Undergraduate text on

Fluid Mechanics

Lecture Notes for

3rd Year Maths & Computing & 2nd Year MSc Students

Course code: MA40011/MA51003

By

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Chapter 1
Basic Concepts

Fluids are two types:
. Incompressible Fluid
. Compressible Fluid

. Liquids are usually incompressible. Their volume do not change when the pressure changes.
. Gasses are compressible, their volume changes when the pressure changes.

. Continuum Hypothesis: We assume that the fluid is uniformly/macroscopically distributed in a region.

. Isotropy: A fluid is said to be isotropic with respect to some properties (say pressure, velocity, density etc.) if that property remains unchanged in all direction.
If that property remains changed, then the fluid is anisotropic.

. Density: The density of fluid is mass per unit volume. Mathematically,

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V}$$

. Specific weight: The specific weight γ of a fluid is defined as the weight per unit volume. Thus,

$$\gamma = \rho g$$

where g is the acceleration due to gravity.

. Pressure: Pressure at a point p is defined as force per unit area.

$$P = \lim_{\delta S \to 0} \frac{\delta F}{\delta S}$$

. Temperature:

. Thermal conductivity: It is given by Fourier's Law.

$$q_n \propto \frac{\partial T}{\partial n} = -k \frac{\partial T}{\partial n}$$

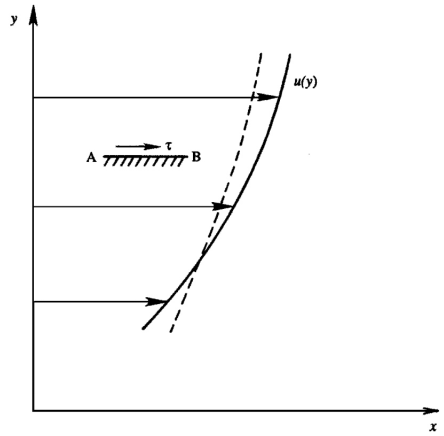
where q_n is the conductive heat flow per unit area, k is thermal conductivity.

. Viscous an Inviscous Fluid: An infinitesimal fluid element is acted upon by two types of force: (1) body force (contact force) and (2) Surface force.

Body force is proportional to mass of the body and surface force is proportional to surface area of the body.

. Viscosity: Viscosity of a fluid is that property which exhibits a certain resistance to alternation of form.

The upper plate moves with the velocity u in the x-direction, whereas the lower plate is stationary.



The fluid at y=0 is at rest and at y=h it is at motion and it is moving with the plate, then the shear stress τ is given by

$$\tau = \mu \frac{du}{dy}$$

where μ is a constant of proportionality and it is called as the viscosity of the fluid.

. Ideal fluid: Shear stress must be 0, which means that $\mu = 0$,

$\implies$ does not exist.

◇ Two types of forces exist on the fluid element-

1. Body force: It is distributed over the entire mass or volume of the element. It is expressed as the per unit mass of the element.

2. Surface force: Two types-

(i) Normal force: Along the normal to the surface area.

(ii) Shear force: Along the plane of the surface area.

$$\tau = \mu \frac{du}{dy}$$

. Laminar/ Stream line flow: A flow in which each fluid particle traces out a defnite curve and the curves traced by any two different particle does not intersect.

.Turbulent flow: Opposite to laminar flow related to Reynolds number. Higher Reynolds no.

. Steady and unsteady flow: A fluid in which properties as pressure, volume etc are independent of time t. i.e,

$$\frac{dP}{dt} = 0$$

Then such flows are called as steady flow.

If dependent of time is called unsteady flow.

.Rotational and Irrotational flow: A flow in which the fluid is rotating about own axis.

$\star$ **Lagrangion Approach:** Initially at $t = 0$, let the fluid particle is at P_0.
The current position is given in terms of initial position.

$$x = f(x_0, y_0, z_0, t)$$

$$y = g(x_0, y_0, z_0, t)$$
$$z = h(x_0, y_0, z_0, t)$$

This is the Lagrangian approach. $\implies \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\frac{dr}{dt} = \sum \frac{dx}{dt}\vec{i}$$

. This approach is very complicated for physical use.
* **Eulerian Approach:** We fix a point in space, say p and we look how a fluid particle is behaving at that point.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\frac{dr}{dt} = \sum \frac{dx}{dt}\vec{i}$$

and

$$\frac{d^2r}{dt^2} = \sum \frac{d^2x}{dt^2}\vec{i}$$

$\star$ **Relation between Lagragion and Eulerian method:**

1. Lagrange $\to$ Euler: Suppose $\phi(x_0, y_0, z_0, t)$ be a physical quantity in Lagrangian description

$$\phi = \phi(x_0, y_0, z_0, t)$$

Since Lagrangian description gives the current position as

$$x = f_1(x_0, y_0, z_0, t)$$
$$y = f_2(x_0, y_0, z_0, t)$$
$$z = f_3(x_0, y_0, z_0, t)$$

Solving we obtain

$$x_0 = g_1(x, y, z, t)$$
$$y_0 = g_2(x, y, z, t)$$
$$z_0 = g_3(x, y, z, t)$$

$\therefore \phi = \phi(x_0, y_0, z_0, t) = \phi(g_1(x, y, z, t), g_2(x, y, z, t), g_3(x, y, z, t))$

2. Euler $\to$ Lagrangian: Suppose $\phi(x, y, z, t)$ is a physical associated with the flow

$$\phi = \phi(x, y, z, t)$$

Eulerian description, we can write,

$$\begin{aligned}\frac{dr}{dt} &= u\vec{i} + v\vec{j} + w\vec{k}\\ &= F_1(x, y, z, t)\vec{i} + F_2(x, y, z, t)\vec{j} + F_3(x, y, z, t)\vec{k}\end{aligned}$$

Also,

$$\frac{dr}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dy}{dt}\vec{k}$$
$$\frac{dx}{dt} = u = F_1(x, y, z, t)$$
$$\frac{dy}{dt} = v = F_2(x, y, z, t)$$
$$\frac{dz}{dt} = w = F_3(x, y, z, t)$$

Solving this differential equation upon integration we get,

$$x(t) = x(t_0) + \int_{t_0}^{t} F_1(x, y, z, t)dt$$
$$y(t) = y(t_0) + \int_{t_0}^{t} F_2(x, y, z, t)dt$$
$$z(t) = z(t_0) + \int_{t_0}^{t} F_3(x, y, z, t)dt$$

$(x(t_0), y(t_0), z(t_0))$ is initial position.

Example 1. The velocity components for a two-dimensonal fluid system can be given in the Eulerian system by

$$u = 2x + 2y + 3t, v = x + y + t/2.$$

Find the displacement of fluid particle in the Lagrangian system.

solution: Given

$$u = 2x + 2y + 3t, v = x + y + t/2. \qquad ......(1)$$

In terms of the displacement x and y, the velocity component u and v may be represented by

$$u = \frac{dx}{dt}, \qquad v = \frac{dy}{dt} \qquad .....(2)$$

From (1) and (2), we have

$$\frac{dx}{dt} = 2x + 2y + 3t, \qquad \frac{dy}{dt} = x + y + t/2 \qquad .....(3)$$

Then equation (3) becomes

$$-x + (D - 1)y = t/2$$

Operating (4) by (D-2), we have

$$-(D - 2)x + (D^2 - 3D + 2)y = (1/2) - t \qquad .....(5)$$

Add (4) and (5), we have,

$$(D2 - 3D)y = (1/2) + 2t \qquad ....(6)$$

$$\therefore C.F. = c_1 + c_2 e^{3t}$$

and

$$\begin{aligned} P.I. &= \frac{1}{D^2 - 3D}(\frac{1}{2} + 2t) \\ &= -\frac{1}{3D}(1 - \frac{1}{3}D)^{-1}(\frac{1}{2} + 2t) \\ &= \frac{1}{3D}(\frac{1}{2} + 2t + \frac{2}{3}) \\ &= -\frac{1}{3}(\frac{2t^2}{2} + \frac{7t}{6}) \\ &= -\frac{t^2}{3} - \frac{7t}{18} \end{aligned}$$

Hence general solution of (6) is

$$y = c_1 + c_2 e^{3t} - (t^2/3) - (7t/18) \qquad ......(7)$$

$$\therefore \frac{dy}{dt} = 3c_2 e^{3t} - \frac{2t}{3} - \frac{7t}{18} \qquad ......(8)$$

Re-writing the second equation

$$x = \frac{dy}{dt} - y - (t/2) \qquad .....(9)$$

Putting the values of y and $\frac{dy}{dt}$ in (9) we get

$$x = -c_1 + 2c_2 e^{3t} + \frac{t^2}{3} - \frac{7t}{9} - \frac{7}{18} \qquad ......(10)$$

Now use the following initial conditions:

$$x = x_0, \qquad y = y - 0, \qquad when \qquad t = t_0 = 0 \qquad ....(11)$$

Using (11), (7) and (10) reduce to

$$y_0 = c_1 + c_2 \qquad and \qquad x_0 = -c_1 + 2c_2 - (7/18) \qquad ......(12)$$

$$c_1 = \frac{2y_0 - x_0}{3} - \frac{7}{54} \qquad and \qquad c_2 = \frac{x_0 + y_0}{3} + \frac{7}{54} \qquad ....(13)$$

Using (13), (10) and (7) give

$$x = \frac{1}{3}x_0 - \frac{2}{3}y_0 + \frac{1}{3}(2x_0 + 2y_0 + 7/9)e^{3t} - \frac{7}{9}t + \frac{1}{3}t^2 - \frac{7}{27}$$

and

$$x = -\frac{1}{3}x_0 + \frac{2}{3}y_0 + \frac{1}{3}(x_0 + y_0 + 7/18)e^{3t} - \frac{7}{18}t + \frac{1}{3}t^2 - \frac{7}{54}$$

This give the desired displacements x and y in the Langrangian system involving the initial position x_0 and y_0 and the time t.

. Recall Vector Analysis:

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\equiv (\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z})$$

$$\vec{\nabla}.\vec{\nabla} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\vec{\nabla}.(f\vec{g}) = \vec{\nabla} f.\vec{g} + f\vec{\nabla}.\vec{g}$$

I. The Divergence theorem (Gauss's theorem): Let S denoted a surface bounding a volume V and n unit vector to the surface. Then

$$\int_S a.n\; dS = \int_V div\; a\; dV$$

II. Stokes's theorem. Let S be a surface bounded by a closed curve C, and n the unite vector normal to the surface. Then,

$$\int_C a.dr = \int_S curl\; a.n\; dS$$

III. Green's theorem. Let ϕ and ψ be two scalar point functions and let S be a surface bounding a volume V. Then

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \int_S (\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\psi}{\partial n})dS$$

1. Cartesian Co-ordinate System:

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\equiv (\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z})$$

2. Spherical Polar System:

$$x = r\sin\theta\cos\phi, y = r\cos\theta\cos\phi$$

$$z = r\cos\theta, r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$$

3. Cylindrical Polar System:

$$x = r\cos\theta, y = r\sin\theta$$

$$z = z, r \geq 0, 0 \leq \theta \leq 2\pi, -\infty \leq \phi \leq \infty$$

Material, Local and Convective derivatives:

Suppose a fluid partial moves from P(x,y,z) at time t to Q(x+δx, y+δy, z+δz) at time t+δt. Further suppose f(x,y,z) be a scalar function associated with some property of the fluid (e.g the pressure or density etc.). Let the change of f due to movement of the fluid particle from P to Q be δf. Then we have

$$\delta f = (\partial f/\partial x)\partial x + (\partial f/\partial y)\partial y + (\partial f/\partial z)\partial z + (\partial f/\partial t)\partial t$$

$$\frac{\delta f}{\delta t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial f}{\partial t} \qquad ....(1)$$

Let

$$\lim_{\delta t \to 0} \frac{\delta f}{\delta t} = \frac{Df}{Dt} \quad or \quad \frac{df}{dt}$$

$$\lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{df}{dt} = v$$

and

$$\lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = u$$

$$\lim_{\delta t \to 0} \frac{\delta z}{\delta t} = \frac{dz}{dt} = w$$

where $\vec{q}$=(u,v,w) is the velocity of the fluid particle at P. Making $\delta t \to 0$.

$$\frac{Df}{Dt} = \vec{u}\frac{\partial f}{\partial x} + \vec{v}\frac{\partial f}{\partial y} + \vec{w}\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} \qquad ....(2)$$

But

$$\vec{q} = u\vec{i} + v\vec{j} + w\vec{k} \qquad ....(3)$$

Using (4) and (2) reduce to

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\vec{q}.\nabla)f \qquad ....(5)$$

$$\therefore \frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{q}.\nabla) \qquad ....(6)$$

$\frac{D}{Dt}$ is called the material (or particle or substantial) derivative. $\frac{\partial}{\partial t}$ is called the local derivative and associated with time variation at a fixed position. $\vec{q}.\nabla$ is called the convection derivative and it is associated with the change of physical quantity f due to motion of the fluid particle.

. **Line of flow:** A line of flow is a line whose direction coincides with the direction of the resultant velocity of the fluid.

. **Stream lines:** A streamline is a curve drawn in the fluid so that its tangent at each point is the direction of motion (i.e, fluid velocity) at a point.
Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector at a point P on a straight line and let $\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$ be the fluid velocity at P. Then $\vec{q}$ is parallel to d$\vec{r}$ at P on the streamline. Thus, the equation of streamlines is given by

$$\vec{q} \times d\vec{r} = 0$$

i.e,

$$(u\vec{i} + v\vec{j} + w\vec{k}) \times (x\vec{i} + y\vec{j} + z\vec{k}) = 0$$

whence

$$vdz - wdy = 0, \qquad wdx - udz = 0, \qquad udy - vdx = 0$$

so that

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad ......(1)$$

(1) gives the stream lines of the fluid at any point P.

Example 1. Obtain the streamlines of a flow u=x, v=-y. OR If the velocity $\vec{q}$ is given by $\vec{q} = x\vec{i} - y\vec{j}$, determine the equations of the streamlines.
Solution: For two dimensional flow (w=0), we have

$$\vec{q} = u\vec{i} + v\vec{j} + w\vec{k} = x\vec{i} - y\vec{j}$$

Streamlines are given by

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{0}$$

So,

$$\frac{dx}{x} - \frac{dy}{-y} = 0 \qquad and \qquad dz = 0$$

$$\Longrightarrow \qquad xy = c_1 \qquad and \qquad z = c_2$$

The streamlines are given by the curves of intersection of $xy = c_1$ *and* $z = c_2$ where c_1 and c_2 being arbitrary constants.

example 2. The velocity components in a three-dimensional flow field for an incompressible fluid are (2x,-y,-z). Is it is possible field? Determine the equations of the streamline passing through the point (1,1,1).
Solution: Streamlines are given by

$$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{-z}$$

Take,

$$\frac{dx}{2x} = \frac{dy}{-y}$$

$$\Longrightarrow \qquad xy^2 = c_1$$

And,

$$\frac{dx}{2x} = \frac{dz}{-z}$$

$$\Longrightarrow \qquad xz^2 = c_2$$

Hence c_1 and c_2 are arbitrary constant.
The required streamline passes through (1,1,1) so that $c_1 = 1$ and $c_2 = 1$. Thus, the desired stream line is given by the intersection of

$$xy^2 = 1 \ \ and \ \ xz^2 = 1.$$

. Path lines:
A path line is the curve or trajectory along which a particular fluid particle travels during its motion.

the different equation of a path line is

$$\frac{d\vec{r}}{dt} = \vec{q}$$

so that,

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v \qquad and \qquad \frac{dz}{dt} = w$$

where ,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \qquad and \qquad \vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$$

EX: The velocity field at a point P in a fluid is given by
$\vec{q} = (x/t, y, 0)$, obtain the path lines.
sol. Here

$$u = x/t, v = y, w = 0$$

The equations of path lines are

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v \qquad and \qquad \frac{dz}{dt} = w$$

i.e,

$$\frac{dx}{dt} = \frac{x}{t}, \qquad \frac{dy}{dt} = y \qquad and \qquad \frac{dz}{dt} = 0 \qquad .....(1)$$

Suppose that (x_0, y_0, z_0) are coordinates of the chosen fluid particle at time $t = t_0$.
From (1)

$$\frac{dx}{dt} = \frac{x}{t} \qquad \Longrightarrow \qquad x = tc_1$$

From initial condition

$$x = \frac{x_0 t}{t_0}$$

Similarly,

$$y = y_0 e^{-t_0} \qquad and \qquad z = z_0$$

Hence the required path lines are given by

$$x = \frac{x_0 t}{t_0}, \qquad y = y_0 e^{-t_0} \qquad and \qquad z = z_0$$

⋆ **Velocity and acceleration of a fluid Partical:**

Let P be the position of the fluid particle at any time t and Q be its position at $t + \delta t$ such that

$$\vec{OP} = \vec{r}, \vec{OQ} = \vec{r} + \vec{\delta r} \implies \vec{PQ} = \vec{\delta r}$$

The velocity $\vec{q}$ at P is given by

$$\vec{q} = \lim_{\delta t \to 0} \frac{(\vec{r} + \delta\vec{r}) - \vec{r}}{(t + \delta t) - t}$$

$$= \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t}$$

$$\implies \qquad \vec{q} = \lim_{\delta t \to 0} \frac{d\vec{r}}{dt}$$

Acceleration of this fluid at P

$$\vec{a} = \frac{d}{dt}(\vec{q})$$

Now,

$$\frac{D}{Dt} = \frac{d}{dt} = (\frac{\partial}{\partial t} + \vec{q}.\vec{\nabla})$$

$$\frac{d}{dt}(\vec{q}) = (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})\vec{q}$$

$$= \tfrac{\partial \vec{q}}{\partial t} + u\tfrac{\partial \vec{q}}{\partial x} + v\tfrac{\partial \vec{q}}{\partial y} + w\tfrac{\partial \vec{q}}{\partial z}$$

Example: If the velocity distribution is $\vec{q} = \vec{i}Ax^2y + \vec{j}By^2zt + \vec{k}Czt^2$, where A, B, C are cost. then find the acceleration and velocity components.

Solution: The acceleration

$$\vec{a} = \frac{\partial \vec{q}}{\partial t} + u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + w\frac{\partial \vec{q}}{\partial z}$$

Also,

$$\vec{q} = u\vec{i} + v\vec{j} + w\vec{k} = \vec{i}Ax^2y + \vec{j}By^2zt + \vec{k}Czt^2$$

$$\therefore \vec{q} = A(2Ax^3y^2 + Bx^2y^2zt)\vec{i} + B(y^2z + 2By^3z^2t^2 + Cy^2zt^3)\vec{j} + C(2zt + Czt^4)\vec{k}$$

The components of acceleration (a_x, a_y, a_z) are given by

$$a_x = A(2Ax^3y^2 + Bx^2y^2zt), \qquad a_y = B(y^2z + 2By^3z^2t^2 + Cy^2zt^3) \qquad a_z = C(2zt + Czt^4)$$

$\star$ **Equation of continuity/ Conservation of mass(By Euler's Method):**

By continuity/ conservation of mass, we mean that the fluid always remain a continuum. When a region of fluid contains no source or sink, then region is considered in accordance with principle of conservation of mass.

fluid in - fluid out + source Sink = fluid accumulation.

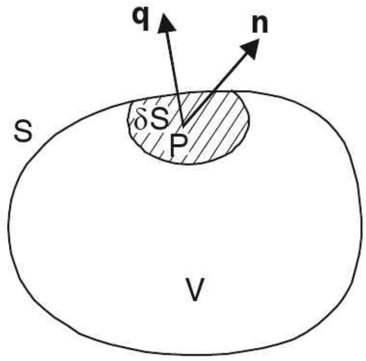
Let S be an arbitrary small closed surface drawn in the compressible fluid enclosing a volume V and S be taken fixed in space. Let δS denoted element of the surface S enclosing P. Let $\vec{n}$ be the unit outward drawn normal at δS and $\vec{q}$ be the fluid velocity at P. Then the normal component of $\vec{q}$ measured outwards from V is $\vec{n}.\vec{q}$. Thus,

Rate of mass flow across δS=$\rho(\vec{n}.\vec{q})\delta S$ $\therefore$ Total rate of mass flow across S

$$= \int_S \rho(\vec{n}.\vec{q})dS = \int_V \nabla.(\rho\vec{q})dV$$

$\therefore$ Total rate of mass flow into V=$-\int_V \nabla.(\rho\vec{q})dV$ ......(1)

The mass of the fluid within S at time t=-$\int_v \rho dV$



$\therefore$ Total rate of mass increase within $S = \frac{\partial}{\partial t}\int_V \rho dV = \int_v \frac{\partial \rho}{\partial t}dV$ ....(2)

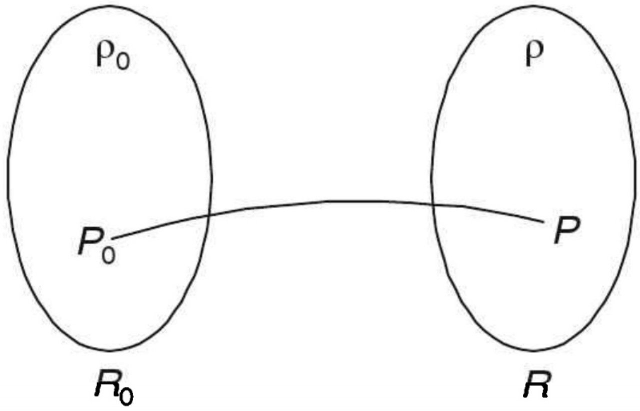
Suppose that the region V of the fluid contains neither sources nor sinks.Then by the law of conservation of fluid mass, the rate of increase of mass of fluid within V must be equal to the total rate of mass flowing into V. From (1) and (2), we get

$$\int_V [\frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{q})]dV = 0$$

Which holds for arbitrary small volumes V, if $\frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{q}) = 0$ ......(3)

Equation (3) is called the equation of continuity, or the conservation of mass and it holds at all points of fluid free from sources and sinks.

⋆ **Equation of continuity(by Lagrange's Method):**



Let R_0 be the region occupied by portion of a fluid at the t=0, and R the region occupied by the same fluid at any time t.
Let (a,b,c) be the initial co-ordinates of a fluid particle . P_0 enclosed in this element and ρ_0 be its density.
Mass of the fluid element at t=0 is $\rho_0\delta a\delta b\delta c$.
Let P be the subsequent position of P_0 at time t and let ρ be density of the fluid there.
Then mass of the fluid element at t is $\rho\delta x\delta y\delta z$.
From the law of conservation of mass,

$$\int\int\int_{R} {}_0\rho_0\delta a\delta b\delta c = \int\int\int_{R}\rho\delta x\delta y\delta z \qquad ......(1)$$

From advance calculus, we have

$$\delta x\delta y\delta z = J\delta a\delta b\delta c \qquad .....(2)$$

Where Jacobian, J=$\frac{\partial(x,y,z)}{\partial(a,b,c)}$
Using (2), (1) may be Re-written as

$$\int\int\int_{R} {}_0\rho_0\delta a\delta b\delta c = \int\int\int_{R} {}_0\rho J\delta a\delta b\delta c$$

$$\int\int\int_{R} {}_0(\rho_0 - \rho J)\delta a\delta b\delta c = 0$$

Which holds for all regions R_0 if $\qquad \rho_0 - \rho J = 0$
which is the equation of continuity in Lagrangian form.

$\star$ **Lagrangian $\Longleftrightarrow$ Eulerian:**

The velocity components in two systems are connected by the equations

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt} \qquad w = \frac{dz}{dt} \qquad ......(1)$$

Also, x=(a,b,c,t), y=y(a,b,c,t) z=z(a,b,c,t) ......(2)

$$\therefore \quad \frac{\partial u}{\partial a} = \frac{\partial}{\partial a}(\frac{dx}{dt}) = \frac{d}{dt}\frac{\partial x}{\partial a}) \qquad so\ that \quad \frac{d}{dt}\frac{\partial x}{\partial a}) = \frac{\partial u}{\partial a} \qquad etc \qquad ....(3)$$

The equation of continuity in the Lagrangian form is

$$\rho J = \rho_0 \qquad ....(4)$$

$$where \qquad J = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{bmatrix} \qquad ......(5)$$

Also, the equating of continuity in the Eulerian form is

$$\frac{d\rho}{dt} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0 \qquad .....(6)$$

Differentiating both sides of (5) w.r.t. 't' and using (3), we get

$$\frac{dJ}{dt} = \begin{bmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{bmatrix} + \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} & \frac{\partial v}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{bmatrix} + \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c} \end{bmatrix} \qquad ......(6)$$

$$\frac{dJ}{dt} = J_1 + J_2 + J_3 \qquad ......(7)$$

Since $\frac{\partial u}{\partial a} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a}$ etc, J_1 can be re-written as (after interchanging its row and columns) Thus, we have

$$J_1 = \begin{bmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial z}\frac{\partial z}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial z}\frac{\partial z}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial z}\frac{\partial z}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{bmatrix}$$

$$= \frac{\partial u}{\partial x}\begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{bmatrix}$$

[The last two determinants vanish because they possess two identical columns]

$$\therefore \qquad J_1 = \frac{\partial u}{\partial x}J, \qquad Using(5)$$

Similarly, we have

$$J_2 = \frac{\partial v}{\partial y} J \qquad and \qquad J_3 = \frac{\partial w}{\partial z} J$$

$$\therefore (7)\ becomes \qquad \frac{dJ}{dt} = J(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) \qquad ......(8)$$

Derivation of Eulerian form from Lagragin form:
From (4),

$$\frac{d\rho}{dt}(\rho J) = \frac{d}{dt}(\rho_0) = 0 \qquad or \qquad \frac{d\rho}{dt} J + \rho \frac{dJ}{dt} = 0$$

$$\therefore \qquad \frac{d\rho}{dt} J + \rho J(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0, \quad Using (8)$$

$$\Longrightarrow \qquad \frac{d\rho}{dt} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0$$

Which is (6) i.e. Eulerian form of equation of continuity.
derivation of Lagrangian form from Eulerian form:
From (6)

$$\Longrightarrow \qquad \frac{d\rho}{dt} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0$$

$$\therefore \qquad \frac{d\rho}{dt} + \rho(\frac{1}{J}\frac{dJ}{dt} = 0, \quad Using (8)$$

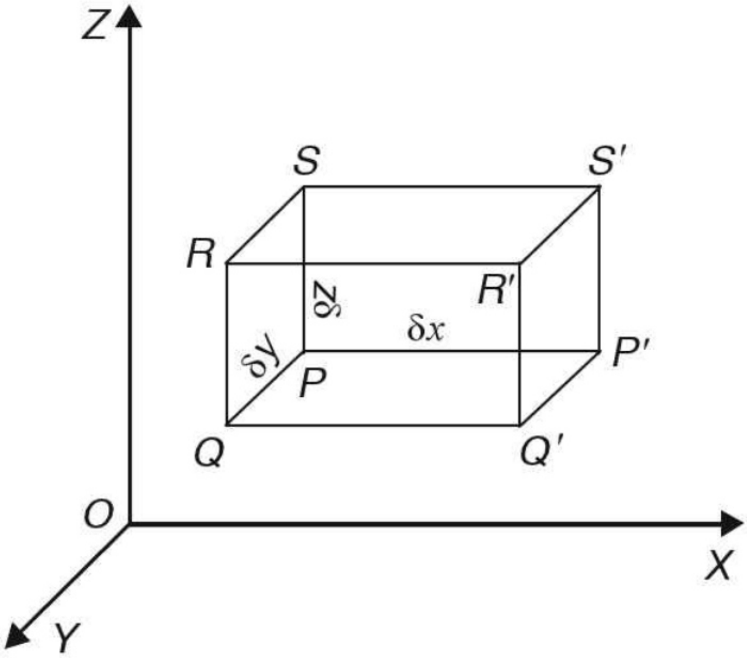
$$or \qquad J\frac{d\rho}{dt} + \rho \frac{dJ}{dt} = 0 \qquad or \qquad \frac{d}{dt}(\rho J) = 0 \qquad .....(9)$$

Integrating (9),

$$\rho J = \rho_0$$

Which is (4) i.e. Lagrangian equation of continuity.

⋆ **Equation of Continuity(Cartesian coordinates:)**
Let there be a fluid particle at P(x,y,z). Let $\rho(x, y, z, t)$ be the density of the fluid at P at any time t and let u, v, w be the velocity components at P parallel to the rectangular coordinate axes. Construct a small parallelpiped with edges $\delta x, \delta y, \delta z$ of lengths parallel to their respective coordinate axes.
We have
Mass of the fluid that passes in through the face PQRS
$= (\rho\delta y\delta z)$u per unit time$=f(x, y, z)$ s ....(1)
$\therefore$ Mass of the fluid that passes out through the opposite face PQRS
$=f(x + \delta x, y, z)$ per unit time $= f(x, y, z) + \delta x \frac{\partial}{\partial x} f(x, y, z) + ...$ ......(2)
$\therefore$ The net gain in per unit time within the element (rectangular parallelepiped) due to flow through the faces PQRS and $P^{'}Q^{'}R^{'}$ by using (1) and (2)
=Mass that enters in through the face PQRS-Mass that leaves through the face $P^{'}Q^{'}R^{'}$
$= f(x, y, z) - [f(x, y, z) + \delta x \frac{\partial}{\partial x} f(x, y, z) + ...]$
$= -\delta x. \frac{\partial}{\partial x}(\rho u \delta y \delta z)$, by (1)
$= -\delta x \delta y \delta z \frac{\partial(\rho u)}{\partial x}$



Similarly, the net gain in mass per unit time within the element due to flow through the faces $PP'S'S$ and $QQ'RR'$ $= -\delta x\delta y\delta z\frac{\partial(\rho v)}{\partial y}$
and the net gain in mass per unit time within the element due to flow through the faces $PP'Q'Q$ and $SS'R'R$ $= -\delta x\delta y\delta z\frac{\partial(\rho w)}{\partial z}$
$\therefore$ Total rate of mass flow into the elementary parallelepiped

$$= -\delta x\delta y\delta z[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}] \qquad ...(3)$$

Again, the mass of the fluid within the chosen element at time t= $\therefore$ Total rate of mass increase within the element $=\rho\delta x\delta y\delta z$

$$= \frac{\partial}{\partial t}(\rho\delta x\delta y\delta z) = (\delta x\delta y\delta z)\frac{\partial\rho}{\partial t} \qquad .....(4)$$

Suppose that the chosen region of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass we have,

$$\delta x\delta y\delta z\frac{\partial\rho}{\partial t} = -\delta x\delta y\delta z[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}]$$

or

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

or,

$$\frac{D\rho}{Dt} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0$$

Which is the desired equation of continuity in cartesian coordinates.

$\star$ **Circulation and Vorticity:** Circulation and vorticity are two are two primary measures of rotation of fluid.

. **circulation:** It is a scalar integral quantity which measures the rotations of a fluid particle along a close curve.

The circulation C, about a closed curve /contour in a fluid is given by

$$C = \oint_c \vec{q}.d\vec{r}$$

. **Vorticity:** The Vorticity is a type of pseudo-vector field that describes the local spinning motion of a fluid element/continuum near some point (the tendency of fluid to rotate)

The vorticity vector fluid field $\vec{\omega}$ is defined as,

$$\vec{\omega} = \vec{\nabla} \times \vec{q}$$

$\star$ **Relation between Circulation and Vorticity:**

Stokes theorem states that "circulation about any closed contour is equal to the normal component of vorticity over the area enclosed by contour".

$$C = \oint_c \vec{q}.d\vec{r} = \oint_s (\vec{\nabla} \times \vec{q}).\vec{n}ds$$

$\star$ **Vortex Line:** A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector $\vec{\Omega}$.

Let $\vec{\Omega} = \Omega_x\vec{i} + \Omega_y\vec{j} + \Omega_z\vec{k}$ and let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector of a point P on a vortex line. Then Ω is parallel to $d\vec{r}$ at P on the vortex line. Hence

$$\vec{\Omega} \times d\vec{r} = 0$$

i.e,

$$(\Omega_x\vec{i} + \Omega_y\vec{j} + \Omega_z\vec{k}) \times (dx\vec{i} + dy\vec{j} + dz\vec{k}) = 0$$

$$\implies \qquad \Omega_x dz - \Omega_z dy = 0, \qquad \Omega_z dx - \Omega_x dz = 0, \qquad and \qquad \Omega_x dy - \Omega_y dx = 0$$

$$\implies \qquad \frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$

This gives the disired equations of vortex lines.

Example: Test whether the motion specified by $\vec{q} = \frac{k^2(x\vec{j} - y\vec{i})}{x^2+y^2}$ (k= constant), is a possible motion for an incompressible fluid. If so, determine the equation of steamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.

Solution: Let $\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$. Then here

$$u = -\frac{k^2y}{x^2+y^2}, \qquad v = \frac{k^2x}{x^2+y^2}, \qquad w = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{2k^2xy}{(x^2+y^2)^2}, \qquad \frac{\partial v}{\partial y} = -\frac{2k^2xy}{(x^2+y^2)^2}, \qquad \frac{\partial w}{\partial z} = 0$$

The equation of streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\implies \qquad \frac{dx}{-\frac{k^2y}{x^2+y^2}} = \frac{dy}{\frac{k^2x}{x^2+y^2}} = \frac{dz}{0}$$

$$\therefore \quad dz = 0 \quad \implies \quad z = c_1 \qquad ....(1)$$

And

$$\frac{dx}{-y} = \frac{dy}{x} \quad \implies \quad X^2 + y^2 = c_2 \qquad ......(2)$$

(1) and (2) together give the steamlines. Clearly, the stremlines are circles whose centers are on the z-axis, their planes being perpendicular to this axis.
Again

$$curl\vec{q} = k^2[\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2}]\vec{k} = 0$$

hence the flow is of the potential and we can find velocity potential $\phi(x, y, z)$ suh that $\vec{q} = -\nabla\phi$. Thus, we have

$$\frac{\partial\phi}{\partial y} = -u = \frac{k^2y}{x^2+y^2} \qquad ......(3)$$

,

$$\frac{\partial\phi}{\partial y} = -v = -\frac{k^2x}{x^2+y^2}, \qquad ......(4)$$

$$\frac{\partial\phi}{\partial z} = -w = 0 \qquad ....(5)$$

From equation (5) we have,

$$\phi = \phi(x, y)$$

Integrating (3)

$$\phi(x, y) = k^2tan^{-1}\frac{x}{y} + f(y)$$

$$\therefore \quad f^{'}(y) = 0 \implies \quad f(y) = cons.$$

$$\phi(x, y) = k^2tan^{-1}\tfrac{x}{y}, \text{ omitted cons.}$$

The equipotentials are given by

$$k^2tan^{-1}\frac{x}{y} = cons. = k^2tan^{-1}c$$

or, $\qquad x = cy$, c is a cons.
Which are planes through the z-axis. They are intersected by the streamlines.

Example: Show that the velocity poremtial $\phi = \frac{a}{2} \times (x^2 + y^2 - 2z^2$ satisfies the Laplace equation. Also determine the streamlines.
Solution : We know that the velocity $\vec{q}$ of the fluid is given by

$$\vec{q} = -\nabla\phi = -(a/2) \times (2x\vec{i} + 2y\vec{j} - 4z\vec{k}) = u\vec{i} + v\vec{j} + w\vec{k}, \quad say$$

The equation of streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad or \qquad \frac{2dx}{x} = \frac{2dy}{y} = \frac{dz}{-z}$$

$$\therefore \quad \frac{2dx}{x} = \frac{2dy}{y} \quad \Longrightarrow x = c_1 y \qquad ....(1)$$

$$And \quad \frac{2dy}{y} = \frac{dz}{-z} \quad \Longrightarrow y^2 z = c_2 \qquad ....(2)$$

(1) and (2) together gives the equation of stremlines, c_1 and c_2 being arbitary constant of integration.
Now given that $\qquad \phi = \frac{a}{2} \times (x^2 + y^2 - 2z^2)$

$$\Longrightarrow \qquad \frac{\partial\phi}{\partial x} = ax, \qquad \frac{\partial\phi}{\partial y} = ay, \qquad \frac{\partial\phi}{\partial z} = -2az$$

$$\Longrightarrow \qquad \frac{\partial^2\phi}{\partial x^2} = a, \qquad \frac{\partial^2\phi}{\partial y^2} = a, \qquad \frac{\partial^2\phi}{\partial z^2} = -2a$$

$$\Longrightarrow \qquad \nabla^2\phi = 0$$

showing that ϕ satisfy the Laplace equation.

⋆ **Streak lines:** A steak line is a line on which lie all those fluid particles that at some earlier instant passed through a certain point of space. Thus, a streak line presents the instantaneous picture of the position of all fluid particles, which have passed through a given point at some previous time.

The equation of the streak lines at a time t can be derived by Lagragian method, Suppose that a fluid particle (x_0, y_0, z_0) passes through a fixed point (x_1, y_1, z_1) in the course of time. Then using Lagrangian method of description, we have

$$f_1(x_0, y_0, z_0, t) = x_1 \qquad f_2(x_0, y_0, z_0, t) = x_2 \qquad f_3(x_0, y_0, z_0, t) = x_3 \qquad ....(1)$$

Solving (1) for x_0, y_0, z_0 we have

$$x_0 = g_1(x_1, y_1, z_1, t) \qquad y_0 = g_2(x_1, y_1, z_1, t) \qquad z_0 = g_3(x_1, y_1, z_1, t) \qquad ....(2)$$

Now a streak line is the locus of the positions (x,y,z) of the particles which have passed through the fixed point (x_1, y_1, z_1). Hence the equation of streak line at a time t is given by

$$x_1 = h_1(x_0, y_0, z_0, t) \qquad y_1 = h_2(x_0, y_0, z_0, t) \qquad z_1 = h_3(x_0, y_0, z_0, t) \qquad ....(3)$$

Substituting the values (x_0, y_0, z_0) in (3), the desired equation of streak line passing through (x_1, y_1, z_1) at a time t is given by

$$x_1 = h_1(g_1, g_2, g_3, t) \qquad y_1 = h_2(g_1, g_2, g_3, t) \qquad z_1 = h_3(g_1, g_2, g_3, t) \qquad ....(4)$$

⋆ **Linear Strain Rate:**

A study of the dynamics of fluid flows involves determination of the forces on an element, which depend on the amount and nature of its deformation, or strain.The deformation of a fluid is similar to that of a solid, where one defines normal strain as the change in length per unit length of a linear element, and shear strain as change of a $\pi/2$ angle.Consider first the linear or normal strain rate of a fluid element in the

$$\frac{1}{\delta x_1}\frac{D}{Dt}(\delta x_1) = \frac{1}{dt}\frac{A'B' - AB}{AB}$$

$$= \tfrac{1}{\delta t}\tfrac{1}{\delta x_1}[\delta x_1 + \tfrac{\delta u_1}{\delta x_1}\delta x_1 dt - \delta x_1]$$

$$= \tfrac{\partial u_1}{\partial x_1}$$

In general, the linear strain rate in the α direction is

$$\frac{\partial u_\alpha}{\partial x_\alpha}, \alpha = 1, 2, 3$$

The total normal strain,

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \tfrac{\partial u_\alpha}{\partial x_\alpha}, \ \alpha \text{ is the dummy variable.}$$

Alternatively:

Consider a volume element of sides $\delta x_1, \delta x_2, \delta x_3$

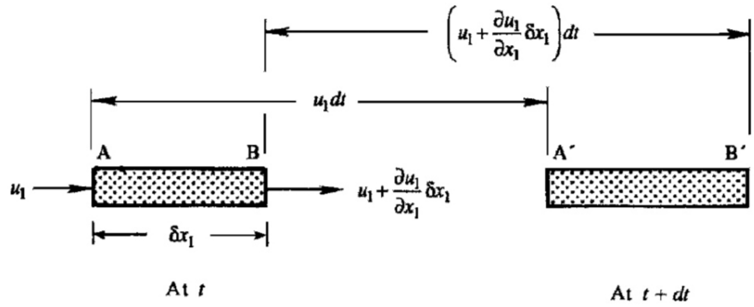
Define change of rate of per unit volume is

$$\frac{1}{\delta x_1\delta x_2\delta x_3}\frac{D}{Dt}(\delta x_1\delta x_2\delta x_3) = \frac{1}{\delta x_1\delta x_2\delta x_3}\frac{D}{Dt}(\delta x_1)\delta x_2\delta x_3 + \frac{1}{\delta x_1\delta x_2\delta x_3}\frac{D}{Dt}(\delta x_2)\delta x_1\delta x_3$$

$$+\tfrac{1}{\delta x_1\delta x_2\delta x_3}\tfrac{D}{Dt}(\delta x_3)\delta x_1\delta x_2$$

$$\implies \frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \tfrac{\partial u_\alpha}{\partial x_\alpha}, \alpha = 1, 2, 3$$



Linear strain rate. Here, $A'B' = AB + BB' - AA'$.

In addition to undergoing normal strain rates, a fluid element may also simply deform in shape. The shear strain rate of an element is defined as the rate of decrease of the angle formed by two mutually perpendicular lines on the element. The shear strain so calculated depends on the orientation of the line pair. Figure shows the position of an element with sides parallel to the coordinate axes at time t , and its subsequent position at $t + dt$. The rate of shear strain is

$$\varepsilon_{ij} = e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), i, j = 1, 2, 3$$

called cauchy strain tensor.
if $i = j$ Then $\varepsilon \implies$ principal stress.

⋆ **Stream Function:**
The volumetric strain rate given is given by

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \tfrac{\partial u_\alpha}{\partial x_\alpha}, \alpha = 1, 2, 3$$

The $\frac{D}{Dt}$ signifies that a specific fluid particle is followed, so that the volume of a particle is inversely proportional to its density. Substituting $\delta\infty\rho^{-1}$ we obtain

$$-\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

An another kind of continuity equation which mean that the fluid flow has no void in it. The density of fluid particles does not change appreciably along the fluid path under certain conditions, the most important of which is that the flow speed should be small compared with the speed of sound in the medium. This is called the Boussinesq approximation.
From the above equation,

$$-\frac{1}{\rho}\frac{D\rho}{Dt} = 0$$

$$\implies \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$$

In many cases the continuity equation consists of two terms only, say

$$\implies \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Solution of this eqution is

$$u(x,y) = \frac{\partial \psi}{\partial y}$$
$$v(x,y) = -\frac{\partial \psi}{\partial x}$$

$\psi(x,y,z) = c$ is called Stream function for every value of c gives one stream line.

Q: what is physical significance!

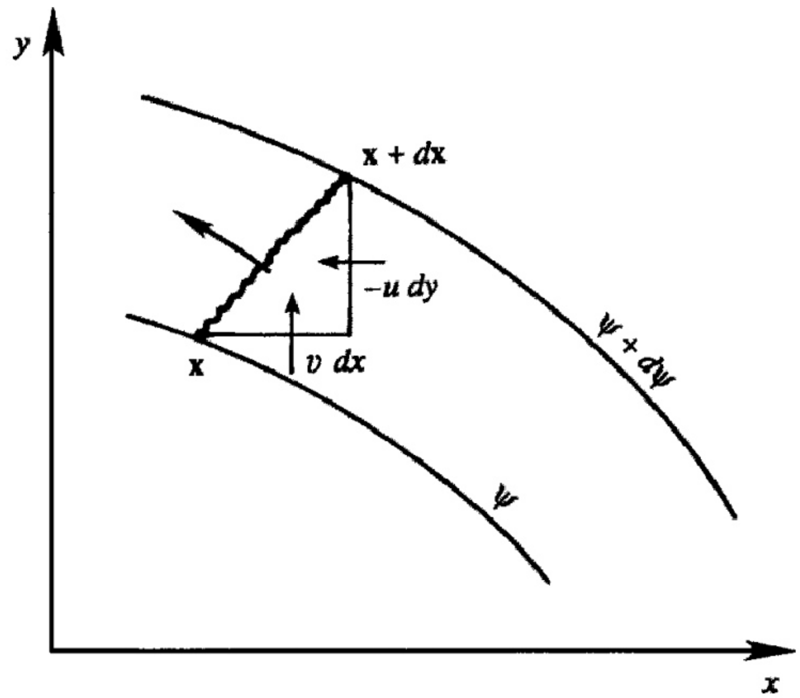
the Streamlines of flow is given by

$$\frac{dx}{u} = \frac{dy}{v}$$
$$\implies vdx - udy = 0$$
$$\implies -\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} = 0$$
$$\implies d\psi = 0$$

Consider the two dimensional flow: Consider an arbitrary fluid element. Here we have shown a case in which both dx and dy are positive. The volume rate of flow across such a line element is

$$vdx + (-u)dy = -\frac{\partial \psi}{\partial x}dx - \frac{\partial \psi}{\partial y}dy = -d\psi$$

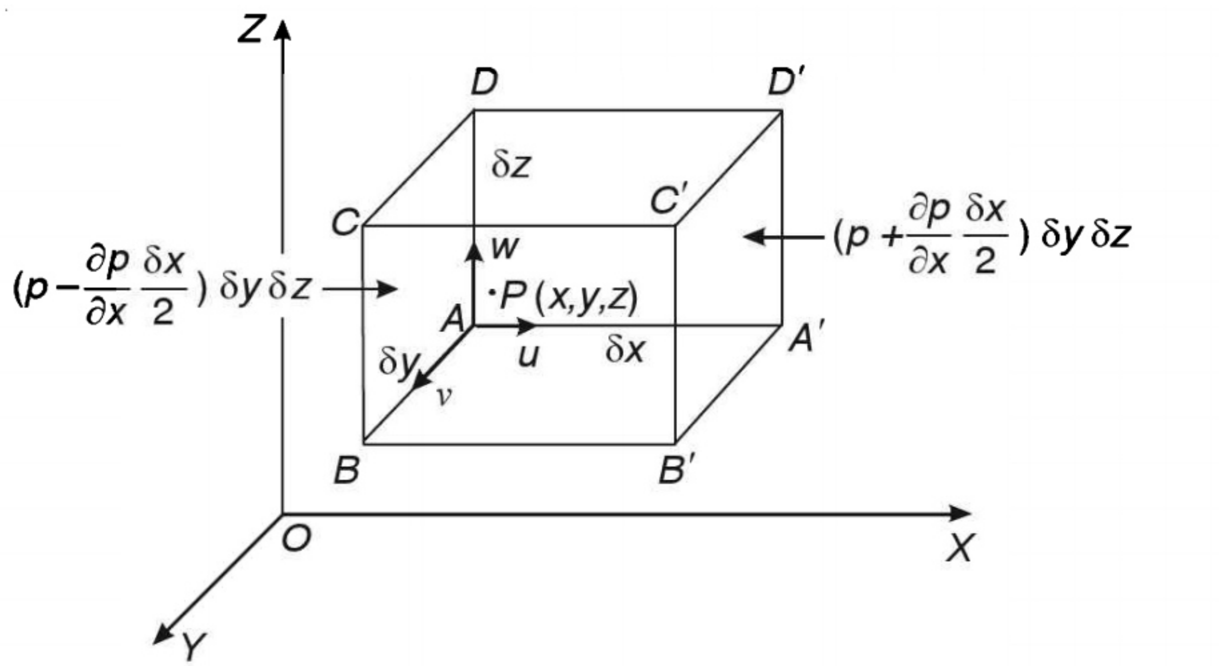
showing that the volume flow rate between a pair of streamlines is numerically equal to the difference in their ψ values. The sign of ψ is such that, facing the direction of motion, ψ increase to the left. This can be seen from the definition equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$,according to which the derivative of ψ in a certain direction gives the velocity component in a direction $\pi/2$ clockwise from the direction of differentiation. This requires that ψ in Figure must increase downward if the flow is from right to left.



Euler's equations of motion:

Let p be the pressure and ρ be density at a point P(x,y,z) in an inviscid (perfect) fluid. Consider an elementary parallelepiped with edges of lengths $\delta x, \delta y, \delta z$ parallel to their respective coordinate axis having P at its center as shown in figure. Let (u,v,w) be the components of velocity and (X,Y,Z) be the components of external force per unit mass at time t at P. Then if p=f(x,y,z), we have

Force on the plane through P parallel to ABCD = p$\delta y \delta z$



$\therefore$ Force on the face ABCD$=f(x-\frac{1}{2}\delta x, y, z)\delta y\delta z$

$$= [f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + ...]\delta y\delta z$$

And Force on the face ABCD$=f(x+\frac{1}{2}\delta x, y, z)\delta y\delta z = [f+\frac{1}{2}\delta x\frac{\partial f}{\partial x}+.......]\delta y\delta z$

$\therefore$ The net force in x-direction due to forces on ABCD and $A'B'C'D'$

$$= [f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + ...]\delta y\delta z - [f + \frac{1}{2}\delta x \frac{\partial f}{\partial x} + ...]\delta y\delta z$$

$$= -(\frac{\partial f}{\partial x})\delta x\delta y\delta z$$

$$= -(\frac{\partial p}{\partial x})\delta x\delta y\delta z \qquad as \qquad p = f(x,y,z)$$

The mass of the element is $\rho\delta x\delta y\delta z$. Hence the external force on the element in x-direction is $X\rho\delta x\delta y\delta z$. Du/Dt is the total acceleration of the element in x-direction.

By Newton's second law of motion,

$$\rho\delta x\delta y\delta z\frac{Du}{Dt} = X\rho\delta x\delta y\delta z - (\frac{\partial p}{\partial x})\delta x\delta y\delta z$$

i.e,

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Similarly,

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

From all those equations we get

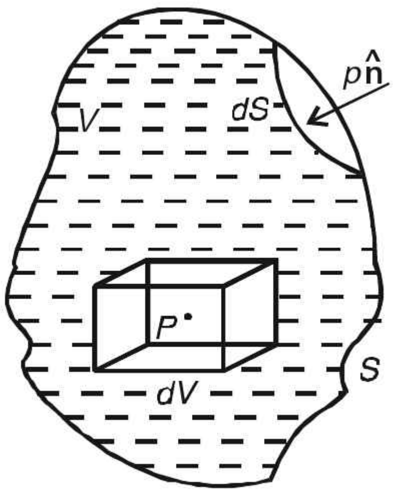
$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p$$

Which is called the Euler's equation of motion.

The equation of motion of an inviscid fluid (Vector method):
Consider any arbitrary closed surface S drawn in the region occupied by the incompressible fluid and moving with it, so that it contains the same fluid particles at every instant.
Total force acting on this mass of fluid = the rate of change in liner momentum.
The mass of fluid under consideration is subjected to following two forces:
(i) the normal pressure thrusts on the boundary.
(ii) the external force $\vec{F}$ per unit mass.

Let ρ be the density of the fluid particle P within the closed surface and let dV the volume enclosing P. The mass of element ρdV will always remain constant. Let $\vec{q}$ be the velocity of the fluid particle P, then the momentum $\vec{M}$ of the volume V is given by

$$\vec{M} = \int_V \vec{q}\rho dV$$



$$\therefore \quad \frac{DM}{Dt} = \frac{D}{Dt}\int_V \vec{q}\rho dV = \int_V \frac{D\vec{q}}{Dt}\rho dV - \int_V \vec{q}\frac{D}{Dt}(\rho dV)$$

$$\Longrightarrow \quad \frac{DM}{Dt} = \int_V \frac{D\vec{q}}{Dt}\rho dV$$

note that the second integral vanishes because the mass ρdV remain constant for all time The material derivative given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{q}.\nabla)$$

Finally, if p be normal pressure thrust at a point of the surface element DS, the total force on the surface S

$$= \int_S p(-\vec{n})dS$$

$$= -\int_V \nabla p dV$$

$\therefore$ The total force acting on the volume V

$$= \int_V \vec{F}\rho dV - \int_V \nabla p dV = \int_V (\vec{F}\rho - \nabla p)dV$$

By the Newton second Law we have,

$$\int_V (\vec{F}\rho - \nabla p)dV = \int_V \frac{D\vec{q}}{Dt}\rho dV \quad or \quad \int_V (\frac{D\vec{q}}{Dt}\rho - \rho\vec{F} + \nabla p)dV = 0$$

Since the volume V enclosed by surface S is arbitrary, so

$$\frac{D\vec{q}}{Dt}\rho - \rho\vec{F} + \nabla p = 0 \quad or \quad \frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p$$

Which is known as Euler's eqution of motion. It is also known as the eqution of motion by flux method.

Deduction of Lamb's hydrodynamical equation :

We may write as

$$\frac{\partial\vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} = \vec{F} - \frac{1}{\rho}\nabla\rho \qquad .....(1)$$

But

$$\nabla(\vec{q}.\vec{q}) = 2[\vec{q} \times curl\vec{q} + (\vec{q}.\nabla)\vec{q}]$$

So that

$$\nabla(\vec{q}.\vec{q}) = \frac{1}{2} \times \nabla(\vec{q}.\vec{q}) - \vec{q} \times curl\vec{q} \qquad ....(2)$$

Using (1) and (2) reduces to

$$\frac{\partial\vec{q}}{\partial t} + \nabla(\vec{q}^2) + curl\vec{q} \times \vec{q} = \vec{F} - \frac{1}{\rho}\nabla p$$

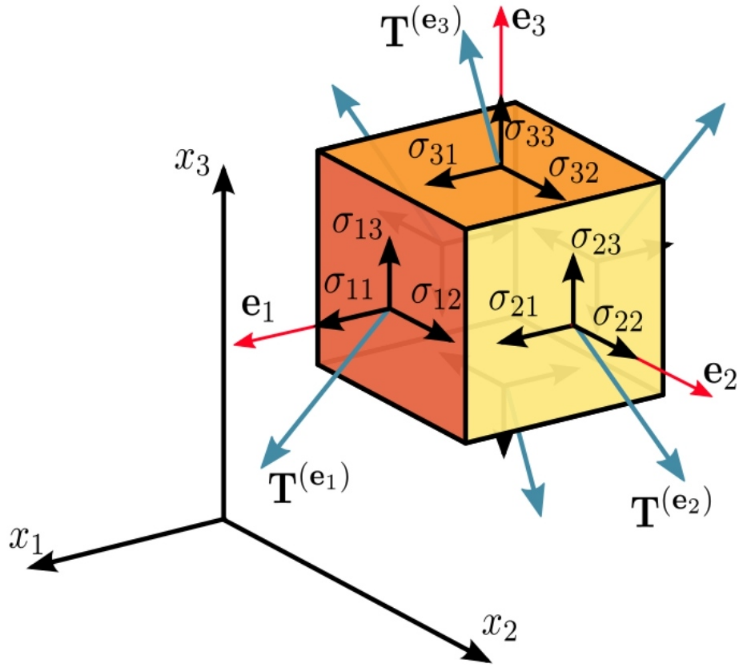
Now the vorticity vector Ω is given by $\Omega = curl\vec{q}$

$$\therefore \quad \frac{\partial\vec{q}}{\partial t} + \nabla(\vec{q}^2) + \Omega \times \vec{q} = \vec{F} - \frac{1}{\rho}\nabla p$$

Which is known as Lamb's hydrodynamical equation.

⋆ **Stress at a point(Cauchy Stress Tensor):**

In continuum mechanics, the Cauchy stress tensor σ , true stress tensor, or simply called the stress tensor is a second order tensor named after Augustin-Louis Cauchy. The tensor consists of nine components σ_{ij} that completely define the state of stress at a point inside a material in the deformed state, placement, or configuration. The tensor relates a unit-length direction vector n to the traction vector T(n) across an imaginary surface perpendicular to n:



$$T^n = \vec{n}\vec{\sigma} \qquad\qquad T_j^{(n)} = \sigma_{ij} n_i$$

where,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

. **Cauchy's postulate:**

According to the Cauchy Postulate, the stress vector $T^{(n)}$ remains unchanged for all surfaces passing through the point P and having the same normal vector n at P. i.e., having a common tangent at P. This means that the stress vector is a function of the normal vector n only, and is not influenced by the curvature of the internal surfaces.

. **Cauchy's fundamental lemma:**

A consequence of Cauchy's postulate is Cauchy's Fundamental Lemma, also called the Cauchy reciprocal theorem, which states that the stress vectors acting on opposite sides of the same surface are equal in magnitude and opposite in direction. Cauchy's fundamental lemma is equivalent to Newton's third law of motion of action and reaction, and is expressed as:

$$-T(n) = T(-n)$$

⋆ **Levi-Civita symbol:**
This symbol is used quite often in tensor analysis, vector analysis

$$\varepsilon_{i_1 i_2 i_3 .....i_n} = (-1)^p \varepsilon_{123...n}$$

where $i_1, i_2, i_3, ..., i_n$ are distinct and in order fashion 1,2,3,...,n and

$$\varepsilon_{123...n} = 1$$

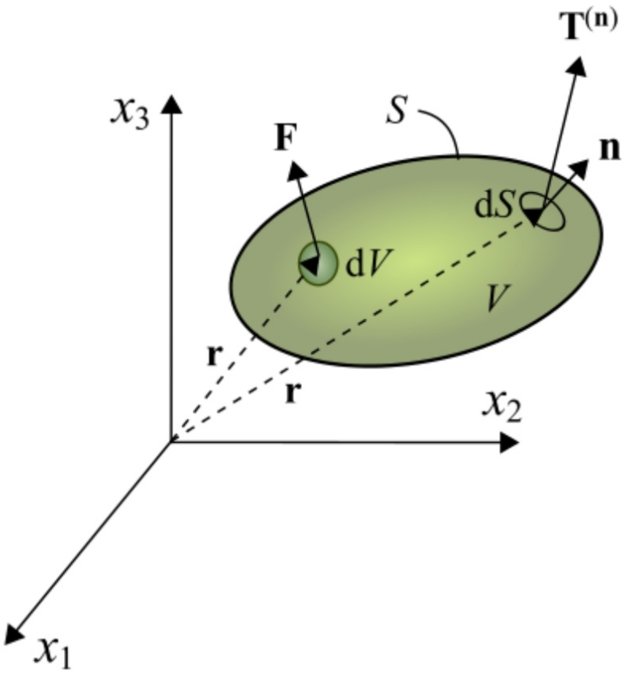
Cauchy's first law of motion
According to the principle of conservation of linear momentum, if the continuum body is in static equilibrium it can be demonstrated that the components of the Cauchy stress tensor in every material point in the body satisfy the equilibrium equations.

$$\sigma_{ji,j} + F_i = 0$$

For example, for a hydrostatic fluid in equilibrium conditions, the stress tensor takes on the form:

$$\sigma_{ij} = -p\delta_{ij},$$

where p is the hydrostatic pressure, and δ_{ij} is the kronecker delta.



Derivation of equilibrium equations: Consider a continuum body (see Figure 4) occupying a volume V, having a surface area S, with defined traction or surface forces $T_i^{(n)}$ per unit area acting on every point of the body surface, and body forces F_i per unit of volume on every point within the volume V. Thus, if the body is in equilibrium the resultant force acting on the volume is zero, thus:

$$\int_S T_i^{(n)} dS + \int_V F_i dV = 0$$

By definition the stress vector is $T_i^{(n)} = \sigma_{ji} n_j$, then

$$\int_S \sigma_{ji} n_j \, dS + \int_V F_i \, dV = 0$$

Using the Gauss's divergence theorem to convert a surface integral to a volume integral gives

$$\int_V \sigma_{ji,j} \, dV + \int_V F_i dV = 0$$

$$\int_V (\sigma_{ji,j} + F_i) dV = 0$$

For an arbitrary volume the integral vanishes, and we have the equilibrium equations

$$\sigma_{ji,j} + F_i = 0$$

Cauchy's second law of motion: According to the principle of conservation of angular momentum, equilibrium requires that the summation of moments with respect to an arbitrary point is zero, which leads to the conclusion that the stress tensor is symmetric, thus having only six independent stress components, instead of the original nine:

$$\sigma_{ij} = \sigma_{ji}$$

Derivation of symmetry of the stress tensor Summing moments about point O (Figure 4) the resultant moment is zero as the body is in equilibrium. Thus,

$$M_O = \int_S (\mathbf{r} \times \mathbf{T}) dS + \int_V (\mathbf{r} \times \mathbf{F}) dV = 0$$

$$0 = \int_S \varepsilon_{ijk} x_j T_k^{(n)} dS + \int_V \varepsilon_{ijk} x_j F_k dV$$

where $\vec{r}$ is the position vector and is expressed as

$$\mathbf{r} = x_j \mathbf{e}_j$$

Knowing that $T_k^{(n)} = \sigma_{mk} n_m$ and using Gauss's divergence theorem to change from a surface integral to a volume integral, we have

$$
\begin{aligned}
0 &= \int_S \varepsilon_{ijk}x_j\sigma_{mk}n_m\,dS + \int_V \varepsilon_{ijk}x_jF_k\,dV \\
&= \int_V (\varepsilon_{ijk}x_j\sigma_{mk})_{,m}dV + \int_V \varepsilon_{ijk}x_jF_k\,dV \\
&= \int_V (\varepsilon_{ijk}x_{j,m}\sigma_{mk} + \varepsilon_{ijk}x_j\sigma_{mk,m})dV + \int_V \varepsilon_{ijk}x_jF_k\,dV \\
&= \int_V (\varepsilon_{ijk}x_{j,m}\sigma_{mk})dV + \int_V \varepsilon_{ijk}x_j(\sigma_{mk,m} + F_k)dV
\end{aligned}
$$

The second integral is zero as it contains the equilibrium equations. This leaves the first integral, where $x_{j,m} = \delta_{jm}$, therefore

$$\int_V (\varepsilon_{ijk}\sigma_{jk})dV = 0\int_V (\varepsilon_{ijk}\sigma_{jk})dV = 0$$

For an arbitrary volume V, we then have

$\varepsilon_{ijk}\sigma_{jk} = 0$ which is satisfied at every point within the body. Expanding this equation we have

$\sigma_{12} = \sigma_{21}, \sigma_{23} = \sigma_{32}, and \sigma_{13} = \sigma_{31}$

or in general $\sigma_{ij} = \sigma_{ji}$

This proves that the stress tensor is symmetric.

$\star$ **Principal stresses and stress invariant:**

At every point in a stressed body there are at least three planes, called principal planes, with normal vectors n , called principal directions, where the corresponding stress vector is perpendicular to the plane, i.e., parallel or in the same direction as the normal vector n , and where there are no normal shear stresses τ_n. The three stresses normal to these principal planes are called principal stresses.

A stress vector parallel to the normal unit vector $\vec{n}$ is given by:

$$\mathbf{T^{(n)}} = \lambda\mathbf{n} = \sigma_\mathrm{n}\mathbf{n}$$

where λ is a constant of proportionality, and in this particular case corresponds to the magnitudes σ_n of the normal stress vectors or principal stresses.

Knowing that $T_i^{(n)} = \sigma_{ij}n_j$ and $n_i = \delta_{ij}n_j$, we have

$$
\begin{aligned}
T_i^{(n)} &= \lambda n_i \\
\sigma_{ij}n_j &= \lambda n_i \\
\sigma_{ij}n_j - \lambda n_i &= 0 \\
(\sigma_{ij} - \lambda\delta_{ij})\,n_j &= 0
\end{aligned}
$$

This is a homogeneous system, i.e. equal to zero, of three linear equations where n_j are the unknowns. To obtain a nontrivial (non-zero) solution for n_j, the determinant matrix of the coefficients must be equal to zero, i.e. the system is singular. Thus,

$$|\sigma_{ij} - \lambda\delta_{ij}| = \begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

Expanding the determinant leads to the characteristic equation

$$|\sigma_{ij} - \lambda\delta_{ij}| = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$$

where

$$\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
&= \sigma_{kk} = \mathrm{tr}(\boldsymbol{\sigma}) \\
I_2 &= \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} \\
&= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\
&= \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right) = \frac{1}{2}\left[\left(\mathrm{tr}(\boldsymbol{\sigma})\right)^2 - \mathrm{tr}(\boldsymbol{\sigma}^2)\right] \\
I_3 &= \det(\sigma_{ij}) = \det(\boldsymbol{\sigma}) \\
&= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{31}^2\sigma_{22}
\end{aligned}$$

The characteristic equation has three real roots λ_i, i.e. not imaginary due to the symmetry of the stress tensor. The $\sigma_1 = \max(\lambda_1, \lambda_2, \lambda_3), \sigma_3 = \min(\lambda_1, \lambda_2, \lambda_3) \quad and \quad \sigma_2 = I_1 - \sigma_1 - \sigma_3$, are the principal stresses, functions of the eigenvalues λ_i.

Stress deviator tensor: The stress tensor $sigma_{ij}$ can be expressed as the sum of two other stress tensors:

1. a mean hydrostatic stress tensor or volumetric stress tensor or mean normal stress tensor, $\pi\delta_{ij}$, which tends to change the volume of the stressed body; and 2. a deviatoric component called the stress deviator tensor, s_{ij}, which tends to distort it. So:

$$\sigma_{ij} = s_{ij} + \pi\delta_{ij},$$

where π is the mean stress given by

$$\pi = \frac{\sigma_{kk}}{3} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = \tfrac{1}{3}I_1.$$

Pressure p is generally defined as negative one-third the trace of the stress tensor minus any stress the divergence of the velocity contributes with, i.e.

$$p = \lambda\,\nabla \cdot \vec{u} - \pi = \lambda\frac{\partial u_k}{\partial x_k} - \pi = \sum_k \lambda\frac{\partial u_k}{\partial x_k} - \pi,$$

where λ is a proportionality constant, ∇ is the divergence operator, x_k is the k:th Cartesian coordinate, $\vec{u}$ is the velocity and u_k is the k:th Cartesian component of $\vec{u}$.

The deviatoric stress tensor can be obtained by subtracting the hydrostatic stress tensor from the Cauchy stress tensor:

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij},$$

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_{11}-\pi & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\pi & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\pi \end{bmatrix}.$$

.Example:

The stress tensor is given by

$$\sigma = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

and the normal is

$$\vec{n} = 2/3\vec{i} - 2/3\vec{j} + 1/3\vec{k}$$

Then the stress vector at that point P.
Sol: We know

$$T^{\vec{(n)}} = \sigma.\vec{n} = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix} . \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 4 \\ -10/3 \\ 0 \end{pmatrix}$$
$$= 4\vec{i} - 10/3\vec{j} + 0\vec{k}$$

Ex:

Find out the principal stresses of

$$\sigma = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

sol. The principal stress are given by

$$\det(\sigma - \lambda I) = 0$$
$$\implies \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 0-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{vmatrix} = 0$$
$$\implies (\lambda+2)(-\lambda^2+5\lambda-4) = 0$$
$$\implies -(\lambda+2)(\lambda-4)(\lambda-1) = 0$$
$$\implies \lambda = 1, -2, 4$$
$$\therefore \sigma_{11} = 4, \sigma_{22} = 1, \sigma_{33} = -2$$

Integral of Euler's equation of motion. Bernoulli's equation. Pressure equation:

When a velocity potential exists (so that the motion is irrotational) and the external forces are derivable from a potential function, the equations of motion can always be integrated. Let ϕ be the velocity potential and V be the force potential. Then, by definition, we get

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}, \qquad w = -\frac{\partial \phi}{\partial z}, \qquad ....(1)$$

$$X = -\frac{\partial V}{\partial x}, \qquad Y = -\frac{\partial V}{\partial y}, \qquad Z = -\frac{\partial V}{\partial z}, \qquad ....(2)$$

And

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \qquad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}, \qquad ....(3)$$

Then well known Euler's dynamical equation are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Using (1), (2) and (3) these can be written as

$$-\frac{\partial^2 \phi}{\partial t \partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$-\frac{\partial^2 \phi}{\partial t \partial y} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$-\frac{\partial^2 \phi}{\partial t \partial z} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Re-writing equation, we get

$$-\frac{\partial}{\partial x}(\frac{\partial \phi}{\partial t}) + \frac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2 + w^2) = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ....(4)$$

$$-\frac{\partial}{\partial y}(\frac{\partial \phi}{\partial t}) + \frac{1}{2}\frac{\partial}{\partial y}(u^2 + v^2 + w^2) = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad ....(5)$$

$$-\frac{\partial}{\partial z}(\frac{\partial \phi}{\partial t}) + \frac{1}{2}\frac{\partial}{\partial z}(u^2 + v^2 + w^2) = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ....(6)$$

Now,

$$d(\frac{\partial \phi}{\partial t}) = \frac{\partial}{dx}(\frac{\partial \phi}{\partial t})dx + \frac{\partial}{\partial y}(\frac{\partial \phi}{\partial t})dy + \frac{\partial}{\partial z}(\frac{\partial \phi}{\partial t})dz \qquad ....(7)$$

$$dV = (\frac{\partial V}{\partial x})dx + (\frac{\partial V}{\partial y})dy + (\frac{\partial V}{\partial y})dy \qquad ....(8)$$

$$dp = (\frac{\partial p}{\partial x})dx + (\frac{\partial p}{\partial y})dy + (\frac{\partial p}{\partial y})dy \qquad ....(9)$$

$$d(u^2+v^2+w^2) = \frac{\partial}{\partial x}(u^2+v^2+w^2) + \frac{\partial}{\partial y}(u^2+v^2+w^2) + \frac{\partial}{\partial z}(u^2+v^2+w^2) \qquad ....(10)$$

Multipling (4), (5) and (7) by dx, dy and dz respectively, then adding and using (7), (8), (9) and (10) we have

$$-d(\frac{\partial \phi}{\partial t}) + \frac{1}{2}d(u^2+v^2+w^2) = -dV - \frac{1}{\rho}dp$$

$$-d(\frac{\partial \phi}{\partial t}) + \frac{1}{2}dq^2 + dV + \frac{1}{\rho}dp = 0 \qquad ....(11)$$

where $q^2 = u^2+v^2+w^2$ If ρ is a function of p, integration of (11) gives

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V + \int \frac{1}{\rho}dp = F(t) \qquad ....(12)$$

Where F(t) is and arbitrary function of t arising from integration in which t is regarded as constant. (12) is Bernoulli,s equation in its most general form. Equation (12) is also known as pressure equation.
Special case 1: let the fluid be homogeneous and inelastic (ρ is cons.). The Bernoulli's equation for unsteady and irrotational motion is given by

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V + \frac{1}{\rho}p = F(t) \qquad ....(13)$$

Special case 2: If motion be steady i.e, $\frac{\partial \phi}{\partial t} = 0$, the Bernoulli's equation for steady irrotational motion of an incompressible fluid is given by

$$\frac{1}{2}q^2 + V + \frac{1}{\rho}p = C \qquad ....(14)$$

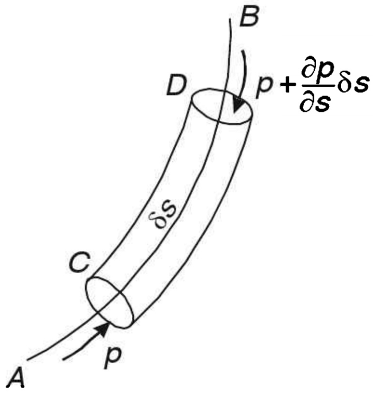
where C is a absolute constant.

Bernoulli's theorem.(Steady motion with no velocity potential and conservative field of force:
When the motion is steady and the velocity potential does not exist, we have

$$\frac{1}{2}q^2 + V + \int \frac{1}{\rho}dp = C,$$

where V is the force potential from which the external forces are derivable.
proof. Consider a streamline AB in the fluid. let δs be an element of this stream line and CD be the small cylinder of cross-sectional area α and δs as axis. If q be the velocity and S be the component of external force per unit mass in direction of the stream line, then by Newton's second law of motion, we have

$$\rho\alpha\delta s.\frac{Dq}{Dt} = \rho\alpha\delta s.S + p\alpha - (p + \frac{\partial p}{\partial s}\delta s)\alpha$$



$$\implies \qquad \frac{Dq}{Dt} = S - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

$$\implies \qquad \frac{\partial q}{\partial t} + q\frac{\partial q}{\partial s}S - \frac{1}{\rho}\frac{\partial p}{\partial s} \qquad ......(1)$$

If motion be steady $\frac{\partial q}{\partial t} = 0$, and if the external force have a potential function V such that $S = -\frac{\partial V}{\partial s}$ (1) reduces to

$$\frac{1}{2}\frac{\partial q^2}{\partial s} + \frac{\partial V}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} = 0 \qquad ....(2)$$

If ρ is a function of p, integration of (2) along the streamline AB yields

$$\frac{1}{2}q^2 + V + \int \frac{1}{\rho}dp = C \qquad ....(3)$$

Where C is a constant whose value depends on the particular chosen streamline.

. **Internal energy:** In the thermodynamics the internal energy of a system is the total energy contained within the system. It is the energy necessary to create/prepare a given state.

. **Potential energy:** In physics potential energy is the energy held by an object because of its position relative to other object, stresses within itself or other factors.

. **Kinetic Energy:** The energy due to the motion of the body $= 1/2mv^2$

$\star$ **The energy equation:**

Statement: The rate of change of total energy (Kinetic, potential and intrinsic) of any portion of a compressible inviscid fluid as it moves about is equal to the rate at which works is being done by the pressure on the boundary. The potential due to the extraneous force is supposed to be independent of time.

proof: Consider any arbitrary closed surface S drawn in the region occupied by the inviscid fluid and Let V be the volume of the fluid within S. Let ρ be the density of the fluid particle P within S and dV be the volume element surrounding P. Let $\vec{q}(\vec{r}, t)$ be the velocity of P. Then, the Eulier,s equation of motion is

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p \qquad .....(1)$$

Let the external force be conservative so that there exists a force potential Ω which is independent of time. Thus

$$\vec{F} = -\nabla\Omega \qquad and \qquad \frac{\partial\Omega}{\partial t} = 0$$

Using above and (1) we get,

$$\rho(\vec{q}.\frac{d\vec{q}}{dt}) = -\vec{q}.\nabla p - \rho(\vec{q}.\nabla)\Omega$$

$$\implies \qquad \rho[\frac{d}{dt}(\frac{1}{2}\vec{q}^2) + (\vec{q}.\nabla)\Omega] = -\vec{q}.\nabla p \qquad .....(2)$$

But

$$\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + (\vec{q}.\nabla)\Omega \quad since \quad \frac{\partial\Omega}{\partial t} = 0$$

Hence, equation (2) becomes

$$\rho\frac{d}{dt}(\frac{1}{2}\vec{q}^2 + \Omega) = -\vec{q}.\nabla p \qquad .....(3)$$

Since,

$$\frac{d(\rho V}{dt}) = 0 \qquad ......(4)$$

Integrating both side of (3) over V, we have

$$\int_V \rho\frac{d}{dt}(\frac{1}{2}\vec{q}^2 + \Omega)dV = -\int_V (\vec{q}.\nabla p)dV$$

$$\implies \quad \int_V [\frac{d}{dt}(\frac{1}{2}\rho\vec{q}^2)\rho dV + \frac{1}{2}\vec{q}^2\frac{d}{dt}(\rho dV)] + \int_V \frac{\partial}{dt}(\rho\Omega dV) = -\int_V (\vec{q}.\nabla p)dV \qquad ......(5)$$

Let T,W and I denoted the Kinetic, potential and intrinsic(internal) energies respectively. Then,

$$T = \int_V \frac{1}{2}\rho\vec{q}^2 dV, \qquad W = \int_V \rho\Omega dV, \qquad I = \int_V \rho E dV \qquad ....(6)$$

Where E is the intrinsic energy per unit mass.
We have

$$\vec{q}.\nabla p = \nabla.(p\vec{q}) - p\nabla.\vec{q}$$

$$\therefore R.H.S \;\; of \;\; (4) = -\int_V \nabla.(p\vec{q})dV - \int_V p\nabla.\vec{q}dV = \int_S p\vec{q}.\vec{n}dS + \int_S p\nabla.\vec{q}dV \qquad ....(7)$$

Where $\vec{n}$ is unit inward normal and dS is the element of the fluid surface S.
We prove that

$$\int_V p\nabla.\vec{q}dV = -\frac{dI}{dt} \qquad ....(8)$$

Now, E is define as the work done by the unit mass of the fluid against external pressure P (assuming that there exist a relation between pressure and density) from its actual state to some standard state in which p_0 and ρ_0 are the values of pressure and density respectively.

$$\therefore \quad E = \int_V^{V_0} pdV, \qquad where \quad V\rho = I, \qquad i.e, \quad V = \frac{1}{\rho}$$

$$\Longrightarrow \quad E = \int_\rho^{\rho_0} pd(\frac{1}{\rho}) = -\int_\rho^{\rho_0} (\frac{p}{\rho^2})d\rho \qquad .....(9)$$

$$\Longrightarrow \quad \frac{dE}{d\rho} = \frac{p}{\rho^2} \qquad and \qquad \frac{dE}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt}$$

$$\Longrightarrow \quad \int_V \frac{dE}{dt}\rho dV = \int_V \frac{p}{\rho^2}\frac{d\rho}{dt}dV \qquad ....(10)$$

But

$$\frac{d}{dt}(E\rho dV) = \frac{dE}{dt}\rho dV + E\frac{d}{dt}(\rho dV)$$

$$\therefore \qquad \frac{d}{dt}(E\rho dV) = \frac{dE}{dt}\rho dV \quad , by(4) \qquad ....(11)$$

Also by equation of continuity,

$$\frac{d\rho}{dt} = -\rho\nabla.\vec{q} \qquad ....(12)$$

Using (11) and (12), (10) reduces to

$$\frac{d}{dt}\int_V E\rho dV = -\int_V p\nabla.qdV \qquad or \qquad \frac{dI}{dt} = -\int_V p\nabla.qdV \quad , by(6)$$

Which prove (8).
Again the rate of work done by fluid pressure on an element on an element δS of S is $p\delta S\vec{n}.\vec{q}$.
Hence the net rate at which work is being done by the fluid pressure is

$$\int_S p.\vec{q}.\vec{n}dS = R \quad , (say) \qquad ....(13)$$

Using (8) and (13), (7) reduced to

$$R.H.S.\ of(4) = R - \frac{dI}{dt} \qquad ....(14)$$

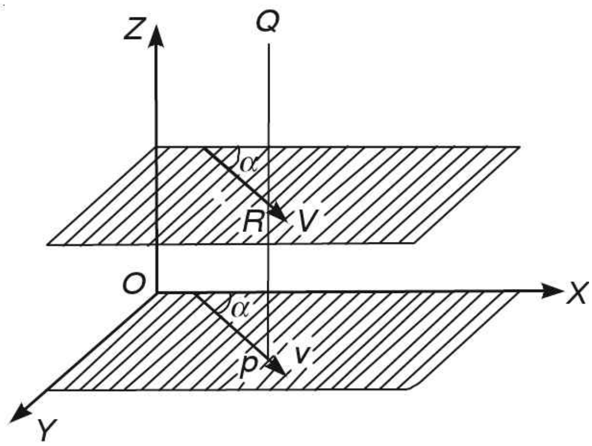
Hence Using (6) and (14), (4) reduces to

$$\frac{d}{dt}(T + W + I) = R \qquad ....(15)$$

Which is the desired energy equation.

⋆ **Two Dimensional flow:**

Suppose the plane under consideration is xy plane and Let P be any arbitrary point in that plane. We drawn a straight line PQ from Q be another point on the plane parallel to xy plane, and lying on PQ.



If $\vec{q}$ be the velocity of the fluid in xy plane which makes an angle α then $\vec{q}$ will be the velocity in the parallel plane and of some magnitude making an α with the x-axis.

Stream function or current function:

Let u and v be the component of velocity in two-dimentional motion. Then the differentional equation of lines of flow or streamlines is

$$\frac{dx}{u} = \frac{dy}{v} \qquad or \qquad vdx - udy = 0 \qquad ....(1)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad or \qquad \frac{\partial v}{\partial y} = \frac{\partial(-u)}{\partial x} \qquad ....(2)$$

(2) shows that L.H.S. of (1) must be an exact differential, $d\psi$ (say) Thus, we have

$$vdx - udy = d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \qquad ....(3)$$

$$so\ that \qquad u = -\frac{\partial \psi}{\partial y} \qquad and \quad v = \frac{\partial \psi}{\partial x} \qquad ......(4)$$

This function is known as the stream function. By (1) and (3) the streamlines are given by $d\psi$=0 i.e., by the equation $\psi = c$, where c is arbitrary constant. Thus the stream function is constant along streamline.

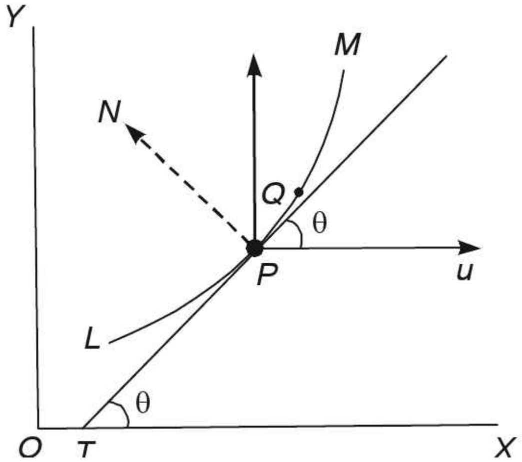
Physical signification of steam function:
Let LM be any curve in the x-y plane and Let ψ_1 and ψ_2 be the stream functions at L and M respectively. Let P be the arbitrary point on LM such that arc LP=s and let Q be the neighbouring point on LM such that arc LQ=s+δs. Let θ be the angle between tangent at P and x-axis. If u and v be the velocity-components at P, then

velocity at P along inward drawn normal PN = $vcos\theta - usin\theta$ .....(1)

When ψ is the stream function, then we have

$$u = -\frac{\partial\psi}{\partial y} \qquad and \quad v = \frac{\partial\psi}{\partial x} \qquad ......(2)$$

$$Again, \qquad cos\theta = \frac{dx}{ds} \qquad and \quad sin\theta = \frac{dy}{ds} \qquad ....(3)$$



Using (1), we get

flux across PQ from right to left =(v cos θ -u sin θ)δs

$\therefore$ Total flux across curve LM from right to left

$$= \int_{LM} (vcos\theta - usin\theta)ds = \int_{LM} (\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds})ds, \ using \ (2) \ and \ (3)$$

$$= \int_{LM} (\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy) = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

Thus a property of the current function is that the difference of its values at two points represents the flow across any line joining the points.

Remark 1. The current function ψ at any point can also be defined as the flux (i.e. rate of flow of fluid) across a curve LP where L is some fixed point in the plane.

Remark 2. Since the velocity normal to δs will contribute to the flux across δs whereas the velocity along tangent to t δs will not contribute towards flux across δs, we have

flux across $\delta s = \delta s \times$ normal velocity

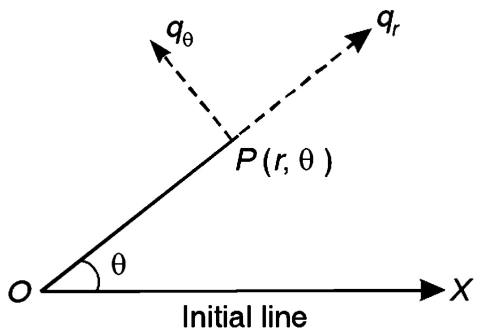
or $(\psi + \delta\psi) - \psi = \delta s \times$ normal to left across δs

or Velocity from right to left across $\delta s = \frac{\partial \psi}{\partial s}$

Remark 3. Velocity components in terms of ψ in plane-polar coordinate (r, θ) can be obtained by using the method outlined in remark 2 above. Let q_r and q_θ be the velocity components in the directions of r and θ increasing respectively. then

q_r=Velocity from right to left across r$\delta\theta$

$=\lim_{\delta r \to 0} \frac{\delta\psi}{r\delta\theta} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$



And q_θ=Velocity from right to left across δr

$=\lim_{\delta r \to 0} \frac{\delta\psi}{\delta r} = \frac{\partial\psi}{\partial r}$

Thus,

$$q_r = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad and \quad q_\theta = \frac{\partial\psi}{\partial r}$$

Spring components in terms of ψ:

We know that the velocity components u and v are functions of x,y and t and w=o in two-dimentional flow. Hence the spin components (ξ, η, ζ) are given by

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \qquad 2\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

and

$$2\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$

Let the motion be irrotational so That $\zeta = 0$ also. then obtain

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad or \quad \nabla^2\psi = 0$$

Showing that ψ satisfies Laplace's equation.

Complex potential:

Let $w = \phi + \vec{i}\psi$ be taken as a function of $x + \vec{i}y$ i.e., z. Thus, suppose that w=$f(z)$ i.e.

$$\phi + \vec{i}\psi = f(x + \vec{i}y) \qquad ....(1)$$

$$\Longrightarrow \qquad \frac{\partial \phi}{\partial x} + \vec{i}\frac{\partial \psi}{\partial x} = f^{'}(x + \vec{i}y)$$

$$And \qquad \frac{\partial \phi}{\partial y} + \vec{i}\frac{\partial \psi}{\partial y} = \vec{i}f^{'}(x + \vec{i}y) \qquad ....(2)$$

$$Or \qquad \frac{\partial \phi}{\partial y} + \vec{i}\frac{\partial \psi}{\partial y} = \vec{i}(\frac{\partial \phi}{\partial x} + \vec{i}\frac{\partial \psi}{\partial x})$$

Equating real and imaginary parts, we get

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad and \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Which are Cauchy-Riemann equations. Then w is an analytic function of z and w is known as the complex potential.

. Properties of the stream function:

1. The stream function ψ is constant along any streamline.
2. For a continuous flow (no sources or sinks), the volume flow rate across any closed path is equal to zero.
3. For two incompressible flow patterns, the algebraic sum of the stream functions is equal to another stream function obtained if the two flow patterns are super-imposed.
4. The rate of change of stream function with distance is directly proportional to the velocity component perpendicular to the direction of change.

Complex potential for some uniform flows :

(i) consider w=$i$kz ....(1)

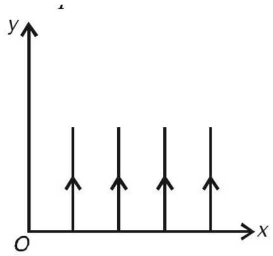
where k is a real and positive constants.

(1) $\Longrightarrow \quad \frac{dw}{dz} = -u + iv = ik \Longrightarrow u = 0$ and v=k,

which is clearly a uniform flow parallel to y-axis.

Hence the complex potential for a uniform flow whose magnitude of the stream is V in the positive y-direction is given by

$$w = iVz$$

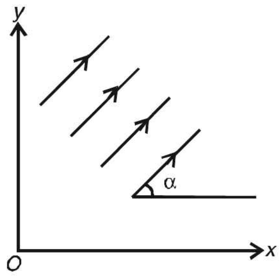


(ii) Consider $\quad w=-Ke^{-i\alpha}z, \quad$ ......(2)
Where k and α are real constants.
From(2),

$$\frac{dw}{dz} = -u + \vec{i}v = -Ke^{-\vec{i}\alpha}$$

$$\Longrightarrow \quad u = k\cos\alpha \quad \textit{and} \quad v = k\sin\alpha$$

which is corresponding to a uniform flow inclined at an angle α to the x-axis.

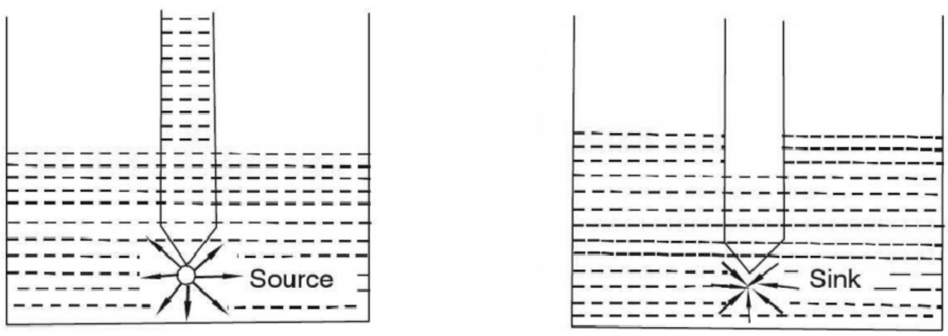


Hence the complex potential for a uniform flow whose magnitude is V and which is inclined is an angle α to x-axis is given by

$$w = Ve^{-i\alpha}z$$

Sources and sinks.

If the motion of a fluid consists of a fluid consists of symmetrical radial flow in all directions proceeding from a point, the point is known as simple source. If, however, the flow is such that the fluid is directed radially inwards to a point from all directions in a symmetrical manner, then the point is known as a simple sink.



Source + Sink = Doublet/Dipole.

⋆ **Strength:** Consider a source at the origin. Then the mass m the fluid coming out from the origin in a unit time is known as the strength of the source. Similarly, in a tank at the origin, the amount of fluid going into the sink in a unit time is called the strength of the sink.

Source and sinks in two-dimensions.

In two-dimensions a source of strength m is such that the flow across any small curve surrounding is $2\pi m$. Sink is regarded as source of strength -m.

Consider a circle of radius r with source at its centre. Then radial velocity q_r is given by

$$q_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad and \quad q_\theta = \frac{\partial \psi}{\partial r} \qquad ....(1)$$

$$or \quad q_r = -\frac{\partial \phi}{\partial r} \qquad as \quad \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad .....(2)$$

Then the flow across the circle is $2\pi r q_r$. Hence we have

$$2\pi r q_r = 2\pi m \qquad or \qquad r q_r = m \qquad ....(3)$$

$$or \qquad r(-\frac{1}{r}\frac{\partial \psi}{\partial \theta}) = m, \ \ by(1)$$

Integrating and omitting constant of integration, we get

$$\psi = m\theta \qquad ....(4)$$

Using (2) and (3) we get,

$$\phi = -m \log r \qquad ....(5)$$

Equation (4) shows that the streamlines are $\theta=$ constant, i.e., straight lines radiating from the source. Again (5) shows that the curves of equi-velocity potential are r=constant, i.e., concentric circles with center at source.

Double (or dipole) in two dimensions:

A combination of a source of strength m and a sink of strength -m at a small distance δs apart, where in the limit m is taken infinity great and δs infinitely small but so that the product mδs remains finite and equal to ν, is called a doublet of strength μ, and the line δs taken in the sense from -m to +m is taken as the axis of the doublet.

Complex potential due to a doublet in two-dimensions:

Let A,B denote the positions of the sink and source and P be any point. Let AP=r, BP=r+δr and $\angle PAB = \theta$. let ϕ be the velocity potential due to this doublet. Then

$$\phi = m \log r - m \log r - m \log(r + \delta r) = -m \log \frac{r + \delta r}{r}$$

$$\Longrightarrow \qquad \phi = -m \log(1 + \frac{\delta r}{r})$$

$$\therefore \quad \phi = -m\frac{\delta r}{r} \qquad ......(1)$$

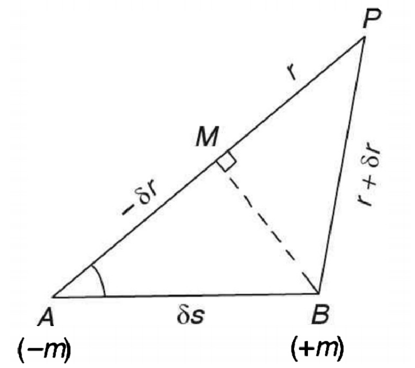
Let BM be perpendicular drawn from B on AP. then,

$$AM = AP - MP = r - (r + \delta r) = -\delta r$$

$$\therefore \quad \cos\theta = \frac{AM}{AB} = -\frac{\delta r}{r} \qquad so \ \ that \quad \delta r = -\delta s \cos\theta$$

$$\therefore From(1), \qquad \phi = \frac{\mu \cos\theta}{r} \qquad ....(2)$$

where $\mu = m\delta s$=strength of the doublet.



$$\therefore From(2), \qquad \frac{\partial \phi}{\partial \theta} = -\frac{\mu \cos\theta}{r^2}$$

$$\implies \quad \frac{1}{r}\frac{\partial \psi}{\partial r} = -\frac{\mu\cos\theta}{r^2} \qquad as \quad \frac{\partial \phi}{\partial \theta} = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

$$\implies \qquad \frac{\partial \psi}{\partial r} = -\frac{\mu\cos\theta}{r}$$

Integration it with respect to θ, we get

$$\psi = -\frac{\mu\sin\theta}{r} + f(r) \qquad ......(3)$$

$$Now, \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} \qquad .....(4)$$

Using (2) and (3), (4) reduce to

$$\frac{1}{r}\left(-\frac{\mu\sin\theta}{r}\right) = -\left[\frac{\mu\sin\theta}{r^2} + f'(r)\right]$$

or $f'(r) = 0 \qquad \implies \quad f(r) = constant$

$$\therefore \psi = -\frac{\mu\sin\theta}{r} \qquad .....(5)$$

Using (2) and (5), the complex potential due to a doublet is given by

$$w = \phi + i\psi = \frac{\mu}{r}(\cos\theta - i\sin\theta) = \frac{\mu}{re^{i\theta}} = \frac{\mu}{z}$$

Note 1: Equi-potential curves are given by ϕ=constant.

$$\implies \qquad \frac{\mu}{r}\cos\theta = constant \qquad or \quad \frac{\cos\theta}{r} = C$$

$$\therefore \qquad r\cos\theta = Cr^2 \qquad or \quad x = C(x^2 + y^2)$$

Which represent circles touching the y-axis at origin.

Note 2: Streamlines are given by ψ=constant. i.e., by

$$\Longrightarrow \qquad \frac{-\mu}{r}\sin\theta = constant \qquad or \qquad \frac{\sin\theta}{r} = C^{'}$$

$$\therefore \qquad r\sin\theta = C^{'}r^2 \qquad or \quad x = C^{'}(x^2+y^2)$$

Which represent circles touching the x-axis at origin.

Note 3: If the doublet makes an angle an angle θ with x-axis, we have to write $\theta-\alpha$ for θ so that

$$w = \frac{\mu}{re^{i(\theta-\alpha)}} = \frac{\mu e^{\vec{i}\alpha}}{re^{i\theta}} = \frac{\mu e^{i\alpha}}{z}$$

Note 4: If doublets of strengths $\mu_1, \mu_2, \mu_3, ...$ are situated at $z = z_1, z_2, z_3, ...$ and their axes making angles $\alpha_1, \alpha_2, \alpha_3, ...$ with x-axis then complex potential due to the above system is given by

$$w = \frac{\mu_1 e^{i\alpha_1}}{z-z_1} + \frac{\mu_2 e^{i\alpha_2}}{z-z_2} + \frac{\mu_3 e^{i\alpha_3}}{z-z_3} + ...$$

Example: what arrangement of sources and sinks will give rise to the function $w = \log(z-\frac{a^2}{z})$. Drawn a rough sketch of the streamlines. Prove the two of the streamlines subdivide into the circle $r = a$ and axis of y.

Solution. Given

$$w = \log(z-\frac{a^2}{z}) = \log(z-a) + \log(z+a) - \log z$$

Which shows that there are two sinks of unit strength at the points z=a and z=-a and a source of unit strength at origin. Since $w = \phi + i\psi$ and $z = x+iy$, we obtain

$$\phi + i\psi = \log(x+iy-a) + \log(x+iy+a) - \log x + iy$$

Equation imaginary parts on both sides, we have

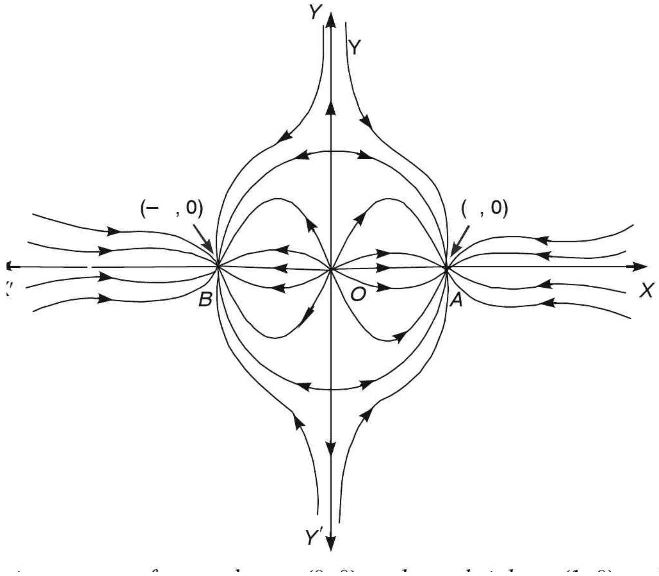
$$\begin{aligned}
\psi &= \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x} \\
&= \tan^{-1}\frac{\frac{y}{x-a}+\frac{y}{x+a}}{1-\tan^{-1}\frac{y}{x-a}.\frac{y}{x+a}} - \tan^{-1}\frac{y}{x} \\
&= \tan^{-1}\frac{2xy}{x^2-y^2-a^2} - \tan^{-1}\frac{y}{x} \\
&= tan^{-1}\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}
\end{aligned}$$

The desired streamlines are given by $\psi = constant = \tan^{-1}(C)$, i.e.,

$$\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = C$$

when C=0, (1) reduces to y=0. Thus x-axis is a streamline. Again, when $C \to \infty$, (1) reduces to $x(x^2 + y^2 - a^2 = 0$, i.e., x=0 and $x(x^2 + y^2 = a^2$ or r=a, which is a streamlines.

Hence the rough sketch of the streamlines is is shown in following figure.

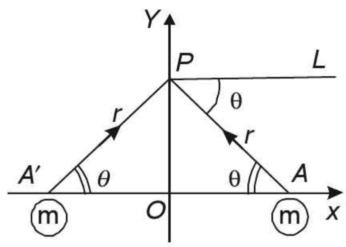


In this figure there is a source of unit strength at origin O and there are two sinks each of unit strength at A (a,0) and (-a,0).

Images.
If in a liquid surface S can be drawn across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this surface is known as the image of the system with regard to the surface. Moreover, if the surface S is a treated as a rigid boundary and the liquid removed from one side of it, the motion on the other will remain unchanged.

As there is no flow across the surface, it must be a streamline. Thus the fluid flows tangentially to the surface and hence the normal velocity of the fluid at any point of the surface is zero.

Image of a source with respect to a line.
Suppose that image of the source m at A(a,0) on x-axis is required with respect to OY. Take an equal source at $A'(-a, 0)$. Let P be any point on OY such that AP=A'P=r. Then the velocity at P due to source at A is $\frac{m}{r}$ along AP and velocity at P due to source A' is $\frac{m}{r}$ along A'P.



Let PL be perpendicular to OY. Then we see that

Resultant velocity at P due to source at A and A' along PL

$$= \frac{m}{r}\cos\theta - \frac{m}{r}\cos\theta = 0$$

showing that there will be no flow across OY. Hence by definition, the image of a simple source with respect to a line in two-dimensional is an equal source equidistant from the line opposite to the source.

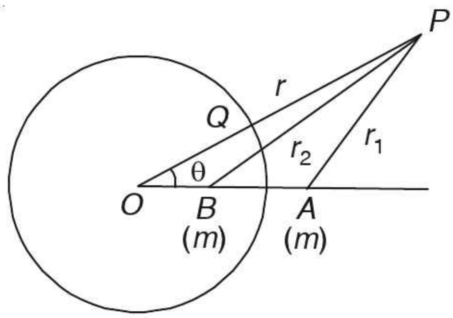
Image of a source with regard to a circle.

Let us determine the image of a source of strength m at a point A with respect to the circle with O as center. Let OA=f and let B be inverse point of A with respect to the circle. If a be the radius of the circle, then OA.OB=a^2 so that OB=$\frac{a^2}{f}$. Let P(z) be an arbitrary point in the plane of the circle.

Let there be a source of strength m at B. If w be the complex potential due to the source at A and B, then we get

$$w = -m\log(z-f) - m\log(z-\frac{a^2}{f})$$

$$= -m[\log(r\cos\theta - f + \vec{i}\sin\theta) + \log(r\cos\theta - \frac{a^2}{f} + \vec{i}\sin\theta)]$$



Writing $w = \phi + \vec{i}\psi$ and equating real parts, we get

$$\phi = -\frac{m}{2}[\log((r\cos\theta - f)^2 + (\sin\theta)^2) + \log((r\cos\theta - \frac{a^2}{f})^2 + (\sin\theta)^2)]$$

$$\phi = -\frac{m}{2}[\log(r^2 + f^2 - 2fr\cos\theta) + \log(r^2 + \frac{a^4}{f^2} - \frac{2ra^2}{f}\cos\theta)]$$

$$\therefore \quad \frac{\partial\phi}{\partial r} = -\frac{m}{2}[\frac{2(r - f\cos\theta}{(r^2 + f^2 - 2fr\cos\theta)} + \frac{2r - \frac{2a^2}{f}\cos\theta}{(r^2 + \frac{a^4}{f^2} - \frac{2ra^2}{f}\cos\theta)}]$$

Hence the normal velocity at any point Q on the circle,

$$-(\frac{\partial\phi}{\partial r})_{r=a} = \frac{m}{2}[\frac{2(a - f\cos\theta}{(a^2 + f^2 - 2fa\cos\theta)} + \frac{2a - \frac{2a^2}{f}\cos\theta}{(a^2 + \frac{a^4}{f^2} - \frac{2a.a^2}{f}\cos\theta)}]$$

$$= m[\frac{a - f\cos\theta + \frac{f^2}{a} - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta}] = \frac{m}{2}$$

Hence the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.

The Milne-Thomson circle theorem or simply the circle theorem.

Statement: Let f(z) be the complex potential for having no rigid boundaries and such that there are no singularities of flow within the circle $|z| = a$. Then, on introducing the solid circular cylinder $|z| = a$ into the flow, the new complex potential is given by

$w = f(z) + \bar{f}(\frac{a^2}{z})\ \ for\ \ |z| \geq a$

The Theorem of Blasius.

In a steady two-dimensional irrotational flow motion of an incompressible fluid under no external forced given by the complex potential w=f(z), if the pressure thrust on the fixed cylinder of any shape are represented by a force (X,Y) and a couple of moment M about the origen of co-ordinates, Then

$$X - iY = \frac{1}{2}i\rho \int_C (\frac{dw}{dz})^2 dz, \qquad M = Real\ \ part\ \ of\ \ [-\frac{1}{2}i\rho \int_C z(\frac{dw}{dz})^2]$$

Where ρ is the fluid density and integrals are taken round the contour C of the cylinder.

proof. Figure shows the section C of the cylinder in the plane XOY. Let p(x,y) and Q$(x+\delta x, y+\delta y)$ be two neighbouring points on C such that arc PQ=δs.If θ be the angle which the tangent PT at P on the contour C makes with x-axis,

$$\cos\theta = \frac{dx}{ds} \qquad \sin\theta = \frac{dy}{ds} \qquad ....(1)$$

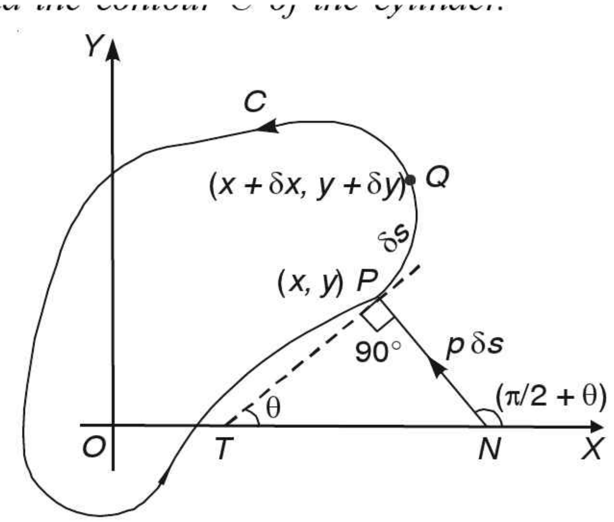
and normal at P makes an angle $(\theta + \frac{\pi}{2})$ with x-axis. now, if p be the pressure at P, the force on unit length of the section δs is pδs normal to C.Then by (1), we have

$$X = \int_C p\cos(\theta + \frac{\pi}{2})ds = -\int_C p\sin\theta ds = -\int_C pdy, using(1) \qquad ....(2)$$

$$Y = \int_C p\sin(\theta + \frac{\pi}{2})ds = -\int_C p\cos\theta ds = -\int_C pdx, using(1) \qquad ....(3)$$

$$M = \int_C [x.p\sin(\theta\frac{\pi}{2})ds - yp\cos(\theta\frac{\pi}{2})ds] = \int_C p(x\cos\theta ds + y\sin\theta ds)$$

$$M = \int_C p(xdx + ydy), by(1) \qquad ....(4)$$



Now Bernoulli's equation in this context is

$$\frac{1}{2}q^2 + \frac{p}{\rho} = B \qquad so\ that \quad p = \rho B - \frac{1}{2}q^2 \qquad .....(5)$$

where q is the fluid velocity, ρ the velocity. Since ρ is constant for an incompressible fluid, take $\rho B = A$(a constant). Again $\vec{q}^2$=$u^2 + v^2$ where u and v are the velocity components. Then (5) $\implies$

$$p = A - \frac{\rho}{2} \times (u^2 + v^2) \qquad ...(6)$$

$$Also, \qquad \frac{dw}{dz} = -u + iv \qquad or \qquad -\frac{dw}{dz} = u + iv \qquad ....(7)$$

Using (6), (2), (3) and (4) reduce to

$$X = -\int_C [A - \frac{1}{2}\rho(u^2 + v^2)]dy = \frac{1}{2}\rho \int_C (u^2 + v^2)dy \qquad ....(8)$$

$$Y = \int_C [A - \frac{1}{2}\rho(u^2 + v^2)]dx = -\frac{1}{2}\rho \int_C (u^2 + v^2)dx \qquad ....(9)$$

$$and \quad M = \int_C [A - \frac{1}{2}\rho(u^2 + v^2)](xdx + ydy) = -\frac{1}{2}\int_c (u^2 + v^2)(xdx + ydy) \qquad ....(10)$$

While simplifying (8),(9) and (10), we get

$$\int_c dy = \int_C dx = \int_C xdx = \int_C xdy = 0$$

Now,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dx + idy}{u + iv} = \frac{dx - idy}{u - iv}$$

or,

$$\frac{dx - idy}{dx + idy} = \frac{u - iv}{u + iv} = \frac{(u - iv)^2}{u^2 + v^2}$$

$$\therefore \qquad (u - iv)^2(dx + idy) = (u^2 + v^2)(dx - idy) \qquad .....(11)$$

From (8) and (9), we have

$$X - iY = \frac{1}{2}\rho \int_c (u^2 + v^2)(dy + idx) = \frac{1}{2}\rho i \int_c (u^2 + v^2)(dx + \frac{1}{i}dy)$$

$$= \tfrac{1}{2}\rho i \textstyle\int_C (u^2 + v^2)(dx - idy) = \tfrac{1}{2}\rho i \int_C (u - iv)^2(dx + idy)$$
$$= \tfrac{1}{2}\rho i \textstyle\int_C (\frac{dw}{dz}$$

Re-writing (10), we have

$$M = Real\ part\ of\ \ -\frac{1}{2}\rho \int_c (x + iy)(u^2 + v^2)(dx - idy)$$

$$M = Real\ part\ of\ \ -\frac{1}{2}\rho \int_c (x + iy)(u - iv)^2(dx + idy)$$

$$M = Real\ part\ of\ \ -[\frac{1}{2}\rho \int_c z(\frac{dw}{dz})^2 dz]$$

Flow and Circulation.
If A and P be any two points in a fluid, then the value of the integral

$$\int_A^P (udx + vdy + wdz),$$

taken along any path from A to P, called the flow along the path from A to P.

when a velocity potential ϕ exists, the flow from A to P is

$$= -\int_A^P (\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz) = -\int_A^P d\phi = \phi_A - \phi_P.$$

The flow round a closed curve is known as the circulation round the curve. Let C be closed curve and Γ be the circulation. Also, let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$ Then,

$$\Gamma = \int_C (udx + vdy + wdz) = \int_C \vec{q}.dr,$$

where the line integral is taken round C in a counter clockwise direction and $\vec{q}$ is the velocity vector.

Stokes's theorem. Let S be a surface bounded by a closed curve C, and n the unite vector normal to the surface.And $\vec{q}$ be velocity, Ω be vorticity Then,

$$\int_C \vec{q}.dr = \int_S curl\ \vec{q}.n\ dS \qquad i.e., \quad \Gamma = \int_S \Omega.\vec{n}dS$$

where Γ be the circulation.

Kelvin's circulation theorem.
When the external forces are conservative and derivable from a single valued potential function and the density is a function of pressure only, the circulation in any closed circuit moving with the fluid is constant for all time.

proof. Let C be closed circuit moving with the fluid so that C always consists of the same fluid particles. Let $\vec{q}$ be the fluid velocity at any point P of the circuit and let $\vec{r}$ be its position vector. Then the circulation along the closed circuit C is given by

$$\Gamma = \int_C \vec{q}.dr \qquad or \qquad \frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_C \vec{q}.dr \qquad ......(1)$$

Since the above integration is performed at constant time, reversing the order of integration and differentiation is justified. Then (1) may be re-written as

$$\frac{D\Gamma}{Dt} = \int_C \frac{D}{Dt}(\vec{q}.dr) \qquad ......(2)$$

$$But \qquad \frac{D}{Dt}(\vec{q}.dr) = \frac{D\vec{q}}{Dt}.dr + \vec{q}.d\vec{q} \qquad ......(3)$$

The Euler's equation of motion is

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p \qquad ....(4)$$

Let the external forces be conservative and derivable from a singe valued potential function V. Then $\vec{F}$=-∇V and (4) becomes

$$\frac{D\vec{q}}{Dt} = -\nabla V - \frac{1}{\rho}\nabla p$$

$$\therefore \quad \frac{D\vec{q}}{Dt}.dr = -\nabla V.dr - \frac{1}{\rho}\nabla p.dr = -dV - \frac{dp}{\rho} \qquad .....(5)$$

$$Also \qquad \vec{q}.d\vec{q} = \frac{1}{2}d(\vec{q}.\vec{q}) = \frac{1}{2}dq^2 \qquad ....(6)$$

where $\vec{q}$ denoted the magnitude of the velocity vector $\vec{q}$. Using (5) and (6), (3) reduces to

$$\frac{D}{Dt}(\vec{q}.dr) = -dV - \frac{dp}{\rho} + \frac{1}{2}dq^2 \qquad .....(7)$$

Using (7) and assuming that ρ is a single-valued function of p.(2) reduces to

$$\frac{D\Gamma}{Dt} = [\frac{1}{q^2} - V - \int_C \frac{dp}{\rho}]_C \qquad ......(8)$$

Where $[\,]_C$ denotes Change in the quantity enclosed within brackets on moving once round C. Since $\vec{q}$, V and p are single-valued function of $\vec{r}$, R.H.S of (8) vanishes. Equation (9) gives the rate of change of flow along any closed circuit moving with the fluid. Thus, it follows that the circulation in any closed circuit moving with the fluid is constant for all time.

Permanence of irrotational motion.

When the external forces are conservative and derivable from a single valued potential, and density is a function of a pressure only, then the motion of an inviscid fluid, if once irrotational, remains irrotational even afterwards.

proof. From Stokes' theorem, the circulation is given by

$$\Gamma = \int_C \vec{q}.dr = \int_S Curl\vec{q}.d\vec{S} \qquad ......(1)$$

At any time t, let the motion be irrotational so that curl$\vec{q}$=0. then (1) shows that $\Gamma = 0$ at that instant. Hence it follows Kelvin's circulation theorem that $\Gamma = 0$ for all time. Hence at any subsequent time, (1) shows that

$$\int_S Curl\vec{q}.d\vec{S} = 0 \qquad ......(2)$$

Since S is arbitrary, (2) shows that $Curl\vec{q} = 0$ at all subsequent time.
i.e., the motion remains irrotational even afterwards.

Green's Theorem. If $\phi, \phi^{'}$ are both single-valued and continuously differential scalar point functions such that $\nabla\phi$ and $\nabla\phi^{'}$ are also continuously differential, then

$$\int_V (\nabla\phi.\nabla\phi^{'})dV = -\int_S \phi\frac{\partial\phi^{'}}{\partial n}dS - \int_V \phi\nabla^2\phi^{'}dV$$

$$= -\int_S \phi^{'}\frac{\partial\phi}{\partial n}dS - \int_V \phi^{'}\nabla^2\phi dV,$$

Where S is closed surface bounding any simply-connected region, δn is an element of inwards normal at a point on S, and V is the volume enclosed by S.
Deduction. Let $\phi = \phi^{'}$ and let ϕ be the velocity potential of a liquid motion within S. Then $\nabla^2\phi = 0$ and hence Green's theorem gives

$$\int_V (\nabla\phi.\nabla\phi)dV = -\int_S \phi\frac{\partial\phi}{\partial n}dS$$

$$\Longrightarrow \quad \int_V [(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial z})^2]dV = -\int_S \phi\frac{\partial\phi}{\partial n}dS \qquad ....(1)$$

Let $\vec{q}$ be the velocity and ρ the density of the liquid, Then (1) reduce to

$$\frac{1}{2}\rho\int_V dV = -\frac{1}{2}\rho\int_S \frac{\partial\phi}{\partial n}dS \qquad ....(2)$$

Clearly, the L.H.S of (4) represents the kinetic energy T of the liquid within S. Hence (4) reduces to

$$T = -\frac{1}{2}\rho\int_S \frac{\partial\phi}{\partial n}dS \qquad ....(3)$$

Now $\rho\phi$ is the impulsive pressure that would set up the motion instantaneously from rest, and -$\frac{\partial\phi}{\partial n}$ is the inward normal velocity at the surface. Hence (5) shows that the kinetic energy set up by impulses, and half the velocity of its point of application. From (5), we also find that the kinetic energy of given mass of liquid moving irrotationally in simply-connected region depends only on the motion of its boundaries.

Suppose on the boundary $\frac{\partial\phi}{\partial n} = 0$. Then (2) reduce to

$$\int_V \vec{q}^2 dV = 0 \qquad ....(4)$$

Since $\vec{q}^2$ is positive, (4) implies that $\vec{q} = 0$ everywhere. Hence the liquid is at rest. Thus a cyclic irrotational motion is impossible in liquid bounded by fixed rigid boundary.

Acyclic and cyclic motions.
The motion in which the velocity potential is single-valued is called acyclic whereas the motion in which the velocity potential is not a single-valued is called cyclic.

Some Uniqueness Theorem:
We shall use the following equivalence of the expressions for the kinetic Energy

$$T = \frac{1}{2}\rho\int_V \vec{q}^2 dV = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}dS \qquad ....(1)$$

Where the symbols have their usual meaning.

Theorem 1: There cannot be two different forms of acyclic irrotational motions of a confined mass of incompressible inviscid liquid, when the boundaries have prescribed velocities.
Proof. If possible, let ϕ_1, ϕ_2 be the velocity potentials of two different motions subject to the condition

$$\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} \ \ at\ each\ point\ of\ S \qquad ......(2)$$

$$Also \qquad \nabla^2\phi_1 = \nabla^2\phi_2 \qquad ....(3)$$

$$Let \qquad \phi = \phi_1 - \phi_2, \qquad then \qquad \nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$$

Hence ϕ is a solution of Laplace's equation and so it is represents irrotational motion in which

$$\frac{\partial \phi}{\partial n} = \frac{\partial}{\partial n}(\phi_1 - \phi_2) = \frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n} = 0, \ \ by(2)$$

Hence $\vec{q} = 0$ by (1). But $\vec{q}^2 = (\nabla\phi)^2$. Thus, we have
$(\nabla\phi)^2 = 0$ so that $\phi=$ constant or $\phi_1 - \phi_2$=constant. Since the constant is of no significance, it follows that the two motions are same.

Theorem 2: There cannot be two different forms of irrotational motion for a given confined mass of incompressible inviscid liquid, whose boundaries are subject to the given impulses.
Proof. If possible, let ϕ_1, ϕ_2 be the velocity potentials of two different motions subject to the condition

$$\rho\phi_1 = \rho\phi_2 \ \ at\ each\ point\ of\ S \qquad ......(4)$$

$$Also \qquad \nabla^2\phi_1 = \nabla^2\phi_2 \qquad ....(5)$$

$$Let \qquad \phi = \phi_1 - \phi_2, \qquad then \qquad \nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$$

Hence ϕ is a solution of Laplace's equation and so it is represents irrotational motion in which

$$\rho = \rho\phi_1 - \rho\phi_2 = 0, \ \ by(4)$$

Hence $\vec{q} = 0$ by (1). But $\vec{q}^2 = (\nabla\phi)^2$. Thus, we have
$(\nabla\phi)^2 = 0$ so that $\phi=$ constant or $\phi_1 - \phi_2$=constant. Showing that the two motions are the same.

Kelvin,s minimum energy theorem.
The irrotational motion of fluid occupying a simply connected region has less kinetic energy then any other motion consistent with same normal velocity of the boundary.
Proof. Let T_1 be the kinetic energy $\vec{q_1}$ the fluid velocity of the actual irrotational motion with a velocity of the actual irrotational motion with a velocity potential ϕ

Then $\qquad \vec{q_1} = -\nabla\phi \qquad ......(1)$

Let T_2 be the kinetic energy, $\vec{q_2}$, the velocity of any other possible state of motion consistent with the same normal velocity of the motions give

$$\nabla.\vec{q_1} = 0 \qquad and \qquad \nabla.\vec{q_2} \qquad .....(2)$$

Let $\vec{n}$ denoted the unit normal at a point of S. Then using the fact that the boundary has the same normal velocity in both motions, we have

$$\vec{n}.\vec{q_1} = \vec{n}.\vec{q_2} \qquad .....(3)$$

Now,

$$T_1 = \frac{1}{2}\rho \int_V \vec{q_1}^2 dV \qquad and \qquad T_2 = \frac{1}{2}\rho \int_V \vec{q_2}^2 dV$$

$$\therefore \quad T_2 - T_1 = \frac{1}{2}\rho \int_V (\vec{q_2}^2 - \vec{q_1}^2) dV = \frac{1}{2}\rho \int_V (2\vec{q_1}.(\vec{q_2} - \vec{q_1}) + (\vec{q_2} - \vec{q_1})^2) dV$$

$$= -\rho \int_V (\nabla\phi).(\vec{q_2} - \vec{q_1}) dV + \frac{1}{2}\rho \int_V (\vec{q_2} - \vec{q_1})^2 dV, by (1) \quad .....(4)$$

But $\nabla.[\phi(\vec{q_2} - \vec{q_1})] = \phi[\nabla.(\vec{q_2} - \vec{q_1})] + (\nabla\phi).(\vec{q_2} - \vec{q_1}) = (\nabla\phi).(\vec{q_2} - \vec{q_1})$, using (2)

$$\therefore \quad \int_V (\nabla\phi).(\vec{q_2} - \vec{q_1}) dV = \int_V \nabla.[\phi(\vec{q_2} - \vec{q_1})] dV = \int_S \phi\vec{n}.(\vec{q_2} - \vec{q_1}) dS, by \ divergence \ Thorem$$

Thus,

$$\int_V (\nabla\phi).(\vec{q_2} - \vec{q_1}) dV = 0, by (3) \qquad .....(5)$$

Making use of (5), (4) reduces to

$$T_2 - T_1 = \frac{1}{2}\rho \int_V (\vec{q_2} - \vec{q_1})^2 dV$$

Since R.H.S. of (6) is non-negative, we have $T_2 - T_1 \geq$, i.e., $T_2 \geq T_1$. Hence the result.

General motion of a cylinder.

We want to study two dimensional motion produced by the motion of a cylinder in an infinite mass of liquid at rest at infinity, or when a cylinder is inserted in a steady stream. For the sake of simplicity we shall suppose the cylinder to be of unit length, and the liquid and the cylinder to be confined between two smooth parallel planes at right angle to the axis of the cylinder.

We know that the velocity potential ϕ and the stream function ψ are connected with the help of C-R equation, namely

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad and \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad ....(1)$$

In view of (1), the complex potential

$$w(z) = \phi(x, y) + i\psi(x, y) \qquad ......(2)$$

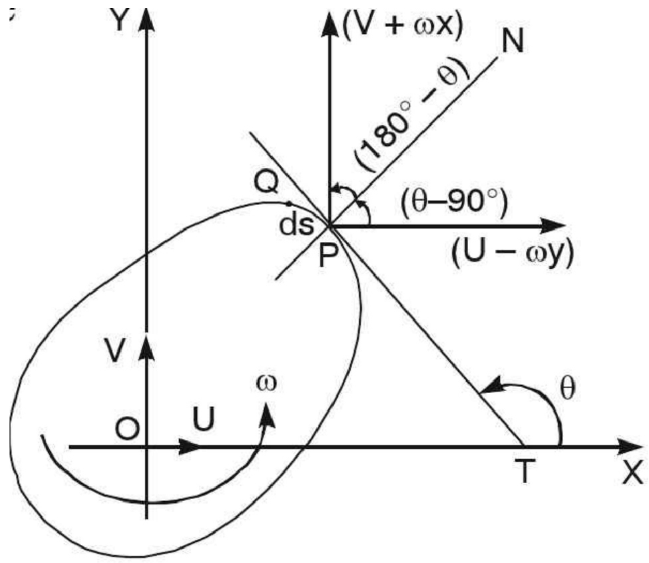
Can be determined by finding only ψ. Then stream function ψ must satisfy the Laplace's equation $\nabla^2\psi = 0$ at all points of the liquid and must also satisfy the boundary conditions:

(i) Since the liquid is at rest at infinity, we must have $\frac{\partial\psi}{\partial x} = 0$ $\frac{\partial\psi}{\partial y} = 0$ at infinity.

(ii) At any fixed boundary the normal velocity must coincide with a streamline $\psi=$ constant.

(iii) At any boundary of the moving cylinder, the normal component of the velocity of the liquid must be equal to the normal component of velocity of the cylinder.

We know express the condition (iii) by a formula for ψ as follows:



Let a point O of the cross-section of any cylinder be taken as origin. Let U and V be the velocities parallel to the axis of X and Y at O and let the cylinder turn with angular velocity ω. If P(x,Y) be any point on the surface of the cylinder, then the velocity component of P are $U-\omega y$ and $V+\omega x$. If θ is the inclination of the tangent at P with OX, then from Differential Calculus,

$$\cos\theta = \frac{dx}{ds} \qquad and \qquad \sin\theta = \frac{dy}{ds}, \qquad .....(3)$$

Where are PQ=ds, P and Q being two neighbouring points.

$$\begin{aligned}\therefore \text{ The outward normal velocity at P} &= (U-\omega y)\cos(\theta-90^o) + (V+\omega x)\cos(\theta-180^o)\\ &= (U-\omega y)\sin\theta - (V+\omega x)\cos\theta\\ &= (U-\omega y)\tfrac{dy}{ds} - (V+\omega x)\tfrac{dx}{ds} \qquad ....(4)\end{aligned}$$

Also the velocity of the liquid in the direction of the outward normal is -$\frac{\partial\psi}{\partial s}$.

On equating the above two expression for the normal component of velocities in accordance with condition (iii), we have

$$-\frac{\partial\psi}{\partial s} = (U-\omega y)\frac{dy}{ds} - (V+\omega x)\frac{dx}{ds} \qquad ....(5)$$

On integrating (5) along the arc, we get

$$\psi = Vx - Uy + \frac{1}{2}\times\omega(x^2+y^2) + C \qquad ...(6)$$

where C is an arbitrary constant.

Note that (6) holds for rotational or irrotational motion. thus (6) is the condition for the most general type of motion of motion of a cylinder of arbitrary cross-section.

If the cylinder is rotating about a fixed axis with angular velocity (so that U=V=0), then (6) reduce to

$$\psi = \frac{1}{2}\times\omega(x^2+y^2) + C \qquad ...(7)$$

$$or, \qquad \psi = \frac{1}{2} \times \omega z\bar{z} + C \qquad ...(8)$$

Suppose the equation of the cross-section of the boundary of the cylinder be of the from

$$z\bar{z} = f(z) + f(\bar{z})$$

Then the complex potential satisfying the boundary condition (8) is

$$w = i\omega f(z) \qquad ......(9)$$

Next, let the cylinder move along the x-axis with velocity U without rotation (so that V=0 and $\omega = 0$). Then (6) reduce to

$$\psi = -Uy + C \qquad .....(10)$$

Similarly, if the cylinder move along the y-axis with velocity V without rotation. Then (6) reduce to

$$\psi = -Uy + C \qquad .....(11)$$

Kinetic energy.

In any type of motion of a cylinder moving in liquid at rest at infinity, the kinetic energy is given by,

$$T = -\frac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n} dS \qquad ....(1)$$

Where ds is the elementary surface and integration is taken round a closed surface.

Suppose the liquid is confined between two smooth planes at unit distance apart. Let ds be the elementary arc of the cylinder so the dS=1.ds=ds. Then (1) reduce to

$$T = -\frac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n} dS \qquad ....(2)$$

where the integration is now round the perimeter of the cross-section of the cylinder. But $-\frac{\partial \phi}{\partial n}$ is normal velocity outwards, and $\frac{\partial \psi}{\partial s}$ is the normal velocity inwards, so that

$\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial s}$. Then (2) to

$$T = -\frac{1}{2}\rho \int_S \phi d\psi \qquad ....(3)$$

Motion of a circular cylinder:

To determine the motion of a circular cylinder moving in an infinite mass of the liquid at rest at infinity, with velocity U in the direction of x-axis.

To find the velocity potential ϕ that will satisfy the given boundary conditions, we have the following consideration:

(i) ϕ satisfies the laplace's equation $\delta^2\phi = 0$ at every point of the liquid. In polar coordinates in two dimensions $\delta^2\phi = 0$ takes the form

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad ......(1)$$

We know that (1) has solutions of the forms

$$t^n \cos n\theta \qquad \textit{and} \qquad t^n \sin n\theta$$

Where n is any integer, positive or negative. Hence the sum of any number of terms of the form

$$A_n t^n \cos n\theta \qquad \textit{and} \qquad B_n t^n \sin n\theta$$

is also a solution.

(ii) Normal velocity at any point of the cylinder = velocity of the liquid at that point in that direction.

$$i.e., \qquad -\frac{\partial\phi}{\partial r} = U\cos\theta, \qquad when\ \ r = a \qquad ......(2)$$

(iii) Since the liquid is at rest at infinity, velocity must be zero there. Thus,

$$-\frac{\partial\phi}{\partial r} = 0 \qquad and \qquad -\frac{1}{r}\frac{\partial\phi}{\partial r} = 0 \quad at \quad r = \infty \qquad ....(3)$$

The above considerations suggest that we must assume the following suitable form of ϕ

$$\phi = Ar\cos\theta + \frac{B}{r}\cos\theta \qquad ....(4)$$

$$\Longrightarrow \qquad -\frac{\partial\phi}{\partial r} = -(A - \frac{B}{r^2})\cos\theta \qquad ....(5)$$

Putting r=a in (5) and using (2), we get

$$U\cos\theta = -(A - \frac{B}{a^2})\cos\theta \qquad so\ that\ \ A - \frac{B}{a^2} = -U \qquad ....(6)$$

Putting r=∞ in (5) and using (3), we get A=0

Then (6) gives B=Ua^2. Hence (4) reduces to

$$\phi = (\frac{Ua^2}{r})\cos\theta \qquad ......(7)$$

It may be noted that (7) also satisfies the second condition given by (3). hence (7) gives the required velocity potential. But

$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial r} \qquad ......(8)$$

$$\therefore \qquad \frac{\partial \psi}{\partial r} = -(\frac{Ua^2}{r})\sin\theta \qquad ....(9)$$

Which gives the stream function of the motion. The complex potential w($= \phi + \vec{i}\psi$) is given by

$$w = (\frac{Ua^2}{r})(\cos\theta + i\sin\theta) = \frac{Ua^2}{r}e^{i\theta} = \frac{Ua^2}{z} \qquad ....(10)$$

Liquid streaming past a fixed circular cylinder.

Let the cylinder be at rest and the liquid flow past the cylinder with velocity U in the negative direction of of x-axis. This motion may be deduced from the previous result by imposing a velocity -U parallel to the x-axis on both the cylinder and the liquid. The cylinder is then reduced to rest and we must add to the velocity potential a term Ux (i.e., Urcos θ) to account for the additional velocity, consequently a term Ursin θ must be added to ψ.Thus, we have

$$\phi = U(r + \frac{a^2}{r})\cos\theta, \qquad \phi = U(r - \frac{a^2}{r})\sin\theta \qquad ......(1)$$

And

$$w = \phi + \vec{i}\psi = U(r\cos\theta + \vec{i}r\sin\theta) + \frac{Ua^2}{r}(\cos\theta + \vec{i}\sin\theta) = Uz + \frac{Ua^2}{z} \qquad .....(2)$$

Example. A circular cylinder of radius a is moving with velocity U along the x-axis; show that the motion produced by the cylinder in a mass of fluid at rest is given by the complex function

$$w = \phi + i\psi = \frac{Ua^2}{z - Ut} \qquad where \ z = x + iy$$

Find the magnitude and direction of velocity in the fluid and that for a marked particle of the fluid, whose polar coordinates are (r,θ) referred to the centre of the cylinder as origin,

$$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = \frac{u}{r^2}(\frac{a^2}{r^2}e^{i\theta} - e^{-i\theta}) \qquad and \qquad (r - \frac{a^2}{r})\sin\theta = b$$

Hence prove that the path of such a particle is the elastic curve given by, $\rho(y - \frac{b}{2}) = \frac{a^2}{4}$, Where ρ is the radius of curvature of the path.

Solution. Let O' be the centre of the circular cross-section of the cylinder at any time t. Then coordinates of O' are (Ut,0). Now, the complex potential of the fluid motion, referred to O' as origin, is $\frac{Ua^2}{z}$. Hence, when referred to the fixed origin O, the complex potential is given by

$$w = \phi + i\psi = \frac{Ua^2}{z - Ut} \qquad ....(1)$$

Which proves the fist part of the problem.

$$From(1), \qquad -\frac{dw}{dz} = \frac{Ua^2}{z - Ut}$$

$$\Longrightarrow \qquad -(-u + iv) = \frac{Ua^2}{r^2 e^{2i\theta}}, \qquad where \quad z - Ut = re^{i\theta}$$

$$\implies \quad (u - iv) = \frac{Ua^2}{r^2}(\cos 2\theta - i\sin 2\theta)$$

$$u = \frac{Ua^2}{r^2}\cos 2\theta \qquad and \qquad v = \frac{Ua^2}{r^2}\sin 2\theta$$

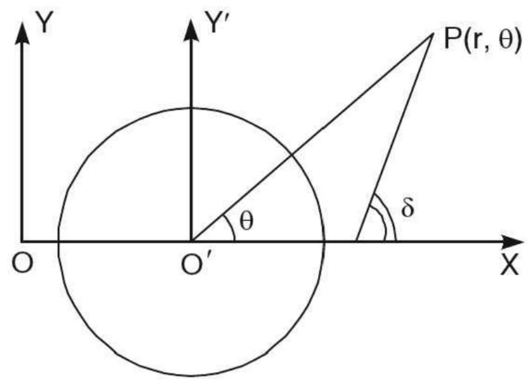
And q=the magnitude of velocity $= \sqrt{u^2+v^2} = \frac{Ua^2}{r^2}$

With δ= direction of the velocity $= \tan^{-1}\frac{v}{u} = 2\theta$

Now consider fixed axes OX, OY at the instant when the centre of the cylinder is at O'. To determine the path of any particle referred to the cylinder, we reduce the cylinder to rest. Hence, relative to the cylinder, the complex potential of the motion at any point P(r,θ) is

$$\therefore \quad w = Uz + \frac{Ua^2}{z} = Ure^{i\theta} + \frac{Ua^2}{r}e^{-i\theta}$$

$$\therefore \qquad \phi = U(r + \frac{a^2}{r})\cos\theta \qquad and \quad \psi = U(r - \frac{a^2}{r})\sin\theta \qquad ......(2)$$



Then at the instant under consideration, the particle is moving along the streamline ψ=constant. i.e.,

$$U(r - \frac{a^2}{r})\sin\theta = Ub \qquad or \qquad (r - \frac{a^2}{r})\sin\theta = b \qquad .....(3)$$

$$Again, \qquad \frac{dr}{dt} = -\frac{\partial\phi}{\partial r} = -U\cos\theta + \frac{Ua^2}{r^2}\cos\theta, \; by(2)$$

$$And \qquad r\frac{d\theta}{dt} = -\frac{1}{r}\frac{\partial\phi}{\partial r} = U\sin\theta + \frac{Ua^2}{r^2}\sin\theta, \; by(2)$$

$$Then, \qquad \frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = -\frac{U\cos\theta}{r} + \frac{Ua^2\cos\theta}{r^3} + (\frac{U\sin\theta}{r} + \frac{Ua^2\sin\theta}{r^3})$$

$$= \frac{U}{r}[\frac{a^2}{r^2}(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)]$$

$$= \frac{U}{r}(\frac{a^2}{r^2}e^{i\theta} - e{-}i\theta) \qquad .....(4)$$

From(3) and (4), the required second result follows.

Let (x,y) be the co-ordinates of P referred to O as origin and let (r,θ) be the polar coordinates of P referred to O' as origin. Then

$$x = Ut + r\cos\theta \qquad and \qquad y = r\cos\theta \qquad .....(5)$$

Let ρ be the radius be curvature of the path. Then

$$\frac{1}{\rho} = \frac{d\delta}{ds} = \frac{d\delta}{dy}\frac{dy}{ds} = \frac{d\delta}{d\Theta}\frac{d\delta}{dy}\frac{dy}{ds} = 2\sin 2\theta\frac{d\theta}{dy} \qquad .....(6)$$

$$Since \qquad \delta = 2\theta \qquad and \quad \frac{dy}{ds} = \sin\delta = \sin 2\theta$$

From (3) and (5), we have

$$y - \frac{a^2}{y} \times \sin^2\theta = b \qquad .....(7)$$

Differentiating (7) w.r.t. 'y', we get

$$1 + \frac{a^2}{y^2}\sin^2\theta - \frac{2a^2}{y}\sin\theta\cos\theta\frac{d\theta}{dy} = 0$$

$$\Longrightarrow \qquad \sin 2\theta\frac{d\theta}{dy} = \frac{y}{a^2}(1 + \frac{y-b}{y}), \;\; by(7) \qquad ....(8)$$

Using (8), (6) reduces to

$$\frac{1}{\rho} = \frac{2y}{a^2}(2 - \frac{b}{y}) \qquad or \qquad \rho(y - \frac{b}{2}) = \frac{a^2}{4}$$

To find complex potential due to circulation about a circular cylinder. Let k be the constant circulation about cylinder. Then the suitable form of ϕ may be obtain by equating to k the circulation round a circle of radius t. Thus, we have

$$(-\frac{1}{r}\frac{\partial\phi}{\partial\theta})(\pi r) = k \qquad i.e., \qquad \frac{\partial\phi}{\partial\theta} = -\frac{k}{2\pi}$$

$$so\ that \qquad \phi = -\frac{k\theta}{2\pi}$$

Since ϕ and ψ are conjugate functions, we have

$$\psi = \frac{k}{2\pi}\log r$$

Thus, the complex potential due to the circulation about a circular cylinder is given by

$$w = \phi + i\psi = -\frac{k\theta}{2\pi} + \frac{ik}{2k}\log r = \frac{ik}{2\pi}(\log r + i\theta) = \frac{ik}{2\pi}\log(re^{i\theta})$$

$$Thus, \qquad w = \frac{ik}{2\pi}\log z$$

Streaming and circulation about a fixed circular cylinder.

We know that the complex potential w_1 due to the circulation of strength k about the cylinder is given by

$$w_1 = \frac{ik}{2\pi}\log z \qquad .....(1)$$

Again, the complex potential w_2 for streaming past a fixed cylinder of radius a, with velocity U, in negative direction of x-axis is given by

$$w_2 = Uz + (\frac{Ua^2}{z}) \qquad ....(2)$$

Hence the complex potential $w(= \phi + i\psi)$ due to the combined effects at any point $z = re^{i\theta}$ is given by

$$w = w_1 + w_2 = U(z + \frac{a^2}{z}) + \frac{ik}{2\pi}\log z \qquad .....(3)$$

$$\Longrightarrow \qquad w = U(re^{i\theta} + a^2e^{-\theta}) + (\frac{ik}{2\pi}) \times \log(re^{i\theta})$$

$$\Longrightarrow \qquad \phi + i\psi = U[r\cos\theta + i\sin\theta + \frac{a^2}{r}(\cos\theta - i\sin\theta)] + \frac{k}{2\pi}\log r$$

$$so\ \ that \qquad \phi = U(r + \frac{a^2}{r})\cos\theta - \frac{k\theta}{2\pi} \quad and \quad \psi = U(r - \frac{a^2}{r})\sin\theta + \frac{k}{2\pi}\log r$$

Since the velocity will be only tangent at the boundary of the cylinder, -$\frac{\partial\phi}{\partial r}$=0 and hence the magnitude of the velocity q is given by

$$q = | -\frac{1}{r}\frac{\partial\phi}{\partial\theta}|_{r=a} = 2U\sin\theta + \frac{k}{2\pi a} \qquad ......(4)$$

If there were no circulation (k=0) there would be points of zero velocity on the cylinder at $\theta = 0$ and $\theta = \pi$,the formal being the point at which the no-coming stream divides. However, in the presence of circulation, the stagnation (or critical) points are given by $q = 0$, i.e.,

$$\sin\theta = -\frac{k}{4\pi Ua} \qquad ....(5)$$

and such points exist when

$$|k| \leq 4\pi Ua \qquad ......(6)$$

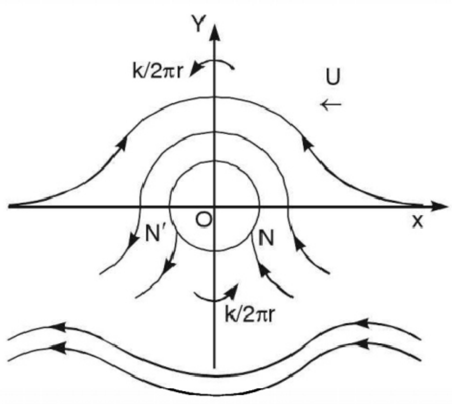


Fig. (*i*) |k| < 4 πaU

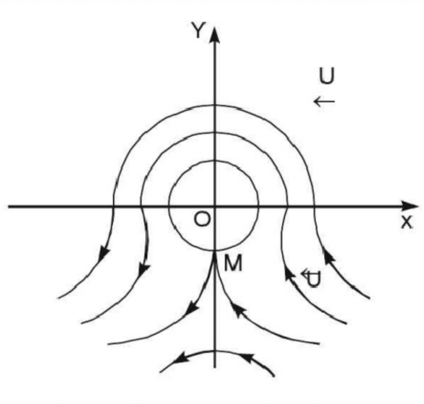


Fig. (*ii*) |k| = 4 πaU

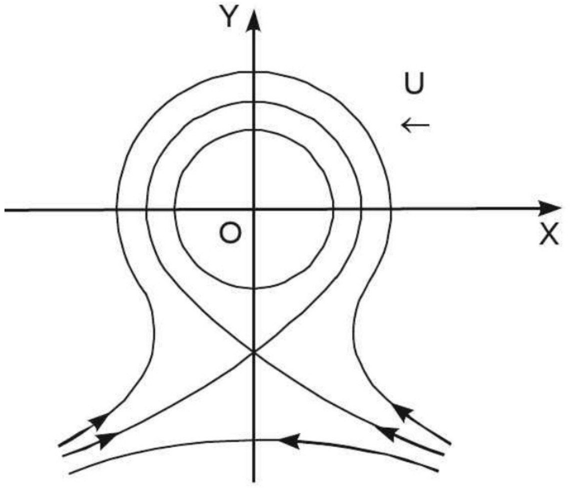


Fig. (*iii*) | k | > 4πaU

The lines of flow are then as shown in figure (i) N, N' being the stagnation points.
When, $|k| = 4\pi aU$, the stagnation points N and N' coincide at the bottom point M of the cylinder as shown in figure (ii).
When $|k| > 4\pi aU$, there are no stagnation points on the cylinder but there is such a point below the cylinder on the axis of y as shown in figure (iii).
(**Remark.** From this discussion, it follows that any point on the circumference might be made a critical point by a suitable choice of the ratio $\frac{k}{U}$. This fact employed in the theory of areofoils.)

We now determine the pressure at points of the cylinder. The pressure at P is given by Bernoulli's theorem

$$\frac{p}{\rho} = F(t) - \frac{q^2}{2} \qquad .....(7)$$

Let π be the pressure at infinity. Then p=π and q=U, so that

$$\frac{\pi}{\rho} = F(t) - \frac{1}{2}U^2 \qquad or \qquad F(t) = \frac{\pi}{\rho} + \frac{1}{2}U^2 \qquad .....(8)$$

Using (8), (7) reduces to

$$\frac{p}{\rho} = \frac{\pi}{\rho} + \frac{1}{2}U^2 - \frac{1}{2}q^2$$

$$\Longrightarrow \qquad p = \pi + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho(2U\sin\theta + \frac{k}{2\pi a})^2 \qquad ......(9)$$

If X,Y be the components of thrust on the cylinder, we have

$$X = -\int_0^{2\pi} p\cos\theta.(ad\theta) \qquad and \qquad Y = -\int_0^{2\pi} p\sin\theta.(ad\theta) \qquad ......(1)$$

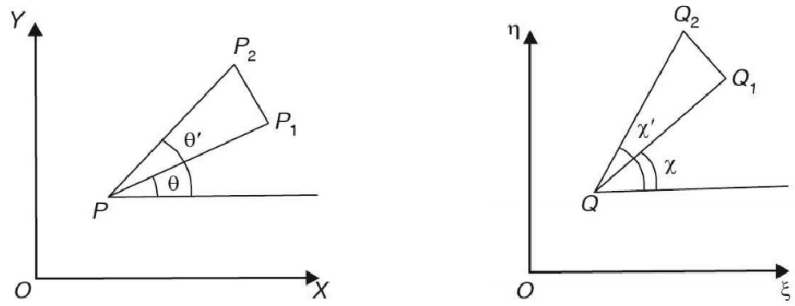
Using (9), (10) reduces to (after simplication)

$$X = 0, \qquad and \qquad Y = \rho kU \qquad ......(11)$$

showing that the cylinder experiences an upward lift. This effect may be attributed to circulation phenomenon.

Conformal Mapping:
A conformal mapping is a mapping which preserves angle in both magnitude and orientation.

Let f(z) be a single-valued differentiable complex function within a closed contour C in xy-plane. Let $\zeta = \xi + \vec{i}\eta$ be another complex variable such that $\zeta = f(z)$.
Then corresponding to each point in Z- plane or on C, there will be a point in ζ- plane or on $C^{'}$. The necessary condition for such map to exist is $f'(z) \neq 0$ at any point in z-plane or on C. This also means that $\frac{d\zeta}{dz}$ must exist independent of the direction of δz.



Let P, P_1, P_2 and Q, Q_1, Q_2 be two sets of points in Z- plans and ζ- plane respectively.

$$\frac{\zeta_1 - \zeta}{Z_1 - Z} = \frac{f(z_1) - f(z)}{z_1 - z}$$

and

$$\frac{\zeta_2 - \zeta}{Z_2 - Z} = \frac{f(z_2) - f(z)}{z_2 - z}$$

when $P_1 \to P$ and $P_2 \to P$

$$\frac{\zeta_1 - \zeta}{Z_1 - Z} = f^{'}(z) \quad and \quad \frac{\zeta_2 - \zeta}{Z_2 - Z} = f^{'}(z) \qquad (Very\ nearly)$$

$$\implies \quad \frac{\zeta_1 - \zeta}{Z_1 - Z} = \frac{\zeta_2 - \zeta}{Z_2 - Z} = f^{'}(z) = \frac{d\zeta}{dz}$$

$$\implies \quad \frac{QQ_1 e^{i\chi}}{PP_1 e^{i\theta}} = \frac{QQ_2 e^{i\chi^{'}}}{PP_2 e^{i\theta^{'}}}$$

$$\implies \quad \frac{QQ_1}{PP_1} e^{i(\chi - \theta)} = \frac{QQ_2}{PP_2} e^{i(\chi^{'} - \theta^{'})}$$

$$\implies \quad e^{i(\chi - \theta)} = e^{i(\chi^{'} - \theta^{'})} \quad (as\ P_1 \to P,\ P_2 \to P \implies Q_1 \to Q,\ Q_2 \to Q)$$

$$\implies \quad \chi - \theta = \chi^{'} - \theta^{'}$$

$$\implies \quad \chi - \chi^{'} = \theta - \theta^{'}$$

$$\implies \quad \angle Q_1 Q Q_2 = \angle P_1 P P_2$$

$$Also, \quad \frac{QQ_1}{PP_1} = \frac{QQ_2}{PP_2} = |f^{'}(z)| = |\frac{d\zeta}{dz}$$

$\implies$ $\Delta P_1 P P_2$ and $\Delta Q_1 Q Q_2$ are similar triangles. This establishes the similarity of the corresponding infinitesimal element of the two planes.

Such a relation have between the two planes.

$$\zeta = f(z) = \xi + \vec{i}\eta, \qquad z = x + \vec{i}y$$

$$\Longrightarrow \qquad \frac{d\zeta}{dz} = f^{'}(z) = \frac{d}{dz}(\xi + \vec{i}\eta) = \xi_x + \vec{i}\eta_x$$

From C-R equation

$$\xi_x = \eta_y \qquad and \quad \xi_y = -\eta_x$$

$$\therefore \quad \frac{d\zeta}{dz} = f^{'}(z) = \frac{d}{dz}(\xi + \vec{i}\eta) = \xi_x + \vec{i}\eta_x = \vec{i}(\eta_x - \vec{i}\eta_y)$$

$$\Longrightarrow \qquad |f^{'}(z)| = \sqrt{(\frac{\partial \eta}{\partial x})^2 + (\frac{\partial \eta}{\partial y})^2}$$

$\omega = F_1(\zeta) = \phi + \vec{i}\psi$, where ϕ and ψ are velocity and current function of any motion within the contour $C^{'}$ in ζ- plane. Then, within $C^{'}$

$$\phi + \vec{i}\psi = F_1(\xi + \vec{i}\eta) = f_1(\xi, \eta) + \vec{i}\bar{f}_1(\xi, \eta)$$

and $C^{'}$ is given by $\psi = \bar{f}_1(\xi, \eta) = constant$
Similarly,

$$\omega = F_1(\zeta) = F_1(\xi + \vec{i}\eta) = F_1(\xi(x, y) + \vec{i}\eta(x, y))$$

$$\Longrightarrow \qquad \phi + \vec{i}\psi = F_2(x, y) = f_2(x, y) + \vec{i}\bar{f}_2(x, y)$$

In this case the contour is

$$\psi(x, y) = \bar{f}_2(x, y) = constant$$

This shows that ψ and ϕ are same in the both planes.
The velocity q_1 and q_2 must also be same

$$|\bar{q_1}| = |\frac{dw}{dz}|, \quad |\bar{q_2}| = |\frac{dw}{d\zeta}|$$

$$\Longrightarrow \qquad |\bar{q_1}|^2 = |\frac{dw}{dz}|^2, \quad |\bar{q_2}|^2 = |\frac{dw}{d\zeta}|^2$$

$$\Longrightarrow \qquad |\bar{q_1}|^2 = |\bar{q_2}|^2$$

$$\Longrightarrow \qquad |\frac{dw}{dz}|^2 = |\frac{dw}{d\zeta}|^2 \qquad ....(1)$$

$$\begin{aligned} \frac{\Delta Q_1 Q Q_2}{\Delta P_1 P P_2} &= \frac{\frac{1}{2} QQ_1 \; QQ_2 \sin\angle Q_1 Q Q_2}{\frac{1}{2} PP_1 \; PP_2 \sin\angle P_1 P P_2} \\ &= \frac{QQ_1 \; QQ_2}{PP_1 \; PP_2} \\ &= |f^{'}(z)|^2 \\ &= \eta_x^2 + \eta_y^2 = h^2 \quad (say) \end{aligned}$$

$$\therefore \; \Delta Q_1 Q Q_2 = h^2 \Delta P_1 P P_2$$

Area of triangle in ζ and Z plans are of ratio h^2 to 1

$$\Longrightarrow \quad d\xi d\eta = h^2 dx dy$$

From (1)

$$|\bar{q_2}|^2 = |\frac{dw}{d\zeta}|^2 = |\frac{dw}{dz}|^2|\frac{dz}{d\zeta}|^2 = \frac{|\bar{q_1}|^2}{h^2}$$

$$\implies \quad |\bar{q_2}|^2 d\xi d\eta = |\bar{q_1}|^2 dxdy$$

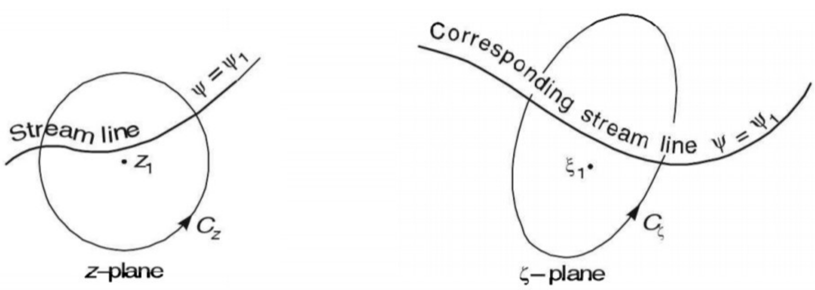
$$\implies \quad \frac{1}{2}\int \rho|\bar{q_2}|^2 d\xi d\eta = \frac{1}{2}\int \rho|\bar{q_1}|^2 dxdy$$

$$\implies \quad T_\zeta = T_Z$$

This says that K.E is same in both Plane.

Transformation of Source:

Let there be a source of strength m at z_1 and ζ_1 be the corresponding point in ζ- plane. Let these be regular points of the transformation. Then a small curve C_z may be drawn to encircle z_1 and similarly C_ζ is drawn to encircle ζ_1



Since we know the value of the stream function is independent of the domain considered, we have

$$\int_{C_z} d\psi = \int_{C_\zeta} d\psi$$

But

$$\int_{C_z} d\psi = \int_{C_z} (\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy)$$

$$= \int_{C_z} (-vdx + udy)$$

$$= \int_{C_z} (udy - vdx)$$

=the total flow across the contour C_z

=Sum of source of strength across the contour C_z

Theorem. Under conformal transformation a uniform line source maps into another uniform line source of the same strength.

Proof. Let there be a uniform line Source of strength m per unit length through the point z=z_0 and suppose that conformal trams $\zeta = f(z)$.

Let C_{z_0} be the curve around z_0 in Z-plane and C_{z_0} is mapped into C_{z_0} in ζ-plane is same and has the form

$$\omega = \phi + \vec{i}\psi \;\; in \;\; Z - plane$$
$$= \phi' + \vec{i}\psi' \;\; in \;\; \zeta - plane$$

Then $\phi = \phi'$ and $\psi = psi'$. Since ψ is the same at the corresponding points $z = z_0$ and $\zeta = \zeta_0$. We have

$$\int_{C_{z_0}} d\psi = \int_{C_{\zeta_0}} d\psi \qquad .....(1)$$

But in Z-plane

$$\omega = -\log(z - z_0)$$
$$\implies \qquad d\omega = -\frac{m \;\, dz}{z - z_0}$$
$$\implies \qquad \int_{C_{z_0}} d\omega = -\int_{C_{z_0}} \frac{m \;\, dz}{z - z_0} \qquad (by \;\; cauchy's \;\; Residue \;\; theorem)$$
$$\implies \qquad \int_{C_{z_0}} = -2\pi m \qquad ......(2)$$

(2) indicates the volume of fluid crossing unit thickness of C_{z_0} per unit time.
From (1) and (2)

$$\int_{C_{\rho_0}} d\psi' = 2\pi m$$

$\implies$ the volume of the fluid crossing unit thickness of C_{ζ_0} per unit time
The conformal mapping preserves the strength of the simple source.

Cor. A uniform doublet maps into a uniform doublet of same strength. (Exercise)

Kutta-Joukowski Theorem: When a cylinder of any shape is placed in a uniform stream of speed U, the resultant thrust on the cylinder is a lift of magnitude kρU per unit length and at right angles to the stream, k is the circulation around cylinder.

Proof. Let there be a fixed cylinder of some from in the finite region of the plane, it's cross section containing the origin. The disturbance of the stream caused by the cylinder can be represented at a great distance in the form

$$w = \frac{A}{z} + \frac{B}{z^2} + ... \qquad ....(1)$$

where A,B,.... depends on U and k. Let the direction of the stream make an angle α with x-axis. Then complex potential w_2 due to uniform stream velocity U is given by

$$w_2 = Ue^{i\alpha}z \qquad .....(2)$$

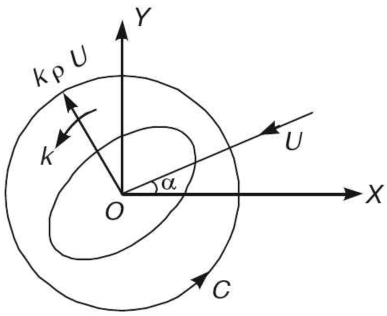
Again complex potential due to circulation

$$w_3 = \frac{ik}{2\pi} \times \log z \qquad ......(3)$$

The complete complex is

$$w = w_1 + w_2 + w_3$$

$$= (Ue^{i\alpha}z) + (\frac{ik}{2\pi} \times \log z) + (\frac{A}{z} + \frac{B}{z^2} + ...) \qquad ....(4)$$



By Blasius theorem, the force (X,Y) exerted on the cylinder is

$$X - \vec{i}Y = \frac{1}{2}\int_c (\frac{dw}{dz})^2 dz$$

$$= \frac{1}{2} i\rho \int_c [-\vec{i}\alpha U e^{-i\alpha z} + \frac{ik}{2\pi z} + A - \frac{B}{z^2} - \frac{C}{z^2}......]^2 dz$$

$$= \frac{1}{2}\vec{i}\rho \int_C (-\vec{i}\alpha U e^{i\alpha z})^2 + \frac{1}{2}\vec{i}\rho \int_c [\frac{ik}{2\rho z} + A - \frac{B}{z^2} - \frac{C}{z^2}]^2$$

$$+2[-i\alpha U e^{-i\alpha z}(\frac{ik}{2\pi z} + A - \frac{B}{z^2}....)]dz$$

$$= \frac{1}{2}\vec{i}\vec{\rho}(2\pi\vec{i}) \qquad (sum\ of\ residues\ at\ z = 0)$$

Since z=0 is the pole of $\frac{dw}{dz}$ inside C the residue at z=0 $\quad = \frac{ikUe^{-i\alpha}}{\pi}$

$$X - \vec{i}Y = \frac{1}{2}\vec{i}\pi(2\pi\vec{i})\frac{ikUe^{-i\alpha}}{\pi}$$

$$= -\vec{i}\rho kUe^{-i\alpha}$$

$$\implies \quad X - iY = -\vec{i}\rho kU(\cos\alpha - \vec{i}\sin\alpha)$$

$\therefore$

$$X = -kU\rho\sin\alpha$$

and

$$Y = k\rho U\cos\alpha$$

$$Resultant\ lift = \sqrt{X^2 + Y^2}$$

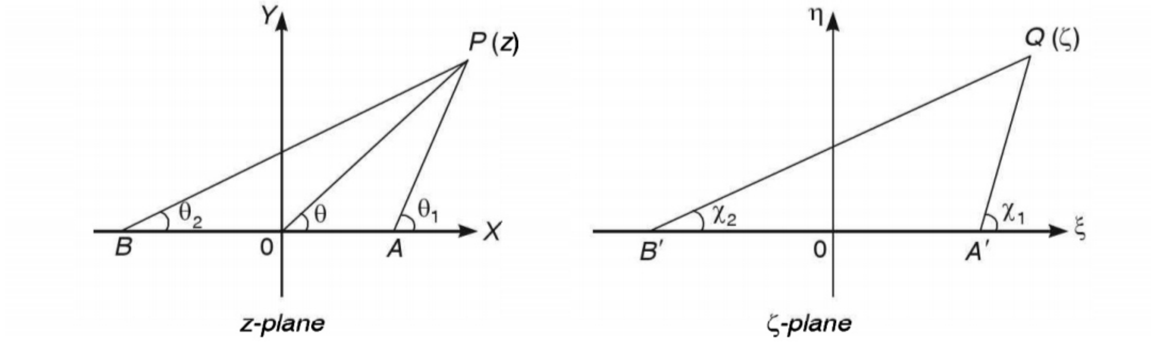
$$= \sqrt{(\rho kU)^2(\cos^2\alpha\sin^2\alpha} = \rho kU$$

Which always acts right angle to the Cylinder.

Joukowski Transformation, Joukwski aerofoils and hypothesis:
Consider the Joukowski transformation

$$\zeta = z + \frac{a^2}{z}$$

Now let A and B be the points at z=-a,a. These points are getting mapping to A' and B' respectively, i.e, $\zeta = -2a$ and $\zeta = 2a$.



$$\begin{aligned}\zeta - 2a &= z - 2a + \frac{a^2}{z} \\ &= \frac{(z-a)^2}{z} \qquad .....(1)\end{aligned}$$

Similarly,

$$\zeta + 2a = \frac{(z+a)^2}{z} \qquad .....(2)$$

From (2)

$$\begin{aligned} A'Qe^{i\phi_1} &= \frac{(APe^{i\theta_1})^2}{OPe^{i\theta}} \\ \implies \quad A'Qe^{i\phi_1} &= \frac{(AP)^2}{OP}e^{i(\theta_1-\theta)} \\ \implies \quad AQ &= \frac{AP^2}{OP} \ \ And \ \ e^{i\phi_1} = e^{i(2\theta_1-\theta)} \\ \implies \quad \phi_1 &= 2\theta_1 - \theta \qquad .....(3)\end{aligned}$$

Again (1) $\implies$

$$\begin{aligned} B'Q &= \frac{BP^2}{OP} \ \ and \ \ \phi = 2\theta_1 - \theta \qquad ...(4) \\ \therefore \quad \angle A'QB' &= \phi_1 - \phi_2 \\ &= 2(\theta_1 - \theta_2) \\ &= 2\angle APB \qquad ...(5)\end{aligned}$$

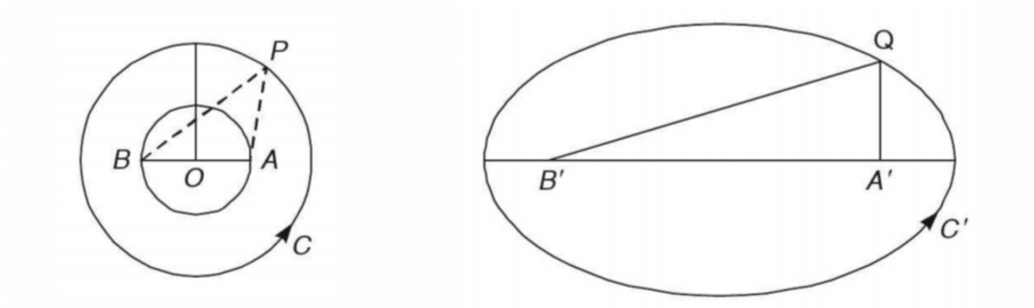
From (3) and (4)

$$\begin{aligned} A'Q + B'Q &= \frac{AP^2 + BP^2}{OP} \\ & \frac{2(OP^2 + OA^2}{OP}, \end{aligned}$$

Since OP is the of median of ΔAPB i.e, $AP^2 + BP^2 = 2(Op^2 + OA^2)$

And when $|z|$ is very large, $\zeta = z$ approximately So that the distant parts of the planes are same.

Consider a circle C With centre Oat origin in Z-plane. If P be any point on this circle C then $A^{'}Q + B^{'}Q = constant$. Because OA and OP are both constants.

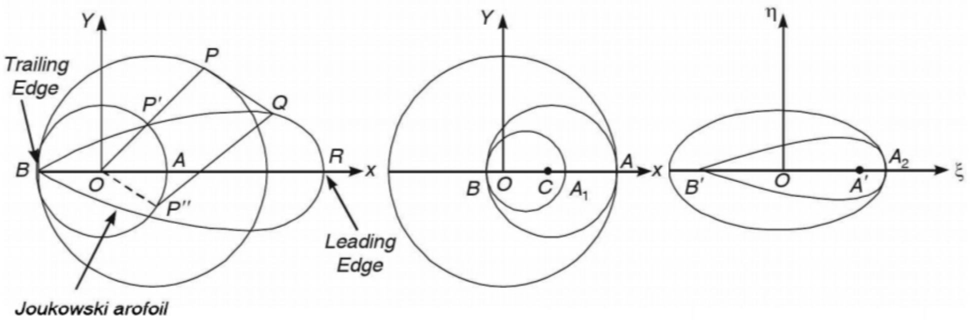


It follows that Q will describe an ellipse $C^{'}$ with $A^{'}$ and $B^{'}$ as foci.

Similarly, if P lies on a bigger circle, Q will describe a bigger ellipse, which shows that the points exterior to circle C will be mapped into points exterior to $C^{'}$.

Hence Joukowski transformation transforms a circle in the z-plane with centers at the origin into confocal ellipses in the $\zeta-$ plane.

Now consider the circle with AB as a diameter. If P be a point on this circle then $\angle APB = \frac{\pi}{2}$ and hence $\angle A^{'}QB^{'} = \pi$, So that Q lies on the line $A^{'}B^{'}$. Hence the circle with diameter AB is mapped into the straight line $A^{'}B^{'}$ of length 4a.



Let P be the point "z" on the bigger circle and AB and $P^{'}$ be its inverse point w.r.t circle AB and $P^{"}$ be the reflection of $P^{'}$ on X-axis, then $P^{"}$ be the point $\frac{a^2}{z}$. Then we drawn a parallelogram with OP with OP and $OP^{"}$ as adjacent sides and OQ be the diagonal such that Q be the point $\zeta = z + \frac{a^2}{z}$.

The locus of the point Q is a fish-shaped contour which touches the line BA on both sides. Such contour is known as Joukowski aerofoil, the point B is known as the trailing edge and R as leading edge. Also

$$\frac{d\zeta}{dz} = 1 - \frac{a^2}{z^2}$$

$$so, \qquad \frac{d\zeta}{dz} = 0 \implies z = \pm a$$

And z=a is mapped $\zeta = 2a$ and z=-a is mapped $\zeta = -2a$

But we need to know $\frac{d\zeta}{dz}$ at B. So, either $\frac{d\zeta}{dz} = 0$ or $\frac{d\zeta}{dz} = \infty$

If $\vec{q'}$ be the velocity at B for the circle and $\vec{q}$ be the velocity at B of the aerofoil, then

$$|\vec{q'}| = |\frac{dw}{d\zeta}| = |\frac{dw}{dz}||\frac{dz}{d\zeta}| = |\vec{q}||\frac{dz}{d\zeta}|$$

If $\frac{d\zeta}{dz} = 0$ then $\frac{dz}{d\zeta} = \infty \implies \vec{q'} = 0$ at B.

$\therefore \frac{d\zeta}{dz} = \infty$ at the trailing edge B.

Hence to avoid infinite velocity at the trailing edge of the aerofoil the velocity at B is taken as zero . i.e, B is taken as stagnation point of the flow in Z-plane. This is Joukowski hypothesis.

Aerofoils: The aerofoil has a fish type profile. It is employed in the construction of modern airplanes. Such an aerofoil has blunt leading edge and a sharp trailing edge. The flow around the aerofoil depends on following assumption.

(i) The air behaves as an incompressible fluid.

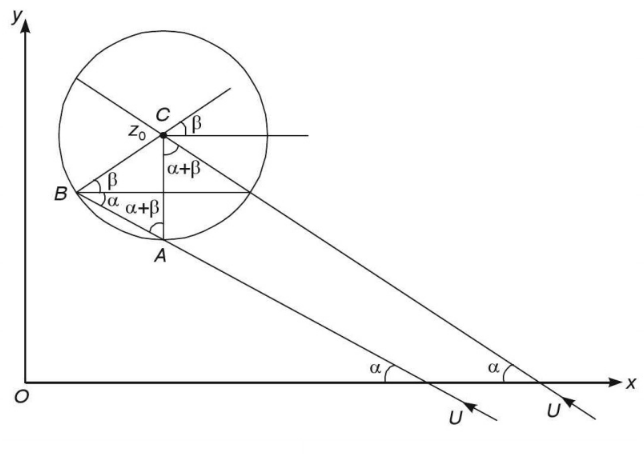
(ii) The aerofoils a cylinder whose cross-section is a curve of fish type.

(iii) the flow is two-dimensional irrotational cyclic motion.

Flow past a circle:

Let U be the velocity of the stream at infinity its direction making an angle α with the $'-ve'$ direction of x-axis. k be the circulation around the circle, whose center is at z_0 and radius b. Then we have

$$w = Ue^{i\alpha}(z - z_0) + \frac{Ub^2}{z - z_0}e^{-i\alpha} + \frac{ik}{2\pi}\log(z - z_0)$$



To find the Stagnation point

$$\frac{dw}{dz} = 0$$

$$\implies \quad Ue^{i\alpha} - \frac{Ub^2}{(z - z_0)^2}e^{-i\alpha} + \frac{ik}{2\pi(z - z_0)} = 0 \qquad .....(1)$$

Taking the Stagnation point as

$$z = z_0 + be^{i(\pi+\beta)} = z_0 - be^{i\beta}$$

Then (1) reduces to

$$U(e^{i\alpha} - e^{-(\alpha+2\beta)}) - \frac{ik}{2\pi b}e^{-i\beta} = 0$$

$$\implies \quad k = 4\pi bu\sin(\alpha+\beta)$$

$$\implies \quad k < 4\pi bU, \ \ since \ \ \sin(\alpha+\beta) < 1$$

The Stagnation points given by

$$z - z_0 = be^{i(\pi+\beta)} = t$$

From (1)

$$Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{t^2} + \frac{ik}{2\pi t} = 0$$

$$\implies \quad t^2 2\pi Ue^{i\alpha} + ikt - 2\pi Ub^2e^{-i\alpha} = 0$$

$$\implies \quad 2\pi Ue^{i\alpha}t^2 + 4\pi bU\sin(\alpha+\beta)it - 2\pi Ub^2e^{-i\alpha} = 0 \qquad [since \ \ k = 4\pi bU\sin(\alpha+\beta)]$$

$$\therefore \quad t = be^{-i\alpha}[-i\sin(\alpha+\beta) \pm \cos(\alpha+\beta)]$$

$$t = be^{-i(2\alpha+\beta)} \ \ or, \ \ -be^{i\beta}$$

Therefore, the points

$$t_1 = be^{i(2\pi-(2\alpha+\beta))} \quad and \ \ t_2 = be^{i(\pi\beta)}$$

Thus $z_0 = be^{i(\pi\beta)}$ gives the point B and $z - z_0 = be^{i(2\pi-(2\alpha+\beta))}$ gives the point A as shown in the figure.

Flow around a circle:

$$w = Ue^{i\alpha}(z - z_0) + \frac{Ub^2e^{i\alpha}}{z - z_0} + \frac{ik}{2\pi}\log(z - z_0)$$

$$\frac{dw}{dz} = Ue^{i\alpha} - \frac{Ub^2e^{i\alpha}}{(z-z_0)^2} + \frac{ik}{2\pi(z-z_0)} \qquad .....(1)$$

For stagnation point, $\frac{dw}{dz} = 0$
Let us take the Stagnation points as

$$z = z_0 + bei(\pi+\beta) = z_0 - be^{i\beta}$$

$$\implies \quad z - z_0 = -be^{i\beta}$$

Then (1) reduces to

$$Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{b^2e^{2i\beta}} + \frac{ik}{2\pi(-be^{i\beta}} = 0$$

$$\implies \quad 2\pi bU[e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}] = ik$$

$$\implies \quad k = 4\pi bU\sin(\alpha+\beta)$$

$$\implies \quad k < 4\pi bU \quad (since \ \ b, U > 0)$$

The Stagnation points are given by

$$z - z_0 = be^{i(\pi+\beta)} = t$$

From (1)

$$Ue^{i\alpha} - \frac{Ub^2e^{i\alpha}}{t^2} + \frac{ik}{2\pi t} = 0$$

$$\implies \quad t = be^{i\alpha}[-i\sin(\alpha+\beta) \pm \cos\alpha+\beta], \ \ putting \ \ k = 4\pi bU\sin(\alpha+\beta)$$

$$\implies \quad t = be^{-i(2\alpha+\beta)} \ \ or, \ \ -be^{i\beta}$$

Therefore, the points

$$t_1 = be^{i(2\pi-(2\alpha+\beta))} \quad and \ \ t_2 = be^{i(\pi\beta)}$$

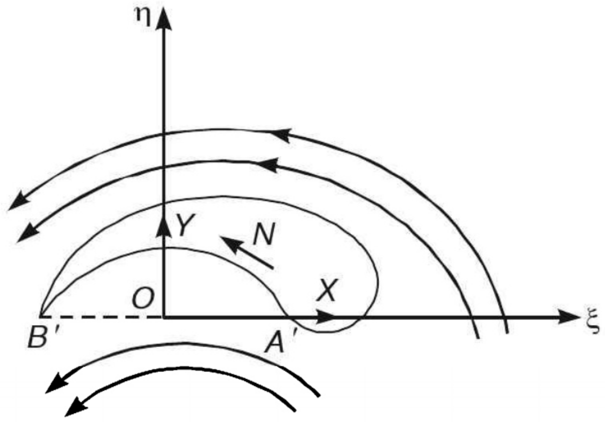
Thus $z_0 = be^{i(\pi\beta)}$ gives the point B and $z - z_0 = be^{i(2\pi-(2\alpha+\beta))}$ gives the point A as shown in the figure.

Flow past an aerofoil:

From Joukowski transformation $\zeta = z + \frac{a^2}{z}$ ....(1) transforms a circle to an aerofoil.

Then the Stagnation point P will be transformed into trailing edge $B^{'}$ and the other Stagnation point A will be transformed to A'. $B^{'}$ will lie on the -ve side of ζ-axis such that $OB^{'}$=2a.

Let U be the velocity of the stream at infinity, its direction making an angle α with ζ-axis. If we take the origin of both Z and ζ planes to coincide, then B and $B^{'}$ will lie on the left side of the origin O.



By Blassius theorem

$$X - iY = \frac{i\rho}{2}\int_{C_\zeta}(\frac{dw}{d\zeta})^2 d\zeta \qquad ......(2)$$

From (1)

$$\frac{dw}{d\zeta} = \frac{\frac{dw}{dz}}{\frac{d\zeta}{dz}} = \frac{\frac{dw}{dz}}{1-\frac{a^2}{z^2}}$$

$$\implies \quad (\frac{dw}{d\zeta})^2 = \frac{(\frac{dw}{dz})^2}{(1-\frac{a^2}{z^2})^2} \qquad .....(3)$$

Again

$$w = Ue^{i\alpha}(z-z_0) + \frac{Ub^2e^{i\alpha}}{z-z_0} + \frac{ik}{2\pi}\log(z-z_0)$$

$$\frac{dw}{dz} = Ue^{i\alpha} - \frac{Ub^2e^{i\alpha}}{(z-z_0)^2} + \frac{ik}{2\pi(z-z_0)}$$

$$\therefore \quad (\frac{dw}{dz})^2(1-\frac{a^2}{z^2})^2 = \frac{1}{(1-\frac{a^2}{z^2})^2}[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{(z-z_0)^2} + \frac{ik}{2\pi(z-z_0)}]^2$$

$$= \frac{1}{(1-\frac{a^2}{z^2})^2}[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{z^2(1-z_0/z)^2} + \frac{ik}{2z\pi(1-z_0/z)}]^2$$

$$= (1+\frac{a^2}{z^2}+....)[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{z^2}(1+\frac{2z_0}{z}+....) + \frac{ik}{2\pi z}(1+\frac{z_0}{z}+....)]^2$$

By (2)

$$X - iY = \frac{i\rho}{2}\int_{C_\zeta} \frac{(\frac{dw}{dz})^2}{(1-\frac{a^2}{z^2})^2} d(z+\frac{a^2}{z})$$

$$= \frac{i\rho}{2}\int_{C_\zeta} \frac{(\frac{dw}{dz})^2}{(1-\frac{a^2}{z^2})^2}(1-\frac{a^2}{z^2})dz$$

$$= \frac{i\rho}{2}\int_{C_\zeta} \frac{(\frac{dw}{dz})^2}{(1-\frac{a^2}{z^2})}dz$$

$$= \frac{i\rho}{2}2\pi i[Sum\ \ of\ \ residues\ \ of\ \ \frac{(\frac{dw}{dz})^2}{(1-\frac{a^2}{z^2})}]$$

$$= -\pi\rho \times (2Ue^{i\alpha}\frac{ik}{2\pi})$$

$$= -i\rho k U e^{i\alpha}$$

$$\implies \quad X - iY = \rho k U \sin\alpha - i\rho k U \cos\alpha$$

$$\implies \quad X = \rho k U \sin\alpha$$

$$And\ \ Y = i\rho k U \cos\alpha$$

$$\therefore \quad F = \sqrt{X^2+Y^2} = \rho k U$$

$$(\frac{dw}{d\zeta})^2\zeta d\zeta = \frac{z+\frac{a^2}{z}}{1-\frac{a^2}{z^2}}(\frac{dw}{dz})^2dz$$

Now,

$$\frac{z+\frac{a^2}{z}}{1-\frac{a^2}{z^2}}(\frac{dw}{dz})^2 = (z+\frac{a^2}{z})(1-\frac{a^2}{z^2})^{-1}[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{z^2}(1+\frac{2z_0}{z}+....) + \frac{ik}{2\pi z}(1+\frac{z_0}{z}+....)]^2$$

$$= (z+\frac{a^2}{z}(1+\frac{a^2}{z}+....)[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{z^2}(1+\frac{2z_0}{z}+....) + \frac{ik}{2\pi z}(1+\frac{z_0}{z}+....)]^2$$

$$\implies \quad Co-efficient\ \ of\ \ \frac{1}{z} = -2U^2b^2 - \frac{k^2}{4\pi^2} + \frac{kiUe^{i\alpha}z_0}{\pi} + 2a^2U^2e^{2i\alpha}$$

By Blasius Theorem,
Moment of force,

$$M = Real\ \ part\ \ of[-\frac{\rho}{2}\times Coeff\ \ of\ \ \frac{1}{2}\times 2\pi i]$$

$$M = Real\ \ part\ \ of[-\frac{\rho}{2}\times(-2U^2b^2 - \frac{k^2}{4\pi^2} + \frac{kiUe^{i\alpha}z_0}{\pi} + 2a^2U^2e^{2i\alpha})\times 2\pi i]$$

Taking $z_0 = Ce^{i\gamma}$, $C > 0$
Then

$$M = Real\ \ part\ \ of[-\frac{\rho}{2} \times (-2U^2b^2 - \frac{k^2}{4\pi^2} + \frac{kiUe^{i\alpha}Ce^{i\gamma}}{\pi} + 2a^2U^2e^{2i\alpha}) \times 2\pi i]$$

$$= 2\pi U^2[2bc\sin(\alpha+\beta)\cos(\alpha+\beta) + a^2\sin 2\alpha]$$

This is the moment at z_0.
The moment at the trailing edge B will be $aY + M$
$\therefore$ Required Moment at B

$$= 4a\pi bU\rho\cos\alpha + 2\pi U^2[2bc\sin(\alpha+\beta)\cos(\alpha+\beta) + a^2\sin 2\alpha]$$

Navier-Stokes equation:
With P(x,y,z) as the center and edges of length $\delta x, \delta y, \delta z$ parallel to co-ordinate axes. We construct an elementary rectangular parallelopiped.

Let us consider the fluid motion is viscous. WE assume the fluid element is moving and its mass is $\rho\delta x\delta y\delta z$. Let the co-ordinates of P_1 and P_2 be $(x - \frac{\delta x}{2}, y, z)$ and $(x + \frac{\delta x}{2}, y, z)$ respectively.

At P the force components parallel to OX, OY and OZ on the rectangular surface ABCD of area $\delta y\delta z$ through P and having $\vec{i}$ as the unit normal

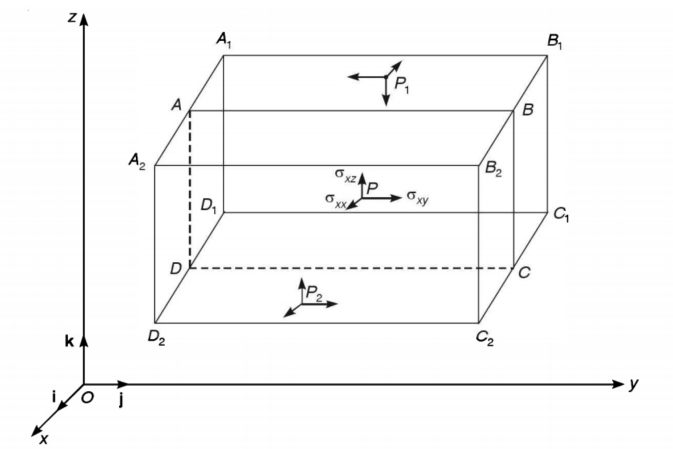
$$= \sigma_{xx}\delta y\delta z; \sigma_{xy}\delta y\delta z; \sigma_{xz}\delta y\delta z \qquad ....(1)$$

At point $P_2(x + \frac{\delta x}{2}, y, z)$ the components of the force on $A_2B_2C_2D_2$ is

$$[(\sigma_{xx} + \frac{\delta x}{2}\frac{\partial\sigma_{xx}}{\partial x})\delta y\delta z; (\sigma_{xy} + \frac{\delta x}{2}\frac{\partial\sigma_{xy}}{\partial x})\delta y\delta z; (\sigma_{xz} + \frac{\delta x}{2}\frac{\partial\sigma_{xz}}{\partial x})\delta y\delta z] \qquad ....(2)$$

At point $P_1(x - \frac{\delta x}{2}, y, z)$ the components of the force on $A_1B_1C_1D_1$ is

$$[-(\sigma_{xx} + \frac{\delta x}{2}\frac{\partial\sigma_{xx}}{\partial x})\delta y\delta z; -(\sigma_{xy} + \frac{\delta x}{2}\frac{\partial\sigma_{xy}}{\partial x})\delta y\delta z; -(\sigma_{xz} + \frac{\delta x}{2}\frac{\partial\sigma_{xz}}{\partial x})\delta y\delta z] \qquad ....(3)$$



Hence the forces on parallel planes $A_2B_2C_2D_2$ and $A_1B_1C_1D_1$ passing through P_1 and P_2 are equivalent to single force at P with components

$$= [\frac{\partial\sigma_{xx}}{\partial x}\delta x\delta y\delta z; \frac{\partial\sigma_{xy}}{\partial x}\delta x\delta y\delta z; \frac{\partial\sigma_{xz}}{\partial x}\delta x\delta y\delta z] \qquad ......(4)$$

Together with couples whose moment are

$$-\sigma_{xz}\delta x\delta y\delta z = -(\sigma_{xz}\delta y\delta z)\delta x \text{ (about OY)}$$

$$\text{And } (\sigma_{xy}\delta y\delta z)\delta x = \sigma_{xy}\delta x\delta y\delta z \text{ (About OZ)}$$

Again the forces on the parallel planes perpendicular to Y-axes are equivalent to a single force at P with component

$$= [\frac{\partial\sigma_{yx}}{\partial y}\delta x\delta y\delta z; \frac{\partial\sigma_{yy}}{\partial y}\delta x\delta y\delta z; \frac{\partial\sigma_{yz}}{\partial y}\delta x\delta y\delta z]$$

Together with couples whose moment are

$$-\sigma_{yx}\delta x\delta y\delta z \text{ (about OZ)}$$
$$\text{and } \sigma_{yz}\delta x\delta y\delta z \text{ (about OX)}$$

Again the forces on the parallel planes perpendicular to Z-axes are equivalent to a single force at P with component

$$= [\frac{\partial\sigma_{zx}}{\partial z}\delta x\delta y\delta z; \frac{\partial\sigma_{zy}}{\partial z}\delta x\delta y\delta z; \frac{\partial\sigma_{zz}}{\partial z}\delta x\delta y\delta z]$$

Together with couples whose moment are

$$-\sigma_{zy}\delta x\delta y\delta z \text{ (about OX)}$$
$$\text{and } \sigma_{zx}\delta x\delta y\delta z \text{ (about OY)}$$

Thus the surface forces on all the six faces of the rectangular parallelopiped are equivalent to a single force at P having components

$$\vec{S} = ((\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z}), (\frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z}), (\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}))\delta x\delta y\delta z$$

Together with a vector couple having components

$$= ((\sigma_{yz} - \sigma_{zy})\delta x\delta y\delta z, (\sigma_{zx} - \sigma_{xz})\delta x\delta y\delta z, (\sigma_{xy} - \sigma_{yx})\delta x\delta y\delta z)$$
$$= (0, 0, 0) \text{ (since Cauchy stress tensor is symmetric)}$$

Let $\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$ be the velocity at P and $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ be the body force/ External force per unit mass.

The total force is given by = Body force + Surface force

$$= \vec{F} + \vec{S}$$

Total force along $\vec{i}$- direction

$$= (\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z})\delta x\delta y\delta z + \rho\vec{F_x}\delta x\delta y\delta z$$

Total force along $\vec{j}$- direction

$$= (\frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z})\delta x\delta y\delta z + \rho\vec{F_y}\delta x\delta y\delta z$$

Total force along $\vec{k}$- direction

$$= (\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}))\delta x\delta y\delta z + \rho\vec{F_z}\delta x\delta y\delta z$$

By newton's 2nd Law

$$\rho\delta x\delta y\delta z\frac{Du}{Dt} = (\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z})\delta x\delta y\delta z + \rho\vec{F_x}\delta x\delta y\delta z$$

$$\implies \quad \rho\frac{Du}{Dt} = (\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z})\delta x\delta y\delta z + \rho\vec{F_x} \quad (since\ \delta x\delta y\delta z \neq 0)$$

Similarly,

$$\rho\frac{Dv}{Dt}=(\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z})\delta x\delta y\delta z+\rho\vec{F_y}$$

$$And\quad \rho\frac{Dw}{Dt}=(\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z})\delta x\delta y\delta z+\rho\vec{F_z}$$

From costitutive law of Newtonion fluid, by Stoke's law of fluid motion

$$\sigma_{xx}=2\mu\frac{\partial u}{\partial x}-\frac{2\mu}{3}\vec{\nabla}.\vec{q}-p$$

$$\sigma_{yy}=2\mu\frac{\partial v}{\partial y}-\frac{2\mu}{3}\vec{\nabla}.\vec{q}-p$$

$$\sigma_{zz}=2\mu\frac{\partial w}{\partial z}-\frac{2\mu}{3}\vec{\nabla}.\vec{q}-p$$

$$\sigma_{xy}=\sigma_{yx}=\mu(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})$$

$$\sigma_{yz}=\sigma_{zy}=\mu(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})$$

$$\sigma_{zx}=\sigma_{xz}=\mu(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x})$$

where p=pressure, μ=viscocity Now,

$$\rho\frac{Du}{Dt}=\rho\vec{F_x}+(2\mu\frac{\partial^2u}{\partial^2x}-\frac{2\mu}{3}\frac{\partial}{\partial x}(\vec{\nabla}.\vec{q})-\frac{\partial p}{\partial x})+\mu(\frac{\partial^2u}{\partial y^2}+\frac{\partial^2v}{\partial y\partial x})+\mu(\frac{\partial^2u}{\partial z^2}+\frac{\partial^2w}{\partial z\partial x})$$

$$\implies\quad \rho\frac{Du}{Dt}=\rho\vec{F_x}-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}[\mu(2\frac{\partial u}{\partial x}-\frac{2}{3}\vec{\nabla}.\vec{q}]+\frac{\partial}{\partial y}[\mu(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})]+\frac{\partial}{\partial z}[\mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]$$

$$\rho\frac{Dv}{Dt}=\rho\vec{F_y}-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}[\mu(2\frac{\partial v}{\partial y}-\frac{2}{3}\vec{\nabla}.\vec{q}]+\frac{\partial}{\partial z}[\mu(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z})]+\frac{\partial}{\partial x}[\mu(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})]$$

$$\rho\frac{Dw}{Dt}=\rho\vec{F_z}-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}[\mu(2\frac{\partial w}{\partial z}-\frac{2}{3}\vec{\nabla}.\vec{q}]+\frac{\partial}{\partial x}[\mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]+\frac{\partial}{\partial y}[\mu(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z})]$$

The above three equations are called the $Navier-Stokes$ equations of motion for a viscous compressible fluid in cartesian coordinates.

Navier-Stokes eqution for viscous incompressible fluid,

$$\frac{d\vec{q}}{dt}=\vec{F}-\frac{1}{\rho}\nabla p+\nu\nabla^2\vec{q}$$

$\nu=\frac{\mu}{\rho}$ is kinemetics of viscosity

Dissipation of energy:

Dissipation of energy is that energy which is dissipated in a viscous liquid in motion on account of the internal friction.

problem: Determination of rate of dissipation of energy of a fluid due to viscosity.

proof. suppose we follow a particle of a viscous, incompressible fluid of density ρ and volume δV such that the mass is $\rho\delta V$. It is moving with velocity $\vec{q}$ at any time t.

Then the

$$K.E = \frac{1}{2}\rho\delta V\vec{q}^2$$

Hence the rate of change of energy as the particle moves with time is given by

$$\frac{D}{Dt}(\frac{1}{2}\rho\delta V\vec{q}^2) = \rho\delta V\vec{q}.\frac{D\vec{q}}{Dt} \qquad ......(1)$$

Let the total volume of the fluid be V and let S be the total surface area

$$\frac{dT}{dt} = \frac{d}{dt}(\int_V \frac{1}{2}\rho\vec{q}^2\delta V) = \rho\int_V \vec{q}.\frac{D\vec{q}}{Dt}dV \qquad ....(2)$$

Now from Navier-Stokes eqution for viscous incompressible fluid,

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q} \qquad ......(3)$$

$\nu = \frac{\mu}{\rho}$ is kinemetics of viscosity.

From (2) and (3), we have

$$\frac{dT}{dt} = \rho\int_V \vec{q}.[\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q}]dV$$

$$= \int_V \vec{q}.(\rho\vec{F})dV - \int_V \vec{q}.\nabla p dV + \int_V \vec{q}.\rho\nu(\nabla^2\vec{q})dV \qquad ....(4)$$

1st term of R.H.S of (4) represents the rate at which the external force $\vec{F}$ is doing work throughout the mass of the fluid, which the 2nd term represents the rate at which pressure is doing work at the boundary.

For ideal fluid, the work by the force $\vec{F}$ in the volume V and work by the pressure at the boundary are same.

$$\therefore \quad \frac{dT}{dt} = D = \rho\int_V \frac{\mu}{\rho}\vec{q}.\nabla^2\vec{q}dV$$

$$D = \rho\int_V \mu\vec{q}.\nabla^2\vec{q}dV$$

where D is the dissipation of energy. Now, if the flow is rotational such that $\vec{\Omega}$ represents the vorticity. Then

$$\vec{\Omega} = \vec{\nabla}\times\vec{q}$$

We know

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{q}) = \vec{\nabla}(\vec{\nabla}.\vec{q}) - \nabla^2\vec{q} = -\nabla^2\vec{q} \;\; (since\ fluid\ is\ incompressible)$$

$$\implies \quad \vec{\nabla}\times\vec{q} = -\nabla^2\vec{q}$$

$$\implies \quad \vec{q}.(\nabla^2\vec{q}) = -\vec{q}.(\vec{\nabla}\times\vec{q})$$

Again,

$$\nabla.(\vec{q}\times\Omega) = \vec{\Omega}.(\vec{\nabla}\times\vec{q}) - \vec{q}.(\nabla\vec{\times}\Omega)$$

$$\implies \quad \nabla.(\vec{q}\times\Omega) = \vec{\Omega}^2 - \vec{q}.(\nabla\vec{\times}\Omega)$$

This gives

$$D = \int_V \mu[\vec{\nabla}.(\vec{q} \times \Omega) - |\Omega|^2]dV$$
$$= \mu \int_V \vec{\nabla}.(\vec{q} \times \Omega)dV - \int_V \mu|\Omega|^2 dV$$
$$= \mu \int_V (\vec{q} \times \Omega).\vec{n}dS - \mu \int_V |\Omega|^2 dV$$

If we assume "$no - slip$" condition i.e, $\vec{q}.\vec{n} = 0$, then the first term will vanish, and

$$D = -\mu \int_V |\Omega|^2 dV$$

$$= -\mu \int_V (\xi^2 + \eta^2 + \zeta^2)dV$$

$$|D| = \mu \int_V (\xi^2 + \eta^2 + \zeta^2)dV$$

Where

$$\vec{\Omega} = \xi\vec{i} + \eta\vec{j} + \zeta\vec{k}$$

Limitations of the Navier Stokes equation:

The main limitations of the Navier Stokes equation

(1) They are not applicable for flow of non-Newtonian fluids.

(2) In derivation of the Navier-stokes equations we regarded fluid as a continuum.

(3) These equations are non-linear in nature and getting a single solution in which convective terms interact in a general manner with viscous term.

(4) Due to idealizations such as infinite plates, fully developed parallel flow in a pipe, even limited number of exact solutions of these equations are valid only in a particular region in a real situation.

Steady laminar flow between two parallel plates. Plane Couette flow.

Consider the steady laminar flow of viscous incompressible fluid between two infinite parallel plates separated by a distance h.

Let x be the direction of flow, y the direction perpendicular to the flow, and the width of the plates is large compared with h and hence the flow may be treated as two dimensional (i.e, $\frac{\partial}{\partial x} = 0$). Let the plates be long enough in the x-direction for the flow to be parallel. Here we take the velocity components v and w to be zero everywhere. Moreover the flow being steady, the flow variables are independent of time ($\frac{\partial}{\partial z} = 0$). Furthermore, the equation of continuity [namely $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$] reduces to $\frac{\partial u}{\partial x} = 0$ so that u=u(y).

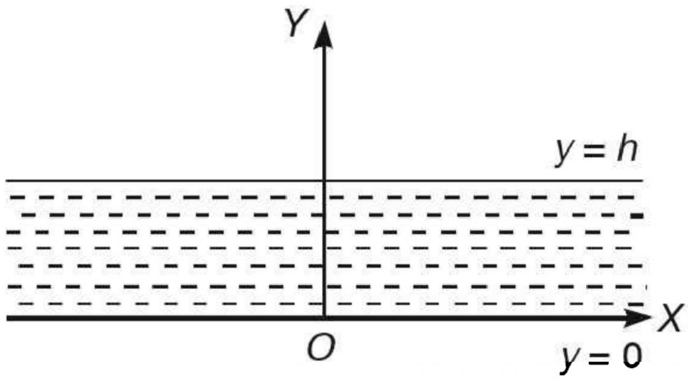
Thus for the present problem

$$u = u(y), \quad v = 0, \quad w = 0, \quad \frac{\partial}{\partial z} = 0, \quad \frac{\partial}{\partial t} = 0 \qquad ....(1)$$

For the present two-dimensional flow in absense of body forces, the Navier Stoke equations for x and y-directions are:

$$0 = (-\frac{\partial p}{\partial x}) + \mu(\frac{d^2u}{dy^2}) \qquad .....(2)$$

$$0 = -\frac{\partial p}{\partial y} \qquad ......(3)$$



Equation (3) shows that the pressure does depend on y. Hence p is a function of x along and so (2) reduces to

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \qquad .....(4)$$

Differentiating both sides of (4) w.r.t x

$$\frac{d}{dx}(\frac{dp}{dx}) = 0$$

So that

$$\frac{dp}{dx} = constant = P \quad (say) \qquad ......(5)$$

Then (4) reduces to

$$\frac{d^2u}{dy^2} = \frac{P}{\mu} \qquad .....(6)$$

Integrating (6),

$$\frac{du}{dy} = \frac{1}{\mu}y + A \qquad ....(7)$$

Integrating (7),

$$u = Ay + B + \frac{P}{2\mu}y^2 \qquad .....(8)$$

Where A and B are arbitrary constants to be determined by the boundary conditions of the problem under consideration.

Boundary condition.

For the plane Couettle flow, P=0. Again, the plate y=0 is kept at rest and the plate y=h is allow to move with velocity V. Then the no Slip condition gives rise to the boundary conditions

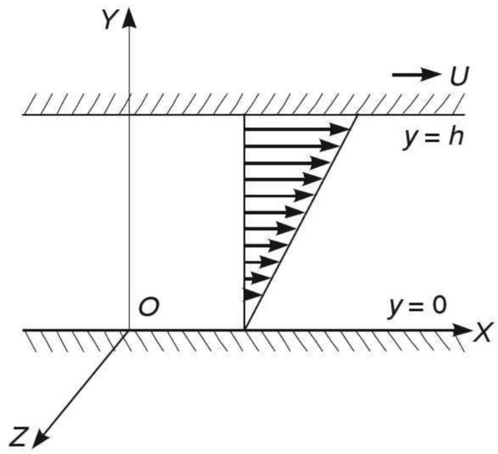
$$u = 0 \quad at \;\; y = 0, \quad and \quad u = U \quad at \quad y = h \qquad\qquad ......(9)$$

Using (9), (8) yields

$$0 = B \quad and \quad U = Ah + B$$

So that

$$B = 0, A = \frac{U}{h}$$



The velocity distribution is liner as shown in the adjoining figure. Now the skin friction (or drag per unit area, i.e., the shearing stress at the pleats) σ_{yx} is given by

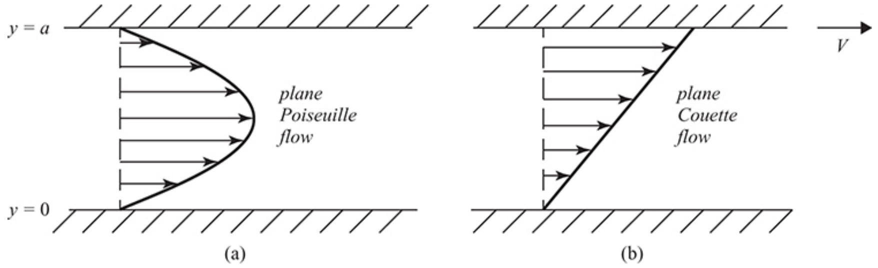
$$\sigma_{yx} = \mu(\frac{du}{dy}) = \mu\frac{U}{h}$$

Plane poiseuille flow.

Consider the steady laminar flow of viscous incompressible fluid between two infinite parallel plates separated by a distance h. Flow previous model

$$v = 0, \qquad w = 0$$

$$\implies \quad \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$$



Flow is incompressile

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial x}=0 \implies \qquad u=u(y)$$

$$\therefore \qquad u=u(y), v=w=0, \frac{\partial}{\partial z}=0, \frac{\partial}{\partial t}=0$$

The Navier Stokes equations for x and y directions takes the form

$$0=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{dy^2}$$

$$\implies 0=-\frac{\partial p}{\partial y}$$

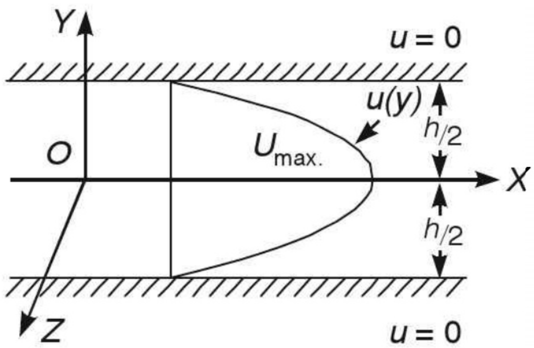
$$\implies p=f(x)=function \quad of \quad x$$

$$\therefore \frac{d^2u}{dy^2}=\frac{1}{\mu}\frac{dp}{dx}=\frac{P}{\mu}, \quad P=\frac{dp}{dx}$$

$$u(y)=\frac{p}{2\mu}y^2+Ay+B$$

The velocity profile is parabola. This is so called plane poiseuille flow. For this the plates are fixed and the fluid is kept in motion by a pressure gradient P.
For example. Let two plates be situated at $y=-\frac{h}{2}$ and $y=\frac{h}{2}$ as shown in figure.

$$\therefore u=0 \quad at \quad y=-\frac{h}{2},\frac{h}{2}$$



Thus,

$$0=u(-\frac{1}{2})=\frac{P}{2\mu}\frac{h^2}{4}-\frac{Ah}{2}+B$$

$$0=u(\frac{1}{2})=\frac{P}{2\mu}\frac{h^2}{4}+\frac{Ah}{2}+B$$

$$\implies \qquad A=0 \quad and \quad B=-\frac{h^2P}{8\mu}$$

$$\therefore \qquad u(y)=\frac{P}{2\mu}(y^2-\frac{h^2}{4})$$

To find u_{max}

$$|u(y)| = |\frac{P}{2\mu}||(y^2 - \frac{h^2}{4})| \qquad .....(1)$$

The max of (1) attained when y=0

$$u_{max}(y) = \frac{|P|}{2\mu}\frac{h^2}{4}$$

The average velocity,

$$u_a = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} udy = -\frac{h^2 P}{8\mu}\int_{-\frac{h}{2}}^{\frac{h}{2}}(1 - \frac{4y^2}{h^2})dy$$
$$= \frac{1}{h}u_{max}[\frac{y}{2} - \frac{4y^3}{3h^2}]_{-\frac{h}{2}}^{\frac{h}{2}}$$

Thus,

$$u_a = \frac{2}{3}u_{max}$$
$$\therefore \quad P = \frac{12\mu u_a}{h^2}$$