Ex let $a = t_1 < t_2 < \dots < t_n = b$ be the nodes with weights $w_1, w_2, \dots w_n$

Define $Q_n : C[a,b] \longrightarrow K$ by

$$Q_n x = \sum_{j=1}^{n} w_j x(t_j) \approx \int_a^b x(t)\,dt$$

$\forall x \in C[a,b]$

Let $Qx = \int_a^b x(t)\,dt$

$\|Qx\|_\infty \le (b-a)\|x\|_\infty$.

if $x_0(t) = 1$, then $Qx_0 = \int_a^b x_0(t)\,dt = b-a$

$\Rightarrow \|Q\| = b-a$

Now

$$|Q_n x| = \left|\sum_{i=1}^{n} w_i x(t_i)\right|$$

$$\le \underbrace{\sum_{i=1}^{n} |w_i|}\ \|x\|_\infty$$

$$\|Q_n x\|_\infty \leq \alpha \|x\|_\infty \Rightarrow \|Q_n\| \leq \sum_{j=1}^{n} |w_j|$$

$\therefore Q_n : C[a,b] \to K$ is a bounded/continuous linear functional.

Let $x_0 \in C[a,b]$, $\|x_0\|_\infty = 1$

$$x_0(t_j) = \operatorname{Sgn}(w_j)$$

$j = 1, 2 \ldots n$

$$Q_n x_0 = \sum_{j=1}^{n} w_j x_0(t_j)$$

$$= \sum_{j=1}^{n} w_j \operatorname{Sgn}(w_j)$$

$$= \sum_{j=1}^{n} |w_j|$$

$$\therefore \|Q_n\| = \sum_{j=1}^{n} |w_j|.$$

Hence by previous theorem, we have the following theorem.

Theorem

Let $Q_n, Q : C[a,b] \longrightarrow \mathbb{K}$ be defined as above, and let E be a subset of $C[a,b]$ such that $\overline{\text{span}\, E} = C[a,b]$ w.r.t $\|\cdot\|_\infty$.

If there exists $c > 0$ such that

$$\sum_{j=1}^{n} |w_j| \leq c < \infty, \quad \forall n \in \mathbb{N}$$

and $\{Q_n x\}$ converges to Qx for every $x \in E$, then $Q_n x \longrightarrow Qx$ for every $x \in X = C[a,b]$.

Note :- In the above theorem, we may take $E = \{ x_k(t) = t^{k-1}, k = 1, 2, 3 \dots \}$

Also if w_j are real and non-negative, then with $x_0(t) = 1$, $\forall t \in [a, b]$, we have

$$Q_n x_0 = \sum_{j=1}^{n} w_j = \sum_{j=1}^{n} |w_j| \quad \text{--- (1)}$$

So $Q_n x_0 \to \int_a^b x_0(t)\,dt$ implies the boundedness of $\sum_{i=1}^{n} |w_i|$.

Hence we have

Theorem: Let $x_k(t) = t^{k-1}, k = 1, 2, 3 \dots$ and $w_j > 0$, $\forall j = 1, 2 \dots n$.

If $Q_n(x_k) \to \int_a^b x_k(t)\,dt$, $\forall k \in \mathbb{N}$, then $Q_n x = \int_a^b x(t)\,dt$, $\forall x \in C[a, b]$.

Let $\{P_0(t), P_1(t), P_2(t), \ldots \ldots\}$ be the set of Legendre polynomials obtained by orthogonalizing the set $\{1, t, t^2, t^3, \ldots\}$ w.r.t L^2 - inner product on the set of polynomials $P[a, b]$.

[Inner product space :—

Let $V(F)$ be a vector space over the field F. An inner product on V is a function $\langle \cdot , \cdot \rangle : V \times V \longrightarrow F$ such that

(1) $\langle v, v \rangle \geq 0, \forall v \in V$

$\langle v, v \rangle = 0 \Leftrightarrow v = 0$

(ii) $\langle v, u \rangle = \overline{\langle u, v \rangle}$, $\forall u, v \in V$

(iii) $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$

$\forall \alpha, \beta \in F$

$u, v, w \in V$.

Then we say

$(V, \langle \cdot, \cdot \rangle)$ is an inner product space

Ex: $X = C[a, b]$ define $\langle \cdot, \cdot \rangle : X \times X \to K$

by $\forall f, g \in C[a, b]$,

$$\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} \, dt$$

$$\langle f, f \rangle = \int_a^b f(t) \overline{f(t)} \, dt = \int_a^b |f(t)|^2 \geq 0$$

$$\langle f, f \rangle = 0 \iff f = 0$$

$$\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} \, dt$$

$$= \overline{\int_a^b g(t)\overline{f(t)}}$$

$$= \overline{\langle g, f \rangle}$$

$$\langle \alpha f + \beta g, h \rangle = \int_a^b (\alpha f + \beta g)(t)\,\overline{h(t)}\,dt$$

$$= \alpha \int_a^b f(t)\overline{h(t)} + \beta \int_a^b g(t)\overline{h(t)}\,dt$$

$$= \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

$\therefore \langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\,dt$ is an inner product on $C[a,b]$.

$(C[a,b], \langle \cdot, \cdot \rangle)$ is an inner product space.

We say f is orthogonal to g if $\langle f, g \rangle = 0$, we write

$f \perp g$

$$\langle P_i, P_j \rangle = 0 \quad \forall\, i \neq j$$

$$= \int_a^b P_i(t) P_j(t)\, dt = 0 \quad \forall\, i \neq j$$

$P_i \in (-1, +1)$

We also have

(1) $\text{Span}\{1, t, t^2, \ldots, t^n\} = \text{Span}\{P_0(t), P_1(t), P_2(t), \ldots, P_n(t)\}$

(2) If $q(t)$ is any polynomial of degree $\leq n-1$, then

$$\int_a^b P_n(t) \cdot q(t)\, dt = 0,$$

where $P_n(t)$ is Legendre polynomial of degree n.

Let $t_1, t_2 \ldots t_n$ be the zeros of Legendre polynomial P_n of degree n.

Then $t_1, t_2 \ldots t_n$ are real and distinct, and lie in (a, b).

Define for any $x \in C[a, b]$

$$L_n x(t) = \sum_{j=1}^{n} l_j(t)\, x(t_j)$$

where

$$l_j(t) = \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{(t - t_i)}{(t_j - t_i)}, \quad \forall\, t \in [a, b]$$

degree of l_j is $n-1$ and

and $l_j(t_i) = \delta_{ij} \;\forall\, i, j$

$$L_n x(t_i) = \sum_{j=1}^{n} l_j(t_i)\, x(t_i)$$

$$= \sum_{j=1}^{n} \delta_{ij}\, x(t_j)$$

$$= x(t_i), \quad i = 1,2,\ldots,n$$

$$L_n^2 x(t) = L_n(L_n x(t))$$

$$= L_n\left(\sum_{j=1}^{n} l_j(t)\, x(t_j)\right)$$

$$= \sum_{j=1}^{n} x(t_j)\, L_n l_j(t)$$

$$= \sum_{j=1}^{n} x(t_j)\, l_j(t)$$

$$= L_n x(t) \quad \forall t$$

$$\therefore L_n^2 x = L_n x, \quad \forall x \in C[a,b]$$

$$\Rightarrow L_n^2 = L_n$$

$$\left[\because L_n l_j(t) = \sum_{i=1}^{n} l_i(t) \cdot l_j(t_i) \right.$$

$$= \sum_{i=1}^{n} l_i(t)\,\delta_{ij}$$

$$= l_j(t) \,]$$

$\therefore L_n : C[a,b] \longrightarrow X_n$

is a projection operator

where X_n is space of all

polynomials of degree $\leq n-1$.

define

$Q_n : C[a,b] \longrightarrow K$, by

$$Q_n x = \int_a^b L_n x(t)\,dt$$

$$= \int_a^b \sum_{i=1}^{n} x(t_i)\, l_i(t)\,dt$$

$$= \sum_{i=1}^{n} \left(\int_a^b l_i(t)\,dt \right) x(t_i)$$

n

$$= \sum_{i=1}^{n} w_i x(t_i)$$

where $w_i = \int_a^b l_i(t)\,dt$

The operator

$$Q_n x = \sum_{i=1}^{n} w_i x(t_i), \quad w_i = \int_a^b l_i(t)\,dt$$

is called Gauss quadrature formulae

Now we prove

$Q_n P = P$, for all polynomials of degree $\leq n-1$, and $w_i > 0$, $i = 1, 2, \dots n$

So let $P(t)$ be any polynomial of degree atmost $n-1$.

Then since $L_n P(t)$ is also a polynomial of degree atmost $n-1$, and $L_n p(t_i) = p(t_i)$, $i = 1, 2 \dots n$.

$\Rightarrow \left(L_n p - p\right](t_i) = 0$, $i = 1, 2 \dots n$

Thus $L_n p - p$ is a polynomial of degree $n-1$, vanishing at n points $t_1, t_2 \dots t_n$.

$\therefore L_n p - p = 0$

$\Rightarrow L_n p = p$.

$$\therefore Q_n p = \int_a^b L_n p(t)\,dt = \int_a^b p(t)\,dt = Q(p)$$

Now we prove $w_j > 0 \ \forall j = 1, 2 \dots n$.

To prove this, first we prove

$$Q_n(f) = \int_a^b f(t)\,dt \quad \forall f \in P_{2n-1}$$

let $f(t)$ be a polynomial of

degree atmost $2n-1$.

Since $p_n \in P_n$ is a polynomial of degree n, there exists polynomials $q(t)$ and $r(t)$ such that

$$f(t) = p_n(t)\,q(t) + r(t)$$

where $r(t)$ is a polynomial of degree atmost $n-1$.

$$\therefore \int_a^b f(t)\,dt = \underbrace{\int_a^b p_n(t)\,q(t)\,dt}_{=0} + \int_a^b r(t)\,dt$$

$$\Rightarrow \int_a^b f(t)\,dt = \int_a^b r(t)\,dt \quad \text{--- } (*)$$

$[\because \langle P_n, q\rangle = \int_a P_n(t) q(t) = 0,$

for all polynomials $q(t)$ of degree atmost $n-1$, since P_n is a Legendre polynomial of degree $n]$

Also $f(t_i) = P_n(t_i) q(t_i) + r(t_i)$

$$= r(t_i) \quad i=1,2 \cdots n$$

$[\because P_n(t_i) = 0]$

$$\therefore Q_n f = \sum_{i=1}^{n} \omega_i f(t_i) = \sum_{i=1}^{n} \omega_i r(t_i)$$

$$= \int_a^b r(t)\,dt$$

$$= \int_a^b f(t)\,dt$$

(by $*$)

$$\therefore Q_n f = \int_a^b f(t)\,dt, \quad \forall f \in P_{2n-1}.$$

Now $[l_j(t)]^2$ is a polynomial of

degree atmost $2n-1$

and $\quad l_j(t_i) = \delta_{ij}$

$$\therefore \quad 0 < \int_a^b [l_j(t)]^2 dt = Q_n(l_j^2)$$

$$= \sum_{i=1}^{n} w_i \underbrace{(l_j(t_i))}_{\delta_{ij}}^2$$

$$= w_j$$

$$\therefore w_j = \int_a^b l_j(t)\,dt > 0 \quad \forall j = 1, 2 \dots n.$$

Thus by previous theorem.

$$Q_n x = \sum_{i=1}^{n} w_i x(t_i)$$

$$\longrightarrow \int_a^b x(t)\,dt = Qx.$$

$$\forall x \in C[a,b]$$

—— // ——

Completeness of $BL(X, Y)$:—

Theorem: Let X and Y be n.l.s. If Y is a Banach space, then $BL(X, Y)$ is also a Banach space. In particular dual of a n.l.s X is complete.
i.e $X' = BL(X, K)$ is complete

Proof:

Suppose Y is a Banach space, and $\{A_n\}$ be a Cauchy sequence in $BL(X, Y)$.

Claim: $\{A_n\}$ is convergent in $BL(X, Y)$.

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ $\ni$

$$\|A_n - A_m\| < \epsilon, \quad \forall\, n, m \geq n_0.$$

Then for each $x \in X$, we have

$$\|A_n x - A_m x\| = \|(A_n - A_m)x\|$$

$$\leq \|A_n - A_m\| \, \|x\|$$

$$< \epsilon \|x\|, \quad \forall\, n, m \geq n_0$$

$\Rightarrow \{A_n x\}$ is a Cauchy sequence in Y.

Since Y is a Banach space, $\{A_n x\}$ converges in Y.

Also $\{A_n\}$ is a Cauchy sequence in $BL(X,Y)$, implies $\{\|A_n\|\}$ is bounded.

Define $A : X \to Y$ by

$$Ax = \lim_{n\to\infty} A_n x, \quad \text{then}$$

A is linear and

$$\|A\| \leq \limsup_{n} \|A_n\| < \infty$$

$$\Rightarrow A \in BL(X, Y).$$

Now for each $x \in X$, $m, n \geq n_0$ and fixed m, we have

$$\|(A - A_m)x\| = \lim_{n\to\infty} \|(A_n - A_m)x\|$$

$$\leq \left(\limsup_{m,n} \|A_n - A_m\|\right) \|x\|$$

$$< \epsilon \|x\|$$

$$\Rightarrow \|A - A_m\| \leq \epsilon, \quad \forall\, m \geq n_0$$

$\therefore\ A_n \to A$ in $BL(X, Y)$

$\Rightarrow BL(X, Y)$ is complete.

Let X and Y be n.l.s and $BL(X, Y)$ be set of all bounded linear operators from X to Y.

Let $\mathcal{A} = \{A_i \,/\, A_i \in BL(X, Y)\}$ be a family of bounded operators from X to Y.

We say $\mathcal{A}$ is pointwise bounded on X, if for each $x \in X$, $\exists\ M_x > 0$ such that

$$\|Ax\| \leq M_x \|x\|, \ \forall\ A \in \mathcal{A}.$$

We say $\mathcal{A}$ is uniformly bounded

if $\{ \|A\| / A \in \mathcal{A} \}$ is

a bounded set.

$\left[\{ \|A_1\|, \|A_2\|, \|A_3\| \ldots \ldots \} \right.$

is a bounded subset of $\mathbb{R}$,

i.e., $\exists\, M > 0 \ni$

$\|A_n\| \leq M \;\; \forall n.$

$\left[\|A_n x\| \leq M \|x\|, \; \forall x \in X \right]$

$\forall A_n \in \mathcal{A}.$

$\Rightarrow \dfrac{\|A_n x\|}{\|x\|} \leq M, \quad x \neq 0$

$\left. \Rightarrow \|A_n\| \leq M, \; \forall n. \right]$

—————— // ——————