Ex: ℓ^p, $1 \leq p \leq \infty$ is a Banach space.

Sol: for $p = \infty$, ℓ^∞ is a Banach space, already proved.

So let $1 \leq p < \infty$.

Let $\{x_n\}$ be a Cauchy sequence in ℓ^p and ~~let~~ let $\epsilon > 0$ be given.

We show that, there exists $x \in \ell^p$ $\ni$ $\|x_n - x\|_p \to 0$ as $n \to \infty$

Since $\{x_n\}$ is a Cauchy sequence there exists $n_0 \in \mathbb{N}$ $\ni$

$$\|x_n - x_m\|_p < \epsilon, \ \forall\, n, m \geq n_0.$$

In particular, for each $k \in \mathbb{N}$, we have

$$|x_n(j)-x_m(j)|^p \le \sum_{j=1}^{k} |x_m(j)-x_n(j)|^p \le \|x_n - x_m\|_p^p < \epsilon^p, \quad \forall\, n, m \ge n_0$$

$\Rightarrow$ For each $j \in \mathbb{N}$, $\{x_n(j)\}$ is a Cauchy sequence in the field $\mathbb{K}$.

Since $\mathbb{K}$ is complete, there exists $\alpha_j \in \mathbb{K}$ $\ni$ $x_n(j) \longrightarrow \alpha_j$

i.e., $|x_n(j) - \alpha_j| \longrightarrow 0$ as $n \to \infty$.

Define $x(j) = \alpha_j$, $j \in \mathbb{N}$.

Then

$$\sum_{j=1}^{k} |x_n(j) - x(j)|^p = \lim_{m\to\infty} \sum_{j=1}^{k} |x_n(j) - x_m(j)|^p$$

$$< \|x_n - x_m\|^p < \epsilon^p$$

$$\Rightarrow \sum_{j=1}^{k} |x_n(j) - x(j)|^p < \epsilon^p, \quad \forall\, n \geq n_0.$$

Since this is true for all $k \in \mathbb{N}$, we have letting $k \to \infty$,

$$\sum_{j=1}^{\infty} |x_n(j) - x(j)|^p < \epsilon^p, \quad \forall\, n \geq n_0$$

$$\Rightarrow \|x_n - x\|_p < \epsilon, \quad \forall\, n \geq n_0$$

Also $\|x\|_p \leq \underbrace{\|x - x_n\|_p}_{\to 0} + \underbrace{\|x_n\|_p}_{< \infty}$

$$\Rightarrow x \in \ell^p, \quad 1 \leq p < \infty.$$

$\therefore$ ℓ^p, $1 \leq p \leq \infty$ is a Banach space

$$[\, x \in \ell^p, \quad x = (x(1), x(2), x(3), \dots) \,]$$

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x(j)|^p \right)^{1/p} < \infty$$

$$\|x_n - x_m\|_p < \epsilon$$

$$\Rightarrow \sum_{j=1}^{\infty} |x_n(j) - x_m(j)|^p < \epsilon^p$$

$$\underline{|x_n(j) - x_m(j)|^p} < \boxed{\sum_{j=1}^{k} |x_n(j) - x_m(j)|^p \le \sum_{j=1}^{\infty} |x_n(j) - x_m(j)|^p < \epsilon^p}$$

$$\Rightarrow |x_n(j) - x_m(j)| < \epsilon$$

$\{x_n(j)\}$ is Cauchy in K.

$$x_n(j) \longrightarrow \alpha = x_0(j)$$

Fix n,

$$\sum_{j=1}^{k} |x_n(j) - x(j)|^p = \lim_{m\to\infty} \sum_{j=1}^{k} |x_n(j) - x_m(j)|^p$$

$$\leq \lim_{m\to\infty} \sum_{j=1}^{\infty} |x_n(j) - x_m(j)|^p$$

$$= \lim_{m\to\infty} \|x_n - x_m\|_p^p$$

$$\leq \lim_{m\to\infty} \epsilon^p$$

$$= \epsilon^p$$

$$\sum_{j=1}^{k} |x_n(j) - x(j)|^p \leq \epsilon^p, \quad \forall\, n \geq n_0$$

letting $k \to \infty$,

$$\sum_{j=1}^{\infty} |x_n(j) - x(j)|^p < \lim_{k\to\infty} \epsilon^p = \epsilon^p$$

$$\Rightarrow \|x_n - x\|_p^p < \epsilon^p$$

$$\Rightarrow \|x_n - x\|_p < \epsilon, \quad \forall\, n \geq n_0$$

$\because \|x\|_p \le \underbrace{\|x - x_n\|_p}_{< \epsilon} + \underbrace{\|x_n\|_p}_{< \infty}$

$< \infty$

$\Longrightarrow x \in \ell^p.$

$\therefore \ell^p, 1 \le p \le \infty$ is a Banach space.

Problem:

(i) The space

$c = \{ x = (x(1), x(2), \dots \dots) / x$ is cgt in $\ell^\infty \}$

is a closed subspace of ℓ^∞.

(ii) $c_0 = \{x = (x(1), x(2), \dots) \mid x(n) \to 0 \text{ as } n \to \infty\}$

is a closed subspace of l^∞.

These are closed subspaces of l^∞; But l^∞ is a Banach space.

$\Rightarrow$ c_0, c both are Banach space.

* c_{00} is not a ~~not~~ a closed subspace of l^∞

$\because x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, 0, 0, \dots) \in c_{00}$

and $x_n \to x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{n}, \frac{1}{n+1}, \dots)$

$\notin C_{00}$

$\therefore$ ~~C_{00}~~ C_{00} is not a closed subspace of a Banach space ℓ^∞.

$\therefore$ C_{00} is not a Banach space.

$$\Big[\, C_{00} = \{ x = (x(1), x(2), \ldots x(n) \ldots) \in \ell^\infty \,/\, x \text{ containing only finitely non-zero terms} \}$$

$$x = (x(1), x(2) \ldots\ldots) \in C_{00}$$

$$= (\underbrace{x(1), x(2) \ldots x(n)}_{\text{finite}}, 0, 0 \ldots 0)$$

Ex:

$C^1[a,b] = \{ x \in C[a,b] \,/\, x$ is differentiable and x' is continuous on [a,b]}

is the space of all K-valued continuous functions on [a,b].

$C^1[a,b]$ contains all polynomials.

$\therefore$ $C^1[a,b]$ is ~~d~~ dense in $C[a,b]$

[$\because$ ~~$x \in C[a,b]$~~ $x \in C[a,b]$

$\exists$ { p_n } of polynomials

$\ni$ $\|p_n - x\|_\infty \to 0$]

Also $C^1[a,b] \neq C[a,b]$

$\Rightarrow$ $C^1[a,b]$ is not closed in $C[a,b]$.

Thus $C^1[a,b]$ is not a closed subspace of a

Banach space $(C[a,b], \|\cdot\|_\infty)$.

$\Rightarrow$ $C^1[a,b]$ is not Banach space w.r.t $\|\cdot\|_\infty$.

Now for $x \in C^1[a,b]$, let

$$\|x\|_{1,\infty} = \max\{\|x\|_\infty, \|x'\|_\infty\}$$

Then clearly $(C^1[a,b], \|\cdot\|_{1,\infty})$ is a n.l.s.

Claim $(C^1[a,b], \|\cdot\|_{1,\infty})$ is a Banach space.

Let $\{x_n\}$ be a Cauchy sequence in $C^1[a,b]$. Then given any $\epsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ $\ni$

$$\|x_n - x_m\|_{1,\infty} < \epsilon, \quad \forall\, n, m \geq n_0$$

$\Rightarrow \max\{ \|x_n - x_m\|_\infty , \|x_n' - x_m'\|_\infty \}$

$\Rightarrow \|x_n - x_m\|_\infty < \epsilon , \|x_n' - x_m'\|_\infty \leq \epsilon < \epsilon$

$\Rightarrow \{x_n\}, \{x_n'\}$ both are Cauchy sequence in $(C[a,b], \|\cdot\|_\infty)$.

But $(C[a,b], \|\cdot\|_\infty)$ is a Banach space.

$\therefore \exists\ x, y \in C[a,b]$ such that $\underline{\|x_n - x\|_\infty \to 0}$ and $\underline{\|x_n' - y\|_\infty} \to 0$ as $n \to \infty$.

By a well known result (Rudin, Real analysis, theorem 7.17), that x is differentiable and $\quad x' = y$.

[$\therefore$ $x_n \to x$, $\quad x_n' \to y$ uniformly

$\Rightarrow$ x is dble, $\quad x' = y$]

Then

$$\|x_n - x\|_{1,\infty} = \max\{\|x_n - x\|_\infty, \|x_n' - x'\|_\infty\} \longrightarrow 0$$

$\Rightarrow$ $x_n \to x \in C^1[a,b]$.

$\therefore$ $(C^1[a,b], \|\cdot\|_{1,\infty})$ is a Banach space.

[$\|x\|_{1,\infty} = \max\{\|x\|_\infty, \|x'\|_\infty\}$

$$= \max\left\{\sup_{t\in[a,b]} |x(t)|, \sup_{t\in[a,b]} |x'(t)|\right\}$$