Inner product Spaces

Let X be a linear space over the field $\mathbb{K}$.
An inner product on X is a function
$\langle \cdot , \cdot \rangle : X \times X \longrightarrow \mathbb{K}$ such that
for all $x, y, z \in X$ and $k \in \mathbb{K}$,
we have

(i) positive-definiteness
$$\langle x, x \rangle \geq 0 , \ \forall x \in X$$
and $\langle x, x \rangle = 0 \iff x = 0$.

(ii) Linearity in the first variable.
$$\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$
$$\langle kx, y \rangle = k \langle x, y \rangle$$

(iii) Conjugate Symmetry:
$$\langle y, x \rangle = \overline{\langle x, y \rangle} .$$

A linear space X with an inner product
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an it is called an Inner product space (I.P.S) and it is denoted by
$(X, \langle \cdot , \cdot \rangle)$.

Note :- An inner product is Conjugate linear in the second variable

i.e.,

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

and

$$\langle x, ky \rangle = \bar{k} \langle x, y \rangle .$$

Ex : let $X = K^n$

For $x = (x(1), x(2), \ldots x(n))$
$y = (y(1), y(2), \ldots y(n))$ $\Big\} \in X,$

define

$$\langle x, y \rangle = \sum_{j=1}^{n} x(j) \overline{y(j)} .$$

Then

$\langle \cdot , \cdot \rangle$ is an inner product on X.

(verify all the properties).

Lemma. Let $\langle \cdot,\cdot \rangle$ be an inner product on a linear space X.

(a) Polarization Identity:

For all $x, y \in X$,

$$4\langle x,y\rangle = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle.$$

(b) Let $x \in X$, then $\langle x, y\rangle = 0 \ \forall y \in X \iff x = 0$.

(c) Schwarz-Inequality:

For all $x, y \in X$,

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle \langle y,y\rangle,$$

where equality holds iff $\{x, y\}$ is L.D.

Proof :(a) For all $x, y \in X$,

$$\langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle$$
$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle.$$

$\therefore$

$$\langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \quad \text{—— (1)}$$

lly

$$\langle x-y, x-y\rangle = \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \quad \text{—— (2)}$$

(1) − (2)

$$\langle x+y, x+y\rangle - \langle x-y, x-y\rangle$$

$$= 2\langle x,y\rangle + 2\langle y,x\rangle \quad \text{—— (3)}$$

Replace y by iy and multiply with i to the Eq. (3), we get

$$i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle$$

$$= 2i\langle x, iy\rangle + 2i\langle iy, x\rangle$$

$$= 2i\bar{i}\langle x, y\rangle + 2ii\langle y, x\rangle$$

$$= 2\langle x, y\rangle - 2\langle y, x\rangle \quad -(4)$$

Adding (3) & (4) we get

$$\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle = 4\langle x, y\rangle.$$

b) If $x = 0$, then

$$0 + \langle 0, y\rangle = \langle 0+0, y\rangle$$

$$= \langle 0, y\rangle + \langle 0, y\rangle.$$

$$\Rightarrow \langle 0, y\rangle = 0, \quad \forall y \in X.$$

Conversely, let $\langle x, y\rangle = 0$, $\forall y \in X$.

Then in particular for $y = x$, we get

$\langle x, x\rangle = 0 \Rightarrow x = 0$, by positive-definiteness.

(c) Let $x, y \in X$ and consider

$$z = \langle y, y\rangle x - \langle x, y\rangle y.$$

Then

$$0 \le \langle z, z\rangle$$

$$= \big\langle \langle y,y\rangle x - \langle x,y\rangle y \,,\; \langle y,y\rangle x - \langle x,y\rangle y \big\rangle$$

$$= \langle y,y\rangle \big\langle x \,,\; \langle y,y\rangle x - \langle x,y\rangle y \big\rangle$$
$$- \langle x,y\rangle \big\langle y \,,\; \langle y,y\rangle x - \langle x,y\rangle y \big\rangle$$

$$= \langle y,y\rangle \left[\overline{\langle y,y\rangle}\, \langle x,x\rangle - \overline{\langle x,y\rangle}\, \langle x,y\rangle \right]$$
$$- \langle x,y\rangle \left[\overline{\langle y,y\rangle}\, \langle y,x\rangle - \overline{\langle x,y\rangle}\, \langle y,y\rangle \right]$$

$$= \langle y,y\rangle \overline{\langle y,y\rangle} \langle x,x\rangle - \langle y,y\rangle \overline{\langle x,y\rangle} \langle x,y\rangle$$
$$- \langle x,y\rangle \cancel{\overline{\langle y,y\rangle}} \langle y,x\rangle + \langle x,y\rangle \cancel{\overline{\langle x,y\rangle}} \langle y,y\rangle$$

$$= \langle y,y\rangle \left[\langle x,x\rangle \langle y,y\rangle - \overline{\langle x,y\rangle}\, \langle x,y\rangle \right]$$

$$= \langle y,y\rangle \left[\langle x,x\rangle \langle y,y\rangle - |\langle x,y\rangle|^2 \right] \quad \text{—(1)}$$

So if $\langle y,y\rangle > 0$, then

$$\langle x,x\rangle \langle y,y\rangle - |\langle x,y\rangle|^2 \geq 0$$

$$\Rightarrow |\langle x,y\rangle|^2 \leq \langle x,x\rangle \langle y,y\rangle.$$

If $\langle y,y\rangle = 0 \Rightarrow y = 0$

$$\Rightarrow \langle x,y\rangle = 0$$

$$\therefore |\langle x,y\rangle|^2 = 0 = \langle x,x\rangle \langle y,y\rangle.$$

Now, let $|\langle x,y\rangle|^2 = \langle x,x\rangle \langle y,y\rangle$,

then from (1), $\langle z,z\rangle = 0$

$$\Rightarrow z = 0$$

$$\Rightarrow \langle y,y\rangle x - \langle x,y\rangle y = 0$$

$$\Rightarrow x = \frac{\langle x,y\rangle}{\langle y,y\rangle} y$$

$\therefore \{x,y\}$ is L.D.

Conversely, let $\{x,y\}$ is L.D.

then $y = kx$, $k \in \mathbb{K}$.

Then

$$|\langle x, y\rangle|^2 = \langle x, y\rangle \overline{\langle x, y\rangle}$$
$$= \langle x, kx\rangle \overline{\langle x, kx\rangle}$$
$$= \bar{k} \langle x, x\rangle \cdot \overline{\bar{k} \langle x, x\rangle}$$
$$= |k|^2 |\langle x, x\rangle|^2$$

and

$$\langle x, x\rangle \langle y, y\rangle = \langle x, x\rangle \langle kx, kx\rangle$$
$$= |k|^2 |\langle x, x\rangle|^2 .$$

$$\therefore \quad |\langle x, y\rangle|^2 = \langle x, x\rangle \langle y, y\rangle .$$

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Theorem: let $\langle \cdot, \cdot\rangle$ be an inner product on a linear space X. For $x \in X$, define $\|x\| = \langle x, x\rangle^{1/2}$, the nonnegative square root of $\langle x, x\rangle$. Then

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|, \ \forall x, y \in X.$$

The function $\|\cdot\| : X \longrightarrow \mathbb{K}$ is a norm on X, i.e., for all

$x, y \in X$ and $k \in K$, we have

$\|x\| \geq 0$ & $\|x\| = 0 \Leftrightarrow x = 0$

$\|x+y\| \leq \|x\| + \|y\|$

$\|kx\| = |k| \, \|x\|$.

Also the following holds.

(a) if $\|x_n - x\| \to 0$ & $\|y_n - y\| \to 0$

then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

(b) (Parallelogram law) For all $x, y \in X$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right].$$

Proof: Let $x, y \in X$.

Then by Schwarz-Inequality, we have

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

$$= \|x\|^2 \|y\|^2$$

$$\Rightarrow |\langle x, y\rangle| \leq \|x\| \|y\|.$$

Also

$$\|x\| = \sqrt{\langle x, x\rangle} \geq 0.$$

$$\|x\| = 0 \iff \sqrt{\langle x, x\rangle} = 0$$

$$\iff \langle x, x\rangle = 0$$

$$\iff x = 0$$ (by definition of $\langle \cdot, \cdot \rangle$).

Also

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$$

$$= \|x\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2$$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|, \ \forall x, y \in X.$

Finally for $k \in K$,

$$\|kx\|^2 = \langle kx, kx\rangle = k\bar{k}\langle x, x\rangle$$
$$= |k|^2 \langle x, x\rangle$$
$$= |k|^2 \|x\|^2$$

$\Rightarrow \|kx\| = |k|\,\|x\|$

$\therefore \|\cdot\|$ is a norm on an inner product space X.

(9) let $\|x_n - x\| \to 0$ & $\|y_n - y\| \to 0$

Thus

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle$$

$$+ \langle x_n, y\rangle - \langle x, y\rangle|$$

$$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle|$$

$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$

$$\leq \underbrace{\underbrace{\|x_n\|}_{<\infty} \underbrace{\|y_n - y\|}_{\to 0}}_{\to 0} + \underbrace{\|x_n - x\|}_{\to 0} \underbrace{\|y\|}_{<\infty}$$

(by Schwarz inequality).

$[\because \|x_n\| = \|x_n - x + x\|$

$$\leq \underbrace{\|x_n - x\|}_{\to 0} + \underbrace{\|x\|}_{<\infty}$$

$$< \infty$$

$\|y_n\| < \infty$

$\therefore \langle x_n, y_n\rangle \to \langle x, y\rangle$.

(b) Let $x, y \in X$, then

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$

$$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$$

$$+ \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2$$

$$= 2\left[\|x\|^2 + \|y\|^2\right].$$

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