Lemma:- Let M be a subspace of a n.l.s N and $x_0 \notin M$. Let $M_0 = \text{Span}\{x_0, M\}$. Let f be a continuous linear functional defined on M. Then there exists a functional f_0 on M_0 such that $\|f_0\| = \|f\|$ and $f_0|_M = f$.

Proof: First assume that N is real n.l.s. and let $M_0 = \text{Span}\{x_0, M\}$. Then every $y \in M_0$ is of the form $y = x + \alpha x_0$, $\alpha \in K$, $x \in M$

Now Define $f_0 : M_0 \longrightarrow K$ by $f_0(x + \alpha x_0) = f(x) + \alpha r_0$, where r_0 is any real number.

We show that f_0 is an extention of f and $\|f\| = \|f_0\|$.

let $x_1, y_1 \in M_0 = \mathrm{Span}\{x_0, M\}$

$\therefore x_1 = x + \alpha x_0, \quad y_1 = y + \beta x_0$

where $x, y \in M$.

For any $a, b \in K = \mathbb{R}$, consider

$$f_0(ax_1 + by_1) = f_0(ax + by + (a\alpha + b\beta)x_0)$$

$$= f(ax + by) + (a\alpha + b\beta)r_0$$

$$[\because ax + by \in M]$$

$$= af(x) + bf(y) + a\alpha r_0 + b\beta r_0$$

$$= a[f(x) + \alpha r_0] + b[f(y) + \beta r_0]$$

$$= a f_0(x_1) + b f_0(y_1).$$

$\therefore f_0 : M_0 \to K$ is linear functional.

Also, for any $y \in M$, we can write

$$y = y + 0 \cdot x_0 \in M_0 = \mathrm{Span}\{x_0, M\}$$

$$\Rightarrow f_0(y) = f_0(y + 0x_0) = f(y) + 0 \cdot r_0 = f(y), \ \forall y \in M.$$

$\Rightarrow f_0|_M = f.$

Claim: $\|f_0\| = \|f\|.$

If $\alpha = 0$, then clearly we have

$$\|f\| = \|f_0\|.$$

So we assume $\alpha \neq 0$.

Since M is a subspace of M_0,

$$\sup_{x \in M} \left\{ \frac{|f_0(x)|}{\|x\|} \,\middle|\, x \neq 0 \right\}$$

$$\leq \sup_{x \in M_0} \left\{ \frac{|f_0(x)|}{\|x\|} \,\middle|\, x \neq 0 \right\}$$

$$\Rightarrow \sup_{x \in M} \left\{ \frac{|f(x)|}{\|x\|} \,\middle|\, x \neq 0 \right\} \leq \sup_{x \in M_0} \left\{ \frac{|f_0(x)|}{\|x\|} \,\middle|\, x \neq 0 \right\}$$

$$\Rightarrow \quad \|f\| \leq \|f_0\| \quad \text{---(1)}$$

In the above we used $f_0(x) = f(x)$ $\forall x \in M$.

let $x_1, x_2 \in M$, then

$$f(x_2) - f(x_1) = f(x_2 - x_1)$$
$$\leq |f(x_2 - x_1)|$$
$$\leq \|f\| \, \|x_2 - x_1\|$$
$$= \|f\| \left[\|x_2 + x_0 - (x_0 + x_1)\| \right]$$
$$\leq \|f\| \left\{ \|x_2 + x_0\| + \|x_0 + x_1\| \right\}$$

$\Rightarrow$

$$-f(x_1) - \|f\| \, \|x_1 + x_0\| \leq -f(x_2) + \|f\| \, \|x_2 + x_0\| \quad \text{—(1)}$$

If we keep x_2 fixed and vary $x_1 \in M$, then L.H.S of (1) is bounded above.

$\therefore$ Supremum of L.H.S exists.

Illy if we keep x_1 fixed and vary $x_2 \in M$, then R.H.S of (1) is bounded below. Hence infimum on R.H.S exists.

$\therefore$

$$\sup_{y \in M} \left\{ -f(y) - \|f\| \|y + x_0\| \right\}$$

$$\leq \inf_{y \in M} \left\{ -f(y) + \|f\| \|y + x_0\| \right\}$$

So choose a real number r_0, such that

$$\sup_{y \in M} \left\{ -f(y) - \|f\| \|y + x_0\| \right\} \leq r_0$$

$$\leq \inf_{y \in M} \left\{ -f(y) + \|f\| \|y + x_0\| \right\}$$

Let $y = x/\alpha$ in the above, we have

$$\sup_{x \in M} \left\{ -f(x/\alpha) - \|f\| \left\|\frac{x}{\alpha} + x_0\right\| \right\}$$

$$\leq r_0 \leq \inf_{x \in M} \left\{ -f(x/\alpha) + \|f\| \left\|\frac{x}{\alpha} + x_0\right\| \right\} \quad \text{--- (2)}$$

If $\alpha > 0$, then from R.H.S of (2), We have

$$r_0 \leq -f(x/\alpha) + \|f\| \, \|\tfrac{x}{\alpha} + x_0\|$$

$$= -\frac{1}{\alpha} f(x) + \frac{\|f\|}{\alpha} \|x + \alpha x_0\|$$

$$\Rightarrow f(x) + \alpha r_0 \leq \|f\| \, \|x + \alpha x_0\|$$

$$\Rightarrow f_0(x + \alpha x_0) = f(x) + \alpha r_0 \leq \|f\| \, \|x + \alpha x_0\|$$

If $z = x + \alpha x_0 \in M_0 = [x_0, M]$,

Then we have

$$f_0(z) = f_0(x + \alpha x_0),$$

So from above, we have

$$f_0(z) = f(x + \alpha x_0) \leq \|f\| \, \|z\|$$

$$\therefore |f_0(z)| \leq \|f\| \, \|z\|$$

$$\Rightarrow \|f_0\| \leq \|f\| \quad (\text{for the case } \alpha > 0)$$

If $\alpha < 0$, then from L.H.S of (2), we have

$$-f(x/\alpha) - \|f\| \left\|\frac{x}{\alpha} + x_0\right\| \leq r_0$$

$$\Rightarrow -\frac{1}{\alpha} f(x) - \frac{1}{|\alpha|} \|f\| \|x + \alpha x_0\| \leq r_0$$

$$\Rightarrow -\frac{1}{\alpha} f(x) + \frac{1}{\alpha} \|f\| \|x + \alpha x_0\| \leq r_0 \quad \left[\because \frac{1}{|\alpha|} = \frac{-1}{\alpha},\ \alpha < 0\right]$$

Multiplying on both side with α and since $\alpha < 0$, above inequality reverses.

$$\therefore -f(x) + \|f\| \|x + \alpha x_0\| \geq r_0 \alpha$$

$$\Rightarrow \|f\| \|x + \alpha x_0\| \geq f(x) + \alpha r_0$$

$$= f_0(x + \alpha x_0)$$

$$\Rightarrow \|f\| \|z\| \geq f_0(z), \ \forall z \in M_0$$

replacing z by $-z$, we get

$$\|f\| \, \|-z\| \geq f_0(-z)$$

$$\Rightarrow -f_0(z) \leq \|f\| \, \|z\|$$

$\therefore$ we get

$$|f_0(z)| \leq \|f\| \, \|z\|$$

$$\Rightarrow \|f_0\| \leq \|f\| \quad \text{for } \alpha < 0 \text{ also.}$$

$$\therefore \|f_0\| \leq \|f\| \quad \text{for } \alpha \neq 0, \ \alpha \in \mathbb{R}. \quad \text{—(3)}$$

$\therefore$ from ① & ③, we get

$$\|f\| = \|f_0\|$$