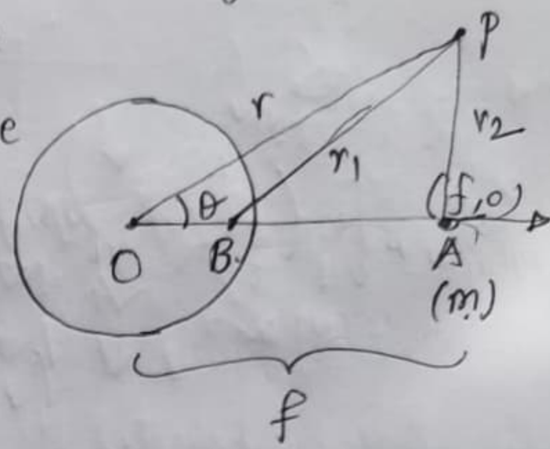
§ Image of a source w.r.t. a circle: 15.09.21

Let us consider/determine the image of a source of strength m at a point A w.r.t. the circle C with center at origin O.

Let $OA = f$ and a be the radius of the circle. Let B be the inverse pt. ~~image~~ of A (~~inverse point~~) w.r.t. the circle. The complex potential at the point $P(z)$, when the source is alone, is



$$w(z) = f(z) = -m \log_e (z-f)$$

Then $\bar{f}(z) = -m \log(z-f)$.

$$\Rightarrow \bar{f}\left(\frac{a^2}{z}\right) = -m \log_e \left(\frac{a^2}{z} - f\right).$$

Since the circle is introduced, i.e., $|z| = a$ is present, by Circle theorem,

$$\hat{w}(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

$$= -m\left[\log(z-f) + \log_e\left(\frac{a^2}{z} - f\right)\right].$$

$$= -m\left[\log(z-f) + \log_e(a^2 - zf) - \log_e z\right].$$

$$= -m\left[\log(z-f) - \log z + \log(-f)\left(z - \frac{a^2}{f}\right)\right]$$

$$= -m\log(z-f) + m\log z - m\log\left(z - \frac{a^2}{f}\right) - m\log_e(-f)$$

we have $-m\log(-f)$, the constant $-m\log(-f)$, real or complex, is immaterial from the view point of analysis the flow.

$$\hat{w}(z) = -m\log(z-f) - m\log\left(z-\frac{a^2}{f}\right) + m\log z$$

$$+ \underbrace{\text{const.}}_{(\text{omit})}$$

The complex potential of the flow consists of a source of strength m at A $(z=f)$, a sink of strength m at $O(z=0)$ and a source of strength m at $B\left(\frac{a^2}{f}\right)$

Since B is the inverse pt. of a circle, we have

$$|OA|\cdot|OB| = a^2 \Rightarrow f\cdot|OB| = a^2 \Rightarrow |OB| = \frac{a^2}{f}$$

Let $Q(z = ae^{i\theta})$ be any pt. on the circle C. Then.

$$w = -m\log(ae^{i\theta}-f) - m\log\left(ae^{i\theta}-\frac{a^2}{f}\right)$$

$$+ m\log(ae^{i\theta})$$

$$= -m\log\left\{(ae^{i\theta}-f)\left(ae^{i\theta}-\frac{a^2}{f}\right)\right\} + m\log(ae^{i\theta})$$

Equating the imaginary part,

$$\psi = -m\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta - f}\right) - m\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta - \frac{a^2}{f}}\right)$$

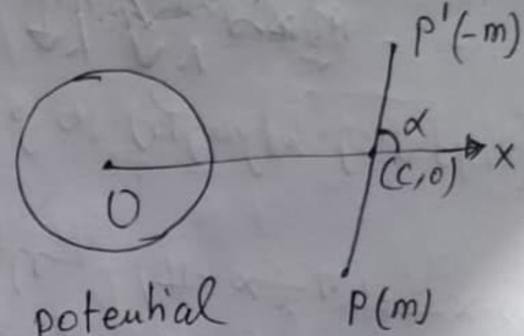
$$+ m\theta$$

$$\Rightarrow \psi = -m\tan^{-1}(\tan\theta) + m\theta \qquad \left[\tan^{-1}A+\tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}\right]$$

$$\Rightarrow \psi = -m\theta + m\theta = 0$$

Therefore $\psi=0$ is the required stream function / streamline, i.e., the circle is a streamline.

§ Image of a doublet w.r.t. a circle:



Let a doublet of strength μ be placed at $z=c$ and the axis of the doublet makes an angle α with +ve x-axis. The complex potential due to the doublet is

$$f(z) = \frac{\mu e^{i\alpha}}{z-c}$$

$$\Rightarrow \bar{f}(z) = \frac{\mu e^{-i\alpha}}{z-c} \Rightarrow \bar{f}\left(\frac{a^2}{z}\right) = \frac{\mu e^{-i\alpha}}{\left(\frac{a^2}{z}-c\right)}$$

Let the circle $|z|=a$ be introduced within the flow;

$$w(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right) = \frac{\mu e^{i\alpha}}{z-c} + \frac{\mu e^{-i\alpha}}{\frac{a^2}{z}-c}$$

$$= \frac{\mu e^{i\alpha}}{z-c} - \frac{\mu e^{i(\pi-\alpha)}}{\frac{a^2}{z}-c}$$

$$= \frac{\mu e^{i\alpha}}{z-c} + \frac{\mu z e^{i(\pi-\alpha)}}{c\left(z-\frac{a^2}{c}\right)}$$

$$= \frac{\mu e^{i\alpha}}{z-c} + \frac{\mu e^{i(\pi-\alpha)}}{c} \cdot \frac{\{z - \frac{a^2}{c} + \frac{a^2}{c}\}}{z - \frac{a^2}{c}}$$

$$= \frac{\mu e^{i\alpha}}{z-c} + \frac{\mu e^{i(\pi-\alpha)}}{c} + \frac{\mu a^2}{c^2} \frac{e^{i(\pi-\alpha)}}{z-\frac{a^2}{c}}$$

neglect

$$\cong \frac{\mu e^{i\alpha}}{z-c} + \frac{\mu a^2}{c^2} \frac{e^{i(\pi-\alpha)}}{z - \frac{a^2}{c}}$$

$\frac{\pi}{2} - (\pi - \theta)$
$= \theta - \frac{\pi}{2}$

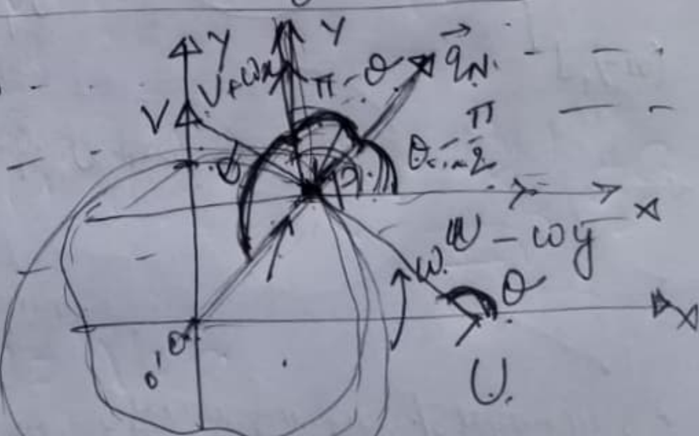
The image of a doublet w.r.t. a circle consists of

(i) a doublet of strength μ at $z=c$ inclined at an angle α

(ii) a doublet of " $\frac{\mu a^2}{c^2}$ at an inverse pt. $z = \frac{a^2}{c}$ inclined at an angle $(\pi-\alpha)$ with x-axis.

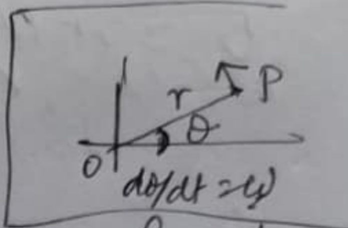
§ General motion of fluid around a cylinder (2D flow):

Let a point of the crossection of the cylinder chosen at the origin O and at origin let U and V be the fluid velocities along x and y-axes, respectively. Let ω be the angular velocity of rotation



(i) The fluid at infinity is at rest because there is no effect of the cylinder at the fluid particles far away. We will have

$$\frac{\partial \varphi}{\partial x} = 0 \ \& \ \frac{\partial \varphi}{\partial y} = 0 \quad \text{at infinity}$$



(ii) ~~At~~ any fixed pt. on the boundary the normal velocity ~~must be zero~~. At boundary of the moving cylinder the normal component of the velocity of the fluid must be equal to the normal component of the velocity of the cylinder.

The velocity components at P are given by ($\vec{q}$ at P)

$$u = U + \frac{dx}{dt} \quad \text{and} \quad v = V + \frac{dy}{dt} \quad -(1)$$

Since $x = r\cos\theta$ and $y = r\sin\theta$

$$\Rightarrow \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt} \quad \text{and} \quad \frac{dy}{dt} = r\cos\theta\frac{d\theta}{dt}$$

$$= -\underbrace{r\omega\sin\theta} \quad \text{and} \quad = \underbrace{r\omega\cos\theta}$$

$$= -\omega y \qquad\qquad = \omega x$$

$$(1) \Rightarrow \quad u = U - \omega y \quad \text{and} \quad v = V + \omega x \quad -(\text{ii})$$

From Diff. Calculus: $\cos\theta = \frac{dx}{ds}$ and $\sin\theta = \frac{dy}{ds}$ $-(\text{iii})$

∴ (ii)

Therefore the outward normal velocity at P

$= (\text{velocity along } x\text{-axis at } P) \cos(\theta - \frac{\pi}{2}) +$

$(\text{velocity along } y\text{-axis at } P) \cos(\pi - \theta)$

$= (U - \omega y) \sin\theta - (V + \omega x) \cos\theta$

$= (U - \omega y) \frac{dy}{ds} - (V + \omega x) \frac{dx}{ds}$ — (III)

The normal velocity of the ~~veloc~~ fluid at $P(x, y)$ on the surface of the cylinder is $-\frac{\partial \psi}{\partial s}$ — (IV)

From (III) & (IV)

$$-\frac{\partial \psi}{\partial s} = (U - \omega y) \frac{dy}{ds} - (V + \omega x) \frac{dx}{ds}$$

$$\Rightarrow \quad \partial \psi = (V + \omega x)\, dx - (U - \omega y)\, dy$$

$$\Rightarrow \quad \psi(x, y) = \frac{\omega}{2}(x^2 + y^2) + (Vx - Uy) + C$$ (V)

Const.

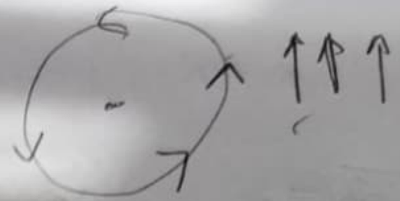
(i) There is only rotation: $U = V = 0$

$$\psi(x, y) = \frac{\omega}{2}(x^2 + y^2) + C$$

The streamline $\psi(x, y) = c \Rightarrow \frac{\omega}{2}(x^2 + y^2) + C = C_1$

$\Rightarrow \quad x^2+y^2 = C$

Circle



(vi) There is no rotation $\omega=0, V=0$ from (v):

$$\psi(x,y) = -Uy \Rightarrow \psi(x,y)=c \Rightarrow -uy=c$$

$\Rightarrow y = \text{Const.}$

uniform.

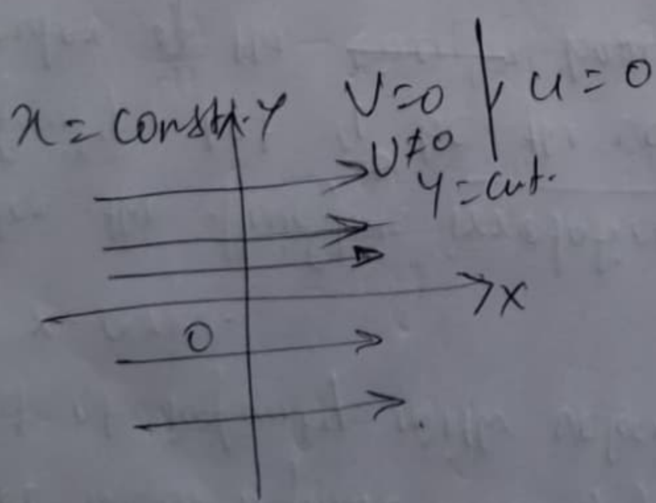
(iii) " " " $\omega=0,\ U=0$

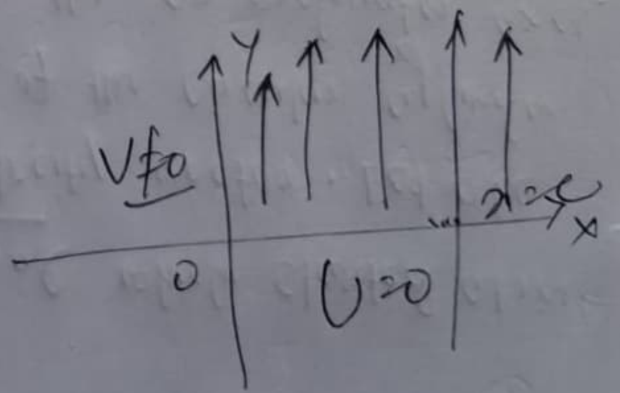
$$\psi(x,y) = Vx \Rightarrow \psi=\text{Cst} \Rightarrow x=\text{Cnt}$$

→ uniform flow

(iv) $\omega=0,\ U=0,\ V=0 \Rightarrow \underline{\underline{\psi=0}}$

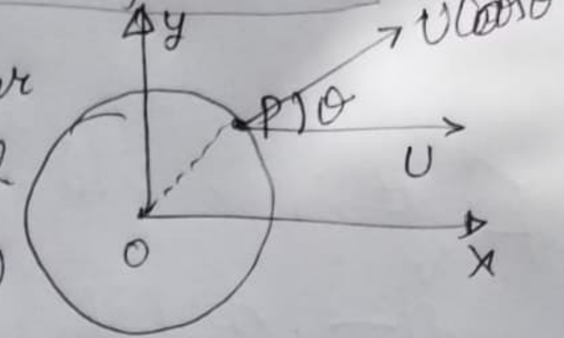
(v) $\omega=0, \quad \psi(x,y)=\text{const} \Rightarrow Vx-Uy=\text{Cnt}$





§ Motion of circular cylinder in an uniform stream:

"To determine the motion of a circular cylinder in an infinite mass of fluid at rest at infinity with velocity U $\parallel$ to x-axis.



Consider the fluid flow is irrotational which started at rest at infinity. Let $\vec{q}$ be the velocity vector. Let O be the center of the ~~circular~~ base of the circular cylinder which is taken as origin of the co-ordinates axes.

There exists a ϕ s.t. $\vec{q} = -\nabla\phi$

$\nabla^2\phi = 0$.

$\phi(x,y) = ax + by$ $\Rightarrow \vec{\nabla}\cdot\vec{q} = 0 \Rightarrow \nabla^2\phi = 0$ — (I)

(r,θ)

$\Rightarrow \phi \neq 0$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

$$\Rightarrow \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \quad \text{— (II)}$$

$$\phi(r,\theta) = R(r)\,\Phi(\theta)$$

The solution of (II) has the forms,

$r^n \cos n\theta$ and $r^n \sin n\theta$, $n \in \mathbb{Z}$

Hence the sum of any number of terms of the form $A_n r^n \cos n\theta$ or, $B_n r^n \sin n\theta$, i.e.,

$$\phi(r,\theta) = A_n r^n \cos n\theta + B_n r^n \sin n\theta \quad \text{--- (VI)}$$