Mathematical Model of an Assignment Problem:

	J_1	J_2	J_3	J_4
M_1	c_{11}	c_{12}	c_{13}	c_{14}
M_2	c_{21}	c_{22}	c_{23}	c_{24}
M_3	c_{31}	c_{32}	c_{33}	c_{34}
M_4	c_{41}	c_{42}	c_{43}	c_{44}

Model:

$$\min: Z = \sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}\, x_{ij}$$

subject to $\sum_{j=1}^{4} x_{ij} = 1, \quad i = 1,2,3,4$

constraints $\begin{cases} \sum_{i=1}^{4} x_{ij} = 1, \quad j = 1,2,3,4 \end{cases}$

$x_{ij} = 0/1$ (Binary)

$i = 1,2,3,4$

$j = 1,2,3,4$

It has 24 Feasible solution.

One job is to be assigned to a Machine so as to minimizing ~~the~~ total Processing time.

If we subtract (or add) a constant (k) from a row or column of an assignment matrix the solution of the problem remain unchanged.

Now

$$\min: Z = \sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}x_{ij}$$

subject to all the constraints

$$\Rightarrow \min: Z' = \sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}x_{ij} \pm k$$

subject to all the constraints.

$$Z - k = \sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}x_{ij} - k$$

$$\Rightarrow Z' = \sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}x_{ij} - k(x_{11}+x_{12}+x_{13}+x_{14}) \quad = 1$$

$$= (c_{11}-k)x_{11} + (c_{12}-k)x_{12} + (c_{13}-k)x_{13} + (c_{14}-k)x_{14} + \sum_{i=2}^{4}\sum_{j=1}^{4} c_{ij}.x_{ij}$$

$$\min: Z' = c'_{11}x_{11} + c'_{12}x_{12} + c'_{13}x_{13} + c'_{14}x_{14} + \sum_{i=2}^{4}\sum_{j=1}^{4} c_{ij}x_{ij}$$

subject to all the constraints.

If we add add/subtract a constant(k) objective function value will be changed. However the solution (X^*) will not change.

Example 1:

	J_1	J_2	J_3	
M_1	2	9	4	$R_1 - 2$
M_2	7	5	3	$R_2 - 3$
M_3	6	1	8	$R_3 - 1$

$k_1 = 2, k_2 = 3, k_3 = 1$

$k_1 + k_2 + k_3 = 6$ units

	J_1	J_2	J_3
M_1	[0]	7	2
M_2	4	2	[0]
M_3	5	[0]	7

$k_1+k_2+k_3=6$

optimal

$M_1 \rightarrow J_1$

$M_2 \rightarrow J_3$

$M_3 \rightarrow J_2$

Every Row has a zero.

Every column has a zero.

Apply Hungarian Algorithm.

$x_{11}=1$, $x_{23}=1$, $x_{32}=1$

min: $Z=6$ units.

It has only one optimal solution. Let us consider some problem which has more optimal solution. See the next Example

Example 2:

	J_1	J_2	J_3	J_4	
M_1	2	4	2	4	$K_1 = 2$
M_2	4	2	4	2	$K_2 = 2$
M_3	3	6	3	6	$K_3 = 3$
M_4	6	3	6	3	$K_4 = 3$

Row 1 - 2
Row 2 - 2
Row 3 - 3
Row 4 - 3

	J_1	J_2	J_3	J_4
M_1	0	2	0	2
M_2	2	0	2	0
M_3	0	3	0	3
M_4	3	0	3	0

Every Row and every column has two zeros.

	J_1	J_2	J_3	J_4
M_1	[0]	2	~~0~~	2
M_2	2	[0]	2	~~0~~
M_3	~~0~~	3	[0]	3
M_4	3	~~0~~	3	[0]

$x_{11}=1$; $x_{22}=1$, $x_{33}=1$, $x_{44}=1$

$M_1 \rightarrow J_1$, $M_2 \rightarrow J_2$, $M_3 \rightarrow J_3$, $M_4 \rightarrow J_4$

min: $Z = 2+2+3+3 = 10$ units

	J_1	J_2	J_3	J_4
M_1	~~0~~	2	[0]	2
M_2	2	~~0~~	2	[0]
M_3	[0]	3	~~0~~	3
M_4	3	[0]	3	~~0~~

$M_1 \rightarrow J_3$, $M_2 \rightarrow J_4$, $M_3 \rightarrow J_1$, $M_4 \rightarrow J_2$

$x_{13}=1$, $x_{24}=1$, $x_{31}=1$, $x_{42}=1$

Alternate optimal Solution.

Example 3

	J_1	J_2	J_3	J_3	
M_1	4	2	2	2	Row 1 − 2
M_2	3	5	3	3	Row 2 − 3
M_3	1	1	6	1	Row 3 − 1
M_4	3	3	3	5	Row 4 − 3

	J_1	J_2	J_3	J_4	
M_1	2	[0]	~~0~~	~~0~~	$x_{12} = 1$
M_2	[0]	2	~~0~~	~~0~~	$x_{21} = 1$
M_3	~~0~~	~~0~~	5	[0]	$x_{34} = 1$
M_4	~~0~~	~~0~~	[0]	2	$x_{43} = 1$

Every raw/column has three zeros.

$M_1 \to J_2$, $M_2 \to J_1$, $M_3 \to J_4$, $M_4 \to J_3$

min: $Z = 9$ units

Example 3. has three optimal solution. Find the two other optimal solutions.

Example 4:

	J_1	J_2	J_3	J_4	J_5	
M_1	2	2	2	1	2	$R_1 - 1$
M_2	2	3	1	2	4	$R_2 - 1$
M_3	2	0	1	1	1	$R_3 - 0$
M_4	2	3	4	3	3	$R_4 - 2$
M_5	3	2	1	2	1	$R_5 - 1$

	1	2	3	4	5	
M_1	1	1	1	[0]	1	$M_1 \to J_4$
M_2	1	2	[0]	1	3	$M_2 \to J_3$
M_3	2	[0]	1	1	1	$M_3 \to J_2$
M_4	[0]	1	2	1	1	$M_4 \to J_1$
M_5	2	1	✗	1	[0]	$M_5 \to J_5$

$x_{14} = 1$
$x_{23} = 1$
$x_{32} = 1$
$x_{41} = 1$
$x_{55} = 1$

min: $z = 5$

Example 4:

	J_1	J_2	J_3	J_4	
M_1	5	8	7	7	$R_1 - 5$
M_2	8	6	6	8	$R_2 - 6$
M_3	7	8	11	10	$R_3 - 7$
M_4	7	6	4	4	$R_4 - 4$

	J_1	J_2	J_3	J_4
M_1	0	3	2	2
M_2	2	0	⊘	2
M_3	⊘	1	4	3
M_4	3	2	0	⊘

Minimum number of horizontal and vertical lines are drawn to cover all the zero elements

	J_1	J_2	J_3	J_4
M_1	0	3	2	2
M_2	2	0	0	2
M_3	0	1	4	3
M_4	3	2	0	0

	J_1	J_2	J_3	J_4
M_1	[0]	2	1	1
M_2	~~3~~	~~0~~	[0]	2
M_3	~~0~~	[0]	3	2
M_4	4	2	~~0~~	[0]

$x_{11}=1,\ x_{23}=1,\ x_{32}=1,\ x_{44}=1$

$Z = 5+6+8+4 = 23$ min

$M_1 \to J_1,\ M_2 \to J_3,\ M_3 \to J_2$

$M_4 \to J_4$