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Note :- A linear operator $F : X \to Y$ is shown to be discontinuous by showing that there exists a bounded set $E \subseteq X$ such that $\{F(x) \mid x \in E\}$ is not bounded in Y.

OR

Produce a bounded sequence $\{x_n\}$ in X such that $\{Ax_n\}$ is unbounded in Y.

Ex :- $X = C^1[0,1]$ with $\|\cdot\|_\infty$

Define $f : X \longrightarrow K$ by

$$f(x) = x'(1), \quad \forall x \in X.$$

Clearly f is linear map, but it is not continuous. Since for the sequence $x_n(t) = t^n$, $t \in [0,1]$

Then $\|x_n\|_\infty = 1$ and

$f(x_n) = |x_n'(1)| = n$, $\forall n \in \mathbb{N}$

Then $\{f(x_n)\}$ is unbounded

$\|f(x_n)\| = n \longrightarrow \infty$

Thus there is no constant α

$\ni \|f(x_n)\| \le \alpha \|x_n\| \quad \forall n$

$\therefore f : X \longrightarrow K$ is discontinuous.

(2) let $X = C^1[0,1]$ with $\|\cdot\|_\infty$

$Y = C[0,1]$ with $\|\cdot\|_\infty$

Define $A : X \longrightarrow Y$ by

$Ax(t) = x'(t)$, $\forall t \in [0,1]$.

$\because x_n(t) = t^n \Rightarrow \|x_n\|_\infty = 1$

and $\|Ax_n\|_\infty = \sup_{t \in [0,1]} |Ax_n(t)|$

$$= \sup_{t\in[0,1]} |x_n'(t)|$$

$$= \sup_{t\in[0,1]} |n t^{n-1}| = n$$

$\therefore \|Ax_n\|_\infty = n \longrightarrow \infty$.

$\therefore$ A is unbounded operator.

Ex: $X = c_{00}$, with $\|\cdot\|_\infty$

$f : X \longrightarrow K$ by

$$f(x) = \sum_{j=1}^{\infty} x(j)$$

$\forall\, x = (x(1), x(2) \cdots \cdots) \in X$.

Let $x_n = (\underbrace{1, 1, 1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$

$$\|x_n\|_\infty = \sup |x_n(i)| = 1$$

and $$|f(x_n)| = \left|\sum_{j=1}^{\infty} x_n(j)\right| \leq \sum_{j=1}^{n} |x_n(j)| = n$$

$$|f(x_n)| = n \longrightarrow \infty \text{ as } n \to \infty$$

Thus for a bounded sequence $\{x_n\}$, $\{f(x_n)\}$ is unbounded.

$\therefore$ f is discontinuous.

* A linear map on a linear space X may be continuous w.r.t some norm on X, but may be discontinuous w.r.t some other norm on X.

Ex: $X = c_{00}$ and $f : X \to \mathbb{K}$

by $f(x) = \sum_{j=1}^{\infty} x(j)$, $x \in X$.

Then

$$|f(x)| = \left|\sum_{j=1}^{\infty} x(j)\right| \leq \sum_{j=1}^{\infty} |x(j)|$$

$$= \|x\|_1$$

$\therefore$ $|f(x)| \leq 1 \cdot \|x\|_1$

$\Rightarrow$ f is bounded w.r.t $\|\cdot\|_1$

But f is discontinuous w.r.t

$\|\cdot\|_2$.

$\because$ $x_n = (1, \frac{1}{2}, \frac{1}{3} \cdots \frac{1}{n}, 0, 0 \cdots) \in c_{00}$

Then $\|x_n\|_2^2 = \sum_{j=1}^{n} \frac{1}{j^2} \leq \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} < \infty$

But $f(x_n) = \sum_{j=1}^{n} \frac{1}{j} \longrightarrow \infty$ as $n \to \infty$

$|f(x_n)| \longrightarrow \infty$ as $n \to \infty$.

$\therefore$ f is discontinuous w.r.t $\|\cdot\|_2$.

Consider another linear map

$f_1 : c_{00} \longrightarrow \mathbb{K}$ by

$$f_1(x) = \sum_{j=1}^{\infty} \frac{x(j)}{j}, \quad x \in c_{00}$$

Then

$$|f_1(x)| = \left| \sum_{j=1}^{\infty} \frac{x(j)}{j} \right|$$

$$\leq \sum_{j=1}^{\infty} \frac{1}{j} |x(j)|$$

$$\leq \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \right)^{1/2} \cdot \left(\sum_{j=1}^{\infty} |x(j)|^2 \right)^{1/2}$$

$$\le \frac{\pi}{\sqrt{6}} \|x\|_2$$

$\therefore f_1$ is continuous w.r.t $\|\cdot\|_2$.

But f_1 is discontinuous w.r.t $\|\cdot\|_\infty$.

Let $x_n = (\underbrace{1, 1, 1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$

Then $\|x_n\|_\infty = 1$

$$|f_1(x_n)| = \sum_{j=1}^{n} \frac{1}{j} \longrightarrow \infty \text{ as } n \to \infty$$

$\therefore f_1$ is discontinuous

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Ex: Consider the infinite matrix of scalars (a_{ij}), $a_{ij} \in K$ $\forall\, i, j$

For $x = (x(1), x(2), x(3) \dots) \in F(\mathbb{N}, \mathbb{K})$,
the set of all functions from $\mathbb{N}$ to $\mathbb{K}$.

define $A : X \longrightarrow Y$, $X, Y \subseteq F(\mathbb{N}, \mathbb{K})$

$$Ax(i) = \sum_{j=1}^{\infty} a_{ij} x(j), \quad i = 1, 2, 3 \dots$$

$x \in X$

$$Ax = A(x(1), x(2), x(3) \dots)$$
$$= (Ax(1), Ax(2), Ax(3) \dots)$$

Assume $\sum_{j=1}^{\infty} |a_{ij}| |x(j)| < \infty$ $\forall x \in X$

Let $\alpha = \sup_{j} \sum_{i=1}^{\infty} |a_{ij}| < \infty$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{i1} & a_{i2} & a_{ij} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

Now for all $x \in \ell^1 = X = Y$ we have

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|\,|x(j)|$$

$$= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{ij}| \right) |x(j)|$$

Now

$$\|Ax\|_{\ell^1} = \sum_{i=1}^{\infty} |Ax(i)|$$

$$= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{ij} x(j) \right|$$

$$\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|\,|x(j)|$$

$$= \sum_{j=1}^{\infty} \Big(\underbrace{\sum_{i=1}^{\infty} |a_{ij}|}_{\alpha_j} \Big) |x(j)|$$

$$\leq \left(\sup_j \sum_{i=1}^{\infty} |a_{ij}| \right) \cdot \sum_{j=1}^{\infty} |x(j)|$$

$$= \alpha \, \|x\|_{\ell^1}$$

$\therefore \|Ax\|_{\ell^1} \le \alpha \|x\|_{\ell^1}, \; \forall x \in \ell^1.$

$\Rightarrow Ax \in \ell^1$ and

$A : \ell^1 \to \ell^1$ is a bounded linear map.

Next assume that (a_{ij}) is an infinite matrix of scalars such that

$$\beta = \sup_i \sum_{j \ge 1}^{\infty} |a_{ij}| < \infty .$$

$$\because Ax(i) = \sum_{j \ge 1}^{\infty} a_{ij} x(j), \quad i \in \mathbb{N}$$

Now take $X = Y = \ell^\infty$. Then for any $x \in \ell^\infty$,

$$\|Ax\|_{\infty} = \sup_i |Ax(i)|$$

Now $|Ax(i)| = \left|\sum_{j\geq 1} a_{ij}x(j)\right|$

$$\leq \sum_{j\geq 1}^{\infty} |a_{ij}|\,|x(j)|$$

$$\leq \left(\sup_i \sum_{j\geq 1}^{\infty} |a_{ij}|\right) \sup_j |x(j)|$$

$$= \beta \|x\|_\infty .$$

$\therefore\ |Ax(i)| \leq \beta \|x\|_\infty \quad \forall\, x \in \ell^\infty$

$\Rightarrow \sup_i |Ax(i)| \leq \beta \|x\|_\infty$

$\Rightarrow \|Ax\|_\infty \leq \beta \|x\|_\infty .$

$\therefore$ $Ax \in \ell^\infty$ and $A : \ell^\infty \to \ell^\infty$ is a bounded linear map.

* Next for the infinite matrix (a_{ij}),

Let $$\alpha_{p,q} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|^q \right)^{p/q}$$

$$\beta = \sup_i \sum_{j=1}^{\infty} |a_{ij}|$$

$$\gamma = \sup_j \sum_{i=1}^{\infty} |a_{ij}|$$

where $1 < p < \infty$ and q is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

Assume that
$$\min\left\{ \alpha_{p,q}^{1/p},\ \beta^{\frac{1}{q}} \gamma^{1/p} \right\} < \infty.$$

Then $$Ax(i) = \sum_{j=1}^{\infty} a_{ij} x(j), \quad i \in \mathbb{N}$$

defines a bounded linear operator

on ℓ^p and

$$\|Ax\|_{\ell^p} \leq \min\left\{ \alpha_{p,q}^{1/p} , \beta^{1/q} \gamma^{1/p} \right\} \|x\|_{\ell^p}$$

Sol: For any $x \in \ell^p$, by Holder's inequality, we have

$$\sum_{j=1}^{\infty} |a_{ij}| \, |x(j)| \leq \left(\sum_{j=1}^{\infty} |a_{ij}|^q \right)^{1/q} \|x\|_p \quad \text{--- } (*)$$

Now

$$|Ax(i)|^p = \left| \sum_{j=1}^{\infty} a_{ij} x(j) \right|^p$$

$$\leq \left[\sum_{j=1}^{\infty} |a_{ij}| \, |x(j)| \right]^p$$

$$\leq \left[\left(\sum_{j=1}^{\infty} |a_{ij}|^q \right)^{1/q} \|x\|_p \right]^p$$

$$= \left(\sum_{j=1}^{\infty} |a_{ij}|^q \right)^{p/q} \cdot \|x\|_p^p$$

$$\Rightarrow \sum_{i=1}^{\infty} |Ax(i)|^p \leq \sum_{j=1}^{\infty} \left(\sum_{j=}^{\infty} |a_{ij}|^q \right)^{p/q} \cdot \|x\|_p^p$$

$$= \alpha_{p,q} \cdot \|x\|_p^p$$

$$\Rightarrow \|Ax\|_p^p \leq \alpha_{p,q} \|x\|_p^p$$

$$\Rightarrow \|Ax\|_p \leq \alpha_{p,q}^{1/p} \|x\|_p \quad \forall x \in \ell^p \quad \text{(1)}$$

$\Rightarrow$ $Ax \in \ell^p$ and $A : \ell^p \to \ell^p$ is a bounded linear operator.

Again using Holder inequality for any $x \in \ell^p$, we have

$$\sum_{j=1}^{\infty} |a_{ij}| |x(j)| = \sum_{j=1}^{\infty} |a_{ij}|^{1/q} |a_{ij}|^{1/p} |x(j)|$$

$$\leq \left(\sum_{j=1}^{\infty} |a_{ij}| \right)^{1/q} \cdot \left(\sum_{j=1}^{\infty} |a_{ij}| |x(j)|^p \right)^{1/p}$$

$$\Rightarrow \left(\sum_{j=1}^{\infty} |a_{ij}|\,|x(j)|\right)^{p} \leq \left(\sum_{j=1}^{\infty} |a_{ij}|\right)^{p/q} \times \sum_{j=1}^{\infty} |a_{ij}|\,|x(j)|^{p}$$

$$\leq \left(\sup_{i} \sum_{j=1}^{\infty} |a_{ij}|\right)^{p/q} \cdot \sum_{j=1}^{\infty} |a_{ij}|\,|x(j)|^{p}$$

$$= \beta^{p/q} \cdot \sum_{j=1}^{\infty} |a_{ij}|\,|x(j)|^{p} \quad (**)$$

$$\therefore \sum_{i=1}^{\infty} |Ax(i)|^{p} = \sum_{i=1}^{\infty} \left|\sum_{j=1}^{\infty} a_{ij}\, x(j)\right|^{p}$$

$$\leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|\,|x(j)|\right)^{p} \quad (\text{by } (**))$$

$$\leq \beta^{p/q} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|\,|x(j)|^{p}\right)$$

$$= \beta^{p/q} \cdot \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{ij}|\,|x(j)|^{p}\right)$$

$$\leq \beta^{p/q}\left(\sup_j \sum_{i=1}^{\infty} |a_{ij}|\right) \cdot \sum_{j=1}^{\infty} |x(j)|^p$$

$$= \beta^{p/q}\, \gamma\, \|x\|_p^p$$

$$\Rightarrow \|Ax\|_p^p \leq \beta^{p/q}\, \gamma\, \|x\|_p^p$$

$$\Rightarrow \|Ax\|_p \leq \beta^{1/q} \cdot \gamma^{1/p}\, \|x\|_p \quad \text{—} (2)$$

$\therefore$ From (1) & (2) we get

$$\|Ax\|_p \leq \min\left\{\alpha_{p,q}^{1/p},\ \beta^{1/q}\gamma^{1/p}\right\} \|x\|_p$$

$\therefore$ A is a bounded linear map on ℓ^p, $1 \leq p \leq \infty$.

$$\|Ax\|_p \le \alpha_{p,\Sigma}^{1/2} \|x\|_p$$

$$\|Ax\|_p \le \beta^{1/2} \gamma^{1/p} \|x\|_p$$

$$\therefore \|Ax\|_p \le \min\left\{ \alpha_{p,\Sigma}^{1/2},\ \beta^{\frac{1}{2}} \gamma^{\frac{1}{p}} \right\} \|x\|_p$$