02/1/2020

DISCRETE MATHEMATICS

Before Midsem
- Graph Theory
- Basic Combinatorics

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Books:-
→ Introductory Graph Theory - Gary Chartrand
→ Discrete Mathematics - K. Rosen
→ Mathematics for Computer Science (MIT)

- Graphs: - Vertices, edges, cycle, connectedness, trees,
- Minimal Connector Problem (Algo)
  (MST).
- Counting Spanning trees
  - Prüfer Sequence
  - Cayley's formula
  - Matrix-Tree Theorem

- Bipartite Graphs, Matchings
- Eulerian Graphs, Hamiltonian Graphs
  (Ore's theorem)

- Planer Graphs

If we can redraw a graph in which no two edges are intersecting, are planer graphs.
for any planer Graph-
no. of vertices - no. of edges + no. of regions = 2

adjacent

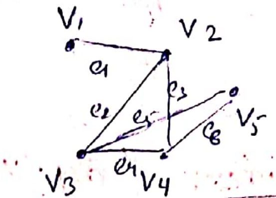
Graph Theory

Def: (simple Graph)

$G = (V, E)$ — vertex set, edge set

$V = \{v_1, v_2, \ldots, v_n\}$

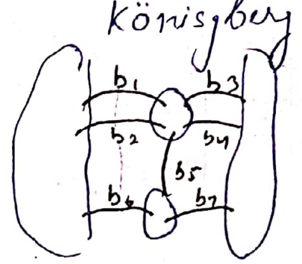
$E = \{e_1, e_2, \ldots, e_6\}$



loop

multigraph — can contain multiple edges b/w 2 vertices.

Euler lived in Königsberg



river Pregel

Starting from one edge, travel each Bridge only once.

(*) Perron-Frobenius Theorem

A graph G consists of a finite set V, whose members are called vertices of G, and a set E of 2-subsets of V, whose members are called edges.

(*) $|V|$ = Order of the graph G

$|E|$ = Size of the graph G.

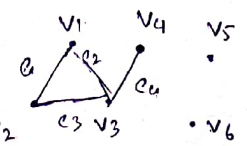
$$G = (V, E) = (V(G), E(G))$$

(*) If $e \in E(G)$, $e = v_i v_j$, then we say the edge e joins the vertices v_i & v_j.

(*) Two vertices v_i & v_j are adjacent in G, if there is edge b/w them, if $v_i v_j \in E(G)$.

(*) If $e = v_i v_j \in E(G)$, then two vertices v_i & v_j are incident with the edge e.

Example:



$V = \{v_1, v_2, v_3 \ldots v_6\}$

$E = \{e_1, e_2, e_3, e_4\}$

(*) v_1 & v_2, v_2 & v_3, v_1 & v_3, v_3 & v_4 are adjacent

* v_1 & v_4 are not adjacent.
* v_1 & v_2 is incident with e_1
  v_2 & v_3 " " e_2
  v_3 & v_1 " " e_3 &
  v_4 & v_3 " " e_4

Degree of a vertex: $(\deg_G(v))$: The degree of vertex v in a Graph G is the number of edges of G incident with the vertex v. Equivalently, no. of vertices which are adjacent to v in G.

In the prev. Graph, G.
$\deg_G(v_1) = 2$
$\deg_G(v_2) = 2$
$\deg_G(v_3) = 3$
$\deg_G(v_4) = 1$

Theorem (First theorem of Graph Theory):

For any Graph G, the sum of the degrees of the vertices of G equals to the twice of the no. of edges.

$V = \{v_1, v_2 \ldots v_n\}$ $E = \{e_1, e_2 \ldots, e_m\}$

$$\boxed{\sum_{i=1}^{n} \deg_G(v_i) = 2 \times m}$$ → Fundamental Th. of Graph Th.

Proof: When summing the degree of the vertices of a graph G, we count each edge of G twice.

Def: Odd Vertex: $\deg_G(v)$ is odd
Similarly, $v \in G$ is an even vertex if $\deg_G(v)$ is even

$V = \{v_1, v_2 \ldots v_n\}$
degree sequence $= \{\deg(v_1), \deg(v_2), \ldots \deg(v_n)\}$

Theorem: Every graph contains an even number of odd vertices.

Proof: Let $\{v_1, v_2 \ldots v_n\}$ be the vertex set of G & it contains K odd vertices.
Without loss of generality (wlog)

let us assume $v_1, v_2, v_3 \ldots v_K$ are odd vertices.
(wlog → we are relabelling $v_{i_1}, v_{i_2}, v_{i_3}, \ldots v_{i_K}$ as $v_1, v_2 - v_K$ such that there is no change)

$$\sum_{i=1}^{n} \deg(v_i) = 2m$$

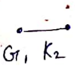
$$\sum_{i=1}^{K} \deg(v_i) + \sum_{i=K+1}^{n} \deg(v_i) = 2m$$

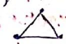
$\sum_{i=K+1}^{n} \deg(v_i)$ → even

$$\sum_{i=1}^{K} \deg(v_i) = \underbrace{2m - \sum_{i=K+1}^{n} \deg(v_i)}_{\text{even}}$$

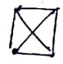
⇒ K is even number.

Def: Complete Graph: A graph is said to be a complete graph, if b/w any 2 vertices of G, there is an edge b/w them.


G_1, K_2


G_2 K_3

Notation for complete graph K_n n = no. of vertices

G_3

K_4

$\mathbb{N} \cup \{0\}$
$= \{1, 2 \ldots\}$

$\mathbb{Z}$
$= \{0, \pm 1, \pm 2 \ldots\}$

$A = \{x_1, x_2 \ldots x_n\}$, $B = \{y_1, y_2, \ldots y_m\}$.

$\phi : A \longrightarrow B$. ϕ → one-one onto.

2 infinite sets, have the same no. of elements if there is a bijection b/w them.

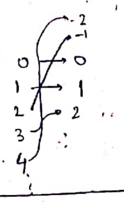
9/1/20 $G = (V, E)$, * Vertex set V; $|V|$ → order of G
* Edge set E, $|E|$ → size of G.
* $e = v_iv_j \in E(G)$ $i \neq j$
  → v_i & v_j are adjacent
  → e & v_i are incident.

Scanned with CamScanner

$1 \longrightarrow 1$
$2 \longrightarrow -1$
$3 \longrightarrow 2$
$4 \longrightarrow -2$

$\phi : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{Z}$

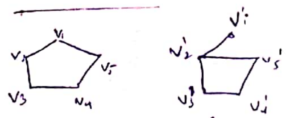
$$f(n) = \begin{cases} 0 & \\ -n/2 & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$



Ⓐ if $f: X \longrightarrow \{1, 2, 3, \dots n\}$.
- bijection $\mathbb{N} = n$

$g: X \longrightarrow \mathbb{N} \cup \{0\} \xrightarrow{f} \mathbb{Z}$
bijection, $|X| = |\mathbb{N} \cup \{0\}|$

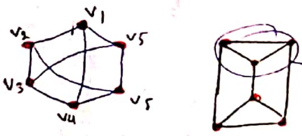
Isomorphism: let $\overset{(V_1, E_1)}{G_1}$ & $\overset{(V_2, E_2)}{G_2}$ be 2 graphs.
G_1 & G_2 are said to be isomorphic if
$\exists$ (there exists) $\phi: V_1 \longrightarrow V_2$ such that v_i & v_j are adjacent in G_1
↳ bijection iff $\phi(v_i)$ & $\phi(v_j)$ are adjacent in G_2



G_1 G_2

v_1' is adjacent to only one vertex in G_2. ∴ it's pre-image in G_1 should have any such vertex. But there is no such vertex G.

∴ $G_1, G_2 \rightarrow$ not isomorphic



G_1 G_2

There is no 3 mutually adjacent pairs in G_1,
So, $G_1, G_2 \rightarrow$ not isomorphic

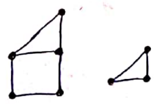
Theorem: If G_1 & G_2 are isomorphic, then the degrees of the vertices of G are exactly same as that of G_2 (up to permutation)
rearrangement

⊛ Converse need not be true i.e. if 2 graphs have same degree of vertices, it is not necessary that both are

Proof: :- let there be a vertex in G_1 & $\deg(v_i) = k$, & let WLOG, $v_1, v_2, v_3, \dots v_k$ are adjacent to v_i.
$\phi(v_i)$ is adjacent with $\phi(v_1), \dots \phi(v_k)$
& these are the only vertices adjacent to $\phi(v_i)$.
(because the preimage must also have equal no. of adj. vert)
∴ $\deg(\phi(v_i)) = \deg(v_i)$

Subgraph: $G = (V, E)$. H is a subgraph of G
if, $V(H) \subseteq V(G)$
$E(H) \subseteq E(G)$

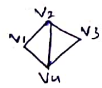
eg ①



G H

→ ~~Graph are~~ H is a subgraph of G.

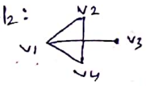
Ⓘ G:



H_1:

→ subgraph

H_2:



not a subgraph

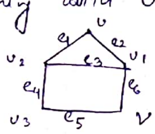
(it must be a labelled graph.)

$E(H_2) \nsubseteq E(G)$

Spanning Subgraph:
A subgraph H of G is said to be spanning if $V(H) = V(G)$.

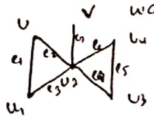
Walk: alternating sequence of vertices & edges of G beginning with u & ending at v.

~~Trial~~:
~~Path~~:



ex: $u\, e_1\, u_2\, e_4\, u_3\, e_5\, v\, e_6\, u_1\, e_6\, v$
(we can start from any & end at any vertex)

Trial: u-v ~~path~~ walk with no edge repitition. but vertex can repeat

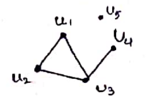


$u\, e_1\, u_1\, e_3\, u_2\, e_4\, u_3\, e_5\, u_4\, e_6\, u_2\, e_7\, v$

Path: u-v trial with no vertex repetition.

Connected Graphs:-

⊛ Two vertices u & v of a Graph are said to be connected if there is a u-v path existing in G.

eg:- 

u_1 & u_4 connected

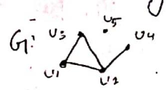
u_1 & u_5 → not connected

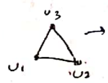
⊛ A graph G is said to be connected if for any 2 vertices u, & v of graph G, there exists a u-v path in G.

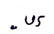
↳ ie→ any 2 vertices are connected.

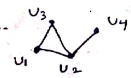
⊛ Disconnected = not connected.

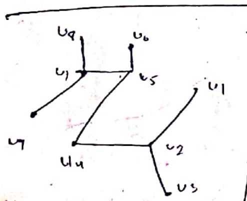
⊛ A connected subgraph H of G is said to be a component of G if H is not ~~containing~~ contained in any connected subgraph of G having more vertices or more edges than H.

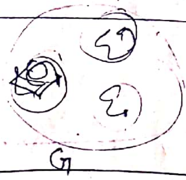
eg. G: 

H:  → not a component

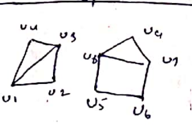
H:  → component

H:  → component

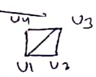




G



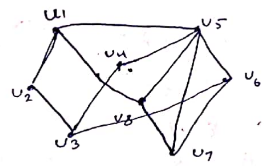
G

H: 

↳ component

Circuit: A u-v trial in which u=v ~~&~~ which (contains atleast 3 edges is called a) circuit

Cycle: A u-v path in which u=v is called a cycle.



$u_1 \to u_2 \to u_3 \to u_4 \to u_5 \to u_8$

$u_1 \leftarrow u_5 \leftarrow u_6 \leftarrow u_7$

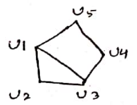
↳ Circuit

$u_1 \to u_2 \to u_3 \to u_4 \to u_5 \to u_1$

↳ cycle

$u_1 \to u_8 \to u_5 \to u_1$

↳ cycle.

⊛ 

$u_1 \to u_4$ Paths

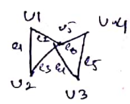
$u_1 \to u_5 \to u_4$

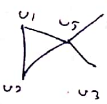
$u_1 \to u_3 \to u_4$

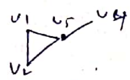
$u_1 \to u_2 \to u_3 \to u_4$

⊛ For an edge e of G, G−e is a subgraph of G having the same vertex set as that of G, & having all the edges of G other than e.

⊛ If v is a vertex of G, then G−v is the subgraph of G whose vertex set consists of all the vertices of G except v and the edge set (of G−v) consists of all the edges of G ~~except~~ except those incident with vertex v.

eg 

$G - e_5$ → 

$G - u_3$ → 

★ **Bridge:** It is an edge in a connected graph G, if $G-e$ is disconnected. (or, no. of components increases)

★ A vertex v of a connected graph G is called a cut vertex if $G-v$ is disconnected. (or increase the no. of components)

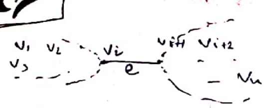
Remark:

ⓐ G is connected & e is bridge, then $G-e$ has exactly 2 components.

$G = G(V, E)$

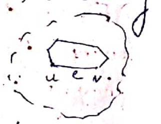
$V_1 = \{v_1, v_2, v_3, \dots v_i\}$

$V_2 = \{v_{i+1}, v_{i+2}, \dots v_n\}$



ⓑ G is connected & v is a cut vertex, then $G-v$ has at least 2 components.

Theorem: Let G be a connected graph. An edge e of G is a bridge iff e does not lie on any cycle.



Proof:

($\Longrightarrow$) e is a bridge of G.

we assume the 1st part & prove the 2nd

So $G-e$ should be disconnected G

u_1, v_1 be the vertices of $G-e$ so that u_1 & v_1 are not connected

Claim: e does not lie on any cycle

On contrary, if e lies in a cycle, say c; $uxy \cdots wvu$

then $u_1 \cdots uvw \cdots v \cdots v_1$ is a $u_1 v_1$ path in $G-e$.

This is a contradiction to the assumption.

Thus the proof.

For the converse:

edge e does not lie in a cycle.

Terrence Tao

Claim: e is a bridge

On contrary, if e is not a bridge. Then $G-e$ is connected. So, $\exists$ a u-v path in $G-e$, say $u v_1 v_2 \dots v_k v$. Now the cycle $u v_1 \dots v_k v u$ is produced in G. ie, e is in a cycle of G.

This is not possible.

$\therefore$ e must be a bridge.

15/1/2020 **Tree:** Any connected graph that has no cycles is called a tree.

Forest: Any graph that does not contain cycles is called a forest.

G - n-vertices
↳ at most ${}^nC_2$ edges

Theorem: If G is a tree of order p, and size q, then $q = p-1$.

proof: (Proof by Strong mathematical induction)

~~If we have a graph on at (n-1) edges.~~

When $p=1$, the proof is clear.

Assume that the result is true for all trees whose number of vertices is at most $(k-1)$ $(p=k-1)$.

Let G be a tree on k-vertices.

As G is a tree, so any edge e of G is a bridge in the graph.

So, the graph $(G-e)$ has exactly 2 components → as G_1 & G_2

Assume G_1 is of the order of p_1 & G_2 is of order p_2.

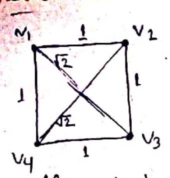
By M. induction size of G_1 is $p_1 - 1$. & size of G_2 is $p_2 - 1$

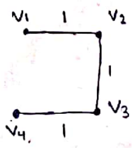
Size of G is: $(p_1 - 1) + (p_2 - 1) + 1$
$= p_1 + p_2 - 1 = p - 1$

ie, $\boxed{q = p-1}$

Minimal Connector Problem / Minimal Spanning Tree

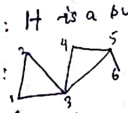
Problem





Subgraph : H is a subgraph of G, $V(G) \supseteq V(H)$ & $E(G) \supseteq E(H)$.

eg G:



$V(H) = \{1, 2, 3\}$
$E(H) = \{(12)\ (23)\}$.

Spanning Subgraph: $H \to$ spanning subgraph if $V(G) = V(H)$ & $E(H) \subseteq E(G)$.

Spanning Tree :- H – spanning tree
$V(H) = V(G)$, H is a tree $E(H) \subseteq E(G)$.

Problem: Find a spanning tree T of G with the sum of the weights of edges is minimum.

(*) Weight of tree T: $w(T) = \sum_{e \in E(T)} w(e)$.

Greedy Algorithm:- Construction of tree with minimum weight).

(*) Choose an edge e_1 of T from G so that $w(e_1)$ is minimum among all other edges of G.

(*) Choose the second edge e_2 of T from $G - \{e_1\}$, so that $w(e_2)$ is minimum among all the edges of $G - \{e_1\}$

(*) Choose the third edge e_3 of T from $G - \{e_1, e_2\}$ so that $w(e_3)$ is min. among $e \in G - \{e_1, e_2\}$. & $\{e_1, e_2, e_3\}$ does not produce a cycle in T.

We continue this process until we obtain a spanning tree T of G.

A spanning tree obtained using the process is called a minimal spanning tree.

Theorem: Let G be a connected graph with weighted edges & let T be a minimal spanning tree. Then T is a spanning tree whose weight is minimum.

Proof: Let G be a graph on n-vertices. Let T be a minimal spanning tree (tree which we obtained by greedy algo.)

Let $e_1, e_2, e_- \dots e_{n-1}$ be the edges of T.

Let T_0 be a spanning tree of G with minimum weight. If T_0 & T are identical, then we are done.

Otherwise $\exists$ an edge which is in T but not in T_0.

WLOG, let e_i be the first edge in T which is not in T_0. $(1 \leq i \leq n-1)$.

Add the edge e_i to T_0, call the new graph as G_0.

Suppose $e_i = uv$. Then the edge e_i in G produces a cycle C in G_0. So $\exists$ an edge e_0 in C (cycle), which is not in T.

consider the graph $T_0' = G_0 - e_0$.

It is clear that T_0' is a spanning tree.

$$w(T_0') = w(T_0) + w(e_i) - w(e_0)$$

$$0 \geq w(T_0) - w(T_0') = w(e_0) - w(e_i)$$

$$\Rightarrow w(e_0) \leq w(e_i)$$ (& since we had both e_i & e_0 to choose from & we chose e_i the $e_i <= e_0$)

as $\begin{pmatrix} e_i \in E(T) \\ e_0 \notin E(T \end{pmatrix} \Rightarrow w(e_0) = w(e_i)$

Note that the number of edges common to T_0' & T exceeds the no. of edges common to T_0 & T by one edge namely, e_i. By continuing this process, we finally arrive at a tree with minimum value which is identical to T.

Recap:

Minimal Spanning Tree (T) :- output of Greedy Algo

Spanning Tree T_0, $w(T_0)$ is min.

To prove: $w(T) = w(T_0)$

T_0 $\qquad$ T $e_1, e_2, \dots, e_i, e_{\dots}$ e_{n-1}

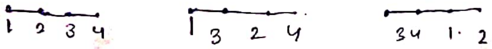
$T_0 \neq T$, $e_i \notin E(T_0)$ $\qquad$ $e_i \in E(T)$

in $T_0 \cup e_i$ $\qquad$ C – cycle induced by e_i

$e_0 \in C \subseteq T_0 \cup e_i$ such that $e_0 \notin E(T)$

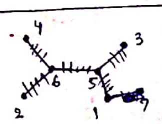
- $T_0' = (T_0 \cup e_i) - e_0$

Count labelled trees:- (Prüfer Sequence)



Labelled tree on n ⟶ $(a_1, a_2, \dots, a_{n-2})$. $a_i \in \{1, 2, \dots\}$

number of sequence - n^{n-2}



$\{1,2,3,4,5,6,7\}$

$a_1 = 6 \Rightarrow$ adjacent to b_1

$a_2 = 5$

$a_3 = 6$

$a_4 = 5$

$a_5 = 1$

$b_1 = 2$ → min. vertex of degree 1

$b_2 = 3$

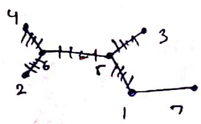
$b_3 = 4$

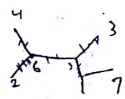
$b_4 = 6$

$b_5 = 5$

a tree should have atleast 2 pendent vertex.

v_i → pendent vertex = $\deg(v_i) = 1$.





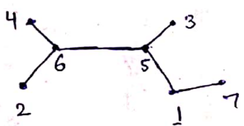
2	6
3	5
4	6
6	5
5	1

b_1	2	a_1	6
b_2	3	a_2	5
b_3	4	a_3	6
b_4	6	a_4	5
b_5	5	a_5	1

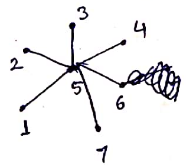
Reverse

• Prufer Seq (6, 5, 6, 5, 1) → tree on 7 vertices.

$\{1, 2, 3, 4, 5, 6, 7\}$ ⟶ smallest element which does not belong to trees uncut set in prufer sequence

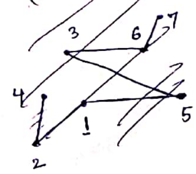


~~(7, 2, 7, 5, 7)~~ $\{1, 2, 3, 4, 5, 6, 7\}$

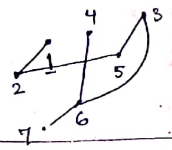


In the end, join 2 elements that are left

~~(1, 3, 6, 7, 3)~~ ~~{1, 2, 3, 4, 5, 6, 7}~~



(1, 3, 6, 2, 3) $\{1, 2, 3, 4, 5, 6, 7\}$



$b_1 = 1$

$b_2 = 2$

$b_3 = 4$

$b_4 = 5$

$b_5 =$

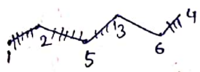
$a_1 = 2$

$a_2 = 5$

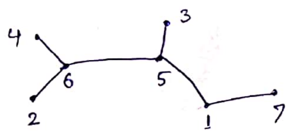
$a_3 = 6$

$a_4 = 3$

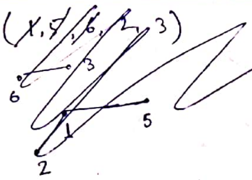
$a_5 =$



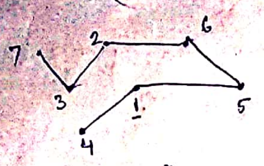
(6, 5, 6, 5, 1) $\{1, 2, 3, 4, 5, 6, 7\}$

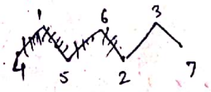


(1, 3, 6, 2, 3) ~~{1, 2, 3, 4, 5, 6, 7}~~



(~~1~~, ~~3~~, ~~6~~, ~~2~~, ~~5~~) {~~1~~, ~~2~~, 3, ~~4~~, ~~5~~, ~~6~~, 7}





$b_1 = 4$ $a_1 = 1$
$b_2 = 1$ $a_2 = 5$
$b_3 = 5$ $a_3 = 6$
$b_4 = 6$ $a_4 = 2$
$b_5 = 2$ $a_5 = 3$

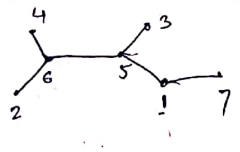
T_1 & T_2 ~~are~~ on n vertices are 2 trees with $(a_1, \ldots a_{n-2})$ as prufer sequence. then we have to prove that $T_1 = T_2$

"deg. of a vertex v in T = # no. of times v appears in the prufer's sequence +1

(~~max of~~) → {1, 2, 3, -- n}
occe... the number of   each element occurs 1.

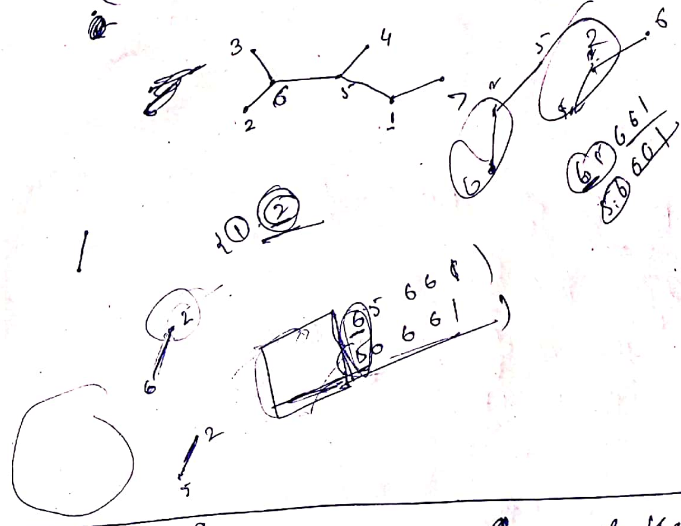
If $(a_1, a_2, \ldots a_{n-2})$ & $(b_1, b_2, \ldots b_{n-2})$ are two diff. prufer sequences, then the trees obtained T_1, & T_2 from $(a_1, a_2 \ldots a_{n-2})$ & $(b_1, \ldots b_{n-2})$ respectively are different.

(~~6~~, ~~5~~, ~~6~~, ~~5~~, 1) → {1, ~~2~~, ~~3~~, ~~4~~, ~~5~~, ~~6~~, 7}



~~(1, 5, 6, 5, 6)~~ ~~{1, 2, 3, 4, 5, 6, 7}~~

(~~6~~, ~~6~~, ~~5~~, ~~5~~, 1   {~~1~~, ~~2~~, ~~3~~, ~~4~~, ~~5~~, ~~6~~, 7}



Proof: case(i) If no. of occurences of one of the a_i's is different in $(a_1, a_2 \ldots a_{n-2})$ & $(b_1, b_2 \; b_{n-2})$. Then deg. of a_i is different in T_1 & T_2

case(ii) If $(a_1, a_2, \ldots a_{n-2})$ $(b_1, \ldots b_{n-2})$ are the same upto permutation. i.e, $a_i = b_{\sigma(i)}$ where $\sigma : \{1, \ldots n-2\} \to \{1 \ldots n-2\}$ σ is a bijection.
wlog, $a_i = b_i \quad \forall \; 1 \le i \le k-1$.
$a_k \neq b_k$.

$(\cancel{a_1}, \cancel{a_2}, \cancel{a_3} \ldots \cancel{a_{k-1}}\ a_k, \ldots a_{n-2}) \to \{1, 2, \ldots n\}$

$(\cancel{b_1}\ \cancel{b_2} \ldots \cancel{b_{k-1}}\ b_k, \ldots b_{n-2}) \to \{1, 2, \ldots n\}$

both are same upto $(k-1)$

$\{1, 2, \ldots n\}$

remaining from a → S_a

the β. b → S_b

never be in T_2 — a_s — a_k → T_1

as role of a_s is done we will not count a_s

never be in T_1 — b_k — a_s — T_2

Number of labelled spanning trees in K_n :-

Cayley's Formula:-

The no. of spanning trees in K_n is n^{n-2}.

$F_n = F_{n-1} + F_{n-2}$.

$F_0 = 0,\ F_1 = 1$.

$e \in E(G)$

$G - e$ → Just delete the edge e from G.

$G \setminus e$ → delete the edge e and merge the end vertices of e.

$\tau(G)$ → denotes the number of spanning trees in G.

Matrix-Tree-theorem

G - graph on n vertices $\{v_1, v_2, \ldots v_n\}$

① $A(G)$ - adjacency matrix
- $n \times n$ matrix + symmetric

$$A(G)_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

② Degree matrix

$$D(G)_{ij} = \begin{cases} \delta_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

③ Laplacian Matrix

$L(G) = D(G) - A(G)$

$L(G)$ → symmetric

(a) no. of spanning trees in G is det | minor wrt any entry of L.

→ determinant of minor

A symmetric → $A = A^T$

A is positive semidefinite if $x^T A x \geq 0\ \forall\ x$
↳ symmetric. x → n vector.

$A x = \lambda x \quad , x \neq 0$

$x^T A x = \lambda x^T x$

$\Rightarrow \lambda = \frac{x^T A x}{x^T x} \geq 0$

$L(G)$ is positive semidefinite

$\lambda_1 \leq \underbrace{\lambda_2}_{\text{(algebraic connectivity)}} \leq \cdots \leq \lambda_n$ eigen values of $L(G)$

$\lambda_1 = 0$ $\quad e = (1, 1, \ldots, 1)^T$

$L(G)\, e = 0 \cdot e$

$\therefore$ 0 is always an eigen value

$\lambda_2 \neq 0 \iff G$ is connected

$G \rightarrow$

$$L(G) = \begin{array}{cc} G_1 & G_2 \\ \left[\begin{array}{c|c} L(G_1) & 0 \\ \hline 0 & L(G_2) \end{array}\right] \end{array}$$

$L(G_1)\, e^1 = 0$

$L(G_2)\, e^2 = 0$

$\begin{pmatrix} e^1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ e^2 \end{pmatrix}$

$$L(G)\begin{pmatrix} e^1 \\ 0 \end{pmatrix} = L(G)\begin{pmatrix} 0 \\ e^2 \end{pmatrix} = 0$$

Vertex Connectivity:- $(\alpha(G))$

Minimum number of vertices whose removal makes the resultant graph disconnected.

1965. M. Fiedler.

$\alpha(G) \geq \lambda_2(G)$

Algebraic Connectivity of Graphs – paper

Spectral Graph Theory

non-pendent vertex → $deg \geq 2$

$\circledast$ Pendent vertex

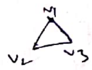
Bipartite Graph:-

If the vertex set of graph G can be split into two disjoint sets A & B so that each edge of G joins a vertex of A & a vertex of B, then G is called a Bipartite graph.

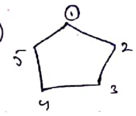
eg ①

$\{1, 3\}$
$\{2, 4\}$

② $\{v_1\}$, $\{v_2, v_3\}$ ✗

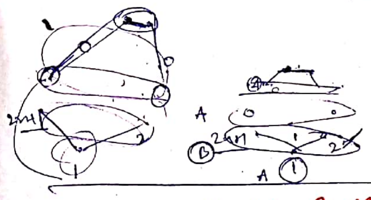


③



$A = \{1, 3, 4\}$ ✗ → so not possible

$B = \{2, 5\}$



Complete Bipartite Graph $K_{m,n}$.

Bipartite graph in which each vertex in A is joined to each vertex in B by an edge.

$K_{2,3}$

$K_{3,3}$

G – connected graph
v_i, v_j two vertices of G.
$d(v_i, v_j)$:= length of the shortest path

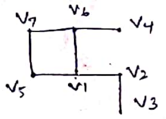
Theorem:- A graph G is bipartite iff G does not contain any odd cycle.

Proof: let G be bipartite.
Suppose that G contains an odd cycle.
Then, we can find sets A & B so that every edge of G has one end point in A & other end point in B, which is a contradiction.
So G does not contain any odd vertices.

Conversely:
Assume G does not contain any odd cycle
Claim:- G is bipartite
If G does not contain any odd cycle then → it will be a bipartite.

If G contains even cycles. Fix $v_i \in G$
$A = \{v_j : d(v_i, v_j) \text{ is even}\}$
$B = \{v_j : d(v_i, v_j) \text{ is odd}\}$

eg.



$A = \{v_7, v_4, v_3, v_1\}$

$B = \{v_5, v_6, v_2\}$

A, B gives a bipartition)
let u_i & u_j be in A such that
u_i is adjacent to u_j $(u_i \sim u_j)$
this will create an odd cycle.

v_i --- u_i u_j --- v_i
A A A A

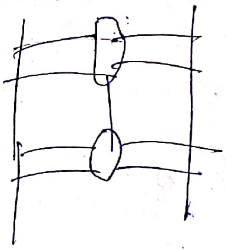
$(v_i — u_i\, u_j — v_i)$ – circuit
odd circuit

* Take any circuit of odd length
* It will always contain a cycle.

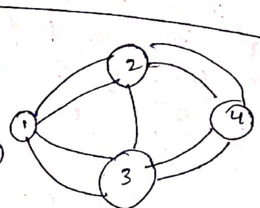
Exercise: Any closed walk of odd length contains an odd cycle.

By exercise, we can conclude G contains an odd cycle
thus it is not bip.

②



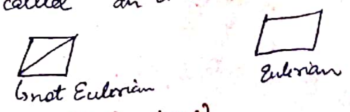
①



① for each vertex other than we need to have one edge to enter & one edge to go out.

Eulerian Graph:

A connected graph G is Eulerian if $\exists$ a closed trial containing every edge of G. Such a trial is called an Eulerian trial.



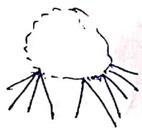
Theorem (Euler, 1736)

A connected graph is Eulerian iff the degree of each vertex of G is even.

Proof:-

($\Rightarrow$) If G is Eulerian, then there is a Eulerian trial P in G. Whenever P passes through a vertex, there is a contribution of 2 towards the degree of that vertex. As every edge appears exactly once in P, so degree of each vertex is even.

($\Leftarrow$) As degree of each vertex of G is even, so G does contain a cycle C. Consider $G-C$.



Exc— Complete the proof.

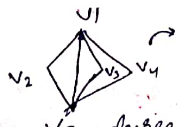
Hamiltonian graph:-

A Graph G is said to be Hamiltonian if $\exists$ a cycle in G containing all the vertices of G. Such cycle is called a Hamiltonian cycle.

Example, cycles, complete gr.

$v_1 v_3 v_4 v_5 v_2 v_1$

— H.G



→ not Hamiltonian

v_2, v_3, v_4 have degree 2 so the edges v_1v_2, v_3v_1, v_1v_4 must be in.

If there is a Hamiltonian cycle C.

As degree of each vertex in C is 2.

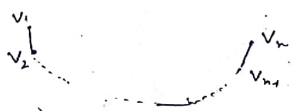
So C can't Hamiltonian.

Theorem:-(Ore's theorem)

If G is a simple graph on n vertices, and $deg(u)+deg(v) \geq n$ for each pair of non-adjacent vertices $(u \,\&\, v)$, then G is Hamiltonian.

Proof:- On contrary assume G is not hamiltonian. By adding extra edges if needed, we can assume, G is non-Hamiltonian, but addition of any edge to G produces a Hamiltonian cycle.

So, the graph contains a path $v_1, v_2, \cdots v_n$ which contains all the vertices of G.



As the vertices v_1 & v_n are not adjacent, by hypothesis.

$deg(v_1) + deg(v_n) \geq n$

$A = \{v_i : 2 \leq i \leq n, v_i \sim v_1\}$

$B = \{v_i : 2 \leq i \leq n, v_{i-1} \sim v_n\}$

$|A| = dg(v_1)$

$|B| = dg(v_n)$

$\exists$ a common element in $A \& B$, i.e.

i.e. $\exists$ $v_i \in G$, such that $v_i \in A \cap B$.

i.e. $v_i \sim v_1$

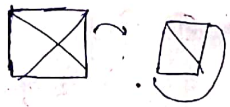
$v_{i-1} \sim v_n$

Cor (Dirac, 1952)

If G is a simple graph with n vertices & if $dg(v) \geq n/2$ for each vertex of G, then G is Hamiltonian.

Planer Graph:

Def. A planer graph is a graph that can be drawn in the plane in such a way that no two edges intersect except at a vertex.



Region:- Let G be a connected plane graph & consider the parts of the plane remaining after we remove the edges & vertices of G.

Theorem (Euler's Theorem)

Let G be a connected planer graph with p vertices, q edges & r regions. Then $p - q + r = 2$

> A planer graph is said to be a plane if it is already drawn in the plane such that no two edges are crossing.

Proof: By mathematical induction on q

*) $q = 0$, $p = 1$, $r = 1$.

$p - q + r = 1 - 0 + 1 = 2$

Assume the result is true for all graphs with $(k-1)$ edges.

(i) G be a graph with K edges.

(*) If G is tree, then (one region)

$p - (p-1) + 1 = 2$

(*) If G is not a tree. $\exists$ an edge in G which is in any cycle,

For $G - e$, by induction,

$p - (q-1) + (r-1) = 2$

$\Rightarrow p - q + r = 2$

Theorem:- Let G be a connected planer graph with p vertices & q edges, where $p \geq 3$. Then

$q \leq 3p - 6$

Proof:

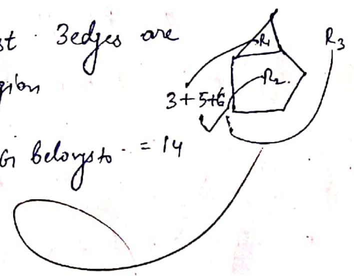
$p = 3$, $q \leq 3$.

if $p \geq 4$.

N - Sum of the number of edges lying in the boundary of each region

As there are atleast 3 edges are there in each region

$N \geq 3r$.



As each node of G belongs to at most 2 regions.

$N \leq 2q$.

$3r \leq N \leq 2q$.

$3r \leq 2q$

By Euler's formula for plane graph

$(p - q + r = 2)$

$r = 2 - p + q)$

$3(2 - p + q) \leq 2q$

$6 - 3p + 3q \leq 2q$

$\boxed{q \leq 3p - 6}$

$K_5 \to p - 5$

complete graph on 5 vertices

$q = 10$. $3p - 6 = 9$.

$(q \leq 9)$ for planer

$K_{3,3}$

$p = 6$

$q = 9$

$q = 3(6) - 6$
$= 12$

$q \le 12$

↳ But it is not planar.

thus it is necessary but not sufficient

So, if it is a bipartite graph

there are atleast 4 edges in each region

$N \ge 4r$

$r = 2 - p + q$

$4r \le 2q$

$2r \le q$

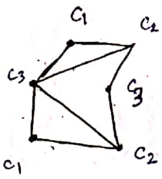
$2(2 - p + q) \le q$

$4 - 2p + 2q \le q$

$q \le 2p - 4$

$2p - 4 = 12 - 4$
$= 8$

Theorem: K_5 & $K_{3,3}$ are non planar graphs



→ minimum no. of colours required to colour G so that no two adjacent vertices receives the same colour.

Chromatic number: ($\chi(G)$)

For a graph G.

$\chi(G) \le 1 + \Delta(G)$ | $\Delta(G)$ - maximum degree of G.

— 2 colors needed

Five Colour Theorem

Four Colour Theorem

05/2/20

Combinatorics

① Suppose there are 10 seats in a row & 2 students have to sit in that row. In how many ways they can sit.

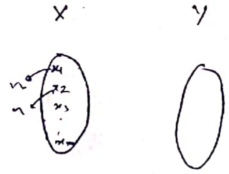
$\overline{S_1}\ \overline{S_2}\ -\ \cdots\ -\ \overline{S_{10}}$

$\rightarrow 10\cdot 9 = 90$

(Product rule)

② $|X| = m$ $|Y| = n$

n^m no. of functi^n $X \to Y$

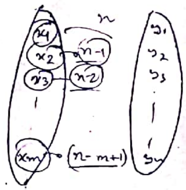


one-one functions – a function s.t. $x \neq y \Rightarrow f(x) \neq f(y)$.

③ No. of one-one functions from the set X to the set Y.

no. of ~ $= n\cdot(n-1)(n-2)\cdots(n-m+1)$

$= {}^nP_m = \dfrac{n!}{(n-m)!}$



④ If X → set with m elements. Then no. of subsets of A

2^m

$X = \{x_1, x_2, x_3, \cdots, x_n\}$

in subset / not in subset ②

Ex. $S = \{(y_1, y_2, \cdots y_n) : y_i = 0 \text{ or } 1\}$

no. of elements $= 2^n$

$f : S \to P(X)$, which is a bijection

$P(X)$ ← Power set of X (set of all subsets).

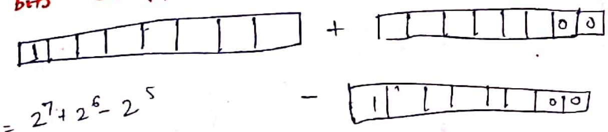
Principle of Inclusion & Exclusion.

Suppose that a task can be done in n_1 or n_2 ways.

$|A \cup B| = |A| + |B| - |A \cap B|$

but some of the n_1 ways to do the task are the same as that of some of (n_2) other ways of doing the task. Then the number of ways of doing the task = add the no. of ways to do it in one way, & the no. of ways - 2nd way and subtract the number of ways to do the task in both ways, among n_1 & n_2.

① How many bit strings of length 8 are there so that it starts with a bit 1. or end with two bits 00?.



$= 2^7 + 2^6 - 2^5$

$= 160$

Pigeon-Hole Principle:-

(Dirichlet Drawer principle)

3 Cages
4 pigeons

If k is a +ve integer and (k+1) or more objects are placed into k boxes, then atleast one box containing two or more of the objects.

→ then minimum 1 cage should have more than 1 pigeon

① Show that for every +ve integer n, there is a multiple of n that has only 0's and 1's in its decimal expansion.

Proof: 1, 11, 111, 1111, . . . 11111 ← (n+1) terms

divide each of them by n

Possible remainders are 0 to (n-1)

if 0 is a remainder → then we are done.

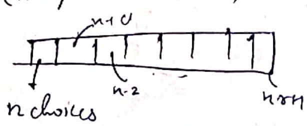
otherwise, by pigeon-hole principle, there exists 2 numbers from the above which have same remainder.

Permutation:- A permutation of a set of distinct object's is an ordered arrangement of these objects. An ordered arrangement of r objects is called r-permutation.

$P(n, r)$ - number of r-permutations of a set with n distinct elements.

$$P(n,r) = n \times (n-1) \times (n-2) \cdots (n-r+1)$$

$$= \frac{n!}{(n-r)!}$$



Combinations:- An r-combination of elements of a set in an unordered selection of r objects is just a subset of S with r-elements.

Notation $\binom{n}{r} = C(n, r)$ = # r-combinations of a set with n-distinct elements.

Theorem: $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$

Corollary:- $C(n,r) = C(n, n-r)$.

Defn: A combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same object but in different ways.

Theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j, \quad n \geq 0$$

Proof:

$$(x+y)^n = (x+y)(x+y)(\cdot) \cdots (x+y) \quad \text{(*)}$$

Coeff of $x^{n-j}y^j$ $(j = 0, 1, 2, \ldots n)$

in ^nC_j ways we can choose x^{n-j} from (*)

Prob: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Sol: $\binom{n}{k}$ → # of k-element subset of an n-element set (k-combination).

So $\sum_{k=0}^{n} \binom{n}{k}$ = # of subsets of an n-element subset

$= 2^n$

Cor: 1) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ (choose $x=1$, $y=-1$ in there)

2) $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$ [choose $x=2, y=1$ in theo].

Pascal's Identity: For $n > 0$, $n \geq k$.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof: Fix an element 'a' in the $(n+1)$ element set.

$\binom{n+1}{k}$ = # of k-element subset of the $(n+1)$ element subsets containing the element a.

+ # of k-element subset of the $(n+1)$ elements not containing a.

$= \binom{n}{k-1} + \binom{n}{k}$ — discarding a, we have to choose k element from n elements

Vander monde's Identity:

$m, n \geq 0$, $r \leq \min\{m, n\}$.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Choose 0 elements from m & r elements from k

+ " 1 " & (r-1) " " "

+ " 2 - - -

+ " r " & 0 ... "

$\binom{m}{r-k}\binom{n}{k}$ = # no. of ways of choosing $(r-k)$ elements from m elements & k elements from n elements

$$\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k} = \binom{m+n}{r}$$

Corrollary :- $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

Combinations with Repetitions :-

Notes in a box: 10, 20, 50, 100, 200, 500, 2000,
In how many ways can we choose 5 notes from this box?

10	20	50	100	200	500	2000

*| * || * || * | * → 10 20 100 500 200

↳ two bars means skipping

Q - no. of ways of placing 5 stars & 6 bars together.

$= {}^{(5+(7-1))}C_5$ (fix 5 places for ⊛ from 11 places)

$= {}^{11}C_6 = {}^{11}C_5$

Theorem: The no. of r-combinations from a set with n elements when repetition of elements is allowed is $C(n+r-1, r) = C(n+r-1, n-1)$.

Proof:

1	2	...	n

n - boxes for n elements
r - objects, say ⊛
The no. of ways of placing r ⊛'s & (n-1) boxes $= C(n-1+r, r) = C(n+r-1, n-1)$

Explaination :

Let us represent a choice a r-combination with repetitions using ⊛ & |'s.
First draw (n-1) bars corresponding to the boundaries of the n-boxes (as given earlier) & draw r no. of ⊛'s between these (n-1) bars (including before the 1st bar & after the last bar).
It is easy to see that the configuration given by (n-1) bars & r ⊛'s gives a r-combination of n elements set with repetitions, vice versa.

1) How many sol^n does the eq^n $x_1+x_2+x_3 = 11$ have where x_1, x_2 & x_3 are non ~~neve~~ integers

$C(11+3-1, 2) = C(13, 2)$
$= {}^{13}C_2 = {}^{13}C_{11}$

| | | | | | | | | | | |

- | |

Put 2 |'s in 13 places

→ same problem for $x_1+x_2+x_3 \le 11$

Explainaction :

Let us represent a choice a r- combination with repetitions using ⊛ & |'s

First draw (n-1) bars corresponding to the boundaries of the n- boxes (as given earlier) & draw r no. of ⊛'s between these (n-1) bars (including before the 1st bar & after the last bar),

It is easy to see that the configuration given by (n-1) bars & r ⊛'s gives a r-combination of n elements set with repetitions, Vice versa.

1) How many sol^n does the eq^n $x_1+x_2+x_3=11$ have where x_1, x_2 & x_3 are non -ve integers

$C(11+3-1, 2) = C(13, 2)$

$= {}^{13}C_2 = {}^{13}C_{11}$

| | | | | | | | | | |

| |

Put 2 1's in 13 places

→ same problem for $x_1+x_2+x_3 \le 11$

Recurrence Relation:-

Def^n: A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses 'a_n' in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$. for all integers n with $n \ge n_0$, where n_0 is a +ve integer.

Solution: A sequence is called a solution of a recurrence relation if its terms satisfies the

eg. Fibonacci sequence

$F_n = F_{n-1} + F_{n-2}$

$F_0 = 1, F_1 = 1$

Recurrence relation.

$a_n = F(a_0, a_1, a_2, \dots, a_{n-1})$

Ex:

① The number of bacteria in a colony doubles every hour. If a colony begins with 5 bacteria, how many will be there in n hours.

Sol:- $a_n = 2a_{n-1}$ (a_n - no. of bacterias in n hours.)

$= 2\cdot 2\cdot a_{n-2} = 2\cdot 2\cdot 2\cdot a_{n-3}$

$= 2^n a_{n-n} = 2^n \cdot a_0 = 2^n(5)$

Re Relation :- $a_n = 2a_{n-1}$, $a_0 = 5$

Sol :- $\boxed{a_n = 2^n \cdot 5}$

② Determine whether the sequence $\{a_n\}$ where $a_n = 3n$, for every non negative integer n, is a solⁿ of $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \dots$

Answer the same question where take $a_n = 2^n$ (or) $a_n = 5$

Ans.

① $2a_{n-1} - a_{n-2} = 2(3(n-1)) - (3(n-2))$

$= 6n - 6 - 3n + 6$

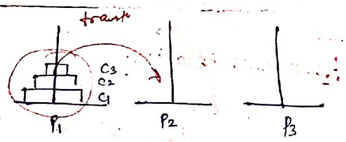
$= 3n = a_n$

② (2^n) $2 \times 2^{n-1} - 2^{n-2} = 2^{n-2}(4-1) = 3\cdot 2^{n-2} \neq a_n$

③ (5) $2\times 5 - 5 = 5 = a_n$

Tower of Hanoi :

H_n = min. no. of steps required to transfer



$H_n = 2H_{n-1} + 1$

$H_1 = 1$, $\boxed{H_n = 2^n - 1}$

① Take $(n-1)$ & put in $P_3 \to H_{n-1}$

② Take C_1 & put in P_2 — 1

③ Put $(n-1)$ from P_3 to P_2 — H_{n-1}

Defⁿ: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$

where, $c_1, c_2, \dots c_k$ are real numbers & $c_k \neq 0$

Def: A linear homogenous recurrence relation of degree K with constant coefficients given by

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$;

where $c_1, c_2, \dots c_k$ are real numbers & $c_k \neq 0$

Initial Value Problem (IVP):

A sequence satisfying the recurrence relation is uniquely determined by

① this recurrence relation, and

② the K initial conditions, $a_0 = C_0$, $a_1 = C_1, \dots a_{k-1} = C_{k-1}$

If $a_n = r^n$ is a solⁿ of ①

then, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ — ①

iff $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$ (take r^{n-k} from both sides)

ie. $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$ — ②

↳ characteristic Equation.

$a = r^n$ is a solⁿ of ① iff r solves ②

Let us consider linear homogeneous recurrence relations of degree two.

Theorem: Let r_1 & r_2 be the real distinct roots of $r^2 - c_1 r - c_2 = 0$

Then the sequence $\{a_n\}$ is a solⁿ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, $n = 0, 1, 2, \dots$ where α_1 & α_2 are constants

Proof: $c_1 a_{n-1} + c_2 a_{n-2} = c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2})$

$= \alpha_1(c_1 r_1^{n-1} + c_2 r_1^{n-2}) + \alpha_2(c_1 r_2^{n-1} + c_2 r_2^{n-2})$

$= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)$

$\because r^2 = c_1 r + c_2$

$= \alpha_1 r_1^{n-2} \cdot r_1^2 + \alpha_2 r_2^{n-2} \cdot r_2^2$

$= \alpha_1 r_1^n + \alpha_2 r_2^n = a_n$

Let $\{a_n\}$ be a solⁿ of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $a_0 = C_0$, $a_1 = C_1$

By previous theorem, the above prob. has unique solⁿ.

$$d_0 = \alpha_1 + \alpha_2$$
$$d_1 = \alpha_1 r_1 + \alpha_2 r_2$$

$$d_0 r_1 = \alpha_1 r_1 + \alpha_2 r_1$$
$$- \; d_1 = \alpha_1 r_1 + \alpha_2 r_2$$
$$d_0 r_1 - d_1 = \alpha_2 (r_1 - r_2)$$

$$\alpha_2 = \frac{d_0 r_1 - d_1}{(r_1 - r_2)} \qquad \alpha_1 = \frac{d_0 r_2 - d_1}{(r_2 - r_1)}$$

Solve the recurrence relation $a_n = c_1 a_{n-1} + 2a_{n-2}$

$a_0 = 2, \; a_1 = 7.$

Ans.

$c_1 = 1, \; c_2 = c_2$

$r^2 - r - 2 = 0$

$r = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = -1, 2.$

$r_1 = -1, \; r_2 = 2 \qquad d_0 = 2, \; d_1 = 7.$

$\alpha_1 = \frac{2.(-1) - 7}{-1-2} = \frac{-9}{-3} = 3$

$\alpha_2 = \frac{2(2) - 7}{-2+1} = \frac{-3}{3} = -1$

$a_n = 3 \cdot 2^n - 1 \cdot (-1)^n$

Fibonacci: $\quad F_n = F_{n-1} + F_{n-2} \qquad a_0 = 0, \; a_1 = 1$
$=d_0 \qquad = d_1$

$c_1 = c_2 = 1 \qquad r^2 - r - 1 = 0 \qquad r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Theorem. Let c_1 & c_2 be real numbers with $c_2 \neq 0$. Suppose $r_0^2 - c_1 r - c_2 = 0$ has only one root. Then, a sequence $\{a_n\}$ is a solⁿ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff. $a_n = \alpha_1 r_0^n + \alpha_2 n \cdot r_0^n$ for $n = 0, 1, 2 \cdots,$ α_1 & α_2 are const.

*) Solve: $a_n = 2a_{n-1} - a_{n-2}$

Ans. $a_n = c_1 a_{n-1} + c_2 a_{n-2} \qquad c_1 = 2, \; c_2 = -1$

$r^2 - 2r + 1 = 0 \qquad r_0 = r_0 = 1$

$$\boxed{a_n = \alpha_1 + n\alpha_2} \qquad a_0 = 0, \; a_1 = 3$$

$a_0 = \alpha_1 \cdot 0 = 0$

$a_1 = \alpha_1 + \alpha_2 = 3$

$\alpha_2 = 3$

$$\boxed{a_n = 3n}$$

1/12/20

Linear Nonhomogeneous Relation with constant coefficients:

*) $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n).$ where c_i's are constants & $F(n)$ is a function of n alone.

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is called the associated homogeneous recurrence relation.

Examples: ① $a_n = a_{n-1} + 2^n$

② $a_n = a_{n-1} + a_{n-2} + n^2 + 1$

③ $a_n = a_{n-3} + a_{n-2} + n!$

Theorem: First solve of Homogeneous part + particular solution } non homogeneous recurrence relation.

If $\{a_n^{(p)}\}$ is a particular solⁿ of the non-homogeneous linear recurrence relation with constant coefficients.

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$

then, every solⁿ is of the form $\{a_n^{(h)} + a_n^{(p)}\}$, where $\{a_n^{(h)}\}$ is a solⁿ of the associated homogeneous recurrence relation, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$.

Proof: Substitute $\{a_n^{(h)}, a_n^{(p)}\}$ in the given eqⁿ.

If $\{a_n^{(h)}\}$ solves the homogeneous part & $\{a_n^{(p)}\}$ is a particular solⁿ (p.s). then $\{a_n^{(p)} + a_n^{(h)}\}$ solves the associated non homogen rec. relation

Proof: $a_n^{(p)} = a_{n-1}^{(p)} \cdot c_1 + c_2 a_{n-2}^{(p)} + \dots + c_k a_{n-k}^{(p)} + F(n)$

Let $\{b_n\}$ be any sol^n of the nonhomogeneous recurrence relation. Claim: $b_n = a_n^{(h)} + a_n^{(p)}$

Consider $\{b_n - a_n^{(p)}\}$, then $b_n - c_1 b_{n-1} - c_2 b_{n-2} - \dots - c_k b_{n-k} = F(n)$

$- \; (a_n^{(p)} - c_1 a_{n-1}^{(p)} - c_2 a_{n-2}^{(p)} - \dots - c_k a_{n-k}^{(p)} = F(n))$

$(b_n - a_n^{(p)}) - c_1(b_{n-1} - a_{n-1}^{(p)}) - \dots - c_k(b_{n-k} - a_{n-k}^{(p)}) = 0$

∴ $b_n - a_n^{(p)} = a_n^{(h)}$ solves the homogeneous part.

i.e. $\boxed{b_n = a_n^{(h)} + a_n^{(p)}}$

---

① **Find all solutions of the recurrence relation** $a_n = 3a_{n-1} + 2n$, what is the sol^n with $a_1 = 3$.

Sol: $a_n^{(h)} = 3^n$

Assume $a_n^{(p)} = cn + d$

$cn + d = 3(c(n-1) + d) + 2n$

$cn + d = 3cn - 3c + 3d + 2n$

$2cn - 3c + 2d + 2n = 0$

$n(2c+2) + (2d - 3c) = 0$

$c = -1$. $\quad 2d - 3c = 0$

$d = \frac{3c}{2} = -3/2$

∴ $a_n^{(p)} = -n - \frac{3}{2}$

$a_n = a_n^{(h)} + a_n^{(p)}$

$\boxed{a_n = \alpha 3^n - n - \frac{3}{2}}$

$a_1 = 3$

$a_1 = \alpha \cdot 3 - 1 - \frac{3}{2} = 3$

$\alpha \cdot 3 = \frac{8+3}{2}$

$\alpha = \frac{11}{6}$

∴ $\boxed{a_n = \frac{11}{6} 3^n - n - \frac{3}{2}}$

② Find all the sol^n of the recurrence relation.
$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Sol^n: $a_n = a_n^{(h)} + a_n^{(p)}$.

$a_n^{(h)} = \alpha_1 3^n + \alpha_2 2^n$ $\quad | \quad r^2 - 5r + 6 = 0$

For the non homogeneous part

$F(n) = 7^n$ ∴ Guess sol^n is $c \cdot 7^n$.

$c 7^n = 5 \cdot c 7^{n-1} - 6 c 7^{n-2} + 7^n$

$c \cdot 7^2 = c \cdot 5(7) - 6c + 7^2$

$49c = 35c - 6c + 49$

$20c = 49$

$c = 49/20$

Theorem: (Guessing the form of particular solution). Suppose that $\{a_n\}$ satisfies the linear ...

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$.

where $c_1, c_2, c_3, \dots, c_k$ are constants & $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n$, where $b_0, b_1, b_2, \dots, b_t$ & s are reals.

Case ① When s is not a root of the characteristic equation of the associated linear homogeneous rec. relation, then there is a particular sol^n of the form $(p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$

Case ② When s is a root of the characteristic equation of the associated linear homogeneous recurrence relation with multiplicity m, there is a particular sol^n of the form $n^m (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$.

---

① Compute a particular sol^n of $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ when

① $F(n) = 3^n \longrightarrow (p_0) 3^n \cdot n^2 \to p_0 = 1/2$

② $F(n) = n \cdot 3^n \longrightarrow (p_0 + p_1 n) 3^n \cdot n^2$

③ $F(n) = n^2 2^n \longrightarrow (p_0 + p_1 n + p_2 n^2) 2^n$

④ $F(n) = (n^2+1) 3^n \to (p_0 + p_1 n + p_2 n^2) \cdot 3^n \cdot n^2$

$a_n^{(h)} = \alpha_1 3^n + \alpha_2 n 3^n$ $\quad | \quad r^2 - 6r + 9$, $r = 3, 3$.

$(p_0 + p_1 n) \cdot 3^n \cdot n^2 = 6 \cdot (p_0 + p_1(n-1)) \cdot 3^{n-1} (n-1)^2 - 9 \cdot (p_0 + p_1(n-2)) \cdot 3^{n-2} (n-2)^2 + n \cdot 3^n$

$9 \cdot n^2 (p_0 + p_1 n) = 6 \cdot (p_0 + p_1 n - p_1) \cdot 3 \cdot (n-1)^2 - 9 \cdot (p_0 + p_1 n - 2p_1)(n-2)^2 + 9n$

Generating Function:-

The Generating function for the sequence $a_0, a_1, \dots a_n \dots$ of reals numbers is the infinite series $G(x) = \sum_{k=0}^{s} a_k x^k$

$= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

$|x| < 1$

(1) $\{1, 1, 1, \dots\}$ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

(2) $\{1, 2, 3, \dots\}$ $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

(3) $(1, 1, 1, 1, 1, 1) \rightarrow$ Generating f^n - $1 + x + x^2 + x^3 + x^4 + x^5$

$= \frac{1-x^6}{1-x}$

(4) $1, a, a^2, \dots$ - G.F. $\frac{1}{1-ax}$, $|x| < \frac{1}{|a|}$

Remark: When we use the generating functions (take them for granted) to solve counting problems, we treat them as a formal power series, we do not consider about convergence issues.

Facts: $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$

(a) $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) \cdot x^k$.

(b) $f(x) \cdot g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$

(c) $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$

Using generating functions to solve recurrence relations:

(1) Solve the recurrence relation $a_k = 3a_{k-1}$, (*) initial condition $a_0 = 2$

Sol: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be a sol' of (*)

$x\,G(x) = \sum_{k=1}^{\infty} a_{k-1} x^k$

$G(x) - 3x\,G(x) = \sum_{k=0}^{\infty} a_k x^k - 3 \sum_{k=1}^{\infty} a_{k-1} x^k$

$= a_0 + \sum_{k=1}^{\infty} \overbrace{\cancel{(a_k - 3a_{k-1})}}^{0} x^k$, $\because (a_k = 3a_{k-1})$

$= 2$

$\boxed{G(x) = \frac{2}{1-3x}} = 2(1 + (3x) + (3x)^2 + (3x)^3 + \dots)$

$a_k = 2 \cdot 3^k$

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(1) Solve the r.r. $a_n = 8a_{n-1} + 10^{n-1}$ initial condition $a_0 = 1$, $a_1 = 9$.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be a sol' of (*).

$\sum_{n=0}^{\infty} a_n x^n = 8 \sum_{n=1}^{\infty} a_{n-1} \cdot x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n$

$G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n$

$= \sum_{n=1}^{\infty} (8a_{n-1} x^n + 10^{n-1} x^n)$

$= 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1} x^{n-1}$

$G(x) - 1 = 8x \cdot G(x) + x \cdot \frac{1}{1-10x}$

$\Rightarrow G(x)(1-8x) = \frac{1-10x+x}{1-10x}$

$G(x) = \frac{1-9x}{(1-8x)(1-10x)} = \frac{1}{2}\left[\frac{1}{1-8x} + \frac{1}{1-10x}\right]$.

$G(x) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n) x^n$

$\Rightarrow a_n = \frac{1}{2}(8^n + 10^n)$

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Enumerative Combinatorics Vol. I & Vol. II