$P(x,y)\, p + Q(x,y)\, q = R(x,y,z)$$

Ex $\quad e^x p - yx q = xz^2$

3. **Quasi-linear eqn.**

$$P(x,y,z)\, p + Q(x,y,z)\, q = R(x,y,z)$$

Ex $\quad (x^2+z^2)\, p - xy q = z^3 x + y^2$

4. Non-linear eqn.

PDE of this form $f(x,y,z,p,q) = 0$ which do not come under the above three types are said to be non-linear eqns. Ex $pq = z$

## Classification of integrals

A solution or integral of a diff. eqn. is a relation between the variables by & means of which and the derivatives obtained from there, the diff. eqn. is satisfied.

A soln. $z = z(x,y)$ when interpreted as a surface in 3-D space is called an integral surface of the pde.

$$f(x,y,z,p,q) = 0 \quad \text{--- (1)}$$

(a) Complete integral

A two parameter family of solutions

$$z = F(x,y,a,b) \quad [F(x,y,z,a,b)=0]$$

is called complete $F(x,y,z,a,b)=0$ integral of (1).

(b) General integral

Any relation of the type $F(u,v)=0$ involving an arb. fn. $F$ connecting two known fns. $u$ & $v$ of $x, y$ & $z$ providing a soln. of a 1st order pde is called a general soln. or general integral.

(c) Singular integral Eliminating a & b from $F(x,y,z,a,b)=0$ $f=0$ $\frac{\partial F}{\partial a}=0$ $\frac{\partial F}{\partial b}=0$

## Theorem

The general solution of the quasi-linear eqn, (or Lagrange's eqn.)

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z) \quad -(1)$$

where $P$, $Q$ and $R$ are continuously differentiable fns. of $x$, $y$ and $z$ is

$$F(u,v) = 0 \quad -(2)$$

where $F$ is an arbitrary differentiable $f^n$ of $u$ and $v$ and

$$u(x,y,z) = c_1 \text{ and } v(x,y,z) = c_2 \quad -(3)$$

are two independent solutions of the system

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \quad (4)$$

**Proof** We have already shown that by eliminating $F$ from (2), we arrive at

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q$$
$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \quad -(5)$$

Hence (2) is a G.S. of (5).

By differentiating (3),

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0$$

Solving the above eqn. for dx, dy, dz

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \quad \text{— (6)}$$

Comparing (6) & (4),

$$\frac{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}{P} = \frac{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}{Q} = \frac{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}}{R}$$

$$= k \text{ (say)} \quad \text{— (7)}$$

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = kP \quad \text{and so on}$$

Substituting in (5),

$$k(Pp + Qq) = kR \Rightarrow Pp + Qq = R$$

i.e. the given eqn (1).

Eqn (4) are known as

Lagrange's auxiliary (or subsidiary) eqns for (1).

## Working rule for solving $Pp + Qq = R$

Step 1. Put the given pde in the standard form

$$Pp + Qq = R \quad — (1)$$

Step 2. Write down the Lagrange's auxiliary eqn. for (1)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad — (2)$$

Step 3. Solve (2) by using any method.

Let $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ be two independent solⁿs. of (2).

Step 4. The general solution (or integral) of (1) is then written in one of the following three equivalent forms

$$\phi(u,v) = 0 \quad \text{or} \quad u = \phi(v) \quad \text{or} \quad v = \phi(u).$$

## Four rules for getting two independent solⁿs. of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ — (1)

Type 1 Suppose that one of the variables is either absent or cancels out from any two fractions of the eqn. (1). Then an integral can be obtained by usual methods. The same procedure can be repeated with another set of two fractions of (1).

Ex Solve $\frac{y^2 z}{x} p + xzq = y^2$

Soln Lagrange's subsidiary eqn. is

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Taking the first two fractions

$$x^2 z\, dx = y^2 z\, dy$$

$$\Rightarrow x^2 dx - y^2 dy = 0$$

$$\Rightarrow \text{i.e. } x^3 - y^3 = c_1$$

Taking the first and the last fraction

$$xy^2\, dx = y^2 z\, dz$$

$$\Rightarrow x\,dx - z\,dz = 0$$

$$\text{i.e. } x^2 - z^2 = c_2$$

G. S. is $\phi(x^3 - y^3, x^2 - z^2) = 0$, $\phi$ being an arbitrary $f^n$.

$x^2 - z^2 = \phi(x^3 - y^3)$ → another form.

Ex Solve $zp = -x$

Solⁿ: $zp + 0q = -x$

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$$

Taking first and last $-x\,dx = z\,dz$ i.e. $x^2 + z^2 = c_1$

The 2nd one gives $dy = 0$ i.e. $y = c_2$

G.S. $\phi(x^2+z^2, y) = 0$

Ex Solve $y^2 p - xyq = x(z-2y)$

Solⁿ: $$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$2x\,dx + 2y\,dy = 0$ i.e. $x^2+y^2 = c_1$ [from 1st two]

Taking last two

$$\frac{dz}{dy} = -\frac{z-2y}{y}$$

$$\Rightarrow \frac{dz}{dy} + \frac{1}{y}z = 2$$

I.F. $= e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

$z.y = \int 2y\,dy + c_2$

$\Rightarrow zy - y^2 = c_2$

$\phi(x^2+y^2, zy-y^2) = 0$ is the desired G.S.

where $\phi$ is an arbitrary function.

# Lecture - 14

21.8.2017

**Type 2** for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ — (1)

One integral of (1) is known by using the rule of type 1 and the other integral is obtained by using the first integral.

**Ex** Solve $p + 3q = 5z + \tan(y - 3x)$

**Soln** $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$

From the 1st two, $dy - 3dx = 0$

i.e. $y - 3x = c_1$

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$$

$$\therefore x - \frac{1}{5}\ln(5z + \tan c_1) = \frac{1}{5}c_2$$

$$\Rightarrow 5x - \ln[5z + \tan(y-3x)] = c_2$$

G.S. $5x - \ln[5z - \tan(y-3x)] = \phi(y-3x)$

Ex $py + qx = xyz^2(x^2-y^2)$

Solⁿ: $\cancel{\frac{dy}{}}\ \frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)}$

$x\,dx - y\,dy = 0$

i.e. $x^2-y^2 = c_1$

$$\frac{dy}{x} = \frac{dz}{xyz^2 c_1}$$

$\Rightarrow c_1 y\,dy - z^{-2}\,dz = 0$

$\Rightarrow c_1 \frac{y^2}{2} + \frac{1}{z} = \frac{1}{2}c_2$

$\Rightarrow y^2(x^2-y^2) + \frac{2}{z} = c_2$ G.S. $y^2(x^2-y^2)+\frac{2}{z} = \phi(x^2-y^2)$

G.S. $y^2(x^2-y^2) + \frac{2}{z} = \phi(x^2-y^2)$

Ex $xzp + yzq = xy$

Solⁿ: $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$

$\frac{dx}{x} - \frac{dy}{y} = 0$ i.e. $\frac{x}{y} = c_1$ $x = c_1 y$

From 2nd & 3rd, $\frac{1}{yz}\,dy = \left(\frac{1}{c_1 y^2}\right) dz$

$\Rightarrow c_1 y\,dy - z\,dz = 0$

$\frac{1}{2}c_1 y^2 - \frac{1}{2}z^2 = \frac{1}{2}c_2$

$\Rightarrow c_1 y^2 - z^2 = c_2$ $xy - z^2 = c_2$

G.S. $\phi(xy - z^2, \frac{x}{y}) = 0$

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CLASS TEST / LABORATORY TEST

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Signature of the Invigilator

| EXAMINATION ( Mid Semester / End Semester ) | | SEMESTER ( Autumn / Spring ) | |
|---|---|---|---|
| Roll Number | | Section | | Name | Koeli Ghoshal |
| Subject Number | MA20103 | Subject Name | PDE |

Lecture 14 (continued) 21-8-2017

**Type 3** for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$P_1, Q_1, R_1 \to$ either consts or $f^n$ of $x, y, z$

each fraction $= \dfrac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}$

$P_1 P + Q_1 Q + R_1 R = 0$

Ex Solve $\{(b-c)/a\} yzp + \{(c-a)/b\} zxq = \{(a-b)/c\} xy$

Sol$^n$ $\dfrac{a\,dx}{(b-c)yz} = \dfrac{b\,dy}{(c-a)zx} = \dfrac{c\,dz}{(a-b)xy}$

Choosing $x, y, z$ as multipliers, each fraction

$$= \frac{ax\,dx + by\,dy + cz\,dz}{xyz[(b-c)+(c-a)+(a-b)]} = \frac{ax\,dx + by\,dy + cz\,dz}{0}$$

i.e. $ax\,dx + by\,dy + cz\,dz = 0 \Rightarrow ax^2 + by^2 + cz^2 = c_1$

Choosing $ax, by, cz$ as multipliers, each fraction

$$= \frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{xyz[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{0}$$

$\Rightarrow a^2x\,dx + b^2y\,dy + c^2z\,dz = 0 \quad \therefore a^2x^2 + b^2y^2 + c^2z^2 = c_2$

G.S. $\phi(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2) = 0$

Ex Solve $(mz-ny)p+(nx-lz)q=ly-mx$

Solⁿ
$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx} \quad -(1)$$

Choosing $x, y, z$ as multipliers, each fraction of (1)

$$=\frac{x\,dx+y\,dy+z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz=0$$
$$x^2+y^2+z^2=c_1$$

Choosing $l, m, n$ as multipliers, each fraction of (1)

$$=\frac{l\,dx+m\,dy+n\,dz}{l(mx-ny)+m(nx-lz)+n(ly-mx)}$$

$$=\frac{l\,dx+m\,dy+n\,dz}{0}$$

$\therefore l\,dx+m\,dy+n\,dz=0$ so that $lx+my+nz=c_2$

G.S. $\phi(x^2+y^2+z^2,\ lx+my+nz)=0$

$\phi$ being an arb. $f^n$.

**Ex** Solve $(z^2-2yz-y^2)p + (xy+zx)q = xy-zx$

**Solⁿ:** $$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} \quad -(1)$$

Taking the last two fractions of (1),

$$(y-z)\,dy = (y+z)\,dz \rightarrow y^2-z^2-2yz = c_1 \quad -(2)$$

Choosing $x, y, z$ as multipliers, each fraction of (1),

$$= \frac{x\,dx+y\,dy+z\,dz}{x(z^2-2yz-y^2)+xy(y+z)+xz(y-z)} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$\Rightarrow x\,dx+y\,dy+z\,dz=0$ so that $x^2+y^2+z^2=c_2 \quad -(3)$

From (1) & (2), the reqd. G.S is

$$\phi\,(y^2-z^2-2yz,\ x^2+y^2+z^2)=0$$

**Ex** Solve $y^2p + x^2q = x^2y^2z^2$

**Solⁿ.** $$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2} \quad -(1)$$

$x^2 \lambda y^4 \pm 1\frac{3}{z^2}$ First two,

$$x^3-y^3=c_1 \quad -(2)$$

Choosing $x^2, y^2, -\frac{2}{z^2}$ as multipliers, each fraction of (1)

$$= \frac{x^2\,dx+y^2\,dy-\frac{2}{z^2}\,dz}{0}$$ so that

$$3x^2\,dx+3y^2\,dy-\frac{6}{z^2}\,dz=0$$

$$x^3+y^3+6\left(\frac{1}{z}\right)=c_2 \quad -(3)$$

G.S. $\phi\left[x^3-y^3,\ x^3+y^3+6\left(\frac{1}{z}\right)\right]=0$

Lecture – 15 & 16 23.8.2017

Type 4 for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ —(1)

$P_1, Q_1, R_1 \rightarrow$ either constants or $f^n$s of $x, y, z$

each fraction $= \dfrac{P_1\,dx + Q_1\,dy + R_1\,dz}{P_1P + Q_1Q + R_1R}$

Numerator is exact differential of the denominator.

$P_2, Q_2, R_2$ $\quad \dfrac{P_2\,dx + Q_2\,dy + R_2\,dz}{P_2P + Q_2Q + R_2R}$

Ex Solve $(y+z)p + (z+x)q = x+y$

Solⁿ: Lagrange's subsidiary eqn.

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad \text{—(1)}$$

Choosing $1, -1, 0$ as multipliers, each fraction of (1)

$$= \frac{d(x-y)}{-(x-y)} \quad \text{—(2)}$$

Again choosing $0, 1, -1$ as multipliers, each fraction of (1) $\dfrac{d(y-z)}{-(y-z)}$ —(3)

Choosing $1, 1, 1$ as multipliers, each fraction of (1)

$$= \frac{d(x+y+z)}{2(x+y+z)} \quad \text{—(4)}$$

From (2), (3) & (4),

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \quad -(5)$$

From first two fractions, $\frac{x-y}{y-z} = c_1$

From first and third fraction,

$$(x-y)^2 (x+y+z) = c_2$$

$$\phi\left[(x-y)^2(x+y+z), \frac{x-y}{y-z}\right] = 0$$

~~Soln~~ Ex Solve $(1+y)p + (1+x)q = z$

Sol$^n$ $$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z} \quad -(1)$$

First two, $(1+x)^2 - (1+y)^2 = c_1 \quad -(2)$

Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{1+y+1+x} = \frac{d(2+x+y)}{2+x+y} \quad -(3)$$

Combining the last fraction of (1) with (3),

$$\frac{2+x+y}{z} = c_2 \quad -(4)$$

G.S. $$\phi\left[(1+x)^2 - (1-y)^2, \frac{2+x+y}{z}\right] = 0$$

Ex Solve $\quad p+q = x+y+z$

Sol$^n$ $\quad \dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{x+y+z}$

From first two, $\quad x-y = c_1$

Choosing 1, 1, 1 as multipliers, each fraction

$$= \frac{dx+dy+dz}{1+1+(x+y+z)} = \frac{d(2+x+y+z)}{2+x+y+z}$$

$$\frac{dx}{1} = \frac{d(2+x+y+z)}{2+x+y+z}$$

$$\ln(2+x+y+z) - \ln c_2 = x$$

$$\Rightarrow \quad e^{-x}(2+x+y+z) = c_2$$

$$\phi\left[x-y,\ e^{-x}(2+x+y+z)\right] = 0$$

## Application of Lagrange's method of solution

### To determine the integral surface passing through a given curve

### Method I

Let $Pp + Qq = R$ —(1)

be the given eqn. Let its auxiliary eqn. gives two independent solutions

$$u(x,y,z) = c_1 \quad \& \quad v(x,y,z) = c_2 \quad -(2)$$

Suppose we wish to obtain the integral surface which passes through the curve whose eqn. is parametric form is given by

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad -(3)$$

where $t$ is a parameter. Then (2) may be expressed as $u[x(t), y(t), z(t)] = c_1$ $v[x(t), y(t), z(t)] = c_2$ —(4)

We eliminate $t$ from (4) and get a relation involving $c_1$ & $c_2$. Finally we replace $c_1$ & $c_2$ with the help of (2) and obtain the reqd. integral surface.

Ex Find the integral surface of the pde

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains $x+y=0$, $z=1$.

Solⁿ

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

$$\frac{x\,dx+y\,dy}{x^2z-y^2z} = \frac{dz}{(x^2-y^2)z}$$

$$x^2+y^2-2z = c_2$$

$yz\,dx + xz\,dy + xy\,dz = 0$

$\cancel{d(x,y,}$

$d(xyz) = 0$

$xyz = c_1$

$x=t,\quad y=-t,\quad z=1$

$-t^2 = c_1 \qquad 2t^2-2 = c_2$

$2c_1 + c_2 + 2 = 0$

$2xyz + x^2+y^2 - 2z+2 = 0$

Method 2

Let $Pp + Qq = R$ —(1) be the given eqn.

Let the Lagrange's eqns. give the following two independent integrals

$$u(x,y,z) = c_1 \qquad v(x,y,z) = c_2 \quad \text{—(2)}$$

Suppose we wish to obtain the integral surface passing through the curve which is determined by the two eqn. $\phi(x,y,z)=0$ and $\psi(x,y,z)=0$ —(3)
We eliminate $x, y, z$ from four eqns of (2) & (3) and obtain a relation between $c_1$ & $c_2$. Finally we replace $c_1$ by $u(x,y,z)$ and $c_2$ by $v(x,y,z)$ in that relation and obtain the desired integral surface.

Same problem by 2nd method

$$xyz = c_1 \qquad x^2+y^2-2z = c_2$$

$$x+y=0 \qquad z=1$$

$$xy = c_1 \qquad x^2+y^2-2 = c_2$$

$$\Downarrow$$

$$(x+y)^2 - 2xy - 2 = c_2$$

$$\Rightarrow \quad -2c_1 - 2 = c_2$$

$$\Rightarrow 2c_1 + c_2 + 2 = 0$$

$$2xyz + x^2+y^2 - 2z+2 = 0$$

Ex Find the eqn. of surface satisfying $4yzp+q+2y=0$ passing through $y^2+z^2=1$, $x+z=2$.

$$y^2+z^2+x+z-3=0$$

**Ex** Find the eqn. of the integral surface of $(x^2-yz)p + (y^2-zx)q = z^2-xy$ passing through $x=1,\ y=0$.

**Sol$^n$** Multipliers $1,-1,0,\quad 0,1,-1,\quad 1,1,1$

$x,y,z$

$$\frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{(y-z)(y+z+x)} \Rightarrow \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)}$$

$$\Rightarrow \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0 \qquad \frac{x-y}{y-z} = c_1$$

$$\text{each fraction} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\therefore \frac{xdx+ydy+zdz}{x+y+z} = dx+dy+dz$$

$$\Rightarrow 2(x+y+z)\,d(x+y+z) - (2xdx+2ydy+2zdz) = 0$$

$$(x+y+z)^2 - (x^2+y^2+z^2) = 2c_2$$

$$\Rightarrow xy+yz+zx = c_2$$

$$-\frac{1}{z} = c_1 \qquad z = c_2 \qquad -\frac{1}{z}\times z = c_1c_2$$

$$\Rightarrow c_1c_2 + 1 = 0$$

$$\Rightarrow \frac{x-y}{y-z}(xy+yz+zx) + 1 = 0$$

$$\Rightarrow (x-y)(xy+yz+zx) + y - z = 0$$

## Compatibility condition

Two 1st order PDEs are said to be compatible if they have a common solution.

$$f(x,y,z,p,q)=0 \quad \text{---}(1)$$
$$g(x,y,z,p,q)=0 \quad \text{---}(2)$$

## Lecture - 17

28.8.2017

### Theorem

The eqns. $f(x,y,z,p,q)=0$ and $g(x,y,z,p,q)=0$ are compatible on a domain $D$ if

(i) $J = \dfrac{\partial(f,g)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial q} \end{vmatrix} \neq 0$ on $D$ ---(3)

(ii) $p$ and $q$ can be explicitly solved from (1) and (2) as $p=\phi(x,y,z)$ and $q=\psi(x,y,z)$ ---(

Further the eqn. $dz=\phi(x,y,z)\,dx+\psi(x,y,z)\,dy$ is integrable.

### Theorem

A necessary and sufficient condition for the integrability of the eqn. $dz=\phi(x,y,z)\,dx+\psi(x,y,z)$ is

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q)} =$$

## A particular case

To show that first order PDEs $p = P(x,y)$ & $q = Q(x,y)$ are compatible if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

**Proof** $dz = p\,dx + q\,dy$

$= P(x,y)\,dx + Q(x,y)\,dy$

$P(x,y)\,dx + Q(x,y)\,dy$ is completely integrable.

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

**Ex** Test for compatibility

$\frac{\partial z}{\partial x} = 5x - 7y$, $\frac{\partial z}{\partial y} = 6x + 8y$

**Sol^n** $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ Not compatible.

**Ex** $\frac{\partial z}{\partial x} = (x+y)^2$ $\frac{\partial z}{\partial y} = x^2 + 2xy - y^2$

**Sol^n** $\frac{\partial P}{\partial y} = 2x + 2y = \frac{\partial Q}{\partial x}$

$\therefore$ Compatible.

$dz = (x^2 + 2xy + y^2)\,dx + (x^2 + 2xy - y^2)\,dy$

$= x^2 dx + (2xy\,dx + x^2 dy) + (y^2 dx + 2xy\,dy) - y^2 dy$

$z = \frac{x^3}{3} + x^2 y + xy^2 - \frac{y^3}{3} + c$