Continuity of a linear map

Let X and Y be normed linear spaces. A linear operator $F : X \to Y$ is said to be continuous at $x \in X$ if $x_n \to x$ in $X \Rightarrow F(x_n) \to F(x)$ in Y

or

given any $\epsilon > 0$ $\exists$ $\delta > 0$ such that $u \in X$,

$$\|x - u\| < \delta \Rightarrow \|F(x) - F(u)\| < \epsilon$$

Theorem: Let X and Y be n.l.s. If X is finite dimensional, then every linear map $F : X \to Y$ is continuous.

Proof: If $X = \{0\}$ then

there is nothing to prove.

Assume $X \neq \{0\}$.

Let $\dim X = m$ with basis $\{u_1, u_2, \ldots u_m\}$.

Let $\{x_n\}$ be a sequence in X.

Then $x_n = \sum_{j=1}^{m} k_{nj} u_j$. $k_{nj} \in K$

If $x_n = \sum_{j=1}^{m} k_{nj} u_j \longrightarrow x = \sum_{j=1}^{m} k_j u_j$,

then $k_{nj} \longrightarrow k_j$, $j = 1, 2, \ldots m$

[by last class]

Now

$$F(x_n) = F\left(\sum_{j=1}^{m} k_{nj} u_j\right)$$

$$= \sum_{j=1}^{m} k_{nj} F(u_j)$$

$$\longrightarrow \sum_{j=1}^{m} k_j F(u_j)$$

$$= F\left(\sum_{j=1}^{m} k_j u_j\right)$$

$$= F(x).$$

Thus $x_n \to x \Rightarrow F(x_n) \to F(x)$

So F is continuous at $x \in X$.

$\because$ x is arbitary element of X,

it follows that F is continuous at every $x \in X$.

(1) $x_n \to x$, $y_n \to y$

$\Rightarrow x_n + y_n \to x + y$

(2) $x_n \to x \Rightarrow kx_n \to kx$

$$\|x_n + y_n - (x+y)\| = \|x_n - x + y_n - y\|$$

$$\leq \|x_n - x\| + \|y_n - y\| \to 0$$

$$\|kx_n - kx\| = |k|\,\|x_n - x\| \longrightarrow 0$$

(3) If $k_n \to k$ & $x_n \to x$

$\Rightarrow k_n x_n \longrightarrow kx$.

$$\because \|k_n x_n - kx\| = \|k_n x_n - k_n x + k_n x - kx\|$$

$$\leq \underbrace{|k_n|}_{<\infty}\,\underbrace{\|x_n - x\|}_{\to 0} + \underbrace{|k_n - k|}_{\to 0}\,\underbrace{\|x\|}_{\infty} \longrightarrow 0$$

$$\Big[\, |k_n| \leq \underbrace{|k_n - k|}_{\to 0} + |k| < \infty \,\Big]$$

* We say a linear map F is bounded on $\overline{U(0,r)}$, $r>0$ of n.l.s X if $\exists\, \beta > 0 \ni$

$$\|F(x)\| \leq \beta, \ \forall x \in \overline{U(0,r)}$$

Theorm: Let X and Y be n.l.s and $F: X \longrightarrow Y$ be a linear map. If F is bounded on $\overline{U(0,r)}$, $r>0$ then there exists $\alpha > 0$ such that

$$\|F(x)\| \leq \alpha \|x\|, \forall x \in X.$$

Proof: Let $F: X \longrightarrow Y$ be bounded on $\overline{U(0,r)} = \{x \in X / \|x\| \leq r\}$

Then there exists $\beta > 0$ $\ni$

$$\|F(x)\| \leq \beta, \forall x \in \overline{U(0,r)}. \quad \circledast$$

If $x = 0$, then clearly

$$0 = \|F(x)\| \leq \alpha \|x\| = 0, \ [\text{here } \beta = \alpha]$$

So, let $0 \neq x \in X$.

Then for $r > 0$,

$$\left\| \frac{rx}{\|x\|} \right\| = r \frac{\|x\|}{\|x\|} = r$$

$$\Rightarrow \frac{rx}{\|x\|} \in \overline{U(0,r)}.$$

$$\Rightarrow \left\| F\left(\frac{rx}{\|x\|}\right) \right\| \leq \beta \quad \text{by } (*)$$

$$\Rightarrow \frac{r}{\|x\|} \|F(x)\| \leq \beta$$

$$\Rightarrow \|F(x)\| \leq \frac{\beta}{r} \|x\|, \ \forall x \in X$$

$$\Rightarrow \|F(x)\| \leq \alpha \|x\|, \ \forall x \in X, \quad \alpha = \beta/r.$$

Theorem: Let $F: X \longrightarrow Y$ be a linear map. Then F is continuous on X iff there exists $\alpha > 0$ such that $\|F(x)\| \leq \alpha \|x\|, \forall x \in X$.

Proof: Suppose $F: X \to Y$ is continuous.

Claim: $\exists\, \alpha > 0 \ni \|F(x)\| \leq \alpha \|x\|$ $\forall x \in X$.

Suppose there exists no $\alpha > 0$ such that $\|F(x)\| \leq \alpha \|x\|$, $\forall x \in X$

Then for each $n \in \mathbb{N}$, we can find an element $x_n \in X$ such that

$$\|F(x_n)\| > n \|x_n\|.$$

$$\Rightarrow \left\| F\left(\frac{x_n}{n\|x_n\|}\right) \right\| > 1$$

Now let $y_n = \dfrac{x_n}{n\|x_n\|}$, then

$$\|y_n - 0\| = \left\| \frac{x_n}{n\|x_n\|} - 0 \right\| = \frac{1}{n} \to 0$$

as $n \to \infty$

But $\|F(y_n) - F(0)\| = \|F(y_n)\| > 1$

Thus $y_n \to 0$, but $F(y_n) \not\to 0$.

F is not continuous at the origin, which is contradiction to F is continuous on X.

$\therefore$ There must exists some $\alpha > 0$ $\ni \|F(x)\| \le \alpha \|x\|, \forall x \in X$.

Conversely, let there exists $\alpha > 0$ such that $\|F(x)\| \le \alpha \|x\|, \forall x \in X$.

Let $\{x_n\}$ be a sequence in X such that $x_n \to 0$.

Then $\|F(x_n)\| \le \alpha \|x_n\| \to 0$

$\Rightarrow$ F is continuous at the origin — (**)

Now let $\{x_n\}$ be a sequence in X $\ni$ $x_n \to x \in X$.

Then $x_n - x \to 0$ as $n \to \infty$.

$\Rightarrow F(x_n - x) \to 0$ as $n \to \infty$ by (**)

$\Rightarrow F(x_n) - F(x) \to 0$ as $n \to \infty$

$\Rightarrow F(x_n) \to F(x)$ as $n \to \infty$.

$\Rightarrow F$ is continuous at every $x \in X$.

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Def: A linear map $F : X \to Y$ is said to be bounded on X if there exists some $\alpha > 0$ such that $\|F(x)\| \leq \alpha \|x\|$, $\forall x \in X$

Note:— By above theorem, we see that $F: X \to Y$ is bounded iff F is continuous on X.

Problem: Prove that $F: X \to Y$ is continuous on X iff it is continuous at the origin.

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