Functional Analysis

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LEC 1 :

① **Definition of norm** :

(i) $\|x\| \geq 0 \quad \forall x \in X$ and $\|x\| = 0 \iff x = 0$

(ii) $\|\alpha x\| = |\alpha| \times \|x\|$ for $\alpha \in K$

(iii) $\|x+y\| \leq \|x\| + \|y\|$

② On K^n, define the norms :

$$\|x\|_1 = \sum_{i=1}^{n} |x(i)| \quad , \text{ and}$$

$\forall x \in K^n$

$$\|x\|_\infty = \max \{ |x(i)| \ ; \ i = 1, 2, \dots, n \}$$

Then, $(K^n, \|\cdot\|_1)$ and $(K^n, \|\cdot\|_\infty)$ are normed linear spaces.

③ On $C[a,b]$, define the norms :

$$\|x\|_1 = \int_a^b |x(t)| \cdot dt \quad , \text{ and}$$

$\forall x \in C[a,b]$

$$\|x\|_\infty = \sup_{t \in [a,b]} \{ |x(t)| \}$$

Then $(C[a,b], \|\cdot\|_1)$ and $(C[a,b], \|\cdot\|_\infty)$ are normed linear spaces.

④ **Theorem** :

(i) For any $x, y \in K^n$ [Cauchy Schwarz Inequality]

$$\sum_{i=1}^{n} |x(i)\, y(i)| \leq \left(\sum_{i=1}^{n} |x(i)|^2 \right)^{1/2} \times \left(\sum_{i=1}^{n} |y(i)|^2 \right)^{1/2}$$

(ii) For any $x, y \in C[a,b]$

$$\int_a^b |x(t)\, y(t)| \cdot dt \leq \left(\int_a^b |x(t)|^2 \cdot dt \right)^{1/2} \times \left(\int_a^b |y(t)|^2 \cdot dt \right)^{1/2}$$

- Consequence of the theorem :

(i) On K^n, define the norm :

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x(i)|^2 \right)^{1/2}$$

Then $(K^n, \|\cdot\|_2)$ is a normed linear space.

(ii) On $C[a,b]$, define the norm :

$$\|x\|_2 = \left(\int_a^b |x(t)|^2 . dt \right)^{1/2}$$

Then $(C[a,b], \|\cdot\|_2)$ is also a normed linear space.

(Extension)

(5) Lemma : Let p and q be real no.s satisfying $\frac{1}{p} + \frac{1}{q} = 1$.
Then for every positive real no.s a and b,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

(Using the lemma)

(6) Theorem : (Holder's Inequality)

Let p and q be positive real no.s s.t. $\frac{1}{p} + \frac{1}{q} = 1$.
Then, $\forall\ x, y \in K^n$,

$$\sum_{i=1}^{n} |x(i)\, y(i)| \le \left(\sum_{i=1}^{n} |x(i)|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y(i)|^q \right)^{1/q}$$

i.e. $\sum_{i=1}^{n} |x(i)\, y(i)| \le \|x\|_p \times \|y\|_q$

◆ Lec 2 :

- Consequence :

For $1 \leq p \leq \infty$ let $\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}$.

Then $\|\cdot\|_p$ is a norm on K^n.

Imp. i.e. $(K^n, \|\cdot\|_p)$ is a n.l.s for $1 \leq p \leq \infty$

* $p=2$, i.e. $\|\cdot\|_2$ is called Euclidean norm on K^n.

① Remark : For $n>1$ and $0<p<1$,

$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}$ is not a norm on K^n.

$\left[\because \|e_1+e_2\|_p = 2^{1/p} > 2 = \|e_1\|_p + \|e_2\|_p\right]$

② Definition of $B(\Omega)$ → set of K-valued bounded functions on Ω.

Also denoted by : $B(\Omega) = \ell^\infty(\Omega)$

For $x \in B(\Omega)$, define :

$\|x\|_\infty = \sup_{t \in \Omega} |x(t)|$

Then $(B(\Omega), \|\cdot\|_\infty)$ is a n.l.s

Corollary : $C([a,b])$ and $R([a,b])$ are subspaces of $B([a,b])$

→ So, $\|\cdot\|_\infty$ is also a norm on $C([a,b])$ and $R([a,b])$.

i.e. $(C([a,b]), \|\cdot\|_\infty)$ and $(R([a,b]), \|\cdot\|_\infty)$ are n.l.s.

③ Theorem : (Holder's Inequality - Continuous case)

Let p and q be positive reals s.t. $\frac{1}{p}+\frac{1}{q}=1$. Then $\forall\, x,y \in C[a,b]$ we have :

$$\int_a^b |x(t)\,y(t)|\cdot dt \leq \left(\int_a^b |x(t)|^p \cdot dt\right)^{1/p} \left(\int_a^b |y(t)|^q \cdot dt\right)^{1/q}$$

i.e. $\int_a^b |x(t) \cdot y(t)| \cdot dt \leq \|x\|_p \cdot \|y\|_q$

- Consequence :

For $x \in C[a,b]$, $\|x\|_p = \left(\int_a^b |x(t)|^p \cdot dt \right)^{1/p}$

for $1 \leq p < \infty$

Imp. Then, $(C[a,b] , \|\cdot\|_p)$ is a n.l.s for $1 \leq p \leq \infty$.

* Note : $\frac{1}{p} + \frac{1}{q} = 1 \rightarrow$ then p is called the conjugate exponent of q and vice versa.

◆ **Lec 3 :**

① $F(\Omega)$ = space of all K-valued functions on Ω
$B(\Omega)$ = space of all K-valued bounded functions on Ω.

② Define : $\ell^p = \left\{ x \in F(\mathbb{N}) \;\middle|\; \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p} < \infty \right\}$

Define norm on ℓ^p as : For $1 \le p \le \infty$, define

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p} ; & x \in \ell^p, \; 1 \le p < \infty \\ \sup\{|x(i)| ; i \in \mathbb{N}\} ; & x \in \ell^p, \; p = \infty \end{cases}$$

Imp. Then, $(\ell^p, \|\cdot\|_p)$ is a n.l.s for $1 \le p \le \infty$.

* Remark : For $0 < p < 1$, $\|x\|_p = \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p}$ is not a norm on ℓ^p. [Similar case as K^n for $n \ge 2$]

③ Define : $c = \{ x \in F(\mathbb{N}) \mid x(n) \text{ converges as } n \to \infty \}$
$c_0 = \{ x \in F(\mathbb{N}) \mid x(n) \to 0 \text{ as } n \to \infty \}$
$c_{00} = \{ x \in F(\mathbb{N}) \mid \exists\, k \in \mathbb{N} \text{ s.t. } x(n) = 0 \;\forall n \ge k \}$

i.e. c = set of all convergent sequences
c_0 = set of all sequences converging to 0
c_{00} = set of all sequences with finitely many non-zero terms.

④ $c_{00} \subsetneq \ell^p \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty$ [For $1 \le p < \infty$]

(i) $x = (1, -1, 1, -1, \dots) \in \ell^\infty$ but $x \notin c$
(ii) $x = (1, \frac{1}{2}, \frac{1}{3}, \dots) \in c_0$ but $x \notin \ell^1$ as $\sum_j \frac{1}{j}$ is divergent
(iii) $x = (1, \frac{1}{2^2}, \frac{1}{3^2}, \dots) \in \ell^1$ but $x \notin c_{00}$

⑤ Thm : For $1 \le p < r < \infty$, $\ell^p \subset \ell^r$ and $\ell^p \subset \ell^\infty$.

Jensen's Inequality : For $1 \le p < r < \infty$, $\|x\|_r \le \|x\|_p$

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Lec 4 :

① **Lemma** : X be a linear space and Y be a n.l.s with norm $\|\cdot\|_Y$, and $T: X \to Y$ be a one-one linear map. Then, $\|x\|_X = \|T(x)\|_Y$ is a norm on X.

② **Banach Spaces** : A n.l.s X over field K is called a Banach space if X is complete in the metric $d(x,y) = \|x-y\|$ induced by the norm.

i.e. A complete n.l.s is called a Banach Space.

i.e. If every Cauchy sequence $\{x_n\}$ in a n.l.s X, is also a convergent sequence in X, i.e. $\|x_n - x\| \to 0$ as $n \to \infty$ and $x \in X$, then X is a Banach Space.

③ **Note** : A subset Y of a complete metric space X, is complete iff it is closed in X. So :

A subset Y of a Banach Space X, is a Banach space iff Y is a closed subspace of X.

Imp. ④ $(K^n, \|\cdot\|_p)$ is a Banach Space $\forall\, n \in \mathbb{N}$ and $1 \le p \le \infty$

Imp. ⑤ $(B(\Omega), \|\cdot\|_\infty)$ is a Banach space where $\Omega \neq \phi$

and $\|x\|_\infty = \sup_{t \in \Omega} |x(t)|$ for $x \in B(\Omega)$

→ i.e. ℓ^∞ is a Banach Space

⑥ Let Ω be a metric space. $C(\Omega)$ = space of K-valued continuous functions defined on Ω

Denote: $C_b(\Omega) = C(\Omega) \cap B(\Omega)$

So, $C_b(\Omega)$: space of all K-valued continuous and bounded functions on Ω.

Then, $C_b(\Omega)$ is a closed subspace of $B(\Omega)$

$\therefore$ $C_b(\Omega)$ is also a Banach Space.

Lec 5 :

Imp. (1) ℓ^p is a Banach space for $1 \le p \le \infty$.

(2) $c = \{x = (x(1), x(2), \dots) \mid x$ is convergent in $\ell^\infty\}$ is a closed subspace of ℓ^∞.

(3) $c_0 = \{x \in F(\mathbb{N}) \mid x(n) \to 0$ as $n \to \infty\}$ is a closed subspace of ℓ^∞.

$\therefore$ c and c_0 are both Banach spaces [as ℓ^∞ is a Banach space]

Note: c_{00} is not a closed subspace of ℓ^∞.

[$x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots 0) \in c_{00}$ and

$x_n \to x = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots) \notin c_{00}$]

$\therefore$ c_{00} is not a Banach Space w.r.t $\|\cdot\|_\infty$.

(4) $C'[a,b] = \{x \in C[a,b] \mid x$ is differentiable and x' is continuous on $[a,b]\}$

* $C'[a,b]$ contains all polynomials. Thus, $C'[a,b]$ is dense in $C[a,b]$

[$\because$ $x \in C[a,b]$ then $\exists \{P_n\}$ of polynomials such that $\|P_n - x\|_\infty \to 0$]

Also, $C'[a,b] \neq C[a,b]$ $\Rightarrow$ $C'[a,b]$ is not closed in $C[a,b]$

Imp. $\therefore$ $C'[a,b]$ is not a Banach Space w.r.t $\|\cdot\|_\infty$

(5) For $x \in C'[a,b]$, define :

$$\|x\|_{1,\infty} = \max\{\|x\|_\infty, \|x'\|_\infty\} = \max\left\{\sup_{t\in[a,b]} |x(t)|, \sup_{t\in[a,b]} |x'(t)|\right\}$$

Then, $(C'[a,b], \|\cdot\|_{1,\infty})$ is a Banach Space

$\to$ (First show n.l.s then show complete)

Lec 6 :

① $X = P[a,b]$ of all polynomials on $[a,b]$ is not a Banach space w.r.t $\|\cdot\|_\infty$.

[$\because$ $P[a,b]$ is dense in $C[a,b]$ w.r.t $\|\cdot\|_\infty$]

Then $P[a,b]$ is not closed, otherwise: $P[a,b] = \overline{P[a,b]} = C[a,b]$

* Also, $P[a,b]$ is not a Banach space w.r.t $\|\cdot\|_p$, $1 \le p < \infty$ [Dense w.r.t $\|\cdot\|_p$ also]

Imp. ~~Thus, $P[a,b]$ is not a Banach space w.r.t $\|\cdot\|_p$, $1 \le p \le \infty$~~

Imp. ② $X = c_{00}$ with $\|\cdot\|_p$, $1 \le p \le \infty$ is not a Banach space.

(For $p = \infty$ we have already seen this earlier)

* Note: $\overline{c_{00}} = c_0$ w.r.t $\|\cdot\|_\infty$.

And, $\overline{c_{00}} = \ell^p$ w.r.t $\|\cdot\|_p$

③ Baire Category Theorem: If X is a $\underbrace{\text{complete metric space}}_{\text{(Banach space)}}$ and $\{X_n\}$ is a sequence of subsets of X such that: $X = \bigcup_{n=1}^{\infty} X_n$, then there exists some $j \in \mathbb{N}$ such that interior of $\overline{X_j}$ is non-empty [i.e. $\overline{X_j}^{\,\circ} \neq \phi$]

④ Lemma: The interior of a proper subspace of a normed linear space X is empty.

⑤ (Application of Baire Category Theorem and above lemma)

Theorem: A Banach space cannot have a denumerable basis.

[denumerable = countably infinite]

- Consequence: If a linear space is a Banach space w.r.t any norm then it cannot have a denumerable basis.

Imp. e.g. $X = C[a,b]$ with $\|\cdot\|_1$ is not a Banach space, but $(C[a,b], \|\cdot\|_\infty)$ is a Banach space.

So, $C[a,b]$ cannot have a denumerable basis.

Lec 7 :

* A Banach space X has either finite basis or uncountable basis. If a linear space X has a denumerable basis then no norm on X makes it a Banach space.

① e.g. $X = P$ → space of all polynomials with coefficients in K.
$X = P$ has a basis $\{u_j(t) = t^j \,;\, j = 0, 1, 2, \dots\}$, which is denumerable.

Imp. So, P is not a Banach space w.r.t any norm on it.

② $X = c_{00}$ is also not a Banach space w.r.t any norm on it.

Imp. $\{e_1, e_2, \dots\}$ with $e_i(j) = \delta_{ij}$ is a denumerable basis for c_{00}.

③ Schauder Basis: A countable subset $\{x_1, x_2, \dots\}$ of a Banach space X is called a Schauder basis if $\|x_n\| = 1$ $\forall n \in \mathbb{N}$ and if for every $x \in X$, there are unique scalars $k_1, k_2, \dots$ in K such that $x = \sum_{j=1}^{\infty} k_j x_j$

* Suppose $(X, \|\cdot\|)$ is a finite dimensional normed linear space, and $\{x_1, x_2, \dots, x_n\}$ be a basis for X. Then X is a Banach space and $\left\{\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \dots, \frac{x_n}{\|x_n\|}\right\}$ is a Schauder basis for X.

Imp. i.e. every finite dimensional normed linear space is a Banach space.

Note: $X = \ell^p$, $p = 1, 2, \dots$
$\{e_1, e_2, \dots\}$ is not a basis for ℓ^p, but it is a Schauder basis for ℓ^p.

④ Construction of a Schauder basis for $C[0,1]$.

$y_0(t) = t$, $y_1(t) = 1 - t$, $y_2(t) = \begin{cases} 2t \,; & 0 \le t \le \frac{1}{2} \\ 2 - 2t \,; & \frac{1}{2} < t \le 1 \\ 0 \,; & \text{otherwise} \end{cases}$

And, $y_{2^n+j}(t) = y_2(2^n t - j + 1)$

Let $x_n = y_n|_{[0,1]}$ then $\{x_1, x_2, \dots\}$ is a Schauder basis for $C[0,1]$

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(5) Equivalent norms : X be a n.l.s with norms $\|\cdot\|$ and $\|\cdot\|_*$.
These norms are equivalent if $\exists\ c_1 > 0$ and $c_2 > 0$ such that
$$c_1 \|x\| \le \|x\|_* \le c_2 \|x\| \quad \forall x \in X$$

(6) If $\|\cdot\|$ and $\|\cdot\|_*$ are two equivalent norms ~~then~~ on a n.l.s X. Then X is a Banach space w.r.t $\|\cdot\|$ iff X is a Banach space w.r.t $\|\cdot\|_*$.

Note: If $\|x_n\| \to 0 \Rightarrow \|x_n\|_* \to 0$, then $\|\cdot\|$ is stronger than $\|\cdot\|_*$.

Theorem :
(7) $\|\cdot\|$ is stronger than $\|\cdot\|_*$ if $\exists\ \alpha > 0$ such that
$\|x\|_* \le \alpha \|x\|$, $\forall x \in X$.

(8) $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent on K^n.

$\|x\|_1 \le \sqrt{n}\, \|x\|_2 \le \sqrt{n}\, \|x\|_1 \Rightarrow \|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\, \|x\|_\infty \Rightarrow \|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent.

Imp. (9) On ℓ^1, $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$. $\left[\|x\|_2 \le \|x\|_1\right]$
But these two are not equivalent.
$$x_n = (\underbrace{\tfrac{1}{n}, \tfrac{1}{n}, \dots, \tfrac{1}{n}}_{n \text{ times}}, 0, 0, \dots) \in \ell^1$$
Then $\|x_n\|_1 = 1$ and $\|x_n\|_2 = \frac{1}{\sqrt{n}}$. Then, $\nexists$ any constant c s.t. $c \times 1 \le \frac{1}{\sqrt{n}}$ $\forall n$.

Imp. (10) On ℓ^2, $\|\cdot\|_2$ is stronger than $\|\cdot\|_\infty$.
But they are not equivalent [Take the same x_n as above]

Lec 8 :

① Definition of cosets and quotient space $\frac{X}{Y}$ where X is a n.l.s and Y is a closed subspace of X.

$$\frac{X}{Y} = \{x + Y \mid x \in X\}$$

Define operations: $(x_1 + Y) + (x_2 + Y) = x_1 + x_2 + Y$
$k(x + Y) = kx + Y$

$\frac{X}{Y}$ is a linear space with these operations

* Define norm on $\frac{X}{Y}$: $|||\cdot|||$ on $\frac{X}{Y}$

$$|||x + Y||| = \inf\{\|x + y\| \mid y \in Y\}$$

② X be a n.l.s. A series $\sum_{n=1}^{\infty} x_n$ in X is said to be absolutely summable if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

A series $\sum_{n=1}^{\infty} x_n$ is called summable in X if $s_n = \sum_{j=1}^{n} x_j$ and $s_n \to x \in X$.

③ ~~A n.l.s X is a Banach space iff every absolutely summable series is summable in X.~~

Theorem

④ Theorem :

Let X be a n.l.s and Y be a closed subspace of X. Then: X is a Banach space iff Y and $\frac{X}{Y}$ are Banach spaces in their induced norms respectively.

(Proof in Lec - 9)

* Compact set = closed and Bounded
* A space X is compact if each of its open covers has a finite subcover.

◆ Lec 9 :

① Riesz Lemma :

X be a n.l.s and Y be a closed subspace of X and $Y \neq X$. Let $r \in \mathbb{R}$ such that $0 < r < 1$. Then $\exists\, x_r \in X$ such that $\|x_r\| = 1$ and $r \leq dist(x_r, Y) \leq 1$

[where, $dist(x_r, Y) = \inf \{ \|x_r - y\| \mid y \in Y \}$

Lec 10 :

① **Lemma** : X be a n.l.s and Y be a subspace of X.

(i) For $x \in X$, $y \in Y$, $k \in K$, $\|kx+y\| \geq |k| \times dist(x, Y)$

(ii) Let Y be finite dimensional. Then Y is closed.
Y is complete.

* **Remark:** An infinite dimensional subspace of a n.l.s X need not be closed in X.

Imp. e.g. $X = \ell^\infty$ and $Y = C_{00}$. $Y \subset X$ but Y is not closed in X.

let $x_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots) \in C_{00}$

But $x_n \to x = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots) \notin C_{00}$.

② **Theorem** : X be a n.l.s. Then the following are equivalent :

(i) Every closed and bounded ~~set~~ subset of X is compact.

(ii) The subset $\{x \in X \mid \|x\| \leq 1\}$ of X is compact.

(iii) X is finite dimensional.

Lec 11 :

① Continuity of linear map : X and Y be n.l.s. A linear map $F: X \to Y$ is called continuous at $x \in X$ if $x_n \to x$ in $X \Rightarrow F(x_n) \to F(x)$ in Y.

OR

Given any $\varepsilon > 0$, $\exists\, \delta > 0$ such that $u \in X$,

$\|x - u\| < \delta \Rightarrow \|F(x) - F(u)\| < \varepsilon$

Imp. ② ~~Theorem : X and Y be n.l.s. If X is finite dimensional, then every linear map F: X→Y is continuous on X.~~

Imp. ③ Theorem : $F: X \to Y$ be a linear map. If F is bounded on $\overline{U(0,r)}$, $r > 0$, then $\exists\, \alpha > 0$ s.t. $\|F(x)\|_Y \leq \alpha \|x\|_X \;\; \forall x \in X$

④ ~~Theorem: F: X→Y be a linear map. Then F is continuous on X~~ iff there exists $\alpha > 0$ such that $\|F(x)\|_Y \leq \alpha \|x\|_X$, $\forall x \in X$.

* From above theorems, $F: X \to Y$ is bounded iff F is continuous on X.

Q. ~~Prove that F: X → Y is continuous on X iff it is continuous at the origin.~~

Lec 12 :

① Theorem: F : X → Y be a linear map. Then F is bounded iff F maps bounded sets in X to bounded sets in Y.

② ~~If F : X → Y is continuous linear map, then it is uniformly continuous.~~

Combining all the theorems above :

Imp. ③ Theorem: F : X → Y be a linear map. Then the following are equivalent :

(i) F is continuous at the origin

(ii) F is continuous at every $x \in X$

(iii) F is uniformly continuous on X.

(iv) $\exists\ \alpha > 0$ such that $\|F(x)\| \leq \alpha \|x\|$, $\forall\, x \in X$

(v) $\{F(x) \mid \|x\| = 1,\ x \in X\}$ is a bounded set in Y

(vi) For every bounded set $E \subseteq X$, the set $F(E) = \{F(x) \mid x \in E\}$ is bounded in Y.

④ Theorem: X and Y be n.l.s and F : X → Y be a linear map. Let Z(F) be a null space of X. Then F is continuous iff Z(F) is closed ~~and and~~ in X and $\tilde{F} : \frac{X}{Z(F)} \to Y$ defined by $\tilde{F}(x + Z(F)) = F(x)$ is continuous.

Lec 13 :

- A linear map $F: X \to Y$ is shown to be discontinuous by showing that there exists a bounded set $E \subseteq X$ such that $\{F(x) / x \in E\}$ is not bounded in Y.

OR

Produce a bounded sequence $\{x_n\}$ in X such that $\{F(x_n)\}$ is unbounded in Y.

① $X = C'[0,1]$ with $\|\cdot\|_\infty$. Define $f: X \to K$ by

$f(x) = x'(1) \quad \forall x \in X$.

Then f is a linear map but not continuous.

② $X = C'[0,1]$ with $\|\cdot\|_\infty$ and $Y = C[0,1]$ with $\|\cdot\|_\infty$.

Define: $A: X \to Y$ by : $Ax(t) = x'(t)$; $\forall t \in [0,1]$

Then A is unbounded map, thus discontinuous.

[For both ① and ②, take $x_n(t) = t^n \Rightarrow \|x_n\|_\infty = 1$]

③ $X = C_{00}$ with $\|\cdot\|_\infty$ $\quad f: X \to K$ by $f(x) = \sum_{j=1}^{\infty} x(j)$

Then f is discontinuous. [Take $x_n = (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$

Then for a bounded sequence $\{x_n\}$, $\{f(x_n)\}$ is unbounded]

* NOTE: A linear map on X may be continuous w.r.t some norm on X but discontinuous w.r.t some other norm on X.

e.g. this map is continuous w.r.t $\|\cdot\|_1$ on C_{00} but discontinuous w.r.t. $\|\cdot\|_2$ or $\|\cdot\|_\infty$ on C_{00}.

④ Consider $f_1: C_{00} \to K$ by $f_1(x) = \sum_{j=1}^{\infty} \frac{x(j)}{j}$; $x \in C_{00}$

Then f_1 is continuous w.r.t $\|\cdot\|_2$, but f_1 is discontinuous w.r.t $\|\cdot\|_\infty$.

⑤ Consider the infinite matrix of scalars (a_{ij}) and define :

$Ax(i) = \sum_{j=1}^{\infty} a_{ij}\, x(j)$.

Then $A: \ell^1 \to \ell^1$ is a bounded linear map.

$A: \ell^\infty \to \ell^\infty$ is also a bounded linear map.

In fact, $A: \ell^p \to \ell^p$ is a bounded linear map, for $1 \le p \le \infty$

⑤ $X = (C[a,b], \|\cdot\|_\infty)$ and $k(\cdot,\cdot) \in C([a,b]\times[a,b])$ s.t. $Ax(s) = \int_a^b k(s,t)\cdot x(t)\,dt$. Then $Ax \in C[a,b]$ and A ~~[illegible]~~ is a bounded linear operator/map.

Lec 14 :

① A linear map from a n.l.s X to a n.l.s Y is continuous iff it maps bounded sets in X to bounded sets in Y.

We call such a map is a bounded linear map.

* The set of all bounded linear maps is denoted by $BL(X,Y)$ or $B(X,Y)$.

* Set of all bounded linear operators $= BL(X,X)$ or $BL(X)$ or $B(X)$

* Set of all bounded linear functionals ~~[illegible]~~ :

$$X' = BL(X, K)$$

~~② A linear map $F: X \to Y$ is bounded below if $\exists\, \beta > 0$ s.t.~~

~~$\beta \|x\| \le \|F(x)\|$, $\forall x \in X$~~

③ Q. Show that $BL(X,Y)$ is a linear space under the pointwise operations :

for $x \in X$, $(F+G)(x) = F(x) + G(x)$ & $(\alpha F)(x) = \alpha F(x)$, $\alpha \in K$.

Theorem:

④ For $F \in BL(X,Y)$, define :

$$\|F\| = \sup\{\|F(x)\| \mid x \in X,\ \|x\| \le 1\}.$$

Then $\|\cdot\|$ is a norm on X called the operator norm.

Imp. * $\boxed{\text{For all } x \in X,\ \|F(x)\| \le \|F\| \cdot \|x\|}$

In fact, $\|F\| = \inf\{\alpha \ge 0 \mid \|F(x)\| \le \alpha\|x\|, \forall x \in X\}$

Also, if $X \ne \{0\}$,

$$\|F\| = \sup\{\|F(x)\| \mid x \in X,\ \|x\| = 1\}$$
$$= \sup\{\|F(x)\| \mid x \in X,\ \|x\| < 1\}$$

$\therefore$ $BL(X,Y)$ is a n.l.s with $\|F\| = \sup\{\|F(x)\| \mid x \in X, \|x\| \le 1\}$

[Proof contd. in Lec 15]

⓪ $X = (C[a,b], \|\cdot\|_\infty)$ and $k(\cdot,\cdot) \in C([a,b]\times[a,b])$ s.t.

$Ax(s) = \int_a^b k(s,t)\, x(t)\, dt$. Then:

$\|A\|_\infty = \sup_{s\in[a,b]} \int_a^b |k(s,t)|\, dt$

Lec 15 :

① Let $A : X \to Y$ be a linear map between two n.l.s X and Y. If $A \in BL(X,Y)$, then null space of A, $N(A)$ is a closed subspace of X. (Converse need not be true)

② $X = C'[0,1]$ and $Y = C[0,1]$ both w.r.t $\|\cdot\|_\infty$

[e.g. for converse not true]

Let $A : X \to Y$ be defined as : $Ax = x'$, $\forall x \in X$.

Then, $N(A) = \{x \in X \mid Ax = x' = 0\}$ = set of all constant functions which is a closed subspace of X.

But A is unbounded $\Rightarrow$ A is ~~not~~ not continuous.

~~(However, such a situation will not arise in case~~ of linear functionals) [For linear functionals, the converse is also true]

③ Theorem : X be a n.l.s and $f : X \to K$ be a non-zero linear functional on X such that null space $N(f)$ is closed. Then, f is continuous and for every $x_0 \in X - N(f)$,

$$\|f\| = \frac{|f(x_0)|}{\text{dist}(x_0, N(f))}$$

[Proof in next lec → Lec 16]

Note that, if $N(f)$ is closed then f is continuous is written in the statement. It's converse is also true and is already mentioned in general, in ① here : i.e. if $A \in BL(X,Y)$ [here $BL(X,K)$] then null space $N(A)$ is closed.

For proof of this, see test 2 solutions. Proof is easy.

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Lec 16 :

① Example: $X = C'[0,1]$ with $\|\cdot\|_\infty$ and $f: X \to K$ be defined as $f(x) = x'(1)$; $\forall x \in X$.

We know that f is discontinuous linear functional on X.
$\Rightarrow N(f)$ is not closed [By last theorem]

(Both are proved using sequences also)

Example: $X = C_{00}$ with $\|\cdot\|_\infty$. For $x \in X$, $f: X \to K$ be defined by $f(x) = \sum_{j=1}^{\infty} x(j)$. Then f is not continuous linear functional.

$\Rightarrow N(f)$ is not closed

② Suppose X and Y be n.l.s and $\{A_n\}$ be a sequence of operators/maps from X to Y i.e. $A_n \in L(X,Y)$. If $\{A_n x\}$ converges for every $x \in X$, then the function $A: X \to Y$ defined by:
$Ax = \lim_{n\to\infty} A_n x$, $x \in X$ is also a linear operator, i.e. $A \in L(X,Y)$.

* If each A_n is a bounded operator/map, what can you say about the boundedness of A?

→ The ans. is negative [in the case each A_n is bounded]

Example: $X = C_{00}$ with $\|\cdot\|_\infty$ and $f_n: X \to K$ defined as $f_n(x) = \sum_{j=1}^{n} x(j)$ $\forall n \in \mathbb{N}$, $\forall x \in X$.

Then $\|f_n\| = n$ $\Rightarrow$ Each f_n is bounded.
$\Rightarrow$ Each f_n is continuous linear functional.

But f is linear, but not continuous.

* By imposing boundedness of $\{\|A_n\|\}$, we ~~can~~ can obtain continuity of A.

③ Theorem: X and Y be n.l.s and $\{A_n\}$ be a sequence in $BL(X,Y)$ such that ~~{..}~~ $\{A_n x\}$ converges in Y for each $x \in X$. If $\{\|A_n\|\}$ is a bounded sequence, then $A: X \to Y$ defined by:
$Ax = \lim_{n\to\infty} A_n x$, $x \in X$ also belongs to $BL(X,Y)$ and:

$$\|A\| \le \liminf_{n\to\infty} \|A_n\|$$

④ Q. If $\|A_n x - Ax\| \to 0$ for each $x \in X$ then what can you say about $\|A_n - A\| \to 0$?? where $A_n \in BL(X, Y)$

$$\|A_n - A\| = \sup\{\|A_n x - Ax\| \mid x \in X, \|x\| \leq 1\}$$

→ The ans. is negative.

Example: $X = \ell^2$ and $\forall n \in \mathbb{N}$, let $A_n : \ell^2 \to \ell^2$ defined as:

$$A_n x(j) = \begin{cases} x(j) & ; j \leq n \\ 0 & ; j > n \end{cases}$$ i.e. $A_n x = (x(1), x(2), \ldots, x(n), 0, 0, 0, \ldots)$

Then for every $x \in \ell^2$, $\|A_n x - x\|_{\ell^2}^2 = \sum_{j=n+1}^{\infty} |x(j)|^2 \to 0$ as $n \to \infty$

$\therefore A_n x \to Ix$ for each $x \in \ell^2$.

But, $\|A_n - I\|_{\ell^2} \geq 1 \Rightarrow \|A_n - I\| \not\to 0$

⑤ Theorem: Let X be a n.l.s and Y be a Banach space and $\{A_n\}$ be a sequence in $BL(X, Y)$ such that $\{\|A_n\|\}$ is bounded subset of $\mathbb{R}$. Suppose $E \subseteq X$ is such that span E is dense in X. If $\{A_n x\}$ converges for every $x \in E$, then $\{A_n x\}$ converges for every $x \in X$, and if $Ax = \lim_{n\to\infty} A_n x$ then $A \in BL(X, Y)$ and

$$\|A\| \leq \liminf_n \|A_n\| < \infty.$$

Lec 17 :

(1) **Theorem :** Let $Q_n, Q : C[a,b] \to K$ be defined as:

$$Q_n x = \sum_{j=1}^{n} w_j \cdot x(t_j) \simeq \int_a^b x(t) \cdot dt \quad \forall x \in C[a,b]$$

and $$Qx = \int_a^b x(t) \cdot dt$$

where, $a = t_1 < t_2 < t_3 < \dots < t_n = b$ be the nodes with weights $w_1, w_2, \dots, w_n$.

. Let E be a subset of $C[a,b]$ such that span $E = C[a,b]$ w.r.t $\|\cdot\|_\infty$. If there exists $c > 0$ such that $\sum_{j=1}^{n} |w_j| \le c < \infty, \forall n \in \mathbb{N}$ and $\{Q_n x\}$ converges to Qx for every $x \in E$, then $Q_n x \to Qx$ for every $x \in X = C[a,b]$.

(2) **Theorem:** Let $x_k(t) = t^{k-1}, k = 1,2,3, \dots$ and $w_j > 0, \forall j = 1,2,\dots,n$

If $Q_n(x_k) \to \int_a^b x_k(t) \cdot dt, \forall k \in \mathbb{N}$ then:

$$Q_n(x) = \int_a^b x(t) \cdot dt, \quad \forall x \in C[a,b]$$

(3) * Let $\{P_0(t), P_1(t), \dots\}$ be the set of Legendre polynomials obtained by orthogonalizing the set $\{1, t, t^2, \dots\}$ w.r.t L^2- inner product on the set of polynomials $P[a,b]$.

* Let $t_1, t_2, \dots, t_n$ be the zeros of Legendre polynomial P_n of degree n. Then $t_i \in \mathbb{R}$ and are distinct and lie in (a,b).

Define for any $x \in C[a,b]$, $L_n x(t) = \sum_{j=1}^{n} l_j(t) x(t_j)$

where $$l_j(t) = \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{(t - t_i)}{(t_j - t_i)}, \quad \forall t \in [a,b]$$

* degree of l_j is $(n-1)$ and $l_j(t_i) = \delta_{ij} \quad \forall i,j$ and

$$L_n x(t_i) = \sum_{j=1}^{n} l_j(t_i) x(t_j) = \sum_{j=1}^{n} \delta_{ij} x(t_j)$$
$$= x(t_i) \quad ; i = 1,2,3, \dots, n$$

And, $L_n^2 x(t) = L_n x(t) \Rightarrow L_n^2 = L_n$

④ Define $Q_n : C[a,b] \to K$ by

$$Q_n x = \int_a^b L_n x(t) \cdot dt$$

$$= \int_a^b \sum_{i=1}^{n} l_i(t) \cdot x(t_i) \cdot dt = \sum_{i=1}^{n} \left(\int_a^b l_i(t) \cdot dt \right) x(t_i)$$

So, $Q_n x = \sum_{i=1}^{n} w_i \, x(t_i)$; where $w_i = \int_a^b l_i(t) \cdot dt$

This operator Q_n is called Gauss quadrature formula.

Imp. * $Q_n P = P$ for all polynomials of degree $\leq n-1$ and $w_i > 0$.

⑤ [Completeness of $BL(X,Y)$]

<u>Theorem</u>: Let X and Y be n.l.s. If Y is a Banach space, then $BL(X,Y)$ is also a Banach space. In particular, the dual of a n.l.s X is complete, (i.e. $X' = BL(X,K)$ is complete)

⑥ (Definition) Let $\mathcal{A} = \{ A_i \mid A_i \in BL(X,Y) \}$ be a family of bounded operators from X to Y.

Definition of point-wise bounded:

* We say $\mathcal{A}$ is pointwise bounded on X, if for each $x \in X$, $\exists \, M_x > 0$ such that $\|Ax\| \leq M_x \|x\|$, $\forall A \in \mathcal{A}$.

* We say $\mathcal{A}$ is uniformly bounded if $\{ \|A\| \mid A \in \mathcal{A} \}$ is a ~~bounded~~ bounded set,

i.e. $\exists \, M > 0$ s.t. $\|A_n\| \leq M \quad \forall n$

i.e. $\|A_n x\| \leq M \|x\| \quad \forall x \in X, \ \forall A_n \in \mathcal{A}$

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① $\mathcal{A} = \{A \mid A \in BL(X,Y)\}$ is uniformly bounded $\Rightarrow$ is pointwise bounded.

But the converse need not be true. (pointwise bdd. $\not\Rightarrow$ uniformly bdd.)

Example: $X = c_{00}$ with $\|\cdot\|_\infty$. For $x \in c_{00}$, define

$f_n(x) = \sum_{j=1}^{n} x(j)$. Then $\|f_n\| = n$ $\left[x_n = (\underbrace{1,1,\dots,1}_{n \text{ times}},0,0,\dots)\right]$

$f_n(x) = \sum_{j=1}^{n} x(j) \to \sum_{j=1}^{\infty} x(j) = f(x)$.

Then, $f_n(x_n) = n \ \forall n \in \mathbb{N}$ but $\{f(x_n)\}$ is unbounded. $\searrow \infty$

$\Rightarrow \{\|f_n\|\}$ not uniformly bounded.

Theorem:

② If X is finite dimensional, then pointwise bounded implies uniformly bounded.

③ [Uniform Boundedness Principle] Let X be a Banach space and Y be a n.l.s, and $\mathcal{A} \subseteq BL(X,Y)$. If $\mathcal{A}$ is pointwise bounded, then $\mathcal{A}$ is uniformly bounded. [i.e. when X is a Banach space, then pointwise bounded $\Rightarrow$ uniformly bounded]

④ Corollary: (Banach-Steinhauss Theorem):

Let X be a Banach space, Y be a n.l.s and $\{A_n\}$ be a ~~any~~ sequence in $BL(X,Y)$ such that for every $x \in X$, $\{A_n x\}$ converges in Y. Let $A: X \to Y$ defined as: $Ax = \lim_{n\to\infty} A_n x$, $x \in X$.

Then, $\{\|A_n\|\}$ is bounded and $A \in BL(X,Y)$.
$\hookrightarrow$ i.e. uniformly bounded.

⑤ Theorem: X and Y be Banach spaces, and $\{A_n\}$ be a sequence in $BL(X,Y)$. Then $\{A_n x\}$ converges in Y for every $x \in X$ iff $\{\|A_n\|\}$ is bounded and there exists a dense subset D of X such that $\{A_n u\}$ converges in Y for every $u \in D$.

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Lec 19

① **Corollary**: Let X be a Banach space, Y be a n.l.s and $\{A_n\}$ be a sequence in $BL(X,Y)$ such that $\{A_n x\}$ converges for every $x \in X$. Let $A: X \to Y$ be defined as $Ax = \lim_{n\to\infty} A_n x$. Then for every totally bounded subset $S \subseteq X$,

$$\sup_{x\in S} \|A_n x - Ax\| \to 0 \quad \text{as } n \to \infty .$$

② **Closed Operator:**
Let X and Y be n.l.s and X_0 be a subspace of X. A linear operator $A: X \to Y$ is said to be closed operator if for every sequence $\{x_n\}$ in X_0 such that $x_n \to x \in X$ and $Ax_n \to y \in Y$, then $x \in X_0$ and $Ax = y$.

e.g. $X = Y = C[0,1]$ w.r.t $\|\cdot\|_\infty$ and $X_0 = C'[0,1] \subset C[0,1]$

define $A: C'[0,1] \subset C[0,1] \to C[0,1]$ by

$Ax = x'$, $\forall x \in C'[0,1]$

Then, A is a closed operator, but not a bounded operator.

Imp. * A closed operator need not be a bounded operator.
Is every bounded operator a closed operator? $\longrightarrow$ (No. e.g in Lec 20)

③ Let $A: X_0 \subseteq X \to Y$ be a linear map, where X, Y are n.l.s. Then,
$G(A) = \{(x, Ax) \mid x \in X_0\}$ is called the graph of A.

* $G(A)$ is a subspace of the product space $X \times Y$.
[norm on $X\times Y$ is given by: $\|(x,y)\| = \|x\|_X + \|y\|_Y$, $\forall (x,y) \in X\times Y$]

④ **Theorem**: Let X and Y be n.l.s and X_0 be a subspace of X. A linear operator $A: X_0 \to Y$ is a closed linear operator iff its graph $G(A) = \{(x, Ax) \mid x \in X_0\}$ is a closed subspace of $X \times Y$.

Lec 20 :

(1) Example: $X = C[a,b]$ with $\|\cdot\|_\infty$ and $X_0 = C'[a,b] \subseteq X$. Let $A: X_0 \subseteq X \to X$ be defined as: $Ax = x$.

Then clearly, A is bounded operator.

$G(A)$ is not a closed subspace of $X \times X$. $\therefore$ A is not a closed operator.

(2) Theorem: Let $A: X_0 \subseteq X \to Y$ be a bounded linear operator.

(i) If X_0 is closed in X, then A is a closed operator.

(ii) If Y is a Banach space and A is a closed operator, then X_0 is a closed subspace of X.

Conv.

Q. Is every closed operator $A: X_0 \subseteq X \to Y$ with closed subspace X_0 and complete Y a bounded operator? $\longrightarrow$ No.

(3) ~~Theorem: Suppose $A: X_0 \subseteq X \to Y$ be a closed operator. Then:~~

~~(i) $N(A)$ is a closed subspace of X.~~

~~(ii) If A is 1-1, then $A^{-1}: R(A) \to X$ is a closed operator.~~

(4) Theorem: Suppose X is a Banach space and $A: X_0 \subseteq X \to Y$ be one-one, closed operator. If $R(A)$ is not closed in Y, then $A^{-1}: R(A) \subseteq Y \to X$ is unbounded operator.

Example: $X = C'[0,1]$ w.r.t $\|\cdot\|_{1,\infty} = \|x\|_\infty + \|x'\|_\infty$, and $Y = C'[0,1]$ w.r.t $\|\cdot\|_\infty$. Define $A: X \to Y$ by $Ax = x$.

Then, $\|Ax\|_\infty = \|x\|_\infty \leq \|x\|_\infty + \|x'\|_\infty = \|x\|_{1,\infty}$. So A is bounded.

And A^{-1} is unbounded. But A^{-1} is a closed operator.

Imp (5) ~~Theorem: Let $A: X \to Y$ be a one-one bounded operator. Then, $A^{-1}: R(A) \subseteq Y \to X$ is a closed operator.~~

(6) Theorem: Let $A_0: X_0 \subseteq X \to Y$ be a bounded operator, where X_0 is dense in X, and Y is a Banach space. Then there exists a unique $A \in BL(X,Y)$ such that A is extension of A_0. Moreover, $\|A\| = \|A_0\|$ and for $x \in X$, $Ax = \lim_{n\to\infty} Ax_n$, where $\{x_n\}$ is a sequence in X_0 such that $x_n \to x$.

(we'll prove later)

Imp (7) ~~Closed Graph Theorem: If X and Y are Banach spaces, then every closed operator $A: X \to Y$ is a continuous operator.~~

~~(Proof contd. in Lec 21)~~

Lec 21 :

① A linear map $P : X \to X$ is said to be a projection operator if $P^2 = P$, i.e. $Px = x, \forall x \in R(P)$.

* In this case, we write $X = R(P) + N(P)$ and $R(P) \cap N(P) = \{0\}$

② Continuity of Projection operators :
Suppose X is a n.l.s and $P : X \to X$ be a continuous projection operator. Then $N(P)$ is closed subspace of X.

* Here P is called projection onto range $R(P)$ along with $N(P)$.
* $I - P : X \to R(I-P)$ is also a projection.
Since, $R(P) = N(I-P)$ and $I-P$ is continuous, implies $N(I-P)$ is closed, implies $R(P)$ is also closed.

Imp. Thus, if $P: X \to X$ is continuous projection, both $R(P)$ and $N(P)$ are closed ~~sup~~ subspaces of X.

③ Corollary : Let X be a Banach space and $P : X \to X$ be a projection operator. If $N(P)$ and $R(P)$ are closed subspaces of X, then P is continuous.

Lec 22 :

① A map $F : X \to Y$ is said to be open map if for every ~~map~~ open set E in X, its image $F(E)$ is open in Y.
* $F : X \to Y$ is continuous iff for every open set E in Y, its inverse image $F^{-1}(E)$ is open in X.

② Theorem: Let X and Y be n.l.s and $F: X \to Y$ be a linear map. Then F is an open map iff there exists some $\gamma > 0$ such that for every $y \in Y$, there exists some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$

③ * Interior of a proper subspace of a n.l.s is empty.
Prob: $F : X \to Y$ be a linear map. If F is an open map, then F is surjective.

④ Corollary : X and Y be n.l.s and $F : X \to Y$ be a bijective linear map. Then F is open map iff $F^{-1} : Y \to X$ is continuous.

Lec 23 :

① **Theorem:** Let X and Y be n.l.s

(a) If Z is a closed subspace of X, then the quotient map $Q : X \to \frac{X}{Z}$ is continuous and open.

(b) Let $F : X \to Y$ is a linear map such that the null space $Z(F)$ is closed in X. Define $\tilde{F} : \frac{X}{Z(F)} \to Y$ by :

$\tilde{F}(x + Z(F)) = F(x)$, $\forall x \in X$.

Then F is open map iff $\tilde{F}$ is an open map.

* Also $\tilde{F}$ defined above is a bounded linear map, and $\|F\| = \|\tilde{F}\|$

② **Open Mapping Theorem:**

Let X and Y be Banach spaces and $F : X \to Y$ be a linear map which is closed and surjective. Then F is continuous and open map.

③ **Bounded Inverse Theorem:**

Let X and Y be Banach spaces and $F : X \to Y$ be bounded, bijective linear map. Then $F^{-1} \in BL(X, Y)$.

④ Definition of Partial order relation, POSET etc.

Zorn's Lemma: Let X be a non-empty POSET with partial order relation "$\leq$" such that every totally ordered subset of X has an upper bound, then X has a maximal element.

Lec 24 :

① **Definition:** (Maximal Proper Subspace)
A subspace Z of a normed l.s X is called maximal proper subspace of X if $a \in X - Z$, $\text{span}\{a, Z\} = X$.

* A maximal proper subspace is called hyper space.

② **Theorem:** Let f be a non-zero linear functional and a non-zero linear space X. Then the null space $Z(f)$ of f is a hyper space in X. That is, there exists $a \in X - Z(f)$ such that
$X = \text{span}\{a, Z(f)\}$.

③ **Theorem:** If Z is a hyperspace in a linear space X, then there exists a linear functional f on X such that $Z(f) = Z$, where $Z(f)$ is a null space of f.

④ **Theorem:** Let X be a linear space and S be a POSET of all proper subspaces of X. Then ~~the~~ a subspace H of X is hyper space iff it is maximal element of S.

⑤ **Lemma:** Let X be a linear space over the complex field $\mathbb{C}$. Regarding X as a linear space over $\mathbb{R}$, consider a real linear functional $u : X \to \mathbb{R}$.
Define $f(x) = u(x) - i u(ix)$, $x \in X$. Then f is a complex linear functional on X.

⑥ Definition of Convex sets.

Lec 25 :

1. **Lemma :** M be a subspace of a n.l.s N and $x_0 \notin M$. Let $M_0 = \text{span}\{x_0, M\}$. Let f be a continuous linear functional defined on M. Then, there exists a functional f_0 on M_0 such that : $\|f_0\| = \|f\|$ and $f_0|_M = f$.

(Case 1: when N is a real n.l.s → This proof is done in this lec. i.e. Lec 25

Case 2: when N is a complex n.l.s → Proof in lec 26)

Lec 26 :

1. ~~(continuation of previous lemma)~~

2. **Theorem :** (Hahn - Banach extension theorem) :- Let M be a subspace of a n.l.s N and $f \in M'$ (i.e. f is a linear functional on M). Then there exists $f_0 \in N'$ (a linear functional on N) such that :

$$f_0|_M = f \quad \text{and} \quad \|f\| = \|f_0\|$$

[Proof uses Zorn's lemma]

~~Lec 27~~ :

1. **Consequences of Hahn Banach Theorem :**

a) X be a n.l.s over K. Let $0 \neq a \in X$. Then there is some $f \in X'$ such that $f(a) = \|a\|$ and $\|f\| = 1$.

Consequently, $\|a\| = \sup\{|f(a)| \mid f \in X', \|f\| \leq 1\}$

b) Let Y be a subspace of a n.l.s X over K and $a \in X$, but $a \notin \overline{Y}$. Then there is some $f \in X'$ such that $f|_Y = 0$ and $f(a) = \text{dist}(a, \overline{Y})$ and $\|f\| = 1$.

Consequently, $x \in \overline{Y} \iff x \in X$ and $f(x) = 0$, whenever $f \in X'$ and $f|_Y = 0$.

c) Let $\{a_1, a_2, \ldots, a_m\}$ be linearly independent set in a n.l.s X. Then there are $f_1, f_2, \cdots, f_m$ in X' such that

$$f_i(a_j) = \delta_{ij} \; ; \forall \, i,j = 1,2,\ldots, m$$

(2) Let $X = K^2$ with $\|\cdot\|_1$ and $Y = \{ (x(1), x(2)) \mid x(2) = 0 \}$
$= \text{span}\{ a = (1,0) \}$

Define $g : Y \to K$ by $g(x(1), x(2)) = x(1)$.

Then $g(a) = 1 = \|g\|$. So $g \in Y'$.

So by Hahn Banach extension theorem, there exists $f \in X'$ such that $f|_Y = g$ and $\|f\| = 1$.

→ For this, we get the Hahn Banach extension as:

$$f(x(1), x(2)) = x(1) + k_2 \, x(2) \quad \text{with} \quad |k_2| \le 1.$$

(3) $Y = \{ (x(1), x(2)) \mid x(1) = x(2) \} = \text{span}\{ b = (\tfrac{1}{2}, \tfrac{1}{2}) \}$

Define $h : Y \to K$ by $h(x(1), x(2)) = 2x(1)$.

Then $\|h\| = 1$, $h(b) = 1 \Rightarrow h \in Y'$

So, by Hahn Banach extension theorem, there exists $f \in X'$ such that: $f|_Y = h$ and $\|f\| = \|h\| = 1$

→ For this, we get: $f(x(1), x(2)) = x(1) + x(2)$

(4) Theorem: (Unique Hahn Banach extensions)

X be a n.l.s. For every subspace Y of X and every $g \in Y'$, there is a unique Hahn Banach extension of g to X iff X' is strictly convex, i.e. for $f_1 \neq f_2$ in X', with $\|f_1\| = \|f_2\| = 1$, we have $\|f_1 + f_2\| < 2$.

[For proof see Balmohan Limaye book]

Lec 28 :

① Definition of inner product spaces $\langle\cdot,\cdot\rangle : X \times X \to K$ s.t.

(i) (Positive definiteness) : $\langle x,x\rangle \geq 0 \ \forall x \in X$ and $\langle x,x\rangle = 0 \iff x = 0$

(ii) (Linearity in the first variable): $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$

$\langle kx, y\rangle = k\langle x,y\rangle$

(iii) (Conjugate symmetry) : $\langle y,x\rangle = \overline{\langle x,y\rangle}$

Consequence of these three: (Conjugate linear in the second variable)

$\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$ and $\langle x, ky\rangle = \bar{k}\langle x,y\rangle$

② On K^n, $\langle x,y\rangle = \sum_{j=1}^{n} x(j)\overline{y(j)}$ is an inner product.

③ Lemma : $\langle\cdot,\cdot\rangle$ be an inner product on a linear space X.

a) [Polarization Identity] for all $x,y \in X$:

$4\langle x,y\rangle = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle$

b) $\langle x,y\rangle = 0 \ \forall y \in X \iff x = 0$

c) [Cauchy Schwarz Inequality] for all $x,y \in X$,

$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$ where equality holds iff $\{x,y\}$ are linearly dependent.

③ Theorem : Let $\langle\cdot,\cdot\rangle$ be an inner product on a linear space X.

For $x \in X$, define $\|x\| = \sqrt{\langle x,x\rangle}$. Then,

$|\langle x,y\rangle| \leq \|x\| \times \|y\|$, $\forall x,y \in X$. The function $\|\cdot\| : X \to K$ is a norm on X. Also, the following results hold :

(a) If $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ then $\langle x_n, y_n\rangle \to \langle x,y\rangle$

(b) [Parallelogram Law] for all $x,y \in X$,

$\|x+y\|^2 + \|x-y\|^2 = 2 \times [\|x\|^2 + \|y\|^2]$

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① **Hilbert Space :** An inner product space, which is complete in the norm induced by the inner product is called a Hilbert Space. (We denote it by H).

* $H = K^n$ is ~~an~~ a Hilbert space with inner product
$$\langle x, y\rangle = \sum_{i=1}^{n} x(i)\overline{y(i)} \quad \text{and induced norm} \quad \|x\|_2 = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n} |x(i)|^2}$$

② ~~Among all the norms $\|\cdot\|_p$ for $1 \le p \le \infty$ on K^n, only the norm $\|\cdot\|_2$ is induced by the inner product, because:~~

~~If $p \neq 2$, and let $x = (1,0,0,\dots,0)$ and $y = (0,1,0,0,\dots,0)$~~

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2^{2/p} + 2^{2/p} = 2 \times 2^{2/p} = 2^{1+2/p}$$

and $\|x\|_p^2 = 1$, $\|y\|_p^2 = 1$ $\longrightarrow$ Then the Parallelogram law doesn't hold. (So it won't be an Inner Product Space)

③ ℓ^2 is a Hilbert Space w.r.t norm $\|\cdot\|_2$ induced by the inner product $\|x\|_2 = \sqrt{\langle x, x\rangle}$

And again, ℓ^p, $p \neq 2$, $\|\cdot\|_p$ does not satisfy the parallelogram law. So, ℓ^p, $p \neq 2$ is not an Inner Product Space. Hence it is not a Hilbert Space.

④ $X = c_{00}$ and define $\langle x, y\rangle = \sum_{j=1}^{\infty} x(j)\overline{y(j)}$ for $x, y \in X$.

Then $\langle \cdot, \cdot \rangle$ is an inner product however c_{00} is incomplete I.P.S. So it is not a Hilbert Space.

Note that we had studied earlier, c_{00} has a ~~denumer~~ denumerable basis, so c_{00} can't be a Banach space w.r.t any ~~other~~ norm. i.e. incomplete w.r.t every norm.

⑤ **Theorem :** Let X be an I.P.S, then :

(a) [Pythagoras Thm.] Let $\{x_1, x_2, \dots, x_n\}$ be an orthogonal set in X. Then $\|x_1 + x_2 + \dots + x_n\|^2 = \sum_{i=1}^{n} \|x_i\|^2$

(b) E be an orthogonal subset of X and $0 \notin E$. Then E is L.I.

In fact, if E is orthonormal then $\|x-y\| = \sqrt{2}$ $\forall\, x, y \in E$ and $x \neq y$.

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① Given a L.I set in an I.P.S X, we can construct an orthonormal set in X. (By Gram Schmidt Orthogonalisation process).

* e.g. $X = \ell^2$ and let $x_n = (\underbrace{1,1,\dots,1}_{n \text{ times}}, 0,0,\dots,0)$.

Then $\{x_1, x_2, \dots, \cancel{x_n}\dots\}$ is L.I set in ℓ^2.

$x_1 = (1,0,0,\dots)$, $y_1 = x_1$, $\|y_1\| = 1$ so $u_1 = y_1$

$x_2 = (1,1,0,0,\dots)$, $y_2 = x_2 - \langle x_2, u_1\rangle u_1$

$\Rightarrow y_2 = (0,1,0,0,\dots)$ and $\|y_2\| = 1$. So $u_2 = y_2$

... and so on.

② [Bessel's Inequality]

Theorem: Let $\{u_1, u_2, \dots, u_n\}$ be a countable orthonormal set in an I.P.S X and $x \in X$. Then, $\sum_n |\langle x, u_n\rangle|^2 \le \|x\|^2$,

where equality ~~had~~ holds iff $x = \sum_n \langle x, u_n\rangle u_n$

③ Theorem : X be an I.P.S and $\{u_1, u_2, \dots\}$ be a countable orthonormal set in X. Then:

(a) If $\sum_n k_n u_n$ converges to some $x \in X$, then $k_n = \langle x, u_n\rangle$, $\forall n$ and $\sum_n |k_n|^2 < \infty$.

(b) [Riesz - Fischer Theorem] : If X is a Hilbert space and $\sum_n |k_n|^2 < \infty$, then $\sum_n k_n u_n$ converges in X.

④ Theorem: Let $\{u_\alpha\}$ be an orthonormal set in an I.P.S X and $x \in X$. ~~Then~~ Let $E_x = \{u_\alpha \mid \langle x, u_\alpha\rangle \neq 0\}$. Then E_x is a countable set, say $E_x = \{u_1, u_2, \dots\}$.

If E_x is denumerable, ~~then~~ then $\langle x, u_n\rangle \to 0$ as $n \to \infty$.

Further, if X is a Hilbert Space, then $\sum_n \langle x, u_n\rangle u_n$ converges to some $y \in H$ such that $x - y \perp u_\alpha$ $\forall \alpha$.

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① Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H.
Then the following are equivalent :

(i) $\{u_\alpha\}$ is an orthonormal basis for H.

(ii) [Fourier Expansion]:
For every $x \in H$, we have $x = \sum_n \langle x, u_n \rangle u_n$, where $\{u_1, u_2, \dots\} = \{u_\alpha \mid \langle x, u_\alpha \rangle \neq 0\}$

(iii) [Parseval's Formula]:
For every $x \in H$, we have: $\|x\|^2 = \sum_n |\langle x, u_n \rangle|^2$
where $\{u_1, u_2, \dots\} = \{u_\alpha \mid \langle x, u_\alpha \rangle \neq 0\}$

(iv) Span $\{u_\alpha\}$ is dense in H.

(v) If $x \in H$ and $\langle x, u_\alpha \rangle = 0$ $\forall \alpha$, then $x = 0$.

② Projections : X be a linear space and X_1 and X_2 be subspaces of X such that $X = X_1 + X_2$, $X_1 \cap X_2 = \{0\}$
i.e. $X = X_1 \oplus X_2$,

then every $x \in X$ can be written as $x = x_1 + x_2$; $x_1 \in X_1$, $x_2 \in X_2$

Then define $P : X \to X_1$ by $Px = P(x_1 + x_2) = x_1$

Then P is a linear map and $P^2 \equiv P$.

$Pu = P(u+0) = u$, $\forall u \in X_1 = R(P)$

Basically, $X_1 = R(P)$, $X_2 = N(P)$

③ A linear operator $P : X \to X$ is called a projection operator or a projection, if : $Pu = u$ $\forall u \in R(P)$.

* If $P : X \to X$ is a projection with $R(P) = X_1$ and $N(P) = X_2$, we say P is projection onto X_1 along X_2.
And, $(I-P)$ is a projection onto X_2 along X_1 with $R(I-P) = X_2$ and $N(I-P) = X_1$

(4) X be an I.P.S and $P : X \to X$ be a projection. We say P is an orthogonal projection if $R(P) \perp N(P)$

* If P is an orthogonal projection, then for any $x \in X$, we have:

$$x = Px + (I-P)x \qquad [\because Px \in R(P) \quad , (I-P)x \in N(P)]$$

By Pythagoras Theorem, $\|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2 \geq \|Px\|^2$

$$\Rightarrow \|Px\| \leq \|x\| \Rightarrow \|P\| \leq 1 \quad \text{-- (i)}$$

Also, $P = P^2 \Rightarrow \|P\| = \|P.P\| \leq \|P\| \times \|P\| \Rightarrow \|P\| \geq 1$ -- (ii)

So, $\|P\| = 1$

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Lec 32 :

(1) **Theorem : [Projection Theorem]**
Let H be a Hilbert space and F be a non-empty closed subspace of H. Then $H = F + F^{\perp}$.
Equivalently, there's an orthogonal projection onto F.
Moreover, $F^{\perp\perp} = F$.
(This is the generalization of the theorem we had studied in linear algebra for finite dimensional vector spaces)

~~* The projection theorem shows that every Hilbert space H has complementary subspace property, i.e. for every~~ non-empty closed subspace F of H, there is a closed subspace G of H such that : $H = F + G$ and $F \cap G = \{0\}$. Here $G = F^{\perp}$ is a closed subspace of H.

For $f \in X'$, $\|f\| = \sup\{\, |f(x)| \mid x \in X,\ \|x\| \leq 1 \,\}$

$$\Rightarrow \quad |f(x)| \leq \|f\| \times \|x\|$$

(2) **Lemma:** X be an I.P.S and $f \in X'$

(a) Let $\{u_1, u_2, \dots\}$ be an orthonormal set in X.
Then $\sum_n |f(u_n)|^2 \leq \|f\|^2$

(b) ~~Let $\{u_\alpha\}$ be an orthonormal set in X and~~
$E_f = \{u_\alpha \mid f(u_\alpha) \neq 0\}$. Then E_f is countable set, ~~and~~ say $\{u_1, u_2, \dots\}$. If E_f is denumerable, $f(u_n) \to 0$ as $n \to \infty$.

(This theorem is somewhat similar to that in Lec 30)

(3) X be an I.P.S over K. For a fixed $y \in X$, define :
$f : X \to K$ by $f(x) = \langle x, y \rangle \quad \forall x \in X$. Then f is a linear map.
And, $\|f\| = \|y\|$.

④ **Theorem** : [Riesz Representation Theorem] :-

Let H be a Hilbert space and $f \in H'$. Then there is a unique $y \in H$ such that : $f(x) = \langle x, y \rangle$, $\forall\, x \in H$.

In fact, if z is a non-zero element of H such that $z \perp Z(f)$, then $y = \dfrac{\overline{f(z)}}{\langle z, z \rangle} z$

Also, if $\{u_\alpha\}$ is an orthonormal basis for H and $\{u_\alpha \mid f(u_\alpha) \neq 0\} = \{u_1, u_2, \dots\}$ then $y = \sum_n \overline{f(u_n)}\, u_n$

* That unique y satisfies $\|y\| = \|f\|$.