

Duality Theory:

Example - 1

(Primal) max: $Z = c_1 x_1 + c_2 x_2 + c_3 x_3$

(P) subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq b_2$$

$$x_1, x_2, x_3 \geq 0$$

Dual: (D) min: $Z' = b_1 y_1 + b_2 y_2$

st to

$$a_{11}y_1 + a_{21}y_2 \geq c_1$$

$$a_{12}y_1 + a_{22}y_2 \geq c_2$$

$$a_{13}y_1 + a_{23}y_2 \geq c_3$$

$$y_1, y_2 \geq 0$$

Apply Lagrange Multiplier Method
to find the Dual LPP (D)

min: $Z' \geq \max: Z$ (Weak)

min: $Z' = \max: Z$ (Strong
Duality)

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Example: 2Primal (P)

$$\text{max: } z = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$$\text{s.t. } a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$

$$x_1, x_2, x_3 \geq 0$$

Dual (D) min: $z' = b_1 y_1 + b_2 y_2$

$$\text{s.t. } a_{11} y_1 + a_{21} y_2 \geq c_1$$

$$a_{12} y_1 + a_{22} y_2 \geq c_2$$

$$a_{13} y_1 + a_{23} y_2 \geq c_3$$

y_1, y_2 are free

Primal LP can be written as:

$$\text{max: } z = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$$\text{s.t. } a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \leq b_1$$

$$-a_{11} x_1 - a_{12} x_2 - a_{13} x_3 \leq -b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \leq b_2$$

$$-a_{21} x_1 - a_{22} x_2 - a_{23} x_3 \leq -b_2$$

$$x_1, x_2, x_3 \geq 0$$

We introduce four dual variables:

$y'_1, y'_2, y'_3, y'_4 \geq 0$ for constraints

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Now the Dual can be written as:

$$(D) \min: z' = b_1 y'_1 - b_1 y'_2 + b_2 y'_3 - b_2 y'_4$$

$$\text{s.t } a_{11} y'_1 - a_{11} y'_2 + a_{21} y'_3 - a_{21} y'_4 \geq c_1$$

$$a_{12} y'_1 - a_{12} y'_2 + a_{22} y'_3 - a_{22} y'_4 \geq c_2$$

$$a_{13} y'_1 - a_{13} y'_2 + a_{23} y'_3 - a_{23} y'_4 \geq c_3$$

$$\text{where } y'_1, y'_2, y'_3, y'_4 \geq 0$$

$$\text{Let } y_1 = y'_1 - y'_2 \Rightarrow y_1 \text{ is free}$$

$$y_2 = y'_3 - y'_4 \Rightarrow y_2 \text{ is free}$$

$$\text{where } y'_1, y'_2, y'_3, y'_4 \geq 0$$

$$\min: z' = b_1(y'_1 - y'_2) + b_2(y'_3 - y'_4)$$

$$\text{s.t } a_{11}(y'_1 - y'_2) + a_{21}(y'_3 - y'_4) \geq c_1$$

$$a_{12}(y'_1 - y'_2) + a_{22}(y'_3 - y'_4) \geq c_2$$

$$a_{13}(y'_1 - y'_2) + a_{23}(y'_3 - y'_4) \geq c_3$$

$$y'_1, y'_2, y'_3, y'_4 \geq 0$$

$$\Rightarrow \min: z' = b_1 y_1 + b_2 y_2$$

$$\text{s.t } a_{11} y_1 + a_{21} y_2 \geq c_1$$

$$a_{12} y_1 + a_{22} y_2 \geq c_2$$

$$a_{13} y_1 + a_{23} y_2 \geq c_3$$

y_1, y_2
are
free

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Primal : LPP (P)Example 3: min: $Z = c_1 x_1 + c_2 x_2 + c_3 x_3$

s.t. $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$

$x_1, x_2, x_3 \geq 0$

Dual: (LPP)(D) max: $Z' = b_1 y_1 + b_2 y_2$

s.t. $a_{11} y_1 + a_{21} y_2 \leq c_1$

$a_{12} y_1 + a_{22} y_2 \leq c_2$

$a_{13} y_1 + a_{23} y_2 \leq c_3$

where y_1, y_2 are free

use: $y_1 = y'_1 - y'_2$

$y_2 = y'_3 - y'_4$

where $y'_1, y'_2, y'_3, y'_4 \geq 0$

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Primal : LPP (P)Example 3: min: $Z = c_1 x_1 + c_2 x_2 + c_3 x_3$

s.t. $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$

$x_1, x_2, x_3 \geq 0$

Dual: (LPP)(D) max: $Z' = b_1 y_1 + b_2 y_2$

s.t. $a_{11} y_1 + a_{21} y_2 \leq c_1$

$a_{12} y_1 + a_{22} y_2 \leq c_2$

$a_{13} y_1 + a_{23} y_2 \leq c_3$

where y_1, y_2 are free

use: $y_1 = y'_1 - y'_2$

$y_2 = y'_3 - y'_4$

where $y'_1, y'_2, y'_3, y'_4 \geq 0$

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Example 5:
Primal LPP
(P)

$$\text{max: } Z = c_1x_1 + c_2x_2 + c_3x_3 + d$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \leq b_3$$

x_1, x_2, x_3 are free

Dual Program :

$$(D) \quad \text{min: } Z' = b_1y_1 + b_2y_2 + b_3y_3 + d$$

Subject to

$$a_{11}y_1 + a_{21}y_2 + a_{31}y_3 = c_1$$

$$a_{12}y_1 + a_{22}y_2 + a_{32}y_3 = c_2$$

$$a_{13}y_1 + a_{23}y_2 + a_{33}y_3 = c_3$$

$y_1, y_2, y_3 \geq 0$

to be contd....