continuation of Lemma:—

Now let N be a complex n.l.s

let $f \in M'$

Then $f(x) = g(x) - i g(ix)$, where $g = \operatorname{Re} f$.

Now since g is a real linear functional on M, by the above case, we can extend g to a real functional g_0 on $M_0 = [x_0, M]$ such that $\|g_0\| = \|g\|$ and

$$g_0|_M = g.$$

For $x \in M_0$, define

$$f_0(x) = g_0(x) - i g_0(ix)$$

Now for any $x, y \in M_0$, $a, b \in \mathbb{R}$, we have

$$f_0(ax+by) = g_0(ax+by) - i g_0(i(ax+by))$$

$$= a g_0(x) + b g_0(y) - i g_0(iax + iby)$$

$$= a g_0(x) + b g_0(y) - i \{a g_0(ix) + b g_0(iy)\}$$

$$= a[g_0(x) - i g_0(ix)] + b[g_0(y) - i g_0(iy)]$$

$$= a f_0(x) + b f_0(y).$$

Now for any $a, b \in \mathbb{R}$, $a + ib \in \mathbb{C}$.

So

$$f_0((a+ib)x) = g_0((a+ib)x) - i g_0(i(a+ib)x)$$

$$= a g_0(x) + b g_0(ix)$$
$$- i[g_0(iax - bx)]$$

$$= a g_0(x) + b g_0(ix) - i a g_0(ix)$$
$$+ i b g_0(x)$$

$$= a[g_0(x) - i g_0(ix)]$$
$$+ ib[g_0(x) - i g_0(ix)]$$

$$= (a+ib)[g_0(x) - i g_0(ix)]$$

$$= (a+ib) f_0(x).$$

$\therefore f_0 : M_0 \to K$ is a complex linear functional.

Also since $g_0 = g$ on M,

for any $x \in M$, we have

$$f_0(x) = f_0(x + 0 \cdot x_0)$$
$$= g_0(x) - i\, g_0(ix)$$
$$= g(x) - i\, g(ix)$$
$$= f(x), \quad \forall x \in M$$

$$\therefore f_0\big|_M = f.$$

Now we prove that $\|f_0\| = \|f\|$ on M_0.

Let $x \in M_0$ and $f_0(x) = re^{i\theta}$

Then

$$|f_0(x)| = r = e^{-i\theta} \cdot re^{i\theta}$$

$$= e^{-i\theta} f_0(x)$$

$$= f_0(e^{-i\theta}x) \quad [\because f_0 \text{ is a complex linear}]$$

So $f_0(e^{-i\theta}x) = r$, the complex valued functional f_0 is real, so it has only real part

$\therefore$

$$|f_0(x)| = r = g_0(e^{-i\theta}x)$$

$$\leq |g_0(e^{-i\theta}x)|$$

$$\leq \|g_0\| \, \|e^{-i\theta}x\|$$

$$= \|g_0\| \, \|x\| \quad \left[\|e^{-i\theta}x\| = |e^{-i\theta}| \|x\|\right]$$

$\Rightarrow \quad |f_0(x)| \leq \|g_0\| \, \|x\|, \ \forall x \in M_0$

$\Rightarrow \quad \|f_0\| \leq \|g_0\| \quad -(1)$

Also since g_0 is an extension of g, we have

$$\|g_0\| \, \|x\| = \|g\| \, \|x\| \quad [f = g(x) - ig(ix)]$$
$$\leq \|f\| \, \|x\| \quad \Big| \; g_0 = g \text{ on } M \quad \text{—} \circledast$$

$$x \in M$$
$$\Rightarrow |f_0(x)| = |g_0(x)|$$
$$\leq \|g_0\| \, \|x\|$$
$$\leq \|f\| \, \|x\| \text{ by } \circledast$$
$$\Rightarrow |f_0(x)| \leq \|f\| \, \|x\|$$

$$\Rightarrow \|f_0\| \leq \|f\| \quad \text{— (2)}$$

Also we can prove as in the real case that

$$\|f\| \leq \|f_0\| \quad \text{— (3)}$$

$\therefore$ from (2) & (3) we have

$$\|f\| = \|f_0\|$$

— // —

[N is n.l.s , $m \subsetneq N$

$x_0 \in N - m$

$m \subsetneq m_0 = [x_0, m]$

$f \in m' \Rightarrow \exists \; f_0 \in m_0'$

$\ni \; \|f\| = \|f_0\|, \; f_0|_m = f$]

Hahn-Banach extension Theorem:

Let m be a subspace of a n.l.s N and $f \in m'$. Then there exists $f_0 \in N'$ such that

$$f_0|_m = f \text{ and } \|f\| = \|f_0\|$$

Proof: If $m = N$, then there

is nothing to prove.

So let $M \neq N$, then M is a proper subspace of N.

Let $P = \{ (M_1, f_1) \,/\, M_1$ is a subspace of N containing M and $\|f_1\| = \|f\|$ & $f_1|_M = f \}$

Let "$\leq$" be a partial order on P, i.e.,

$$(M_1, f_1) \leq (M_2, f_2)$$

$$\Longleftrightarrow \quad M_1 \subseteq M_2, \; f_1 = f_2 \text{ on } M_1$$

$$\|f_1\| = \|f_2\|$$

$\because \; M \subseteq M, \; f \in M', \; f = f$ on M

$\Rightarrow (M, f) \in P \neq \phi$.

Let
$$S = \{ (M_i, f_i) \mid i = 1, 2, \ldots \} \text{ be}$$
a totally ordered subset of P.

Then S has an upper bound
$$\left(\bigcup_i M_i , F \right) \quad \text{where } F(x) = f_i(x) \quad \forall x \in M_i$$
and $\bigcup_i M_i$ is also a subspace containing M. $[\because M \subseteq M_i \ \forall i]$

This is true for every totally ordered subset of P.

$\therefore$ By Zorn's lemma, P has a maximal element say $(\overline{M}, f_0)$.

Claim : $\overline{M} = N$.

Suppose $\overline{M} \neq N$

. . . . .

Then $\overline{m} \subsetneq N \Rightarrow x_0 \in N - \overline{m}$

$\overline{m}_0 = [x_0, \overline{m}]$

Then by previous lemma, we can extend f_0 in such a way that

$$(\overline{m}, f_0) \leq (\overline{m}_0, F_0)$$

where $\overline{m}_0 = [\overline{m}, x_0]$, $x_0 \in N - \overline{m}$,

and $F_0|_{\overline{m}} = f_0$ & $\|F_0\| = \|f_0\|$

But $(\overline{m}, f_0)$ is a maximal element of P, so we must have $\overline{m} = \overline{m}_0$, which contradicts the fact that $\overline{m}_0 \subsetneq N$.

So our assumption $\overline{m} \neq N$ is wrong

$\therefore \overline{m} = N$

and $f_0|_M = f$ and $\|f\| = \|f_0\|$

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