06/09/2020

Def :- A linear map $F : X \longrightarrow Y$ is said to be bounded if there exists $m > 0$ such that

$$\|F(x)\|_Y \leq m \|x\|_X \quad \forall x \in X;$$

where X & Y are n.l.s

[: $\|F(x)\| \leq \alpha, \quad \forall x \in X$

Suppose E is a bounded set, so we can choose r large enough so that $\underline{E} \subset \overline{U(0,r)}$

if F is bounded on $\overline{U(0,r)}$, then F is also bounded on E

Also we know that F is bounded on $\overline{U(0,r)}$ iff $\exists$ ~~...~~ $\alpha > 0$ $\ni$

$$\|F(x)\| \le \lambda \|x\|, \ \forall x \in X$$

Theorem: let X and Y be n.l.s and $F: X \to Y$ be a linear map. Then F is bounded iff F maps bounded sets in X to bounded sets in Y.

Proof: let F be bounded.

Then there exists $m > 0$ such that

$$\|F(x)\| \le m\|x\|, \ \forall x \in X. \quad (1)$$

Let $B = \overline{U(0,r)}$ be a bounded set in X

Then $x \in B \Rightarrow \|x\| \le r$.

Then from (1), we have

$$\|F(x)\| \le m\|x\| \le mr, \ \forall x \in B$$

$$\Rightarrow \|F(x)\| \le mr < \infty$$

$\Rightarrow F(B)$ is bounded in Y.

Conversely, let F maps bounded sets in X into bounded sets in Y.

claim: F is a bounded linear map.

Let $\overline{U(0,1)}$ be a closed unit ball in X, which is bounded in X.

Then there exists $k>0$ such that

$$\|F(x)\| \le k, \quad \forall x \in \overline{U(0,1)}$$

Now for any $0 \ne x \in X$, let $y = \frac{x}{\|x\|}$,

then $\|y\| = \left\|\frac{x}{\|x\|}\right\| = 1$

$$\Rightarrow y \in \overline{U(0,1)}$$

$$\Rightarrow \|F(y)\| \le k$$

$$\Rightarrow \left\|F\left(\frac{x}{\|x\|}\right)\right\| \le k$$

$$\Rightarrow \|F(x)\| \le k\|x\|, \quad \forall x \in X$$

$\Rightarrow F$ is a bounded linear map

[$\because \overline{U(0,1)} = B$ is a bounded set

Then $F(B)$ is bounded in Y

$\Rightarrow \exists\ k > 0 \ni \|F(x)\| \leq k$

$\forall x \in B$]

* Let X and Y be n.l.s.

If $F: X \longrightarrow Y$ is continuous linear map, then it is uniformly continuous.

Proof: If F is continuous on X, then it is continuous at the origin.

$\therefore$ Given $\epsilon > 0$, $\exists\ \delta > 0 \ni$

$$\|x\| < \delta \Rightarrow \|F(x)\| < \epsilon \quad \text{---} (1)$$

Hence for any $u \in X$, replacing x by $x - u$ in (1), we get

$\|x-u\|<\delta \Rightarrow \|F(x-u)\|<\epsilon$

i.e., $\|F(x)-F(u)\|<\epsilon$

Since δ is independent of $u \in X$, it follows that F is uniformly continuous on X.

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Combining above all the result we have the following theorem:

Theorem: Let X and Y be n.l.s and $F: X \to Y$ be a linear map. Then the following are equivalent:

(i) F is continuous at the origin

(ii) F is continuous at every $x \in X$.

(iii) F is uniformly continuous on X.

(iv) There exists $\alpha > 0$ such that
$$\|F(x)\| \le \alpha \|x\|, \ \forall x \in X.$$

(v) $\{F(x) \mid \|x\| = 1, x \in X\}$ is a bounded set in Y.

(vi) For every bounded set $E \subseteq X$, the set
$$F(E) = \{F(x) \mid x \in E\}$$ is bounded in Y.

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Theorem: Let X and Y be n.l.s and $F : X \longrightarrow Y$ be a linear map. Let $Z(F)$ be a null space of X. Then F is continuous iff $Z(F)$ is closed in X and $\tilde{F} : \frac{X}{Z(F)} \longrightarrow Y$ defined by

$\tilde{F}(x+Z(F)) = F(x)$ is continuous.

Proof: let F be continuous on X.

Then $Z(F) = F^{-1}\{0\}$ is closed in X, since $\{0\}$ is a closed set in Y.

[OR let $x_n \in Z(F) \ni x_n \to x$ in X.

Then $F(x_n) = 0 \ \forall n$

also as F is continuous, $x_n \to x$

$$\Rightarrow F(x_n) \to F(x)$$

$$\underset{0}{\underset{\parallel}{F(x_n)}}$$

$$\Rightarrow F(x) = 0 \Rightarrow x \in Z(F).$$

$\therefore Z(F)$ is closed]

$\therefore \dfrac{X}{Z(F)}$ is a n.l.s.

Then map $\tilde{F} : \dfrac{X}{Z(F)} \to Y$ be

defined by

$$\tilde{F}(x + Z(F)) = F(x), \forall x \in X$$

is a linear map.

$\because F : X \to Y$ is continuous,

$\exists \alpha > 0 \ni \|F(x)\| \leq \alpha \|x\|, \forall x \in X.$

Now let $x \in X$, $z \in Z(F)$. Then

$$\|\tilde{F}(x + Z(F))\| = \|\tilde{F}(x + z + Z(F))\|$$

$$= \|F(x+z)\|$$

$$\leq \alpha \|x+z\|$$

Since above inequality is true

for any $z \in Z(F)$, it follows that

$$\|\tilde{F}(x+Z(F))\| \leq \alpha \inf\{\|x+z\| \mid z \in Z(F)\}$$
$$= \alpha\, |||x+Z(F)|||$$

$\Rightarrow \tilde{F} : \dfrac{X}{Z(F)} \longrightarrow Y$ is continuous.

Conversely, let $Z(F)$ is closed and $\tilde{F} : \dfrac{X}{Z(F)} \longrightarrow Y$ is continuous.

Claim : $F : X \longrightarrow Y$ is continuous.

Consider for any $x \in X$,

$$\|F(x)\| = \|\tilde{F}(x+Z(F))\|$$
$$\leq \alpha\, |||x+Z(F)|||$$
$$\leq \alpha \|x\| \quad [\because 0 \in Z(F)]$$

$\Rightarrow F$ is continuous