14/09/2021

Theorem: let X be a n.l.s and $f : X \rightarrow K$ be a nonzero linear functional on X such that $N(f)$ is closed in X. Then f is continuous, and for every $x_0 \in X - N(f)$,

$$\|f\| = \frac{|f(x_0)|}{\text{dist}(x_0, N(f))}.$$

Proof: Let $x_0 \in X$ such that $f(x_0) \neq 0$. Then for every $x \in X$,

$$x = x - \frac{f(x)}{f(x_0)} x_0 + \frac{f(x)}{f(x_0)} \cdot x_0$$

$$= y + \alpha x_0,$$

where $y = x - \frac{f(x)}{f(x_0)} x_0$, $\alpha = \frac{f(x)}{f(x_0)}$.

Then $f(y) = f\left(x - \frac{f(x)}{f(x_0)} x_0\right)$

$$= f(x) - \frac{f(x)}{f(x_0)} \cdot f(x_0)$$

$$= 0$$

$$\Rightarrow \quad y \in N(f).$$

$\therefore$ $\text{dist}(x, N(f)) = \text{dist}(y + \alpha x_0, N(f))$

$$= \text{dist}(\alpha x_0, N(f))$$

$$= |\alpha| \, \text{dist}(x_0, N(f)) \quad \text{—(1)}$$

$\because$ $x_0 \notin N(f) \Rightarrow \text{dist}(x_0, N(f)) > 0$

$\therefore$ from (1), we have

$$\left| \frac{f(x)}{f(x_0)} \right| = |\alpha| = \frac{\text{dist}(x, N(f))}{\text{dist}(x_0, N(f))}$$

$$\Rightarrow |f(x)| = \frac{|f(x_0)|}{\text{dist}(x_0, N(f))} \text{dist}(x, N(f))$$

$$\leq \frac{|f(x_0)|}{\text{dist}(x_0, N(f))} \cdot \|x\|$$

$$\left[\because \text{dist}(x, N(f)) \leq \|x - 0\| = \|x\|\right]$$

$$\Rightarrow \|f\| \leq \frac{|f(x_0)|}{\text{dist}(x_0, N(f))} \quad \text{---} (2)$$

$\Rightarrow$ f is continuous.

Also for every $u \in N(f)$,

$$|f(x_0)| = |f(x_0) - f(u)|$$

$$= |f(x_0 - u)|$$

$$\leq \|f\| \, \|x_0 - u\| \quad \forall u \in N(f)$$

Taking infimum over all $u \in N(f)$ we get

$$|f(x_0)| \leq \|f\| . \text{diff}\left(x_0, N(f)\right)$$

$$\Rightarrow \frac{|f(x_0)|}{\text{diff}\left(x_0, N(f)\right)} \leq \|f\| \quad \text{—(3)}$$

from ② & ③, we get

$$\|f\| = \frac{|f(x_0)|}{\text{diff}\left(x_0, N(f)\right)} .$$

Ex: $X = C^1[0,1]$ with $\|\cdot\|_\infty$ and $f : X \longrightarrow K$ be defined by $f(x) = x'(1)$, $\forall x \in X$.

We know that f is discontinuous

linear functional on X.

$\Rightarrow$ $N(f)$ is not closed by above theorem.

OR

let $x(t) = t$, $x_n(t) = t - \frac{t^n}{n}$, $\forall t \in [0,1]$, $\forall n \in \mathbb{N}$.

$$\|x_n - x\|_\infty = \frac{1}{n} \longrightarrow 0$$

and $f(x_n) = x_n'(1) = 1 - 1 = 0$

So $\{x_n\}$ is a sequence in $N(f)$,

but $f(x) = 1 \neq 0$

$\Rightarrow x \notin N(f)$.

$\therefore$ $N(f)$ is not closed.

(2) $X = c_{00}$, $\|\cdot\|_\infty$

For $x \in X$, $f : X \longrightarrow K$ be defined by $f(x) = \sum_{j=1}^{\infty} x(j)$

Then f is not continuous linear functional.

$\Rightarrow$ $N(f)$ is not closed.

$$x_n = (-1, \underbrace{\frac{1}{n}, \frac{1}{n}, \cdots \frac{1}{n}}_{n \text{ times}}, 0, 0 \cdots) \in X$$

Then $f(x_n) = -1 + \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}} + 0 \cdots$

$= 0$

$\Rightarrow x_n \in N(f)$.

Let $x = (-1, 0, 0, 0 \cdots) \in X$

Then $\|x_n - x\|_\infty = \frac{1}{n} \longrightarrow 0$ as $n \to \infty$

$[x_n - x = (0, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n}, 0, 0, \dots)]$

but $f(x) = -1 \neq 0$

$\therefore x \notin N(f)$

$\Rightarrow N(f)$ is not closed.

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Suppose X and Y be n.l.s and $\{A_n\}$ be a sequence of operators from X to Y i.e., $\{A_n\} \in L(X, Y)$

If $\{A_n x\}$ Converges for every $x \in X$, then a function $A : X \longrightarrow Y$ defined

by
$$Ax = \lim_{n\to\infty} A_n x, \quad x \in X$$
is also a linear operator.

For any $x, y \in X$, $\alpha, \beta \in K$,

$$A(\alpha x + \beta y) = \lim_{n\to\infty} A_n(\alpha x + \beta y)$$
$$= \lim_{n\to\infty} [\alpha A_n x + \beta A_n y]$$
$$= \alpha \lim_{n\to\infty} A_n x + \beta \lim_{n\to\infty} A_n y$$
$$= \alpha Ax + \beta Ay$$
$$\Rightarrow A \in L(X, Y).$$

If each A_n is a bounded operator, what can you say about boundedness of A?

The answer is negative.

Ex: $X = \ell_\infty$, $\|\cdot\|_\infty$.

for each $n \in \mathbb{N}$, define $f_n : X \to K$ by

$$f_n(x) = \sum_{j=1}^{n} x(j),$$

$x = (x(1), x(2), \ldots) \in X$.

Then

$\|f_n\| = n$

$\left[\because \quad x_n = (1, \underbrace{1, 1, 1, \ldots 1}_{n\text{ times}}, 0, 0, \ldots)\right]$

$\Rightarrow$ each f_n is bounded.

$\Rightarrow$ Each f_n is continuous linear functional.

$$\|f_n\|_\infty = \sup\{ |f_n(x)| \,/\, x \in X, \|x\|_\infty \leq 1 \}$$

$$\because \|x_n\|_\infty = 1 = \sup_{j=1,2,\ldots\infty} |x_n(j)| = 1$$

So $f_n(x_n) = \sum_{j=1}^{n} x_n(j) = n$

$\Rightarrow \|f_n\| = n, \quad n \in \mathbb{N}$

$\{f_n\}$ is a sequence of bounded linear functionals

Also

$$f_n(x) = \sum_{j=1}^{n} x(j) \longrightarrow \sum_{j=1}^{\infty} x(j) = f(x) \text{ as } n \to \infty$$

$[f(x) = \lim_{n\to\infty} f_n(x)$ is discontinuous$]$

but f is linear, but not continuous.

By imposing boundedness of $\{\|A_n\|\}$, we can obtain continuity of A.

Theorem: Let X and Y be n.l.s and $\{A_n\}$ be a sequence in $\underline{BL}(X, Y)$ such that $\{A_n x\}$ converges in Y for each $x \in X$. If $\{\|A_n\|\}$ is bounded sequence, then $A : X \longrightarrow Y$ defined by

$$Ax = \lim_{n \to \infty} A_n x, \quad x \in X$$

is also belongs to $BL(X, Y)$.

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and $\|A\| \leq \liminf_{n\to\infty} \|A_n\|$.

Proof: Clearly $A: X \to Y$ is linear.

Now

$$Ax = \lim_{n\to\infty} A_n x, \quad x \in X$$

$$\Rightarrow \|Ax\| = \|\lim_{n\to\infty} A_n x\|$$

$$= \lim_{n\to\infty} \|A_n x\|$$

$$\leq \left(\lim_{n\to\infty} \|A_n\| \, \|x\|\right)$$

$$\leq \left(\liminf_{n\to\infty} \|A_n\|\right) \|x\|$$

$$\Rightarrow \|A\| \leq \liminf_{n\to\infty} \|A_n\|$$

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$$[\because \ \|F\| = \sup\{\|F(x)\| \,/\, x\in X, \ \|x\|\leq 1\}$$

$$= \sup\left\{\frac{\|F(x)\|}{\|x\|} \,\middle/\, x\in X, \ x\neq 0\right\}$$

$$\geq \frac{\|F(x)\|}{\|x\|}$$

$$\Rightarrow \ \|F(x)\| \leq \|F\|\,\|x\|$$

Also

$$\|x\| = \|x-y+y\|$$

$$\leq \|x-y\| + \|y\|$$

$$\Rightarrow \ \|x\| - \|y\| \leq \|x-y\| \quad -(1)$$

Also $\|y\| = \|y-x+x\|$

$$\leq \|x-y\| + \|x\|$$

$$\Rightarrow \|y\| - \|x\| \leq \|x-y\|$$

$$\Rightarrow -[\|x\| - \|y\|] \leq \|x-y\|$$

$\therefore$ from ① & ②, we get — ②

$$\big| \|x\| - \|y\| \big| \leq \|x-y\|$$

So if $x_n \rightarrow x$, i.e., $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$

$$\big| \|x_n\| - \|x\| \big| \leq \|x_n - x\| \rightarrow 0$$

$$\Rightarrow \|x_n\| \rightarrow \|x\|$$

$\Rightarrow \|\cdot\|$ is Continuous]

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If $\|A_n x - Ax\| \rightarrow 0$ for each $x \in X$, then what can you say about $\|A_n - A\| \longrightarrow 0$?

where $\{A_n\} \in BL(X, Y)$.

$\|A_n - A\| = \sup\{\|A_n x - Ax\| \,/\, x \in \cancel{X}, \|x\| \leq 1\}$.

The answer is negative.

Ex: $X = \ell^2$, for each $n \in N$,

let $A_n : \ell^2 \longrightarrow \ell^2$ by

$$A_n x(j) = \begin{cases} x(j), & j \leq n \\ 0, & j > n \end{cases}$$

i.e.,

$$A_n x = (x(1), x(2), \ldots x(n), 0, 0, \ldots)$$

Then for every $x \in \ell^2$,

$$\|A_n x - x\|_{\ell^2}^2 = \sum_{j=n+1}^{\infty} |x(j)|^2 \longrightarrow 0$$

as $n \to \infty$

$\therefore$ $A_n x \to Ix$, for each $x \in \ell^2$.

Now $e_{n+1} = (0, 0, \cdots \underset{n+1\text{th place}}{1}, 0, 0 \cdots)$

$$\|Ae_{n+1} - e_{n+1}\|_{\ell^2} = 1, \ \forall n \in \mathbb{N}.$$

So $\sup\limits_{\substack{x \in X \\ \|x\|_{\ell^2} \leq 1}} \|A_n x - Ix\|$

$$\geq \|Ae_{n+1} - e_{n+1}\|$$

$$= 1$$

$$\Rightarrow \|A_n - I\|_{\ell^2} \geq 1$$

$$\|A_n - I\| \not\longrightarrow 0$$

even though $A_n x \to Ix$ $\forall x \in X$.

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In the previous theorem, we assumed that $\{A_n x\}$ converges for every $x \in X$. But this can be guaranted by knowing that $\{A_n x\}$ converges for every x in a subset E of X such that $\overline{\operatorname{Span} E} = X$.

Theorem: let X be a n.l.s, Y be a Banach space and $\{A_n\}$ be a sequence in $BL(X, Y)$ such that $\{\|A_n\|\}$ is bounded further of R. Suppose $E \subseteq X$ is such that $\operatorname{span} E$ is dense in X. If $\{A_n x\}$ converges for every $x \in E$, then $\{A_n x\}$ converges for every $x \in X$ and if

$$Ax = \lim_{n \to \infty} A_n x, \text{ then}$$

$A \in BL(X, Y)$ and

$$\|A\| \leq \liminf_{n} \|A_n\| < \infty$$

Proof: Suppose $\{A_n x\}$ converges for every $x \in E$, then $\{A_n x\}$ also converges for every $x \in \operatorname{span} E = D$ (say)

Let $x \in X$ and $\epsilon > 0$ be given.

$\because D = \overline{\text{Span } E} = X$, $\exists\, u \in D \ni$

$\|x - u\| < \epsilon$.

Now for all $m, n \in N$,

$$\|A_n x - A_m x\| = \|A_n x - A_n u + A_n u - A_m u + A_m u - A_m x\|$$

$$\le \|A_n x - A_n u\| + \|A_n u - A_m u\| + \|A_m u - A_m x\|$$

$$\le [\|A_n\| + \|A_m\|]\, \|x - u\| + \|A_n u - A_m u\|$$

$\because \{\|A_n\|\}$ is bounded, $\exists\, \alpha > 0$

$\ni \|A_n\| \le \alpha, \forall n$

$$\therefore \|A_n x - A_m x\| \le [\alpha + \alpha]\, \|x - u\| + \|A_n u - A_m u\|$$

$$\leq 2\alpha\epsilon + \|A_n u - A_m u\| \qquad \text{—— (1)}$$

Now $\{A_n u\}$ converges for every $u \in D$, $\exists\ n_0 \in N \ni$

$$\|A_n u - A_m u\| < \epsilon, \ \forall\ n, m \geq n_0, \ u \in D.$$

Using this in (1), we get

$$\|A_n x - A_m x\| \leq 2\alpha\epsilon + \epsilon$$

$$= (2\alpha + 1)\epsilon$$

$\Rightarrow \{A_n x\}$ Cauchy sequence in in Y

$\because Y$ is Banach space,

$\{A_n x\}$ converges for every $x \in X$.

Define $A : X \to Y$ by

$$Ax = \lim_{n\to\infty} A_n x, \quad x \in X.$$

$$\Rightarrow \quad \|A\| \leq \liminf_{n} \|A_n\| < \infty$$

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