Theorem: Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Then the following are equivalent.

(i) $\{u_\alpha\}$ is an orthonormal basis for H.

(ii) (Fourier expansion): For every $x \in H$, we have $x = \sum_n \langle x, u_n \rangle u_n$, where $\{u_1, u_2, \dots\} = \{u_\alpha \mid \langle x, u_\alpha \rangle \neq 0\}$

(iii) (Parseval's formula): for every $x \in H$, we have

$$\|x\|^2 = \sum_n |\langle x, u_n \rangle|^2.$$

where $\{u_1, u_2, \dots\} = \{u_\alpha \mid \langle x, u_\alpha \rangle \neq 0\}$

(iv) $\text{Span}\{u_\alpha\}$ is dense in H

(v) If $x \in H$ and $\langle x, u_\alpha \rangle = 0, \forall \alpha$

..

Then $x = 0$.

Proof: (i) $\Rightarrow$ (ii)

Let $\{u_\alpha\}$ be an orthonormal basis for H. Let $x \in H$.

Then by previous lemma (last class)

$\sum_n \langle x, u_n \rangle u_n$ converges to some

$y \in H$ and $x - y \perp u_\alpha, \forall \alpha$.

If $y \neq x$, then $u = \dfrac{y-x}{\|y-x\|}$.

Then $\|u\| = 1$ and $u \perp \{u_\alpha\}$

$\Rightarrow \{u\} \cup \{u_\alpha\}$ is an orthonormal set containing $\{u_\alpha\}$ in H, contradicting the maximality of $\{u_\alpha\}$.

$$\therefore\ x = y = \sum_n \langle x, u_n\rangle u_n .$$

(ii) $\Rightarrow$ (iii)

for any $x \in H$, we have

by (ii) $\quad x = \sum_n \langle x, u_n\rangle u_n$

Then

$$\|x\|^2 = \langle x, x\rangle$$

$$= \left\langle \sum_n \langle x, u_n\rangle u_n ,\ \sum_m \langle x, u_m\rangle u_m \right\rangle$$

$$= \sum_n \langle x, u_n\rangle \sum_m \overline{\langle x, u_m\rangle}\ \underbrace{\langle u_n, u_m\rangle}_{\delta_{nm}}$$

$$= \sum_n |\langle x, u_n\rangle|^2 .$$

(ii) $\Rightarrow$ (iv)

For any $x \in H$, we have

$$x = \sum_n \langle x, u_n\rangle u_n .$$

$\because$ For each $m = 1, 2, 3, \cdots$, let

$$x_m = \sum_{n=1}^{m} \langle x, u_n \rangle u_n \in \operatorname{Span}\{u_\alpha\}$$

$$\longrightarrow \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n = x$$

$x_m \in \operatorname{Span}\{u_\alpha\}$ and $x_m \to x$.

$$\therefore \overline{\operatorname{Span}\{u_\alpha\}} = H.$$

(iv) $\Rightarrow$ (v)

given that $\overline{\operatorname{Span}\{u_\alpha\}} = H$.

Let $x \in H$ be such that

$$\langle x, u_\alpha \rangle = 0 \quad \forall \alpha.$$

and let

$$x_m \to x, \quad x_m \in \operatorname{Span}\{u_\alpha\}.$$

$$\because \langle x, u_\alpha \rangle = 0 \ \forall \alpha$$

$$\Rightarrow \langle x, x_m \rangle = 0, \ x_m \in \operatorname{Span}\{u_\alpha\}$$

$\therefore \quad 0 = \langle x, x_n \rangle \longrightarrow \langle x, x \rangle$

$\Rightarrow \quad \|x\|^2 = 0$

$\Rightarrow \quad x = 0.$

(v) $\Rightarrow$ (i)

Given that $\langle x, u_\alpha \rangle = 0 \ \forall \alpha$

$\Rightarrow x = 0.$

Claim: $\{u_\alpha\}$ is an orthonormal basis for H.

Let E be an orthonormal set in H containing $\{u_\alpha\}$.

If $u \in E$ and $u \neq u_\alpha \ \forall \alpha$,

then $\langle u, u_\alpha \rangle = 0 \ \forall \alpha$

$\Rightarrow \quad u = 0$ (by (v))

But $u \in E \Rightarrow \|u\| = 1$, this

Contradiction shows that

$E = \{u_\alpha\}$.

$\therefore$ $\{u_\alpha\}$ is a maximal orthonormal set in H. That is $\{u_\alpha\}$ is an orthonormal basis for H.

Clearly (iii) $\Rightarrow$ (ii)

See the proof of Bessel's inequality.

$$\therefore \begin{tikzcd} (i) \arrow[r, Rightarrow] & (ii) \arrow[d, Rightarrow] & (iii) \arrow[l, Leftrightarrow] \\ (v) \arrow[u, Rightarrow] & (iv) \arrow[l, Rightarrow] & \end{tikzcd}$$

Projections: ——

Let X be a linear space and

X_1 and X_2 be subspaces of X

such that

$$X = X_1 + X_2 \ , \ X_1 \cap X_2 = \{0\}$$

i.e., $X = X_1 \oplus X_2$;

then every $x \in X$ can be written uniquely as

$$x = x_1 + x_2, \quad x_1 \in X_1$$
$$x_2 \in X_2 .$$

Then define $P : X \longrightarrow X_1$ by

$$Px = P(x_1 + x_2) = x_1 .$$

Then P is a linear map.

and for any $u \in X_1$,

we have

$$Pu = P(u+0) = u, \quad \forall\, u \in X_1 = R(P).$$

and for any $v \in X_2$, we have

$$Pv = P(0+v) = 0.$$

$$\therefore X_1 = R(P), \quad X_2 = N(P).$$

and $P^2 = P$

A linear operator $P : X \to X$ is called projection operator or simply a projection if

$$Pu = u, \quad \forall\, u \in R(P)$$

If $P : X \to X$ is a projection with $R(P) = X_1$ and $N(P) = X_2$, We say P is projection onto X_1 along X_2.

and

$I-P$ is a projection onto X_2 along X_1 with

$$R(I-P) = X_2$$

$$N(I-P) = X_1.$$

Note:-

(1) Let X be an I.P.S and $P : X \to X$ be a projection.

We say P is an orthogonal projection if $R(P) \perp N(P)$

[i.e. $X = X_1 \oplus X_2$.

$P : X \to X_1$ is an orthogonal projection and

$$R(P) = X_1 \quad N(P) = X_2$$

$$X_1 \perp X_2]$$

(3) If P is an orthogonal projection, then for any $x \in X_1$ we have

$$x = \underbrace{Px}_{\in R(P)} + \underbrace{(I-P)x}_{\in N(P)}$$

$$\therefore \|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$$

(by Pythagoras theory)

$$\geq \|Px\|^2$$

$\therefore \quad \|Px\|^2 \leq \|x\|^2$

$\Rightarrow \quad \|Px\| \leq \|x\|$

$\Rightarrow \quad \|P\| \leq 1 \quad \text{——} (1)$

Also, since

$$P = P^2$$

$\Rightarrow \quad \|P\| = \|P^2\|$

$$= \|P \cdot P\|$$

$$\leq \|P\| \, \|P\|$$

$\Rightarrow \quad \|P\| \geq 1 \quad \text{——} (2)$

$\therefore$ from (1) & (2), we get

$$\|P\| = 1.$$

[For any $x \in H$

$$\|ABx\| = \|A(Bx)\|$$
$$\leq \|A\| \, \|Bx\|$$
$$\leq \|A\| \, \|B\| \, \|x\|$$
$$\Rightarrow \|AB\| \leq \|A\| \, \|B\| \text{]}$$

(Projection theorem) :—

Let H be a Hilbert space and F be a non-empty closed subspace of H. Then $H = F + F^{\perp}$.

Equivalently, there is an orthogonal projection onto F.

Moreover $F^{\perp\perp} = F$