Maximal proper subspace :—

A subspace Z of a normed linear space X is called maximal proper subspace of X if $a \in X - Z$, $\text{Span}\{a, Z\} = X$.

A maximal proper subspace is called hyper space.

Theorem: Let f be a nonzero linear functional on a nonzero linear space X. Then the null space $Z(f)$ of f is a hyper space in X. That is there exists $a \in X - Z(f)$ such that $X = \text{Span}\{a, Z(f)\}$.

Proof: Since $f \neq 0$, $Z(f)$ is a proper subspace of X.

Let $a \in X$ such that $f(a) \neq 0$.

Now for any $x \in X$, consider

$$z = x - \frac{f(x)}{f(a)} \cdot a \quad \text{——} (1)$$

Then
$$f(z) = f(x) - \frac{f(x)}{f(a)} \cdot f(a)$$
$$= 0$$

$$\Rightarrow z \in Z(f)$$

From (1), we have

$$x = z + \frac{f(x)}{f(a)} \cdot a$$
$$\in \operatorname{Span}\{a, Z(f)\}$$

$$\left[x = x - \frac{f(x)}{f(a)}a + \frac{f(x)}{f(a)}\cdot a = z + \frac{f(x)}{f(a)}\cdot a\right]$$

$\therefore X \subseteq \text{Span}\{a, Z(f)\} \subseteq X$

$\therefore X = \text{Span}\{a, Z(f)\}$.

$\Rightarrow Z(f)$ is a maximal proper subspace of X and hence it is a hyperspace.

Theorem: If Z is a hyperspace in a linear space X, then there exists a linear functional f on X such that $Z(f) = Z$, where $Z(f)$ is a null space of f.

Proof: Let Z be a hyperspace in X, and $a \in X - Z$. Then

$$\operatorname{Span}\{a, Z\} = X.$$

Now for any $x \in X$, we have

$$x = z + \alpha a, \quad \begin{array}{l} z \in Z \\ \alpha \in K \end{array}$$

Define $f : X \to K$ by

$$f(x) = f(z + \alpha a) = \alpha.$$

Then f is a linear and

$$f(0 + 1 \cdot a) = 1, \quad (\because 0 \in Z)$$

and

$$Z(f) = \{x \in X \mid f(x) = 0\}$$

$$= \{u + \alpha a \mid f(u + \alpha a) = 0, \; u \in Z, \alpha \in K\}.$$

$$= \{u + \alpha a \mid \alpha = 0, u \in Z\}$$

$$= \{u \mid u \in Z\}$$

$$= Z$$

Theorem: Let X be a linear space and S be the POSET of all proper subspaces of X. Then a subspace H of X is hyperspace iff it is maximal element of S.

Proof: Let H be a hyperspace and $x_0 \in X - H$. Then

$$\mathrm{Span}\{x_0, H\} = X$$

Let Y be any proper subspace of X

such that $H \subseteq Y$.

Claim: $H = Y$.

If $H \neq Y$, let $y_0 \in Y - H$.

Since $\text{Span}\{x_0, H\} = X$,

so there exists $\alpha_0 \in K$, $u \in H$

such that

$$y_0 = u + \alpha_0 x_0, \quad [\because Y \subseteq X]$$

$$\Rightarrow x_0 = \frac{y_0 - u}{\alpha_0} \in \text{Span}\{y_0, H\} \subseteq Y$$

$$\Rightarrow x_0 \in Y.$$

$$\Rightarrow X = \text{Span}\{x_0, H\} \subseteq \text{Span}\{y_0, H\} \subseteq Y$$

$\Rightarrow X = Y$, this is contradiction to Y is a proper subspace of X.

$\therefore \quad H = X.$

Thus every hyper space is a maximal element of S.

Conversely, let H be a maximal element in S. Let $x_0 \in X - H$.

Then H is a proper subspace of $\operatorname{span}\{x_0, H\}$. Then by maximality of H, we have

$$\operatorname{Span}\{x_0, H\} = X$$

$\Rightarrow$ H is a Hyper space.

Lemma :- Let X be a linear space over the complex field $\mathbb{C}$. Regarding X as a linear space over $\mathbb{R}$, consider a real linear functional $u : X \to \mathbb{R}$.

Define $f(x) = u(x) - i u(ix)$, $x \in X$.

Then f is a complex linear functional on X.

Proof: For any $x, y \in X$,

$$f(x+y) = u(x+y) - i u(i(x+y))$$

$$= u(x) + u(y) - i [u(ix + iy)]$$

$$= u(x) + u(y) - i [1 \cdot u(ix) + 1 u(iy)]$$

[$\because u$ is real linear]

$$= [u(x) - i u(ix)] + [u(y) - i u(iy)]$$

$$= f(x) + f(y)$$

If $\alpha \in \mathbb{R}$, $x \in X$

$$f(\alpha x) = u(\alpha x) - i u(i \alpha x)$$

$$= \alpha u(x) - i\alpha u(ix)$$
$$= \alpha [u(x) - i u(ix)]$$
$$= \alpha f(x)$$

$\Rightarrow$ f is linear over real field.

Now

$$f(ix) = u(ix) - i u(i \cdot ix)$$
$$= u(ix) - i u(-1x)$$
$$= u(ix) - i(-1) u(x)$$
$$= u(ix) + i u(x)$$
$$= i[u(x) - i u(ix)]$$
$$= i f(x).$$

$\Rightarrow$ f is a Complex linear functional.

Convex set :— let X be a linear space over the field K.

A subset E of X is said to be convex set if

$tx + (1-t)y \in E, \quad \forall\, x, y \in E$
$0 < t < 1.$

* $U(x,r)$, $\overline{U(x,r)}$ are convex sets in a n.l.s X.

Let $a, b \in U(x,r)$, $0 < t < 1$

$\Rightarrow \|x-a\| < r, \quad \|x-b\| < r$

Consider

$$\|ta + (1-t)b - x\| = \|ta + (1-t)b - [tx + (1-t)x]\|$$

$$= \|t(a-x) + (1-t)(b-x)\|$$

$$\leq t\|a-x\| + (1-t)\|b-x\|$$

$$< t \cdot r + (1-t)r$$

$$= r.$$

$\Rightarrow ta + (1-t)b \in U(x,r).$

$\therefore$ $U(x, r)$ is a convex set.

Lemma: Let X be a normed linear space. Let E_1 and E_2 be any two subsets of X such that E_1 is open. Then $E_1 + E_2$ is also open set.

Proof: Let $x \in X$ and $x_1 \in E_1$

$\because$ E_1 is open, $\exists\, r > 0 \ni$

$U(x_1, r) \subset E_1$.

Consider

$$U(x + x_1, r) = \{y \in X / \|y - x - x_1\| < r\}$$

$$= \{y \in X / \|(y - x) - x_1\| < r\}$$

$$= \{x + z / \|z - x_1\| < r\}$$

where $z = y - x$.

$$= x + \{z / \|z - x_1\| < r\}$$

$= x + U(x_1, r)$

$\subset E_1 + x.$

$\therefore U(x+x_1, r) \subseteq E_1 + x$

$= \{x + x_1 \mid x_1 \in E_1\}$

$E_1 + x$ is an open set.

$\because E_1 + E_2 = \bigcup \{E_1 + x_2 \mid x_2 \in E_2\}$

$\Rightarrow E_1 + E_2$ is an open set.

* For any $r > 0$, for any open set E, rE is an open set.