25.08.21

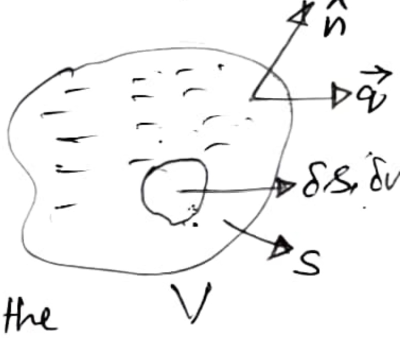
(1)

§ Conservation of mass:

The mathematical expression of the law of conservation of mass is known as equation of continuity.

In general the conservation of mass translates as

fluid in − fluid out + Sources − Sinks = accumulated fluid within the region.



Derivation: Consider a fluid region of volume V and surface area S at any time t_0. Let an infinitesimal small fluid element δs (of volume δv) is considered in V. Let ρ be the fluid density. Throughout the motion the mass of any fluid element remains unchanged as it moves about. Then this shows that the material derivation,

$$\frac{d}{dt}(\rho\,\delta v) = 0. \quad \text{—(1)}$$

which is the equn. of continuity or conservation in its simplest form. ✓

Consider the closed surface S in a fluid medium containing a volume V fixed in space. Let $\vec{q}$ be the fluid velocity and $\hat{n}$ be the unit outward drawn normal. δs be the fluid element with volume δv. The normal component of $\vec{q}$ measured outward from $V = \vec{q}\cdot\hat{n}$. —(1)

Rate of mass flow acceross $\delta s = \rho.(\hat{n}.\vec{q})\, \delta s$.

Total " " entire volume $V = \int_S \rho\, (\hat{n}.\vec{q})\, ds$

$= \int_V \vec{\nabla}.(\rho\vec{q})\, dv$, Gauss-divergence thm.

Total rate mass of the fluid into $V = -\int_V \vec{\nabla}.(\rho\vec{q})\, dv$. — (II)

Also, the rate of change of mass of the fluid within V

$$= \frac{\partial}{\partial t}\int_V \rho\, dv = \int_V \frac{\partial \rho}{\partial t} dv \quad \text{— (III)}$$

By conservation of mass.

$$\int_V \frac{\partial \rho}{\partial t} dv = -\int_V \vec{\nabla}.(\rho\vec{q})\, dv.$$

$$\Rightarrow \int_V \left(\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\vec{q})\right) dv = 0$$

This equn. is true for any arb. V in the fluid region.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\vec{q}) = 0 \quad \text{— (IV)}$$

This is the required equn. of Continuity without any sources or sinks.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\rho \cdot \vec{q} + \rho\, \vec{\nabla} \cdot \vec{q} = 0$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial t} + \vec{q} \cdot \vec{\nabla}\right)\rho + \rho\, \vec{\nabla} \cdot \vec{q} = 0$$

$$\Rightarrow \quad \underbrace{\frac{d\rho}{dt} + \rho\, \vec{\nabla} \cdot \vec{q} = 0}_{\text{equ}^n \text{ of continuity.}}, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{q} \cdot \vec{\nabla}$$

For an incompressible fluid, the density remains unchanged w.r.t. t, i.e., $\frac{d\rho}{dt} = 0$, therefore,

$$\rho\, \vec{\nabla} \cdot \vec{q} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{q} = 0$$

$\vec{q}$ is solenoidal.

In cartesian form:

$$\frac{d\rho}{dt} + \rho\, \vec{\nabla} \cdot \vec{q} = 0$$

$$\Rightarrow \quad \frac{d\rho}{dt} + \rho \left[\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (u, v, w)\right] = 0$$

$$\Rightarrow \quad \frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0.$$

$$\frac{D}{Dt} = \frac{d}{dt}$$

In Polar - Coordinates :

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(\rho r^2 q_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\rho \sin\theta \, q_\theta \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \left(\rho q_\phi \right) = 0,$$

$P(r, \theta, \phi)$, $\vec{q} = (q_r, q_\theta, q_\phi)$.

In Cylinderical co-ordinates: $P(r, \theta, z)$, $\vec{q} = (q_r, q_\theta, q_z)$.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(\rho r q_r \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\rho q_\theta \right) + \frac{\partial}{\partial z} \left(\rho q_z \right) = 0.$$

Ex 1: Let $\vec{q} = 5x\hat{i} + 5y\hat{j} - 10z\hat{k}$. verify if the flow is incompressible ?

Ans: $\vec{\nabla} \cdot \vec{q} = 0$ ✓