Time series analysis.

Time series is a collection of random variables $\{X_t \mid t \in T\}$ where T is an index set representing time.

Distribution of timeseries:

$X_0 = 0 \quad X_i \overset{iid}{\sim} N(0,1)$

$$z_i = \sum_{j=0}^{i} x_j$$

$$\text{Var}(Z_i) = \text{Var}\left(\sum_{j=0}^{i} X_j\right) = i$$

$$\Rightarrow \text{Var}(Z_{60}) = 60$$

and $\text{Var}(Z_{90}) = 90.$

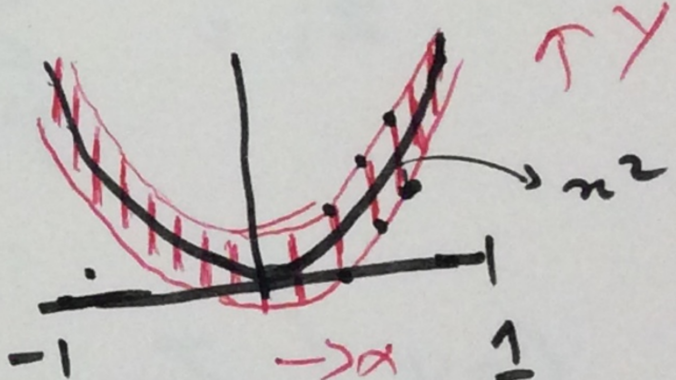
$$\begin{aligned}
&\text{Cov}(Z_{60}, Z_{90}) \\
&= \text{Cov}\left(Z_{60}, Z_{60} + \sum_{j=61}^{90} X_j\right) \\
&= \text{cov}(Z_{60}, Z_{60}) + \underbrace{\text{Cov}\left(Z_{60}, \sum_{j=61}^{90} X_j\right)} \\
&= \text{Var}(Z_{60}) + 0 \\
&= 60
\end{aligned}$$

Note { Even if X_i's are uncorrelated then also this cov. is zero.

* The necessary condition for $\mathrm{cov}\left(Z_{60}, \sum_{j=61}^{90} X_j\right) = 0$ is that X_js are uncorrelated.

* Normally distributed random variables with correlation zero implies independence.

* Correlation or Covariance zero does not imply independence in general.



$X \sim U(-1, 1)$

$Y \mid X \sim U(x^2 - \epsilon, x^2 + \epsilon)$

$E(XY) = E(X^3) = 0 \qquad E(X)E(Y) = 0$

Hence covariance $\underline{(X, Y) = 0}$.

$E(X_i) = 0 \quad V(X_i) = 1$ and they are uncorrelated.

(iid) with finte 2nd order moment

$$Z_t = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i \qquad t \in [0,1]$$

partial sum process.

what will be the distribution of Z_t when $n \uparrow \infty$??

$$Z_t = \frac{\sqrt{[nt]}}{\sqrt{n}} \left(\frac{1}{\sqrt{[nt]}} \sum_{i=1}^{[nt]} X_i \right)$$

when $n \uparrow \infty$

iid

→ indefendent but not identical.

→ uncorrelated ✓

→ dependent ✓ more restricting

$$\boxed{e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$

$\downarrow \sqrt{t}$

$\downarrow$ converse to $N(0,1)$ in distribut by CLT.

Z_t will converge in distribution to

$$\sqrt{t}\, N(0,1) \equiv N(0, t).$$

$Z_t \sim$ following Brownian motion.

$\{x_i\}$ series of numbers.

$$S_k = \sum_{i=1}^{k} x_i \qquad k \in \mathbb{N}.$$

→ partial sum.

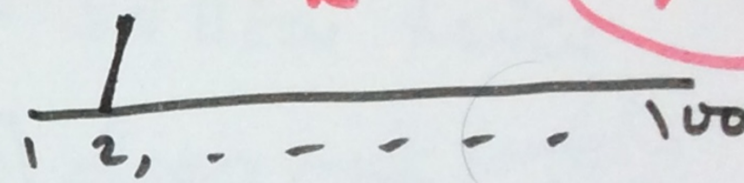
→ Cumulative sum.

$X_k \to k$

$S_k \to k$

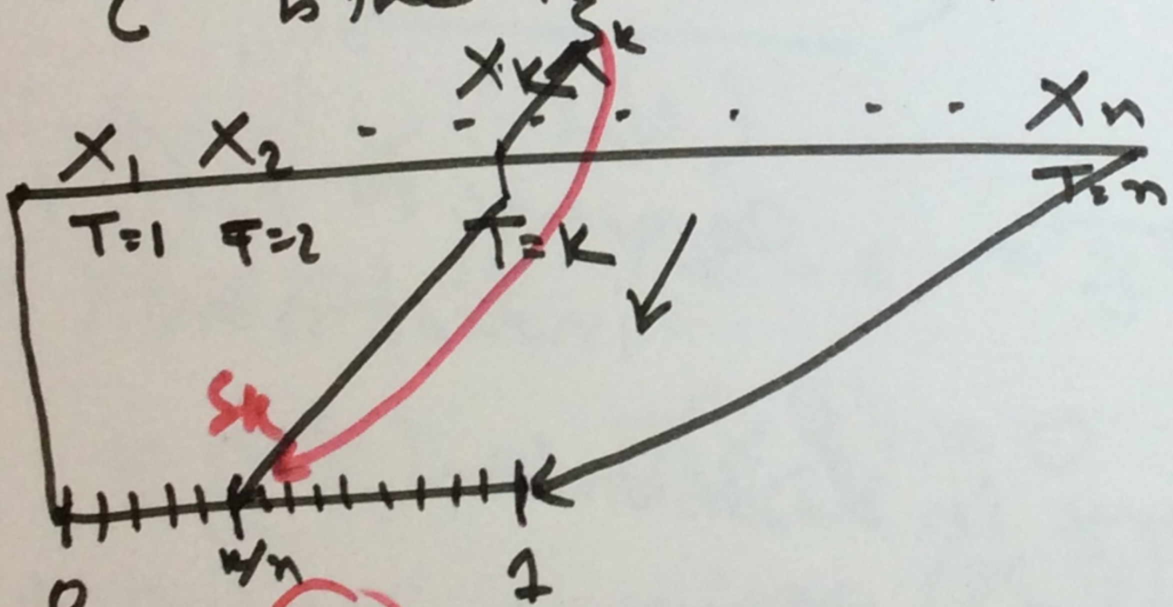
$X_k \to k/n$

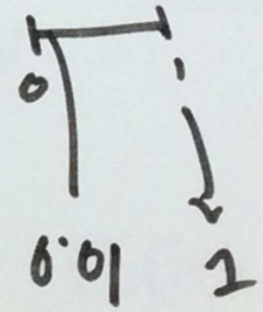
$S_k \to k/n$



n = sample size.

t is the time index.





$$t = \frac{k}{n} \to X_k.$$

1. White noice:

A time series $\{W_t\}$ such that $E(W_t)=0$, $V(W_t)=\sigma_w^2$,

$$W_t \sim WN(0, \sigma_w^2).$$

and they are uncorrelated.

Example: (1) $x_i \overset{iid}{\sim} N(0, \sigma^2)$

(2) $x_i \sim \begin{cases} N(0, \sigma^2) & \text{when } i \text{ is even} \\ \exp(1) - 1 & \text{when } i \text{ is odd} \end{cases}$ > independent.

→ (a) WN need not be normally distribute..

(b) WN need not be iid.

(c) iid seqence is always WN, with mean zero, and finte variance.

/ (d). WN is weakly stationary.

/ (e) WN with normally distibuted r.v.s are strongly stationary.

(2) Binary process:

Consider a time series $\{X_t\}$ such that

$$X_t = \begin{cases} +1 & \text{with prob. } p = 1/2 \\ -1 & \text{with prob. } 1-p = 1/2 \end{cases}$$

$E(X_t) = 0 \qquad V(X_t) = 1$

In general when $p \neq 1/2$

$E(X_t) = 2p - 1 \qquad V = 1 - (2p-1)^2$

$= \underline{4p(1-p)}$

$W \sim \text{bernoulli}(p)$.

$X = \dfrac{W - 0.5}{0.5}$

$X = a + bW$

$V(X) = b^2 V(W)$

$= 4p(1-p)$

(3). Random walk:

Let $\{W_t\}$ be an iid sequence of random variables.

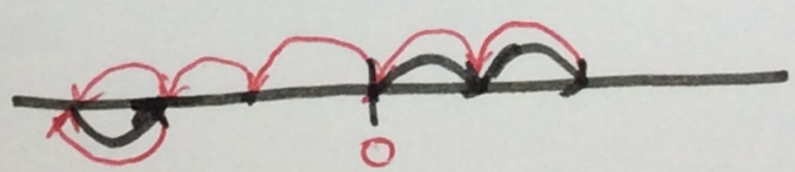
and define $X_0 = 0, \quad X_t = \sum_{i=1}^{t} W_i$ where

$\{W_t\} \sim$ binary process

or $N(0, \sigma^2)$

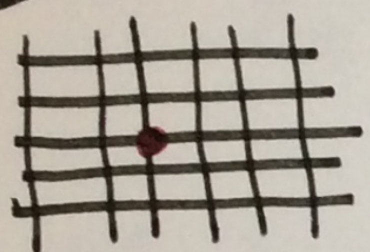
or. Laplace$(0, \lambda)$

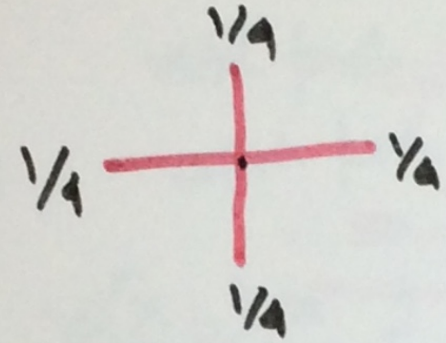
$E(W_t) = 0 \quad Var(W_t) = \sigma_w^2.$



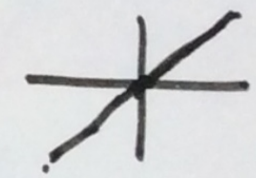
ⓐ $\mathbb{Z}$ $w_t \sim$ Binary process $(1/2)$. then it will eventually comeback to the place where it has started from.

ⓑ $\mathbb{Z}\times\mathbb{Z}$





The random varible eventually comeback to $(0,0)$.



ⓒ $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

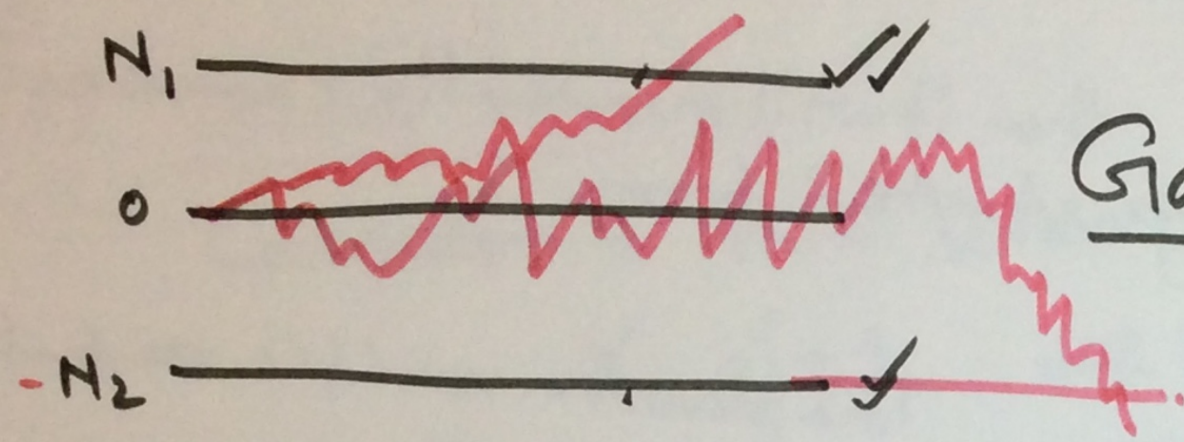
with equal probabity to each $\underline{6}$ ~~8~~ direction. then the random variable may not come back to $(0,0,0)$.

(4) Random walk with drift:

$$\begin{cases} w_i \overset{iid}{\sim} \quad E(w_i)=\delta \quad V(w)=\sigma^2 \\ X_t = \sum_{i=1}^{t} w_i \end{cases}$$

Mean of X_t is a function of t.

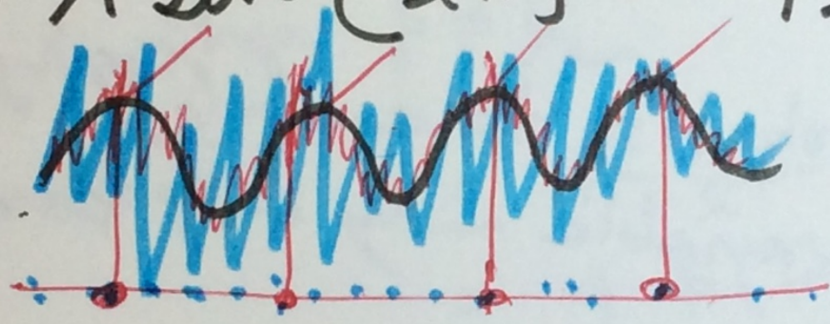
$$\begin{cases} X_t = \underline{t\delta} + \sum_{i=1}^{t} z_i \\ z_i \overset{iid}{\sim} \quad E(z_i)=0 \\ \qquad V(z_i)=\sigma_z^2. \end{cases}$$



Gamblers Ruin problem.

5 Signal with noise.

$$X_t = A \sin(2\pi f t + \varphi) + W_t .$$



6 Moving average process: (order one).

$$W_t \sim WN(0, \sigma_w^2).$$

$$X_t = W_t + \theta W_{t-1}.$$

$$X_t$$

$$\uparrow$$

$$(\theta_1 W_{t-1} + \theta_2 W_t + \theta_3 W_{t+1})$$

7 Auto regressive process : (order one).

$\{W_t\} \sim WN(0, \sigma_w^2)$.

$$X_t = \phi X_{t-1} + W_t . \qquad |\phi| < 1, \ \phi \neq 0$$

$$Y = \beta X + \epsilon .$$

8 Wiener process / Brownian motion.

$\{X_t\}$ is said to follow wiener process ~~of~~ or BM. if.

(i) $X_0 = a$ ($a = 0 \Rightarrow$ standard BM).

(ii) $X_{t+u} - X_t$ is idependently distributed to $X_s \ \forall s \leq t$. $u > 0$.

(iii) $X_{t+u} - X_t \sim N(0, u)$., $u > 0$

(iv) $\{X_t\}$ has continuous path on $t \in$ Time.

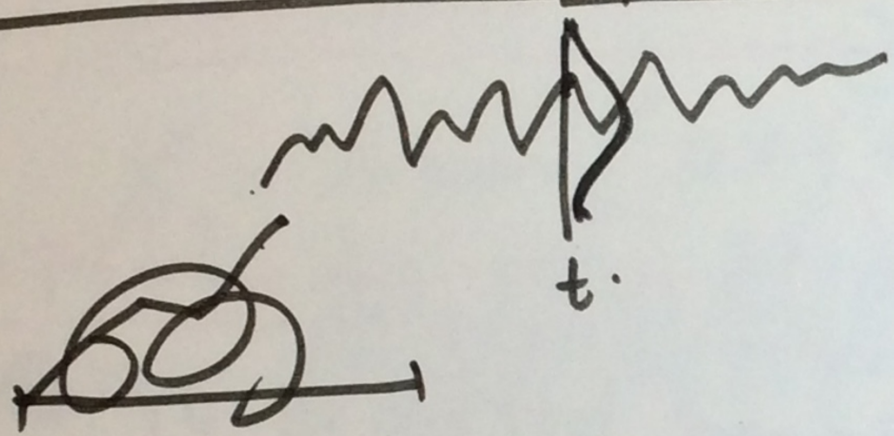
(but not differentiable)

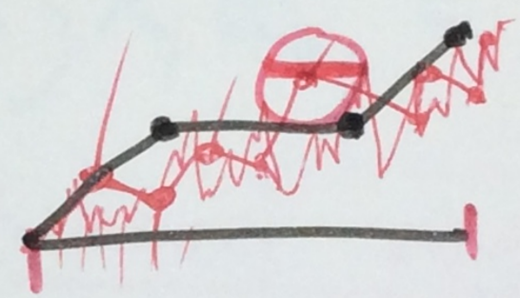
$$\begin{cases} W_k \overset{iid}{\sim} \quad E(W_k)=0 \quad V(W_k)=1. \\ \cancel{X_t} \; Y_n(t) = \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} W_i}_{\to N(0,t)} \longrightarrow X_t \sim BM. \end{cases}$$

For large n.

$t \in [0,1]$.

Continuous time process.





③ Brownian Bridge.

Brownian motion with restriction.

$(t=0, X_t=0)$ and $(t=T, X_t=b)$.

If $(T=1$ and $b=0)$ $\Rightarrow$ standard Brownian bridge.

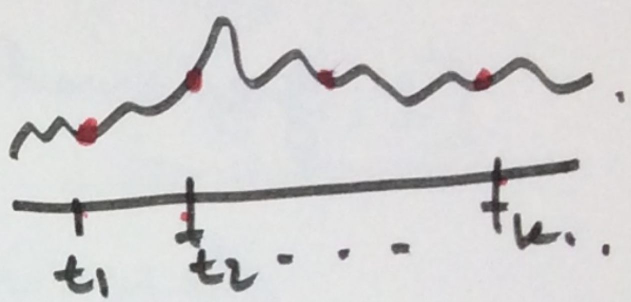
$$B_0(t) = \underbrace{X_t}_{N(0,t)} - t\underbrace{X_1}_{N(0,1)}$$

$\underline{X_t \sim BM \text{ on } [0,1]}$.

$X_t \sim$ Standard BM.

$t < s$

$$\begin{pmatrix} X_t \\ X_s \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} t & t \\ t & s \end{pmatrix} \right)$$



$B_0(t) \sim$ Standard BB.

$$\begin{pmatrix} B_0(t) \\ B_0(s) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} t(1-t) & t(1-s) \\ t(1-s) & s(1-s) \end{pmatrix} \right)$$

Any $k \in \mathbb{N}$ and $\underline{t_1, t_2, t_3, \dots t_k} \in \mathbb{R}$.

As $(X_{t_1}, X_{t_2}, \dots, X_{t_k})$ or $(B_0(t_1), B_0(t_2), \dots B_0(t_k))$

follows multivariate Normal.

X_t and $B_0(t)$ are known a <u>Gaussian process.</u>

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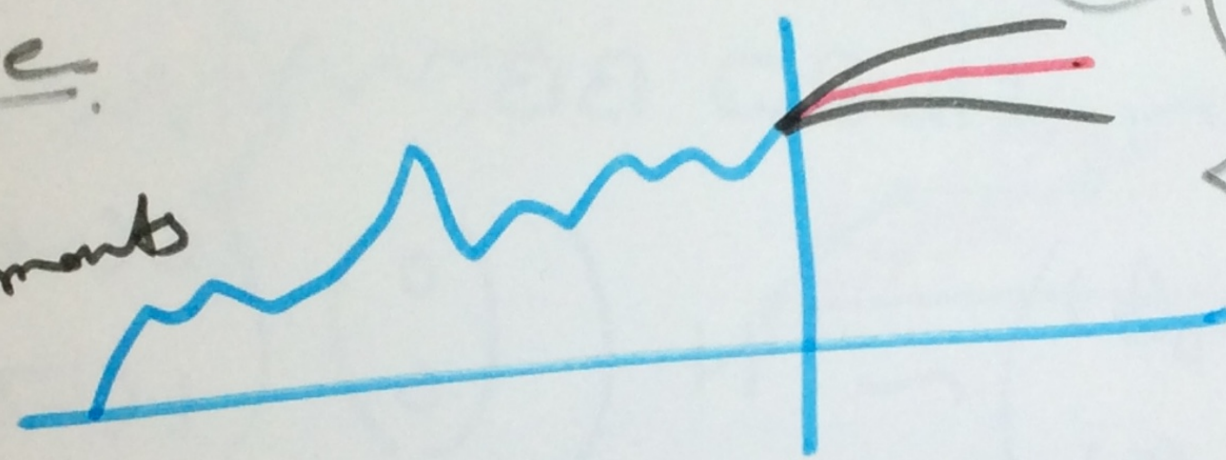
$$X_t = (\delta_0 + \delta_1 t + \delta_2 t^2) + A \sin(2\pi f t) + W_t.$$

general features of a time series we might observe.

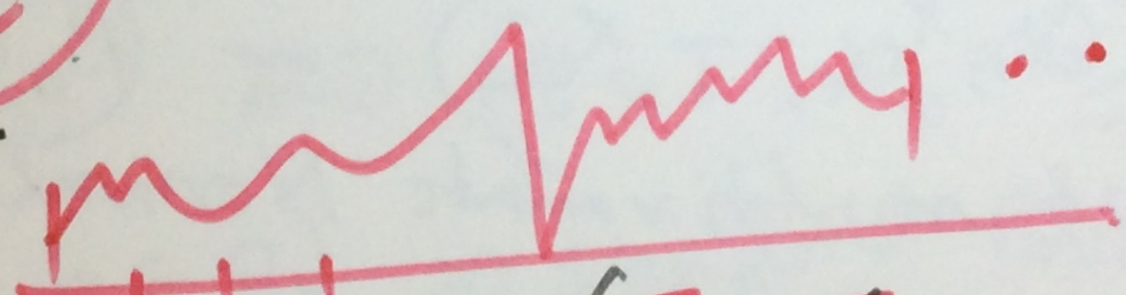


** Even though all moments are same two random variables may have different distn.

Weak
Second order moments are same.

For all $k \in \mathbb{N}$.
$t_1 t_2 \ldots t_k \in \mathbb{R}$
$h \in \mathbb{R}$.

W_t.



Stronger.
Joint distribution of $(Z_{t_1}, Z_{t_2} \ldots Z_{t_k})$
is same as $(Z_{t_1+h}, Z_{t_2+h} \ldots Z_{t_k+h})$.