Let X and Y be any two metric spaces. A map $F: X \to Y$ is said to be open map iff for every open set E in X, its image $F(E)$ is open in Y.

* $F: X \to Y$ is continuous iff for every open set E in Y, its inverse image $F^{-1}(E)$ is open in X.

Theorem: Let X and Y be n.l.s and $F: X \to Y$ be a linear map. Then F is an open map iff there exists some $\gamma > 0$ such that for every $y \in Y$, there exists some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$.

Proof:

Let $F: X \to Y$ be open map.

Then since $U_X(0,1)$ is open in X,

so $F(U_X(0,1))$ is open in Y

As $0 = F(0) \in F(U_X(0,1))$,

there is some $\delta > 0$ such that

$$\overline{U_Y(0,\delta)} \subset F(U_X(0,1)). \quad \text{--- (1)}$$

$$\left[\ \overline{U_Y(0,\delta)} = \{y \in Y \,/\, \|y\| \leq \delta\}\ \right]$$

Now consider $y \in Y$, $y \neq 0$.

Then $\dfrac{\delta y}{\|y\|} \in \overline{U_Y(0,\delta)}$

$$\left[\because\ \left\|\frac{\delta y}{\|y\|}\right\| = \delta\right]$$

$\therefore$ By (1), we see that there exists some $x_1 \in U_X(0,1)$ such that

$$F(x_1) = \frac{\delta y}{\|y\|}$$

So letting $x = \frac{\|y\|}{\delta} x_1$, we get

$$F(x) = F\left(\frac{\|y\|}{\delta} \cdot x_1\right)$$

$$= \frac{\|y\|}{\delta} F(x_1)$$

$$= \frac{\|y\|}{\delta} \cdot \frac{\delta y}{\|y\|} = y.$$

and

$$\|x\| = \left\| \frac{\|y\|}{\delta} x_1 \right\| = \frac{\|y\|}{\delta} \cdot \|x_1\|$$

$$\leq \frac{1}{\delta} \|y\|$$

$$\left[\because x_1 \in U_X(0,1) \Rightarrow \|x_1\| < 1\right]$$

Choosing $\gamma = \frac{1}{\delta}$, we see that

$$\|x\| \leq \gamma \|y\|$$

Conversely, assume that for every $y \in Y$, $\exists$ some $x \in X$ with

$$F(x) = y \quad \text{and} \quad \|x\| \leq \gamma \|y\|,$$

$\gamma > 0$.

Claim: $F : X \to Y$ is open map.

So let E be any open set in X and $x_0 \in E$. Then

$$U_X(x_0, \delta) \subset E, \text{ for some } \delta > 0.$$

Let $y \in Y$ with $\|y - F(x_0)\| < \frac{\delta}{\gamma}$.

i.e., $y \in U_Y\left(F(x_0), \frac{\delta}{\gamma}\right)$ — $(*)$

$\because$ $y - F(x_0) \in Y$, by the assumption, there exists some $x \in X$ such that

$$F(x) = y - F(x_0)$$

and $\|x\| \leq \gamma \|y - F(x_0)\|$

$$\Rightarrow \|x\| \leq \gamma \|y - F(x_0)\| < \gamma \cdot \frac{\delta}{\gamma} = \delta$$

$$\Rightarrow x \in U_X(0, \delta).$$

Also

$$F(x) = y - F(x_0)$$

$$\Rightarrow y = F(x) + F(x_0)$$

$$= F(x + x_0)$$

and $\|x + x_0 - x_0\| = \|x\| < \delta$

$$\Rightarrow x + x_0 \in U_X(x_0, \delta)$$

$$\Rightarrow y = F(x + x_0) \in F(U_X(x_0, \delta))$$

$$\subset F(E)$$

$\Rightarrow \quad y \in F(E).$

$\Rightarrow y \in U_Y\left(f(x_0), \frac{\delta}{\gamma}\right) \subset F(E)$

$\Rightarrow$ $F(E)$ is open in Y

$\Rightarrow$ F is open map

—//—

* Interior of a proper subspace of a n.l.s is empty.

Problem: Let X and Y be n.l.s and $F: X \to Y$ be a linear map. If F is an open map, then F is surjective.

Corollary :— Let X and Y be

n.l.s and $F: X \to Y$ be bijective linear map. Then F is open map iff $F^{-1}: Y \to X$ is continuous.

Proof: $F: X \to Y$ is open map

$\iff$ For every $y \in Y$ $\exists$ $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$, $\gamma > 0$

$\iff$ $\|F^{-1}(y)\| \leq \gamma \|y\|$ $\forall y \in Y$ $\left[\because F(x) = y \Rightarrow F^{-1}y = x\right.$

$\iff$ $\|F^{-1}\| \leq \gamma$.

$\iff$ F^{-1} is continuous.