Given a L.I set in an I.P.S X, we can construct an orthonormal set in X.

Theorem:

Gram-Schmidt orthogonalization :—

let $\{x_1, x_2, \ldots\}$ be L.I set in an I.P.S X. Define $y_1 = x_1$,

$u_1 = \frac{y_1}{\|y_1\|}$ and for $n = 2, 3, \ldots$

let, $y_n = x_n - \sum_{i=1}^{n-1} \langle x_n, u_i \rangle u_i$, $u_n = \frac{y_n}{\|y_n\|}$

Then $\{u_1, u_2, \ldots\}$ is an orthonormal set in X and

$$\operatorname{Span}\{x_1, x_2, x_3 \ldots\} = \operatorname{Span}\{u_1, u_2, u_3, \ldots\}$$

Proof: Since $\{x_1\}$ is L.I, $y_1 = x_1 \neq 0$,

$$\|u_1\| = \left\| \frac{y_1}{\|y_1\|} \right\| = 1$$

and $\mathrm{Span}\{u_1\} = \mathrm{Span}\{x_1\}$

Now for $n \geq 1$, assume that we have defined y_n and u_n as stated above, and proved that

$$\mathrm{Span}\{u_1, u_2, \dots u_n\} = \mathrm{Span}\{x_1, x_2, \dots x_n\}.$$

Define $y_{n+1} = x_{n+1} - \sum_{i=1}^{n} \langle x_{n+1}, u_i \rangle u_i$

$\because \{x_1, x_2 \dots x_n, x_{n+1}\}$ is L.I set,

so $x_{n+1} \notin \mathrm{Span}\{x_1, x_2 \dots x_n\} = \mathrm{Span}\{u_1, u_2, \dots u_n\}$.

$\therefore y_{n+1} \neq 0$, Let $u_{n+1} = \dfrac{y_{n+1}}{\|y_{n+1}\|}$.

Then $\|u_{n+1}\| = 1$ and for all $j \leq n$, we have

$$\langle y_{n+1}, u_j \rangle = \left\langle x_{n+1} - \sum_{i=1}^{n} \langle x_{n+1}, u_i \rangle u_i , u_j \right\rangle$$

$$= \langle x_{n+1}, u_j \rangle - \sum_{i=1}^{n} \langle x_{n+1}, u_i \rangle \underbrace{\langle u_i, u_j \rangle}_{\delta_{ij}}$$

$$= \langle x_{n+1}, u_j \rangle - \langle x_{n+1}, u_j \rangle$$

$$= 0.$$

$$\therefore \langle u_{n+1}, u_j \rangle = \left\langle \frac{y_{n+1}}{\|y_{n+1}\|}, u_j \right\rangle$$

$$= \frac{1}{\|y_{n+1}\|} \langle y_{n+1}, u_j \rangle = 0$$

$\Rightarrow \{u_1, u_2, \dots u_n, u_{n+1}\}$ is an orthonormal set. Also

$$\text{Span}\{u_1, u_2, \dots u_n, u_{n+1}\} = \text{Span}\{x_1, x_2, \dots x_n, u_{n+1}\}$$

$$= \text{Span}\{x_1, x_2, \dots x_n, x_{n+1}\}.$$

Thus by mathematical induction, proof is complete.

Ex: $X = \ell^2$, for $n = 1, 2, 3 \cdots$

let $x_n = (\underbrace{1, 1, 1, \cdots 1}_{n \text{ times}}, 0, 0 \cdots)$.

Then $\{x_1, x_2, x_3, \cdots\}$ is L.I set in ℓ^2.

$x_1 = (1, 0, 0 \cdots)$, $\quad y_1 = x_1$, $\|y_1\| = 1$

$u_1 = \frac{y_1}{\|y_1\|} = (1, 0, 0 \cdots)$

$x_2 = (1, 1, 0, 0 \cdots)$

$y_2 = x_2 - \langle x_2, u_1 \rangle u_1$

Clearly $\langle x_2, u_1 \rangle = 1$

$\therefore \ y_2 = x_2 - \langle x_2, u_1 \rangle u_1 = (1, 1, 0 \cdots) - 1 \cdot (1, 0, 0 \cdots)$

$= (0, 1, 0, \cdots)$

$\|y_2\| = 1$

$\therefore u_2 = \dfrac{y_2}{\|y_2\|} = (0,1,0,\ldots).$

and so on

(Bessel's Inequality):

Theorem: let $\{u_1, u_2, \ldots\}$ be a countable orthonormal set in an I.P.S X and $x \in X$. Then

$$\sum_n |\langle x, u_n\rangle|^2 \leq \|x\|^2,$$

where the equality holds iff $x = \sum_n \langle x, u_n\rangle u_n$.

Proof: For $m = 1, 2, 3, \ldots$, let

$$x_m = \sum_{n=1}^{m} \langle x, u_n\rangle u_n.$$

Since $\{u_1, u_2, \ldots u_m\}$ is an orthonormal set, we have

$$\langle x, x_m\rangle = \left\langle x, \sum_{n=1}^{m} \langle x, u_n\rangle u_n\right\rangle$$

$$= \sum_{n=1}^{m} |\langle x, u_n\rangle|^2$$

$$= \langle x_M, x \rangle$$
$$= \langle x_M, x_M \rangle$$

Hence

$$0 \le \|x - x_M\|^2 = \langle x - x_M, x - x_M \rangle$$
$$= \langle x, x \rangle - \langle x, x_M \rangle - \langle x_M, x \rangle + \langle x_M, x_M \rangle$$
$$= \|x\|^2 - \sum_{n=1}^{M} |\langle x, u_n \rangle|^2 \quad \text{---} \; (*)$$

Letting $m \to \infty$, we get

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \le \|x\|^2$$

This is called Bessel's inequality.

If the equality holds, i.e.,

if $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$, then

from $(*)$

$$0 \leq \|x - x_M\|^2 = 0$$

$$\Rightarrow \|x - x_M\|^2 \longrightarrow 0 \text{ as } M \longrightarrow \infty.$$

i.e., $\|x - x_M\| \longrightarrow 0$

$$\Rightarrow x = \lim_{M \to \infty} x_M = \lim_{M \to \infty} \sum_{n=1}^{M} \langle x, u_n \rangle u_n$$

$$= \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n.$$

Conversely, assume

$x = \sum_n \langle x, u_n \rangle u_n$. Then

$$\|x\|^2 = \langle x, x \rangle$$

$$= \left\langle \sum_n \langle x, u_n \rangle u_n, \sum_p \langle x, u_p \rangle u_p \right\rangle$$

$$= \sum_n |\langle x, u_n \rangle|^2.$$

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Now we discuss the convergence of

a series $\sum_n k_n u_n$ in an I.P.S X where $k_n \in K$ and $\{u_1, u_2, \dots\}$ is an orthonormal set in X.

Theorem: Let X be an IPS, $\{u_1, u_2, u_3 \dots\}$ be a countable orthonormal set in X. Then

(a) If $\sum_n k_n u_n$ converges to some $x \in X$, then $k_n = \langle x, u_n \rangle$, $\forall n$ and $\sum_n |k_n|^2 < \infty$.

(b) (Riesz — Fischer Theorem):- If X is a Hilbert space and $\sum_n |k_n|^2 < \infty$, then $\sum_n k_n u_n$ converges in X.

Proof: Let $x = \sum_n k_n u_n$ $\left(\because \sum_n k_n u_n \to x\right)$.

Then

$$\langle x, u_n \rangle = \Big\langle \sum_j k_j u_j , u_n \Big\rangle$$

$$= \sum_j k_j \langle u_j, u_n \rangle$$

$$= k_n.$$

And by Bessel's Inequality we have

$$\sum_n |k_n|^2 = \sum_n |\langle x, u_n \rangle|^2 \le \|x\|^2 < \infty$$

$$\Rightarrow \sum_n |k_n|^2 < \infty.$$

(b) Now assume that X is a Hilbert space and $\sum_n |k_n|^2 < \infty$.

Claim: $\sum_n k_n u_n$ Converges in X.

For $m = 1, 2, 3, \ldots$, let $x_m = \sum_{n=1}^{m} k_n u_n$.

Then for $j = 1, 2, 3 \ldots$ and $m > j$, we have

$$x_m - x_j = \sum_{n=j+1}^{m} k_n u_n$$

$\therefore \|x_M - x_j\|^2 = \langle x_M - x_j, x_M - x_j \rangle$

$$= \left\langle \sum_{n=j+1}^{M} k_n e_n, \sum_{p=j+1}^{M} k_p e_p \right\rangle$$

$$= \sum_{n=j+1}^{M} |k_n|^2$$

Hence if $\sum_n |k_n|^2 < \infty$, then it follows that $\sum_{n=j+1}^{M} |k_n|^2 \longrightarrow 0$ as $n, M \longrightarrow \infty$.

$\therefore \|x_M - x_j\|^2 \longrightarrow 0$ as $M, j \longrightarrow \infty$.

$\Longrightarrow \{x_m\}$ is a Cauchy Sequence in X. But X is a Hilbert space.

$\therefore x_m \longrightarrow x \in X$.

$\therefore \sum_n k_n e_n$ cgs in X.

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Def (orthonormal basis):

An orthonormal set $\{u_\alpha\}$ in a Hilbert space H is said to be an orthonormal basis for H if it is maximal in the sense that if $\{u_\alpha\}$ is contained in some orthonormal subset E of H, then

$E = \{u_\alpha\}$.

Let H be a Hilbert space and $H \neq \{0\}$. Let $\mathcal{C}$ be a family of orthonormal sets in H.

Then $\mathcal{C} \neq \phi$

$\because H \neq \{0\}$, $\exists\ 0 \neq x \in H$.

Then $\left\{\frac{x}{\|x\|}\right\}$ is an orthonormal set in H.

Then $\mathcal{C}$ is a POSET with set inclusion.

Let T be any totally ordered subfamily of $\mathcal{C}$.

Then $\bigcup_{A \in T} A$ is an upper bound for T.

$\therefore$ By Zorn's Lemma, $\mathcal{C}$ has a maximal element, which is called orthonormal basis.

Theorem: Let $\{u_\alpha\}$ be an orthonormal set in an IPS X and $x \in X$.

Let $E_x = \{u_\alpha / \langle x, u_\alpha \rangle \neq 0\}$.

Then E_x is a countable set, say $E_x = \{u_1, u_2, \ldots\}$.

If E_x is denumerable, then
$$\langle x, u_n \rangle \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Further, if X is a Hilbert space, then $\sum_n \langle x, u_n \rangle u_n$ converges to some $y \in H$ such that $x - y \perp u_\alpha$, $\forall \alpha$.

Proof: If $x = 0$, there is nothing to prove.

So let $x \neq 0$. For $j = 1, 2, 3, \ldots$, let
$$E_j = \{ u_\alpha \mid \|x\| \leq j \, |\langle x, u_\alpha \rangle| \}$$

Fix j, suppose that E_j contains

distinct elements $u_{\alpha_1}, u_{\alpha_2}, \ldots u_{\alpha_m}$.

Then

$$\|x\| \le j \, |\langle x, u_{\alpha_1} \rangle|$$

$$\|x\| \le j \, |\langle x, u_{\alpha_2} \rangle|$$

$$\vdots$$

$$\|x\| \le j \, |\langle x, u_{\alpha_m} \rangle|$$

$$\Rightarrow 0 \le m\|x\|^2 \le j^2 \sum_{n=1}^{m} |\langle x, u_{\alpha_n} \rangle|^2$$

$$\le j^2 \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

$$\le j^2 \|x\|^2 \quad \text{(by Bessel's Inequality)}$$

$$\Rightarrow \quad m \le j^2$$

This shows that each E_j contains atmost j^2 elements.

Also since $E_x = \bigcup_j E_j$, we see that E_x is countable.

Also, if $E_x = \{u_1, u_2, u_3, \ldots\}$ is denumerable, then

$$\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2 < \infty.$$

Hence the nth term of this convergence series converges to zero

i.e., $\langle x, u_n\rangle \to 0$ as $n \to \infty$.

Further if X is a Hilbert space, then by Riesz - Fisher theorem

$\sum_n \langle x, u_n\rangle u_n$ converges to some $y \in X$.

Then for any α,

$\langle y, u_\alpha\rangle = \left\langle \sum_n \langle x, u_n\rangle u_n, u_\alpha \right\rangle$

$$= \sum_n \langle x, u_n \rangle \underbrace{\langle u_n, u_\alpha \rangle}_{\delta_{n\alpha}}$$

$$= \langle x, u_\alpha \rangle$$

$$\Rightarrow \langle x-y, u_\alpha \rangle = 0, \quad \forall \alpha$$

$$\Rightarrow x-y \perp u_\alpha \quad \forall \alpha.$$