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let $\alpha_0 = \inf\{\alpha \geq 0 \mid \|F(x)\| \leq \alpha\|x\|, \forall x \in X\}$

$\beta = \sup\{\|F(x)\| \mid x \in X, \|x\| = 1\}$

$\gamma = \sup\{\|F(x)\| \mid x \in X, \|x\| < 1\}$

Since $\|F\| = \sup\{\|F(x)\| \mid x \in X, \|x\| \leq 1\}$,

we have

$$\beta, \gamma \leq \|F\| \quad \text{—(1)}$$

Now consider $x \in X$ and $0 < r \leq 1$.

Since F is linear, we have

$$\|F(x)\| = \left\| F\left(\frac{rx}{\|x\|}\right) \right\| \frac{\|x\|}{r}$$

$$\leq \sup\{\|F(z)\| \mid z \in X, \|z\| = r\} \frac{\|x\|}{r}$$

—(2)

If $r = 1$ in (2), we get

$$\|F(x)\| \leq \sup\{\|F(z)\| / z \in X, \|z\|=1\} \frac{\|x\|}{1}$$

$$= \beta \|x\|$$

$\therefore \|F(x)\| \leq \beta \|x\|, \ \forall x \in X.$

$$\Rightarrow \inf\{\alpha > 0 / \|F(x)\| \leq \alpha \|x\|_{x \in X}\} \leq \beta$$

$$\Rightarrow \alpha_0 \leq \beta$$

If $r < 1$, from (2), we have

$$\|F(x)\| \leq \sup\{\|F(z)\| / \|z\| < 1\} \frac{\|x\|}{r}$$

$$= \gamma \frac{\|x\|}{r}$$

Letting $r \to 1$, we have

$$\|F(x)\| \leq \gamma \|x\|$$

$$\Rightarrow \alpha_0 \leq \gamma$$

$$\therefore \alpha_0 \leq \min\{\beta, \gamma\} \quad \text{--- (3)}$$

Consider $\alpha > 0$ such that

$$\|F(x)\| \leq \alpha \|x\|, \ \forall \ x \in X$$

Taking supremum overall $x \in X, \|x\| \leq 1$, we have

$$\sup\{\|F(x)\| \,/\, x \in X, \|x\| \leq 1\} \leq \alpha \cdot 1$$

$$\Rightarrow \|F\| \leq \alpha$$

Since α_0 is infimum of all such α's we see that

$$\|F\| \leq \alpha_0 \quad \text{—(4)}$$

Thus from (1) (3) & (4), we get

$$\|F\| \leq \alpha_0 \leq \min\{\beta, \gamma\} \leq \|F\|$$

—//—

Ex: $X = C[a,b]$, $\|\cdot\|_\infty$

$$Ax(s) = \int_a^b k(s,t)x(t)\,dt, \quad k(\cdot,\cdot) \in C([a,b]\times[a,b])$$

Then we proved that

$$\|Ax\|_\infty \leq c\, \|x\|_\infty \, , \quad \text{--- (1)}$$

where $c = \sup\limits_{a \leq s \leq b} \int_a^b |k(s,t)|\, dt$.

For (1) taking supremum over all $x \in X$, $\|x\|_\infty \leq 1$, we get

$$\|A\|_\infty \leq c = \sup_{a \leq s \leq b} \int_a^b |k(s,t)|\, dt \quad \text{--- (1)}$$

Claim: $\|A\|_\infty = c$.

Since $\underline{s} \longrightarrow \int_a^b |k(\underline{s},t)|\, dt$ is continuous on $[a,b]$, so there exists a point $s_0 \in [a,b]$ such that

$$\sup_{a \le s \le b} \int_a^b |k(s,t)|\,dt = \int_a^b |k(s_0,t)|\,dt .$$

Now for given any $\epsilon > 0$, we have

$$\int_a^b \left[|k(s_0,t)| - \epsilon \right] dt = \int_a^b \frac{|k(s_0,t)|^2 - \epsilon^2}{|k(s_0,t)| + \epsilon}\, dt$$

$$\le \int_a^b \frac{|k(s_0,t)|^2}{|k(s_0,t)| + \epsilon}\, dt$$

$$= \int_a^b \frac{k(s_0,t) \cdot \overline{k(s_0,t)}}{|k(s_0,t)| + \epsilon}\, dt$$

$$= \int_a^b k(s_0,t) \cdot \frac{\overline{k(s_0,t)}}{|k(s_0,t)| + \epsilon} \, dt$$

$$= \int_a^b k(s_0,t) \, x_\epsilon(t) \, dt$$

Where $$x_\epsilon(t) = \frac{\overline{k(s_0,t)}}{|k(s_0,t)| + \epsilon}$$

$$\therefore \left| \int_a^b \left[|k(s_0,t)| - \epsilon \right] dt \right| \le \left| \int_a^b k(s_0,t) x_\epsilon(t) \, dt \right|$$

$$= |A x_\epsilon(t)|$$

$$\le |A x_\epsilon(t)|$$

$$\le \|A x_\epsilon\|_\infty$$

$$\le \|A\|_\infty$$

$$\left[\because |x_\epsilon(t)| = \frac{|k(s_0,t)|}{|k(s_0,t)|+\epsilon} < 1\right.$$

$$= \|x_\epsilon\|_\infty < 1$$

$$\therefore \int_a^b \left[|k(s_0,t)| - \epsilon\right] dt \le \|A\|_\infty$$

$$\Rightarrow \int_a^b |k(s_0,t)|\, dt \le \|A\|_\infty + \epsilon(b-a)$$

Letting $\epsilon \to 0$, we get

$$\int_a^b |k(s_0,t)|\, dt = \sup_{a \le s \le b} \int_a^b |k(s,t)|\, dt$$

$$\le \|A\|_\infty \quad \text{---} (2)$$

$\therefore$ from ① & ② we get

$$\|A\|_\infty = \sup_{s \in [a,b]} \int_a^b |k(s,t)|\,dt$$

Let $A : X \longrightarrow Y$ be a linear map between two n.l.s X and Y.

If $A \in BL(X, Y)$, then null space of A, $N(A)$ is a closed subspace of X.

But converse need not be true

Ex: $X = C^1[0,1]$

$Y = C[0,1]$

both with $\|\cdot\|_\infty$.

Let $A : X \longrightarrow Y$ be defined by

$$Ax = x', \quad \forall x \in X$$

Then

$$N(A) = \{x \in X \mid Ax = x' = 0\}$$

$$= \{x \in X \mid x = c\}$$

= set of all constant functions

which is a closed subspace of X.

But A is unbounded

$\because$ $x_n(t) = t^n, \quad t \in [0,1]$

Then $\|x_n\|_\infty = 1$

$\|Ax\|_\infty = n$

$\Rightarrow$ A is not continuous.

However, such a situation will not arise for linear functionals.

Theorem: Let X be a n.l.s and $f: X \to K$ be a nonzero linear functional on X such that null space $N(f)$ is closed. Then f is continuous and for every $x_0 \in X - N(f)$,

$$\|f\| = \frac{|f(x_0)|}{\operatorname{dist}(x_0, N(f))}$$