FUNCTIONAL ANALYSIS

Second Edition

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Preface

This book is intended to serve as a text for a variety of introductory courses in functional analysis, primarily at the Master's level in a mathematics programme. Functional analysis embodies the abstract approach to analysis. It brings out the essence of a problem by clearing out unnecessary details and thus gives a unified treatment of apparently unrelated topics. It highlights the interplay between algebraic structures and distance structures. Since functional analysis provides a major link between mathematics and its applications, scientists and enlightened engineers may also find this book useful. The book is based on lecture courses given by the author mainly at the Indian Institute of Technology Bombay.

Prerequisites. The reader of this book is expected to know set theoretic concepts, elements of linear algebra and rudiments of metric spaces. These topics are reviewed in the first three sections. The fourth section contains a brief treatment of the theory of Lebesgue measure on the real line. While this treatment is not indispensable for understanding the subject matter of the book, it is useful in illustrating various concepts. The book is elementary in the sense that neither general topological spaces nor arbitrary measure spaces are discussed anywhere.

Plan. The general plan of the book is to impose a distance structure on a linear space, exploit it to the fullest and introduce additional features only when one cannot get any further without them. Thus the basic structure of a normed space is introduced in Section 5, continuous linear maps on it are discussed in Section 6 and the important Hahn-Banach theorems are proved in Section 7. Complete normed spaces (that is, Banach spaces) are introduced in Section 8 and the completeness of a norm is exploited to obtain four major theorems, namely the uniform boundedness principle, the closed graph theorem, the open mapping theorem and the bounded inverse theorem in Sections 9, 10 and 11. The consideration of the spectrum of a bounded operator in Section 12 as well as the account of duals and transposes in Section 13 often refer to Banach spaces. While the basic theory of compact operators given in Section 17, 18 and 19 works well on any normed space, the discussion on approximate solutions in Section 20 does need the normed space to be complete. Finally, the geometrically significant structure of an inner product space is introduced in Section 21 and complete inner product spaces (that is, Hilbert spaces) are studied in Sections 22, 23 and 24. Bounded operators on Hilbert spaces are considered in Sections 25 to 28 with special reference to the adjoint operation.

Since some of the readers may be interested only in the richest structure of a Hilbert space, the material in Sections 21 to 28 is kept independent of Sections 5 to 20.

Courses. Depending on the number of hours available for instruction and on the maturity of the audience, several courses can be developed by choosing appropriate sections in the book. Some suggestions:

(i) A course on Banach spaces and operators on them based on Sections 5 to 14 with additional material from Section 15 on weak and weak* convergence, Section 16 on reflexivity, Appendix A on fixed points, or Appendix B on extreme points.

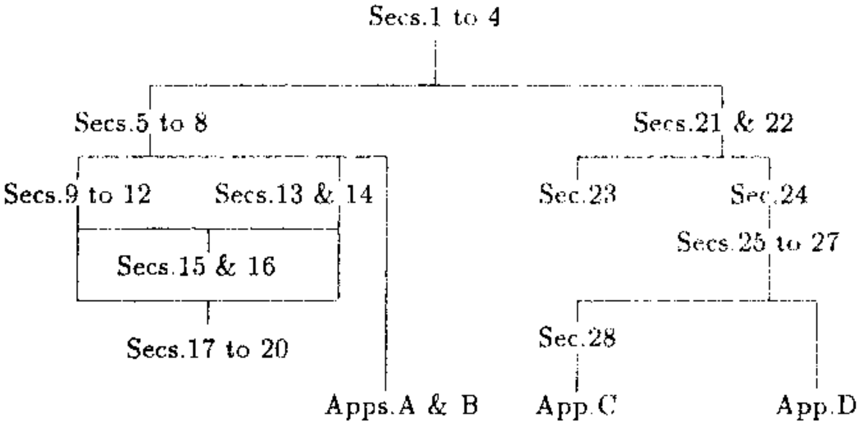
(ii) A course on Hilbert spaces and operators on them based on Sections 21, 22 and 24 to 28 with additional material from Section 23 on approximation and optimization, Appendix C on Sturm-Liouville problems, or Appendix D on unbounded operators and quantum mechanics.

(iii) A course on Banach and Hilbert spaces based on Sections 5 to 8 and 21 to 24 with additional material from Section 23 on approximation and optimization, Appendix A on fixed points, or Appendix B on extreme points.

(iv) A course on bounded operators and compact operators on normed spaces based on Sections 5 to 13 and 17 to 20.

(v) A comprehensive course on Banach and Hilbert spaces, and bounded and compact operators on them based on Sections 5 to 13, 17 to 19, 21, 22 and 24 to 28 with additional material from Sections 14, 15, 16, 20, 23 or Appendices A, B, C, D.

The essential interdependence of various sections and appendices is indicated in the following diagram.



Of course, not everything in a given section need be covered. Judicious choice of material (like selecting only some of the corollaries and one or two applications of an important theorem) is very much called for. Here is how a course should *not* be developed: Include all major theorems in the book and exclude all examples and applications.

Arrangement. The book contains seven chapters and four appendices. Each chapter begins with a brief introduction summarizing its contents, placing them in context and pointing out some novel features. There are four sections in each chapter. A typical section dealing with a major result is organized as follows. It begins with a discussion of a basic feature of the main result. A technical result follows in the form of a lemma. The proof of the main result is accomplished by using this lemma in conjunction with other facts already developed. Two types of examples follow next. One type shows that some of the hypotheses like completeness, finite dimensionality etc. cannot be dropped. The other type shows how in particular cases interesting consequences are obtained when the hypotheses are, in fact, satisfied. Both types of examples are an integral part of that section and sometimes involve numerical calculations. Occasionally, there is a subsection which forms a part of the section by itself and can be omitted if there is not enough time to cover it. The lemmas, theorems, corollaries and examples are numbered for the purpose of cross reference, but the definitions are not. An extensive index at the end of the book can be used to locate the definitions of new terms. A list of symbols which precedes the index may also be helpful. At the end of each section, a long list of problems, based sequentially on the topics covered in that section, is given. All problems are in the form of statements to be established, obviating the need for a separate list of answers. The problems range from the most easy to the very challenging, for which hints are often provided. Results based on these problems are not used later in the text of the book. Thus the reader is not required to solve these problems, although he is strongly urged to attempt as many of them as he can in order to gain insight.

The four appendices at the end of the book are of a different nature as compared to the sections. They point to further areas where the beginnings made in the book can lead. In particular, not all results stated in these appendices are proved and no problems are listed.

Approach. We have in general preferred a geometric approach to an analytic one. It has dictated the kind of proofs given for some major theorems like the Hahn-Banach theorems of Section 7. A few schematic figures, drawn using GLE, are included to help a reader visualize the relevant arguments. Also, the essentially applied approach of constructing a solution, or at least an approximate one, is adopted rather than just proving the existence or the uniqueness of such a solution. For example, see 12.4, Section 20, 23.2, 27.5(b), 28.7.

Applications. Many results proved in the book are applied to diverse areas of mathematics such as classical analysis (generalized limits in 7.12, Fourier series in 9.4, 11.2, 15.5 and 22.8(b), convergence of quadrature formulae in 9.5, summability methods in 9.7, the moment problem of Hausdorff in 14.7, the Fourier-Plancherel transform in 26.6), differential and integral equations (the perturbation technique in Section 10, Fredholm integral equations in 19.3, 19.4 and 28.8(b), Sturm-Liouville problems in Appendix C), probability theory (Helly's selection principle in 15.7), approximation and optimization theory (best approximation in Section 23), fixed point theory (Appendix A), convex programming (Appendix B), as well as to other branches of science such as optimal control theory (quadratic loss control for dynamical systems in 23.6), signal analysis (after 26.6) and quantum mechanics (Appendix D).

Treatment. Several standard books on functional analysis have been consulted to treat specific topics covered in the book. Also, recent research work is cited to indicate the present frontiers of the subject matter, especially of some long-standing open questions.

In order to give a historical perspective, statements of most major results are preceded by the names of mathematicians who discovered them and the years of their discovery.

Coverage. The book contains enough significant material for a first course in functional analysis. Any introductory book has to leave out some of the finest topics. The present book is no exception. For example, the distribution theory, the spectral theory of bounded self-adjoint operators or the theory of Banach algebras do not find a place here. These are all parts of what is known as linear functional analysis. The fast growing subject of nonlinear functional analysis (including calculus in Banach spaces) is far beyond the scope of this book.

Second edition. This edition differs to some extent from the first edition in style as well as in contents. Hopefully, it is more read-able than the first. It gives more motivation and less formulae. The sections in the first edition on the closed graph theorem, the open mapping theorem, the spectrum of a bounded operator, the Fredholm alternative and the integral equations are reorganized. While considering the spectrum of a bounded operator or of a compact operator, the underlying normed space is not required to be complete. The following additions are made: construction of various quadrature formulae in Section 9, relations between the zero spaces and the range spaces of a bounded linear map and its transpose (13.7), the closed range theorem of Banach (13.10), an elementary proof of Eberlein's theorem on reflexivity (16.5), equivalence of boundedness and weak boundedness of a sequence in a Hilbert space (24.8), a discussion of Hilbert-Schmidt operators (28.2) and an entire section on approximate solutions (Section 20). Some results and examples from the first edition are relegated to the problems in the second edition. The lists of problems are now considerably longer. In order to keep the book size manageable, the chapter on the spectral analysis of self-adjoint operators is reluctantly dropped.

Acknowledgments. The book was originally written in 1980 under a project sponsored by the University Grants Commission, New Delhi. During the past fifteen years, several students and teachers have sent me their reactions to the first edition. I am grateful to all of them. The second edition was supported by the Curriculum Development Programme of the Indian Institute of Technology Bombay. Some of my friends, especially M. Thamban Nair and P. Shunmugaraj, have taken great pains to read the new version and suggest improvements. I am indeed indebted to them. I thank C. L. Anthony for processing the manuscript using LaTeX. Sabbatical leave granted by the I. I. T. Bombay and encouragement given by my wife Nirmala were crucial for the completion of this edition.

August 8, 1995 Balmohan V. Limaye

Contents

Chapter I

Preliminaries

This chapter prepares the reader for undertaking a study of functional analysis. Basic notions of set theory are given in Section 1. Linear spaces and metric spaces are introduced in Section 2 and 3, respectively. Korovkin's theorem for positive linear maps is employed to obtain some important approximation results. A review of the theory of Lebesgue measure on the real line is given in Section 4. A brief discussion of Fourier series and integrals concludes this section, which will provide numerous examples of subsequent results.

1 Relations on a Set

As a deductive science, mathematics rests on the theory of sets as its foundation. A 'set' is supposed to be made up of its 'elements'. In other words, certain elements 'belong to' a set while certain others do not. The notation $\in$ and $\notin$ are used to denote 'belongs to' and 'does not belong to'.

It was realized at the turn of this century that not every collection of elements can be allowed to constitute a set without leading to paradoxes. One such famous paradox is due to Russell (1901). Suppose we allow as sets, things T where T could 'belong to' itself. Let S be the set of all things T such that $T \notin T$. Then it is easily seen that $S \in S$ if and only if $S \notin S$, which is clearly paradoxical. To understand this phenomenon better, consider the following real life example. In a town lives a barber who shaves exactly all those who do not shave themselves. The question is whether the barber shaves himself. Answer: The barber shaves himself if and only if he does not shave himself! Here the set S of all those who do not shave themselves is represented by the barber and the question reduces to Russell's paradox.

In order to avoid such paradoxes, some restrictions have to be put on what one should call sets. These are formulated in various axiom systems for the theory of sets. The one that we shall tacitly follow is due to Zermelo and Frankel (1908). (See, for example, [12]). We do not wish to develop such a set theory here. The preceding remarks are made only to put things in a proper perspective.

We shall assume familiarity with the elementary notions in set theory such as a subset ($\subset$), union ($\cup$), intersection ($\cap$), complementation (E^c), and also with finite and infinite sets. A set is called **denumerable** if it is in one to one correspondence with the set of all natural numbers and it is called **countable** if it is either finite or denumerable. We shall denote the empty set by $\emptyset$.

A **relation** on a set X is a subset of the cartesian product $X \times X = \{(x, y) : x, y \in X\}$. We consider three important types of relations on a set. These will be extensively used in the sequel.

A relation R on a set X is called a **function** if $(x, y), (x, z) \in R$ implies that $y = z$. The set $\{x \in X : (x, y) \in R$ for some $y \in X\}$ is called the **domain** of the function and the set $\{y \in X : (x, y) \in R$ for some $x \in X\}$ is called the **range** of the function. Thus a function associates to every element in its domain a unique element in its range. Let now X_1 and X_2 be two sets and $X = X_1 \cup X_2$. If F is a function on X such that the domain of F is X_1 and the range of F is contained in X_2, we shall adopt the usual notation $F : X_1 \to X_2$ with $F(x_1) = x_2$ if and only if $(x_1, x_2) \in F$. If $Y \subset X_1$, then $F_{|Y}$ will denote the **restriction** of F to Y, obtained by restricting x_1 to belong to Y. Also, if $Y \subset X_2$ then $F^{-1}(Y)$ will denote the **inverse image** $\{x_1 \in X : F(x_1) \in Y\}$ of Y under F. If $F(x_1) = F(y_1)$ implies that $x_1 = y_1$, then F is said to be **injective** or **one-to-one**. If for every $x_2 \in X_2$ there is some $x_1 \in X_1$ with $F(x_1) = x_2$, then F is said to be

surjective or F is said to map X_1 **onto** X_2. A function F is said to be **bijective** if it is injective as well as surjective. If F is a bijective and $F(x_1) = x_2$, then $F^{-1}(x_2) = x_1$ defines the **inverse** F^{-1} of F. If $F : X_1 \to X_2$ and $G : X_2 \to X_3$, then the **composition** of F and G is the function $G \circ F : X_1 \to X_3$ given by $(G \circ F)(x_1) = G(F(x_1))$ for $x_1 \in X_1$.

A relation R on a set X is called an **equivalence relation** if (i) R is **reflexive**, that is, $(x, x) \in R$ for every $x \in X$, (ii) R is **symmetric**, that is, $(x, y) \in R$ implies that $(y, x) \in R$ and (iii) R is **transitive**, that is, $(x, y), (y, z) \in R$ implies that $(x, z) \in R$. An equivalence relation R on X partitions the set X into a collection of subsets, each of which is called an **equivalence class**. Two elements x and y are in the same equivalence class if and only if $(x, y) \in R$. We shall write $x \sim y$ if $(x, y) \in R$. The equivalence classes are mutually disjoint and their union is X. For example, the set of all straight lines in the plane can be partitioned into equivalence classes so that two straight lines are equivalent if and only if they are parallel.

A relation R on a set X is called a **partial order** if R is (i) reflexive, (ii) **antisymmetric**, that is, $(x, y), (y, x) \in R$ implies that $x = y$ and (iii) transitive. A partial order R thus differs from an equivalence relation only in symmetry. We shall write $x \leq y$ if $(x, y) \in R$. A **partially ordered set** is a set X together with a partial order on it. For $Y \subset X$ and $x \in X$, if $y \leq x$ for every $y \in Y$, then x is said to be **an upper bound** for Y in X. If $x \leq y$ implies that $x = y$ for every $y \in X$, then x is said to be **maximal** in X. It should be noted that maximal elements may not exist at all or may exist in plenty. Natural numbers with the usual order contain no maximal element. On the other hand, let us consider the set of all branches of a given tree at a given time, and put a partial order on it by letting $b_1 \leq b_2$ if the branch b_2 has grown out of the branch b_1. Then all the most newly formed branches are maximal. The tree in Figure 1 has ten maximal branches. Finally, we define a **totally ordered set** to be a partially ordered set X in which $x, y \in X$ implies that $x \leq y$ or $y \leq x$, that is, any two elements of X are comparable. The natural numbers are totally ordered, but the branches of a tree are not.

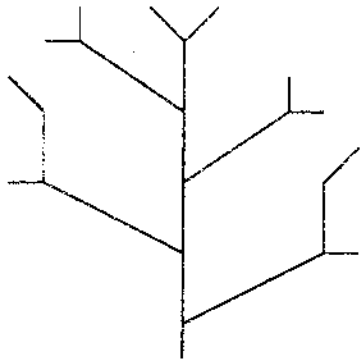


Figure 1

We now state an additional axiom of set theory.

Zorn's lemma
Let X be a nonempty partially ordered set such that every totally ordered subset of X has an upper bound in X. Then X contains a maximal element.

Several remarks are in order. First of all, although this statement is called a lemma, it is an axiom of set theory. In fact, it can be shown to be equivalent to the following.

Axiom of choice
If I is a nonempty set and X_i is a nonempty set for every $i \in I$, then there exists a function $F : I \to \bigcup\{X_i : i \in I\}$ such that $F(i) \in X_i$ for every $i \in I$.

While the selection procedure in the axiom of choice seems to be readily acceptable, it has been proved to be independent of the

Zermelo-Frankel axiom system. It was shown by Gödel (1940) and Cohen (1963) that if the Zermelo-Frankel axiom system is consistent, then it remains consistent together with the axiom of choice and also together with the negation of the axiom of choice.

We shall find life much easier if we assume that the axiom of choice holds. We shall do so by assuming Zorn's lemma to be valid. Of course, it must be pointed out that Zorn's lemma is an existential statement. It asserts the existence of a maximal element in a set under certain conditions, but gives no clue for finding such an element. We are thus at a disadvantage from the point of view of construction procedures. A good account of the constructive methods in analysis (which does not assume Zorn's lemma) can be found in [4].

2 Linear Spaces and Linear Maps

We introduce an algebraic structure on a set X and study functions on X which are well-behaved with respect to this structure.

From now onwards, $\mathbf{K}$ will denote either $\mathbf{R}$, the set of all real numbers or $\mathbf{C}$, the set of all complex numbers. For $k \in \mathbf{C}$, $\operatorname{Re} k$ and $\operatorname{Im} k$ will denote the real part and the imaginary part of k.

A **linear space** (or a **vector space**) over $\mathbf{K}$ is a nonempty set X along with a function $+ : X \times X \to X$, called **addition**, and a function $\cdot : \mathbf{K} \times X \to X$, called **scalar multiplication**, such that for all $x, y, z \in X$ and $k, l \in \mathbf{K}$, we have

$$x + y = y + x,$$

$$x + (y + z) = (x + y) + z,$$

there exists $0 \in X$ such that $x + 0 = x$,

there exists $-x \in X$ such that $x + (-x) = 0$,

$$k \cdot (x + y) = k \cdot x + k \cdot y,$$
$$(k + l) \cdot x = k \cdot x + l \cdot x,$$
$$(kl) \cdot x = k \cdot (l \cdot x),$$
$$1 \cdot x = x.$$

We shall write kx in place of $k \cdot x$. We shall also adopt the following notations. For $x, y \in X, k \in \mathbf{K}$ and subsets E, F of X,

$$x + F = \{x + y : y \in F\}, \quad E + F = \{x + y : x \in E, y \in E\},$$
$$kE = \{kx : x \in E\}.$$

Let X be a linear space (over $\mathbf{K}$). A subset E of X is said to be **convex** if $rx + (1 - r)y \in E$ whenever $x, y \in E$ and $0 < r < 1$. If $E \subset X$, then the smallest convex subset of X containing E is

$$\operatorname{co}(E) = \{r_1 x_1 + \cdots + r_n x_n : x_1, \ldots, x_n \in E, r_1, \ldots, r_n \geq 0, \sum_{j=1}^{n} r_j = 1\}.$$

It is called the **convex hull** of E.

A nonempty subset E of X is said to be a **subspace** of X if $kx + ly \in E$ whenever $x, y \in E$ and $k, l \in \mathbf{K}$. If $\emptyset \neq E \subset X$, then the smallest subspace of X containing E is

$$\operatorname{span} E = \{k_1 x_1 + \cdots + k_n x_n : x_1, \ldots, x_n \in E, k_1, \ldots, k_n \in \mathbf{K}\}$$

It is called the **span** of E. If $\operatorname{span} E = X$, we say that E **spans** X.

A subset E of X is said to be **linearly independent** if for all $x_1, \ldots, x_n \in E$ and $k_1, \ldots, k_n \in \mathbf{K}$, the equation $k_1 x_1 + \cdots + k_n x_n = 0$ implies that $k_1 = \cdots = k_n = 0$. It is called **linearly dependent** if it is not linearly independent, that is, if there exist $x_1, \ldots, x_n \in E$ and $k_1, \ldots, k_n \in \mathbf{K}$ such that $k_1 x_1 + \cdots + k_n x_n = 0$, where at least one k_j is nonzero.

A subset E of X is called a **Hamel basis** or simply a **basis** for X if $\operatorname{span} E = X$ and E is linearly independent. Does every linear space have a basis? Clearly, if x≠{0}, every maximal lineraly independent subset as well as every minimal subset which spans X is a basis for X. But do such subsets of X exist? One may start with a linearly independent subset of X and progressively enlarge it until it spans X, or one may start with a subset which spans X and progressively reduce it until it becomes linearly independent. While it is not at all obvious whether one can obtain a basis for X in this manner, Zorn's lemma guarantees this whenever $X \neq \{0\}$. If, however, $X = \{0\}$, then X has no basis, since the only subsets of X are $\emptyset$ and $\{0\}$, of which $\emptyset$ does not span X and $\{0\}$ is not linearly independent.

2.1 Theorem

Let X be a nonzero linear space. Consider a linearly independent subset L of X and a subset S of X which spans X such that $L \subset S$. Then there exists a basis B for X such that $L \subset B \subset S$.

Proof:

Let $\mathcal{E} = \{E : L \subset E \subset S, E \text{ is linearly independent}\}$. Then $\mathcal{E}$ is nonempty since $L \in \mathcal{E}$. Also, the inclusion $\subset$ is a partial order on $\mathcal{E}$. Further, if $\mathcal{F}$ is a totally ordered subset of $\mathcal{E}$, then $E = \bigcup\{F : F \in \mathcal{F}\}$ is an upper bound for $\mathcal{F}$ in $\mathcal{E}$, because $L \subset E \subset S$ and E is linearly independent since whenever $x_1, \ldots, x_n \in E$, there exists (a linearly independent) $F \in \mathcal{F}$ such that $x_1, \ldots, x_n \in F$ as $\mathcal{F}$ is totally ordered. By Zorn's lemma, $\mathcal{E}$ contains a maximal element B.

Since $B \in \mathcal{E}$, we see that $L \subset B \subset S$ and B is linearly independent. It remains show that B spans X. First we prove that $B \neq \emptyset$. Since $\operatorname{span} S = X \neq \{0\}$, S contains a nonzero element of X. Since B is maximal in $\mathcal{E}$, it follows that $B \neq \emptyset$. Next, we prove that $S \subset \operatorname{span} B$. Suppose for a moment that there is some $x \in S$ with $x \notin \operatorname{span} B$. Then $B \cup \{x\}$ is a linearly independent subset of X. For let $kx + k_1b_1 + \cdots + k_nb_n = 0$ for some $k \in \mathbb{K}$ and $k_1b_1 + \cdots + k_nb_n = y \in \operatorname{span} B$. Hence $kx = -y \in \operatorname{span} B$. Since $x \notin \operatorname{span} B$, we obtain $k = 0$. But then $y = k_1b_1 + \cdots + k_nb_n = 0$ and since B is linearly independent, we obtain $k_1 = \cdots = k_n = 0$. This shows that $B \cup \{x\}$ is an element of $\mathcal{E}$ which is larger than B, contrary to the maximality of B in $\mathcal{E}$. Hence $S \subset \operatorname{span} B$. As $\operatorname{span} S = X$, we have $\operatorname{span} B = X$, as desired. □

2.2 Corollary

Let X be a nonzero linear space.

(a) There exists a basis for X.

(b) Let Y be a subspace of X and C be a basis for Y. Then there is a basis B for X such that $C \subset B$.

Proof:

(a) Let $L = \emptyset$ and $S = X$ in Theorem 2.1.

(b) Let $L = C$ and $S = X$ in Theorem 2.1. □

2.3 Theorem

Let a linear space X have a basis consisting of n elements, $1 \leq n < \infty$. Then every basis for X has n elements. Further, if a subset of n elements in X either spans X or is linearly independent, then it is, in fact, a basis for X.

Proof:

Let $B = \{b_1, \ldots, b_n\}$ be a basis for X consisting of n elements. First, we show that any linearly independent subset of X has at most n elements. Let L be a linearly independent subset of X. If $L \subset B$, then there is nothing to prove. If, on the other hand, there is some $a_1 \in L$ with $a_1 \notin B$, then since the set $\{a_1\}$ is linearly independent and since the set $B \cup \{a_1\}$ spans X, there is a basis B_1 for X such that $\{a_1\} \subset B_1 \subset B \cup \{a_1\}$ by 2.2(*b*). As $a_1 \in X = \operatorname{span} B$ and as B_1 is linearly independent, it follows that $B_1 \neq B \cup \{a_1\}$. Hence at least one element of B does not belong to B_1, so that B_1 has at most n elements. If $L \subset B_1$, we are through. If, one the other hand, there is some $a_2 \in L$ with $a_2 \notin B_1$, then again by 2.2(b), there is a basis

B_2 for X such that $\{a_1, a_2\} \subset B_2 \subset B_1 \cup \{a_2\}$, where at least one element of B_1 does not belong to B_2, so that B_2 also has at most n elements. If $L \subset B_2$, we are through. Otherwise, we may repeat this process. We claim that $L \subset B_k$ for some $k \leq n$, where B_k has at most n elements. For otherwise, there will be $n+1$ elements $a_1, \ldots, a_{n+1}$ in L such that $\{a_1, \ldots, a_n\} \subset B_n$ and $a_{n+1} \notin B_n$. But since B_n has at most n elements, we see that $B_n = \{a_1, \ldots, a_n\}$ and since B_n is a basis for X, a_{n+1} will be a linear combination of $a_1, \ldots, a_n$, contrary to the linear independence of L. Thus L has at most n elements.

Let C be any basis for X. Since C is linearly independent, it has m elements with $m \leq n$. Interchanging the roles of B and C, we see that $n \leq m$. Hence $m = n$, that is, C has the same number of elements as B.

Let now L and S be subsets of X having n elements each such that L is linearly independent and S spans X. By 2.2(*b*), there are bases B_1 and B_2 of X such that $L \subset B_1$ and $B_2 \subset S$. Since B_1 and B_2 must have n elements each, we see that $L = B_1$ and $S = B_2$. Thus L and S are both bases for X. □

If a linear space X has a basis consisting of a finite number of elements, then X is said to be **finite dimensional** and the number of elements in a basis for X is called the **dimension** of X. It follows from Theorem 2.3 that every basis for a finite dimensional linear space has the same (finite) number of elements and hence the dimension is well-defined. The space $\{0\}$ is said to have **zero dimension**. Note that it has no basis !

If a linear space contains an infinite linearly independent subset, then it is said to be **infinite dimensional**. It can be shown by using elaborate cardinality arguments that any two bases for an infinite dimensional linear space are in one to one correspondence. (See [34], Theorem 3.12.) We then define the dimension of X to be the cardinality of a basis for X. It is denoted by dim X.

We shall now give procedures for obtaining new linear spaces from the given ones. A subspace Y of a linear space X is said to be **proper** if $Y \neq X$. A proper subspace Y of X is a linear space along with the induced addition and scalar multiplication. Further, for $x_1, x_2 \in X$, if we let $x_1 \sim x_2$ whenever $x_1 - x_2 \in Y$, then $\sim$ is an equivalence relation on X. Let us denote the equivalence class of $x \in X$ by $x + Y$ and let

$$X/Y = \{x + Y : x \in X\}.$$

For $x_1 + Y, x_2 + Y$ in X/Y and $k \in \mathbf{K}$, define

$$(x_1 + Y) + (x_2 + Y) = (x_1 + x_2) + Y \quad \text{and} \quad k(x_1 + Y) = kx_1 + Y.$$

Then it can be readily seen that X/Y along with these functions is a linear space over $\mathbf{K}$. It is called the **quotient space** of X by Y.

Next, let $X_1, \ldots, X_n$ be linear spaces over $\mathbf{K}$ and consider

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : x_j \in X_j, j = 1, \ldots, n\}.$$

For $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $X_1 \times \cdots \times X_n$ and $k \in \mathbf{K}$, define

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n),$$

and

$$k(x_1, \ldots, x_n) = (kx_1, \ldots, kx_n).$$

Then $X_1 \times \cdots \times X_n$ along with these functions is a linear space over $\mathbf{K}$. It is called the **product space** of $X_1, \ldots, X_n$. The most common example of a product space is $\mathbf{K}^n = \mathbf{K} \times \cdots \times \mathbf{K}$ (n times).

Linear Maps

Let X and Y be linear spaces over $\mathbf{K}$. A **linear map** from X to Y is a function $F : X \to Y$ such that

$$F(k_1x_1 + k_2x_2) = k_1F(x_1) + k_2F(x_2)$$

for all $x_1, x_2 \in X$ and $k_1, k_2 \in \mathbf{K}$. Two important subspaces are associated with such a map. The subspace

$$R(F) = \{y \in Y : F(x) = y \quad \text{for some } x \in X\}$$

of Y is called the **range space** of F. The subspace

$$Z(F) = \{x \in X : F(x) = 0\}$$

of X is called the **zero space** of F. If $Z(F) = X$, we write $F = 0$.

2.4 Theorem

Let X and Y be nonzero linear spaces over $\mathbf{K}$.

(**a**) Let F be a linear map from X to Y. Then F is injective if and only if $Z(F) = \{0\}$, and in that case the inverse function from $R(F)$ to X is linear. Also,

$$\dim X = \dim Z(F) + \dim R(F).$$

(**b**) Let $\dim X = n, 1 \leq n < \infty$. Then there is a bijective linear map from X to Y if and only if $\dim Y = n$. In that case, every injective or surjective linear map from X to Y is, in fact, bijective.

(**c**) Let X_0 be a subspace of X and F_0 be a linear map from X_0 to Y. Then there is a linear map F from X to Y such that $F_{|X_0} = F_0$.

Proof:

(**a**) If F is injective and $F(x) = 0$ for some $x \in X$, then since $F(x) = 0 = F(0)$, we have $x = 0$, that is, $Z(F) = \{0\}$. Conversely, assume that $Z(F) = \{0\}$. If $F(x_1) = F(x_2)$ for some $x_1, x_2 \in X$, then $F(x_1 - x_2) = F(x_1) - F(x_2) = 0$, that is, $x_1 - x_2 \in Z(F)$, so that $x_1 = x_2$. Thus F is injective.

Suppose now that F is injective and let $G : R(F) \to X$ be defined by $G(F(x)) = x$ for all $x \in X$. Then for $x_1, x_2 \in X$ and $k_1, k_2 \in \mathbf{K}$,

$$G(k_1 F(x_1) + k_2 F(x_2)) = G(F(k_1 x_1 + k_2 x_2)) = k_1 x_1 + k_2 x_2,$$

which equals $k_1G(F(x_1)) + k_2G(F(x_2))$. This shows that G is linear.

If $Z(F) = \{0\}$, then $\dim Z(F) = 0$ and F is a bijective linear map from X to $R(F)$, so that $\dim X = \dim Z(F) + \dim R(F)$. Also, if $R(F) = \{0\}$, then $\dim R(F) = 0$ and $Z(F) = X$, so that again $\dim X = \dim Z(F) + \dim R(F)$. Now assume that $Z(F) \neq \{0\}$ and $R(F) \neq \{0\}$. Let L be a basis for $Z(F)$. By 2.2(b), there exists a subset $\{x_t\}$ of X such that $L \bigcup \{x_t\}$ is a basis for X. It is enough to show that $\{F(x_t)\}$ is a basis for $R(F)$. Let $k_1F(x_{t_1}) + \cdots + k_nF(x_{t_n}) = 0$ for some $k_1, \ldots, k_n \in \mathbf{K}$. If $x = k_1x_{t_1} + \cdots + k_nx_{t_n}$, then $x \in \operatorname{span}\{x_t\}$ and since $F(x) = 0$, we also have $x \in Z(F) = \operatorname{span} L$. Since $L \bigcup \{x_t\}$ is linearly independent, it follows that $x = 0$ and in turn $k_1 = \cdots = k_n = 0$. Thus $\{F(x_t)\}$ is a linearly independent set. Next, consider $y \in R(F)$ with $y = F(x)$ for some $x \in X$. Since the set $L \bigcup \{x_t\}$ spans X, we have $x = x_0 + x_1$ for some $x_0 \in \operatorname{span} L$ and $x_1 \in \operatorname{span}\{x_t\}$. Then

$$y = F(x) = F(x_0) + F(x_1) = 0 + F(x_1) \in \operatorname{span}\{F(x_t)\}.$$

Thus $\{F(x_t)\}$ spans $R(F)$, concluding the proof that $\{F(x_t)\}$ is, in fact, a basis for $R(F)$.

(b) Suppose there is a bijective linear map F from X to Y. Since F is injective, $Z(F) = \{0\}$. Hence by (a) above,

$$\dim Y = \dim R(F) = \dim Z(F) + \dim R(F) = \dim X = n.$$

Conversely, let $\dim Y = n$. If $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are bases of X and Y respectively, and we define

$$F(k_1x_1 + \cdots + k_nx_n) = k_1y_1 + \cdots + k_ny_n$$

for $k_1, \ldots, k_n \in \mathbf{K}$, then it is clear that F is linear and bijective.

Let $\dim X = \dim Y = n$. If F is an injective linear map from X to Y, then $Z(F) = \{0\}$, so that

$$\dim R(F) = \dim X - \dim Z(F) = n - 0 = n,$$

and hence $R(F) = Y$, that is, F is surjective as well. Conversely, if F is a surjective linear map from X to Y, then $R(F) = Y$, so that

$$\dim Z(F) = \dim X - \dim R(F) = n - n = 0,$$

so that $Z(F) = \{0\}$, that is, F is injective as well.

(c) If $X_0 = \{0\}$, we let $F = 0$. If $X_0 \neq \{0\}$, let B_0 be a basis for X_0. By 2.2(b), there exists a subset $\{x_t\}$ of X such that $B_0 \bigcup \{x_t\}$ is a basis for X. For $x \in X$ with $x = x_0 + x_1$, $x_0 \in \operatorname{span} B_0$ and $x_1 \in \operatorname{span}\{x_t\}$, define $F(x) = F_0(x_0)$. It is easy to check that $F : X \to Y$ is linear and $F_{|X_0} = F_0$. □

Linear maps from a linear space X to $\mathbf{K}$ (which is a linear space of dimension 1 over $\mathbf{K}$) will be of particular importance to us. They are intimately related to maximal subspaces of X, as we shall presently see. A linear map from X to $\mathbf{K}$ is known as a **linear functional** on X. A maximal proper subspace of X is known as a **hyperspace** in X. Note that a proper subspace Z of X is maximal if and only if the span of $Z \bigcup \{a\}$ equals X for each $a \notin Z$. In this case, for each $x \in X$, there are unique $z \in Z$ and $k \in \mathbf{K}$ such that $x = z + ka$. If Z is a hyperspace in X and $a \in X$, then $a + Z$ is known as a **hyperplane** in X.

The following simple result establishes a connection between the nonzero linear functionals on a linear space and the hyperspaces in it.

2.5 Theorem

Let X be a nonzero linear space.

(a) Let f be a linear functional on X and $f \neq 0$. Then $Z(f)$ is a hyperspace in X. Also, f is determined by $Z(f)$ and the value of f at any one element not in $Z(f)$.

(b) If Z is a hyperspace in X, then there is a linear functional f on X such that $Z = Z(f)$.

Proof:

(a) Since $f \neq 0, Z(f)$ is a proper subspace of X. Let $a \in X$ such that $f(a) \neq 0$. For $x \in X$, consider

$$z = x - \frac{f(x)}{f(a)}a.$$

Then $z \in Z(f)$, so that $\operatorname{span} Z(f) \cup \{a\} = X$. Hence $Z(f)$ is a hyperspace in X. Also, if g is a linear functional on X such that $Z(g) = Z(f)$ and $g(a) = f(a)$, then

$$g(x) = g(z) + \frac{f(x)}{f(a)}g(a) = f(z) + f(x) = f(x).$$

Thus f is determined by $Z(f)$ and $f(a)$.

(b) Let Z be a hyperspace in X and $a \notin Z$. Consider $x \in X$. Then there are unique $z \in Z$ and $k \in \mathbf{K}$ such that $x = z + ka$. Define $f(x) = k$. It is easy to check that f is a linear functional on X and $Z(f) = Z$. □

2.6 Examples

(a) Linear maps on finite dimensional linear spaces

Let X and Y be linear spaces of dimensions n and m respectively, where $1 \leq n, m < \infty$. Consider a basis $\{x_1, \ldots, x_n\}$ for X and a basis $\{y_1, \ldots, y_m\}$ for Y. Let $F : X \to Y$ be a linear map. For each $j = 1, \ldots, n$, there are unique $k_{1,j}, \ldots, k_{m,j}$ in $\mathbf{K}$ such that

$$F(x_j) = k_{1,j}y_1 + \cdots + k_{m,j}y_m.$$

For $x \in X$ with $x = a_1x_1 + \cdots + a_nx_n$, we have

$$F(x) = \sum_{j=1}^{n} a_j F(x_j) = \sum_{j=1}^{n} a_j \left(\sum_{i=1}^{m} k_{i,j} y_i \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_j k_{i,j} \right) y_i.$$

Hence the linear map F is determined by the $m \times n$ matrix $M = (k_{i,j})$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ with respect to the bases $\{x_1, \ldots, x_n\}$

and $\{y_1, \dots y_m\}$ for X and Y. Note that $k_{i,j}$ is the coefficient of y_i in the expression of $F(x_j)$ in terms of $y_1, \dots, y_m$.

Conversely, let $M = (k_{i,j})$ be an $m \times n$ matrix with $k_{i,j} \in \mathbf{K}$. For $x \in X$ with $x = a_1 x_1 + \cdots + a_n x_n$, if we let

$$F(x) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_j k_{i,j} \right) y_i,$$

then it is easy to check F is a linear map from X to Y and it is determined by the matrix M. We then say that the matrix M **defines the** linear map F with respect to the bases $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ for X and Y.

Now $F(x) = 0$ if and only if $\sum_{j=1}^{n} a_j k_{i,j} = 0$ for all $i = 1, \dots, m$. Hence F is injective if and only if the system of m equations

$$\sum_{j=1}^{n} k_{i,j} a_j = 0, \quad i = 1, \dots, m$$

in the n unknowns $a_1, \dots, a_n$ has $(0, \dots, 0)$ as the only solution. When $m = n$, this happens exactly when $\det(k_{i,j}) \neq 0$, that is, when the matrix M is nonsingular.

Suppose that Z is a linear space over $\mathbf{K}, \dim Z = \ell, 1 \le \ell < \infty$ and $\{z_1, \dots, z_\ell\}$ is a basis for Z. Let G be a linear map from Y to Z. If G is determined by the $\ell \times m$ matrix L with respect to the bases $\{y_1, \dots, y_m\}$ and $\{z_1, \dots, z_\ell\}$ for Y and Z, then we see that the linear map $G \circ F$ from X to Z is determined by the $\ell \times n$ matrix LM with respect to the bases $\{x_1, \dots, x_n\}$ and $\{z_1, \dots, z_\ell\}$ for X and Z.

Let now $\{\tilde{x}_1, \dots, \tilde{x}_n\}$ and $\{\tilde{y}_1, \dots, \tilde{y}_m\}$ be another bases for X and Y respectively. Suppose that the linear map F from X to Y is determined by the $m \times n$ matrix $\tilde{M} = (\tilde{k}_{i,j})$ with respect to the new bases $\{\tilde{x}_1, \dots, \tilde{x}_n\}$ and $\{\tilde{y}_1, \dots, \tilde{y}_m\}$ for X and Y. We wish to relate the matrix $\tilde{M}$ to the matrix M. Consider the bijective linear maps $U : X \to X$ and $V : Y \to Y$ given by

$U(k_1 x_1 + \cdots + k_n x_n) = k_1 \tilde{x}_1 + \cdots + k_n \tilde{x}_n, \quad k_1, \dots, k_n \in \mathbf{K},$

$V(h_1 y_1 + \cdots + h_m y_m) = h_1 \tilde{y}_1 + \cdots + h_m \tilde{y}_m, \quad h_1, \dots, h_m \in \mathbf{K},$

and the linear map $\tilde{F} = V^{-1} F U$ from X to Y. Now for $j = 1, \dots, n$, we have

$$F(\tilde{x}_j) = \tilde{k}_{1,j}\tilde{y}_1 + \cdots + \tilde{k}_{m,j}\tilde{y}_m,$$

so that

$$\begin{aligned}\tilde{F}(x_j) &= V^{-1}FU(x_j) = V^{-1}F(\tilde{x}_j) \\ &= \tilde{k}_{1,j}V^{-1}(\tilde{y}_1) + \cdots + \tilde{k}_{m,j}V^{-1}(\tilde{y}_m) \\ &= k_{1,j}y_1 + \cdots + k_{m,j}y_m.\end{aligned}$$

Hence the matrix $\tilde{M}$ defines the map $\tilde{F}$ with respect to the bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ for X and Y. If the map U is determined by the $n \times n$ (nonsingular) matrix P with respect to the old basis $\{x_1, \ldots, x_n\}$ for X and the map V is determined by the $m \times m$ (nonsingular) matrix Q with respect to the old basis $\{y_1, \ldots, y_m\}$ for Y, then the matrix $Q^{-1}MP$ defines the map $V^{-1}FU$ with respect to these bases for X and Y. Since $\tilde{F} = V^{-1}FU$, we see that

$$\tilde{M} = Q^{-1}MP.$$

In particular, if $Y = X$, $y_j = x_j$ and $\tilde{y}_j = \tilde{x}_j$ for $j = 1, \ldots, n$, we obtain the formula for the change of basis:

$$\tilde{M} = P^{-1}MP.$$

The polynomial given by

$$p(t) = \det(M - tI)$$

is known as the **characteristic polynomial** of a linear map F from X to X. It does not depend on the choice of a basis $\{x_1, \ldots, x_n\}$ for X, since if the map F is determined by a matrix $\tilde{M}$ with respect to another basis $\{\tilde{x}_1, \ldots, \tilde{x}_n\}$ for X, then

$$\begin{aligned}\det(\tilde{M} - tI) &= \det((P^{-1}(M - tI)P) \\ &= \det(P^{-1})\det(M - tI)\det(P) \\ &= \det(M - tI).\end{aligned}$$

The simplest example of a linear map from a finite dimensional linear space to a finite dimensional linear space is given by $X = \mathbf{K}^n, Y = \mathbf{K}^m$ and for $x = (x(1), \ldots, x(n))$ in X,

$$F(x) = \begin{bmatrix} k_{1,1} & & k_{1,n} \\ \vdots & & \vdots \\ k_{m,1} & \cdots & k_{m,n} \end{bmatrix} \begin{bmatrix} x(1) \\ \vdots \\ x(n) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n k_{1,j}x(j) \\ \vdots \\ \sum_{j=1}^n k_{m,j}x(j) \end{bmatrix}$$

with $k_{i,j} \in \mathbf{K}$. If $x_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{K}^n$, where 1 occurs only in the jth place and $y_i = (0, \ldots, 0, 1, 0, \ldots 0) \in \mathbf{K}^m$, where 1 occurs only in the ith place, then the map F is determined by the matrix $M = (k_{i,j})$ with respect to the bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ for X and Y.

In particular, if $m = 1$, and for $x \in \mathbf{K}^n$ we let

$$f(x) = k_1 x(1) + \cdots + k_n x(n),$$

where $k_1, \ldots, k_n \in \mathbf{K}$, then f is a linear functional on $\mathbf{K}^n$. In fact, every linear functional on $\mathbf{K}^n$ is of this form. Also, a hyperspace in $\mathbf{K}^n$ is a subspace of dimension $n - 1$ and equals

$$\{(x(1), \ldots, x(n)) \in \mathbf{K}^n : k_1 x(1) + \cdots + k_n x(n) = 0\}$$

for fixed $k_1, \ldots, k_n \in \mathbf{K}$.

(b) Let X denote the linear space of all polynomials in one variable with coefficients in $\mathbf{K}$. For $j = 1, 2, \ldots$ let $x_j(t) = t^{j-1}$. Then $\{x_1, x_2, \ldots\}$ is a basis for X. Let F be a linear map from X to X. Then for each $j = 1, 2, \ldots$, there are unique $k_{1,j}, \ldots, k_{m_j,j}$ in $\mathbf{K}$ such that

$$F(x_j) = k_{1,j}x_1 + \cdots + k_{m_j,j}x_{m_j}.$$

For $x \in X$ with $x = a_1 x_1 + \cdots + a_n x_n$, we have

$$F(x) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_j k_{i,j} \right) x_i,$$

where $m = \max\{m_1, \ldots, m_n\}$ and $k_{i,j} = 0$ if $m_j < i \le m$. Hence the linear map F is determined by the infinite matrix $M = (k_{i,j}), i, j = 1, 2, \ldots$, which is **column-finite**, that is, each of its columns has only a finite number of nonzero entries. (In fact, the jth column has all the entries after the m_jth entry equal to zero.)

Conversely, let $M = (k_{i,j})$ be a column-finite infinite matrix with $k_{i,j} \in \mathbf{K}$. For $x \in X$ with $x = a_1 x_1 + \cdots + a_n x_n$, let

$$F(x) = b_1 x_1 + b_2 x_2 + \cdots$$

with $b_i = \sum_{j=1}^{n} k_{i,j} a_j$ for $i = 1, 2, \ldots$. Since M is column-finite, there exists a positive integer m such that $k_{i,j} = 0$ for all $i > m$ and all $j = 1, \ldots, n$. Hence $b_{m+1} = b_{m+2} = \cdots = 0$, so that $F(x) \in X$. It is easy to check that F is a linear map from X to X.

If we fix $t_0 \in \mathbf{K}$ and let $f(x) = x(t_0)$ for $x \in X$, then f is a linear functional on X. For example, if $t_0 = 1$, then for $x = a_1 x_1 + \cdots + a_n x_n$, we have

$$f(x) = a_1 + \cdots + a_n.$$

(c) Consider the set X of all functions from the interval [0,1] to $\mathbf{K}$. Let $X_1 = \{x \in X : x \text{ is Riemann integrable}\}, Y_1 = \{x \in X : x \text{ is continuous}\}$ and $X_2 = \{x \in X : x \text{ is differentiable}\}$.

Let $F_1 : X_1 \to Y_1$ and $F_2 : X_2 \to X$ be defined by

$$F_1(x)(s) = \int_0^s x(t)\,dt \quad \text{and} \quad F_2(x)(s) = x'(s) \quad \text{for } 0 \le s \le 1.$$

Then F_1 and F_2 are linear maps. For a fixed $s_0 \in [0, 1]$, let

$$f_1(x) = F_1(x)(s_0) \quad \text{and} \quad f_2(x) = F_2(x)(s_0).$$

Then f_1 and f_2 are linear functionals on X_1 and X_2, respectively.

We thus see that many important concepts in analysis like integration and differentiation ca. be treated in a unified manner by considering appropriate linear spaces and linear maps.

3 Metric Spaces and Continuous Functions

We introduce a distance structure on a set X and study functions on X which are well-behaved with respect to this structure.

A **metric** d on a nonempty set X is a function $d : X \times X \to \mathbf{R}$ such that for all $x, y, z \in X$,

$$\begin{aligned} d(x,y) &\geq 0 \quad \text{and} \quad d(x,y) = 0 \text{ if and only if } x = y, \\ d(y,x) &= d(x,y), \\ d(x,y) &\leq d(x,z) + d(z,y). \end{aligned}$$

The last condition is known as the **triangle inequality**. A **metric space** is a nonempty set X along with a metric on it.

As a trivial example of a metric on a nonempty set X, let for $x, y \in X$,

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

This metric is known as the **discrete metric** on X.

Next, let $X = \mathbf{K}^n$, where $n = 1, 2, \ldots$. For $x = (x(1), \ldots, x(n))$ and $y = (y(1), \ldots, y(n))$ in $\mathbf{K}^n$, let

$$d_1(x,y) = \sum_{j=1}^{n} |x(j) - y(j)| \quad \text{and} \quad d_\infty(x,y) = \max_{1 \leq j \leq n} |x(j) - y(j)|.$$

Since $|x(j) - y(j)| \leq |x(j) - z(j)| + |z(j) - y(j)|$ for all $z(j) \in \mathbf{K}$, it is easy to see that d_1 and d_∞ are metrics on $\mathbf{K}^n$.

Also, for $1 < p < \infty$ and $x, y \in \mathbf{K}^n$, let

$$d_p(x,y) = \left(\sum_{j=1}^{n} |x(j) - y(j)|^p \right)^{1/p}$$

To show that d_p is a metric on $\mathbf{K}^n$, we prove the following inequalities.

3.1 Lemma

Let $a_j, b_j \in \mathbf{K}$, $j = 1, \ldots, n$. For $1 < p < \infty$, let q satisfy $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

(a) (Hölder's inequality) For $1 < p < \infty$, we have

$$\sum_{j=1}^{n} |a_j b_j| \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |b_j|^q\right)^{1/q}$$

(b) (Minkowski's inequality) For $1 \leq p < \infty$, we have

$$\left(\sum_{j=1}^{n} |a_j + b_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |b_j|^p\right)^{1/p}$$

Proof:

(a) First we prove an auxiliary result. For $a \geq 0$ and $b \geq 0$, we show that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

If $b = 0$, then this inequality is obvious. Let then $b > 0$. Consider the function

$$x(t) = \frac{1}{q} + \frac{t}{p} - t^{1/p}, \quad t \geq 0.$$

Then $x'(t) = (1 - t^{-1/q})/p$ for all $t \geq 0$. Since $x'(t) < 0$ for all $t < 1$ and $x'(t) > 0$ for all $t > 1$, we obtain $x(t) \geq x(1) = 0$, that is,

$$t^{1/p} \leq \frac{1}{q} + \frac{t}{p}, \quad t \geq 0.$$

Letting $t = a^p/b^q$, we obtain the desired result.

Now consider $\alpha = \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p}$ and $\beta = \left(\sum_{j=1}^{n} |b_j|^q\right)^{1/q}$ If $\alpha = 0$ or $\beta = 0$, then both sides of Hölder's inequality are equal to zero. So assume $\alpha > 0$ and $\beta > 0$. Letting $a = |a_j|/\alpha$ and $b = |b_j|/\beta$ for $j = 1, \ldots, n$ in the auxiliary result proved above, we have

$$\frac{|a_j|}{\alpha}\frac{|b_j|}{\beta} \leq \frac{|a_j|^p}{p\alpha^p} + \frac{|b_j|^q}{q\beta^q}.$$

Hence

$$\begin{aligned}\sum_{j=1}^{n}|a_jb_j| &\leq \alpha\beta\left(\frac{1}{p\alpha^p}\sum_{j=1}^{n}|a_j|^p + \frac{1}{q\beta^q}\sum_{j=1}^{n}|b_j|^q\right)\\ &= \alpha\beta\left(\frac{1}{p}+\frac{1}{q}\right) = \alpha\beta.\end{aligned}$$

We remark that if $p = 2$, then Hölder's inequality can be easily proved as follows:

$$\begin{aligned}\left(\sum_{j=1}^{n}|a_jb_j|\right)^2 &= \sum_{j=1}^{n}|a_jb_j|^2 + 2\sum_{i<j}|a_ib_ia_jb_j|\\ &\leq \sum_{j=1}^{n}|a_jb_j|^2 + \sum_{i<j}(|a_ib_j|^2 + |a_jb_i|^2)\\ &= \left(\sum_{j=1}^{n}|a_j|^2\right)\left(\sum_{j=1}^{n}|b_j|^2\right)\end{aligned}$$

(b) If $p = 1$, then the result is obvious since $|a_j + b_j| \leq |a_j| + |b_j|$ for $j = 1, \ldots, n$. Assume now that $1 < p < \infty$. Then by (a) above,

$$\begin{aligned}\sum_{j=1}^{n}(|a_j| + |b_j|)^p &= \sum_{j=1}^{n}|a_j|(|a_j| + |b_j|)^{p-1} + \sum_{j=1}^{n}|b_j|(|a_j| + |b_j|)^{p-1}\\ &\leq \left(\sum_{j=1}^{n}|a_j|^p\right)^{\frac{1}{p}}\left[\sum_{j=1}^{n}(|a_j| + |b_j|)^{(p-1)q}\right]^{\frac{1}{q}}\\ &\quad + \left(\sum_{j=1}^{n}|b_j|^p\right)^{\frac{1}{p}}\left[\sum_{j=1}^{n}(|a_j| + |b_j|)^{(p-1)q}\right]^{\frac{1}{q}}\\ &= \left[\left(\sum_{j=1}^{n}|a_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n}|b_j|^p\right)^{\frac{1}{p}}\right]\left[\sum_{j=1}^{n}(|a_j| + |b_j|)^p\right]^{\frac{1}{q}}\end{aligned}$$

since $(p-1)q = p$. Again, since $1 - 1/q = 1/p$, we have

$$\left[\sum_{j=1}^{n}(|a_j| + |b_j|)^p\right]^{1/p} \leq \left(\sum_{j=1}^{n}|a_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n}|b_j|^p\right)^{1/p}$$

As $|a_j + b_j| \leq |a_j| + |b_j|$ for all $j = 1, \ldots n$, the desired result follows. □

Coming to the triangle inequality for d_p with $1 < p < \infty$, let $x = (x(1), \ldots, x(n)), y = (y(1), \ldots, y(n)), z = (z(1), \ldots, z(n)) \in \mathbf{K}^n$ and $a_j = x(j) - z(j)$, $b_j = z(j) - y(j)$ for $j = 1, \ldots, n$. By 3.1(b), we have

$$\begin{aligned} d_p(x,y) = \left(\sum_{j=1}^{n} |a_j + b_j|^p\right)^{1/p} &\leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |b_j|^p\right)^{1/p} \\ &= d_p(x,z) + d_p(z,y). \end{aligned}$$

It is easy to see that the other two requirements of a metric are satisfied by d_p.

If $n = 1$, that is, $X = \mathbf{K}$, then all the metrics $d_p, 1 \leq p \leq \infty$, reduce to the **usual metric** given by $d(x,y) = |x - y|$ for $x, y \in \mathbf{K}$.

Next, we generalize these consideration to infinite tuples. For $1 \leq p < \infty$, consider the following set of sequences in $\mathbf{K}$:

$$\ell^p = \left\{(x(1), x(2), \ldots) : x(j) \in \mathbf{K} \text{ and } \sum_{j=1}^{\infty} |x(j)|^p < \infty\right\}.$$

For $x = (x(1), x(2), \ldots)$ and $y = (y(1), y(2), \ldots)$ in ℓ^p, let

$$d_p(x,y) = \left(\sum_{j=1}^{\infty} |x(j) - y(j)|^p\right)^{1/p}.$$

Letting $a_j = x(j)$ and $b_j = -y(j)$ for $j = 1, 2, \ldots$ and letting $n \to \infty$ in Minkowski's inequality, we find that $d_p(x,y) < \infty$ for all x, y in ℓ^p. Further, if $z = (z(1), z(2), \ldots) \in \ell^p$, then letting $a_j = x(j) - z(j)$ and $b_j = z(j) - y(j)$ for $j = 1, 2, \ldots$ and letting $n \to \infty$ in the same inequality, we see that d_p is a metric on ℓ^p.

Finally, consider the set of all bounded sequences in $\mathbf{K}$:

$$\ell^\infty = \{(x(1), x(2), \ldots) : x(j) \in \mathbf{K} \text{ and } \sup_{j=1,2,\ldots} |x(j)| < \infty\}.$$

For $x = (x(1), x(2), \ldots)$ and $y = (y(1), y(2), \ldots)$ in ℓ^∞, let

$$d_\infty(x,y) = \sup_{j=1,2,\ldots} |x(j) - y(j)|.$$

It is immediate that d_∞ is a metric on ℓ^∞.

The metric space ℓ^∞ is a special case of the following example. Consider a set T and let $B(T)$ denote the set of all $\mathbf{K}$-valued bounded functions on T, that is,

$$B(T) = \{x : T \to \mathbf{K}, \sup_{t\in T} |x(t)| < \infty\}.$$

For $x, y \in B(T)$, let

$$d_\infty(x, y) = \sup_{t\in T} |x(t) - y(t)|.$$

Then d_∞ is a metric on $B(T)$, known as the **sup metric**.

Having considered several examples of metric spaces, we shall now discuss some important notions regarding them.

Let d be a metric on X. For $x \in X$ and $r > 0$, the set

$$U_d(x, r) = \{y \in X : d(x, y) < r\}$$

is called the **open ball** about x of radius r. We shall write $U(x, r)$ for $U_d(x, r)$ when there is no ambiguity.

A subset E of X is said to be **bounded** if $E \subset U(x, r)$ for some $x \in X$ and $r > 0$.

A subset E of X is said to be **open** in X if for every $x \in E$, there is some $r > 0$ such that $U(x, r) \subset E$. Clearly, the subsets $\emptyset$ and X of X are open in X. Further, every union of open sets in X and every intersection of a finite number of open sets in X are open in X. Also, a subset of X is open in X if and only if it is a union of open balls in X. If d is the discrete metric, then every subset of X is open. If $X = \mathbf{R}$ and d denotes the usual metric on $\mathbf{R}$, then an open subset in $\mathbf{R}$ is, in fact, a disjoint union of a countable number of open intervals. For if E is open in $\mathbf{R}$, then for $x, y \in E$, let $x \sim y$ whenever there is an open interval $(a, b) = \{s \in \mathbf{R} : a < s < b\}$ such that $\{x, y\} \subset (a, b) \subset E$. Then $\sim$ is an equivalence relation on E. It can be seen that each equivalence class is an open interval in $\mathbf{R}$. Since each nonempty open interval in $\mathbf{R}$ contains a rational number and since the set of all rational numbers is denumerable, it follows that the set of all equivalence classes is countable and E is their disjoint union.

Suppose, that d and d' are two metrics on X. Then d is said to be **stronger** than d' if for every $x \in X$ and every $\epsilon > 0$, there is some $\delta > 0$ such that $U_d(x, \delta) \subset U_{d'}(x, \epsilon)$. This is the case if and only if every open subset of X with respect to d' is also open with respect to d. We say that two metrics are **equivalent** if each is stronger than the other. This is the case if and only if the same subsets of X are open with respect to both. It can be seen that all the metrics d_p, $1 \leq p \leq \infty$, are equivalent on $\mathbf{K}^n$, but the discrete metric is stronger than each of them.

Let $E \subset X$. An element x of X is called an **interior point** of E if there is some $r > 0$ such that $U(x, r) \subset E$. The set of all interior points of E is called the **interior** of E. It will be denoted by E°. Is is the largest open subset of E. Clearly, E is open in X if and only if $E^\circ = E$.

A subset of X is said to be **closed** in X if its complement in X is open in X. Clearly, the subsets X and $\emptyset$ of X are closed in X. Further, every intersection of closed sets in X and every union of a finite number of closed sets in X are closed in X.

Let $E \subset X$. An element $x \in X$ is called a **limit point** (or an **accumulation point**) of E if for every $r > 0$, there is some y in $U(r, x) \bigcap E$ with $y \neq x$. A limit point of E may or may not belong to E. The set of all points and limit points of E is called the **closure** of E. It will be denoted by $\overline{E}$. Thus

$$\overline{E} = \{x \in X : U(x, r) \bigcap E \neq \emptyset \text{ for every } r > 0\}.$$

The closure of E is the smallest closed subset of X containing E. Clearly, E is closed in X if and only if $\overline{E} = E$.

If $Y \subset X$, then d induces a metric on Y in a natural way. A subset F of Y is open (resp., closed) in Y if and only if $F = E \cap Y$ for some E which is open (resp., closed) in X.

A subset E of X is said to be **dense** in X if $\overline{E} = X$. This is the case if and only if $E \cap U(x, r) \neq \emptyset$ for every $x \in X$ and every $r > 0$. A metric space X is called **separable** if it contains a countable dense subset. If X is a separable metric space and $Y \subset X$, then Y is separable in the induced metric. This can be seen as follows. Let $E = \{x_1, x_2, \ldots\}$ be a dense subset of X. If E is contained in Y, then there is nothing to prove. Otherwise, we construct a countable subset of Y whose points are arbitrarily close to the set E. For positive integers n and m, let $U_{n,m} = U(x_n, 1/m)$ and choose $y_{n,m} \in U_{n,m} \cap Y$ whenever it is nonempty. We show that the countable subset $\{y_{n,m}\}$ of Y is dense in Y. Let $y \in Y$ and $r > 0$. Let m be so large that $1/m \leq r/2$ and find $x_n \in U(y, 1/m)$. Then clearly $y \in Y \cap U_{n,m}$ and

$$d(y, y_{n,m}) \leq d(y, x_n) + d(x_n, y_{n,m}) < \frac{1}{m} + \frac{1}{m} \leq \frac{r}{2} + \frac{r}{2} = r.$$

Thus $y_{n,m} \in U(y, r)$. Since $y \in Y$ and $r > 0$ are arbitrary, we see that the set $\{y_{n,m}\}$ is dense in Y.

3.2 Theorem

For $1 \leq p < \infty$, the metric space ℓ^p is separable, but ℓ^∞ is not separable.

Proof:

Let $1 \leq p < \infty$. For $j = 1, 2, \ldots$, let $e_j = (0, \ldots, 0, 1, 0, 0, \ldots)$, where 1 occurs only in the jth place and

$$E = \{k_1 e_1 + \cdots + k_n e_n : n = 1, 2, \ldots, \operatorname{Re} k_j \text{ and } \operatorname{Im} k_j \text{ rational for all } j\}.$$

Since the rational numbers are countable, E is a countable set. We show that E is dense in ℓ^p. Let $x \in \ell^p$ and $r > 0$. As $\sum_{j=1}^{\infty} |x(j)|^p$ is finite, there is some n such that

$$\sum_{j=n+1}^{\infty} |x(j)|^p < \frac{r^p}{2}.$$

Since the rational numbers are dense in $\mathbf{R}$, there are $k_1, \ldots, k_n$ in $\mathbf{K}$ with $\operatorname{Re} k_j$ and $\operatorname{Im} k_j$ rational and

$$|x(j) - k_j|^p < \frac{r^p}{2n}, \quad j = 1, \ldots, n.$$

Consider $y = k_1 e_1 + \ldots + k_n e_n \in E$. Then

$$[d(x,y)]^p = \sum_{j=1}^{n} |x(j) - k_j|^p + \sum_{j=n+1}^{\infty} |x(j)|^p < \frac{r^p}{2} + \frac{r^p}{2} = r^p.$$

Hence $y \in U(x,r)$ and we are through.

On the other hand, ℓ^∞ is not separable. Let $S = \{x \in \ell^\infty : x(j) = 0 \text{ or } 1 \text{ for } j = 1, 2, \ldots\}$. Then $d_\infty(x,y) = 1$ for all $x \neq y$ in S. Let $\{x_1, x_2, \ldots\}$ be any countable subset of ℓ^∞ and $0 < r \leq 1/2$. Then each $U(x_n, r)$ contains at most one element of S for $n = 1, 2, \ldots$. Since S is uncountable, there is some $x \in S$ such that $x \notin U(x_n, r)$ for all $n = 1, 2, \ldots$. In other words, $\{x_1, x_2, \ldots\} \bigcap U(x,r) = \emptyset$, so that the set $\{x_1, x_2, \ldots\}$ cannot be dense in ℓ^∞. $\square$

A **sequence** (x_n) in a set X is a function from the set $\{1, 2, \ldots\}$ to X, the value of the function at n being denoted by x_n. Let d be a metric on X. A sequence (x_n) is said to **converge** in X (with respect to d), if there is some $x \in X$ such that for every $\epsilon > 0$ there is some n_0 with $d(x_n, x) < \epsilon$ for all $n \geq n_0$. It is clear that there is at most one such x and when it exists, we say that (x_n) converges to x in X, write $x_n \to x$ in X or $\lim_{n\to\infty} x_n = x$ and call x the **limit** of (x_n).

If d is the discrete metric on X, then $x_n \to x$ in X if and only if there is some n_0 with $x_n = x_{n_0}$ for all $n \geq n_0$. If $X = \mathbf{R}$ and d is the usual metric, then a sequence (x_n) converges in $\mathbf{R}$ if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x \in \mathbf{R}$ and then $\lim_{n\to\infty} x_n = x$. Recall that if we let for $m = 1, 2, \ldots,$

$$y_m = \sup\{x_m, x_{m+1}, \ldots\} \quad \text{and} \quad z_m = \inf\{x_m, x_{m+1}, \ldots\},$$

then

$$\limsup_{n\to\infty} x_n = \inf\{y_1, y_2, \ldots\} \quad \text{and} \quad \liminf_{n\to\infty} x_n = \sup\{z_1, z_2, \ldots\}.$$

If $X = \mathbf{K}^n$ and $d = d_p, 1 \le p \le \infty$, then $x_n \to x$ in $\mathbf{K}^n$ if and only if $x_n(j) \to x(j)$ in $\mathbf{K}$ for $j = 1, \ldots, n$. On the other hand, if $X = \ell^p$ and $x_n \to x$ in X, then $x_n(j) \to x(j)$ in $\mathbf{K}$ for each $j = 1, 2, \ldots$. However, $(x_n(j))$ may converge in $\mathbf{K}$ for each $j = 1, 2, \ldots$, without (x_n) being convergent in ℓ^p. For example, if $x_n = e_n$ as defined earlier, then $e_n(j) \to 0$ as $n \to \infty$ for each $j = 1, 2, \ldots$, but (e_n) is not convergent in ℓ^p, since if $e_n \to x$, then $x(j) = 0$ for each $j = 1, 2, \ldots$, so that $x = 0$, while $d_p(e_n, 0) = 1$ for each $n = 1, 2, \ldots$. We remark that $x_n \to x$ in $B(T)$ if and only if (x_n) is **uniformly convergent** to x on T, that is, for every $\epsilon > 0$, there is some n_0 such that $|x_n(t) - x(t)| < \epsilon$ for all $t \in T$ and $n \ge n_0$.

If (x_n) is a sequence and $n_1 < n_2 < \cdots$, then (x_{n_m}) is called a **subsequence** of (x_n). It is clear that $x_n \to x$ in X if and only if $x_{n_m} \to x$ for each subsequence (x_{n_m}) of (x_n).

We note that if d and d' are two metrics on X, then d is stronger than d' if and only if whenever $x_n \to x$ in X with respect to d, we have $x_n \to x$ in X with respect to d'. Thus d and d' are equivalent if and only if the same sequences in X are convergent and have the same limits with respect to both.

Sequences can be used to describe the closure of a subset E in a metric space X. In fact, $x \in \overline{E}$ if and only if there is a sequence (x_n) in E such that $x_n \to x$ in X. In particular, E is closed in X if and only if $x_n \to x$ in X and $x_n \in E$ for all n imply that $x \in E$.

Completeness

Let d be a metric on a nonempty set X. A sequence (x_n) in X is said to be **Cauchy** (with respect to d) if for every $\epsilon > 0$, there is some

n_0 such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$. Every Cauchy sequence (x_n) is **bounded**, that is, there is some $x \in X$ and some $\alpha > 0$ such that $d(x_n, x) \leq \alpha$ for all n. However, a bounded sequence in X need not be Cauchy, as the example $X = \mathbf{R}$ with the usual metric and $x_n = (-1)^n, n = 1, 2, \ldots$ shows. Further, if a sequence (x_n) converges in a metric space X, then it is Cauchy. But a Cauchy sequence need not converge in X, as the example $X = (0, 1]$ with the induced usual metric and $x_n = 1/n, n = 1, 2, \ldots$ shows. Here is a simple useful test to check the convergence of a Cauchy sequence. If a Cauchy sequence has a convergent subsequence, then the sequence itself is convergent.

A metric space X is said to be **complete** if every Cauchy sequence in X converges in X. Loosely speaking, we can say that a metric space X is complete if every sequence in X which tries to converge finds some x in X to converge to !

If d is the discrete metric on X and (x_n) is a Cauchy sequence in X, then there is some n_0 such that $x_n = x_{n_0}$ for all $n > n_0$. Hence a discrete metric space is complete. Also, $\mathbf{R}$ with the usual metric is complete. This can be seen as follows. Let (x_n) be a Cauchy sequence in $\mathbf{R}$. Since it is bounded, we have $\limsup_{n \to \infty} x_n = x \in \mathbf{R}$. Then there is a subsequence (x_{n_m}) of (x_n) which converges to x. Hence (x_n) itself converges to x. The completeness of $X = \mathbf{C}$ with the usual metric can be easily deduced from the completeness of $\mathbf{R}$.

If $X = \mathbf{K}^n$ and $d = d_p, 1 \leq p \leq \infty$, then a sequence (x_m) is Cauchy in $\mathbf{K}^n$ if and only if the sequence $(x_m(j))$ is Cauchy in $\mathbf{K}$ for $j = 1, \ldots, n$. It follows that $\mathbf{K}^n$ is complete.

3.3 Theorem

For $1 \leq p \leq \infty$, the metric space ℓ^p is complete.

Proof:

First let $1 \leq p < \infty$ and consider a Cauchy sequence (x_n) in ℓ^p. Let $\epsilon > 0$. There exists n_0 such that $d_p(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$. For

$n, m \geq n_0$ and $i = 1, 2, \ldots$, we have

$$|x_n(i) - x_m(i)| \leq \left(\sum_{j=1}^{i} |x_n(j) - x_m(j)|^p\right)^{1/p} \leq d_p(x_n, x_m) < \epsilon.$$

In particular, $(x_n(i))$ is a Cauchy sequence in $\mathbf{K}$ for each $i = 1, 2, \ldots$. Since $\mathbf{K}$ is complete, let $x_n(i) \to x(i)$ in $\mathbf{K}$ as $n \to \infty$. Keeping $n \geq n_0$ fixed and letting $m \to \infty$ in the inequality given above, we have

$$\left(\sum_{j=1}^{i} |x_n(j) - x(j)|^p\right)^{1/p} \leq \epsilon, \quad i = 1, 2, \ldots.$$

Let $x = (x(1), x(2), \ldots)$. Considering $a_j = x(j) - x_{n_0}(j)$ and $b_j = x_{n_0}(j)$ in Minkowski's inequality (3.1(b)), we see that

$$\left(\sum_{j=1}^{i} |x(j)|^p\right)^{1/p} \leq \epsilon + \left(\sum_{j=1}^{\infty} |x_{n_0}(j)|^p\right)^{1/p}$$

for $i = 1, 2, \ldots$. Hence $x \in \ell^p$. Finally, letting $i \to \infty$ in the second last inequality, we obtain

$$d_p(x_n, x) = \left(\sum_{j=1}^{\infty} |x_n(j) - x(j)|^p\right)^{1/p} \leq \epsilon$$

for all $n \geq n_0$. Hence $x_n \to x$ in ℓ^p. Thus ℓ^p is complete for $1 \leq p < \infty$.

Next, consider a Cauchy sequence (x_n) in ℓ^∞. Let $\epsilon > 0$. There is some n_0 such that for all $n, m \geq n_0$ and $j = 1, 2, \ldots$, we have

$$|x_n(j) - x_m(j)| \leq d_\infty(x_n, x_m) = \sup_{j=1,2,\ldots} |x_n(j) - x_m(j)| < \epsilon.$$

In particular, $(x_n(j))$ is a Cauchy sequence in $\mathbf{K}$ for each $j = 1, 2, \ldots$. Let $x_n(j) \to x(j)$ in $\mathbf{K}$ as $n \to \infty$. Keeping $n \geq n_0$ fixed and letting $m \to \infty$ in the last inequality, we have

$$d_\infty(x_n, x) = \sup_{j=1,2,\ldots} |x_n(j) - x(j)| \leq \epsilon$$

for all $n \geq n_0$. Hence $x_n \to x$ in ℓ^∞. Thus ℓ^∞ is complete. $\square$

In a similar manner, it follows that if T is a set, then the metric space $B(T)$ of all $\mathbf{K}$-valued bounded functions on T along with the sup metric is complete.

We remark that the property of completeness of a metric may not be shared by an equivalent metric. For example, let d denote the usual metric on $(0,1]$ and d' be defined by $d'(x,y) = |\frac{1}{x} - \frac{1}{y}|$ for $x, y \in (0,1]$. Then $x_n \to x$ in $(0,1]$ with respect to d' if and only if $x_n \to x$ in $(0,1]$ with respect to d, so that d and d' are equivalent metrics on $(0,1]$. However, $(0,1]$ is complete with respect to d' but not with respect to d.

If X is a metric space and $E \subset X$, then the closedness of E in X and the completeness of E in the induced metric are related in the following manner. If E is complete, then E is closed in X. Conversely, if X is complete and E is closed in X, then E is complete.

We now prove a very useful result.

3.4 Theorem (Baire, 1899)

Let X be a metric space. Then the intersection of a finite number of dense open subsets of X is dense in X.

If X is complete, then the intersection of a countable number of dense open subsets of X is dense in X.

Proof:

Let $D_1, D_2, \ldots$ be dense open subsets of X. For $x_0 \in X$ and $r_0 > 0$, consider $U_0 = U(x_0, r_0)$. As D_1 is dense in X, let $x_1 \in D_1 \bigcap U_0$. As $D_1 \bigcap U_0$ is open in X, there is some $r_1 > 0$ such that $U_1 = U(x_1, r_1)$ is contained in $D_1 \bigcap U_0$. Proceeding inductively, suppose $U_{n-1} = U(x_{n-1}, r_{n-1})$ and $U_n = U(x_n, r_n)$ are such that $U_n \subset D_n \bigcap U_{n-1}$. As D_{n+1} is dense in X, let $x_{n+1} \in D_{n+1} \bigcap U_n$. As $D_{n+1} \bigcap U_n$ is open in X, there is some $r_{n+1} > 0$ such that $U_{n+1} = U(x_{n+1}, r_{n+1})$ is contained in $D_{n+1} \bigcap U_n$. Thus there are $x_1, x_2, \ldots$ in X and positive numbers $r_1, r_2, \ldots$ such that $U_m = U(x_m, r_m)$ is contained in $D_m \bigcap U_{m-1}$

for $m = 1, 2, \ldots$. Clearly, for a given $n = 1, 2, \ldots, x_n$ belongs to $(\bigcap_{m=1}^{n} D_m) \bigcap U_0$ and hence it is nonempty. Since $x_0 \in X$ and $r_0 > 0$ are arbitrary, we see that $\bigcap_{m=1}^{n} D_m$ is dense in X.

Let now X be complete. We find a sequence (x_m) in X and a sequence (r_m) of positive numbers as above and by decreasing r_m, if necessary, are can assume that $r_m \le 1/m$ as well as $\overline{U}_m \subset D_m \bigcap U_{m-1}$ for $m = 1, 2, \ldots$. Now fix a positive integer m. If $n, j \ge m$, then $x_n, x_j \in U_m = U(x_m, r_m)$, so that

$$d(x_n, x_j) \le d(x_n, x_m) + d(x_m, x_j) < \frac{2}{r_m} \le \frac{2}{m}.$$

Hence the sequence (x_n) is Cauchy in X. Since X is complete, let $x_n \to x$ in X. But since $x_n \in U_m$ for all $n \ge m$, it follows that $x \in \overline{U}_m$. As $\overline{U}_m \subset D_m \bigcap U_0$ for all $m = 1, 2, \ldots$, we see that x belongs to $(\bigcap_{m=1}^{\infty} D_m) \bigcap U_0$ and hence it is nonempty. Again, since $x_0 \in X$ and $r_0 > 0$ are arbitrary, we see that $\bigcap_{m=1}^{\infty} D_m$ is dense in X. $\square$

We remark that in a noncomplete metric space, the intersection of a denumerable number of dense open subsets need not be dense. In fact, it may be empty, as the following example shows. Let $X = \{q_1, q_2, \ldots\}$ denote the set of all rational numbers along with the metric induced by the usual metric on $\mathbf{R}$. If D_m denotes the complement of $\{q_m\}$ in X, then D_m is dense and open in X for each $m = 1, 2, \ldots$, but $\bigcap_{m=1}^{\infty} D_m = \emptyset$.

Baire's theorem implies that in a complete metric space, a countable intersection of dense open subsets is nonempty. Hence if a complete metric space is a countable union of its subsets, then the closure of at least one such subset must have nonempty interior.

Compactness

The concept of compactness is a useful and natural generalization of finiteness. A subset E of a metric space X is said to be **compact** (relative to X) if every cover of E by open subsets of X has a finite subcover. It is easy to see that if $E \subset Y \subset X$, then E is compact relative to Y if and only if E is compact relative to X. Hence we can talk about E being a compact metric space in its own right. Compactness can be described in terms of closed subsets as follows. A family of subsets is said to have the **finite intersection property** if every finite subfamily has a nonempty intersection. Then a metric space X is compact if and only if every family of closed subsets of X having the finite intersection property has itself a nonempty intersection.

To characterize the compactness of a metric space in terms of sequences and to relate compactness to completeness, we introduce the following concept. A subset E of a metric space X is said to be **totally bounded** if for every $\epsilon > 0$, there are $x_1, \ldots, x_n$ in X such that $E \subset U(x_1, \epsilon) \bigcup \cdots \bigcup U(x_n, \epsilon)$. If E is totally bounded, then we can, in fact, find such $x_1, \ldots, x_n$ in E itself. It is easy to see that every subset of a totally bounded set is totally bounded and so is its closure. Note that every totally bounded set is bounded, but a bounded set need not be totally bounded. For example, let $X = \mathbf{R}$ with the metric d given by $d(x, y) = \min\{1, |x - y|\}$ for $x, y \in \mathbf{R}$. Then X is clearly bounded, but not totally bounded since for $0 < \epsilon < 1$ and $x_1, \ldots, x_n \in \mathbf{R}$, $1 + |x_1| + \ldots + |x_n| \not\in U_d(x_1, \epsilon) \bigcup \ldots \bigcup U_d(x_n, \epsilon)$. However, if $X = \mathbf{K}^n$ with the metric d_p, $1 \le p \le \infty$, then it can be shown that every bounded subset of $\mathbf{K}^n$ is, in fact, totally bounded.

3.5 Theorem

Let X be a metric space. The following conditions are equivalent.

(i) X is compact.

(ii) Every sequence in X has a convergent subsequence.

(iii) X is complete and totally bounded.

Proof:

(i) implies (ii): Let (x_n) be a sequence in a compact metric space X. If no subsequence of (x_n) converges in X, then for each $x \in X$, there is some $r_x > 0$ and a positive integer n_x such that $x_n \notin U(x, r_x)$ for all $n \geq n_x$. Since $X = \bigcup\{U(x, r_x) : x \in X\}$ and X is compact, there are $x_1, \ldots, x_m \in X$ such that $X = U(x_1, r_{x_1}) \bigcup \ldots \bigcup U(x_m, r_{x_m})$. But if $n_0 = \max\{n_{x_1}, \ldots, n_{x_m}\}$, then $x_{n_0} \notin U(x_j, r_{x_j})$, $j = 1, \ldots, m$, so that $x_{n_0} \notin X$, which is impossible. Hence (x_n) has a convergent subsequence.

(ii) implies (iii): Suppose that every sequence in X has a convergent subsequence. Since a Cauchy sequence having a convergent subsequence is itself convergent, we see that X is complete. Next, assume for a moment that X is not totally bounded. Then there is some $\epsilon > 0$ such that X cannot be covered by finitely many open balls of radius ϵ. Let $x_1 \in X$. Having chosen $x_1, \ldots, x_n \in X$ inductively, find $x_{n+1} \in X$ such that $x_{n+1} \notin U(x_1, \epsilon) \bigcup \ldots \bigcup U(x_n, \epsilon)$, $n = 1, 2, \ldots$. Then $d(x_n, x_m) \geq \epsilon$ for all $n, m = 1, 2, \ldots$, so that (x_n) cannot have a convergent subsequence, contrary to our assumption. Hence X is totally bounded.

(iii) implies (i): Suppose that X is complete and totally bounded. Assume for a moment that X is not compact. Consider an open cover of X without any finite subcover. Since X is totally bounded, cover X by finitely many open balls of radius 1. Then for at least one of these open balls of radius 1, say $U_0 = U(x_0, 1)$, there is no finite subcover from the given open cover. Being a subset of the totally bounded metric space X, U_0 is itself totally bounded. As before, there is some $x_1 \in U_0$ such that $U_1 = U(x_1, 1/2)$ has no finite subcover from the given open cover of X. Proceeding inductively, we find a sequence (x_n) in X with $x_{n+1} \in U_n = U(x_n, 1/2^n)$ such that U_n has no finite subcover from the given open cover of X. Since

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \leq \sum_{j=n}^{m-1} \frac{1}{2^j} \leq \frac{1}{2^{n-1}}$$

for all $m > n$, we see that (x_n) is a Cauchy sequence in the complete metric space X. Let $x_n \to x$ in X. Then x belongs to an open set E

of the given open cover of X. Let $r > 0$ be such that $U = U(x, r) \subset E$ and choose n so large that $d(x_n, x) < r/2$ as well as $1/2^n < r/2$. Now if $y \in X$ and $d(x_n, y) < 1/2^n$, then

$$d(x, y) \le d(x, x_n) + d(x_n, y) < \frac{r}{2} + \frac{1}{2^n} < r,$$

so that $U_n \subset U \subset E$. Thus $\{E\}$ constitutes a finite subcover for U_n, in contradiction to our construction of U_n. □

3.6 Corollary

Let X be a complete metric space and $E \subset X$.

(a) E is compact if and only if E is closed and totally bounded.

(b) $\overline{E}$ is compact if and only if E is totally bounded.

Proof:

Obvious from 3.5. □

3.7 Corollary

For $1 \le p \le \infty$, consider the metric d_p on $\mathbf{K}^n$.

(a) **(Heine-Borel)** A subset of $\mathbf{K}^n$ is compact if and only if it is closed and bounded.

(b) **(Bolzano-Weierstrass)** Every bounded sequence in $\mathbf{K}^n$ has a convergent subsequence.

Proof:

Since a bounded subset of $\mathbf{K}^n$ is totally bounded, part (a) follows from 3.6(a) and part (b) follows from 3.6(b) and 3.5(ii). □

Continuous Functions

Roughly speaking, a function from a metric space to a metric space is continuous if it sends 'nearby' points to 'nearby' points. If X and

Y are metric spaces with metrics d and e respectively, then a function $F : X \to Y$ is said to be **continuous** at $x_0 \in X$ if for every $\epsilon > 0$, there is some $\delta > 0$ (possibly depending on ϵ and x_0) such that $e(F(x), F(x_0)) < \epsilon$ for all $x \in X$ satisfying $d(x, x_0) < \delta$. Further, F is said to be continuous on X if it is continuous at every point of X. It is easy to see that F is continuous on X if and only if the set $F^{-1}(E)$ is open in X whenever the set E is open in Y. Also, this happens if and only if $F(x_n) \to F(x)$ in Y whenever $x_n \to x$ in X.

We remark that the inverse of a bijective continuous function need not be continuous. (Compare 2.4(a).) As an example, we may consider $X = [0, 2\pi)$, $Y = \{z \in \mathbf{C} : |z| = 1\}$ and $F(t) = \exp(it)$ for $t \in [0, 2\pi)$.

If $F : X \to Y$ is an injective continuous function such that $F^{-1} : R(F) \to X$ is also continuous, then F is called a **homeomorphism**. If there is a homeomorphism from X onto Y, then X and Y are said to be **homeomorphic**.

We give some examples of homeomorphic metric spaces.

1) Let $X = \mathbf{R}$, $Y = (-1, 1)$ and $F(t) = t/(1 + |t|)$ or $F(t) = \dfrac{2}{\pi}\arctan t$ for $t \in \mathbf{R}$.

2) Any X with a metric d, $Y = X$ with an equivalent metric d' and $F(x) = x$ for $x \in X$.

If $F : X \to Y$ satisfies $d(x, x') = e(F(x), F(x'))$ for all $x, x' \in X$, then F is called an **isometry**. If there is an isometry from X onto Y, then X and Y are said to be **isometric**. An isometry is clearly a homeomorphism, but a homeomorphism may not be an isometry as the preceding examples show. Isometric spaces share the same 'metric' properties. This is not so for homeomorphic spaces. For example, $\mathbf{R}$ with the usual metric is a complete metric space which is not totally bounded, while the interval (-1, 1) is a totally bounded metric space which is not complete.

For a nonempty subset E of a metric space X and $x \in E$, we define the **distance** of x from E by

$$\operatorname{dist}(x,E)=\inf\{d(x,y): y\in E\}.$$

Let $d_E(x)=\operatorname{dist}(x,E)$. Since $|d_E(x)-d_E(x')|\le d(x,x')$ for all x,x' in X, it follows that $d_E: X\to \mathbf{R}$ is a continuous function. Note that $d_E(x)=0$ for every $x\in E$, and if E is closed, then $d_E(x)\neq 0$ for every $x\notin E$. This function enables us to give a quick proof of an important separation property of metric spaces.

3.8 Urysohn's lemma (Urysohn, 1925)

Let E_1 and E_2 be disjoint nonempty closed subsets of a metric space X. Then there is a continuous function f from X to $[0,1]$ such that $f_{|E_1}=0$ and $f_{|E_2}=1$.

Proof:
For $x\in X$, let

$$f(x)=\frac{\operatorname{dist}(x,E_1)}{\operatorname{dist}(x,E_1)+\operatorname{dist}(x,E_2)}.\qquad\square$$

3.9 Tietze extension theorem (Tietze, 1915)

Let E be a closed nonempty subset of a metric space X and g be a continuous function from E to $\mathbf{K}$. Then there exists a continuous function f from X to $\mathbf{K}$ such that $f_{|E}=g$. Further, if $|g(x)|\le 1$ for all $x\in E$, then f can be so chosen that $|f(x)|\le 1$ for all $x\in X$.

Sketch of proof:
First assume that $g: E\to[1,2]$ and define

$$f(x)=\begin{cases} g(x), & \text{if } x\in E\\ \inf\{g(y)d(x,y): y\in E\}/\operatorname{dist}(x,E), & \text{if } x\notin E.\end{cases}$$

Then $f: X\to[1,2]$ is a continuous function and $f_{|E}=g$. (See [17], 4.5.1.)

Next, let $g : E \to [-1,1]$. Since the interval $[-1,1]$ is homeomorphic to the interval [1,2], there is a continuous function $f : X \to [-1,1]$ such that $f_{|E} = g$.

Now let $g : E \to \mathbf{R}$. Consider a homeomorphism F from $\mathbf{R}$ onto $(-1,1)$ and let $g_1 = F \circ g$. Since $g_1 : E \to (-1,1) \subset [-1,1]$ is a continuous function, there is a continuous function $f_1 : X \to [-1,1]$ such that $f_{1|E} = g_1$. Let $E_1 = \{x \in X : |f_1(x)| = 1\}$. If $E_1 = \emptyset$, then let $f = F^{-1} \circ f_1$. If $E_1 \neq \emptyset$, we proceed as follows. By 3.8, we find a continuous function $h : X \to [0,1]$ such that $h_{|E_1} = 0$ and $h_{|E} = 1$. Let $h_1 = hf_1$. Note that $h_1 : X \to (-1,1)$ is a continuous function and $h_{1|E} = g_1$. Let $f = F^{-1} \circ h_1$. Then $f : X \to \mathbf{R}$ is a continuous function and $f_{|E} = F^{-1} \circ g_1 = g$, as desired.

Finally, let $g : E \to \mathbf{C}$. Since $\operatorname{Re} g$ and $\operatorname{Im} g$ are continuous functions from E to $\mathbf{R}$, there are continuous functions f_1 and f_2 from X to $\mathbf{R}$ such that $f_{1|E} = \operatorname{Re} g$ and $f_{2|E} = \operatorname{Im} g$. Let $f = f_1 + i f_2$. Then $f : X \to \mathbf{C}$ is a continuous function and $f_{|E} = g$. Suppose now that $|g(x)| \leq 1$ for all $x \in E$. Consider a homeomorphism F from $\{z \in \mathbf{C} : |z| \leq 1\}$ onto $\{z \in \mathbf{C} : -1 \leq \operatorname{Re} z, \operatorname{Im} z \leq 1\}$. Since $\operatorname{Re} F \circ g$ and $\operatorname{Im} F \circ g$ are continuous functions from E to $[-1,1]$, there are continuous functions h_1 and h_2 from X to $[-1,1]$ such that $h_{1|E} = \operatorname{Re} F \circ g$ and $h_{2|E} = \operatorname{Im} F \circ g$. Let $\tilde{f} = F^{-1} \circ (h_1 + i h_2)$. Then $\tilde{f} : X \to \mathbf{C}$ is a continuous function, $\tilde{f}_{|E} = F^{-1} \circ (\operatorname{Re} F \circ g + i \operatorname{Im} F \circ g) = g$ and $|\tilde{f}(x)| \leq 1$ for all $x \in X$, as desired. □

Let us now consider a metric d on a set T and let

$$C(T) = \{x \in B(T) : x \text{ continuous on } T\}.$$

As we have remarked after the proof of 3.3, $B(T)$ is complete in the sup metric

$$d_\infty(x,y) = \sup_{t \in T} |x(t) - y(t)|, \quad x, y \in B(T).$$

We show that the subspace $C(T)$ is closed in $B(T)$. Consider a sequence (x_n) in $C(T)$ such that $x_n \to x$ in $B(T)$. Let $t_0 \in T$ and $\epsilon > 0$.

There is some n_0 such that $d_\infty(x_{n_0}, x) < \epsilon/3$. Since x_{n_0} is continuous at t_0, there is some $\delta > 0$ such that $|x_{n_0}(t) - x_{n_0}(t_0)| < \epsilon/3$ for all $t \in T$ with $d(t, t_0) < \delta$. Then for all such t, we have

$$\begin{aligned} |x(t) - x(t_0)| &\leq |(x - x_{n_0})(t)| + |x_{n_0}(t) - x_{n_0}(t_0)| + |(x_{n_0} - x)(t_0)| \\ &\leq d_\infty(x, x_{n_0}) + \frac{\epsilon}{3} + d_\infty(x_{n_0}, x) < \epsilon. \end{aligned}$$

Hence x is continuous at t_0. Since $t_0 \in T$ is arbitrary, $x \in C(T)$. Thus a uniform limit of a sequence of continuous functions is continuous. This shows that $C(T)$ is closed in the complete metric space $B(T)$. As a result, $C(T)$ is a complete metric space.

We now assume that T is a compact metric space. If S is a metric space and $F : T \to S$ is continuous, then it is easy to see that the range $R(F)$ of F is a compact subset of S. In particular, a continuous function sends each closed subset of T to a closed subset of S. If, in addition, F is injective, then the inverse function from $R(F)$ to T is also continuous, so that F is a homeomorphism.

We note that a continuous function $F : T \to S$ is, in fact, **uniformly continuous**, that is, for every $\epsilon > 0$, there exists some $\delta > 0$ such that $e(F(t), F(t')) < \epsilon$ whenever $d(t, t') < \delta$. This can be seen as follows. Let $t \in T$. By the continuity of F at $t \in T$, there is some δ_t, such that $e(F(t), F(t')) < \epsilon/2$ whenever $d(t, t') < \delta_t$. Let $U_t = U(t, \delta_t/2)$. Then $\bigcup\{U_t : t \in T\}$ is an open cover of T. By the compactness of T, there is a finite subcover $U_{t_1}, \ldots, U_{t_n}$ of T. Then $\delta = \min\{\delta_{t_1}/2, \ldots, \delta_{t_m}/2\}$ works.

Since every $\mathbf{K}$-valued continuous function on T is bounded, we remark that $C(T)$ is just the set of all continuous $\mathbf{K}$-valued functions on T. A subset E of $C(T)$ is bounded in the sup metric if and only if there is some $\alpha > 0$ such that $|x(t)| \leq \alpha$ for all $x \in E$ and all $t \in T$, that is, the functions in E are **uniformly bounded** on T. We say that a subset E of $C(T)$ is bounded at $t \in T$ if there is some $\alpha_t > 0$ such that $|x(t)| \leq \alpha_t$ for all $x \in E$. Clearly, if E is uniformly bounded, then it is bounded at each $t \in T$. The converse holds under a special condition which we now introduce. A subset E of $C(T)$ is said to be

equicontinuous at $t \in T$ if for every $\epsilon > 0$, there is some $\delta > 0$ such that $|x(t) - x(s)| < \epsilon$ for all $x \in E$ and $s \in T$ with $d(s,t) < \delta$, where δ may depend on t, but not on $x \in E$.

3.10 Theorem
Let T be a compact metric space and $E \subset C(T)$. Suppose that E is bounded as well as equicontinuous at each $t \in T$. Then

(a) (**Ascoli, 1883**) E is uniformly bounded on T. In fact, E is totally bounded in the sup metric on $C(T)$.

(b) (**Arzela, 1889**) Every sequence in E contains a uniformly convergent subsequence.

Proof:

(**a**) Let $\epsilon > 0$. Since E is equicontinuous at each $t \in T$ and since T is compact, there are $t_1, \ldots, t_n$ in T and positive numbers $\delta_1, \ldots, \delta_n$ such that $T = \bigcup\{U(t_i, \delta_i) : i = 1, \ldots, n\}$, and $|x(t) - x(t_i)| < \epsilon$ for all $x \in E$ and all $t \in U(t_i, \delta_i)$. Since E is bounded at each $t \in T$, there are $\alpha_1, \ldots, \alpha_n$ such that $|x(t_i)| \leq \alpha_i$ for all $x \in E$. Let $\alpha = \max\{\alpha_1, \ldots, \alpha_n\} + \epsilon$. Then it follows that $|x(t)| \leq \alpha$ for all $x \in E$ and all $t \in T$. Thus E is uniformly bounded on T.

Next, let $\mathbf{K}_\alpha = \{k \in \mathbf{K} : |k| \leq \alpha\}$ and for $x \in E$, define $e(x) = (x(t_1), \ldots, x(t_n)) \in \mathbf{K}_\alpha^n$. It is easy to see that $\mathbf{K}_\alpha^n$ is totally bounded. Hence we can cover it by a finite union of open balls of radius ϵ, say $V_1, \ldots, V_m$. If $j = 1, \ldots, m$ and $V_j \bigcap \{e(x) : x \in E\} \neq \emptyset$, choose $x_j \in E$ such that $e(x_j) \in V_j$. We show that E is the union of open balls of radius 5ϵ about these x_j's. Let $x \in E$. Then $e(x) \in V_j$ for some $j = 1, \ldots, m$. Since $e(x_j) \in V_j$ and the radius of V_j is ϵ, we see that $|x(t_i) - x_j(t_i)| < 2\epsilon$ for all $i = 1, \ldots, n$. Now each $t \in T$ belongs to some $U(t_i, \delta_i), i = 1, \ldots, n$ and then

$$\begin{aligned}
|x(t) - x_j(t)| &\leq |x(t) - x(t_i)| + |x(t_i) - x_j(t_i)| + |x_j(t_i) - x_j(t)| \\
&< \epsilon + 2\epsilon + \epsilon,
\end{aligned}$$

so that $d_\infty(x, x_j) = \sup_{t \in T} |x(t) - x_j(t)| \le 4\epsilon < 5\epsilon$. This proves that E is totally bounded in the sup metric d_∞.

(b) We have noted that $C(T)$ is a complete metric space. By (a) above and by 3.6(b), $\overline{E}$ is compact. Now Theorem 3.5 implies that every sequence in E contains a subsequence which converges in $\overline{E}$ and hence in $C(T)$ with respect to the sup metric, that is, it is uniformly convergent on T. □

Before we conclude this long section, we consider an approximation result for $C([a, b])$. It is both startling and useful. For real-valued continuous functions x and y on $[a, b]$, we shall write $x \le y$ if $x(t) \le y(t)$ for all $t \in [a, b]$. A function $P : C([a, b]) \to C([a, , b])$ is said to be **positive** if $P(x) \ge 0$ whenever $x \ge 0$. Note that if P is positive and also linear, then $P(x) \le P(y)$ whenever $x \le y$.

3.11 Theorem (Korovkin, 1953)
Consider $x_0(t) = 1, x_1(t) = t, x_2(t) = t^2$ for $t \in [a, b]$. For $n = 1, 2, \ldots,$ let $P_n : C([a, b]) \to C([a, b])$ be a positive linear map. If $P_n(x_j) \to x_j$ uniformly on $[a, b]$ for $j = 0, 1, 2$, then $P_n(x) \to x$ uniformly on $[a, b]$ for every x in $C([a, b])$.

Proof:
If $x \in C([a, b])$, then $x = \operatorname{Re} x + i \operatorname{Im} x$, where $\operatorname{Re} x, \operatorname{Im} x \in C([a, b])$ are real-valued. Since $P_n(x) = P_n(\operatorname{Re} x) + iP_n(\operatorname{Im} x)$ for each n, it is enough to prove that $P_n(x) \to x$ when x is real-valued. Let then $x \in C([a, b])$ be real-valued. Since x is bounded, there is some $\alpha \in \mathbf{R}$ such that $|x(t)| \le \alpha$ for all $t \in [a, b]$. For $t, s \in [a, b]$, we have

$$-2\alpha \le x(t) - x(s) \le 2\alpha.$$

Let $\epsilon > 0$. Since x is uniformly continuous on $[a, b]$, there exists some $\delta > 0$ such that for $t, s \in [a, b]$ with $|t - s| < \delta$, we have

$$-\epsilon < x(t) - x(s) < \epsilon.$$

Now fix $s \in [a,b]$ and consider the function $y_s(t) = (t-s)^2, t \in [a,b]$. Then for $|t-s| \geq \delta$, we have $y_s(t) \geq \delta^2$. Combining these inequalities, we see that for all $t \in [a,b]$,

$$-\epsilon - \frac{2\alpha}{\delta^2} y_s(t) \leq x(t) - x(s) \leq \epsilon + \frac{2\alpha}{\delta^2} y_s(t).$$

Since each P_n is positive and linear, we have

$$-\epsilon P_n(x_0) - \frac{2\alpha}{\delta^2} P_n(y_s) \leq P_n(x) - x(s)P_n(x_0) \leq \epsilon P_n(x_0) + \frac{2\alpha}{\delta^2} P_n(y_s).$$

By assumption, $(P_n(x_0)(s))$ converges to 1 uniformly for $s \in [a,b]$. Also, since $y_s = x_2 - 2sx_1 + s^2 x_0$, we have

$$P_n(y_s)(s) = P_n(x_2)(s) - 2sP_n(x_1)(s) + s^2 P_n(x_0)(s).$$

Hence by assumption, $(P_n(y_s)(s))$ converges to $s^2 - 2s \cdot s + s^2 \cdot 1 = 0$ uniformly for $s \in [a,b]$. Thus $(P_n(x)(s))$ converges to $x(s)$ uniformly for $s \in [a,b]$. □

3.12 Corollary (Weierstrass, 1885)

The set of all polynomials in one variable is dense in $C([a,b])$ with the sup metric.

Proof:

Without loss of generality, assume $a = 0$ and $b = 1$. For $n = 1, 2, \ldots$, let

$$B_n(x)(t) = \sum_{k=0}^{n} x(\frac{k}{n}) \binom{n}{k} t^k (1-t)^{n-k}, \quad t \in [0,1].$$

Then each B_n is a positive linear map on $C([0,1])$. (The letter B stands for Bernstein.) Also, it can be verified that for all $n = 1, 2, \ldots$,

$$B_n(x_0) = x_0, \quad B_n(x_1) = x_1, \text{ and } B_n(x_2) = (1 - \frac{1}{n})x_2 + \frac{1}{n}x_1.$$

Hence $B_n(x_j) \to x_j$ for $j = 0, 1, 2$ and the result follows from Korovkin's theorem (3.11). □

3.13 Theorem (Korovkin, 1953)
Let $X = \{x \in C([-\pi,\pi]) : x(\pi) = x(-\pi)\}$. Consider $x_0(t) = 1$, $x_1(t) = \cos t$ and $x_2(t) = \sin t$ for $t \in [-\pi,\pi]$. Let $P_n : X \to X$ be a positive linear map for $n = 1, 2, \ldots$. If $P_n(x_j) \to x_j$ uniformly on $[-\pi,\pi]$ for $j = 0, 1, 2$, then $P_n(x) \to x$ uniformly on $[-\pi,\pi]$ for every $x \in X$.

Proof:
A proof similar to that of 3.11 can be given by considering the function $z_s(t) = \sin^2(t-s)/2$ instead of the function $y_s(t) = (t-s)^2$. $\square$

4 Lebesgue Measure and Integration on R

In this section we shall review the theory of Lebesgue measure and p-integrable functions on $\mathbf{R}$. Spaces of these functions provide some of the most concrete and useful examples of many theorems in functional analysis. Although most of the considerations in this section hold for measures on arbitrary sets, our aim here is not to give a full-fledged account of measure and integration, but to develop the most important example of a measure, namely the Lebesgue measure on $\mathbf{R}$. We also utilize this development to study some elements of Fourier series and Fourier integrals. This section is not a prerequisite for understanding the later chapters, but it will provide many illustrations of results proved in the sequel.

The concept of Lebesgue measure on $\mathbf{R}$ is a generalization of the idea of length of an interval in $\mathbf{R}$ to a wider class of subsets of $\mathbf{R}$.

The **Lebesgue outer measure** of a set $E \subset \mathbf{R}$ is given by

$$m^*(E) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : E \subset \bigcup_{n=1}^{\infty} I_n\right\},$$

where I_n is an open interval in $\mathbf{R}$ and $\ell(I_n)$ denotes the length of I_n.

It is easy to see that $m^*(\emptyset) = 0$, $m^*(E) \geq 0$ for all $E \subset \mathbf{R}$, $m^*(E_1) \leq m^*(E_2)$ for $E_1 \subset E_2 \subset \mathbf{R}$, $m^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)$ for all subsets $E_1, E_2, \ldots$ of $\mathbf{R}$ and $m^*(I) = \ell(I)$ if I is any interval in $\mathbf{R}$. Unfortunately, even when $E_1, E_2, \ldots$ are pairwise disjoint subsets of $\mathbf{R}$, we may not have $m^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m^*(E_n)$. We consider some special subsets of $\mathbf{R}$ for which this property holds.

A set $E \subset \mathbf{R}$ is said to be **(Lebesgue) measurable** if

$$m^*(A) = m^*(A \bigcap E) + m^*(A \bigcap E^c)$$

for every $A \subset \mathbf{R}$, where E^c denotes the complement of E in $\mathbf{R}$. If E is measurable, then $m^*(E)$ is called the **Lebesgue measure** of E and is denoted simply by $m(E)$.

It is not difficult to see that $\mathbf{R}$ and $\emptyset$ are measurable subsets and that complements and countable unions of measurable sets are measurable. Also, open intervals are measurable. Every open set in $\mathbf{R}$ is measurable since it is a (disjoint) countable union of open intervals. It can be checked that the Lebesgue measure m is **countably additive** on measurable sets, that is, if $E_1, E_2, \ldots$ are pairwise disjoint measurable sets, then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n).$$

Finally, the definition of m^* shows that if E is measurable and $\epsilon > 0$, then there exist an open set E_1 and a closed set E_2 in $\mathbf{R}$ such that $E_2 \subset E \subset E_1$, $m(E_1 \bigcap E^c) < \epsilon$ and $m(E \bigcap E_2^c) < \epsilon$. This property is called the **regularity** of the Lebesgue measure m.

Measurable and Simple Functions

An extended real-valued function x on $\mathbf{R}$ is said to be **(Lebesgue) measurable** if $x^{-1}(E)$ is a measurable subset for every open subset E of $\mathbf{R}$ and if the subsets $x^{-1}(\infty)$ and $x^{-1}(-\infty)$ of $\mathbf{R}$ are measurable. A complex-valued function x on $\mathbf{R}$ is said to be **(Lebesgue)** measurable if its real and imaginary parts $\operatorname{Re} x$ and $\operatorname{Im} x$ are both measurable. By a **measurable function** on $\mathbf{R}$, we shall mean either an extended real-valued or a complex valued measurable function. Clearly, a continuous function on $\mathbf{R}$ is measurable. Let x and y be measurable functions. We say that x equals y **almost everywhere** (abbreviated as a.e.) on E if

$$m(\{t \in E : x(t) \neq y(t)\}) = 0.$$

If $x^{-1}(\infty) \bigcap y^{-1}(-\infty) = \emptyset = x^{-1}(-\infty) \bigcap y^{-1}(\infty)$, then $x + y$ is a measurable function. Also, xy is a measurable function (with the convention $\infty \cdot 0 = 0 = -\infty \cdot 0$). Further, the functions $\max\{x, y\}, \min\{x, y\}$ and $|x|$ are measurable. More importantly, if (x_n) is a sequence of measurable functions such that $x_n(t) \to x(t)$ for each $t \in \mathbf{R}$, then x is measurable. For $E \subset \mathbf{R}$, let c_E denote the **characteristic function** of E, that is,

$$c_E(t) = \begin{cases} 1, & \text{if } t \in E \\ 0, & \text{if } t \notin E. \end{cases}$$

It is measurable if and only if E is a measurable subset. A **simple function** is a scalar-valued function on $\mathbf{R}$ whose range is finite. If $r_1, \ldots, r_m$ are the distinct values of such a function s, then $s = \sum_{j=1}^{m} r_j c_{E_j}$, where $E_j = \{t \in \mathbf{R} : s(t) = r_j\}, j = 1, \ldots, m$. Also, s is measurable if and only if $E_1, \ldots, E_m$ are measurable subsets. Let $x : \mathbf{R} \to [0, \infty]$ and for $n = 1, 2, \ldots$, consider the simple function

$$s_n(t) = \begin{cases} (j-1)/2^n, & \text{if } (j-1)/2^n \le x(t) < j/2^n, j = 1, 2, \ldots, n2^n \\ n, & \text{if } x(t) \ge n. \end{cases}$$

Then $0 \le s_1(t) \le s_2(t) \cdots \le x(t)$ and $s_n(t) \to x(t)$ for each $t \in \mathbf{R}$. If x is bounded, the sequence (s_n) converges to x uniformly on $\mathbf{R}$. If $x : \mathbf{R} \to [-\infty, \infty]$, then by considering $x = x^+ - x^-$, where $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$, we see that there exists a sequence of simple functions converging to x at every point of $\mathbf{R}$. Note that if x is measurable, each of these simple functions is measurable.

Let s be a nonnegative simple measurable function on $\mathbf{R}$ with $s = \sum_{j=1}^{m} r_j c_{E_j}$, where $r_1, \ldots, r_n$ are distinct nonnegative numbers. We define

$$\int_{\mathbf{R}} s\,dm = \sum_{j=1}^{m} r_j m(E_j).$$

For a measurable function $x : \mathbf{R} \to [0, \infty]$, define

$$\int_{\mathbf{R}} x\,dm = \sup\left\{\int_{\mathbf{R}} s\,dm : 0 \le s \le x, s \text{ simple and measurable}\right\}.$$

In this case, $\int_{\mathbf{R}} x\,dm = 0$ if and only if $x = 0$ almost everywhere on $\mathbf{R}$. Let now x be an extended real-valued measurable function on $\mathbf{R}$. If at least one of $\int_{\mathbf{R}} x^+\,dm$ and $\int_{\mathbf{R}} x^-\,dm$ is finite, the **Lebesgue integral** of x with respect to the Lebesgue measure m is defined by

$$\int_{\mathbf{R}} x\,dm = \int_{\mathbf{R}} x^+\,dm - \int_{\mathbf{R}} x^-\,dm.$$

If x is a complex-valued measurable function on $\mathbf{R}$, then we define

$$\int_{\mathbf{R}} x\,dm = \int_{\mathbf{R}} \operatorname{Re} x\,dm + i\int_{\mathbf{R}} \operatorname{Im} x\,dm$$

whenever $\int_{\mathbf{R}} \operatorname{Re} x\,dm$ and $\int_{\mathbf{R}} \operatorname{Im} x\,dm$ are well-defined. Then we have

$$\left|\int_{\mathbf{R}} x\,dm\right| \le \int_{\mathbf{R}} |x|\,dm.$$

If x is a measurable function on $\mathbf{R}$ and $\int_{\mathbf{R}} |x|\,dm < \infty$, we say that x is an **integrable** function on $\mathbf{R}$.

If a function x is defined only on a fixed measurable set $E \subset \mathbf{R}$, then we can define analogously the measurability of x on E and the integral of x on E, which will be denoted by $\int_E x\,dm$.

The most important and useful results in this development concern the convergence of integrals of a sequence of measurable functions. We refrain from giving the proofs which can be found in several texts on the Lebesgue theory. (See, for example [52], 1.26 and 1.34.)

4.1 Theorem

Let (x_n) be a sequence of measurable functions on a measurable subset E of $\mathbf{R}$.

(a) (**Monotone convergence theorem**) If $0 \le x_1(t) \le x_2(t) \le \cdots$ and $x_n(t) \to x(t)$ for all $t \in E$, then

$$\int_E x_n \, dm \to \int_E x \, dm.$$

(b) (**Dominated convergence theorem**) If $|x_n(t)| \le y(t)$ for all $n = 1, 2, \ldots$ and $t \in E$, where y is an integrable function on E and if $x_n(t) \to x(t)$ for all $t \in E$, then x_n, x are integrable on E and

$$\int_E x_n \, dm \to \int_E x \, dm.$$

If, in particular, $m(E) < \infty$ and $|x_n(t)| \le \alpha$ for all $n = 1, 2, \ldots$, $t \in E$ and some $\alpha > 0$, then the result 4.1(b) is known as the **bounded convergence theorem.**

We note that if x_1 and x_2 are integrable functions on E, then

$$\int_E (x_1 + x_2) \, dm = \int_E x_1 \, dm + \int_E x_2 \, dm.$$

This result is obvious if x_1 and x_2 are simple measurable functions. The general case is obtained by approximating x_1^+, x_1^-, x_2^+ and x_2^- by nondecreasing sequences of simple measurable functions and applying the monotone convergence theorem (4.1(a)).

Calculus with Lebesgue Measure

Let $E = [a, b]$, a finite closed interval in $\mathbf{R}$. We relate Lebesgue integration with Riemann integration and with differentiation.

Consider a $\mathbf{K}$-valued bounded function x on $[a, b]$. It can be shown that x is Riemann integrable on $[a, b]$ if and only if the set of discontinuities of x on $[a, b]$ is of (Lebesgue) measure zero. In that case,

x is Lebesgue integrable on $[a, b]$ and integral $\int_a^b x(t)\,dt$ equals the integral $\int_a^b x\,dm$. (See, for example, [51], 11.33.)

4.2 Fundamental theorem for Riemann integration

A $\mathbf{K}$-valued function x is differentiable on $[a, b]$ and its derivative x' is continuous on $[a, b]$ if and only if

$$x(t) = x(a) + \int_a^t y(s)\,ds, \quad a \le t \le b,$$

for some continuous function y on $[a, b]$. In that case, $x'(t) = y(t)$ for all $t \in [a, b]$.

For a proof, see [51], 6.20 and 6.21.

In order to obtain a similar result for Lebesgue integration, we introduce the following concept. A $\mathbf{K}$-valued function x on $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ if for every $\epsilon > 0$, there is some $\delta > 0$ such that $\sum_{j=1}^n |x(t_j) - x(s_j)| < \epsilon$ whenever $a \le s_1 < t_1 < \cdots < s_n < t_n \le b$ and $\sum_{j=1}^n (t_j - s_j) < \delta$. Clearly, every absolutely continuous function is uniformly continuous on $[a, b]$. On the other hand, if x is differentiable on $[a, b]$ and its derivative x' is bounded on $[a, b]$, then x is absolutely continuous by the mean value theorem.

4.3 Fundamental theorem for Lebesgue integration

A $\mathbf{K}$-valued function x is absolutely continuous on $[a, b]$ if and only if

$$x(t) = x(a) + \int_a^t y\,dm, \quad a \le t \le b,$$

for some (Lebesgue) integrable function y on $[a, b]$. In that case, $x'(t) = y(t)$ for almost all $t \in [a, b]$.

For a proof, see [50], pp. 103-107.

We now introduce a related concept. A **K**-valued function x is said to be of **bounded variation** if its **total variation**

$$V(x) = \sup\{\sum_{j=1}^{n} |x(t_j) - x(t_{j-1})| : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$$

is finite. It can be shown that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$. Also, if x is of bounded variation on $[a, b]$, then $x'(t)$ exists for almost all $t \in [a, b]$ and x' is (Lebesgue) integrable on $[a, b]$. (See [50], pp. 96-100.) However, a function of bounded variation on $[a, b]$ need not be continuous on $[a, b]$. For example, the characteristic function of the set $[0, 1/2]$ is of bounded variation on $[0, 1]$, but it is not continuous there. Also, a continuous function of bounded variation need not be absolutely continuous on $[a, b]$. For example, Cantor's ternary function is continuous and of bounded variation on $[0, 1]$, but it is not absolutely continuous.

While we shall generally restrict ourselves in what follows to the Lebesgue measure on **R**, we will have to refer on a few occasions to the Lebesgue measure on $\mathbf{R}^2$. It generalizes the idea of area of a rectangle, just as the Lebesgue measure on **R** generalizes the idea of length of an interval. The following result will be used in the sequel. For a proof, we refer the reader to [50], pp. 269-270.

4.4 Theorem (Fubini and Tonelli)

Let $m \times m$ denote the Lebesgue measure on $\mathbf{R}^2$ and $k(.,.)$ be a **K**-valued measurable function on the rectangle $[a, b] \times [c, d]$. If either

$$\iint_{[a,b]\times[c,d]} |k(s,t)|\, d(m\times m)(s,t) < \infty$$

or if $k(s, t) \geq 0$ for all $(s, t) \in [a, b] \times [c, d]$, then $\int_c^d k(s, t)\, dm(t)$ exists for almost every $s \in [a, b]$ and $\int_a^b k(s, t)\, dm(s)$ exists for almost every $t \in [c, d]$. The functions defined by these integrals are integrable on

$[a, b]$ and $[c, d]$ respectively and

$$\iint_{[a,b]\times[c,d]} k(s,t)\, d(m\times m)(s,t) = \int_a^b \left[\int_c^d k(s,t)\, dm(t)\right] dm(s)$$
$$= \int_c^d \left[\int_a^b k(s,t)\, dm(s)\right] dm(t).$$

L^p-spaces

Let E be a measurable subset of $\mathbf{R}$ and $1 \le p < \infty$. A measurable function x on E is said to be **p-integrable** on E if the function $|x|^p$ is integrable on E. [In particular, x is 1-integrable on E if and only if it is integrable on E.] A 2-integrable function is also known as a **square-integrable** function. In order to study spaces of p-integrable functions, we prove the following inequalities.

4.5 Lemma

Let x and y be measurable functions on E.

(a) **(Hölder's inequality)** Let $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then

$$\int_E |xy|\, dm \le \left(\int_E |x|^p\, dm\right)^{1/p} \left(\int_E |y|^q\, dm\right)^{1/q}.$$

(b) **(Minkowski's inequality)**. Let $1 \le p < \infty$. Assume that $m(x^{-1}(\infty) \cap y^{-1}(-\infty)) = 0 = m(x^{-1}(-\infty) \cap y^{-1}(\infty))$. Then

$$\left(\int_E |x+y|^p\, dm\right)^{1/p} \le \left(\int_E |x|^p\, dm\right)^{1/p} + \left(\int_E |y|^p\, dm\right)^{1/p}.$$

Proof:

(a) We note that xy is a measurable function on E and hence $\int_E |xy|\, dm$ is well-defined. Let

$$\alpha = \left(\int_E |x|^p\, dm\right)^{1/p} \quad \text{and} \quad \beta = \left(\int_E |y|^q\, dm\right)^{1/q}$$

If $\alpha = 0$ or $\beta = 0$, then $xy = 0$ almost everywhere on E and hence $\int_E |xy|\,dm = 0$. If $\alpha = \infty$ or $\beta = \infty$, then the inequality obviously holds. Let then $0 < \alpha, \beta < \infty$. The proof proceeds exactly as in 3.1(a) if we replace a_j by $x(t)$, b_j by $y(t)$ and the summation from $j = 1$ to n by integral over E with respect to the Lebesgue measure.

(b) The assumption on x and y shows that $\int_E |x+y|^p\,dm$ is well-defined. The proof proceeds exactly as in 3.1(b). $\square$

Let $1 \le p < \infty$. For p-integrable functions x and y on E, define

$$d_p(x,y) = \left(\int_E |x-y|^p\,dm\right)^{1/p}$$

Note that $m(\{t : |x(t)| = \infty\}) = 0 = m(\{t : |y(t)| = \infty\})$ since $\int_E |x|^p\,dm < \infty$ and $\int_E |y|^p\,dm < \infty$. Hence $d_p(x,y)$ is well-defined, nonnegative and $d_p(x,y) = d_p(y,x)$. Also, Minkowski's inequality (with y replaced by $-y$) shows that d_p satisfies the triangle inequality. However, if $d_p(x,y) = 0$, we can only say that $x = y$ almost everywhere on E.

For measurable functions x and y on E, we let $x \sim y$ if $x = y$ almost everywhere on E. It is easy to see that $\sim$ is an equivalence relation on the set of all measurable functions on E.

For $1 \le p < \infty$, let $L^p(E)$ denote the set of all equivalence classes of p-integrable functions. Clearly, d_p induces a metric on $L^p(E)$.

Let us now consider the case $p = \infty$. A measurable function x is said to be **essentially bounded** on E if there exists some $\beta > 0$ such that $m(\{t \in E : |x(t)| > \beta\}) = 0$, and then β is called an **essential bound** for $|x|$ on E. It can be seen that if $\alpha = \inf\{\beta : \beta$ an essential bound for $|x|$ on $E\}$, then α is itself an essential bound for $|x|$ on E. It is called the **essential supremum** of $|x|$ on E and will be denoted by $\mathrm{essup}_E|x|$. An example of an essentially bounded function on $[0,1]$ which is not bounded is given by $x(t) = n$ if $t = 1/n$, $n = 1, 2, \ldots$ and $x(t) = t$ otherwise. [Note that $\mathrm{essup}_E|x| = 1$.] If x and y are essentially bounded functions on E, then it follows that $\mathrm{essup}_E|x| + \mathrm{essup}_E|y|$ is an essential bound for $|x + y|$ on E and hence

$$\mathrm{essup}_E|x + y| \leq \mathrm{essup}_E|x| + \mathrm{essup}_E|y|.$$

Let $L^\infty(E)$ denote the set of all equivalence classes of essentially bounded functions on E under the equivalence relation $\sim$ introduced above. Then

$$d_\infty(x, y) = \mathrm{essup}_E|x - y|, \quad x, y \in E,$$

induces a metric on $L^\infty(E)$.

4.6 Theorem

For $1 \leq p \leq \infty$, the metric space $L^p(E)$ is complete.

Proof:

First let $1 \leq p < \infty$ and consider a Cauchy sequence (x_n) in $L^p(E)$. To prove the convergence of (x_n), it is enough to show that a subsequence of (x_n) converges. Hence by passing to a subsequence, if necessary, we may assume that $d_p(x_{n+1}, x_n) \leq 1/2^n, n = 1, 2, \ldots$. Let $x_0 = 0$ and for $t \in E$ and $n = 1, 2, \ldots$, consider

$$y_n(t) = \sum_{j=0}^{n} |x_{j+1}(t) - x_j(t)| \quad \text{and} \quad y(t) = \sum_{j=0}^{\infty} |x_{j+1}(t) - x_j(t)|.$$

By Minkowski's inequality (4.5(b)), we have for $n = 1, 2, \ldots$,

$$\left(\int_E |y_n|^p \, dm\right)^{1/p} \leq \sum_{j=0}^{n} d_p(x_{j+1}, x_j) \leq d_p(x_1, x_0) + \sum_{j=1}^{n} \frac{1}{2^j}.$$

The monotone convergence theorem (4.1(a)) shows that

$$\left(\int_E |y|^p \, dm\right)^{1/p} = \lim_{n\to\infty} \left(\int_E |y_n|^p \, dm\right)^{1/p} \leq d_p(x_1, x_0) + 1 < \infty.$$

Hence the function y is finite almost everywhere on E. Now the series $\sum_{j=0}^{\infty}|x_{j+1}(t) - x_j(t)|$ is absolutely summable and hence summable for almost all $t \in E$. For such $t \in E$, we let

$$x(t) = \sum_{j=0}^{\infty} [x_{j+1}(t) - x_j(t)]$$

Noting that $x_n(t) = \sum_{j=0}^{n-1}[x_{j+1}(t) - x_j(t)]$ for all $t \in E$, we have

$$\lim_{n\to\infty} x_n(t) = x(t) \quad \text{and} \quad |x_n(t)| \le \sum_{j=0}^{n-1} |x_{j+1}(t) - x_j(t)| \le y(t)$$

for almost all $t \in E$. Since the function y^p is integrable, the dominated convergence theorem (4.1(a)) gives

$$\int_E |x|^p \, dm = \lim_{n\to\infty} \int_E |x_n|^p \, dm \le \int_E y^p \, dm < \infty.$$

Hence $x \in L^p(E)$. By Minkowski's inequality (4.5(b)), $|x| + y \in L^p(E)$ and $|x - x_n|^p \le (|x| + y)^p$ for all $n = 1, 2, \ldots$. Again, the dominated convergence theorem (4.1(b)) gives

$$d_p(x_n, x) = \left(\int_E |x_n - x|^p \, dm\right)^{1/p} \to 0 \quad \text{as } n \to \infty.$$

Thus the sequence (x_n) converges to x in $L^p(E)$, showing that the metric space $L^p(E)$ is complete.

Next, consider a Cauchy sequence (x_n) in $L^\infty(E)$. Let

$$A_j = \{t \in E : |x_j(t)| > \mathrm{essup}_E |x_j|\}, \quad j = 1, 2, \ldots,$$

$$B_{m,n} = \{t \in E : |x_m(t) - x_n(t)| > \mathrm{essup}_E |x_m - x_n|\}, \quad m, n = 1, 2, \ldots.$$

If F is the union of all A_j and $B_{m,n}$, then $m(F) = 0$ and the sequence (x_n) converges uniformly to a bounded function x on the complement of F in E. Hence $x \in L^\infty(E)$ and $d_\infty(x_n, x) \to 0$ as $n \to \infty$. Thus the sequence (x_n) converges in $L^\infty(E)$, showing that the metric space $L^\infty(E)$ is complete. □

Let us consider some approximation results for functions in $L^p(E)$.

4.7 Theorem

Consider a measurable subset E of $\mathbf{R}$.

(a) If $1 \leq p < \infty$, then the set of all simple measurable functions on E which are zero outside subsets of finite measure is dense in $L^p(E)$.

The set of all simple measurable functions is dense in $L^\infty(E)$.

(b) If $m(E) < \infty$ and $1 \leq p < \infty$, then the set of all bounded continuous functions on E is dense in $L^p(E)$.

(c) If $E = [a, b]$ and $1 \leq p < \infty$, then the set of all step functions on E is dense in $L^p([a, b])$.

(d) The metric space $L^p([a, b])$ is separable for $1 \leq p < \infty$, but the metric space $L^\infty([a, b])$ is not separable.

Proof:

(a) Note that each simple measurable function on E belongs to $L^\infty(E)$ and if such a function is zero outside a set of finite measure, then it is p-integrable.

Consider first a measurable function $x : E \to [0, \infty]$. Then there is a sequence (s_n) of simple measurable functions on E such that $0 \leq s_1(t) \leq s_2(t) \cdots \leq x(t)$ and $s_n(t) \to x(t)$ for all $t \in E$.

If x is p-integrable, then so is each s_n and hence it is zero outside some set of finite measure. Since $0 \leq (x - s_n)^p \leq x^p$ and x^p is integrable, the dominated convergence theorem (4.1(b)) shows that

$$d_p(x, s_n) = \left(\int_E |x - s_n|^p \, dm\right)^{1/p} \to 0 \quad \text{as } n \to \infty.$$

If $x \in L^\infty(E)$, then the sequence (s_n) converges uniformly to x on E, so that $d_\infty(x, s_n) \to 0$ as $n \to \infty$.

If x is an extended real-valued p-integrable function, then we consider $x = x^+ - x^-$, and if x is a complex-valued p-integrable function, then we consider $x = \operatorname{Re} x + i \operatorname{Im} x$ and obtain $d_p(x, s_n) \to 0$ as $n \to \infty$ as before.

(b) Let $m(E) < \infty$. Then every bounded measurable (in particular, continuous) function is p-integrable on E. Consider now a closed subset F of E and its characteristic function c_F. For $n = 1, 2, \ldots$, let

$$x_n(t) = \frac{1}{1 + n\text{dist}(t, F)}, \quad t \in E.$$

As we have seen in Section 3, $\text{dist}(\cdot, F)$ is a continuous function on E. Hence each x_n is continuous on E. Also, $x_n(t) = 1$ for all $t \in F$, whereas $x_n(t) \to 0$ as $n \to \infty$ for all $t \notin F$. Hence $(c_F - x_n)(t) \to 0$ for all $t \in E$. Since $m(E) < \infty$ and $|x_n(t)| \leq 1$ for all $t \in E$, the bounded convergence theorem (4.1(b)) shows that

$$d_p(c_F, x_n) = \left(\int_E |c_F - x_n|^p \, dm\right)^{1/p} \to 0 \quad \text{as } n \to \infty.$$

Next, consider a measurable subset E_0 of E and its characteristic function c_{E_0}. By the regularity of the Lebesgue measure, for every $\epsilon > 0$, there is a closed subset F of E_0 such that $m(E_0 \bigcap F^c) < \epsilon^p$, so that $d_p(c_{E_0}, c_F) < \epsilon$. We can thus approximate c_{E_0} and hence every simple measurable function on E, by bounded continuous functions on E. The result now follows from (a) above.

(c) Consider a (bounded) continuous function x on $[a, b]$. Then x is uniformly continuous on $[a, b]$. Given $\epsilon > 0$, there is a positive integer n such that $|x(s) - x(t)| < \epsilon$ whenever $s, t \in [a, b]$ and $|s - t| < (b - a)/n$. Define a step function y as follows: $y(a) = x(a)$ and $y(t) = x(t_{j-1})$ for $t_{j-1} < t \leq t_j$, where $t_j = [a + j(b - a)]/n$, $j = 1, \ldots, n$. Then

$$d_p(x, y) = \left(\int_a^b |x(t) - y(t)|^p \, dm\right)^{1/p} \leq \left(\sum_{j=1}^n \frac{\epsilon^p (b - a)}{n}\right)^{1/p} = \epsilon(b-a)^{1/p}.$$

We can thus approximate each (bounded) continuous function on $[a, b]$ by step functions. But by (b) above, (bounded) continuous functions on $[a, b]$ are dense in $L^p([a, b])$, and hence so are the step functions.

(d) Let $1 \leq p < \infty$. Each step function can be approximated by step functions which have steps at rational numbers in $[a, b]$ and which take values whose real and imaginary parts are rational numbers.

Since such step functions are countable, it follows that $L^p([a,b])$ is separable.

To see that $L^\infty([a,b])$ is not separable, let c_t denote the characteristic function of the interval $[a,t]$ and $S = \{c_t : t \in [a,b]\}$. Then $d_\infty(c_s, c_t) = 1$ whenever $s \neq t$. Let $\{x_1, x_2, \ldots\}$ be any countable subset of $L^\infty([a,b])$ and let $0 < r \leq 1/2$. Then each $U(x_n, r)$ contains at most one element of S. Since S is uncountable, there is some $x \in S$ such that $x \notin U(x_n, r)$ for all $n = 1, 2, \ldots$. In other words, $\{x_1, x_2, \ldots\} \bigcap U(x, r) = \emptyset$, so that the set $\{x_1, x_2, \ldots\}$ cannot be dense in $L^\infty([a,b])$. □

Fourier Series and Integrals

We consider the problem of representing any integrable 2π-periodic function on **R** in terms of the special 2π-periodic functions $e^{int}, n = 0, \pm 1, \pm 2, \ldots$. Let $x \in L^1([-\pi, \pi])$. For $n = 0, \pm 1, \pm 2, \ldots$, the n**th Fourier coefficient** of x is defined to be

$$\hat{x}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-int} \, dm(t)$$

and the (formal) series $\sum_{n=-\infty}^{\infty} \hat{x}(n) e^{int}$ is called the **Fourier series** of x. For $n = 0, 1, 2, \ldots$, consider the nth partial sum

$$s_n(t) = \sum_{k=-n}^{n} \hat{x}(k) e^{ikt}, \quad t \in [-\pi, \pi].$$

We make some remarks regarding the convergence of the sequence (s_n). Long ago Kolmogorov ([40], 1926) gave an example of a function x in $L^1([-\pi, \pi])$ such that the corresponding sequence $(s_n(t))$ diverges for each $t \in [-\pi, \pi]$. If, however, $x \in L^p([-\pi, \pi])$ for some $p > 1$, then $(s_n(t))$ converges for almost all $t \in [-\pi, \pi]$. (See [8] and [32].) In case x is a function of bounded variation on $[-\pi, \pi]$, then $(s_n(t))$ converges to $[x(t^+) + x(t^-)]/2$ for each $t \in (-\pi, \pi)$ and to $[x(\pi^-) + x(-\pi^+)]/2$ for $t = \pm\pi$. If, in fact, x is a continuous function of bounded variation on $[-\pi, \pi]$ and $x(-\pi) = x(\pi)$, then (s_n) converges to x uniformly on $[-\pi, \pi]$. (See [39], p. 53.) On the other hand, if x is merely a continuous function satisfying $x(-\pi) = x(\pi)$, then $(s_n(0))$ may diverge, as we shall see in 9.4. More generally, given a subset E of $[-\pi, \pi]$ with $m(E) = 0$, there exists a continuous function x on $[-\pi, \pi]$ with $x(-\pi) = x(\pi)$ such that $(s_n(t))$ diverges for each $t \in E$. (See [39], p. 58.)

Concerning the convergence of (s_n) in the metric $d_p, 1 \le p < \infty$, we remark that if $x \in L^1([-\pi, \pi])$, then $(d_1(x, s_n))$ may not tend to zero, but if $x \in L^p([-\pi, \pi)$ with $1 < p < \infty$, then $d_p(x, s_n) \to 0$ as $n \to \infty$. (See [39], p. 50.) We shall prove this result for the case $p = 2$ in 22.8(b).

We obtain an integral representation of each s_n. For $n = 0, 1, 2, \ldots$, define the nth **Dirichlet kernel** by

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}, \quad t \in \mathbf{R}.$$

Then for $t \in [-\pi, \pi]$, we have

$$s_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-n}^{n} x(s) e^{ik(t-s)} \right) dm(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(s) D_n(t-s)\, dm(s)$$

Let us examine the Dirichlet kernels. Notice that $D_n(-t) = D_n(t)$ for all $t \in \mathbf{R}$ and $\int_{-\pi}^{\pi} D_n(t)\, dt = 2\pi$. Next, by considering $D_n(t)e^{it/2} - D_n(t)e^{-it/2}$, we obtain

$$D_n(t) = \begin{cases} 2n+1, & \text{if } t = 0, \pm 2\pi, \pm 4\pi, \ldots \\ \dfrac{\sin(2n+1)t/2}{\sin t/2}, & \text{otherwise.} \end{cases}$$

Note that D_n takes both positive and negative values. Also,

$$\begin{aligned} \int_{-\pi}^{\pi} |D_n(t)|\, dt &\ge 2 \int_{-\pi}^{\pi} \frac{|\sin(2n+1)t/2|}{|t|}\, dt \\ &= 4 \int_{0}^{(2n+1)\pi/2} \frac{|\sin t|}{t}\, dt \end{aligned}$$

$$\begin{aligned} &> 4\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}\frac{|\sin t|}{k\pi}\,dt \\ &= \frac{8}{\pi}\sum_{k=1}^{n}\frac{1}{k}. \end{aligned}$$

Hence $\int_{-\pi}^{\pi}|D_n(t)|\,dt \to \infty$ as $n \to \infty$. This shows that $|D_n|$ is not well-behaved as $n \to \infty$.

An extremely useful approach in this situation is to consider the arithmetic means of the partial sums of the Fourier series and their integral representations. For $m = 1, 2, \ldots$, let

$$a_m = \frac{1}{m}\sum_{n=0}^{m-1} s_n$$

and define the mth **Fejer kernel** by

$$K_m(t) = \frac{1}{m}\sum_{n=0}^{m-1} D_n(t), \quad t \in \mathbf{R}.$$

Then for $t \in [-\pi, \pi]$, we have

$$\begin{aligned} a_m(t) &= \frac{1}{m}\sum_{n=0}^{m-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} x(s)D_n(t-s)\,dm(s) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(s)K_m(t-s)\,dm(s). \end{aligned}$$

Let us examine the Fejer kernels. Note that $K_m(-t) = K_m(t)$ for all $t \in \mathbf{R}$ and $\int_{-\pi}^{\pi} K_m(t)\,dt = 2\pi$. Since $K_1(t) = D_0(t) = 1$ and

$$(m+1)K_{m+1}(t) - mK_m(t) = D_m(t), \quad m = 1, 2, \ldots$$

for all $t \in \mathbf{R}$, it follows by mathematical induction that

$$K_m(t) = \begin{cases} m, & \text{if } t = 0, \pm 2\pi, \pm 4\pi, \ldots \\ \dfrac{1}{m}\left(\dfrac{\sin mt/2}{\sin t/2}\right)^2 = \dfrac{1-\cos mt}{m(1-\cos t)}, & \text{otherwise.} \end{cases}$$

Note that $K_m(t) \geq 0$ for all $t \in \mathbf{R}$ and $m = 1, 2, \ldots$.

4.8 Theorem (Fejer, 1904)

Let x be a continuous $\mathbf{K}$-valued function on $[-\pi,\pi]$ such that $x(\pi) = x(-\pi)$. Then the sequence of arithmetic means of the partial sums of the Fourier series of x converges to x uniformly on $[-\pi,\pi]$.

Proof:
Let $X = \{x \in C([-\pi,\pi]) : x(\pi) = x(-\pi)\}$. For $x \in X$ and $m = 1,2,\ldots,$ let $F_m(x)$ denote the mth arithmetic mean as specified in the theorem. Then

$$F_m(x)(t) = a_m(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(s)K_m(t-s)\,dm(s), \quad t \in [-\pi,\pi].$$

Clearly, F_m is a linear operator. Also, since $K_m(t-s) \geq 0$ for all $t, s \in \mathbf{R}$, we see that $F_m(x)(t) \geq 0$ for all $t \in [-\pi,\pi]$ whenever $x(s) \geq 0$ for all $s \in [-\pi,\pi]$, that is, F_m is a positive operator on X. Let $x_0(t) = 1, x_1(t) = \cos t$ and $x_2(t) = \sin t$ for $t \in [-\pi,\pi]$. Since the Fourier series of x_0, x_1 and x_2 are $1, \cos t = (e^{it}+e^{-it})/2$ and $\sin t = (e^{it}-e^{-it})/2i$ respectively, we see that

$$\begin{aligned} F_m(x_0)(t) &= \frac{1+\cdots+1}{m} = 1, \\ F_m(x_1)(t) &= \frac{0+\cos t+\cdots+\cos t}{m} = \frac{(m-1)\cos t}{m}, \\ F_m(x_2)(t) &= \frac{0+\sin t+\cdots+\sin t}{m} = \frac{(m-1)\sin t}{m} \end{aligned}$$

for all $m = 1,2,\ldots$ and $t \in [-\pi,\pi]$. It is obvious that for $j = 0, 1,$ and 2, the sequence $(F_m(x_j))$ converges to x_j uniformly on $[-\pi,\pi]$. By Korovkin's theorem (3.13), we see that for each $x \in X$, the sequence $(F_m(x))$ converges to x uniformly on $[-\pi,\pi]$, as desired. □

We remark that if $x \in L^p([-\pi,\pi])$ for some p with $1 \leq p < \infty$, then $d_p(x, F_m(x)) \to 0$ as $m \to \infty$, and if $x \in L^\infty([-\pi,\pi])$, then $d_1(xy, F_m(x)y) \to 0$ as $m \to \infty$ for every $y \in L^1([-\pi,\pi])$. (See [28], pp. 17-20. Compare 15.5.)

We shall now consider some interesting properties of the Fourier coefficients of an integrable function on $[-\pi,\pi]$.

4.9 Theorem
Let $x \in L^1([-\pi, \pi])$.

(a) **(Riemann-Lebesgue lemma, 1903)** $\hat{x}(n) \to 0$ as $n \to \pm\infty$.

(b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{\hat{x}(n) - \hat{x}(-n)}{n}$ converges in $\mathbf{R}$.

(c) If $\hat{x}(n) = 0$ for all $n = 0, \pm 1, \pm 2, \ldots$, then $x(t) = 0$ for almost all $t \in [-\pi, \pi]$. In particular, the Fourier coefficients of x determine x.

Proof:

(a) First suppose that $x(t) = k_1 e^{ij_1 t} + \cdots + k_m e^{ij_m t}$, $t \in [-\pi, \pi]$, for some $k_1, \ldots, k_m$ in $\mathbf{K}$ and distinct integers $j_1, \ldots, j_m$. Then $\hat{x}(j_1) = k_1, \ldots, \hat{x}(j_m) = k_m$ and $\hat{x}(n) = 0$ for all other n. In particular, $\hat{x}(n) = 0$ for all n with $|n| > \max\{|j_1|, \ldots |j_m|\}$. Thus the desired result holds for this x. Next, suppose that x is a continuous function on $[-\pi, \pi]$ with $x(\pi) = x(-\pi)$. Since the result holds for each arithmetic mean of the partial sums of the Fourier series of x, it follows from Fejer's theorem (4.8) that the result holds for x as well. Finally, if x is an integrable function on $[-\pi, \pi]$, then a slight modification of the proof of 4.7(b) shows that there is a sequence (x_m) of continuous functions on $[-\pi, \pi]$ with $x_m(\pi) = x_m(-\pi)$ such that $d_1(x_1, x_m) \to 0$ as $m \to \infty$. Hence the result holds for every $x \in L^1([-\pi, \pi])$.

(b) For $s \in [-\pi, \pi]$, define

$$y(s) = \int_{-\pi}^{s} x(t)\, dm(t) - \hat{x}(0)s.$$

Then $y(\pi) = \hat{x}(0)\pi = y(-\pi)$. By the fundamental theorem for Lebesgue integration (4.3), y is absolutely continuous on $[-\pi, \pi]$ and $y' = x - \hat{x}(0)$ almost everywhere on $[-\pi, \pi]$. Note that y is a function of bounded variation and it is continuous at 0. As a consequence of a result quoted in the beginning of our discussion of Fourier series, the partial sums of the Fourier series of y converge to y at $t = 0$, that is,

$y(0) = \lim_{m\to\infty} \sum_{n=-m}^{m} \hat{y}(n)$. Now for each nonzero integer n, we have

$$\begin{aligned}
\hat{y}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} y(s)e^{-ins}\,dm(s)\\
&= \frac{1}{2\pi}\left[y(s)\frac{e^{-ins}}{-in}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} y'(s)\frac{e^{-ins}}{-in}\,dm(s)\\
&= 0 + \frac{1}{2\pi i n}\int_{-\pi}^{\pi} x(s)e^{-ins}\,dm(s) - 0\\
&= \frac{\hat{x}(n)}{in},
\end{aligned}$$

since $y(\pi) = y(-\pi)$ and $e^{-in\pi} = e^{in\pi}$. As

$$\sum_{n=-m}^{m} \hat{y}(n) = \hat{y}(0) + \frac{1}{i}\sum_{n=1}^{m} \frac{\hat{x}(n) - \hat{x}(-n)}{n},$$

we see that the series $\sum_{n=1}^{\infty} \frac{\hat{x}(n) - \hat{x}(-n)}{n}$ converges in $\mathbf{R}$.

(c) Assume that $\hat{x}(n) = 0$ for all $n = 0, \pm 1, \pm 2, \ldots$ and define

$$y(s) = \int_{-\pi}^{s} x(t)\,dm(t)$$

as in (b) above. (Note: $\hat{x}(0) = 0$) Then $\hat{y}(n) = \hat{x}(n)/in = 0$ for all $n = \pm 1, \pm 2, \ldots$, so that the Fourier series of y is just the constant $\hat{y}(0)$. As a result, each arithmetic mean of the partial sums of the Fourier series of y also equals $\hat{y}(0)$. By Fejer's theorem (4.8), we see that $y(t) = \hat{y}(0)$ for all $t \in [-\pi, \pi]$. Hence $x(t) = y'(t) = 0$ for almost all $t \in [-\pi, \pi]$. □

4.10 Corollary

Let $x \in L^1([-\pi, \pi])$. If $\sum_{n=-\infty}^{\infty} |\hat{x}(n)| < \infty$, then

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{x}(n)e^{int}$$

for almost all $t \in [-\pi, \pi]$.

Proof:

Let $\sum_{n=-\infty}^{\infty} |\hat{x}(n)| < \infty$. Then series $\sum_{n=-\infty}^{\infty} \hat{x}(n)e^{int}$ converges uniformly for $t \in [-\pi, \pi]$ to $y(t)$, say. Then y is a continuous function on $[-\pi, \pi]$ and for all $n = 0, \pm 1, \pm 2, \ldots$, we have

$$\hat{y}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(t)e^{-int}\, dm(t) = \sum_{k=-\infty}^{\infty} \frac{\hat{x}(k)}{2\pi} \int_{-\pi}^{\pi} e^{i(k-n)t}\, dm(t) = \hat{x}(n).$$

By 4.9(c), $x(t) = y(t) = \sum_{n=-\infty}^{\infty} \hat{x}(n)e^{int}$ for almost all $t \in [-\pi, \pi]$. $\square$

In analogy with the theory of Fourier series for functions in $L^1([-\pi, \pi])$, we may develop a theory for functions in $L^1(\mathbf{R})$ as follows. Consider $x \in L^1(\mathbf{R})$. For $u \in \mathbf{R}$, define the **Fourier integral of x at u** by

$$\hat{x}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-iut}\, dm(t).$$

The (formal) integral $\int_{-\infty}^{\infty} \hat{x}(u)e^{iut}\, dm(t)$ is called the **Fourier transform of x**. With a great deal of more care than what is needed for the theory of Fourier series, the following theorem can be proved.

4.11 Theorem

Let $x \in L^1(R)$. Then

(a) $\hat{x}$ is a continuous function on $\mathbf{R}$ and $\hat{x}(u) \to 0$ as $u \to \pm\infty$.

(b) If $\hat{x}(u) = 0$ for all $u \in \mathbf{R}$, then $x(t) = 0$ for almost all $t \in \mathbf{R}$.

(c) (**Inversion theorem**) If $\hat{x} \in L^1(\mathbf{R})$, then for almost all $t \in \mathbf{R}$,

$$\sqrt{2\pi}\, x(t) = \int_{-\infty}^{\infty} \hat{x}(u)e^{iut}\, dm(u).$$

For proofs of these results, see [52], 9.6, 9.11 and 9.12. Compare (a) with 4.9(a), (b) with 4.9(c) and (c) with 4.10.

In 26.6 we shall develop a parallel notion of a Fourier-Plancherel transform of a function in $L^2(\mathbf{R})$.

Chapter II

Fundamentals of Normed Spaces

This chapter provides the basic framework for the developments given in Chapters III, IV and V. The concept of a norm on a linear space is introduced and illustrated in Section 5, where the interplay between a linear structure and a metric structure begins. The continuity of linear maps on normed spaces is discussed in Section 6. It gives rise to normed spaces of bounded linear maps. Section 7 deals with a separation theorem and an extension theorem due to Hahn and Banach. While these results are existential in nature and their proofs are not constructive, they are the corner stones of the duality considerations given in Chapter IV. A result of Taylor and Foguel on the uniqueness of Hahn-Banach extensions is noteworthy. Completeness of a normed space needed for various results of Chapter III, is studied in Section 8. A complete normed space is known as a Banach space.

5 Normed Spaces

On a linear space we impose a metric structure which is well-behaved with respect to the linear operations of addition and scalar multiplication. As noted in Section 2, $\mathbf{K}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

Let X be a linear space over $\mathbf{K}$. A **norm** on X is a function $\| \ \|$ from X to $\mathbf{R}$ such that for all $x, y \in X$ and $k \in \mathbf{K}$,

$$\|x\| \geq 0 \quad \text{and} \quad \|x\| = 0 \quad \text{if and only if} \quad x = 0,$$

$$\|x + y\| \leq \|x\| + \|y\|,$$

$$\|kx\| = |k|\,\|x\|.$$

A **normed space** is a linear space with a norm on it. For x a
y in X, let $d(x, y) = \|x - y\|$. It can be easily seen that d is a met
on X. Since $\big| \|x\| - \|y\| \big| \leq \|x - y\|$ for all x and y in X, the funct
$\| \ \|$ is uniformly continuous on X. Further, $x_n \to x$, $y_n \to y$ in X a
$k_n \to k$ in $\mathbf{K}$ imply that $x_n + y_n \to x + y$ and $k_n x_n \to kx$ in X. This
be referred to as the continuity of addition and scalar multiplicati

5.1 Examples

(a) Spaces $\mathbf{K}^n$: For $n = 1$, the absolute value function $|$
a norm on $\mathbf{K}$. Since $\|k\| = |k| \, \|1\|$ for all $k \in \mathbf{K}$, it follows that a
norm on $\mathbf{K}$ is a positive scalar multiple of the absolute value functi
For $n > 1$, there are a variety of norms on $\mathbf{K}^n$. We describe some.

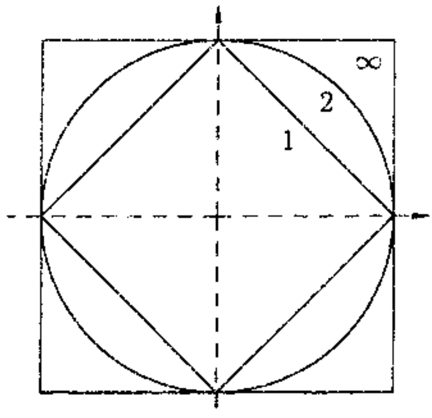


Figure 2

Let $p \geq 1$ be a real number. For $x = (x(1), \ldots, x(n)) \in \mathbf{K}^n$, le

$$\|x\|_p = \left(|x(1)|^p + \cdots + |x(n)|^p\right)^{1/p}.$$

If $p = 1$, then $\| \ \|_1$ is clearly a norm on $\mathbf{K}^n$. If $p > 1$, then lett
$a_j = x(j)$ and $b_j = y(j)$ in Minkowski's inequality (3.1(b)), then
follows that $\|x + y\|_p \leq \|x\|_p + \|y\|_p$. Hence $\| \ \|_p$ is a norm on $\mathbf{I}$
The norm $\| \ \|_2$ is known as the **Euclidean norm**. Next, it can
easily seen that

$$\|x\|_\infty = \max\{|x(1)|, \ldots, |x(n)|\}, \quad x \in \mathbf{K}^n,$$

also defines a norm on $\mathbf{K}^n$.

If $\mathbf{K} = \mathbf{R}$ and $n = 2$, the sets $\{x \in \mathbf{R}^2 : \|x\|_p = 1\}$ for $p = 1, 2$ and ∞ are shown in Figure 2.

Several other norms on $\mathbf{K}^n$ can be constructed by using the methods indicated in Problems 5-4 and 5-5.

(b) Sequence spaces: For $1 \le p < \infty$, consider the set ℓ^p of scalar sequences introduced in Section 3. For $x = (x(1), x(2), \ldots)$ in ℓ^p, let

$$\|x\|_p = \left(|x(1)|^p + |x(2)|^p + \cdots\right)^{1/p}.$$

If $p = 1$, then ℓ^1 is linear space and $\|\ \|_1$ is clearly a norm on it. If $p > 1$, then letting $n \to \infty$ in Minkowski's inequality (3.1(b)), it follows that $\|x + y\|_p \le \|x\|_p + \|y\|_p$, so that ℓ^p is a linear space and $\|\ \|_p$ is a norm on it.

Let $1 \le p < r < \infty$. Consider $x \in \ell^p$ with $\|x\|_p \le 1$. Then $|x(j)| \le 1$, and hence $|x(j)|^r \le |x(j)|^p$ for all $j = 1, 2, \ldots$. This shows that $\|x\|_r \le 1$. Now, if x is any nonzero element of ℓ^p, then by considering $x/\|x\|_p$, it follows that

$$\|x\|_r \le \|x\|_p, \quad \text{if } 1 \le p < r < \infty.$$

This is known as **Jensen's inequality**. It implies that $\ell^p \subset \ell^r$ and if $x_n \to x$ in ℓ^p, then $x_n \to x$ in ℓ^r.

Next, consider the linear space ℓ^∞ of all bounded scalar sequences. It can be easily seen that

$$\|x\|_\infty = \sup\{|x(j)| : j = 1, 2, \ldots\}$$

defines a norm on ℓ^∞. Also, if $p \ge 1$ and $x \in \ell^p$, then $\|x\|_\infty \le \|x\|_p$ Hence $\ell^p \subset \ell^\infty$ and if $x_n \to x$ in ℓ^p, then $x_n \to x$ in ℓ^∞.

The following subspaces will occur in several examples later.

$$\begin{aligned} c &= \{x \in \ell^\infty : x(j) \quad \text{converges in } \mathbf{K} \text{ as } j \to \infty\} \\ c_0 &= \{x \in c : x(j) \to 0 \quad \text{as } j \to \infty\} \\ c_{00} &= \{x \in \ell^p : \text{all but finitely many } x(j)\text{'s are equal to } 0\} \end{aligned}$$

Note that for all $n = 1, 2, \ldots$, $\mathbf{K}^n$ can be considered as a subspace of c_{00}. Also, for $1 \le p < \infty$, $c_{00} \subset \ell^p \subset c_0 \subset c \subset \ell^\infty$, where all the inclusions are proper.

(c) L^p-spaces: Let m denote the Lebesgue measure on $\mathbf{R}$ and E be a measurable subset of $\mathbf{R}$. (See Section 4 for definitions.)

For $1 \le p < \infty$, consider the set $L^p(E)$ of all equivalence classes of p-integrable functions on E. For x in $L^p(E)$, let

$$\|x\|_p = \left(\int_E |x|^p \, dm\right)^{1/p}.$$

If $p = 1$, then $L^1(E)$ is a linear space and $\| \ \|_1$ is clearly a norm on it. If $p > 1$, Minkowski's inequality (4.5(b)) implies that for x, y in $L^p(E)$, $\|x + y\|_p \le \|x\|_p + \|y\|_p$. Hence $L^p(E)$ is a linear space and $\| \ \|_p$ is a norm on it.

Let $1 \le p < r < \infty$. If $m(E) < \infty$ and x is a measurable function on E, we show that

$$\|x\|_p \le [m(E)]^{(r-p)/pr} \|x\|_r.$$

Since $r/p > 1$, we find $q > 1$ such that $p/r + 1/q = 1$. Then by Hölder's inequality (4.5(a)), we have

$$\int_E |x|^p \, dm \ \le \ \left(\int_E (|x|^p)^{r/p} \, dm\right)^{p/r} \left(\int_E 1^q \, dm\right)^{1/q}.$$

Taking pth roots of both sides we obtain the desired inequality, since $1/pq = (r - p)/pr$. It shows that every r-integrable function on E is also p-integrable on E and if $x_n \to x$ in $L^r(E)$, then $x_n \to x$ in $L^p(E)$. For this reason, $L^r(E)$ can be considered as a subset of $L^p(E)$, if $1 \le p < r < \infty$ and $m(E) < \infty$.

Next, consider the set $L^\infty(E)$ of all equivalence classes of essentially bounded measurable functions on E. For $x \in L^\infty(E)$, let

$$\|x\|_\infty = \operatorname{essup}_E |x|.$$

For $x, y \in L^\infty(E)$, it can be seen that $\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$. Hence $L^\infty(E)$ is a linear space and $\|\ \|_\infty$ is a norm on it. If $m(E) < \infty$ and $x \in L^\infty(E)$, then

$$\|x\|_p \le [m(E)]^{1/p}\|x\|_\infty < \infty,$$

so that every essentially bounded measurable function on E is p-integrable on E and if $x_n \to x$ in $L^\infty(E)$, then $x_n \to x$ in $L^p(E)$. For this reason, $L^\infty(E)$ can be considered as a subset of $L^p(E)$, if $1 \le p < \infty$ and $m(E) < \infty$.

If $m(E) = \infty$, there may be no inclusion relation among the spaces $L^p(E)$. For example, consider $E = \mathbf{R}$ and let

$$x_1(t) = \begin{cases} \dfrac{1}{\sqrt{|t|}}, & \text{if } 0 < |t| < 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad x_2(t) = \begin{cases} 0, & \text{if } |t| < 1 \\ \dfrac{1}{t}, & \text{otherwise.} \end{cases}$$

Then x_1 belongs to $L^1(\mathbf{R})$, but not to $L^2(\mathbf{R})$ or to $L^\infty(\mathbf{R})$, while x_2 belongs to $L^2(\mathbf{R})$, but not to $L^1(\mathbf{R})$. Also, the constant function 1 belongs to $L^\infty(\mathbf{R})$, but not to any other $L^p(\mathbf{R})$.

(d) Function spaces: Let T be a set and let $B(T)$ denote the set of all bounded scalar-valued functions on T. For x in $B(T)$, let

$$\|x\|_\infty = \sup\{|x(t)| : t \in T\}.$$

Then $\|\ \|_\infty$ is a norm on $B(T)$. It is known as the **sup norm**.

If T is a metric space, consider the following subspaces of $B(T)$:

$$\begin{aligned} C(T) &= \{x \in B(T) : x \text{ is continuous on } T\} \\ C_0(T) &= \{x \in C(T) : \text{for every } \epsilon > 0, \text{there is a compact set} \\ &\qquad\qquad E \subset T \text{ such that } |x(t)| < \epsilon \text{ for all } t \notin E\} \\ C_c(T) &= \{x \in C(T) : \text{there is a compact set } E \subset T \text{ such that} \\ &\qquad\qquad x(t) = 0 \text{ for all } t \notin E\} \end{aligned}$$

The elements of $C_0(T)$ are known as the continuous functions **vanishing at infinity** and those of $C_c(T)$ are known as the continuous functions with **compact support**. In case T is a compact metric space, $C_c(T) = C_0(T) = C(T)$, which is the set of all scalar-valued continuous functions on T.

Let $T = \{1, 2, \ldots\} \cup \{\infty\}$. For $n, m = 1, 2, \ldots$, define

$$d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|, \quad d(n,\infty) = \frac{1}{n} = d(\infty, n) \quad \text{and} \quad d(\infty,\infty) = 0.$$

Then d is a metric on T and T is compact. Consider a scalar-valued function x on T. Then $x \in C(T)$ if and only if x is continuous at ∞. This happens exactly when $\lim_{n\to\infty} x(n)$ exists and equals $x(\infty)$. Thus $C(T)$ can be identified with the subspace c of ℓ^∞.

Let $T = [a, b]$ and for $k = 1, 2, \ldots$, consider

$$C^k([a,b]) = \left\{x \in C([a,b]) : \text{the } k\text{th derivative } x^{(k)} \in C([a,b])\right\}.$$

Then $C^k([a,b])$ is a subspace of $C([a,b])$. Also, if we let

$$\|x\| = \|x\|_\infty + \|x^{(1)}\|_\infty + \cdots + \|x^{(k)}\|_\infty, \quad x \in C^k([a,b]),$$

then $\| \ \|$ is a norm on $C^k([a,b])$.

(e) Inner product spaces: In Section 21, we shall introduce the concept of an inner product $\langle \ , \ \rangle$ on a linear space X. It behaves like the dot product on $\mathbf{R}^2$. We shall prove that

$$\|x\| = \langle x,\, x\rangle^{1/2}, \quad x \in X,$$

defines a norm on X. Some of the most important normed spaces are obtained in this way: $\mathbf{K}^n$ with the norm $\| \ \|_2$, ℓ^2, $L^2(E)$. Another example is as follows: Let T be a measurable subset of $\mathbf{R}$ such that $m(T) < \infty$. For x and y in $C(T)$, let

$$\langle x,\, y\rangle = \int_T x\overline{y}\, dm.$$

Then

$$\|x\|_2 = \left(\int_T |x|^2\, dm\right)^{1/2}, \quad x \in X,$$

defines a norm on $C(T)$.

In Section 2 we have seen some methods of obtaining new linear spaces from the old. The new ones can be endowed with a norm, as we now show.

5.2 Theorem

(a) Let Y be a subspace of a normed space X. Then Y and its closure $\overline{Y}$ are normed spaces with the induced norm.

(b) Let Y be a closed subspace of a normed space X. For $x + Y$ in the quotient space X/Y, let

$$|||x + Y||| = \inf \{\|x + y\| : y \in Y\}.$$

Then $|||\ |||$ is a norm on X/Y, called the **quotient norm**.

A sequence $(x_n + Y)$ converges to $x + Y$ in X/Y if and only if there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X.

(c) Let $\|\ \|_j$ be a norm on a linear space X_j, $j = 1, \ldots, m$. Fix p such that $1 \leq p \leq \infty$. For $x = (x(1), \ldots, x(m))$ in the product space $X = X_1 \times \cdots \times X_m$, let

$$\|x\|_p = \begin{cases} (\|x(1)\|_1^p + \cdots + \|x(m)\|_m^p)^{1/p}, & \text{if } 1 \leq p < \infty \\ \max\{\|x(1)\|_1, \ldots, \|x(m)\|_m\}, & \text{if } p = \infty. \end{cases}$$

Then $\|\ \|_p$ is a norm on X.

A sequence (x_n) converges to x in X if and only if $(x_n(j))$ converges to $x(j)$ in X_j for every $j = 1, \ldots, m$.

Proof:

(a) The continuity of addition and scalar multiplication shows that $\overline{Y}$ is a subspace of X. It is clear that the norm on X induces a norm on Y, and on $\overline{Y}$.

(b) Let $x \in X$. It is clear that $|||x + Y||| \geq 0$. If $|||x + Y||| = 0$, there is a sequence (y_n) in Y such that $x + y_n \to 0$, that is, $y_n \to -x$.

Since Y is closed, we have $-x \in Y$, that is, $x + Y = 0$ in X/Y. Next, let x_1 and x_2 be in X and $\epsilon > 0$. Find y_1 and y_2 in Y such that

$$\|x_1 + y_1\| < \inf\{\|x_1 + y\| : y \in Y\} + \frac{\epsilon}{2} = |||x_1 + Y||| + \frac{\epsilon}{2},$$
$$\|x_2 + y_2\| < \inf\{\|x_2 + y\| : y \in Y\} + \frac{\epsilon}{2} = |||x_1 + Y||| + \frac{\epsilon}{2}.$$

Then

$$\|x_1+y_1+x_2+y_2\| \le \|x_1+y_1\|+\|x_2+y_2\| \le |||x_1+Y|||+|||x_2+Y|||+\epsilon.$$

Since $y_1 + y_2 \in Y$, we see that

$$|||(x_1 + x_2) + Y||| \le |||x_1 + Y||| + |||x_2 + Y||| + \epsilon,$$

for every $\epsilon > 0$. This shows that

$$|||(x_1 + Y) + (x_2 + Y)||| \le |||x_1 + Y||| + |||x_2 + Y|||.$$

Further, for $x \in X$, and $0 \neq k \in \mathbf{K}$,

$$\begin{aligned} |||k(x + Y)||| &= |||kx + Y||| = \inf\{\|kx + y\| : y \in Y\} \\ &= \inf\{|k|\, \|x + \frac{y}{k}\| : y \in Y\} \\ &= |k| \inf\{\|x + y\| : y \in Y\} \\ &= |k|\, |||x + Y|||. \end{aligned}$$

Hence $|||\ \ |||$ is a norm on X/Y.

Let $(x_n + Y)$ be a sequence in X/Y. If (y_n) is a sequence in Y such that $x_n + y_n \to x$ in X, then

$$|||(x_n + Y) - (x + Y)||| = |||(x_n - x) + Y||| \le \|x_n - x + y_n\|$$

for every n, so that $x_n + Y \to x + Y$ in X/Y. Conversely, assume that $x_n + Y \to x + Y$ in X/Y. Since

$$|||(x_n + Y) - (x + Y)||| = \inf\{\|x_n - x + y\| : y \in Y\},$$

choose $y_n \in Y$ such that $\|x_n - x + y_n\| < |||(x_n + Y) - (x + Y)||| + 1/n$, $n = 1, 2, \ldots$. Then it follows that $x_n - x + y_n \to 0$, that is, $x_n + y_n \to x$ in X as $n \to \infty$.

(c) Let $1 \le p < \infty$. For x and y in X, we have

$$\begin{aligned}
\|x+y\|_p &= (\|x(1)+y(1)\|_1^p + \cdots + \|x(m)+y(m)\|_m^p)^{\frac{1}{p}} \\
&\le [(\|x(1)\|_1 + \|y(1)\|_1)^p + \cdots + (\|x(m)\|_m + \|y(m)\|_m)^p]^{\frac{1}{p}} \\
&\le \left[\sum_{j=1}^{m} \|x(j)\|_j^p\right]^{1/p} + \left[\sum_{j=1}^{m} \|y(j)\|_j^p\right]^{1/p} \\
&= \|x\|_p + \|y\|_p
\end{aligned}$$

by letting $a_j = \|x(j)\|_j$ and $b_j = \|y(j)\|_j$ in Minkowski's inequality (3.1(b)). Also, it is clear that $\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$. Hence $\|\ \|_p$ is a norm on X for each $p, 1 \le p \le \infty$.

It can be readily checked that for a sequence (x_n) in X, $x_n \to x$ in X if and only if $x_n(j) \to x(j)$ in X_j for every $j = 1, \ldots, m$. □

The classical Heine-Borel theorem (3.7(a)) states that a subset E of $\mathbf{K}^n$ is compact if and only if E is closed and bounded. We shall now examine whether this result holds in a normed space X. For this purpose, we first prove two lemmas which are of independent interest.

5.3 Lemma (F. Riesz, 1918)
Let X be a normed space, Y be a closed subspace of X and $Y \ne X$. Let r be a real number such that $0 < r < 1$. Then there exists some $x_r \in X$ such that

$$\|x_r\| = 1 \quad \text{and} \quad r \le \operatorname{dist}(x_r, Y) \le 1.$$

Proof:
Since $Y \ne X$, consider $x \in X$ such that $x \notin Y$. As the subspace Y is closed, we have $\operatorname{dist}(x, Y) > 0$. Also, as $r < 1$, there exists some $y_0 \in Y$ such that $\|x - y_0\| \le \operatorname{dist}(x, Y)/r$. Let

$$x_r = \frac{x - y_0}{\|x - y_0\|}.$$

The choice of x_r is indicated in Figure 3.

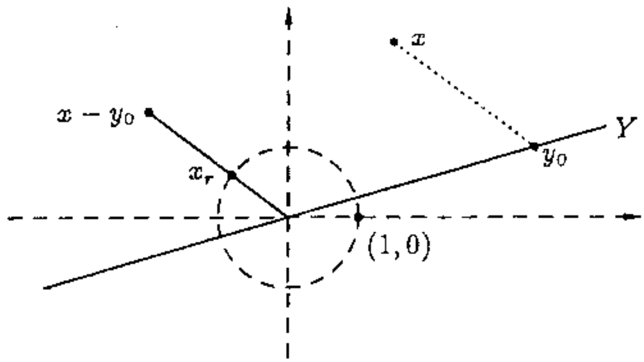


Figure 3

Clearly, $\|x_r\| = 1$. Since $0 \in Y$, we see that $\operatorname{dist}(x_r, Y) \leq 1$. Also, since $y_0 \in Y$, we have

$$\operatorname{dist}(x_r, Y) = \frac{\operatorname{dist}(x - y_0, Y)}{\|x - y_0\|} = \frac{\operatorname{dist}(x, Y)}{\|x - y_0\|} \geq r. \qquad \square$$

The Riesz lemma says that if Y is a closed proper subspace of a normed space X, then there exists a point on the unit sphere of X whose distance from Y is as close to 1 as we please. Does there exist a point on the unit sphere whose distance from Y is exactly 1? In general, the answer is negative (Problem 5-8(a)), but if Y is a finite dimensional subspace of X, or if X is a reflexive normed space, the answer is affirmative. (See Problems 5-8(b), 5-7, 7-3, 24-4 and 16.6(b), 16.7.)

5.4 Lemma

Let X be a normed space and Y be a subspace of X.

(a) For $x \in X, y \in Y$ and $k \in \mathbf{K}$, we have

$$\|kx + y\| \geq |k| \operatorname{dist}(x, Y).$$

(b) Let Y be finite dimensional. Then Y is complete. In particular, it is closed in X.

Let $\{y_1, \ldots, y_m\}$ be a basis for Y and (x_n) be a sequence in Y. If

$$x_n = k_{n,1}y_1 + \cdots + k_{n,m}y_m, \quad n = 1, 2, \ldots,$$

then (x_n) converges to $x = k_1y_1 + \cdots + k_my_m$ if and only if $k_{n,j} \to k_j$ as $n \to \infty$ for each $j = 1, \ldots, m$. Also, (x_n) is bounded if and only if $(k_{n,j})$ is bounded for each $j = 1, \ldots, m$.

Proof:

(a) If $k = 0$, the result is obvious. If $k \neq 0$, then

$$\|kx + y\| = |k|\,\|x + \frac{y}{k}\| \geq |k| \operatorname{dist}(x, Y),$$

since $-y/k \in Y$.

(b) To prove the completeness of a finite dimensional subspace Y of X, we use mathematical induction on the dimension m of Y. Let $m = 1$. Then $Y = \{ky : k \in \mathbf{K}\}$ with $y \neq 0$. If (x_n) is a Cauchy sequence in Y with $x_n = k_ny$, then for all n and p,

$$|k_n - k_p|\,\|y\| = \|x_n - x_p\|.$$

Hence (k_n) is a Cauchy sequence in $\mathbf{K}$, which is complete. If $k_n \to k$ in $\mathbf{K}$, then $x_n \to ky$ in Y. Thus Y is complete. Assume now that every $m-1$ dimensional subspace of X is complete. Let Y be an m dimensional subspace of X and (x_n) be a Cauchy sequence in Y. Let $\{y_1, \ldots, y_m\}$ be a basis for Y. Then the $m-1$ dimensional subspace $Z = \operatorname{span}\{y_2, \ldots, y_m\}$ is complete by the inductive assumption. Now for each $n = 1, 2, \ldots,$

$$x_n = k_ny_1 + z_n$$

for some $k_n \in \mathbf{K}$ and $z_n \in Z$. By (a) above,

$$\|x_n - x_p\| = \|(k_n - k_p)y_1 + z_n - z_p\| \geq |k_n - k_p| \operatorname{dist}(y_1, Z).$$

Since $y_1 \notin Z$ and Z is closed in X, we see that $\text{dist}(y_1, Z) > 0$. Hence (k_n) is a Cauchy sequence in $\mathbf{K}$. As $z_n = x_n - k_n y_1$, it follows that (z_n) is Cauchy in Z, which is complete. If $k_n \to k$ in $\mathbf{K}$ and $z_n \to z$ in Z, we see that $x_n \to ky_1 + z$ in Y. Thus Y is complete. In particular, Y is closed in X.

Next, let $x_n \in Y$ and $x_n = k_{n,1}y_1 + \cdots + k_{n,m}y_m$, $\quad n = 1, 2, \ldots$. If $k_{n,j} \to k_j$ as $n \to \infty$ for each $j = 1, \ldots, m$, let $x = k_1 y_1 + \cdots + k_n y_n$. Then

$$\|x_n - x\| \leq |k_{n,1} - k_1| \, \|y_1\| + \cdots + |k_{n,m} - k_m| \, \|y_m\| \to 0$$

as $n \to \infty$. Thus $x_n \to x$. Conversely, let $x_n \to x = k_1 y_1 + \cdots + k_m y_m$. Then by (a) above,

$$\begin{aligned} \|x_n - x\| &= \|(k_{n,1} - k_1)y_1 + \cdots + (k_{n,m} - k_m)y_m\| \\ &\geq |k_{n,j} - k_j| \operatorname{dist}(y_j, Y_j), \end{aligned}$$

where $Y_j = \text{span}\{y_i : i = 1, \ldots, m, i \neq j\}$ for $j = 1, \ldots, m$. Since $y_j \notin Y_j$ and Y_j is closed in X, we see that $\text{dist}(y_j, Y_j) > 0$. Hence $k_{n,j} \to k_j$ as $n \to \infty$ for each $j = 1, ..., m$.

Finally, if $(k_{n,j})$ is bounded for each $j = 1, \ldots, m$, say $|k_{n,j}| \leq \alpha_j$ for all $n = 1, 2, \ldots$, then

$$\|x_n\| \leq \alpha_1 \|y_1\| + \cdots + \alpha_m \|y_m\|$$

for all $n = 1, 2, \ldots$, that is, (x_n) is bounded. Conversely, let (x_n) be bounded. For each $j = 1, \ldots, m$, let Y_j be defined as above. Then

$$\|x_n\| = \|k_{n,1}y_1 + \cdots + k_{n,m}y_m\| \geq |k_{n,j}| \operatorname{dist}(y_j, Y_j).$$

by (a) above. Hence $(k_{n,j})$ is bounded for each $j = 1, \ldots, m$. □

We remark that an infinite dimensional subspace of a normed space X may not be closed in X. For example, if $X = \ell^\infty$ and $Y = c_{00}$, then Y is a subspace of X, but Y is not closed in X. In fact, $x_n = (1, \ldots, 1/n, 0, 0, \ldots) \in c_{00}$ for each $n = 1, 2, \ldots$, but $x_n \to (1, 1/2, 1/3, \ldots) \notin c_{00}$. It can be seen that the closure c_{00} is c_0.

We are now in a position to prove the result we are looking for.

5.5 Theorem

Let X be a normed space. The following conditions are equivalent.

(i) Every closed and bounded subset of X is compact.

(ii) The subset $\{x \in X : \|x\| \leq 1\}$ of X is compact.

(iii) X is finite dimensional.

Proof:

Since $\{x \in X : \|x\| \leq 1\}$ is closed and bounded, (i) implies (ii).

Suppose now that (ii) holds. Let, if possible, $\{z_1, z_2, \ldots\}$ be an infinite linearly independent subset of X and consider

$$Z_n = \operatorname{span}\{z_1, \ldots, z_n\}, \quad n = 1, 2, \ldots.$$

Being finite dimensional, Z_n is a closed subspace of Z_{n+1} by 5.4(b). Also, $Z_n \neq Z_{n+1}$, since $\{z_1, \ldots, z_{n+1}\}$ is linearly independent. By the Riesz lemma (5.3), there is some $x_n \in Z_{n+1}$ such that

$$\|x_n\| = 1 \quad \text{and} \quad \operatorname{dist}(x_n, Z_n) \geq \frac{1}{2}.$$

Now (x_n) is a sequence in $\{x \in X : \|x\| \leq 1\}$ having no convergent subsequence, because $\|x_n - x_m\| \geq 1/2$ for all $m \neq n$. Hence the set $\{x \in X : \|x\| \leq 1\}$ cannot be compact. Thus (ii) implies (iii).

Finally, suppose that X is finite dimensional and let E be a closed and bounded subset of X. To prove that E is compact, we show that every sequence in E has a subsequence which converges in E. Consider a sequence (x_n) in E. Let $\{y_1, \ldots, y_m\}$ be a basis for X, and

$$x_n = k_{n,1}y_1 + \cdots + k_{n,m}y_m, \quad n = 1, 2, \ldots.$$

Then by 5.4(b), $(k_{n,j})$ is bounded for each $j = 1, \ldots, m$. By the Bolzano-Weierstrass theorem (3.7(b)) for $\mathbf{K}$ and by passing to a subsequence of a subsequence several times, we find $n_1 < n_2 < \cdots$ such that $(k_{n_p,j})$ converges in $\mathbf{K}$ as $p \to \infty$ for each $j = 1, \ldots, m$. By 5.4(b), the subsequence (x_{n_p}) converges to some $x \in X$. Since $x_{n_p} \in E$ and E is closed, we have $x \in E$, that is, (x_{n_p}) converges in E. This shows that (iii) implies (i). □

We shall now study some aspects of the interplay between the linear structure and the metric structure of a normed space X.

Let $x \in X$ and $r > 0$. The closure of the open ball

$$U(x,r) = \{y \in X : \|x - y\| < r\}$$

about x of radius r is, in fact, the closed ball about x of radius r, namely

$$\overline{U}(x,r) = \{y \in X : \|x - y\| \leq r\}.$$

This can be seen as follows. If $y \in X$ and $\|x - y\| = r$, then $ty + (1-t)x$ belongs to $U(x,r)$ for every $t \in (0,1)$ and converges to y as $t \to 1$. Note that both $U(x,r)$ and $\overline{U}(x,r)$ are convex subsets of X.

5.6 Theorem

Let X be a normed space.

(a) If E_1 is open in X and $E_2 \subset X$, then $E_1 + E_2$ is open in X.

(b) Let E be a convex subset of X. Then the interior E^o of E and the closure $\overline{E}$ of E are also convex. If $E^o \neq \emptyset$, then $\overline{E} = \overline{E^o}$.

(c) Let Y be a subspace of X. Then $Y^o \neq \emptyset$ if and only if $Y = X$.

Proof:

(a) Let $x \in X$ and $x_1 \in E_1$. Since E_1 is open, there is some $r > 0$ such that $U(x_1, r) \subset E_1$. But $U(x_1 + x, r) = U(x_1, r) + x \subset E_1 + x$.

Hence $E_1 + x$ is open for every $x \in X$. Since

$$E_1 + E_2 = \bigcup\{E_1 + x_2 : x_2 \in E_2\},$$

it follows that $E_1 + E_2$ is open.

(b) Let $0 < t < 1$. Since E is convex, $tE^\circ + (1-t)E^\circ \subset E$. Also, by (a) above, $tE^\circ + (1-t)E^\circ$ is open. Hence $tE^\circ + (1-t)E^\circ \subset E^\circ$, that is, E° is convex.

Next, let $x, y \in \overline{E}$. Find sequences (x_n) and (y_n) in E such that $x_n \to x$, and $y_n \to y$. Since $tx_n + (1-t)y_n \to tx + (1-t)y$, we see that $tx + (1-t)y \in \overline{E}$. Thus $\overline{E}$ is convex.

Now assume that $E^\circ \neq \emptyset$. Clearly, $\overline{E^\circ} \subset \overline{E}$. To show $\overline{E} \subset \overline{E^\circ}$, it is enough to show that $E \subset \overline{E^\circ}$. Consider $a \in E^\circ$.

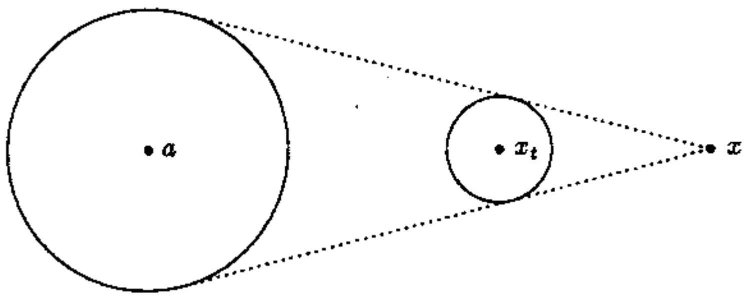


Figure 4

Let $r > 0$ be such that $U(a, r) \subset E$, that is, $a + ry \in E$ for every $y \in X$ with $\|y\| < 1$. Let $x \in E$ and $x_t = ta + (1-t)x$. We claim that $U(x_t, tr) \subset E$, that is, $x_t + try \in E$ for every $y \in X$ with $\|y\| < 1$. This is obvious from the convexity of E, because $x_t + try = t(a + ry) + (1-t)x$, where $a + ry \in E$ and $x \in E$. Thus $x_t \in E^\circ$. Since $x_t \to x$ as $t \to 0$, we have $x \in \overline{E^\circ}$.

(c) If $Y = X$, then $Y^\circ = X \neq \emptyset$. Conversely, let $Y^\circ \neq \emptyset$. Consider $a \in Y^\circ$ and let $r > 0$ be such that $\overline{U}(a, r) \subset Y$. Then

$\overline{U}(0,r) = \overline{U}(a,r) - a \subset Y$. Let $x \in X$. If $x = 0$, then clearly $x \in Y$. If $x \neq 0$, then $\dfrac{rx}{\|x\|} \in \overline{U}(0,r) \subset Y$. Hence $x \in Y$ as well. □

The geometry of the **unit sphere** $\{x \in X : \|x\| = 1\}$ of a nonzero normed space X plays an important role in the analysis of functions on X. Keeping this in mind, we introduce the following concept.

A normed space X is said to be **strictly convex** if for $x \neq y$ in X with $\|x\| = 1 = \|y\|$, we have $\|x + y\| < 2$.

This says that the mid-point $(x+y)/2$ of two distinct points x and y on the unit sphere of X does not lie on the unit sphere of X, but it lies in the open unit ball $U(0,1)$ of X. In particular, no line segment lies on the unit sphere.

It is easy to see that if $n \geq 2$, then $\mathbf{K}^n$ with either of the norms $\| \ \|_1$ and $\| \ \|_\infty$ is not strictly convex. Let

$$x = (1,0,0,\ldots,0), y = (0,1,0,\ldots,0) \quad \text{and} \quad z = (1,1,0,\ldots,0).$$

Then $x \neq y, \|x\|_1 = 1 = \|y\|_1$, but $\|x+y\|_1 = 2$, and $x \neq z, \|x\|_\infty = 1 = \|z\|_\infty$, but $\|x + z\|_\infty = 2$. However, $\mathbf{K}^n$ with the norm $\| \ \|_2$ is strictly convex. For if $|x(1)|^2 + \cdots + |x(n)|^2 = 1 = |y(1)|^2 + \cdots + |y(n)|^2$, and $|x(1) + y(1)|^2 + \cdots + |x(n) + y(n)|^2 = 4$, then the real part of $x(1)\overline{y(1)} + \cdots + x(n)\overline{y(n)}$ equals 1 and hence $|x(1) - y(1)|^2 + \cdots + |x(n) - y(n)|^2 = 0$, that is, $x = y$.

Consider norms $\| \ \|, \| \ \|'$ and the associated metrics $d(x,y) = \|x - y\|, d'(x,y) = \|x - y\|'$ on a linear space X. The norm $\| \ \|$ is said to be **stronger** than (resp., **equivalent** to) the norm $\| \ \|'$ if the metric d is stronger than (resp., equivalent to) the metric d'. The norms $\| \ \|$ and $\| \ \|'$ are said to be **comparable** if one of them is stronger than the other.

5.7 Theorem

Let $\| \ \|$ and $\| \ \|'$ be norms on a linear space X. Then the norm $\| \ \|$

is stronger than the norm $\| \ \|'$ if and only if there is some $\alpha > 0$ such that $\|x\|' \leq \alpha\|x\|$ for all $x \in X$.

Further, the norm $\| \ \|$ is equivalent to the norm $\| \ \|'$ if and only if there are $\alpha > 0$ and $\beta > 0$ such that $\beta\|x\| \leq \|x\|' \leq \alpha\|x\|$ for all $x \in X$.

Proof:
Let $\| \ \|$ be stronger than $\| \ \|'$. Then there is some $r > 0$ such that

$$0 \in \{x \in X : \|x\| < r\} \subset \{x \in X : \|x\|' < 1\}.$$

Let $0 \neq x \in X$ and $\epsilon > 0$. Since $\|\dfrac{rx}{(1+\epsilon)\|x\|}\| < r$, it follows that $\|\dfrac{rx}{(1+\epsilon)\|x\|}\|' < 1$, that is, $r\|x\|' \leq (1+\epsilon)\|x\|$. As this is true for every $\epsilon > 0$, it follows that $\|x\|' \leq \alpha\|x\|$ with $\alpha = 1/r$.

Conversely, let $\|x\|' \leq \alpha\|x\|$ for all $x \in X$. Let (x_n) be a sequence in X such that $d(x_n, x) = \|x_n - x\| \to 0$. Since $\|x_n - x\|' \leq \alpha\|x_n - x\|$ for all n, we see that $d'(x_n, x) = \|x_n - x\| \to 0$. Hence the metric d (resp., the norm $\| \ \|$) is stronger than the metric d' (resp., the norm $\| \ \|'$).

The result about equivalence of norms follows immediately. $\square$

To illustrate Theorem 5.7, consider the norms $\| \ \|_1, \| \ \|_2$ and $\| \ \|_\infty$ on $\mathbf{K}^n$. Then it is easy to see that for all $x \in \mathbf{K}^n$,

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \quad \text{and} \quad \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty.$$

Thus the three norms are equivalent. In fact, it will follow from 6.3(b) that all norms on $\mathbf{K}^n$ are equivalent. On the other hand, if we let for $x \in c_{00}$,

$$\|x\| = \sup_{j=1,2,\ldots} \left\{ (2j-1)|x(2j-1)|, \frac{|x(2j)|}{2j} \right\},$$

then the norms $\| \ \|_\infty$ and $\| \ \|$ on c_{00} are not comparable. For if $x_n = (0, \ldots, 0, 1, 0, 0, \ldots)$, where 1 occurs only in the nth entry, then $\|x_n\|_\infty = 1$, $\|x_{2n-1}\| = 2n-1$ and $\|x_{2n}\| = 1/2n$ for all $n = 1, 2, \ldots$.

It is interesting to note that if X is a separable normed space, then there is an equivalent norm on X in which it is strictly convex. (See Problem 7-18(b).)

Problems

5-1 Let $\| \ \|$ be a norm on a linear space X. If $x, y \in X$ and $\|x+y\| = \|x\| + \|y\|$, then $\|sx+ty\| = s\|x\| + t\|y\|$ for all $s \geq 0, t \geq 0$. (Hint: If $s \geq t$, write $sx + ty = s(x+y) - (s-t)y$.)

5-2 Let $X = \mathbf{K}^3$. For $x = (x(1), x(2), x(3)) \in X$, let

$$\|x\| = \left[(|x(1)|^2 + |x(2)|^2)^{3/2} + |x(3)|^3\right]^{1/3}.$$

Then $\| \ \|$ is a norm on $\mathbf{K}^3$.

5-3 Let $1 \leq p \leq \infty$. For the norms $\| \ \|_p$ on $\mathbf{K}^n$ introduced in 5.1(a), we have

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$$

for every $x \in \mathbf{K}^n$,

$$\{x \in \mathbf{K}^n : \|x\|_p < 1\} \subset \{x \in \mathbf{K}^n : \|x\|_r < 1\}$$

for all $1 \leq p < r \leq \infty$, and

$$\bigcup_{1 \leq p < \infty} \{x \in \mathbf{K}^n : \|x\|_p < 1\} = \{x \in \mathbf{K}^n : \|x\|_\infty < 1\}.$$

5-4 Let $\| \ \|_1, \ldots, \| \ \|_m$ be norms on a linear space X. Let $r_1, \ldots, r_m$ be positive real numbers and $1 \leq p \leq \infty$. For $x \in X$, let

$$\|x\| = \begin{cases} (r_1\|x\|_1^p + \cdots + r_m\|x\|_m^p)^{1/p}, & \text{if } 1 \leq p < \infty \\ \max\{r_1\|x\|_1, \ldots, r_m\|x\|_m\}, & \text{if } p = \infty. \end{cases}$$

Then $\| \ \|$ is a norm on X, and $\|x_n - x\| \to 0$ if and only if $\|x_n - x\|_j \to 0$ for each $j = 1, \ldots, m$.

5-5 Let X be a linear space and E be a subset of X which is convex, **balanced** (that is, $kx \in E$ whenever $x \in E, k \in \mathbf{K}$ and $|k| \leq 1$), **absorbing** (that is, for every $x \in X$, there is $r > 0$ such that $x/r \in E$) and which does not contain any nonzero subspace of X. For $x \in X$, let

$$\|x\| = \inf\{r > 0 : \frac{x}{r} \in E\}.$$

Then $\|\ \|$ is a norm on X, called the **Minkowski gauge** of E. Further,

$$\{x \in X : \|x\| < 1\} \subset E \subset \{x \in X : \|x\| \leq 1\}.$$

In particular, if Γ is a simple closed curve in $\mathbf{R}^2$ enclosing a convex set E such that $(0,0) \in E$ and $(-x(1), -x(2)) \in \Gamma$ whenever $(x(1), x(2)) \in \Gamma$, then there is a norm $\|\ \|$ on $\mathbf{R}^2$ such that $\{(x(1), x(2)) \in \mathbf{R}^2 : \|(x(1), x(2)\| = 1\}$ coincides with Γ.

5-6 Let $\|\ \|_j$ be a norm on a linear space X_j for $j = 1, 2, \ldots$. Consider $X = \{(x(1), x(2), \ldots) : x(j) \in X_j, \sum_{j=1}^{\infty} \|x(j)\|_j^p < \infty$, if $1 \leq p < \infty$ and $\sup_{j=1,2,\ldots} \|x(j)\|_j < \infty$, if $p = \infty\}$. Then X is a linear space and for $x = (x(1), x(2), \ldots) \in X$,

$$\|x\| = \begin{cases} \left(\sum_{j=1}^{\infty} \|x(j)\|_j^p\right)^{1/p}, & \text{if } 1 \leq p < \infty \\ \sup_{j=1,2,\ldots} \|x(j)\|_j, & \text{if } p = \infty \end{cases}$$

defines a norm on X.

5-7 Let X be a normed space. The Riesz lemma with $r = 1$ holds in X if and only if for every closed proper subspace Y of X, there are $x \in X$ and $y_0 \in Y$ such that $\|x - y_0\| = \operatorname{dist}(x, Y) > 0$.

5-8 (a) Let $X = \{x \in C([0,1]) : x(0) = 0\}$ with the sup norm and $Y = \left\{x \in X : \int_0^1 x(t)\,dt = 0\right\}$. Then Y is a closed proper subspace of X. But there is no $x_1 \in X$ with $\|x_1\|_\infty = 1$ and $\operatorname{dist}(x_1, Y) = 1$. (Compare 5.3.)

(b) Let Y be a finite dimensional proper subspace of a normed space X. Then there is some $x \in X$ with $\|x\| = 1$ and $\operatorname{dist}(x, Y) = 1$. (Compare 5.3.)

5-9 Let X be a finite dimensional normed space. Then there is a basis $\{x_1, \ldots, x_m\}$ of X such that for each $j = 1, \ldots, m$,

$$\|x_j\| = 1 \quad \text{and} \quad \|x_j - \sum_{i=1}^{j-1} k_i x_i\| \geq 1$$

for all $k_i \in \mathbf{K}$. (Hint: Problem 5-8(b) and induction) In fact, we can require that for each $j = 1, \ldots, m$,

$$\|x_j\| = 1 \quad \text{and} \quad \|x_j - \sum_{i=1, i\neq j}^{m} k_i x_i\| \geq 1$$

for all $k_i \in \mathbf{K}$. (Hint: Let $\{y_1, \ldots, y_m\}$ be a basis for X. For $k_{i,j} \in \mathbf{K}$, let $f(k_{1,1}, \ldots, k_{m,m}) = \det(k_{i,j}), i, j = 1, \ldots, m$. Find $\ell_{i,j} \in \mathbf{K}$ such that $|f(\ell_{1,1} \ldots, \ell_{m,m})| = \max\{|f(k_{1,1}, \ldots, k_{m,m})| : \|k_{i,1}y_1 + \cdots + k_{i,m}y_m\| \leq 1,$ $i = 1, \ldots, m\}$. Let $x_i = \ell_{i,1}y_1 + \cdots + \ell_{i,m}y_m$, $i = 1, \ldots, m\}$.)

5-10 Let X_1 be a closed subspace and X_2 be a finite dimensional subspace of a normed space X. Then $X_1 + X_2$ is closed in X. (Hint: Proof of 5.4(b)).

5-11 Let X be a normed space such that $\overline{U}(0,1)$ is totally bounded. Then X is finite dimensional. (Hint: Proof of '(ii) implies (iii)' of Theorem 5.5)

5-12 Let $n \geq 2$ and $0 < p < 1$. For $x \in \mathbf{K}^n$, let $\|x\|_p = \left[\sum_{j=1}^{n} |x(j)|^p\right]^{1/p}$. Then the set $\{x \in \mathbf{K}^n : \|x\|_p \leq 1\}$ is not convex and $\|\ \|_p$ is not a norm on $\mathbf{K}^n$.

5-13 Let Y be a subspace of a normed space X. Then Y is nowhere dense in X (that is, the interior of the closure of Y is empty) if and only if Y is not dense in X. If Y is a hyperspace in X, then Y is nowhere dense in X if and only if Y is closed in X.

5-14 Let E_1 be a compact subset and E_2 be a closed subset of a normed space X such that $E_1 \bigcap E_2 = \emptyset$. Then $(E_1 + U(0,r)) \bigcap E_2 = \emptyset$ for some $r > 0$.

5-15 Let $1 \leq p \leq \infty$.

(a) The closed unit ball in ℓ^p is convex, closed and bounded, but not compact.

(b) Let (α_j) be a sequence of positive real numbers. Assume that it is p-summable if $1 \leq p < \infty$ and that it tends to zero if $p = \infty$. Then the set

$$E = \{x \in \ell^p : |x(j)| \leq \alpha_j, j = 1, 2, \ldots\}$$

is convex and compact. It is not contained in any finite dimensional subspace of ℓ^p. The interior of E is empty. If $p = 2$ and $\alpha_j = 1/j, j = 1, 2, \ldots$, then E is known as the **Hilbert cube.**

5-16 Let X be a normed space.

(a) If $E \subset X$ with $E^0 \neq \emptyset$, then E spans X.

(b) If $X = c_{00}$ with the norm $\| \ \|_\infty$ and $E = \{x \in X : |x(j)| \leq 1/j, j = 1, 2, \ldots\}$, then E is a convex subset of X, E spans X, but $E^0 = \emptyset$.

5-17 If $1 < p < \infty$, then ℓ^p with the norm $\| \ \|_p$ is strictly convex. (Hint: If $(\sum_{j=1}^\infty |a_j + b_j|^p)^{1/p} = \left(\sum_{j=1}^\infty |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^\infty |b_j|^p\right)^{1/p}$, then either $b_j = 0$ for all j, or $a_j = tb_j$ for some $t \geq 0$ and all j.)

5-18 Let $X = C([a, b])$.

(a) For $x \in X$, let $\|x\|_1 = \displaystyle\int_a^b |x(t)|\,dt$ and $\|x\|_\infty = \sup\{|x(t)| : t$ in $[a, b]\}$. Then the norm $\| \ \|_\infty$ is stronger than but not equivalent to the norm $\| \ \|_1$.

(b) Let $\| \ \|$ be a norm on X. Then there is a sequence (x_n) in X such that $\|x_n\| = 1$ for all n, but $x_n(t) \to 0$ as $n \to \infty$ for all $t \in [a, b]$.

5-19 Let $X = C^1([a, b])$. For $x \in X$, let

$$\|x\| = \|x\|_\infty + \|x'\|_\infty \quad \text{and} \quad \|x\|_0 = |x(a)| + \|x'\|_\infty.$$

Then $\| \ \|$ and $\| \ \|_0$ are equivalent norms on X. In fact, for every $x \in X$,

$$\|x\|_0 \leq \|x\| \leq (b - a + 1)\|x\|_0,$$

the constant $b - a + 1$ being the least possible. Also, if $x \in X$, then there is a sequence (p_n) of polynomials such that $\|x - p_n\| \to 0$.

5-20 Let $1 \leq p < r \leq \infty$. Then

(a) ℓ^r is not contained in ℓ^p.

(b) The norm $\| \ \|_p$ on c_{00} is stronger than but not equivalent to the norm $\| \ \|_r$.

5-21 Let X denote the linear space of all polynomials in one variable with coefficients in $\mathbf{K}$. For $p \in X$ with $p(t) = a_0 + a_1 t + \cdots + a_n t^n$, let

$$\begin{aligned} \|p\| &= \sup\{|p(t)| : 0 \leq t \leq 1\}, \\ \|p\|_1 &= |a_0| + \cdots + |a_n|, \end{aligned}$$

$$\|p\|_\infty = \max\{|a_0|, \ldots, |a_n|\}.$$

Then $\| \ \|$, $\| \ \|_1$ and $\| \ \|_\infty$ are norms on X, $\|p\| \le \|p\|_1$ and $\|p\|_\infty \le \|p\|_1$ for all $p \in X$. The norms $\| \ \|$ and $\| \ \|_1$ are not equivalent. The norms $\| \ \|$ and $\| \ \|_\infty$ are not comparable.

6 Continuity of Linear Maps

Let X and Y be normed spaces and $F : X \to Y$ be a linear map. Must F be continuous? If not, under what conditions will it be continuous? We answer these questions in this section. We also introduce a norm on the linear space of all continuous linear maps from X to Y. Several examples of continuous and discontinuous linear maps are given.

6.1 Theorem
Let X and Y be normed spaces. If X is finite dimensional, then every linear map from X to Y is continuous. Conversely, if X is infinite dimensional and $Y \neq \{0\}$, then there is a discontinuous linear map from X to Y.

Proof:
Let X be finite dimensional and $F : X \to Y$ be linear. If $X = \{0\}$, there is nothing to prove. Let now $X \neq \{0\}$ and $\{a_1, \ldots, a_m\}$ be a basis for X. For a sequence (x_n) in X, let

$$x_n = k_{n,1}a_1 + \cdots + k_{n,m}a_m,$$

where $k_{n,j} \in \mathbf{K}$, $j = 1, \ldots, m$. If $x_n \to x = k_1a_1 + \cdots + k_ma_m$ in X, then $k_{n,j} \to k_j$ as $n \to \infty$ for each $j = 1, \ldots, m$ by 5.4(b), and

$$\begin{aligned} F(x_n) &= k_{n,1}F(a_1) + \cdots + k_{n,m}F(a_m) \\ &\to k_1F(a_1) + \cdots + k_mF(a_m) = F(x) \end{aligned}$$

by the linearity of F and the continuity of addition and scalar multiplication in Y. Thus every linear map from X to Y is continuous.

Now assume that X is infinite dimensional. Let $\{a_1, a_2, \ldots\}$ be an infinite linearly independent subset of X. Let $x_n = a_n/n\|a_n\|$ for $n = 1, 2, \ldots$. Then $L = \{x_1, x_2, \ldots\}$ is an infinite linearly independent subset of X and $\|x_n\| = 1/n$ for all n. By 2.2(b), there exists a basis B for X such that $L \subset B$. Since $Y \neq \{0\}$, consider $0 \neq b \in Y$, define

$$F(x) = \begin{cases} b, & \text{if } x \in L \\ 0, & \text{if } x \in B, x \notin L, \end{cases}$$

and extend F linearly to all of X. Since $x_n \to 0$ as $n \to \infty$, but $F(x_n) = b \neq 0$ for all n, F is not continuous. Thus there exists a linear map from X to Y which is not continuous. □

For a linear map, the continuity condition can be stated in a variety of apparently weaker or stronger ways.

6.2 Theorem

Let X and Y be normed spaces and $F : X \to Y$ be a linear map. The following conditions are equivalent.

(i) F is bounded on $\overline{U}(0, r)$ for some $r > 0$.

(ii) F is continuous at 0.

(iii) F is continuous on X.

(iv) F is uniformly continuous on X.

(v) $\|F(x)\| \leq \alpha\|x\|$ for all $x \in X$ and some $\alpha > 0$.

(vi) The zero space $Z(F)$ of F is closed in X and the linear map $\tilde{F} : X/Z(F) \to Y$ defined by $\tilde{F}(x + Z(F)) = F(x), x \in X$, is continuous.

Proof:
We show that (i) implies (v). The successive reverse implications are immediate. Let $\|F(x)\| \le \beta$ for all $x \in \overline{U}(0,r), r > 0$. If $x = 0$, then $F(x) = 0$, and if $x \neq 0$, then since $rx/\|x\|$ is in $\overline{U}(0,r)$, we have

$$\|F(x)\| = \frac{\|x\|}{r} \left\|F\left(\frac{rx}{\|x\|}\right)\right\| \le \frac{\beta}{r}\|x\|.$$

Next, we show that F is continuous on X if and only if (vi) holds Let F be continuous on X. Then $Z(F) = F^{-1}(\{0\})$ is closed in X, since $\{0\}$ is closed in Y. Hence $X/Z(F)$ is a normed space in the quotient norm given in 5.2(b). It can be easily seen that the map $\tilde{F} : X/Z(F) \to Y$ given by $\tilde{F}(x + Z(F)) = F(x), x \in X$, is well-defined. By (v), there is some $\alpha > 0$ such that $\|F(x)\| \le \alpha\|x\|$ for all $x \in X$. Now let $x \in X$ and $z \in Z(F)$. Then

$$\|\tilde{F}(x + Z(F))\| = \|\tilde{F}(x + z + Z(F))\| = \|F(x + z)\| \le \alpha\|x + z\|.$$

Since this is true for every $z \in Z(F)$, we see that

$$\|\tilde{F}(x + Z(F))\| \le \alpha \inf\{\|x + z\| : z \in Z(F)\} = \alpha|||x + Z(F)|||.$$

Hence $\tilde{F}$ is continuous. Conversely, assume that $Z(F)$ is closed and the linear map $\tilde{F}$ is continuous. Then for some $\alpha > 0$, we have

$$\|F(x)\| = \|\tilde{F}(x + Z(F))\| \le \alpha|||x + Z(F)||| \le \alpha\|x\|, \quad x \in X.$$

Hence F is continuous again by (v). $\square$

6.3 Corollary

(a) A linear map F from a normed space X to a normed space Y is a homeomorphism if and only if there are $\alpha, \beta > 0$ such that

$$\beta\|x\| \le \|F(x)\| \le \alpha\|x\|$$

for all $x \in X$. In case there is a linear homeomorphism from X onto Y, X is complete if and only if Y is complete.

(b) Let X and Y be normed spaces with X finite dimensional. Then every bijective linear map from X to Y is a homeomorphism. All norms on X are equivalent and X is complete in each. If the dimension of X is n, then there is a linear homeomorphism from $\mathbf{K}^n$ onto X.

Proof:

(a) Let F be a linear map from X to Y and

$$\beta\|x\| \le \|F(x)\| \le \alpha\|x\|$$

for some $\alpha, \beta > 0$ and all $x \in X$. Then F is clearly injective and it is continuous by 6.2(v). The inverse map $F^{-1} : R(F) \to X$ is automatically linear by 2.4(a) and $\|F^{-1}(y)\| \le \|y\|/\beta$ for all $y \in Y$. Hence F^{-1} is also continuous, showing that F is a homeomorphism.

Conversely, let F be a linear homeomorphism from X to Y. By 6.2(v), there are $\alpha, \gamma > 0$ such that

$$\|F(x)\| \le \alpha\|x\| \quad \text{and} \quad \|F^{-1}(y)\| \le \gamma\|y\|$$

for all $x \in X$ and $y \in Y$, that is,

$$\frac{\|x\|}{\gamma} \le \|F(x)\| \le \alpha\|x\|$$

for all $x \in X$, as desired.

Now suppose that there is a linear homeomorphism from X onto Y. Then (x_n) is a Cauchy sequence in X (resp., (x_n) converges in X) if and only if $(F(x_n))$ is a Cauchy sequence in Y (resp., $(F(x_n))$ converges in Y). Hence X is complete if and only if Y is complete.

(b) Let X be finite dimensional and $F : X \to Y$ be linear. Then F is continuous by 6.1. If F is bijective, then Y is also finite dimensional and F^{-1} is linear as well as continuous, that is, F is a homeomorphism.

If $\|\ \|$ and $\|\ \|'$ are norms on a finite dimensional linear space X, then since the identity map $I : X \to X$ is obviously linear and bijective, we have by (a) above

$$\beta\|x\| \leq \|I(x)\|' = \|x\|' \leq \alpha\|x\|$$

for all $x \in X$. Hence the norms $\| \ \|$ and $\| \ \|'$ are equivalent by 5.7.

If X has dimension n, there is a bijective linear map $F : \mathbf{K}^n \to X$. Since F is a homeomorphism and $\mathbf{K}^n$ is complete in any of the norms $\| \ \|_p, 1 \leq p \leq \infty$, it follows that X is also complete in every norm. $\square$

6.4 Corollary

Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range $R(F)$ of F is finite dimensional. Then F is continuous if and only if the zero space $Z(F)$ of F is closed in X.

In particular, a linear functional f on X is continuous if and only if $Z(f)$ is closed in X.

Proof:

If $R(F) = \{0\}$, then there is nothing to prove. Let $R(F) \neq \{0\}$ and $\{y_1, \ldots, y_m\}$ be a basis for $R(F)$. Let $x_j \in X$ be such that $F(x_j) = y_j, j = 1, \ldots, m$. Then

$$\operatorname{span}\{x_1 + Z(F), \ldots, x_m + Z(F)\} = X/Z(F).$$

This can be seen as follows. If $x \in X$ and $F(x) = k_1 y_1 + \cdots + k_m y_m$, then $x - k_1 x_1 - \cdots - k_m x_m \in Z(F)$. Hence

$$\begin{aligned} x + Z(F) &= (k_1 x_1 + \cdots + k_m x_m) + Z(F) \\ &= k_1(x_1 + Z(F)) + \cdots + k_m(x_m + Z(F)). \end{aligned}$$

In particular, $X/Z(F)$ is finite dimensional.

If $Z(F)$ is closed in X, then $X/Z(F)$ is a finite dimensional normed space. Now 6.1 shows that the linear map $\tilde{F} : X/Z(F) \to Y$ given by $\tilde{F}(x + Z(F)) = F(x), x \in X$, is continuous. By 6.2(vi), F is continuous if and only if $Z(F)$ is closed in X.

If f is a linear functional on X, then $Y = \mathbf{K}$ and $R(f)$ is of dimension ≤ 1. Hence f is continuous if and only if $Z(f)$ is closed. $\square$

6.5 Examples

(a) An important example of a discontinuous linear map is given by the operation of differentiation. Let $X = C^1([0,1])$, the space of all continuously differentiable scalar-valued functions on [0,1] and $Y = C([0,1])$, the space of all continuous scalar-valued functions on [0,1], both with the sup norms. For $x \in X$, let $F(x) = x'$, the derivative of x on [0, 1]. Then F is clearly linear. But F is not continuous since if $x_n(t) = t^n, t \in [0,1]$, then $\|x_n\|_\infty = 1$, while $\|F(x_n)\| = \|x_n'\| = n$ for every $n = 1, 2, \ldots$, so that condition (v) of Theorem 6.2 is violated. Note that $Z(F)$ consists of the set of all constant functions on [0,1], which is closed in X, and that $R(F) = Y$, which is infinite dimensional. This shows that we cannot drop the assumption of the finite dimensionality of $R(F)$ from 6.4.

Define $f : X \to \mathbf{K}$ by $f(x) = x'(1), x \in X$. Again, the linear functional f is not continuous. By 6.4, $Z(f) = \{x \in X : x'(1) = 0\}$ cannot be closed. This also follows directly by considering $z_n(t) = t - t^n/n, t \in [0,1]$, and noting that $z_n \to x_1$ in $X, z_n \in Z(f)$, but $x_1 \notin Z(f)$.

(b) A linear map on a linear space X may be continuous with respect to some norm on X, but discontinuous with respect to another norm on X. To illustrate this, let $X = c_{00}$, the space of all scalar sequences which have only finitely many nonzero members. Consider the linear map

$$f_0(x) = x(1) + x(2) + \cdots, \quad x \in X.$$

The map f_0 is continuous with respect to the norm $\| \ \|_1$, since

$$|f_0(x)| \le \sum_{j=1}^{\infty} |x(j)| = \|x\|_1, \quad x \in X.$$

But f_0 is discontinuous with respect to the norm $\| \ \|_2$, since $x_n = (1, 1/2, \ldots, 1/n, 0, 0, \ldots) \in c_{00}$ and

$$\|x_n\|_2^2 = \sum_{j=1}^{n} \frac{1}{j^2} \le \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6},$$

while

$$f_0(x_n) = \sum_{j=1}^{n} \frac{1}{j} \to \infty \quad \text{as} \quad n \to \infty.$$

Similarly, consider the linear map

$$f_1(x) = \sum_{j=1}^{\infty} \frac{x(j)}{j}, \quad x \in X.$$

Then f_1 is continuous with respect to the norm $\| \ \|_2$, since

$$|f_1(x)| \le \sum_{j=1}^{\infty} \frac{|x(j)|}{j} \le \left(\sum_{j=1}^{\infty} \frac{1}{j^2}\right)^{1/2} \left(\sum_{j=1}^{\infty} |x(j)|^2\right)^{1/2}.$$

But f_1 is discontinuous with respect to the norm $\| \ \|_\infty$, since for $x_n = (1, \ldots, 1, 0, 0, \ldots)$, where 1 appears n times, we have

$$\|x_n\|_\infty = 1, \quad \text{while} \quad f_1(x_n) = \sum_{j=1}^{n} \frac{1}{j} \to \infty \quad \text{as} \quad n \to \infty.$$

For further examples of this kind, see Problem 6-17.

(c) A rich supply of linear maps comes from matrices. If $M = (k_{i,j})$ is an $m \times n$ matrix with scalar entries and $x \in \mathbf{K}^n$, let $Mx \in \mathbf{K}^m$ be defined by

$$Mx(i) = \sum_{j=1}^{n} k_{i,j} x(j), \quad i = 1, \ldots, m.$$

This map from $\mathbf{K}^n$ to $\mathbf{K}^m$ is clearly linear and 6.1 shows that it is continuous with respect to all norms on $\mathbf{K}^n$ and $\mathbf{K}^m$.

Analogously, consider an infinite matrix $M = (k_{i,j})$ with scalar entries. We say that M defines a linear map from a sequence space X to a sequence space Y if for every $x = (x(1), x(2), \ldots) \in X$ and each $i = 1, 2, \ldots$, the series $\sum_{j=1}^{\infty} k_{i,j} x(j)$ is convergent and if we let

$$Mx(i) = \sum_{j=1}^{\infty} k_{i,j} x(j),$$

then $Mx \in Y$.

First, let $X = \ell^1 = Y$. Consider the supremum of the column sums of the matrix $|M| = (|k_{i,j}|)$, namely,

$$\alpha_1 = \sup\{\sum_{i=1}^{\infty} |k_{i,j}| : j = 1, 2, \ldots\}.$$

Assume that $\alpha_1 < \infty$. For $x \in \ell^1$, we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |k_{i,j}x(j)| \leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |k_{i,j}|\ |x(j)| \leq \alpha_1 \|x\|_1.$$

Thus for each $i = 1, 2, \ldots, \sum_{j=1}^{\infty} |k_{i,j}x(j)| < \infty$. Hence the series $\sum_{j=1}^{\infty} k_{i,j}x(j)$ is convergent. Also,

$$\|Mx\|_1 = \sum_{i=1}^{\infty} |Mx(i)| = \sum_{i=1}^{\infty} |\sum_{j=1}^{\infty} k_{i,j}x(j)| \leq \alpha_1 \|x\|_1.$$

Thus M defines a continuous linear map from ℓ^1 to ℓ^1, provided $\alpha_1 < \infty$. We shall prove in Section 9 that conversely, if M defines a linear map from ℓ^1 to ℓ^1 as above, then $\alpha_1 < \infty$.

Next, let $X = \ell^\infty = Y$. Consider the supremum of the row sums of the matrix $|M| = (|k_{i,j}|)$, namely,

$$\alpha_\infty = \sup\{\sum_{j=1}^{\infty} |k_{i,j}| : i = 1, 2, \ldots\}.$$

Assume that $\alpha_\infty < \infty$. For $x \in \ell^\infty$ and $i = 1, 2, \ldots,$ we have

$$\sum_{j=1}^{\infty} |k_{i,j}x(j)| \leq \alpha_\infty \|x\|_\infty.$$

Hence the series $\sum_{j=1}^{\infty} k_{i,j}x(j)$ is convergent and

$$\begin{aligned} \|Mx\|_\infty &= \sup\{|Mx(i)| : i = 1, 2, \ldots\} \\ &= \sup\{|\sum_{j=1}^{\infty} k_{i,j}x(j)| : i = 1, 2, \ldots\} \leq \alpha_\infty \|x\|_\infty. \end{aligned}$$

Thus M defines a continuous linear map from ℓ^∞ to ℓ^∞, provided $\alpha_\infty < \infty$. Again, conversely, if M defines a linear map from ℓ^∞ to ℓ^∞, then α_∞n>∞, as we shall see in Section 9.

Finally, let $X = \ell^p = Y, 1 < p < \infty$. Assume that $\alpha_1 < \infty$, and $\alpha_\infty < \infty$. Let $x \in \ell^p$. Then $x \in \ell^\infty$. Since $\alpha_\infty < \infty$, the series $\sum_{j=1}^\infty k_{i,j}x(j)$ is convergent for each $i = 1, 2, \ldots$. Writing

$$|k_{i,j}x(j)| = |k_{i,j}|^{1/q}(|k_{i,j}|^{1/p}|x(j)|),$$

where $1/p + 1/q = 1$ and letting n tend to ∞ in Hölder's inequality (3.1(a)), we have

$$\begin{aligned}
\|Mx\|_p^p &= \sum_{i=1}^\infty |Mx(i)|^p = \sum_{i=1}^\infty |\sum_{j=1}^\infty k_{i,j}x(j)|^p \\
&\le \sum_{i=1}^\infty \left(\sum_{j=1}^\infty |k_{i,j}x(j)|\right)^p \\
&\le \sum_{i=1}^\infty \left(\sum_{j=1}^\infty |k_{i,j}|\right)^{p/q} \left(\sum_{j=1}^\infty |k_{i,j}|\,|x(j)|^p\right) \\
&\le \alpha_\infty^{p/q} \sum_{j=1}^\infty \sum_{i=1}^\infty |k_{i,j}|\,|x(j)|^p \le \alpha_\infty^{p/q}\alpha_1\|x\|_p^p.
\end{aligned}$$

Hence

$$\|Mx\|_p \le \alpha_1^{1/p}\alpha_\infty^{1/q}\|x\|_p.$$

Thus M defines a continuous linear map from ℓ^p to ℓ^p, if $\alpha_1 < \infty$ and $\alpha_\infty < \infty$. We shall presently show that these conditions are not necessary. But first we give another sufficient condition. If $1 < p < \infty$ and $1/p + 1/q = 1$, let

$$\beta_p = \left[\sum_{i=1}^\infty \left(\sum_{j=1}^\infty |k_{i,j}|^q\right)^{p/q}\right]^{1/p}$$

In particular,

$$\beta_2 = \left(\sum_{i=1}^\infty \sum_{j=1}^\infty |k_{i,j}|^2\right)^{1/2}$$

Assume that $\beta_p < \infty$. Let $x \in \ell^p$ For each $i = 1, 2, \ldots$,

$$\sum_{j=1}^\infty |k_{i,j}x(j)| \le \left(\sum_{j=1}^\infty |k_{i,j}|^q\right)^{1/q} \left(\sum_{j=1}^\infty |x(j)|^p\right)^{1/p} < \infty.$$

Hence the series $\sum_{j=1}^{\infty} k_{i,j}x(j)$ is convergent and

$$\|Mx\|_p^p \le \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |k_{i,j}|^q \right)^{p/q} \left(\sum_{j=1}^{\infty} |x(j)|^p \right) \le \beta_p^p \|x\|_p^p.$$

Hence $\|Mx\|_p \le \beta_p \|x\|_p$. Thus M defines a continuous linear map from ℓ^p to ℓ^p, provided $\beta_p < \infty$. Again, this condition is not necessary.

Let

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \dots \\ 0 & 1 & 0 \dots \\ 0 & 0 & 1 \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \dots \\ \frac{1}{2} & 0 & 0 \dots \\ \frac{1}{3} & 0 & 0 \dots \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

For M_1, we have $\alpha_1 < \infty$, $\alpha_\infty < \infty$, but $\beta_p = \infty$, while for M_2, we have $\alpha_1 = \infty = \alpha_\infty$, but $\beta_p < \infty$. Thus both M_1 and M_2 define continuous linear maps from ℓ^2 to ℓ^2, and so does $M_1 + M_2$ for which $\alpha_1 = \alpha_\infty = \beta_p = \infty$!

As a matter of fact, no worthwhile necessary as well as sufficient conditions in terms of the entries of a matrix M are known for M to define a continuous linear map from ℓ^p to $\ell^p, 1 < p < \infty$.

Sufficient conditions for M to define a continuous linear map from ℓ^p to $\ell^r, 1 \le p, r \le \infty$, are given in Problems 6-13 and 6-14.

(d) Analogous to the ℓ^p-case treated in (c) above, consider $X = L^p([a,b]) = Y$. The L^p-analogue of a matrix transformation on ℓ^p is a **Fredholm integral operator** defined as follows. Let $k(.,.)$ be a measurable function on $[a,b] \times [a,b]$. For each $x \in L^p([a,b])$, let

$$F(x)(s) = \int_a^b k(s,t)x(t)dm(t)$$

be well-defined for almost every $s \in [a,b]$ and $F(x) \in L^p([a,b])$. Note that the discrete variables i and j in the ℓ^p-case are replaced by the continuous variables s and t, and summation is replaced by integration. Let

$$\begin{aligned} \alpha_1 &= \operatorname{essup}\{\int_a^b |k(s,t)|dm(s) : t \in [a,b]\}, \\ \alpha_\infty &= \operatorname{essup}\{\int_a^b |k(s,t)|dm(t) : s \in [a,b]\}, \\ \beta_p &= \left[\int_a^b \left(\int_a^b |k(s,t)|^q dm(t)\right)^{p/q} dm(s)\right]^{1/p}, \end{aligned}$$

where $1 < p < \infty$ and $1/p + 1/q = 1$. If $\alpha_1 < \infty$, then F is a continuous linear map from $L^1([a,b])$ to $L^1([a,b])$, and if $\alpha_\infty < \infty$, then F is a continuous linear map from $L^\infty([a,b])$ to $L^\infty([a,b])$. Also, if $\alpha_1 < \infty$ and $\alpha_\infty < \infty$, or if $\beta_p < \infty$, then F is a continuous linear map from $L^p([a,b])$ to $L^p([a,b]), 1 < p < \infty$. The proofs are similar to the proofs in the ℓ^p-case, but additional care must be taken to justify the measurability of functions involved therein. See [26] for a detailed account of the case $p = 2$.

Bounded Linear Maps

It follows from 6.2 that a linear map from a normed space X to a normed space Y is continuous if and only if it maps bounded sets in X onto bounded sets in Y. Hence such a map is known as a **bounded linear map.** The set of all such maps will be denoted by $BL(X,Y)$. A linear map from X to itself is called **an operator** on X. We write $BL(X)$ for the set $BL(X,X)$ of all **bounded operators** on X. Also, we write X' for the set $BL(X,\mathbf{K})$ of all **bounded linear functionals** on X.

If $F \in BL(X,Y)$ and $F \neq 0$, then F is not bounded on X in the usual sense of the word 'bounded'. However, we have

$$\|F(x)\| \leq \alpha\|x\|, \quad x \in X,$$

for some $\alpha > 0$.

We say that a linear map F from X to Y is **bounded below** if

$$\beta\|x\| \leq \|F(x)\|, \quad x \in X,$$

for some $\beta > 0$.

Thus 6.3(a) says that a linear map F is a homeomorphism if and only if it is bounded and bounded below.

It can be easily seen that $BL(X, Y)$ is a linear space under the pointwise operations: For $x \in X$, we have

$$(F + G)(x) = F(x) + G(x) \quad \text{and} \quad (kF)(x) = kF(x).$$

6.6 Theorem

Let X and Y be normed spaces. For $F \in BL(X, Y)$, define

$$\|F\| = \sup\{\|F(x)\| : x \in X, \|x\| \leq 1\}.$$

Then $\| \ \|$ is a norm on $BL(X, Y)$, called the **operator norm**. For all $x \in X$, we have

$$\|F(x)\| \leq \|F\| \, \|x\|.$$

In fact,

$$\|F\| = \inf\{\alpha \geq 0 : \|F(x)\| \leq \alpha\|x\| \text{ for all } x \in X\}.$$

Also, if $X \neq \{0\}$, then

$$\|F\| = \sup\{\|F(x)\| : x \in X, \|x\| = 1\} = \sup\{\|F(x)\| : x \in X, \|x\| < 1\}$$

Proof:

If $X = \{0\}$, then there is nothing to prove. Let $X \neq \{0\}$. It is clear that $\| \ \|$ is a norm on $BL(X, Y)$. Let

$$\begin{aligned}
\alpha_0 &= \inf\{\alpha \geq 0 : \|F(x)\| \leq \alpha\|x\| \text{ for all } x \in X\}, \\
\beta &= \sup\{\|F(x)\| : x \in X, \|x\| = 1\}, \\
\gamma &= \sup\{\|F(x)\| : x \in X, \|x\| < 1\}.
\end{aligned}$$

Then clearly, $\beta \le \|F\|$ and $\gamma \le \|F\|$. Consider a nonzero $x \in X$ and $0 < r \le 1$. Since F is linear, we have

$$\|F(x)\| = \left\|F\left(\frac{rx}{\|x\|}\right)\right\| \frac{\|x\|}{r} \le \sup\{\|F(z)\| : z \in X, \|z\| = r\}\frac{\|x\|}{r}.$$

If $r = 1$, then $\|F(x)\| \le \beta\|x\|$. Hence $\alpha_0 \le \beta$. If $r < 1$, then

$$\|F(x)\| \le \sup\{\|F(z)\| : z \in X, \|z\| < 1\}\frac{\|x\|}{r} = \frac{\gamma\|x\|}{r}$$

Letting $r \to 1$, we see that $\|F(x)\| \le \gamma\|x\|$ for all $x \in X$. Hence $\alpha_0 \le \gamma$.

Finally, we show that $\|F\| \le \alpha_0$. Consider $\alpha \ge 0$ such that $\|F(x)\| \le \alpha\|x\|$ for all $x \in X$. Taking supremum over all $x \in X$ with $\|x\| \le 1$, we obtain $\|F\| \le \alpha$. Since α_0 is the infimum of all such α's, we obtain $\|F\| \le \alpha_0$.

Thus $\|F\| \le \alpha_0 \le \min\{\beta, \gamma\} \le \|F\|$ and the proof is complete. $\square$

We note that the computation of the operator norm of a nonzero element F in $BL(X, Y)$ involves a constrained optimization problem:

'Maximize $\|F(x)\|$, subject to $\|x\| = 1, x \in X$'.

In general, there may not exist $x \in X$ with $\|x\| = 1$ for which $\|F\| = \|F(x)\|$. However, for every $\epsilon > 0$, there is some $x_\epsilon \in X$ such that $\|x_\epsilon\| = 1$ and $\|F\| < \|F(x_\epsilon)\| + \epsilon$. The task of calculating $\|F\|$ is often very difficult. Only in some special cases this can be accomplished. Otherwise one has to be satisfied with some upper bound for $\|F\|$. The following examples illustrate this point.

6.7 Examples

(a) Let $M = (k_{i,j})$ be an $m \times n$ matrix with scalar entries. Consider the linear map from $\mathbf{K}^n$ to $\mathbf{K}^m$ defined by M: For $x \in \mathbf{K}^n$, let

$$Mx(i) = \sum_{j=1}^{n} k_{i,j}x(j), \quad i = 1, \ldots, m.$$

Let $1 \leq p \leq \infty$. Consider the norm $\| \ \|_p$ on $\mathbf{K}^n$ as well as on $\mathbf{K}^m$ and let $\|M\|_p$ denote the operator norm of M. Taking a clue from 6.5(c), we let

$$\begin{aligned} \alpha_1 &= \max_{j=1,\ldots,n} \sum_{i=1}^{m} |k_{i,j}|, \quad \alpha_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |k_{i,j}|, \\ \beta_p &= \left[\sum_{i=1}^{m} \left(\sum_{j=1}^{n} |k_{i,j}|^q \right)^{p/q} \right]^{1/p}, 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

As in 6.5(c), it follows that

$$\|M\|_p \leq \begin{cases} \alpha_1, & \text{if } p = 1 \\ \min\{\alpha_1^{1/p} \alpha_\infty^{1/q}, \beta_p\}, & \text{if } 1 < p < \infty \\ \alpha_\infty, & \text{if } p = \infty. \end{cases}$$

Let $p = 1$. We show that $\|M\|_1 = \alpha_1$. There is an integer a with $1 \leq a \leq n$ such that

$$\alpha_1 = |k_{1,a}| + \cdots + |k_{m,a}|.$$

Define $x \in \mathbf{K}^n$ by

$$x(j) = \delta_{a,j}, \quad j = 1, \ldots, n.$$

Here $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$. Then $\|x\|_1 = 1$ and

$$\|Mx\|_1 = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} k_{i,j} x(j) \right| = \sum_{i=1}^{m} |k_{i,a}| = \alpha_1,$$

proving that $\|M\|_1 = \alpha_1$.

Next, let $p = \infty$. We show that $\|M\|_\infty = \alpha_\infty$. There is an integer b with $1 \leq b \leq m$ such that

$$\alpha_\infty = |k_{b,1}| + \cdots + |k_{b,n}|.$$

Define $x \in \mathbf{K}^n$ by

$$x(j) = \operatorname{sgn}(k_{b,j}), \quad j = 1, \ldots, n.$$

Here $\operatorname{sgn}(z) = \overline{z}/|z|$ if $0 \neq z \in \mathbf{C}$ and $\operatorname{sgn}(z) = 0$ if $z = 0$. If $\alpha_\infty = 0$, then there is nothing to prove. If $\alpha_\infty \neq 0$, then $\|x\|_\infty = 1$ and

$$\|Mx\|_\infty = \max_{i=1,\dots,m} \Big|\sum_{j=1}^{n} k_{i,j}x(j)\Big| \geq \sum_{j=1}^{n} |k_{b,j}| = \alpha_\infty,$$

proving that $\|M\|_\infty = \alpha_\infty$.

Finally, let $1 < p < \infty$. Then $\|M\|_p$ can be less than both $\alpha_1^{1/p}\alpha_\infty^{1/q}$ and β_p. For example, let $m = n = p = 2$ and $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\alpha_1 = 2 = \alpha_\infty$ and $\beta_2 = \sqrt{3}$. Also, $\|M\|_2 = \dfrac{1+\sqrt{5}}{2}$. This can be seen as follows. Let $(x(1), x(2)) \in \mathbf{K}^2$ with $|x(1)|^2 + |x(2)|^2 = 1$. Then

$$\begin{aligned} |x(1) + x(2)|^2 + |x(1)|^2 &\leq (|x(1)| + |x(2)|)^2 + |x(1)|^2 \\ &= 1 + 2|x(1)|\,|x(2)| + |x(1)|^2. \end{aligned}$$

Maximizing $f(s) = 1 + 2s\sqrt{1-s^2} + s^2$ for $s \in [0,1]$ and considering $x(1) = \left(\dfrac{5+\sqrt{5}}{10}\right)^{1/2}, x(2) = \left(\dfrac{5-\sqrt{5}}{10}\right)^{1/2}$, we see that

$$\|M\|_2^2 = \sup\{|x(1) + x(2)|^2 + |x(1)|^2 : |x(1)|^2 + |x(2)|^2 = 1\} = \frac{3+\sqrt{5}}{2}.$$

Hence $\|M\|_2 = \dfrac{1+\sqrt{5}}{2} < \beta_2 < \alpha_1^{1/2}\alpha_\infty^{1/2}$. (See Problem 6-15 for values of p other than 2.)

There is a special case, namely when $m = 1$, in which we have $\|M\|_p = \beta_p$. We leave the proof to Problem 6-10. (Compare 13.2.)

(b) Let $M = (k_{i,j})$ be an infinite matrix with scalar entries. As we have seen in 6.5(c), it defines a continuous linear map from ℓ^1 to ℓ^1, provided $\alpha_1 = \sup\{\sum_{i=1}^\infty |k_{i,j}| : \; j = 1, 2, \dots\} < \infty$, and then $\|Mx\|_1 \leq \alpha_1\|x\|_1$ for all $x \in \ell^1$. Thus $\|M\|_1 \leq \alpha_1$. In fact, $\|M\|_1 = \alpha_1$. This can be seen as follows. Let $\epsilon > 0$. Then there is a positive integer a such that

$$\alpha_1 - \epsilon < \sum_{i=1}^{\infty} |k_{i,a}|.$$

Define $x \in \ell^1$ by

$$x(j) = \delta_{j,n}, \quad j = 1, 2, \ldots .$$

Then $\|x\|_1 = 1$ and

$$\|Mx\|_1 = \sum_{i=1}^{\infty} |k_{i,n}| > \alpha_1 - \epsilon.$$

Thus $\alpha_1 - \epsilon \leq \|M\|_1 \leq \alpha_1$. Since this is true for every $\epsilon > 0$, we have $\|M\|_1 = \alpha_1$.

Next, as we have seen in 6.5(c), M defines a continuous linear map from ℓ^∞ to ℓ^∞, provided $\alpha_\infty = \sup\{\sum_{j=1}^{\infty} |k_{i,j}| : i = 1, 2, \ldots\} < \infty$ and then $\|M\|_\infty \leq \alpha_\infty$. Arguments similar to the one given above and in 6.7(a) show that $\|M\|_\infty = \alpha_\infty$.

If $1 < p < \infty$, we can only say that $\|M\|_p \leq \min\{\alpha_1^{1/p}\alpha_\infty^{1/r}, \beta_p\}$ by what we have seen in 6.5(c).

We shall now consider a special kind of linear functionals which will be of use in Section 14. Let X denote a linear space of scalar-valued function on a set T. A linear functional f on X is said to be **positive** if $f(x) \geq 0$ whenever $x \in X$ and $x(t) \geq 0$ for all $t \in T$. Let 1 denote the function whose value is 1 at all $t \in T$.

6.8 Theorem

Let X denote a subspace of $B(T)$ with the sup norm, $1 \in X$ and f be a linear functional on X. If f is continuous and $\|f\| = f(1)$, then f is positive. Conversely, if $\operatorname{Re} x \in X$ whenever $x \in X$ and if f is positive, then f is continuous and $\|f\| = f(1)$.

Proof:

By 6.6 we have $|f(1)| \leq \|f\| \, \|1\|_\infty = \|f\|$.

Assume that f is continuous and $\|f\| = f(1)$. To prove that f is positive, it is enough to consider $x \in X$ such that $0 \leq x(t) \leq 1$ for all $t \in T$ and show that $f(x) \geq 0$. Let $y = 2x - 1$, so that $-1 \leq y(t) \leq 1$ for all $t \in T$. First we show that $f(y)$ is a real number. If $\mathbf{K} = \mathbf{R}$, there is nothing to prove. Let $\mathbf{K} = \mathbf{C}$ and $f(y) = r + is$, where r and s are real numbers For every real number a, we have

$$|y + ia|^2 = y^2 + a^2 \leq 1 + a^2,$$

so that

$$\begin{aligned}(s + a\|f\|)^2 &= (s + af(1))^2 \leq |r + i(s + af(1))|^2 \\ &\leq |f(y + ia)|^2 \leq \|f\|^2(1 + a^2).\end{aligned}$$

Then $s^2 + 2sa\|f\| \leq \|f\|^2$ for every real number a. Hence we conclude that $s = 0$, that is, $f(y) = r \in \mathbf{R}$. As $\|y\|_\infty \leq 1$, we have $|r| = |f(y)| \leq \|f\|$. Thus

$$f(x) = f(\frac{y + 1}{2}) = \frac{r + \|f\|}{2} \geq 0.$$

Conversely, assume that $\operatorname{Re} x \in X$ whenever $x \in X$ and that f is positive. Let y be a real-valued function in X. Since $0 \leq \|y\|_\infty - y(t)$ for all $t \in T$, we have $0 \leq f(\|y\|_\infty - y) = \|y\|_\infty f(1) - f(y)$. As $f(1)$ is real, $f(y)$ must also be real. Now let x be an element of X and $f(x) = re^{is}$, where r and s are real numbers. Considering $y = e^{-is}x$, $y_1 = \operatorname{Re} y$ and $y_2 = \operatorname{Im} y = \operatorname{Re}(-iy)$, we find that

$$r = f(e^{-is}x) = f(y_1) + if(y_2).$$

But $r, f(y_1)$ and $f(y_2)$ are real, so that $f(y_2) = 0$. The inequalities $-\|x\|_\infty \leq y_1(t) \leq \|x\|_\infty$, $t \in T$, imply that

$$|f(x)| = |f(e^{-is}x)| = |f(y_1)| \leq f(\|x\|_\infty) = \|x\|_\infty f(1).$$

This shows that f is continuous and $\|f\| \leq f(1)$. Since $f(1) \leq |f(1)| \leq \|f\|$, we obtain $\|f\| = f(1)$, as desired. □

Here is a typical example of a positive linear functional on $C([a, b])$. Let y be a nondecreasing function on $[a, b]$ and for $x \in C([a, b])$, define

$$g(x) = \int_a^b x\, dy,$$

the Riemann-Stieltjes integral of x with respect to y. In 14.5 we shall show that every positive linear functional on $C([a,b])$ is of this kind.

Problems

6-1 If X is an infinite dimensional normed space, then it contains a hyperspace which is not closed.

6-2 Let X and Y be normed spaces and $F : X \to Y$ be linear. Then F is continuous if and only if for every Cauchy sequence (x_n) in X, the sequence $(F(x_n))$ is Cauchy in Y.

6-3 Let E be a measurable subset of $\mathbf{R}$ and for $t \in E$, let $x_1(t) = t$. Let $X = \{x \in L^2(E) : x_1 x \in L^2(E)\}$ and $F : X \to L^2(E)$ be defined by $F(x) = x_1 x$. If $E = [a,b]$, then F is continuous, but if $E = \mathbf{R}$, then F is not continuous.

6-4 Let X and Y be normed spaces and Z be a closed subspace of X.

(a) Let $F \in BL(X,Y)$ and $Z \subset Z(F)$. The map $\tilde{F} : X/Z \to Y$ given by $\tilde{F}(x+Z) = F(x), x \in X$, is well-defined, $\tilde{F} \in BL(X/Z, Y)$ and $\|\tilde{F}\| = \|F\|$.

(b) If $\tilde{F} \in BL(X/Z, Y)$ and we let $F(x) = \tilde{F}(x+Z)$ for $x \in X$, then $F \in BL(X,Y)$ and $\|F\| = \|\tilde{F}\|$.

6-5 Let X be a normed space and f be a nonzero linear functional on X. Then f is discontinuous if and only if $Z(f)$ is dense in X.

6-6 Let X and Y denote sequence spaces and $F : X \to Y$ be a continuous linear map. If $X = \ell^p$ with $1 \le p < \infty$, or $X = c_0$, then F is represented by an infinite matrix $(k_{i,j})$ in the sense that for every $x \in X$,

$$F(x)(i) = \sum_{j=1}^{\infty} k_{i,j} x(j), \quad i = 1, 2, \ldots.$$

If $X = c$, then there is an infinite matrix $(k_{i,j})$ and scalars $h_1, h_2, \ldots$ such that for every $x \in X$,

$$F(x)(i) = h_i \ell + \sum_{j=1}^{\infty} k_{i,j}(x(j) - \ell), \quad i = 1, 2, \ldots,$$

where $\ell = \lim_{j\to\infty} x(j)$.

6-7 If $F \in BL(X,Y), F \neq 0$ and $\alpha \geq 0$, then

$$\inf\{\|x\| : x \in X, \|F(x)\| = \alpha\} = \frac{\alpha}{\|F\|}.$$

6-8 **(Ascoli's lemma)** Let X be a normed space and f be a nonzero linear functional on X such that $Z(f)$ is closed in X. Then for $a \in X$ and $k \in \mathbf{K}$, we have

$$\operatorname{dist}(a, \{x \in X : f(x) = k\}) = \frac{|f(a) - k|}{\|f\|}.$$

In particular, if $a \notin Z(f)$, then $\|f\| = |f(a)|/\operatorname{dist}(a, Z(f))$.

6-9 Let X be a normed space and $P \in BL(X)$ satisfy $P^2 = P$. Then $\|P\| = 0$ or $\|P\| \geq 1$. If $Q = I - P$, then $Q^2 = Q$ and $\|P\| - 1 \leq \|Q\| \leq \|P\| + 1$. These lower and upper bounds for $\|Q\|$ can be realized.

6-10 Let f be a linear functional on $\mathbf{K}^n$. Then

$$f(x) = a_1 x(1) + \cdots + a_n x(n)$$

for some $a_j \in \mathbf{K}$ and all $x = (x(1), \ldots, x(n)) \in \mathbf{K}^n$. Consider the norm $\| \ \|_p, 1 \leq p \leq \infty$, on $\mathbf{K}^n$. Then

$$\|f\|_p = \begin{cases} \max\{|a_1|, \ldots, |a_n|\}, & \text{if } p = 1 \\ \left(\sum_{j=1}^n |a_j|^q\right)^{1/q}, & \text{if } 1 < p < \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1 \\ |a_1| + \cdots + |a_n|, & \text{if } p = \infty. \end{cases}$$

6-11 Let an $n \times n$ diagonal matrix $\operatorname{diag}(k_1, \ldots, k_n)$ define a linear map $M : \mathbf{K}^n \to \mathbf{K}^n$ as in 6.7(a). Then for $1 \leq p \leq \infty$,

$$\|M\|_p = \max\{|k_i| : i = 1, \ldots, n\}.$$

6-12 Let $M = (k_{i,j})$ be an infinite matrix with scalar entries whose every column has only a finite number of nonzero entries. Then M defines a linear map from c_{00} to c_{00}. Also, $\|M\|_1 = \alpha_1$ and $\|M\|_\infty = \alpha_\infty$, where α_1 and α_∞ are as defined in 6.5(c).

6-13 Let $M = (k_{i,j})$ be an infinite matrix with scalar entries and

$$\alpha_{p,r} = \begin{cases} \sup_{j=1,2,\ldots}(\sum_{i=1}^{\infty} |k_{i,j}|^r)^{1/r}, & \text{if } p = 1, 1 \le r < \infty \\ \sup_{i=1,2,\ldots}(\sum_{j=1}^{\infty} |k_{i,j}|^q)^{1/q}, & \text{if } 1 < p \le \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1, r = \infty \\ \sup_{i,j=1,2,\ldots} |k_{i,j}|, & \text{if } p = 1, r = \infty. \end{cases}$$

If $\alpha_{p,r} < \infty$, then M defines a continuous linear map from ℓ^p to ℓ^r and its operator norm equals $\alpha_{p,r}$. (Note that $\alpha_{1,1} = \alpha_1$ and $\alpha_{\infty,\infty} = \alpha_\infty$ as defined in 6.5(c). Compare Problem 9-15.)

6-14 Let $M = (k_{i,j})$ be an infinite matrix with scalar entries. If $1 < p \le \infty$ and $1 \le r < \infty$, define

$$\beta_{p,r} = \left[\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |k_{i,j}|^q\right)^{r/q}\right]^{1/r},$$

where $1/p + 1/q = 1$. If $\beta_{p,r} < \infty$, then M defines a continuous linear map from ℓ^p to ℓ^r and its operator norm is at most $\beta_{p,r}$.

6-15 Let $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $1 < p < \infty$ and $1/p + 1/q = 1$. Then $\|M\|_p < \beta_p = (2^{p/q} + 1)^{1/p} < 2 = \alpha_1 = \alpha_\infty$. (Hint: If $0 \le s, 0 \le t$ and $s^p + t^p = 1$, then $s + t \le 2^{1/q}$.)

6-16 Let X denote the linear space of all $m \times n$ matrices with scalar entries. Let $1 \le p, r \le \infty$. For M in X, let

$$\|M\|_{p,r} = \max\{\|Mx\|_r : x \in \mathbf{K}^n, \|x\|_p \le 1\}.$$

Then $\|\ \|_{p,r}$ is a norm on X. Also,

$$\|M\|_{p,r} = \begin{cases} \max_{j=1,\ldots,n} (\sum_{i=1}^{m} |k_{i,j}|^r)^{1/r}, & \text{if } p = 1, 1 \le r < \infty \\ \max_{i=1,\ldots,m} \left(\sum_{j=1}^{n} |k_{i,j}|^q\right)^{1/q}, & \text{if } 1 < p \le \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1, r = \infty \\ \max_{i=1,\ldots,m, j=1,\ldots,n} |k_{i,j}|, & \text{if } p = 1, r = \infty. \end{cases}$$

Let $1 \le u, r < \infty$. For M in X, let

$$\|M\|'_{u,r} = \left[\sum_{i=1}^{m} \left(\sum_{j=1}^{n} |k_{i,j}|^u\right)^{r/u}\right]^{1/r}.$$

Then $\| \ \|'_{u,r}$ is a norm on X. If $1 < p \le \infty, 1/p + 1/q = 1$ and $1 \le r < \infty$, then

$$\frac{1}{n^{1/r}} \|M\|'_{r,r} \le \|M\|_{p,r} \le \|M\|'_{q,r}.$$

If we identify X with $\mathbf{K}^{mn}$, then $\| \ \|_{1,\infty}$ is the norm $\| \ \|_\infty$ on $\mathbf{K}^{mn}$ and for $1 \le r < \infty$, $\| \ \|'_{r,r}$ is the norm $\| \ \|_r$ on $\mathbf{K}^{mn}$. The norm $\| \ \|'_{2,2}$ given by

$$\|M\|'_{2,2} = \left(\sum_{j=1}^{m} \sum_{j=1}^{n} |k_{i,j}|^2 \right)^{1/2}$$

is known as the **Frobenius norm**

6-17 Let $X = c_{00}$ with the norm $\| \ \|_p, 1 \le p \le \infty$. For $r \ge 0$, consider the linear functional f_r on X defined by

$$f_r(x) = \sum_{j=1}^{\infty} \frac{x(j)}{j^r}, \quad x \in X.$$

If $p = 1$, then f_r is continuous and $\|f_r\|_1 = 1$. If $1 < p \le \infty$, then f_r is continuous if and only if $r > 1 - 1/p = 1/q$, and then

$$\|f_r\|_p = \left(\sum_{j=1}^{\infty} \frac{1}{j^{rq}} \right)^{1/q}$$

6-18 The linear functional f on c defined by $f(x) = \lim_{j \to \infty} x(j), x \in c$, is continuous and $\|f\| = 1$.

6-19 Let $X = C([a,b])$ with the sup norm.

(a) For $y \in L^1([a,b])$, define $f \in X'$ by

$$f(x) = \int_a^b xy \, dm, \quad x \in X.$$

Then $\|f\| = \|y\|_1$. (Hint: If $y \in C([a,b])$, let $x_n = n\bar{y} \,/\, (1 + n|y|)$ for $n = 1, 2, \ldots$.)

(b) If $k(.,.)$ is a continuous function on $[a,b] \times [a,b]$ and

$$F(x)(s) = \int_a^b k(s,t) x(t) \, dt, \quad x \in X, \ s \in [a,b],$$

then $F \in BL(X)$ and $\|F\| = \sup\{ \int_a^b |k(s,t)| \, dt : s \in [a,b] \}$.

6-20 Let $X = C([a,b])$ with the norm given by

$$\|x\|_1 = \int_a^b |x(t)|\,dt, \quad x \in X.$$

If $k(.,.)$ is a continuous function on $[a,b]\times[a,b]$ and

$$F(x)(s) = \int_a^b k(s,t)x(t)\,dt, \quad x \in X,\ s \in [a,b],$$

then $F \in BL(X)$ and $\|F\| = \sup\{\int_a^b |k(s,t)|\,ds : t \in [a,b]\}$.

7 Hahn-Banach Theorems

The Hahn-Banach separation theorem and the Hahn-Banach extension theorem are two of the most fundamental results in functional analysis. The first deals with a separation of two disjoint convex subsets of a normed space by a closed hyperplane, while the second ensures a norm-preserving linear extension of a functional on a subspace of a normed space. The first is geometric in nature, while the second has an analytic character. They are closely related to each other since a closed hyperplane in a normed space X is just a translate of a zero space $Z(f)$ of some nonzero f in its dual X'. (See 2.5 and 6.4.)

We shall adopt a geometric approach to prove a basic result (Theorem 7.3) from which both the Hahn-Banach theorems will be deduced. [The extension theorem is often proved by an analytic approach which is outlined in Problem 7-14]. This basic result is first proved for normed spaces over $\mathbf{R}$. A simple technical result (Lemma 7.1) enables us to treat the case of normed spaces over $\mathbf{C}$. We remark that the complex case of the Hahn-Banach theorems was proved by Bohenblust and Sobczyk in 1938, about ten years after Hahn and Banach proved the real case.

7.1 Lemma
Let X be a linear space over $\mathbf{C}$. Regarding X as a linear space over $\mathbf{R}$, consider a real-linear functional $u : X \to \mathbf{R}$. Define

$$f(x) = u(x) - iu(ix), \quad x \in X.$$

Then f is a complex-linear functional on X.

Proof:
As u is real-linear, it is easy to see that f is also real-linear. In fact, f is complex-linear, since

$$f(ix) = u(ix) - iu(-x) = u(ix) + iu(x) = i[u(x) - iu(ix)] = if(x)$$

for all $x \in X$. $\square$

The following result is of crucial importance.

7.2 Lemma
Let X be a linear space over $\mathbf{K}$ and Y be a subspace of X which is not a hyperspace in X. If x_1 and x_2 are in X but not in Y, then there is some x in X such that for all $t \in [0,1]$,

$$tx_1 + (1-t)x \notin Y \quad \text{and} \quad tx_2 + (1-t)x \notin Y.$$

If X is a normed space, then the complement Y^c is connected.

Proof:
If $tx_1 + (1-t)x_2 \notin Y$ for all $t \in (0,1)$, then clearly we can let $x = x_2$. Assume now that there is some $s \in (0,1)$ such that $sx_1 + (1-s)x_2 \in Y$. Then it is easy to see that $\operatorname{span}\{Y, x_1\} = \operatorname{span}\{Y, x_2\}$. Since Y is not a hyperspace in X, $\operatorname{span}\{Y, x_1\} \neq X$. Let $x \in X$ be such that $x \notin \operatorname{span}\{Y, x_1\}$. Let, if possible, $y = tx_1 + (1-t)x$ belong to Y for some $t \in (0,1)$. Then $x = (y - tx_1)/(1-t)$ would belong to $\operatorname{span}\{Y, x_1\}$, contrary to our choice of x. Hence $tx_1 + (1-t)x \notin Y$

for all $t \in (0,1)$. Similarly, since $x \notin \text{span}\{Y, x_2\}$, it follows that $tx_2 + (1-t)x \notin Y$ for all $t \in (0,1)$, as desired. See Figure 5.

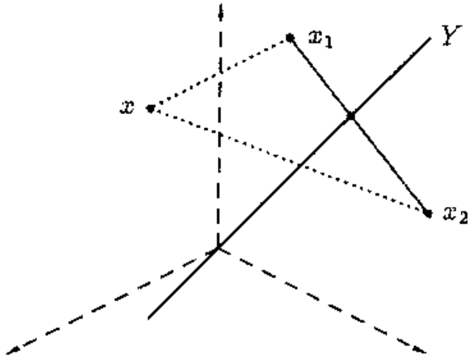


Figure 5

Let X be a normed space and, if possible, $Y^c = S_1 \cup S_2$, where S_1 and S_2 are nonempty disjoint open subsets of Y^c. Consider $x_1 \in S_1$ and $x_2 \in S_2$. By what we have proved above, there is some $x \in X$ such that $-tx_1 + (1+t)x \in Y^c$ for all $t \in [-1,0]$ and $tx_2 + (1-t)x \in Y^c$ for all $t \in [0,1]$. Define $\alpha : [-1,1] \to Y^c$ as follows:

$$\alpha(t) = \begin{cases} -tx_1 + (1+t)x, & \text{if } t \in [-1,0] \\ tx_2 + (1-t)x, & \text{if } t \in [0,1]. \end{cases}$$

Then $\alpha(-1) = x_1 \in S_1$ and $\alpha(1) = x_2 \in S_2$. Since α is a continuous function, $[-1,1]$ will thus be the disjoint union of the nonempty open subsets $\alpha^{-1}(S_1)$ and $\alpha^{-1}(S_2)$, contrary to the connectedness of $[-1,1]$. Hence Y^c is connected. $\square$

7.3 Theorem

Let X be a normed space over $\mathbf{K}$, E be a nonempty open convex subset of X and Y be a subspace of X such that $E \cap Y = \emptyset$. Then there is a closed hyperspace Z in X such that $Y \subset Z$ and $E \cap Z = \emptyset$.

Consequently, there is some $f \in X'$ such that $f(x) = 0$ for every $x \in Y$, but $\text{Re } f(x) \neq 0$ for every $x \in E$.

Proof:
First we assume that $\mathbf{K} = \mathbf{R}$. We shall use transfinite induction in the form of Zorn's lemma. Let

$$\mathcal{W} = \{W : W \text{ is a subspace of } X,\ Y \subset W \text{ and } E \cap W = \emptyset\}.$$

Then $Y \in \mathcal{W}$ by hypothesis. The inclusion $W_1 \subset W_2$ for W_1, W_2 in $\mathcal{W}$ defines a partial order on $\mathcal{W}$. Every totally ordered subset of $\mathcal{W}$ has an upper bound in $\mathcal{W}$, namely its union. By Zorn's lemma, there is a maximal element Z in $\mathcal{W}$. Then Z is a subspace of X, $Y \subset Z$ and $E \cap Z = \emptyset$.

Suppose for a moment that Z is not a hyperspace in X. We shall show that there is some $x_0 \in X$ with $x_0 \notin Z$ and $E \cap \operatorname{span}\{Z, x_0\} = \emptyset$. Consider

$$S = Z + \cup\{rE : r > 0\}.$$

Then

S and $-S$ are open in X: As E is open, rE is open for every $r > 0$, and hence $\cup\{rE : r > 0\}$ is open. By 5.6(a), S is open. Clearly, $-S$ is open as well.

$S \cap -S = \emptyset$: $z_1 + r_1x_1 = -z_2 - r_2x_2$ for some z_1, z_1 in Z, x_1, x_2 in E and $r_1 > 0, r_2 > 0$ imply that

$$\frac{r_1x_1 + r_2x_2}{r_1 + r_2} = -\frac{z_1 + z_2}{r_1 + r_2} \in E \cap Z,$$

since E is convex and Z is a subspace. As $E \cap Z = \emptyset$, this is a contradiction.

$S \cup -S \subset Z^c$: $z_2 = \pm(z_1 + r_1x_1)$ for some z_1, z_2 in Z, x_1 in E and $r_1 > 0$ imply that

$$x_1 = \frac{\pm z_2 - z_1}{r_1} \in E \cap Z.$$

As $E \cap Z = \emptyset$, this is a contradiction.

These three statements show that $S \cup -S \neq Z^c$, for otherwise Z^c will be a disjoint union of two nonempty open subsets and this is impossible since Z^c is connected by Lemma 7.2. Hence there is some $x_0 \in X$ such that $x_0 \notin Z \cup S \cup -S$. We now claim that

$$E \cap \operatorname{span}\{Z, x_0\} = \emptyset.$$

Let, if possible, $x = z + tx_0$ for some $x \in E, z \in Z$ and $t \in \mathbf{R}$. But $t = 0$ implies that $x = z \in E \cap Z, t > 0$ implies that $x_0 = -z/t + x/t \in S$, and $t < 0$ implies that $x_0 = -z/t + x/t \in -S$, each of which is a contradiction.

Thus $\operatorname{span}\{Z, x_0\}$ is an element of $\mathcal{W}$ which is strictly larger than Z, contradicting the maximality of Z in $\mathcal{W}$.

Hence Z must be a hyperspace in X. Since E is a nonempty open set in X and $E \cap Z = \emptyset$, it follows that the closure $\overline{Z}$ of Z cannot equal X. As $Z \subset \overline{Z} \subset X$ and Z is a hyperspace, it follows that $\overline{Z} = Z$, that is, Z is closed in X.

By 2.5(b), there is a linear functional $f : X \to \mathbf{R}$ such that $Z(f) = Z$. Now $x \in Y$ implies that $x \in Z$, that is, $f(x) = 0$, and $x \in E$ implies that $x \notin Z$, that is, $f(x) \neq 0$. Also, 6.4 shows that f is continuous, since $Z(f) = Z$ is closed in X.

Thus the theorem is proved for the case $\mathbf{K} = \mathbf{R}$. Let, now, $\mathbf{K} = \mathbf{C}$. Regarding X as a linear space over $\mathbf{R}$, we find a continuous real-linear functional $u : X \to \mathbf{R}$ such that $u(x) = 0$ for every x in Y and $u(x) \neq 0$ for every $x \in E$. As in Lemma 7.1, let $f(x) = u(x) - iu(ix)$ for $x \in X$, so that $f : X \to \mathbf{C}$ is complex-linear. Clearly, f is continuous. Now $x \in Y$ implies that $ix \in iY = Y$ and hence $f(x) = 0 - i\,0 = 0$. Also, $x \in E$ implies that $\operatorname{Re} f(x) = u(x) \neq 0$. By 2.5(a) and 6.4, $Z = Z(f)$ is a closed hyperspace in the linear space X over $\mathbf{C}$ such that $Y \subset Z$ and $E \cap Z = \emptyset$. $\square$

The preceding result shows that if Y is a proper closed subspace of X, then there is some nonzero f in X' such that $f_{|Y} = 0$. In particular, if $X \neq \{0\}$, then letting $Y = \{0\}$, we obtain $X' \neq \{0\}$.

Before turning to the Hahn-Banach theorems, we point out an interesting property of a nonzero linear functional on a normed space.

7.4 Lemma

Let X be a normed space over $\mathbf{K}$, and f be a nonzero linear functional on X. If E is an open subset of X, then $f(E)$ is an open subset of $\mathbf{K}$.

Proof:

Since f is nonzero, there is some $a \in X$ such that $f(a) = 1$. Note that $a \neq 0$. Let $x \in E$. Since E is open, there is some $r > 0$ such that $U(x, r) \subset E$. If $|k| < r/\|a\|$, then $x - ka \in E$, so that $f(x) - k = f(x - ka) \in f(E)$. Thus

$$\{k' \in \mathbf{K} : |f(x) - k'| < \frac{r}{\|a\|}\} \subset f(E),$$

showing that $f(E)$ is open in $\mathbf{K}$. □

7.5 Hahn-Banach separation theorem (Hahn, 1927 and Banach, 1929). Let X be a normed space over $\mathbf{K}$ and E_1, E_2 be nonempty disjoint convex subsets of X, where E_1 is open in X. Then there is a real hyperplane in X which separates E_1 and E_2 in the following sense: For some $f \in X'$ and $t \in \mathbf{R}$, we have

$$\operatorname{Re} f(x_1) < t \leq \operatorname{Re} f(x_2)$$

for all $x_1 \in E_1$ and $x_2 \in E_2$.

Proof:

The set $E_1 - E_2 = \{x_1 - x_2 : x_1 \in E_1, x_2 \in E_2\}$ is nonempty and convex. It is open by 5.6(a). Also, $0 \notin E_1 - E_2$, since $E_1 \cap E_2 = \emptyset$. Letting $Y = \{0\}$ and $E = E_1 - E_2$ in 7.3, we obtain $f \in X'$ such that $\operatorname{Re} f(x_1 - x_2) \neq 0$ for all $x_1 \in E_1$ and $x_2 \in E_2$. Thus $\operatorname{Re} f(E_1)$ and $\operatorname{Re} f(E_2)$ are disjoint convex subsets of $\mathbf{R}$. Hence they are, in fact, nonoverlapping intervals in $\mathbf{R}$. We can assume that $\operatorname{Re} f(E_1)$ lies to the left of $\operatorname{Re} f(E_2)$, upon replacing f by $-f$, if necessary. Now $\operatorname{Re} f : X \to \mathbf{R}$ is a nonzero continuous real-linear functional and E_1 is an open subset of X. Hence Lemma 7.4 shows that $\operatorname{Re} f(E_1)$ is an open interval. If t is the right end-point of $\operatorname{Re} f(E_1)$, we see that

$$\operatorname{Re} f(x_1) < t \leq \operatorname{Re} f(x_2)$$

for all $x_1 \in E_1$ and $x_2 \in E_2$. □

Geometrically, the preceding separation result says that the set E_1 lies on one side of the real hyperplane $\{x \in X : \operatorname{Re} f(x) = t\}$ and the set E_2 lies on the other, since

$$E_1 \subset \{x \in X : \operatorname{Re} f(x) < t\} \quad \text{and} \quad E_2 \subset \{x \in X : \operatorname{Re} f(x) \geq t\}.$$

7.6 Corollary
Let E be a nonempty convex subset of a normed space X over $\mathbf{K}$.

(a) If $a \in X$ but $a \notin \overline{E}$, then there are $f \in X'$ and $t \in \mathbf{R}$ such that $\operatorname{Re} f(x) \leq t < \operatorname{Re} f(a)$ for all $x \in \overline{E}$.

(b) If $E^o \neq \emptyset$ and b belongs to the boundary of E in X, then there is a nonzero $f \in X'$ such that $\operatorname{Re} f(x) \leq \operatorname{Re} f(b)$ for all $x \in \overline{E}$.

Proof:

(a) The closure $\overline{E}$ of the convex set E is convex by 5.6(b). If $a \notin \overline{E}$, find $r > 0$ such that $U(a, r) \cap \overline{E} = \emptyset$. Letting $E_1 = U(a, r)$ and $E_2 = \overline{E}$ in Theorem 7.5, we obtain some $g \in X'$ and $s \in \mathbf{R}$ such that $\operatorname{Re} g(a) < s \leq \operatorname{Re} g(x)$ for all $x \in \overline{E}$. Then let $f = -g$ and $t = -s$.

(b) The interior E^o of the convex set E is convex by 5.6(b). If b belongs to the boundary of E in X, then $b \notin E^o$. If $E^o \neq \emptyset$, letting $E_1 = E^o$, and $E_2 = \{b\}$ in Theorem 7.5, we obtain some $f \in X'$ and $t \in \mathbf{R}$ such that $\operatorname{Re} f(x) < t \leq \operatorname{Re} f(b)$ for all $x \in E^o$. Hence $f \neq 0$ and $\operatorname{Re} f(x) \leq \operatorname{Re} f(b)$ for all $x \in \overline{E^o}$. But when $E^o \neq \emptyset$, we have $\overline{E^o} = \overline{E}$ by 5.6(b). Hence the result. □

Let X be a normed space. A convex subset E of X with a nonempty interior is called a **convex body**. Let b be in the boundary of a convex body E. If $f \in X', f \neq 0$ and $\operatorname{Re} f(x) \leq \operatorname{Re} f(b)$ for all $x \in E$, then f is called a **support functional** for E at b and the corresponding real hyperplane $\{x \in X : \operatorname{Re} f(x) = \operatorname{Re} f(b)\}$ is called a **support hyperplane** for E at b. Our result 7.6(b) guarantees the existence of such a support hyperplane.

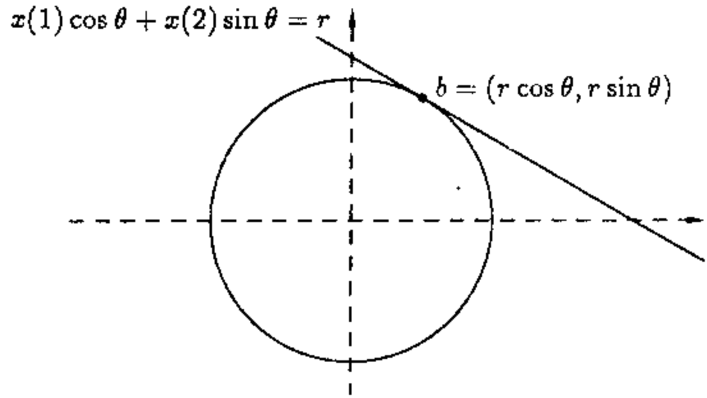


Figure 6

In particular, let $r > 0$ and $E = \overline{U}(0, r)$. Then $\{x \in X : \|x\| = r\}$ is the boundary of the convex body E. Thus for every $b \in X$ with $\|b\| = r$, there is a nonzero $f \in X'$ such that

$$\operatorname{Re} f(x) \leq \operatorname{Re} f(b)$$

for all $x \in X$ with $\|x\| \leq r$. For example, let $X = \mathbf{R}^2$ with the norm $\| \ \|_2$. If $b = (r\cos\theta, r\sin\theta)$, then $f(x(1), x(2)) = x(1)\cos\theta + x(2)\sin\theta$ is a support functional for $\{x \in \mathbf{R}^2 : \|x\|_2 \leq r\}$ at b, and $\{(x(1), x(2)) : x(1)\cos\theta + x(2)\sin\theta = r\}$ is the corresponding support hyperplane.This is illustrated in Figure 6.

We now move on to the extension theorem of Hahn and Banach. Let us consider a normed space X, subspace Y of X and a continuous linear functional g on Y. As we have seen in 2.4(c), g can be extended linearly to X by extending a basis for Y to a basis for X. Also, since

g is uniformly continuous on Y by 6.2(iv), it can be extended continuously to the closure $\overline{Y}$ of Y, and then the Tietze extension theorem (3.9) can be employed to extend g continuously to X. However, it is far from obvious how to extend g linearly and continuously to X. This is accomplished by the Hahn-Banach extension theorem. We shall deduce it from Theorem 7.3 by using the following result.

7.7 Lemma

Let X be a normed space over $\mathbf{K}$, $f \in X'$ and $f \neq 0$. Let $a \in X$ with $f(a) = 1$ and $r > 0$. Then

$$U(a,r) \cap Z(f) = \emptyset \quad \text{if and only if} \quad \|f\| \leq \frac{1}{r}.$$

Proof:

Let $U(a,r) \cap Z(f) = \emptyset$. If $x \in X$ and $f(x) \neq 0$, then $a - x/f(x)$ is in $Z(f)$, so that $\|a - (a - x/f(x))\| \geq r$, that is, $\|x\| \geq r|f(x)|$. Hence $\|f\| \leq 1/r$.

Conversely, let $\|f\| \leq 1/r$. If $x \in U(a,r)$, that is, $\|x - a\| < r$, then

$$|f(x) - 1| = |f(x) - f(a)| = |f(x-a)| \leq \|f\| \; \|x - a\| < 1,$$

so that $f(x) \neq 0$. Hence $U(a,r) \cap Z(f) = \emptyset$. □

This result also follows from Problem 6-8 with $k = 0$.

7.8 Hahn-Banach extension theorem

Let X be a normed space over $\mathbf{K}, Y$ be a subspace of X and $g \in Y'$. Then there is some $f \in X'$ such that $f_{|Y} = g$ and $\|f\| = \|g\|$.

Proof:

If $g = 0$ on Y, then we can let $f = 0$ on X. Now assume that g is not identically zero on Y, and let $a \in Y$ be such that $g(a) = 1$. Letting

$r = 1/\|g\|$ in Lemma 7.7, we see that

$$U_Y(a, \frac{1}{\|g\|}) \cap Z(g) = \emptyset.$$

Let $E = U_X(a, 1/\|g\|)$. Then E is a nonempty open convex subset of X, and it is disjoint from the subspace $Z(g)$ of X. By Theorem 7.3, there is some $f \in X'$ such that $f(y) = 0$ for every $y \in Z(g)$ but $\operatorname{Re} f(x) \neq 0$ for every $x \in E$. Consequently, $f(x) \neq 0$ for every $x \in E$.

Note that $f(a) \neq 0$ since $a \in E$. Replacing f by $f/f(a)$, we may assume that $f(a) = 1$. Now

$$f_{|Y} : Y \to \mathbf{K}, \quad Z(g) \subset Z(f_{|Y}) \quad \text{and} \quad g(a) = 1 = f_{|Y}(a).$$

Let $y \in Y$. Then $y - g(y)a \in Z(g) \subset Z(f_{|Y})$, so that $f(y) - g(y) = 0$. Hence $f_{|Y} = g$ on Y and $\|g\| = \|f_{|Y}\| \leq \|f\|$. On the other hand,

$$U_X(a, \frac{1}{\|g\|}) \cap Z(f) = \emptyset.$$

By Lemma 7.7, we have $\|f\| \leq 1/(1/\|g\|) = \|g\|$. Thus $\|f\| = \|g\|$. $\square$

For a subspace Y of a normed space X and $g \in Y'$, a **Hahn-Banach extension** of g to X is an element f of X' such that $f_{|Y} = g$ and $\|f\| = \|g\|$. Theorem 7.8 guarantees that there is at least one Hahn-Banach extension of every $g \in Y'$. We shall consider the question of the uniqueness of a Hahn-Banach extension in 7.11.

7.9 Examples

(a) Let T be a metric space. Consider the linear space $B(T)$ of all bounded $\mathbf{K}$-valued functions on T with the sup norm and the subspace $C(T)$ of $B(T)$ consisting of all bounded continuous functions.

Let g be a positive linear functional on $C(T)$. Then g is continuous. In fact, if we let $X = C(T)$ in 6.8, then $\|g\| = g(1)$. Let f be a linear functional on $B(T)$ such that $f_{|C(T)} = g$. Then f is a Hahn-Banach extension of g to $B(T)$ if and only if $\|f\| = \|g\| = g(1) = f(1)$, that is, if and only if f is a positive functional, again by 6.8 with $X = B(T)$.

(b) Let $X = \mathbf{K}^2$ with the norm $\| \ \|_1$. Consider

$$Y = \{(x(1), x(2)) \in X : x(2) = 0\}.$$

Define $g \in Y'$ by

$$g(x(1), x(2)) = x(1).$$

Then it is clear that $\|g\| = 1 = g(a)$, where $a = (1,0) \in Y$. Since $Y =$ span $\{a\}$, we see that a function f on X is a Hahn-Banach extension of g to X if and only if f is linear on X and $\|f\| = 1 = f(a)$. Now, if f is linear on X, then $f(x(1), x(2)) = k_1 x(1) + k_2 x(2)$ for all $(x(1), x(2))$ in $\mathbf{K}^2$ and some fixed k_1 and k_2 in $\mathbf{K}$. Then $\|f\| = \max\{|k_1|, |k_2|\}$ as in Example 6.7(a), and $f(a) = 1$ if and only if $k_1 = 1$. Thus the Hahn-Banach extensions of g to X are given by

$$f(x(1), x(2)) = x(1) + k_2 x(2),$$

where $k_2 \in \mathbf{K}$ with $|k_2| \leq 1$. We note that the real hyperplane $\{x \in X : \mathrm{Re}\,(x(1) + k_2 x(2)) = 1\}$ is a support hyperplane for $\{x \in X : \|x\|_1 \leq 1\}$ at $a = (1,0)$. See Figure 7 for the case $\mathbf{K} = \mathbf{R}$.

Next, consider

$$Z = \{(x(1), x(2)) \in X : x(1) = x(2)\}.$$

Define $h \in Z'$ by

$$h(x(1), x(2)) = 2x(1).$$

Then it is clear that $\|h\| = 1 = h(b)$, where $b = (1/2, 1/2) \in Z$. Since $Z =$ span $\{b\}$, we see that a function f on X is a Hahn-Banach extension of h to X if and only if f is linear on X and $\|f\| = 1 = f(b)$. As before, this is the case if and only if $f(x(1), x(2)) = k_1 x(1) + k_2 x(2)$ with $\max\{|k_1|, |k_2|\} = 1$ and $k_1 + k_2 = 2$, that is, $k_1 = 1 = k_2$. Thus the only Hahn-Banach extension of h to X is given by

$$f(x(1), x(2)) = x(1) + x(2).$$

We note that the real hyperplane $\{x \in X : \mathrm{Re}(x(1) + x(2)) = 1\}$ is a support hyperplane for $\{x \in X : \|x\|_1 \leq 1\}$ at $b = (1/2, 1/2)$. See Figure 7 for the case $\mathbf{K} = \mathbf{R}$.

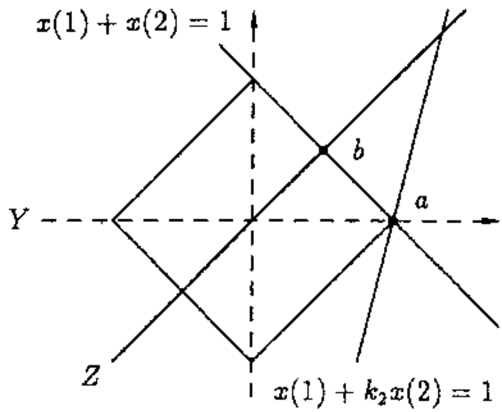


Figure 7

Similar considerations hold for the norm $\| \ \|_\infty$. See Problem 7-8.

The Hahn-Banach extension theorem has important consequences. We give some of them below.

7.10 Theorem

Let X be a normed space over $\mathbf{K}$.

(a) Let $0 \neq a \in X$. Then there is some $f \in X'$ such that $f(a) = \|a\|$ and $\|f\| = 1$. Consequently,

$$\|a\| = \sup\{|f(a)| : f \in X', \|f\| \leq 1\}.$$

(b) Let Y be a subspace of X and $a \in X$ but $a \notin \overline{Y}$. Then there is some $f \in X'$ such that $f_{|Y} = 0$, $f(a) = \mathrm{dist}(a, \overline{Y})$ and $\|f\| = 1$. Consequently, $x \in \overline{Y}$ if and only if $x \in X$ and $f(x) = 0$ whenever $f \in X'$ and $f_{|Y} = 0$.

(c) Let $\{a_1, \ldots, a_m\}$ be a linearly independent set in X. Then there are $f_1, \ldots, f_m$ in X' such that $f_j(a_i) = \delta_{i,j}$, $1 \leq i, j \leq m$.

Proof:

(a) Consider $Y = \{ka : k \in \mathbf{K}\}$ and $g(ka) = k\|a\|, k \in \mathbf{K}$. Then Y is a subspace of X, $g \in Y'$ and $\|g\| = 1$. Hence by the Hahn-Banach extension theorem (7.8), there is some $f \in X'$ such that $f_{|Y} = g$ and $\|f\| = \|g\| = 1$.

(b) $\overline{Y}$ is a closed subspace of X by 5.2(a). Consider the quotient space $X/\overline{Y}$. Since $a \notin \overline{Y}$, the coset $a + \overline{Y}$ is not zero in $X/\overline{Y}$. By part (a), there is some $\tilde{f} \in (X/\overline{Y})'$ such that $\tilde{f}(a + \overline{Y}) = |||a + \overline{Y}|||$ and $\|\tilde{f}\| = 1$. Define $f : X \to \mathbf{K}$ by $f(x) = \tilde{f}(x + \overline{Y})$ for $x \in X$. Then $f = 0$ on Y and $f(a) = |||a + \overline{Y}||| = \operatorname{dist}(a, \overline{Y})$. Also, $\|f\| = \|\tilde{f}\| = 1$. This can be seen as follows. For all $x \in X$,

$$|f(x)| = |\tilde{f}(x + \overline{Y})| \leq \|\tilde{f}\| \; |||x + \overline{Y}||| \leq \|\tilde{f}\| \, \|x\|$$

and for all $y \in \overline{Y}$,

$$|\tilde{f}(x + \overline{Y})| = |\tilde{f}(x + y + \overline{Y})| = |f(x + y)| \leq \|f\| \, \|x + y\|,$$

so that $|\tilde{f}(x + \overline{Y})| \leq \|f\| \inf\{\|x + y\| : y \in Y\} = \|f\| \; |||x + \overline{Y}|||$.

(c) For $j = 1, \ldots, m$, let $Y_j = \operatorname{span}\{a_i : 1 \leq i \leq m, i \neq j\}$. The finite dimensional subspace Y_j is closed in X by 5.4(b). Since $a_j \notin Y_j = \overline{Y}_j$, there is some $g_j \in X'$ such that $g_j = 0$ on Y_j, and $g_j(a_j) = \operatorname{dist}(a_j, Y_j) \neq 0$ by (b) above. Let $f_j = g_j / \operatorname{dist}(a_j, Y_j)$. Then $f_j \in X'$ and $f_j(a_i) = \delta_{i,j}, 1 \leq i, j \leq m$. □

Part (b) of 7.10 is often used in approximation theory in the following way. Suppose we wish to show that an element a of a normed space X can be approached by elements of a subspace Y of X. Then it is enough to show that $f(a) = 0$ whenever $f \in X'$ and $f_{|Y} = 0$. Of course, for this purpose, it is useful to know all $f \in X'$. We shall take up this question in Sections 13, 14 and 24. See 7.12 and Problem 14-2 for applications of 7.10(b).

Part (c) of 7.10 can be compared with 16.2.

Unique Hahn-Banach Extensions

If Y is a dense subspace of a normed space X and g is a continuous linear functional on Y, then the uniform continuity of g (guaranteed by Theorem 6.2) enables us to conclude that g has a unique continuous linear extension f to X, and also that $\|f\| = \|g\|$. Thus g has a unique Hahn-Banach extension to X. If, however, Y is not dense in X, a Hahn-Banach extension of g to X may not be unique in general, as the first example in 7.9(b) shows. Normed spaces which admit unique Hahn-Banach extensions can be characterized in terms of the geometry of the unit balls of their duals as follows.

7.11 Theorem (Taylor-Foguel, 1958)
Let X be a normed space. For every subspace Y of X and every $g \in Y'$, there is a unique Hahn-Banach extension of g to X if and only if X' is **strictly convex**, that is, for $f_1 \neq f_2$ in X' with $\|f_1\| = 1 = \|f_2\|$, we have $\|f_1 + f_2\| < 2$.

Proof:
Assume that X' is strictly convex. Let Y be a subspace of $X, g \in Y'$, and let f_1 and f_2 be Hahn-Banach extensions of g to X. If $g = 0$ on Y, then $\|f_1\| = \|f_2\| = \|g\| = 0$, so that $f_1 = 0 = f_2$. Let, then, $g \neq 0$. Without loss of generality, assume that $\|g\| = 1$. It is easy to see that $(f_1 + f_2)/2$ is a continuous linear extension of g to X and $\|(f_1 + f_2)/2\| = \|g\| = 1$. Since $\|f_1\| = \|f_2\| = \|g\| = 1$, the strict convexity of X' shows that $f_1 = f_2$.

Conversely, assume that X' is not strictly convex. Then there are $f_1 \neq f_2$ in X' such that $\|f_1\| = 1 = \|f_2\|$ and $\|f_1 + f_2\| = 2$. Let (x_n) be a sequence in X such that $\|x_n\| = 1$ for each n and $|f_1(x_n) + f_2(x_n)| \to 2$. By the parallelogram law for elements of $\mathbf{K}$,

$$|f_1(x_n) + f_2(x_n)|^2 + |f_1(x_n) - f_2(x_n)|^2 = 2|f_1(x_n)|^2 + 2|f_2(x_n)|^2.$$

Since $\|f_1\| = 1 = \|f_2\|$, we see that $|f_1(x_n)| \leq 1$ and $|f_2(x_n)| \leq 1$.

Hence the right side of the equality above is less than or equal to 4. But the first term on the left side tends to 4 as $n \to \infty$. Thus

$$|f_1(x_n) - f_2(x_n)| \to 0, \quad |f_1(x_n)| \to 1 \quad \text{and} \quad |f_2(x_n)| \to 1.$$

Let $Y = \{x \in X : f_1(x) = f_2(x)\}$ and define $g \in Y'$ by $g = f_{1|Y} = f_{2|Y}$. We show that $\|g\| = 1$. First, $\|g\| = \|f_{1|Y}\| \leq \|f_1\| = 1$. To prove the reverse inequality, consider $a \in X$ such that $f_1(a) \neq f_2(a)$. For $n = 1, 2, \ldots$, let

$$z_n = x_n - \frac{f_1(x_n) - f_2(x_n)}{f_1(a) - f_2(a)} a.$$

It is easy to check that $f_1(z_n) = f_2(z_n)$, that is, $z_n \in Y$ for each n. Since $f_1(x_n) - f_2(x_n) \to 0$, we see that $\|x_n - z_n\| \to 0$. As $\|x_n\| = 1$, we have $\|z_n\| \to 1$. Also, we see that $|f_1(x_n) - f_1(z_n)| \to 0$. As $|f_1(x_n)| \to 1$, we have $|f_1(z_n)| \to 1$. For all large n, let $y_n = z_n/\|z_n\|$. Then y_n belongs to Y, $\|y_n\| = 1$ and $|g(y_n)| = |f_1(y_n)| = |f_1(z_n)|/\|z_n\| \to 1$ as $n \to \infty$. This shows that $\|g\| = 1$. Thus f_1 and f_2 are two distinct Hahn-Banach extensions of g to X $\square$

Banach Limits

Let c denote the linear space of all convergent scalar sequences with the norm $\| \|_\infty$. Consider the 'limit' functional $g : c \to \mathbf{K}$ defined by

$$g(x) = \lim_{j \to \infty} x(j), \quad x = (x(1), x(2), \ldots) \in c.$$

In an attempt to generalize the concept of a limit, we note the following properties of the functional g:

1) g is linear
2) g is continuous and $\|g\| = 1$
3) $g(a) = 1$, where $a = (1, 1, \ldots)$
4) $g(x) = g(\tau(x))$ for all $x \in c$, where $\tau(x)(j) = x(j + 1)$ for $j = 1, 2, \ldots$.

Considering c as a subspace of ℓ^∞, the normed space of all bounded scalar sequences, let f be a Hahn-Banach extension of g to ℓ^∞. Then $f(x) = g(x) = \lim_{j\to\infty} x(j)$ for every $x \in c$, and the properties 1), 2) and 3) hold for f as they do for g. We shall show that g admits a Hahn-Banach extension to ℓ^∞ possessing also the property 4). Such a functional is called a **Banach limit**.

7.12 Theorem

There exists a linear functional f on ℓ^∞ such that $\|f\| = 1 = f(a)$ and $f(\tau(x)) = f(x)$ for all $x \in \ell^\infty$, where $a = (1, 1, \dots)$ and $\tau(x)(j) = x(j+1)$ for $j = 1, 2, \dots$.

Every such functional f satisfies

$$f(x) = \lim_{j\to\infty} x(j), \quad x \in c,$$

so that f is a Banach limit.

Proof:

Let $Y = \{x - \tau(x) : x \in \ell^\infty\}$. Then Y is a subspace of ℓ^∞. We show that $\mathrm{dist}(a, \overline{Y}) = 1$. Clearly, $\mathrm{dist}(a, \overline{Y}) \le \|a - 0\|_\infty = 1$. Were $\mathrm{dist}(a, \overline{Y}) < 1$, there would exist $x \in \ell^\infty$ and some $\delta > 0$ such that

$$\|a - (x - \tau(x))\|_\infty = 1 - \delta.$$

Then for $j = 1, 2, \dots,$

$$\mathrm{Re}\,(1 - x(j) + x(j+1)) \le |1 - x(j) + \tau(x)(j)| \le \|a - x + \tau(x)\|_\infty = 1 - \delta,$$

so that $\mathrm{Re}\, x(j+1) \le \mathrm{Re}\, x(j) - \delta$. This shows that $\mathrm{Re}\, x(j+1) \le \mathrm{Re}\, x(1) - j\delta$ for $j = 1, 2, \dots$, and hence $\mathrm{Re}\, x(j) \to -\infty$ as $j \to \infty$, contrary to $x \in \ell^\infty$. Thus $\mathrm{dist}(a, \overline{Y}) = 1$.

Now by 7.10(b), there is some $f \in (\ell^\infty)'$ such that $f = 0$ on Y, $f(a) = \mathrm{dist}(a, \overline{Y}) = 1$ and $\|f\| = 1$. Since $x - \tau(x) \in Y$ for all $x \in X$, we see that $f(x - \tau(x)) = 0$, that is, $f(x) = f(\tau(x))$. (See Problem 7-20 for an alternative proof.)

Consider now any linear functional f on ℓ^∞ such that $\|f\| = 1 = f(a)$ and $f(x) = f(\tau(x))$ for all $x \in \ell^\infty$. Let $x \in c$ and $\lim_{j\to\infty} x(j) = \ell$. For $\epsilon > 0$, there is a positive integer m such that $|x(j) - \ell| < \epsilon$ for all $j \geq m$. Let $y = (x(m), x(m+1), \ldots)$. Then

$$f(x) = f(\tau(x)) = f(\tau^2(x)) = \cdots = f(\tau^{m-1}(x)) = f(y).$$

Since

$$y - \ell\, a = (x(m) - \ell, x(m+1) - \ell, \ldots),$$

we see that $\|y - \ell\, a\|_\infty \leq \epsilon$, and since $\|f\| = 1 = f(a)$, we have

$$|f(x) - \ell| = |f(y) - \ell| = |f(y - \ell\, a)| \leq \|y - \ell\, a\|_\infty \leq \epsilon.$$

Since this is true for every $\epsilon > 0$, $f(x) = \ell = \lim_{j\to\infty} x(j)$. $\square$

We remark that the functional $g(x) = \lim_{j\to\infty} x(j), x \in c$, is positive, that is, $g(x) \geq 0$ if $x \in c$ and $x(j) \geq 0$ for all j. Also, every Banach limit f is a positive functional on ℓ^∞, as can be seen by letting $T = \{1, 2, \ldots\}$ and $X = B(T) = \ell^\infty$ in 6.8 and noting that $\|f\| = f(1)$.

A bounded sequence $x = (x(1), x(2), \ldots)$ is said to be **almost convergent** if all Banach limits have the same value at x. For example, the sequence $a_0 = (1, 0, 1, 0, \ldots)$ is not convergent, but it is almost convergent. Noting that $a_0 + \tau(a_0) = a$, we have

$$1 = f(a) = f(a_0) + f(\tau(a_0)) = 2f(a_0),$$

that is, $f(a_0) = 1/2$ for every Banach limit f.

Problems

X denotes a normed space over $\mathbf{K}$, unless otherwise stated.

7-1 Let X be a linear space over $\mathbf{C}$, and f be a complex-linear functional on X. Then $\operatorname{Re} f$ is a real-linear functional on X, regarded as a linear space over $\mathbf{R}$. Further, $\operatorname{Re} f$ determines f as follows:

$$f(x) = \operatorname{Re} f(x) - i\operatorname{Re} f(ix), \quad x \in X.$$

If $\| \ \|$ is a norm on X, then $\|\operatorname{Re} f\| = \|f\|$.

7-2 Let X be a linear space over $\mathbf{C}$. If Z is a complex hyperspace in X and $a \in X$, but $a \notin Z$, then $Z + \{it\,a : t \in \mathbf{R}\}$ is a real hyperspace in X. If Z_r is a real hyperspace in X, then $Z_r \cap iZ_r$ is a complex hyperspace in X. For example, if $X = B(T)$ and $t \in T$, then $Z_r = \{x \in X : x(t) \in \mathbf{R}\}$ is a real hyperspace in $B(T)$, and $Z_r \cap iZ_r = \{x \in X : x(t) = 0\}$ is a complex hyperspace in X.

7-3 The Riesz lemma (5.3) with $r = 1$ holds in X if and only if every $f \in X'$ attains its norm on the unit sphere of X.

7-4 Let E_1 and E_2 be nonempty disjoint convex subsets of X, with E_1 compact and E_2 closed in X. then for some $f \in X'$ and t_1, t_2 in $\mathbf{R}$

$$\operatorname{Re} f(x_1) \leq t_1 < t_2 \leq \operatorname{Re} f(x_2)$$

for all $x_1 \in E_1$ and $x_2 \in E_2$. (Hint: Problem 5-14 and Theorem 7.5)

7-5 If $f \in X'$ and $t \in \mathbf{R}$, then $\{x \in X : t \leq \operatorname{Re} f(x)\}$ is known as a **real half-space** in X. Let E be a closed convex subset of X. Then E is the intersection of all the real half-spaces containing it.

7-6 Let E be a convex subset of X and $x \in X$. Then $x \in \overline{E}$ if and only if $\operatorname{Re} f(x) \geq 1$ for all $f \in X'$ such that $\operatorname{Re} f \geq 1$ on E and $\operatorname{Re} f(x) \leq 1$ for all $f \in X'$ such that $\operatorname{Re} f \leq 1$ on E. (Compare 7.10(b).)

7-7 Let E_1 and E_2 be nonempty disjoint convex subsets of X. Assume that E_1 has an **internal point** (that is, there is some $a_1 \in E_1$ such that the set $E_1 - a_1 = \{x_1 - a_1 : x_1 \in E_1\}$ is absorbing). Then there is a nonzero linear functional $f : X \to \mathbf{K}$ such that $\operatorname{Re} f(x_1) \leq \operatorname{Re} f(x_2)$ for all $x_1 \in E_1$ and $x_2 \in E_2$. (Hint: If $a_2 \in E_2$, let $E = (E_1 - a_1) - (E_2 - a_2)$ and for $x \in X$, $p(x) = \inf\{t > 0 : x \in tE\}$ in Problem 7-14(b).)

7-8 Let $X = \mathbf{K}^2$ with the norm $\| \ \|_\infty$.

(a) Consider $Y = \{(x(1), x(2)) \in X : x(2) = 0\}$ and define $g \in Y'$ by $g(x(1), x(2)) = x(1)$. The only Hahn-Banach extension of g to X is given by $f(x(1), x(2)) = x(1)$.

(b) Consider $Z = \{(x(1), (x(2)) \in X : x(1) = x(2)\}$, and define $g \in Z'$ by $g(x(1), x(2)) = x(1)$. The Hahn-Banach extensions of g to X are given by $f(x(1), x(2)) = tx(1) + (1 - t)x(2)$, where $t \in [0, 1]$ is fixed.

7-9 Let $0 \neq a \in X$ and $0 \neq f \in X'$. Then f is a support functional for $\overline{U}(0, \|a\|)$ at a if and only if $\|f\| \, \|a\| = f(a)$.

7-10 Let $0 \neq a \in X$. There is a unique $f \in X'$ such that $\|f\| = 1$ and $f(a) = \|a\|$ if and only if there is a unique support functional f for $\overline{U}(0, \|a\|)$ at a such that $f(a) = \|a\|$ if and only if there is a unique support hyperplane for $\overline{U}(0, \|a\|)$ at a.

7-11 (**Helly, 1912**) For each s in some index set S, let $x_s \in X$ and $k_s \in \mathbf{K}$. Let $\alpha \geq 0$. Then there exists some $f \in X'$ such that $f(x_s) = k_s$ for each $s \in S$ if and only if $|\Sigma_s h_s k_s| \leq \alpha \|\Sigma_s h_s x_s\|$, where the sum $\sum_s$ is finite with $s \in S$ and $h_s \in \mathbf{K}$.

7-12 Let $X = C([a, b])$ with the sup norm, and Y be the subspace of X consisting of all constant functions and $g(y) = y(a)$ for $y \in Y$. For a nondecreasing function on $[a, b]$ such that $z(b) - z(a) = 1$, define

$$f_z(x) = \int_a^b x \, dz, \quad x \in X.$$

Then f_z is a Hahn-Banach extension of g. In particular, if for a fixed t in $[a, b]$, z_t denotes the characteristic function of $[a, b]$, then $f_{z_t} = x(t), x \in X$.

7-13 Let Y be a subspace of a normed space X, and $g \in Y'$. Then the set of all Hahn-Banach extensions of g to X is a nonempty, convex, closed and bounded subset of X'. Its interior is empty. It may not be compact.

7-14 Let X be a linear space over $\mathbf{R}$, and let $p : X \to \mathbf{R}$ satisfy

$$p(x + y) \leq p(x) + p(y), \quad p(tx) = tp(x)$$

for all x and y in X, and $t \geq 0$. Let Y be a subspace of X, and $g : Y \to \mathbf{R}$ be a linear functional such that $g(y) \leq p(y)$ for all $y \in Y$.

7-20 Let $Y = \{x \in \ell^\infty : \lim_{n\to\infty}[x(1) + \cdots + x(n)]/n \text{ exists }\}$. For $x \in Y$, define $g(x) = \lim_{n\to\infty}[x(1)+\cdots+x(n)]/n$. Let f be a Hahn-Banach extension of g to ℓ^∞. Then $\|f\| = 1 = f((1,1,\ldots)), f(\tau(x)) = f(x)$ for all $x \in \ell^\infty$ and $f(x) = \lim_{n\to\infty} x(n)$ for all $x \in c$, so that f is a Banach limit. (Compare 7.12. Hint: For all $x \in \ell^\infty, x - \tau(x) \in Y$ and $g(z - \tau(x)) = 0$)

7-21 Let $\mathbf{K} = \mathbf{R}$, and f be a Banach limit. Then for every $x \in \ell^\infty$,

$$\liminf_{n\to\infty} x(n) \le f(x) \le \limsup_{n\to\infty} x(n).$$

8 Banach Spaces

It was Banach who first recognized and extensively exploited the completeness of a normed space. In his epoch-making book [2] of 1932, he modestly refers to complete normed spaces as spaces of type B, since they were already being named after him.

A normed space X over $\mathbf{K}$ is called a **Banach space** if X is complete in the metric $d(x, y) = \|x - y\|$ induced by the norm $\|\ \|$.

Since a subset of a complete metric space X is complete if and only if it is closed in X, it follows that a subspace Y of a Banach space X is a Banach space if and only if Y is closed in X.

Also, it follows from 6.3(a) that if X is a Banach space, and if there is a linear homeomorphism from X onto a normed space Y, then Y is also a Banach space.

We have already seen that the following normed spaces are complete, that is, they are Banach spaces.

1) $\mathbf{K}^n$ with any of the norms $\|\ \|_p, 1 \le p \le \infty$,
2) Finite dimensional normed spaces (6.3(b)),
3) $\ell^p, 1 \le p \le \infty$, with the norm $\|\ \|_p$ (3.3),

4) $L^p(E), 1 \leq p \leq \infty$, with the norm $\| \quad \|_p$, where E is a Lebesgue measurable subset of $\mathbf{R}$ (4.6),

5) $B(T)$, where T is a set and $C(T)$, where T is a metric space (Section 3).

Here are some further examples.

As a subspace of ℓ^∞, c_{00} is not closed. To see this, let $x_n = (1, 1/2, \ldots, 1/n, 0, 0, \ldots)$. Then $x_n \in c_{00}$ for each n, but $x_n \to x = (1, 1/2, 1/3, \ldots)$, which is not in c_{00}. Hence c_{00} is not a Banach space. On the other hand, it can be easily seen that the subspace c_0 is closed in ℓ^∞. Hence c_0 is a Banach space.

The subspace $C([a,b])$ is not closed in $L^p([a,b]), 1 \leq p < \infty$, as can be seen by noting that for $a < c < b$, the characteristic function of $[a,c]$ is in the closure of $C([a,b])$, but is not in $C([a,b])$. In fact, $C([a,b])$ is a proper dense subspace of $L^p([a,b])$ by 4.7(b). Hence $C([a,b])$, considered as a subspace of $L^p([a,b]), 1 \leq p < \infty$, is not a Banach space.

Let T be a metric space and consider the subspaces $C_0(T)$ and $C_c(T)$ of $C(T)$, introduced in 5.1(d). Let $x_n \in C_0(T)$ and $x_n \to x$ in $C(T)$. For $\epsilon > 0$, find a positive integer n such that $\|x_n - x\|_\infty < \epsilon/2$. Since $x_n \in C_0(T)$, there exists a compact subset E of T such that $|x_n(t)| < \epsilon/2$ if $t \notin E$. Then $|x(t)| < \epsilon$ for all $t \notin E$, so that $x \in C_0(T)$. Thus $C_0(T)$ is closed in $C(T)$. Hence $C_0(T)$ is a Banach space. On the other hand, $C_c(T)$ may not be closed in $C_0(T)$, as the following example shows. Let $x(t) = e^{-|t|}$ for $t \in \mathbf{R}$. For $n = 1, 2, \ldots$, let $x_n(t) = x(t)$ if $|t| \leq n, x_n(t) = 0$ if $|t| \geq n+1$, and let x_n equal a polynomial of degree 1 in $[n, n+1]$ and in $[-(n+1), n]$ such that x_n is continuous on $\mathbf{R}$. Then $x_n \in C_c(\mathbf{R}), \|x_n - x\|_\infty \to 0, x \in C_0(\mathbf{R})$, but $x \notin C_c(\mathbf{R})$. Thus $C_c(\mathbf{R})$ is not a Banach space. If, however, a metric space T is compact, then $C_c(T) = C(T)$ is a Banach space.

We shall now characterize Banach spaces among normed spaces. For this purpose we use the additive structure of a normed space. Let

$x_n \in X, n = 1, 2, \ldots$. We say that the series $\sum_{n=1}^{\infty} x_n$ is **summable** in X if the sequence (s_m) of its partial sums given by $s_m = \sum_{n=1}^{m} x_n$ converges in X. If (s_m) converges to s in X, then we write $s = \sum_{n=1}^{\infty} x_n$ and say that s is the **sum** of the series. A series $\sum_{n=1}^{\infty} x_n$ is said to be **absolutely summable** if $\sum_{n=1}^{\infty} \|x_n\| < \infty$. We know that every absolutely summable series in $\mathbf{K}$ is summable. In fact, we have the following characterization of a Banach space in terms of summable series.

8.1 Theorem

A normed space X is a Banach space if and only if every absolutely summable series of elements in X is summable in X.

Proof:

Let X be a Banach space, $x_n \in X$ and $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then the sequence (s_m) of the partial sums of $\sum_{n=1}^{\infty} x_n$ is Cauchy in X, since

$$\|s_{m+j} - s_m\| = \|x_{m+1} + \cdots + x_{m+j}\| \leq \|x_{m+1}\| + \cdots + \|x_{m+j}\|$$

for all $m, j = 1, 2, \ldots$. Since X is complete, (s_m) converges in X, that is, $\sum_{n=1}^{\infty} x_n$ is summable in X.

Conversely, let every absolutely summable series be summable in X. Let (s_n) be a Cauchy sequence in X. Let m_1 be a positive integer such that $\|s_m - s_{m_1}\| \leq 1$ for all $m \geq m_1$. Define inductively $m_2, m_3, \ldots$ such that $m_n < m_{n+1}$ and for all $m \geq m_n$,

$$\|s_m - s_{m_n}\| \leq \frac{1}{n^2}.$$

Let $x_n = s_{m_{n+1}} - s_{m_n}$ for $n = 1, 2, \ldots$. Then $\sum_{n=1}^{\infty} \|x_n\| \leq \sum_{n=1}^{\infty} 1/n^2$. Hence, by assumption, $\sum_{n=1}^{\infty} x_n$ is summable in X. Since $s_{m_n} = s_{m_1} + \sum_{j=1}^{n-1} x_j$, it follows that the subsequence (s_{m_n}) of the Cauchy sequence (s_m) converges in X. Hence (s_m) itself converges in X. This shows that X is complete. □

In Sections 5 and 6 we have seen how to construct new normed spaces from the old by considering subspaces, quotient spaces, product spaces and spaces of bounded linear maps. We now investigate whether these new normed spaces are Banach spaces.

8.2 Theorem

(a) Let X be a normed space and Y be a closed subspace of X. Then X is a Banach space if and only if Y and X/Y are Banach spaces in the induced norm and the quotient norm, respectively.

(b) Let $X_1, \ldots, X_m$ be normed spaces and $X = X_1 \times \cdots \times X_m$. Then $X_1, \ldots, X_m$ are all Banach spaces if and only if X is a Banach space in one of the norms $\| \ \|_p$ given in 5.2(c).

(c) Let X and Y be normed spaces and $X \neq \{0\}$. Then $BL(X, Y)$ is a Banach space in the operator norm if and only if Y is a Banach space.

(d) The dual X' of every normed space X is a Banach space.

Proof:

(a) Let X be a Banach space. Then Y is a Banach space since it is closed in X. To show X/Y is a Banach space, consider a sequence $(x_n + Y)$ in X/Y such that $\sum_{n=1}^{\infty} |||x_n + Y||| < \infty$. By the definition of the quotient norm $||| \ |||$, there is some $y_n \in Y$ such that

$$\|x_n + y_n\| < |||x_n + Y||| + \frac{1}{n^2}, \quad n = 1, 2, \ldots.$$

Then $\sum_{n=1}^{\infty} \|x_n + y_n\| < \infty$. Since X is a Banach space, 8.1 shows that $\sum_{n=1}^{\infty}(x_n + y_n) = s \in X$. Now for $m = 1, 2, \ldots,$

$$\begin{aligned} |||\sum_{n=1}^{m}(x_n + Y) - (s + Y)||| &= |||\sum_{n=1}^{m}(x_n + y_n) - s + Y||| \\ &\leq \|\sum_{n=1}^{m}(x_n + y_n) - s\|. \end{aligned}$$

This shows that $\sum_{n=1}^{\infty}(x_n + Y) = s + Y \in X/Y$. Again by 8.1, we see that X/Y is a Banach space.

Conversely, assume that Y and X/Y are Banach spaces. Consider a Cauchy sequence (x_n) in X. Since

$$|||(x_n + Y) - (x_m + Y)||| = |||(x_n - x_m) + Y||| \leq \|x_n - x_m\|$$

for all $n, m = 1, 2, \ldots$, we see that $(x_n + Y)$ is a Cauchy sequence in X/Y. Let $x_n + Y \to x + Y$ in X/Y. Then by 5.2(b), there is a sequence (y_n) in Y such that $x_n + y_n \to x$ in X. Since $y_n - y_m = y_n + x_n - x - x_n + x_m - x_m - y_m + x$ and

$$\|y_n - y_m\| \leq \|y_n + x_n - x\| + \|x_n - x_m\| + \|x_m + y_m - x\|$$

for all $n, m = 1, 2, \ldots$, it follows that (y_n) is a Cauchy sequence in Y. Let $y_n \to y$ in Y. then $x_n = (x_n + y_n) - y_n \to x - y$ in X. This shows that X is a Banach space.

(b) Let $X_1, \ldots, X_m$ be Banach spaces. If (x_n) is a Cauchy sequence in $X = X_1 \times \cdots \times X_m$, then $(x_n(j))$ is a Cauchy sequence in X_j for each $j = 1, \ldots, m$. Let $x_n(j) \to x(j)$ in X_j for each $j = 1, \ldots, m$. Then by 5.2(c), $x_n \to x = (x(1), \ldots, x(m))$ in X. Hence X is a Banach space.

Conversely, assume that $X = X_1 \times \cdots \times X_m$ is a Banach space. Fix $j, 1 \leq j \leq m$. Let $(x_n(j))$ be a Cauchy sequence in X_j and let

$$x_n = (0, \cdots, 0, x_n(j), 0, \cdots, 0) \in X.$$

Clearly, (x_n) is Cauchy in X. Let $x_n \to x$ in X. Then again by 5.2(c), $x_n(j) \to x(j)$ in X_j. Hence X_j is a Banach space for each $j = 1, \ldots, m$.

(c) Let Y be a Banach space, and let (F_n) be a Cauchy sequence in $BL(X, Y)$. For every $\epsilon > 0$, there is a positive integer n_0 such that

$$\|F_n(x) - F_m(x)\| \leq \|F_n - F_m\| \; \|x\| < \epsilon \|x\|$$

for all $x \in X$ and all $n, m \geq n_0$. Hence for each fixed $x \in X$, $(F_n(x))$ is a Cauchy sequence in Y, and so it converges in Y to $F(x)$, say.

It is easy to see that $F : X \to Y$ is linear. Letting $m \to \infty$ in the inequality $\|F_n(x) - F_m(x)\| < \epsilon\|x\|$, we see that

$$\|F_n(x) - F(x)\| \le \epsilon\|x\|$$

for all $x \in X$ and all $n \ge n_0$. Since $F = (F - F_n) + F_n$, we see that $F \in BL(X,Y)$ and $\|F_n - F\| \to 0$ as $n \to \infty$, that is, (F_n) converges to F in $BL(X,Y)$. Thus $BL(X,Y)$ is a Banach space.

Conversely, let $X \ne \{0\}$ and $BL(X,Y)$ be a Banach space. Consider $a \in X, a \ne 0$. We see from 7.10(a) that there is some $f \in X'$ with $f(a) = \|a\| \ne 0$ and $\|f\| = 1$. Consider a Cauchy sequence (y_n) in Y and define $F_n(x) = f(x)y_n$ for $x \in X$. Then it is easy to see that $F_n \in BL(X,Y)$ and

$$\|F_n - F_m\| = \|y_n - y_m\|$$

for all $n, m = 1, 2, \ldots$. As (F_n) is a Cauchy sequence in $BL(X,Y)$, let $F_n \to F$ in $BL(X,Y)$. Now $y_n = F_n(a)/\|a\| \to F(a)/\|a\|$ in Y. Hence the sequence (y_n) converges in Y. Thus Y is a Banach space.

(d) Let X be a normed space. Then $X' = BL(X, \mathbf{K})$ is a Banach space by (c) above. □

We now prove some useful results about continuous linear extensions and about convergence of continuous linear maps when the range space is a Banach space.

8.3 Theorem

Let X be a normed space and Y be a Banach space.

(a) Let X_0 be a dense subspace of X and $F_0 \in BL(X_0, Y)$. Then there is a unique $F \in BL(X,Y)$ such that $F_{|X_0} = F_0$. Also, $\|F\| = \|F_0\|$.

(b) Consider $F_n \in BL(X,Y)$ such that $\|F_n\| \le \alpha$ for all n and some $\alpha > 0$. Let E be a subset of X whose span is dense in X. Assume that $(F_n(x))$ converges in Y for every $x \in E$. Then there is a unique $F \in BL(X,Y)$ such that $F_n(x) \to F(x)$ in Y for every $x \in X$.

Proof:

(a) Let $x \in X$. Then there is a sequence (x_n) in X_0 such that $x_n \to x$. Since

$$\|F_0(x_n) - F_0(x_m)\| \le \|F_0\|\,\|x_n - x_m\|$$

for all $n, m = 1, 2, \ldots$, we see that $(F_0(x_n))$ is a Cauchy sequence in Y. As Y is a Banach space, let $F_0(x_n) \to y$ in Y and define $F(x) = y$. Then it is easy to check that $F : X \to Y$ is well-defined, linear and continuous. In fact, $\|F\| = \|F_0\|$, since for all $x \in X$,

$$\|F(x)\| = \lim_{n\to\infty} \|F(x_n)\| \le \|F_{|X_0}\| \lim_{n\to\infty} \|x_n\| = \|F_0\|\,\|x\|.$$

The uniqueness of F follows from the denseness of X_0 in X.

(b) Let X_0 denote the span of E. Since each F_n is linear and $(F_n(x))$ converges for every $x \in E$, it follows that $(F_n(x))$ converges for every $x \in X_0$. For $x \in X_0$, let $F_0(x) = \lim_{n\to\infty} F_n(x)$, so that

$$\|F_0(x)\| = \lim_{n\to\infty} \|F_n(x)\| \le \alpha\|x\|.$$

This shows that $F_0 \in BL(X_0, Y)$. By (a), there is a unique F in $BL(X,Y)$ such that $F_{|X_0} = F_0$. Let, now, $x \in X$ and $\epsilon > 0$. Find $x_0 \in X_0$ such that $\|x - x_0\| < \epsilon$ and a positive integer n_0 such that $\|F_n(x_0) - F(x_0)\| < \epsilon$ for all $n \ge n_0$. Then

$$\begin{aligned}\|F_n(x) - F(x)\| &\le \|F_n(x - x_0)\| + \|(F_n - F)(x_0)\| + \|F(x_0 - x)\| \\ &\le (\|F_n\| + \|F\|)\|x - x_0\| + \|F_n(x_0) - F(x_0)\| \\ &\le (\alpha + \|F\| + 1)\epsilon\end{aligned}$$

for all $n \ge n_0$. This shows that $F_n(x) \to F(x)$ for every $x \in X$. □

We shall see in Section 9 that the following converse of 8.3(b) is false in general, but holds good if X is a Banach space: If F_n belongs to $BL(X,Y)$ and $(F_n(x))$ converges for every $x \in X$, then there exist some $\alpha > 0$ such that $\|F_n\| \leq \alpha$ for all $n = 1, 2, \ldots$.

We now show that a normed space can be embedded as a dense subspace of a Banach space. Also, we use the extension result 8.3(a) to conclude that such a Banach space is essentially unique.

Let X be a normed space and $x \in X$. Define $j_x : X' \to \mathbf{K}$ by

$$j_x(f) = f(x), \quad f \in X'.$$

Then it is clear that j_x is linear and $\|j_x\| = \|x\|$ by Corollary 7.10(a) of the Hahn-Banach extension theorem. In particular, $j_x \in (X')'$. We shall denote the **second dual** $(X')'$ of X by X''. Define $J : X \to X''$ by

$$J(x) = j_x, \quad x \in X.$$

Then J is linear and $\|J(x)\| = \|x\|$ for each $x \in X$, that is, J is a linear isometry of X into X''. It is called the **canonical embedding** of X into X''.

Let X_c denote the closure of $J(X)$ in X''. [Note that if X is a Banach space, then $J(X)$ is a closed subspace of X'', and hence $X_c = J(X)$.] It follows from 8.2(d) that X'' is a Banach space, and since X_c is closed in X'', we see that X_c is a Banach space. Thus J is a linear isometry of X into the Banach space X_c in which $J(X)$ is dense. The Banach space X_c is called the **completion** of the normed space X. It is unique in the following sense. If X_1 is a Banach space and $J_1 : X \to X_1$ is a linear isometry such that $J_1(X)$ is dense in X_1, then there is a surjective linear isometry $F : X_c \to X_1$. This can be seen as follows. Define $F_0 : J(X) \to X_1$ by

$$F_0(J(x)) = J_1(x), \quad x \in X.$$

Then $F_0 \in BL(J(X), X_1)$, where $J(X)$ is dense in X_c and X_1 is a Banach space. Schematically,

$$\begin{array}{ccccc} X & \xrightarrow{J} & J(X) & \longrightarrow & X_c \\ & J_1 \searrow & F_0 \downarrow & \swarrow F & \\ & & X_1 & & \end{array}$$

By 8.3(a), F_0 extends to some $F \in BL(X_c, X_1)$. Since $\|F_0(J(x))\| = \|J_1(x)\| = \|x\| = \|J(x)\|$ for all $x \in X$, it can be seen that F is also a linear isometry. Also, F is surjective since $F(X_c)$ is both closed as well as dense in X_1.

As an example, let $X = C([a, b])$ with the norm given by

$$\|x\|_1 = \int_a^b |x(t)|dt, \quad x \in X.$$

Since X is dense in $L^1([a, b])$ by 4.7(b) and $L^1([a, b])$ is a Banach space, we see that $L^1([a, b])$ can be identified with the completion of X.

Before concluding this section, we prove a peculiar result regarding the number of elements in the basis of a Banach space.

8.4 Theorem

A Banach space cannot have a denumerable (Hamel) basis.

Proof:

Assume for a moment that a Banach space X has a denumerable basis $\{x_1, x_2, \ldots\}$. For $m = 1, 2, \ldots$, let $Y_m = \text{span}\{x_1, \ldots, x_m\}$. Being a finite dimensional subspace, Y_m is closed in X by 5.4(b). Thus the complement D_m of Y_m in X is open in X. Since $x_{m+1} \notin Y_m$, we see that $Y_m \neq X$ and hence the interior Y_m° of Y_m is empty by 5.6(c). This says that D_m is dense in X. Since X is complete, the intersection $\bigcap_{m=1}^{\infty} D_m$ of the dense open sets $D_m, m = 1, 2, \ldots$ must be dense in X by Baire's theorem (3.4). But since $\text{span}\{x_1, x_2, \ldots\} = X$, we see that $\bigcap_{m=1}^{\infty} D_m = \emptyset$ and arrive at a contradiction. $\square$

This result shows that a Banach space X is either finite dimensional, or else it has a (Hamel) basis which is uncountable. An infinite dimensional separable Banach space has, in fact, a basis which is in one to one correspondence with the set of all real numbers. (See [45], 1934.) Thus from the view point of constructive analysis, the concept of a (Hamel) basis is not suitable for an infinite dimensional Banach space. Let us then relax the requirement of a (Hamel) basis that every element of X be a **finite** linear combination of the basis elements and admit denumerable linear combinations. This leads us to the following useful concept.

Let X be a normed space. A countable subset $\{x_1, x_2, \ldots\}$ of X is called a **Schauder basis** for X if $\|x_n\| = 1$ for each n and if for every $x \in X$, there are unique $k_1, k_2, \ldots$ in $\mathbf{K}$ such that $x = \sum_n k_n x_n$.

Of course, if X is finite dimensional and $\{x_1, \ldots, x_m\}$ is a (Hamel) basis for X, then $\{x_1/\|x_1\|, \ldots, x_m/\|x_m\|\}$ is a Schauder basis for X. For example, if $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 occurs only in the jth place, then $\{e_1, \ldots, e_n\}$ is a Schauder basis for $\mathbf{K}^n$. It will be called the **standard** basis for $\mathbf{K}^n$. Also, if $e_j = (0, \ldots, 0, 1, 0, 0, \ldots)$, where 1 occurs only in the jth place, $j = 1, 2, \ldots$, then the set $\{e_1, e_2, \ldots\}$ is a Schauder basis for c_{00}, for $\ell^p, 1 \le p < \infty$, as well as for c_0. It will be called the **standard Schauder basis**. Problems 8-14 and 8-15 give Schauder bases for $L^p([0,1]), 1 \le p < \infty$, (consisting of step functions) and for $C([0,1])$ (consisting of saw-tooth functions).

If there is a Schauder basis $\{x_1, x_2, \ldots\}$ for a normed space X, then X must be separable, since $\{k_1 x_{n_1} + \cdots + k_m x_{n_m} : m \ge 1, k_j \in \mathbf{K}$ and $\operatorname{Re} k_j$ and $\operatorname{Im} k_j$ rational, $j = 1, \ldots, m\}$ is a countable dense subset of X. All the classical separable Banach spaces have Schauder bases, although finding them can be a difficult task. We shall see in Section 22 that every separable Hilbert space has a Schauder basis. Whether every separable Banach space has a Schauder basis was an open question for a long time. It was settled in the negative by Enflo ([21], 1973), who found a closed subspace of c_0 which has no Schauder basis.

If $\{x_1, x_2, \dots\}$ is a Schauder basis for X, then for $n = 1, 2, \dots$, define $f_n : X \to \mathbf{K}$ by

$$f_n(x) = a_n, \quad \text{for } x = \sum_{n=1}^{\infty} a_n x_n \in X.$$

The uniqueness condition in the definition of a Schauder basis shows that each f_n is well-defined and linear on X. It is called the nth **coefficient functional** on X corresponding to the Schauder basis $\{x_1, x_2, \dots\}$ for X. If X is finite dimensional, then 5.4(b) implies that each f_n is a continuous linear functional. In fact, we shall see in 11.4 that if X is a Banach space, then each f_n is a continuous linear functional and $\|f_n\| \le \alpha$ for all $n = 1, 2, \dots$ and some $\alpha > 0$.

Problems

8-1 For $k = 1, 2, \dots$, let $X = C^k([a, b])$, the linear space of all k times continuously differentiable functions on $[a, b]$. For $x \in X$, let

$$\|x\| = \|x\|_\infty + \|x^{(1)}\|_\infty + \cdots + \|x^{(k)}\|_\infty.$$

Then X is a Banach space in the norm $\|\ \|$. The subspace $C^{k+1}([a, b])$ of $C^k([a, b])$ is not a Banach space in the induced norm. In fact, for every $x \in X$, there is a sequence (p_n) of polynomials such that $\|x - p_n\| \to 0$. (Compare Problem 5-19.)

8-2 Let Z denote the space of all scalar-valued functions of bounded variation on $[a, b]$. For $z \in Z$, let

$$\|z\| = |z(a)| + V(z),$$

where $V(z)$ denotes the total variation of z. Then $\|\ \|$ is a norm on Z and Z is a Banach space. Also, $z_n \to z$ in Z if and only if $z_n(a) \to z(a)$ and $V(z_n - z) \to 0$. In that case, $\|z_n - z\|_\infty \to 0$. (Hint: If (z_n) is a sequence in Z and $z_n(t) \to z(t)$ for every $t \in [a, b]$, then $V(z) \le \liminf V(z_n)$).

8-3 Let D denote the open unit disk in $\mathbf{C}$.

(a) Let $A = \{x \in C(\overline{D}) : x \text{ analytic on } D\}$. For $x \in A$, let

$$\|x\| = \max\{|x(e^{it})| : t \in [-\pi, \pi]\}.$$

Then A is a Banach space in the norm $\| \ \|$.

(b) For $x \in C(D)$, let

$$\|x\|_p = \begin{cases} \sup_{0 \le r < 1} \left(\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} |x(re^{it})|^p dt \right)^{1/p}, & \text{if } 1 \le p < \infty \\ \sup_{0 \le r < 1} \max\{|x(re^{it})| : t \in [-\pi, \pi]\}, & \text{if } p = \infty. \end{cases}$$

Then $H^p(D) = \{x \in C(D) : x \text{ analytic in } D, \|x\|_p < \infty\}$ is a Banach space.

8-4 Let X be a normed space, and $x_n \in X, n = 1, 2, \ldots$. If X is a Banach space and $\limsup \|x_n\|^{1/n} < 1$, then $\sum_{n=1}^{\infty} x_n$ is summable in X. If $\limsup \|x_n\|^{1/n} > 1$, then $\sum_{n=1}^{\infty} x_n$ is not summable in X.

8-5 Let X be a Banach space. If a series $\sum_{n=1}^{\infty} x_n$ is absolutely summable in X, then it is **unconditionally summable** in X, that is, $\sum_{n=1}^{\infty} x_{p(n)}$ is summable in X for every permutation p of the natural numbers. The converse is true if X is finite dimensional, but is not true if $X = c_0$ (Hint: Consider $x_n = e_n/n$.) [In fact, Dvoretsky and Rogers proved in 1950 that if the converse holds, then X must be finite dimensional. See [16], p. 59.

8-6 Let X be a Banach space. For $A \in BL(X)$, the series $\sum_{n=0}^{\infty} A^n/n!$ is summable in $BL(X)$. Let $\exp A$ denote its sum. If A and B are in $BL(X)$, and $AB = BA$, then $\exp(A + B) = (\exp A)(\exp B)$.

8-7 Let $\| \ \|$ be a norm on the linear space X of all polynomials in one variable with coefficients in **K**. (For example, see Problem 5-21.) Then there is a sequence (p_n) in X such that $\sum_{n=1}^{\infty} \|p_n\| < \infty$, but if $q_m = \sum_{n=1}^{m} p_n$, then (q_m) does not converge in X.

8-8 If Y is a proper dense subspace of a Banach space X, then Y is not a Banach space in the induced norm.

8-9 Let $\| \ \|_j$ be a norm on a linear space X_j for $j = 1, 2, \ldots$, and for a fixed p with $1 \le p \le \infty$, let X be the product space defined in Problem 5-6. Then X is a Banach space if and only if each X_j is a Banach space. (Hint: Proof of 3.3)

8-10 The linear space c_{00} cannot be a Banach space in any norm. The completion of c_{00} equipped with the norm $\| \ \|_p$ can be identified with ℓ^p if $1 \leq p < \infty$, and with c_0 if $p = \infty$.

8-11 For $1 \leq p < \infty$ and $k = 1, 2, \ldots$, let $W^{k,p} = \{x \in C^{k-1}([a,b]) : x^{(k-1)}$ is absolutely continuous on $[a,b]$ and $x^{(k)} \in L^p([a,b])\}$. Define

$$\|x\|_{k,p} = \left(\sum_{j=0}^{k} \int_a^b |x^{(j)}|^p \, dm \right)^{1/p}, \quad x \in W^{k,p}.$$

Then $W^{k,p}$ is a Banach space, known as the **Sobolev space** of order (k,p) on $[a,b]$. It is the completion of $C^k([a,b])$ with the norm $\| \ \|_{k,p}$. (Compare 21.3(d).)

8-12 Let $\{x_1, x_2, \ldots\}$ be a Schauder basis for a Banach space X. If $k_n \in \mathbf{K}$, and $\sum_{n=1}^{\infty} |k_n| < \infty$, then there is a unique $x \in X$ such that $f_n(x) = k_n$ for each coefficient functional $f_n, n = 1, 2, \ldots$. The converse does not hold.

8-13 Let $a = (1, 1, \ldots)$. Then $\{a, e_1, e_2, \ldots\}$ is a Schauder basis for the subspace c of ℓ^∞.

8-14 Let $1 \leq p < \infty$. For $t \in [0,1]$, let $x_1(t) = 1$,

$$x_2(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1/2 \\ -1, & \text{if } 1/2 < t \leq 1 \end{cases}$$

and for $n = 1, 2, \ldots, j = 1, \ldots, 2^n$,

$$x_{2^n+j}(t) = \begin{cases} 2^{n/p}, & \text{if } (2j-2)/2^{n+1} \leq t \leq (2j-1)/2^{n+1} \\ -2^{n/p}, & \text{if } (2j-1)/2^{n+1} < t \leq 2j/2^{n+1} \\ 0, & \text{otherwise.} \end{cases}$$

Then the **Haar system** $\{x_1, x_2, x_3, \ldots\}$ is a Schauder basis for $L^p([0,1])$. Each x_n is a step function.

8-15 **(Schauder)** For $t \in \mathbf{R}$, let $y_0(t) = t, y_1(t) = 1 - t$,

$$y_2(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 1/2 \\ 2 - 2t, & \text{if } 1/2 < t \leq 1 \\ 0, & \text{if } t < 0 \text{ or } t > 1 \end{cases}$$

and $y_{2^n+j}(t) = y_2(2^n t - j + 1)$ for $n = 1, 2, ..., j = 1, 2, ..., 2^n$. If $x_n = y_n|_{[0,1]}$, then $\{x_0, x_1, x_2, \ldots\}$ is a Schauder basis for $C([0,1])$. Each x_n is a nonnegative piecewise linear continuous function, known as a **saw-tooth function**.

8-16 Let $\{x_1, x_2, \ldots\}$ be a Schauder basis for a normed space X and for $m = 1, 2, \ldots$, let

$$F_m(x) = \sum_{n=1}^{m} f_n(x) x_n, \quad x \in X,$$

where f_n is the nth coefficient functional corresponding to $\{x_1, x_2, \ldots\}$. Then $F_p F_q = F_m$, where $m = \min(p, q)$ for all $p, q = 1, 2, \ldots$. Further, $\|F_m(x)\| \leq \|F_{m+1}(x)\|$ for all $x \in X$ and $m = 1, 2, \ldots$ if and only if $\|F_m\| \leq 1$ for all $m = 1, 2, \ldots$.

Chapter III

Bounded Linear Maps on Banach Spaces

This chapter belongs to the core of functional analysis. The most celebrated results for bounded linear maps on Banach spaces are the uniform boundedness principle, the closed graph theorem, the open mapping theorem and the bounded inverse theorem. They are proved in Sections 9, 10 and 11. Several useful applications of these results are also given. In Section 12, a set of scalars, known as spectrum, is associated with a bounded linear map. It involves the (non)invertibility of such a map. When the scalars are complex numbers, the spectral radius formula is proved. It illustrates the interplay between the algebraic structure and the distance structure in a striking manner.

9 Uniform Boundedness Principle

A set of continuous functions from a metric space to a metric space can be bounded at each point without being uniformly bounded. For example, let

$$x_n(t) = \begin{cases} n^2 t, & \text{if} \quad 0 \le t \le \dfrac{1}{n} \\ \dfrac{1}{t}, & \text{if} \quad \dfrac{1}{n} < t \le 1 \end{cases}$$

for $n = 1, 2, \ldots$. Then $x_n(t) \in C([0,1]), x_n(0) = 0$ and $|x_n(t)| \le 1/t$ for each $t \in [0,1]$. But $x_n(1/n) = n$ for $n = 1, 2, \ldots$. Hence the set $\{x_n\}$ is bounded at each point of $[0,1]$, but it is not uniformly bounded on $[0,1]$. In this connection, we recall Ascoli's theorem (3.10(a)) which says that if T is a compact metric space and if a subset of $C(T)$ is bounded as well as equicontinuous at each $t \in T$, then it is uniformly bounded on T. In the present section, we show that if a set of continuous linear maps from a Banach space X to a normed space Y is bounded at each $x \in X$, then it is uniformly bounded on each bounded subset of X.

9.1 Theorem (Uniform boundedness principle)
Let X be a Banach space, Y be a normed space and $\mathcal{F}$ be a subset of $BL(X,Y)$ such that for each $x \in X$, the set $\{F(x) : F \in \mathcal{F}\}$ is bounded in Y. Then for each bounded subset E of X, the set $\{F(x) : x \in E, F \in \mathcal{F}\}$ is bounded in Y, that is, $\mathcal{F}$ is uniformly bounded on E. In particular, $\sup\{\|F\| : F \in \mathcal{F}\} < \infty$.

Proof:
For $n = 1, 2, \ldots$, let

$$D_n = \{x \in X : \|F(x)\| > n \text{ for some } F \in \mathcal{F}\}.$$

For each $F \in \mathcal{F}$, the function $x \to \|F(x)\|$ is continuous on X, so that the set $\{x \in X : \|F(x)\| > n\}$ is open in X. Since D_n is the union of all these sets, it follows that D_n is open in X.

Let $x \in X$. Then, by hypothesis, $\|F(x)\| \leq n$ for all $F \in \mathcal{F}$ and some positive integer n, that is, $x \notin D_n$. Thus $\bigcap_{n=1}^{\infty} D_n = \emptyset$. Consequently, $\bigcap_{n=1}^{\infty} D_n$ cannot be dense in X.

Since X is a Banach space, Baire's theorem (3.4) implies that some D_m must not be dense in X. Then there is some $a \in X$ and some $r > 0$ such that $\overline{U}_X(a,r) \bigcap D_m = \emptyset$, that is, if $y \in X$ and $\|y - a\| \leq r$, then $\|F(y)\| \leq m$ for all $F \in \mathcal{F}$.

Let E be a bounded subset of X, $\|x\| \leq \alpha$ for all $x \in E$ and some $\alpha > 0$. Consider $x \in E$, $F \in \mathcal{F}$. Then $\|F(\frac{rx}{\alpha} + a)\| \leq m$, so that

$$\begin{aligned} \|F(x)\| = \frac{\alpha}{r}\|F(\frac{rx}{\alpha})\| &= \frac{\alpha}{r}\|F(\frac{rx}{\alpha} + a) - F(a)\| \\ &\leq \frac{\alpha}{r}\left[\|F(\frac{rx}{\alpha} + a)\| + \|F(a)\|\right] \leq \frac{2\alpha m}{r}. \end{aligned}$$

Thus $\sup\{\|F(x)\| : x \in E, F \in \mathcal{F}\} \leq 2\alpha m/r$, that is, $\mathcal{F}$ is uniformly bounded on E.

If we let $E = \overline{U}_X(0,1)$, it follows that

$$\begin{aligned}\sup\{\|F\| : F \in \mathcal{F}\} &= \sup\{\|F(x)\| : x \in X, \|x\| \leq 1, F \in \mathcal{F}\}\\ &\leq \frac{2m}{r} < \infty.\end{aligned}$$

This completes the proof. $\square$

Several remarks are in order.

1) Geometrically, this result says that either each F in $\mathcal{F}$ maps a given bounded subset of a Banach space X into a fixed ball in the normed space Y, or else there is some $x \in X$ such that no ball in Y contains all $F(x)$ with $F \in \mathcal{F}$.

2) Actually, only the following properties of $F \in \mathcal{F}$ are used in the proof of 9.1. The function $x \mapsto \|F(x)\|$ is continuous from X to nonnegative real numbers, $\|F(x+y)\| \leq \|F(x)\| + \|F(y)\|$ and $\|F(kx)\| = |k|\|F(x)\|$ for all x and y in X and $k \in \mathbf{K}$. A function having these properties is known as a **continuous seminorm** on X. Hence considering $E = \overline{U}_X(0,1)$, we obtain the following result.

Let X be a Banach space and $\mathcal{P}$ be a set of continuous seminorms on X such that the set $\{p(x) : p \in \mathcal{P}\}$ is bounded for each x in X. Then there is a constant α such that $p(x) \leq \alpha$ for all $p \in \mathcal{P}$ and all $x \in X$ with $\|x\| \leq 1$.

3) The proof of 9.1 can be modified to yield a version of the uniform boundedness principle for continuous 'affine' functions defined on a bounded, complete, convex subset of a normed space X. (See Problem 9-1.)

4) Let X and Y be normed spaces and $\mathcal{F} \subset BL(X,Y)$. Then $X_0 = \{x \in X : \mathcal{F} \text{ is bounded at } x\}$ is a subspace of X. Suppose $X_0 \neq X$. By 5.6(c), the interior of X_0 is empty, that is, the complement of X_0 is dense in X. Thus if $\mathcal{F}$ is unbounded at some $x \in X$, then, in fact, $\mathcal{F}$ is unbounded at each x in a dense subset of X.

9.2 Corollary

Let X be a Banach space, Y be a normed space and (F_n) be a sequence in $BL(X,Y)$ such that the sequence $(F_n(x))$ converges in Y for every $x \in X$. For $x \in X$, define

$$F(x) = \lim_{n\to\infty} F_n(x).$$

(a) (Banach-Steinhaus theorem, 1927) F is a bounded linear map from X to Y and

$$\|F\| \le \liminf_{n\to\infty} \|F_n\| \le \sup\{\|F_n\| : n = 1, 2, \ldots\} < \infty.$$

(b) Let E be a totally bounded subset of X. Then $(F_n(x))$ converges to $F(x)$ uniformly for $x \in E$.

Proof:

Let $\alpha = \sup\{\|F_n\| : n = 1, 2, \ldots\}$.

(a) It is clear that F is linear. For every $x \in X$, the set $\{\|F_n(x)\| : n = 1, 2, \ldots\}$ is bounded because $(F_n(x))$ is a convergent sequence in Y. By 9.1, the set $\{\|F_n\| : n = 1, 2, \ldots\}$ is bounded. For $x \in X$,

$$\|F(x)\| = \lim_{n\to\infty} \|F_n(x)\| \le (\liminf_{n\to\infty} \|F_n\|)\|x\|,$$

so that F is continuous and $\|F\| \le \liminf_{n\to\infty} \|F_n\| \le \alpha < \infty$.

(b) Let $\epsilon > 0$. Since E is totally bounded, there are $x_1, \ldots, x_m$ in E such that

$$E \subset U(x_1, \epsilon) \bigcup \cdots \bigcup U(x_m, \epsilon).$$

Now $F_n(x_j) \to F(x_j)$ as $n \to \infty$ for each $j = 1, \ldots, m$. Find n_0 such that

$$\|F_n(x_j) - F(x_j)\| < \epsilon$$

for all $n \ge n_0$ and all $j = 1, \ldots, m$. Let $x \in E$ and choose x_j in X such that $\|x - x_j\| < \epsilon$. Then for all $n \ge n_0$, we have

$$\begin{aligned}
\|F_n(x) - F(x)\| &\le \|F_n(x - x_j)\| + \|(F_n - F)(x_j)\| + \|F(x_j - x)\| \\
&\le (\|F_n\| + \|F\|)\|x - x_j\| + \|F_n(x_j) - F(x_j)\| \\
&\le (2\alpha + 1)\epsilon.
\end{aligned}$$

Hence $(F_n(x))$ converges to $F(x)$ uniformly for $x \in E$. □

Again, we make some remarks.

1) The hypothesis of completeness on X cannot be dropped from the results 9.1 and 9.2(a). For example, let $X = c_{00}$ with the sup norm and define

$$f_n(x) = \sum_{j=1}^{n} x(j), \quad x \in X.$$

Then $f_n \in X' = BL(X, \mathbf{K})$. In fact, $\|f_n\| = n$ for all $n = 1, 2, \ldots$. Now fix $x \in X$ and find m_x such that $x(j) = 0$ for all $j > m_x$. Then

$$|f_n(x)| \leq m_x \|x\|_\infty$$

for all $n = 1, 2, \ldots$. However, the set $\{\|f_n\| : n = 1, 2, \ldots\}$ is unbounded, showing that 9.1 does not hold. Also, if we let

$$f_0(x) = \sum_{j=1}^{\infty} x(j), \quad x \in X,$$

then $f_n(x) = f_0(x)$ for all $n > m_x$. Hence $f_n(x) \to f_0(x)$ for all $x \in X$. But $f_0 \notin X'$, as can be seen very easily. Hence 9.2 (a) does not hold. This is because c_{00} is not complete in the sup norm.

2) In 9.2(a), strict inequality can occur in $\|F\| \leq \liminf_{n\to\infty} \|F_n\|$. For example, if $X = \ell^1$ and $f_n(x) = x(n)$ for $x \in X$, then $f_n \in X'$ and $f_n(x) \to f(x) = 0$ for each $x \in X$. Since $\|f_n\| = 1$ for each n, we have

$$0 = \|f\| < \liminf \|f_n\| = 1.$$

Let $E = \{e_n : n = 1, \ldots\}$. It follows that $(f_n(x))$ does not converge to $f(x)$ uniformly for $x \in E$. Hence 9.2(b) does not hold if E is a bounded but not a totally bounded subset of a Banach space X.

9.3 Theorem

(a) (Resonance theorem) Let X be a normed space and E be a subset of X. Then E is bounded in X if and only if $f(E)$ is bounded in $\mathbf{K}$ for every $f \in X'$.

(b) Let X and Y be normed spaces and $F : X \to Y$ be linear Then F is continuous if and only if $g \circ F$ is continuous for every $g \in Y'$

Proof:

(a) Let E be bounded in X, that is, $\|x\| \leq \alpha$ for all $x \in E$ and some $\alpha > 0$. If $f \in X'$, then

$$|f(x)| \leq \|f\| \, \|x\| \leq \|f\| \alpha$$

for all $x \in E$, so that $f(E)$ is bounded in $\mathbf{K}$.

Conversely, assume that $f(E)$ is bounded in $\mathbf{K}$ for every f in X'. Let $J : X \to (X')'$ be the canonical embedding introduced in Section 8: $J(x)(f) = f(x), x \in X, f \in X'$. Then $\{J(x) : x \in E\}$ is a subset of $BL(X', \mathbf{K})$, where X' is a Banach space by 8.3(c). For each fixed $f \in X'$, the set $\{|J(x)(f)| : x \in E\} = \{|f(x)| : x \in E\}$ is bounded. Hence by the uniform boundedness principle (9.1) we see that the set $\{\|J(x)\| : x \in E\}$ is bounded. Since $\|J(x)\| = \|x\|$ for each $x \in X$, it follows that E is a bounded subset of X.

(b) Let $F : X \to Y$ be continuous. Since every $g \in Y'$ is continuous, we see that $g \circ F$ is continuous. Conversely, let $g \circ F$ be continuous for every $g \in Y'$. Let $E = \{F(x) : x \in X, \|x\| \leq 1\} \subset Y$. Then $g(E) = g \circ F(\overline{U}_X)$ is bounded for every $g \in Y'$. Hence by (a) above, E is a bounded subset of Y, that is, the linear map F is bounded on $\overline{U}_X$. Thus F is continuous by 6.2(i). □

Here are some comments on the results given above.

1) The statement 9.3(a) says that the boundedness of $f(E)$ for every $f \in X'$ implies the boundedness of E in X. However, the convergence of $(f(x_n))$ for every $f \in X'$ may not imply the convergence of (x_n) in X. For example, let $X = \ell^2$ and $e_n = (0, \ldots, 0, 1, 0, 0, \ldots), n = 1, 2, \ldots$. By 13.2, we can conclude that $(f(e_n))$ converges to 0 for every $f \in X'$. But obviously (e_n) does not converge in X, since $\|e_n - e_m\|_2 = \sqrt{2}$ for all $n \neq m$. A similar comment can be made about

the differentiability of a function with values in a normed space. (See Problem 14-8.) However, the analyticity of a Banach space-valued function F defined on an open subset of $\mathbf{C}$ can be deduced from the analyticity of the function $g \circ F$ for every g in the dual of that Banach space. (See Problem 9-9.)

2) Let X and Y be Banach spaces and $F : X \to Y$ be linear. Then 9.3(b) can be improved as follows. If $\{g_s\}$ is a subset of Y' such that $g_s \circ F$ is continuous for every s and for each $y \in Y$, there is some s with $g_s(y) \neq 0$, then F is continuous. (See Problem 10-8.)

We shall now give several applications of the uniform boundedness principle to various problems arising in analysis. These are of two kinds. Some yield negative results like the divergence of some Fourier series of continuous 2π-periodic functions, or the divergence of the Newton-Cotes quadrature formulae. On the other hand, some yield positive results like the convergence of the Gauss-Legendre quadrature formulae, or the convergence of some summability methods.

Divergence of Fourier Series of Continuous Functions

In Section 4 we have defined the Fourier series of a function in $L^1([-\pi,\pi]$ and mentioned several results regarding its convergence. We shall now use the uniform boundedness principle (9.1) to prove one of them.

9.4 Theorem

Let $X = \{x \in C([-\pi,\pi]) : x(\pi) = x(-\pi)\}$ with the sup norm. Then the Fourier series of every x in a dense subset of X diverges at 0.

Proof:

For $x \in X$ and $m = 1, 2, \ldots$, let $f_m(x) = \sum\limits_{n=-m}^{m} \hat{x}(n)$. Consider the

Dirichlet kernel $x_m(t) = \sum_{n=-m}^{m} e^{int}, m = 0, 1, 2, \ldots$ and $t \in [-\pi, \pi]$. Then

$$2\pi f_m(x) = \int_{-\pi}^{\pi} x(s)x_m(s)ds, \quad x \in X.$$

Fix m. It is clear that f_m is a linear functional on X and

$$2\pi|f_m(x)| \leq \|x_m\|_1 \|x\|_\infty, \quad x \in X,$$

where $\| \ \|_1$ is the usual norm on $L^1([-\pi, \pi])$. Hence $2\pi\|f_m\| \leq \|x_m\|_1$. To prove the reverse inequality, note that x_m is a real-valued function and let

$$E_m = \{t \in [-\pi, \pi] : x_m(t) \geq 0\}.$$

For $n = 1, 2, \ldots$, consider

$$x_{m,n}(t) = \frac{1 - n\text{dist}(t, E_m)}{1 + n\text{dist}(t, E_m)}, \quad t \in [-\pi, \pi].$$

Then it is clear that $x_{m,n} \in C([-\pi, \pi]), x_{m,n}(t) = 1$ if $t \in E_m$, and $x_{m,n}(t) \to -1$ as $n \to \infty$ if $t \notin E_m$. Also, since $x_m(-t) = x_m(t)$ for all t, we see that $\text{dist}(\pi, E_m) = \text{dist}(-\pi, E_m)$ and hence $x_{m,n}(\pi) = x_{m,n}(-\pi)$. Thus $x_{m,n} \in X$ and $x_{m,n}(s)x_m(s) \to |x_m(s)|$ as $n \to \infty$ for every $s \in [-\pi, \pi]$. Since $\|x_{m,n}\|_\infty \leq 1$, the bounded convergence theorem (4.1(b)) implies that

$$2\pi f_m(x_{m,n}) = \int_{-\pi}^{\pi} x_{m,n}(s)x_m(s)\, ds \to \int_{-\pi}^{\pi} |x_m(s)|ds = \|x_m\|_1,$$

as $n \to \infty$. Hence $2\pi\|f_m\| = \|x_m\|_1$ for each $m = 0, 1, \ldots$.

We have seen in Section 4 that $\|x_m\|_1 \to \infty$. Thus the set $\{\|f_m\| : m = 0, 1, \ldots\}$ is not bounded. The uniform boundedness principle (9.1) now shows that there is some $x \in X$ such that the set $\{|f_m(x)| : m = 0, 1, \ldots\}$ is not bounded. Hence the Fourier series of x diverges at 0. Thus $X_0 = \{x \in X :$ the Fourier series of x converges at $0\}$ is a proper subspace of X. By 5.3(c), the complement of X_0 is dense in X, that is, the Fourier series of every x in a dense subset of X diverges at 0. □

The proof given above illustrates the power of functional analytic methods in treating problems of classical analysis. It must be pointed out, however, that these methods are seldom constructive and do not, in general, yield concrete examples. Although we have shown above that there are a lot of continuous 2π-periodic functions on $\mathbf{R}$ whose Fourier series diverge at 0, we have given no indication whatsoever of how to construct a single such function. Specific examples of such functions were known since long and to satisfy readers' curiosity, we cite one due to Fejer (1911). For $m, n = 1, 2, \ldots$, let

$$Q(m, n, t) = \sin mt \sum_{j=1}^{n} \frac{\sin jt}{j} \qquad \text{and} \qquad x(t) = \sum_{p=1}^{\infty} \frac{Q(m_p, n_p, t)}{p^2},$$

where $m_p/2 = n_p = 2^{p^3}$.

While we are on this topic, we point out an interesting result of Pal (1914) and Bohr (1935): If $x \in C([-\pi, \pi])$ and $x(\pi) = x(-\pi)$, then there is a homeomorphism h from $[-\pi, \pi]$ onto $[-\pi, \pi]$ such that the Fourier series of $x \circ h$ converges uniformly on $[-\pi, \pi]$.

Quadrature Formulae

Let $X = C([a, b])$ with the sup norm and define $f : X \to \mathbf{K}$ by

$$f(x) = \int_a^b x(t)dt, \quad x \in X.$$

Although it is of great importance to find the value of the linear functional f at a given continuous function x, it is seldom possible to do so exactly. Note that the Riemann sum $\sum_{j=1}^{m}(s_j - s_{j-1})x(t_j)$, where $a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$ and $s_{j-1} \le t_j \le s_j$, converges to $f(x)$ if $\max\{|s_j - s_{j-1}| : j = 1, \ldots, m\} \to 0$. Similar sums are used to calculate $f(x)$ approximately. A **quadrature formula** is a functional $g : X \to \mathbf{K}$ given by

$$g(x) = \sum_{j=1}^{k} w_j x(t_j),$$

where the **nodes** $t_1, \ldots, t_m$ satisfy $a \le t_1 < \cdots < t_m \le b$ and the **weights** $w_1, \ldots, w_m$ are in $\mathbf{K}$.

It is easy to see that the linear functional f is continuous and $\|f\| = (b-a)$. We now calculate the norm of a quadrature formula g. Since

$$|g(x)| \le \left(\sum_{j=1}^{m} |w_j|\right) \|x\|_\infty$$

for all $x \in X$, we see that $\|g\| \le \sum_{j=1}^m |w_j|$. On the other hand, define $x \in X$ by letting

$$x(t_j) = \operatorname{sgn} w_j, \quad j = 1, \ldots, m,$$

$$x(t) = \begin{cases} x(t_1), & \text{if } \ a \le t < t_1 \\ \alpha_j t + \beta_j, & \text{if } \ t_{j-1} < t < t_j, j = 2, \ldots, m \\ x(t_m), & \text{if } \ t_m < t \le b, \end{cases}$$

where α_j and β_j are so chosen that x is continuous on $[a,b]$. Then it is easy to see that $\|x\|_\infty = 1$ (unless, of course all the w_j's are zero, in which case $g = 0$) and $g(x) = \sum_{j=1}^m |w_j|$. Thus

$$\|g\| = \sum_{j=1}^{m} |w_j|.$$

Given a sequence (g_n) of quadrature formulae, we ask whether $g_n(x)$ converges to $f(x) = \int_a^b x(t)\,dt$ for every $x \in C([a,b])$. The following result gives a set of necessary and sufficient conditions.

9.5 Theorem (Polya, 1933)

Let (g_n) be a sequence of quadrature formulae:

$$g_n(x) = \sum_{j=1}^{m_n} w_j^{(n)} x(t_j^{(n)}), \quad x \in C([a,b]).$$

Then $g_n(x) \to \int_a^b x(t)dt$ for every $x \in C([a,b])$ if and only if

1) $g_n(x) \to \int_a^b x(t)dt$ for every x in a set E whose span is dense in $C([a,b])$ and

2) $\sum_{j=1}^{m_n} |w_j^{(n)}| \le \alpha < \infty$ for all $n = 1, 2, \ldots$ and some $\alpha > 0$.

Proof:
Suppose that $g_n(x) \to \int_a^b x(t)dt$ for every $x \in C([a,b])$. Then condition 1) is satisfied with $E = C([a,b])$. Also, we have seen that

$$\|g_n\| = \sum_{j=1}^{m_n} |w_j^{(n)}|, \qquad n = 1, 2, \ldots.$$

Thus condition 2) follows from the Banach-Steinhaus theorem 9.2(a).

Conversely, if conditions 1) and 2) hold, then 8.3(b) shows that for every $x \in C([a,b])$, $g_n(x) \to g(x)$, where g is a continuous linear functional on $C([a,b])$. It is obvious that $g(x) = \int_a^b x(t)dt$ for all x in $C([a,b])$. □

A special case is worth mentioning. If all the weights in a sequence (g_n) of quadrature formulae are nonnegative, then condition 2) of Theorem 9.5 is automatically satisfied if $g_n(1) \to \int_a^b dt$, that is, $\sum_{j=1}^{m_n} w_j^{(n)} \to (b-a)$, since $|w_j^{(n)}| = w_j^{(n)}$ for all n and $j = 1, \ldots, m_n$.

Let $x_k(t) = t^k, t \in [a,b]$ for $k = 0, 1, 2, \ldots$ and $E = \{x_0, x_1, \cdots\}$. Then span E is the set of all polynomials in one variable and it is dense in $C([a,b])$ by Weierstrass' theorem (3.12). By 9.5,

$$g_n(x) = \sum_{j=1}^{m_n} w_j^{(n)} x(t_j^{(n)}) \to \int_a^b x(t)dt$$

for every $x \in C([a,b])$ if and only if

$$g_n(x_k) = \sum_{j=1}^{m_n} w_j^{(n)} (t_j^{(n)})^k \to \int_a^b t^k\,dt = \frac{b^{k+1} - a^{k+1}}{k+1}$$

for every $k = 0, 1, 2, \ldots$ and $\sum_{j=1}^{m_n} |w_j^{(n)}| \le \alpha < \infty$. This latter condition is redundant if $w_j^{(n)} \ge 0$ for all n and j, as we have just seen.

One of the simplest ways of ensuring that $g_n(x_k) \to \int_a^b t^k dt$ for all $k = 0, 1, \ldots$ is to make sure that for each fixed k, $g_n(x_k) = \int_a^b t^k dt$ for all large enough n. Let us see how such quadrature formulae can be constructed.

Let $t_1, \ldots, t_m$ be distinct points in $[a, b]$. Then for $j = 1, \ldots, m$, there is a unique polynomial ℓ_j of degree $m-1$ such that $\ell_j(t_i) = \delta_{i,j}$ for $i = 1, \ldots, m$. In fact,

$$\ell_j(t) = \frac{\prod_{i=1, i\neq j}^{m}(t - t_i)}{\prod_{i=1, i\neq j}^{m}(t_j - t_i)}, \quad t \in [a, b].$$

It is known as the jth **Lagrange polynomial** with respect to $t_1, \ldots, t_m$

9.6 Theorem

(a) Let distinct nodes $t_1, \ldots, t_m$ in $[a, b]$ be given. Then for weights $w_1, \ldots, w_m$ in $\mathbf{K}$, we have

$$\sum_{j=1}^{m} w_j t_j^k = \int_a^b t^k dt \quad \text{for } k = 0, \ldots, m-1$$

if and only if

$$w_j = \int_a^b \ell_j(t) dt, \quad j = 1, \ldots, m.$$

(b) For distinct nodes $t_1, \ldots, t_m$ in $[a, b]$ and weights $w_1, \ldots, w_m$ in $\mathbf{K}$, we have

$$\sum_{j=1}^{m} w_j t_j^k = \int_a^b t^k dt \quad \text{for } k = 0, \ldots, 2m-1$$

if and only if

$$\int_a^b t^k \prod_{i=1}^{m}(t - t_i) dt = 0 \quad \text{for } k = 0, \ldots, m-1$$

and

$$w_j = \int_a^b \ell_j(t) dt, \quad j = 1, \ldots, m.$$

In that case,

$$w_j = \int_a^b \ell_j^2(t) dt > 0.$$

Proof:

(a) Let $\sum_{j=1}^{m} w_j t_j^k = \int_a^b t^k dt$ for $k = 0, \ldots, m-1$. Then for every polynomial p of degree $\leq m-1$, we have

$$\int_a^b p(t)dt = \sum_{j=1}^{m} w_j p(t_j).$$

Now ℓ_j is a polynomial of degree $m-1$ and $\ell_j(t_i) = \delta_{i,j}, i, j = 1, \ldots, m$. Hence

$$\int_a^b \ell_j(t)dt = \sum_{i=1}^{m} w_i \ell_j(t_i) = w_j, \quad j = 1, \ldots, m.$$

Conversely, let $w_j = \int_a^b \ell_j(t)dt, j = 1, \ldots, m$. For $k = 0, \ldots, m-1$,

$$\sum_{j=1}^{m} w_j t_j^k = \sum_{j=1}^{m} \left(\int_a^b \ell_j(t)dt \right) t_j^k = \int_a^b \left(\sum_{j=1}^{m} t_j^k \ell_j(t) \right) dt.$$

Since $\sum_{j=1}^{m} t_j^k \ell_j(t)$ and t^k are both polynomials of degree $\leq m-1$ and since they agree at the m points $t_1, \ldots, t_m$, they must be identical. Thus

$$\sum_{j=1}^{m} w_j t_j^k = \int_a^b t^k dt, \quad k = 0, \ldots, m-1.$$

(b) Let $\sum_{j=1}^{m} w_j t_j^k = \int_a^b t^k dt$ for $k = 0, \ldots, 2m-1$. Then for every polynomial p of degree $\leq 2m-1$, we have

$$\int_a^b p(t)dt = \sum_{j=1}^{m} w_j p(t_j).$$

Now $t^k \prod_{i=1}^{m}(t - t_i)$ is a polynomial of degree $\leq 2m-1$ for each $k = 0, \ldots, m-1$ and it is zero at each $t_1, \ldots, t_m$. Hence

$$\int_a^b t^k \prod_{i=1}^{m}(t - t_i)dt = \sum_{j=1}^{m} w_j 0 = 0.$$

Also, $w_j = \int_a^b \ell_j(t)dt, j = 1, \ldots, m$ by (a) above.

Conversely, let $\int_a^b t^k \prod_{i=1}^m (t - t_i)dt = 0$ for each $k = 0, \ldots, m-1$ and $w_j = \int_a^b \ell_j(t)dt, j = 1, \ldots, m$. Fix k with $0 \leq k \leq 2m-1$. Then

$$t^k = q(t) \prod_{i=1}^m (t - t_i) + r(t)$$

for some polynomials q and r of degree $\leq m-1$. By our assumption,

$$\int_a^b q(t) \prod_{i=1}^m (t - t_i)dt = 0$$

and by (a) above,

$$\int_a^b r(t)dt = \sum_{j=1}^m w_j r(t_j).$$

But $r(t_j) = t_j^k, j = 1, \ldots, m$. Thus we see that

$$\int_a^b t^k dt = 0 + \sum_{j=1}^m w_j t_j^k,$$

as desired.

Finally, since ℓ_j^2 is a nonzero polynomial of degree $2m-2$ (which is less than $2m-1$) and since $\ell_j^2(t_i) = \delta_{i,j}, i, j = 1, \ldots, m$, we have

$$0 < \int_a^b \ell_j^2(t)dt = \sum_{i=1}^m w_i \ell_j^2(t_i) = w_j, \quad j = 1, \ldots, m. \qquad \square$$

Part (a) of this result shows that if we want a quadrature formula $g(x) = \sum_{j=1}^m w_j x(t_j)$ to integrate exactly the m functions $x_k(t) = t^k, k = 0, \ldots, m-1$ on $[a, b]$, that is,

$$\sum_{j=1}^m w_j t_j^k = \int_a^b t^k dt, \quad k = 0, \ldots, m-1,$$

then we can choose the m nodes $t_1, \ldots, t_m$ arbitrarily in $[a, b]$, but then the weights $w_1, \ldots, w_m$ must be chosen by the rule $w_j = \int_a^b \ell_j(t)dt$ for $j = 1, \ldots, m$, where ℓ_j is the Lagrange polynomial with respect to the nodes $t_1, \ldots, t_m$. As an illustration, consider the following important special case.

Let $a = 0, b = 1$ and for $m = 2, 3, \ldots$,

$$t_j^{(m)} = \frac{j-1}{m-1}, \quad j = 1, \ldots, m.$$

Thus the m nodes $t_1^{(m)}, \ldots, t_m^{(m)}$ are equally spaced in $[0,1]$ for each $m = 2, 3, \ldots$. Letting

$$\ell_j^{(m)}(t) = \prod_{i=1, i\neq j}^{m} \left(t - \frac{i-1}{m-1}\right) \left[\prod_{i=1, i\neq j}^{m} \left(\frac{j-1}{m-1} - \frac{i-1}{m-1}\right)\right]^{-1},$$

we obtain the **Newton-Cotes quadrature formulae**

$$C_m(x) = \sum_{j=1}^{m} w_j^{(m)} x\left(\frac{j-1}{m-1}\right), \quad m = 2, 3, \ldots,$$

where $w_j^{(m)} = \int_a^b \ell_j^{(m)}(t)dt$. For instance, if $m = 2$, then 0 and 1 are the nodes, $\ell_1^{(2)}(t) = 1 - t$ and $\ell_2^{(2)}(t) = t$, so that

$$C_2(x) = \frac{x(0) + x(1)}{2},$$

which is the **trapezoidal rule**. If $m = 3$, then $0, \frac{1}{2}$ and 1 are the nodes, $\ell_1^{(3)}(t) = 2(t - \frac{1}{2})(t-1) = \ell_3^{(3)}(t)$, and $\ell_2^{(3)}(t) = -4t(t-1)$, so that

$$C_3(x) = \frac{x(0) + 4x(\frac{1}{2}) + x(1)}{6},$$

which is the **Simpson rule**.

Fix $k = 0, 1, 2, \ldots$. Since we have chosen the weights as per part (a) of Theorem 9.6, we have

$$C_m(x_k) = \sum_{j=1}^{m} w_j^{(m)} \left(\frac{j-1}{m-1}\right)^k = \int_0^1 t^k dt$$

for all natural numbers $m \geq k+1$. It is not difficult to show that for $m = 2, 3, \ldots$ and $j = 1, \ldots, m$,

$$w_j^{(m)} = \frac{(-1)^{m-j}}{(j-1)!(m-j)!(m-1)} \int_0^{m-1} \prod_{i=1, i\neq j}^{m} (s - i + 1)ds.$$

By estimating the integrals from 0 to 3, from 3 to $m-4$ and from $m-4$ to $m-1$ separately, it is shown in [42], pp. 84-86 that $|w_{m+1}^{(2m+1)}|$ is of the order of $(2m)!/m!m![m \log 2m]^2$. Hence $|w_{m+1}^{(2m+1)}|$ and, in turn, $\sum_{j=1}^{m} |w_j^{(m)}|$ tend to infinity as $m \to \infty$. By Polya's theorem (9.5), there is some $x \in C([0,1])$ for which the Newton-Cotes quadrature formulae $C_m(x)$ do not converge to $\int_0^1 x(t)dt$. This also shows that condition 2) of Polya's theorem cannot be dropped.

Next, part (b) of Theorem 9.6 shows that if we want a quadrature formula $\sum_{j=1}^{m} w_j x(t_j)$ to integrate the $2m$ functions $x_k(t) = t^k, k = 0, \ldots, 2m-1$, exactly on $[a,b]$, then we must choose the m nodes $t_1, \ldots, t_m$ such that

$$\int_a^b p(t)(t-t_1)\cdots(t-t_m)dt = 0$$

for all polynomials p of degree $\leq m-1$ and then the weights $w_1, \ldots, w_m$ must be chosen by the rule $w_j = \int_a^b \ell_j(t)dt$ for $j = 1, \ldots, m$, as in part (a). These weights are automatically positive. A polynomial having these $t_1, \ldots, t_m$ as its roots is known as a **Legendre polynomial** of degree m. In Section 22, we shall show that these polynomials can be found iteratively by applying the Gram-Schmidt orthonormalization process to the linearly independent set $\{x_0, x_1, x_2, \ldots\}$.

For the sake of convenience, let $a = -1$ and $b = 1$. Let p_m denote the Legendre polynomial of degree m such that $p_m(1) = 1$. By using a recurrence relation among the coefficients of p_m, it can be shown that

$$\begin{aligned} p_m(t) = \frac{1}{2^m}\Bigg[\frac{(2m)!}{0!m!m!}t^m &- \frac{(2m-2)!}{1!(m-1)!(m-2)!}t^{m-2} \\ &+ \frac{(2m-4)!}{2!(m-2)!(m-3)!}t^{m-4} + \cdots\Bigg]. \end{aligned}$$

In particular,

$$p_1(t) = t, \quad p_2(t) = \frac{3t^2-1}{2}, \quad p_3(t) = \frac{5t^3-3t}{2}.$$

For $m = 1, 2, \ldots,$ let $t_1^{(m)}, \ldots, t_m^{(m)}$ denote the (distinct) roots of p_m.

Letting

$$\ell_j^{(m)}(t) = \prod_{i=1, i\neq j}^{m} \left(t - t_i^{(m)}\right) \left[\prod_{i=1, i\neq j}^{m} \left(t_j^{(m)} - t_i^{(m)}\right)\right]^{-1}$$

we obtain the **Gauss-Legendre quadrature formulae**

$$G_m(x) = \sum_{j=1}^{m} w_j^{(m)} x(t_j^{(m)}), \quad m = 1, 2, \ldots,$$

where $w_j^{(m)} = \int_{-1}^{1} \ell_j^{(m)}(t)dt$. Fix $k = 0, 1, 2, \ldots$. Since we have chosen the nodes and the weights as per part (b) of Theorem 9.6, we have

$$G_m(x_k) = \sum_{j=1}^{m} w_j^{(m)} \left[t_j^{(m)}\right]^k = \int_{-1}^{1} t^k dt$$

for all natural numbers m satisfying $(k+1)/2 \leq m$, so that $(G_m(x_k))$ converges to $\int_{-1}^{1} t^k dt$ as $m \to \infty$. Also, since the weights $w_j^{(m)}$ are all positive, $\sum_{j=1}^{m} |w_j^{(m)}| = \sum_{j=1}^{m} w_j^{(m)} = G_m(x_0) = 2$. By Polya's theorem, we see that

$$G_m(x) \to \int_{-1}^{1} x(t)dt \quad \text{as } m \to \infty$$

for every $x \in C([-1, 1])$.

We remark that in order to employ the Gauss-Legendre quadrature formulae, it is necessary to find the roots of the Legendre polynomials in $[-1, 1]$, which is a formidable task. Instead, it is often preferable to subdivide the interval $[-1, 1]$ and apply a low order Gauss-Legendre quadrature formula, say with $m = 2$ or 4, to each of the subintervals. (See Problem 9-13.)

Matrix Transformations and Summability Methods

Consider an infinite matrix $M = (k_{i,j})$ with scalar entries. Recall from 6.5(c) that M defines a linear map from a sequence space X to a sequence space Y if for every sequence $x = (x(1), x(2), \ldots) \in X$, the series $\sum_{j=1}^{\infty} k_{i,j}x(j)$ converges in $\mathbf{K}$ and if $Mx(i)$ denotes its sum, then $Mx = (Mx(1), Mx(2), \ldots) \in Y$. (If we consider the sequence x as an infinite column with jth entry equal to $x(j)$, then Mx is simply the infinite column given by the matrix product of M and x.) This linear map is known as a **matrix transformation** and we shall denote it by M itself.

Let X and Y be sequence spaces such that X contains c_{00}. Let M define a linear map from X to Y. Now $e_j \in X$ for every $j = 1, 2, \ldots$ and $Me_j(i) = k_{i,j}$ for all $i = 1, 2, \ldots$. Thus Me_j is the jth column of the matrix M and it belongs to Y. If $Y \subset \ell^r, 1 \le r \le \infty$, then we see that every column of M is in ℓ^r. We use the uniform boundedness principle to show that if X is a closed subspace of $\ell^p, 1 \le p \le \infty$, then every row of M is in ℓ^q, where $1/p + 1/q = 1$. Fix $i = 1, 2, \ldots$. Let

$$g_{i,n}(x) = \sum_{j=1}^{n} k_{i,j}x(j), \quad x \in X,$$

for $n = 1, 2, \ldots$. Then $g_{i,n} \in X'$ and $\|g_{i,n}\| \le \|(k_{i,1}, \ldots, k_{i,n})\|_q$. In fact, we can show that $\|g_{i,n}\| = \|(k_{i,1}, \ldots, k_{i,n})\|_q$. (See Problem 6-10 and 13.2.) Since $(g_{i,n}(x))$ converges for every $x \in X$ and X is a Banach space, 9.2(a) shows that $\|g_{i,n}\| \le \alpha$ for all $n = 1, 2, \ldots$ and some $\alpha > 0$, that is, the ith row $(k_{i,1}, k_{i,2}, \ldots)$ belongs to ℓ^q.

Let us consider some special cases. Let $X = \ell^1 = Y$. We have shown in 6.5(c) that if the supremum of the column sums of the matrix $|M| = (|k_{i,j}|)$ is finite, that is,

$$\alpha_1 = \sup\left\{\sum_{i=1}^{\infty} |k_{i,j}| : j = 1, 2, \ldots\right\} < \infty,$$

then M defines a (continuous) linear map from ℓ^1 to ℓ^1. We now prove the converse. Assume that $Mx \in \ell^1$ for every $x \in \ell^1$. Fix $i = 1, 2, \ldots$. As we have just seen, the ith row of M belongs to ℓ^∞, that is,

$$\sup\{|k_{i,j}| : j = 1, 2, \ldots\} < \infty.$$

If we let

$$f_i(x) = \sum_{j=1}^{\infty} k_{i,j} x(j), \quad x \in \ell^1,$$

then $f_i \in (\ell^1)'$ and $\|f_i\| \leq \sup\{|k_{i,j}| : j = 1, 2, \ldots\}$. For each $n = 1, 2, \ldots$, let

$$p_n(x) = \sum_{i=1}^{n} |f_i(x)|, \quad x \in \ell^1.$$

Then $p_n(x) \geq 0$ for all $x \in \ell^1$. Since each f_i is continuous, so is each p_n. Also, it is easy to see that for all x, y in X and $k \in \mathbf{K}$,

$$p_n(x + y) \leq p_n(x) + p_n(y) \quad \text{and} \quad p_n(kx) = |k| p_n(x).$$

Thus p_n is a continuous seminorm on ℓ^1. Further,

$$p_n(x) = \sum_{i=1}^{n} \Big| \sum_{j=1}^{\infty} k_{i,j} x(j) \Big| \leq \sum_{i=1}^{\infty} \Big| \sum_{j=1}^{\infty} k_{i,j} x(j) \Big| = \|Mx\|_1 < \infty.$$

By the second remark after the proof of the uniform boundedness principle (9.1), there is some $\alpha > 0$ such that $p_n(x) \leq \alpha$ for all $n = 1, 2, \ldots$ and all $x \in \ell^1$ with $\|x\|_1 \leq 1$. Thus for each $x \in \ell^1$ with $\|x\|_1 \leq 1$, we have

$$\sum_{i=1}^{\infty} |f_i(x)| = \sup\{p_n(x) : n = 1, 2, \ldots\} \leq \alpha.$$

In particular, letting $x = e_j = (0, \ldots, 0, 1, 0, \ldots)$ and noting that $|f_i(e_j)| = |k_{i,j}|$ for $i = 1, 2, \ldots$, we obtain

$$\sum_{i=1}^{\infty} |k_{i,j}| \leq \alpha, \quad j = 1, 2, \ldots.$$

Hence $\alpha_1 < \infty$, as desired. This result was first proved by K. Knopp and Lorentz in 1949.

Similarly we have noted in 6.5(c) that if the supremum of the row sums of the matrix $|M| = (|k_{i,j}|)$ is finite, that is,

$$\alpha_\infty = \sup\left\{ \sum_{j=1}^{\infty} |k_{i,j}| : i = 1, 2, \ldots \right\} < \infty,$$

then M defines a (continuous) linear map from ℓ^∞ to ℓ^∞. To prove the converse, assume that $Mx \in \ell^\infty$ for every $x \in \ell^\infty$. Fix $i = 1, 2, \ldots$. Then the i th row of M belongs to ℓ^1, that is,

$$r_i = \sum_{j=1}^{\infty} |k_{i,j}| < \infty.$$

If we let

$$f_i(x) = \sum_{j=1}^{\infty} k_{i,j} x(j), \quad x \in \ell^\infty,$$

then $f_i \in (\ell^\infty)'$ and $\|f_i\| \le r_i$. In fact, if we let $x(j) = \operatorname{sgn} k_{i,j}$, $j = 1, 2, \ldots$, then $\|x\|_\infty \le 1$ and $f_i(x) = r_i$. Thus $\|f_i\| = r_i$, $i = 1, 2, \ldots$.

Now for each fixed $x \in \ell^\infty$,

$$|f_i(x)| = |Mx(i)| \le \|M(x)\|_\infty, \quad i = 1, 2, \ldots.$$

Hence by the uniform boundedness principle (9.1),

$$\alpha_\infty = \sup\{r_i : i = 1, 2, \ldots\} = \sup\{\|f_i\| : i = 1, 2, \ldots\} < \infty.$$

As we have mentioned in Section 6, no worthwhile necessary and sufficient conditions are known for M to define a linear map from ℓ^p to ℓ^p, if $1 < p < \infty$.

We now consider the subspace c of ℓ^∞, which consists of all convergent scalar sequences and seek necessary and sufficient conditions on the entries $k_{i,j}$ of a matrix M, so that

$$Mx \in c \text{ and } \lim_{i\to\infty} Mx(i) = \lim_{i\to\infty} x(i) \text{ for every } x \in c.$$

We shall later comment on the significance of this problem.

9.7 Theorem (Silverman and Toeplitz, 1911)

Let $k_{i,j} \in \mathbf{K}$ be such that

1) $\alpha_\infty = \sup\left\{\sum_{j=1}^{\infty} |k_{i,j}| : i = 1, 2, \ldots\right\} < \infty$,
2) $\lim_{i\to\infty} k_{i,j} = 0$ for every $j = 1, 2, \ldots$,
3) $\lim_{i\to\infty} \sum_{j=1}^{\infty} k_{i,j} = 1$.

Then $M = (k_{i,j})$ defines a linear map from c to c and

$$\lim_{i\to\infty} Mx(i) = \lim_{i\to\infty} x(i)$$

for every $x \in c$. The converse also holds.

Proof:
Recall that c_0 is the subspace of c consisting of all sequences which converge to zero. First we show that M maps c_0 into c_0. Let $x \in c_0$. Given $\epsilon > 0$, find j_0 such that

$$|x(j)| < \epsilon \quad \text{for all } j > j_0,$$

and because of condition 2), find i_0 such that

$$\sum_{j=1}^{j_0} |k_{i,j}| < \epsilon \quad \text{for all } i > i_0.$$

Note that $Mx(i) = \sum_{j=1}^{\infty} k_{i,j}x(j)$ is well-defined for each i because of condition 1). Now

$$\begin{aligned} |Mx(i)| &\leq \sum_{j=1}^{j_0} |k_{i,j}|\,|x(j)| + \sum_{j=j_0+1}^{\infty} |k_{i,j}|\,|x(j)| \\ &< \epsilon(\|x\|_\infty + \alpha_\infty) \end{aligned}$$

for all $i > i_0$. Hence $Mx \in c_0$.

Next, let $x \in c$. If $e = (1, 1, \ldots)$, $\ell = \lim_{i\to\infty} x(i)$ and $y = x - \ell e$, then $x = y + \ell e$ with $y \in c_0$. Since

$$Me(i) = \sum_{j=1}^{\infty} k_{i,j}, \quad i = 1, 2, \ldots,$$

condition 3) shows that $Me \in c$ and $\lim_{i\to\infty} Me(i) = 1$. Hence

$$Mx = My + \ell Me$$

belongs to c and

$$\lim_{i\to\infty} Mx(i) = \lim_{i\to\infty} My(i) + \ell \lim_{i\to\infty} Me(i) = 0 + \ell = \ell.$$

Conversely, assume that for every $x \in c, M(x)$ belongs to c and $\lim_{i\to\infty} Mx(i) = \lim_{i\to\infty} x(i)$. To prove that condition 1) holds, we note that every row of M is in ℓ^1, as shown in the beginning of this subsection. Fix $i = 1, 2, \ldots$. Then $\sum_{j=1}^{\infty} |k_{i,j}| < \infty$. Let

$$f_i(x) = \sum_{j=1}^{\infty} k_{i,j} x(j), \quad x \in c.$$

Then $f_i \in c'$ and $\|f_i\| \le \sum_{j=1}^{\infty} |k_{i,j}|$. In fact, equality holds here. For $n = 1, 2, \ldots$, let

$$x_n(j) = \begin{cases} \operatorname{sgn} k_{i,j}, & \text{if } 1 \le j \le n \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_i(x_n) = \sum_{j=1}^{n} |k_{i,j}| \to \sum_{j=1}^{\infty} |k_{i,j}|,$$

as $n \to \infty$. Now for every $x \in c, f_i(x) = Mx(i)$ and by assumption, the sequence $(f_i(x))$ converges to ℓ, where $\ell = \lim_{i\to\infty} x(i)$. Again by the Banach-Steinhaus theorem 9.2(a),

$$\alpha_\infty = \sup\left\{\sum_{j=1}^{\infty} |k_{i,j}| : i = 1, 2, \ldots\right\} = \sup\{\|f_i\| : i = 1, 2, \ldots\} < \infty.$$

Hence condition 1) holds. Next, $e_j = (0, \ldots, 0, 1, 0, \ldots) \in c$ for $j = 1, 2, \ldots$ and since

$$\lim_{i\to\infty} Me_j(i) = \lim_{i\to\infty} e_j(i) = 0,$$

we have

$$\lim_{i\to\infty} k_{i,j} = \lim_{i\to\infty} Me_j(i) = 0.$$

Hence condition 2) holds. Finally, $e = (1, 1, \ldots) \in c$ and since

$$\lim_{i\to\infty} Me(i) = \lim_{i\to\infty} e(i) = 1,$$

we have

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} k_{i,j} = \lim_{i\to\infty} Me(i) = 1.$$

Hence condition 3) holds. This completes the proof. □

The three conditions on an infinite matrix $M = (k_{i,j})$ given above can be rephrased as follows. The first states that the row sums of the matrix $|M| = (|k_{i,j}|)$ are bounded, the second states that the entries in each column of M tend to 0 and the third states that the row sums of the matrix M tend to 1. A matrix satisfying these conditions is called a **Toeplitz matrix**, since Toeplitz was the first to prove the converse part of Theorem 9.7 (without the help of the Banach-Steinhaus theorem). See the proof of Theorem 2 in Chapter III of [27].

As in the case of the weights of quadrature formulae of 9.5, we note that if all the entries of a matrix are nonnegative, then the first condition of Theorem 9.7 follows automatically from the third.

We now comment on the use of Toeplitz matrices in summation of series. Recall that a scalar series $\sum_{n=1}^{\infty} a_n$ is **summable** (in the usual sense) if the sequence (s_i), $s_i = a_1 + \cdots + a_i$, of its partial sums converges in $\mathbf{K}$. If $s_i \to s$, then s is called the **sum** of the series. Consider now an infinite matrix $M = (k_{i,j})$. We shall say that **the** series $\sum_{n=1}^{\infty} a_n$ is M-**summable**, if for every $i = 1, 2, \ldots$, the series $\sum_{j=1}^{\infty} k_{i,j} s_j$ converges in $\mathbf{K}$ to, say, t_i and if the sequence (t_i) converges in $\mathbf{K}$ to, say, t. In that case t is called the M-**sum** of the series. A **summability method** corresponding to a matrix M is said to be **permanent** if every series which is summable in the usual sense is M-summable and if its M-sum coincides with the usual sum. It is clear that if M is a Toeplitz matrix, then the corresponding method of summability is permanent. The converse also holds. Given a sequence $(x(1), x(2), \ldots) \in c$, let $a_n = x(n) - x(n-1)$ with $x(0) = 0$. Then the series $\sum_{n=1}^{\infty} a_n$ has its jth partial sum equal to $x(j), j = 1, 2, \ldots$. By the converse part of Theorem 9.7, M is a Toeplitz matrix.

The usual summability method corresponds to the identity matrix. The method of summation by the arithmetic means of the partial sums corresponds to the matrix

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & . & . & . \\ 1/2 & 1/2 & 0 & . & . & . \\ 1/3 & 1/3 & 1/3 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Here $k_{i,j} = 1/i$ for $j \leq i$ and $k_{i,j} = 0$ for $i < j$, so that

$$t_i = \frac{s_1 + \cdots + s_i}{i}, \quad i = 1, 2, \ldots.$$

This is known as the **Cesaro summability method**. It is permanent since M_1 is easily seen to be a Toeplitz matrix. Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}$. The sequence of its partial sums is $(1, 0, 1, 0, \ldots)$. Clearly, the series diverges in the usual sense. But $t_i = 1/2$, if i is even and $t_i = (i+1)/2i$, if i is odd. Since $t_i \to 1/2$ as $i \to \infty$, we see that the M_1-sum of this divergent series is $1/2$. Also, we have seen in 9.4 that there are continuous 2π-periodic functions whose Fourier series do not converge in the usual sense. However, Fejer's theorem (4.8) shows that the Fourier series of every continuous 2π-periodic function is M_1-summable.

Problems

9-1 Let X and Y be normed spaces and E be a bounded, complete, convex subset of X. A map F from E to Y is called **affine** if

$$F(tx + (1-t)y) = tF(x) + (1-t)F(y)$$

for all $0 < t < 1$ and $x, y \in E$. Let $\mathcal{F}$ be a set of continuous affine maps from E to Y. Then either the set $\{\|F(x)\| : F \in \mathcal{F}\}$ is unbounded for each x in some dense subset of E, or else $\mathcal{F}$ is uniformly bounded on E.

9-2 Let $1 \le p \le \infty$ and $X = c_{00}$ with the norm $\| \ \|_p$. For $n = 1, 2, \ldots$, let

$$f_n(x) = nx(n), \quad x \in X.$$

Then $f_n(x) \to 0$ for every $x \in X$, but $\|f_n\| = n \to \infty$. (Compare 9.2(a).)

9-3 Let X be a normed space and (x_n) be a sequence in X such that $(f(x_n))$ converges in $\mathbf{K}$ for every $f \in X'$. Then the sequence (x_n) is bounded. For $f \in X'$, let $F(f) = \lim_{n\to\infty} f(x_n)$. Then $F \in X''$ and $\|F\| \le \liminf_{n\to\infty} \|x_n\|$.

9-4 **(Condensation of singularities)** Let X be a Banach space and Y be a normed space. For $n, m = 1, 2, \ldots$, let $F_{n,m} \in BL(X, Y)$. If for each $m = 1, 2, \ldots$, there exists some $x_m \in X$ such that the set $\{\|F_{n,m}(x_m)\| : n = 1, 2, \ldots,\}$ is unbounded, then there is a dense subset D of X such that the set $\{\|F_{n,m}(x)\| : n = 1, 2, \ldots,\}$ is unbounded for every $x \in D$ and every $m = 1, 2, \ldots$.

9-5 **(Continuity of bilinear maps)** Let X, Y and Z be linear spaces and $F : X \times Y \to Z$ be a **bilinear map**, that is, for each fixed $x \in X$, the map $F_x : Y \to Z$ defined by $F_x(y) = F(x, y)$ is linear and for each fixed $y \in Y$, the map $F^y : X \to Z$ defined by $F^y(x) = F(x, y)$ is linear. Let X and Y be normed spaces and one of them be a Banach space. If F_x and F_y are continuous for each fixed $x \in X$ and $y \in Y$, then

$$\|F(x, y)\| \le \alpha \|x\| \, \|y\|$$

for all $x \in X, y \in Y$ and some $\alpha > 0$. In particular, if $x_n \to x$ in X and $y_n \to y$ in Y, then $F(x_n, y_n) \to F(x, y)$ in Z.

9-6 Let X and Y be Banach spaces and $F_n \in BL(X, Y), n = 1, 2, \ldots$. Then there is some $F \in BL(X, Y)$ such that $F_n(x) \to F(x)$ for every $x \in X$ if and only if $(F_n(x))$ converges for every x in some set E whose span is dense in X and the set $\{\|F_n\| : n = 1, 2, \ldots\}$ is bounded.

9-7 Let X be a Banach space. Let P_n in $BL(X)$ be such that $P_n^2 = P_n$ and $R(P_n) \subset R(P_m)$ for each $n = 1, 2, \ldots$ and all large enough m. Then $P_n(x) \to x$ for every $x \in X$ if and only if $\bigcup\{R(P_n) : n = 1, 2, \ldots,\}$ is dense in X and the set $\{\|P_n\| : n = 1, 2, \ldots\}$ is bounded.

9-8 Let X be a Banach space and Y, Z be normed spaces. Let F_n, F be linear maps from X to Y and G_n, G be linear maps from Z to X. If each F_n is continuous, $F_n(x) \to F(x)$ for every $x \in X$ and $G_n(z) \to G(z)$ for every $z \in Z$, then $F_nG_n(z) \to FG(z)$ for every $z \in Z$.

9-9 **(Dunford's theorem, 1938)** Let X be a Banach space over $\mathbf{C}$ and D be an open subset of $\mathbf{C}$. If $F : D \to X$ is such that $f o F$ is analytic for every $f \in X'$. Then for every $z_0 \in D$, $\lim_{z \to z_0}[F(z) - F(z_0)]/(z - z_0)$ exists in X. (Hint: For $z \neq z_0$ and $w \neq z_0$, let

$$G(z, w) = \frac{F(z) - F(z_0)}{z - z_0} - \frac{F(w) - F(z_0)}{w - z_0}.$$

Let $f \in X'$. By Cauchy's integral formula,

$$f(G(z, w)) = \frac{1}{2\pi i} \int_{|s - z_0| = r} \frac{(z - w) f(F(s))}{(s - z)(s - w)(s - z_0)} ds.$$

By 9.2(a), the set $E = \{F(s) \in X : |s - z_0| = r\}$ is bounded.)

9-10 There exists a dense subset D of $X = \{x \in C([-\pi, \pi]) : x(\pi) = x(-\pi)\}$ such that the Fourier series of every $x \in D$ diverges at every rational number in $[-\pi, \pi]$.

9-11 Let $R([a, b])$ denote the set of all scalar-valued Riemann integrable functions on $[a, b]$ with the sup norm. Let E be a subset of $R([a, b])$ such that if $t_1, \ldots, t_m$ are distinct points in $[a, b]$ and $k_1, \ldots, k_m$ are in $\mathbf{K}$ with $|k_j| = 0$ or 1, then there is some $x \in E$ with $\|x\| \leq 1$ and $x(t_j) = k_j, j = 1, \ldots, m$. Let Y denote the closure of span E in $R([a, b])$. For $n = 1, 2, \ldots$ and $x \in R([a, b])$, let

$$f_n(x) = \sum_{j=1}^{m_n} w_j^{(n)} x(t_j^{(n)}) \quad \text{and} \quad f(x) = \int_a^b x(t) dt.$$

Then $f_n(x) \to f(x)$ for every $x \in Y$ if and only if $f_n(x) \to f(x)$ for every $x \in E$ and $\sum_{j=1}^{m_n} |w_j^{(n)}| \leq \alpha < \infty$ for all $n = 1, 2, \ldots$.

9-12 Let $C^1([a, b])$ denote the set of all continuously differentiable function on $[a, b]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$. For $x \in C^1([a, b])$, let $g(x) = \sum_{j=1}^{m} w_j x(t_j)$. If $t_{m+1} = b$ and $\alpha(g) = \sum_{j=1}^{m} |\sum_{i=1}^{j} w_i|(t_{j+1} - t_j) + |\sum_{j=1}^{m} w_j|$, then $\|g\| \leq \alpha(g) \leq (b - a + 2)\|g\|$.

If (g_n) is a sequence of quadrature formulae, then $g_n(x) \to \int_a^b x(t)dt$ for every $x \in C^1([a,b])$ if and only if $g_n(x_k) \to \int_a^b t^k dt$ for $k = 0, 1, 2, \ldots$ and $\sum_{j=1}^{m_n} (t_{j+1}^{(n)} - t_j^{(n)}) |\sum_{i=1}^{j} w_i| \leq \alpha < \infty$ for all $n = 1, 2, \ldots$ and some $\alpha > 0$, where $x_k(t) = t^k$, $t \in [a,b]$. (Hint: Proof of 9.5, Problem 5-19)

9-13 Let $a = 0, b = 1$ and $g(x) = \sum_{j=1}^{m} w_j x(t_j)$ be a quadrature formula. The corresponding **compound quadrature formulae** are defined by

$$g_n(x) = \frac{1}{n} \sum_{r=1}^{n} \sum_{j=1}^{m} w_j \, x(\frac{r - 1 + t_j}{n}), \qquad x \in C([0,1]).$$

Then $g_n(x) \to \int_0^1 x(t)dt$ for every $x \in C([0,1])$ if and only if $\sum_{j=1}^{m} w_j = 1$.

9-14 Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. A sequence $y = (y(1), y(2), \ldots)$ belongs to ℓ^q if and only if $\sum_{j=1}^{\infty} x(j)y(j)$ converges for every $x \in \ell^p$.

9-15 Let $M = (k_{i,j})$ be an infinite matrix and $\alpha_{p,r}$ be defined as in Problem 6-13. Then M defines a linear map from ℓ^p to ℓ^r if and only if $\alpha_{p,r} < \infty$.

9-16 Let $M = (k_{i,j})$ be an infinite matrix, $1 < p \leq \infty$ and $1 \leq r < \infty$. Let X be a subspace of ℓ^p containing c_{00} and Y be a subspace of ℓ^r.

(a) If X is closed in ℓ^p and M defines a linear map from X to Y, then the rows of M form a bounded subset of $\ell^q, 1/p + 1/q = 1$, that is,

$$\sup_{i=1,2,\ldots} \left\{ \sum_{j=1}^{\infty} |k_{i,j}|^q \right\} < \infty.$$

(b) If M defines a continuous linear map from X to Y, then the columns of M form a bounded subset of ℓ^r, that is,

$$\sup_{j=1,2,\ldots} \left\{ \sum_{i=1}^{\infty} |k_{i,j}|^r \right\} < \infty.$$

9-17 An infinite matrix $M = (k_{i,j})$ defines a linear map from c_0 to c_0 if and only if

1) $\alpha_\infty = \sup\{\sum_{j=1}^{\infty} |k_{i,j}| : i = 1, 2, \ldots, \} < \infty$ and
2) $\lim_{i \to \infty} k_{i,j} = 0$ for every $j = 1, 2, \ldots$.

9-18 Let $0 < r < 1$ and consider the **Euler-Knopp matrix** $M_r = (k_{i,j})$, where

$$k_{i,j} = \begin{cases} \binom{i}{j-1} r^{j-1}(1-r)^{i-j+1}, & \text{if } 1 \le j \le i+1 \\ 0, & \text{if } i+1 < j. \end{cases}$$

Then M_r is a Toeplitz matrix. If $|1 - r + rz| < 1$, then the series $\sum_{n=1}^{\infty} z^{n-1}$ is M_r-summable and its M_r-sum
is $1/(1-z)$. In particular, the M_r-sum of $\sum_{n=1}^{\infty}(-1)^{n-1}$ is $1/2$.

9-19 **(Semicontinuous analogue of the Silverman-Toeplitz theorem)** Let $a < b \le \infty$ and $Y = \{y : [a,b) \to \mathbf{K} : \lim_{s\to b} y(s) \text{ exists}\}$. For $s \in [a,b)$, and $j = 1, 2, \ldots$, let $k(s,j) \in \mathbf{K}$ be such that

1) $\sup\{\sum_{j=1}^{\infty} |k(s,j)| : s \in [a,b)\} < \infty$,

2) $\lim_{s\to b} k(s,j) = 0$ for every $j = 1, 2, \ldots$ and

3) $\lim_{s\to b} \sum_{j=1}^{\infty} k(s,j) = 1$. For x in c and $s \in [a,b)$, let $Mx(s) = \sum_{j=1}^{\infty} k(s,j)x(j)$. Then M maps c into Y and $\lim_{s\to b} Mx(s) = \lim_{i\to\infty} x(i)$. The converse also holds.

9-20 Let $x \in c$ and

$$M_b x(s) = \sum_{j=1}^{\infty} \frac{e^{-s} s^{j-1} x(j)}{(j-1)!}, \quad s \in [0, \infty).$$

Then $\lim_{s\to\infty} M_b x(s) = \lim_{i\to\infty} x(i)$. The semicontinuous matrix $M_b = e^{-s}s^{j-1}/(j-1)!)$, $s \in [0,\infty), j = 1, 2, \ldots$, is known as the **Borel matrix.**

9-21 **(Kojima-Schur)** An infinite matrix $M = (k_{i,j})$ maps c into c if and only if

1) $\alpha_\infty = \sup\{\sum_{j=1}^{\infty} |k_{i,j}| : i = 1, 2, \ldots\} < \infty$ and

2) $\lim_{i\to\infty} \sum_{j=m}^{\infty} k_{i,j}$ exists for each $m = 1, 2, \ldots$.

The set of all such matrices is a linear space under elementwise addition and scalar multiplication and the set of Toeplitz matrices is a closed convex subset of it.

9-22 Let γ denote the set of all sequences $x = (x(1), x(2), \ldots)$ such that $\sum_{n=1}^{\infty} x(n)$ is summable in $\mathbf{K}$. An infinite matrix $L = (\ell_{i,j})$ maps γ into c and $\lim_{i\to\infty} Lx(i) = \sum_{n=1}^{\infty} x(n)$ if and only if

1) $\sup\{\sum_{j=1}^{\infty} |\ell_{i,j} - \ell_{i,j+1}| : i = 1, 2, \ldots\} < \infty$ and

2) $\lim_{i\to\infty} \ell_{i,j} = 1$ for every $j = 1, 2, \ldots$.

(Hint: Use Abel's partial summation for $\sum_{j=1}^{\infty} \ell_{i,j} x(j)$. Note that the matrix $M = (\ell_{i,j} - \ell_{i,j+1})$ maps c_0 into c_0.)

9-23 Let γ be as defined in Problem 9-22. For $x \in \gamma$, let

$$M_a x(s) = \sum_{j=1}^{\infty} s^{j-1} x(j), \qquad 0 \leq s < 1.$$

Then $\lim_{s\to 1} M_a x(s) = \sum_{n=1}^{\infty} x(n)$. The semicontinuous matrix $M_a = (s^{j-1})$, $s \in [0, 1)$, $j = 1, 2, \ldots$, is known as the Abel matrix.

10 Closed Graph and Open Mapping Theorems

In 6.2 we have given several necessary and sufficient conditions for a linear map from a normed space X to a normed space Y to be continuous. We shall now give another such condition when X and Y are Banach spaces.

Let X and Y be metric spaces and F be a map from X to Y. Then F is continuous if $x_n \to x$ in X implies that $F(x_n) \to F(x)$ in Y. A map F is said to be **closed** if $x_n \to x$ in X and $F(x_n) \to y$ in Y imply that $y = F(x)$.

Clearly, a continuous map is closed. However, a closed map may not be continuous. For example, let $X = \mathbf{R} = Y$ and $F(t) = 1/t$ if $t \neq 0$ and $F(0) = 0$.

It can be easily seen that F is closed if and only if the set

$$\mathrm{Gr}\,(F) = \{(x, F(x)) \in X \times Y : x \in X\}$$

is closed in $X \times Y$, equipped with the metric

$$d((x_1, y_1), (x_2, y_2)) = \left[d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)\right]^{1/2}$$

The set $\mathrm{Gr}(F)$ is sometimes called the **graph** of F. [Note that according to the definition of a function given in Section 1, $\mathrm{Gr}(F)$ is the function F itself!]

If a closed map F is bijective, then its inverse F^{-1} is also a closed map. To see this, let $y_n \to y$ in Y and $F^{-1}(y_n) \to x$ in X. Letting $x_n = F^{-1}(y_n)$, we have $x_n \to x$ in X and $F(x_n) \to y$ in Y. Since F is closed, $y = F(x)$, that is, $x = F^{-1}(y)$ as desired. Alternatively, one can observe that

$$\mathrm{Gr}(F^{-1}) = \{(y, F^{-1}(y)) \in Y \times X : y \in Y\},$$

and $(y, x) \in \mathrm{Gr}(F^{-1})$ if and only if $(x, y) \in \mathrm{Gr}(F)$. Hence $\mathrm{Gr}(F^{-1})$ is closed in $Y \times X$ whenever $\mathrm{Gr}(F)$ is closed in $X \times Y$. In this regard, it is worth noting that the inverse of a bijective continuous map may not be continuous. For example, let $X = [0, 2\pi), Y = \{z \in \mathbf{C} : |z| = 1\}$ and $F(\theta) = \exp(i\theta)$ for $\theta \in X$. Then F is a bijective continuous map. But F^{-1} is not continuous, since Y is compact while X is not.

Before stating our main theorem, we prove a crucial preliminary result.

10.1 Lemma

Let X be a linear space over $\mathbf{K}$. Consider subsets U and V of X, and $k \in \mathbf{K}$ such that $U \subset V + kU$. Then for every $x \in U$, there is a sequence (v_n) in V such that

$$x - (v_1 + kv_2 + \cdots + k^{n-1}v_n) \in k^n U, \qquad n = 1, 2, \ldots.$$

Proof:

Let $x \in U$. Since $U \subset V + kU$, there is some $v_1 \in V$ such that $x - v_1$ is in kU. Let $n \geq 1$ and assume that we have found $v_1, \ldots, v_n$ in V as stated in the lemma. Then $x = v_1 + kv_2 + \cdots + k^{n-1}v_n + k^n u$ for some $u \in U$. Since $u = v_{n+1} + ku_0$ for some $v_{n+1} \in V$ and $u_0 \in U$, we see that

$$x - (v_1 + kv_2 + \cdots + k^{n-1}v_n + k^n v_{n+1}) \in k^{n+1}U.$$

Thus we inductively obtain a sequence (v_n) in V with the stated property. $\square$

10.2 Closed graph theorem (Banach, 1932)
Let X and Y be Banach spaces and $F : X \to Y$ be a closed linear map. Then F is continuous.

Proof:
By condition (i) of 6.2, it is enough to prove that F is bounded on some neighborhood of 0. For each positive integer n, let

$$V_n = \{x \in X : \|F(x)\| \leq n\}.$$

We prove that some V_n contains a neighborhood of 0 in X. Now

$$X = \bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} \overline{V}_n,$$

where $\overline{V}_n$ denotes the closure of the set V_n in X. Hence

$$\bigcap_{n=1}^{\infty} (\overline{V}_n)^c = \emptyset,$$

where $(\overline{V}_n)^c$ denotes the complement of the set $\overline{V}_n$ in X. Since X is a Banach space, one of the open sets $(\overline{V}_n)^c$ must not be dense in X by Baire's theorem (3.4). Hence we find a positive integer p, some $x_0 \in X$ and $\delta > 0$ such that $U(x_0, \delta) \subset \overline{V}_p$. We shall show that $U(0, \delta) \subset V_{4p}$.

First, note that $U(0, \delta) \subset \overline{V}_{2p}$. For, if $x \in X$ with $\|x\| < \delta$, then $x + x_0 \in U(x_0, \delta) \subset \overline{V}_p$. Also, $x_0 \in \overline{V}_p$. If (v_n) and (w_n) are sequences in V_p such that $v_n \to x + x_0$ and $w_n \to x_0$, then $v_n - w_n \to x$, where $v_n - w_n \in V_{2p}$ since

$$\|F(v_n - w_n)\| \leq \|F(v_n)\| + \|F(w_n)\| \leq 2p.$$

Thus $x \in \overline{V}_{2p}$. In particular, for every $x \in U(0, \delta)$, there is some x_1 in V_{2p} such that $\|x - x_1\| < \delta/2$. Hence

$$U(0, \delta) \subset V_{2p} + \frac{1}{2}U(0, \delta).$$

Consider $x \in U(0, \delta)$. Letting $U = U(0, \delta), V = V_{2p}$ and $k = 1/2$ in 10.1, we see that there is a sequence (v_n) in V_{2p} such that

$$x - \left(v_1 + \frac{v_2}{2} + \cdots + \frac{v_n}{2^{n-1}}\right) \in \frac{1}{2^n} U(0, \delta)$$

for each $n = 1, 2, \ldots$. Let

$$w_n = v_1 + \frac{v_2}{2} + \cdots + \frac{v_n}{2^{n-1}}, \quad n = 1, 2, \ldots.$$

Since $\|x - w_n\| < \delta/2^n$, it follows that $w_n \to x$ in X. Also, for all $n > m$, we have

$$\|F(w_n) - F(w_m)\| = \|F(\sum_{j=m+1}^{n} \frac{v_j}{2^{j-1}})\| \leq \sum_{j=m+1}^{n} \frac{\|F(v_j)\|}{2^{j-1}} \leq \frac{4p}{2^m}.$$

Hence $(F(w_n))$ is a Cauchy sequence in Y. As Y is a Banach space, $(F(w_n))$ converges in Y, and as F is a closed map, we see that $F(w_n) \to F(x)$ in Y. If we let $m = 0$ and $w_0 = 0$, then $\|F(w_n)\| \leq 4p$ for all $n \geq 1$ by the calculation given above. Hence

$$\|F(x)\| = \lim_{n \to \infty} \|F(w_n)\| \leq 4p.$$

Since $x \in U(0, \delta)$ is arbitrary, we see that $U(0, \delta) \subset V_{4p}$. Thus the linear map F is bounded on the neighborhood $U(0, \delta)$ of 0. □

We consider an interesting consequence of the closed graph theorem. A linear map P from a linear space X to itself is called a **projection** if $P^2 = P$. If P is a projection, then so is $I - P$ and $R(P) = Z(I - P), Z(P) = R(I - P)$. It follows that

$$X = R(P) + Z(P) \quad \text{and} \quad R(P) \cap Z(P) = \{0\}$$

for every projection P defined on X. Conversely, if Y and Z are subspaces of X such that $X = Y + Z$ and $Y \cap Z = \{0\}$, then for every $x \in X$ there are unique $y \in Y$ and $z \in Z$ such that $x = y + z$, so that the linear map given by $P(x) = y$ is a projection. It is called the **projection onto Y along Z**.

The closedness and the continuity of a projection can be determined by the closedness of its range space and zero space.

10.3 Theorem

Let X be a normed space and $P : X \to X$ be a projection. Then P is a closed map if and only if the subspaces $R(P)$ and $Z(P)$ are closed in X. In that case, P is, in fact, continuous, if X is a Banach space.

Proof:

Let P be closed, $y_n \in R(P), z_n \in Z(P)$ and $y_n \to y, z_n \to z$ in X. Then $P(y_n) = y_n \to y, P(z_n) = 0 \to 0$ in X, so that $P(y) = y$ and $P(z) = 0$. Thus $y \in R(P)$ and $z \in Z(P)$, showing that the subspaces $R(P)$ and $Z(P)$ are closed in X.

Conversely, let $R(P)$ and $Z(P)$ be closed in $X, x_n \to x$ and $P(x_n) \to y$ in X. Since $R(P)$ is closed and $P(x_n) \in R(P)$, we see that $y \in R(P)$. Also, since $Z(P)$ is closed and $x_n - P(x_n) \in Z(P)$, we see that $x - y$ is in $Z(P)$. Thus $x = y + z$ with $y \in R(P)$ and $z = x - y \in Z(P)$. Hence $P(x) = y$, showing that P is a closed map.

If X is a Banach space and the subspaces $R(P)$ and $Z(P)$ are closed, then by the closed graph theorem (10.2), the closed map P is, in fact, continuous. $\square$

We remark that if X is linear space and Y is a subspace of X, then there exists a projection P defined on X such that $R(P) = Y$. For, if a (Hamel) basis $\{y_s\}$ for Y is extended to a basis $\{y_s\} \cup \{z_t\}$ for X by 2.2(b) and we let $Z = \operatorname{span}\{z_t\}$, then clearly $X = Y + Z$ and $Y \cap Z = \{0\}$, showing that there is a projection onto Y along Z. Now if X is a normed space and Y is a closed subspace of X, does there exist a closed projection defined on X such that $R(P) = Y$? By 10.3, such a projection exists if and only if there is a closed subspace Z of X such that $X = Y + Z$ and $Y \cap Z = \{0\}$. In that case, Z is called a **closed complement** of Y in X. It is known that c_0 has no closed complement in ℓ^∞ and that $C([0, 1])$ has no closed complement in $B([0,1])$. (See [57] and [23].) On the other hand, if Y is a finite dimensional subspace of a normed space X, then there exists a closed complement of Y in X. (Compare Problem 10-19.)

We pass on to consider a closely related result known as the open mapping theorem. A map F from a metric space X to a metric space Y is said to be **open** if for every open set E in X, its image $F(E)$ is open in Y. Note that a map F is continuous if and only if for every open set E in Y, its inverse image $F^{-1}(E)$ is open in X.

First we prove some preliminary results.

10.4 Theorem

Let X and Y be normed spaces and $F : X \to Y$ be linear. Then F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma\|y\|$.

In particular, if a linear map is open, then it is surjective.

Proof:

Let F be an open map. Since $U_X(0,1)$ is open in X, the set $F(U_X(0,1))$ is open in Y. As $0 = F(0) \in F(U_X(0,1))$, there is some $\delta > 0$ such that $\overline{U}_Y(0,\delta) \subset F(U_X(0,1))$. Consider $y \in Y, y \neq 0$. Then $\delta y/\|y\|$ belongs to $\overline{U}_Y(0,\delta)$. Hence there is some $x_1 \in U_X(0,1)$ such that $F(x_1) = \delta y/\|y\|$. Letting $x = \|y\|x_1/\delta$, we see that $F(x) = y$ and $\|x\| < \|y\|/\delta$. Thus we can let $\gamma = 1/\delta$.

Conversely, assume that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma\|y\|$ for some fixed $\gamma > 0$. Consider an open set E in X and $x_0 \in E$. Then $U_X(x_0,\delta) \subset E$ for some $\delta > 0$. Let $y \in Y$ with $\|y - F(x_0)\| < \delta/\gamma$. By hypothesis, there is some $x \in X$ such that $F(x) = y - F(x_0)$ and $\|x\| \leq \gamma\|y - F(x_0)\|$. Then $y = F(x) + F(x_0) = F(x + x_0)$, where $x + x_0 \in U_X(x_0,\delta) \subset E$, since $\|x\| < \delta$. Thus $U_Y(F(x_0),\delta/\gamma) \subset F(E)$. Hence $F(E)$ is an open set in Y. We conclude that F is an open map. $\square$

10.5 Theorem

Let X and Y be normed spaces.

(a) If Z is a closed subspace of X, then the quotient map Q from X to X/Z is continuous and open.

(b) Let $F : X \to Y$ be a linear map such that the subspace $Z(F)$ is closed in X. Define $\tilde{F} : X/Z(F) \to Y$ by $\tilde{F}(x + Z(F)) = F(x)$ for $x \in X$. Then F is an open map if and only if $\tilde{F}$ is an open map.

Proof:

(a) The map Q is continuous because $|||Q(x)||| = |||x + Z||| \leq \|x\|$ for all $x \in X$. To show that the linear map Q is open, we use the result given in 10.4. Consider any $\epsilon > 0$. Let $x + Z \in X/Z$. Then

$$\inf\{\|x + z\| : z \in Z\} = |||x + Z||| < (1 + \epsilon)|||x + Z|||,$$

so that there is some $z_0 \in Z$ with $\|x + z_0\| < (1 + \epsilon)|||x + Z|||$. Since $Q(x + z_0) = x + Z$, we can let $\gamma = 1 + \epsilon$ in 10.4 and conclude that Q is an open map.

(b) Since $Z(F)$ is a closed subspace of X, $X/Z(F)$ is a normed space in the quotient norm. Let $Q : X \to X/Z(F)$ be the quotient map. For $E \subset X$, we have $F(E) = \tilde{F}(Q(E))$. As the map Q is open by part (a), it follows that F is open whenever $\tilde{F}$ is open. For $\tilde{E} \subset X/Z(F)$, we have $\tilde{F}(\tilde{E}) = F(Q^{-1}(\tilde{E}))$, since the map Q is surjective. As the map Q is continuous by part (a), it follows that $\tilde{F}$ is open whenever F is open. $\square$

10.6 Open mapping theorem (Banach, 1932)

Let X and Y be Banach spaces and $F : X \to Y$ be a linear map which is closed and surjective. Then F is continuous and open.

Proof:

By the closed graph theorem (10.2), F is continuous. Hence 6.2(vi) shows that $Z(F)$ is closed in X and that the map $\tilde{F} : X/Z(F) \to Y$

is continuous, where $\tilde{F}(x + Z(F)) = F(x), x \in X$. In particular, $\tilde{F}$ is a closed map. Clearly, it is injective. Also, since the map F is surjective, so is $\tilde{F}$. Thus $\tilde{F}$ is a bijective closed linear map. Hence its inverse map $\tilde{G} : Y \to X/Z(F)$ is closed and linear (2.4(a)). As Y and $X/Z(F)$ are Banach spaces (8.2(a)), the closed graph theorem shows that $\tilde{G}$ is continuous, that is, $\tilde{F}$ is open. By 10.5(b), F is an open map. □

We consider an application of the open mapping theorem to solutions of operator equations. Let X and Y be Banach spaces and $F \in BL(X, Y)$. Suppose that for every $y \in Y$, the operator equation

$$F(x) = y$$

has a solution in X, that is, the map F is surjective. Then by 10.6 and 10.4, there exists some $\gamma > 0$ such that for every $y \in Y$, the above-mentioned operator equation has, in fact, a solution x in X whose norm is at most $\gamma\|y\|$. This estimate on the norm of a solution in terms of the norm of the so-called free term of the equation is important in many situations.

One such situation occurs when although a unique solution is known to exist in X for every y in Y, one is able to find such a solution only for y belonging to a specified dense subset E of Y. If $y \in Y$ but $y \notin E$, then one may find a sequence (y_n) in E such that $y_n \to y$ and a sequence (x_n) in X such that $F(x_n) = y_n, n = 1, 2, \ldots$. A natural question arises whether the sequence (x_n) will converge to the unique element x of X such that $F(x) = y$. The answer is in the affirmative. For

$$F(x - x_n) = F(x) - F(x_n) = y - y_n,$$

so that $\|x - x_n\| \leq \gamma\|y - y_n\| \to 0$ as $n \to \infty$. Hence x_n can be called an **approximate solution** of $F(x) = y$. This also shows that the solution $x \in X$ corresponding to $y \in Y$ depends continuously on

y. We have, therefore, established the validity of the perturbation technique used in the theory of operator equations. It consists of changing the free term a little bit and admitting a small change in the solution.

Let us describe a concrete case. Consider an mth order nonhomogeneous linear ordinary differential equation with variable coefficients:

$$a_m(t)x^{(m)}(t) + \cdots + a_0(t)x(t) = y(t), \quad t \in [a, b],$$

where each $a_j \in C([a, b])$ and $a_m(t) \neq 0$ for every $t \in [a, b]$. Also, consider the initial conditions

$$x(a) = x'(a) = \cdots = x^{(k)}(a) = 0,$$

where $0 \leq k \leq m - 1$. It is well known ([11], Theorem 8 in Chapter 6) that for every $y \in C([a, b])$, there is a solution of the above-mentioned differential equation which satisfies the initial conditions and such a solution is unique if $k = m - 1$.

Suppose that $k = m - 1$. Let $Y = C([a, b])$ and

$$X = \{x \in C^{(m)}([a, b]) : x(a) = \cdots = x^{(m-1)}(a) = 0\}.$$

For $x \in X$, let

$$F(x) = a_m x^{(m)} + \cdots + a_0 x.$$

Then $F : X \to Y$ is linear and bijective. Also, if we consider the sup norm $\| \ \|_\infty$ on Y and the norm given by

$$\|x\| = \|x\|_\infty + \cdots + \|x^{(m)}\|_\infty$$

on X, then X and Y are Banach spaces and $F \in BL(X, Y)$ since

$$\|F(x)\|_\infty \leq (\|a_m\|_\infty + \cdots + \|a_0\|_\infty)\|x\|, \quad x \in X.$$

Let E denote the set of all polynomials on $[a, b]$ and suppose that for every $p \in E$ we are able to find the unique element x of X such that $F(x) = p$.

For convenience, let $a = 0$ and $b = 1$. Consider a continuous function y on $[0, 1]$ which is not a polynomial. For $n = 1, 2, \ldots$, let

$$p_n(t) = \sum_{k=0}^{n} y(\frac{k}{n}) \binom{n}{k} t^k(1-t)^{n-k}, \quad t \in [0,1].$$

As shown in the proof of 3.12, $\|p_n - y\|_\infty \to 0$ as $n \to \infty$. For each n, find $x_n \in X$ such that

$$a_m x_n^{(m)} + \cdots + a_0 x_n = p_n.$$

By what we have seen before, it follows that the sequence (x_n) converges in X to the unique element x in X such that

$$a_m x^{(m)} + \cdots + a_0 x = y.$$

In other words, the approximate solution x_n of the initial value problem converges to the exact solution X uniformly on $[a, b]$, and so does the jth derivative $x_n^{(j)}$ to $x^{(j)}, j = 1, \ldots, m$, as $n \to \infty$.

Similarly, consider a multi-point boundary value problem

$$a_m(t)x^{(m)}(t) + \cdots + a_0(t)x(t) = y(t), \quad x(t_1) = \cdots = x(t_m) = 0,$$

where $a = t_1 < \cdots < t_m = b$ and $a_0, \ldots, a_m$ are continuous functions on $[a, b]$. Assume that for every $y \in \mathbf{C}([a, b])$, there is a unique solution $x \in C([a, b])$. Approximations to such a solution can be found as in the case of an initial value problem. The theory of numerical solution of differential problems is based on this technique.

10.7 Examples

We give several examples to show that the closed graph theorem and the open mapping theorem may not hold if the normed spaces X and/or Y are not Banach spaces.

(a) Let $X = Y = c_{00}$ with the norm $\| \ \|_\infty$. We have seen in Section 6 that c_{00} is not a Banach space. For $x = (x(1), x(2), \ldots)$ in X, let

$$F(x)(j) = j\, x(j), \quad j = 1, 2, \ldots.$$

Then F is a linear map from X to Y. Also, F is closed. To see this, let $x_n \to x$ in X and $F(x_n) \to y$ in Y. Then for every fixed $j = 1, 2, \ldots, x_n(j) \to x(j)$ as $n \to \infty$, so that $j\, x_n(j) = F(x_n)(j)$ converges to $j\, x(j)$ as well as to $y(j)$. Thus $y(j) = j\, x(j)$ for all $j = 1, 2, \ldots$, that is, $y = F(x)$. However, F is not continuous, because if we let $x_n = (0, \ldots, 0, 1, 0, 0, \ldots)$, where 1 occurs only in the nth entry, then $\|x_n\|_\infty = 1$ but $\|F(x_n)\|_\infty = n \to \infty$ as $n \to \infty$.

The inverse map $F^{-1} : Y \to X$ given by

$$F^{-1}(y)(j) = \frac{y(j)}{j}, \quad y \in X, \quad j = 1, 2, \ldots,$$

is continuous since $\|F^{-1}(y)\|_\infty \leq \|y\|_\infty$ for all $y \in Y$. Thus F^{-1} is a closed linear map, which is also surjective. But it is not an open map since F is not continuous.

Next, consider $P : X \to X$ defined by

$$P(x)(2j-1) = x(2j-1) + j\, x(2j) \quad \text{and} \quad P(x)(2j) = 0$$

for $x \in X$ and $j = 1, 2, \ldots$. Then P is linear and $P^2 = P$. Also, $R(P) = \{x \in X : x(2j) = 0 \text{ for } j = 1, 2, \ldots\}$ and $Z(P) = \{x \in X : x(2j-1) + j\, x(2j) = 0 \text{ for } j = 1, 2, \ldots\}$ are both closed subspaces of X. Hence P is a closed map by 10.3. However, P is not continuous since if we let $x_n = (1, \ldots, 1, 0, 0, \ldots)$, where 1 occurs in the first $2n$ entries, then $\|x_n\|_\infty = 1$, but $\|P(x_n)\|_\infty = n + 1$, which tends to infinity as $n \to \infty$.

(b) Let $X = C^1([a, b])$ and $Y = C([a, b])$, both with the sup norm $\|\ \|_\infty$. Then Y is a Banach space. Since X is a proper dense subspace of Y by 3.12, X is not a Banach space. For $x \in X$, let $F(x) = x'$, the derivative of x. Then $F : X \to Y$ is clearly linear. Also, it is a closed map. To see this, let $x_n \to x$ in X and $x_n' \to y$ in Y. Since convergence in the sup norm is nothing but uniform convergence, an elementary theorem in analysis shows that x is differentiable on $[a, b]$ and $x' = y$, that is, $y = F(x)$. However, F is not continuous, because if we let $x_n(t) = [(t-a)/(b-a)]^n, t \in [a,b]$, then $\|x_n\|_\infty \le 1$ but $\|F(x_n)\|_\infty = n \to \infty$ as $n \to \infty$. This is a prime example of a closed linear map which is not continuous. Many differential operators fall in this category.

(c) Let $X = C^1([a,b])$ with the norm given by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $Y = C^1([a,b])$ with the sup norm. Then it can be seen that X is a Banach space but Y is not. For $x \in X$, let $F(x) = x$. Then $F : X \to Y$ is clearly linear. Also, it is continuous since $\|F(x)\|_\infty = \|x\|_\infty \le \|x\|_\infty + \|x'\|_\infty = \|x\|$ for all $x \in X$. However, F is not an open map since the inverse map $F^{-1} : Y \to X$ is discontinuous.

(d) We describe a situation where X is not a Banach space, Y is a Banach space and the open mapping theorem fails.

Consider an infinite dimensional separable Banach space Y. [For example, $Y = \ell^p$ for some p with $1 \le p < \infty$.] Let $\{y_s : s \in S\}$ be a (Hamel) basis for Y with $\|y_s\| = 1$ for all $s \in S$. By 8.4, S is an uncountable set. Let X denote the set of all scalar-valued functions on S which vanish at all but a finite number of elements of S. Then X is a linear space over $\mathbb{K}$ under pointwise addition and scalar multiplication. For $x \in X$, let

$$\|x\| = \sum_{s \in S} |x(s)|.$$

Then $\|\ \|$ is a norm on X. Note that X is not separable. This can be seen as follows. For a fixed $s \in S$, let $x_s(t) = \delta_{s,t}, t \in S$. Then $\{x_s : s \in S\}$ is an uncountable set in X and $\|x_{s_1} - x_{s_2}\| = 2$ whenever $s_1 \ne s_2$. For $x \in X$, define

$$F(x) = \sum_{s \in S} x(s) y_s.$$

Then $F : X \to Y$ is a linear map. It is continuous since

$$\|F(x)\| \le \sum_{s \in S} |x(s)|\, \|y_s\| = \sum_{s \in S} |x(s)| = \|x\|$$

for all $x \in X$. Also, it can be seen that F is bijective. However, F is not an open map since otherwise F would be a homeomorphism from a nonseparable normed space X onto a separable normed space Y! This, in fact, implies that X is not a Banach space. (Compare Problem 22-18.)

Since $F : X \to Y$ is a closed bijective map, it follows that its inverse $F^{-1} : Y \to X$ is also a closed map. However, F^{-1} is not continuous since F is not an open map. Thus we see that the closed graph theorem may not hold if the range space is not complete although the domain space is complete.

Problems

10-1 Let X be a metric space.

(a) If Y and Z are metric spaces, $F : X \to Y$ is continuous and $G : Y \to Z$ is closed, then $G \circ F : X \to Z$ is closed.

(b) If Y is a normed space, $F : X \to Y$ is continuous and $G : X \to Y$ is closed, then $F + G : X \to Y$ is closed.

(c) If Y is a metric space, $F : X \to Y$ is closed and $R(F) \subset Z \subset Y$, then $F : X \to Z$ is closed.

10-2 Let X and Y be normed spaces and $F : X \to Y$ be linear. Let $\tilde{F} : X/Z(F) \to Y$ be defined by $\tilde{F}(x + Z(F)) = F(x), x \in X$. Then F is a closed map if and only if $Z(F)$ is closed in X and $\tilde{F}$ is a closed map. (Compare 6.2(vi) and 10.5(b).)

10-3 Let X be a normed space and $f : X \to \mathbf{K}$ be linear. Then f is closed if and only if f is continuous. (Hint: 6.4 and Problem 10-2)

10-4 Let X be an infinite dimensional normed space. If Y is a nonzero normed space, then there is some $F : X \to Y$ which is linear but not closed.

10-5 Let X be a Banach space, Y be a normed space and $F : X \to Y$ be linear. Assume that $F(x_n) \to F(x)$ whenever $x_n \to x$ in X and $(F(x_n))$ is Cauchy in Y. Then F is continuous. (Hint: Proof of 10.2, or consider the completion of Y.)

10-6 Let F be a closed linear map from a Banach space X to a normed space Y. If F is not continuous, then Y is not complete, and $F : X \to Y_c$ is not a closed map, where Y_c denotes the completion of Y.

10-7 The uniform boundedness principle (9.1) for functionals can be deduced from the closed graph theorem (10.2).

10-8 Let X and Y be Banach spaces and $F : X \to Y$ be linear. Let $\{g_s\} \subset Y'$ such that for every nonzero y in Y, there is some s with $g_s(y) \neq 0$. Then F is continuous if and only if $g_s \circ F$ is continuous for every s. (Compare 9.3(b). Hint: 10.2)

10-9 Let X and Y be Banach spaces, X_0 be a subspace of X and $F : X_0 \to Y$ be linear. Assume that $\mathrm{Gr}(F) = \{(x, F(x)) : x \in X_0\}$ is closed in $X \times Y$, where $X \times Y$ is equipped with the norm $\|(x,y)\|_p$, $1 \le p \le \infty$, introduced in 5.2(c). For $x \in X_0$, let

$$\|x\|_F = \|(x, F(x))\|_p.$$

Then X_0 is a Banach space with the norm $\| \ \|_F$ and $F \in BL(X_0, Y)$. The norm $\| \ \|_F$ is called a **norm of the graph.**

10-10 Let X and Y be Banach spaces and $F : X \to Y$ be closed and linear. For $x \in X$, define $\|x\|_F = \|x\| + \|F(x)\|$. Then $\| \ \|$ and $\| \ \|_F$ are equivalent norms on X.

10-11 Let X and Y be Banach spaces, and Z be a normed space. Consider $G \in BL(X, Z)$ and $H \in BL(Y, Z)$. Suppose that for every $x \in X$, there is a unique $y \in Y$ such that $G(x) = H(y)$ and define $F(x) = y$. Then $F \in BL(X, Y)$.

10-12 Let an infinite matrix M define a linear map from ℓ^p to ℓ^r, where $1 \le p, r \le \infty$. Then $M \in BL(\ell^p, \ell^r)$. (Hint: For each $i = 1, 2, \ldots$, the ith row of M is in ℓ^q, where $1/p + 1/q = 1$.)

10-13 Let $1 \le p, r \le \infty$ and $1/p + 1/q = 1$. Let $k(.,.)$ be a measurable function on $[a,b] \times [a,b]$ such that $k(s,.) \in L^q([a,b])$ for almost every $s \in [a,b]$.

(Such a function is known as a **Carleman kernel.**) For $x \in L^p([a,b])$,

$$F(x)(s) = \int_a^b k(s,t)x(t)\,dm(t)$$

exists for almost every $s \in [a,b]$. If $F(x) \in L^r([a,b])$ for each $x \in L^p([a,b])$, then $F \in BL(L^p([a,b]), L^r([a,b]))$.

10-14 Let E be a set of integers and

$$C_E = \{x \in C([-\pi,\pi]) : \hat{x}(j) = 0 \quad \text{for all } j \notin E\},$$

where $\hat{x}(j)$ is the jth Fourier coefficient of x. Then C_E is closed subspace of $C([-\pi,\pi])$. If $\sum_{j=-\infty}^{\infty} |\hat{x}(j)| < \infty$ for every $x \in C_E$, then E is called a **Sidon set**. For such a set, $\sum_{j=-\infty}^{\infty} |\hat{x}(j)| \leq \alpha\|x\|_\infty$ for all $x \in C_E$ and some $\alpha > 0$.

10-15 Let $X = L^2([a,b])$ with the norm $\| \ \|_2$ and $Y = C([a,b])$ with the norm $\| \ \|_\infty$. If $F \in BL(X)$ and $R(F) \subset Y$, then $F_{|Y} \in BL(Y)$.

10-16 A **Banach function space** on a set T is a linear space X of scalar-valued functions on T with a complete norm $\| \ \|$ on X such that for each fixed $t \in T$, the linear functional $f_t(x) = x(t), x \in X$, is continuous. Let X and Y be Banach function spaces on T and let $x_0 : T \to \mathbf{K}$ be such that $x_0 x \in Y$ for every $x \in X$. Then the map $F : X \to Y$ defined by $F(x) = x_0 x, x \in X$, is continuous. (Hint: 10.2).

10-17 Let $1 \leq p \leq \infty$ and define $\sigma : \ell^p \to \ell^p$ by

$$\sigma((x(1), x(2), \ldots)) = (0, x(1), x(2), \ldots).$$

Then a linear map $F : \ell^p \to \ell^p$ is continuous if $F \circ \sigma = \sigma \circ F$. (Hint: For $x \in \ell^p$ and $m = 1, 2, \ldots, x = \sum_{j=1}^{m} x(j)e_j + \sigma^m\tau^m(x)$ and $F(\sigma^m\tau^m(x))(m) = \sigma^m F(\tau^m(x))(m) = 0$, where $\tau((x(1), x(2), \ldots)) = (x(2), x(3), \ldots)$.)

10-18 Let Y and Z be closed subspaces of a Banach space X such that $Y \cap Z = \{0\}$. Then $Y + Z$ is closed in X if and only if $\delta\|y\| \leq \operatorname{dist}(y, Z)$ for some $\delta > 0$ and all $y \in Y$. (Hint: Consider the projection P from $Y + Z$ onto Y along Z.)

10-19 Let Y be a finite dimensional subspace of a normed space X. Then there is a continuous projection P defined on X such that $R(P) = Y$. (Hint: 7.10(c)).

10-20 Let X be a normed space and Y be a finite dimensional normed space. If $F : X \to Y$ is linear and surjective, then F is an open map. (Compare 7.4 and its proof. Hint: By 6.3(b), assume $Y = \mathbf{K}^n$ with the norm $\| \ \|_\infty$. Alternatively, let $\{y_1, \ldots, y_n\}$ be a basis of Y, $Y_j = \mathrm{span}\,\{y_i : 1 \le i \le n, i \ne j\}, d_j = \mathrm{dist}(y_j, Y_j)$ and $F(a_j) = y_j$ for $j = 1, \ldots, n$. If $U(x,r) \subset E$, then $U(F(x), \frac{r}{d}) \subset F(E)$, where $d = \left[\frac{\|a_1\|}{d_1} + \cdots + \frac{\|a_n\|}{d_n}\right]$.)

10-21 The open mapping theorem can be directly proved by using 10.1. (Hint: $Y = \bigcup_{n=1}^{\infty} \overline{V}_n$, where $V_n = \{F(x) : x \in X, \|x\| \le n\}$ and $U_Y(0,\delta)$ is contained in $V_{2p} + \frac{1}{2}U_Y(0,\delta)$ for some positive integer p and $\delta > 0$.)

10-22 Let X be a Banach space and Y be a normed space. Let $F : X \to Y$ be a closed linear map such that $R(F)$ is not a countable union of nowhere dense subsets of Y. Then $R(F) = Y$. In fact, F is an open map. Further, F is continuous if and only if Y is a Banach space. (Hint: Problem 10-21, 6.2(vi), 6.3(a), 10.2)

10-23 The closed graph theorem can be deduced from the open mapping theorem. (Hint: $X \times Y$ is a Banach space and the map $(x, F(x)) \to x \in X$ is one-to-one and onto.)

10-24 Let X be a Banach space and Y, Z be closed subspaces of X such that $X = Y + Z$. Then there is some $\alpha > 0$ such that for every $x \in X$, there are $y \in Y$ and $z \in Z$ with $x = y + z$ and $\|y\| + \|z\| < \alpha\|x\|$.

10-25 Let X and Y be Banach spaces and $F \in BL(X,Y)$ be surjective. Let $y_n \to y$ in Y. If $F(x) = y$, then there is a sequence (x_n) in X such that $F(x_n) = y_n$ for each n and $x_n \to x$ in X.

10-26 Let a_0, a_1 and a_2 be continuous scalar-valued functions on $[0,1]$ where $a_2(t) \ne 0$ for all $t \in [0,1]$. For $r = 0, 1, 2, \ldots$, let z_r denote the unique function in $C^2([0,1])$ such that

$$a_2(t)z_r''(t) + a_1(t)z_r'(t) + a_0(t)z_r = t^r, \quad t \in [0,1],$$

and $z_r(0) = z_r'(0) = 0$. For $y \in C([0,1])$ and $n = 1, 2, \ldots$, let

$$x_n(t) = \sum_{r=0}^{n} \left[\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} y\left(\frac{j}{n}\right)\right] \binom{n}{r} z_r(t), \quad t \in [0,1].$$

Then (x_n) converges uniformly on $[0,1]$ to x,

$$a_2(t)x''(t) + a_1(t)x'(t) + a_0(t)x(t) = y(t), \quad t \in [0,1],$$

and $x(0) = x'(0) = 0$. Also, (x_n') and (x_n'') converge uniformly on $[0,1]$ to x' and x'', respectively. [Hint: 10.5, 10.6 and the Bernstein polynomials considered in the proof of 3.12 and in Section 14]

10-27 Let m be a positive integer and $X = C^m([a,b])$ with $\|x\| = \|x\|_\infty + \cdots + \|x^{(m-1)}\|_\infty, x \in X$. Let $Y = C([a,b])$ with the sup norm and $F(x) = x^{(m)}$ for $x \in X$. Then F is a closed linear map from X to Y but F is not continuous.

11 Bounded Inverse Theorem

In this section we study some consequences of the closed graph theorem (10.2). The following special case of the open mapping theorem (10.6) is of great importance.

11.1 Bounded inverse theorem
Let X and Y be Banach spaces. Let $F \in BL(X,Y)$ be bijective. Then $F^{-1} \in BL(Y,X)$.

Proof:
By 2.4(a), F^{-1} is a linear map from Y to X. Since F is continuous, it is closed. Hence F^{-1} is also closed as we have seen in the beginning of Section 10. Since Y and X are Banach spaces, the closed graph theorem implies that F^{-1} is continuous. Thus $F^{-1} \in BL(Y,X)$. □

This result shows that just as the inverse of a bijective linear map from a linear space to a linear space is linear and the inverse of a bijective closed map from a metric space to a metric space is closed, the inverse of a bijective, linear and continuous map from a Banach space to a Banach space is linear and continuous. This may not hold for maps on noncomplete normed spaces. See parts (c) and (d) of 10.7. As another example, let $X = c_{00}$ with $\| \ \|_1$ and $Y = c_{00}$ with $\| \ \|_\infty$. If $F(x) = x$ for $x \in X$, then $F : X \to Y$ is bijective, linear and continuous. But F^{-1} is not continuous since for $x_n = (1, \ldots, 1, 0, 0, \ldots)$, we have $\|x_n\|_\infty = 1$ and $\|F^{-1}(x_n)\| = \|x_n\|_1 = n$ for all $n = 1, 2, \ldots$.

The bounded inverse theorem has applications of two kinds. Let X and Y be Banach spaces and F be an injective continuous linear map from X into Y. If $F^{-1} : R(F) \to X$ is known to be discontinuous, then F cannot be surjective, that is, there is some $y \in Y$ such that the operator equation $F(x) = y$ has no solution in X. This is a negative result. On the other hand, if F is known to be surjective, then F^{-1} is continuous, that is, the solution of the operator equation $F(x) = y$ depends continuously on y. This is a positive result.

The Riemann-Lebesgue lemma states that if $x \in L^1([-\pi, \pi])$, then its nth Fourier coefficient $\hat{x}(n) \to 0$ as $n \to \pm\infty$ (4.9(a)). Conversely, if $k_n \in \mathbf{K}$ for $n = 0, \pm 1, \pm 2, \ldots$ and $k_n \to 0$ as $n \to \pm\infty$, does there exist some $x \in L^1([-\pi, \pi])$ such that $\hat{x}(n) = k_n$ for each n? The bounded inverse theorem gives a negative answer.

11.2 Theorem

There are scalars $k_n \in \mathbf{K}, n = 0, \pm 1, \pm 2, \ldots$ such that $k_n \to 0$ as $n \to \pm\infty$, but there is no $x \in L^1([-\pi, \pi])$ such that $\hat{x}(n) = k_n$ for all $n = 0, \pm 1, \pm 2, \ldots$.

Proof:

Let $X = L^1([-\pi, \pi])$ with the norm given by

$$\|x\|_1 = \int_{-\pi}^{\pi} |x|\, dm, \quad x \in X,$$

and $Y = \{(\cdots, y(-1), y(0), y(1), \cdots) : y(n) \in \mathbf{K}$ and $y(n) \to 0$ as

$n \to \pm\infty\}$ with the norm given by

$$\|y\|_\infty = \sup\{|y(n)| : n = 0, \pm 1, \pm 2, \cdots\}, \quad y \in Y.$$

Then X and Y are Banach spaces. (See 4.6 for the completeness of X. The completeness of Y follows in the same way as the completeness of c_0, proved in Section 8.)

For $x \in X$, define

$$F(x) = (\cdots, \hat{x}(-1), \hat{x}(0), \hat{x}(1), \cdots),$$

where $\hat{x}(n)$ is the nth Fourier coefficient of x. The Riemann-Lebesgue lemma 4.9(a) says that $F(x) \in Y$ for every $x \in X$. Thus F maps X into Y. Clearly, F is linear. Also, if $F(x) = 0$, that is, $\hat{x}(n) = 0$ for all n, then $x = 0$ almost everywhere on $[-\pi, \pi]$ by 4.9(c). Thus F is injective. Further, F is continuous since for all $x \in X$ and $n = 0, \pm 1, \pm 2, \ldots$,

$$2\pi|\hat{x}(n)| = \left|\int_{-\pi}^{\pi} x(t)e^{-int}\, dm(t)\right| \leq \int_{-\pi}^{\pi} |x|\, dm = \|x\|_1,$$

so that $\|F(x)\|_\infty = \sup\{|\hat{x}(n)| : n = 0, \pm 1, \pm 2, \ldots\} \leq \|x\|_1/2\pi$.

However, F^{-1} is not continuous. This can be seen as follows. For $m = 1, 2, \ldots$, consider the mth Dirichlet kernel

$$x_m(t) = \sum_{n=-m}^{m} e^{int}, \quad t \in [-\pi, \pi].$$

Then $F(x_m) = (\cdots, 0, 0, 1, \cdots, 1, 0, 0, \cdots) = y_m$ say, where 1 occurs only in the $-m$th to mth entries. Now $\|y_m\|_\infty = 1$ for all $m = 1, 2, \ldots$, but $\|F^{-1}(y_m)\|_1 = \|x_m\|_1 \to \infty$, as we have seen in Section 4.

The bounded inverse theorem (11.1) implies that F cannot be surjective, that is, there is some $y \in Y$ such that $F(x) \neq y$ for any $x \in X$. In other words, there is no $x \in L^1([-\pi, \pi])$ such that $\hat{x}(n) = y(n)$ for all $n = 0, \pm 1, \pm 2, \ldots$, although $y(n) \to 0$ as $n \to \pm\infty$. Letting $k_n = y(n)$ we obtain the desired result. □

A stronger version of this result is given in Problem 11-3. Our functional analytic method, however, does not concretely produce a scalar sequence (k_n) which tends to zero as $n \to \pm\infty$, but is not the sequence of Fourier coefficients of any function in $L^1([-\pi,\pi])$. An example of such a sequence is given below. Let

$$k_n = 0 \text{ for } n = \ldots, -2, -1, 0, 1 \quad \text{and} \quad k_n = \frac{1}{\log n} \text{ for } n = 2, 3, \ldots.$$

Then $k_n \to 0$ as $n \to \pm\infty$. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{\hat{x}(n) - \hat{x}(-n)}{n}$ converges for every $x \in L^1([-\pi,\pi])$ (4.9(b)), while the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges, there is no $x \in L^1([-\pi,\pi])$ such that $\hat{x}(n) = k_n$ for all n.

The proof of 4.10 shows that if $k_n \in \mathbf{K}$ with $\sum_{n=-\infty}^{\infty} |k_n| < \infty$, then there is a continuous function x on $[-\pi,\pi]$ such that $x(-\pi) = x(\pi)$ and $\hat{x}(n) = k_n$ for all n. Later we shall see in 22.8(b) that if $k_n \in \mathbf{K}$ with $\sum_{n=-\infty}^{\infty} |k_n|^2 < \infty$, then there is a square-integrable function x on $[-\pi,\pi]$ such that $\hat{x}(n) = k_n$ for all n.

We move on to consider an application of the bounded inverse theorem which is of a positive nature.

11.3 Two-norm theorem

Let X be a Banach space in the norm $\| \ \|$. Then a norm $\| \ \|'$ on the linear space X is equivalent to the norm $\| \ \|$ if and only if X is also a Banach space in the norm $\| \ \|'$ and the norm $\| \ \|'$ is comparable to the norm $\| \ \|$.

Proof:

If the norms $\| \ \|$ and $\| \ \|'$ are equivalent, then clearly they are comparable and since X complete in the norm $\| \ \|$, it follows from 6.3(a) that X is complete in the norm $\| \ \|'$.

Conversely, assume that X is a Banach space in the norm $\| \ \|'$ and the norm $\| \ \|'$ is comparable to the norm $\| \ \|$. We can assume, without loss of generality, that the norm $\| \ \|$ is stronger than the norm $\| \ \|'$. By 5.7, there is some $\alpha > 0$ such that $\|x\|' \le \alpha\|x\|$ for all $x \in X$. Let Y denote the linear space X with the norm $\| \ \|'$ and consider the identity map $I : X \to Y$. Clearly, I is bijective, linear and continuous. By the bounded inverse theorem (11.1), $I^{-1} : Y \to X$ is also continuous, that is, $\|x\| \le \gamma\|x\|'$ for all $x \in X$ and some $\gamma > 0$. Letting $\beta = 1/\gamma$, we see that

$$\beta\|x\| \le \|x\|' \le \alpha\|x\|, \quad x \in X.$$

Again by 5.7, the norms $\| \ \|$ and $\| \ \|'$ are equivalent. □

This result shows that two comparable complete norms on a linear space are, in fact, equivalent. It raises the following two questions.

Let X be a Banach space in the norm $\| \ \|$.

1) Is there a norm $\| \ \|'$ on X which is comparable to the norm $\| \ \|$, but in which X is not complete?
2) Is there a norm $\| \ \|'$ on X in which X complete but which is not comparable to the norm $\| \ \|$?

An affirmative answer to the first question is easily obtained by letting $X = C([0,1])$, and

$$\|x\|_\infty = \sup\{|x(t)| : 0 \le t \le 1\}, \quad \|x\|_1 = \int_0^1 |x(t)|\,dt$$

for $x \in X$. Then X is complete in the norm $\| \ \|_\infty$, $\|x\|_1 \le \|x\|_\infty$ for all $x \in X$, but X is not complete in the norm $\| \ \|_1$. In fact, the Banach space $L^1([0,1])$ is the closure of its subspace X with the induced norm $\| \ \|_1$. (See 4.7(b).)

An affirmative answer to the second question is obtained as follows. Let Y be an infinite dimensional Banach space over $\mathbf{K}$ and f be a discontinuous linear functional on Y. (See 6.1.) Let

$$X = \{(y,k) : y \in Y, k \in \mathbf{K}\}$$

and for $x = (y, k) \in X$,

$$\|x\| = \|y\| + |k|, \quad \|x\|' = \|y\| + |f(y) - k|.$$

It is not difficult to check that $\| \ \|$ and $\| \ \|'$ are complete norms on X. However, they are not comparable. This can be seen as follows. Since f is a discontinuous linear functional on Y, there is a sequence (y_n) in Y such that $\|y_n\| \leq 1$, but $|f(y_n)| \to \infty$ as $n \to \infty$. (See 6.2(v).) Then $\|(y_n, 0)\| = \|y_n\| \leq 1$, but $\|(y_n, 0)\|' = \|y_n\| + |f(y_n)| \to \infty$ as $n \to \infty$, while $\|(y_n, f(y_n)\|' = \|y_n\| \leq 1$, but $\|(y_n, f(y_n)\| = \|y_n\| + |f(y_n)| \to \infty$ as $n \to \infty$.

We now consider an application of the two-norm theorem. In Section 8 we have defined a Schauder basis $\{x_1, x_2, \ldots\}$ for a normed space X. Recall that if $x = \sum_{j=1}^{\infty} a_j x_j$, then $f_j(x) = a_j$ yields the jth coefficient functional corresponding to the Schauder basis $\{x_1, x_2 \ldots\}$.

11.4 Theorem

The coefficient functionals corresponding to a Schauder basis for a Banach space X are continuous. In fact, they form a bounded subset of its dual X'.

Proof:
Let $\| \ \|$ denote the norm on the Banach space X. If $\{x_1, x_2 \ldots\}$ is a Schauder basis for X and $f_1, f_2, \ldots$ are the corresponding coefficient functionals, then for $m = 1, 2, \ldots,$ define

$$F_m(x) = f_1(x)x_1 + \cdots + f_m(x)x_m, \quad x \in X.$$

Let $x \in X$. Since $F_m(x) \to x$ as $m \to \infty$, we see that the set $\{F_m(x) : m = 1, 2, \ldots\}$ is bounded in X. Let

$$\|x\|' = \sup\{\|F_m(x)\| : m = 1, 2, \ldots\}, \quad x \in X.$$

Then for all $x \in X$,

$$|f_1(x)| = \|f_1(x)x_1\| = \|F_1(x)\| \leq \|x\|'$$

and for $j = 2, 3, \ldots,$

$$|f_j(x)| = \|f_j(x)x_j\| = \|F_j(x) - F_{j-1}(x)\| \leq 2\|x\|'.$$

To see that $\|\ \|'$ is a norm on X, we simply note that if $\|x\|' = 0$, then $f_j(x) = 0$ for all $j = 1, 2, \ldots$, so that $x = 0$. The other properties of a norm follow immediately.

Also, for all $x \in X$, we have

$$\|x\| = \|\sum_{j=1}^{\infty} f_j(x)x_j\| = \lim_{m\to\infty} \|\sum_{j=1}^{m} f_j(x)x_j\| = \lim_{m\to\infty} \|F_m(x)\| \leq \|x\|'.$$

Thus the norms $\|\ \|$ and $\|\ \|'$ are comparable.

We show that X is complete in the norm $\|\ \|'$ as well. Let (y_n) be a Cauchy sequence in X with respect to the norm $\|\ \|'$. Since $\|y_n - y_k\| \leq \|y_n - y_k\|'$, it follows that (y_n) is Cauchy with respect to the norm $\|\ \|$. As X is complete in the norm $\|\ \|$, let $y \in X$ be such that $\|y_n - y\| \to 0$. We claim that $\|y_n - y\|' \to 0$.

First note that for all n, k and j,

$$|f_j(y_n) - f_j(y_k)| = |f_j(y_n - y_k)| \leq 2\|y_n - y_k\|'.$$

Hence for each fixed j, the sequence $(f_j(y_k))$ is Cauchy in $\mathbf{K}$. Let $f_j(y_k) \to a_j \in \mathbf{K}$ as $k \to \infty$. Then for each fixed m,

$$F_m(y_k) = f_1(y_k)x_1 + \cdots + f_m(y_k)x_m \to_1 x_1 + \cdots + a_m x_m$$

as $k \to \infty$.

Let $\epsilon > 0$. Since $\|y_n - y_k\|' \to 0$ as $n, k \to \infty$ and $\|y_n - y\| \to 0$ as $n \to \infty$, there is some n_0 such that

$$\|y_n - y_k\|' < \epsilon \quad \text{and} \quad \|y_n - y\| < \epsilon \quad \text{for all } n, k \geq n_0.$$

Fix $m = 1, 2, \ldots$. For all $n, k \geq n_0$, we have

$$\|F_m(y_n) - F_m(y_k)\| = \|F_m(y_n - y_k)\| \leq \|y_n - y_k\|' < \epsilon.$$

Keeping $n \geq n_0$ fixed and letting $k \to \infty$, we have

$$\|F_m(y_n) - \sum_{j=1}^{m} a_j x_j\| \leq \epsilon \quad \text{for all } m = 1, 2, \ldots \text{ and } n \geq n_0.$$

Since $F_m(y_{n_0}) \to y_{n_0}$, there is some m_0 such that

$$\|y_{n_0} - F_m(y_{n_0})\| < \epsilon \quad \text{for all } m \geq m_0.$$

Writing

$$y - \sum_{j=1}^{m} a_j x_j = y - y_{n_0} + y_{n_0} - F_m(y_{n_0}) + F_m(y_{n_0}) - \sum_{j=1}^{m} a_j x_j,$$

we see that for all $m \geq m_0$,

$$\|y - \sum_{j=1}^{m} a_j x_j\| \leq \epsilon + \epsilon + \epsilon = 3\epsilon.$$

Hence $y = \sum_{j=1}^{\infty} a_j x_j$. Finally, for all $n \geq n_0$ we have

$$\|y_n - y\|' = \sup_{m=1,2,\ldots} \|F_m(y_n - y)\| = \sup_{m=1,2,\ldots} \|F_m(y_n) - \sum_{j=1}^{m} a_j x_j\| \leq \epsilon.$$

Thus $\|y_n - y\|' \to 0$ as $n \to \infty$, showing that X is complete in the norm $\|\ \ \|'$.

Since the complete norms $\|\ \ \|$ and $\|\ \ \|'$ on X are comparable, the two-norm theorem (11.3) shows that they are equivalent. Hence there is some $\alpha > 0$ such that

$$\sup_{m=1,2,\ldots} \|F_m(x)\| = \|x\|' \leq \alpha \|x\| \quad \text{for all } x \in X.$$

Thus $F_m \in BL(X)$ and $\|F_m\| \leq \alpha$ for all $m = 1, 2, \ldots$. In particular,

$$|f_1(x)| = \|F_1(x)\| \leq \alpha \|x\|$$

and for $j = 1, 2, \ldots,$

$$|f_j(x)| = \|F_j(x) - F_{j-1}(x)\| \leq (\|F_j\| + \|F_{j-1}\|)\|x\| \leq 2\alpha \|x\|$$

for all $x \in X$. Hence $f_j \in X'$ and $\|f_j\| \le 2\alpha$ for all $j = 1, 2, \ldots$. □

11.5 Example

The two-norm theorem allows us to obtain characterizations (up to equivalence) of some well known norms. For example, consider the usual norm on $X = L^1([-\pi, \pi])$ given by

$$\|x\|_1 = \int_{-\pi}^{\pi} |x| \, dm, \quad x \in X.$$

It is complete, and if $\|x_n - x\|_1 \to 0$, then the jth Fourier coefficient $\hat{x}_n(j)$ of x_n tends to the jth Fourier coefficient $\hat{x}(j)$ of x for every $j = 0, \pm 1, \pm 2, \ldots$. Let $\| \ \|'$ be any complete norm on X such that if $\|x_n - x\|' \to 0$, then $\hat{x}_n(j) \to \hat{x}(j)$ for each j. We show that the norms $\| \ \|_1$ and $\| \ \|'$ are equivalent. Consider the identity map I from X with the norm $\| \ \|_1$ to X with the norm $\| \ \|'$. To see that I is a closed map, let $\|x_n - x\|_1 \to 0$ and $\|x_n - y\|' = \|I(x_n) - y\|' \to 0$. Then

$$\hat{y}(j) = \lim_{j \to \infty} \hat{x}_n(j) = \hat{x}(j) \quad \text{for } j = 0, \pm 1, \pm 2, \ldots.$$

By 4.9(c), $y = x$ almost everywhere on $[-\pi, \pi]$, that is, $I(x) = y$. The closed graph theorem (10.2) implies that I is a continuous map. Hence there is some $\alpha > 0$ such that $\|x\|' \le \alpha \|x\|_1$ for all $x \in X$. The complete and comparable norms $\| \ \|_1$ and $\| \ \|'$ are equivalent.

Similar examples are given in Problems 11-5, 11-6 and 11-7.

Problems

11-1 Let X and Y be Banach spaces and $F \in BL(X, Y)$. Then $R(F)$ is linearly homeomorphic to $X/Z(F)$ if and only if $R(F)$ is closed in Y. (Hint: 8.2(a) and 11.1)

11-2 Let X be a separable Banach space. Then there is a closed subspace Z of ℓ^1 such that X is linearly homeomorphic to ℓ^1/Z. (Hint: If $\{z_j\}$ is a dense subset of $\overline{U}(0,1)$, consider $F : \ell^1 \to X$ defined by $F(x) = \sum_{j=1}^{\infty} x(j) z_j$.)

11-3 Let $Y = \{(\ldots, y(-1), y(0), y(1), \ldots) : y(n) \in \mathbf{K}$ and $y(n) \to 0$ as $n \to \pm\infty\}$ with $\|y\|_\infty = \sup\{|y(n)| : n = 0, \pm 1, \pm 2, \ldots\}$ for $y \in Y$. Then Y is not a countable union of nowhere dense subsets of Y. Let $Z = \{y \in Y :$ there is some $x \in L^1([-\pi, \pi])$ such that $\hat{x}(n) = y(n)$ for all $n = 0, \pm 1, \pm 2, \ldots\}$. Then the interior of Z is empty and Z is a countable union of nowhere dense subsets of Y. (Compare 11.2. Hint: Problem 10-22)

11-4 For $x \in L^1(\mathbf{R})$, consider the Fourier transform of x :

$$\hat{x}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-iut}\, dm(t), \quad u \in \mathbf{R}.$$

Then there is some $y \in C_0(\mathbf{R})$ such that $y \neq \hat{x}$ for any $x \in L^1(\mathbf{R})$. (Hint: For $n = 1, 2, \ldots$, let $x_n(t) = (\sin t \sin nt)/t^2$, if $0 \neq t \in \mathbf{R}$ and $x_n(0) = n$. Then $x_n \in L^1(\mathbf{R}), \|x_n\|_1 \to \infty$, but $\|\hat{x}_n\|_\infty \leq \alpha$ for all n and small $\alpha > 0$.)

11-5 Let X denote the sequence space $\ell^p (1 \leq p \leq \infty)$, or c, or c_0. Let $\| \ \|'$ be a complete norm on X such that if $\|x_n - x\|' \to 0$, then $x_n(j) \to x(j)$ for every $j = 1, 2, \ldots$. Then $\| \ \|$ is equivalent to the usual norm $\| \ \|_p$ on X.

11-6 Let $\| \ \|'$ be a complete norm on $C([a, b])$ such that if $\|x_n - x\|' \to 0$, then $x_n(t) \to x(t)$ for every $t \in [a, b]$. Then $\| \ \|'$ is equivalent to the sup norm on $C([a, b])$.

11-7 Let $\| \ \|'$ be a complete norm on $C^1([a, b])$ such that if $\|x_n - x\|' \to 0$, then $x_n(a) \to x(a)$ and $x_n'(t) \to x'(t)$ for every $t \in [a, b]$. Then $\| \ \|'$ is equivalent to the norm on $C^1([a, b])$ given by $\|x\| = \|x\|_\infty + \|x'\|_\infty$. In particular, if $\|x\|' = |x(a)| + \|x'\|_\infty$, then $\| \ \|'$ is equivalent to $\| \ \|$. (Compare Problem 5-19).

11-8 Let X be a Banach space with a Schauder basis $\{x_1, x_2, \ldots\}$. For $j = 1, 2, \ldots$, let X_j denote the closure of span $\{x_k : k = 1, 2, \ldots, k \neq j\}$. Then $X_j = Z(f_j)$, where f_j is the jth coefficient functional. Also, $\|f_j\| = 1/\text{dist}(x_j, X_j)$, and $\text{dist}(x_j, X_j) \geq \delta$ for all $j = 1, 2, \ldots$ and some $\delta > 0$. (Hint: 11.4 and Problem 6-8)

11-9 Let X be a Banach space and $\{x_1, x_2, \ldots\}$ be a Schauder basis for X. Then there is an equivalent norm $\| \ \|'$ on X such that for every

$x = \sum_{n=1}^{\infty} a_n x_n$ in X, we have

$$\Big\| \sum_{n=1}^{m} a_n x_n \Big\|' \le \Big\| \sum_{n=1}^{m+1} a_n x_n \Big\|'$$

for all $m = 1, 2, \ldots$, that is, $\{x_1, x_2, \ldots\}$ is a **monotone Schauder basis** for X with respect to the norm $\| \ \|'$. (Hint: Proof of 11.4 and Problem 8-16)

11-10 A countable set $\{x_1, x_2, \ldots\}$ in a normed space X is said to be an **absolute Schauder basis** for X if $\|x_n\| = 1$ for each n and if for every $x \in X$, there are unique $k_1, k_2, \ldots$ in $\mathbf{K}$ such that

$$x = \sum_n k_n x_n \quad \text{and} \quad \sum_n |k_n| < \infty.$$

Every Banach space with an absolute Schauder basis is linearly homeomorphic to ℓ^1 with the norm $\| \ \|_1$.

12 Spectrum of a Bounded Operator

Let X be a linear space over $\mathbf{K}$. By an **operator** on X, we mean a linear map from X to X. If A and B are operators on X, then so is their composition $A \circ B$. For ease in notation, we shall denote $A \circ B$ simply by AB. We shall denote the identity operator on X by I. For operators A, B, C on X and a scalar k, we have

$$A(B + C) = AB + AC, \quad (A + B)C = AC + BC,$$

$$A(BC) = (AB)C. \quad k(AB) = (kA)B = A(kB), \quad AI = A = IA.$$

In general, AB may not equal BA. For example, if $X = \mathbf{K}^2$ and for $x = (x(1), x(2)) \in X$,

$$A(x) = (0, x(1)) \quad \text{and} \quad B(x) = (x(2), 0),$$

then $AB(x) = (0, x(2))$, while $BA(x) = (x(1), 0)$.

Consider an operator A on X. If A is bijective, that is, if for every $y \in X$, there is a unique $x \in X$ such that $A(x) = y$, and if we let $B(y) = x$, then it follows that B is an operator on X and $AB = I = BA$. Conversely, if there is an operator B on X such that $AB = I = BA$, then we see that A is bijective. Clearly, such a map B is unique. In fact, if $AB = I = CA$ for some maps B and C from X to X, then

$$C = CI = C(AB) = (CA)B = IB = B.$$

This operator B is called the **inverse** of A, and is denoted by A^{-1}. If A^{-1} exists and B is an operator on X such that $AB = I$ or $BA = I$, then we must have $B = A^{-1}$. As a particular case, let $X = \mathbf{K}^n$ and M an $n \times n$ matrix whose determinant is not zero. Then for every $y \in X$, there is a unique $x \in X$ such that $Mx = y$. Thus M^{-1} exists. If N is an $n \times n$ matrix such that MN is the identity matrix, then N is the inverse of M.

Further, it is easy to see that A^{-1} and B^{-1} exist if and only if $(AB)^{-1}$ and $(BA)^{-1}$ exist. In that case,

$$(AB)^{-1} = B^{-1}A^{-1} \quad \text{and} \quad B(AB)^{-1} = A^{-1} = (BA)^{-1}B.$$

Let now X be a normed space. If A and B belong to $BL(X)$, then AB belongs to $BL(X)$. In fact, since

$$\|AB(x)\| = \|A(B(x))\| \leq \|A\| \, \|B(x)\| \leq \|A\| \, \|B\| \, \|x\|$$

for all $x \in X$, we see that $\|AB\| \leq \|A\| \, \|B\|$. Consequently, we have

$$\|A^n\| \leq \|A\|^n, \quad n = 1, 2, \ldots.$$

Also, the composition is continuous in the sense that if $A_n \to A$ and $B_n \to B$ in $BL(X)$, then $A_n B_n \to AB$ in $BL(X)$, because

$$\begin{aligned} \|A_n B_n - AB\| &\leq \|A_n B_n - A_n B\| + \|A_n B - AB\| \\ &\leq \|A_n\| \, \|B_n - B\| + \|A_n - A\| \, \|B\|. \end{aligned}$$

Let $A \in BL(X)$. We say that A is **invertible** (in $BL(X)$) if A^{-1} exists and belongs to $BL(X)$. We have the following characterizations of an invertible operator.

12.1 Theorem

Let X be a normed space and $A \in BL(X)$. Then A is invertible if and only if A is bounded below and surjective.

Let X be a Banach space. Then

(a) A is invertible if and only if A is bijective.

(b) A is invertible if and only if A is bounded below and the range of A is dense in X.

Proof:
Let A be invertible. Then it is bounded below since

$$\|x\| = \|A^{-1}(A(x))\| \leq \|A^{-1}\| \, \|A(x)\|$$

for all $x \in X$. Also, A is clearly surjective.

Conversely, assume that A surjective and there is some $\beta > 0$ such that

$$\beta\|x\| \leq \|A(x)\|, \quad x \in X.$$

Then A is injective as well, so that A^{-1} exists and is linear. Also, letting $x = A^{-1}(y)$ we see that

$$\beta\|A^{-1}(y)\| \leq \|A(A^{-1}(y))\| = \|y\|$$

for all $y \in X$. Hence A^{-1} belongs to $BL(X)$ and $\|A^{-1}\| \leq 1/\beta$. Thus A is invertible.

Now assume that X is a Banach space.

(a) Let A be bijective. Then A^{-1} exists and is linear. Also, A^{-1} belongs to $BL(X)$ by the bounde ` inverse theorem (11.1).

(b) Let A be bounded below and $R(A)$ be dense in X. To conclude that A is invertible, it is enough to show that A is surjective. Let

$y \in X$. Since $R(A)$ is dense in X, there is a sequence $(A(x_n))$ in $R(A)$ such that $A(x_n) \to y$ in X. As A is bounded below, there is some $\beta > 0$ such that $\beta\|x\| \leq \|A(x)\|$ for all $x \in X$. Then

$$\beta\|x_n - x_m\| \leq \|A(x_n - x_m)\| = \|A(x_n) - A(x_m)\|$$

for all $n, m = 1, 2, \ldots$. This shows that (x_n) is a Cauchy sequence in X. Since X is a Banach space, $x_n \to x$ in X, so that $A(x_n) \to A(x)$ by the continuity of A. Hence $y = A(x) \in R(A)$. Thus $R(A) = X$, that is, A is surjective. $\square$

Note that 12.1(a) is a consequence of the bounded inverse theorem (11.1), whereas 12.1(b) has an independent proof.

Let $A \in BL(X)$. If $k \in \mathbf{K}$, then the invertibility of the bounded operator $A - kI$ plays a crucial role in the solution of the operator equation

$$A(x) - kx = y$$

for a fixed $y \in X$. We are thus led to the following definition.

The set $\rho(A) = \{k \in \mathbf{K} : A - kI \text{ is invertible}\}$ is called the **resolvent set** of A. The complement $\sigma(A)$ of this set in $\mathbf{K}$ is called the **spectrum** of A. Thus

$$\sigma(A) = \{k \in \mathbf{K} : A - kI \text{ is not invertible}\}.$$

A scalar belonging to $\sigma(A)$ is known as a **spectral value** of A. A spectral value k of A is an undesirable scalar as far as the solution of the operator equation $A(x) - kx = y$ is concerned. Of course, it is important to know what these undesirable scalars are. We shall see in Section 28 that certain bounded operators (known as compact self-adjoint operators) can be represented in terms of their spectra. It is, therefore, appropriate to undertake a study of the spectrum of a bounded operator.

How can one find some scalars in $\sigma(A)$? If $A - kI$ is not injective, or more generally, if it is not bounded below, then it must not be invertible, so that $k \in \sigma(A)$. We are thus led to the following relatively more tractable parts of $\sigma(A)$.

The **eigenspectrum** $\sigma_e(A)$ of A consists of all k in $\mathbf{K}$ such that $A - kI$ is not injective. Thus $k \in \sigma_e(A)$ if and only if there is some nonzero x in X such that $A(x) = kx$. Then k is called an **eigenvalue** of A and x is called a corresponding **eigenvector** of A. The subspace $Z(A - kI)$ is known as the **eigenspace** of A corresponding the eigenvalue k.

The **approximate eigenspectrum** $\sigma_a(A)$ of A consists of all k in $\mathbf{K}$ such that $A - kI$ is not bounded below. Thus $k \in \sigma_a(A)$ if and only if there is a sequence (x_n) in X such that $\|x_n\| = 1$ for each n and $\|A(x_n) - kx_n\| \to 0$ as $n \to \infty$. Then k is called an **approximate eigenvalue** of A. If $k \in \sigma_e(A)$ and x is a corresponding eigenvector, then letting $x_n = x/\|x\|$ for all n, we conclude that $k \in \sigma_a(A)$. Hence

$$\sigma_e(A) \subset \sigma_a(A) \subset \sigma(A).$$

We now consider a class of bounded operators for which the inclusions stated above are, in fact, equalities. An operator A on a linear space X is said to be of **finite rank** if the range of A is finite dimensional.

12.2 Theorem

Let X be a normed space and $A \in BL(X)$ be of finite rank. Then

$$\sigma_e(A) = \sigma_a(A) = \sigma(A).$$

Proof:

It is enough to show that $\sigma(A) \subset \sigma_e(A)$. Let $k \notin \sigma_e(A)$, that is, let $A - kI$ be injective. We show that $A - kI$ is invertible.

Case 1: X is finite dimensional. If the dimension of X is n, then it follows from 2.4(a) that

$$\dim Z(A - kI) + \dim R(A - kI) = n.$$

Since $A - kI$ is injective, we see that $\dim Z(A - kI) = 0$. Hence $R(A - kI)$ is an n dimensional subspace of X, that is, $R(A - kI) = X$. Then $A - kI$ is invertible by 6.1

Case 2: X is infinite dimensional. In this case $k \neq 0$, since otherwise A would be an injective operator on X and if $\{x_1, x_2, \ldots\}$ is an infinite linearly independent subset of X, then $\{A(x_1), A(x_2), \ldots\}$ would be an infinite linearly independent subset of $R(A)$, contradicting its finite dimensionality.

Let B denote the restriction $(A - kI)_{|R(A)} : R(A) \to R(A)$. Since $A - kI$ is injective, so is B. As we have seen in Case 1, B is then surjective as well. Now consider $y \in X$. Then $A(y) \in R(A)$. Hence there is some $u \in R(A)$ with $B(u) = A(y)$, that is, $A(u) - ku = A(y)$, or $A(u - y) = ku$. Letting $x = (u - y)/k$, we find that $A(x) = u = kx + y$, that is, $(A - kI)(x) = y$. Thus $A - kI$ is surjective.

Next, we claim that $A - kI$ is bounded below. Otherwise, there is a sequence (x_n) in X such that $\|A(x_n) - kx_n\| \to 0$ as $n \to \infty$ and $\|x_n\| = 1$ for each n. Then $(A(x_n))$ is a bounded sequence in the finite dimensional normed space $R(A)$. By 5.5, there is a convergent subsequence $(A(x_{n_j}))$. If $A(x_{n_j}) \to y$, then $kx_{n_j} \to y$ as well. Since $\|y\| = |k| \lim_{j\to\infty} \|x_{n_j}\| = |k| \neq 0$, we see that $y \neq 0$. But

$$A(y) = A(\lim_{j\to\infty} kx_{n_j}) = k \lim_{j\to\infty} A(x_{n_j}) = ky,$$

and since $A - kI$ is injective, we must have $y = 0$. This contradiction justifies our claim.

Now $A - kI$ is invertible by 12.1. □

In 12.7 we shall give examples to show that $\sigma_a(A) \not\subset \sigma_e(A)$ and $\sigma(A) \not\subset \sigma_a(A)$, in general.

Where can we locate eigenvalues and spectral values of a bounded operator? An upper bound for the moduli of eigenvalues can be easily obtained as follows. Let $k \in \sigma_e(A)$ and $A(x) = kx$ for some nonzero

$x \in X$. Then $A^n(x) = k^n x$ for all $n = 1, 2, \ldots$, so that

$$|k|^n \|x\| = \|A^n(x)\| \leq \|A^n\| \, \|x\|.$$

Hence

$$|k| \leq \inf_{n=1,2,\ldots} \|A^n\|^{1/n} \leq \|A\|.$$

A similar result holds for every approximate eigenvalue k. (See Problem 12-8.)

To treat the case of a spectral value, we need to assume that X is a Banach space. First note that by 12.1, $k \in \sigma(A)$ if and only if $A - kI$ is not bijective, and also if and only if either $A - kI$ is not bounded below or $R(A - kI)$ is not dense in X.

Now we give a sufficient condition for the invertibility of a bounded operator on a Banach space. It also yields a series expansion of the inverse operator.

12.3 Lemma

Let X be a Banach space, $A \in BL(X)$ and $\|A^p\| < 1$ for some positive integer p. Then the bounded operator $I - A$ is invertible. Also,

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n \quad \text{and} \quad \|(I - A)^{-1}\| \leq \frac{1 + \|A\| + \cdots + \|A^{p-1}\|}{1 - \|A^p\|}.$$

Proof:

Since $\|A^{pn+j}\| \leq \|A^p\|^n \|A^j\|$ for all $n = 0, 1, 2, \ldots,\ j = 1, \ldots, p-1$ and since $\|A^p\| < 1$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \|A^n\| &= \sum_{n=0}^{\infty} \|A^{pn}\| + \sum_{n=0}^{\infty} \|A^{pn+1}\| + \cdots + \sum_{n=0}^{\infty} \|A^{pn+p-1}\| \\
&\leq \sum_{n=0}^{\infty} \|A^p\|^n (1 + \|A\| + \cdots + \|A^{p-1}\|) \\
&= \frac{1 + \|A\| + \cdots + \|A^{p-1}\|}{1 - \|A^p\|} < \infty.
\end{aligned}$$

As X is a Banach space, so is $BL(X)$ by 8.2(c). Hence it follows from 8.1 that the absolutely summable series $\sum_{n=0}^{\infty} A^n$ is summable in

$BL(X)$. Let

$$B = \sum_{n=0}^{\infty} A^n \quad \text{and} \quad B_m = \sum_{n=0}^{m} A^n, m = 0, 1, 2, \ldots.$$

Then it is easy to see that

$$B_m(I - A) = I - A^{m+1} = (I - A)B_m, \quad m = 0, 1, 2, \ldots.$$

Now $A^{m+1} \to 0$ in $BL(X)$ as $m \to \infty$ since the series $\sum_{n=0}^{\infty} A^n$ is summable in $BL(X)$. Hence letting $m \to \infty$, we obtain $(I - A)B = I = B(I - A)$, so that

$$(I - A)^{-1} = B = \sum_{n=0}^{\infty} A^n.$$

Hence A is invertible and

$$\|(I - A)^{-1}\| \le \sum_{n=0}^{\infty} \|A^n\| \le \frac{1 + \|A\| + \cdots + \|A^{p-1}\|}{1 - \|A^p\|}.$$

□

12.4 Example

Theorem 12.3 can be useful in finding the inverse of $I - A$ if we can calculate $A^n, n = 1, 2, \ldots$ and if one of these powers can be shown to have norm less than 1. We give an illustration of this kind.

Let $X = C([0, 1])$ and $k(.,.)$ be a scalar-valued continuous function on $[0, 1] \times [0, 1]$. For $x \in X$, let

$$A(x)(s) = \int_0^1 k(s, t)x(t)\, dt, \quad s \in [0, 1].$$

Then $A \in BL(X)$ and

$$\|A\| \le \sup\{|k(s, t)| : 0 \le s, t \le 1\} = \|k\|_\infty.$$

We say that A is a **Fredholm integral operator** on X with continuous kernel $k(.,.)$. If B is another Fredholm integral operator on

X with continuous kernel $h(.,.)$, then it is easy to see that AB is a Fredholm operator on X with continuous kernel $k \circ h$, where

$$k \circ h(s,t) = \int_0^1 k(s,u)h(u,t)\,du, \quad 0 \le s,t \le 1.$$

Also,

$$\|AB\| \le \|k \circ h\|_\infty \le \|k\|_\infty \|h\|_\infty.$$

Letting $h = k$ and applying this result repeatedly, we find that for $n = 2, 3, \ldots, A^n$ is a Fredholm operator on X with kernel

$$k^{(n)}(s,t) = \int_0^1 \cdots \int_0^1 k(s,u_1)k(u_1,u_2)\cdots k(u_{n-1},t)\,du_1\cdots du_{n-1},$$

which is known as the nth **iterated kernel**. We have

$$k^{(m+n)} = k^{(m)} \circ k^{(n)}, \quad m, n = 1, 2, \ldots.$$

Also,

$$\|A^n\| \le \|k^{(n)}\|_\infty \le \|k\|_\infty^n, \quad n = 1, 2, \ldots.$$

Let $k(.,.)$ be a kernel such that $\|k^{(p)}\|_\infty < 1$ for some positive integer p. Then $\|A^p\| < 1$. By Lemma 12.3, we have for all $y \in X$ and $s \in [0,1]$,

$$(I - A)^{-1}(y)(s) = \sum_{n=0}^{\infty} A^n(y)(s) = y(s) + \sum_{n=1}^{\infty} \int_0^1 k^{(n)}(s,t)y(t)\,dt.$$

We note that the series $\sum_{n=1}^{\infty} k^{(n)}(s,t)$ converges uniformly and absolutely for $(s,t) \in [0,1]\times[0,1]$ since for $n = 1, 2, \ldots$ and $j = 1, \ldots, p-1$,

$$\|k^{(pn+j)}\|_\infty \le \|k^{(pn)}\|_\infty \|k^{(j)}\|_\infty \le \|k^{(p)}\|_\infty^n \|k^{(j)}\|_\infty.$$

Since $\|k^{(p)}\|_\infty < 1$,

$$h(s,t) = \sum_{n=1}^{\infty} k^{(n)}(s,t), \quad 0 \le s,t \le 1,$$

is a continuous function on $[0,1]\times[0,1]$. Also, we can interchange the order of summation and integration in the expression given above for $(I - A)^{-1}(y)(s)$ and obtain

$$(I - A)^{-1}(y)(s) = y(s) + \int_0^1 h(s,t)y(t)\,dt, \quad 0 \le s \le 1.$$

In other words,

$$(I - A)^{-1} = I + B,$$

where B is a Fredholm integral operator on X with kernel $h(.,.)$. The kernel $h(.,.)$ is known as the **reciprocal kernel**, or the **resolvent kernel**.

12.5 Theorem

Let X be a Banach space.

(a) If $A, B \in BL(X)$, A is invertible and $\epsilon = \|(A - B)A^{-1}\| < 1$, then B is invertible,

$$B^{-1} = A^{-1} \sum_{n=0}^{\infty} [(A - B)A^{-1}]^n,$$

$$\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \epsilon} \quad \text{and} \quad \|B^{-1} - A^{-1}\| \leq \frac{\|A^{-1}\|\epsilon}{1 - \epsilon}.$$

(b) The set of all invertible operators is open in $BL(X)$ and the map $A \mapsto A^{-1}$ is continuous on this set.

Proof:

(a) In Lemma 12.3, replace A by $(A - B)A^{-1}$ and put $p = 1$ to conclude that $I - (A - B)A^{-1} = BA^{-1}$ is invertible and its inverse in given by

$$(BA^{-1})^{-1} = \sum_{n=0}^{\infty} [(A - B)A^{-1}]^n.$$

Since A is invertible, it follows that $B = (BA^{-1})A$ is invertible and

$$B^{-1} = A^{-1}(BA^{-1})^{-1} = A^{-1} \sum_{n=0}^{\infty} [(A - B)A^{-1}]^n.$$

Now $\epsilon = \|(A - B)A^{-1}\| < 1$. Hence

$$\|B^{-1}\| \leq \|A^{-1}\| \sum_{n=0}^{\infty} \|(A - B)A^{-1}\|^n = \frac{\|A^{-1}\|}{1 - \epsilon}.$$

Also, since $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$, we have

$$\|B^{-1} - A^{-1}\| \le \|B^{-1}\| \, \|(A - B)A^{-1}\| \le \frac{\|A^{-1}\|\epsilon}{1 - \epsilon}.$$

(b) If $A, B \in BL(X)$, A is invertible and $\|B - A\| < 1/\|A^{-1}\|$, then

$$\|(A - B)A^{-1}\| \le \|B - A\| \, \|A^{-1}\| < 1,$$

so that B is invertible by part (a). Thus the open ball about A of radius $1/\|A^{-1}\|$ is contained in the set of all invertible operators. This set is, therefore, open in $BL(X)$.

Next, consider a sequence (A_n) of invertible operators such that $A_n \to A$ in $BL(X)$. Then $\epsilon_n = \|(A - A_n)A^{-1}\| \le \|A - A_n\| \, \|A^{-1}\| \to 0$ as $n \to \infty$. Thus there is some n_0 such that $\epsilon_n < 1$ for all $n \ge n_0$. Again by part (a), we have

$$\|A_n^{-1} - A^{-1}\| \le \frac{\|A^{-1}\|\epsilon_n}{1 - \epsilon_n} \quad \text{for all } n \ge n_0.$$

Letting $n \to \infty$, we see that $A_n^{-1} \to A^{-1}$ in $BL(X)$. Hence inversion is a continuous map. □

We now proceed to prove several facts about the spectrum of a bounded operator on a Banach space.

12.6 Theorem

Let X be a Banach space over $\mathbf{K}$ and $A \in BL(X)$.

(a) (Neumann expansion) Let $k \in \mathbf{K}$ such that $|k|^p > \|A^p\|$ for some positive integer p. Then $k \notin \sigma(A)$ and

$$(A - kI)^{-1} = -\sum_{n=0}^{\infty} \frac{A^n}{k^{n+1}}.$$

Consequently, for every $k \in \sigma(A)$, we have

$$|k| \le \inf_{n=1,2,\ldots} \|A^n\|^{1/n} \le \|A\|.$$

(b) $\sigma(A)$ is a compact subset of $\mathbf{K}$.

Proof:

(a) Note that $k \neq 0$ and $A - kI = -k(I - \dfrac{A}{k})$. By Lemma 12.3 we see that $A - kI$ is invertible and

$$(A - kI)^{-1} = -\frac{1}{k}(I - \frac{A}{k})^{-1} = -\frac{1}{k}\sum_{n=0}^{\infty}\frac{A^n}{k^n} = -\sum_{n=0}^{\infty}\frac{A^n}{k^{n+1}}.$$

Let now $k \in \sigma(A)$. Then $|k|^p \leq \|A^p\|$ for every positive integer p. Thus

$$|k| \leq \inf_{p=1,2,\ldots} \|A^p\|^{1/p} \leq \|A\|,$$

since $\|A^p\| \leq \|A\|^p$ for all p.

(b) Part (a) shows that $\sigma(A)$ is a bounded subset of $\mathbf{K}$. In fact, $\sigma(A)$ is contained in $\{k \in \mathbf{K} : |k| \leq \|A\|\}$. We now show that $\sigma(A)$ is closed in $\mathbf{K}$. Let $k_n \in \sigma(A)$ and $k_n \to k$ in $\mathbf{K}$. Then $A - k_n I \to A - kI$ in $BL(X)$. Since the set of all noninvertible operators is closed in $BL(X)$ by 12.5(b), we see that $A - kI$ is not invertible, that is, $k \in \sigma(A)$. Being a closed and bounded subset of $\mathbf{K}$, $\sigma(A)$ is compact. □

12.7 Examples

(a) Let $X = \mathbf{K}^n$ with a given norm, and M be an $n \times n$ matrix with scalar entries. Then M defines an operator on X, which is continuous by 6.1. For $k \in \mathbf{K}$, the matrix $M - kI$ is invertible if and only if $\det(M - kI) \neq 0$. Thus $k \in \sigma(M)$ if and only if k is a root of the characteristic polynomial $p(t) = \det(M - tI)$. (See 2.6(a).) By 12.2, a spectral value of M is just an eigenvalue of M. Since the characteristic polynomial of M is of degree n, there are at most n distinct eigenvalues of M. If $\mathbf{K} = \mathbf{C}$, then the fundamental theorem of algebra implies that there is at least one eigenvalue of M. On the other hand, if $\mathbf{K} = \mathbf{R}$, then M may have no eigenvalues, and so $\sigma(M)$ may be empty. A simple example is provided by the matrix $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, whose characteristic polynomial is $t^2 + 1$.

If M is a triangular matrix and $k_1, \ldots, k_n$ are the diagonal entries, then we have

$$\det(M - tI) = (t - k_1)\cdots(t - k_n),$$

and hence $\sigma(M) = \sigma_e(M) = \{k_1, \ldots, k_n\}$. If M is not a triangular matrix, then the problem of finding its eigenvalues poses great difficulties. Algorithms have been developed to reduce M to an 'approximately triangular' matrix by similarity transformations. The most notable among these is the QR algorithm. (See, for example, 4.6 of [56].) It enables us to obtain approximations of all eigenvalues of M.

If k is an eigenvalue of M, then $|k| \leq \|M\|$, where $\| \ \|$ is the operator norm on $BL(\mathbf{K}^n)$ induced by the given norm on $\mathbf{K}^n$. Various choices of these norms yield upper bounds for the eigenspectrum of M. Let $M = (k_{i,j}), i, j = 1, \ldots, n$, α_1 (resp., α_∞) denote the maximum of the column sums (resp., the row sums) of the matrix $(|k_{i,j}|)$ and

$$\beta_2 = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|k_{i,j}|^2\right)^{1/2}$$

Then $|k| \leq \min\{\alpha_1, \alpha_\infty, \beta_2\}$ for every $k \in \sigma_e(M)$. Problem 12-15 gives a well-known inclusion theorem for eigenvalues of M.

If A is a finite rank operator on a linear space X, then the problem of finding nonzero eigenvalues of A can be reduced to a matrix eigenvalue problem. See 20.2(b).

(b) Let $X = \ell^p$ with the norm $\| \ \|_p, 1 \leq p \leq \infty$. For $x = (x(1), x(2), \ldots) \in X$, let

$$A(x) = (x(1), \frac{x(2)}{2}, \frac{x(3)}{3}, \cdots).$$

Then $A \in BL(X)$ and $\|A\|_p = 1$. Since $A(e_n) = e_n/n$, we see that $1/n$ is an eigenvalue of A with e_n as a corresponding eigenvector, $n = 1, 2, \ldots$. Since $A(x) = 0$ implies $x = 0$, we note that 0 is not an eigenvalue of A. However, since $\|A(e_n)\|_p \to 0$ as $n \to \infty$ and

$\|e_n\|_p = 1$ for each n, we see that A is not bounded below, that is, 0 is an approximate eigenvalue of A.

Now consider $k \in \mathbf{K}$ such that $k \notin \{0, 1, 1/2, \ldots\}$. Then there is some $\delta > 0$ such that $|k - \frac{1}{n}| \geq \delta$ for all $n = 1, 2, \ldots$. For $y = (y(1), y(2), \ldots) \in X$, define

$$B_k(y) = \left(\frac{y(1)}{1-k}, \frac{y(2)}{\frac{1}{2}-k}, \cdots\right).$$

Since

$$\delta |B_k(y)(n)| \leq \left|\frac{1}{n} - k\right| \, |B_k(y)(n)| = |y(n)|, \quad n = 1, 2, \ldots,$$

$B_k(y) \in X$ for all $y \in X$. It is easy to see that $B_k \in BL(X)$ and $(A - kI)B_k = I = B_k(A - kI)$. Hence $A - kI$ is invertible. Thus

$$\sigma_e(A) = \{1, \frac{1}{2}, \ldots\} \quad \text{and} \quad \sigma_a(A) = \{0, 1, \frac{1}{2}, \ldots\} = \sigma(A).$$

Next, consider the **right shift operator** on X given by

$$C(x) = (0, x(1), x(2), \ldots), \quad x = (x(1), x(2), \ldots) \in X.$$

Then $\|C(x)\|_p = \|x\|_p$ for all $x \in X$, so that $C \in BL(X)$ and $\|C\|_p = 1$. For $k \in \mathbf{K}$, we have

$$(C - kI)(x) = (-kx(1), x(1) - kx(2), x(2) - kx(3), \ldots).$$

By considering the cases $k = 0$ and $k \neq 0$ separately, we see that if $(C - kI)(x) = 0$, then $x = 0$. Hence no k in $\mathbf{K}$ is an eigenvalue of C, that is, $\sigma_e(C) = \emptyset$.

Now suppose that $(C - kI)(x) = e_1$ for some $x \in X$. Then

$$-kx(1) = 1 \quad \text{and} \quad x(n) - kx(n+1) = 0 \text{ for } n = 1, 2, \ldots,$$

so that $x(n) = -1/k^n$ for $n = 1, 2, \ldots$. If $|k| < 1$, then $x(n) \to \infty$ as $n \to \infty$, showing that x cannot belong to X. Thus there is no $x \in X$ such that $(C - kI)x = e_1$ if $|k| < 1$. Hence for every $k \in \mathbf{K}$ with

$|k| < 1, C - kI$ is not surjective and $k \in \sigma(C)$. On the other hand, $|k| \le \|C\|_p = 1$ for every $k \in \sigma(C)$ by 12.6(a). Thus

$$\{k \in \mathbf{K} : |k| < 1\} \subset \sigma(C) \subset \{k \in \mathbf{K} : |k| \le 1\}.$$

Since $\sigma(C)$ is closed by 12.6(b), we conclude that

$$\sigma(C) = \{k \in \mathbf{K} : |k| \le 1\}.$$

Let $|k| > 1$. We calculate the inverse of $C - kI$. It can be easily seen that for all $n = 1, 2, \ldots,$

$$C^n(x) = (0, \ldots, 0, x(1), x(2), \ldots), \quad x \in X,$$

where the first n entries are zero. By the Neumann expansion (12.6(a)),

$$(C - kI)^{-1} = -\sum_{n=0}^{\infty} \frac{C^n}{k^{n+1}},$$

so that for every $y \in X$ and $j = 1, 2, \ldots,$ we have

$$(C - kI)^{-1}(y)(j) = -\left(\frac{y(j)}{k} + \cdots + \frac{y(1)}{k^j}\right).$$

Finally, we shall show that

$$\sigma_a(C) = \{k \in \mathbf{K} : |k| = 1\}.$$

Let $k \in \mathbf{K}$ with $|k| < 1$. For $x \in X$, we have

$$\|C(x) - kx\|_p \ge \|C(x)\|_p - |k|\,\|x\|_p = (1 - |k|)\|x\|_p.$$

Thus $C - kI$ is bounded below, that is, $k \notin \sigma_a(C)$.

Next, let $k \in \mathbf{K}$ with $|k| = 1$. If $1 \le p < \infty$, let

$$x_n = n^{-1/p}(1, \overline{k}, \ldots, (\overline{k})^{n-1}, 0, 0, \ldots)$$

for $n = 1, 2, \ldots$. Then $\|x_n\|_p = 1$, but

$$\|C(x_n) - kx_n\|_p = \|n^{-1/p}(-k, 0, \ldots, 0, (\overline{k})^{n-1}, 0, 0, \ldots)\|_p = \frac{2^{1/p}}{n^{1/p}},$$

which tends to zero as $n \to \infty$. If $p = \infty$, let

$$x_n = n^{-1}(1, 2\overline{k}, \ldots, n(\overline{k})^{n-1}, (n-1)(\overline{k})^n, \ldots, 2(\overline{k})^{2n-3}, (\overline{k})^{2n-2}, 0, 0, \ldots)$$

for $n = 1, 2, \ldots$. Then again $\|x_n\|_\infty = 1$, but $\|C(x_n) - kx_n\|_\infty$ equals

$$\|n^{-1}(-k, -1, -\overline{k}, \ldots, -(\overline{k})^{n-2}, (\overline{k})^{n-1}, \ldots, (\overline{k})^{2n-2}, 0, 0, \ldots)\|_\infty,$$

which tends to zero as $n \to \infty$. Hence $k \in \sigma_a(A)$.

Thus we have shown that

$$\sigma_e(C) = \emptyset, \sigma_a(C) = \{k \in \mathbf{K} : |k| = 1\} \text{ and } \sigma(C) = \{k \in \mathbf{K} : |k| \leq 1\}.$$

(c) Let $X = C([a,b])$ with the sup norm. Fix $x_0 \in X$ and for $x \in X$, let

$$A(x) = x_0 x.$$

Then $A \in BL(X)$ and $\|A\| = \|x_0\|_\infty$. For $k \in \mathbf{K}$, we have

$$(A - kI)(x) = (x_0 - k)x, \quad x \in X.$$

First assume that $k \neq x_0(t)$ for any $t \in [a,b]$. Then the function $1/(x_0 - k)$ belongs to X. For $x \in X$, let

$$B_k(x) = \frac{x}{x_0 - k}.$$

Then $B_k \in BL(X)$ and $(A - kI)B_k = I = B_k(A - kI)$. This shows that $k \notin \sigma(A)$.

Next, assume that $k = x_0(t_0)$ for some $t_0 \in [a,b]$. For $n = 1, 2, \ldots$, define $x_n \in X$ by

$$x_n(t) = \begin{cases} 1 - n|t - t_0|, & \text{if } |t - t_0| \leq 1/n \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|x_n\|_\infty = 1$. We show that $\|A(x_n) - kx_n\|_\infty \to 0$ as $n \to \infty$. Let $\epsilon > 0$. Since x_0 is continuous at t_0, there is some $\delta > 0$ such that

$|x_0(t) - x_0(t_0)| < \epsilon$ for all $t \in [a, b]$ with $|t - t_0| < \delta$. Choose n_0 such that $n_0 > 1/\delta$. Then for all $n \geq n_0$ and all $t \in [a, b]$, we have

$$|A(x_n)(t) - kx_n(t)| = |x_0(t) - x_0(t_0)|\,|x_n(t)| < \epsilon.$$

This shows that $k \in \sigma_a(A)$. Thus we have shown that

$$\sigma_a(A) = \{x_0(t) : t \in [a, b]\} = \sigma(A).$$

Which spectral values of A are eigenvalues? Suppose $k \in \mathbf{K}$ is an eigenvalue of A. Then there is some nonzero $x \in X$ such that $A(x) = kx$, that is, $(x_0(t) - k)x(t) = 0$ for all $t \in [a, b]$. Since $x \neq 0$, there is some $t_0 \in [a, b]$ such that $x(t_0) \neq 0$. As x is continuous on $[a, b]$, $x(t) \neq 0$ for all t in some neighborhood of t_0. This shows that $x_0(t) = k$ for all t in this neighborhood. Conversely, suppose that $x_0(t) = k$ for all t in some neighborhood $(t_0 - \delta, t_0 + \delta) \bigcap [a, b]$ of a point $t_0 \in [a, b]$. Define $x \in X$ by

$$x(t) = \begin{cases} 1, & \text{if } t \in [t_0 - \delta/2, t_0 + \delta/2] \bigcap [a, b] \\ 0, & \text{if } t \in [a, b] \text{ and } |t - t_0| \geq \delta \\ 2(t_0 - t + \delta)/\delta, & \text{if } t \in [a, b] \text{ and } t_0 + \delta/2 < t < t_0 + \delta \\ 2(t - t_0 + \delta)/\delta, & \text{if } t \in [a, b] \text{ and } t_0 - \delta < t < t_0 - \delta/2. \end{cases}$$

Then $x \neq 0$ and $(x_0(t) - k)x(t) = 0$ for all $t \in [a, b]$, that is, $A(x) = kx$. Hence k is an eigenvalue of A. Thus

$$\sigma_e(A) = \{k \in \mathbf{K} : x_0(t) = k \text{ for all } t \text{ in some } (c, d), a \leq c < d \leq b\}.$$

Let X be a normed space over $\mathbf{K}$ and $A \in BL(X)$. The **spectral radius** of A is defined by

$$r_\sigma(A) = \sup\{|k| : k \in \sigma(A)\}.$$

The result 12.6(a) can now be stated as follows:
If X is a Banach space over $\mathbf{K}$ and $A \in BL(X)$, then

$$r_\sigma(A) \leq \inf_{n=1,2,\ldots} \|A^n\|^{1/n} \leq \|A\|.$$

We have earlier given an example of a 2×2 matrix which defines an operator on $\mathbf{R}^2$ whose spectrum is empty. Even when the spectrum is nonempty, we may have $r_\sigma(A) < \inf_{n=1,2,\ldots} \|A^n\|^{1/n}$. For example, let $X = \mathbf{R}^3$ and

$$M = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then it can be seen that $\sigma(M) = \{0\}$, so that $r_\sigma(A) = 0$. But since $Mx = (x(2), -x(1), 0), M^2x = -(x(1), x(2), 0)$ for all $x \in \mathbf{R}^3$ and $M^3 = -M$, it follows that $\|M^n\|_p = 1$ for all $n = 1, 2, \ldots$ $(1 \le p \le \infty)$. Thus $\inf_{n=1,2,\ldots} \|M^n\|_p^{1/n} = 1$.

We shall now show that if X is a Banach space over $\mathbf{C}$ and $A \in BL(X)$, then indeed $\sigma(A) \ne \emptyset$ and $r_\sigma(A) = \inf_{n=1,2,\ldots} \|A^n\|^{1/n}$. These results employ Liouville's theorem and Laurent's theorem from complex analysis in conjunction with the Hahn-Banach extension theorem and the uniform boundedness principle which we have studied earlier. Since these results will not be needed in the sequel, they could be skipped without losing continuity in our treatment.

12.8 Theorem

Let X be a nonzero Banach space over $\mathbf{C}$ and $A \in BL(X)$. Then

(a) **(Gelfand-Mazur theorem, 1941)** $\sigma(A)$ is nonempty.

(b) **(Spectral radius formula, Gelfand, 1941)**

$$r_\sigma(A) = \inf_{n=1,2,\ldots} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

Proof:

Let $f \in (BL(X))'$. For z in the resolvent set $\rho(A)$ of A, define

$$w_f(z) = f((A - zI)^{-1}).$$

By 12.5(b), $\rho(A)$ is open in $\mathbf{C}$. We show that w_f is an analytic function on $\rho(A)$. Consider $z_0 \in \rho(A)$. Now $(A - z_0I) - (A - zI) = (z - z_0)I$.

Multiplying on the left by $(A - z_0I)^{-1}$ and on the right by $(A - zI)^{-1}$ we obtain

$$(A - zI)^{-1} - (A - z_0I)^{-1} = (z - z_0)(A - z_0I)^{-1}(A - zI)^{-1}$$

Hence

$$\begin{aligned}\lim_{z\to z_0} \frac{w_f(z) - w_f(z_0)}{z - z_0} &= \lim_{z\to z_0} f\left(\frac{(A - zI)^{-1} - (A - z_0I)^{-1}}{z - z_0}\right) \\ &= \lim_{z\to z_0} f((A - z_0I)^{-1}(A - zI)^{-1}) \\ &= f((A - z_0I)^{-2})\end{aligned}$$

by the linearity and the continuity of f, and by the continuity of inversion proved in 12.5(b). Hence w_f is analytic on $\rho(A)$.

(a) Let, if possible, $\sigma(A) = \emptyset$. Then $\rho(A) = \mathbf{C}$, so that w_f is analytic on $\mathbf{C}$, that is, w_f is an entire function. Also, for $|z| > \|A\|$,

$$\|(A - zI)^{-1}\| = \left\|\frac{1}{z}\left(\frac{A}{z} - I\right)^{-1}\right\| \le \frac{1}{|z| - \|A\|} \to 0$$

as $|z| \to \infty$. As f is continuous, $w_f(z) = f((A - zI)^{-1}) \to 0$ as $|z| \to \infty$. Thus w_f is bounded on $\mathbf{C}$. By Liouville's theorem, w_f must be a constant function on $\mathbf{C}$, and this constant must be zero. In particular, $f(A^{-1}) = w_f(0) = 0$. Since this is true for every f in $(BL(X))'$, it follows from 7.10(a) (a consequence of the Hahn-Banach extension theorem) that $A^{-1} = 0$, which is impossible. Thus $\sigma(A) \neq \emptyset$.

(b) Since $\sigma(A) \neq \emptyset$ by (a) above and since $\sigma(A)$ is a bounded subset of $\mathbf{C}$ by 12.6(a), we see that $0 \le r_\sigma(A) < \infty$. Now the set $D = \{z \in \mathbf{C} : |z| > r_\sigma(A)\}$ is contained in $\rho(A)$. Hence w_f is analytic on D for every $f \in (BL(X))'$.

If $|z| > \|A\|$, then by 12.6(a) we have the Neumann expansion

$$(A - zI)^{-1} = -\sum_{n=0}^{\infty} \frac{A^n}{z^{n+1}}.$$

Again by the continuity and the linearity of f, we obtain the Laurent expansion

$$w_f(z) = f((A - zI)^{-1}) = -\sum_{n=0}^{\infty} \frac{f(A^n)}{z^{n+1}}, \quad |z| > \|A\|.$$

Since $r_\sigma(A) \leq \|A\|$ and since the function w_f is analytic in D, it follows that the preceding expansion of $w_f(z)$ in negative powers of z is, in fact, valid in D by the uniqueness of the Laurent expansion. Now fix $z \in D$. Then $(f(A^n)/z^{n+1})$ is a bounded sequence in $\mathbf{C}$ for every $f \in (BL(X))'$. By the consequence 9.3(a) of the uniform boundedness principle, the set $\{A^n/z^{n+1} : n = 1, 2, \ldots\}$ is bounded in $BL(X)$. Hence there is some $\alpha > 0$ such that

$$\|A^n\| \leq \alpha |z|^{n+1}, \quad n = 1, 2, \ldots.$$

Taking nth roots, we have

$$\limsup_{n\to\infty} \|A^n\|^{1/n} \leq \limsup_{n\to\infty} \alpha^{1/n} |z|^{(n+1)/n} = |z|.$$

Since this holds for every $z \in \mathbf{C}$ with $|z| > r_\sigma(A)$, it follows that $\limsup_{n\to\infty} \|A^n\|^{1/n} \leq r_\sigma(A)$. Now 12.6(a) yields

$$r_\sigma(A) \leq \inf_{n=1,2,\ldots} \|A^n\|^{1/n} \leq \limsup_{n\to\infty} \|A^n\|^{1/n} \leq r_\sigma(A).$$

Hence $\lim_{n\to\infty} \|A^n\|^{1/n}$ exists and equals $r_\sigma(A)$, which is also equal to $\inf_{n=1,2,\ldots} \|A^n\|^{1/n}$. □

The spectral radius formula gives an excellent illustration of the interplay between the algebraic structure and the metric structure on the normed space $BL(X)$. The real number $\sup\{|z| : z \in \mathbf{C}, A - zI$ is not invertible in $BL(X)\}$ is obtained by using the algebraic structure of $BL(X)$, while the real number $\lim_{n\to\infty} \|A^n\|^{1/n}$ is obtained by using the metric structure of $BL(X)$. The spectral radius formula says that these two numbers are equal!

Problems

X denotes a normed space over $\mathbf{K}$ and $A \in BL(X)$, unless otherwise stated.

12-1 (a) If A is invertible, then $\sigma(A^{-1}) = \{k^{-1} : k \in \sigma(A)\}$.

(b) Let p be a polynomial in one variable. If $k \in \sigma(A)$, then $p(k)$ is in $\sigma(p(A))$. If $\mathbf{K} = \mathbf{C}$, then $\sigma(p(A)) = \{p(z) : z \in \sigma(A)\}$.

12-2 If A is invertible, then $\liminf_{n\to\infty} \|A^n\|^{1/n} > 0$.

12-3 If $B \in BL(X)$ and $k \neq 0$, then $k \in \sigma(AB)$ if and only if $k \in \sigma(BA)$. (*Hint*: If $I - AB$ is invertible, then $I + B(I - AB)^{-1}A$ is the inverse of $I - BA$.)

12-4 If M and N are similar matrices (that is, $N = P^{-1}MP$ for some nonsingular matrix P), then $\sigma_e(N) = \sigma_e(M)$.

12-5 Let $M = (k_{i,j})$, $i, j = 1, \ldots, n$, define an operator on $\mathbf{C}^n$. Then $\sum_{j=1}^{n} k_{j,j}$ is the sum of the eigenvalues of M, and $\det(k_{i,j})$ is the product of the eigenvalues of M.

12-6 Let $\{k_s : s \in S\}$ be a set of distinct eigenvalues of A. If x_s is an eigenvector of A corresponding to k_s, then the set $\{x_s : s \in S\}$ is linearly independent in X. (Hint: Induction)

12-7 Let (k_n) be a sequence of eigenvalues of A. If $k_n \to k$ in $\mathbf{K}$, then k is an approximate eigenvalue of A. However, an approximate eigenvalue of A may not be a limit of a sequence of eigenvalues of A.

12-8 If $k \in \sigma_a(A)$, then $|k| \leq \inf_{n=1,2,\ldots} \|A^n\|^{1/n}$. Further, $\sigma_a(A)$ is a compact subset of $\mathbf{K}$. (Hint: If $k \in \sigma_a(A)$, then $|k| \leq \|A\|$ and $k^n \in \sigma_a(A^n)$, since $A^n - k^n I = (A^{n-1} + kA^{n-2} + \cdots + k^{n-2}A + k^{n-1}I)(A - kI)$.)

12-9 Let X be a Banach space and assume that $\|A\| < 1$. Let $y \in X$ Then there is a unique $x \in X$ such that $x = y + A(x)$. Further, if $x_0 \in X$ and we let $x_n = y + A(x_{n-1})$ for $n = 1, 2, \ldots$, then $x_n \to x$. In fact,

$$\|x_n - x\| \leq \|A\|^n \|x_0 - x\| \quad \text{for } n = 0, 1, 2, \ldots.$$

If, in particular, $x_0 = y$, then

$$\|x_n - x\| \le \frac{\|A\|^{n+1}\|y\|}{1 - \|A\|} \quad \text{for } n = 0, 1, 2, \ldots .$$

12-10 Let $X = C([0,1])$ and A denote a Fredholm integral operator with continuous kernel $k(.,.)$. Let p be a positive integer and $k^{(p)}$ denote the pth iterated kernel. For every $\mu \in \mathbf{K}$ with $|\mu| < 1/\|k^{(p)}\|_\infty^{1/p}$, the series $\sum_{n=1}^{\infty} \mu^n k^{(n)}(s,t)$ converges uniformly and absolutely for $(s,t) \in [0,1]\times[0,1]$ to a continuous function $h_\mu(.,.)$ on $[0,1]\times[0,1]$ and

$$(I - \mu A)^{-1} = I + B_\mu,$$

where B_μ is the Fredholm integral operator with the reciprocal kernel $h_\mu(.,.)$. (Hint: Replace A by μA in 12.4.)

12-11 Let $X = C([0,1])$, and $k(s,t) = st,\ s,t \in [0,1]$. For $n = 2, 3, \ldots$, the nth iterated kernel is $k^{(n)}(s,t) = st/3^{n-1}, s,t \in [0,1]$. For $|\mu| < 3$, the reciprocal kernel is $h_\mu(s,t) = 3\mu st/(3-\mu), s,t \in [0,1]$. For every $y \in X$, the unique solution of

$$x(s) - \mu s \int_0^1 t\, x(t)\, dm(t) = y(s), \quad s \in [0,1],$$

is given by

$$x(s) = y(s) + \frac{3\mu s}{3-\mu} \int_0^1 t\, y(t)\, dm(t), \quad s \in [0,1].$$

12-12 **(Volterra kernel, Volterra operator)** Consider a measurable function $k(.,.)$ on $[0,1]\times[0,1]$ such that $k(s,t) = 0$ if $0 \le s \le t \le 1$. Assume that $|k(s,t)| \le \alpha$ for all $s,t \in [0,1]$.

(a) If $k^{(n)}(.,.)$ denotes the nth iterated kernel, then $k^{(n)}(s,t) = 0$ if $0 \le s \le t \le 1$ and $|k^{(n)}(s,t)| \le \alpha^n (s-t)^{n-1}/(n-1)!$ if $0 \le t < s \le 1$, for all $n = 1, 2, \ldots$. (Hint: Induction on n)

(b) Let $X = L^p([0,1]), 1 \le p \le \infty$, and A denote the integral operator with kernel $k(.,.)$. Then $\|A^n\| \le \alpha^n/n!$ for all $n = 1, 2, \ldots$. Further, if $0 \ne \mu \in \mathbf{K}$, then $I - \mu A$ is invertible and $(I - \mu A)^{-1} = \sum_{n=0}^{\infty} \mu^n A^n$.

12-13 Let X be a Banach space, A invertible in $BL(X)$, $B \in BL(X)$ and $\epsilon = \|(A-B)A^{-1}\| < 1$.

(a) If $A(x) = y$ and $B(u) = v$, then

$$\|x - u\| \leq \frac{\|A^{-1}\|}{1 - \epsilon} \min\{\epsilon\|v\| + (1 - \epsilon)\|y - v\|, \epsilon\|y\| + \|y - v\|\}.$$

(b) If $\epsilon \leq \delta/(1 + \delta)$ for some $\delta < 1$, then B is invertible and

$$\|(A - B)B^{-1}\| \leq \delta,\ \|A^{-1}\| \leq \frac{\|B^{-1}\|}{1 - \delta},\ \|A^{-1} - B^{-1}\| \leq \frac{\|B^{-1}\|\delta}{1 - \delta}.$$

12-14 Let X be a Banach space, $A, B \in BL(X)$, A invertible and $\|[A^{-1}(A - B)]^p\| < 1$ for some positive integer p. Then B is invertible and

$$B^{-1} = \left(\sum_{n=0}^{\infty} [A^{-1}(A - B)]^n\right) A^{-1}.$$

12-15 **(Gershgorin's theorem).** If $M = (k_{i,j})$ is an $n \times n$ matrix, then $\sigma_e(M)$ is contained in $\bigcup_{j=1}^{n} E_j$ as well as in $\bigcup_{i=1}^{n} F_i$, where $E_j = \{k \text{ in } \mathbf{K} : |k - k_{j,j}| \leq \sum_{i \neq j} |k_{i,j}|\}$ and $F_i = \{k \text{ in } \mathbf{K} : |k - k_{i,i}| \leq \sum_{j \neq i} |k_{i,j}|$. (Hint: Let $k \in \sigma_e(M), k \neq k_{1,1}, \ldots k_{n,n}$ and $D = \operatorname{diag}(k_{1,1}, \ldots, k_{n,n})$. If $C = (D - M)(D - kI)^{-1}$, then $\|C\|_1 \geq 1$ and if $C = (D - kI)^{-1}(D - M)$, then $\|C\|_\infty \geq 1$ by 12.5(a) and Problem 12-14 with $p = 1$.)

12-16 Let $M = (k_{i,j}), i, j = 1, \ldots, n$, be a **diagonally dominant** matrix (that is, $|k_{i,i}| > \sum_{j \neq i} |k_{i,j}|$ for each $i = 1, \ldots, n$, or $|k_{j,j}| > \sum_{i \neq j} |k_{i,j}|$ for each $j = 1, \ldots, n$). Then M is invertible. (Hint: Problem 12-15)

12-17 Let E be a nonempty compact subset of $\mathbf{K}$. There is an infinite diagonal matrix $M \in BL(\ell^p), 1 \leq p \leq \infty$, such that $\sigma(M) = E$. (Hint: Consider a countable dense subset of E.)

12-18 Let $X \neq \{0\}$ and $P \in BL(X)$ be a projection. If $P = 0$, then $\sigma(P) = \{0\}$, if $P = I$, then $\sigma(P) = \{1\}$, and if $0 \neq P \neq I$, then $\sigma(P) = \{0, 1\}$. In fact, if $0 \neq k \neq 1$, then

$$(P - kI)^{-1} = \frac{P - (1 - k)I}{k(1 - k)}.$$

12-19 Let $X = \ell^p, 1 \leq p \leq \infty$, or c, or c_0.

(a) For $x \in X$, let $A(x) = (0, x(1), x(2)/2, x(3)/3, \ldots)$. Then

$$\sigma_e(A) = \emptyset \quad \text{and} \quad \sigma_a(A) = \{0\} = \sigma(A).$$

(b) Consider the **left shift operator** on X given by

$$A(x) = (x(2), x(3), \ldots), \quad x = (x(1), x(2), \ldots).$$

Then

$$\sigma_a(A) = \{k \in \mathbf{K} : |k| \leq 1\} = \sigma(A).$$

If $X = \ell^p, 1 \leq p < \infty$ or $X = c_0$, then $\sigma_e(A) = \{k \in \mathbf{K} : |k| < 1\}$. If $X = c$, then $\sigma_e(A) = \{k \in \mathbf{K} : |k| < 1\} \cup \{1\}$. If $X = \ell^\infty$, then $\sigma_e(A) = \sigma(A)$.

12-20 Let A and B denote the left and the right shift operators on ℓ^2, respectively. Let $X = \ell^p \times \ell^2$ and for $(x, y) \in X$, let

$$C(x, y)) = (A(x), (B + 3I)(y)).$$

Then $\sigma(C) = \{k : |k| \leq 1 \text{ or } |k - 3| \leq 1\}$ and $\sigma_e(C) = \{k : |k - 3| < 1\}$.

12-21 Let $X = \mathbf{R}^2$ with the norm $\| \ \|_2$ and for a fixed $t \in [0, 2\pi)$, let

$$M_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Then $\|(M_t)^n\|_2 = 1$ for all $n = 1, 2, \ldots, \sigma(M_0) = \{1\}, \sigma(M_\pi) = \{-1\}$ and $\sigma(M_t) = \emptyset$ if $t \in (0, \pi) \cup (\pi, 2\pi)$.

12-22 If k belongs to the boundary of $\sigma(A)$, then $k \in \sigma_a(A)$. If X is a Banach space over $\mathbf{C}$, then $\sigma_a(A) \neq \emptyset$ and

$$\sup\{|k| : k \in \sigma_a(A)\} = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

12-23 Let X be a Banach space over $\mathbf{C}$, $A, B \in BL(X)$ and $AB = BA$. Then $r_\sigma(AB) \leq r_\sigma(A) r_\sigma(B)$. This may not hold if $AB \neq BA$.

12-24 For $x = (x(1), x(2), \ldots) \in \ell^1$, let

$$A(x) = (0, x(1), 2x(2), x(3), 2x(4), \ldots).$$

Then the sequence $(\|A^n\|^{1/n})$ is not monotone. Also, $\lim_{n\to\infty} \|A^n\|^{1/n} = \sqrt{2}$.

12-25 If $\mathbf{K} = \mathbf{C}$, then $\sigma(A)$ is nonempty, $\lim_{n\to\infty} \|A^n\|^{1/n}$ exists and is less than or equal to $r_\sigma(A)$.

Chapter IV

Spaces of Bounded Linear Functionals

This chapter deals with the duality between a normed space and the space of all bounded linear functionals on it. Section 13 introduces the dual of a normed space and the transpose of a bounded linear map on it. The zero spaces and the range spaces of a bounded linear map and its transpose are related. In Section 14, the duals of the important normed spaces $L^p([a,b])$ and $C([a,b])$ are described. Some weaker concepts of convergence in a normed space and in its dual are considered in Section 15. A reflexive normed space is one for which the canonical embedding in the second dual is surjective. This concept is introduced in Section 16 and its connections with weak convergence and with the geometry of the normed space are explored. In particular, an elementary proof of Eberlein's theorem is given.

13 Duals and Transposes

Let X be a normed space over $\mathbf{K}$. The linear space X' of all continuous linear functionals on X with the norm given by

$$\|x'\| = \sup\{|x'(x)| : x \in X, \|x\| \le 1\}, \quad x' \in X',$$

is called the **normed dual** or simply the **dual** of X. The very definition of a function shows that $x_1' = x_2'$ in X' if and only if $x_1'(x) = x_2'(x)$ for all $x \in X$. On the other hand, the consequence 7.10(a) of the Hahn-Banach extension theorem shows that $x_1 = x_2$ in X if and only if $x'(x_1) = x'(x_2)$ for all $x' \in X'$ and that

$$\|x\| = \sup\{|x'(x)| : x' \in X', \|x'\| \le 1\}, \quad x \in X$$

in analogy with the definition of $\|x'\|$ stated above. This interchangeability between X and X' explains the nomenclature '(normed) dual' for X'.

We have already studied some aspects of the dual X'. For example, we have seen in 8.2(c) that the dual X' of every normed space X is a Banach space. Further, we have shown in Section 8 that every normed space X can be canonically embedded in the dual $(X')'$ of X'. We have used this fact to obtain the completion of X. In Section 9, we have proved that a subset E of X is bounded in X if and only if $x'(E)$ is a bounded subset of $\mathbf{K}$ for every $x' \in X'$. In this way, a question about a subset of X can be reduced to a question about subsets of $\mathbf{K}$ by means of the dual space X'.

Here are two additional facts about X and X'.

13.1 Theorem

Let X be a normed space.

(a) Let X_0 be a dense subspace of X. For $x' \in X'$, let $F(x')$ denote the restriction of x' to X_0. Then the map F is a linear isometry from X' onto X_0'.

(b) If X' is separable, then so is X.

Proof:

(a) Let $x' \in X'$. It is clear that $F(x')$ belongs to X_0', $\|F(x')\| = \|x'\|$ and that the map F is linear. By 8.3(a), every $x_0' \in X_0'$ has a unique norm-preserving extension to X. Hence F is surjective.

(b) Let X' be separable. If $X = \{0\}$, there is nothing to prove. Let $X \neq \{0\}$. Just after defining the separability of a metric space in Section 3, we have seen that every subset of a separable metric space is separable. Hence let $\{x_1', x_2', \ldots\}$ be a countable dense subset of $\{x' \in X' : \|x'\| = 1\}$. By 6.6,

$$\|x_n'\| = \sup\{|x_n'(x)| : x \in X, \|x\| = 1\}$$

for $n = 1, 2, \ldots$, so that there is some $x_n \in X$ with

$$\|x_n\| = 1 \quad \text{and} \quad |x_n'(x_n)| > \frac{1}{2}.$$

Let $D = \{k_1 x_1 + \cdots + k_m x_m : m = 1, 2, \ldots, \operatorname{Re} k_j \text{ and } \operatorname{Im} k_j$ are rational for $j = 1, \ldots, m\}$. Then D is a countable subset of X, and it is clearly dense in the subspace $Y = \operatorname{span}\{x_1, x_2, \ldots\}$ of X. Now we show that Y is dense in X. Let $x' \in X'$ be such that $x'(x_n) = 0$ for all $n = 1, 2, \ldots$. By 7.10(b), it is enough to show that $x' = 0$. If $x' \neq 0$, then $f = x'/\|x'\|$ has norm 1 and hence $\|x_n' - f\| < 1/2$ for some n. But since $x'(x_n) = 0$ and $\|x_n\| = 1$, we see that

$$\frac{1}{2} < |x_n'(x_n)| = |x_n'(x_n) - f(x_n)| \leq \|x_n' - f\| < \frac{1}{2},$$

which is a contradiction. Thus D is a countable dense subset of X, showing that X is separable. $\square$

If X is a finite dimensional normed space, then its dual X' has the same dimension as X. For if $\{x_1, \ldots, x_m\}$ is an ordered basis for X, then by 7.10(c), there are $x_1', \ldots, x_m'$ in X' such that

$$x_j'(x_i) = \delta_{i,j}, \quad 1 \leq i, j \leq m.$$

It is easy to see that $\{x_1', \ldots, x_m'\}$ is an ordered basis for X', and is uniquely determined by the given basis $\{x_1, \ldots, x_m\}$ for X. It is called the **dual basis** for X' relative to the basis $\{x_1, \ldots, x_m\}$ for X. Since the finite dimensions of X and X' are equal, 6.3(b) shows that X' is linearly homeomorphic to X. In general, X' is not linearly isometric to X. This will be clear from 13.3(a). Problem 13-5 is also noteworthy in this regard.

13.2 Theorem

Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. For a fixed $y \in \ell^q$, let

$$f_y(x) = \sum_{j=1}^{\infty} x(j)y(j), \qquad x \in \ell^p.$$

Then $f_y \in (\ell^p)'$ and $\|f_y\| = \|y\|_q$. The map $F : \ell^q \to (\ell^p)'$ defined by

$$F(y) = f_y, \qquad y \in \ell^q,$$

is a linear isometry from ℓ^q into $(\ell^p)'$

If $1 \le p < \infty$, then F is surjective. In fact, if $f \in (\ell^p)'$ and $y = (f(e_1), f(e_2), \ldots)$, then $y \in \ell^q$ and $f = F(y)$.

Proof:
Let $y \in \ell^q$. For $x \in \ell^p$, we have

$$\sum_{j=1}^{\infty} |x(j)y(j)| \le \|x\|_p \|y\|_q.$$

This is obvious if $p = 1$ or ∞ and follows by letting $n \to \infty$ in Hölder's inequality (3.1(a)) if $1 < p < \infty$. Hence f_y is well-defined, linear and $\|f_y\| \le \|y\|_q$. Now we prove $\|y\|_q \le \|f_y\|$. If $y = 0$, there is nothing to prove. Assume, therefore, that $y \ne 0$.

Let $p = 1$, so that $q = \infty$. Then for $j = 1, 2, \ldots,$

$$|y(j)| = |f_y(e_j)| \le \|f_y\| \, \|e_j\|_1 = \|f_y\|,$$

showing that $\|y\|_\infty \le \|f_y\|$.

Next, let $1 < p \le \infty$, so that $1 \le q < \infty$. Let

$$x(j) = \operatorname{sgn} y(j) |y(j)|^{q-1},$$

so that $x(j)y(j) = |y(j)|^q$ for each $j = 1, 2, \ldots$. If $p = \infty$, then $q = 1, \|x\|_\infty \le 1$ and $\|y\|_1 = f_y(x) \le \|f_y\|$. If $1 < p < \infty$, then $|x(j)|^p = |y(j)|^q$ for each $j = 1, 2, \ldots$, as $pq - p = q$. Hence $x \in \ell^p$ and

$$\sum_{j=1}^{\infty} |y(j)|^q = \sum_{j=1}^{\infty} x(j)y(j) = f_y(x).$$

But

$$f_y(x) \le \|f_y\| \, \|x\|_p = \|f_y\| \left(\sum_{j=1}^{\infty} |y(j)|^q \right)^{1/p},$$

so that $(\sum_{j=1}^{\infty}|y(j)|^q)^{1-1/p} \leq \|f_y\|$, that is, $\|y\|_q \leq \|f_y\|$, again since $1-1/p = 1/q$. Thus $\|f_y\| = \|y\|_q$. If we let $F(y) = f_y$, $y \in \ell^q$, then F is a linear isometry from ℓ^q into $(\ell^p)'$ for $1 \leq p \leq \infty$.

Let $1 \leq p < \infty$. To show that F is surjective, consider $f \in (\ell^p)'$ and let $y = (f(e_1), f(e_2), \ldots)$.

If $p = 1$, then $|y(j)| \leq \|f\|\,\|e_j\|_1 = \|f\|$, so that $y \in \ell^\infty$. Let $1 < p < \infty$ and for $n = 1, 2, \ldots$, define $y_n = (y(1), \ldots, y(n), 0, 0, \ldots)$. Then $y_n \in \ell^q$. By what we have seen above, $\|y_n\|_q \leq \|f_{y_n}\|$. Now $\|f_{y_n}\| = \sup\{|\sum_{j=1}^{n} x(j)y(j)| : x \in \ell^p, \|x\|_p \leq 1\}$. Consider $x \in \ell^p$ with $\|x\|_p \leq 1$ and define $x_n = (x(1), \ldots, x(n), 0, 0, \ldots)$. Then x_n belongs to ℓ^p, $\|x_n\|_p \leq \|x\|_p \leq 1$ and

$$f(x_n) = \sum_{j=1}^{n} x(j)f(e_j) = \sum_{j=1}^{n} x(j)y(j) = f_{y_n}(x).$$

Thus $\|f_{y_n}\| \leq \|f\| = \sup\{|f(x)| : x \in \ell^p, \|x\|_p \leq 1\}$, so that

$$\left(\sum_{j=1}^{\infty}|y(j)|^q\right)^{1/q} = \lim_{n\to\infty}\|y_n\|_q \leq \limsup_{n\to\infty}\|f_{y_n}\| \leq \|f\| < \infty,$$

that is, $y \in \ell^q$.

Now let $x \in \ell^p$. Since $p < \infty$, we see that $x = \lim_{n\to\infty}\sum_{j=1}^{n} x(j)e_j$. Hence by the continuity and the linearity of f,

$$f(x) = \lim_{n\to\infty} f\left(\sum_{j=1}^{n} x(j)e_j\right) = \sum_{j=1}^{\infty} x(j)f(e_j) = \sum_{j=1}^{\infty} x(j)y(j) = f_y(x).$$

Thus $f = f_y$, that is, $F(y) = f$, showing that F is surjective. □

Since ℓ^1 is separable but ℓ^∞ is not (3.2), the preceding result shows that the dual of a separable normed space need not be separable. Thus the converse of 13.1(a) does not hold.

13.3 Corollary

Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$.

(a) The dual of $\mathbf{K}^n$ with the norm $\| \ \|_p$ is linearly isometric to $\mathbf{K}^n$ with the norm $\| \ \|_q$.

(b) The dual of c_{00} with the norm $\| \ \|_p$ is linearly isometric to ℓ^q.

(c) The dual of c_0 with the norm $\| \ \|_\infty$ is linearly isometric to ℓ^1.

Proof:

(a) Replace the summation $\sum_{j=1}^{\infty}$ by the summation $\sum_{j=1}^{n}$ in the proof of 13.2.

(b) If $1 \le p < \infty$, then c_{00} is a dense subspace of ℓ^p, so that the dual of c_{00} is linearly isometric to ℓ^q by 13.1(a) and 13.2.

Let $p = \infty$, so that $q = 1$. Consider $y \in \ell^1$. Define

$$f_y(x) = \sum_{j=1}^{\infty} x(j)y(j), \quad x \in c_{00}.$$

Then, as in the proof of 13.2, $f_y \in (c_{00})'$ and $\|f_y\| \le \|y\|_1$. For $n = 1, 2, \ldots$, define

$$x_n(j) = \begin{cases} \operatorname{sgn} y(j), & \text{if } 1 \le j \le n \\ 0, & \text{if } j > n. \end{cases}$$

Then $x_n \in c_{00}$, $\|x_n\|_\infty \le 1$ and

$$f_y(x_n) = \sum_{j=1}^{\infty} x_n(j)y(j) = \sum_{j=1}^{n} |y(j)|.$$

Hence $\sum_{j=1}^{n} |y(j)| \le \|f_y\|$ for all $n = 1, 2, \ldots$. Thus $\|f_y\| = \sum_{j=1}^{\infty} |y(j)| = \|y\|_1$, and the map $F : \ell^1 \to (c_{00})'$ given by $F(y) = f_y$ is a linear isometry from ℓ^1 into $(c_{00})'$. To prove F is surjective, consider f in $(c_{00})'$ and let $y = (f(e_1), f(e_2), \cdots)$. By defining x_n for $n = 1, 2, \ldots$ as above, we see that

$$\|f\| \ge f(x_n) = \sum_{j=1}^{n} x_n(j)y(j) = \sum_{j=1}^{n} |y(j)|, \quad n = 1, 2, \ldots,$$

so that $y \in \ell^1$. If $x \in c_{00}$, then $x = \sum_{j=1}^n x(j)e_j$ for some n and hence

$$f(x) = \sum_{j=1}^n x(j)f(e_j) = \sum_{j=1}^n x(j)y(j) = f_y(x).$$

Thus $f = f_y$, that is, $F(y) = f$, showing that f is surjective.

(c) Since c_{00} is dense in c_0, we use 13.1(a) and (b) above. □

We remark that the linear isometry $F : \ell^1 \to (\ell^\infty)'$ given in 13.2 is not surjective. This can be seen as follows. Note that c_0 is a closed subspace of ℓ^∞ and if $a = (1, 1, \ldots)$, then $a \notin c_0$. By the consequence 7.10(b) of the Hahn-Banach extension theorem, there is some $f \in (\ell^\infty)'$ such that $f(x) = 0$ for every $x \in c_0$ and $f(a) \neq 0$. (For example, we can take f to be a Banach limit, as discussed in Section 7.) Were $f = F(y)$ for some $y \in \ell^1$, then we would have $y(j) = F(y)(e_j) = f(e_j) = 0$ for all $j = 1, 2, \ldots$, that is, $y = 0$, so that $f = F(0) = 0$, a contradiction. In fact, there is no homeomorphism from ℓ^1 onto $(\ell^\infty)'$, because ℓ^1 is separable but ℓ^∞ is not (3.2) and this would contradict 13.1(b).

Having considered the dual of a normed space X, we now turn to a similar concept for a bounded linear operator on X.

Let X and Y be normed spaces and $F \in BL(X, Y)$. Define a map $F' : Y' \to X'$ as follows. For $y' \in Y'$ and $x \in X$, let

$$F'(y')(x) = y'(F(x)).$$

Clearly, F' is linear. It is continuous since $\|F'(y')\| \leq \|y'\| \, \|F\|$ for all $y' \in Y'$. The map F' is called the **transpose** of F. The reason for this nomenclature will be clear from the following examples.

13.4 Examples

(a) Let X be a normed space of dimension n and $x_1, \ldots, x_n$ constitute an ordered basis for X. Let Y be a normed space of dimension m and $y_1, \ldots, y_m$ constitute an ordered basis for Y. Let

$F \in BL(X, Y)$ be defined by an $m \times n$ matrix $M = (k_{i,j})$ with respect to these ordered bases, that is,

$$F(x_j) = k_{1,j}y_1 + \cdots + k_{m,j}y_m, \qquad j = 1, \ldots, n.$$

Let $x_1', \ldots, x_n'$ constitute the ordered dual basis for X' relative to the ordered basis $x_1, \ldots, x_n$ for X, and $y_1', \ldots, y_m'$ constitute the ordered dual basis for Y' relative to the ordered basis $y_1, \ldots, y_m$ for Y. Let an $n \times m$ matrix $(k_{i,j}')$ define the linear map $F' : Y' \to X'$, with respect to these bases, that is,

$$F'(y_j') = k_{1,j}'x_1' + \cdots + k_{n,j}'x_n', \qquad j = 1, \ldots, m.$$

Then

$$k_{i,j}' = F'(y_j')(x_i) = y_j'(F(x_i)) = y_j'(k_{1,i}y_1 + \cdots + k_{m,i}y_m) = k_{j,i}$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Thus the matrix $(k_{i,j}')$ is the transpose M^t of the matrix $M = (k_{i,j})$.

(b) Let $1 \le p < \infty, 1 \le r < \infty$ and $F \in BL(\ell^p, \ell^r)$. Then the infinite matrix $M = (k_{i,j})$ with $k_{i,j} = F(e_j)(i)$ defines the bounded linear map F, as we have seen in 6.5(c).

Now $F' \in BL((\ell^r)', (\ell^p)')$. Let $1/p + 1/q = 1$ and $1/r + 1/s = 1$. By 13.2(b), the maps

$$F_1 : \ell^q \to (\ell^p)' \quad \text{and} \quad F_2 : \ell^s \to (\ell^r)'$$

given by

$$F_1(y)(x) = \sum_{j=1}^{\infty} x(j)y(j) = F_2(y)(x)$$

are surjective linear isometries since $p < \infty$ and $r < \infty$. Let the map $G : \ell^s \to \ell^q$ be defined by

$$G = F_1^{-1}F'F_2.$$

Schematically,

$$\begin{array}{ccc}
(\ell^r)' & \xrightarrow{F'} & (\ell^p)' \\
F_2 \uparrow & & \uparrow F_1 \\
\ell^s & \xrightarrow{G} & \ell^q
\end{array}$$

Note that if $z \in \ell^q$, then

$$z(i) = F_1(z)(e_i) \quad \text{for all } i = 1, 2, \ldots.$$

Fix $y \in \ell^s$. Then $G(y) \in \ell^q$ and

$$\begin{aligned}
G(y)(i) &= F_1(G(y))(e_i) = F'(F_2(y))(e_i) \\
&= F_2(y)(F(e_i)) = \sum_{j=1}^{\infty} F(e_i)(j)y(j) = \sum_{j=1}^{\infty} k_{j,i}y(j)
\end{aligned}$$

for all $i = 1, 2, \ldots$. Thus the infinite matrix $M^t = (k_{j,i})$, which is the transpose of the matrix $M = (k_{i,j})$, defines the bounded linear map $G : \ell^s \to \ell^q$, which can be identified with F'.

For examples of transposes of some bounded linear maps on $L^p([a, b])$ and on $C([a, b])$, we refer to Problems 14-9, 14-17 and 14-18.

13.5 Theorem

Let X, Y and Z be normed spaces.

(a) Let F_1 and F_2 be in $BL(X, Y)$, and $k \in \mathbf{K}$. Then

$$(F_1 + F_2)' = F_1' + F_2', \quad (kF_1)' = kF_1'.$$

Let $F \in BL(X, Y)$ and $G \in BL(Y, Z)$. Then

$$(GF)' = F'G'.$$

(b) Let $F \in BL(X, Y)$. Then

$$\|F'\| = \|F\| = \|F''\| \quad \text{and} \quad F''J_X = J_Y F,$$

where J_X and J_Y are the canonical embeddings of X and Y into X and Y'', respectively.

Proof:

(a) For $y' \in Y'$ and $x \in X$,

$$\begin{aligned}(F_1 + F_2)'(y')(x) &= y'((F_1 + F_2)(x)) = y'(F_1(x)) + y'(F_2(x)) \\ &= F_1'(y')(x) + F_2'(y')(x) = (F_1' + F_2')(y')(x).\end{aligned}$$

Hence $(F_1 + F_2)' = F_1' + F_2'$. Similarly, $(kF_1)' = kF_1'$.

For $z' \in Z'$ and $x \in X$,

$$(GF)'(z')(x) = z'(GF(x)) = F'(z'G)(x) = F'(G'(x')(x)$$

Hence $(GF)' = F'G'$.

(b) By 7.10(a),

$$\begin{aligned}\|F'\| &= \sup\{\|F'(y')\| : y' \in Y', \|y'\| \le 1\} \\ &= \sup\{|y'(F(x))| : y' \in Y', \|y'\| \le 1, x \in X, \|x\| \le 1\} \\ &= \sup\{\|F(x)\| : x \in X, \|x\| \le 1\}.\end{aligned}$$

Hence $\|F'\| = \|F\|$. In turn, $\|F''\| = \|(F')'\| = \|F'\|$.

For $x'' \in X''$ and $y' \in Y'$, we have $F''(x'')(y') = x''(F'(y'))$. In particular, if $x \in X$ and $x'' = J_X(x)$, then

$$\begin{aligned}F''(J_X(x))(y') &= J_X(x)(F'(y')) = F'(y')(x) \\ &= y'(F(x)) = J_Y(F(x))(y')\end{aligned}$$

for every $y' \in Y'$. Hence $F''J_X = J_Y F$. Schematically,

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "J_X"'] & Y \arrow[d, "J_Y"] \\ X'' \arrow[r, "F''"] & Y'' \end{tikzcd}$$

□

13.6 Example

Let $X = c_{00} = Y$, with the norm $\|\ \|_\infty$. Then by 13.3(b), X' is linearly isometric to ℓ^1, and by 13.2, X'' is linearly isometric to ℓ^∞. The completion of c_{00} (that is, the closure of $J(c_{00})$ in $(c_{00})''$) is linearly isometric to c_0. Let $F \in BL(c_{00})$. Then F'' can be thought of as a norm-preserving linear extension of F to ℓ^∞.

There is a dichotomy between the zero spaces and the range spaces of F and F', which we now explore.

13.7 Theorem

Let X and Y be normed spaces and $F \in BL(X, Y)$. Then

(a) $Z(F) = \{x \in X : x'(x) = 0 \text{ for all } x' \in R(F')\}$.

(b) $Z(F') = \{y' \in Y' : y'(y) = 0 \text{ for all } y \in R(F)\}$.

In particular, F' is one-to-one if and only if $R(F)$ is dense in Y.

(c) $R(F) \subset \{y \in Y : y'(y) = 0 \text{ for all } y' \in Z(F')\}$,

where equality holds if and only if $R(F)$ is closed in Y.

(d) $R(F') \subset \{x' \in X' : x'(x) = 0 \text{ for all } x \in Z(F)\}$,

where equality holds if X and Y are Banach spaces and $R(F)$ is closed in Y.

Proof:

(a) Let $x \in X$. By 7.10(a), $F(x) = 0$ if and only if $F'(y')(x) = y'(F(x)) = 0$ for every $y' \in Y'$.

(b) Let $y' \in Y'$. Then $F'(y') = 0$ if and only if $y'(F(x)) = F'(y')(x) = 0$ for every $x \in X$.

Now F' is one-to-one, that is, $Z(F') = \{0\}$ if and only if $y' = 0$ whenever $y'(y) = 0$ for every $y \in R(F)$. By 7.10(b), this happens if and only if the closure of $R(F)$ is Y, that is, $R(F)$ is dense in Y.

(c) Let $y \in R(F)$ and $y = F(x)$ for some $x \in X$. If $y' \in Z(F')$, then $y'(y) = y'(F(x)) = F'(y')(x) = 0$. Hence

$$R(F) \subset \{y \in Y : y'(y) = 0 \text{ for all } y' \in Z(F')\}.$$

If equality holds in this inclusion, then $R(F)$ is closed in Y since $R(F) = \bigcap\{Z(y') : y' \in Z(F')\}$, and each $Z(y')$ is a closed subspace of Y. Conversely, assume that $R(F)$ is closed in Y. Let $y_0 \notin R(F)$. Then by 7.10(b), there is some $y' \in Y'$ such that $y'(y_0) \neq 0$, but $y'(y) = 0$ for every $y \in R(F)$. In particular, $F'(y')(x) = y'(F(x)) = 0$ for all $x \in X$, that is, $y' \in Z(F')$. This shows that $y_0 \notin \{y \in Y : y'(y) = 0$ for all $y' \in Z'(F')\}$. Thus equality holds in the inclusion mentioned above.

(d) Let $x' \in R(F')$ and $x' = F'(y')$ for some $y' \in Y'$. If $x \in Z(F)$, then $x'(x) = F'(y')(x) = y'(F(x)) = y'(0) = 0$. Hence

$$R(F') \subset \{x' \in X' : x'(x) = 0 \text{ for all } x \in Z(F)\}.$$

Now assume that $R(F)$ is closed in Y, and that X and Y are Banach spaces. We show that equality holds in the inclusion stated above. Let $x' \in X'$ be such that $x'(x) = 0$ whenever $F(x) = 0$. We must find $y' \in Y'$ such that $F'(y') = x'$, that is, $y'(F(x)) = x'(x)$ for every $x \in X$. Define $g : R(F) \to \mathbf{K}$ by $g(y) = x'(x)$, if $y = F(x)$. Since $x'(x) = 0$ for all $x \in Z(F)$, it is easy to check that g is well-defined and linear. Also, the map $F : X \to R(F)$ is linear, continuous and surjective, where X is a Banach space and so is the closed subspace $R(F)$ of the Banach space Y. Hence by the open mapping theorem (10.6) and by 10.4, there is some $\gamma > 0$ such that for every $y \in R(F)$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma\|y\|$, so that

$$|g(y)| = |x'(x)| \leq \|x'\| \, \|x\| \leq \gamma\|x'\| \, \|y\|.$$

This shows that g is a continuous linear functional on $R(F)$. By the Hahn-Banach extension theorem 7.8, there is some $y' \in Y'$ such that $y'_{|R(F)} = g$. Then $F'(y')(x) = y'(F(x)) = g(F(x)) = x'(x)$ for every $x \in X$, as desired. □

It follows from 13.7(a) that if $R(F')$ is dense in X', then F is injective. The converse, however, does not hold. (Compare 13.7(b).) For example, let $X = Y = \ell^1$ and for $x \in X, F(x)(j) = x(j)/j, j = 1, 2, \ldots$. Then $F \in BL(X)$ and F is clearly injective. Now consider $x_0' \in X'$ defined by

$$x_0'(x) = \sum_{j=1}^{\infty} x(j), \quad x = (x(1), x(2), \ldots) \in X.$$

For $x' \in X'$, we have

$$\begin{aligned} \|F'(x') - x_0'\| &= \sup\{|F'(x')(x) - x_0'(x)| : x \in X, \|x\| \le 1\} \\ &\ge \sup\{|F'(x')(e_j) - x_0'(e_j)| : j = 1, 2, \ldots\}, \end{aligned}$$

where $e_j = (0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 occurring only in the jth place. Since for $j = 1, 2, \ldots$,

$$\begin{aligned} |F'(x')(e_j) - x_0'(e_j)| &= |x'(F(e_j)) - 1| = |x'(\tfrac{e_j}{j}) - 1| \\ &\ge 1 - \frac{|x'(e_j)|}{j} \ge 1 - \frac{\|x'\|}{j}, \end{aligned}$$

we see that $\|F'(x') - x_0\| \ge 1$ for every $x' \in X'$. Thus x_0' does not belong to the closure of $R(F')$.

13.8 Corollary

Let X be a normed space and $A \in BL(X)$. Then $\sigma(A') \subset \sigma(A)$.

If X is a Banach space, then

$$\sigma(A) = \sigma_a(A) \bigcup \sigma_e(A') = \sigma(A').$$

Proof:

Let $k \in \mathbf{K}$ be such that $A - kI$ is invertible. If $(A - kI)B = I = B(A - kI)$ for some $B \in BL(X)$, then by 13.5(a), $B'(A' - kI) = I = (A' - kI)B'$, showing that $A' - kI$ is invertible. Thus $\sigma(A') \subset \sigma(A)$.

Let X be a Banach space. By 12.1(b), $k \in \sigma(A)$ if and only if either $A - kI$ is not bounded below or $R(A - kI)$ is not dense in X.

To say $A - kI$ is not bounded below is to say that $k \in \sigma_a(A)$. Also, by 13.7(b), $R(A - kI)$ is dense in X if and only if $A' - kI$ is one-to-one, that is, $k \notin \sigma_e(A')$. Thus $\sigma(A) = \sigma_a(A) \bigcup \sigma_e(A')$.

Finally, to conclude $\sigma(A) = \sigma(A')$, it is enough to show that $\sigma_a(A) \subset \sigma(A')$. Let $k \notin \sigma(A')$, that is, $A' - kI$ is invertible. If $x \in X$, then by 7.10(a), there is some $x' \in X'$ such that $x'(x) = \|x\|$ and $\|x'\| = 1$, so that

$$\begin{aligned} \|x\| = |x'(x)| &= |(A' - kI)(A' - kI)^{-1}(x')(x)| \\ &= |(A' - kI)^{-1}(x')((A - kI)(x))| \\ &\leq \|(A' - kI)^{-1}\| \, \|A(x) - kx\|. \end{aligned}$$

Thus $A - kI$ is bounded below, that is, $k \notin \sigma_a(A)$. □

This result is a consequence of the fact that the range of a map $F \in BL(X, Y)$ is dense in Y if and only if its transpose F' is one-to-one (13.7(b)). We now give a necessary and sufficient condition for the range of F to equal Y.

13.9 Theorem
Let X and Y be Banach spaces, and $F \in BL(X, Y)$. Then $R(F) = Y$ if and only if F' is bounded below.

Proof:
Let $R(F) = Y$. Since X and Y are Banach spaces, F is an open map by 10.6. Hence 10.4 shows that there is some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with

$$F(x) = y \quad \text{and} \quad \|x\| < \gamma\|y\|.$$

Let $y' \in Y'$. Then for every $y \in Y$,

$$|y'(y)| = |y'(F(x))| = |F'(y')(x)| \leq \|F'(y')\| \, \|x\| < \gamma\|F'(y')\| \, \|y\|.$$

Hence $\|y'\| < \gamma\|F'(y')\|$ for all $y' \in Y'$, showing that F' is bounded below.

Conversely, assume that F' is bounded below, that is,

$$\beta\|y'\| \leq \|F'(y')\|$$

for some $\beta > 0$ and all $y' \in Y'$. We first claim that for every $y \in Y$ with $\|y\| \leq 1$, there is some $x \in X$ such that,

$$\|y - F(x)\| < \frac{1}{2} \quad \text{and} \quad \|x\| \leq \frac{1}{\beta}.$$

Suppose for a moment that this is not the case. Then there is some $y_0 \in Y$ with $\|y_0\| \leq 1$ and $\|y_0 - F(x)\| \geq 1/2$ whenever $\|x\| \leq 1/\beta$. Then

$$U_Y(y_0, \frac{1}{2}) \bigcap \{F(x) : \|x\| \leq \frac{1}{\beta}\} = \emptyset,$$

where $U_Y(y_0, 1/2)$ is a nonempty open convex subset of Y and $\{F(x) : \|x\| \leq 1/\beta\}$ is a nonempty convex subset of Y. By 7.5, there exist some $f \in Y'$ and $t \in \mathbf{R}$ such that

$$\text{Re } f(y_0) < t \leq \text{Re } f(F(x))$$

for all $x \in X$ with $\|x\| \leq 1/\beta$. Let $y' = -f$ and $s = -t$. Then

$$\text{Re}\, y'(F(x)) \leq s < \text{Re}\, y'(y_0)$$

for all $x \in X$ with $\|x\| \leq 1/\beta$. Replacing x by $\text{sgn}\, y'(F(x))\, x$, we have

$$|y'(F(x))| \leq s < \text{Re}\, y'(y_0)$$

for all $x \in X$ with $\|x\| \leq 1/\beta$. (See Problem 7-1.) Again, replacing x by x/β, we have

$$|F'(y')(x)| = |y'(F(x))| \leq \beta s < \beta\, \text{Re}\, y'(y_0) \leq \beta\|y'\|$$

for all $x \in X$ with $\|x\| \leq 1$. Hence

$$\|F'(y')\| < \beta\|y'\| \leq \|F'(y')\|,$$

a contradiction. Hence our claim is justified.

Thus if we let

$$U = \{y \in Y : \|y\| < 1\} \quad \text{and} \quad V = \{F(x) : x \in X, \|x\| \le 1/\beta\},$$

then we have shown that

$$U \subset V + \frac{1}{2}U.$$

Let $y \in U$. By 10.1, there is a sequence (v_n) in V such that

$$y - \left(v_1 + \frac{v_2}{2} + \cdots + \frac{v_n}{2^{n-1}}\right) \in \frac{1}{2^n}U$$

for $n = 1, 2, \ldots$. If $v_n = F(x_n)$ with $\|x_n\| \le 1/\beta$, we see that

$$y = \sum_{n=1}^{\infty} \frac{v_n}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{F(x_n)}{2^{n-1}}.$$

Since

$$\beta \sum_{n=1}^{\infty} \left\|\frac{x_n}{2^{n-1}}\right\| \le \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} < \infty,$$

and X is a Banach space, 8.1 shows that the series $\sum_{n=1}^{\infty} x_n/2^{n-1}$ is summable to some x in X. As F is continuous, we have

$$F(x) = \sum_{n=1}^{\infty} \frac{F(x_n)}{2^{n-1}} = y.$$

Thus $\{y \in Y : \|y\| < 1\} \subset R(F)$. Since $R(F)$ is a subspace, it follows that $Y = R(F)$. □

Theorem 13.9 is perhaps the most significant result of this section. It is useful in establishing the existence of a solution of an operator equation $F(x) = y$, where X and Y are Banach spaces and F belongs to $BL(X, Y)$. To prove that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$, it is enough to prove an inequality involving the transpose F' of F : $\beta\|y'\| \le \|F'(y')\|$ for some $\beta > 0$ and all $y' \in Y'$. Compare 25.5(c).

We conclude this section with a consequence of 13.7 and 13.9.

13.10 Theorem (Closed range theorem of Banach, 1932)
Let X and Y be Banach spaces, and $F \in BL(X,Y)$. Then $R(F)$ is closed in Y if and only if $R(F')$ is closed in X'. In that case,

$$\begin{aligned} R(F) &= \{y \in Y : y'(y) = 0 \text{ for all } y' \in Z(F')\}, \\ R(F') &= \{x' \in X' : x'(x) = 0 \text{ for all } x \in Z(F)\}. \end{aligned}$$

Proof:
Let $R(F)$ be closed in Y. Then by 13.7(d),

$$R(F') = \bigcap_{x \in Z(F)} \{x' \in X' : x'(x) = 0\}.$$

Since the set $\{x' \in X' : x'(x) = 0\}$ is closed in X' for each $x \in Z(F)$, it follows that $R(F')$ is closed in X'.

Conversely, let $R(F')$ be closed in X'. Let Y_1 denote the closure of $R(F)$ in Y, and $F_1 : X \to Y_1$ be defined by $F_1(x) = F(x)$. Then $R(F_1) = R(F)$ is dense in Y_1, so that $F_1' : Y_1' \to X'$ is one-to-one by 13.7(b). We claim that

$$R(F_1') = R(F').$$

Let $y_1' \in Y_1'$. By the Hahn-Banach extension theorem 7.8, there is some $y' \in Y'$ such that $y'_{|Y_1} = y_1'$. Then for every $x \in X$,

$$F'(y')(x) = y'(F(x)) = y_1'(F(x)) = y_1(F_1(x)) = F_1'(y_1')(x).$$

Hence $F'(y') = F_1'(y_1')$, showing that $R(F_1') \subset R(F')$. On the other hand, let $y' \in Y'$. Then $y_1' = y'_{|Y_1} \in Y_1'$ and $F'(y') = F_1'(y_1')$ as above, showing that $R(F') \subset R(F_1')$.

Thus $R(F_1') = R(F')$ is a closed subspace of the Banach space X'. Now the linear map $F_1' : Y_1' \to R(F_1')$ is continuous and one-to-one, where Y_1' and $R(F_1')$ are Banach spaces. Therefore F_1' is a homeomorphism by 10.4. In particular, it is bounded below. Hence $F_1 : X \to Y_1$ is surjective by 13.9 and $R(F) = R(F_1) = Y_1$. Thus $R(F)$ equals its own closure Y_1, that is, $R(F)$ is closed. Finally,

$$R(F) = \{y \in Y : y'(y) = 0 \text{ for all } y' \in Z(F')\}$$

by 13.7(c). $\square$

Problems

13-1 Let X and Y be normed spaces. For $F \in BL(X,Y)$, we have

$$\|F\| = \sup\{|y'(F(x))| : x \in X, \|x\| \le 1, y' \in Y', \|y'\| \le 1\}.$$

13-2 Let Y be a subspace of a normed space X. For $x' \in X'$, let $F(x') = x'_{|Y}$. Then F is a surjective linear map from X' to Y' such that $\|F(x')\| \le \|x'\|$ for all $x' \in X'$. In fact, for every $y' \in Y'$, there is some $x' \in X'$ such that $F(x') = y'$ and $\|x'\| = \|y'\|$. (Hint: 7.8)

13-3 Let X be a normed space. For a subset E of X, the subset

$$E^a = \{x' \in X' : x'(x) = 0 \text{ for every } x \in E\}$$

of X' is called the **annihilator** of E. Then E^a is a closed subspace of X'. If Y is a closed subspace of X, then Y' is linearly isometric to X'/Y^a and $(X/Y)'$ is linearly isometric to Y^a.

13-4 Consider a norm $\|\ \ \|_j$ on a linear space $X_j, j = 1, 2, \ldots$. Fix $1 \le p \le \infty$. Let X be the product normed space as defined in Problem 5-6. Let q satisfy $1/p + 1/q = 1$. For $f_j \in X_j'$, let $\|f_j\|_j = \sup\{|f_j(x_j)| : x_j \in X_j, \|x_j\|_j \le 1\}, j = 1, 2, \ldots$. Let Y be the set of all $(f_1, f_2, \ldots)$ such that $f_j \in X_j', \sum_j \|f_j\|_j^q < \infty$ if $1 \le q < \infty$ and $\sup_j \|f_j\|_j < \infty$ if $q = \infty$.

For $(f_1, f_2, \ldots) \in Y$, define

$$\|(f_1, f_2, \ldots)\| = \begin{cases} (\|f_1\|_1^q + \|f_2\|_2^q + \cdots)^{1/2}, & \text{if } 1 \le q < \infty \\ \sup\{\|f_1\|_1, \|f_2\|_2, \cdots\}, & \text{if } q = \infty. \end{cases}$$

Then the map $F((f_1, f_2, \cdots))(x(1), x(2), \ldots) = \sum_j f_j(x(j))$ defines a linear isometry from the normed space Y to the normed dual X' of X. It is surjective if $1 \leq p < \infty$. If $p = \infty$, then it is surjective if and only if there is a positive integer n such that $X_j = \{0\}$ for all $j \geq n$. (Hint: Proof of 13.2)

13-5 Let $\|\ \|$ be a norm on $\mathbf{R}^2$ such that the set $\{(x, y) \in \mathbf{R}^2 : \|(x, y)\| = 1\}$ consists of a polygon with vertices at $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})$. Consider the dual norm

$$\|(a, b)\|' = \sup\{|xa + yb| : \|(x, y)\| \leq 1\}$$

for $(a, b) \in \mathbf{R}^2$. Then the set $\{(a, b) \in \mathbf{R}^2 : \|(a, b)\|' = 1\}$ also consists of a polygon with vertices at $(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})$, where (a_j, b_j) is the unique solution of the two equations

$$x_j a + y_j b = 1 = x_{j+1} a + y_{j+1} b$$

in the unknowns a and b for $j = 0, \ldots, n-1$ with $x_n = x_0$ and $y_n = y_0$.

13-6 Let $X = c$. For a fixed $y = (y(0), y(1), y(2), \ldots)$ in ℓ^1, let

$$f_y(x) = y(0) \lim_{j \to \infty} x(j) + \sum_{j=1}^{\infty} x(j) y(j), \quad x \in X.$$

Then $f_y \in X'$ and $\|f_y\| = \|y\|_1$. The map $F : \ell^1 \to c'$ defined by $F(y) = f_y$, $y \in \ell^1$, is a linear isometry from ℓ^1 onto c'. (Hint: Problem 8-13)

13-7 ℓ^1 is not linearly homeomorphic to ℓ^p if $1 < p \leq \infty$.

13-8 Let Y be a finite dimensional subspace of a normed space X and let Z be a closed subspace of X such that $X = Y + Z$ and $Y \bigcap Z = \{0\}$. If $\{x_1, \ldots, x_m\}$ is a basis for Y, then there are unique $x_1', \ldots, x_m'$ in X' with

$$x_j'(x_i) = \delta_{i,j} \text{ for } 1 \leq i, j \leq m \quad \text{and} \quad x_j' = 0 \text{ on } Z.$$

(Compare the discussion of projections in Section 10.)

13-9 Let X be a Banach space, Y and Z be closed subspaces of X such that $X = Y + Z$ and $Y \bigcap Z = \{0\}$, and let P be the (continuous) projection onto Y along Z. Then P' is the (continuous) projection onto Z^a along Y^a.

13-10 A normed space X is said to be **smooth** if for every $a \in X$ with $\|a\| = 1$, there is a unique support hyperplane for $\overline{U}(0,1)$ at a. If X' is strictly convex, then X is smooth and if X' is smooth, then X is strictly convex). (Hint: Problem 7-10)

13-11 For a Banach space X and $A \in BL(X)$, $\sigma(A) = \sigma_e(A) \bigcup \sigma_a(A')$. (Hint: 12.1(a) and 13.9)

13-12 Let $X = \ell^2 \times \ell^2$ and C denote the operator on X defined in Problem 12-20. Then neither $\sigma_e(C)$ is contained in $\sigma_e(C')$ nor $\sigma_e(C')$ is contained in $\sigma_e(C)$, although $\sigma(C) = \sigma(C')$.

13-13 Let $F \in BL(X,Y)$ and $y \in Y'$.

(a) $\|F'(y')\| \leq \|F\| \, \|y'\|$.

(b) If there is some $\alpha > 0$ such that for every $y \in Y$, there is a sequence (x_n) in X with $F(x_n) \to y$ and $\|x_n\| \leq \alpha \|y\|$, then $\|y'\| \leq \alpha \|F'(y')\|$.

In particular, if F satisfies $\beta \|x\| \leq \|F(x)\| \leq \alpha \|x\|$ for all $x \in X$ and some $\alpha, \beta > 0$ and if F is surjective, then F' satisfies $\beta \|y'\| \leq \|F'(y')\| \leq \alpha \|y'\|$ for all $y' \in Y'$ and $F'$is surjective.

13-14 Let X and Y be normed spaces and $F \in BL(X,Y)$. Assume that F' is bounded below: $\beta \|y'\| \leq \|F'(y')\|$ for all $y' \in Y'$ and some $\beta > 0$. Then $\overline{U}_Y(0,\beta)$ is contained in the closure of $F(\overline{U}_X(0,1))$.

13-15 Let X and Y be Banach spaces and $F \in BL(X,Y)$. Then $R(F) = Y$ if and only if F' is injective and $R(F')$ is closed in X'.

14 Duals of $L^p([a,b])$ and $C([a,b])$

In the last section we have shown that the dual of ℓ^p is linearly isometric to ℓ^q if $1 \leq p < \infty$ and $1/p + 1/q = 1$, while the dual of c_0 is linearly isometric to ℓ^1. In the present section we treat the analogous spaces $L^p([a,b])$ and $C([a,b])$. The role played by the standard Schauder basis $\{e_1, e_2, \ldots\}$ in the last section will be taken over by $\{c_t : t \in (a, b]\}$, where c_t is the characteristic function of the subinterval $(a, t]$ of $[a, b]$.

We begin with a result which establishes a linear isometry from $L^q([a, b])$ into the dual of $L^p([a, b])$, where $1 \le p \le \infty$ and $1/p + 1/q = 1$. We write L^p for $L^p([a, b])$ for the sake of convenience.

14.1 Theorem

Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. For a fixed $y \in L^q$, define $f_y : L^p \to \mathbf{K}$ by

$$f_y(x) = \int_a^b xy\, dm, \quad x \in L^p.$$

Then $f_y \in (L^p)'$ and $\|f_y\| = \|y\|_q$. The map $F : L^q \to (L^p)'$ defined by

$$F(y) = f_y, \qquad y \in L^q,$$

is a linear isometry from L^q to $(L^p)'$.

Proof:

Let $y \in L^q$. Clearly, f_y is linear. Also,

$$|f_y(x)| \le \int_a^b |x|\, |y| dm \le \|x\|_p \|y\|_q$$

for all $x \in L^p$. This is obvious if $p = 1$ or ∞ and follows from Hölder's inequality (4.5(a)), if $1 < p < \infty$. Hence $f_y \in (L^p)'$ and $\|f_y\| \le \|y\|_q$. Now we prove $\|y\|_q \le \|f_y\|$. If $y = 0$, there is nothing to prove. Assume, therefore, that $y \ne 0$.

First let $p = 1$, so that $q = \infty$. To show $\|y\|_\infty \le \|f_y\|$, that is, $|y(t)| \le \|f_y\|$ for almost all $t \in [a, b]$, we let

$$E_n = \{t \in [a, b] : |y(t)| > \|f_y\| + \frac{1}{n}\},$$

for $n = 1, 2, \ldots$, and $E = \{t \in [a, b] : |y(t)| > \|f_y\|\}$. Since $E = \bigcup_{n=1}^{\infty} E_n$, we have $m(E) \le \sum_{n=1}^{\infty} m(E_n)$. Hence it is enough to prove that $m(E_n) = 0$ for each n. Let

$$x_n = (\operatorname{sgn} y) c_{E_n},$$

where c_{E_n} is the characteristic function of E_n. Then $x_n y = |y|c_{E_n}$, $|x_n| = c_{E_n}$ and $\|x_n\|_1 = m(E_n)$. Hence

$$(\|f_y\| + \frac{1}{n})m(E_n) \le \int_{E_n} |y|dm = \int_a^b x_n y\, dm = f_y(x_n) \le \|f_y\| m(E_n).$$

Thus $m(E_n) = 0$ for each n, as desired.

Next, let $1 < p \le \infty$, so that $1 \le q < \infty$. Let

$$x = (\operatorname{sgn} y)|y|^{q-1}.$$

Then $xy = |y|^q$. If $p = \infty$, then $\|x\|_\infty = 1$ and $\|y\|_1 = f_y(x) \le \|f_y\|$. If $1 < p < \infty$, then $|x|^p = |y|^q$, since $pq - p = q$, so that $x \in L^p$ and

$$\int_a^b |y|^q\, dm = \int_a^b xy\, dm = f_y(x).$$

But

$$f_y(x) \le \|f_y\|\,\|x\|_p = \|f_y\|(\int_a^b |y|^q dm)^{1/p},$$

so that $(\int_a^b |y|^q dm)^{1-1/p} \le \|f_y\|$, that is, $\|y\|_q \le \|f_y\|$, again since $1 - 1/p = 1/q$.

It is now clear that $F(y) = f_y, y \in L^q$, defines a linear isometry from L^q to $(L^p)'$. $\square$

The following result is crucial in establishing that the linear isometry from L^q to $(L^p)'$ given in 14.1 is surjective if $p < \infty$, and also in describing the dual of $C([a,b])$.

14.2 Lemma

Let $c_a = 0$ and for each $t \in (a,b]$, let c_t denote the characteristic function of $(a,t]$. Let X denote the linear space of all finite linear combinations of $\{c_t : t \in [a,b]\}$ and for a linear functional f on X, let

$$z(t) = f(c_t), \qquad t \in [a,b].$$

(a) Let $1 \le p < \infty$ and consider the norm on X given by

$$\|x\|_p = (\int_a^b |x|^p dm)^{1/p}, \qquad x \in X.$$

If $f \in X'$, then the function z is absolutely continuous on $[a, b]$.

(b) Consider the sup norm on X given by

$$\|x\|_\infty = \sup \{|x(t)| : t \in [a, b]\}, \qquad x \in X.$$

If $f \in X'$, then the function z is of bounded variation and the total variation of z on $[a, b]$ is at most $\|f\|$.

Proof:

(a) Let $1 \le p < \infty$. Consider a finite collection $\{(t_j, t_j') : j = 1, \ldots, n\}$ of nonoverlapping intervals in $[a, b]$, and let $k_j = \operatorname{sgn}[z(t_j') - z(t_j)]$. We note that

$$\sum_{j=1}^{n} k_j [z(t_j') - z(t_j)] = \sum_{j=1}^{n} |z(t_j') - z(t_j)|.$$

Letting

$$x = \sum_{j=1}^{n} k_j (c_{t_j'} - c_{t_j}),$$

and noting that $\|x\|_p \le \left(\sum_{j=1}^{n} (t_j' - t_j) \right)^{1/p}$, we obtain

$$\sum_{j=1}^{n} |z(t_j') - z(t_j)| = f(x) \le \|f\| \, \|x\|_p \le \|f\| \left(\sum_{j=1}^{n} (t_j' - t_j) \right)^{1/p}$$

If $f = 0$, then $z(t) = 0$ for all t. Assume now that $f \neq 0$. Let $\epsilon > 0$. If $\sum_{j=1}^{n} (t_j' - t_j) < \epsilon^p / \|f\|^p$, it follows that $\sum_{j=1}^{n} |z(t_j') - z(t_j)| < \epsilon$. Hence the function z is absolutely continuous on $[a, b]$.

(b) Given a partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$, let

$$x = \sum_{j=1}^{n} k_j (c_{t_j} - c_{t_{j-1}}),$$

where $k_j = \operatorname{sgn}[z(t_j) - z(t_{j-1})]$, $j = 1, \ldots, n$. Then again

$$\sum_{j=1}^{n} |z(t_j) - z(t_{j-1})| = f(x) \le \|f\| \, \|x\|_\infty \le \|f\|.$$

Hence z is a function of bounded variation on $[a,b]$ and the total variation of z on $[a,b]$ is at most $\|f\|$. □

14.3 Theorem (Riesz representation theorem for L^p)
Let $1 \le p < \infty$ and $1/p + 1/q = 1$. Then the linear isometry from L^q to $(L^p)'$ given in 14.1 is surjective.

Proof:
Let $f \in (L^p)'$, and $z(t) = f(c_t)$, where c_t is the characteristic function of $(a,t]$ for $t \in (a,b]$ and $c_a = 0$. Then $z(a) = f(c_a) = f(0) = 0$, and z is absolutely continuous on $[a,b]$ by 14.2. The fundamental theorem for Lebesgue integration (4.3) shows that z' exists almost everywhere on $[a,b]$, $z' \in L^1$ and

$$z(t) = z(a) + \int_a^t z'\,dm = \int_a^t z'\,dm.$$

Let $y = z'$. We show that $y \in L^q$ and $F(y) = f_y = f$, that is,

$$f(x) = \int_a^b xy\,dm \tag{$*$}$$

for all $x \in L^p$, where $1/p + 1/q = 1$.

For $a \le s < t \le b$, let $c_{s,t}$ denote the characteristic function of the interval $(s,t]$. Then

$$\begin{aligned} f(c_{s,t}) &= f(c_t - c_s) = f(c_t) - f(c_s) \\ &= z(t) - z(s) = \int_s^t z'\,dm = \int_a^b c_{s,t}\,y\,dm. \end{aligned}$$

Thus equation (*) holds for $x = c_{s,t}$. Since the characteristic function of a subinterval of $[a,b]$ equals some $c_{s,t}$, except possibly at the two end point of the subinterval, it follows that equation (*) holds for such a function. Now every open subset of $\mathbf{R}$ is a countable disjoint union of open intervals. Hence the regularity of the Lebesgue measure (discussed in Section 4) shows that equation (*) holds when x is the characteristic function of a measurable subset of $[a,b]$, and when x is a simple measurable function.

Now let $x \in L^p$ and assume that $x \geq 0$. Then there is a sequence (x_n) of simple measurable functions such that $0 \leq x_1(t) \leq x_2(t) \leq \cdots \leq x(t)$, and $x_n(t) \to x(t)$ for every $t \in [a,b]$ as well as $\|x_n - x\|_p \to 0$. (See 4.7(a).) Let $(\operatorname{Re} y)^+ = \max\{\operatorname{Re} y, 0\}$. By the monotone convergence theorem (4.1(a)), we have

$$\lim_{n\to\infty} \int_a^b x_n (\operatorname{Re} y)^+ \, dm = \int_a^b x (\operatorname{Re} y)^+ \, dm.$$

Since $\|x_n - x\|_p \to 0$, there is some $\alpha > 0$ such that $\|x_n\|_p \leq \alpha$ for all $n = 1, 2, \ldots$. If $E = \{t \in [a,b] : \operatorname{Re} y \geq 0\}$ and c_E is the characteristic function of E, then

$$\int_a^b x_n (\operatorname{Re} y)^+ \, dm = \int_a^b x_n (\operatorname{Re} y)^+ c_E \, dm = \operatorname{Re} \int_a^b x_n y c_E \, dm.$$

Since $x_n c_E$ is a simple measurable function, we have

$$\operatorname{Re} \int_a^b x_n c_E y \, dm = \operatorname{Re} f(x_n c_E).$$

Hence

$$0 \leq \int_a^b x_n (\operatorname{Re} y)^+ \, dm = \operatorname{Re} f(x_n c_E) \leq |f(x_n c_E)| \leq \|f\| \alpha < \infty.$$

Similarly, considering $(\operatorname{Re} y)^-, (\operatorname{Im} y)^+$ and $(\operatorname{Im} y)^-$, and noting that $y = (\operatorname{Re} y)^+ - (\operatorname{Re} y)^- + i(\operatorname{Im} y)^+ - i(\operatorname{Im} y)^-$, we have

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \int_s^b x_n y \, dm = \int_a^b xy \, dm.$$

Thus equation $(*)$ holds for all $x \in L^p$ with $x \geq 0$. Finally, for $x \in L^p$,

$$x = (\operatorname{Re} x)^+ - (\operatorname{Re} x)^- + i(\operatorname{Im} x)^+ - i(\operatorname{Im} x)^-.$$

Hence equation $(*)$ holds for all $x \in L^p$.

It remains to show that $y \in L^q$, where $1/p + 1/q = 1$.

First, let $p = 1$, so that $q = \infty$. If $t \in [a,b]$ and $z'(t)$ exists, then

$$\begin{aligned} |y(t)| = |z'(t)| &= \lim_{h\to 0} \left| \frac{z(t+h) - z(t)}{h} \right| = \lim_{h\to 0} \frac{|f(c_{t+h} - c_t)|}{|h|} \\ &\leq \limsup_{h\to 0} \frac{\|f\|}{|h|} \|c_{t+h} - c_t\|_1 = \|f\|, \end{aligned}$$

since $\|c_{t+h} - c_t\|_1 = |\int_t^{t+h} dm| = |h|$. Thus

$$\|y\|_\infty \leq \|f\| < \infty.$$

Next, let $1 < p < \infty$. For $n = 1, 2, \ldots$, let $D_n = \{t \in [a,b] : |y(t)| \leq n\}$, and c_n denote the characteristic function of D_n. Referring to the proof of Theorem 14.1 for the case $1 < p < \infty$, and considering

$$x = (\operatorname{sgn} y)|y|^{q-1}, \quad x_n = xc_n,$$

we find that $\int_a^b |x_n|^p\, dm = \int_a^b |y|^q c_n\, dm \leq n^q(b-a) < \infty$ and

$$\left(\int_a^b |y|^q c_n dm\right)^{1/q} \leq \|f\|, \quad n = 1, 2, \ldots.$$

Since (c_n) converges monotonically to the constant function 1 on $[a,b]$, it follows from 4.1(a) that

$$\|y\|_q = \left(\int_a^b |y|^q dm\right)^{1/q} \leq \|f\| < \infty. \qquad \square$$

The Riesz representation theorem for L^p is often proved by employing the Radon-Nikodym theorem of measure theory. We have avoided an appeal to this theorem at the cost of a slightly laborious proof.

We remark that the linear isometry $F : L^1 \to (L^\infty)'$ given by $F(y) = f_y$, where

$$f_y(x) = \int_a^b xy\, dm, \quad x \in L^\infty,$$

is not surjective. This can be seen as follows. For $x \in C([a,b])$ with the sup norm $\|\ \|_\infty$, let $g(x) = x(b)$. Then $g \in (C([a,b]))'$ and $\|g\| = 1$. Let f be a Hahn-Banach extension of g to L^∞. We show that $f \neq f_y$ for any $y \in L^1$. For $n = 1, 2, \ldots$, let $b_n = b - (b-a)/n$ and

$$x_n(t) = \begin{cases} 0, & \text{if } t \in [a,\ b_n] \\ \dfrac{t - b_n}{b - b_n}, & \text{if } t \in [b_n, b]. \end{cases}$$

Then $x_n \in C([a,b]), \|x_n\|_\infty \leq 1$, $x_n(t) \to 0$ for all $t \in [a,b)$ and $x_n(b) = 1$. Hence for $y \in L^1, f_y(x_n) = \int_a^b x_n y \, dm \to 0$ by the dominated convergence theorem (4.1(b)), while $f(x_n) = g(x_n) = 1$, so that $f \neq f_y$. In fact, there is no homeomorphism from L^1 onto L^∞, because L^1 is separable but L^∞ is not (4.7(d)), and this would contradict 13.1(b).

We point out the reason why our argument in 14.3 for the surjectivity of the linear isometry does not hold for $p = \infty$. If $f \in (L^\infty)'$ and $z(t) = f(c_t)$ for $t \in (a,b]$ and $z(a) = 0$, then the function z is of bounded variation by 14.2(b). As a result, z' exists almost everywhere on $[a,b]$ and $z' \in L^1$, as pointed out in Section 4. However, the function z may not be absolutely continuous on $[a,b]$. As a result, it is not possible to conclude that
$f(c_t) = z(t) = \int_a^t z' \, dm = \int_a^b c_t z' \, dm = f_{z'}(c_t)$. (See Problem 14-5.)

In this case, one may think of showing that

$$f(x) = \int_a^b x \, dz, \quad x \in L^\infty.$$

But the integral on the right may not be defined for all $x \in L^\infty$. (Again, see Problem 14-5.) If, however, $x \in C([a,b])$, then $\int_a^b x \, dz$ exists. In fact, $\int_a^b x \, dz$ is the limit of the Riemann-Stieltjes sum

$$\sum_{j=1}^{n} x(s_j)[z(t_j) - z(t_{j-1})],$$

as the mesh of the partition $a = t_0 < t_1 < \ldots < t_n = b$ tends to zero and $t_{j-1} \leq s_j \leq t_j, j = 1, \ldots, n$. Further,

$$\left| \int_a^b x \, dz \right| \leq \|x\|_\infty V(z),$$

where $\|x\|_\infty = \sup\{|x(t)| : t \in [a,b]\}$ and $V(z)$ is the total variation of z on $[a,b]$.

Let $BV([a,b])$ denote the linear space of all $\mathbf{K}$-valued functions of bounded variation on $[a,b]$. For $z \in BV([a,b])$, consider

$$\|z\| = |z(a)| + V(z).$$

Then $\| \ \|$ is a norm on $BV([a,b])$. For a fixed $z \in BV([a,b])$, define $f_z : C([a,b]) \to \mathbf{K}$ by

$$f_z(x) = \int_a^b x\,dz, \qquad x \in C([a,b]).$$

Then $f_z \in (C([a,b]))'$ and $\|f_z\| \le \|z\|$. However, $\|f_z\|$ may not equal $\|z\|$. For example, if $w = z + 1$, then $f_w = f_z$, but $\|w\| = \|z\| + 1$, so that either $\|f_z\| \neq \|z\|$ or $\|f_w\| \neq \|w\|$.

This also shows that distinct functions of bounded variation can give rise to the same linear functional on $C([a,b])$. In order to mend these matters, we introduce the following concept.

A function z of bounded variation on $[a,b]$ is said to be **normalized** if $z(a) = 0$ and z is right continuous on (a,b). Let $NBV([a,b])$ denote the set of all normalized functions of bounded variation on $[a,b]$. It is a linear space and the total variation gives a norm on it.

14.4 Lemma

Let $z \in BV([a,b])$. Then there is a unique $y \in NBV([a,b])$ such that

$$\int_a^b x\,dz = \int_a^b x\,dy$$

for all $x \in C([a,b])$. In fact,

$$y(t) = \begin{cases} 0, & \text{if } t = a \\ z(t^+) - z(a), & \text{if } t \in (a,b) \\ z(b) - z(a), & \text{if } t = b. \end{cases}$$

Moreover, $V(y) \le V(z)$.

Proof:

Let $y : [a,b] \to \mathbf{K}$ be defined as above. Note that the right limit $z(t^+)$ exists for every $t \in (a,b)$, because Re z and Im z are real-valued functions of bounded variation, and hence each of them is a difference of two monotonically increasing functions. This also shows that z has only a countable number of discontinuities in $[a,b]$.

Let $\epsilon > 0$. We show that $V(y) \le V(z) + \epsilon$. Consider a partition $a = t_0 < t_1 \cdots < t_{n-1} < t_n = b$ of $[a, b]$. Choose points $s_1, \ldots, s_{n-1}$ in (a, b) which satisfy

$$t_j < s_j, \quad |z(t_j^+) - z(s_j)| < \frac{\epsilon}{2n}, \quad j = 1, \ldots, n-1.$$

Let $s_0 = a$ and $s_n = b$. Then

$$\begin{aligned}
|y(t_1) - y(t_0)| &\le |z(t_1^+) - z(s_1)| + |z(s_1) - z(s_0)|, \\
|y(t_j) - y(t_{j-1})| &\le |z(t_j^+) - z(s_j)| + |z(s_j) - z(s_{j-1})| \\
&\qquad + |z(s_{j-1}) - z(t_{j-1}^+)|, \quad j = 2, \ldots, n-1, \\
|y(t_n) - y(t_{n-1})| &\le |z(s_n) - z(s_{n-1})| + |z(s_{n-1}) - z(t_{n-1}^+)|.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=1}^{n} |y(t_j) - y(t_{j-1})| &\le \sum_{j=1}^{n} |z(s_j) - z(s_{j-1})| + \frac{\epsilon + 2(n-2)\epsilon + \epsilon}{2n} \\
&< \sum_{j=1}^{n} |z(s_j) - z(s_{j-1})| + \epsilon.
\end{aligned}$$

Since this is true for every partition of $[a, b]$, $V(y) \le V(z) + \epsilon$. As $\epsilon > 0$ is arbitrary, $V(y) \le V(z)$. In particular, y is of bounded variation on $[a, b]$. The definition of y shows that, in fact, $y \in NBV([a, b])$.

Next, let $x \in C([a, b])$. Apart from the subtraction of the constant $z(a)$, the function y agrees with the function z, except possibly at the points of discontinuities of z. Since these points are countable, they can be avoided while forming the Riemann-Stieltjes sum

$$\sum_{j=1}^{n} x(t_j)[z(t_j) - z(t_{j-1})],$$

which approximates $\int_a^b x\,dz$. Since such a sum equals $\sum_{j=1}^{n} x(t_j)[y(t_j) - y(t_{j-1})]$ and approximates $\int_a^b x\,dy$, we see that $\int_a^b x\,dz = \int_a^b x\,dy$.

To prove the uniqueness of y, let $y_0 \in NBV([a, b])$ be such that $\int_a^b x\,dz = \int_a^b x\,dy_0$ for all $x \in C([a, b])$ and $w = y - y_0$. Then $w(a) =$

$y(a) - y_0(a) = 0 - 0 = 0$. Also, since $1 \in C([a,b])$,

$$w(b) = w(b) - w(a) = \int_a^b dw = \int_a^b dy - \int_a^b dy_0 = 0.$$

Next, let $c \in (a,b)$. For a sufficiently small positive h, let

$$x(t) = \begin{cases} 1, & \text{if} \quad a \le t \le c \\ 1 - \dfrac{t-c}{h}, & \text{if} \quad c < t \le c+h \\ 0, & \text{if} \quad c+h < t \le b. \end{cases}$$

Then $x \in C([a,b])$ and $|x(t)| \le 1$ for all $t \in [a,b]$. Since

$$0 = \int_a^b x\, dy - \int_a^b x\, dy_0 = \int_a^b x\, dw = \int_a^c dw + \int_c^{c+h} (1 - \frac{t-c}{h}) dw,$$

we have

$$w(c) = \int_a^c dw = -\int_c^{c+h} (1 - \frac{t-c}{h}) dw.$$

It follows that

$$|w(c)| \le V_c^{c+h},$$

where V_c^{c+h} denotes the total variation of w on $[c, c+h]$. As w is right continuous at c, its total variation function $v(t) = V_a^t$, $t \in [a,b]$, is also right continuous at c. Let $\epsilon > 0$. There is some $\delta > 0$ such that for $0 < h < \delta$,

$$|w(c)| \le V_c^{c+h} = v(c+h) - v(c) < \epsilon.$$

Hence $w(c) = 0$. Thus $w = 0$, that is, $y_0 = y$. □

14.5 Theorem (Riesz representation theorem for $C([a,b])$)
For a fixed $y \in NBV([a,b])$, define $f_y : C([a,b]) \to \mathbf{K}$ by

$$f_y(x) = \int_a^b x\, dy, \quad x \in C([a,b]).$$

Consider the sup norm $\| \ \|_\infty$ on $C([a,b])$. Then $f_y \in (C([a,b]))'$ and $\|f_y\| = V(y)$. Also, f_y is a positive functional on $C([a,b])$ if and only if y is a nondecreasing function on $[a,b]$.

The map $F : NBV([a,b]) \to (C([a,b]))'$ defined by

$$F(y) = f_y, \quad y \in NBV([a,b]),$$

is a linear isometry from $NBV([a,b])$ onto $(C([a,b]))'$.

Proof:
It is clear that f_y is a linear functional on $C([a,b])$. Since

$$|f_y(x)| = \left|\int_a^b x\,dy\right| \leq \|x\|_\infty V(y)$$

for all $x \in C([a,b])$, it follows that $f_y \in (C([a,b]))'$, and $\|f_y\| \leq V(y)$. Also, if y is a nondecreasing function on $[a,b]$ and $x \geq 0$ on $[a,b]$, then $\int_a^b x\,dy \geq 0$ by the definition of the Riemann-Stieltjes integral, so that f_y is a positive functional on $C([a,b])$.

Further, it is obvious that the map F given by $F(y) = f_y$ is a linear map from $NBV([a,b])$ to $(C([a,b]))'$. The uniqueness part of 14.4 shows that the map F is one-to-one. To complete our proof, therefore, we let $g \in (C([a,b]))'$ and show that there is some $y \in NBV([a,b])$ with $f_y = g, V(y) = \|g\|$ and if, in particular, g is a positive functional, then y is a nondecreasing function.

Consider the linear space $B([a,b])$ of all bounded $\mathbf{K}$-valued functions on $[a,b]$ with the sup norm $\|\ \|_\infty$, of which $C([a,b])$ is a subspace. By the Hahn-Banach extension theorem 7.8, there is a continuous linear functional f on $B([a,b])$ such that $f_{|C([a,b])} = g$ and $\|f\| = \|g\|$. Let $c_a = 0$, and c_t denote the characteristic function of $(a,t]$ for $t \in (a,b]$. Let

$$X = \operatorname{span}\{c_t : t \in [a,b]\} \subset B([a,b])$$

and

$$z(t) = f(c_t), \quad t \in [a,b].$$

It follows from 14.2(b) that z is a function of bounded variation on $[a,b]$ and

$$V(z) \leq \|f_{|X}\| \leq \|f\|.$$

We show that

$$g(x) = \int_a^b x\,dz$$

for all $x \in C([a,b])$. For $x \in C([a,b])$ and a partition $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ of $[a,b]$, let

$$u = \sum_{j=1}^{n} x(t_j)(c_{t_j} - c_{t_{j-1}}).$$

Then $u \in X$ and

$$f(u) = \sum_{j=1}^{n} x(t_j)[z(t_j) - z(t_{j-1})].$$

Let $\epsilon > 0$. Since x is uniformly continuous on $[a,b]$ and since $\int_a^b x\,dz$ is the limit of the Riemann-Stieltjes sum, we can choose a partition of $[a,b]$ so fine that

$$\|u - x\|_\infty < \epsilon \quad \text{and} \quad \left|\int_a^b x\,dz - f(u)\right| < \epsilon.$$

Then

$$\begin{aligned} \left|\int_a^b x\,dz - g(x)\right| &\le \left|\int_a^b x\,dz - f(u)\right| + |f(u) - f(x)| \\ &< \epsilon + \|f\|\,\|u - x\|_\infty < \epsilon(1 + \|f\|). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $g(x) = \int_a^b x\,dz$.

By 14.4, there is some $y \in NBV([a,b])$ such that $V(y) \le V(z)$ and $\int_a^b x\,dy = \int_a^b x\,dz = g(x)$ for all $x \in C([a,b])$. Hence $f_y = g$ and

$$\|g\| = \|f_y\| \le V(y) \le V(z) \le \|f\| = \|g\|,$$

that is, $V(y) = \|g\|$. If, in particular, g is a positive functional on $C([a,b])$, then every Hahn-Banach extension of g to $B([a,b])$ is also positive by 7.9(a). Thus f is a positive functional, so that for all $s \le t$ in $[a,b]$,

$$z(t) - z(s) = f(c_t) - f(c_s) = f(c_t - c_s) \ge 0$$

since $c_t - c_s \geq 0$ on $[a, b]$. This shows that z is a nondecreasing function on $[a, b]$. But then the normalized function of bounded variation as constructed in 14.4 from z is also nondecreasing, that is, y is nondecreasing. This completes the proof. □

We now give an application of these results to a classical problem.

Moment Sequences

If z is a function of bounded variation on $[0, 1]$, then for $n = 0, 1, 2, \ldots$, the Riemann-Stieltjes integral

$$\int_0^1 t^n dz(t)$$

is called the nth **moment** of z. The moments of a distribution function of a random variable play an important role in statistics. A sequence of scalars $\mu(n), n = 0, 1, 2, \ldots$, is called a **moment sequence** if there is some $z \in BV([0, 1])$ whose nth moment is $\mu(n), n = 0, 1, 2, \ldots$. For example, if α is a positive number, then $1/(n + \alpha), n = 0, 1, 2, \ldots$, is a moment sequence since letting $z(t) = t^\alpha/\alpha, t \in [0, 1]$, we see that

$$\int_0^1 t^n dz(t) = \int_0^1 t^{n+\alpha-1} dt = \frac{1}{n + \alpha}, \quad n = 0, 1, 2, \ldots.$$

Similarly, if $0 < r \leq 1$, then $(r^n), n = 0, 1, 2, \ldots$, is a moment sequence since letting z to be the characteristic function of $[r, 1]$, we see that

$$\int_0^1 t^n dz(t) = r^n, n = 0, 1, 2, \ldots.$$

If $\mu(n)$ is the nth moment of $z \in BV([0, 1])$, then $|\mu(n)| \leq V(z), n = 0, 1, 2, \ldots$. Hence every moment sequence is bounded. In fact, it is convergent, as indicated in Problem 14-19. Thus not every scalar sequence can be a moment sequence. The problem of determining which sequences are moment sequences is known as the **moment problem of Hausdorff** or the **little moment problem**.

To tackle this problem, we introduce the **forward difference operator** $\triangle$ as follows. Let X denote the linear space of all scalar sequences $(\mu(n)), n = 0, 1, 2, \ldots$ and let $\tau : X \to X$ be defined (as in Section 7) by

$$\tau(\mu)(n) = \mu(n+1), \quad \mu \in X, \quad n = 0, 1, 2, \ldots$$

Let I denote the identity map from X to X and define

$$\Delta = \tau - I.$$

Thus for all $\mu \in X$ and $n = 0, 1, 2, \ldots,$

$$\Delta(\mu)(n) = \mu(n+1) - \mu(n).$$

For $r = 0, 1, 2, \ldots,$ we have

$$\Delta^r = (\tau - I)^r = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \tau^j,$$

that is, for all $\mu \in X$ and $n = 0, 1, 2, \ldots,$

$$\Delta^r(\mu)(n) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \mu(n+j).$$

In particular,

$$\Delta^r(\mu)(0) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \mu(j).$$

Next, let us consider the linear space $P([0,1])$ of all scalar-valued polynomial functions on [0,1]. For $m = 0, 1, 2, \ldots,$ let

$$p_m(t) = t^m, \quad t \in [0,1].$$

Then $P([0,1]) = \text{span}\,\{p_m : m = 0, 1, 2, \ldots\}$. By the Weierstrass theorem (3.12), $P([0,1])$ is dense in $C([0,1])$ with the sup norm $\| \ \|_\infty$. In fact, we showed that $\|B_n(x) - x\|_\infty \to 0$ for each $x \in C([0,1])$, where the nth **Bernstein polynomial** $B_n(x)$ is given by

$$B_n(x)(t) = \sum_{j=0}^{n} x(\frac{j}{n}) \binom{n}{j} t^j (1-t)^{n-j}, \quad t \in [0,1], \; n = 0, 1, 2, \ldots.$$

We express $B_n(x)$ as a linear combination of $p_0, p_1, p_2, \ldots$. Since

$$(1-t)^{n-j} = \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} t^k = \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} p_k(t),$$

and $p_j p_k = p_{j+k}$, we have

$$B_n(x) = \sum_{j=0}^{n} x(\frac{j}{n}) \binom{n}{j} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} p_{j+k}.$$

As $\binom{n}{j}\binom{n-j}{k} = \binom{j+k}{j}\binom{n}{j+k}$, we put $j+k=r$ and obtain

$$B_n(x) = \sum_{r=0}^{n} \left[\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} x(\frac{j}{n}) \right] \binom{n}{r} p_r.$$

In particular, $B_n(x)$ is a polynomial of degree at most n. Also,

$$B_n(p_0)(t) = \sum_{j=0}^{n} \binom{n}{j} t^j (1-t)^{n-j} = [t + (1-t)]^n = 1 = p_0(t), \quad t \in [0,1].$$

For a nonnegative integer m, consider the subspace

$$P_{(m)} = \{p \in P([0,1]) : p \text{ is of degree } \leq m\}$$

of $P([0,1])$. Let $p \in P_{(m)}$. We show that $B_n(p) \in P_{(m)}$ for each $n = 1, 2, \ldots$. If $n \leq m$, then clearly $B_n(p) \in P_{(n)} \subset P_{(m)}$.

Now fix $n \geq m+1$. Consider the sequence $(\mu_p(j))$ defined by

$$\mu_p(j) = p(\frac{j}{n}), \quad j = 0, 1, 2, \ldots.$$

Noting the expressions for $\triangle^r(\mu)(0)$ and $B_n(x)$, we obtain

$$B_n(p) = \sum_{r=0}^{n} (\triangle^r \mu_p)(0) \binom{n}{r} p_r.$$

Since p is a polynomial of degree at most m, it follows that

$$(\triangle\mu_p)(j) = p(\frac{j+1}{n}) - p(\frac{j}{n}) = k_0 + k_1 j + \cdots + k_{m-1} j^{m-1}, \quad j = 0, 1, 2, \ldots,$$

for some scalars $k_0, \ldots, k_{m-1}$. Proceeding similarly, we conclude that $(\triangle^m \mu_p)(j)$ equals a constant for $j = 0, 1, 2, \ldots$ and for each $r \geq m+1$, we have

$$(\triangle^r \mu_p)(j) = 0, \quad j = 0, 1, 2, \ldots.$$

In particular, $(\triangle^r \mu_p)(0) = 0$ for all $r \geq m+1$, so that

$$B_n(p) = \sum_{r=0}^{m} (\triangle^r \mu_p)(0) \binom{n}{r} p_r.$$

Hence $B_n(p) \in P_{(m)}$, as desired.

We now prove a simple but useful result.

14.6 Lemma

Let g be a linear functional on $P([0,1])$ and $\mu(n) = g(p_n)$ for $n = 0, 1, 2, \ldots$. If $x_{j,r}(t) = t^j(1-t)^r$ for $t \in [0,1]$, then

$$g(x_{j,r}) = (-1)^r \Delta^r(\mu)(j), \quad j, r = 0, 1, 2, \ldots.$$

Proof:
We have

$$x_{j,r} = p_j \sum_{k=0}^{r} (-1)^k \binom{r}{k} p_k = \sum_{k=0}^{r} (-1)^k \binom{r}{k} p_{j+k}$$

Hence

$$g(x_{j,r}) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} g(p_{j+k}) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \mu(j+k),$$

which equals $(-1)^r \Delta^r(\mu)(j)$. □

We are now in a position to give criteria for a sequence to be a moment sequence.

14.7 Theorem (Hausdorff, 1921)
Let $(\mu(n)), n = 0, 1, 2, \ldots$ be a sequence of scalars. Then the following conditions are equivalent.

(i) $(\mu(n))$ is a moment sequence.

(ii) For $n = 0, 1, 2, \ldots$ and $j = 0, \ldots, n$, let

$$\alpha_{n,j} = \binom{n}{j} (-1)^{n-j} \Delta^{n-j}(\mu)(j).$$

Then $\sum_{j=0}^{n} |\alpha_{n,j}| \leq \alpha$ for all n and some $\alpha > 0$.

(iii) The linear functional $g : P([0, 1]) \to \mathbf{K}$ defined by

$$g(k_0 p_0 + \cdots + k_n p_n) = k_0 \mu(0) + \cdots + k_n \mu(n),$$

is continuous, where $n = 0, 1, 2, \ldots$ and $k_0, \ldots, k_n \in \mathbf{K}$.

Further, there is a nondecreasing function on $[0, 1]$ whose nth moment is $\mu(n)$ if and only if $\alpha_{n,j} \geq 0$ for all $n = 0, 1, 2, \ldots$ and $j = 0, \ldots, n$. This is the case if and only if the linear functional g is positive.

Proof:

(i) implies (ii): Let $z \in BV([0, 1])$ be such that the nth moment of z is $\mu(n), n = 0, 1, 2, \ldots$. Then

$$g(p) = \int_0^1 p \, dz, \quad z \in P([0, 1]),$$

defines a linear functional g on $P([0, 1])$ such that $g(p_n) = \mu(n)$ for $n = 0, 1, 2, \ldots$. By Lemma 14.6, $\alpha_{n,j} = \binom{n}{j} g(x_{j,n-j})$ for $n = 0, 1, 2, \ldots$ and $j = 0, \ldots, n$. Since $x_{j,n-j} \geq 0$ on $[0, 1]$, it follows that

$$|\alpha_{n,j}| \leq \binom{n}{j} \int_0^1 x_{j,n-j} dv_z,$$

where $v_z(t)$ is the total variation of z on $[0,t]$. But for $n = 0, 1, 2, \ldots$,

$$\sum_{j=0}^{n} \binom{n}{j} x_{j,n-j} = B_n(1) = 1,$$

as we have seen before. Hence

$$\sum_{j=0}^{n} |\alpha_{n,j}| \le \int_0^1 B_n(1) dv_z = V(z),$$

where $V(z)$ is the total variation of z on $[0,1]$.

Note that if z is nondecreasing then since $x_{j,n-j} \ge 0$, we have

$$\alpha_{n,j} = \binom{n}{j} \int_0^1 x_{j,n-j}\, dz \ge 0$$

for all $n = 0, 1, 2, \ldots$ and $j = 0, \ldots, n$.

(ii) implies (iii): For a nonnegative integer m, let g_m denote the restriction of g to $P_{(m)}$. Since g is linear on $P([0,1])$ and $P_{(m)}$ is a finite dimensional subspace of $P([0,1])$, it follows from 6.1 that g_m is continuous.

Let $p \in P([0,1])$. Since

$$B_n(p) = \sum_{j=0}^{n} p(\frac{j}{n}) \binom{n}{j} x_{j,n-j},$$

and $g(p_n) = \mu(n)$ for $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned} g(B_n(p)) &= \sum_{j=0}^{n} p(\frac{j}{n}) \binom{n}{j} g(x_{j,n-j}) \\ &= \sum_{j=0}^{n} p(\frac{j}{n}) \binom{n}{j} (-1)^{n-j} \Delta^{n-j}(\mu)(j) \\ &= \sum_{j=0}^{n} p(\frac{j}{n}) \alpha_{n,j} \end{aligned}$$

by Lemma 14.6. Hence

$$|g(B_n(p))| \le \sum_{j=0}^{n} |p(\frac{j}{n})||\alpha_{n,j}| \le ||p||_\infty \sum_{j=0}^{n} |\alpha_{n,j}| \le \alpha ||p||_\infty.$$

Now let the degree of p be m. Then $B_n(p) \in P_{(m)}$ for all $n = 1, 2, \ldots$, as we have seen before. Since $\|B_n(p) - p\|_\infty \to 0$ and g_m is continuous,

$$|g(p)| = |g_m(p)| = \lim_{n\to\infty} |g_m(B_n(p))| \leq \alpha \|p\|_\infty.$$

This shows that g is continuous on $P([0,1])$.

If $\alpha_{n,j} \geq 0$ for all n and j, and if $p \geq 0$ on $[0,1]$ then

$$g(p) = \lim_{n\to\infty} g(B_n(p)) = \lim_{n\to\infty} \sum_{j=0}^{n} p(\frac{j}{n})\alpha_{n,j} \geq 0,$$

that is, g is a positive functional. In this case,

$$\sum_{j=0}^{n} |\alpha_{n,j}| = \sum_{j=0}^{n} \alpha_{n,j} = \int_0^1 B_n(1)\, dz = \int_0^1 dz = \mu(0).$$

(iii) implies (i): Since $P([0,1])$ is dense in $C([0,1])$ with the sup norm $\| \ \|_\infty$, 8.3(a) shows that there is some $f \in (C([0,1]))'$ with $f_{|P([0,1])} = g$ and $\|f\| = \|g\|$. By 14.5, there is some $z \in NBV([0,1])$ such that

$$f(x) = \int_0^1 x\, dz, \quad x \in C([0,1]).$$

In particular, for $n = 0, 1, 2, \ldots$,

$$\mu(n) = g(p_n) = f(p_n) = \int_0^1 t^n\, dz(t),$$

that is, $\mu(n)$ is the nth moment of z.

If the functional g is positive and $x \in C([0,1])$ with $x \geq 0$ on $[0,1]$, then $B_n(x) \geq 0$ on $[0,1]$ for all n and we have

$$f(x) = \lim_{n\to\infty} f(B_n(x)) = \lim_{n\to\infty} g(B_n(x)) \geq 0,$$

that is, f is a positive functional on $C([0,1])$. As in 14.5, this implies that there is a nondecreasing function z such that $f = f_z$. In particular, $\mu(n)$ is the nth moment of a nondecreasing function z. □

If $(\mu(n))$ is a moment sequence, then there is, in fact, a unique $y \in NBV([0,1])$ such that $\mu(n)$ is the nth moment of y. This follows from Lemma 14.4 by noting that $P([0,1]) = \text{span}\{p_n : n = 0,1,2,\ldots\}$ is dense in $C([0,1])$ with the sup norm $\|\ \|_\infty$.

Before we conclude, we point out that the equations

$$\mu(n) = \int_0^1 t^n dz(t), \quad n = 0,1,2,\ldots,$$

may be regarded as the discrete analogue of the Laplace transformation

$$\mu(s) = \int_0^\infty e^{-su} d\alpha(u), \quad s \in \mathbf{K},$$

where $\alpha : [0,\infty) \to \mathbf{K}$ is of bounded variation on every subinterval of $[0,\infty)$. This can be seen by replacing the nonnegative integer n by the variable s and making the change of variable $t = e^{-u}, u \in [0,\infty)$, so that $\alpha(u) = -z(e^{-u})$. Results similar to Hausdorff's theorem 14.7 characterize the functions that can be expressed as Laplace integrals. (See [59], p. 306.)

Problems

14-1 Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. Consider a $\mathbf{K}$-valued function y on $[a,b]$. Then $y \in L^q([a,b])$ if and only if $xy \in L^1([a,b])$ for every x in $L^p([a,b])$. (Compare Problem 9-14. Hint: Define $y_n(t) = y(t)$ if $|y(t)| \le n$ and $y_n(t) = 0$, if $|y(t)| > n$. Use 14.1 and 9.2(a) for $f_{y_n}, n = 1,2,\ldots$.)

14-2 Let $\{t_1, t_2, \ldots\}$ be a dense subset of $[a,b]$, c_j denote the characteristic function of $(a, t_j]$ and $X = \text{span}\{c_j : j = 1,2,\ldots\}$. Then X is dense in $L^p([a,b])$ for each $p, 1 \le p < \infty$.

14-3 Let $1 \le p < \infty$ and $1/p + 1/q = 1, X = C([a,b])$ with the norm $\|\ \|_p$, and for $y \in L^q([a,b])$, let

$$f_y(x) = \int_a^b xy\, dm, \quad x \in X.$$

Then $f_y \in X'$ and $\|f_y\| = \|y\|_q$. The map $F : L^q \to X'$ given by $F(y) = f_y$ is a surjective linear isometry.

14-4 Let $1 \le p < \infty$. For $t \in (a,b]$, let c_t denote the characteristic function of $(a,t]$ and $c_a = 0$. Let $X = \operatorname{span}\{c_t : t \in [a,b]\}$ and

$$\|x\| = \left(\int_a^b |x|^p\, dm\right)^{1/p}, \quad x \in X.$$

Let $g \in X'$. If $z(t) = g(c_t), t \in [a,b]$, then z' exists almost everywhere on $[a,b], \|z'\|_q = \|g\|$ and $\int_a^b xz'\, dm = g(x)$ for every $x \in X$. Further, X' is linearly isometric to $L^q([a,b])$.

14-5 For $t \in (0,1]$, let c_t denote the characteristic function of $(0,t]$ and $X = \operatorname{span}\{c_t : t \in (0,1]\}$. For $x \in X$, let $\|x\|_\infty = \sup\{|x(t)| : t \in [0,1]\}$. If $x = k_1 c_{t_1} + \cdots + k_n c_{t_n}$, $0 < t_1 < \cdots < t_m \le 1/2 < t_{m+1} < \cdots < t_n \le 1$, let

$$g(x) = k_{m+1} + \cdots + k_n.$$

Then $g \in X'$. Let f be a Hahn-Banach extension of g to $L^\infty([a,b])$ and

$$z(a) = 0, \quad z(t) = f(c_t), \quad t \in (a,b].$$

Then z is of bounded variation but not absolutely continuous on $[a,b]$. Also, $z'(t) = 0$ for each $t \ne 1/2$ in $[0,1]$, $f(c_t) \ne \int_a^t z'\, dm$ for each $t \in (1/2,1]$ and $\int_a^b c_{1/2}\, dz$ is not defined.

14-6 For $1 \le p \le \infty$, let $L^p = L^p([-\pi,\pi]), H^p = \{x \in L^p : \hat{x}(n) = 0$ for $n = -1,-2,\ldots\}$ and $H_0^p = \{x \in H^p : \hat{x}(0) = 0\}$. If $1 \le p < \infty$, then $(H^p)'$ is linearly isometric to L^q/H_0^q, and if $1 < p \le \infty$, then H^p is linearly isometric to $(L^q/H_0^q)'$, where $1/p + 1/q = 1$. (Hint: Problem 13-3. If $1 \le p < \infty$, H^p is the closure of the span of $\{x_0, x_1, x_2, \ldots\}$ in L^p and H_0^p in the closure of the span of $\{x_1, x_2, \ldots\}$ in L^p, where $x_n(t) = e^{int}, -\pi \le t \le \pi$.)

14-7 Let $E = (-\infty,\infty)$ or $[a,\infty)$ or $(-\infty,b]$, where a and b are real numbers. Let $1 \le p \le \infty, 1/p + 1/q = 1, X = L^p(E)$ and $Y = L^q(E)$. For a fixed $y \in Y$, define $f_y : X \to \mathbf{K}$ by

$$f_y(x) = \int_E xy\, dm, \quad x \in X.$$

Then $f_y \in X'$ and $\|f_y\| = \|y\|_q$. The map $F : Y \to X'$ defined by $F(y) = f_y, y \in Y$, is a linear isometry from Y to X'. It is surjective if $1 \le p < \infty$. (Hint: $E = \bigcup_{k=1}^\infty [a_k, b_k]$ with $[a_k, b_k] \subset [a_{k+1}, b_{k+1}]$. Use 14.1 and 14.3).

14-8 Let $1 \le p < \infty, X = L^p(\mathbf{R})$ and $0 \ne x_0 \in X$. Define $h : \mathbf{R} \to X$ by $h(0) = 0$ and for $s \ne 0$ in $\mathbf{R}$,

$$h(s)(t) = se^{it/s}x_0(t), \quad t \in \mathbf{R}.$$

Then for every $f \in X'$, the derivative of $f \circ h$ at 0 equals 0, but $[h(s) - h(0)]/s$ does not converge in X as $s \to 0$. (Compare Problem 9-9.)

14-9 Let $k(.,.)$ be a Lebesgue measurable function on $[a,b]\times[a,b]$, and let α_1 and α_∞ be defined as in 6.5(d). Let $1 \le p < \infty$. If $p = 1$, assume that $\alpha_1 < \infty$, and if $1 < p < \infty$, assume that $\alpha_1 < \infty$ and $\alpha_\infty < \infty$. Let $A : L^p \to L^p$ be defined by

$$A(x)(s) = \int_a^b k(s,t)\, x(t)\, dm(t), \quad x \in L^p, s \in [a,b].$$

If $1/p + 1/q = 1$ and F is the linear isometry from L^q onto $(L^p)'$ given in 14.1 and 14.3, then

$$(F^{-1}A'F)(y)(s) = \int_a^b k(t,s)\, y(t)\, dm(t), \quad y \in L^q, s \in [a,b].$$

(Compare 13.4(b) and 19.3.)

14-10 For a fixed $t \in [a,b]$, let $f_t \in (C([a,b]))'$ be defined by

$$f_t(x) = x(t), \quad x \in C([a,b]).$$

Let y_t be the function in $NBV([a,b])$ which represents f_t as in Theorem 14.5. If $t = a$, then y_t is the characteristic function of $(a,b]$, and if $a < t \le b$, then y_t is the characteristic function of $[t,b]$. If $a \le t_1 < t_2 < \cdots < t_{n-1} < t_n \le b$ and

$$f(x) = k_1 x(t_1) + \cdots + k_m x(t_m), \quad x \in C([a,b]),$$

then the function in $NBV([a,b])$ corresponding to $f \in (C([a,b]))'$ is a step function.

14-11 The duals of $P([a,b]), C^\infty([a,b])$ and $C^m([a,b]), m = 1,2,\ldots$, with the sup norm are linearly isometric to $NBV([a,b])$.

14-12 $NBV([a,b])$ with the total variation norm is a nonseparable Banach space, while $C([a,b])$ with the sup norm is a separable Banach space (Compare 14.5 and 13.1(b).)

14-13 Let $z \in BV([a,b])$. Then $\int_a^b x\,dz = 0$ for every $x \in C([a,b])$ if and only if $z(a) = z(t^-) = z(t^+) = z(b)$ for every $t \in (a,b)$.

14-14 For $t \in [a,b]$, let $\tilde{c}_t$ denote the characteristic function of $[a,t]$. Let $X = \operatorname{span}\{\tilde{c}_t : t \in [a,b]\}$ with

$$\|x\|_\infty = \sup\{|x(t)| : t \in [a,b]\}, \quad z \in X.$$

Let $Z = BV([a,b])$ with

$$\|z\| = |z(a)| + V(z), \quad z \in Z.$$

For a fixed $z \in Z$, define $f_z : X \to \mathbf{K}$ by

$$f_z(k_1\tilde{c}_{t_1} + \cdots + k_m\tilde{c}_{t_m}) = k_1 z(t_1) + \cdots + k_m(t_m).$$

Then $f_z \in X'$ and $\|f_z\| = \|z\|$. The map $F : Z \to X'$ defined by $F(z) = f_z$ is a linear isometry from Z onto X'. Hence $BV([a,b])$ is a Banach space.

14-15 Let T be a compact metric space and $f \in C(T)'$ be such that $f \neq 0$ and $f(xy) = f(x)f(y)$ for all x, y in $C(T)$. Then $f(1) = \|f\| = 1$ and f is a positive functional. In fact, there is some $t \in T$ such that $f(x) = x(t)$ for all $x \in C(T)$.

14-16 Let $X = C^1([a,b])$ with $\|x\| = |x(a)| + \|x'\|_\infty$ for $x \in X$ and $Y = \mathbf{K} \times NBV([a,b])$ with $\|(k,y)\| = \max\{|k|, V(y)\}$. For $(k,y) \in Y$, define $f_{k,y} : X \to \mathbf{K}$ by

$$f_{k,y}(x) = kx(a) + \int_a^b x'\,dy, \quad x \in X.$$

Then $f_{k,y} \in X'$ and $\|f_{k,y}\| = \|(k,y)\|$. The map $F : Y \to X'$ defined by

$$F((k,y)) = f_{k,y}, \quad (k,y) \in Y,$$

is a linear isometry from Y onto X'. (Hint: For $w \in C([a,b])$ and $t \in [a,b]$, define $G(w)(t) = \int_a^t w(s)\,ds$. For $f \in X'$, define $g : C([a,b]) \to \mathbf{K}$ by $g(w) = f(G(w))$. Then use 14.5 and its proof.)

In particular, if $z_1, z_2 \in BV([a,b])$, then there is a unique $y \in NBV([a,b])$ such that

$$\int_a^b x\,dz_1 + \int_a^b x'\,dz_2 = [z_1(b) - z_1(a)]x(a) + \int_a^b x'\,dy$$

for all $x \in X$. In fact, $y = y_1 + y_2$, where $y_1(t) = z_1(b)(t-a) - \int_a^t z_1(s)\,ds$ for $t \in [a,b]$ and y_2 is the unique function in $NBV([a,b])$ given by 14.4 with $z = z_2$.

14-17 Let $k(.,.)$ be a continuous function on $[a,b]\times[a,b]$. Let $A : C([a,b]) \to C([a,b])$ be defined by

$$A(x)(s) = \int_a^b k(s,t)\,x(t)\,dt, \quad x \in C([a,b]), s \in [a,b].$$

If F is the linear isometry from $NBV([a,b])$ onto $(C([a,b]))'$ as in 14.5, then

$$(F^{-1}A'F)(y)(s) = \int_a^s \left[\int_a^b k(t,u)\;dy(t)\right] du.$$

14-18 For a fixed $t \in [a,b]$, let $A : C([a,b]) \to C([a,t])$ be defined by $A(x) = x|_{[a,t]}$. If F is the linear isometry from $NBV([a,b])$ onto $(C([a,b]))'$ and F_t is the linear isometry from $NBV([a,t])$ onto $(C([a,t]))'$ as in 14.5, then for every y ε NBV ([a,t]), we have

$$(F^{-1}A'F_t)(y)(s) = \begin{cases} y(s), & \text{if } a \le s \le t \\ y(t), & \text{if } t < s \le b. \end{cases}$$

14-19 Let $z \in BV([0,1])$ and $\mu(n) = \int_0^1 t^n dz(t)$ for $n = 0,1,2,\ldots$. Then the sequence $(\mu(n))$ is convergent. (Hint: If z is nondecreasing, then the sequence $(\mu(n))$ is nonincreasing and bounded below by 0.) [In fact, $\mu(n) \to z(1) - z(1^-)$ as $n \to \infty$.]

14-20 A sequence of scalars $(\mu(n)), n = 0,1,2,\ldots$, is a moment sequence if and only if $\mu(n) = \mu_1(n) - \mu_2(n) + i\mu_3(n) - i\mu_4(n)$, where $i = \sqrt{-1}$ and

$$(-1)^{n-j}\Delta^{n-j}(\mu_k)(j) \ge 0$$

for all $k = 1,\ldots,4, n = 0,1,2,\ldots$ and $j = 0,\ldots,n$.

14-21 Let $y \in NBV([a,b])$ and $\mu(n) = \int_0^1 t^n\,dy(t)$ for $n = 0,1,2,\ldots$. Then

$$V(y) = \sup\{\sum_{j=0}^n \binom{n}{j} |\Delta^{n-j}(\mu)(j)| : n = 0,1,2,\ldots\},$$

where Δ is the forward difference operator.

14-22 Let $\mu(n)$ be the nth moment of an absolutely continuous function z on $[a, b], n = 0, 1, 2, \ldots$. Let $\alpha_{n,j}$ be defined as in (ii) of Theorem 14.7. If $z' \in L^p([0,1])$, with $1 \leq p < \infty$, then

$$(n+1)^{p-1} \sum_{j=0}^{n} |\alpha_{n,j}|^p \leq \|z'\|_p^p < \infty$$

and if $z' \in L^\infty(0,1])$, then

$$(n+1)|\alpha_{n,j}| \leq \|z'\|_\infty < \infty$$

for $n = 0, 1, 2, \ldots$. (Hint: $\alpha_{n,j} = \int_0^1 \binom{n}{j} t^j (1-t)^{n-j} z'(t)dt$ and $(n+1)\int_0^1 \binom{n}{j} t^j(1-t)^{n-j}dt = 1$) [If $1 < p \leq \infty$, then the converse also holds but it is difficult to prove. See [59], pp. 109-111.

15 Weak and Weak* Convergence

In this section, we introduce new modes of convergence of sequences in a normed space X and in its dual X'. They are, in general, not as strong as the norm convergence. For this reason, they often prove to be more freely available and useful.

A sequence (x_n) in a normed space X is said to be **weak convergent** (or **converge weakly**) if there is some x in X such that $x'(x_n) \to x'(x)$ in K for every $x' \in X'$. In that case, we write $x_n \xrightarrow{w} x$ in X.

Let us discuss some aspects of the notion of weak convergence. First, if $x_n \xrightarrow{w} x$ in X and also $x_n \xrightarrow{w} y$ in X, then $y = x$. For if $y \neq x$, then by 7.10(a), there is some $x' \in X'$ such that $x'(y - x) = \|y - x\| \neq 0$, while

$$x'(y-x) = x'(y) - x'(x) = \lim_{n\to\infty} x'(x_n) - \lim_{n\to\infty} x'(x_n) = 0,$$

a contradiction. Hence, if $x_n \xrightarrow{w} x$ in X, then we say that x is the **weak limit** of (x_n) in X.

Next, if $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ in X, then it is easy to see that $x_n + y_n \xrightarrow{w} x + y$ in X, and if $k_n \to k$ in K, then $k_n x_n \xrightarrow{w} kx$ in X. Thus the linear space operations of addition and scalar multiplication n X are compatible with the weak convergence of sequences in X.

Further, if $x_n \to x$ in X, that is, $\|x_n - x\| \to 0$, then $x_n \xrightarrow{w} x$ in X, because

$$|x'(x_n) - x'(x)| = |x'(x_n - x)| \leq \|x'\| \, \|x_n - x\|$$

for every $x' \in X'$. This fact explains the nomenclature 'weak convergence'.

In general, $x_n \xrightarrow{w} x$ in X does not imply that $x_n \to x$ in X. We give some examples to illustrate this feature.

Let $X = \ell^p, 1 < p < \infty$, and $e_n = (0, \ldots, 0, 1, 0, 0, \ldots)$, where 1 occurs only in the nth entry. Then $\|e_n\|_p = 1$ for each n, so that $e_n \not\to 0$ in X. If $x' \in X'$, then by 13.3(b), there is some $y \in \ell^q$ with $1/p + 1/q = 1$ such that

$$x'(x) = \sum_{j=1}^{\infty} x(j)y(j), \quad x \in X.$$

Since $\sum_{n=1}^{\infty} |y(n)|^q < \infty$, $x'(e_n) = y(n) \to 0$ as $n \to \infty$ and hence $e_n \xrightarrow{w} 0$ in X. [Note that (e_n) is not weak convergent to 0 in ℓ^1. This can be seen by considering the functional $x'(x) = \sum_{j=1}^{\infty} x(j), x \in \ell^1$.]

As another example, let $X = L^p([-\pi, \pi]), 1 \leq p < \infty$, and $x_n(t) = e^{int}, t \in [-\pi, \pi], n = 1, 2, \ldots$. Then $\|x_n\|_p = (2\pi)^{1/p}$ for each n, so that $x_n \not\to 0$ in X. If $x' \in X'$, then by 14.3, there is some $y \in L^q([-\pi, \pi])$ with $1/p + 1/q = 1$ such that

$$x'(x) = \int_{-\pi}^{\pi} xy \, dm, \quad x \in X.$$

Then $x'(x_n) = \int_{-\pi}^{\pi} y(t)e^{int} \, dm(t) = 2\pi\hat{y}(-n) \to 0$ as $n \to \infty$ by the Riemann-Lebesgue lemma (4.9(a)) applied to $y \in L^q([-\pi, \pi])$, which is a subset of $L^1([-\pi, \pi])$. Hence $x_n \xrightarrow{w} 0$ in X.

See Problem 15-3 for an example of a sequence (x_n) in $X = C([a,b])$ with the sup norm such that $x_n \xrightarrow{w} x$ in X, but $x_n \not\to x$ in X.

The examples given above make it plain that in order to determine the weak convergence of a sequence (x_n) in X, we must have a pretty good idea of the elements of X'.

15.1 Theorem

Let X be a normed space and (x_n) be a sequence in X. Then (x_n) is weak convergent in X if and only if (i) (x_n) is a bounded sequence in X and (ii) there is some $x \in X$ such that $x'(x_n) \to x'(x)$ for every x' in some subset of X' whose span is dense in X'.

In that case, for every subsequence (x_{n_k}) of (x_n), x belongs to the closure of $\operatorname{co}(\{x_{n_1}, x_{n_2}, \ldots\})$ and $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.

Proof:

Let $x_n \xrightarrow{w} x$ in X and $E = \{x_n : n = 1, 2, \ldots\}$. For every $x' \in X'$, $x'(E)$ is a bounded subset of $\mathbf{K}$, since $(x'(x_n))$ is a convergent sequence. By 9.3(a), E is a bounded subset of X, that is, condition (i) holds. Also, condition (ii) is clearly satisfied.

Conversely, assume that conditions (i) and (ii) hold. Define maps F_n and F from X' to $\mathbf{K}$ by

$$F_n(x') = x'(x_n) \quad \text{and} \quad F(x') = x'(x), \quad x' \in X'.$$

Then F_n and F are linear. Also, $\|F_n\| \le \|x_n\|$ and $\|F\| \le \|x\|$. By 8.3(b), we see that $F_n(x') \to F(x')$ for every $x' \in X'$, that is, $x_n \xrightarrow{w} x$.

Let (x_{n_k}) be a subsequence of (x_n) and C denote the closure of $\operatorname{co}(\{x_{n_1}, x_{n_2}, \ldots\})$. If $x \notin C$, then there is some $\delta > 0$ such that $U(x, \delta) \bigcap C = \emptyset$. By the Hahn-Banach separation theorem (7.5), there are $x' \in X'$ and $\alpha \in \mathbf{R}$ such that

$$\operatorname{Re} x'(x) < \alpha \le \operatorname{Re} x\,(x_{n_k}), \quad k = 1, 2, \ldots.$$

This contradicts $x'(x_{n_k}) \to x'(x)$ and shows that $x \in C$.

Next, let $r = \liminf_{n\to\infty} \|x_n\|$ and (x_{n_k}) be a subsequence of (x_n) such that $\|x_{n_k}\| \to r$ as $k \to \infty$. Let $\epsilon > 0$. Then there is a positive integer k_0 such that $\|x_{n_k}\| \le r + \epsilon$ for all $k \ge k_0$. By what we have proved above, x belongs to the closure of $\text{co}\,(\{x_{n_k} : k = k_0, k_0 + 1, \ldots\})$. Hence

$$\|x\| \le \sup\{\|x_{n_k}\| : k = k_0, k_0 + 1, \ldots\} \le r + \epsilon.$$

Since this is true for every $\epsilon > 0$, we see that $\|x\| \le r$. □

It follows from 15.1 that if $x_n \xrightarrow{w} x$ in X, then there is a sequence (y_n) of convex combinations of $x_1, x_2, \ldots$ such that $\|y_n - x\| \to 0$.

15.2 Examples

(a) Let X be a finite dimensional normed space. Then $x_n \xrightarrow{w} x$ in X if and only if $x_n \to x$ in X. This can be seen as follows. Let $\{y_1, \ldots, y_m\}$ be a basis for X. Then for all $x \in X$, we have

$$x = f_1(x)y_1 + \cdots + f_m(x)y_m,$$

where $f_j \in X'$ by 6.1. Now, if $x_n \xrightarrow{w} x$ in X, then $f_j(x_n) \to f_j(x)$ as $n \to \infty$ for each $j = 1, \ldots, m$. Hence $x_n \to x$ in X by the continuity of scalar multiplication and addition.

(b) Let $X = \ell^1$. We establish **Schur's lemma** which states that $x_n \xrightarrow{w} x$ in X if and only if $x_n \to x$ in X. If this were not the case, there would exist a sequence (x_n) in ℓ^1 such that $x_n \xrightarrow{w} 0$ in ℓ^1, but $x_n \not\to 0$ in ℓ^1. By passing to a subsequence of (x_n), if necessary, we can assume that

$$\|x_n\|_1 = \sum_{j=1}^{\infty} |x_n(j)| \ge \epsilon$$

for some $\epsilon > 0$ and $n = 1, 2, \ldots$. For $j = 1, 2, \ldots$, let f_j denote the jth **coordinate functional**: $f_j(x) = x(j), x \in \ell^1$. Then for $j = 1, 2, \ldots$,

$$x_n(j) = f_j(x_n) \to 0 \text{ as } n \to \infty.$$

Set $n_0 = m_0 = 1$ and inductively define sequences (n_k) and (m_k) of natural numbers as follows. Having defined n_{k-1} and m_{k-1}, let n_k be the smallest integer greater than n_{k-1} such that

$$\sum_{j=1}^{m_{k-1}} |x_{n_k}(j)| < \frac{\epsilon}{5},$$

and m_k be the smallest integer greater than m_{k-1} such that

$$\sum_{j=m_k+1}^{\infty} |x_{n_k}(j)| < \frac{\epsilon}{5}.$$

Define

$$y(j) = \begin{cases} 1, & \text{if } j = 1 \\ \operatorname{sgn} x_{n_k}(j), & \text{if } m_{k-1} + 1 \le j \le m_k,\ k = 1, 2, \ldots \end{cases}$$

Then $y \in \ell^\infty$ and $\|y\|_\infty \le 1$. Define $f_y : \ell^1 \to \mathbf{K}$ by

$$f_y(x) = \sum_{j=1}^{\infty} x(j)y(j), \quad x \in \ell^1.$$

Then $f_y \in (\ell^1)'$. Now, for each fixed $k = 1, 2, \ldots,$

$$x_{n_k}(j)y(j) = |x_{n_k}(j)| \quad \text{for } m_{k-1} + 1 \le j \le m_k,$$

so that

$$\left|f_y(x_{n_k}) - \sum_{j=1}^{\infty} |x_{n_k}(j)|\,\right| \le 2\sum_{j=1}^{m_{k-1}} |x_{n_k}(j)| + 2\sum_{j=m_k+1}^{\infty} |x_{n_k}(j)| < \frac{4\epsilon}{5}$$

and

$$|f_y(x_{n_k})| \ge \|x_{n_k}\|_1 - \frac{4\epsilon}{5} \ge \frac{\epsilon}{5}.$$

But since $x_n \xrightarrow{w} 0$, we see that $x_{n_k} \xrightarrow{w} 0$ and, in particular, $f_y(x_{n_k}) \to 0$ as $k \to \infty$. This contradiction completes the proof.

(c) Let $X = \ell^p, 1 < p < \infty$. Then $x_n \xrightarrow{w} x$ in X if and only if (x_n) is a bounded sequence in X and $x_n(j) \to x(j)$ as $n \to \infty$ for each $j = 1, 2, \ldots$. This follows from 15.1 since the span of the set of all coordinate functionals $f_j, j = 1, 2, \ldots$ is dense in X'. This can be seen as follows. Let $1/p + 1/q = 1$ and consider the surjective linear isometry $F : \ell^q \to X'$ given by

$$F(y)(x) = \sum_{j=1}^{\infty} x(j)y(j), \quad y \in \ell^q \text{ and } x \in \ell^p.$$

Since $1 < q < \infty, \{e_1, e_2, \ldots\}$ is a Schauder basis for ℓ^q and $F(e_j) = f_j, j = 1, 2, \ldots$. Hence if $f \in X'$ and $F(y) = f$ with $y \in \ell^q$, then

$$f = F(y) = F\left(\sum_{j=1}^{\infty} y(j)e_j\right) = \sum_{j=1}^{\infty} y(j)f_j = \lim_{n\to\infty} \sum_{j=1}^{n} y(j)f_j.$$

Thus the span of $\{f_1, f_2 \ldots\}$ is dense in X'.

(d) Let $X = L^1([a, b])$. Then $x_n \xrightarrow{w} x$ in X if and only if (x_n) is a bounded sequence in X and

$$\int_E x_n \, dm \to \int_E x \, dm$$

for every measurable subset E of $[a, b]$. This follows from 15.1 since the span of the set $S = \{f_E : E \text{ is a measurable subset of } [a, b]\}$ is dense in X', where

$$f_E(x) = \int_E x \, dm, \quad x \in X.$$

Note that X' is linearly isometric to $L^\infty([a, b])$ by 14.3 and the set of all simple measurable functions is dense in $L^\infty([a, b])$ by 4.7(a).

(e) Let $X = L^p([a, b]), 1 < p < \infty$. Then $x_n \xrightarrow{w} x$ in X if and only if (x_n) is a bounded sequence in X and

$$\int_c^d x_n \, dm \to \int_c^d x \, dm$$

for every subinterval $[c, d]$ of $[a, b]$. This follows from 15.1 since the span of the set $S = \{f_{c,d} : [c, d] \subset [a, b]\}$, where

$$f_{c,d}(x) = \int_c^d x \, dm, \quad x \in X,$$

is dense in X'. Note that X' is linearly isometric to $L^q([a, b])$ by 14.3, where $1/p + 1/q = 1$ and the set of all step functions is dense in $L^q([a, b])$ by 4.7(c), since $1 < q < \infty$.

(f) Let $X = C([a, b])$ with the sup norm. Then $x_n \xrightarrow{w} x$ in X if and only if (x_n) is a bounded sequence in X and $x_n(t) \to x(t)$ for each $t \in [a, b]$. This can be seen as follows. Let $x_n \xrightarrow{w} x$ in X. Then the sequence (x_n) is bounded by 15.1 and if for a fixed $t \in [a, b]$, we let

$$f_t(x) = x(t), \quad x \in X,$$

then $f_t \in X'$, so that $x_n(t) = f_t(x_n) \to f_t(x) = x(t)$ as $n \to \infty$. Conversely, assume that the sequence (x_n) is bounded and $x_n(t) \to x(t)$ for each $t \in [a, b]$. If $f \in X'$, then by 14.5, there is some y in $NBV([a, b])$ such that

$$f(x) = \int_a^b x\, dy, \quad x \in X.$$

Now $y = y_1 - y_2 + iy_3 - iy_4$, where each y_j is a nondecreasing function on $[a, b]$. By the bounded convergence theorem for the Lebesgue-Stieltjes measure induced by y_j on $[a, b]$, we see that

$$\int_a^b x_n\, dy_j \to \int_a^b x\, dy_j \quad \text{for } j = 1, \ldots, 4,$$

so that

$$f(x_n) = \int_a^b x_n\, dy \to \int_a^b x\, dy = f(x).$$

(Compare 4.1(b).) Thus $x_n \xrightarrow{w} x$ in X. □

We now turn to a mode of convergence of sequences in the dual X' of a normed space X. A sequence (x'_n) in X' is said to be **weak* convergent** if there is some $x' \in X'$ such that $x'_n(x) \to x'(x)$ in $\mathbf{K}$ for every $x \in X$. In that case, we write $x'_n \xrightarrow{w^*} x'$ in X'. The nomenclature 'weak* convergence' comes from the fact that the dual of X is sometimes denoted by X^*.

It is obvious that if $x'_n \xrightarrow{w*} x'$ and also $x'_n \xrightarrow{w*} y'$ in X', then $y' = x'$. Hence, if $x'_n \xrightarrow{w*} x'$, then we say that x' is the **weak* limit** of (x'_n) in X'. As in the case of the weak convergence in X, it is easy to see that the linear space operations of addition and scalar multiplication in X' are compatible with the weak* convergence of sequences in X'.

Further, if $x'_n \to x'$ in X', that is, $\|x'_n - x'\| \to 0$, then $x'_n \xrightarrow{w*} x'$ in X' because

$$|x'_n(x) - x'(x)| = |(x'_n - x')(x)| \leq \|x'_n - x'\| \, \|x\|$$

for every $x \in X$.

In general, $x'_n \xrightarrow{w*} x'$ in X' does not imply that $x'_n \to x'$ in X'. For example, if $X = \ell^p, 1 \leq p < \infty$, then X' is linearly isometric to ℓ^q, where $1/p + 1/q = 1$, and it can be seen that $e_n \xrightarrow{w*} 0$ in ℓ^q, but $e_n \not\to 0$ in ℓ^q. Again, if $X = L^p([-\pi, \pi]), 1 \leq p < \infty$, then X' is linearly isometric to $L^q([-\pi, \pi])$, where $1/p + 1/q = 1$, and if we let $y_n(t) = e^{int}$ for t in $[-\pi, \pi]$, then it follows by the Riemann-Lebesgue lemma (4.9(a)) that $y_n \xrightarrow{w*} 0$. However, $y_n \not\to 0$ in $L^q([-\pi, \pi])$. See Problem 15-18 for an example of a sequence (y_n) in $NBV([a, b])$, regarded as the dual of $C([a, b])$, such that $y_n \xrightarrow{w*} 0$, but $y_n \not\to 0$ in $NBV([a, b])$.

Since X'' is the dual of the normed space X', we see that $x'_n \xrightarrow{w} x'$ in X' if and only if $x''(x'_n) \to x''(x')$ for every $x'' \in X''$. Consider the canonical embedding $J : X \to X''$. If $x'_n \xrightarrow{w} x'$ in X', then for every $x \in X$, we have

$$x'_n(x) = J(x)(x'_n) \to J(x)(x') = x'(x),$$

that is, $x' \xrightarrow{w*} x'$ in X'.

Thus we have three modes of convergence in the normed dual X': norm convergence, weak convergence and weak* convergence. Norm convergence implies weak convergence and weak convergence implies weak* convergence.

We give an example of a Banach space X such that the three modes of convergence in X' are distinct. Let $X = \ell^1$. By 13.2, X'

can be identified with ℓ^∞. Consider the sequence (e_n) in ℓ^∞. Since $\|e_n\|_\infty = 1$ for each n, we see that $\|e_n\|_\infty \not\to 0$. However, we show that $e_n \xrightarrow{w} 0$ in ℓ^∞. Suppose for a moment that this is not the case. Then there exists some $f \in (\ell^\infty)'$ such that $f(e_n) \not\to 0$. Hence there are positive integers $n_1 < n_2 < \cdots$ and some $\delta > 0$ such that $|f(e_{n_j})| \geq \delta > 0$ for each $j = 1, 2, \ldots$. For $m = 1, 2, \ldots$, define

$$x_m = \operatorname{sgn} f(e_{n_1})e_{n_1} + \cdots + \operatorname{sgn} f(e_{n_m})e_{n_m}.$$

Then for each m, $\|x_m\|_\infty \leq 1$, but

$$f(x_m) = |f(e_{n_1})| + \cdots + |f(e_{n_m})| \geq m\delta,$$

which tends to ∞ as $m \to \infty$. This contradicts the continuity of f on ℓ^∞. Hence $e_n \xrightarrow{w} 0$ in ℓ^∞. Next, consider the sequence (a_n) in ℓ^∞, where $a_n = (1, \ldots, 1, 0, 0, \ldots)$ with 1 occurring only in the first n entries and let $a = (1, 1, \ldots)$. Then (a_n) is not weak convergent to a. To see this, find $f \in (\ell^\infty)'$, such that $f(x) = 0$ for every $x \in c_0$ and $f(a) \neq 0$ by appealing to 7.10(b). Since each a_n is in c_0, we see that $f(a_n) = 0$. Thus $f(a_n) \not\to f(a)$. However, we show that $a_n \xrightarrow{w^*} a$ in ℓ^∞. Let F denote the linear isometry from ℓ^∞ to $(\ell^1)'$ considered in 13.2. Then for each $x \in \ell^1$,

$$F(a_n)(x) = \sum_{j=1}^{\infty} x(j)a_n(j) = \sum_{j=1}^{n} x(j),$$

while

$$F(a)(x) = \sum_{j=1}^{\infty} x(j)a(j) = \sum_{j=1}^{\infty} x(j).$$

Hence $F(a_n)(x) \to F(a)(x)$ for every $x \in \ell^1$, that is, $a_n \xrightarrow{w^*} a$. Thus, in general, weak convergence is weaker than norm convergence, and weak* convergence is even weaker than the weak convergence !

15.3 Theorem

Let (x_n') be a sequence in a normed space X. If (i) (x_n') is bounded and (ii) $(x_n'(x))$ is a Cauchy sequence in $\mathbf{K}$ for each x in a subset of X

whose span is dense in X, then (x_n') is weak* convergent in X'. The converse holds if X is a Banach space.

If $x_n' \overset{w*}{\to} x'$ in X', then $\|x'\| \le \liminf_{n\to\infty} \|x_n'\|$.

Proof:
If conditions (i) and (ii) are satisfied, then by 8.3(b) with $Y = \mathbf{K}$, there is some $x' \in BL(X,K) = X'$ such that $x_n'(x) \to x'(x)$ for all $x \in X$. Hence $x_n' \overset{w*}{\to} x'$.

Conversely, assume that $x_n' \overset{w*}{\to} x'$. Then for $x \in X$, we have

$$|x'(x)| = \lim_{n\to\infty} |x_n'(x)| \le (\liminf_{n\to\infty} \|x_n'\|)\|x\|,$$

so that $\|x'\| \le \liminf_{n\to\infty} \|x_n'\| \le \sup_{n=1,2,\ldots} \|x_n'\|$.

If X is a Banach space, then $\sup_{n=1,2,\ldots} \|x_n'\| < \infty$ by the Banach-Steinhaus theorem (9.2(a)). Hence condition (i) holds. Also, condition (ii) is clearly satisfied. □

We remark that if X is not a Banach space, then we may have $x_n' \overset{w*}{\to} x'$ in X' without the sequence (x_n') being bounded. Let $X = c_{00}$ with the norm $\| \ \|_p$, $1 \le p \le \infty$. For $n = 1, 2, \ldots$, define

$$x_n'(x) = nx(n), \quad x \in X.$$

Then $x_n' \overset{w*}{\to} 0$ in X', but $\|x_n'\| = n \to \infty$ as $n \to \infty$. Note that (x_n') is not weak convergent in X' by 15.1.

In Problem 15-17, we give necessary and sufficient conditions for the weak* convergence of a sequence in X' when X is a finite dimensional normed space, when $X = \ell^p$ or $L^p([a,b])$ with $1 \le p < \infty$ and when $X = C([a,b])$ with the sup norm.

Bolzano-Weierstrass Property

The classical Bolzano-Weierstrass theorem says that every bounded sequence in $\mathbf{K}$ has a convergent subsequence. If X is a normed space, it can be seen from 5.5 that every bounded sequence in X has a convergent subsequence if and only if X is finite dimensional. Thus, when X is an infinite dimensional normed space, the classical Bolzano-Weierstrass property does not hold for the norm convergence of sequences in X. It is, therefore, worthwhile to investigate whether every bounded sequence in X has a weak convergent subsequence, and whether every bounded sequence in X' has a weak* convergent subsequence.

Since the weak convergence in ℓ^1 is the same as the norm convergence in ℓ^1 by 15.2(b) and since ℓ^1 is infinite dimensional, it follows that not every bounded sequence in ℓ^1 has a weak convergent subsequence. In fact, $\|e_n\|_1 = 1$ for all n, but (e_n) does not have a weak convergent subsequence since $\|e_n - e_m\|_1 = 2$ for all $n \neq m$. Alternatively, if (e_{n_k}) is a subsequence of (e_n) and $e_{n_k} \xrightarrow{w} x_0$ in ℓ^1, then for each $j = 1, 2, \ldots, e_{n_k}(j) \to x_0(j)$ as $k \to \infty$, so that $x_0 = 0$, but if we let $f(x) = \sum_{j=1}^{\infty} x(j)$ for $x \in \ell^1$, then $f \in \ell^1$ and $f(e_{n_k}) \to 1$, whereas $f(x_0) = f(0) = 0$. This contradiction shows that (e_n) has no weak convergent subsequence in ℓ^1. As another example of this situation, let $X = c_0$ with the norm $\| \ \|_\infty$ and for $n = 1, 2, \ldots$, let $a_n = (1, \ldots, 1, 0, 0, \ldots)$, with 1 occurring only in the first n entries. Then $\|a_n\|_\infty = 1$ for all n. Suppose there is a subsequence (a_{n_k}) of (a_n) and $x \in c_0$ such that $a_{n_k} \xrightarrow{w} x$. Then for every $j = 1, 2, \ldots, a_{n_k}(j) \to x(j)$ as $k \to \infty$. But for each fixed j, $a_{n_k}(j) = 1$ for all k with $n_k \geq j$. Hence $x(j) = 1$ for each $j = 1, 2, \ldots$. This is not possible since $x \in c_0$. Thus no subsequence of (a_n) is weak convergent in X, although (a_n) is a bounded sequence in X.

In the next section we shall introduce reflexive normed spaces and show that they are characterized by the Bolzano-Weierstrass property for weak convergence.

Next, to see that not every bounded sequence in X' has a weak* convergent subsequence, let $X = \ell^\infty$ and for $n = 1, 2, \ldots$, define

$$f_n(x) = x(n), \quad x \in X.$$

Then $f_n \in X$ and $\|f_n\| = 1$ for all n. Let (f_{n_k}) be a subsequence of (f_n). Define $x \in X$ by

$$x(j) = \begin{cases} 1, & \text{if } j = n_k \text{ and } k \text{ is odd} \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_{n_k}(x) = x(n_k) = 1$ for every odd k and $f_{n_k}(x) = x(n_k) = 0$ for every even k. Since $(f_{n_k}(x))$ does not converge in $\mathbf{K}$, (f_{n_k}) cannot be weak* convergent in X'.

After giving all these negative results, we prove a remarkably useful positive result.

15.4 Theorem (Banach, 1932)
Let X be a separable normed space. Then every bounded sequence in X' has a weak* convergent subsequence.

Proof:
Let $\{x_1, x_2, \ldots\}$ be a countable dense subset of X. Consider a bounded sequence (x_n') in X'. Then $(x_n'(x_1))$ is a bounded sequence in $\mathbf{K}$. By the classical Bolzano-Weierstrass theorem, there is a convergent subsequence $(x_{n,1}'(x_1))$ of $(x_n'(x_1))$. Similarly, there is a convergent subsequence $(x_{n,2}'(x_2))$ of $(x_{n,1}'(x_2))$. Inductively, for each $k = 2, 3, \ldots$, there is a convergent subsequence $(x_{n,k}'(x_k))$ of $(x_{n,k-1}'(x_k))$. Define $y_n' = x_{n,n}'$ for $n = 1, 2, \ldots$. Then (y_n') is a subsequence of the bounded sequence (x_n') and $(y_n'(x_k))$ converges for each $k = 1, 2, \ldots$. Since $\{x_1, x_2, \ldots\}$ is dense in X, it follows from 15.3 that (y_n') is weak* convergent in X'. □

Theorems 15.3 and 15.4 imply that the closed unit ball of the dual of a separable normed space X is **weak* sequentially compact**, that is, if $x_n' \in X'$ with $\|x_n'\| \leq 1$, then there is a subsequence (x_{n_k}') of (x_n') such that $x_{n_k}' \xrightarrow{w^*} x'$, where $\|x'\| \leq \liminf_{k\to\infty} \|x_{n_k}'\| \leq 1$.

If X is an arbitrary normed space, then this result no longer holds, as we have seen before. However, if we replace sequences by nets, it can be proved that every bounded net in X' has a weak* convergent subnet. This is known as the **Banach-Alaoglu theorem (1940)**. Since we have restricted ourselves to metric topologies in this book, we refer the reader to 3.15 of [53] for a proof of this result.

We now consider some applications of 15.4.

15.5 Theorem

Let $k_n \in \mathbf{K}$ for $n = 0, \pm 1, \pm 2, \ldots$. Define

$$s_m(t) = \sum_{n=-m}^{m} k_n e^{int}, \quad t \in [-\pi, \pi],\ m = 0, 1, 2, \ldots$$

and

$$a_m = \frac{s_0 + \cdots + s_{m-1}}{m}, \quad m = 1, 2, \ldots.$$

Let $1 < q \leq \infty$ and assume that the sequence (a_m) is bounded in $L^q([-\pi, \pi])$. Then there is some $y \in L^q([-\pi, \pi])$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} y(t) e^{-int}\, dm(t) = k_n, \quad n = 0, \pm 1, \pm 2, \ldots,$$

that is, $\sum_{n=-\infty}^{\infty} k_n e^{int}$ is the Fourier series of some y in $L^q([-\pi, \pi])$.

Proof:

For $x \in L^1([-\pi, \pi])$ and $n = 0, \pm 1, \pm 2, \ldots$, let

$$\hat{x}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-int}\, dm(t)$$

denote the nth Fourier coefficient of x. Then for each $n = 0, \pm 1, \pm 2, \ldots$ and $m > |n|$, we have

$$\hat{a}_m(n) = \frac{m - |n|}{m} k_n.$$

Let p satisfy $1/p + 1/q = 1$, so that $1 \leq p < \infty$. Consider $X = L^p([-\pi, \pi])$ and for $m = 1, 2, \ldots$, define

$$x'_m(x) = \int_{-\pi}^{\pi} x a_m\, dm, \quad x \in X.$$

Then $|x'_m(x)| \le \|x\|_p \|a_m\|_q$ by Hölder's inequality (4.5(a)). Since the sequence $(\|a_m\|_q)$ is assumed to be bounded, we see that (x'_m) is a bounded sequence in X'. By the Banach-Alaoglu theorem (15.4), there is a subsequence (x'_{m_j}) of (x'_m) which is weak* convergent to x' in X'. By the Riesz representation theorem (14.3), there is some y in $L^q(-\pi, \pi])$ such that

$$x'(x) = \int_{-\pi}^{\pi} xy\,dm, \quad x \in X.$$

For $n = 0, \pm 1, \pm 2, \ldots$, let $x_n(t) = e^{-int}/2\pi, t \in [-\pi, \pi]$. Then $x'_m(x_n) = \hat{a}_m(n)$ and $x'(x_n) = \hat{y}(n)$ for all m and n. Hence for each fixed $n = 0, \pm 1, \pm 2, \ldots$, we obtain

$$\hat{y}(n) = \lim_{j \to \infty} \hat{a}_{m_j}(n) = \lim_{j \to \infty} \frac{m_j - |n|}{m_j} k_n = k_n. \qquad \square$$

It can be seen from Problem 15-21 that the preceding result does not hold for the case $q = 1$. Problem 15-22 gives analogous result for this case.

15.6 Theorem

Let (z_n) be a sequence of nondecreasing functions on $[a, b]$ such that $\alpha \le z_n(t) \le \beta$ for some constants α, β, all $n = 1, 2, \ldots$ and $t \in [a, b]$. Then there is a nondecreasing function z on $[a, b]$ such that z is right continuous on (a, b) and for some subsequence (z_{n_j}) of (z_n), we have $z_{n_j}(a) \to z(a), z_{n_j}(b) \to z(b)$ and $z_{n_j}(t) \to z(t)$ for every $t \in (a, b)$ at which z is continuous.

Proof:

Consider $X = C([a, b])$. For $n = 1, 2, \ldots$, let $y_n = z_n - z_n(a)$ and

$$x'_n(x) = \int_a^b x\,dy_n, \quad x \in X.$$

Then $|x'_n(x)| \le \|x\|_\infty [z_n(b) - z_n(a)] \le (\beta - \alpha)\|x\|_\infty$. Hence (x'_n) is a bounded sequence in X'. By the Banach-Alaoglu theorem (15.4), there is a subsequence (x'_{n_j}) of (x'_n) which is weak* convergent to some $x' \in X'$. By the Riesz representation theorem (14.5), there is some $y \in NBV([a,b])$ such that

$$x'(x) = \int_{-\pi}^{\pi} x\,dy, \quad x \in X.$$

The very definition of $NBV([a,b])$ shows that $y(a) = 0$ and y is right continuous on (a,b) . Also, for all $x \in X$ with $x \geq 0$, we have

$$\int_a^b x\,dy = \lim_{j\to\infty} \int_a^b x\,dy_{n_j} \geq 0,$$

since each y_{n_j} is nondecreasing. Hence y is nondecreasing by 14.5.

Next, let $t \in (a,b)$ and $\epsilon > 0$ be such that $t + \epsilon < b$. Define

$$x_\epsilon(s) = \begin{cases} 1, & \text{if } a \leq s \leq t \\ \dfrac{(\epsilon - s + t)}{\epsilon}, & \text{if } t \leq s \leq t + \epsilon \\ 0, & \text{if } t + \epsilon \leq s \leq b. \end{cases}$$

Then $x_\epsilon \in C([a,b])$ and $0 \leq x_\epsilon \leq 1$. Now

$$y_{n_j}(t) = \int_a^t dy_{n_j} = \int_a^t x_\epsilon\,dy_{n_j} \leq \int_a^b x_\epsilon\,dy_{n_j}$$

and

$$\int_a^b x_\epsilon\,dy = \int_a^{t+\epsilon} x_\epsilon\,dy \leq \int_a^{t+\epsilon} dy = y(t+\epsilon).$$

Hence

$$\limsup_{j\to\infty} y_{n_j}(t) \leq \lim_{j\to\infty} \int_a^b x_\epsilon\,dy_{n_j} = \int_a^b x_\epsilon\,dy \leq y(t+\epsilon).$$

Similarly, we can show that for every $\epsilon > 0$ with $a < t - \epsilon$,

$$y(t-\epsilon) \leq \liminf_{j\to\infty} y_{n_j}(t).$$

Since these inequalities hold for all sufficiently small positive ϵ,

$$\lim_{\epsilon\to 0} y(t-\epsilon) \leq \liminf_{j\to\infty} y_{n_j}(t) \leq \limsup_{j\to\infty} y_{n_j}(t) \leq \lim_{\epsilon\to 0} y(t+\epsilon).$$

As y is a nondecreasing function on $[a, b]$, we see that

$$y(t^-) = \lim_{\epsilon \to 0} y(t - \epsilon) \quad \text{and} \quad \lim_{\epsilon \to 0} y(t + \epsilon) = y(t^+).$$

If y is continuous at $t \in (a, b)$, then $y(t^-) = y(t) = y(t^+)$, so that $\lim_{j \to \infty} y_{n_j}(t)$ exists and equals $y(t)$.

Finally, we note that $(z_{n_j}(a))$ is a bounded sequence in $\mathbf{R}$ and hence we can assume, by passing to a subsequence if necessary, that $z_{n_j}(a) \to c$ in $\mathbf{R}$. Let $z = y + c$. Then z is a nondecreasing function on $[a, b]$, it is right continuous on (a, b) and

$$\lim_{j \to \infty} z_{n_j}(t) = \lim_{j \to \infty} [y_{n_j}(t) + z_{n_j}(a)] = y(t) + c = z(t)$$

for every $t \in (a, b)$ at which z is continuous. Also, since $y(a) = 0$, we have $z_{n_j}(a) \to c = y(a) + c = z(a)$. Further, since the constant function 1 belongs to X,

$$y_{n_j}(b) = \int_a^b dy_{n_j} \to \int_a^b dy = y(b),$$

so that $z_{n_j}(b) = y_{n_j}(b) + z_{n_j}(a) \to y(b) + c = z(b)$. $\square$

We note that a nondecreasing function on $[a, b]$ can be discontinuous only at countably many points. Hence the subsequence (z_{n_j}) of Theorem 15.6 converges at all but a countably many points of $[a, b]$.

In probability theory, a **cumulative distribution function** is defined as a nondecreasing right continuous function F from $\mathbf{R}$ to $\mathbf{R}$ such that $\lim_{t \to -\infty} F(t) = 0$ and $\lim_{t \to \infty} F(t) = 1$. If f is a random variable, that is, a real-valued measurable function on a probability space and if for a fixed $t \in \mathbf{R}$, $F(t)$ denotes the probability that f is less than or equal to t, then F is a cumulative distribution function. [Note: Every cumulative distribution function arises in this way!]

15.7 Corollary (Helly's selection principle)

Let $a < b$ and (F_n) be a sequence of cumulative distribution functions such that $F_n(a) \to 0$ and $F_n(b) \to 1$ as $n \to \infty$. Then there is a cumulative distribution function F such that for some subsequence (F_{n_j}) of (F_n), we have $F_{n_j}(t) \to F(t)$ for every $t \in \mathbf{R}$ at which F is continuous, $F(t) = 0$ for all $t < a$ and $F(t) = 1$ for all $t \geq b$.

Proof:
Let $z_n = F_{n|[a,b]}$ for $n = 1, 2, \ldots$. Then (z_n) is a sequence of nondecreasing right continuous functions on $[a, b]$. Also, $0 \leq z_n(t) \leq 1$ for all $n = 1, 2, \ldots$ and $t \in [a, b]$. By 15.6, there is a nondecreasing function z on $[a, b]$ such that z is right continuous on (a, b) and for some subsequence (z_{n_j}) of (z_n), we have $z_{n_j}(t) \to z(t)$ for every $t \in (a, b)$ at which z is continuous. Define

$$F(t) = \begin{cases} 0, & \text{if } t < a \\ z(a^+), & \text{if } t = a \\ z(t), & \text{if } a < t < b \\ 1, & \text{if } b \leq t. \end{cases}$$

It is easy to see that F is a cumulative distribution function. Let $t \leq a$. Then $0 \leq F_n(t) \leq F_n(a) \to 0$ as $n \to \infty$. If $t < a$, then $F(t) = 0$ by the definition of F. If $t = a$ and F is continuous at a, then $F(a) = 0$. Hence $F_n(t) \to F(t)$ as $n \to \infty$. Let $t \geq b$. Then $1 \geq F_n(t) \geq F_n(b) \to 1$ as $n \to \infty$. Since $F(t) = 1$, we see that $F_n(t) \to F(t)$ as $n \to \infty$. Let $a < t < b$ and assume that F is continuous at t. Then z is continuous at t and $F_{n_j}(t) = z_{n_j}(t) \to z(t) = F(t)$ as $j \to \infty$. $\square$

Problems

15-1 Let X be a normed space.

(a) If E is a closed convex subset of X, (x_n) is a sequence in E and $x_n \xrightarrow{w} x$ in X, then $x \in E$. (Hint: 15.1)

(b) Let Y be a closed subspace of X. If $x_n \xrightarrow{w} x$ in X, then $x_n + Y \xrightarrow{w} x + Y$ in X/Y.

15-2 Let $1 \leq p \leq \infty$ and $X = \ell^p$. Then $e_n \not\to 0$ in X, and $e_n \xrightarrow{w} 0$ in X if and only if $1 < p \leq \infty$.

15-3 Let $X = C([a,b])$ with the sup norm. Fix $t_0 \in [a,b)$. For each positive integer n with $t_0 + 2/n < b$, let

$$x_n(t) = \begin{cases} 0, & \text{if } a \le t \le t_0 \text{ or } t_0 + 2/n \le t \le b \\ n(t - t_0), & \text{if } t_0 \le t \le t_0 + 1/n \\ n(2/n - t + t_0), & \text{if } t_0 + 1/n \le t \le t_0 + 2/n. \end{cases}$$

Then $x_n \xrightarrow{w} 0$ in X, but $x_n \not\to 0$ in X.

15-4 Let X be a Banach space and Y be a normed space. Let (F_n) be a sequence in $BL(X,Y)$ such that for each fixed $x \in X, (F_n(x))$ is weak convergent in Y. If $F_n(x) \xrightarrow{w} y$ in Y, let $F(x) = y$. Then $F \in BL(X,Y)$ and

$$\|F\| \le \liminf_{n\to\infty} \|F_n\| \le \sup_{n=1,2,\dots} \|F_n\| < \infty.$$

(Compare 9.2(a). Hint: 9.1 and 15.1.)

15-5 Let X and Y be normed spaces and $F : X \to Y$ be a linear map. Consider the following conditions:

(i) Whenever $x_n \to x$ in X, we have $F(x_n) \to F(x)$ in Y.

(ii) Whenever $x_n \xrightarrow{w} x$ in X, we have $F(x_n) \xrightarrow{w} F(x)$ in Y.

(iii) Whenever $x_n \to x$ in X, we have $F(x_n) \xrightarrow{w} F(x)$ in Y.

(iv) Whenever$x_n \xrightarrow{w} x$ in X, we have $F(x_n) \to F(x)$ in Y.

Conditions (i), (ii) and (iii) are equivalent. Condition (iv) implies condition (ii), but not conversely.

15-6 (a) Let $X = c_{00}$ or c_0 with the norm $\| \ \|_\infty$. Then $x_n \xrightarrow{w} x$ in X if and only if (x_n) is a bounded sequence in X and $x_n(j) \to x(j)$ as $n \to \infty$ for each $j = 1, 2, \dots$.

(b) Let $X = c$ with the norm $\| \ \|_\infty$. Then $x_n \xrightarrow{w} x$ in X if and only if (x_n) is a bounded sequence in $X, x_n(j) \to x(j)$ as $n \to \infty$ for each $j = 1, 2, \dots$ and $\lim_{j\to\infty} x_n(j) \to \lim_{j\to\infty} x(j)$ as $n \to \infty$. (Hint: Problem 13-6)

15-7 Let (x_n) be a sequence in $L^1([a,b])$ such that $x_n \ge 0$ and $x_n \xrightarrow{w} 0$ in $L^1([a,b])$. Then $x_n \to 0$ in $L^1([a,b])$. (Compare 15.2(b), (d).)

15-8 Let $X = C^1([a,b])$ with $\|x\| = |x(a)| + \|x'\|_\infty$. Then $x_n \xrightarrow{w} x$ in X if and only if $x_n(a) \to x(a), x_n'(t) \to x'(t)$ for each $t \in [a,b]$ and $\|x_n'\| \le \alpha$ for all n and some $\alpha > 0$. (Compare 15.2(f). Hint: Problem 14-16)

15-9 Let $X = c_{00}$ with the norm $\|\ \|_p, 1 \le p < \infty$. For $n = 1, 2, \ldots$, let $x_n'(x) = x(n), x \in X$. Then $x_n' \xrightarrow{w} 0$ but $x_n' \not\to 0$, while $nx_n' \xrightarrow{w*} 0$ but $nx_n' \not\xrightarrow{w} 0$ in X'.

15-10 Let (x_n) be a sequence in a normed space X and for $m = 1, 2, \ldots$, let E_m denote the closure of $\operatorname{co}(\{x_m, x_{m+1}, \ldots\})$. Then (x_n) is weak convergent in X if and only if $(x'(x_n))$ is a Cauchy sequence in $\mathbf{K}$ for each $x' \in X'$ and $\bigcap_{m=1} E_m \neq \emptyset$. In that case, $\bigcap_{m=1}^{\infty} E_m$ is a singleton set.

15-11 **(Weak basis theorem of Karlin, 1948)** Let X be a normed space. A countable subset $\{x_1, x_2, \ldots\}$ of X is called a **weak Schauder basis** for X if $\|x_j\| = 1$ for each j and for every $x \in X$, there are unique $k_j \in \mathbf{K}, j = 1, 2, \ldots$ such that $\sum_{j=1}^n k_j x_j \xrightarrow{w} x$ as $n \to \infty$. If X is a Banach space, then $\{x_1, x_2, \ldots\}$ is a weak Schauder basis for X if and only if it is a Schauder basis for X.

15-12 Let X be a normed space. A countable subset $\{x_1', x_2', \ldots\}$ of X' is called a **weak* Schauder basis** for X' if $\|x_j'\| = 1$ for each j and for every $x' \in X'$, there are unique $k_j \in \mathbf{K}, j = 1, 2, \ldots$ such that $\sum_{j=1}^n k_j x_j' \xrightarrow{w*} x'$ as $n \to \infty$. Let X be a Banach space and $\{x_1, x_2, \ldots\}$ be a Schauder basis for X with coefficient functionals $\{f_1, f_2, \ldots\}$. If $x_j' = f_j/\|f_j\|, j = 1, 2, \ldots$, then $\{x_1', x_2', \ldots\}$ is a weak* Schauder basis for X' with coefficient functionals $\{\|f_1\| J(x_1), \|f_2\| J(x_2), \ldots\}$.

15-13 Let X (resp., Y) be a Banach space with a Schauder basis $\{x_1, x_2, \ldots\}$ (resp., $\{y_1, y_2, \ldots\}$) and f_j (resp., g_j) be the corresponding jth coefficient functional, $j = 1, 2, \ldots$. Let $F \in BL(X, Y)$ be defined by a matrix $(k_{i,j})$ with respect to these Schauder bases, that is, $g_i(F(x_j)) = k_{i,j}, i, j = 1, 2, \ldots$. Let $x_j' = f_j/\|f_j\|$ and $y_j' = g_j/\|g_j\|, j = 1, 2, \ldots$. If $F' \in BL(Y', X')$ is defined by a matrix $(k_{i,j}')$ with respect to the weak* Schauder bases $\{y_1', y_2', \ldots\}$ and $\{x_1', x_2', \ldots\}$ of Y' and X' respectively, then

$$k_{i,j}' = \frac{\|f_i\|}{\|g_j\|} k_{j,i}, \quad i, j = 1, 2, \ldots.$$

15-14 (a) Let $X = c_0$ with the norm $\| \ \|_\infty$ and for $n = 1, 2, \ldots, x_n = (1, \ldots, 1, 0, 0, \ldots)$, with 1 occurring only in the first n entries. Then X is a Banach space, $(x'(x_n))$ converges in $\mathbf{K}$ for every $x' \in X'$, but no subsequence of (x_n) is a weak convergent.

(b) Let $X = c_{00}$ with the norm $\| \ \|_\infty$ and for $n = 1, 2, \ldots x'_n(x) = \sum_{j=1}^{n} x(j), x \in X$. Then $(x'_n(x))$ converges in $\mathbf{K}$ for every $x \in X$, but no subsequence of (x'_n) is weak* convergent.

15-15 Let $X_1 = c_0$ and $X_2 = c$ with the norm $\| \ \|_\infty$. Then both X'_1 and X'_2 are linearly isometric to ℓ^1. However, $e_n \overset{w*}{\to} 0$ in ℓ^1, considered as the dual of c_0, but $e_n \overset{w*}{\not\to} 0$ in ℓ^1, considered as the dual of c.

15-16 Let $X = c_0$ with the norm $\| \ \|_\infty$ and for $n = 1, 2, \ldots$, let $x'_n(x) = x(n), x \in X$. Then $x'_n \overset{w*}{\to} 0$ in X' but 0 does not belong to the closure of $\operatorname{co}(\{x'_1, x'_2, \ldots\})$ in X'. (Compare 15.1.)

15-17 (a) Let X be a finite dimensional normed space. Then $x'_n \overset{w*}{\to} x'$ in X' if and only if $x'_n \to x'$ in X'.

(b) Let $1 < q \leq \infty$. Then $y_n \overset{w*}{\to} y$ in ℓ^q, considered as the dual of ℓ^p with $1/p + 1/q = 1$ if and only if (y_n) is a bounded sequence in ℓ^q and $y_n(j) \to y(j)$ as $n \to \infty$ for each $j = 1, 2, \ldots$.

(c) Let $1 < q \leq \infty$. Then $y_n \overset{w*}{\to} y$ in $L^q([a, b])$, considered as the dual of $L^p([a, b])$ with $1/p + 1/q = 1$ if and only if (y_n) is a bounded sequence in $L^q([a, b])$ and $\int_c^d y_n \, dm \to \int_d^c y \, dm$ for every subinterval $[c, d]$ of $[a, b]$.

(d) Let $y_n, y \in NBV([a, b])$. If $(V(y_n))$ is bounded, $y_n(b) \to y(b)$ and $y_n(t) \to y(t)$ for all t in a dense subset of (a, b), then $y_n \overset{w*}{\to} y$ in $NBV([a, b])$, considered as the dual of $C([a, b])$. Converse holds if each y_n is nondecreasing. (Hint: Compare the proof of 15.6.)

15-18 Let $y_n(t) = 1$ if $0 < t < 1/n$ and $y_n(t) = 0$ if $t = 0$ or $1/n < t \leq 1$. Then $y_n \overset{w*}{\to} 0$ in $NBV([0, 1])$, but $V(y_n) = 2$ for each n.

15-19 Let $t_0 \in [a, b]$ and for $n = 1, 2, \ldots$, let y_n be a nonnegative measurable function on $[a, b]$ such that $y_n(t) = 0$ for all t satisfying $|t - t_0| \geq 1/n$

and $\int_a^b y_n(t)\, dm(t) = 1$. Let $X = C([a,b])$ with the sup norm and define

$$x_n'(x) = \int_a^b x y_n \, dm, \quad x \in X.$$

If $x'(x) = x(t_0)$ for $x \in X$, then $x_n' \xrightarrow{w*} x'$ in X'.

15-20 Let X be a normed space such that X' is separable. If (x_n) is a bounded sequence in X, then there is a subsequence (x_{n_k}) such that $(x'(x_{n_k}))$ converges in $\mathbf{K}$ for every $x' \in X'$. (Compare 15.3. Hint: Proof of 15.4.)

15-21 Let $k_n = 1$ for $n = 0, \pm 1, \pm 2, \ldots$ and define s_m and a_m as in 15.5. Then the sequence (a_m) is bounded in $L^1([-\pi,\pi])$, but there is no y in $L^1([-\pi,\pi])$ such that $\hat{y}(n) = k_n$ for all n. (Compare 15.5. Hint: a_m is the mth Fejer kernel, so that $\|a_m\|_1 = 2\pi$. Also, $\hat{y}(n) \to 0$ by 4.9(a).)

15-22 Let k_n, s_m and a_m be as in 15.5. If the sequence (a_m) is bounded in $L^1([-\pi,\pi])$, then there is some $y \in NBV([-\pi,\pi])$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int}\, dy(t) = k_n, \quad n = 0 \pm 1, \pm 2, \ldots.$$

(Hint: Consider $C([-\pi,\pi])$ and use 14.5.)

15-23 For $n = 1, 2, \ldots$, let $z_n(t) = 0$ if $t \le 0$, $z_n(t) = nt$ if $0 < t < 1/n$ and $z_n(t) = 1$ if $t \ge 1/n$. Then each z_n is a cumulative distribution function with $z_n(0) = 0$ and $z_n(1) = 1$. Let $z(t) = 0$ if $t < 0$ and $z(t) = 1$ if $t \ge 0$. Then z is a cumulative distribution function which is discontinuous only at 0, $z_n(t) \to z(t)$ for all $t \neq 0$, but $z_n(0) \not\to z(0)$. (Compare 15.7.)

16 Reflexivity

In Section 8 we have seen how to embed a normed space X in its second dual X''. For a fixed $x \in X$, define $J(x) : X' \to \mathbf{K}$ by

$$J(x)(x') = x'(x), \quad x' \in X'.$$

Then $J(x) \in X''$, $\|J(x)\| = \|x\|$ and the map $J : X \to X''$ is linear. Since X'' is a Banach space by 8.3(c), it follows that the subspace $J(X)$ is closed in X'' if and only if X is a Banach space.

We say that a normed space X is **reflexive** if the canonical embedding J is surjective, that is, if x'' is a continuous linear functional on X', then there is some $x \in X$ such that $x'' = J(x)$. Although this requirement is set-theoretic in nature, it has deep connections with weak convergence in X and with the geometry of X. We shall explore some of these connections in this section.

We emphasize that X is reflexive if it is linearly isometric to its second dual X'' under the particular map J defined above. James has given an example ([35], 1951) of a normed space X which is linearly isometric to X'' but is not reflexive.

16.1 Theorem

Let X be a reflexive normed space. Then

(a) X is Banach and it remains reflexive in any equivalent norm.

(b) X' is reflexive.

(c) Every closed subspace of X is reflexive.

(d) X is separable if and only if X' is separable.

Proof:

(a) The dual X' and, in turn, the second dual X'' of the normed space X are Banach spaces by 8.2(c). Being reflexive, X is linearly isometric to X'', so that X is a Banach space. Also, in any equivalent norm on X, the dual X' and the second dual X'' remain unchanged, so that X remains reflexive.

(b) Let $J' : X' \to X'''$ be the canonical embedding of X' into its second dual X'''. Knowing that the canonical embedding $J : X \to X''$ is surjective, we show that J' is surjective. Let $x''' \in X'''$. Define

$x' \in X'$ by

$$x'(x) = x'''(J(x)), \quad x \in X.$$

Consider $x'' \in X''$. Then there is some $x \in X$ such that $J(x) = x''$, and

$$J'(x')(x'') = x''(x') = J(x)(x') = x'(x) = x'''(J(x)) = x'''(x'').$$

Thus $J'(x') = x'''$, showing that J' is surjective.

(c) Let Y be a closed subspace of X and $J_Y : Y \to Y''$ be the canonical embedding of Y surjective Y''. Knowing that the canonical embedding $J : X \to X''$ is surjective, we show that J_Y is surjective. Let $y'' \in Y''$. Define $x'' \in X''$ by

$$x''(x') = y''(x'_{|Y}), \quad x' \in X'.$$

Now there is a unique $x \in X$ such that $J(x) = x''$. In fact, $x \in Y$. For if $x' \in X'$ with $x'_{|Y} = 0$, then

$$x'(x) = J(x)(x') = x''(x') = y''(x'_{|Y}) = y''(0) = 0.$$

Hence by 7.10(b), $x \in \overline{Y} = Y$. Consider $y' \in Y'$. Let $x' \in X'$ be such that $x'_{|Y} = y'$. (For example, x' can be a Hahn-Banach extension of y' to X.) Since $x \in Y$, we have

$$\begin{aligned} J_Y(x)(y') &= y'(x) = x'(x) = J(x)(x') = x''(x') \\ &= y''(x'_{|Y}) = y''(y'). \end{aligned}$$

Thus $J_Y(x) = y''$, showing that J_Y is surjective.

(d) We have already seen in 13.1(b) that if X' is separable, then so is X. Conversely, assume that X is separable. Since X is reflexive, X'' is isometric to X. Hence $X'' = (X')'$ is separable. By 13.1(b) again, we see that X' is separable. □

We make some remarks on the results given above. While a reflexive normed space is a Banach space and its dual is also reflexive, neither a Banach space nor a normed space whose dual is reflexive need be reflexive. However, a Banach space whose dual is reflexive is always reflexive. Just as every closed subspace Y of a reflexive normed space X is itself reflexive, the quotient space X/Y is also reflexive. Also, the product space of reflexive spaces is reflexive. We leave the proofs of these facts as well as of the fact that if X is reflexive then weak convergence in X' coincides with weak* convergence in X' to the reader. (See Problems 16-4 to 16-8.)

We now consider some finite dimensional situations. First we prove a preliminary result which should be compared with 7.10(c).

16.2 Lemma

Let X be a normed space and $\{x_1', \ldots, x_m'\}$ be a linearly independent subset of X'. Then there are $x_1, \ldots, x_m$ in X such that $x_j'(x_i) = \delta_{i,j}$ for $i, j = 1, \ldots, m$.

Proof:

We use induction on m. If $m = 1$, then since $\{x_1'\}$ is linearly independent, let $x_0 \in X$ with $x_1'(x_0) \neq 0$ and $x_1 = x_0/x_1'(x_0)$. Next, assume that the result is true for $m = k$. Let $\{x_1', \ldots, x_{k+1}'\}$ be a linearly independent subset of X'. Since its subset $\{x_1', \ldots, x_k'\}$ is linearly independent, there are $y_1, \ldots, y_k$ in X such that $x_j'(y_i) = \delta_{i,j}$ for $1 \leq i, j \leq k$. We claim that there is some $x_0 \in X$ with $x_j'(x_0) = 0$ for $1 \leq j \leq k$, but $x_{k+1}'(x_0) \neq 0$. For $x \in X$, let $y = x_1'(x)y_1 + \cdots + x_k'(x)y_k$. Then $(x - y) \in \bigcap_{j=1}^k Z(x_j')$. Were $\bigcap_{j=1}^k Z(x_j') \subset Z(x_{k+1}')$, then

$$x_{k+1}'(x) = x_{k+1}'(x-y) + x_{k+1}'(y) = x_1'(x)x_{k+1}'(y_1) + \cdots + x_k'(x)x_{k+1}'(y_k),$$

so that

$$x_{k+1}' = x_{k+1}'(y_1)x_1' + \cdots + x_{k+1}'(y_k)x_k' \in \operatorname{span}\{x_1', \ldots, x_k'\},$$

contrary to the linear independence of $\{x_1', \ldots, x_{k+1}'\}$. Hence our claim is justified. Now let $x_{k+1} = x_0/x_{k+1}'(x_0)$ and for $i = 1, \ldots, k$, let

$x_i = y_i - x'_{k+1}(y_i)x_{k+1}$. Then it can be easily checked that $x'_j(x_i) = \delta_{i,j}$ for $i, j = 1, \ldots, k+1$. □

16.3 Theorem (Helly, 1912)

Let X be a normed space.

(a) Consider $x'_1, \ldots, x'_m$ in X', $k_1, \ldots, k_m$ in $\mathbf{K}$ and $\alpha \geq 0$. Then for every $\epsilon > 0$, there is some $x_\epsilon \in X$ such that $x'_j(x_\epsilon) = k_j$ for each $j = 1, \ldots, m$ and $\|x_\epsilon\| < \alpha + \epsilon$ if and only if

$$|\sum_{j=1}^{m} h_j k_j| \leq \alpha \|\sum_{j=1}^{m} h_j x'_j\|$$

for all $h_1, \ldots, h_m$ in $\mathbf{K}$.

(b) Let S be a finite dimensional subspace of X' and $x'' \in X''$. If $\epsilon > 0$, then there is some $x_\epsilon \in X$ such that

$$x''_{|S} = J(x_\epsilon)_{|S} \quad \text{and} \quad \|x_\epsilon\| < \|x''\| + \epsilon.$$

Proof:

(a) Suppose that for every $\epsilon > 0$, there is some $x_\epsilon \in X$ such that $x'_j(x_\epsilon) = k_j$ for each $j = 1, \ldots, m$ and $\|x_\epsilon\| < \alpha + \epsilon$. Fix $h_1, \ldots, h_m$ in $\mathbf{K}$. Then

$$\begin{aligned} |\sum_{j=1}^{m} h_j k_j| &= |\sum_{j=1}^{m} h_j x'_j(x_\epsilon)| = \left| \left(\sum_{j=1}^{m} h_j x'_j\right)(x_\epsilon)\right| \\ &\leq \|\sum_{j=1}^{m} h_j x'_j\| \, \|x_\epsilon\| < (\alpha + \epsilon)\|\sum_{j=1}^{m} h_j x'_j\|. \end{aligned}$$

As this is true for every $\epsilon > 0$, we see that $|\sum_{j=1}^{m} h_j k_j| \leq \alpha \|\sum_{j=1}^{m} h_j x'_j\|$.

Conversely, suppose that for all $h_1, \ldots, h_m$ in $\mathbf{K}$, $|\sum_{j=1}^{m} h_j k_j| \leq \alpha \|\sum_{j=1}^{m} h_j x'_j\|$. First we note that $\{x'_1, \ldots, x'_m\}$ can be assumed to be a linearly independent set. To see this, let $x'_1, \ldots, x'_n$ with $n \leq m$ be a maximal linearly independent subset of $\{x'_1, \ldots, x'_m\}$. Given $\epsilon > 0$, let $x_\epsilon \in X$ be such that $\|x_\epsilon\| \leq \alpha + \epsilon$ and $x'_j(x_\epsilon) = k_j$ for $j = 1, \ldots, n$.

If $n < \ell \leq m$, then $x'_\ell = h_1 x'_1 + \cdots + h_n x'_n$ for some $h_1, \ldots, h_n$ in $\mathbf{K}$. Hence

$$x'_\ell(x_\epsilon) = h_1 x'_1(x_\epsilon) + \cdots + h_n x'_n(x_\epsilon) = h_1 k_1 + \cdots + h_n k_n.$$

But

$$|k_\ell - \sum_{j=1}^{n} h_j k_j| \leq \alpha \|x'_\ell - \sum_{j=1}^{n} h_j x'_j\| = 0,$$

so that $x'_\ell(x_\epsilon) = k_\ell$, as well.

Consider the map $F : X \to \mathbf{K}^m$ given by

$$F(x) = (x'_1(x), \ldots, x'_m(x)).$$

Clearly, F is a linear map. Also, it is surjective. To see this, consider $(h_1, \ldots, h_m) \in \mathbf{K}^m$. By Lemma 16.2, there are $x_1, \ldots, x_m$ in X such that $x'_j(x_i) = \delta_{i,j}, 1 \leq i, j \leq m$. If we let $x = h_1 x_1 + \ldots + h_m x_m$, then it follows that $F(x) = (h_1, \ldots, h_m)$. Now a modification of the proof of 7.4 shows that F maps each open subset of X onto an open subset of $\mathbf{K}^m$. (See Problem 10-20. Compare 10.6.)

Let $\epsilon > 0$ and consider $U_\epsilon = \{x \in X : \|x\| < \alpha + \epsilon\}$. We wish to show that there is some $x_\epsilon \in U_\epsilon$ with $F(x_\epsilon) = (k_1, \ldots, k_m)$. Suppose this is not the case. Then $(k_1, \ldots, k_m)$ does not belong to the open convex set $F(U_\epsilon)$. By the Hahn-Banach separation theorem (7.5) for $\mathbf{K}^m$, there is a (continuous) linear functional f on $\mathbf{K}^m$ such that

$$\operatorname{Re} f((x'_1(x), \ldots, x'_m(x)) \leq \operatorname{Re} f((k_1, \ldots, k_m))$$

for all $x \in U_\epsilon$. By 13.3(a), there is some $(h_1, \ldots, h_m) \in \mathbf{K}^m$ such that

$$f((c_1, \ldots, c_m)) = c_1 h_1 + \cdots + c_m h_m$$

for all $(c_1, \ldots, c_m) \in \mathbf{K}^m$. Hence

$$\operatorname{Re}[h_1 x'_1(x) + \cdots + h_m x'_m(x)] \leq \operatorname{Re}(h_1 k_1 + \cdots + h_m k_m)$$

for all $x \in U_\epsilon$. If $h_1 x'_1(x) + \cdots + h_m x'_m(x) = re^{i\theta}$ with $r \geq 0$ and $-\pi < \theta \leq \pi$, then by considering $xe^{-i\theta}$ in place of x, it follows that

$$|h_1 x'_1(x) + \cdots + h_m x'_m(x)| \leq \operatorname{Re}(h_1 k_1 + \cdots + h_m k_m)$$

for all $x \in U_\epsilon$. But

$$\sup\{|\sum_{j=1}^{m} h_j x_j'(x)| : x \in U_\epsilon\} = (\alpha + \epsilon)\|\sum_{j=1}^{m} h_j x_j'\|.$$

Hence

$$(\alpha + \epsilon)\|\sum_{j=1}^{m} h_j x_j'\| \leq \operatorname{Re} \sum_{j=1}^{m} h_j k_j \leq |\sum_{j=1}^{m} h_j k_j| \leq \alpha\|\sum_{j=1}^{m} h_j x_j'\|.$$

This contradiction shows that there must be some $x_\epsilon \in U_\epsilon$ with $F(x_\epsilon) = (k_1, \ldots, k_m)$, as desired.

(b) Let $\{x_1', \ldots, x_m'\}$ be a basis for the finite dimensional subspace S of X' and let $k_j = x''(x_j'), j = 1, \ldots, m$. Then for all $h_1, \ldots, h_m$ in $\mathbf{K}$, we have

$$|\sum_{j=1}^{m} h_j k_j| = |\sum_{j=1}^{m} h_j x''(x_j')| = |x''\left(\sum_{j=1}^{m} h_j x_j'\right)| \leq \|x''\|\,\|\sum_{j=1}^{n} h_j x_j'\|.$$

Letting $\alpha = \|x''\|$ in (a) above, we see that for every $\epsilon > 0$, there is some $x_\epsilon \in X$ such that $\|x_\epsilon\| \leq \|x''\| + \epsilon$ and for $j = 1, \ldots, m$,

$$J(x_\epsilon)(x_j') = x_j'(x_\epsilon) = k_j = x''(x_j'),$$

that is, $J(x_\epsilon)_{|S} = x''_{|S}$, as desired. □

Part (a) of 16.3 should be compared with the result in Problem 7-11, also known after Helly. Part (b) of 16.3 says that if we restrict ourselves to a finite dimensional subspace of X', then we are close to reflexivity. We have deduced part (b) from part (a). On the other hand, part (a) can be deduced from part (b) by employing Problem 7-11.

16.4 Examples

(a) Let X be a finite dimensional normed space. Then X' and X'' are normed spaces of the same finite dimension (See 7.10(c).) Since the canonical embedding $J : X \to X''$ is linear and injective, it is also surjective by 2.4(b). Hence X is reflexive.

(b) The sequence space ℓ^1 is not reflexive, because $(\ell^1)'$ is isometric to ℓ^∞ by 13.2, and ℓ^1 is separable but ℓ^∞ is not by 3.2. Hence 16.1(d) applies.

If $1 < p < \infty$, then ℓ^p is reflexive. This can be seen as follows. Let $1 < q < \infty$ be such that $1/p + 1/q = 1$. By 13.2, the maps $F : \ell^q \to (\ell^p)'$ and $G : \ell^p \to (\ell^q)'$ defined by

$$F(y)(x) = \sum_{j=1}^{\infty} x(j)y(j) = G(x)(y), \quad x \in \ell^p, y \in \ell^q,$$

are surjective linear isometries. Let $x'' \in (\ell^p)''$. Then $x'' \circ F \in (\ell^q)'$ and there is a unique $x \in \ell^p$ such that $G(x) = x'' \circ F$. Also, if $x' \in (\ell^p)'$, there is a unique $y \in \ell^q$ such that $F(y) = x'$, so that

$$x''(x') = x''(F(y)) = G(x)(y) = F(y)(x) = x'(x) = J(x)(x').$$

Thus the canonical embedding $J : \ell^p \to (\ell^p)''$ is surjective.

The normed space $X = c_0$ is not reflexive, because 13.3(c) shows that X' is linearly isometric to ℓ^1, which is not reflexive. Hence 16.1(b) applies.

The normed space ℓ^∞ is not reflexive, because it has a closed subspace c_0 which is not reflexive. Hence 16.1(c) applies. For an alternative proof, see Problem 16-5.

(c) Consider $L^p([a,b]), 1 \le p \le \infty$. Then the results about ℓ^p for various values of p remain valid for $L^p([a,b])$ as well. We merely have to appeal to 14.3 and 4.7(d) instead of 13.2 and 3.2, and consider $X = C([a,b])$ instead of $X = c_0$ with the norm $\| \ \|_\infty$. Note that $X = C([a,b])$ is not reflexive. For X' is isometric to $NBV([a,b])$ by 14.5, and X is separable but $NBV([a,b])$ is not. (See Problem 14-12.) Hence 16.1(d) applies. □

We now establish a relationship between reflexivity and weak convergence.

16.5 Theorem (Eberlein, 1947)
Let X be a normed space. Then X is reflexive if and only if every bounded sequence in X has a weak convergent subsequence.

Proof:
Let X be reflexive. Consider a bounded sequence (x_n) in X.

First suppose that X is separable. Since the canonical embedding $J : X \to X''$ is an isometry, we see that $(J(x_n))$ is a bounded sequence in X''. Since $X'' = (X')'$ and X' is separable by 16.1(d), it follows from 15.4 that there is a subsequence $(J(x_{n_k}))$ which is weak* convergent to some $x'' \in X''$. Let $x'' = J(x)$ for some $x \in X$, since the map J is surjective. Then for every $x' \in X'$, we have

$$x'(x_{n_k}) = J(x_{n_k})(x') \to J(x)(x') = x'(x),$$

that is, $x_{n_k} \xrightarrow{w} x$ in X.

If X is not separable, let Y denote the closure of span $\{x_1, x_2, \ldots\}$. By 16.1(c), Y is a reflexive space and it is clearly separable. By what we have just seen, the bounded sequence (x_n) in Y has a subsequence (x_{n_k}) such that $y'(x_{n_k}) \to y'(x)$ for some $x \in Y$ and each $y' \in Y'$. Now for $x' \in X'$, let $y' = x'_{|Y} \in Y'$. Then

$$x'(x_{n_k}) = y'(x_{n_k}) \to y'(x) = x'(x),$$

that is, $x_{n_k} \xrightarrow{w} x$ in X.

Thus if X is reflexive, then every bounded sequence in X has a weak convergent subsequence.

Conversely, assume that X is not reflexive. Since the map J is not surjective, $J(X)$ is a proper closed subspace of X''. By 7.10(b), there is some $x_0''' \in X'''$ such that $\|x_0'''\| = 1$ and $x_0'''(J(x)) = 0$ for every $x \in X$. Let $0 < r < 1$. Choose $x_0'' \in X''$ such that $\|x_0''\| < 1$ and $|x_0'''(x_0'')| > r$. Then $\|x_0''\| > r$.

We use mathematical induction to show that there are sequences (x_n) in X and (x_n') in X' such that $\|x_n\| \le 1, \|x_n'\| \le 1, x_0''(x_m') = r, x_m'(x_n) = 0$ if $n < m$ and $x_m'(x_n) = r$ if $n \ge m$ for all $n, m = 1, 2, \ldots$.

Since $\|x_0''\| > r$, there is some $x_1' \in X'$ such that $\|x_1'\| \le 1$ and $x_0''(x_1') = r$. Then $\|x_1'\| > r$. Hence there is some $x_1 \in X$ such that $\|x_1\| \le 1$ and $x_1'(x_1) = r$. This is the first step of the induction. Suppose that we have chosen $x_1, \ldots, x_{m-1}$ in X and $x_1', \ldots, x_{m-1}'$ in X' as desired. Now consider the finite subset $\{J(x_1), \ldots, J(x_{m-1}), x_0''\}$ of X''. If we let $k_1 = \cdots = k_{m-1} = 0, k_m = r$ and $\alpha = r/|x_0'''(x_0'')|$, then for all $h_1, \ldots, h_m$ in $\mathbf{K}$, we have

$$\begin{aligned}
|h_1k_1 + \cdots + h_mk_m| &= r|h_m| \\
&= \alpha|x_0'''(x_0'')|\,|h_m| \\
&= \alpha|x_0'''[h_1J(x_1) + \cdots + h_{m-1}J(x_{m-1}) + h_mx_0'']| \\
&\le \alpha\|h_1J(x_1) + \cdots + h_{m-1}J(x_{m-1}) + h_mx_0''\|,
\end{aligned}$$

since $x_0'''(J(x_n)) = 0$ for each $n = 1, \ldots, m-1$ and $\|x_0'''\| = 1$. As $\alpha < 1$, it follows from 16.3(a) that there is some $x_m' \in X'$ with $\|x_m'\| \le 1$ and $x_m'(x_n) = J(x_n)(x_m') = 0$ for each $n = 1, \ldots, m-1$ and $x_0''(x_m') = r$.

Next, consider the finite subset $\{x_1', \ldots, x_m'\}$ of X'. If we let $k_1 = \cdots = k_m = r$ and $\alpha = \|x_0''\|$, then for all $h_1, \ldots, h_m$ in $\mathbf{K}$, we have

$$\begin{aligned}
|h_1k_1 + \cdots + h_mk_m| &= |h_1r + \cdots + h_mr| \\
&= |x_0''(h_1x_1' + \cdots h_mx_m')| \\
&\le \alpha\|h_1x_1' + \cdots + h_mx_m'\|,
\end{aligned}$$

since $x_0''(x_n') = r$ for each $n = 1, \ldots, m$. As $\alpha < 1$, again it follows from 16.3 (a) that there is some $x_m \in X$ with $\|x_m\| \le 1$ and $x_n'(x_m) = r$ for each $n = 1, \ldots, m$. Thus we have proved the existence of sequences (x_n) in X and (x_n') in X' such that $\|x_n\| \le 1, \|x_n'\| \le 1$ and

$$x_n'(x_m) = \begin{cases} r, & \text{if } n \le m \\ 0, & \text{if } m < n. \end{cases}$$

We claim that the bounded sequence (x_n) in X has no weak convergent subsequence. Let, if possible, $x_{n_k} \xrightarrow{w} x$ in X. Then by 15.1, x belongs to the closure of the convex hull of $\{x_{n_1}, x_{n_2}, \ldots\}$. Find $y = \alpha_1 x_{n_1} + \cdots + \alpha_j x_{n_j}$ with $\alpha_1 \geq 0, \ldots \alpha_j \geq 0$ and $\alpha_1 + \cdots + \alpha_j = 1$ such that $\|x - y\| < r/2$. Since the sequence $x_{n_{j+1}}, x_{n_{j+2}}, \ldots$ is also weak convergent to x, the element x belongs to the closure of the convex hull of $\{x_{n_{j+1}}, x_{n_{j+2}}, \ldots\}$. Find $z = \beta_1 x_{n_{j+1}} + \cdots + \beta_k x_{n_{j+k}}$ with $\beta_1 \geq 0, \ldots, \beta_k \geq 0$ and $\beta_1 + \cdots + \beta_k = 1$ such that $\|x - z\| < r/2$. Now

$$\begin{aligned} x'_{n_{j+1}}(z - y) &= \beta_1 x'_{n_{j+1}}(x_{n_{j+1}}) + \cdots + \beta_k x'_{n_{j+1}}(x_{n_{j+k}}) \\ &\quad - \alpha_1 x'_{n_{j+1}}(x_{n_1}) - \cdots - \alpha_j x'_{n_{j+1}}(x_{n_j}) \\ &= (\beta_1 + \cdots + \beta_k) r - (\alpha_1 + \cdots + \alpha_j) 0 \\ &= r, \end{aligned}$$

but

$$\begin{aligned} |x'_{n_{j+1}}(z - y)| &\leq |x'_{n_{j+1}}(z - x)| + |x'_{n_{j+1}}(x - y)| \\ &\leq \|x'_{n_{j+1}}\|(\|z - x\| + \|x - y\|) \\ &< r, \end{aligned}$$

since $\|x'_{n_{j+1}}\| \leq 1$, $\|z - x\| < r/2$ and $\|x - y\| < r/2$. This contradiction justifies our claim and completes the proof. □

The preceding result can be rephrased as follows: A normed space is reflexive if and only if its closed unit ball is '**weak sequentially compact**'. Recall from 15.1 that if $x_n \xrightarrow{w} x$ in X, then $\|x\| \leq \liminf_{n \to \infty} \|x_n\|$.

16.6 Corollary

Let X be a nonzero reflexive space. Then the following conditions hold.

(a) Every nonempty closed convex subset of X contains an element of minimal norm.

(b) For every proper closed subspace Y of X, there is some $x_1 \in X$ such that $\|x_1\| = 1 = \text{dist}(x_1, Y)$.

(c) For every continuous linear functional f on X, there is some $x_1 \in X$ such that $\|x_1\| = 1$ and $|f(x_1)| = \|f\|$.

Proof:

(a) Let E be a nonempty closed convex subset of X and let

$$r = \inf\{\|x\| : x \in E\}.$$

Then $0 \leq r < \infty$. We show that there is some $x_0 \in E$ with $\|x_0\| = r$. Find $x_n \in E$ with $\|x_n\| \leq r + 1/n, n = 1, 2, \ldots$. Then (x_n) is a bounded sequence in X. By (ii), there is a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{w} x_0$ in X. Now 15.1 shows that x_0 belongs to the closure C of the convex hull of the set $\{x_{n_1}, x_{n_2}, \ldots\}$. But $C \subset E$, since E is convex and closed, so that $x_0 \in E$ and $r \leq \|x_0\|$. On the other hand, 15.1 also shows that

$$\|x_0\| \leq \liminf_{k\to\infty} \|x_{n_k}\| \leq \lim_{k\to\infty} (r + \frac{1}{n_k}) = r.$$

Thus $\|x_0\| = r$.

(b) Let Y be a proper closed subspace of $X, a \in X$ and $a \notin Y$. Then $E = \{a - y : y \in Y\}$ is a nonempty closed convex subset of X. By (a), there is some $y_0 \in Y$ such that $\|a - y_0\| \leq \|a - y\|$ for all $y \in Y$. Hence

$$\|a - y_0\| = \text{dist}(a, Y) = \text{dist}(a - y_0, Y).$$

As $a - y_0 \neq 0$, let $x_1 = (a - y_0)/\|a - y_0\|$. Then $\|x_1\| = 1 = \text{dist}(x_1, Y)$.

(c) Let f be a continuous linear functional on X. If $f = 0$, then for every $x \in X$ with $\|x\| = 1$, we have $\|f\| = 0 = |f(x)|$. Let $f \neq 0$. Then $Z(f)$ is a proper closed subspace of X. By (b), there is some

$x_1 \in X$ such that $\|x_1\| = 1 = \mathrm{dist}(x_1, Z(f))$. In particular, $f(x_1) \neq 0$. If $x \in X$, then $y = x - f(x)x_1/f(x_1)$ belongs to $Z(f)$. Now

$$\begin{aligned} |f(x)| &= |f(x)|\mathrm{dist}(x_1, Z(f)) = \mathrm{dist}(f(x)x_1, Z(f)) \\ &\leq \|f(x)x_1 + f(x_1)y\| = |f(x_1)|\,\|x\|. \end{aligned}$$

Hence $\|f\| \leq |f(x_1)|$. But since $\|x_1\| = 1$, we have $|f(x_1)| \leq \|f\|$. Thus $\|f\| = |f(x_1)|$. (Compare Problem 6-8.) □

James showed that if X is a Banach space and 16.6(c) holds, then X is reflexive. In ([37], 1972) he gave an elementary proof of this result which is based only on 16.3(a) and Problem 16-9. The proof of Corollary 16.6 then shows that the reflexivity of a Banach space is equivalent to each of the conditions 16.6(a), 16.6(b) and 16.6(c). James also showed ([36], 1971) that the completeness assumption on X cannot be dropped from this result.

The statement 16.6(b) is just the Riesz lemma (5.3) for $r = 1$. It is equivalent to the statement 16.6(c) for any normed space X, as indicated in Problem 7-3.

16.7 Examples

Let $X = \ell^1$. For a fixed $y \in \ell^\infty$, let

$$f_y(x) = \sum_{j=1}^{\infty} x(j)y(j), \quad x \in X.$$

We have seen in 13.2 that

$$X' = \{f_y : y \in \ell^\infty\}.$$

Hence the canonical embedding $J : X \to X''$ is given by

$$J(x)(f_y) = \sum_{j=1}^{\infty} x(j)y(j), \quad x \in X, y \in \ell^\infty.$$

We have remarked after 13.3 that there is some nonzero $y' \in (\ell^\infty)'$ such that $y'(y) = 0$ for all $y \in c_0$. (For example, let y' be a Banach limit (7.12)). Define $x'' : X' \to \mathbf{K}$ by

$$x''(f_y) = y'(y), \quad y \in \ell^\infty.$$

Then $0 \neq x'' \in X''$. Were $x'' = J(x)$ for some $x \in X$, then

$$x(n) = J(x)(f_{e_n}) = x''(f_{e_n}) = y'(e_n) = \lim_{j\to\infty} e_n(j) = 0$$

for all $n = 1, 2, \ldots$, so that $x = 0$ and in turn $x'' = J(0) = 0$. Thus $x'' \notin J(X)$, showing that the map J is not surjective.

While considering the Bolzano-Weierstrass property in Section 15, we have noted that the bounded sequence (e_n) in ℓ^1 does not have a weak convergent subsequence, in agreement with Eberlein's theorem (16.5).

We now illustrate how parts (a), (b) and (c) of 16.6 fail to hold for the nonreflexive space ℓ^1.

Let (k_n) be a sequence in $\mathbf{K}$ such that $\sup\{|k_n| : n = 1, 2, \ldots\} = 1$, but $|k_n| < 1$ for each $n = 1, 2, \ldots$. [For example, $k_n = (n-1)/n$ for $n = 1, 2, \ldots$.] Define $f : \ell^1 \to \mathbf{K}$ by

$$f(x) = \sum_{j=1}^{\infty} k_j x(j), \quad x \in \ell^1.$$

Then f is a continuous linear functional on ℓ^1 and $\|f\| = \sup\{|k_n| : n = 1, 2, \ldots\} = 1$, as we have seen in 6.7(b).

Consider a nonzero element x of ℓ^1. Then $x(n) \neq 0$ for some n. Since $|k_n| < 1$, we have

$$|f(x)| \leq \sum_{j=1}^{\infty} |k_j|\,|x(j)| < \sum_{j=1}^{\infty} |x(j)| = \|x\|_1.$$

This shows that $|f(x)| < 1$ for every $x \in \ell^1$ with $\|x\|_1$. Thus f does not attain its norm on the unit sphere of ℓ^1 and 16.6(c) does not hold for ℓ^1.

Let $Y = Z(f)$. Then Y is a proper closed subspace of ℓ^1. Let $x \in \ell^1$ with $\|x\|_1 = 1$. Since $|f(x)| < 1 = \sup\{|k_n| : n = 1, 2, \ldots\}$, we see that $|f(x)| < |k_m|$ for some m. Then

$$y = x - \frac{f(x)}{k_m} e_m$$

belongs to Y and

$$\|x - y\| = \frac{|f(x)|}{|k_m|} < 1$$

Hence $\mathrm{dist}(x, Y) \leq \|x - y\| < 1$. Thus 16.6(b) does not hold for ℓ^1

Finally, let $E = \{x \in \ell^1 : f(x) = 1\}$. Then E is a nonempty closed convex subset of ℓ^1. Let $x_n = e_n/k_n$, whenever $k_n \neq 0$. Then $x_n \in E$ and

$$\inf_{x \in E} \|x\|_1 \leq \inf_n \|x_n\|_1 = \frac{1}{\sup_n |k_n|} = 1$$

On the other hand, $1 = |f(x)| < \|x\|$ for every $x \in E$, as we have mentioned above. Thus E does not contain an element of minimal norm and 16.6(a) does not hold for ℓ^1.

See Problems 5-8(a) and 16-14 for further examples.

Uniform Convexity

We discuss an interesting geometric condition which implies reflexivity of a Banach space. In Section 5 we have noted that the closed unit ball of a normed space X is a convex subset of X. We have also introduced the concept of strict convexity of X, which says that the mid-point of the segment joining two points on the unit sphere of X does not lie on the unit sphere of X. Now we consider a yet stronger requirement on X and show that it implies the reflexivity of X.

A normed space X is said to be **uniformly convex** if for every $\epsilon > 0$, there exists some $\delta > 0$ such that for all x and y in X with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, we have $\|x + y\| \leq 2(1 - \delta)$.

This concept can be interpreted geometrically as follows. Given $\epsilon > 0$, there is some $\delta > 0$ such that if x and y are in the closed unit ball of X and they are at a distance at least ϵ from each other, then their mid-point lies at a distance at least δ from the unit sphere. [Here δ may depend on ϵ.] It is clear that a uniformly convex space is strictly convex. Conversely, if X is finite dimensional and strictly convex, then X is uniformly convex. This can be seen as follows. Let $\epsilon > 0$ and

$$E = \{(x, y) \in X \times X : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}.$$

Then E is a closed and bounded subset of $X \times X$ and so it is compact by 5.5. For $(x, y) \in E$, let

$$f(x, y) = 2 - \|x + y\|.$$

Now f is a continuous and strictly positive function on E. Hence there is some $\delta > 0$ such that $f(x, y) \geq 2\delta$ for all $(x, y) \in E$. This implies the uniform convexity of X. In general, however, a strictly convex normed space may not be uniformly convex. Let $X = c_{00}$ and define a norm on X inductively as follows. For $n = 1, 2, \ldots$, let

$$X_n = \{x \in X : x(j) = 0 \text{ for all } j > n\}.$$

For $x \in X_1$, let

$$\|x\| = |x(1)|.$$

Assume that $\|x\|$ is defined for all $x \in X_{n-1}$. If $x \in X_n$, then $x = y_{n-1} + x(n)e_n$ for some $y_{n-1} \in X_{n-1}$. Define

$$\|x\| = \left(\|y_{n-1}\|^n + |x(n)|^n\right)^{1/n}.$$

It can be checked inductively that $\|\ \|$ is a strictly convex norm on X. (Compare Problems 5-2 and 5-4.) For $n = 1, 2, \ldots$, let

$$x_n = \frac{e_1 + e_n}{2^{1/n}} \quad \text{and} \quad y_n = \frac{-e_1 + e_n}{2^{1/n}},$$

so that

$$\|x_n\| = 1 = \|y_n\| \quad \text{and} \quad \|x_n + y_n\| = 2^{(n-1)/n} = \|x_n - y_n\|.$$

Then $\|x_n - y_n\| \geq 1$ for all n, while $\|x_n + y_n\| \to 2$ as $n \to \infty$. Hence X is not uniformly convex.

It can be easily seen that the normed spaces $\ell^1, c, \ell^\infty, L^1([a,b])$ $C([a,b])$ and $L^\infty([a,b])$ are not strictly convex. It was proved by Clarkson ([10], 1936) that the normed spaces ℓ^p and $L^p([a.b])$ with $1 < p < \infty$ are uniformly convex.

We proceed to point out some consequences of uniform convexity.

16.8 Lemma

Let X be a uniformly convex normed space and (x_n) be a sequence in X such that $\|x_n\| \to 1$ and $\|x_n + x_m\| \to 2$ as $n, m \to \infty$. Then $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$, that is, (x_n) is a Cauchy sequence.

Proof:

Suppose (x_n) is not a Cauchy sequence in X. Then for some $\epsilon > 0$ and for every positive integer n_0, there are $n, m \geq n_0$ with

$$\|x_n - x_m\| \geq 4\epsilon.$$

This implies that for a given $x \in X$ and a positive integer m_0, there is some $m > m_0$ with

$$\|x_m - x\| \geq 2\epsilon.$$

Since $\|x_n\| \to 1$ as $n \to \infty$, we see that for each $j = 1, 2, \ldots$, there is a positive integer n_j such that

$$\|x_n\| \leq 1 + \frac{1}{j} \quad \text{for all } n \geq n_j.$$

Choose $m_1 = n_1$. Then $\|x_{m_1}\| \leq 1 + 1 = 2$. Letting $m_0 = \max\{m_1, n_2\}$ and $x = x_{m_1}$, we see that there is some $m_2 > m_0$ with

$$\|x_{m_2} - x_{m_1}\| \geq 2\epsilon.$$

Note that $\|x_{m_2}\| \leq 1 + 1/2$ since $m_2 > n_2$. In this manner, we find a subsequence (x_{m_j}) of (x_m) such that for each $j = 1, 2, \ldots$,

$$\|x_{m_{j+1}} - x_{m_j}\| \geq 2\epsilon \quad \text{and} \quad \|x_{m_j}\| \leq 1 + \frac{1}{j}.$$

By the uniform convexity of X, there is some $\delta > 0$ such that $\|x+y\| \le 2(1-\delta)$ whenever x and y are in X, $\|x\| \le 1$, $\|y\| \le 1$ and $\|x-y\| \ge \epsilon$. If we let $y_j = x_{m_j}$ for $j = 1, 2, \ldots$, then

$$\left\|\frac{y_j}{1+1/j}\right\| \le 1, \left\|\frac{y_{j+1}}{1+1/j}\right\| \le 1 \text{ and } \left\|\frac{y_{j+1}-y_j}{1+1/j}\right\| \ge \frac{2\epsilon}{1+1/j} \ge \epsilon,$$

so that

$$\left\|\frac{y_{j+1}+y_j}{1+1/j}\right\| \le 2(1-\delta).$$

Thus $\limsup_{j\to\infty} \|y_{j+1}+y_j\| \le 2(1-\delta) < 2$. This is in contradiction with $\|x_n + x_m\| \to 2$ as $n, m \to \infty$. Hence (x_n) is a Cauchy sequence in X. $\square$

16.9 Theorem (Milman, 1938)

Let X be a Banach space which is uniformly convex in some equivalent norm. Then X is reflexive.

Proof:

As we have seen in 16.1(a), a reflexive normed space remains reflexive in an equivalent norm. Hence we can assume, without loss of generality, that X is a uniformly convex Banach space in the given norm $\|\ \|$ on X.

Let $x'' \in X''$. Without loss of generality, we assume that $\|x''\| = 1$. To show that there is some $x \in X$ with $J(x) = x''$, first we find a sequence (x_n') in X' such that

$$\|x_n'\| = 1 \quad \text{and} \quad |x''(x_n')| > 1 - \frac{1}{n} \quad \text{for } n = 1, 2, \ldots.$$

For a fixed n, let $S_n = \operatorname{span}\{x_1', \ldots, x_n'\}$ and $\epsilon_n = 1/n$ in Helly's theorem (16.3(b)) and find $x_n \in X$ such that

$$x''_{|S_n} = J(x_n)_{|S_n} \quad \text{and} \quad \|x_n\| < 1 + \frac{1}{n}.$$

Then for $n = 1, 2, \ldots$ and $m = 1, \ldots, n$,

$$x''(x_m') = J(x_n)(x_m') = x_m'(x_n),$$

so that

$$1-\frac{1}{n} < |x''(x_n')| = |x_n'(x_n)| \le \|x_n\| < 1+\frac{1}{n}$$

and

$$2-\frac{2}{n} < |2x''(x_n')| = |x_m'(x_n)+x_m'(x_m)| \le \|x_n+x_m\| \le 2+\frac{1}{n}+\frac{1}{m}.$$

Thus we have

$$\lim_{n\to\infty} \|x_n\| = 1 \quad \text{and} \quad \lim_{n,m\to\infty} \|x_n+x_m\| = 2$$

By 16.8, (x_n) is a Cauchy sequence in X. Since X is a Banach space, let $x_n \to x$ in X. Then $\|x\| = 1$. Also, since $x''(x_m') = x_m'(x_n)$ for all $n \ge m$, the continuity of x_m' shows that

$$x''(x_m') = x_m'(x), \quad m = 1, 2, \ldots.$$

To complete the proof, consider $x' \in X'$. Replacing S_n by the span of $\{x', x_1', \ldots, x_n'\}$ in the previous argument, we find some $y \in X$ such that $\|y\| = 1$ and

$$x''(x') = x'(y) \quad \text{and} \quad x''(x_m') = x_m'(y), \quad m = 1, 2, \ldots.$$

We claim that $y = x$. Now

$$\|x+y\| \ge |x_m'(x+y)| = 2|x''(x_m')| > 2 - \frac{2}{m}$$

for all $m = 1, 2, \ldots$, so that $\|x+y\| \ge 2$. Since $\|x\|, \|y\| = 1$, the strict convexity of X implies that $y = x$ and hence $x''(x') = x'(y) = x'(x)$ for all $x' \in X'$, that is, $x'' = J(x)$. Thus X is reflexive. □

We remark that the converse of Milman's theorem is false. Day ([15], 1941) has given the following example of a reflexive normed space X which is not uniformly convex in any equivalent norm. For $1 < p < \infty$, let X be the set of all $x = (x_1, x_2, \ldots)$ with $x_n \in \mathbf{K}^n$ and

$$\|x\| = \left(\sum_{n=1}^{\infty} \|x_n\|_\infty^p\right)^{1/p} < \infty.$$

In Section 5 we have mentioned that some of the most important normed spaces are obtained from inner product spaces. We shall take up their study in Section 21 and show that they are uniformly convex. If the norm induced by an inner product on X is complete, then X is reflexive by Milman's theorem. We shall also prove this result independently in Section 24.

Problems

16-1 Let X be a reflexive normed space. Then X is strictly convex (resp., smooth) if and only if X' is smooth (resp., strictly convex). (Hint: Problem 13-10)

16-2 Let X be a reflexive normed space and $\{x_1, x_2, \ldots\}$ be a Schauder basis for X with coefficient functionals $\{f_1, f_2, \ldots\}$. If $x_n' = f_n/\|f_n\|, n = 1, 2, \ldots$, then $\{x_1', x_2', \ldots,\}$ is a Schauder basis for X' with coefficient functionals $\{\|f_1\|J(x_1), \|f_2\|J(x_2), \ldots\}$. (Hint: Problems 15-11 and 15-12)

16-3 Let $X = C([a,b]), a < c < b$, and w be the characteristic function of $(c, b]$. For $y \in NBV([a,b])$, let

$$f_y(x) = \int_a^b x\,dy, \;\; x \in X \quad \text{and} \quad x''(f_y) = \int_a^b w\,dy, \;\; y \in NBV([a,b]).$$

Then $x'' \in X''$, but $x'' \notin J(X)$.

16-4 (a) The linear space $X = c_{00}$ with the norm $\|\ \|_p, 1 \le p \le \infty$, is not reflexive, but X' is reflexive if $1 < p < \infty$.

(b) The Banach space c with the norm $\|\ \|_\infty$ is not reflexive.

16-5 A Banach space X is reflexive if and only if X' is reflexive. In particular, the normed spaces ℓ^∞, and $L^\infty([a,b])$ and $NBV([a,b])$ are not reflexive. (Hint: If X is a Banach space and X' is reflexive, then $J(X)$ is closed and dense in X''. Alternatively, use 16.5. The Banach spaces ℓ^1, $L^1([a,b])$ and $C([a,b])$ are not reflexive.)

16-6 Let Y be a closed subspace of a normed space X. Then X is reflexive if and only if Y and X/Y are reflexive.

16-7 Let $\| \ \|_j$ be a norm on a linear space X_j for $j = 1, 2, \ldots$. Fix $1 \le p \le \infty$. Let X be the product normed space as defined in Problem 5-6. If $1 < p < \infty$, then X is reflexive if and only if each X_j is reflexive. If $p = 1$ or ∞, then X is reflexive if and only if there is a positive integer n such that $X_j = \{0\}$ for all $j > n$ and $X_1, \ldots, X_n$ are reflexive. (Hint: 16.4 (b) and 16.1(c)).

16-8 Let X be a normed space. Then $x_n \xrightarrow{w} x$ in X if and only if $J(x_n) \xrightarrow{w^*} J(x)$ in $(X')'$. If X is reflexive, then $x_n' \xrightarrow{w} x'$ in X' if and only if $x_n' \xrightarrow{w^*} x'$ in X', and in that case, x' belongs to the closure of the convex hull of $\{x_1', x_2', \ldots\}$. (Compare 15.1 and Problem 15-16.)

16-9 A normed space X is reflexive if and only if every separable closed subspace of X is reflexive (Hint: 16.5)

16-10 Let X be a reflexive normed space, (x_n) be a sequence in X and for $m = 1, 2, \ldots$, let E_m denote the closure of the convex hull of $\{x_m, x_{m+1} \ldots\}$. Then $x_n \xrightarrow{w} x$ in X if and only if (x_n) is a bounded sequence and $\bigcap_{m=1}^{\infty} E_m = \{x\}$. (Hint: Problem 15-10 and 16.5)

16-11 Let X be a reflexive normed space and $F \in BL(X, Y)$. If E is a closed, bounded and convex subset of X, then $F(E)$ is a closed, bounded and convex subset of Y. In particular, $F(\overline{U}_X)$ is closed in Y. (Hint: 16.5, 15.1 and Problem 15-5)

16-12 Let X be a reflexive normed space and E be a nonempty closed convex subset of X. Then for every $x \in X$, there is some $y \in E$ such that $\|x - y\| = \operatorname{dist}(x, E)$, that is, there is a **best approximation** to x from E. (Compare 23.5).

16-13 Let T be a compact metric space. Then $C(T)$ with the sup norm is reflexive if and only if T has only a finite number of points. (Hint: 16.5)

16-14 Let (k_j) be a sequence in $\mathbf{K}$ with $\sum_{j=1}^{\infty} |k_j| = 1$, but $\sum_{j=1}^{n} |k_j| < 1$ for each $n = 1, 2, \ldots$. [For example, $k_j = 1/2^j, j = 1, 2, \ldots$]

(a) Let $f(x) = \sum_{j=1}^{\infty} k_j x(j)$ for $x \in c_0$. Then f is a continuous linear functional on c_0, but it does not attain its norm on the unit sphere of c_0.

(b) Let $Y = \{x \in c_0 : \sum_{j=1}^{\infty} k_j x(j) = 0\}$. Then Y is a proper closed subspace of c_0, but $\mathrm{dist}(x, Y) < 1$ for every $x \in c_0$ with $\|x\|_\infty = 1$.

(c) Let $E = \{x \in c_0 : \sum_{j=1}^{\infty} k_j x(j) = 1\}$. Then E is a closed convex subset of c_0, but it does not contain an element of minimal norm. (Compare 16.5, 16.6 and 16.7. Hint: If $x \in E$, then $|x(n)| < 1$ and $k_n \neq 0$ for some n.)

16-15 (a) For $n = 2, 3, \ldots, \mathbf{K}^n$ with the norm $\|\ \|_1$ is reflexive but not strictly convex.

(b) ℓ^1 is strictly convex but not reflexive in some norm which is equivalent to the norm $\|\ \|_1$. (Hint: Problem 7-18(b))

(c) ℓ^1 is not uniformly convex in any norm which is equivalent to the norm $\|\ \|_1$. (Hint: 16.9)

16-16 A subspace of ℓ^1 is reflexive if and only if it is finite dimensional. (Hint: 16.5, 15.2(b) and 5.5)

16-17 Let X be a uniformly convex normed space, $x \in X$ and (x_n) be a sequence in X.

(a) If $\|x\| = 1, \|x_n\| \to 1$ and $\|x_n + x\| \to 2$, then $\|x_n - x\| \to 0$.

(b) $x_n \to x$ in X if and only if $x_n \xrightarrow{w} x$ in X and $\limsup_{n\to\infty} \|x_n\| \leq \|x\|$. (Hint: 16.8, 15.1 and 7.6)

16-18 Let X be a normed space and $x' \in X'$. Then there is at least one, at most one, or exactly one $x \in X$ such that $\|x\| = 1$ and $\|x'\| = x'(x)$, provided X is a reflexive space, a strictly convex space, or a uniformly convex Banach space, respectively.

Chapter V

Compact Operators on Normed Spaces

This chapter focuses on a natural and useful generalization of bounded linear maps having a finite dimensional range. The concept of a compact linear map is introduced in Section 17. Its relations with weak convergence and reflexivity are noted. The spectrum of a compact operator on a normed space is of a very special nature, which is described in Section 18. The properties of the spectrum of a compact operator are employed in Section 19 to obtain criteria for the existence and the uniqueness of solutions of some equations involving that operator. Section 20 is devoted to constructing approximate solutions of these equations by considering finite rank approximations of a compact operator. A reduction of the finite rank problem to a finite dimensional problem is also given.

17 Compact Linear Maps

We have seen in 6.2 that a linear map F from a normed space X to a normed space Y is continuous if and only if it sends the open unit ball U in X to a bounded subset of Y. Then the closure of $F(U)$ is a closed and bounded subset of Y. Analogously, we introduce a property of linear maps which is, in general, stronger than continuity.

A linear map F from a normed space X to a normed space Y is said to be **compact** if the closure of $F(U)$ is a compact subset of Y. A compact linear map is also known as a **completely continuous** linear map in view of a result we shall prove in 17.5(a).

Since a compact subset of Y is bounded, a compact linear map is always continuous. However, not every continuous linear map is compact. For example, if X is an infinite dimensional normed space, then the identity map I on X is clearly linear and continuous, but it is not compact because the closure of $I(U)$ is the closed unit ball in X which cannot be compact by 5.5.

We now consider some salient features of compact linear maps. For this purpose we recall from 3.5 that a subset E of a metric space is compact if and only if every sequence in E has a subsequence which converges in E. Further, this is the case if and only if E is complete and totally bounded.

17.1 Theorem

Let X and Y be normed spaces and $F : X \to Y$ be linear.

(a) F is a compact map if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

(b) If F is a compact map, then $F(U)$ is a totally bounded subset of Y. Conversely, if Y is Banach and $F(U)$ is a totally bounded subset of Y, then F is a compact map.

(c) If F is continuous and of finite rank, then F is a compact map and $R(F)$ is closed in Y. Conversely, if X and Y are Banach spaces, F is a compact map and $R(F)$ is closed in Y, then F is continuous and of finite rank.

Proof:

(a) Let F be a compact map and (x_n) be a sequence in X such that $\|x_n\| < \alpha$ for some $\alpha > 0$ and all $n = 1, 2, \ldots$. Then $F(x_n/\alpha) \in F(U)$. Since $\overline{F(U)}$ is compact, 3.5 shows that a subsequence of $(F(x_n/\alpha))$ converges in $\overline{F(U)} \subset Y$. As a result, a subsequence of $(F(x_n))$ converges in Y.

Conversely, assume that for every bounded sequence (x_n) in X, $(F(x_n))$ has a convergent subsequence. To show that $\overline{F(U)}$ is compact, consider a sequence (y_n) in it. Find $x_n \in U$ such that $\|y_n - F(x_n)\| < 1/n$ for $n = 1, 2, \ldots$. Let (x_{n_j}) be a subsequence of (x_n) such that $(F(x_{n_j}))$ converges to some $y \in Y$. It follows that the subsequence (y_{n_j}) of (y_n) also converges to y, which belongs to $\overline{F(U)}$. Again by 3.5, $\overline{F(U)}$ is compact, that is, F is a compact map.

(b) If F is a compact map, then by 3.5 $\overline{F(U)}$ is totally bounded, and so is its subset $F(U)$.

Conversely, if Y is a Banach space and $F(U)$ is a totally bounded subset of it, then its closure $\overline{F(U)}$ is complete and totally bounded. Again by 3.5, $\overline{F(U)}$ is compact, that is, F is a compact map.

(c) Let $F \in BL(X, Y)$ be of finite rank. Since $R(F)$ is finite dimensional, it is a closed subspace of Y by 5.4(b). Since F is continuous, $\overline{F(U)}$ is a bounded subset of Y by 6.2. Thus $\overline{F(U)}$ is a closed and bounded subset of the finite dimensional space $R(F)$. By 5.5, $\overline{F(U)}$ is compact, that is, F is a compact map.

Conversely, assume that X and Y are Banach spaces, F is a compact map such that $R(F)$ is closed in Y. Then F is continuous. Also, $R(F)$ is a Banach space and $F : X \to R(F)$ is surjective. By the open mapping theorem (10.6), $F(U)$ is open in $R(F)$. Hence there is some $\delta > 0$ such that

$$E = \{y \in R(F) : \|y\| < \delta\} \subset F(U).$$

Since $R(F)$ is closed, we have

$$\{y \in R(F) : \|y\| \leq \delta\} = \overline{E} \subset \overline{F(U)} \subset R(F).$$

As $\overline{F(U)}$ is compact, we find that the closed ball of radius δ about 0 in the normed space $R(F)$ is compact. It follows from 5.5 that $R(F)$ is finite dimensional, that is, F is of finite rank. $\square$

We shall denote the set of all compact linear maps from a normed space X to a normed space Y by $CL(X, Y)$. We write $CL(X)$ for $CL(X, X)$. By 17.1(c), we see that $CL(X, \mathbf{K}) = X'$.

17.2 Theorem

(a) Let $k \in \mathbf{K}$ and $F, G \in CL(X, Y)$. Then kF and $F + G$ belong to $CL(X, Y)$.

(b) Let $F \in BL(X, Y), G \in BL(Y, Z)$ and one of them be compact. Then $GF \in CL(X, Z)$.

(c) Let Y be a Banach space, $F_n \in CL(X, Y)$, $F \in BL(X, Y)$ and $\|F_n - F\| \to 0$. Then $F \in CL(X, Y)$.

Proof:

(a) Let (x_n) be a bounded sequence in X. Since $F \in CL(X, Y)$, there is a subsequence (x_{n_j}) of (x_n) such that $(F(x_{n_j}))$ converges in Y. Clearly, $(kF(x_{n_j}))$ converges in Y. Again, since $G \in CL(X, Y)$, there is a subsequence (z_j) of (x_{n_j}) such that $(G(z_j))$ converges in Y. Then (z_j) is a subsequence of (x_n) and $(F + G)(z_j)$ converges in Y. By 17.1(a), we see that kF and $F + G$ are in $CL(X, Y)$.

(b) Let (x_n) be a bounded sequence in X. If $F \in CL(X, Y)$, let $(F(x_{n_j}))$ converge in Y. Since G is continuous, $(G(F(x_{n_j}))$ converges in Z. If $G \in CL(Y, Z)$, then since $(F(x_n))$ is a bounded sequence in Y, a subsequence of $(G(F(x_n)))$ converges in Z. Again by 17.1(a), $GF \in CL(X, Z)$.

(c) By 17.1(b), it is enough to show that $F(U)$ is a totally bounded subset of Y. Let $\epsilon > 0$. Find a positive integer n with $\|F_n - F\| < \epsilon/3$. Now $F_n(U)$ is totally bounded since $F_n \in CL(X, Y)$. Hence there are $x_1, \ldots, x_m$ in U such that

$$F_n(U) \subset \bigcup_{j=1}^{m} U(F_n(x_j), \epsilon/3).$$

For $x \in U$, find some x_j such that $F_n(x) \in U(F_n(x_j), \epsilon/3)$, so that

$$\begin{aligned}\|F(x) - F(x_j)\| &\leq \|(F - F_n)(x)\| + \|F_n(x) - F_n(x_j)\| \\ &\quad + \|(F_n - F)(x_j)\|\end{aligned}$$

$$\begin{aligned} &< \|F - F_n\| \|x\| + \frac{\epsilon}{3} + \|F_n - F\| \|x_j\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus

$$F(U) \subset \bigcup_{j=1}^{m} U(F(x_j), \epsilon),$$

proving that $F(U)$ is totally bounded. $\square$

These results show that $CL(X,Y)$ is a subspace of $BL(X,Y)$ and $CL(X)$ is, in fact, a two-sided ideal of $BL(X)$. Further, if Y is a Banach space then $CL(X,Y)$ is closed in $BL(X,Y)$.

We now consider the compactness of the transpose of a bounded linear map.

17.3 Theorem (Schauder, 1930)
Let X and Y be normed spaces and $F \in BL(X,Y)$. If $F \in CL(X,Y)$, then $F' \in CL(Y',X')$. The converse holds if Y is a Banach space.

Proof:
Let $F \in CL(X,Y)$. Consider a bounded sequence (y_n') in Y'. For y and z in Y,

$$|y_n'(y) - y_n'(z)| \le \|y_n'\| \, \|y - z\| \le \alpha \|y - z\|$$

for some $\alpha > 0$ and all $n = 1, 2, \ldots$. Let $T = \overline{F(U)}$. Then $\{y'_{n|T} : n = 1, 2, \ldots\}$ is a set of uniformly bounded equicontinuous functions on the compact metric space T. By Arzela's theorem (3.10(b)), $(y'_{n|T})$ has a subsequence $(z'_{j|T})$ which converges uniformly on T. For $i, j = 1, 2, \ldots$, we have by 6.6

$$\begin{aligned} \|F'(z_i') - F'(z_j')\| &= \sup\{|F'(z_i' - z_j')(x)| : x \in U\} \\ &= \sup\{|(z_i' - z_j')(F(x))| : x \in U\} \\ &\le \sup\{|z_i'(y) - z_j'(y)| : y \in T\}. \end{aligned}$$

Since the sequence $(z'_{j,p})$ is uniformly Cauchy on T, we see that $(F'(z'_j))$ is a Cauchy sequence in X'. It must converge in X' as X' is a Banach space by 8.2(d). We have thus shown that $(F'(y'_n))$ has a convergent subsequence. By 17.1(a), $F' \in CL(Y', X')$.

Conversely, assume that Y is a Banach space, $F \in BL(X, Y)$ and $F' \in CL(Y', X')$. Then $F'' \in CL(X'', Y'')$ by the argument given above. Now consider the canonical embeddings $J_X : X \to X''$ and $J_Y : Y \to Y''$ introduced in Section 8. Since $F'' J_X = J_Y F$ by 13.5(b), we see that $J_Y(F(U)) = \{F''(J_X(x)) : x \in X, \|x\| \le 1\}$ is contained in $\{F''(x'') : x'' \in X'', \|x''\| \le 1\}$. This last set is totally bounded in Y'' since F'' is a compact map. As a result, $J_Y(F(U))$ is a totally bounded subset of Y''. Since J_Y is an isometry, $F(U)$ is a totally bounded subset of Y. As Y is a Banach space, 17.1(b) shows that $F \in CL(X, Y)$. $\square$

17.4 Examples

We have seen in 17.1(c) that every continuous linear map of finite rank from X to Y is compact. In particular, every $x' \in X'$ is a compact map. Further, if X is finite dimensional, then every linear map F from X to Y is compact since F is continuous by 6.1 and F is clearly of finite rank. Also, 17.2(c) shows that if Y is a Banach space, each $F_n \in BL(X, Y)$ is of finite rank and $\|F_n - F\| \to 0$, then F is a compact map. Quite often this is how the compactness of a linear map is established. In fact, if Y is a Banach space with a Schauder basis, then every $F \in CL(X, Y)$ arises in this way. (See Problem 17-8.) In 1932, Banach inquired whether every compact operator on every separable Banach space is a norm limit of a sequence of continuous operators of finite rank. This question remained unanswered till 1973, when Enflo [21] settled it in the negative. In 1974, Alexander [1] showed that if $2 < p < \infty$, then there is a compact operator A on a closed subspace of ℓ^p which is not the norm limit of a sequence of continuous operators of finite rank.

(a) An $m \times n$ matrix defines a linear map from $\mathbf{K}^n$ to $\mathbf{K}^m$ which is clearly a compact map.

Let us now consider an infinite matrix $M = (k_{i,j})$. We give conditions under which it defines a compact linear map from ℓ^p to ℓ^p.

Let $p = 1$. For $j = 1, 2, \ldots$, define

$$\gamma(j) = \sum_{i=1}^{\infty} |k_{i,j}|.$$

If $\gamma(j) < \infty$ for each j and $\gamma(j) \to 0$ as $j \to \infty$, then $M \in CL(\ell^1)$. This can be seen as follows. For $n = 1, 2, \ldots$ let M_n denote the infinite matrix whose first n columns are the same as those of the matrix M and whose all other columns are zero. Our discussion in 6.5(c) shows that M, M_n and $M - M_n$ define bounded linear maps from ℓ^1 to ℓ^1. Also,

$$\|M - M_n\| \le \sup\{\gamma(j) : j = n+1, n+2, \ldots\} \to 0 \text{ as } n \to \infty.$$

Further, each M_n is of finite rank since for all $x \in \ell^1$,

$$M_n x(i) = \sum_{j=1}^{n} k_{i,j} x(j), \quad i = 1, 2, \ldots,$$

that is, $M_n x = x(1)Me_1 + \cdots + x(n)Me_n$. Since ℓ^1 is a Banach space, M is a compact map by 17.2(c). That the condition "$\gamma(j) \to 0$ as $j \to \infty$" is not necessary for the compactness of M can be seen by considering $M = (k_{i,j})$, where $k_{1,j} = 1$ and $k_{i,j} = 0$ if $j = 1, 2, \ldots, i = 2, 3, \ldots$. Note that the range of M equals span $\{e_1\}$.

Next, let $p = \infty$. For $i = 1, 2, \ldots$, define

$$\delta(i) = \sum_{j=1}^{\infty} |k_{i,j}|.$$

If $\delta(i) < \infty$ for each i and $\delta(i) \to 0$ as $i \to \infty$, then $M \in CL(\ell^\infty)$. To prove this, consider the transpose $N = (k_{j,i})$ of the matrix M. Then $N \in CL(\ell^1)$ by our result for the case $p = 1$. Now 13.4(b) shows that the transpose of N, namely M, defines a bounded linear map from ℓ^∞ to ℓ^∞, and by the Riesz-Schauder theorem 17.3, it must be a compact map. We also give an independent argument for showing that $M \in CL(\ell^\infty)$ as follows. For $n = 1, 2, \ldots$, let M_n denote the infinite matrix whose first n rows are the same as those of the matrix M and whose all other rows are zero. Then our discussion in 6.5(c) shows that M, M_n and $M - M_n$ define bounded linear maps from ℓ^∞ to ℓ^∞. Also,

$$\|M - M_n\| \leq \sup\{\delta(i) : i = n+1, n+2, \ldots\} \to 0 \quad \text{as } n \to \infty.$$

Further, each M_n is of finite rank since for all $x \in \ell^\infty$,

$$(M_n x)(i) = \begin{cases} \sum_{j=1}^{\infty} k_{i,j} x(j), & \text{if } 1 \leq i \leq n \\ 0 & \text{if } i > n, \end{cases}$$

that is, $M_n x = (Mx)(1)e_1 + \ldots + (Mx)(n)e_n$. Since ℓ^∞ is a Banach space, M is a compact map again by 17.2(c). That the condition "$\delta(i) \to 0$ as $i \to \infty$" is not necessary for the compactness of M can be seen by considering $M = (k_{i,j})$, where $k_{i,1} = 1$ and $k_{i,j} = 0$ if $i = 1, 2, \ldots, j = 2, 3, \ldots$. Note that the range of M equals span $\{Me_1\}$.

Finally, let $1 < p < \infty$. If one of the sequences $(\gamma(j))$ and $(\delta(i))$ is bounded and the other tends to zero, then $M \in CL(\ell^p)$. To prove this, assume that $(\gamma(j))$ is bounded and $\delta(i) \to 0$ as $i \to \infty$. Considering the matrix M_n given in the case $p = \infty$, we find (as in 6.5(c)) that

$$\|M - M_n\|_p \leq \left[\sup_{j=1,2,\ldots} \gamma(j)\right]^{1/p} \left[\sup_{i=n+1,n+2,\ldots} \delta(i)\right]^{1/q},$$

where $1/p + 1/q = 1$. Again, since ℓ^p is a Banach space, we obtain the desired result by 17.2(c). Similar argument holds if $\gamma(j) \to 0$ as $j \to \infty$ and $(\delta(i))$ is bounded.

We have seen in 6.5(c) that if $1 < p < \infty$ and

$$\beta_p = \left[\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |k_{i,j}|^q\right)^{p/q}\right]^{1/p} < \infty,$$

where $1/p + 1/q = 1$, then $M \in BL(\ell^p)$. In fact, $M \in CL(\ell^p)$ This follows by considering the matrix M_n given in the case $p = \infty$ above and noting (as in 6 5(c)) that

$$\|M - M_n\|_p^p \leq \sum_{i=n+1}^{\infty} \left(\sum_{j=1}^{\infty} |k_{i,j}|^q \right)^{p/q} \to 0 \quad \text{as } n \to \infty.$$

That this condition is not necessary for the compactness of M can be seen by considering $M = \text{ diag } (1, 1/2^{1/p}, 1/3^{1/p}, \ldots)$. Note that if $M_n = \text{ diag } (1, 1/2^{1/p}, \ldots, 1/n^{1/p}, 0, 0, \ldots)$, then $\|M - M_n\|_p^p$ equals $1/(n+1)$ and hence it tends to 0 as $n \to \infty$.

Other sufficient conditions for the compactness of M are given in Problems 17-11, 17-12 and 17-13. Also, some necessary conditions are given in Problem 17-14.

(b) In analogy with a linear map defined by an infinite matrix $(k_{i,j})$, let us consider a Fredholm integral operator defined by a kernel $k(s,t)$:

$$F(x)(s) = \int_a^b k(s,t)x(t)\, dm(t), \quad s \in [a, b].$$

First, assume that $k(.,.)$ is continuous on $[a, b] \times [a, b]$. Let x be an integrable function on $[a, b]$. If $s_n \to s$ in $[a, b]$, then the Lebesgue dominated convergence theorem shows that $F(x)(s_n) \to F(x)(s)$. Hence $F(x)$ is continuous on $[a, b]$. We prove that if (x_n) is a sequence of integrable functions on $[a, b]$ such that $\|x_n\|_1 \leq \alpha$ for some $\alpha > 0$ and all $n = 1, 2, \ldots$, then $(F(x_n))$ has a subsequence which converges uniformly on $[a, b]$. Let $|k(s, t)| \leq \beta$ for all $s, t \in [a, b]$. Then $|F(x_n)(s)| \leq \beta\alpha$ for all $n = 1, 2, \ldots$ and $s \in [a, b]$. Thus $(F(x_n))$ is uniformly bounded. Also, it is equicontinuous on $[a, b]$, because

$$|F(x_n)(s) - F(x_n)(u)| \leq \int_a^b |k(s,t) - k(u,t)|\, |x_n(t)|\, dm(t)$$

for all $s, u \in [a, b]$ and $n = 1, 2 \ldots$. Let $\epsilon > 0$. Since $k(.,.)$ is uniformly continuous on $[a, b] \times [a, b]$, there is some $\delta > 0$ such that $|k(s, t) - k(u, v)| < \epsilon$ if $|s - u| < \delta$ and $|t - v| < \delta$. Hence

$$|F(x_n)(s) - F(x_n)(u)| \leq \epsilon\alpha \quad \text{if } |s - u| < \delta.$$

Now, Arzela's theorem (3.10(b)) shows that $(F(x_n))$ has a uniformly convergent subsequence.

Let us denote by X one of the normed spaces $C([a,b])$ or $L^p([a,b])$, where $1 \le p \le \infty$, and similarly for Y. If $x \in X$, then x is integrable on $[a,b]$, and if (x_n) is a bounded sequence in X, then $\|x_n\|_1 \le \alpha$ for some $\alpha > 0$ and all $n = 1, 2, \ldots$. On the other hand, if y is continuous on $[a,b]$, then $y \in Y$, and if (y_n) is a uniformly convergent sequence in $C([a,b])$, then (y_n) converges in Y. Hence for every bounded sequence (x_n) in X, $(F(x_n))$ has a convergent subsequence in Y by what we have proved above. Thus F is a compact linear map from X to Y, provided the kernel $k(.,.)$ is continuous on $[a,b]\times[a,b]$.

We give another condition for the compactness of F. Let $X = L^p([a,b])$ with $1 < p \le \infty$ and $Y = L^q([a,b])$, where $1/p + 1/q = 1$. Assume that $k(.,.) \in L^q([a,b]\times[a,b])$. Then for $x \in X$ and $s \in [a,b]$,

$$|F(x)(s)| \le \int_a^b |k(s,t)|\,|x(t)|\,dm(t) \le \|x\|_p \left(\int_a^b |k(s,t)|^q\,dm(t)\right)^{1/q}$$

and hence

$$\|F(x)\|_q^q = \int_a^b |F(x)(s)|^q\,dm(s) \le \|x\|_p^q \int_a^b \int_a^b |k(s,t)|^q\,dm(t)\,dm(s),$$

showing that $\|F\| \le \|k\|_q$. Since $1 \le q < \infty$, there is sequence (k_n) in $C([a,b]\times[a,b])$ such that $\|k_n - k\|_q \to 0$ as $n \to \infty$. (Compare 4.7(b).) If we let

$$F_n(x)(s) = \int_a^b k_n(s,t)x(t)\,dm(t), \quad s \in [a,b],$$

then it follows that

$$\|F - F_n\| \le \|k - k_n\|_q \to 0 \quad \text{as } n \to \infty.$$

Since each F_n is a compact linear map from $L^p([a,b])$ to $L^q([a,b])$ and $L^q([a,b])$ is a Banach space, we see by 17.2(c) that F is also a compact linear map.

Other sufficient conditions for the compactness of F are given in Problems 17-16, 17-17 and 17-18.

Before concluding this section we discuss how compactness of linear maps is related to weak convergence and reflexivity.

17.5 Theorem

Let X and Y be normed spaces and $F : X \to Y$ be linear.

(a) Let $F \in CL(X, Y)$. If $x_n \xrightarrow{w} x$ in X, then $F(x_n) \to F(x)$ in Y.

(b) Let X be reflexive and $F(x_n) \to F(x)$ in Y whenever $x_n \xrightarrow{w} x$ in X. Then $F \in CL(X, Y)$.

Proof:

(a) Let $x_n \xrightarrow{w} x$ in X. By 15.1(a), (x_n) is a bounded sequence in X. Suppose for a moment that $F(x_n) \not\to F(x)$. Then there is some $\epsilon > 0$ and a subsequence (x_{n_j}) such that $\|F(x_{n_j}) - F(x)\| \geq \epsilon$ for all $j = 1, 2, \ldots$. Since F is a compact linear map and (x_{n_j}) is a bounded sequence in X, there is a subsequence (z_j) of (x_{n_j}) such that $(F(z_j))$ converges to some y in Y. Then $\|y - F(x)\| \geq \epsilon$, so that $y \neq F(x)$. On the other hand, if $y' \in Y'$, then $y' \circ F \in X'$ and since $z_j \xrightarrow{w} x$ in X, we see that

$$y'(F(x)) = \lim_{j\to\infty} (y' \circ F)(z_j) = y'(\lim_{j\to\infty} F(z_j)) = y'(y).$$

Thus $y'(y - F(x)) = 0$ for every $y' \in Y'$. By 7.10(a), we must have $y = F(x)$. This contradiction proves that $F(x_n) \to F(x)$ in Y.

(b) Let (x_n) be a bounded sequence in X. Since X is reflexive, Eberlein's theorem (16.5) shows that (x_n) has a weak convergent subsequence (x_{n_j}). Let $x_{n_j} \xrightarrow{w} x$ in X. Then by our hypothesis, $F(x_{n_j}) \to F(x)$ in Y. Thus for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y. Hence F is a compact map by 17.1(a). $\square$

We remark that the requirement of the reflexivity of X cannot be dropped from 17.5(b). For example, if F denotes the identity map from ℓ^1 to ℓ^1, then Schur's test (15.2(b)) shows that $F(x_n) \to F(x)$ whenever $x_n \xrightarrow{w} x$ in ℓ^1. However, the identity map F is not compact since ℓ^1 is infinite dimensional.

An interesting consequence of 17.5(a) is given in Problem 17-23.

Problems

17-1 Let X and Y be normed spaces and $F \in CL(X, Y)$.

(a) Let Z be a subspace of X and $W = \overline{F(Z)} \subset Y$. Define $G : Z \to W$ by $G(z) = F(z)$ for $z \in Z$. Then $G \in CL(Z, W)$.

(b) F may not belong to $CL(X, F(X))$.

(c) $F(X)$ is a separable subspace of Y.

17-2 Let X and Y be normed spaces and $F \in CL(X, Y)$. If X_c denotes the closure of $J_X(X)$ in X'', and F_c denotes the restriction of F'' to X_c, then $F_c \in CL(X_c, J_Y(Y))$. Thus a compact linear map from X to Y has a compact linear extension from the completion of X to Y itself.

17-3 Let X be a Banach space and $A \in CL(X)$ with $R(A)$ infinite dimensional. Then $E_1 = \{(x, 0) : x \in X\}$ and $E_2 = \{(x, A(x)) : x \in X\}$ are closed subspaces of $X \times X$, but the subspace $E_1 + E_2$ is not closed in $X \times X$. (Hint: $R(A)$ is not closed in $X, E_1 + E_2 = \{(x_1, y_2) : x_1 \in X$ and $y_2 \in R(A)\}$)

17-4 Let X be a Banach space and $P \in BL(X)$ be a projection. Then $P \in CL(X)$ if and only if P is of finite rank.

17-5 Let X be an infinite dimensional normed space and $A \in CL(X)$. Let p be a polynomial in one variable. Then $p(A) \in CL(X)$ if and only if $p(0) = 0$.

17-6 Let X be a normed space and $A : X \to X$ be linear. Assume that $(A - k_0 I)^{-1} \in CL(X)$ for some $k_0 \in \mathbf{K}$. For $k \in \mathbf{K}, (A - kI)^{-1} \in CL(X)$ if and only if $(A - kI)^{-1} \in BL(X)$. (Hint: $(A - kI)^{-1} - (A - k_0 I)^{-1} = (k - k_0)(A - kI)^{-1}(A - k_0 I)^{-1}$)

17-7 Let X and Y be normed spaces and $X \neq \{0\}$. Then Y is a Banach space if and only if $CL(X,Y)$ is a Banach space in the operator norm. (Hint: 17.2(c) and the proof of 8.2(c))

17-8 Let X be a normed space and Y be a Banach space. If Y admits a Schauder basis and $F \in CL(X,Y)$, then there is a sequence (F_n) of finite rank maps in $BL(X,Y)$ such that $\|F_n - F\| \to 0$ as $n \to \infty$. (Hint: 11.4, 9.2(b) and 17.2(c))

17-9 Let $M = \text{diag}\,(k_1, k_2 \ldots)$ and X be a sequence space.

(a) If M defines a map in $CL(X)$, then $k_n \to 0$ as $n \to \infty$.

(b) If X is a Banach space and $k_n \to 0$ as $n \to \infty$, then M defines a map in $CL(X)$.

17-10 Let X be a sequence space which is complete and (w_n) be a sequence in $\mathbf{K}$. For $x = (x(1), x(2), \ldots) \in X$, let

$$A(x) = (0, w_1x(1), w_2x(2), \ldots) \quad \text{and} \quad B(x) = (w_2x(2), w_3x(3), \ldots).$$

Then $A \in CL(X)$ if and only if $w_n \to 0$ if and only if $B \in CL(X)$. [The maps A and B are known as the **weighted shift operators**.]

17-11 Let $M = (k_{i,j})$ be an infinite matrix. For $1 \leq r \leq \infty$ and $j = 1, 2, \ldots$, define

$$\gamma_r(j) = \begin{cases} \left(\sum_{i=1}^{\infty} |k_{i,j}|^r \right)^{1/r}, & \text{if } 1 \leq r < \infty \\ \sup_{i=1,2,\ldots} |k_{i,j}|, & \text{if } r = \infty. \end{cases}$$

If $\gamma_r(j) \to 0$ as $j \to \infty$, then M defines a compact map from ℓ^1 to ℓ^r and its norm equals $\sup_{j=1,2,\ldots} \gamma_r(j)$. (Hint: Problem 6-13 and 17.4(a))

17-12 Let $M = (k_{i,j})$ be an infinite matrix. For $1 \leq p \leq \infty$ and $i = 1, 2, \ldots$, define

$$\delta_p(i) = \begin{cases} \left(\sum_{j=1}^{\infty} |k_{i,j}|^q \right)^{1/q}, & \text{if } 1 < p \leq \infty, 1/p + 1/q = 1 \\ \sup_{j=1,2,\ldots} |k_{i,j}|, & \text{if } p = 1. \end{cases}$$

If $\delta_p(i) \to 0$ as $i \to \infty$, then M defines a compact map from ℓ^p to ℓ^∞ and its norm equals $\sup_{i=1,2,\ldots} \delta_p(i)$. (Hint: Problem 6-13 and 17.4(a))

17-13 Let $M = (k_{i,j})$ be an infinite matrix, $1 < p \le \infty$ and $1 \le r < \infty$. Let $\beta_{p,r}$ be as defined in Problem 6-14. If $\beta_{p,r} < \infty$, then M defines a compact linear map from ℓ^p to ℓ^r. (Hint: Problem 6-14 and 17.4(a))

If $1 < p < \infty$ and $1 \le r < p/(p-1)$, it is enough to assume that

$$\beta'_{p,r} = \left[\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}|k_{i,j}|^r\right)^{q/r}\right]^{1/q} < \infty,$$

and then the norm of the map is at most $\beta'_{p,r} (\le \beta_{p,r})$. (Hint: Consider the transpose of M as in 13.4(b). Special care is needed for the case $r = 1$.)

17-14 Let an infinite matrix $M = (k_{i,j})$ define a compact linear map from ℓ^p to ℓ^r, $1 \le p, r \le \infty$.

(a) If $1 < p \le \infty$, then the sequence of columns of M tends to zero in ℓ^r, that is, $\|(k_{1,j}, k_{2,j}, \ldots)\|_r \to 0$ as $j \to \infty$. This may not hold if $p = 1$.

(b) If $1 \le p, r < \infty$, then the sequence of rows of M tends to zero in ℓ^q, $1/p + 1/q = 1$, that is, $\|(k_{i,1}, k_{i,2}, \ldots)\|_q \to 0$ as $i \to \infty$. This may not hold if $r = \infty$. (Hint: 13.4(b)) (Compare Problems 9-16, 17-11, 17-12.)

17-15 Consider $\phi \in L^1([-a,a])$ and extend it to $\mathbf{R}$ so as to have period $2a$. Let $X = L^p([-a,a])$, $1 \le p < \infty$, and $x \in X$. Define the **convolution**

$$(\phi * x)(s) = \int_{-a}^{a} \phi(s-t)x(t)\,dm(t), \quad -a \le s \le a.$$

Then $\|\phi * x\|_p \le \|\phi\|_1 \|x\|_p$. If we let $A(x) = \phi * x$, then $A \in CL(X)$. (Hint: $A(x)(s) = \displaystyle\int_{s-a}^{s+a} \phi(t)x(s-t)\,dm(t)$ for $s \in [-a,a]$. Find $\phi_n \in C([-a,a])$ such that $\phi_n(-a) = \phi_n(a)$ and $\|\phi_n - \phi\|_1 \to 0$.)

17-16 Let $k(.,.)$ be a measurable function on $[a,b]\times[a,b]$ which satisfies

(i) $\alpha_\infty = \sup\{\displaystyle\int_a^b |k(s,t)|\,dm(t) : s \in [a,b]\} < \infty$ and

(ii) $\displaystyle\lim_{\delta\to 0}\int_a^b |k(s+\delta,t) - k(s,t)|\,dm(t) = 0$ for every $s \in [a,b]$.

Let $X = L^\infty([a,b])$. If $Y = C([a,b])$ or $L^r([a,b])$, $1 \le r \le \infty$, then the Fredholm integral operator with kernel $k(.,.)$ is in $CL(X,Y)$.

17-17 Let $X = L^p([a,b]), Y = L^r([a,b])$ and $k(.,.) \in L^u([a,b]\times[a,b])$, where $1 < p \le \infty, 1 \le r < \infty$ and $1 \le u \le \infty$. Let F denote the Fredholm integral operator with kernel $k(.,.)$. If $1/p + 1/q = 1$ and $q, r \le u$, then F is in $CL(X,Y)$ and $\|F\|_{p,r} \le (b-a)^\alpha \|k\|_u$, where $\alpha = \dfrac{1}{q} + \dfrac{1}{r} - \dfrac{2}{u}$. In particular, if $k(.,.)$ is a bounded measurable function on $[a,b]\times[a,b]$, then $F \in CL(X,Y)$ for all $1 < p \le \infty$ and $1 \le r < \infty$. (Compare 17.4(b). Hint: For the case $u = \infty$, consider $v = \max\{q, r\} < \infty$.)

17-18 **(Weakly singular kernel)** Let $h(.,.)$ be a bounded measurable function on $[a,b]\times[a,b], 0 \le \alpha < 1$ and

$$k(s,t) = \frac{h(s,t)}{|s-t|^\alpha}, \quad s,t \in [a,b],\ s \ne t.$$

Let F denote the Fredholm integral operator with kernel $k(.,.)$.

(a) Let $1 < p \le \infty, 1/p + 1/q = 1, 1 \le r < \infty$ and $u = \max\{q, r\}$. If $\alpha u < 1$, then $F \in CL(L^p([a,b]), L^r([a,b]))$. (Hint: $\displaystyle\int_a^b dt/|s-t|^\alpha \le A_\alpha$ for all $s \in [a,b]$, where $A_\alpha = 2^\alpha (b-a)^{1-\alpha}/(1-\alpha)$)

(b) If $1 \le p \le \infty$, then $F \in BL(L^p)$ and $\|F\| \le \|h\|_\infty A_\alpha$. In fact, if $1 < p < \infty$, then $F \in CL(L^p)$. (Hint: Find β such that $1 < \beta < p$ and $\alpha\beta < 1$. For $n = 1, 2, \ldots$, let $k_n(s,t) = k(s,t)$ if $|k(s,t)| \le n$ and $k_n(s,t) = 0$ if $|k(s,t)| > n$. If F_n is the operator with kernel $k_n(.,.)$, then $\|F - F_n\| \le \|h\|_\infty^\beta A_{\alpha\beta}/n^{\beta-1} \to 0$ as $n \to \infty$.)

17-19 Let $k(.,.) \in C([a,b]\times[a,b])$. For $y \in NBV([a,b])$, let

$$B(y)(s) = \int_a^s \left[\int_a^b k(t,u)\,dy(t)\right] du, \quad s \in [a,b].$$

Then $B \in CL(NBV([a,b]))$. (Hint: Problem 14-17)

17-20 Let $X = L^2((0,\infty))$ and for $x \in X$, consider

$$A(x)(s) = \frac{1}{s}\int_0^s x(s)\,dm(t), \quad s \in (0,\infty).$$

Then $A \in BL(X)$ and $\|A\| \le 2$, but $A \notin CL(X)$. (Hint: 17.5(a) with $x_n(t) = n$ for $0 < t \le 1/n^2$, $x_n(t) = 0$ for $t > 1/n^2$)

17-21 Let X be a reflexive normed space. Then $CL(X, \ell^1) = BL(X, \ell^1)$. (Hint: 15.2(b) and 17.5(b))

17-22 Let X and Y be normed spaces with X reflexive and $F \in CL(X,Y)$. If E is a closed, bounded, convex subset of X, then $F(E)$ is a compact convex subset of Y. In particular, a bounded linear map F from X to Y is compact if and only if $F(\overline{U})$ is a compact subset of Y. (Hint: Problem 16-11)

17-23 Let X be a nonzero Banach space. If X is reflexive, then for every normed space Y, each $F \in CL(X,Y)$ attains its norm on the unit sphere of X. (Hint: 16.5 and 17.5(a)) [The converse also holds, since the converse of 16.6(c) holds.]

18 Spectrum of a Compact Operator

In this section we develop the Riesz-Schauder theory of the spectrum of a compact operator on a normed space X over $\mathbf{K}$. We show that this spectrum resembles the spectrum of a finite matrix except for the number 0. We begin our study with some preliminary results of independent interest. Recall that $CL(X)$ denotes the set of all compact operators on X.

18.1 Lemma

Let X be a normed space, $A \in CL(X)$ and $0 \neq k \in \mathbf{K}$. If (x_n) is a bounded sequence in X such that $A(x_n) - kx_n \to y$ in X, then there is a subsequence (x_{n_j}) of (x_n) such that $x_{n_j} \to x$ in X and $A(x) - kx = y$.

Proof:

By 17.1(a), there is a subsequence (x_{n_j}) of (x_n) such that $(A(x_{n_j}))$ converges to some z in X. Then

$$kx_{n_j} = kx_{n_j} - A(x_{n_j}) + A(x_{n_j}) \to -y + z,$$

so that $x_{n_j} \to (z-y)/k = x$, say. Also, since A is continuous,

$$A(x) - kx = \lim_{j\to\infty}[A(x_{n_j}) - kx_{n_j}] = z - [-y+z] = y. \qquad \square$$

This results shows that if $A \in CL(X)$, $k \neq 0$ and (x_n) is a bounded sequence of approximate solutions of the equation $A(x) - kx = y$, then a subsequence of (x_n) converges to an exact solution of this equation.

The following result, which is based on the Riesz lemma (5.3), is crucial in the analysis of the spectrum of a compact operator.

18.2 Lemma

Let X be a normed space and $A : X \to X$.

(a) Let $0 \neq k \in \mathbf{K}$ and Y be a proper closed subspace of X such that $(A - kI)(X) \subset Y$. Then there is some $x \in X$ such that $\|x\| = 1$ and for all $y \in Y$,

$$\|A(x) - A(y)\| \geq \frac{|k|}{2}.$$

(b) Let $A \in CL(X)$ and $k_0, k_1, \ldots$ be scalars with $|k_n| \geq \delta$ for some $\delta > 0$ and $n = 0, 1, 2, \ldots$. Let $Y_0, Y_1, \ldots, Z_0, Z_1, \ldots$ be closed subspaces of X such that for $n = 0, 1, \ldots,$

$$Y_{n+1} \subset Y_n, \quad (A - k_nI)(Y_n) \subset Y_{n+1},$$

$$Z_n \subset Z_{n+1}, \quad (A - k_{n+1}I)(Z_{n+1}) \subset Z_n.$$

Then there are nonnegative integers m and ℓ such that

$$Y_{m+1} = Y_m \quad \text{and} \quad Z_{\ell+1} = Z_\ell.$$

Proof:

(a) First we note that $A(Y) \subset Y$ since $A(y) = [A(y) - ky] + ky$ belongs to Y for all $y \in Y$. Now by the Riesz lemma (5.3), there is some $x \in X$ such that $\|x\| = 1$ and $\operatorname{dist}(x, Y) \geq 1/2$. Consider $y \in Y$. Since $A(x) - kx \in Y$ and $A(y) \in Y$, we have

$$\begin{aligned} \|A(x) - A(y)\| &= \|kx - [kx - A(x) + A(y)]\| \\ &= |k|\,\|x - \frac{1}{k}[kx - A(x) + A(y)]\| \\ &\geq |k|\operatorname{dist}(x, Y) \geq \frac{|k|}{2}. \end{aligned}$$

(b) Assume for a moment that Y_{n+1} is a proper closed subspace of Y_n for each $n = 0, 1, \ldots$. By (a) above, there is some $y_n \in Y_n$ such that $\|y_n\| = 1$ and for all $y \in Y_{n+1}$,

$$\|A(y_n) - A(y)\| \geq \frac{|k_n|}{2} \geq \frac{\delta}{2}, \quad n = 0, 1, \ldots$$

It follows that (y_n) is a bounded sequence in X and

$$\|A(y_n) - A(y_m)\| \geq \frac{\delta}{2}, \quad n, m = 0, 1, \ldots \text{ with } n \neq m.$$

As a consequence, $(A(y_n))$ cannot have a convergent subsequence, contrary to the compactness of the operator A. Hence there is some nonnegative integer m such that $Y_{m+1} = Y_m$.

It can similarly be seen that there is some nonnegative integer ℓ such that $Z_{\ell+1} = Z_\ell$. □

We are now ready to prove an important result about the spectrum of a compact operator. It should be compared with Theorem 12.2 which was proved for bounded operators of finite rank. Recall that $\sigma(A), \sigma_e(A)$ and $\sigma_a(A)$ denote respectively the spectrum, the eigenspectrum and the approximate eigenspectrum of A.

18.3 Theorem

Let X be a normed space and $A \in CL(X)$.

(a) Every nonzero spectral value of A is an eigenvalue of A, so that

$$\{k : k \in \sigma_e(A), k \neq 0\} = \{k : k \in \sigma(A), k \neq 0\}.$$

(b) If X is infinite dimensional, then $0 \in \sigma_a(A)$.

(c) $\sigma_a(A) = \sigma(A)$.

Proof:

(a) Let $0 \neq k \in \mathbb{K}$. Assume that k is not an eigenvalue of A, that is, $A - kI$ is injective. We prove that k is not a spectral value of A, that is, $A - kI$ is invertible.

As a first step, we claim that the operator $A - kI$ is bounded below. Otherwise, we can find a sequence (x_n) in X such that $\|x_n\| = 1$ for each n and $\|(A - kI)(x_n)\| \to 0$ as $n \to \infty$. By Lemma 18.1, there is a subsequence (x_{n_j}) of (x_n) such that $x_{n_j} \to x$ in X and $A(x) - kx = 0$. Since $A - kI$ is injective, we see that $x = 0$. But $\|x\| = \lim_{j\to\infty} \|x_{n_j}\| = 1$, and we obtain a contradiction. Thus $A - kI$ is bounded below.

Next, we show that $A - kI$ is surjective, that is, $R(A - kI) = X$. Note that $R(A - kI)$ is a closed subspace X. This can be seen as follows. Let $(A(x_n) - kx_n)$ be a sequence in $R(A - kI)$ which converges to some $y \in X$. Then $((A - kI)(x_n))$ is a bounded sequence in X and since $A - kI$ is bounded below, we see that (x_n) is also a bounded sequence in X. By Lemma 18.1, there is a subsequence (x_{n_j}) of (x_n) such that $x_{n_j} \to x$ in X and $A(x) - kx = y$. Thus $y \in R(A - kI)$, showing that $R(A - kI)$ is closed in X.

Now let $Y_n = R((A - kI)^n)$ for $n = 0, 1, \ldots$. Then each Y_n is closed in X . This is obvious for $n = 0$ since $Y_0 = X$ and we have just proved this for $n = 1$. If $n \geq 2$, then $(A - kI)^n = A_n - k_n I$, where

$$A_n = A^n + \binom{n}{1}(-k)A^{n-1} + \cdots + \binom{n}{n-1}(-k)^{n-1}A, \quad k_n = -(-k)^n.$$

By 17.2 (a) and (b), $A_n \in CL(X)$, and clearly $k_n \neq 0$. Further, since $A - kI$ is injective, it follows that $A_n - k_n I = (A - kI)^n$ is also injective. Hence we can replace A by A_n, and k by k_n in the argument given above and conclude that $R(A_n - k_n I) = Y_n$ is a closed subspace of X.

Since $Y_{n+1} \subset Y_n$ and $Y_{n+1} = (A - kI)(Y_n)$, part (b) of Lemma

18.2 shows that there is a nonnegative integer m with $Y_{m+1} = Y_m$. If $m = 0$, then $Y_1 = Y_0$. If $m > 0$, we claim that $Y_m = Y_{m-1}$. Let $y \in Y_{m-1}$, that is, $y = (A - kI)^{m-1}(x)$ for some $x \in X$. Then $(A-kI)(y) = (A-kI)^m(x) \in Y_m = Y_{m+1}$, so that there is some $z \in X$ with $(A-kI)(y) = (A-kI)^{m+1}(z)$. Since $(A-kI)(y-(A-kI)^m(z)) = 0$ and since $A-kI$ is injective, it follows that $y-(A-kI)^m(z) = 0$, that is, $y = (A-kI)^m(z) \in Y_m$. Thus $Y_m = Y_{m-1}$. Proceeding similarly, if $m > 1$, we see that $Y_{m+1} = Y_m = Y_{m-1} = Y_{m-2} = \cdots = Y_1 = Y_0$. But $Y_1 = R(A-kI)$ and $Y_0 = X$. Hence $A-kI$ is surjective.

Being bounded below and surjective, $A-kI$ is invertible by 12.1. Hence every nonzero spectral value of A is an eigenvalue of A. Since $\sigma_e(A) \subset \sigma(A)$ always, the proof of (a) is complete.

(b) Let X be infinite dimensional. Consider an infinite linearly independent subset $\{x_1, x_2, \ldots\}$ of X, and let $Z_n = \operatorname{span}\{x_1, \ldots, x_n\}, n = 1, 2, \ldots$. Then Z_n is a proper subspace of Z_{n+1}, and being finite dimensional, each Z_n is closed in X by 5.4(b). By the Riesz lemma (5.3), there is some $z_{n+1} \in Z_{n+1}$ such that $\|z_{n+1}\| = 1$ and $\operatorname{dist}(z_{n+1}, Z_n) \geq 1/2$. Assume for a moment that A is bounded below : $\beta\|x\| \leq \|A(x)\|$ for all $x \in X$ and some $\beta > 0$. Then for all $n, m = 1, 2, \ldots$ and $n \neq m$, we have

$$\|A(z_n) - A(z_m)\| \geq \beta\|z_n - z_m\| \geq \frac{\beta}{2},$$

so that $(A(z_n))$ cannot have a convergent subsequence, contrary to the compactness of A. Hence A is not bounded below, that is, $0 \in \sigma_a(A)$.

(c) If X is finite dimensional, then A is necessarily of finite rank. Hence $\sigma_a(A) = \sigma(A)$ by 12.2.

If X is infinite dimensional, then $0 \in \sigma_a(A)$ by (b) above. Also, since $\sigma_e(A) \subset \sigma_a(A) \subset \sigma(A)$ always, it follows from (a) above that $\sigma_a(A) = \sigma(A)$. □

Let X be a normed space over $\mathbf{C}$ and $A \in CL(X)$. If X is finite dimensional and $\{x_1, \ldots, x_n\}$ is a basis for X, then A is defined by an

$n \times n$ matrix with complex entries. The characteristic polynomial of this matrix has at least one complex root which is then an eigenvalue of A. If X is infinite dimensional, then by 18.3(b), 0 is an approximate eigenvalue of A. Thus in all cases,

$$\sigma(A) = \sigma_a(A) \neq \emptyset.$$

This establishes the nonemptiness of the spectrum of a compact operator on a normed space over $\mathbf{C}$. [Compare the Gelfand-Mazur theorem (12.8(a)).]

The result 18.3(a) tells us that to analyze the spectrum of a compact operator, it is sufficient to study its eigenspectrum. We now take up this study.

18.4 Lemma

Let X be a linear space, $A : X \to X$ linear and $A(x_n) = k_n x_n$ for some $0 \neq x_n \in X$ and $k_n \in \mathbf{K}, n = 1, 2, \ldots$.

(a) Let $k_n \neq k_m$ whenever $n \neq m$. Then $\{x_1, x_2, \ldots\}$ is a linearly independent subset of X.

(b) Let X be a normed space, $A \in CL(X)$ and the set $\{x_1, x_2, \ldots\}$ be linearly independent and infinite. Then $k_n \to 0$ as $n \to \infty$.

Proof:

(a) Since $x_1 \neq 0$, the set $\{x_1\}$ is linearly independent. Let $n = 2, 3, \ldots$, and assume that the set $\{x_1 \ldots, x_n\}$ is linearly independent. Let, if possible, $x_{n+1} = c_1 x_1 + \cdots + c_n x_n$ for some $c_1, \ldots, c_n$ in $\mathbf{K}$. Then

$$k_{n+1} x_{n+1} = c_1 k_{n+1} x_1 + \cdots + c_n k_{n+1} x_n$$

and also

$$k_{n+1} x_{n+1} = A(x_{n+1}) = \sum_{j=1}^{n} c_j A(x_j) = c_1 k_1 x_1 + \cdots + c_n k_n x_n.$$

Subtraction gives $c_1(k_1 - k_{n+1})x_1 + \cdots + c_n(k_n - k_{n+1})x_n = 0$. Since the set $\{x_1, \ldots, x_n\}$ is linearly independent, $c_j(k_j - k_{n+1}) = 0$ for each j. As $x_{n+1} \neq 0$, we see that $c_j \neq 0$ for some $j, 1 \leq j \leq n$, so that $k_{n+1} = k_j$. But this is impossible. Thus the set $\{x_1, \ldots, x_{n+1}\}$ is linearly independent. By mathematical induction, we see that the set $\{x_1, x_2, \ldots\}$ is linearly independent.

(b) For $n = 1, 2, \ldots$, let $Z_n = \operatorname{span}\{x_1, \ldots, x_n\}$. Since x_{n+1} does not belong to Z_n, Z_n is a proper subspace of Z_{n+1}. Also, Z_n is closed in X by 5.4(b), and $(A - k_{n+1}I)(Z_{n+1}) \subset Z_n$ since $(A - k_{n+1}I)(x_{n+1}) = 0$. If $k_n \not\to 0$ as $n \to \infty$, we can assume by passing to a subsequence that $|k_n| \geq \delta > 0$ for all $n = 1, 2, \ldots$. Now Lemma 18.2(b) shows that $Z_{\ell+1} = Z_\ell$ for some positive integer ℓ, which contradicts the fact that Z_ℓ is a proper subspace of $Z_{\ell+1}$. Hence $k_n \to 0$ as $n \to \infty$. □

18.5 Theorem

Let X be a normed space and $A \in CL(X)$.

(a) The eigenspectrum and the spectrum of A are countable sets and have 0 as the only possible limit point. In particular, if $\{k_1, k_2 \ldots\}$ is an infinite set of eigenvalues of A, then $k_n \to 0$ as $n \to \infty$.

(b) Every eigenspace of A corresponding to a nonzero eigenvalue of A is finite dimensional.

Proof:

(a) Since $\{k : k \in \sigma(A), k \neq 0\} = \{k : k \in \sigma_e(A), k \neq 0\}$ by 18.3(a), we only need to show that the set $\sigma_e(A)$ is countable and 0 is the only possible limit point of it. For $\delta > 0$, let

$$E_\delta = \{k \in \sigma_e(A) : |k| \geq \delta\}.$$

Suppose for a moment that E_δ is an infinite set for some $\delta > 0$. Let $k_n \in E_\delta$ for $n = 1, 2, \ldots$ with $k_n \neq k_m$ whenever $n \neq m$. If x_n is an eigenvector of A corresponding to the eigenvalue k_n, then by 18.4(a), the set $\{x_1, x_2, \ldots\}$ is linearly independent, and consequently $k_n \to 0$ as $n \to \infty$ by 18.4(b). But this is impossible since $|k_n| \geq \delta$ for each n. Thus E_δ is a finite set for each $\delta > 0$. Since $\sigma_e(A) = \bigcup_{n=1}^{\infty} E_{1/n}$, it follows that $\sigma_e(A)$ is a countable set and that $\sigma_e(A)$ has no limit point except possibly the number 0.

Further, $\sigma_e(A)$ is a bounded subset of $\mathbf{K}$ since $|k| \le \|A\|$ for every $k \in \sigma_e(A)$. If $\{k_1, k_2 \ldots\}$ is an infinite subset of $\sigma_e(A)$, then it must have a limit point by the Bolzano-Weierstrass theorem for $\mathbf{K}$. As the only possible limit point is 0, we see that $k_n \to 0$ as $n \to \infty$.

(b) Let $0 \ne k \in \sigma_e(A)$. Suppose for a moment that $\{x_1, x_2 \ldots\}$ is an infinite linearly independent set of eigenvectors of A corresponding to k. Letting $k_n = k$ for $n = 1, 2, \ldots$ in 18.4(b), we find that the sequence (k_n) must tend to zero. But this is impossible since $k \ne 0$. Thus the eigenspace of A corresponding to k is finite dimensional. $\square$

We now consider the spectrum of the transpose of a compact operator. If A is a bounded operator on a normed space X, we have seen in 13.8 that $\sigma(A') \subset \sigma(A)$, where equality holds provided X is a Banach space. We, however, cannot make similar statements for $\sigma_e(A')$ and $\sigma_e(A)$ as the example $X = \ell^2, A((x(1), x(2), \ldots)) = (0, x(1), x(2), \ldots)$ shows . (See 12.7(b), 13.4(b) and Problem 12-19(b).) On the other hand, the situation for a compact operator is indeed pleasant.

18.6 Theorem

Let X be a normed space and $A \in CL(X)$. Then

(a) $\dim Z(A' - kI) = \dim Z(A - kI) < \infty$ for $0 \ne k \in \mathbf{K}$,

(b) $\{k : k \in \sigma_e(A'), k \ne 0\} = \{k : k \in \sigma_e(A), k \ne 0\}$,

(c) $\sigma(A') = \sigma(A)$.

Proof:

(a) By 17.3, $A' \in CL(X')$. Let $k \ne 0$. Then 18.5(b) shows that the dimension m of $Z(A - kI)$ and the dimension m' of $Z(A' - kI)$ are both finite.

First we show that $m' \le m$

If $m = 0$, that is, $k \notin \sigma_e(A)$, then by 18.3(a) we see that $k \notin \sigma(A)$. Since $\sigma(A') \subset \sigma(A)$ by 13.8, we have $k \notin \sigma(A')$. In particular, $A' - kI$ is injective, that is, $m' = 0$.

Next, let $m \geq 1$. Consider a basis $\{x_1, \ldots, x_m\}$ of $Z(A - kI)$. By the consequence 7.10(c) of the Hahn-Banach theorem, there are $x_1', \ldots, x_m'$ in X' such that

$$x_j'(x_i) = \delta_{i,j}, \quad i, j = 1, \ldots, m.$$

Let, if possible, $\{y_1', \ldots, y_{m+1}'\}$ be a linearly independent subset of $Z(A' - kI)$ containing $m + 1$ elements. By 16.2, there are $y_1, \ldots, y_{m+1}$ in X such that

$$y_j'(y_i) = \delta_{i,j}, \quad i, j = 1, \ldots, m + 1.$$

Consider the map $B : X \to X$ defined by

$$B(x) = \sum_{i=1}^{m} x_i'(x) y_i, \quad x \in X.$$

Since $B \in BL(X)$ and B is of finite rank, B is a compact operator on X by 17.1(c). Since A is also compact, 17.2(a) shows that $A - B$ is a compact operator. We show that $A - B - kI$ is injective but not surjective and obtain a contradiction to 18.3(a). For this purpose, we note that $y_j' \in Z(A' - kI)$ and hence

$$\begin{aligned} y_j'(A - B - kI)(x) &= (A' - kI)(y_j')(x) - y_j'(\sum_{i=1}^{m} x_i'(x) y_i) \\ &= 0 - \sum_{i=1}^{m} x_i'(x) y_j'(y_i) \\ &= \begin{cases} -x_j'(x), & \text{if } 1 \leq j \leq m \\ 0, & \text{if } j = m + 1. \end{cases} \end{aligned}$$

Now let $x \in X$ satisfy $(A - B - kI)(x) = 0$. Then it follows that $-x_j'(x) = y_j'(A - B - kI)(x) = y_j'(0) = 0$ for $1 \leq j \leq m$, and in turn, $B(x) = 0$. Hence $(A - kI)(x) = 0$, that is, $x \in Z(A - kI)$. Since $\{x_1, \ldots, x_m\}$ is a basis of $Z(A - kI)$, we have

$$x = k_1 x_1 + \cdots + k_m x_m$$

for some $k_1, \ldots, k_m$ in $\mathbf{K}$. But

$$0 = x_j'(x) = x_j'(k_1 x_1 + \cdots + k_m x_m) = k_j, \quad j = 1, \ldots, m,$$

so that $x = 0x_1 + \cdots + 0x_m = 0$. Thus $A - B - kI$ is injective. Next, we claim that $y_{m+1} \notin R(A - B - kI)$. For, if $y_{m+1} = (A - B - kI)(x)$ for some $x \in X$, then

$$1 = y_{m+1}'(y_{m+1}) = y_{m+1}'((A - B - kI)(x)) = 0,$$

as we have noted above. Hence $A - B - kI$ is not surjective.

Thus a linearly independent subset of $Z(A' - kI)$ can have at most m elements, that is, $m' \le m$.

To obtain $m \le m'$, we argue as follows. Let m'' denote the dimension of $Z(A'' - kI)$. Considering the compact operator A' in place of A, we find that $m'' \le m'$. If J denotes the canonical embedding of X into X'' considered in Section 8, then $A''J = JA$ by 13.5(b). Hence $J(Z(A - kI)) \subset Z(A'' - kI)$, so that $m \le m''$. Thus $m \le m'' \le m'$. Consequently, $m' = m$, as desired.

(b) Let $0 \ne k \in \mathbf{K}$. Part (a) shows that $Z(A - kI) \ne \{0\}$ if and only if $Z(A' - kI) \ne \{0\}$, that is, $k \in \sigma_e(A)$ if and only if $k \in \sigma_e(A')$

(c) Since A and A' are compact operators, we have by 18.3(a)

$$\{k : k \in \sigma(A), k \ne 0\} = \{k : k \in \sigma_e(A), k \ne 0\},$$
$$\{k : k \in \sigma(A'), k \ne 0\} = \{k : k \in \sigma_e(A'), k \ne 0\}.$$

It follows from (b) above that

$$\{k : k \in \sigma(A'), k \ne 0\} = \{k : k \in \sigma(A), k \ne 0\}.$$

If X is finite dimensional, then we see from 13.4(a) and $\det M = \det M^t$ that $0 \in \sigma(A')$ if and only if $0 \in \sigma(A)$. If X is infinite dimensional, then X' is also infinite dimensional and hence $0 \in \sigma_a(A)$ as well as $0 \in \sigma_a(A')$ by 18.3(b). Thus in both cases, $\sigma(A') = \sigma(A)$. $\square$

This result completes our study of the spectrum of a compact operator. As we have seen, such a spectrum is very much like the spectrum of a finite matrix, except for the number zero. We now give some examples to illustrate the peculiarity of the number 0 as a spectral value of a compact operator. We also consider some special compact operators whose spectra can be found by searching for their eigenvalues.

18.7 Examples

(a) As an element of the spectrum of a compact operator on an infinite dimensional normed space, the number $k = 0$ has a special status.

1. $k = 0$ can be a spectral value of a compact operator A without being its eigenvalue. Also, 0 can be the limit point of the spectrum of A. For example, let $X = \ell^p, 1 \leq p \leq \infty$, and

$$A(x) = (x(1), \frac{x(2)}{2}, \ldots) \quad \text{for } x = (x(1), x(2), \ldots) \in \ell^p.$$

Then A is a compact operator (See 17.4(a) and Problem 17-9(b).) Since A is clearly injective, 0 is not an eigenvalue of A, but since A is not bounded below, 0 is a spectral value of A. Also, $k_n = 1/n$ is an eigenvalue of A and $k_n \to 0$ as $n \to \infty$.

2. The eigenspace of a compact operator corresponding to the eigenvalue 0 can be infinite dimensional. The simplest example is provided by the zero operator on an infinite dimensional normed space.

3. $k = 0$ can be an eigenvalue of a compact operator A without being an eigenvalue of the transpose A' of A, and vice versa. For example, let $X = \ell^p, 1 < p < \infty$, and A denote the compact operator on X defined by

$$A(x) = (x(2), \frac{x(3)}{2}, \ldots) \quad \text{for } x = (x(1), x(2), \ldots) \in \ell^p.$$

As we have seen in 13.4(b), A' can be identified with the compact operator B on $\ell^q, 1/p + 1/q = 1$, defined by

$$B(x) = (0, x(1), \frac{x(2)}{2}, \ldots) \quad \text{for } x = (x(1), x(2), \ldots) \in \ell^q.$$

Since $A((1,0,0,\ldots)) = (0,0,\ldots)$, we see that 0 is an eigenvalue of A. But since B is clearly injective, 0 is not an eigenvalue of B. Also, since the transpose B' of B can be identified with A itself, we obtain a situation where 0 is not an eigenvalue of a compact operator B although it is an eigenvalue of its transpose B'.

(b) Let $X = L^2([a,b])$ and for $x \in X$, define

$$A(x)(s) = \int_a^s x(t)\, dm(t), \quad s \in [a,b].$$

Our discussion in 17.4(b) shows that $A \in CL(X)$ since for $x \in X$, we have

$$A(x)(s) = \int_a^b k(s,t)x(t)\, dm(t), \quad s \in [a,b],$$

where

$$k(s,t) = \begin{cases} 0, & \text{if } a \le s \le t \le b \\ 1, & \text{if } a \le t < s \le b, \end{cases}$$

so that $k(.,.) \in L^2([a,b]\times[a,b])$. The operator A is known as the **Volterra integration operator.** [It is a special case of the Volterra operator considered in Problem 12-12.]

We show that $\sigma_e(A) = \emptyset$. Let $x \in X$ and $k \in \mathbf{K}$ such that $A(x) = kx$. Then

$$kx(s) = \int_a^s x(t)\, dm(t) \quad \text{for almost all } s \in [a,b].$$

Since $x \in L^2([a,b]) \subset L^1([a,b])$, the fundamental theorem for Lebesgue integration (4.3) shows that the right hand side is an absolutely continuous function of s, and $kx'(s) = x(s)$ for almost all $s \in [a,b]$. If $k = 0$, then $x = 0$ a.e. on $[a,b]$. If $k \neq 0$, then it follows that x is (absolutely) continuous on $[a,b]$, and in turn, $x \in C^1([a,b])$. Hence for all $s \in [a,b]$,

$$kx'(s) = x(s).$$

Since x satisfies this Bernoulli linear differential equation, we see that $x(s) = ce^{s/k}$ for all $s \in [a,b]$ and some $c \in \mathbf{K}$. But $kx(a) = 0$. Hence $c = 0$, and in turn, $x = 0$. This completes the proof that $\sigma_e(A) = \emptyset$. Since A is compact and X is infinite dimensional, $0 \in \sigma_a(A)$ by 18.3(b). Thus $\sigma_e(A) = \emptyset$, but $\sigma_a(A) = \{0\} = \sigma(A)$.

(c) Let $X = C([0,1])$ or $L^p([0,1]), 1 \le p \le \infty$. For $x \in X$, let

$$A(x)(s) = (1-s)\int_0^s t\,x(t)\,dm(t) + s\int_s^1 (1-t)x(t)\,dm(t), \quad s \in [0,1].$$

Then $A \in CL(X)$, since it is a Fredholm integral operator with a continuous kernel given by

$$k(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1 \\ s(1-t), & \text{if } 0 \le s \le t \le 1. \end{cases}$$

(See 17.4(b).) Let $0 \neq x \in \mathbf{X}$ and $0 \neq k \in \mathrm{K}$ be such that $A(x) = kx$. Then $\mathbf{k} \in R$ for all $s \in$ [a,b],

$$kx(s) = (1-s)\int_0^s t\,x(t)\,dm(t) + s\int_s^1 (1-t)x(t)\,dm(t).$$

Putting $s = 0$ and $s = 1$, we note that $x(0) = 0 = x(1)$. Since $t\,x(t)$ and $(1-t)x(t)$ are integrable functions of $t \in [0,1]$, it follows by 4.3 that the right hand side of the equation given above is an absolutely continuous function of $s \in [0,1]$. Hence x is (absolutely) continuous on $[0,1]$. This implies that $t\,x(t)$ and $(1-t)x(t)$ are continuous functions of t on $[0,1]$. Thus the right hand side is, in fact, a continuously differentiable function of s and we have for all $s \in [0,1]$,

$$\begin{aligned} kx'(s) &= (1-s)s\,x(s) - \int_0^s t\,x(t)\,dm(t) - s(1-s)x(s) \\ &\quad + \int_s^1 (1-t)x(t)\,dm(t) \\ &= -\int_0^s t\,x(t)\,dm(t) + \int_s^1 (1-t)x(t)\,dm(t). \end{aligned}$$

This shows that x' is a continuously differentiable function and for all $s \in [0,1]$, we have

$$kx''(s) = -s\,x(s) - (1-s)x(s) = -x(s).$$

The differential equation $kx'' + x = 0$ has a nonzero solution satisfying $x(0) = 0 = x(1)$ if and only if $k = 1/n^2\pi^2, n = 1, 2, \ldots$, and in that case, its most general solution is given by $x(s) = c \sin n\pi s, s \in [0, 1]$, where $c \in \mathbf{K}$. Let $k_n = 1/n^2\pi^2, n = 1, 2, \ldots$ and $x_n(s) = \sin n\pi s$ for $s \in [0, 1]$. Then each k_n is an eigenvalue of A and the corresponding eigenspace $Z(A - k_n I) = \operatorname{span}\{x_n\}$ is one dimensional .

Next, 0 is not an eigenvalue of A. For if $A(x) = 0$ for some $x \in X$, then by differentiating the expression for $A(x)(s)$ with respect to s two times, we see, as above, that $x(s) = 0$ for all $s \in [0, 1]$. On the other hand, since A is compact and X is infinite dimensional, 0 is an approximate eigenvalue of A by 18.3(b). Thus

$$\sigma_e(A) = \{\frac{1}{\pi^2}, \frac{1}{4\pi^2}, \ldots\} \quad \text{and} \quad \sigma_a(A) = \sigma(A) = \{0, \frac{1}{\pi^2}, \frac{1}{4\pi^2}, \ldots\}.$$

Before we conclude this section, we remark that the calculation of the eigenspectrum of a compact operator is, in general, an extremely difficult task. Eigenvalue problems for differential operators are even more difficult to solve. They are often converted into eigenvalue problems for integral operators by using a procedure which is reverse to the one we used in parts (b) and (c) of 18.7.

It is often possible to approximate a compact operator by bounded operators of finite rank, as we shall show in Section 20. Then the nonzero eigenvalues of the compact operator can be approximated by the nonzero eigenvalues (or their arithmetic means) of the finite rank operators. We refer the interested reader to [9] or [43].

Problems

X denotes a normed space over $\mathbf{K}$, unless otherwise stated.

18-1 **(Perturbation by a compact operator)** Let $A \in BL(X)$ and $B \in CL(X)$. If k is a spectral value of A but not an eigenvalue of A, then it is a spectral value of $A + B$. (Hint: If $A + B - kI$ is invertible and $A - kI$ is injective, then $I - (A + B - kI)^{-1}B$ is injective, since $A - kI = (A + B - kI)[I - (A + B - kI)^{-1}B]$.)

18-2 Let $A \in CL(X)$ and $k \neq 0$. Then $A - kI$ is injective if and only if it is surjective.

18-3 Let $A \in CL(X)$ and $k \neq 0$. Then there is a constant $\gamma > 0$ such that for every $y \in R(A - kI)$ there is some $x \in X$ with $A(x) - kx = y$ and $\|x\| < \gamma\|y\|$. (Hint: If X is a Banach space, use 10.4 and 10.6. In general, there is some $\gamma > 0$ such that $\operatorname{dist}(x, Z(A - kI)) < \gamma\|A(x) - kx\|, x \in X$.)

18-4 Let M be an infinite triangular matrix with all its diagonal entries equal to 0. If M defines a compact operator on ℓ^2, then $\sigma(M) = \{0\}$. This may not hold if M defines a bounded noncompact operator on ℓ^2.

18-5 **(Weighted shift operator)** Let X be a sequence space which is Banach. Let (k_n) be a sequence in $\mathbf{K}$ converging to 0. For $x \in X$, let

$$A(x) = (0, k_1 x(1), k_2(2), \ldots).$$

Then $\sigma(A) = \{0\}$. Further, $0 \in \sigma_e(A)$ if and only if $k_n = 0$ for some n.

18-6 Let $X = L^2([0,1])$. For $x \in X$, let

$$A(x)(s) = \begin{cases} \displaystyle\int_0^s x(t)\,dm(t), & \text{if } 0 \leq s \leq 1/2 \\[2ex] \displaystyle\int_0^{1-s} x(t)\,dm(t), & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

Then $\sigma_e(A) = \{0\} = \sigma(A)$ and the eigenspace $Z(A)$ is infinite dimensional.

18-7 Let $X = C([-1,1])$ or $L^2([-1,1])$. For $x \in X$, let

$$A(x)(s) = \int_{-s}^{s} x(t)\,dm(t), \quad s \in [-1,1].$$

Then $A \in CL(X), \sigma_e(A) = \{0\} = \sigma(A)$ and the eigenspace $Z(A)$ is infinite dimensional. (Hint: $A^2 = 0$ since $A(x)$ is an odd function for every $x \in X$, and if x is itself an odd function, then $A(x) = 0$.)

18-8 Let $X = C([0,1])$. For $x \in X$, let $A(x)(s) = \int_0^s [x(t)/\sqrt{s^2 - t^2}]\, dt$ if $0 < s \leq 1$ and $A(x)(0) = \pi x(0)/2$. Then $A \in BL(X)$ and $\|A\| = \pi/2$, but $A \notin CL(X)$. (Hint: $[0, \pi/2] \subset \sigma_e(A)$)

18-9 Let $X = L^2([0,1])$. For $x \in X$, let

$$A(x)(s) = \int_0^1 \min\{s,t\} x(t)\, dm(t), \quad s \in [0,1].$$

Then $A \in CL(X), \sigma_e(A) = \{k_1, k_2, \ldots\}$, where $k_n = 4/(2n-1)^2\pi^2$ and $Z(A - k_nI) = \operatorname{span}\{x_n\}$, where $x_n(s) = \sin(2n-1)\pi s/2$, $0 \leq s \leq 1, n = 1, 2, \ldots$. Further, $\sigma(A) = \{k_1, k_2, \ldots\} \cup \{0\}$.

18-10 Let $X = L^2([0,1])$ and A denote the Volterra integration operator. (See 18.7(b).) Then A' can be identified with the operator B defined by

$$B(x)(s) = \int_s^1 x(t)\, dm(t), \quad x \in X, \quad s \in [0,1].$$

Let $C = BA$. Then C is a Fredholm integral operator with kernel $k(s,t) = 1 - \max\{s,t\}, 0 \leq s,t \leq 1, \sigma_e(C) = \{k_1, k_2, \ldots\}$, where $k_n = 4/(2n-1)\pi^2$ and $Z(C - k_nI) = \operatorname{span}\{x_n\}$, where $x_n(s) = \cos(2n-1)\pi s/2$, $0 \leq s \leq 1$.

18-11 **(Power compact operator)** Let $A \in BL(X)$ be such that A^p is compact for some positive integer p. [For example, let $X = \ell^p, 1 \leq p \leq \infty$, and $A(x) = (0, x(1), 0, x(3), \ldots)$ or $A(x) = (x(2), 0, x(4), 0, \ldots)$ for $x \in X$.] Lemma 18.1 and Theorems 18.3, 18.5 and 18.6 hold for A.

18-12 Let $A \in CL(X)$ and $0 \neq k \in \mathbf{K}$.

(a) For $n = 0, 1, 2, \ldots, Z_n = Z((A - kI)^n)$ is finite dimensional and $Y_n = R((A - kI)^n)$ is closed in X. (Compare the proof of 19.1(c).)

(b) There is a smallest nonnegative integer ℓ such that $Z_\ell = Z_{\ell+1}$. Also, ℓ is the smallest nonnegative integer such that $Y_\ell = Y_{\ell+1}$. (Hint: $Z_n = Z_{n+1}$ if and only if $Y_n = Y_{n+1}$ by Problem 18-2). [If $k \in \sigma_e(A)$, then the dimension of $Z(A - kI)$ is called the **geometric multiplicity** of k, the positive integer ℓ is called the **ascent** of k and the dimension of $Z((A - kI)^\ell)$ is called the **algebraic multiplicity** of k.]

18-13 Let $A \in CL(X)$ and $0 \neq k \in \mathbf{K}$. Then the (finite) dimensions of $Z((A - kI)^n)$ and $Z((A' - kI)^n)$ are equal for each $n = 0, 1, 2, \ldots$. Further, the geometric multiplicity, the ascent and the algebraic multiplicity of k as eigenvalue of A is the same as the geometric multiplicity, the ascent and the algebraic multiplicity of k as an eigenvalue of A'. (Hint: 18.6(a))

19 Fredholm Alternative

In this section we utilize the properties of the spectrum of a compact operator A given in the last section to solve some equations involving A. To motivate our discussion, we state the following three results about the solution of a finite number of linear equations in the same number of unknowns.

Consider a nonhomogeneous system of n equations in the unknowns $x(1), \ldots, x(n)$:

$$\begin{array}{ccccccc} k_{1,1}x(1) & + & \cdots & + & k_{1,n}x(n) & = & y(1) \\ \vdots & & & & \vdots & & \vdots \\ k_{n,1}x(1) & + & \cdots & + & k_{n,n}x(n) & = & y(n), \end{array} \qquad \cdots [y]$$

where $y = (y(1), \ldots, y(n))$ is a given element of $\mathbf{K}^n$. Let $M = (k_{i,j})$ denote the coefficient matrix of this system and $M^t = (k_{j,i})$ denote its transpose.

(a) The system $[y]$ has a unique solution for every fixed $y \in \mathbf{K}^n$ if and only if the associated homogeneous system

$$\begin{array}{ccccccc} k_{1,1}x(1) & + & \cdots & + & k_{1,n}x(n) & = & 0 \\ \vdots & & & & \vdots & & \vdots \\ k_{0,1}x(1) & + & \cdots & + & k_{n,n}x(n) & = & 0 \end{array} \qquad \cdots [0]$$

has $x(1) = \cdots = x(n) = 0$ as the only solution. [This follows by noting that the matrix M is invertible if and only if $Mx = 0$ implies $x = 0$.]

(b) The homogeneous system [0] has a nonzero solution if and only if the transposed homogeneous system

$$\begin{array}{ccccccc} k_{1,1}x(1) & + & \cdots & + & k_{n,1}x(n) & = & 0 \\ \vdots & & & & \vdots & & \vdots \\ k_{1,n}x(1) & + & \cdots & + & k_{n,n}x(n) & = & 0 \end{array} \qquad \cdots [0]^t$$

has a nonzero solution. Further, the maximum number of linearly independent solutions of the systems [0] and $[0]^t$ are the same. [This follows by noting that if r is the rank of M and s is the rank of M^t, then the maximum number of linearly independent solutions of the system [0] (resp., the system $[0]^t$) equals $n - r$ (resp., $n - s$) and that $r = s$.]

(c) For a given $y \in \mathbf{K}^n$, the system $[y]$ has a solution if and only if

$$x(1)y(1) + \cdots + x(n)y(n) = 0$$

for every solution $x = (x(1), \ldots, x(n))$ of the homogeneous system $[0]^t$. In that case, the general solution of the system $[y]$ is obtained by adding a finite linear combination of solutions of the homogeneous system [0] to a (fixed) particular solution of the system $[y]$. [This follows by noting that the system $[y]$ has a solution if and only if the rank of M equals the rank of the augmented matrix

$$\begin{bmatrix} k_{1,1} & \cdots & k_{1,n} & y(1) \\ \vdots & & \vdots & \vdots \\ k_{n,1} & \cdots & k_{n,n} & y(n) \end{bmatrix}$$

This is the case if and only if y belongs to the span of the columns of the matrix M, that is, $f(y) = 0$ for every linear functional f on $\mathbf{K}^n$ which vanishes on the columns of M.]

Solutions of the system $[y]$, when they exist, can be found by the elimination process of Gauss or by the orthogonalization method of Householder (See [56], 1.7 and 3.2.)

The elegant results from linear algebra stated above do not have a straight forward generalization to a denumerable system of linear equations in denumerable unknowns.

For example, consider a system whose coefficient matrix $M = (k_{i,j})$ is given by $k_{j+1,j} = 1/j$ and $k_{i,j} = 0$ for all other $i, j = 1, 2, \ldots$. The associated homogeneous system $[0]$ has $x(1) = x(2) = \cdots = 0$ as the only solution. Yet, if $y = (1, 0, 0, \cdots)$, then the nonhomogeneous system $[y]$ has no solution. Also, the transposed homogeneous system $[0]^t$ has a nonzero solution, namely, $x = (1, 0, 0, \ldots)$. However, it can be seen that the nature of solutions of the denumerable system of equations having $I - M$ as its coefficient matrix is very similar to the nature of solutions of a finite system. Note that M defines a compact operator on $\ell^p, 1 \le p \le \infty$. We are thus led to the following results.

19.1 Theorem (Fredholm alternative, 1917)

Let X be a normed space over $\mathbf{K}$ and A be a compact operator on X.

(a) Exactly one of the following alternatives holds:

(i) For every $y \in X$, there is a unique $x \in X$ such that $x - A(x) = y$.

(ii) There exists some nonzero $x \in X$ such that $x - A(x) = 0$.

If the alternative (ii) holds, then the maximum number of linearly independent solutions of the homogeneous equation $x - A(x) = 0$ is finite.

(b) The homogeneous equation $x - A(x) = 0$ has a nonzero solution in X if and only if the transposed homogeneous equation $x' - A'(x') = 0$ has a nonzero solution in X'.

Moreover, the maximum number of linearly independent solutions of these two equations is the same.

(c) For a given $y \in X$, the equation $x - A(x) = y$ possesses a solution in X if and only if

$$x_j'(y) = 0, \quad j = 1, \ldots, m,$$

where $x_1', \ldots, x_m'$ constitute a basis for the solution space of the transposed homogeneous equation $x' - A'(x') = 0$.

Moreover, if x_0 is a particular solution of $x - A(x) = y$, then the general solution is given by

$$x = x_0 + k_1 x_1 + \cdots + k_m x_m,$$

where $k_1, \ldots, k_m$ are scalars and $x_1, \ldots, x_m$ constitute a basis for the solution space of the equation $x - A(x) = 0$.

Proof:

(a) The alternative (ii) holds exactly when 1 is an eigenvalue of A. If 1 is not an eigenvalue of A, then by 18.3(a), $I - A$ is an invertible operator on A, so that it is bijective, that is, the alternative (i) holds.

If 1 is an eigenvalue of A, then by 18.5(b), the corresponding eigenspace $Z(I - A) = \{x \in X : x - A(x) = 0\}$ is finite dimensional. Hence (a) follows.

(b) By 18.6(a), the subspace $Z(I - A')$ of X' has the same (finite) dimension as the subspace $Z(I - A)$ of X. Hence (b) follows.

(c) Let $y \in X$. The equation $x - A(x) = y$ has a solution in X if and only if $y \in R(I - A)$. We first show that $R(I - A)$ is closed in X. If $I - A$ is injective, we have already seen this in the proof of 18.3(a). If $I - A$ is not injective, this proof can be modified as follows. By 18.5(b), $Z(I - A)$ is a finite dimensional subspace of X. Let $x_1, \ldots, x_m$ constitute a basis for $Z(I - A)$. Then by 7.10(c), there are $x_1', \ldots, x_m'$ in X' such that $x_j'(x_i) = \delta_{i,j}, i, j = 1, \ldots, m$. Let $Y = \{x \in X : x_j'(x) = 0 \text{ for each } j = 1, \ldots, m\}$. Then it can be seen that Y is a closed subspace of X with $X = Y + Z(I - A)$ and $Y \bigcap Z(I - A) = \{0\}$. Define $C : Y \to X$ by $C(x) = x - A(x)$. Then C is injective and $R(C) = R(I - A)$. As in the proof of 18.3(a), we find that C is bounded below and that $R(I - A) = R(C)$ is a closed subspace of X.

Letting $F = I - A$ in 13.7(c), we obtain

$$R(I - A) = \{y \in X : x'(y) = 0 \text{ for every } x' \in Z(I - A')\}.$$

Now the dimension of $Z(I - A')$ equals m by 18.6(a). Let $x'_1, \ldots, x'_m$ constitute a basis for $Z(I - A')$. Then we see that $y \in R(I - A)$ if and only if $x'_1(y) = \cdots = x'_m(y) = 0$.

Finally, let $x_0 \in X$ be such that $x_0 - A(x_0) = y$. If $x \in X$ with $x - A(x) = y$, then $x - x_0 - A(x - x_0) = y - y = 0$, that is, $x - x_0 \in Z(I - A)$. Since $x_1, \ldots, x_m$ constitute a basis of $Z(I - A)$, we have $x - x_0 = k_1 x_1 + \cdots + k_m x_m$ for some $k_1, \ldots, k_m$ in $\mathbf{K}$. $\square$

We remark that the operator equation

$$x - \mu A(x) = y,$$

where A is a compact operator and μ is a nonzero scalar can be treated by employing Theorem 19.1, because μA is a compact operator. Then we can assert that exactly one of the following alternatives holds:
(i) For every $y \in X$, there is a unique $x \in X$ such that $x - \mu A(x) = y$.
(ii) There exists a nonzero $x \in X$ such that $x - \mu A(x) = 0$.

Note that the second alternative holds exactly when $1/\mu$ is an eigenvalue of A. By 18.5(a), this alternative holds for only a countable number of nonzero scalars μ, say for $\mu = \mu_1, \mu_2 \ldots$. Further, if the set $\{\mu_1, \mu_2, \ldots\}$ is infinite, then $\mu_n \to \infty$ as $n \to \infty$.

Thus we see that except for a countable number of scalars μ, the equation $x - \mu A(x) = y$ has a unique solution in X for every given $y \in X$. Further, any bounded subset of the scalars contains only a finite number of such exceptional scalars.

We shall now consider some consequences of Theorem 19.1 for solving denumerable linear systems and integral equations. These results are comparable to the elegant results for solutions of finite linear systems considered in beginning of this section. The only difference is that instead of the equation $Mx = y$, we now consider the equation $(I - M)x = y$.

19.2 Theorem

Let $M = (k_{i,j})$ be an infinite matrix and $X = \ell^p, 1 \le p < \infty$. If $p = 1$, assume that $\gamma(j) = \sum_{i=1}^{\infty} |k_{i,j}| < \infty$ for each $j = 1, 2, \ldots$ and $\gamma(j) \to 0$ as $j \to \infty$, while if $1 < p < \infty$, assume that

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |k_{i,j}|^q \right)^{p/q} < \infty, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

(a) Either for every $y = (y(1), y(2), \ldots) \in \ell^p$, there is a unique $x = (x(1), x(2), \ldots) \in \ell^p$ such that

$$\begin{array}{rcrcrcl}
(1 - k_{1,1})x(1) & - & k_{1,2}x(2) & - & k_{1,3}x(3) & - & \ldots = y(1) \\
-k_{2,1}x(1) & + & (1 - k_{2,2})x(2) & - & k_{2,3}x(3) & - & \ldots = y(2) \\
-k_{3,1}x(1) & - & k_{3,2}x(2) & + & (1 - k_{3,3})x(3) & - & \ldots = y(3) \\
\vdots & & \vdots & & \vdots & & \qquad \vdots
\end{array}$$

or there is some nonzero $x = (x(1), x(2) \ldots) \in \ell^p$ such that

$$\begin{array}{rcrcrcl}
(1 - k_{1,1})x(1) & = & k_{1,2}x(2) & + & k_{1,3}x(3) & + & \ldots \\
(1 - k_{2,2})x(2) & = & k_{2,1}x(1) & + & k_{2,3}x(3) & + & \ldots \\
(1 - k_{3,3})x(3) & = & k_{3,1}x(1) & + & k_{3,2}x(2) & + & \ldots \\
\vdots & & \vdots & & \vdots & &
\end{array} \qquad \cdots [0]$$

The homogeneous system [0] can never have an infinite number of linearly independent solutions.

(b) The maximum number of linearly independent solutions in ℓ^p of the homogeneous system [0] is equal to the maximum number of linearly independent solutions in $\ell^q, 1/p + 1/q = 1$, of the transposed homogeneous system

$$\begin{array}{rcrcrcl}
(1 - k_{1,1})x(1) & = & k_{2,1}x(2) & + & k_{3,1}x(3) & + & \ldots \\
(1 - k_{2,2})x(2) & = & k_{1,2}x(1) & + & k_{3,2}x(3) & + & \ldots \\
(1 - k_{3,3})x(3) & = & k_{1,3}x(1) & + & k_{2,3}x(2) & + & \ldots \\
\vdots & & \vdots & & \vdots & &
\end{array} \qquad \cdots [0]^t$$

(c) The system $[y]$ possesses a solution $x \in \ell^p$ for just those $y = (y(1), y(2), \ldots) \in \ell^p$ which satisfy

$$x_j(1)y(1) + x_j(2)y(2) + x_j(3)y(3) + \ldots = 0, \quad j = 1, \ldots, m,$$

where $\{x_1, \ldots, x_m\}$ is a basis of the solution space in ℓ^q of the transposed homogeneous system $[0]^t$.

Proof:
As we have seen in 17.4(a), the infinite matrix M defines a compact operator on X. Also, 13.4(b) shows that the matrix $M^t = (k_{j,i})$ defines the transpose of this compact operator. Hence the desired results follow directly from 19.1. □

We now turn to a continuous analog of the matrix equation $x - Mx = y$. Consider the integral equation

$$x(s) - \int_a^b k(s,t)x(t)\,dm(t) = y(s), \quad a \le s \le b.$$

We seek a solution x of this equation when the so-called 'free term' y is given. This integral equation is known as the **Fredholm integral equation of the second kind.** [The equation

$$\int_a^b k(s,t)x(t)\,dm(t) = y(s), \quad a \le s \le b$$

is known as the **Fredholm integral equation of the first kind**. We shall not, however, deal with this equation.]

19.3 Theorem (Fredholm, 1903)
Let $k(.,.) \in L^2([a,b] \times [a,b])$.

(a) Exactly one of the following alternatives holds:

(i) For every $y \in L^2([a,b])$, there is a unique $x \in L^2([a,b])$ such that for almost all $s \in [a,b]$,

$$x(s) - \int_a^b k(s,t)x(t)\,dm(t) = y(s) \qquad \cdots [y]$$

(ii) There exists some nonzero $x \in L^2([a,b])$ such that for almost all $s \in [a,b]$,

$$x(s) = \int_a^b k(s,t)x(t)\,dm(t) \qquad \cdots [0]$$

If the alternative (ii) holds, then the maximum number of linearly independent solutions of the homogeneous equation [0] is finite.

(b) The homogeneous equation [0] has a nonzero solution in $L^2([a,b])$ if and only if the transposed homogeneous equation

$$z(s) = \int_a^b k(t,s)z(t)\,dm(t), \qquad \cdots [0]^t$$

has a nonzero solution in $L^2([a,b])$.

(c) For a given $y \in L^2([a,b])$, the integral equation [y] has a solution in $L^2([a,b])$ if and only if

$$\int_a^b yz_j\,dm = 0, \quad j = 1,\ldots,m,$$

where $z_1,\ldots,z_m$ constitute a basis for the solution space of the transposed homogeneous equation $[0]^t$.

Proof:
For $x \in L^2([a,b])$, define

$$A(x)(s) = \int_a^b k(s,t)x(t)\,dm(t), \quad s \in [a,b].$$

We have seen in 17.4(b) that A is a compact operator on $L^2([a,b])$. Hence part (a) follows immediately from 19.1(a).

To obtain parts (b) and (c) from 19.1(b) and 19.1(c), we argue as follows.

Let $F : L^2([a,b]) \to (L^2([a,b]))'$ denote the surjective linear isometry given by the Riesz representation theorem (14.3). Thus for a fixed $z \in L^2([a,b])$,

$$F(z)(x) = \int_a^b xz\,dm, \quad x \in L^2([a,b]).$$

Then A' can be identified with the operator $B = F^{-1}A'F$ on $L^2([a,b])$. Schematically,

$$\begin{array}{ccc}
(L^2([a,b]))' & \xrightarrow{A'} & (L^2([a,b]))' \\
F\uparrow & & \uparrow F \\
L^2([a,b]) & \xrightarrow{B} & L^2([a,b]).
\end{array}$$

(For a matrix operator on $\ell^p, 1 \le p < \infty$, we have given a similar reasoning in 13.4.(b).) Fix $y \in L^2([a,b])$. Then $B(y) \in L^2([a,b])$ and for all x in $L^2([a,b])$, we have

$$\begin{aligned}
\int_a^b x\, B(y)\, dm &= F(B(y))(x) = A'(F(y))(x) \\
&= F(y)(A(x)) = \int_a^b A(x) y\, dm \\
&= \int_a^b \left[\int_a^b k(t,u)x(u)dm(u)\right] y(t)\, dm(t) \\
&= \int_a^b x(u) \left[\int_a^b k(t,u)y(t)\ dm(t)\right] dm(u).
\end{aligned}$$

The last equality follows from 4.4, because

$$\begin{aligned}
&\int_a^b \int_a^b |k(t,u)x(u)y(t)|\ dm(t)\ dm(u) \\
&\le \left[\int_a^b \int_a^b |k(t,u)|^2\ dm(t)\ dm(u)\right]^{\frac{1}{2}} \left[\int_a^b \int_a^b |x(u)|^2 |y(t)|^2\ dm(u)\ dm(t)\right] \\
&= \left[\int_a^b \int_a^b |k(t,u)|^2\ dm(t)\ dm(u)\right]^{\frac{1}{2}} \left[\int_a^b |x|^2\, dm\right]^{\frac{1}{2}} \left[\int_a^b |y|^2\, dm\right]^{\frac{1}{2}},
\end{aligned}$$

which is finite, since $k(.,.) \in L^2([a,b] \times [a,b])$ and $x, y \in L^2([a,b])$. (See Hölder's inequality 4.5(a).)

It follows that for almost all $u \in [a,b]$,

$$B(y)(u) = \int_a^b k(t,u)y(t)\, dm(t).$$

Now parts (b) and (c) follow from 19.1(b) and 19.1(c) respectively, if we let $x' = F(z)$ and note that $x' - A'(x') = F(z) - A'(F(z)) = 0$ if and only if $z - (F^{-1}A'F)(z) = 0$, that is, $z - B(z) = 0$. □

19.4 Corollary

Let $k(.,.) \in C([a,b]\times[a,b])$. Then the results in 19.3 remain valid if we replace $L^2([a,b])$ by $C([a,b])$ throughout. Each equation appearing in these results then holds for all $s \in [a,b]$.

Proof:

The proof of 19.3(a) remains unchanged since $A : C([a,b]) \to C([a,b])$ is a compact operator, as we have seen in 17.4(b). As for parts (b) and (c) of 19.3, note that $A(x) \in C([a,b])$ for all $x \in L^2([a,b])$. This follows from the inequality

$$|A(x)(s) - A(x)(s_0)| \leq \int_a^b |k(s,t) - k(s_0,t)|\, |x(t)|\, dm(t)$$

and the uniform continuity of $k(.,.)$ on $[a,b]\times[a,b]$. Similar reasoning shows that $B(z) \in C([a,b])$ for all $z \in L^2([a,b])$, where

$$B(z)(u) = \int_a^b k(t,u)z(t)\, dm(t), \quad u \in [a,b].$$

Hence if $x - A(x) = 0$ for some $x \in L^2([a,b])$, then $x \in C([a,b])$, and if $z - B(z) = 0$ for some $z \in L^2([a,b])$, then $z \in C([a,b])$. Consequently, the results for $L^2([a,b])$ proved in 19.3 hold for $C([a,b])$ as well. □

We mention that the results of 19.4 have a useful application to some problems in potential theory. Let Γ be a smooth simple closed curve in $\mathbf{R}^2$ and let y be a continuous real-valued function on Γ. The **interior** (resp., **exterior**) **Dirichlet Problem** for Γ consists of finding a harmonic function u on the interior (resp., exterior) of Γ whose boundary values coincide with the values of the given function y on Γ. The **interior** (resp., **exterior**) **Neumann problem** for Γ consists of finding a harmonic function u on the interior (resp., exterior) of Γ

such that the normal derivative of u coincides with y on Γ. By using integrals which involve potentials of single and double layers and employing 19.4, it can be shown that the interior and the exterior Dirichlet problems as well as the exterior Neumann problem for Γ always have a solution, while the interior Neumann problem has a solution for precisely those continuous functions y which satisfy $\int_\Gamma y(s)ds = 0$. The interested reader is referred to an excellent treatment of these problems given in [48], pp. 190-194.

19.5 Example

Consider the kernel

$$k(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1 \\ s(1-t), & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Let $X = L^2([0,1])$ or $C([0,1])$. For $0 \ne \mu \in \mathbf{K}$. For a given $y \in X$, consider the integral equation

$$x(s) - \mu \int_0^1 k(s,t)x(t)\,dm(t) = y(s) \qquad \cdots [y]$$

In 18.7(c), we have seen that the eigenvalues of the compact operator A given by

$$A(x)(s) = \int_0^1 k(s,t)x(t)\,dm(t), \quad x \in X, \quad s \in [a,b],$$

are $\{1/n^2\pi^2 : n = 1,2,\ldots\}$ and also that $Z(A - I/n^2\pi^2) = \operatorname{span}\{x_n\}$, where $x_n(s) = \sin n\pi s, s \in [0,1]$. Also, since $k(s,t) = k(t,s)$ for all $s,t \in [0,1]$, it follows that A' can be identified with A itself. Hence we obtain the following results from 19.3 and 19.4.

(i) If $\mu \ne n^2\pi^2$ for any $n = 1,2,\ldots$, then for every $y \in X$, there is a unique solution $x \in X$ of the integral equation $[y]$.

(ii) If $\mu = n_0^2\pi^2$ for some $n_0 = 1,2,\ldots$, then the integral equation $[y]$ has a solution in X if and only if

$$\int_0^1 y(t)\sin n_0\pi t\,dm(t) = 0.$$

Also, if x_0 is such a solution, then the general solution is given by

$$x(s) = x_0(s) + c \sin n_0 \pi s, \quad s \in [0, 1],$$

where $c \in \mathbf{K}$.

Before concluding this section, it is essential to remark that the results of this section tell us about the existence and uniqueness of solutions of operator equations, but they do not say how to find a solution. Calculating an exact solution is of then a formidable task. It can be accomplished only in rare cases. In the next section, we shall see how approximate solutions can be obtained. [In case $|\mu|$ is small, the unique solution of $x - \mu A(x) = y$ can be obtained by using the Neumann series expansion:

$$x = (I - \mu A)^{-1}(y) = \sum_{n=0}^{\infty} \mu^n A^n(y).$$

In this case also, it is not always easy to find $A^n(y)$ for $n = 2, 3, \ldots$. Compare 12.4.]

Problems

19-1 If the alternative (i) holds in 19.1(a), then the unique element $x \in X$ satisfying $x - A(x) = y$ depends continuously on $y \in X$.

19-2 Let X be a normed space and $A \in BL(X)$ be of finite rank. Then the results in 19.1 can be proved for A without appealing to the spectral theory for compact operators given in Section 18. (Hint: 12.2, 20.2)

19-3 Let X be a Banach space and $A \in CL(X)$. For a given $y' \in X'$, the equation $x' - A'(x') = y'$ possesses a solution in X' if and only if

$$y'(x_j) = 0, \quad j = 1, \ldots, m,$$

where $x_1, \ldots, x_m$ constitute a basis for the solution space of the homogeneous equation $x - A(x) = 0$.

Moreover, if x_0' is a particular solution of $x' - A(x') = y'$, then the general solution is given by

$$x' = x_0' + k_1 x_1' + \cdots + k_m x_m',$$

where $k_1, \ldots, k_m$ are scalars and $x_1', \ldots, x_m'$ constitute a basis for the solution space of the equation $x' - A'(x') = 0$. (Compare 19.1(c). Hint: 13.7(d))

19-4 Let $M = (1/ij), i, j = 1, 2, \ldots$ and $\mu \in \mathbf{K}, \mu \neq 0$. If $\mu \neq 6/\pi^2$ then for every $y \in \ell^2$, there is a unique $x \in \ell^2$ such that $x - \mu M x = y$. If $\mu = 6/\pi^2$, then $x_0 - \mu M x_0 = 0$ if and only if x_0 is a scalar multiple of $(1, 1/2, 1/3, \ldots)$, and for $y \in \ell^2$, there is some $x \in \ell^2$ such that $x - \mu M x = y$ if and only if $\sum_{j=1}^{\infty} y(j)/j = 0$.

19-5 Let X be a normed space. Then the Fredholm alternative (19.1) holds for a power compact operator A on X. (Hint: Problem 18-11)

19-6 Let $X = L^2([0,1])$ or $C([0,1])$ and $0 \neq \mu \in \mathbf{K}$. Fix $y \in X$ and consider the integral equation $[y]$:

$$x(s) - \mu \int_0^s t\, x(t)dt - \mu s \int_s^1 x(t)dt = y(s), \quad 0 \leq s \leq 1.$$

(a) If $\mu \neq (2n-1)^2\pi^2/4$ for any $n = 1, 2, \ldots$, then for every $y \in X$, there is a unique solution x in X of the integral equation $[y]$.

(b) If $\mu = (2n_0 - 1)^2\pi^2/4$ for some $n_0 = 1, 2, \ldots$, then the integral equation $[y]$ has a solution in X if and only if

$$\int_0^1 y(t) \sin{(2n_0 - 1)}\frac{\pi t}{2}\, dm(t) = 0.$$

Also, if x_0 is such a solution, then the general solution is given by

$$x(s) = x_0(s) + c \sin{(2n_0 - 1)}\frac{\pi s}{2},$$

where $c \in \mathbf{K}$.(Hint: Problem 18-9)

19-7 Let $X = C([0,1])$. For $y \in X$, consider the integral equation $[y]$:

$$x(s) - \mu \left[(1-s)\int_0^s x(t)\, dt - \int_s^1 (1-t)x(t)\, dt\right] = y(s), \quad 0 \leq s \leq 1.$$

(a) If $\mu = 1$, then for every given $y \in X$, there is a unique solution x in X of the integral equation [y].

(b) If $\mu = \pi^2/4$ and $y(s) = s$ for $0 \le s \le 1$, then there is no solution in X of the integral equation [y].

(c) If $\mu = \pi^2/4$ and $y(s) = \cos 3\pi s/2$ for $0 \le s \le 1$, then there are infinitely many solutions in X of the integral equation [y]. (Hint: Problem 18.10)

20 Approximate Solutions

In the last section we saw that if A is a compact operator on a normed space X and if $x = 0$ is the only element of X satisfying $x - A(x) = 0$, then for every $y \in X$, there is a unique $x \in X$ such that

$$x - A(x) = y.$$

In the present section we consider the question of finding approximations to the exact solution x. In the process, we deal with finite rank operators and give a procedure to reduce equations and eigenvalue problems involving these operators to matrix equations and matrix eigenvalue problems.

First we show that if an operator $A_0 \in BL(X)$ is close to the operator A and if an element $y_0 \in X$ is close to the free term y, then the element $x_0 \in X$ satisfying $x_0 - A_0(x_0) = y_0$ is close to the element x satisfying $x - A(x) = y$, that is , x_0 can be called an approximate solution of the equation $x - A(x) = y$.

20.1 Theorem

Let X be a Banach space and A be a compact operator on X such that $x = 0$ is the only element of X satisfying $x - A(x) = 0$. Then

$I - A$ is invertible. Let $A_0 \in BL(X)$ satisfy

$$\epsilon = \|(A - A_0)(I - A)^{-1}\| < 1.$$

Then for given $y, y_0 \in X$, there are unique $x, x_0 \in X$ such that

$$x - A(x) = y, \quad x_0 - A_0(x_0) = y_0$$

and

$$\|x - x_0\| \leq \frac{\|(I - A)^{-1}\|}{1 - \epsilon}(\epsilon\|y\| + \|y - y_0\|).$$

Proof:
Since A is compact and $I - A$ is injective, it follows from 18.3(a) that $I - A$ is invertible. As X is a Banach space and

$$\|[(I - A) - (I - A_0)](I - A)^{-1}\| = \epsilon < 1,$$

it follows from 12.5(a) that $I - A_0$ is invertible and

$$\|(I - A_0)^{-1}\| \leq \frac{\|(I - A)^{-1}\|}{1 - \epsilon}, \quad \|(I - A)^{-1} - (I - A_0)^{-1}\| \leq \frac{\epsilon\|(I - A)^{-1}\|}{1 - \epsilon}$$

Let $y, y_0 \in X$. Since $I - A$ and $I - A_0$ are invertible, there are unique $x, x_0 \in X$ such that $x - A(x) = y$ and $x_0 - A_0(x_0) = y_0$. Also,

$$\begin{aligned} x - x_0 &= (I - A)^{-1}(y) - (I - A_0)^{-1}(y_0) \\ &= [(I - A)^{-1} - (I - A_0)^{-1}](y) + (I - A_0)^{-1}(y - y_0). \end{aligned}$$

Hence

$$\|x - x_0\| \leq \frac{\epsilon\|(I - A)^{-1}\|}{1 - \epsilon}\|y\| + \frac{\|(I - A)^{-1}\|}{1 - \epsilon}\|y - y_0\|,$$

as desired. □

Theorem 20.1 can be compared with Problem 12-13.

In order to employ the preceding result, we must be able to find an operator $A_0 \in BL(X)$ such that (i) the equation $x_0 - A_0(x_0) = y_0$

is solvable explicitly and (ii) $\|(A - A_0)(I - A)^{-1}\|$ is small. We shall show that these requirements can often be fulfilled by an appropriately chosen bounded operator A_0 of finite rank.

First we prove that if A_0 is an operator of finite rank, then the solution of the equation $x_0 - A_0(x_0) = y_0$ can be reduced to the solution of a finite system of linear equations which can then be solved by standard methods. Next, when the operator A is compact, we give several ways of constructing a bounded linear operator A_0 of finite rank such that $\|A - A_0\|$ is arbitrarily small. In particular, if $\|A - A_0\|$ is less than $1/\|(I - A)^{-1}\|$, then $\epsilon = \|(A - A_0)(I - A)^{-1}\| < 1$ and then the result 20.1 becomes applicable.

20.2 Theorem (Whitley, 1986)
Let A_0 be an operator of finite rank on a linear space X over $\mathbf{K}$ given by

$$A_0(x) = f_1(x)x_1 + \cdots + f_m(x)x_m, \quad x \in X,$$

where $x_1, \cdots, x_m$ are in X and $f_1, \ldots, f_m$ are linear functionals on X. Let

$$M = \begin{bmatrix} f_1(x_1) & \cdots & f_1(x_m) \\ \vdots & & \vdots \\ f_m(x_1) & \cdots & f_m(x_m) \end{bmatrix}.$$

(a) Consider $y_0 \in X$ and let $v_0 = (f_1(y_0), \ldots, f_m(y_0))$. Then

$$x_0 - A_0(x_0) = y_0 \quad \text{and} \quad u_0 = (f_1(x_0), \ldots, f_m(x_0))$$

if and only if

$$u_0 - Mu_0 = v_0 \quad \text{and} \quad x_0 = y_0 + u_0(1)x_1 + \cdots + u_0(m)x_m.$$

(b) Let $0 \neq k \in \mathbf{K}$. Then k is an eigenvalue of A_0 if and only if k is an eigenvalue of M. Further, if u_0 (resp., x_0) is an eigenvector of M (resp., A_0) corresponding to k, then $x_0 = u_0(1)x_1 + \cdots + u_0(m)x_m$ (resp., $u_0 = (f_1(x_0), \cdots, f_m(x_0))$ is an eigenvector of A_0 (resp., M) corresponding to k.

Proof:

(a) Let $x_0 - A_0(x_0) = y_0$ and $u_0 = (f_1(x_0), \ldots, f_m(x_0))$. Then for $i = 1, \ldots, m$,

$$
\begin{aligned}
(Mu_0)(i) &= f_i(x_1)f_1(x_0) + \cdots + f_i(x_m)f_m(x_0) \\
&= f_i(f_1(x_0)x_1 + \cdots + f_m(x_0)x_m) \\
&= f_i(A_0(x_0)) \\
&= f_i(x_0 - y_0) \\
&= u_0(i) - v_0(i),
\end{aligned}
$$

that is, $u_0 - Mu_0 = v_0$, as desired. Also,

$$
\begin{aligned}
x_0 &= y_0 + A_0(x_0) \\
&= y_0 + f_1(x_0)x_1 + \cdots + f_m(x_0)x_m \\
&= y_0 + u_0(1)x_1 + \cdots + u_0(m)x_m.
\end{aligned}
$$

Conversely, let $u_0 - Mu_0 = v_0$ and $x_0 = y_0 + u_0(1)x_1 + \cdots + u_0(m)x_m$. Then

$$
\begin{aligned}
A_0(x_0) &= f_1(x_0)x_1 + \cdots + f_m(x_0)x_m \\
&= \left[f_1(y_0) + \sum_{j=1}^{m} u_0(j)f_1(x_j)\right] x_1 + \cdots \\
&\quad + \left[f_m(y_0) + \sum_{j=1}^{m} u_0(j)f_m(x_j)\right] x_m \\
&= [v_0(1) + (Mu_0)(1)]x_1 + \cdots + [v_0(m) + (Mu_0)(m)]x_m \\
&= u_0(1)x_1 + \cdots + u_0(m)x_m \\
&= x_0 - y_0,
\end{aligned}
$$

as desired. Also, for $i = 1, \ldots, m$,

$$
\begin{aligned}
u_0(i) &= v_0(i) + (Mu_0)(i) \\
&= v_0(i) + f_i(x_1)u_0(1) + \cdots + f_i(x_m)u_0(m) \\
&= v_0(i) + f_i(u_0(1)x_1 + \cdots + u_0(m)x_m) \\
&= v_0(i) + f_i(x_0 - y_0) \\
&= f_i(x_0),
\end{aligned}
$$

that is, $u_0 = (f_1(x_0), \ldots, f_m(x_0))$.

(b) Since $k \neq 0$, let $\mu = 1/k$. Replacing A_0 by μA_0 and letting $y_0 = 0$ in (a) above, we see that

$$x_0 - \mu A_0(x_0) = 0 \quad \text{and} \quad u_0 = (f_1(x_0), \ldots, f_m(x_0))$$

if and only if

$$u_0 - \mu M u_0 = 0 \quad \text{and} \quad x_0 = u_0(1)x_1 + \cdots + u_0(m)x_m.$$

Hence $A_0(x_0) = kx_0$ with $x_0 \neq 0$ if and only if $Mu_0 = ku_0$ with $u_0 \neq 0$. Thus k is an eigenvalue of A_0 if and only if k is an eigenvalue of M. Also, the eigenvectors of A_0 and M corresponding to k are related by $u_0 = (f_1(x_0), \ldots, f_m(x_0))$ and $x_0 = u_0(1)x_1 + \cdots + u_0(m)x_m$. $\square$

Letting $k = 1$ in 20.2(b), we see that $A_0(x) = x$ has a nonzero solution in X if and only if $Mu = u$ has a nonzero solution in $\mathbf{K}^m$. Also, for a given $y_0 \in X$, the general solution of $x - A_0(x) = y_0$ is given by $x = y_0 + u(1)x_1 + \cdots + u(m)x_m$, where $u = (u(1), \ldots, u(m))$ is the general solution of $u - Mu = (f_1(y_0), \ldots, f_m(y_0))$. Thus the problem of solving the operator equation $x - A_0(x) = y_0$ is reduced to solving the matrix equation $u - Mu = v_0$, where $v_0 = (f_1(y_0), \ldots, f_m(y_0))$. (See [58], Lemma 1.)

We now proceed to describe several methods of approximating a compact operator by bounded operators of finite rank. First we describe some methods related to projections.

20.3 Theorem

Let X be a Banach space and A be a compact operator on X. For $n = 1, 2, \ldots$, let $P_n \in BL(X)$ be a projection of finite rank and

$$A_n^P = P_n A, \quad A_n^S = AP_n, \quad A_n^G = P_n A P_n.$$

If $P_n(x) \to x$ in X for every $x \in X$, then $\|A_n^P - A\| \to 0$. If, in addition, $P_n'(x') \to x'$ in X' for every $x' \in X'$, then $\|A_n^S - A\| \to 0$ and $\|A_n^G - A\| \to 0$.

Proof:
Let $P_n(x) \to x$ in X for every $x \in X$. Then it follows that $A_n^P(x) \to A(x)$ in X for every $x \in X$. Since $A \in CL(X)$, the set $E = \{A(x) : x \in X, \|x\| \leq 1\}$ is totally bounded. As X is a Banach space, $(P_n(y))$ converges to y uniformly on E by 9.2(b). Hence

$$\|A_n^P - A\| = \|(P_n - I)A\| = \sup\{\|(P_n - I)(y)\| : y \in E\} \to 0,$$

as $n \to \infty$.

Assume now that $P_n'(x') \to x'$ in X' for every $x' \in X'$ as well. By 17.3, A' is a compact operator on X and

$$(A_n^S)' = (AP_n)' = P_n'A'.$$

Replacing A by A' and P_n by P_n' and recalling 13.5(b), we see that

$$\|A_n^S - A\| = \|(A_n^S - A)'\| = \|P_n'A' - A'\| \to 0$$

by what we have just seen. Also,

$$\begin{aligned}
\|A_n^G - A\| &= \|P_nAP_n - P_nA + P_nA - A\| \\
&\leq \|P_n(AP_n - A)\| + \|P_nA - A\| \\
&\leq \|P_n\| \, \|A_n^S - A\| + \|A_n^P - A\|,
\end{aligned}$$

which tends to zero as $n \to \infty$, since the sequence $(\|P_n\|)$ is bounded by 9.2(a). □

We mention that the superscripts P, S and G in A_n^P, A_n^S and A_n^G stand for Projection, Sloan and Galerkin respectively, the last two being the names of the mathematicians who introduced those methods. We now describe several ways of constructing bounded projections P_n of finite rank such that $P_n(x) \to x$ for every $x \in X$.

1. Truncation of a Schauder expansion

Let X be a Banach space with a Schauder basis $\{x_1, x_2, \ldots,\}$. Let $f_1, f_2, \ldots$ be the corresponding coefficient functionals. (See Section 8.) For $n = 1, 2, \ldots$, define

$$P_n(x) = \sum_{k=1}^{n} f_k(x)x_k, \quad x \in X.$$

By 11.4, each $f_k \in X'$, and hence each $P_n \in BL(X)$. It is clear that $P_n^2 = P_n$ and that each P_n is of finite rank. The very definition of a Schauder basis implies that $P_n(x) \to x$ in X for every $x \in X$. Hence $\|A - A_n^P\| \to 0$, if $A \in CL(X)$.

As a special case, let H be a (separable) Hilbert space and let $\{u_1, u_2, \ldots\}$ be an orthonormal basis for H. (See Section 22 for definition and existence of an orthonormal basis.) Then for $n = 1, 2, \ldots$,

$$P_n(x) = \sum_{k=1}^{n} \langle x, u_k \rangle u_k, \quad x \in H,$$

where $\langle\ ,\ \rangle$ is the inner product on H. Note that each P_n is obtained by truncating the Fourier expansion of $x \in H$. (See 22.7(ii).) Since H' can be identified with H by 24.2, and P_n' can be identified with P_n, we obtain $\|A - A_n^S\| \to 0$ and $\|A - A_n^G\| \to 0$, in addition to $\|A - A_n^P\| \to 0$.

2. Piecewise linear interpolation

Let $X = C([a, b])$ with the sup norm. For $n = 1, 2, \ldots$, consider n nodes $t_1^{(n)}, \ldots, t_n^{(n)}$ in $[a, b]$:

$$a = t_0^{(n)} \le t_1^{(n)} < \cdots < t_n^{(n)} \le t_{n+1}^{(n)} = b.$$

For $j = 1, \ldots, n$, let $u_j^{(n)} \in C([a, b])$ be such that

(i) $u_j^{(n)}(t_i^{(n)}) = \delta_{i,j}, \quad i = 1, \ldots, n,$

(ii) $u_1^{(n)}(a) = 1, u_j^{(n)}(a) = 0$ for $j = 2, \ldots, n,$

$u_n^{(n)}(b) = 1, u_j^{(n)}(b) = 0$ for $j = 1, \ldots, n-1,$

(iii) $u_j^{(n)}$ is linear on each of the subintervals $[t_k^{(n)}, t_{k+1}^{(n)}], k = 0, \ldots, n$.

The functions $u_1^{(n)}, \ldots, u_n^{(n)}$ are known as the **hat functions** because of the shapes of their graphs. Let $t \in [a, b]$. Then $u_j^{(n)}(t) \geq 0$ for all $j = 1, \ldots, n$. If $t \in [t_k^{(n)}, t_{k+1}^{(n)}]$, then $u_k^{(n)}(t) + u_{k+1}^{(n)}(t) = 1$ and $u_j^{(n)}(t) = 0$ for all $j \neq k, k+1$. Thus $u_1^{(n)}(t) + \cdots + u_n^{(n)}(t) = 1$.

For $x \in C([a, b])$, define

$$P_n(x) = \sum_{j=1}^{n} x(t_j^{(n)}) u_j^{(n)}.$$

Then P_n is called a **piecewise linear interpolatory projection**. Let $h_n = \max\{t_{j+1}^{(n)} - t_j^{(n)} : j = 0, \ldots, n\}$ denote the mesh of the partition of $[a, b]$ by the given nodes. We show that $P_n(x) \to x$ in $C([a, b])$ for every $x \in C([a, b])$, provided $h_n \to 0$ as $n \to \infty$. Fix $x \in C([a, b])$ and let $\epsilon > 0$. By the uniform continuity of x on $[a, b]$, there is some $\delta > 0$ such that $|x(s) - x(t)| < \epsilon$ whenever $|s - t| < \delta$. Choose n_0 such that $h_n < \delta$ for all $n \geq n_0$. Consider $n \geq n_0$ and $t \in [a, b]$. If $u_j^{(n)}(t) \neq 0$, then $t \in [t_{j-1}^{(n)}, t_{j+1}^{(n)}]$, so that $|t_j^{(n)} - t| \leq h_n < \delta$ and $|x(t_j^{(n)}) - x(t)| < \epsilon$. Hence

$$\begin{aligned}
|P_n(x)(t) - x(t)| &= \left|\sum_{j=1}^{n} \left(x(t_j^{(n)}) - x(t)\right) u_j^{(n)}(t)\right| \\
&\leq \sum_{j=1}^{n} |x(t_j^{(n)}) - x(t)| u_j^{(n)}(t) \\
&\leq \sum_{j=1}^{n} \epsilon\, u_j^{(n)}(t) \\
&= \epsilon.
\end{aligned}$$

Thus $\|P_n(x) - x\|_\infty \to 0$, provided $h_n \to 0$.

We pass on to consider a special method for integral operators. Let X denote either $C([a, b])$ or $L^2([a, b])$. A kernel $k_0(.,.)$ is said to be **degenerate** if

$$k_0(s, t) = \sum_{i=1}^{m} x_i(s) y_i(t), \quad s, t \in [a, b],$$

where x_i and y_i belong to X, $i = 1, \ldots, m$. The Fredholm integral operator A_0 with a degenerate kernel $k_0(.,.)$ is given by

$$\begin{aligned} A_0(x)(s) &= \int_a^b k_0(s,t)x(t)\,dm(t) \\ &= \sum_{i=1}^m \left[\int_a^b y_i(t)x(t)\,dm(t)\right] x_i(s) \end{aligned}$$

for $s \in [a,b]$. It is clear that $A_0 \in BL(X)$. Also, A_0 is of finite rank because $R(A_0) \subset \operatorname{span}\{x_1, \ldots, x_m\}$.

20.4 Theorem

Let $X = C([a,b])$ (resp., $L^2([a,b])$) and $k(.,.) \in C([a,b]\times[a,b])$ (resp., $L^2([a,b]\times[a,b])$). Let $(k_n(.,.))$ be a sequence of degenerate kernels in $C([a,b]\times[a,b])$ (respectively, $L^2([a,b]\times[a,b])$) such that $\|k - k_n\|_\infty \to 0$ (resp., $\|k - k_n\|_2 \to 0$). If A and A_n^D are the Fredholm integral operators with kernels $k(.,.)$ and $k_n(.,.)$ respectively, then $\|A - A_n\| \to 0$, where $\|\ \|$ denotes the operator norm on $BL(X)$.

Proof:

Let $X = C([a,b])$. Then it is easy to see that for every $x \in X$,

$$\|(A - A_n^D)(x)\|_\infty \leq (b-a)\|k - k_n\|_\infty \|x\|_\infty.$$

Hence $\|A - A_n^D\| \leq (b-a)\|k - k_n\|_\infty \to 0$ as $n \to \infty$.

Next, let $X = L^2([a,b])$. Letting $p = 2 = q$ in 17.4(b), we see that $\|A\| \leq \|k\|_2$. Hence $\|A - A_n^D\| \leq \|k - k_n\|_2 \to 0$ as $n \to \infty$. □

We now describe several ways of constructing degenerate kernels $k_n(.,.), n = 1, 2, \ldots$, which approximate a given kernel $k(.,.)$.

1. Bernstein polynomials in two variables

Let $a = 0$ and $b = 1$ for simplicity, and $k(.,.) \in C([0,1]\times[0,1])$. For $n = 0, 1, 2, \ldots$, and $s, t \in [0,1]$, let

$$k_n(s,t) = \sum_{i,j=0}^n k\left(\frac{i}{n}, \frac{j}{n}\right) \binom{n}{i} \binom{n}{j} s^i (1-s)^{n-i} t^j (1-t)^{n-j}.$$

By Weierstrass' approximation theorem for polynomials in two variables (which can be proved in a manner similar to 3.12), we see that $\|k - k_n\|_\infty \to 0$.

2. Interpolation in the second variable

Let $k(\cdot,\cdot) \in C([a,b]\times[a,b])$ and for each fixed $s \in [a,b]$,

$$k_s(t) = k(s,t), \quad t \in [a,b].$$

For $n = 1, 2, \ldots$, consider the piecewise linear interpolatory projection P_n introduced earlier and define

$$k_n(s,t) = P_n(k_s)(t) = \sum_{j=1}^{n} k(s, t_j^{(n)}) u_j^{(n)}(t), \quad s,t \in [a,b],$$

where $u_1^{(n)}, \ldots, u_n^{(n)}$ are the corresponding hat functions. Let $\epsilon > 0$. By the uniform continuity of the function $k(\cdot,\cdot)$ on $[a,b]\times[a,b]$, there is some $\delta > 0$ such that $|k_s(t_1) - k_s(t_2)| < \epsilon$, whenever $s \in [a,b]$ and $|t_1 - t_2| < \delta$. Assume that the mesh h_n of the partition of $[a,b]$ tends to zero as $n \to \infty$ and choose n_0 such that $h_n < \delta$ for all $n \geq n_0$. If $n \geq n_0$, then as we have seen earlier

$$|k_n(s,t) - k(s,t)| = |P_n(k_s)(t) - k_s(t)| < \epsilon$$

for all $s,t \in [a,b]$. Thus $\|k - k_n\|_\infty \to 0$, provided $h_n \to 0$.

3. Truncation of a Taylor series expansion

If $k(\cdot,\cdot)$ has a uniformly and absolutely convergent double Taylor series expansion

$$k(s,t) = \sum_{i,j=0}^{\infty} k_{i,j}(s - s_0)^i (t - t_0)^j, \quad s,t \in [a,b]$$

about $(s_0, t_0) \in \mathbf{R}^2$, then for $n = 0, 1, 2, \ldots$, let

$$k_n(s,t) = \sum_{i,j=0}^{n} k_{i,j}(s - s_0)^i (t - t_0)^j, \quad s,t \in [a,b].$$

It follows that $\|k - k_n\|_\infty \to 0$. A simple example is given by

$$k(s,t) = e^{st} = \sum_{j=0}^{\infty} \frac{s^j t^j}{j!}, \quad s,t \in [a,b].$$

4. Truncation of a Fourier expansion

Let $k(.,.) \in L^2([a,b]\times[a,b])$ and $\{v_1, v_2, \ldots\}$ be an orthonormal basis for $L^2([a,b])$. (See 22.8(b), 22.8(c) and Problems 22-13, 22-14, 22-15.) For $i,j = 1,2,\ldots$, let $w_{i,j}(s,t) = v_i(s)\overline{v_j(t)}, s,t \in [a,b]$. Then $\{w_{i,j} : i,j = 1,2,\ldots\}$ is an orthonormal basis for $L^2([a,b]\times[a,b])$, as we shall see in 22.8(d). By 22.7(ii),

$$k = \sum_{i,j} \langle k,\, w_{i,j}\rangle w_{i,j}.$$

For $n = 1,2,\ldots$ and $s,t \in [a,b]$, let

$$k_n(s,t) = \sum_{i,j=1}^{n} \langle k,\, w_{i,j}\rangle w_{i,j}(s,t) = \sum_{i,j=1}^{n} \langle k,\, w_{i,j}\rangle v_i(s)\overline{v_j(t)},$$

where

$$\langle k,\, w_{i,j}\rangle = \int_a^b \int_a^b k(s,t)\overline{v_i(s)}v_j(t)\, dm(t)\, dm(s), \quad i,j = 1,2,\ldots.$$

Then $k_n(.,.)$ is a degenerate kernel and $\|k - k_n\|_2 \to 0$ as $n \to \infty$.

20.5 Examples

(a) For $i,j = 1,2,\ldots$, let $k_{i,j} \in \mathbf{K}$ satisfy $\sum_{i,j=1}^{\infty} |k_{i,j}|^2 < \infty$. Assume that the only square-summable solution of the denumerable homogeneous system

$$\begin{array}{ccccccccc} x(1) & - & k_{1,1}x(1) & - & k_{1,2}x(2) & - & \cdots & = & 0 \\ x(2) & - & k_{2,1}x(1) & - & k_{2,2}x(2) & - & \cdots & = & 0 \\ \vdots & & \vdots & & \vdots & & & & \vdots \end{array} \qquad \cdots [0]$$

is the zero solution. Let $y = (y(1), y(2), \ldots) \in \ell^2$.

For $n = 1, 2, \ldots$, consider the following system of n linear equations in n unknowns:

$$\begin{array}{ccccccccc} x(1) & - & k_{1,1}x(1) & - & \cdots & - & k_{1,n}x(n) & = & y(1) \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ x(n) & - & k_{n,1}x(1) & - & \cdots & - & k_{n,n}x(n) & = & y(n) \end{array} \qquad \cdots [y]_n$$

Then for all sufficiently large n, the system $[y]_n$ has a unique solution $x_n \in \mathbf{K}^n$. If we let $x_n(i) = 0$ for $i = n+1, n+2, \ldots$, then $x_n \in \ell^2$ and the sequence (x_n) converges in ℓ^2 to the unique solution $x = (x(1), x(2), \ldots)$ of the denumerable system

$$\begin{array}{ccccccccc} x(1) & - & k_{1,1}x(1) & - & k_{1,2}x(2) & - & \cdots & = & y(1) \\ x(2) & - & k_{2,1}x(1) & - & k_{2,2}x(2) & - & \cdots & = & y(2) \\ \vdots & & \vdots & & \vdots & & & & \vdots \end{array}$$

In fact,

$$\|x - x_n\| \le \frac{\|(I-A)^{-1}\|_2}{1-\epsilon_n}(\epsilon_n\|y\|_2 + \|y - y_n\|_2),$$

provided $\epsilon_n = \|A - A_n\| < 1/\|(I-A)^{-1}\|$, where

$$A(x)(i) = \sum_{j=1}^{\infty} k_{i,j}x(j), \quad x \in \ell^2, i = 1, 2, \ldots,$$

$$A_n(x)(i) = \begin{cases} \sum_{j=1}^{n} k_{i,j}x(j), & \text{if } i = 1, \ldots, n \\ 0, & \text{if } i > n. \end{cases}$$

These results follow from 20.1 and 20.3 if we note that A is a compact operator (17.4(a)) and if

$$P_n(x) = (x(1), \ldots, x(n), 0, 0, \ldots), \quad x \in X,$$

then $A_n = P_nAP_n = A_n^G$. Since P_n is obtained by truncating the Fourier expansion of $x \in \ell^2$, we see that $P_n(x) \to x$ for every x in ℓ^2 and $P_n'(x') \to x'$ for every x' in $(\ell^2)'$. Hence $\|A - A_n^G\|_2 \to 0$. We remark that $(x(1), \ldots, x(n))$ is a solution of the system $[y]_n$ if and only if $(x(1), \ldots, x(n), 0, 0, \ldots)$ is a solution of the system $x - A_n(x) = (y(1), \ldots, y(n), 0, 0, \ldots)$, $x \in \ell^2$.

Let us illustrate this procedure in a specific case. Let the matrix $(k_{i,j})$ be

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \end{bmatrix}$$

Then $\sum_{i,j=1}^{\infty} |k_{i,j}|^2 = \sum_{i=1}^{\infty} 1/i^2 = \pi^2/6 < \infty$. Also, $A(x) = x$ means

$$(x(2), x(3)/2, x(4)/3, \ldots) = (x(1), x(2), x(3), \ldots),$$

so that $x(2) = x(1), x(3) = 2x(1), x(4) = 3!x(1)$, etc. For $x \in \ell^2$, this is possible only for $x = 0$.

Let $y = (1/2, 1/3, 1/4 \ldots) \in \ell^2$. To find approximate solutions of the equation $x - A(x) = y$, we fix a positive integer n, and solve

$$x(1) - x(2) = \frac{1}{2}, \qquad x(2) - \frac{x(3)}{2} = \frac{1}{3}, \ldots,$$

$$x(n-1) - \frac{x(n)}{n-1} = \frac{1}{n}, \qquad x(n) = \frac{1}{n+1}.$$

If $x_n \in \mathbf{K}^n$ is the solution of this system, we have

$$x_n(j) = \frac{1}{j+1} + \frac{x_n(j+1)}{j} \text{ for } j = 1 \ldots, n-1 \text{ and } x_n(n) = \frac{1}{n+1}.$$

By back substitution, we obtain $x_n(j) = \dfrac{1}{j} - \dfrac{(j-1)!}{(n+1)!}$ for $j = 1, \ldots, n$. Letting $x_n(j) = 0$ for $j = n+1, n+2, \ldots$, we see that $x_n \to (1, 1/2, 1/3, \ldots)$ in ℓ^2 . Thus we have found the unique solution $x = (1, 1/2, 1/3, \ldots)$ in ℓ^2 of the equation $x - A(x) = (1/2, 1/3, 1/4, \ldots)$.

(b) Let $k(s,t) = s^2t + st^2$ for $s, t \in [0,1]$ and consider the integral equation

$$x(s) - \mu \int_0^1 (s^2t + st^2)x(t)\,dt = y(s), \quad s \in [0,1],$$

where $\mu \in \mathbf{K}$ and $x, y \in C([0,1])$. Since $k(.,.)$ is a degenerate kernel, the Fredholm integral operator given by

$$A_\mu(x)(s) = \int_0^1 \mu(s^2t + st^2)x(t)\,dt, \quad x \in C([0,1]), \quad s \in [0,1],$$

is of finite rank. In fact,

$$A_\mu(x) = f_1(x)x_1 + f_2(x)x_2,$$

where $x_1(s) = s^2$, $x_2(s) = s$ for $s \in [0,1]$, and

$$f_1(x) = \mu \int_0^1 t\,x(t)\,dt, \quad f_2(x) = \mu \int_0^1 t^2 x(t)\,dt$$

for all $x \in C([0,1])$. Then

$$M_\mu = \begin{bmatrix} f_1(x_1) & f_1(x_2) \\ f_2(x_1) & f_2(x_2) \end{bmatrix} = \mu \begin{bmatrix} 1/4 & 1/3 \\ 1/5 & 1/4 \end{bmatrix}.$$

It follows from 20.2 that $A_\mu(x) = x$ has a nonzero solution in X if and only if $M_\mu u = u$ has a nonzero solution $u \in \mathbf{K}^2$, that is, $\det(I - M_\mu) = (1 - \mu/4)^2 - \mu^2/15 = 0$, or $\mu = -60 \pm 16\sqrt{15}$.

Let $\mu \neq -60 \pm 16\sqrt{15}$. Consider $y_0 \in C([0,1])$. Then by 20.2, the unique element x_0 of $C([0,1])$ satisfying $x_0 - A_\mu(x_0) = y_0$ is given by

$$\begin{aligned} x_0(s) &= y_0(s) + u_0(1)x_1(s) + u_0(2)x_2(s) \\ &= y_0(s) + u_0(1)s^2 + u_0(2)s \end{aligned}$$

for $s \in [0,1]$, where $u_0 = (u_0(1), u_0(2))$ satisfies

$$u_0 - M_\mu u_0 = v_0 = (f_1(y_0), f_2(y_0)),$$

that is,

$$\begin{aligned} u_0(1) - \mu\left[\frac{u_0(1)}{4} + \frac{u_0(2)}{3}\right] &= \mu \int_0^1 t\,y_0(t)dt, \\ u_0(2) - \mu\left[\frac{u_0(1)}{5} + \frac{u_0(2)}{4}\right] &= \mu \int_0^1 t^2 y_0(t)\,dt. \end{aligned}$$

If $\mu = -60+16\sqrt{15}$, then the homogeneous equation $u - M_\mu u = 0$ has $u = c(1, \sqrt{3/5})$ as the general solution, where c is a scalar. Hence

$$x(s) = c[u(1)x_1(s) + u(2)x_2(s)] = c(s^2 + \sqrt{3/5}\,s), \quad s \in [0,1],$$

is the general solution of $x - A_\mu(x) = 0$.

Since $k(s,t) = k(t,s)$ for all $s,t \in [0,1]$, it follows from 19.3(c) that the integral equation

$$x(s) - (-60 + 16\sqrt{15}) \int_0^1 (s^2 t + st^2)x(t)\,dt = y(s), \quad s \in [0,1],$$

has a solution for exactly those $y \in C([0,1])$ which satisfy

$$\int_0^1 (t^2 + \sqrt{3/5}\,t)y(t)\,dt = 0.$$

For such a function y, the general solution is given by

$$x(s) = x_0(s) + c(s^2 + \sqrt{3/5}\,s), \quad s \in [0,1],$$

where $x_0(s)$ is a particular solution and c is a scalar. Note that x_0 can be computed by solving $u_0 - M_\mu u_0 = (f_1(y), f_2(y))$ as in the previous case.

Similar considerations apply if $\mu = -60 - 16\sqrt{15}$.

(c) Let $X = C([0,1])$ and

$$k(s,t) = \sin\frac{st}{2}, \quad s,t \in [0,1].$$

We have the double Fourier series expansion

$$k(s,t) = \frac{st}{2} - \frac{s^3t^3}{3!8} + \frac{s^5t^5}{5!32} - \cdots,$$

which converges uniformly and absolutely for $s,t \in [0,1]$. As a first approximation of $k(s,t)$, consider

$$k_0(s,t) = \frac{st}{2}, \quad s,t \in [0,1].$$

Since

$$u - \frac{u^3}{3!} \le \sin u \le u - \frac{u^3}{3!} + \frac{u^5}{5!}$$

for all $u \ge 0$, we have

$$\|k\|_\infty = \sup\{|\sin \frac{st}{2}| : 0 \le s, t \le 1\} \le \sin \frac{1}{2} \le \frac{48}{100}$$

and

$$\|k_0 - k\|_\infty = \sup\{|\frac{st}{2} - \sin \frac{st}{2}| : 0 \le s, t \le 1\} \le \frac{1}{2} - \sin \frac{1}{2} \le \frac{1}{48}.$$

Let A and A_0 denote the Fredholm integral operators on $C([0,1])$ with kernels $k(.,.)$ and $k_0(.,.)$ respectively. Then $\|A\| \le \|k\|_\infty \le 48/100$, which is less than 1. Hence $I - A$ is invertible and

$$\|(I-A)^{-1}\| \le \frac{1}{1 - \|A\|} \le \frac{100}{52}.$$

Also, since $\|A - A_0\| \le \|k - k_0\|_\infty \le 1/48$, we have

$$\epsilon = \|(A - A_0)(I - A)^{-1}\| \le \|A - A_0\| \, \|(I-A)^{-1}\| \le \frac{1}{48}\frac{100}{52} = \frac{25}{624}.$$

Hence 20.1 shows that for every $y \in C([0,1])$, there is a unique x in $C([0,1])$ such that $x - A(x) = y$, and that for every $y_0 \in C([0,1])$, there is a unique $x_0 \in C([0,1])$ such that $x_0 - A_0(x_0) = y_0$ with

$$\begin{aligned} \|x - x_0\|_\infty &\le \frac{100}{52}\frac{1}{1 - \frac{25}{624}}\left(\frac{25}{624}\|y\|_\infty + \|y - y_0\|_\infty\right) \\ &= \frac{25}{7787}(25\|y\|_\infty + 624\|y - y_0\|_\infty) \\ &\le 0.081\|y\|_\infty + 2.004\|y - y_0\|_\infty. \end{aligned}$$

We illustrate the procedure of finding an approximate solution with the help of the degenerate kernel $k_0(s,t) = st/2$. Let $x(s) = 1$ for all $s \in [0,1]$. Then $x - A(x) = y$, where

$$y(s) = 1 - \int_0^1 \sin \frac{st}{2}\, dt = \begin{cases} 1 - \frac{2}{s}(1 - \cos \frac{2}{s}), & \text{if } s \ne 0 \\ 1 & \text{if } s = 0. \end{cases}$$

Now $k_0(s,t) = x_1(s)y_1(t)$, where $x_1(s) = s/2$ and $y_1(t) = t$ for s, t in $[0,1]$.

Let $y_0 = y$ and for $x \in C([0,1])$,

$$A_0(x)(s) = \int_0^1 x_1(s)y_1(t)x(t)\,dt = f_1(x)x_1(s), \quad s \in [0,1],$$

where $f_1(x) = \int_0^1 y_1(t)x(t)\,dt = \int_0^1 t\,x(t)\,dt$. Since $\int_0^1 (t^2/2)\,dt = 1/6$, we see that $M = [f_1(x_1)] = [1/6]$ in 20.2. Hence the function x_0 in $C([0,1])$ satisfying $x_0 - A_0(x_0) = y$ is given by

$$x_0(s) = y(s) + u_0(1)x_1(s), \quad s \in [0,1],$$

where $u_0(1)$ satisfies

$$u_0(1) - M u_0(1) = f_1(y) = \int_0^1 t\,y(t)\,dt,$$

that is,

$$u_0(1)(1 - \frac{1}{6}) = \int_0^1 t\left[1 - \frac{2}{t}(1 - \cos\frac{t}{2})\right] dt.$$

This gives

$$u_0(1) = \frac{6}{5}\left(-\frac{3}{2} + 4\sin\frac{1}{2}\right),$$

so that for $0 < s \le 1$, we have

$$\begin{aligned} x_0(s) &= 1 + \frac{2}{s}\left(\cos\frac{s}{2} - 1\right) + \frac{6}{5}\left(-\frac{3}{2} + 4\sin\frac{1}{2}\right)\frac{s}{2} \\ &= 1 + \frac{2}{s}\left(-\frac{s^2}{8} + \frac{s^4}{384} - \cdots\right) - \frac{9s}{10} + \frac{12s}{5}\sin\frac{1}{2} \\ &= 1 + \left(\frac{12}{5}\sin\frac{1}{2} - \frac{23}{20}\right)s + \frac{s^3}{192} - \cdots. \end{aligned}$$

We have thus found an approximate solution x_0 of the operator equation $x - A_0(x) = y$ explicitly. Now $x(s) = 1, s \in [0,1]$, is the exact solution and

$$\begin{aligned} |x(s) - x_0(s)| &\le \frac{12}{5}\left(\sin\frac{1}{2} - \frac{23}{48}\right)s + \frac{s^3}{192} \\ &\le 0.002\,s + 0.006\,s^3, \end{aligned}$$

since $23/48 \leq \sin(1/2) \leq 48/100$. (Note: $1/2 - (1/2)^3/3! = 23/48$.) Thus

$$\|x - x_0\|_\infty \leq 0.008.$$

Our earlier estimate, which was obtained without calculating x_0 explicitly, merely gives

$$\|x - x_0\|_\infty \leq 0.081\|y\|_\infty + 2.004\|y - y_0\|_\infty \leq 0.081,$$

since it can be easily checked that $|y(s)| \leq 1$ for all $s \in [0,1]$ and since we have taken $y_0 = y$.

Problems

20-1 Let X be a Banach space and $A, A_0 \in BL(X)$. For given $y, y_0 \in X$, there are unique $x, x_0 \in X$ such that $x - A(x) = y$ and $x_0 - A_0(x_0) = y_0$ in each of the following cases.

(a) A_0 is of finite rank, 1 is not an eigenvalue of A_0 and $\delta = \|(A_0 - A)(I - A_0)^{-1}\| < 1$. In this case,

$$\|x - x_0\| \leq \frac{\|(I - A_0)^{-1}\|}{1 - \delta}(\delta\|y_0\| + \|y - y_0\|).$$

(b) A is compact, 1 is not an eigenvalue of A and $\epsilon = \|(A - A_0)(I - A)^{-1}\| < 1$. In this case,

$$\|x - x_0\| \leq \frac{\|(I - A)^{-1}\|}{1 - \epsilon}[\epsilon\|y_0\| + (1 - \epsilon)\|y - y_0\|].$$

(c) Let $\|A - A_0\| = \beta, 1 - \|A\| = \gamma$ and $\beta < \gamma$. In this case,

$$\|x - x_0\| \leq \frac{\min\{\beta\|y\| + \gamma\|y - y_0\|, \beta\|y_0\| + (\gamma - \beta)\|y - y_0\|\}}{\gamma(\gamma - \beta)}.$$

(Hint: 12.3, 12.5(a), Problem 12-13 and the proof of 20.1)

20-2 Let $A_0 = (1/ij), i, j = 1, 2, \ldots$ and $y_0 \in \ell^2$. Then the unique $x_0 \in \ell^2$ satisfying $x_0 - A_0(x_0) = y_0$ is given by

$$x_0 = y_0 + \left[\frac{6}{6 - \pi^2}\sum_{j=1}^{\infty}\frac{y_0(j)}{j}\right](1, \frac{1}{2}, \frac{1}{3}, \ldots).$$

(Hint: $A_0(x) = f_1(x)x_1$, where $f_1(x) = \sum_{j=1}^{\infty} x(j)/j$ for $x \in \ell^2$ and $x_1 = (1, 1/2, 1/3, \ldots)$ as in 20.2(a))

20-3 Let $X = C([a,b])$ or $L^2([a,b])$. Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ be in X. Fix $0 \neq \mu \in \mathbf{K}$. For $y \in X$, consider the integral equation

$$x(s) - \mu \sum_{i=1}^{m} \left(\int_a^b y_i(t)x(t)dm(t) \right) x_i(s) = y(s) \qquad \cdots [y]$$

For every $y \in X$, it has a unique solution in X if and only if $1/\mu$ is not an eigenvalue of $M = (k_{i,j})$, where

$$k_{i,j} = \int_a^b y_i(t)x_j(t)dm(t), \quad i, j = 1, \ldots, m.$$

If $1/\mu$ is an eigenvalue of M, then the integral equation has a solution for precisely those functions y in X which satisfy

$$\left[\int_a^b y_1(t)y(t)dm(t) \right] z(1) + \cdots + \left[\int_a^b y_m(t)y(t)dm(t) \right] z(m) = 0$$

for every $z = (z(1), \ldots, z(m))$ satisfying $z = \mu M^t z$.

Every solution of the integral equation $[y]$ is given by

$$x(s) = y(s) + u(1)x_1 + \cdots + u(m)x_m,$$

where $u = (u(1), \ldots, u(m))$ satisfies

$$u - \mu M = \mu \left[\int_a^b y_1(t)y(t)dm(t), \ldots, \int_a^b y_m(t)y(t)\, dm(t) \right].$$

20-4 Let $X = C([0,1])$ or $L^2([0,1])$. Fix $0 \neq \mu \in \mathbf{K}$. The integral equation

$$x(s) - \mu \int_0^1 (s-t)x(t)dt = s, \quad s \in [0,1],$$

has no solution if $\mu^2 + 12 = 0$. If $\mu^2 + 12 \neq 0$, then it has a unique solution given by

$$x(s) = \frac{6(\mu+2)s - 4\mu}{\mu^2 + 12}, \quad s \in [0,1].$$

20-5 Let $k(.,.) \in C([a,b] \times [a,b])$ and A denote the corresponding Fredholm integral operator. If $k_n(.,.)$ denotes the nth degenerate kernel obtained by interpolating in the first variable in a piecewise linear way, then

- $A_n^D = P_n A = A_n^P$, where P_n is the nth piecewise linear interpolatory projection.

20-6 Let

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & 1 & -\frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{1}{5} & 1 & -\frac{1}{3} & 0 & \dots \\ 0 & 0 & \frac{1}{6} & 1 & -\frac{1}{4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Then for every $y \in \ell^2$, there is a unique $x \in \ell^2$ such that $Lx = y$. Let $y \in \ell^2$ be defined by $y(1) = 1/6$ and $y(j) = 1/(j+1)$ for $j = 2, 3, \dots$. The approximate solutions of $Lx = y$ obtained by truncating L to $1\times1, 2\times2, 3\times3$ and 4×4 matrices are given (correct up to four decimal places) by $x_1 = (.1667, 0, 0, \dots), x_2 = (.4, .2333, 0, 0, \dots), x_3 = (.4753, .3086, .1883, 0, 0, \dots)$ and $x_4 = (.4951, .3284, 2378, .0299, 0, 0, \dots)$, while the exact solution is $x = (1/2, 1/3, 1/4, \dots) = (.5, .3333, .25, \dots)$.

20-7 (a) **(Erdös-Turan)** For $n = 1, 2, \dots$, let $t_1^{(n)}, \dots, t_n^{(n)}$ be the roots in $[-1, 1]$ of the nth Legendre polynomial. For $x \in C([-1, 1])$, let

$$P_n(x)(t) = \sum_{k=1}^{n} x(t_k^{(n)})\ell_k^{(n)}(t), \quad t \in [-1, 1],$$

where $\ell_k^{(n)}, k = 1, \dots, n$, are the Lagrange interpolatory polynomials introduced in 9.6(a). Then $\|P_n(x) - x\|_2 \to 0$ as $n \to \infty$.

(b) **(Vainikko)** Let A be a compact operator on $L^2([-1, 1])$ such that $R(A) \subset C([-1, 1])$ and let $A_n^P = P_n A$. Then $\|A - A_n^P\| \to 0$ as $n \to \infty$.

20-8 In 20.5(c), let $k_0(s,t) = (st/2) - (s^3t^3/48)$ instead of $k_0(s,t) = st/2$ for $s, t \in [0, 1]$. Then $\|x - x_0\|_\infty \le 0.00097$.

20-9 Let $X = C([0,1]), k(\cdot,\cdot) \in C([0,1]) \times [0,1])$ and A be the Fredholm integral operator with kernel $k(\cdot,\cdot)$.

(a) If $|\mu| < 1/\|k\|_\infty$, then for every $y \in X$, there is a unique $x \in X$ such that $x - \mu A(x) = y$. Further, if we let

$$x_n(s) = y(s) + \sum_{j=1}^{n} \mu^j \left[\int_0^1 k^{(j)}(s,t) y(t) dm(t)\right], \quad s \in [0, 1],$$

where $k^{(j)}(.,.)$ is the jth iterated kernel, then

$$\|x_n - x\|_\infty \le \frac{\|y\|_\infty |\mu|^{n+1} \|k\|_\infty^{n+1}}{1 - |\mu| \, \|k\|_\infty}$$

(b) If $k(s,t) = 0$ for all $s \le t$ and $0 \ne \mu \in \mathbf{K}$, then

$$\|x_n - x\|_\infty \le \|y\|_\infty \sum_{j=n+1}^{\infty} \frac{|\mu|^j \, \|k\|_\infty^j}{j!}$$

(Hint: 12.4 and Problem 12-12)

Chapter VI

Geometry of Hilbert Spaces

This chapter aims at providing a geometric structure to a linear space. The basic concept of an inner product is introduced in Section 21. An inner product induces a norm on the linear space. If such a space is complete, then it is known as a Hilbert space. The notion of orthogonality of two elements of an inner product space is discussed in Section 22. It leads to an expansion of an element of a Hilbert space in terms of an orthonormal basis. The problem of finding a best approximation from a given subset of an inner product space X to an element x in X is considered in Section 23. If the given subset is a space F of X, then the problem reduces to finding an element y in F such that $x - y$ is orthogonal to F. Satisfactory answers are obtained when the subspace F is finite dimensional or has a finite codimension. Two major theorems known as the projection theorem and the Riesz representation theorem are proved in Section 24. The former allows the decomposition of a Hilbert space into a closed subspace and its orthogonal complement, while the latter shows that every continuous linear functional on a Hilbert space is obtained by taking inner products with a fixed element of that space. This simplifies the considerations of weak convergence and weak boundedness and yields a uniform boundedness principle for continuous linear functionals on a Hilbert space.

21 Inner Product Spaces

From this section onwards, we study a new structure on a linear space. This structure will be useful for introducing geometric concepts like perpendicularity in the setting of a linear space.

We begin our study by considering the dot product of two vectors $x = (x(1), x(2))$ and $y = (y(1), y(2))$ in $\mathbf{R}^2$, namely,

$$x \cdot y = x(1)y(1) + x(2)y(2).$$

If $x \neq 0$ and $y \neq 0$, then the 'cosine rule' gives

$$x \cdot y = \sqrt{x(1)^2 + x(2)^2}\,\sqrt{y(1)^2 + y(2)^2}\,\cos\theta,$$

where θ is the angle between x and y satisfying $0 \leq \theta \leq \pi$. Thus the dot product of two nonzero vectors in $\mathbf{R}^2$ is intimately connected with the angle between them.

By requiring some crucial properties of the dot product to hold, we shall define a similar operation on an arbitrary linear space over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. While stating our definitions and results we make the convention that the bar over a scalar k (denoting the conjugate $\overline{k}$ of k) as well as all the terms involving $i = \sqrt{-1}$ are to be dropped when $\mathbf{K} = \mathbf{R}$. Also, for $k \in \mathbf{C}$, we write $k \geq 0$ when k is a nonnegative real number.

Let X be a linear space over $\mathbf{K}$. An **inner product** on X is a function $\langle\ ,\ \rangle$ from $X \times X$ to $\mathbf{K}$ such that for all x, y, z in X and k in $\mathbf{K}$, we have

(i) positive-definiteness:

$$\langle x,\, x\rangle \geq 0 \quad \text{and} \quad \langle x,\, x\rangle = 0 \text{ if and only if } x = 0,$$

(ii) linearity in the first variable:

$$\langle x + y,\, z\rangle = \langle x,\, z\rangle + \langle y,\, z\rangle \quad \text{and} \quad \langle kx,\, y\rangle = k\langle x,\, y\rangle,$$

(iii) conjugate-symmetry:

$$\langle y,\, x\rangle = \overline{\langle x,\, y\rangle}.$$

It follows that an inner product is **conjugate-linear in the second variable**:

$$\langle x,\, y + z\rangle = \langle x,\, y\rangle + \langle x,\, z\rangle \quad \text{and} \quad \langle x,\, ky\rangle = \overline{k}\langle x,\, y\rangle.$$

An **inner product space** is a linear space with an inner product on it.

21.1 Lemma

Let $\langle\ ,\ \rangle$ be an inner product on a linear space X.

(a) (Polarization identity) For all $x, y \in X$,

$$\begin{aligned} 4\langle x,\, y\rangle &= \langle x+y,\, x+y\rangle - \langle x-y,\, x-y\rangle \\ &\qquad + i\langle x+iy,\, x+iy\rangle - i\langle x-iy,\, x-iy\rangle. \end{aligned}$$

(b) Let $x \in X$. Then $\langle x,\, y\rangle = 0$ for all $y \in X$ if and only if $x = 0$.

(c) (Schwarz inequality) For all $x, y \in X$,

$$|\langle x,\, y\rangle|^2 \le \langle x,\, x\rangle\langle y,\, y\rangle,$$

where equality holds if and only if the set $\{x, y\}$ is linearly dependent.

Proof:

(a) Since $\langle\ ,\ \rangle$ is linear in the first variable and conjugate-linear in the second, it is easy to check that the right side of the desired identity reduces to $4\langle x,\, y\rangle$.

(b) If $x = 0$, then $\langle 0,\, y\rangle = \langle 0+0,\, y\rangle = \langle 0,\, y\rangle + \langle 0,\, y\rangle$, so that $\langle 0,\, y\rangle = 0$ for all $y \in X$. Conversely, let $\langle x,\, y\rangle = 0$ for all $y \in X$. Then, in particular, we have $\langle x,\, x\rangle = 0$. By the positive-definiteness of $\langle\ ,\ \rangle$, we see that $x = 0$.

(c) Let $x, y \in X$ and consider $z = \langle y,\, y\rangle x - \langle x,\, y\rangle y$. Then

$$\begin{aligned} 0 \le \langle z,\, z\rangle &= \langle y,\, y\rangle^2\langle x,\, x\rangle - \langle y,\, y\rangle\overline{\langle x,\, y\rangle}\langle x,\, y\rangle \\ &\qquad -\langle x,\, y\rangle\langle y,\, y\rangle\langle y,\, x\rangle + \langle x,\, y\rangle\overline{\langle x,\, y\rangle}\langle y,\, y\rangle \\ &= \langle y,\, y\rangle\left(\langle x,\, x\rangle\langle y,\, y\rangle - |\langle x,\, y\rangle|^2\right). \end{aligned}$$

If $\langle y,\, y\rangle > 0$, then it follows that $\langle x,\, x\rangle\langle y,\, y\rangle - |\langle x,\, y\rangle|^2 \ge 0$, as desired. If $\langle y,\, y\rangle = 0$, then $y = 0$ and hence $\langle x,\, y\rangle = 0$ by (b) above, so that

$|\langle x,\, y\rangle|^2 = 0 = \langle x,\, x\rangle\langle y,\, y\rangle$. The elements x, y and z are schematically shown in Figure 8.

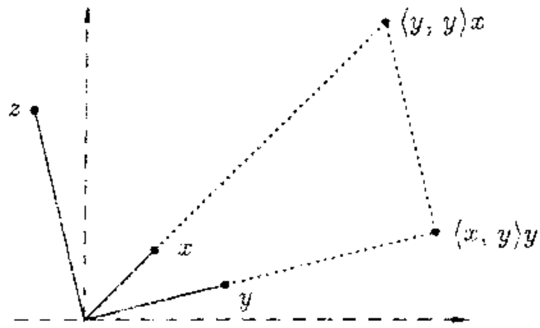


Figure 8

Next, let $|\langle x,\, y\rangle|^2 = \langle x,\, x\rangle\langle y,\, y\rangle$. Then we have $\langle z,\, z\rangle = 0$, so that $z = 0$, that is, $\langle y,\, y\rangle x = \langle x,\, y\rangle y$. Hence the set $\{x, y\}$ is linearly dependent. Conversely, if the set $\{x, y\}$ is linearly dependent, then either $x = ky$ or $y = kx$ for some $k \in \mathbf{K}$. In this case, we can readily see that $|\langle x,\, y\rangle|^2 = \langle x,\, x\rangle\langle y,\, y\rangle$. $\square$

The polarization identity shows that an inner product $\langle\ ,\ \rangle$ on a linear space X is determined by the diagonal entries $\langle z,\, z\rangle, z \in X$. The result 21.1(b) implies that if $x, y \in X$ and $\langle x,\, z\rangle = \langle y,\, z\rangle$ for all $z \in X$, then $x = y$, that is, an element x in X is determined by the scalars $\langle x,\, y\rangle, y \in X$.

We are now in a position to show that an inner product $\langle\ ,\ \rangle$ on a linear space X induces a norm $\|\ \ \|$ on X in a canonical way. The Schwarz inequality (21.1(c)) is crucially used here. We note some interesting geometric properties of such a norm. We also prove that if X is endowed with the metric $d(x, y) = \|x - y\|$, $x, y \in X$, then the function $\langle\ ,\ \rangle$ is continuous on $X \times X$.

21.2 Theorem

Let $\langle\ ,\ \rangle$ be an inner product on a linear space X. For $x \in X$, define $\|x\| = \langle x,\, x\rangle^{1/2}$, the nonnegative square root of $\langle x,\, x\rangle$. Then

$$|\langle x,\, y\rangle| \le \|x\|\, \|y\| \quad \text{for all } x, y \in X.$$

The function $\|\ \| : X \to \mathbf{K}$ is a norm on X, that is, for all $x, y \in X$ and $k \in \mathbf{K}$, we have

$$\begin{aligned} \|x\| \ge 0 \quad \text{and} \quad \|x\| = 0 \quad \text{if and only if} \quad x = 0, \\ \|x+y\| \le \|x\| + \|y\|, \\ \|kx\| = |k|\, \|x\|. \end{aligned}$$

Also, the following results hold.

(a) If $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, then $\langle x_n,\, y_n\rangle \to \langle x,\, y\rangle$.

(b) (Parallelogram law) For all $x, y \in X$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

(c) The linear space X is **uniformly convex** in the norm $\|\ \|$, that is, for every $\epsilon > 0$ there is some $\delta > 0$ such that for all $x, y \in X$ with $\|x\| \le 1, \|y\| \le 1$ and $\|x - y\| \ge \epsilon$, we have $\|x+y\| \le 2 - 2\delta$.

Proof:
Let $x, y \in X$. The Schwarz inequality (21.1(c)) says that $|\langle x,\, y\rangle|^2 \le \|x\|^2\|y\|^2$, that is, $|\langle x,\, y\rangle| \le \|x\|\, \|y\|$. Next, $\|x\| = \langle x,\, x\rangle^{1/2} \ge 0$ and $\|x\| = 0$ if and only if $\langle x,\, x\rangle = 0$, that is , $x = 0$ by the positive-definiteness of $\langle\ ,\ \rangle$.

Also,

$$\begin{aligned} \|x+y\|^2 &= \langle x+y,\, x+y\rangle \\ &= \langle x,\, x\rangle + \langle x,\, y\rangle + \langle y,\, x\rangle + \langle y,\, y\rangle \\ &= \|x\|^2 + 2\mathrm{Re}\,\langle x,\, y\rangle + \|y\|^2 \\ &\le \|x\|^2 + 2|\langle x,\, y\rangle| + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Thus $\|x+y\| \le \|x\| + \|y\|$. Finally, for $k \in \mathbf{K}$, we have

$$\|kx\|^2 = \langle kx,\, kx\rangle = k\bar{k}\langle x,\, x\rangle = |k|^2\|x\|^2,$$

so that $\|kx\| = |k|\,\|x\|$. Hence $\|\ \ \|$ is a norm on X.

(a) Let $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. Then

$$\begin{aligned} |\langle x_n,\, y_n\rangle - \langle x,\, y\rangle| &\le |\langle x_n,\, y_n\rangle - \langle x_n,\, y\rangle| + |\langle x_n,\, y\rangle - \langle x,\, y\rangle| \\ &= |\langle x_n,\, y_n - y\rangle| + |\langle x_n - x,\, y\rangle| \\ &\le \|x_n\|\,\|y_n - y\| + \|x_n - x\|\,\|y\| \end{aligned}$$

by the Schwarz inequality (21.1(c)). Since $(\|x_n\|)$ is a bounded sequence, we see that $\langle x_n,\, y_n\rangle \to \langle x,\, y\rangle$.

(b) Let $x, y \in X$. Then

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \langle x+y,\, x+y\rangle + \langle x-y,\, x-y\rangle \\ &= \langle x,\, x\rangle + \langle x,\, y\rangle + \langle y,\, x\rangle + \langle y,\, y\rangle \\ &\quad + \langle x,\, x\rangle - \langle x,\, y\rangle - \langle y,\, x\rangle + \langle y,\, y\rangle \\ &= 2\langle x,\, x\rangle + 2\langle y,\, y\rangle \\ &= 2\left(\|x\|^2 + \|y\|^2\right). \end{aligned}$$

(c) Let $\epsilon > 0$. Consider $x, y \in X$ with $\|x\| \le 1, \|y\| \le 1$ and $\|x-y\| \ge \epsilon$. Then $\epsilon \le 2$. The parallelogram law gives

$$\|x+y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) - \|x-y\|^2 \le 4 - \epsilon^2.$$

Hence $\|x+y\| \le \sqrt{4-\epsilon^2} = 2 - 2\delta$, if we let $\delta = 1 - (1-\epsilon^2/4)^{1/2}$, which is a positive number. □

If $\|\ \ \|$ is a norm on a linear space X which satisfies the parallelogram law (21.2(b)) and if we define for $x, y \in X$,

$$\langle x,\, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

in accordance with the polarization identity (21.1(a)), then it can be shown that $\langle\ ,\ \rangle$ is an inner product on the linear space X and it satisfies $\langle x, x\rangle^{1/2} = \|x\|$ for all $x \in X$. This result, proved by Jordan and von Neumann, characterizes inner product spaces among all normed spaces. Since we shall have no occasion to use this result , we leave it to Problem 21-4.

An inner product space which is complete in the norm induced by the inner product is called a **Hilbert space**. (Equivalently, the result of Jordan and von Neumann shows that a Hilbert space is a complete normed space in which the norm satisfies the parallelogram law.) In the sequel we reserve the letter H for a Hilbert space. Note that an inner product space X can be **completed** to a Hilbert space H, that is, we can find a Hilbert space H and a linear map $F : X \to H$ such that $\langle F(x), F(y)\rangle = \langle x, y\rangle$ for all $x, y \in X$ and $R(F)$ is dense in H. A construction of such a completion along with its essential uniqueness is outlined in Problem 21-9. A result in Section 8 which employs the canonical embedding J of X into its second dual X'' can also be used for this purpose by noting that if $J(x_n) \to x''$ and $J(y_n) \to y''$ in X'', then we can define $\langle x'', y''\rangle = \lim_{n\to\infty}\langle x_n, y_n\rangle$.

It is easy to see that a subspace of a Hilbert space is a Hilbert space with the induced inner product if and only if the subspace is closed. Also, it is possible to construct a product Hilbert space from a sequence of Hilbert spaces in the same way as ℓ^2 is constructed from the sequence $\mathbf{K}, \mathbf{K}, \ldots$. (See Problem 21-10.) The quotient of a Hilbert space by one of its closed subspaces is again a Hilbert space. (See Problem 24-12(c).)

21.3 Examples

(a) Let $H = \mathbf{K}^n$. For $x = (x(1), \ldots, x(n))$, $y = (y(1), \ldots, y(n))$ in H, define

$$\langle x, y\rangle = \sum_{j=1}^{n} x(j)\overline{y(j)}.$$

It is easy to see that $\langle\ ,\ \rangle$ is an inner product on H and

$$\|x\|_2 = \langle x,\, x\rangle^{1/2} = \sqrt{x(1)^2 + \cdots + x(n)^2}, \quad x \in H.$$

We have seen in Section 3 that H is complete in the metric $d_2(x,y) = \|x - y\|_2$. Hence H is a Hilbert space. Among all the norms $\|\ \ \|_p, 1 \le p \le \infty$, on $\mathbf{K}^n (n \ge 2)$, only the norm $\|\ \ \|_2$ is induced by an inner product, because if $p \neq 2$ and we let $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$, then

$$\|x + y\|_p^2 + \|x - y\|_p^2 = 2^{1+(2/p)} \neq 4 = 2(\|x\|_p^2 + \|y\|_p^2),$$

so that the parallelogram law 21.2(b) does not hold.

Another example of an inner product on $\mathbf{K}^2$ is given by

$$\langle x,\, y\rangle = \frac{x(1)\overline{y(1)}}{a^2} + \frac{x(2)\overline{y(2)}}{b^2},$$

where a and b are real numbers. (See Problems 21-6 and 21-7.)

Here is an example of a function on $\mathbf{K}^4 \times \mathbf{K}^4$ which is linear in the first variable and **conjugate-symmetric** but is not an inner product:

$$\langle x,\, y\rangle_M = x(1)\overline{y(1)} + x(2)\overline{y(2)} + x(3)\overline{y(3)} - x(4)\overline{y(4)}.$$

The linear space $\mathbf{R}^4$ with the function $\langle\ ,\ \rangle_M$ from $\mathbf{R}^4 \times \mathbf{R}^4$ to $\mathbf{R}$ is known as the **Minkowski space.** A vector $x \in \mathbf{R}^4$ is said to be **space-like** if $\langle x,\, x\rangle_M > 0$, **light-like** if $\langle x,\, x\rangle_M = 0$ and **time-like** if $\langle x,\, x\rangle_M < 0$. This space plays a fundamental role in the theory of relativity.

(b) Let $X = c_{00}$, the linear space of all scalar sequences each of which has only a finite number of nonzero entries. For x and y in X, define

$$\langle x,\, y\rangle = \sum_{j=1}^{\infty} x(j)\overline{y(j)}.$$

It is easy to see that $\langle\ ,\ \rangle$ is an inner product on X and

$$\|x\|_2 = \langle x,\, x\rangle^{1/2} = \left(\sum_{j=1}^{\infty} |x(j)|^2\right)^{1/2}, \quad x \in X.$$

Let $H = \ell^2$. For $x = (x(1), x(2), \ldots)$ and $y = (y(1), y(2), \ldots)$ in H, define

$$\langle x,\, y\rangle = \sum_{j=1}^{\infty} x(j)\overline{y(j)}.$$

Since $2|x(j)|\;|y(j)| \le |x(j)|^2 + |y(j)|^2$ for each $j = 1, 2, \ldots$, we have

$$\sum_{j=1}^{\infty} |x(j)\overline{y(j)}| \le \frac{1}{2}\left(\sum_{j=1}^{\infty} |x(j)|^2 + \sum_{j=1}^{\infty} |y(j)|^2\right) < \infty,$$

so that the series $\sum_{j=1}^{\infty} x(j)\overline{y(j)}$ converges (absolutely). Hence the function $\langle\;,\;\rangle$ on $H \times H$ is well-defined. It is now easy to see that $\langle\;,\;\rangle$ is an inner product on H and

$$\|x\|_2 = \langle x,\, x\rangle^{1/2} = \left(\sum_{j=1}^{\infty} |x(j)|^2\right)^{1/2}, \quad x \in H.$$

As we have seen in 3.3, H is complete in the metric $d_2(x, y) = \|x - y\|_2$. Thus H is a Hilbert space. It is easy to see that $X = c_{00}$ is dense in $H = \ell^2$. However, $X \ne H$. Hence X is not closed in H. Thus X is an inner product space which is not a Hilbert space and the completion of X is H.

Letting $x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, 0, \ldots)$ and appealing to the parallelogram law 21.2(b), we find that among all the ℓ^p-spaces, $1 \le p \le \infty$, only ℓ^2 is an inner product space.

(c) Let $X = C([a, b])$, the linear space of all scalar-valued continuous functions on $[a, b]$. For x and y in X, define

$$\langle x,\, y\rangle = \int_a^b x(t)\overline{y(t)}\, dt.$$

It is easy to see that $\langle\;,\;\rangle$ is an inner product on X and

$$\|x\|_2 = \langle x,\, x\rangle^{1/2} = \left(\int_a^b |x(t)|^2\, dt\right)^{1/2}, \quad x \in X.$$

Let $H = L^2([a, b])$, the linear space of all equivalence classes of scalar-valued square-integrable functions on $[a, b]$, obtained by identifying functions which are equal almost everywhere. (See Section 4.)

For x and y in H, let

$$\langle x, y\rangle = \int_a^b x\bar{y}\, dm.$$

Since $2|x(t)|\,|y(t)| \leq |x(t)|^2 + |y(t)|^2$ for each $t \in [a, b]$, we have

$$\int_a^b |x\bar{y}|\, dm \leq \frac{1}{2}\left(\int_a^b |x|^2\, dm + \int_a^b |y|^2\, dm\right) < \infty,$$

so that the function $x\bar{y}$ is integrable on $[a, b]$. Hence the function $\langle\ ,\ \rangle$ is well-defined on $H \times H$. It is easy to see that $\langle\ ,\ \rangle$ is an inner product on H and

$$\|x\|_2 = \langle x, x\rangle = \left(\int_a^b |x|^2\, dm\right)^{1/2}, \quad x \in H.$$

As we have seen in 4.6, H is complete in the metric $d_2(x, y) = \|x-y\|_2$. Thus H is a Hilbert space. By 4.7(b), $X = C([a, b])$ is dense in $H = L^2([a, b])$. However, $X \neq H$. Hence X is not closed in H. This shows that X is an inner product space which is not a Hilbert space, and that the completion of X is H. Among all the normed spaces $L^p([0, 1]), 1 \leq p \leq \infty$, only the space $L^2([0, 1])$ is an inner product space, because if $p \neq 2$, and we let $x(t) = t$ and $y(t) = 1 - t$ for $t \in [0, 1]$, then

$$\|x + y\|_p^2 + \|x - y\|_p^2 = 1 + \frac{1}{(p + 1)^{2/p}} \neq \frac{4}{(p + 1)^{2/p}} = 2\left(\|x\|_p^2 + \|y\|_p^2\right),$$

so that the parallelogram law 21.2(b) does not hold. In a like manner, we see that if E is a measurable subset of $\mathbf{R}$, and for $x, y \in L^2(E)$ we let

$$\langle x, y\rangle = \int_E x\overline{y}\, dm,$$

then $\langle\ ,\ \rangle$ is an inner product on $L^2(E)$, and it is complete in the norm given by

$$\|x\|_2 = \left(\int_E |x|^2 dm\right)^{1/2}, \quad x \in L^2(E).$$

(d) Let $X = C^1([a,b])$, the linear space of all scalar-valued continuously differentiable functions on $[a,b]$. For x and y in X, define

$$\langle x,\, y\rangle_a = x(a)\overline{y}(a) + \int_a^b x'(t)\overline{y}'(t)\,dt.$$

Clearly, $\langle\ ,\ \rangle_a$ is an inner product on X and

$$\|x\|_a^2 = \langle x,\, x\rangle_a = |x(a)|^2 + \int_a^b |x'(t)|^2\,dt, \quad x \in X.$$

Let $H = \{x \in C([a,b]) :\ x$ is absolutely continuous on $[a,b]$ and $x' \in L^2([a,b])\}$. For x and y in H, let

$$\langle x,\, y\rangle_a = x(a)\overline{y}(a) + \int_a^b x'\overline{y}'\,dm.$$

It can be easily seen that $\langle\ ,\ \rangle_a$ is an inner proudct on H and

$$\|x\|_a^2 = \langle x,\, x\rangle_a = |x(a)|^2 + \int_a^b |x'|^2\,dm, \quad x \in H.$$

Let (x_n) be a Cauchy sequence in H. Then the sequence $(x_n(a))$ is Cauchy in K and the sequence (x_n') is Cauchy in $L^2([a,b])$. Since K and $L^2([a,b])$ are complete, let $x_n(a) \to k$ in K and $x_n' \to y$ in $L^2([a,b])$. Note that

$$x_n(t) = x_n(a) + \int_a^t x_n'\,dm, \quad t \in [a,b], n = 1, 2, \ldots,$$

by the fundamental theorem for Lebesgue integration (4.3). Accordingly, define

$$x(t) = k + \int_a^t y\,dm, \quad t \in [a,b].$$

By 4.3, x is absolutely continuous on $[a,b]$ and $x' = y \in L^2([a,b])$, so that $x \in H$. Also, $x_n(a) \to k = x(a)$ in K and $x_n' \to y = x'$ in $L^2([a,b])$. Hence $x_n \to x$ in H. This shows that H is complete.

Next, we show that $X = C^1([a,b])$ is dense in H. Consider $x \in H$. By the fundamental theorem for Lebesgue integration (4.3), we have

$$x(t) = x(a) + \int_a^t x'\,dm, \quad t \in [a,b].$$

Since $x' \in L^2([a,b])$ and $C([a,b])$ is dense in $L^2([a,b])$ by 4.7(b), let $y_n \in C([a,b])$ such that $y_n \to x'$ in $L^2([a,b])$. For $n = 1, 2, \ldots$, define

$$x_n(t) = x(a) + \int_a^t y_n \, dm, \quad t \in [a,b].$$

By the fundamental theorem for Riemann integration (4.2), x_n is in $C^1([a,b])$ and $x_n' = y_n \in C([a,b])$, so that $x_n \in X$ for each n. Also, $x_n(a) = x(a)$ for each n and $x_n' = y_n \to x'$ in $L^2([a,b])$. Hence (x_n) converges to x in H. This shows that X is dense in H.

Since the function defined by

$$x(t) = |t - \frac{a+b}{2}|, \quad t \in [a,b],$$

is in H but not in X, we see that $X \neq H$. Hence X is not closed in H. Thus X is an inner product space which is not a Hilbert space, and the completion of X is H.

Problems

21-1 Let X be a linear space over $\mathbf{K}$. Consider a function $\langle\ ,\ \rangle : X \times X \to \mathbf{K}$ which is linear in the first variable, conjugate-symmetric and satisfies $\langle x, x\rangle \geq 0$ for all $x \in X$. Then $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$ for all $x, y \in X$. (Compare 21.1(c). Hint: If $\langle x, x\rangle = 0 = \langle y, y\rangle$, then $\langle x+y, x+y\rangle + \langle x-y, x-y\rangle = 0$.)

21-2 **(Generalized polarization identity)** Let $\mathbf{K} = \mathbb{C}$ and $A : X \to X$ be a linear map. Then

$$\begin{aligned} 4\langle A(x), y\rangle &= \langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle \\ &\quad + i\,\langle A(x+iy), x+iy\rangle - i\langle A(x-iy), x-iy\rangle. \end{aligned}$$

21-3 Let $\langle\ ,\ \rangle$ be an inner product on a linear space X. For $x \neq 0$ and $y \neq 0$ in X, define the angle between x and y as follows:

$$\theta_{x,y} = \arccos \frac{\operatorname{Re}\langle x, y\rangle}{\sqrt{\langle x, x\rangle\langle y, y\rangle}}, \quad 0 \leq \theta_{x,y} \leq \pi.$$

Then $\theta_{x,y}$ is well-defined and satisfies the **law of cosines**

$$\langle x, x\rangle + \langle y, y\rangle - \langle x - y, x - y\rangle = 2\langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}\cos\theta_{x,y}.$$

In particular, if $x \neq 0$ and $y \neq 0$, then $\langle x - y, x - y\rangle = \langle x, x\rangle + \langle y, y\rangle$ if and only if $\theta_{x,y} = \pi/2$.

21-4 **(Jordan and von Neumann, 1935)** Let $\|\ \ \|$ be a norm on a linear space X which satisfies the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \quad x, y \in X.$$

For $x, y \in X$, define

$$\langle x, y\rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - y\|^2\right).$$

Then $\langle\ ,\ \rangle$ is the unique inner product on X satisfying $\sqrt{\langle x, x\rangle} = \|x\|$ for all $x \in X$. (Hint: For $\mathbf{K} = \mathbf{R}$, $\langle x + z, y\rangle = 2\langle x, \frac{y}{2}\rangle + 2\langle z, \frac{y}{2}\rangle$ follows from

$$\begin{aligned} 4\langle x + z, y\rangle &= \|x + z + y\|^2 - \|x + z - y\|^2 \\ &= 2\left(\|x + \frac{y}{2}\|^2 - \|x - \frac{y}{2}\|^2 + \|z + \frac{y}{2}\|^2 - \|z - \frac{y}{2}\|^2.\right) \end{aligned}$$

21-5 Let X be a normed space. If on every two dimensional subspace Y of X, there is an inner product $\langle\ ,\ \rangle_Y$ such that $\langle y, y\rangle_Y = \|y\|^2$ for all $y \in Y$, then there is an inner product $\langle\ ,\ \rangle$ on X such that $\langle x, x\rangle = \|x\|^2$ for all $x \in X$. (Hint: Problem 21-4 and 21.2(b))

21-6 Let X be a normed space over $\mathbf{R}$. The norm satisfies the parallelogram law if and only if in every plane through the origin, the set of all elements having norm equal to 1 forms an ellipse with its center at the origin.

21-7 Let $\langle\ ,\ \rangle$ be an inner product on a linear space X and $T : X \to X$ be a linear one-to-one map. Let

$$\langle x, y\rangle_T = \langle T(x), T(y)\rangle, \quad x, y \in X.$$

Then $\langle\ ,\ \rangle_T$ is an inner product on X.

In particular, a function $\langle\ ,\ \rangle : \mathbf{K}^2 \times \mathbf{K}^2 \to \mathbf{K}$ is an inner product on $\mathbf{K}^2$ if and only if there are $\alpha > 0, \gamma > 0$ and $\beta \in \mathbf{K}$ with $|\beta|^2 < \alpha\gamma$ such that

$$\langle x, y\rangle = \alpha x(1)\overline{y(1)} + \beta x(1)\overline{y(2)} + \overline{\beta}x(2)\overline{y(1)} + \gamma x(2)\overline{y(2)}$$

for all $x, y \in \mathbf{K}^2$. Thus if E is an ellipse in $\mathbf{R}^2$ with its center at $(0,0)$, then there is an inner proudct $\langle\ ,\ \rangle$ on $\mathbf{R}^2$ such that $E = \{x \in \mathbf{R}^2 : \langle x,\, x\rangle = 1\}$.

21-8 Let X and Y be inner product spaces. Then a linear map $F : X \to Y$ satisfies $\langle F(x),\, F(y)\rangle = \langle x,\, y\rangle$ for all $x, y \in X$ if and only if it satisfies $\|F(x)\| = \|x\|$ for all $x \in X$, where the norms on X and Y are induced by the respective inner products.

21-9 Let X be an inner product space. Let H denote the set of all Cauchy sequences in X modulo the equivalence relation

$$(x_n) \sim (y_n) \quad \text{if and only if} \quad \lim_{n\to\infty} \langle x_n - y_n,\, x_n - y_n\rangle = 0.$$

If we let $(x_n) + (y_n) = (x_n + y_n), k(x_n) = (kx_n)$ and

$$\langle (x_n),\, (y_n)\rangle = \lim_{n\to\infty} \langle x_n,\, y_n\rangle,$$

then H is a Hilbert space with inner product $\langle\ ,\ \rangle$. Further, the linear map $F : X \to H$ given by $F(x) = (x, x, \ldots)$ satisfies $\langle F(x),\, F(y)\rangle = \langle x,\, y\rangle$ for all $x, y \in X$, and $R(F)$ is dense in H. The Hilbert space H is unique up to a linear bijection which preserves inner products.

21-10 For $j = 1, 2, \ldots$, let $\langle\ ,\ \rangle_j$ be an inner product on a linear space X_j over $\mathbf{K}$. Let X denote the set $\{(x(1), x(2), \ldots) : x(j) \in X_j, j = 1, 2, \ldots$ and $\sum_{j=1}^{\infty} \langle x(j),\, x(j)\rangle_j < \infty\}$ with pointwise addition and scalar multiplication. Let $\langle x,\, y\rangle = \sum_{j=1}^{\infty} \langle x(j),\, y(j)\rangle_j$, $x, y \in X$. Then X is a linear space over $\mathbf{K}$ and $\langle\ ,\ \rangle$ is an inner product on X. Further, X is a Hilbert space if and only if each X_j is a Hilbert space, $j = 1, 2, \ldots$. (Compare Problems 5-6 and 8-9 for the case $p = 2$. Hint: Proof of the completeness of ℓ^2 given in 3.3)

21-11 For $x, y \in C([a, b])$, define

$$\langle x,\, y\rangle_0 = \int_a^b \int_a^b x(s)\overline{y}(t)\, ds\, dt, \quad \langle x,\, y\rangle = \int_a^b \int_a^b \frac{\sin(s-t)}{s-t} x(s)\overline{y}(t) ds\, dt.$$

Then $\langle\ ,\ \rangle_0$ is not an inner product, but $\langle\ ,\ \rangle$ is an inner product on $C([a, b])$. (Hint: $\sin(s-t) = \frac{1}{2}(s-t)\displaystyle\int_{-1}^{1} e^{iu(s-t)}\, du$)

21-12 Let $X = C^m([a, b])$, the linear space of all m times continuously differentiable scalar-valued functions as $[a,b]$. For $x,y \in X$, let

$$\langle x,\, y\rangle_a = \sum_{j=0}^{m-1} x^{(j)}(a)\overline{y}^{(j)}(a) + \int_a^b x^{(m)}(t)\overline{y}^{(m)}(t)\,dt,$$

and $\|x\|_a^2 = \langle x,\, x\rangle_a$. Then $\langle\ ,\ \rangle_a$ is an inner product on X and the completion of X is $H = \{x \in C^{m-1}([a,b]) : x^{(m-1)}$ is absolutely continuous on $[a,b]$ and $x^{(m)} \in L^2([a,b])\}$ along with the inner proudct $\langle\ ,\ \rangle_a$ on H.

21-13 Let X and H be as in 21.3(d). For $x, y \in H$, let

$$\langle x,\, y\rangle = \int_a^b (x\overline{y} + x'\overline{y}')\,dm$$

and $\|x\|^2 = \langle x,\, x\rangle$. Then $\langle\ ,\ \rangle$ is an inner product on H and the norm $\|\ \|$ is equivalent to the norm $\|\ \|_a$ given in 21.3(d). (Hint: For $y \in L^2(([a,b])$ and $t \in [a,b]$, we have $|\int_a^t y\,dm| \le \sqrt{b-a}\|y\|_2$. If x is absolutely continuous on $[a,b]$ and $x' \in L^2([a,b])$, then $\|x\|_2 \le \sqrt{b-a}\,|x(a)| + (b-a)\|x'\|_2$ and $\sqrt{b-a}\,|x(a)| \le \|x\|_2 + (b-a)\|x'\|_2$.)

22 Orthonormal Sets

Two vectors in $\mathbf{R}^2$ are perpendicular if their dot product is zero. Since an inner product on a linear space is a generalization of the dot product on $\mathbf{R}^2$, we can introduce the concept of perpendicularity for elements of X as follows. Let X be an inner product space over $\mathbf{K}$. For x and y in X, we say that x and y are **orthogonal** (to each other) if $\langle x,\, y\rangle = 0$. In that case, we write $x \perp y$. For subsets E and F of X, we write $E \perp F$ if $x \perp y$ for all $x \in E$ and $y \in F$. We say that a subset E of X is **orthogonal** if $x \perp y$ for all $x \neq y$ in E. If, in addition, $\langle x,\, x\rangle = 1$ for all $x \in E$, we say that E is **orthonormal**. We shall write $\|x\| = \langle x,\, x\rangle^{1/2}$ for $x \in X$. Closely associated with perpendicularity is the famous theorem of Pythagoras.

22.1 Theorem

Let X be an inner product space.

(a) (Pythagoras) Let $\{x_1, \ldots, x_n\}$ be an orthogonal set in X. Then

$$\|x_1 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2.$$

(b) Let E be an orthogonal subset of X and $0 \notin E$. Then E is linearly independent. If, in fact, E is orthonormal, then $\|x - y\| = \sqrt{2}$ for all $x \neq y$ in E.

Proof:

(a) Since $\langle x_j, x_k \rangle = 0$ for $j \neq k$, we have

$$\begin{aligned} \|x_1 + \cdots + x_n\|^2 &= \langle x_1 + \cdots + x_n, x_1 + \cdots + x_n \rangle \\ &= \sum_{j,k=1}^{n} \langle x_j, x_k \rangle = \sum_{j=1}^{n} \langle x_j, x_j \rangle \\ &= \|x_1\|^2 + \cdots + \|x_n\|^2. \end{aligned}$$

(b) Let $x_1, \ldots, x_n$ belong to E and $k_1, \ldots, k_n$ to $\mathbf{K}$ such that $k_1 x_1 + \cdots + k_n x_n = 0$. For each fixed $j = 1, \ldots, n$, we have

$$k_j \langle x_j, x_j \rangle = \langle k_1 x_1 + \cdots + k_n x_n, x_j \rangle = \langle 0, x_j \rangle = 0.$$

Since $0 \notin E$, we see that $x_j \neq 0$, that is, $\langle x_j, x_j \rangle \neq 0$. Hence $k_j = 0$. Thus E is linearly independent. If, in fact, E is orthonormal, then for $x \neq y$ in E, we have

$$\|x - y\|^2 = \langle x - y, x - y \rangle = \langle x, x \rangle + \langle y, y \rangle = 2. \qquad \square$$

It follows that every orthonormal set in X is linearly independent. Conversely, given a countable linearly independent set in X, we can algorithmically construct an orthonormal set, retaining the span of the elements at each step.

22.2 Theorem (Gram-Schmidt orthonormalization)
Let $\{x_1, x_2, \ldots\}$ be a linearly independent subset of an inner product space X. Define $y_1 = x_1$, $u_1 = \dfrac{y_1}{\|y_1\|}$ and for $n = 2, 3, \ldots$,

$$y_n = x_n - \langle x_n, u_1\rangle u_1 - \cdots - \langle x_n, u_{n-1}\rangle u_{n-1}, \quad u_n = \frac{y_n}{\|y_n\|}.$$

Then $\{u_1, u_2, \ldots\}$ is an orthonormal set in X and for $n = 1, 2, \ldots$

$$\operatorname{span}\{u_1, \ldots, u_n\} = \operatorname{span}\{x_1, \ldots, x_n\}.$$

Proof:
As $\{x_1\}$ is a linearly independent set, we see that $y_1 = x_1 \neq 0$, $\|u_1\| = \|y_1\|/\|y_1\| = 1$ and $\operatorname{span}\{u_1\} = \operatorname{span}\{x_1\}$.

For $n \geq 1$, assume that we have defined y_n and u_n as stated above, and proved that $\{u_1, \ldots, u_n\}$ is an orthonormal set satisfying $\operatorname{span}\{u_1, \ldots, u_n\} = \operatorname{span}\{x_1, \ldots, x_n\}$. Define

$$y_{n+1} = x_{n+1} - \langle x_{n+1}, u_1\rangle u_1 - \cdots - \langle x_{n+1}, u_n\rangle u_n.$$

As $\{x_1, \ldots, x_{n+1}\}$ is a linearly independent set, x_{n+1} does not belong to $\operatorname{span}\{x_1, \ldots, x_n\} = \operatorname{span}\{u_1, \ldots, u_n\}$. Hence $y_{n+1} \neq 0$. Let $u_{n+1} = y_{n+1}/\|y_{n+1}\|$. Then $\|u_{n+1}\| = 1$ and for all $j \leq n$, we have

$$\begin{aligned}
\langle y_{n+1}, u_j\rangle &= \langle x_{n+1}, u_j\rangle - \sum_{k=1}^{n} \langle x_{n+1}, u_k\rangle \langle u_k, u_j\rangle \\
&= \langle x_{n+1}, u_j\rangle - \langle x_{n+1}, u_j\rangle \\
&= 0,
\end{aligned}$$

since $\langle u_k, u_j\rangle = 0$ for all $k \neq j, k = 1, \ldots, n$. Thus $\langle u_{n+1}, u_j\rangle = \langle y_{n+1}, u_j\rangle/\|y_{n+1}\| = 0$ for $j = 1, \ldots, n$ as well. Hence $\{u_1, \ldots, u_{n+1}\}$ is an orthonormal set. Also,

$$\operatorname{span}\{u_1, \ldots, u_{n+1}\} = \operatorname{span}\{x_1, \ldots, x_n, u_{n+1}\} = \operatorname{span}\{x_1, \ldots, x_{n+1}\}.$$

By mathematical induction, the proof is complete. $\square$

22.3 Examples

(a) Let $X = \ell^2$. For $n = 1, 2, \ldots$, let $x_n = (1, \ldots, 1, 0, 0, \ldots)$, where 1 occurs only in the first n entries. It can be easily seen that the Gram-Schmidt orthonormalization yields

$$y_n = (0, \ldots, 0, 1, 0, 0, \ldots) = u_n,$$

where 1 occurs only in the nth entry.

(b) (Orthonormal polynomials) Let $-\infty \le a < b \le \infty$ and let w be a positive continuous function on (a, b). Let $L^2_w(a, b)$ denote the linear space of all equivalence classes of scalar-valued measurable functions x on (a, b) such that

$$\int_a^b |x|^2 w \; dm < \infty,$$

where two functions belong to the same equivalence class if they agree almost everywhere on (a, b). For x and y in $L^2_w(a, b)$, let

$$\langle x, \, y\rangle_w = \int_a^b x\overline{y} \, w \; dm.$$

Since $2|x\overline{y}| \, w \le |x|^2 \, w + |y|^2 \, w$, the function $\langle \; , \; \rangle_w$ is well-defined on $L^2_w(a, b) \times L^2_w(a, b)$. Clearly, it is an inner product on $L^2_w(a, b)$. The function w is known as a **weight function** on (a, b). Assume that it satisfies

$$\int_a^b t^{2n} w(t) \, dt < \infty, \quad n = 0, 1, 2, \ldots,$$

that is, if we let $x_n(t) = t^n$ for $n = 0, 1, 2, \ldots$ and $t \in (a, b)$, then $x_n \in L^2_w(a, b)$ for each n. Noting that a (finite) linear combination of $x_0, x_1, \ldots$ is a polynomial in t and has only a finite number of roots in (a, b), we conclude that $\{x_0, x_1, \ldots\}$ is a linearly independent subset of $L^2_w(a, b)$.

By the Gram-Schmidt orthonormalization described in 22.2, we obtain an orthonormal set $\{u_0, u_1, \ldots\}$ in $L^2_w(a, b)$ such that

$$\text{span}\{u_0, \ldots, u_n\} = \text{span}\{x_0, \ldots, x_n\}, \quad n = 0, 1, \ldots.$$

Clearly, each u_n is a polynomial in t of degree n. It is known as the **orthonormal polynomial** of degree n with respect to the weight function w on (a, b).

For various choices of an interval (a, b) and a weight function w on it, we obtain many classical systems of polynomials.

1. $(a, b) = (-1, 1), w(t) = 1$: **Legendre polynomials**
2. $(a, b) = (-1, 1), w(t) = 1/\sqrt{1-t^2}$: **Tchebychev polynomials of the first kind**
3. $(a, b) = (-1, 1), w(t) = \sqrt{1-t^2}$: **Tchebychev polynomials of the second kind**
4. $(a, b) = (0, \infty), w(t) = e^{-t}$: **Laguerre polynomials**
5. $(a, b) = (-\infty, \infty), w(t) = e^{-t^2}$: **Hermite polynomials**

Formulae for these and several other polynomials, their peculiar properties and recursion relations among them have been extensively studied. (For example, see [55].) We restrict ourselves to calculating the first three Legendre polynomials.

Let $-1 \le t \le 1$. We have $y_0(t) = x_0(t) = 1$, so that $\|y_0\|_2^2 = \int_{-1}^{1} ds = 2$ and $u_0 = y_0/\|y_0\| = 1/\sqrt{2}$. Next,

$$y_1(t) = x_1(t) - \langle x_1,\, u_0\rangle u_0(t) = t - \left(\int_{-1}^{1} \frac{s}{\sqrt{2}}\, ds\right) \frac{1}{\sqrt{2}} = t,$$

so that $\|y_1\|_2^2 = \int_{-1}^{1} s^2\, ds = 2/3$ and $u_1(t) = y_1(t)/\|y_1\|_2 = \sqrt{3}\,t/\sqrt{2}$. Further,

$$\begin{aligned} y_2(t) &= x_2(t) - \langle x_2,\, u_0\rangle u_0(t) - \langle x_2,\, u_1\rangle u_1(t) \\ &= t^2 - \left(\int_{-1}^{1} \frac{s^2}{\sqrt{2}}\, ds\right) \frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} \frac{\sqrt{3}}{\sqrt{2}} s^3\, ds\right) \frac{\sqrt{3}}{\sqrt{2}} t \\ &= t^2 - \frac{1}{3}, \end{aligned}$$

so that $\|y_2\|_2^2 = \int_{-1}^{1} (s^2 - 1/3)^2\, ds = 8/45$ and $u_2(t) = y_2(t)/\|y_2\| = \sqrt{10}(3t^2 - 1)/4$.

We now undertake a deeper study of orthonormal sets. We shall often have to deal with countable orthonormal sets. The notation $\sum_n$ will stand for summation over $n = 1, 2, \ldots$, which may be finite or infinite.

We begin our study with a basic inequality which generalizes the Schwarz inequality (21.1(c)).

22.4 Lemma (Bessel's inequality)
Let $\{u_1, u_2, \ldots\}$ be a countable orthonormal set in an inner product space X and $x \in X$. Then

$$\sum_n |\langle x,\, u_n\rangle|^2 \leq \|x\|^2,$$

where equality holds if and only if $x = \sum_n \langle x,\, u_n\rangle u_n$.

Proof:
For $m = 1, 2, \ldots$, let $x_m = \sum_{n=1}^m \langle x,\, u_n\rangle u_n$. Since $\{u_1, \ldots, u_m\}$ is an orthonormal set, we have

$$\langle x,\, x_m\rangle = \langle x_m,\, x\rangle = \langle x_m,\, x_m\rangle = \sum_{n=1}^m |\langle x,\, u_n\rangle|^2.$$

Hence

$$0 \leq \|x - x_m\|^2 = \langle x - x_m,\, x - x_m\rangle = \|x\|^2 - \sum_{n=1}^m |\langle x,\, u_n\rangle|^2.$$

Letting $m \to \infty$ (if the set $\{u_1, u_2, \ldots\}$ is denumerable), we obtain

$$\sum_n |\langle x,\, u_n\rangle|^2 \leq \|x\|^2.$$

If equality holds here, that is, $\sum_{n=1}^m |\langle x,\, u_n\rangle|^2 \to \|x\|^2$ as $m \to \infty$, then $\|x - x_m\|^2 \to 0$, that is,

$$x = \lim_{m\to\infty} x_m = \sum_n \langle x,\, u_n\rangle u_n.$$

Conversely, if x is given as above, then

$$\|x\|^2 = \langle x,\, x\rangle = \langle \sum_n \langle x,\, u_n\rangle u_n,\, \sum_n \langle x,\, u_n\rangle u_n\rangle = \sum_n |\langle x,\, u_n\rangle|^2$$

by the continuity, the linearity in the first variable and the conjugate-linearity in the second variable of the inner product $\langle\ ,\ \rangle$. $\square$

Let us now study the convergence of a series $\sum_{n=1}^{\infty} k_n u_n$ in an inner product space X, where $k_n \in \mathbf{K}$ and $\{u_1, u_2, \ldots\}$ is an orthonormal set in X.

22.5 Theorem

Let X be an inner product space, $\{u_1, u_2, \ldots\}$ be a countable orthonormal set in X and $k_1, k_2, \ldots$ belong to $\mathbf{K}$.

(a) If $\sum_n k_n u_n$ converges to some x in X, then $\langle x,\ u_n\rangle = k_n$ for each n and $\sum_n |k_n|^2 < \infty$.

(b) (Riesz-Fischer Theorem, 1907) If X is a Hilbert space and $\sum_n |k_n|^2 < \infty$, then $\sum_n k_n u_n$ converges in X.

Proof:

(a) If $x = \sum_n k_n u_n$, then the orthonormality of the set $\{u_1, u_2, \ldots\}$ shows that $\langle x,\ u_n\rangle = k_n$ for each n. Now by 22.4,

$$\sum_n |k_n|^2 = \sum_n |\langle x,\ u_n\rangle|^2 = \|x\|^2 < \infty.$$

(b) For $m = 1, 2, \ldots$, let $x_m = \sum_{n=1}^{m} k_n u_n$. Then for $j = 1, 2, \ldots$ and $m > j$, we have $x_m - x_j = \sum_{n=j+1}^{m} k_n u_n$ and

$$\|x_m - x_j\|^2 = \langle x_m - x_j,\ x_m - x_j\rangle = \sum_{n=j+1}^{m} |k_n|^2$$

again by the orthonormality of the set $\{u_1, u_2, \ldots\}$. If $\sum_n |k_n|^2 < \infty$, then it follows that (x_m) is a Cauchy sequence in X , and if X is a Hilbert space, that is, if X is complete, then (x_m) converges to some x in X, as desired. $\square$

We remark that if $\{u_1, u_2, \ldots\}$ is an orthonormal set in a Hilbert space H and a sequence (k_n) in $\mathbf{K}$ satisfies $\sum_n |k_n| < \infty$, then by 8.1, the series $\sum_n k_n u_n$ converges in H. But the condition $\sum_n |k_n| < \infty$ is not necessary for the convergence of the series $\sum_n k_n u_n$ in H. In fact, as we have seen above, $\sum_n |k_n|^2 < \infty$ is a necessary and sufficient condition for such a convergence.

If $\sum_n |k_n|^2 < \infty$ and we let $x = \sum_n k_n u_n$, then x belongs to H and satisfies $\langle x, u_n \rangle = k_n$ for each n. However, x need not be the only element of H having this property. In fact, if $v \in H$ and $\langle v, u_n \rangle = 0$ for each n, then $x + v$ is also an element of H satisfying $\langle x + v, u_n \rangle = k_n$ for each n.

As a case in point, consider $H = L^2([-\pi, \pi])$ and for $n = 1, 2, \ldots$,

$$u_n(t) = \frac{\sin nt}{\sqrt{\pi}} \quad \text{and} \quad v_n(t) = \frac{\cos nt}{\sqrt{\pi}}, \quad t \in [-\pi, \pi].$$

Since $\{u_1, u_2, \ldots\}$ is an orthonormal set in H and $\sum_{n=1}^{\infty} 1/n^2 < \infty$,

$$x = \sum_{n=1}^{\infty} \frac{u_n}{n}$$

is an element of H satisfying $\langle x, u_n \rangle = 1/n$ for each n. If (h_n) is a sequence of scalars with $\sum_{n=1}^{\infty} |h_n|^2 < \infty$ and we let

$$v = \sum_{n=1}^{\infty} h_n v_n,$$

then $x + v \in H$ and $\langle x + v, u_n \rangle = 1/n$ for each n.

We observe that if any element x of H is determined by the requirement $\langle x, u_n \rangle = k_n$ for each n, then the orthonormal set $\{u_1, u_2, \ldots\}$ cannot be enlarged to a larger orthonormal set. These considerations lead us to the following important concept.

An orthonormal set $\{u_\alpha\}$ in a Hilbert space H is said to be an **orthonormal basis** for H if it is maximal in the sense that if $\{u_\alpha\}$ is contained in some orthonormal subset E of H, then, in fact, $E = \{u_\alpha\}$.

Let H be a Hilbert space and $H \neq \{0\}$. Then H does contain orthonormal sets. For example, if x is a nonzero element of H, then $\{x/\|x\|\}$ is surely an orthonormal set in H. To see that H contains an orthonormal basis, and in fact, every orthonormal set E in H can be extended to an orthonormal basis for H, we argue as follows. Let $\mathcal{E}$ denote the set of all orthonormal sets in H containing E. Then set inclusion gives a partial order on $\mathcal{E}$. It is easy to see that every totally ordered subset of $\mathcal{E}$ has an upper bound in $\mathcal{E}$, namely the union of its members. An application of Zorn's lemma then yields a maximal element of $\mathcal{E}$, which is clearly an orthonormal basis for H.

Before giving several necessary and sufficient conditions under which an orthonormal set is, in fact, an orthonormal basis, we prove some preliminary results.

22.6 Lemma

Let $\{u_\alpha\}$ be an orthonormal set in an inner product space X and $x \in X$. Let

$$E_x = \{u_\alpha : \langle x, u_\alpha\rangle \neq 0\}.$$

Then E_x is a countable set, say $E_x = \{u_1, u_2, \ldots\}$. If E_x is denumerable, then $\langle x, u_n\rangle \to 0$ as $n \to \infty$.

Further, if X is a Hilbert space, then $\sum_n \langle x, u_n\rangle u_n$ converges in X to some y such that $x - y \perp u_\alpha$ for every α.

Proof:

If $x = 0$, there is nothing to prove. Let $x \neq 0$. For $j = 1, 2, \ldots$, let

$$E_j = \{u_\alpha : \|x\| \leq j|\langle x, u_\alpha\rangle|\}.$$

Fix j. Suppose that E_j contains distinct elements $u_{\alpha_1}, \ldots, u_{\alpha_m}$. Then

$$0 < m\|x\|^2 \leq j^2 \sum_{n=1}^{m} |\langle x, u_{\alpha_n}\rangle|^2.$$

But by Bessel's inequality,

$$\sum_{n=1}^{m} |\langle x, u_{\alpha_n}\rangle|^2 \leq \|x\|^2.$$

This shows that $m \le j^2$. Thus E_j contains at most j^2 elements. Since E_x is the union of all E_j's, $j = 1, 2, \ldots$, we see that E_x a countable set. Also, if $E_x = \{u_1, u_2, \ldots\}$ is denumerable, then

$$\sum_{n=1}^{\infty} |\langle x,\, u_n\rangle|^2 \le \|x\|^2 < \infty$$

by 22.4, so that the nth term $|\langle x,\, u_n\rangle|^2$ of this convergent series tends to zero as $n \to \infty$.

Further, if X is a Hilbert space, then by the Riesz-Fisher theorem (22.5(b)), $\sum_n \langle x,\, u_n\rangle u_n$ converges to some y in X and

$$\langle y,\, u_\alpha\rangle = \langle \sum_n \langle x,\, u_n\rangle u_n,\, u_\alpha\rangle = \sum_n \langle x,\, u_n\rangle\langle u_n,\, u_\alpha\rangle = \langle x,\, u_\alpha\rangle$$

for each α, that is, $x - y \perp u_\alpha$ for each α. □

22.7 Theorem

Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Then the following conditions are equivalent.

(i) $\{u_\alpha\}$ is an orthonormal basis for H.

(ii) **(Fourier expansion)** For every $x \in H$, we have

$$x = \sum_n \langle x,\, u_n\rangle u_n,$$

where $\{u_1, u_2, \ldots\} = \{u_\alpha : \langle x,\, u_\alpha\rangle \ne 0\}$.

(iii) **(Parseval formula)** For every $x \in H$, we have

$$\|x\|^2 = \sum_n |\langle x,\, u_n\rangle|^2,$$

where $\{u_1, u_2, \ldots\} = \{u_\alpha : \langle x,\, u_\alpha\rangle \ne 0\}$.

(iv) $\operatorname{span}\{u_\alpha\}$ is dense in H.

(v) If $x \in H$ and $\langle x,\, u_\alpha\rangle = 0$ for all α, then $x = 0$.

Proof:

(i) implies (ii): Let $\{u_\alpha\}$ be a maximal orthonormal set in H. Consider $x \in H$. By 22.6, $\sum_n \langle x, u_n\rangle u_n = y$ for some $y \in H$ with $x - y \perp \{u_\alpha\}$. If $y \neq x$, let $u = (y - x)/\|y - x\|$. Then $\|u\| = 1$ and $u \perp \{u_\alpha\}$, so that $\{u_\alpha\} \bigcup \{u\}$ is an orthonormal set in H, contradicting the maximality of the set $\{u_\alpha\}$. Thus $x = y = \sum_n \langle x, u_n\rangle u_n$.

Conditions (ii) and (iii) are equivalent by the equality criterion given in 22.4.

Condition (ii) implies condition (iv) because $\sum_{n=1}^{m} \langle x, u_n\rangle u_n$ belongs to span$\{u_\alpha\}$ for each $m = 1, 2, \ldots$.

(iv) implies (v): Let $x \in H$ be such that $\langle x, u_\alpha\rangle = 0$ for all α, and let $x_m \to x$, where $x_m \in$ span$\{u_\alpha\}$. Since $0 = \langle x_m, x\rangle \to \langle x, x\rangle$, we see that $\langle x, x\rangle = 0$, that is, $x = 0$.

(v) implies (i): Let E be an orthonormal set in H containing $\{u_\alpha\}$. If $u \in E$ and $u \neq u_\alpha$ for all α, then $\langle u, u_\alpha\rangle = 0$ by the orthogonality of E, so that $u = 0$ by (v). However, being an element of the orthonormal set E, we must have $\|u\| = 1$. This contradiction shows that $E = \{u_\alpha\}$. Thus $\{u_\alpha\}$ is a maximal orthonormal set in H, that is, an orthonormal basis for H. $\square$

Note that only the part '(i) implies (ii)' of this theorem uses the completeness of H via the Riesz-Fischer theorem 22.5(b). That the completeness assumption cannot be dropped from the Riesz-Fischer theorem can be seen by letting

$$X = c_{00}, \quad u_n = (0, \ldots, 0, 1, 0, 0, \ldots) \quad \text{and} \quad k_n = \frac{1}{n} \text{ for } n = 1, 2, \ldots,$$

where only the nth entry of u_n is 1. Dixmier ([18], 1953) gave an example of an (incomplete) inner product space X which is larger than the closure of the linear span of any orthonormal subset. Hence the completeness assumption cannot be dropped from Theorem 22.7

either.

Conditions (iv) and (v) of 22.7 can be used to check whether a given orthonormal set in H is, in fact, an orthonormal basis for H. If that is the case, then conditions (ii) and (iii) yield useful information for each $x \in H$.

22.8 Examples

(a) Let $H = \ell^2$ and for $n = 1, 2, \ldots,$

$$u_n = (0, \ldots, 0, 1, 0, 0, \ldots),$$

where 1 occurs only in the nth entry. Then $\{u_1, u_2, \ldots\}$ is an orthonormal set in H. If $x \in H$ and $x(n) = \langle x, u_n \rangle = 0$ for all n, then $x = 0$. Hence $\{u_n : n = 1, 2, \ldots\}$ is an orthonormal basis for H by 22.7(v).

(b) Let $H = L^2([-\pi, \pi])$ and for $n = 0, \pm 1, \pm 2, \ldots,$

$$u_n(t) = \frac{e^{int}}{\sqrt{2\pi}}, \quad t \in [-\pi, \pi].$$

Then $\{u_n : n = 0, \pm 1, \pm 2, \ldots\}$ is an orthonormal set in H. For $x \in H$ and $n = 0, \pm 1, \pm 2, \ldots$, we have

$$\langle x, u_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x(t) e^{-int} \, dm(t) = \sqrt{2\pi}\, \hat{x}(n),$$

where $\hat{x}(n)$ is the nth Fourier coefficient of x, as introduced in Section 4. We have proved in 4.9(c) that if $\hat{x}(n) = 0$ for all $n = 0, \pm 1, \pm 2, \ldots$, then $x = 0$ a.e. on $[-\pi, \pi]$, that is, if $\langle x, u_n \rangle = 0$ for all n, then $x = 0$. Hence $\{u_n : n = 0, \pm 1, \pm 2, \ldots\}$ is an orthonormal basis for H by 22.7(v). This is one of the most useful orthonormal bases, well-known from the classical times. In fact, much of the Hilbert space theory is modeled after this example.

Let us see what our general results say in this particular case. For every $x \in H$, the Fourier expansion 22.7(ii)

$$x = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \hat{x}(n) u_n$$

means that the sequence (s_m) of partial sums of the Fourier series of a square-integrable function x on $[-\pi, \pi]$ converges to the function in the mean square:

$$\int_{-\pi}^{\pi} |x - s_m|^2 \, dm \to 0, \quad \text{where } s_m = \sqrt{2\pi} \sum_{n=-m}^{m} \hat{x}(n) u_n.$$

This positive result should be contrasted with the negative result 9.4 regarding the lack of pointwise convergence of the Fourier series of a continuous function on $[-\pi, \pi]$.

The Parseval formula 22.7(iii) is reduced to

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |x|^2 \, dm = \sum_{n=-\infty}^{\infty} |\hat{x}(n)|^2, \quad x \in H.$$

The results 22.5(a) and 22.5(b) (Riesz-Fischer theorem) say that if (k_n) is a square-summable sequence of scalars, then there is a square-integrable function x on $[-\pi, \pi]$ whose nth Fourier coefficient is k_n for all n. This elegant result should be compared with our earlier result 15.5, where boundedness of the arithmetic means of the partial sums was assumed.

Finally, the result $\langle x, u_n \rangle \to 0$ as $n \to \infty$ proved in 22.6 simply restates the Riemann-Lebesgue lemma (4.9(a)) for square-integrable functions. Note that the converse is false. As can be seen from 11.2, there is a scalar sequence (k_n) such that $k_n \to 0$ as $n \to \pm\infty$, but for no x in $L^1([-\pi, \pi])$, we have $\hat{x}(n) = k_n$ for all $n = 0, \pm 1, \pm 2, \ldots$.

Another useful orthonormal basis for $H = L^2([-\pi, \pi])$ is obtained by noting that $e^{int} = \cos nt + i \sin nt$, $\cos nt = (e^{int} + e^{-int})/2$ and $\sin nt = (e^{int} - e^{-int})/2i$ for $t \in [-\pi, \pi], n = 1, 2, \ldots$. Then

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nt}{\sqrt{\pi}}, \frac{\sin nt}{\sqrt{\pi}} : n = 1, 2, \ldots \right\}$$

is an orthonormal basis for H. If we let

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) \, dm(t), \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x(t) \cos nt \, dm(t),$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin nt\, dm(t), \quad n = 1, 2, \ldots,$$

then the Fourier expansion for $x \in H$ becomes

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$

where the series converges in the mean square. Also, the Parseval formula for $x \in H$ becomes

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |x|^2\, dm = 2|a_0|^2 + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right).$$

(c) Let $H = L^2([-1,1])$ and for $n = 0, 1, 2, \ldots$, let u_n denote the Legendre polynomial of degree n, obtained by applying the Gram-Schmidt orthonormalization to the linearly independent set $\{x_0, x_1, \ldots\}$ where $x_n(t) = t^n, t \in [-1,1]$. Then $\{u_0, u_1, \ldots\}$ is an orthonormal set in H.

Let $x \in H$ and $\epsilon > 0$. As the set of all continuous functions on $[-1,1]$ is dense in H (4.7(b)), there is some $y \in C([-1,1])$ such that $\|x - y\|_2 < \epsilon$. By the Weierstrass theorem (3.12), there is a polynomial p such that $|y(t) - p(t)| < \epsilon$ for all $t \in [-1,1]$. Then

$$\|y - p\|_2^2 = \int_{-1}^{1} |y(t) - p(t)|^2\, dt \le 2\epsilon^2.$$

Thus $\|x - p\|_2 \le \|x - y\|_2 + \|y - p\|_2 < (1 + \sqrt{2})\epsilon$. This shows that the set of all polynomials on $[-1,1]$ is dense in H. Since

$$\begin{aligned}\operatorname{span}\{u_0, \ldots, u_n\} &= \operatorname{span}\{x_0, \ldots, x_n\} \\ &= \{p : p \text{ is a polynomial of degree} \le n\},\end{aligned}$$

we see that $\operatorname{span}\{u_0, u_1, \ldots\}$ is the set of all polynomials on $[-1,1]$. By 22.7(iv), $\{u_0, u_1, \ldots\}$ is an orthonormal basis for H.

For a more general result regarding orthonormal polynomials with respect to a weight function w on a finite interval (a,b), see Problem 22-14. Also, if $u_0, u_1, \ldots$ denote the Laguerre polynomials and $w_1(t) =$

$e^{-t/2}$ for $t \in [0, \infty)$, then $\{u_n w_1 : n = 0, 1, 2, \ldots\}$ is an orthonormal basis for $L^2([0, \infty))$, and if $v_0, v_1, \ldots$ denote the Hermite polynomials and $w_2(t) = e^{-t^2/2}$ for $t \in (-\infty, \infty)$, then $\{v_n w_2 : n = 0, 1, 2, \ldots\}$ is an orthonormal basis for $L^2(-\infty, \infty)$. The reader is referred to [13], pp. 95-97 for an ingenious proof due to von Neumann.

(d) Let $H = L^2([a, b] \times [a, b])$ with the inner product

$$\langle\langle x, y \rangle\rangle = \iint_{[a,b]\times[a,b]} x\overline{y}\, d(m\times m), \quad x, y \in H.$$

Consider orthonormal bases $\{u_i\}$ and $\{v_j\}$ for $L^2([a, b])$ and for $i, j = 1, 2, \ldots$, let

$$w_{i,j}(s, t) = u_i(s)\overline{v_j(t)}, \quad s, t \in [a, b].$$

We show that $\{w_{i,j}\}$ is an orthonormal basis for H. For $i, j, k, \ell = 1, 2, \ldots$, it is easy to see that

$$\langle\langle w_{i,j}, w_{k,\ell} \rangle\rangle = \langle u_i, u_k \rangle \langle v_\ell, v_j \rangle.$$

Since $\langle u_i, u_k \rangle = \delta_{i,k}$ and $\langle v_\ell, v_j \rangle = \delta_{\ell,j}$, we see that $\{w_{i,j}\}$ is an orthonormal set in H.

Next, consider $x \in H$ such that $\langle\langle x, w_{i,j} \rangle\rangle = 0$ for all $i, j = 1, 2, \ldots$. For a fixed $s \in [a, b]$, let $x_s(t) = x(s, t)$, $t \in [a, b]$. Now

$$\begin{aligned}\langle\langle x, x \rangle\rangle &= \iint_{[a,b]\times[a,b]} |x(s,t)|^2 d(m\times m)(s,t) \\ &= \int_a^b \left(\int_a^b |x_s(t)|^2\, dm(t) \right) dm(s)\end{aligned}$$

by Fubini's theorem (4.4). It follows that $x_s \in L^2([a, b])$ for almost every $s \in [a, b]$, and since $\{v_j\}$ is an orthonormal basis for $L^2([a, b])$, we have

$$\int_a^b |x_s(t)|^2\, dm(t) = \|\overline{x_s}\|_2^2 = \sum_j |\langle \overline{x_s}, v_j \rangle|^2$$

by Parseval's formula (22.7(iii)). Thus

$$\langle\langle x, x \rangle\rangle = \int_a^b \left(\sum_j |\langle \overline{x_s}, v_j \rangle|^2 \right) dm(s) = \sum_j \left(\int_a^b |\langle \overline{x_s}, v_j \rangle|^2\, dm(s) \right).$$

Fix $j = 1, 2, \ldots$ and let $y_j(s) = \langle v_j, \overline{x_s} \rangle$ for $s \in [a, b]$.

Then $y_j \in L^2([a, b])$ and for all $i = 1, 2, \ldots$,

$$\begin{aligned} 0 = \langle\langle x, w_{i,j} \rangle\rangle &= \iint_{[a,b]\times[a,b]} x(s,t)\overline{u_i(s)}v_j(t)\,d(m\times m)(s,t) \\ &= \int_a^b \left[\int_a^b x(s,t)v_j(t)\,dm(t)\right]\overline{u_i(s)}\,dm(s) \\ &= \int_a^b y_j(s)\overline{u_i(s)}\,dm(s). \end{aligned}$$

Since $\{u_i\}$ is an orthonormal basis for $L^2([a, b])$, 22.7(v) shows that $y_j = 0$. As this is true for each $j = 1, 2, \ldots$, we see that

$$\langle\langle x, x \rangle\rangle = \sum_j \|y_j\|_2^2 = 0.$$

Again by 22.7(v), we see that $\{w_{i,j}\}$ is an orthonormal basis for H.

(e) We give an example of an uncountable orthonormal basis for a Hilbert space.

For $r_1, \ldots, r_m \in \mathbf{R}$ and $c_1, \ldots, c_m \in \mathbf{C}$, let

$$p(t) = c_1 e^{ir_1 t} + \cdots + c_m e^{ir_m t}, \quad t \in \mathbf{R}.$$

Then p is called a **trigonometric polynomial** on $\mathbf{R}$. If q is also a trigonometric polynomial on $\mathbf{R}$ and

$$q(t) = d_1 e^{is_1 t} + \cdots + d_n e^{is_n t}, \quad t \in \mathbf{R},$$

then

$$p(t)\overline{q(t)} = \sum_{j=1}^{m}\sum_{k=1}^{n} c_j \overline{d}_k e^{i(r_j - s_k)t}, \quad t \in \mathbf{R}.$$

For $r \in \mathbf{R}$, it is easy to see that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{irt}\,dt = \begin{cases} 1, & \text{if } r = 0 \\ 0, & \text{if } r \neq 0. \end{cases}$$

Hence

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} p(t)\overline{q(t)}\,dt = \sum_j \sum_k c_j \overline{d}_k,$$

where the summation is taken over all subscripts j and k for which $r_j = s_k$. Let

$$\langle p, q\rangle = \sum_j \sum_k c_j \bar{d}_k.$$

Then $\langle\ ,\ \rangle$ is an inner product on the complex linear space X of all trigonometric polynomials on $\mathbf{R}$. Let H denote the completion of the inner product space X. Then H is a Hilbert space and X can be considered as a dense subspace of H. For a fixed $r \in \mathbf{R}$, let

$$u_r(t) = e^{irt}, \quad t \in \mathbf{R}.$$

Then $\{u_r : r \in \mathbf{R}\}$ is an uncountable orthonormal set in H. Since $\operatorname{span}\{u_r\} = X$ and X is dense in H, we see that $\{u_r\}$ is an orthonormal basis for H by 22.7(iv). □

Before concluding this section, we give some interesting characterizations of a Hilbert space having a countable orthonormal basis.

22.9 Theorem

Let H be a nonzero Hilbert space over $\mathbf{K}$. Then the following conditions are equivalent.

(i) H has a countable orthonormal basis.

(ii) H is linearly isometric to $\mathbf{K}^n$ for some n, or to ℓ^2.

(iii) H is separable.

Proof:

(i) implies (ii): Let $\{u_1, u_2, \ldots\}$ be a countable orthonormal basis for H. For $x \in H$, let

$$F(x) = (\langle x, u_1\rangle, \langle x, u_2\rangle, \ldots).$$

If $\{u_1, u_2, \ldots\}$ is a finite set having n elements, then F is a linear map from H to $\mathbf{K}^n$. If $\{u_1, u_2, \ldots\}$ is a denumerable set, then Bessel's inequality (22.4) shows that F is a linear map from H to ℓ^2. If we put the norm $\|\ \|_2$ on the range space of F, the Parseval formula (22.7(iii)) shows that

$$\|x\|^2 = \sum_n |\langle x, u_n\rangle|^2 = \|F(x)\|_2^2$$

for every $x \in H$. Hence F is an isometry. Finally, F is surjective by 22.5(a) and (b).

Condition (ii) implies condition (iii) since $\mathbf{K}^n$ as well as ℓ^2 are separable (3.2).

(iii) implies (i): Let $\{z_1, z_2, \ldots\}$ be a countable dense subset of H. Let x_1 be the first nonzero element among $z_1, z_2, \ldots$. Next, let x_2 be the first element among $z_2, z_3, \ldots$ which does not belong to span$\{x_1\}$. Having defined $x_1, \ldots, x_n$ in a similar manner, let x_{n+1} be the first element among $z_{n+1}, z_{n+2}, \ldots$ which does not belong to span$\{x_1, \ldots, x_n\}$. The inductively defined set $\{x_1, x_2, \ldots\}$ is linearly independent and span$\{x_1, x_2, \ldots\} =$ span$\{z_1, z_2, \ldots\}$.

By the Gram-Schmidt orthonormalization (22.2), find an orthonormal set $\{u_1, u_2, \ldots\}$ such that span$\{u_1, u_2, \ldots\} =$ span$\{x_1, x_2, \ldots\}$. It follows that span$\{u_1, u_2, \ldots\}$ is dense in H. Hence $\{u_1, u_2, \ldots\}$ is, in fact, a (countable) orthonormal basis for H by 22.7(iv). □

It is clear from the proof given above that a Hilbert space H has a finite orthonormal basis having n elements if and only if H is linearly isometric to $\mathbf{K}^n$ with the norm $\|\ \|_2$, and H has a denumerable orthonormal basis if and only if H is linearly isometric to ℓ^2. In particular, if $H = L^2([-\pi, \pi])$ and for $x \in H$, we let

$$F(x) = \sqrt{2\pi}(\ldots, \hat{x}(-2), \hat{x}(-1), \hat{x}(0), \hat{x}(-1), \hat{x}(2), \ldots),$$

where $\hat{x}(n)$ is the nth Fourier coefficient of x, then F is a linear isometry from H onto the linear space of all doubly infinite square-summable sequences. It is clear that this space is linearly isometric to ℓ^2. Finally, we point out that a countable orthonormal basis for H is, in particular, a Schauder basis for H. (See Section 8 for definition.) Thus we have shown above that every separable Hilbert space has a Schauder basis. Note that not every separable Banach space has a Schauder basis, as we have seen in Section 8.

Problems

X (resp., H) denotes an inner product space (resp., a Hilbert space) over $\mathbf{K}$, unless otherwise stated.

22-1 Let $\{x_1, \ldots, x_n\}$ be an orthogonal set in X and $k_1, \ldots, k_n$ be scalars having absolute value 1. Then $\|k_1x_1 + \cdots + k_nx_n\| = \|x_1 + \cdots + x_n\|$.

22-2 For $x, y \in X$, $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if and only if $\operatorname{Re}\langle x, y\rangle = 0$. In particular, if $\mathbf{K} = \mathbf{C}$ and $\dim X \geqslant 2$, then there are $x, y \in X$ such that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ without x being orthogonal to y. (Compare 22.1(a). Hint: 22.2)

22-3 Let $E \subset X$ be closed under scalar multiplication and $x \in X$. Then $x \perp E$ if and only if $\operatorname{dist}(x, E) = \|x\|$.

22-4 (a) The closed unit ball $\{x \in X : \|x\| \leq 1\}$ is compact if and only if X is finite dimensional.

(b) Every finite dimensional subspace of X is closed in X.

22-5 A Hilbert space cannot have a denumerable (Hamel) basis. (Compare 8.4, 22.9(i). Hint: 22.2 If $\{u_1, u_2, \ldots\}$ is a denumerable orthonormal set in H, then $\sum_{n=1}^{\infty} u_n/n$ does not belong to span $\{u_1, u_2, \ldots\}$.)

22-6 **(QR factorization)** Let A be an $m \times n$ matrix with entries in $\mathbf{K}$ such that the n columns of A are linearly independent in $\mathbf{K}^m$. Then there exist a unique $m \times n$ matrix Q and a unique $n \times n$ matrix R such that $A = QR$, where the n columns of Q are orthonormal in $\mathbf{K}^m$ and R is upper triangular with positive diagonal entries. (Hint: 22.2)

22-7 (a) If E and F are closed subsets of H and $E \perp F$, then $E + F$ is closed in H.

(b) If H is infinite dimensional, then there are closed subspaces E and F of H such that $E \cap F = \{0\}$, but $E + F$ is not closed in H. (Hint: If $\{u_1, u_2, \ldots\}$ is a denumerable orthonormal subset of H, let E be the closure of span $\{u_{2n-1} : n = 1, 2, \ldots\}$ and F be the closure of span $\{u_{2n-1} + u_{2n}/n : n = 1, 2, \ldots\}$.)

22-8 Let (x_n) be a sequence in H.

(a) If $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges in H.

(b) Let $\{x_1, x_2, \ldots\}$ be an orthogonal set in H. Then $\sum_{n=1}^{\infty} x_n$ converges in H if and only if $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$ if and only if there is some x in H such that $\langle x, x_n \rangle = \|x_n\|^2$ for all n.

22-9 Let $\{u_\alpha\}$ be an orthonormal set in H. For $x \in H$, let $E_x = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$. Then for every enumeration $u_1, u_2, \ldots$ of the countable set E_x, the series $\sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$ converges in H and its sum does not depend on the particular enumeration $u_1, u_2, \ldots$.

22-10 Let $F_1, F_2, \ldots$ be closed subspaces of H such that $F_n \perp F_m$ for all $n \neq m$. Let F denote the closure of span $\bigcup_n F_n$ and consider $x \in F$. Then for each $n = 1, 2, \ldots$, there is a unique $x_n \in F_n$ such that $x = \sum_n x_n$. (Hint: 22.7(ii) and Problem 22-8(b))

22-11 Let $\{u_\alpha\}$ be an orthonormal set in X.

(a) **(Fourier expansion, Parseval formula)** If x belongs to the closure of span $\{u_\alpha\}$, then

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n \quad \text{and} \quad \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2,$$

where $\{u_1, u_2, \ldots\} = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$.

(b) span $\{u_\alpha\}$ is dense in X if and only if every x in X has a Fourier expansion as above if and only if for every x, y in X, the **Parseval identity**

$$\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, u_n \rangle \langle u_n, y \rangle$$

holds, where $\{u_1, u_2, \ldots\} = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0 \text{ and } \langle y, u_\alpha \rangle \neq 0\}$.

22-12 Let $\{x_1, x_2, \ldots\}$ be an orthonormal basis for H. Then an orthonormal set $\{y_1, y_2, \ldots\}$ in H is an orthonormal basis for H if and only if $\sum_{m=1}^{\infty} |\langle y_m, x_n\rangle|^2 = 1$ for each $n = 1, 2, \ldots$.

22-13 Let $H = L^2([0,1])$. Then $\{1, \sqrt{2}\cos \pi t, \sqrt{2}\cos 2\pi t, \ldots\}$ as well as $\{\sqrt{2}\sin \pi t, \sqrt{2}\sin 3\pi t, \ldots\}$ are orthonormal bases for H.

22-14 Let (a,b) be a finite open interval and w be a weight function on (a,b) as defined in 22.3(b). Let $u_0, u_1, \ldots$ be the orthonormal polynomials with respect to w. Then $\{u_n\sqrt{w} : n = 0, 1, \ldots\}$ is an orthonormal basis for $L^2([a,b])$.

22-15 The Haar system given in Problem 8-14 is an orthonormal basis for $L^2([0,1])$.

22-16 An **almost periodic continuous function** on $\mathbf{R}$ is defined to be a uniform limit of a sequence of trigonometric polynomials on $\mathbf{R}$. Let $AP(\mathbf{R})$ denote the linear space of all such functions. Then for $x, y \in AP(\mathbf{R})$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\overline{y(t)}\, dt$$

exists and defines an inner product on $AP(\mathbf{R})$. For a fixed $r \in \mathbf{R}$, let $u_r(t) = e^{irt}, t \in \mathbf{R}$. Then $\{u_r : r \in \mathbf{R}\}$ is an uncountable maximal orthonormal subset of $AP(\mathbf{R})$.

22-17 If H has a denumerable orthonormal basis, then every orthonormal basis for H is denumerable.

22-18 Let T be a set and H denote the set of all scalar-valued functions x on T such that the set $\{t \in T : x(t) \neq 0\}$ is countable and $\sum_{t\in T} |x(t)|^2 < \infty$. For $x, y \in H$, let

$$\langle x, y\rangle = \sum_{t\in T} x(t)\overline{y(t)}.$$

Then H is a Hilbert space with the inner product $\langle\ ,\ \rangle$. For a fixed $t \in T$, let $u_t(s) = \delta_{s,t},\ s \in T$. Then the subset $\{u_t : t \in T\}$ of H is in one-to-one correspondence with T and constitutes an orthonormal basis for H. Further, H is separable if and only if the set T is countable. [Note: If $T = \{1, \ldots, n\}$, then $H = \mathbf{K}^n$ and if $T = \{1, 2, \ldots\}$, then $H = \ell^2$.]

23 Approximation and Optimization

This section is important for applications of the geometric properties of an inner product space. However, it can be skipped without affecting our development of the theory. Here we deal with some problems of approximation which have a bearing on optimization subject to certain constraints. We consider the following question. What best can be done if we want to come close to a given element of an inner product space X while having to remain in a given subset of X? To formalize our study, we introduce the following notion.

Let X be an inner product space and E be a subset of X. Given an element x of X, an element y of E is said to be a **best approximation** from E to x if $\|x - y\| \leq \|x - z\|$ for all $z \in E$, that is, $\|x - y\| = \operatorname{dist}(x, E)$. Such an element y is also known as an **optimal solution** of the following problem:

$$\text{` Minimize } \|x - z\|, \text{ subject to } z \in E. \text{ '}$$

Then $x - y$ is known as an **optimal error**.

Three natural questions arise. Does a best approximation from E to x exist? Can there be more than one best approximations from E to x? How can one find a best approximation from E to x? We now address ourselves to these questions. We also seek a usable characterization of a best approximation from a subspace of X to an element of X.

23.1 Theorem

Let X be an inner product space.

(a) Let $E \subset X$ and $x \in \overline{E}$. Then there exists a best approximation from E to x if and only if $x \in E$.

(b) If $E \subset X$ is convex, then there exists at most one best approximation from E to any $x \in X$..

(c) Let F be a subspace of X and $x \in X$. Then $y \in F$ is a best approximation from F to x if and only if $x - y \perp F$ and in that case

$$\operatorname{dist}(x, F) = \langle x, x - y\rangle^{1/2}$$

Proof:

(a) If $x \in E$, then clearly x is a best approximation to itself from E. Conversely, let $y \in E$ be a best approximation from E to x . Then $\|x - y\| = \operatorname{dist}(x, E) = 0$, that is, $x = y$. Hence $x \in E$.

(b) Let y_1 and y_2 be best approximations from a convex subset E of X to $x \in X$. By the parallelogram law (21.2(b)) for $x - y_1$ and $x - y_2$, we have

$$2\|x - y_1\|^2 + 2\|x - y_2\|^2 = \|2x - y_1 - y_2\|^2 + \|y_2 - y_1\|^2.$$

Let $d = \operatorname{dist}(x, E)$. Then $\|x - y_1\| = d = \|x - y_2\|$. Also, since $(y_1 + y_2)/2 \in E$, we have $\|x - (y_1 + y_2)/2\| \geq d$. Hence

$$0 \leq \|y_2 - y_1\|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0,$$

that is, $y_2 = y_1$, showing that there is at most one best approximation from E to x.

(c) Let $y \in F$ such that $x - y \perp F$. Consider $z \in F$. Since F is a subspace, $y - z \in F$, so that $x - y \perp y - z$. By the Pythagoras theorem (22.1(a)),

$$\|x - z\|^2 = \|(x - y) + (y - z)\|^2 = \|x - y\|^2 + \|y - z\|^2.$$

Hence $\|x - y\| \leq \|x - z\|$ for all $z \in F$, that is, y is a best approximation from F to x.

Conversely, let $y \in F$ be a best approximation from F to x. To show that $x - y \perp F$, we may consider $z \in F$ with $\|z\| = 1$. Then

$w = y + \langle x - y, z \rangle z$ belongs to F, since F is a subspace. Hence

$$\begin{aligned} \|x - y\|^2 &\le \|x - w\|^2 = \langle x - w, x - w \rangle \\ &= \langle x - y, x - y \rangle - |\langle x - y, z \rangle|^2 \langle z, z \rangle \\ &= \|x - y\|^2 - |\langle x - y, z \rangle|^2. \end{aligned}$$

Thus $\langle x - y, z \rangle = 0$, that is, $x - y \perp z$. (Compare Problem 22-3.) Further, since $y \perp x - y$, we have

$$[\operatorname{dist}(x, F)]^2 = \|x - y\|^2 = \langle x - y, x - y \rangle = \langle x, x - y \rangle. \qquad \square$$

The result 23.1(a) shows that if a subset E of X is not closed, then there is some x in X such that there is no best approximation from E to x, while 23.1(b) shows that if E is convex, then there cannot be two best approximations from E to any $x \in X$. We shall show in 23.5 that if E is a nonempty closed convex subset of a complete inner product space X, then for each $x \in X$, there is a unique best approximation from E to x.

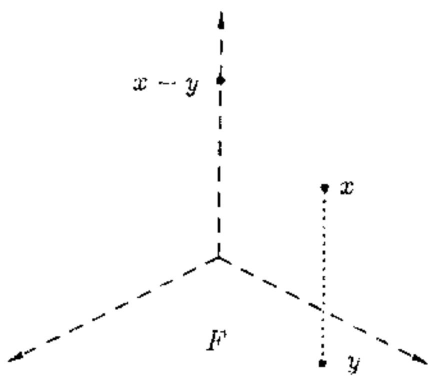


Figure 9

While the result 23.1(c) does not guarantee the existence of a best approximation from a subspace F of an inner product space X to

$x \in X$, it does give a clue for finding it. It tells us to look for some $y \in F$ such that $x - y \perp F$. (See Problem 23-14 for an analogue of this result for convex sets.) The elements $x, y, x - y$ and the subspace F are schematically shown in Figure 9: y is obtained by dropping a perpendicular from x to F.

We shall now show that if F is a finite dimensional subspace of X, then best approximations from F always exist and they can be written down explicitly. The same holds for a subset consisting of all elements of X satisfying a finite number of linear constraints.

23.2 Theorem

Let X be an inner product space, $\{x_1, \ldots, x_m\}$ be a linearly independent subset of X and $x \in X$.

(a) Let $F = \operatorname{span}\{x_1, \ldots, x_m\}$. Then the **unique best approximation from F to x is given by**

$$y = k_1 x_1 + \cdots + k_m x_m,$$

where $k_1, \ldots, k_m$ form the unique solution of the **normal equations**:

$$\begin{array}{rcl} \langle x_1, x_1\rangle k_1 + \cdots + \langle x_m, x_1\rangle k_m & = & \langle x, x_1\rangle \\ \vdots \qquad\qquad \vdots & & \vdots \\ \langle x_1, x_m\rangle k_1 + \cdots + \langle x_m, x_m\rangle k_m & = & \langle x, x_m\rangle. \end{array}$$

Also,

$$\operatorname{dist}(x, F) = \langle x, x - k_1 x_1 - \cdots - k_m x_m\rangle^{1/2}.$$

(b) Let $c_1, \ldots, c_m$ be scalars and

$$E = \{y \in X : \langle y, x_j\rangle = c_j \text{ for } j = 1, \ldots, m\}.$$

Then the unique best approximation from E to x is given by

$$y = x + k_1 x_1 + \cdots + k_m x_m,$$

where $k_1, \ldots, k_m$ form the unique solution of the equations

$$\begin{array}{rcl} \langle x_1, x_1\rangle k_1 + \cdots + \langle x_m, x_1\rangle k_m & = & c_1 - \langle x, x_1\rangle \\ \vdots \qquad\qquad \vdots & & \vdots \\ \langle x_1, x_m\rangle k_1 + \cdots + \langle x_m, x_m\rangle k_m & = & c_m - \langle x, x_m\rangle. \end{array}$$

Also,

$$\operatorname{dist}(x, E) = \|k_1 x_1 + \cdots + k_m x_m\|.$$

Proof:
First we show that the m columns of the matrix

$$M = \begin{bmatrix} \langle x_1, x_1\rangle & \cdots & \langle x_m, x_1\rangle \\ \vdots & & \vdots \\ \langle x_1, x_m\rangle & \cdots & \langle x_m, x_m\rangle \end{bmatrix}$$

are linearly independent elements of K^m, so that M is invertible. If

$$\begin{array}{rcl} a_1\langle x_1, x_1\rangle + \cdots + a_m\langle x_m, x_1\rangle & = & 0 \\ \vdots \qquad\qquad \vdots & & \vdots \\ a_1\langle x_1, x_m\rangle + \cdots + a_m\langle x_m, x_m\rangle & = & 0 \end{array}$$

for some scalars $a_1, \ldots, a_m$, then

$$\begin{aligned} \|a_1 x_1 + \cdots + a_m x_m\|^2 &= \langle a_1 x_1 + \cdots + a_m x_m, a_1 x_1 + \cdots + a_m x_m\rangle \\ &= \sum_{j=1}^{m} \overline{a}_j \langle a_1 x_1 + \cdots + a_m x_m, x_j\rangle \\ &= \sum_{j=1}^{m} \overline{a}_j \cdot 0 = 0, \end{aligned}$$

that is, $a_1 x_1 + \cdots + a_m x_m = 0$. But since $\{x_1, \ldots, x_m\}$ is a linearly independent subset of X, we see that $a_1 = \cdots = a_m = 0$.

(a) By 23.1(c), an element $y = k_1 x_1 + \cdots + k_m x_m$ of F is a best approximation from F to x if and only if $x - y \perp F$, that is, $\langle y, x_j\rangle = \langle x, x_j\rangle$ for each $j = 1, \ldots, m$. This means

$$k_1\langle x_1, x_j\rangle + \cdots + k_m\langle x_m, x_j\rangle = \langle x, x_j\rangle, \quad j = 1, \ldots, m.$$

These are the normal equations stated in the theorem. We have shown above that the coefficient matrix M of this system of m equations in the m unknowns $k_1, \ldots, k_m$ is invertible. Hence there are unique scalars $k_1, \ldots, k_m$ which satisfy the normal equations. Then $y = k_1x_1 + \cdots + k_mx_m$ is the unique best approximation to x from F. Further, as we have seen in 23.1(c),

$$[\operatorname{dist}(x, F)]^2 = \langle x,\, x - y\rangle = \langle x,\, x - k_1x_1 - \cdots - k_mx_m\rangle.$$

(b) The nonhomogeneous system

$$\langle x_1,\, x_j\rangle k_1 + \cdots + \langle x_m,\, x_j\rangle k_m = c_j - \langle x,\, x_j\rangle, \quad j = 1, \ldots, m$$

of m linear equations in m unknowns has a unique solution $k_1, \ldots, k_m$, since its coefficient matrix M is invertible. Let $y = x + k_1x_1 + \cdots + k_mx_m$. Then it follows that $\langle y,\, x_j\rangle = c_j$ for $j = 1, \ldots, m$, that is, $y \in E$. To show that y is the unique best approximation from E to x, we show that 0 is the unique best approximation from $E - y$ to $x - y$. But this follows from 23.1(b) and 23.1(c) by noting that $E - y$ is a subspace of X and $x - y \perp E - y$, because for every $z \in E$, we have

$$\begin{aligned}
\langle z - y,\, x - y\rangle &= \langle z - y,\, -k_1x_1 - \cdots - k_mx_m\rangle \\
&= -\overline{k}_1\langle z - y,\, x_1\rangle - \cdots - \overline{k}_n\langle z - y,\, x_m\rangle \\
&= -\overline{k}_1(c_1 - c_1) - \cdots - \overline{k}_n(c_m - c_m) \\
&= 0.
\end{aligned}$$

Also, as we have noted in 23.1(c),

$$[\operatorname{dist}(x, E)]^2 = [\operatorname{dist}(x - y, E - y)]^2 = \langle x - y,\, x - y - 0\rangle = \|x - y\|^2,$$

so that $\operatorname{dist}(x, E) = \|x - y\| = \|k_1x_1 + \cdots + k_mx_m\|$. □

The unique solutions of the systems of equations in parts (a) and (b) of the preceding theorem can be written down by using Cramer's rule and by considering the **Gram matrix**

$$G(x_1, \ldots, x_m) = \begin{bmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_m, x_1 \rangle \\ \vdots & & \vdots \\ \langle x_1, x_m \rangle & \cdots & \langle x_m, x_m \rangle \end{bmatrix},$$

where $x_1, \ldots, x_m$ are in X. (For example, see Problem 23-3.) However, this is seldom done in practice, because it may involve too many computations and because the procedure may not be numerically stable. For example, if $X = C([0,1])$, and $x_m(t) = t^{m-1}, m = 1, \ldots,$ then the (i,j)th entry of the matrix $G(x_1, \ldots, x_m)$ is

$$\langle x_j, x_i \rangle = \int_0^1 t^{i+j-2}\, dt = \frac{1}{i+j-1}, \quad i, j = 1, 2, \ldots.$$

This matrix, called the **Hilbert matrix**, is known to be numerically intractable. If $1 \le n_1 < n_2 < \cdots$ are positive integers and $F_k = \operatorname{span}\{x_{n_1}, \ldots, x_{n_k}\}$, then the formula for $\operatorname{dist}(x, F_k)$ in terms of the Gram matrices given in Problem 23-3 can be used to prove an interesting theorem of Muntz (1914):

$$\operatorname{span}\{t^{n_1}, t^{n_2}, \ldots\} \text{ is dense in } L^2([0,1]) \text{ if and only if } \sum_{j=1}^{\infty} \frac{1}{n^j} = \infty.$$

(See Problem 23-4.)

We now proceed to consider best approximations from an infinite dimensional subspace F of an inner product space X. The method of reducing the question to solving a finite system of linear equations is now inadequate. Also, if we are to guarantee a best approximation from F to every $x \in X$, then by 23.1(a), we must assume that F is closed in X.

Suppose that $\{x_1, x_2, \ldots\}$ is an infinite linearly independent subset of X, and let F denote the closure of its span. One may attempt to find a best approximation from F to $x \in X$ as follows. For $m = 1, 2, \ldots,$ let $F_m = \operatorname{span}\{x_1, \ldots, x_m\}$, and let y_m denote the unique best approximation from F_m to x. If $y_m \to y$ in X, then it is legitimate to expect that y is a best approximation from F to x. The convergence of the sequence (y_m) can be achieved by assuming that the inner product space X is complete. Yet, there is another difficulty. To find y_m as in 23.2(a), one needs to solve a system of m equations in m unknowns with coefficient matrix

$$M_m = \begin{bmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_m, x_1 \rangle \\ \vdots & & \vdots \\ \langle x_1, x_m \rangle & \cdots & \langle x_m, x_m \rangle \end{bmatrix}, \quad m = 1, 2, \ldots.$$

It can be seen that the solution of this $m \times m$ system is in no way useful for solving the next $(m+1) \times (m+1)$ system with coefficient matrix M_{m+1}. Hence the process of obtaining y_m can be long and tedious. Instead, if we orthonormalize the set $\{x_1, \ldots, x_m\}$ by the Gram-Schmidt process 22.2, then this difficulty can be overcome. If the set $\{x_1, \ldots, x_m\}$ is orthonormal, then the coefficient matrix $G(x_1, \ldots, x_m)$ reduces to the $m \times m$ identity matrix and the coefficients of $x_1, \ldots, x_m$ in the expression for y_{m+1} remain unaltered. Because of this permanence of form, the technique of orthonormalization in an inner product space proves to be very powerful.

23.3 Theorem

Let H be a Hilbert space, $\{x_1, x_2, \ldots\}$ be a linearly independent subset of H and let F denote the closure of its span. Let $u_1, u_2, \ldots$ be obtained from $x_1, x_2, \ldots$ by the Gram-Schmidt orthonormalization 22.2. For $m = 1, 2, \ldots$ and $x \in H$, let

$$F_m = \operatorname{span}\{x_1, \ldots, x_m\} \quad \text{and} \quad y_m = \sum_{n=1}^{m} \langle x, u_n \rangle u_n.$$

Then y_m is the unique best approximations from F_m to x. The sequence (y_m) converges to

$$y = \sum_n \langle x, u_n \rangle u_n,$$

which is the unique best approximation from F to x. Also,

$$\operatorname{dist}(x, F) = \left(\|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\, u_n\rangle|^2\right)^{1/2}$$

Proof:
Fix a positive integer m. Then for $n = 1, \ldots, m$,

$$\langle x - y_m,\, u_n\rangle = \langle x,\, u_n\rangle - \langle \sum_{j=1}^{m} \langle x,\, u_j\rangle u_j,\, u_n\rangle = \langle x,\, u_n\rangle - \langle x,\, u_n\rangle = 0,$$

so that $x - y_m \perp \operatorname{span}\{u_1, \ldots, u_m\} = \operatorname{span}\{x_1, \ldots, x_m\}$ by 22.2. Since $y_m \in F_m$ and $x - y_m \perp F_m$, it follows by 23.1(b) and 23.1(c) that y_m is the unique best approximation from F_m to x. [Alternatively, the normal equations given in 23.2 reduce to $k_1 = \langle x,\, u_1\rangle, \ldots, k_m = \langle x,\, u_m\rangle$. Hence $y_m = k_1u_1 + \cdots + k_mu_m = \sum_{n=1}^{m} \langle x,\, u_n\rangle u_n$ is the unique best approximation from $F_m = \operatorname{span}\{u_1, \ldots, u_m\}$ to x.]

Since H is a Hilbert space, Lemma 22.6 shows that the series $\sum_n \langle x,\, u_n\rangle u_n$ converges to some y in H with $x - y \perp u_n$ for each $n = 1, 2, \ldots$. As F is the closure of $\operatorname{span}\{x_1, x_2, \ldots\}$, we see that $y \in F$ and $x - y \perp F$. It follows again by 23.1(b) and 23.1(c) that y is the unique best approximation from F to x and

$$[\operatorname{dist}(x, F)]^2 = \langle x,\, x - y\rangle = \|x\|^2 - \sum_n |\langle x,\, u_n\rangle|^2. \qquad \square$$

23.4 Examples

(a) Least square approximation problems in analysis can be cast as best approximation problems in appropriate inner product spaces. Suppose that x is a scalar-valued continuous function on $[a, b]$ and we want to find a polynomial p of degree at most m such that

$$\int_a^b |x(t) - p(t)|^2\, dt$$

is minimized. Let $X = C([a, b])$ with the inner product induced from $L^2([a, b])$ and let P_m be the subspace of X of all polynomials of degree at most m. Then 23.2(a) and 23.3 give methods for finding such a unique polynomial p.

Let us take a concrete case. Let $a = -1, b = 1$ and $x(t) = e^t$ for $-1 \le t \le 1$. First, let $m = 1$. If $x_0(t) = 1$ and $x_1(t) = t$, then $P_1 = \operatorname{span}\{x_0, x_1\}$. It can be easily seen that

$$\langle x_0,\, x_0\rangle = 2, \quad \langle x_0,\, x_1\rangle = 0, \quad \langle x_1,\, x_1\rangle = \frac{2}{3},$$

$$\langle x,\, x_0\rangle = e - e^{-1}, \quad \langle x,\, x_1\rangle = 2e^{-1}, \quad \langle x,\, x\rangle = \frac{1}{2}(e^2 - e^{-2}).$$

The two normal equations of 23.2(a) are

$$2k_0 + 0k_1 = e - e^{-1} \quad \text{and} \quad 0k_0 + \frac{2}{3}k_1 = 2e^{-1}.$$

Hence $k_0 = (e - e^{-1})/2$ and $k_1 = 3e^{-1}$, so that

$$p_1(t) = k_0 x_0(t) + k_1 x_1(t) = \frac{1}{2}(e - e^{-1}) + 3e^{-1}t, \quad -1 \le t \le 1,$$

is the best linear least square approximation to x. Also,

$$[\operatorname{dist}(x, P_1)]^2 = \frac{1}{2}(e^2 - e^{-2}) - \frac{1}{2}(e - e^{-1})^2 - 6e^{-2} = 1 - 7e^{-2}.$$

If we wish to find the best quadratic least square approximation to x, then we must additionally consider $x_2(t) = t^2$, find $\langle x_0,\, x_2\rangle, \langle x_1,\, x_2\rangle$, $\langle x_2,\, x_2\rangle, \langle x,\, x_2\rangle$ and solve the three corresponding normal equations again. This is the method outlined in 23.2(a).

On the other hand, as in 23.3, we may orthonormalize the functions x_0, x_1 and x_2 to obtain the Legendre polynomials

$$u_0(t) = \frac{1}{\sqrt{2}}, \quad u_1(t) = \frac{\sqrt{3}}{\sqrt{2}}\, t \quad \text{and} \quad u_2(t) = \frac{3\sqrt{10}}{4}\left(t^2 - \frac{1}{3}\right).$$

(See 22.3(b).) Since

$$\langle x,\, u_0\rangle = \frac{1}{\sqrt{2}}(e - e^{-1}),\ \langle x,\, u_1\rangle = \sqrt{6}\, e^{-1} \text{ and } \langle x,\, u_2\rangle = \frac{\sqrt{5}}{\sqrt{2}}(e - 7e^{-1}),$$

the best linear and the best quadratic least square approximations to x are given by

$$p_1(t) = \frac{1}{\sqrt{2}}(e - e^{-1})\frac{1}{\sqrt{2}} + \sqrt{6}e^{-1}\frac{\sqrt{3}}{\sqrt{2}}\,t = \frac{1}{2}(e - e^{-1}) + 3e^{-1}t,$$

$$p_2(t) = p_1(t) + \frac{\sqrt{5}}{\sqrt{2}}(e - 7e^{-1})\frac{3\sqrt{10}}{4}(t^2 - \frac{1}{3}) = p_1(t) + \frac{15}{4}(e - 7e^{-1})(t^2 - \frac{1}{3}),$$

respectively. Also,

$$[\mathrm{dist}(x, P_2)]^2 = [\mathrm{dist}(x, P_1)]^2 - \frac{5}{2}(e - 7e^{-1})^2 = 36 - \frac{5}{2}e^2 - \frac{259}{2}e^{-2}.$$

Notice that in this method one makes only a bit of extra effort to obtain p_2 from p_1, and $\mathrm{dist}(x, P_2)$ from $\mathrm{dist}(x, P_1)$. [Roughly, $\mathrm{dist}(x, P_1)$ and $\mathrm{dist}(x, P_2)$ equal 0.229 and 0.038, respectively.]

(b) Consider the optimization problem

$$\text{` Minimize } \int_0^1 |y|^2\,dm, \quad \text{subject to}$$

$$y \in L^2([0,1]),\ \int_0^1 t\,y(t)\,dm(t) = 1 \text{ and } \int_0^1 t^2 y(t)\,dm(t) = 2.\text{'}$$

Thus we need to find a best approximation to 0 from the set

$$E = \{y \in L^2([0,1]) : \int_0^1 t\,y(t)\,dm(t) = 1 \text{ and } \int_0^1 t^2 y(t)\,dm(t) = 2\}.$$

Let $x_1(t) = t$ and $x_2(t) = t^2, t \in [0,1]$. Since

$$\langle x_1, x_1\rangle = \frac{1}{3}, \quad \langle x_1, x_2\rangle = \frac{1}{4} \quad \text{and} \quad \langle x_2, x_2\rangle = \frac{1}{5},$$

23.2(b) shows that the unique best approximation to 0 from the set E is $y = 0 + k_1x_1 + k_2x_2$, where

$$\frac{1}{3}k_1 + \frac{1}{4}k_2 = 1 \quad \text{and} \quad \frac{1}{4}k_1 + \frac{1}{5}k_2 = 2.$$

Hence $k_1 = -72, k_2 = 100$, so that $y(t) = -72t + 100t^2,\ t \in [0,1]$.

For a minimization problem which combines the results 23.2(a) and 23.2(b), see Problem 23-13.

(c) Let $H = L^2([-\pi, \pi])$ and for $n = 1, 2, \ldots,$

$$u_n(t) = \frac{\sin nt}{\sqrt{\pi}}, \quad t \in [-\pi, \pi].$$

Let F denote the closure of span $\{u_1, u_2, \ldots\}$. If $x \in H$ has the Fourier expansion

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt), \quad t \in [-\pi, \pi],$$

as in 22.8(b), then the unique best approximation from F to x is

$$y(t) = \sum_{n=1}^{\infty} b_n \sin nt, \quad t \in [-\pi, \pi].$$

As a particular case, let $x(t) = t + t^2, t \in [-\pi, \pi]$. Since

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (t + t^2) \sin nt\, dt = \frac{2}{n}(-1)^{n+1},$$

we see that the best approximation from F to x is given by

$$y(t) = 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \cdots\right), \quad t \in [-\pi, \pi].$$

Note that $y(t) = t, t \in [-\pi, \pi]$. [Alternatively, $y \in F$ since y is an odd function and $x - y \perp F$ since $x - y$ is an even function. Hence by 23.1(c), y is the unique best approximation from F to x.] □

We now prove the existence and the uniqueness of a best approximation from a nonempty closed convex subset of a Hilbert space. The proof is nonconstructive, but it is coordinate-free in the sense that it does not employ any orthonormal basis.

23.5 Theorem

Let E be a nonempty closed convex subset of a Hilbert space H. Then for each $x \in H$, there exists a unique best approximation from E to x. In particular, there is a unique element in E of minimal norm.

Proof:
Let $x \in H$ and $d = \operatorname{dist}(x, E)$. Then there is a sequence (y_n) in E such that $\|x - y_n\| \to d$. Applying the parallelogram law 21.2(b) to $x - y_n$ and $x - y_m$, we obtain

$$2\|x - y_n\|^2 + 2\|x - y_m\|^2 = \|2x - y_n - y_m\|^2 + \|y_m - y_n\|^2$$

for all $m, n = 1, 2, \ldots$. Since E is convex, $(y_n + y_m)/2$ belongs to E and $\|x - (y_n + y_m)/2\| \geq d$. Hence

$$\|y_m - y_n\|^2 \leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0.$$

This shows that (y_n) is a Cauchy sequence in E. Since H is complete and E is closed in H, we see that $y_n \to y$ in E. It is clear that

$$\|x - y\| = \lim_{n \to \infty} \|x - y_n\| = d.$$

Thus y is a best approximation from E to x. The uniqueness of y follows from 23.1(b), the proof being very similar to the argument given above.

The unique best approximation from E to 0 is the unique element in E of minimal norm. □

For a characterization of the best approximation from a nonempty closed convex subset E of a Hilbert space to $x \in H$, see Problem 23-14. If E is, in fact, a closed subspace of H, then 23.1(c) shows that the best approximation from E to x is given by the unique $y \in E$ for which $x - y \perp E$.

23.6 Example
We describe in brief a quadratic loss control problem for a dynamical system under a linear dynamical constraint. Consider

$$\dot{x}(t) = \alpha x(t) + \beta u(t) \quad \text{for } 0 \leq t \leq 1, \quad x(0) = k_0.$$

Here the real number $x(t)$ denotes the 'state' of the system at time t and the real number $u(t)$ denotes the control input at time t; the dot denotes differentiation with respect to t and α, β are real constants. Starting with the initial state k_0, the system produces the state $x(t)$ when the control $u(t)$ is applied at time t.

The problem of minimizing the quadratic objective functional

$$Q(x,u) = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]\,dm(t).$$

consists of making a suitable choice of the control $u(t)$ so that a compromise is achieved in keeping the states $x(t)$ of the system small and at the same time conserving the total control energy $\int_0^1 u^2(t)\,dt$ of the system.

We can easily integrate the differential equation governing the system and use the initial condition to obtain

$$x(t) = e^{\alpha t}\left[k_0 + \beta\int_0^t e^{-\alpha s}u(s)\,dm(s)\right], \quad 0 \le t \le 1.$$

The problem can now be reformulated as follows.

Let $\mathbf{K} = \mathbf{R}$ and $H = L^2([0,1]) \times L^2([0,1])$. For (x_1, u_1) and (x_2, u_2) in H, define

$$\langle (x_1,u_1),\, (x_2,u_2)\rangle = \int_0^1 [x_1(t)x_2(t) + u_1(t)u_2(t)]\,dm(t).$$

Then $\langle\ ,\ \rangle$ is an inner product on H and H is complete in the norm given by

$$\|(x,u)\| = \left\{\int_0^1 [x^2(t) + u^2(t)]\,dm(t)\right\}^{1/2},$$

which equals $[2Q(x,u)]^{1/2}$. The problem is to minimize $\|(x,u)\|$, subject to the conditions

$$x, u \in L^2([0,1]) \quad \text{and} \quad x = h(u),$$

where

$$h(u)(t) = e^{\alpha t}\left[k_0 + \beta\int_0^t e^{-\alpha s}u(s)\,dm(s)\right]$$

for almost all $t \in [0,1]$.

If $k_0 = 0$, then the unique solution of this problem is $x = 0 = u$. In general, let

$$E = \left\{(h(u), u) : u \in L^2([0,1])\right\}.$$

It is easy to see that E is a nonempty convex subset of the Hilbert space H. To show that E is closed in H, we first note that $(x_n, u_n) \to (x, u)$ in H if and only if $\|x_n - x\|_2 \to 0$ and $\|u_n - u\|_2 \to 0$. Also,

$$h(u_n)(t) - h(u)(t) = \beta e^{\alpha t} \int_0^t e^{-\alpha s}[u_n(s) - u(s)]\, dm(s),$$

so that Hölder's inequality (4.5(a)) gives

$$\begin{aligned} |h(u_n)(t) - h(u)(t)|^2 &\leq \beta^2 e^{2\alpha t} \int_0^t e^{-2\alpha s}\, ds \int_0^t |u_n(s) - u(s)|^2\, dm(s) \\ &\leq \frac{\beta^2}{2\alpha}(e^{2\alpha t} - 1)\|u_n - u\|_2^2 \end{aligned}$$

for almost all $t \in [0,1]$. Integrating over $[0,1]$, we obtain

$$\|h(u_n) - h(u)\|_2 \leq \gamma \|u_n - u\|_2,$$

where $\gamma = |\beta|\sqrt{e^{2\alpha} - 1 - 2\alpha}/2|\alpha|$. Thus $h(u_n) \to h(u)$ in $L^2([0,1])$ whenever $u_n \to u$ in $L^2([0,1])$. Hence if $(h(u_n), u_n) \to (x, u)$ in H, then $x = h(u)$. This shows that E is closed in H.

Our minimization problem thus consists of finding a best approximation to 0 from the nonempty closed convex subset E of the Hilbert space H. By 23.5, it has a unique solution. In order to find this solution one may proceed as follows.

Let $u_0 \in L^2([0,1])$ and $x_0 = h(u_0)$. Then

$$E = (x_0, u_0) + F,$$

where $F = \{(h_0(u), u) : u \in L^2([0,1])\}$ with

$$h_0(u)(t) = \beta e^{\alpha t} \int_0^t e^{-\alpha s} u(s)\, dm(s)$$

for almost all $t \in [0,1]$. It is easy to see that F is a nonempty closed subspace of H. Consider a linearly independent subset $\{u_1, u_2, \ldots\}$

of $L^2([0,1])$ whose span is dense in $L^2([0,1])$. Then the span of the linearly independent set $\{(h_0(u_1), u_1), (h_0(u_2), u_2), \ldots\}$ is dense in F. Now 23.3 can be employed to obtain a best approximation from F to (x_0, u_0), which is then the unique best approximation to 0 from E.

Problems

X denotes an inner product space over $\mathbf{K}$, unless otherwise stated.

23-1 Let $\{x_1, x_2, \ldots\}$ be a linearly independent subset of X. Let $y_1, y_2, \ldots$ and $u_1, u_2, \ldots$ be as given in the Gram-Schmidt orthonormalization 22.2. Then y_n is the optimal error of the problem

'Minimize $\|x_n - z\|$, subject to $z \in \operatorname{span}\{x_1, \ldots, x_{n-1}\}$,'

and u_n is the normalized optimal error.

23-2 Let w be a weight function on (a,b). Let x be a measurable function on (a,b) such that $\int_a^b |x|^2 w\, dm < \infty$. Let u_j be the orthonormal polynomial of degree j with respect to the weight function w and $\langle x, u_j\rangle_w = \int_a^b x\overline{u}_j w\, dm, j = 0, \ldots, m$. Then the polynomial p of degree at most m which minimizes $\int_a^b |x - p|^2 w\, dm$ is given by $\sum_{j=0}^m \langle x, u_j\rangle_w u_j$.

23-3 Let $\{x_1, \ldots, x_m\}$ be a linearly independent subset of X and $x \in X$. Then the unique best approximation from $F = \operatorname{span}\{x_1, \ldots, x_m\}$ to x is given by

$$y = \sum_{j=1}^m \frac{\det G_j(x_1, \ldots, x_m)}{\det G(x_1, \ldots, x_m)} x_j,$$

where $G_j(x_1, \ldots, x_m)$ is obtained from the Gram matrix $G(x_1, \ldots, x_m)$ by replacing its jth column by $\langle x, x_1\rangle, \ldots, \langle x, x_m\rangle$. Also,

$$[\operatorname{dist}(x, F)]^2 = \frac{\det G(x_1, \ldots, x_m, x)}{\det G(x_1, \ldots, x_m)}.$$

23-4 **(Muntz, 1914)** Let $1 \le n_1 < n_2 < \ldots$ be positive integers. Let $x_n(t) = t^n$ for $t \in [0,1]$ and $n = 1, 2, \ldots$. Then span $\{x_{n_1}, x_{n_2}, \ldots\}$ is dense in

$L^2([0,1])$ if and only if $\sum_{j=1}^{\infty} 1/n_j = \infty$. (Hint: Let $F_k = \text{span}\,\{x_{n_1}, \ldots, x_{n_k}\}$. For each fixed $m \neq n_1, n_2, \ldots$,

$$\operatorname{dist}(x_m, F_k) = \frac{\prod_{j=1}^{k}(n_j - m)}{\sqrt{2m+1}\prod_{j=1}^{k}(n_j + m + 1)},$$

$\operatorname{dist}(x_m, F_k) \to 0$ if and only if $\sum_{j=1}^{k} \log|(n_j - m)/(n_j + m + 1)| \to -\infty$ as $k \to \infty$.)

23-5 The set of all polynomials having only even degree terms is dense in $L^2([0,1])$, but the set of all polynomials whose terms are of degrees $2, 4, 8, \ldots$ is not dense in $L^2([0,1])$. (Hint: Problem 23-4)

23-6 Let F be a closed subspace of a Hilbert space H and $\{u_\alpha\}$ be an orthonormal basis for F. Let $x \in H$ and $\{u_1, u_2, \ldots\} = \{u_\alpha : (x,\, u_\alpha) \neq 0\}$. Then $y = \sum_n (x,\, u_n) u_n$ is the unique best approximation from F to x. (Hint: 22.6 and 23.1(c). Compare 23.3.)

23-7 Let $X = C([-1,1]), x(t) = 1 - t^2, x_0(t) = 1$ and $x_1(t) = \cos \pi t$, for t in $[0,1]$. The best approximation from $\text{span}\,\{x_0, x_1\}$ is $2/3 + 4x_1/\pi^2$ to x.

23-8 The element $x \in L^2([-\pi,\pi])$ of minimal norm satisfying

$$\int_{-\pi}^{\pi} t\, x(t)\, dm(t) = 1 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin t\, x(t)\, dm(t) = 2$$

is given by $x(t) = [9t/2 + (3 - 2\pi^2)\sin t]/\pi(6 - \pi^2)$ for $t \in [-\pi, \pi]$.

23-9 Let $0 \neq x_1 \in X$ and $c_1 \in \mathbf{K}$. Then the element $x \in X$ of minimal norm satisfying $(x,\, x_1) = c_1$ is $c_1 x_1/(x_1,\, x_1)$.

23-10 Let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be in X and $c_1, \ldots, c_m$ be scalars. If $E = \{y \in \text{span}\,\{x_1, \ldots, x_n\} : (y,\, y_j) = c_j, j = 1, \ldots, m\}$ is nonempty, then there is a unique best approximation from E to every $x \in X$. (Hint: If $x_0 \in E$, then let $F = E - x_0$ in 23.2(a).)

23-11 Let $\{u_1, u_2, \ldots\} \bigcup \{v_\alpha\}$ be an orthonormal set in H, and let F and G respectively denote the closures of span $\{u_1, u_2, \ldots\} \bigcup \{v_\alpha\}$ and span $\{v_\alpha\}$. If $x \in H$ and y is the best approximation from F to x, then $y - \sum_n (x,\, u_n) u_n$ is the best approximation from G to x.

23-12 Let F be a subspace of X and G be a subspace of F. Let $x \in X$. If y is a best approximation from F to x and z is a best approximation from G to y, then z is a best approximation from G to x.

23-13 The unique polynomial p of degree at most 2 which minimizes $\int_{-1}^{1} |e^t - p(t)|^2\, dt$ and satisfies $\int_{-1}^{1} p(t)\, dt = 0$ is given by

$$p(t) = 3e^{-1}t + \frac{15}{4}(e - 7e^{-1})(t^2 - \frac{1}{3}), \quad -1 \le t \le 1.$$

(Hint: 23.4(a), Problems 23-11, 23-12)

23-14 Let E be a convex subset of X and $x \in X$. Then y is a best approximation from E to x if and only if $\operatorname{Re}\langle x - y, z - y\rangle \le 0$ for all $z \in E$. If y and w are best approximations from E to x and z respectively, then $\|y - w\| \le \|x - z\|$. (Hint: For $z \in E$ and $0 < r \le 1$, let $w = rz + (1 - r)y$, so that $\|x - w\|^2 = \|x - y\|^2 + r^2\|y - z\|^2 + 2r\operatorname{Re}\langle x - y, y - z\rangle$. Consider $r = 1$ as well as $r \to 0$.)

23-15 Let $\{x_1, \ldots, x_m\}$ be a linearly independent subset of X and

$$E = \{k_1, x_1 + \cdots + k_m x_m : k_j \ge 0 \text{ for } j = 1, \ldots, m\}.$$

Consider $x \in X$. Then the best approximation from E to x is given by $y = \alpha_1 x_1 + \cdots + \alpha_m x_m$, where $\alpha_1 \ge 0, \ldots, \alpha_m \ge 0$ and for each $i = 1, \ldots, m$,

$$\operatorname{Re}\langle x, x_i\rangle \le \sum_{j=1}^{m} \alpha_j \operatorname{Re}\langle x_j, x_i\rangle,$$

where equality holds whenever $\alpha_i > 0$.

In particular, if the set $\{x_1, \ldots, x_m\}$ is orthonormal, then

$$y = \sum_{j=1}^{\ell} \operatorname{Re}\langle x, x_{n_j}\rangle x_{n_j},$$

where $\{x_{n_1}, \ldots, x_{n_\ell}\} = \{x_i : \operatorname{Re}\langle x, x_i\rangle > 0, 1 \le i \le m\}$.

24 Projection and Riesz Representation Theorems

For a subset E of an inner product space X, let

$$E^\perp = \{y \in X : y \perp x \text{ for every } x \in E\}.$$

The linearity and the continuity of the inner product in the first variable imply that $E^\perp$ is a closed subspace of X. Further, if F denotes the closure of the span of E, then $F^\perp = E^\perp$ by the conjugate-linearity and the continuity of the inner product in the second variable. Also, it is clear that $E \bigcap E^\perp \subset \{0\}$. We shall prove that if X is complete and F is a nonempty closed subspace of X, then every element x of X can be written as the sum of an element y of F and an element z of $F^\perp$. Since $F \bigcap F^\perp = \{0\}$, it follows that $y \in F$ and $z \in F^\perp$ are unique. If we let $P(x) = y$, it is clear that P is a linear map from X to X and satisfies $P^2 = P$. Since $R(P) = F$ and $Z(P) = F^\perp$, we see that $R(P) \perp Z(P)$. For this reason, P is called the **orthogonal projection** onto the closed subspace F.

24.1 Projection theorem

Let H be a Hilbert space and F be a nonempty closed subspace of H. Then $H = F + F^\perp$. Equivalently, there is an orthogonal projection onto F. Moreover, $F^{\perp\perp} = F$.

Proof:

Let $x \in H$. We shall show that there is some $y \in F$ which satisfies $x - y \perp F$. Then $x = y + z$, where $z = x - y \in F^\perp$, so that $H = F + F^\perp$.

Let y denote the best approximation from F to x as given in 23.5. Then $y \in F$ and $x - y \perp F$ by 23.1(c). See Figure 9 in Section 23.

Alternatively, we proceed as follows. If $F = \{0\}$, then $F^\perp = H$ and there is nothing to prove. Let then $F \neq \{0\}$. Since F is a closed subspace of H, F is a Hilbert space. As we have seen in Section 22, there is an orthonormal basis $\{u_\alpha\}$ for F. Let $x \in H$. By 22.6, $\{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$ is a countable set $\{u_1, u_2, \ldots\}$ say, and the series $\sum_n \langle x, u_n \rangle u_n$ converges to some y in H such that $x - y \perp u_\alpha$ for every α. Since each $u_n \in F$ and F is a closed subspace, it follows that $y \in F$. Also, since $\{u_\alpha\}$ is an orthonormal basis for F, 22.7(iv) shows that F is the closure of span $\{u_\alpha\}$. Hence $x - y \in F^\perp$.

Finally, we prove that $F^{\perp\perp} = F$. Let $x \in F$. For every $z \in F^\perp$, we have $\langle x, z \rangle = 0$. Hence that $x \in (F^\perp)^\perp = F^{\perp\perp}$. Conversely, let $x \in F^{\perp\perp}$. Then $x = y + z$ for some $y \in F$ and $z \in F^\perp$. By what we have just seen, $y \in F^{\perp\perp}$. Thus $z = x - y \in F^{\perp\perp}$, so that $z \in F^\perp \bigcap F^{\perp\perp}$. Hence $z = 0$, that is, $x = y \in F$. □

The projection theorem shows that every Hilbert space H has the **complemented subspace property**, that is, for every nonempty closed subspace F of H, there is a closed subspace G of H such that $F + G = H$ and $F \bigcap G = \{0\}$. In fact, we can let $G = F^\perp$. The closed subspace $F^\perp$ is called the **orthogonal complement** of the closed subspace F. The complemented subspace property characterizes Hilbert spaces among all complete normed spaces in the following sense. Let X be a complete normed space such that for every nonempty closed subspace Y of X, there is a closed subspace Z of X such that $X = Y + Z$ and $Y \bigcap Z = \{0\}$. Then there is a linear homeomorphism from X onto some Hilbert space. This result is due to Lindenstrauss and Tzafriri ([44], 1974).

Continuous Linear Functionals

Let X be an inner product space over $\mathbf{K}$ and f be a linear functional on X. Let f be continuous on X. Then f is continuous at 0 and $f(0) = 0$. Hence there is some $\delta > 0$ such that $|f(x)| \leq 1$ whenever

$x \in X$ and $\|x\| \le \delta$. The linearity of f shows that $|f(x)| \le \|x\|/\delta$ for all $x \in X$. Conversely, if $|f(x)| \le \alpha\|x\|$ for some $\alpha > 0$ and all $x \in X$, then for all x and y in X, we have

$$|f(x) - f(y)| = |f(x-y)| \le \alpha\|x-y\|,$$

showing that f is (uniformly) continuous on X.

The set X' of all continuous linear functionals on X is called the **dual** of X. It is easy to see that X' is a linear space under pointwise addition and scalar multiplication. For $f \in X'$, let

$$\|f\| = \sup\{|f(x)| : x \in X, \|x\| \le 1\}.$$

Then $\| \ \|$ is a norm on X'. We shall later show that there is an inner product $\langle \ , \ \rangle'$ on X' such that $\langle f, f\rangle' = \|f\|^2$ for all $f \in X'$.

We have the basic inequality

$$|f(x)| \le \|f\| \, \|x\|, \quad f \in X', x \in X.$$

It implies that if $X \neq \{0\}$, then

$$\|f\| = \sup\{|f(x)| : x \in X, \|x\| = 1\}.$$

We shall now prove a result regarding a continuous linear functional f on X, which is analogous to the Bessel inequality (22.4) and its consequence 22.6. Note that it gives a lower bound for $\|f\|$.

24.2 Lemma

Let X be an inner product space and $f \in X'$.

(a) Let $\{u_1, u_2, \ldots\}$ be an orthonormal set in X. Then

$$\sum_n |f(u_n)|^2 \le \|f\|^2.$$

(b) Let $\{u_\alpha\}$ be an orthonormal set in X and

$$E_f = \{u_\alpha : f(u_\alpha) \neq 0\}.$$

Then E_f is a countable set, say $\{u_1, u_2, \ldots\}$. If E_f is denumerable, then $f(u_n) \to 0$ as $n \to \infty$.

Proof:

(a) For $m = 1, 2, \ldots$, let

$$y_m = \sum_{n=1}^{m} \overline{f(u_n)} u_n.$$

Since $\{u_1, \ldots, u_m\}$ is an orthonormal set,

$$\|y_m\|^2 = \langle y_m, y_m \rangle = \sum_{n=1}^{m} |f(u_n)|^2 = \beta_m, \text{ say.}$$

Since $f(y_m) = \sum_{n=1}^{m} \overline{f(u_n)} f(u_n) = \beta_m$ and $|f(y_m)| \leq \|f\| \; \|y_m\|$, we see that $\beta_m \leq \|f\| \sqrt{\beta_m}$, that is, $\beta_m \leq \|f\|^2$. Letting $m \to \infty$ (if the set is denumerable), we obtain

$$\sum_n |f(u_n)|^2 \leq \|f\|^2.$$

(b) If $f = 0$, then $E_f = \emptyset$. Let $f \neq 0$. For $j = 1, 2, \ldots$, let

$$E_j = \{u_\alpha : \|f\| \leq j|f(u_\alpha)|\}.$$

Fix j. Suppose E_j contains distinct elements $u_{\alpha_1}, \ldots, u_{\alpha_m}$. Then

$$m\|f\|^2 \leq j^2 \sum_{n=1}^{m} |f(u_{\alpha_n})|^2 \leq j^2 \|f\|^2$$

by (a) above. Since $\|f\| \neq 0$, this shows that $m \leq j^2$. Thus E_j contains at most j^2 elements. Since $E_f = \bigcup_j E_j$, we see that E_f is a countable set. Also, if E_f is denumerable, then

$$\sum_{n=1}^{\infty} |f(u_n)|^2 \leq \|f\|^2 < \infty,$$

so that the nth term $|f(u_n)|^2$ of this convergent series tends to zero as $n \to \infty$. □

Let us consider some examples of continuous linear functionals on X. Fix $y \in X$ and define

$$f(x) = \langle x, y \rangle, \quad x \in X.$$

The linearity and the continuity of the inner product in the first variable imply that $f \in X'$. Let us calculate $\|f\|$. By the Schwarz inequality 21.1(c), we have

$$|f(x)| \leq \|x\| \, \|y\|, \quad x \in X.$$

Hence $\|f\| \leq \|y\|$. If $y = 0$, then clearly $f = 0$, and $\|f\| = 0 = \|y\|$. If $y \neq 0$, let $x = y/\|y\|$. Then $\|x\| = 1$ and

$$f(x) = \langle \frac{y}{\|y\|}, y \rangle = \|y\|,$$

showing that $\|f\| = \|y\|$.

We shall now show that if X is complete, then this procedure gives all the continuous linear functionals on X. We give two proofs. The first is based on the projection theorem (24.1), while the second uses the existence of an orthonormal basis for X along with the Fourier expansion 22.7(ii).

24.3 Riesz representation theorem (F. Riesz and M. Frechet, 1907)

Let H be a Hilbert space and $f \in H'$. Then there is a unique $y \in H$ such that

$$f(x) = \langle x, y \rangle, \quad x \in H.$$

In fact, if z is a nonzero element of H such that $z \perp Z(f)$, then

$$y = \frac{\overline{f(z)}\, z}{\langle z, z \rangle}.$$

Also, if $\{u_\alpha\}$ is an orthonormal basis for H and $\{u_\alpha : f(u_\alpha) \neq 0\} = \{u_1, u_2, \ldots\}$ as in 24.2(b), then

$$y = \sum_n \overline{f(u_n)} u_n.$$

Proof:
If $f = 0$, then let $y = 0$, so that for all $x \in H$, we have

$$f(x) = 0 = \langle x, 0\rangle.$$

Let $f \neq 0$. Since $Z(f)$ is a closed subspace of H, the projection theorem (24.1) shows that $H = Z(f) + Z(f)^{\perp}$. As $Z(f) \neq H$, consider a nonzero element z in $Z(f)^{\perp}$. Let $x \in \mathbf{H}$ By 2.5(a), $Z(f)$ is a hyperspace in H. Hence

$$x = w + kz$$

for some $w \in Z(f)$ and $k \in \mathbf{K}$. Then

$$\langle x, z\rangle = \langle w, z\rangle + k\langle z, z\rangle = k\langle z, z\rangle,$$

so that $k = \langle x, z\rangle/\langle z, z\rangle$. Hence

$$f(x) = f(w) + kf(z) = kf(z) = \frac{\langle x, z\rangle}{\langle z, z\rangle}f(z) = \langle x, \frac{\overline{f(z)}}{\langle z, z\rangle}z\rangle.$$

Thus we let $y = \overline{f(z)}z/\langle z, z\rangle$.

Alternatively, we proceed as follows. Let $\{u_\alpha\}$ be an orthonormal basis for H and $\{u_\alpha : f(u_\alpha) \neq 0\} = \{u_1, u_2, \ldots\}$, as in 24.2(b). Then

$$\sum_n |f(u_n)|^2 \leq \|f\|^2 < \infty$$

by 24.2(a). Since H is a Hilbert space, the Riesz-Fischer theorem (22.5(b)) shows that the series $\sum_n \overline{f(u_n)}u_n$ converges in H. Let

$$y = \sum_n \overline{f(u_n)}u_n.$$

We claim that $f(x) = \langle x, y\rangle$ for all $x \in H$. Let $x \in H$ and $\{u_\alpha : \langle x, u_\alpha\rangle \neq 0\} = \{v_1, v_2 \ldots\}$ by 22.6. The Fourier expansion (22.7(ii)) shows that

$$x = \sum_m \langle x, v_m\rangle v_m.$$

Hence

$$f(x) = \sum_m \langle x, v_m\rangle f(v_m).$$

On the other hand,

$$\langle x,\, y\rangle = \sum_m \langle x,\, v_m\rangle\langle v_m,\, y\rangle.$$

We show that $\langle v_m,\, y\rangle = f(v_m)$ for each $m = 1, 2, \ldots$. Fix m. Then

$$\langle v_m,\, y\rangle = \langle v_m, \sum_n \overline{f(u_n)}u_n\rangle = \sum_n f(u_n)\langle v_m,\, u_n\rangle.$$

Now, if $v_m = u_{n_0}$ for some n_0, then $\langle v_m,\, y\rangle = f(u_{n_0}) = f(v_m)$. Next, let $v_m \neq u_n$ for any n. Then $\langle v_m,\, y\rangle = 0$, and also $f(v_m) = 0$, since $v_m \notin \{u_1, u_2 \ldots\} = \{u_\alpha : f(u_\alpha) \neq 0\}$. It, therefore, follows that $f(x) = \langle x,\, y\rangle$.

Finally, let us prove the uniqueness of $y \in H$. If $f(x) = \langle x,\, y_1\rangle$ for all $x \in H$ and some $y_1 \in H$ as well, then letting $x = y - y_1$, we obtain $\langle y - y_1,\, y\rangle = f(y - y_1) = \langle y - y_1,\, y_1\rangle$. Hence $\langle y - y_1,\, y - y_1\rangle = 0$, that is, $y_1 = y$. □

For $f \in H'$, the unique element $y \in H$ such that $f(x) = \langle x,\, y\rangle$ for all $x \in X$ is known as the **representer** of f. It satisfies $\|y\| = \|f\|$, as we have already seen.

24.4 Examples

(a) The Riesz representation theorem for $H = \mathbf{K}^n$ simply states the familiar fact that every (continuous) linear functional f on $\mathbf{K}^n$ is given by

$$f(x) = k_1x(1) + \cdots + k_nx(n), \quad x \in \mathbf{K}^n,$$

for fixed $k_1, \ldots, k_n$ in $\mathbf{K}$. For if y is the representer of f, then we can let $k_j = \overline{y(j)}, j = 1, \ldots, n$.

The following extensions of this finite dimensional result are noteworthy. Every continuous linear functional f on ℓ^2 is given by

$$f(x) = k_1x(1) + k_2x(2) + \cdots, \quad x \in \ell^2,$$

for a fixed square-summable sequence (k_n) in $\mathbf{K}$. For again if y is the representer of f, then we can let $k_j = \overline{y(j)}, j = 1, 2, \ldots$. (Compare 13.2.)

Let E be a measurable subset of $\mathbf{R}$. Every continuous linear functional f on $L^2(E)$ is given by

$$f(x) = \int_a^b k(t)x(t)\,dm(t), \quad x \in L^2(E),$$

for a fixed square-integrable function k on E. For again if y is the representer of f, then we can let $k(t) = \overline{y(t)}$ for $t \in E$. (Compare 14.1 for the particular case $E = [a, b]$.)

If $H = \{x \in C([a,b]) : x$ absolutely continuous on $[a, b]$ and $x' \in L^2([a,b])\}$, then every continuous linear functional f on H is given by

$$f(x) = k(a)x(a) + \int_a^b k'(t)x'(t)\,dm(t), \quad x \in H,$$

for a fixed absolutely continuous function k on $[a, b]$ such that k' belongs to $L^2([a,b])$. This follows immediately from the Riesz representation theorem 24.3 if we recall the definition of the inner product on the Hilbert space H given in 21.3(d).

(b) The projection theorem and the Riesz representation theorem do not hold for an incomplete inner product space. (See Problem 24-2.)

For example, let $X = c_{00}$, the linear space of all scalar sequences having only a finite number of nonzero entries with the inner product defined in 21.3(b).

Define $f : X \to \mathbf{K}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{x(n)}{n}, \quad x \in X.$$

Then f is linear. By Hölder's inequality (4.5(a)),

$$|f(x)|^2 \le \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)\left(\sum_{n=1}^{\infty} |x(n)|^2\right) = \frac{\pi^2}{6}\|x\|^2.$$

Hence f is continuous and $\|f\| \le \pi/\sqrt{6}$.

Since $f \neq 0, F = Z(f)$ is a proper closed subspace of X. Let $z \in F^{\perp}$. Since $z \in c_{00}$, we see that $z = (z(1), \ldots, z(m), 0, 0, \ldots)$ for some positive integer m. For $1 \leq n \leq m$, define $x_n \in X$ by letting $x_n(n) = 1, x_n(m+1) = -(m+1)/n$ and $x_n(j) = 0$ for all $j \neq n$ or $m+1$. Then $f(x_n) = 1/n - (m+1)/n(m+1) = 0$, so that $x_n \in F$ and $z(n) = \langle z, x_n \rangle = 0$. Thus $F^{\perp} = \{0\}$. This shows that $F + F^{\perp} \neq X$, that is, the projection theorem does not hold for X.

Also, $f \in X'$ has no representer in X. To see this, let $y \in X$. Were $f(x) = \langle x, y \rangle$ for all $x \in X$, then by letting $e_n = (0, \ldots, 0, 1, 0, 0, \ldots)$, where 1 occurs only in the nth entry, we see that $y(n) = \langle e_n, y \rangle = f(e_n) = 1/n \neq 0$ for all $n = 1, 2, \ldots$. But this is impossible since $y \in c_{00}$. This shows that the Riesz representation theorem does not hold for X.

We give some consequences of the Riesz representation theorem.

24.5 Theorem

Let H be a Hilbert space.

(a) For $f \in H'$, let y_f be the representer of f in H. Then the map $T : H' \to H$ given by $T(f) = y_f$ is a surjective conjugate-linear isometry.

(b) For f and g in H', define

$$\langle f, g \rangle' = \langle T(g), T(f) \rangle.$$

Then $\langle\ ,\ \rangle'$ is an inner product on $H', \langle f, f \rangle' = \|f\|^2$ for all $f \in H'$ and H' is a Hilbert space.

(c) For $y \in H$, define $j_y : H' \to \mathbf{K}$ by

$$j_y(f) = f(y), \quad f \in H'.$$

Then j_y is a continuous linear functional on H' and the map J from H to H'' defined by

$$J(y) = j_y, \quad y \in H,$$

is a surjective linear isometry, that is, H is **reflexive**.

Proof:

(a) For f and g in H', we have

$$(f+g)(x) = f(x) + g(x) = \langle x,\, y_f\rangle + \langle x,\, y_g\rangle = \langle x,\, y_f + y_g\rangle$$

for all $x \in H$. Hence $y_f + y_g$ is the representer of $f + g \in H'$, that is, $T(f+g) = T(f) + T(g)$. Similarly, for $f \in H'$ and $k \in \mathbf{K}$, we have

$$(kf)(x) = kf(x) = k\langle x,\, y_f\rangle = \langle x,\, \overline{k}y_f\rangle$$

for all $x \in H$. Hence $\overline{k}y_f$ is the representer of $kf \in H'$, that is, $T(kf) = \overline{k}T(f)$. Thus the map $T : H' \to H$ is conjugate-linear. To see that T is surjective, consider $y \in H$ and let $f(x) = \langle x,\, y\rangle$ for all $x \in X$. Then $y = T(f)$. We have already seen that $\|T(f)\| = \|y_f\| = \|f\|$, so that T is an isometry.

(b) For all $f \in H'$, we have

$$\langle f,\, f\rangle' = \langle T(f),\, T(f)\rangle \geq 0,$$

and $\langle f,\, f\rangle' = 0$ if and only if $\|T(f)\| = \|y_f\| = \|f\| = 0$, that is, $f = 0$. Also, for f, g, h in H' and $k \in \mathbf{K}$,

$$\begin{aligned} \langle f+g,\, h\rangle' &= \langle T(h),\, T(f+g)\rangle = \langle T(h),\, T(f) + T(g)\rangle \\ &= \langle T(h),\, T(f)\rangle + \langle T(h),\, T(g)\rangle = \langle f,\, h\rangle' + \langle f,\, g\rangle'. \end{aligned}$$

Similarly, it can be seen that $\langle kf,\, h\rangle' = k\langle f,\, h\rangle'$ and $\langle f,\, h\rangle' = \overline{\langle h,\, f\rangle'}$. Thus $\langle\ ,\ \rangle'$ is an inner product on H'. Since H is complete and the map $T : H' \to H$ is a surjective isometry, we see that H' is also complete, that is, H' is a Hilbert space.

(c) Let $y \in H$. Clearly, $j_y : H' \to \mathbf{K}$ is linear and

$$|j_y(f)| = |f(y)| \leq \|y\|\, \|f\|$$

for all $f \in H'$. Hence j_y is continuous and $\|j_y\| \le \|y\|$. In fact, if we define $f \in H'$ by $f(x) = \langle x, y \rangle$ for all $x \in X$, then

$$|j_y(f)| = \langle y, y \rangle = \|y\|^2,$$

so that $\|j_y\| = \|y\|$.

It can be easily verified that $J : H \to H''$ is linear. To see that J is surjective, consider $\phi \in H''$. Applying the Riesz representation theorem (24.3) to the continuous linear functional ϕ on the Hilbert space H', we obtain a representer $g \in H'$ of ϕ. Then

$$\phi(f) = \langle f, g \rangle' = \langle y_g, y_f \rangle = f(y_g) = J(y_g)(f)$$

for all $f \in H'$. Thus $\phi = J(y_g)$. Also, J is an isometry since $\|J(y)\| = \|j_y\| = \|y\|$, as we have already seen. □

24.6 Theorem (Unique Hahn-Banach extension)
Let H be a Hilbert space, G be a subspace of H and g be a continuous linear functional on G. Then there is a unique continuous linear functional f on H such that $f_{|G} = g$ and $\|f\| = \|g\|$.

Proof:
Let F denote the closure of G in H. If $x \in F$, then there is a sequence (x_n) in G such that $\|x_n - x\| \to 0$. Since

$$|g(x_n) - g(x_m)| = |g(x_n - x_m)| \le \|g\| \, \|x_n - x_m\|$$

for all n and m, we see that $(g(x_n))$ is a Cauchy sequence in $\mathbf{K}$. Let $g(x_n) \to k$ in $\mathbf{K}$. If (y_n) is another sequence in G with $\|y_n - x\| \to 0$, then

$$|g(x_n) - g(y_n)| = |g(x_n - y_n)| \le \|g\| \, \|x_n - y_n\| \le \|g\|(\|x_n - x\| + \|x - y_n\|).$$

This implies that $g(y_n) \to k$ as well. Define $\tilde{g}(x) = k$. It can be seen that $\tilde{g} : F \to \mathbf{K}$ is linear, continuous and satisfies $\|\tilde{g}\| = \|g\|$.

Applying the Riesz representation theorem 24.3 to the Hilbert space F and the continuous linear functional $\tilde{g}$ on F, we obtain a representer $y \in F$ of $\tilde{g}$, that is,

$$\tilde{g}(x) = \langle x, y \rangle, \quad x \in F.$$

Define $f : H \to \mathbf{K}$ by

$$f(x) = \langle x, y \rangle, \quad x \in H.$$

Then f is linear $f_{|G} = g$ and $\|f\| = \|y\| = \|\tilde{g}\| = \|g\|$.

To prove the uniqueness of the extension f of g, consider $h \in H'$ such that $h_{|G} = g$ and $\|h\| = \|g\|$. Since G is dense in F and h is continuous on F, we see that $h_{|F} = \tilde{g}$. In particular, $h(y) = \tilde{g}(y)$. Let $z \in H$ be the representer of h. Then $\|z\| = \|h\| = \|g\| = \|y\|$ and $\langle y, z \rangle = h(y) = \tilde{g}(y) = \langle y, y \rangle$. Hence

$$\|y - z\|^2 = \|y\|^2 - 2\mathrm{Re}\,\langle y, z \rangle + \|z\|^2 = 2\|y\|^2 - 2\mathrm{Re}\,\langle y, y \rangle = 0,$$

so that $z = y$. Thus for all $x \in H$, we have

$$h(x) = \langle x, z \rangle = \langle x, y \rangle = f(x),$$

that is, $h = f$. □

Weak Convergence and Weak Boundedness

Let (x_n) be a sequence in a Hilbert space H. We say that (x_n) converges to x in H if $\|x_n - x\| = \langle x_n - x, x_n - x \rangle^{1/2} \to 0$ as $n \to \infty$, and write $x_n \to x$. We now introduce another concept of convergence in H. We say that (x_n) is **weak convergent** (or, **converges weakly** to x in H if $\langle x_n, y\rangle \to \langle x, y\rangle$ as $n \to \infty$ for every $y \in H$, and then write $x_n \xrightarrow{w} x$. (The Riesz representation theorem (24.3) shows that the present definition of weak convergence agrees with an earlier definition given in Section 15.)

The continuity of the inner product in the first variable implies that $x_n \xrightarrow{w} x$ whenever $x_n \to x$. The converse, however, does not hold for any infinite dimensional Hilbert space H. If (u_n) is an infinite orthonormal sequence in H, then $\langle u_n, y\rangle \to 0$ as $n \to \infty$ for every y in H by 22.6, but $\|u_n\| = 1$ for every n. Thus $u_n \xrightarrow{w} 0$, but $u_n \not\to 0$. In fact, (u_n) has no convergent subsequence since $\|u_n - u_m\| = \sqrt{2}$ for all $n \neq m$. For example, if $H = \ell^2$, we can let $u_n = (0, \ldots, 0, 1, 0, 0, \ldots)$, where 1 occurs only in the nth entry, and if $H = L^2([-\pi, \pi])$, we can let $u_n(t) = e^{int}/\sqrt{2\pi}$ for $t \in [-\pi, \pi], n = 1, 2, \ldots$. [Note that if H is finite dimensional, then there is a linear isometry from H onto $\mathbf{K}^m$ for some positive integer m (22.9), and hence $x_n \to x$ in H if and only if $x_n \xrightarrow{w} x$ in H.]

Our next result makes precise the relationship between $x_n \to x$ and $x_n \xrightarrow{w} x$. We also give an infinite dimensional version of the classical Bolzano-Weierstrass theorem, which can be compared with Eberlein's theorem (16.5) in view of 24.3 and 24.5(c).

24.7 Theorem

Let (x_n) be a sequence in a Hilbert space H.

(a) $x_n \to x$ if and only if $x_n \xrightarrow{w} x$ and $\limsup_{n\to\infty} \|x_n\| \leq \|x\|$.

(b) If (x_n) is bounded, then it has a weak convergent subsequence.

Proof:

(a) Let $x_n \to x$. Then $x_n \xrightarrow{w} x$, as we have already noted. Also, $\limsup_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|x_n\| = \|x\|$.

Conversely, let $x_n \xrightarrow{w} x$ and $\limsup_{n\to\infty} \|x_n\| \leq \|x\|$. For each n,

$$0 \leq \|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \|x_n\|^2 + \|x\|^2 - 2\mathrm{Re}\,\langle x_n, x\rangle.$$

Since $\limsup_{n\to\infty} \|x_n\|^2 \le \|x\|^2$ and Re $\langle x_n,\, x\rangle \to \langle x,\, x\rangle = \|x\|^2$,

$$0 \le \limsup_{n\to\infty} \|x_n - x\|^2 \le \|x\|^2 + \|x\|^2 - 2\|x\|^2 = 0,$$

so that $\lim_{n\to\infty} \|x_n - x\|$ exists and equals 0. Thus $x_n \to x$.

(b) Assume that $\|x_n\| \le \alpha$ for all n and some $\alpha > 0$. By the Schwarz inequality (21.1(c)), $|\langle x_n,\, x_1\rangle| \le \|x_n\|\ \|x_1\| \le \alpha^2$ for all n. The classical Bolzano-Weierstrass theorem for $\mathbf{K}$ shows that the bounded sequence $(\langle x_n,\, x_1\rangle)$ has a convergent subsequence $(\langle x_{n,1},\, x_1\rangle)$. Again, the bounded sequence $(\langle x_{n,1},\, x_2\rangle)$ has a convergent subsequence $(\langle x_{n,2},\, x_2\rangle)$, and so on. Consider now the diagonal sequence $(x_{n,n})$. We show that for every $x \in H$, the sequence $(\langle x_{n,n},\, x\rangle)$ converges in $\mathbf{K}$. If $x = x_m$ for some m, then for each $n > m$, $(\langle x_{n,n},\, x\rangle)$ is a subsequence of the convergent sequence $(\langle x_{n,m},\, x_m\rangle)$, and hence it converges in $\mathbf{K}$. As a result, if $x \in \operatorname{span}\{x_1, x_2, \ldots\}$, then $(\langle x_{n,n},\, x\rangle)$ converges in $\mathbf{K}$. Let F denote the closure of $\operatorname{span}\{x_1, x_2, \ldots\}$. Consider $x \in F$. Let (y_p) be a sequence in $\operatorname{span}\{x_1, x_2, \ldots\}$ such that $y_p \to x$ as $p \to \infty$. Then for all n, m and p, we have

$$\begin{aligned}
|\langle x_{n,n},\, x\rangle - \langle x_{m,m},\, x\rangle| &= |\langle x_{n,n} - x_{m,m},\, x\rangle| \\
&\le |\langle x_{n,n} - x_{m,m},\, x - y_p\rangle| \\
&\quad + |\langle x_{n,n} - x_{m,m},\, y_p\rangle| \\
&\le \|x_{n,n} - x_{m,m}\|\,\|x - y_p\| \\
&\quad + |\langle x_{n,n} - x_{m,m},\, y_p\rangle| \\
&\le 2\alpha\|x - y_p\| + |\langle x_{n,n} - x_{m,m},\, y_p\rangle|.
\end{aligned}$$

Since $\|x - y_p\| \to 0$ as $p \to \infty$ and $|\langle x_{n,n} - x_{m,m},\, y_p\rangle| \to 0$ as $n, m \to \infty$ for each p, we see that $(\langle x_{n,n},\, x\rangle)$ is a Cauchy sequence in $\mathbf{K}$. Hence it converges in $\mathbf{K}$. Next, let $x \in F^\perp$. Then $\langle x_{n,n},\, x\rangle = 0$ for all n since $x_{n,n} \in F$. Thus $(\langle x_{n,n},\, x\rangle)$ converges to 0.

By the projection theorem (24.1), we have $H = F + F^\perp$. Hence $(\langle x_{n,n},\, x\rangle)$ converges for every $x \in H$. Define

$$f(x) = \lim_{n\to\infty} \langle x,\, x_{n,n}\rangle, \quad x \in H.$$

Then f is a linear functional on H and since

$$|f(x)| = \lim_{n\to\infty} |\langle x,\, x_{n,n}\rangle| \le \alpha\|x\|, \quad x \in H,$$

we see that f is continuous. By the Riesz representation theorem, let y be the representer of f. Then $\langle x,\, x_{n,n}\rangle \to f(x) = \langle x,\, y\rangle$ for all $x \in H$. Thus $x_{n,n} \xrightarrow{w} y$, where $(x_{n,n})$ is a subsequence of (x_n). □

Analogous to weak convergence, we define weak boundedness as follows. Let E be a subset of a Hilbert space H. We say that E is **weak bounded** if for every $y \in H$, there is some $\alpha_y \ge 0$ such that $|\langle x,\, y\rangle| \le \alpha_y$ for all $x \in E$.

We have the following surprising result.

24.8 Theorem

A subset of a Hilbert space is weak bounded if and only if it is bounded.

Proof:

Let E be a subset of H.

If E is bounded, that is, $\|x\| \le \alpha$ for all $x \in E$ and some $\alpha \ge 0$, then for each $y \in H$,

$$|\langle x,\, y\rangle| \le \|x\|\;\|y\| \le \alpha\|y\|, \quad x \in E,$$

so that E is weak bounded.

Conversely, assume that E is weak bounded. For each $y \in H$, let $|\langle x,\, y\rangle| \le \alpha_y$ for all $x \in E$ and some $\alpha_y > 0$. First consider a finite dimensional subspace F of H. By 22.2, there is a basis $\{y_1, \ldots, y_n\}$ of F such that $\langle y_i,\, y_j\rangle = \delta_{i,j}$ for $i, j = 1, \ldots, n$. For $x \in H$, let

$$P_F(x) = \langle x,\, y_1\rangle y_1 + \cdots + \langle x,\, y_n\rangle y_n.$$

Then $(x - P_F(x)) \perp P_F(y)$ for all $x, y \in H$. If $x \in E$, we have by Pythagoras' theorem (22.1(a)),

$$\|P_F(x)\|^2 = |\langle x,\, y_1\rangle|^2 + \cdots + |\langle x,\, y_n\rangle|^2 \le \alpha_{y_1}^2 + \cdots + \alpha_{y_n}^2.$$

This shows that the set $\{P_F(x) : x \in E\}$ is bounded in H.

Assume, for a moment, that E is an unbounded set in H. Then there is some $x_1 \in E$ such that $\|x_1\| \geq 1$. Let $z_1 = x_1, F_1 = \text{span}\{z_1\}$ and $P_1 = P_{F_1}$. Since $\{P_1(x) : x \in E\}$ is bounded and E is unbounded, the set $\{x - P_1(x) : x \in E\}$ is also unbounded. Hence there is some $x_2 \in E$ such that

$$\|x_2 - P_1(x_2)\| \geq 2\left[2 + \frac{\alpha_{z_1}}{\|z_1\|}\right].$$

Let $z_2 = x_2 - P_1(x_2), F_2 = \text{span}\{z_1, x_2, z_2\}$ and $P_2 = P_{F_2}$. Proceeding inductively, we define for $m = 2, 3, \ldots,$

$$F_m = \text{span}\{z_1, x_2, z_2, \ldots, x_m, z_m\}$$

and $P_m = P_{F_m}$. Then we can find $x_{m+1} \in E$ such that

$$\|x_{m+1} - P_m(x_{m+1})\| \geq (m+1)\left[m + 1 + \sum_{n=1}^{m} \frac{\alpha_{z_n}}{\|z_n\|}\right]$$

and let $z_{m+1} = x_{m+1} - P_m(x_{m+1})$. It is clear that

$$\langle x_{m+1}, z_{m+1}\rangle = \langle z_{m+1}, z_{m+1}\rangle \quad \text{for } m = 1, 2, \ldots,$$

$$\langle x_{m+1}, z_n\rangle = 0 \quad \text{for } n = m+2, m+3, \ldots,$$

because $P_m(x_{m+1}) \perp z_{m+1}, x_{m+1} \in F_{m+1}$ and $z_{m+2} \perp F_{m+1}$, etc.

Let $u_n = z_n/\|z_n\|$ for $n = 1, 2, \ldots$. Since $z_1, \ldots, z_m$ belong toF_m and $z_{m+1} \perp F_m$, we see that $\{u_1, u_2, \ldots\}$ is an orthonormal set in H. Since H is a Hilbert space, the Riesz-Fischer theorem (22.5(b)) shows that the series $\sum_{n=1}^{\infty} u_n/n$ converges to some y in H. Then for $m = 1, 2, \ldots,$

$$\begin{aligned}
|\langle x_{m+1}, y\rangle| &= |\langle x_{m+1}, \sum_{n=1}^{m+1} \frac{u_n}{n}\rangle| \\
&\geq |\langle x_{m+1}, \frac{u_{m+1}}{m+1}\rangle| - \sum_{n=1}^{m} |\langle x_{m+1}, \frac{u_n}{n}\rangle| \\
&\geq \frac{|\langle z_{m+1}, z_{m+1}\rangle|}{(m+1)\|z_{m+1}\|} - \sum_{n=1}^{m} \frac{\alpha_{z_n}}{n\|z_n\|} \\
&\geq m + 1,
\end{aligned}$$

as can be easily seen from our choice of the element x_{m+1} belonging to E. Hence each $x_{m+1} \in E$ and $\langle x_{m+1}, y\rangle \to \infty$ as $m \to \infty$, contrary to the weak boundedness of the set E.

Thus every weak bounded set in H is bounded. □

In view of the Riesz representation theorem, the result 24.8 can be stated as follows. Let $\mathcal{F}$ be a set of continuous linear functionals on H. Then $\{f(y) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbf{K}$ for each $y \in H$ if and only if the set $\{\|f\| : f \in \mathcal{F}\}$ is bounded. This is known as the **uniform boundedness principle** for continuous linear functionals on H. (Compare 9.1 as well as 9.3(a).)

For some characterizations of weak convergence in H, see Problems 16-10 and 24-15.

Problems

H denotes a Hilbert space over $\mathbf{K}$, unless otherwise stated.

24-1 (a) If $E \subset H$, then $E^{\perp\perp}$ is the closure of the span of E.

(b) If F_1 and F_2 are closed subspaces of H, then $(F_1 \cap F_2)^{\perp}$ equals the closure of $F_1^{\perp} + F_2^{\perp}$.

24-2 Let X be an inner product space. Then the projection theorem (24.1) holds for X if and only if the Riesz representation theorem (24.3) holds for X if and only if X is complete. (Hint: Proofs of 24.3 and 24.1. If H is the completion of X and $z \notin X, z \in H$, let $f(x) = \langle x, z\rangle$ for $x \in X$.)

24-3 Let F be a finite dimensional subspace of an inner product space X. Then $X = F + F^{\perp}$ and $F^{\perp\perp} = F$.

24-4 (**Riesz Lemma** for $r = 1$) Let F be a closed subspace of H and $F \neq H$. Then there exists some $x \in H$ such that $\|x\| = 1, \operatorname{dist}(x, F) = 1$ and $\|x - y\| = \sqrt{2}$ for every $y \in F$ with $\|y\| = 1$. This result may not hold in an inner product space X. (Hint: 24.1 or the proof of 5.3)

24-5 Let $H = L^2([a,b])$ and $z, w \in H$. For $x \in H$, define

$$f(x) = \int_a^b z(t)\left[\int_a^t w(s)x(s)\,dm(s)\right]dm(t).$$

Then $f \in H'$ and the representer of f is given by

$$y(t) = \overline{w(t)}\int_t^b \overline{z(s)}\,dm(s), \quad t \in [a,b].$$

24-6 Let T be a set and H be a Hilbert space of functions on T. Let $k(\cdot,\cdot)$ be a function on $T \times T$. For a fixed $t \in T$, let $k_t(s) = k(s,t)$, $s \in T$. Then $k(\cdot,\cdot)$ is said to be a reproducing kernel for H if

(i) for every fixed $t \in T$, $k_t \in H$ and

(ii) for every fixed $x \in H$, $x(t) = \langle x, k_t\rangle$ for all $t \in T$.

There is a reproducing kernel for H if and only if for every fixed $t \in T$, the evaluation functional $f_t(x) = x(t)$, $x \in H$, is continuous on H. In that case, for every fixed $t \in T$,

$$k(t,t) = \sup\{|x(t)|^2 : x \in H, \langle x, x\rangle \le 1\}.$$

Also, if there is some $x \in H$ with $x(t) \neq 0$, then this supremum is attained only at $x_\theta = e^{i\theta}k_t/\sqrt{k(t,t)}$, $0 \le \theta < 2\pi$.

24-7 Let $H = \{x \in C([a,b]) : x$ is absolutely continuous and x' belongs to $L^2([a,b])\}$. Fix $z, w \in L^2([a,b])$. For $x \in H$, let

$$f(x) = \int_a^b (xz + x'w)\,dm.$$

Also, let $H_0 = \{x \in H : x(a) = 0 = x(b)\}$ and $f_0 = f_{|H_0}$.

(a) Consider the inner product $\langle\ ,\ \rangle_a$ on H given in 21.3(d). Then $f \in H'$ and the representer of f is given by

$$y(t) = u(a) + \int_a^t u(s)ds + v(t), \quad t \in [a,b],$$

where

$$u(s) = \int_s^b \bar{z}\,dm \ \text{ for } s \in [a,b] \quad \text{and} \quad v(t) = \int_a^t \overline{w}\,dm \ \text{ for } t \in [a,b].$$

Also, $f_0 \in H_0'$ and the representer of f_0 is given by

$$y_0(t) = \int_a^t u(s)ds + v(t) - \left[\int_a^b u(s)ds + v(b)\right]\frac{t-a}{b-a}, \quad t \in [a,b].$$

(b) Consider the inner product $\langle\ ,\ \rangle$ on H given in Problem 21-13. Then $f \in H'$ and the representer of f is the unique element $y \in H$ with

$y' - \overline{w} \in H$, $(y' - \overline{w})' = y - \overline{z}$ a.e. on $[a,b]$, $y'(a) = \overline{w}(a)$ and $y'(b) = \overline{w}(b)$.

Also, $f_0 \in H_0'$ and the representer of f_0 is the unique element $y_0 \in H$ with

$$y_0' - \overline{w} \in H,\ (y_0' - \overline{w})' = y_0 - \overline{z} \ \text{ a.e. on } [a,b],\ y_0(a) = 0 = y_0(b).$$

Further, if $g(x) = x(a)$ for $x \in H$, then $g \in H'$ and the representer of g is given by $(e^{a+t} + e^{a+2b-t})/(e^{2b} - e^{2a})$, $t \in [a,b]$.

24-8 Let $q \in L^1([a,b])$ and $k_1, k_2 \in \mathbf{K}$. Then there is a unique function y on $[a,b]$ such that y and y' are absolutely continuous, $y'' - y = q$ a.e. on $[a,b]$, $y'(a) = k_1$ and $y'(b) = k_2$. (Hint: Choose α and β such that $\alpha a + \beta = k_1$ and $\alpha b + \beta = k_2 - \int_a^b q\,dm$. Let $w(t) = \int_a^t \overline{q}\,dm + \overline{\alpha}t + \beta$ in the first part of Problem 24-7(b)). Also, there is a unique function y on $[a,b]$ such that y and y' are absolutely continuous, $y'' - y = q$ a.e. on $[a,b]$, $y(a) = k_1$ and $y(b) = k_2$. (Hint: choose α and β such that $\alpha a + \beta = k_1$ and $\alpha b + \beta = k_2$. Let $w(t) = \int_a^t [\overline{q}(s) + \overline{\alpha}s + \overline{\beta}]dm(s)$ and $y(t) = y_0(t) + \alpha t + \beta$ in the second part of Problem 24-7(b).)

24-9 Let F be a closed subspace of H. If $a \in H$ but $a \notin F$, then there is a unique $f \in H'$ such that $f(a) = \operatorname{dist}(a,F)$, $f(x) = 0$ for all $x \in F$ and $\|f\| = 1$. (Compare 7.10(b). Hint: 24.1 and 24.3)

24-10 Let $f \in H'$. If $g \in H'$, $\|g\| = \|f\|$ and $g(x) = f(x)$ for some nonzero $x \in Z(f)^\perp$, then $g = f$. (Hint: Proof of 24.6)

24-11 Let $a_0, a_1, \ldots, a_m$ be in $C^m([a,b])$. For $x \in C^m((a,b))$, let $D(x) = -ix'$ and

$$A(x) = a_0x + a_1D(x) + \cdots + a_mD^m(x).$$

Let $Z = C_c^\infty((a,b))$, the linear space of all infinitely differentiable functions on (a,b) each of which vanishes outside some closed subinterval of (a,b). For a given $y \in L^2((a,b))$, a weak solution of the differential equation $A(x) = y$ is an element $x_0 \in L^2((a,b))$ such that for every $z \in Z$, we have

$$\langle y,\, z\rangle = \langle x_0,\, B(z)\rangle, \quad \text{where } B(z) = \bar{a}_0 z + D(\bar{a}_1 z) + \cdots + D^m(\bar{a}_m z).$$

Let $y \in L^2((a,b))$. A necessary and sufficient condition for the existence of a weak solution of $A(x) = y$ is that

$$|\langle y,\, z\rangle| \leq \alpha \|B(z)\|_2$$

for all $z \in Z$ and some $\alpha > 0$. (Hint: Consider $G = \{w \in L^2((a,b)) : w = B(z)$ for some $z \in Z\}$ and$g(w) = \langle z,\, y\rangle$ for $w \in G$. Use 24.6 and 24.3.)

24-12 Let X be an inner product space.

(a) There is an inner product $\langle\ ,\ \rangle'$ on X' such that $\langle f,\, f\rangle' = \|f\|^2$ for every $f \in X'$. For $f, g \in X'$, $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$.

(b) Let Y be a subspace of X and g be a continuous linear functional on Y. Then there is a unique continuous linear functional f on X such that $f_{|Y} = g$ and $\|f\| = \|g\|$.

(c) Let Y be a closed subspace of X. There is an inner product $\langle\ ,\ \rangle_q$ on the quotient space X/Y such that $\langle x+Y,\, x+Y\rangle_q = \inf\{\langle x+y,\, x+y\rangle : y \in Y\}$ for every $x \in X$. Further, X is a Hilbert space if and only Y and X/Y are Hilbert spaces. (Compare 8.2(a). Hint: Consider the completion of X and use 24.1.)

24-13 Let $X = c_{00}$, the linear space of all scalar sequences which have only a finite number of nonzero entries with the inner product defined in 21.3(b).

(a) Let $x_n = (1, 1/2, \cdots, 1/n, 0, 0, \cdots), n = 1, 2, \ldots$. Then (x_n) is a bounded sequence in X. Let (x_{n_k}) be a subsequence of (x_n). There is no $x \in X$ such that $\langle x_{n_k},\, y\rangle \to \langle x,\, y\rangle$ for every $y \in X$. (Compare 24.7(b).)

(b) Let E be the set of all $(1, \ldots, 1, 0, 0 \ldots)$, where 1 occurs only in the first n entries. Then for each $y \in X$, the set $\{\langle x,\, y\rangle : x \in E\}$ is bounded, but E is not a bounded subset of X. (Compare 24.8).

24-14 $x_n \to x$ in H if and only if $\langle x_n,\, y\rangle \to \langle x,\, y\rangle$ uniformly for $y \in H$ with $\|y\| \leq 1$.

24-15 Let (x_n) be a sequence in H and $\{u_\alpha\}$ be an orthonormal basis for H. Then $x_n \xrightarrow{w} x$ in H if and only if (x_n) is bounded and $(\langle x_n, u_\alpha\rangle)$ is a Cauchy sequence for each α. (Hint: 24.8, 22.7 and 24.3)

24-16 Let $\{x_1, x_2, \ldots\}$ be an orthogonal set in H. Then $\sum_n x_n$ converges in H if and only if $\sum_n \langle x_n, y\rangle$ converges in $\mathbf{K}$ for every $y \in H$. (Hint: Problem 22.8(b), 24.8).

24-17 Let X be an inner product space and $\{u_1, u_2, \ldots\}$ be an orthonormal set in X. Let $k_1, k_2, \ldots$ be scalars.

(a) If $\sum_j |k_j|^2 < \infty$ and we let $f(x) = \sum_j k_j \langle x, u_j\rangle$ for $x \in X$, then $f \in X'$ and $\|f\|^2 \le \sum_j |k_j|^2$.

(b) If X is complete, the set $\{u_1, u_2, \ldots\}$ be infinite and $\sum_j k_j\langle x, u_j\rangle$ converges in $\mathbf{K}$ for every $x \in H$, then $\sum_j |k_j|^2 < \infty$.

24-18 **(Banach-Saks)** If $x_n \xrightarrow{w} x$ in H, then there is a subsequence (x_{n_k}) of (x_n) such that the arithmetic mean $\dfrac{1}{m}\displaystyle\sum_{k=1}^{m} x_{n_k} \to x$ in H.

24-19 Let $H = L^2([0,1])$ and (x_n) be a sequence in H. Then $x_n \xrightarrow{w} 0$ in H if and only if (x_n) is bounded in H and $\displaystyle\int_{-\beta}^{\beta} |\hat{x}_n|^2\, dm \to 0$ for every $\beta > 0$, where

$$\hat{x}_n(u) = \frac{1}{\sqrt{2\pi}}\int_0^1 x(s)e^{-ius}\, dm(s), \quad u \in \mathbf{R}.$$

(Hint: For $x, y \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, we have $\hat{x}, \hat{y} \in L^2(\mathbf{R})$ and as in Section 26, $\displaystyle\int_{-\infty}^{\infty} x(t)\,\overline{y(t)}\, dm(t) = \int_{-\infty}^{\infty} \hat{x}(u)\overline{\hat{y}(u)}\, dm(u)$.)

Chapter VII

Bounded Operators on Hilbert Spaces

This chapter presents a detailed study of bounded linear maps from a Hilbert space to itself. The adjoint of such a bounded operator is introduced in Section 25. It corresponds to the conjugate transpose of a matrix in a finite dimensional situation. The consideration of an adjoint yields easy proofs of the bounded inverse theorem, the closed graph theorem and the open mapping theorem (considered earlier in Sections 10 and 11), as indicated in the problems on this section. Several special kinds of operators that behave well with their adjoints are considered in Section 26. They are the normal, self-adjoint and unitary operators. A positive operator is related to a generalized Schwarz inequality. The Fourier-Plancherel transform gives an example of a unitary operator. In Section 27, two subsets of scalars, known as the spectrum and the numerical range, are associated with a bounded operator on a Hilbert space. They give useful information about the operator, especially when the operator is normal or self-adjoint. The Ritz method for approximating the smallest and the largest spectral value of a self-adjoint operator is given. This section also includes a version of the finite dimensional spectral theorem for normal operators when the scalars are complex numbers and for self-adjoint operators when the scalars are real numbers. In Section 28, a natural generalization of a finite dimensional self-adjoint operator is considered in the form of a compact self-adjoint operator. Such an operator can be described in terms of its nonzero eigenvalues and corresponding eigenvectors, thus yielding explicit solutions of equations involving that operator. This gives a stronger form of the Fredholm alternative considered earlier in Section 19. The special case of a Fredholm integral equation is treated. It is useful in solving some ordinary linear differential equations with linear boundary conditions. This will be considered Appendix C.

25 Bounded Operators and Adjoints

By an **operator** A on an inner product space X over $\mathbf{K}$, we mean a linear map A from X to X. It is said to be **bounded** if $\|A(x)\| \leq \alpha\|x\|$ for all $x \in X$ and some $\alpha > 0$, where $\|x\| = \sqrt{\langle x, x\rangle}$. A bounded operator A is uniformly continuous on X, since for all x and y in X, $\|A(x) - A(y)\| \leq \alpha\|x - y\|$. Conversely, if A is a linear map from X to X and A is continuous at 0, then A is a bounded operator on X. This can be seen as follows. Let $\epsilon > 0$. There is some $\delta > 0$ such that $\|A(x)\| < \epsilon$ whenever $x \in X$ and $\|x\| \leq \delta$. For $x \in X$ with $x \neq 0$, consider $y = \delta x/\|x\|$. Then $\|y\| \leq \delta$ and hence

$$\|A(x)\| = \frac{\|x\|\,\|A(y)\|}{\delta} \leq \frac{\epsilon}{\delta}\|x\|,$$

so that we can let $\alpha = \epsilon/\delta$.

The set of all bounded operators on X is denoted by $BL(X)$. It can be seen that if $A, B \in BL(X)$ and $k \in \mathbf{K}$, then $A + B, kA$ and AB belong to $BL(X)$. We say that $A \in BL(X)$ is **invertible** if there is some $B \in BL(X)$ such that $AB = I = BA$, where I is the identity operator on X.

For $A \in BL(X)$, let

$$\|A\| = \sup\{\|A(x)\| : x \in X, \|x\| \leq 1\}.$$

Then it follows that $\|\ \|$ is a norm on $BL(X)$ and we have

$$\|A(x)\| \leq \|A\|\,\|x\|$$

for all $x \in X$, an inequality of fundamental importance. For A, B in $BL(X)$ and x in X, we have

$$\|AB(x)\| \leq \|A\| \, \|B(x)\| \leq \|A\| \, \|B\| \, \|x\|,$$

so that $\|AB\| \leq \|A\| \, \|B\|$.

If (A_n) and (B_n) are sequences in $BL(X)$ such that $A_n \to A$ and $B_n \to B$ in $BL(X)$, then we see that $A_n + B_n \to A + B$ and $A_n B_n \to AB$, because

$$\begin{aligned} \|A_n + B_n - A - B\| &\leq \|A_n - A\| + \|B_n - B\|, \\ \|A_n B_n - AB\| &\leq \|A_n B_n - A_n B\| + \|A_n B - AB\| \\ &\leq \|A_n\| \, \|B_n - B\| + \|A_n - A\| \, \|B\|. \end{aligned}$$

Calculation of the norm of a bounded operator A on X is, in general, a very demanding task. It calls for maximizing $\|A(x)\|$, subject to $x \in X$ with $\|x\| \leq 1$. Note that for $x \in X$, we have

$$\|A(x)\| = \sup\{|\langle A(x), y\rangle| : y \in X, \|y\| \leq 1\}.$$

This follows by noting $|\langle A(x), y\rangle| \leq \|A(x)\| \, \|y\|$ for all $y \in X$ and $|\langle A(x), y\rangle| = \|A(x)\|$ if we let $y = A(x)/\|A(x)\|$, provided $A(x) \neq 0$. Hence we have

$$\|A\| = \sup\{|\langle A(x), y\rangle| : x, y \in X, \|x\| \leq 1, \|y\| \leq 1\}.$$

In 27.7(b) we shall give another expression for $\|A\|$ which is of considerable theoretical significance.

25.1 Examples

(a) Consider a countable orthonormal set $\{u_1, u_2, \ldots\}$ in a Hilbert space H. Let $(k_{i,j}), i, j = 1, 2 \ldots$ be a matrix with scalar entries such that for each fixed $x \in H$, the series $\sum_j k_{i,j}\langle x, u_j\rangle$ converges in $\mathbf{K}$ for each $i = 1, 2, \ldots$, and if $f_i(x)$ denotes the sum of this series, then the series $\sum_i f_i(x)u_i$ converges in H. If we let

$$A(x) = \sum_i f_i(x)u_i, \quad x \in H,$$

then A is an operator on H. We say that the matrix $(k_{i,j})$ defines the operator A on H with respect to $u_1, u_2, \ldots$. Note that

$$k_{i,j} = f_i(u_j) = \langle A(u_j),\, u_i\rangle$$

for all $i, j = 1, 2, \ldots$, and

$$\sum_j k_{i,j}\langle x,\, u_j\rangle = f_i(x) = \langle A(x),\, u_i\rangle$$

for all $i = 1, 2, \ldots$ and $x \in H$. Using matrix multiplication, these relations can be written as follows:

$$\begin{bmatrix} k_{1,1} & k_{1,2} & \cdot & \cdot & \cdot \\ k_{2,1} & k_{2,2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \langle x,\, u_1\rangle \\ \langle x,\, u_2\rangle \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \langle A(x),\, u_1\rangle \\ \langle A(x),\, u_2\rangle \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

The orthonormality of $u_1, u_2, \ldots$ shows that

$$\|A(x)\|^2 = \langle A(x),\, A(x)\rangle = \sum_i |f_i(x)|^2, \quad x \in H.$$

Hence the operator A defined by the matrix $(k_{i,j})$ is bounded if and only if

$$\sum_i \Big|\sum_j k_{i,j}\langle x,\, u_j\rangle\Big|^2 \le \alpha^2\|x\|^2$$

for all $x \in H$ and some $\alpha > 0$. In that case, $\|A\| \le \alpha$.

We shall now give some sufficient conditions under which a matrix $(k_{i,j})$ defines a bounded operator on H with respect to the orthonormal set $\{u_1, u_2, \ldots\}$ in H.

Firstly, assume that

$$\beta_2 = \left(\sum_{i,j} |k_{i,j}|^2\right)^{1/2} < \infty.$$

Fix $x \in H$. For each $i = 1, 2, \ldots$, Hölder's inequality (3.1(a)) and Bessel's inequality (22.4) yield

$$\sum_j |k_{i,j}\langle x, u_j\rangle| \le \left(\sum_j |k_{i,j}|^2\right)^{1/2} \left(\sum_j |\langle x, u_j\rangle|^2\right)^{1/2} \le \beta_2 \|x\|.$$

Hence the series $\sum_j k_{i,j}\langle x, u_j\rangle$ converges in $\mathbf{K}$ to some $f_i(x)$ for each $i = 1, 2, \ldots$. Also,

$$\sum_i |f_i(x)|^2 \le \sum_i \left(\sum_j |k_{i,j}|^2\right) \left(\sum_j |\langle x, u_j\rangle|^2\right) \le \|x\|^2 \beta_2^2$$

for all $x \in H$. The series $\sum_i f_i(x) u_i$ converges in H for each x in H by the Riesz-Fischer theorem (22.5(b)). Hence the matrix $(k_{i,j})$ defines an operator A with respect to $u_1, u_2, \ldots$ and $\|A\| \le \beta_2$.

Secondly, assume that

$$\alpha_1 = \sup_j \sum_i |k_{i,j}| < \infty \quad \text{and} \quad \alpha_\infty = \sup_i \sum_j |k_{i,j}| < \infty.$$

The following estimates show, as above, that the matrix $(k_{i,j})$ defines an operator A on H and $\|A\| \le \sqrt{\alpha_1 \alpha_\infty}$. This result is known as the **Schur test**. Fix $x \in H$. For each $i = 1, 2 \ldots$,

$$\sum_j |k_{i,j}\langle x, u_j\rangle| \le \alpha_\infty \sup_j |\langle x, u_j\rangle| \le \alpha_\infty \|x\| < \infty.$$

For $i, j = 1, 2, \ldots$,

$$|k_{i,j}\langle x, u_j\rangle| = |k_{i,j}|^{1/2} \left(|k_{i,j}|^{1/2} |\langle x, u_j\rangle|\right),$$

so that Hölder's inequality (3.1(a)) yields

$$\begin{aligned} \sum_i |f_i(x)|^2 &\le \sum_i (\sum_j |k_{i,j}|)(\sum_j |k_{i,j}|\, |\langle x, u_j\rangle|^2) \\ &\le \alpha_\infty \sum_i \sum_j |k_{i,j}|\, |\langle x, u_j\rangle|^2 \le \alpha_1 \alpha_\infty \|x\|^2. \end{aligned}$$

We remark that neither the condition '$\beta_2 < \infty$' nor the conditions '$\alpha_1 < \infty, \alpha_\infty < \infty$' are necessary for a matrix $(k_{i,j})$ to define a bounded operator. For example, if $k_{i,j} = \delta_{i,j}$ for $i, j = 1, 2 \ldots$, then $\beta_2 = \infty$, but $\alpha_1 < \infty$ and $\alpha_\infty < \infty$, while if $k_{1,j} = 1/j = k_{j,1}$ for $j = 1, 2, \ldots$ and $k_{i,j} = 0$ if $i, j > 1$, then $\beta_2 < \infty$, but $\alpha_1 = \infty = \alpha_\infty$.

Conversely, if a matrix $(k_{i,j})$ defines a bounded operator A on H with respect to $\{u_1, u_2, \ldots\}$, what conditions must be satisfied by its entries?

Firstly, we must have

$$\gamma = \sup_j \sum_i |k_{i,j}|^2 < \infty.$$

This follows if we observe that for each $j = 1, 2, \ldots$,

$$\sum_i |k_{i,j}|^2 = \sum_i |\langle A(u_j),\, u_i\rangle|^2 \le \|A(u_j)\|^2 \le \|A\|^2 < \infty$$

by Bessel's inequality for the element $A(u_j)$ of H.

Secondly,, we must have

$$\delta = \sup_i \sum_j |k_{i,j}|^2 < \infty.$$

This will follow from the results in 25.4(a). [We remark that the conditions '$\gamma < \infty$ and $\delta < \infty$' are not sufficient for a matrix $(k_{i,j})$ to define a bounded operator. For example, let M denote the infinite matrix having $n \times n$ diagonal blocks of entries $1/\sqrt{n}, n = 1, 2, \ldots$ and all other entries equal to 0. Then $\gamma = 1 = \delta$, but if $x_n = (0, \ldots, 0, 1/\sqrt{n}, \ldots, 1/\sqrt{n}, 0, 0, \ldots)$, where $1/\sqrt{n}$ occurs in the n entries $[(n-1)n+2]/2, \ldots, (n+1)n/2$, then $\|x_n\|_2 = 1$ but $\|Mx_n\|_2 = \sqrt{n} \to \infty$ as $n \to \infty$.]

Finally, we remark that if $\{u_1, u_2, \ldots\}$ is an orthonormal basis for H, then each $A \in BL(H)$ is defined by the matrix $(\langle A(u_j),\, u_i\rangle)$ with respect to this basis. To see this, consider the Fourier expansion (22.7(ii))

$$x = \sum_j \langle x,\, u_j\rangle u_j, \quad x \in H.$$

By the continuity and the linearity of A, we have

$$\langle A(x),\, u_i\rangle = \sum_j \langle x,\, u_j\rangle\langle A(u_j),\, u_i\rangle = f_i(x), \text{ say.}$$

Hence we have the Fourier expansion

$$A(x) = \sum_i \langle A(x),\, u_i\rangle u_i = \sum_i f_i(x)u_i, \quad x \in H.$$

Thus A is defined by the matrix $(\langle A(u_j),\, u_i\rangle)$ with respect to $u_1, u_2, \ldots$. On the other hand, if an orthonormal set $\{u_1, u_2, \ldots\}$ is not an orthonormal basis for H, then there is some $A \in BL(H)$ which is not defined by a matrix with respect to $\{u_1, u_2, \ldots\}$. To see this, consider $u \in H$ such that $\|u\| = 1$ and $u \perp u_j$ for each $j = 1, 2, \ldots$. Let $A(x) = \langle x,\, u\rangle u$ for $x \in H$. Were A defined by a matrix $(k_{i,j})$ with respect to $u_1, u_2, \ldots$, we must have $k_{i,j} = \langle A(u_j),\, u_i\rangle = \langle 0,\, u_i\rangle = 0$, and in turn $A(x) = 0$ for all $x \in H$. But $A(u) = u \neq 0$.

(b) If $H = \ell^2, u_j = (0, \ldots 0, 1, 0, 0, \ldots)$ for $j = 1, 2, \ldots$, and if a matrix $(k_{i,j})$ defines a bounded operator A on H, then for each $x = (x(1), x(2), \ldots)$ in H, we have

$$A(x)(i) = \sum_{j=1}^{\infty} k_{i,j}x(j), \quad i = 1, 2, \ldots.$$

We consider a continuous analog of this result. Let $H = L^2([a, b])$. In place of a matrix $(k_{i,j}), i, j = 1, 2, \ldots$, consider a measurable function $k(.,.)$ on $[a, b]\times[a, b]$. For $x \in H$, suppose that

$$A(x)(s) = \int_a^b k(s, t)x(t)\, dm(t)$$

is well-defined for almost every $s \in [a, b]$ and $A(x) \in H$. Let

$$\alpha_1 = \operatorname{essup}\left\{\int_a^b |k(s, t)|\, dm(s) : t \in [a, b]\right\},$$

$$\alpha_\infty = \operatorname{essup}\left\{\int_a^b |k(s, t)|\, dm(t) : s \in [a, b]\right\},$$

$$\beta_2 = \left[\int_a^b \int_a^b |k(s, t)|^2 dm(t)\, dm(s)\right]^{1/2}.$$

It can be shown that if $\alpha_1 < \infty$ and $\alpha_\infty < \infty$, or if $\beta_2 < \infty$, then the kernel $k(.,.)$ defines a bounded operator A and $\|A\| \le \min\{\sqrt{\alpha_1\alpha_\infty}, \beta_2\}$. The proof is similar to the considerations in (a) above, but proper attention must be paid to the measurability of various functions that arise in the proof. The operator A is known as a **Fredholm integral operator** on H with kernel $k(.,.)$. (Compare 6.5(d) and 12.4.) See [26] for a detailed account of such operators.

The following result will be useful in defining the adjoint of a bounded operator on H.

25.2 Theorem
Let H be a Hilbert space and $A \in BL(H)$. Then there is a unique $B \in BL(H)$ such that for all $x, y \in H$,

$$\langle A(x),\, y\rangle = \langle x,\, B(y)\rangle.$$

Proof:
Fix $y \in H$ and consider the map $f_y : H \to \mathbf{K}$ defined by

$$f_y(x) = \langle A(x),\, y\rangle, \qquad x \in H.$$

Clearly, f_y is a linear functional. Also, since

$$|f_y(x)| \le \|A(x)\|\,\|y\| \le \|A\|\,\|y\|\,\|x\|$$

for all $x \in H$, we see that f_y is continuous and $\|f_y\| \le \|A\|\,\|y\|$. By the Riesz representation theorem (24.3), there is a unique $z \in H$ such that $f_y(x) = \langle x,\, z\rangle$ for all $x \in H$. Define $B(y) = z$. It is easy to see that B is an operator on H. Also, since

$$\|B(y)\| = \|z\| = \|f_y\| \le \|A\|\,\|y\|$$

for all $y \in H$, we see that B is continuous. Clearly, for all $x, y \in H$,

$$\langle A(x),\, y\rangle = f_y(x) = \langle x,\, B(y)\rangle.$$

Further, for each fixed $y \in H$, this condition determines the element $B(y)$ of H by 21.1(b). Hence the map B is unique $\square$

Let $A \in BL(H)$. The unique element B of $BL(H)$ which satisfies $\langle A(x), y\rangle = \langle x, B(y)\rangle$ for all $x, y \in H$ is called the **adjoint** of A. It is denoted by A^*.

We remark that if X is an inner product space which is not complete and $A \in BL(X)$, then there may not exist $B \in BL(X)$ such that $\langle A(x), y\rangle = \langle x, B(y)\rangle$ for all $x, y \in H$. For example, let $X = c_{00}$ the linear space of all scalar sequences having only a finite number of nonzero entries with the inner product defined in 21.3(b). For $x \in X$, let

$$A(x) = \left(\sum_{j=1}^{\infty} \frac{x(j)}{j}, 0, 0, \dots\right).$$

Then $A \in BL(X)$. In fact, $\|A\| \le \left(\sum_{j=1}^{\infty} 1/j^2\right)^{1/2} = \pi/\sqrt{6}$. For $n = 1, 2, \dots$, let $u_n = (0, \dots, 0, 1, 0, 0, \dots)$, where 1 occurs only in the nth entry. If $B \in BL(X)$ and $\langle A(x), y\rangle = \langle x, B(y)\rangle$ for all $x, y \in H$, then

$$\overline{B(u_1)(n)} = \langle u_n, B(u_1)\rangle = \langle A(u_n), u_1\rangle = \frac{1}{n} \neq 0$$

for all $n = 1, 2, \dots$. But this is impossible since $B(u_1) \in c_{00}$.

We now study the behavior of the adjoint operation on $BL(H)$ with respect to the operations of addition, scalar multiplication and composition. We also relate the norms of A, A^* and A^*A.

25.3 Theorem

Let H be a Hilbert space. Consider $A, B \in BL(H)$ and $k \in \mathbf{K}$. Then

(a) $(A+B)^* = A^*+B^*, (kA)^* = \bar{k}A^*, (AB)^* = B^*A^*, (A^*)^* = A$. Further, A is invertible if and only if A^* is invertible, and in that case $(A^*)^{-1} = (A^{-1})^*$.

(b) $\|A^*\| = \|A\|$ and $\|A^*A\| = \|A\|^2 = \|AA^*\|$.

Proof:

(a) For $x, y \in H$,

$$\begin{aligned}\langle (A+B)x,\, y\rangle &= \langle A(x),\, y\rangle + \langle B(x),\, y\rangle \\ &= \langle x,\, A^*(y)\rangle + \langle x,\, B^*(y)\rangle \\ &= \langle x,\, (A^*+B^*)y\rangle.\end{aligned}$$

By the uniqueness part in the definition of an adjoint, we see that $(A+B)^* = A^* + B^*$. Similarly, we find that $(kA)^* = \overline{k}A^*, (AB)^* = B^*A^*$ and $(A^*)^* = A$.

If A is invertible, then $AC = I = CA$ for some $C \in BL(H)$. Then $C^*A^* = I^* = A^*C^*$. Since $I^* = I$, we see that A^* is invertible and $(A^*)^{-1} = C^* = (A^{-1})^*$. Conversely, if A^* is invertible, then $(A^*)^* = A$ is invertible.

(b) For $x, y \in H$, we have

$$|\langle A^*(x),\, y\rangle| = |\langle x,\, A(y)\rangle| = |\langle A(y),\, x\rangle|.$$

Taking supremum over all $x, y \in H$ with $\|x\| \leq 1$ and $\|y\| \leq 1$, we find that $\|A^*\| = \|A\|$. Next,

$$\|A^*A\| \leq \|A^*\|\, \|A\| = \|A\|^2.$$

On the other hand, for $x \in H$, we have

$$\|A(x)\|^2 = \langle A(x),\, A(x)\rangle = \langle A^*A(x),\, x\rangle \leq \|A^*A\|\, \|x\|^2.$$

Taking supremum over all $x \in H$ with $\|x\| \leq 1$, we find that $\|A\|^2 \leq \|A^*A\|$. Thus $\|A^*A\| = \|A\|^2$. Replacing A by A^* and noting that $(A^*)^* = A$, we obtain $\|AA^*\| = \|A\|^2$. $\square$

25.4 Examples

(a) Let H be a separable Hilbert space and $u_1, u_2, \ldots$ constitute an orthonormal basis for H. We have seen in 25.1(a) that each A in $BL(H)$ is defined by the matrix $M = (k_{i,j})$, where $k_{i,j} = \langle A(u_j), u_i\rangle$, $i, j = 1, 2, \ldots$. Since $A^* \in BL(H)$ and

$$\langle A^*(u_j), u_i\rangle = \langle u_j, A(u_i)\rangle = \overline{\langle A(u_i), u_j\rangle}$$

for $i, j = 1, 2, \ldots$, it follows that A^* is defined by the matrix $(\overline{k}_{j,i})$ with respect to the same orthonormal basis $u_1, u_2, \ldots$ of H. The matrix $(\overline{k}_{j,i})$ is the **conjugate transpose** $\overline{M}^t$ of the matrix $M = (k_{i,j})$. Problem 25-13 shows that the elements $u_1, u_2, \ldots$ of H need to be orthogonal to each other and Problem 25-14 shows that we ought to consider the **conjugate** of the transpose of $(k_{i,j})$ not the transpose itself.

It follows that if a matrix $(k_{i,j})$ defines a bounded operator A on H with respect to an orthonormal basis $u_1, u_2, \ldots$ for H, then the conjugate transpose of the matrix $(k_{i,j})$ defines the bounded operator A^* on H and hence

$$\sup_i \sum_j |k_{i,j}|^2 = \sup_j \sum_i |\overline{k}_{j,i}|^2 < \infty,$$

as mentioned earlier in 25.1(a).

If $A \in BL(H)$ is defined by a diagonal matrix diag $(k_1, k_2, \ldots)$, then A^* is also defined by a diagonal matrix, namely diag $(\overline{k}_1, \overline{k}_2, \ldots)$. If H is infinite dimensional and $A \in BL(H)$ is the **right shift operator** satisfying $A(u_j) = u_{j+1}$, $j = 1, 2, \ldots$, then A and A^* are defined by the matrices

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

respectively, with respect to $u_1, u_2, \ldots$. Hence the adjoint A^* of the right shift operator A is the **left shift operator** which satisfies $A^*(u_1) = 0$ and $A^*(u_j) = u_{j-1}$, $j = 2, 3, \ldots$.

(b) Let $H = L^2([a,b])$ and $k(.,.)$ be a measurable function on $[a,b]\times[a,b]$ such that either $\beta_2 < \infty$, or $\alpha_1 < \infty$ and $\alpha_\infty < \infty$, where β_1, α_1 and α_∞ are defined in 25.1(b). Now

$$\begin{aligned} A(x)(s) &= \int_a^b k(s,t)x(t)\,dm(t), \quad x \in H, s \in [a,b], \\ B(x)(s) &= \int_a^b \overline{k(t,s)}x(t)\,dm(t), \quad x \in H, s \in [a,b], \end{aligned}$$

define bounded operators A and B on H. We show that $A^* = B$. For $x, y \in H$, we have

$$\langle A(x), y\rangle = \int_a^b \left[\int_a^b k(s,t)x(t)\,dm(t)\right]\overline{y}(s)dm(s),$$

while

$$\langle x, B(y)\rangle = \int_a^b x(t)\left[\int_a^b k(u,t)\overline{y(u)}\,dm(u)\right]dm(t).$$

By Fubini's theorem (4.4), these two repeated integrals will be equal if we show that

$$\alpha = \int_a^b \int_a^b |k(s,t)x(t)\overline{y(s)}|dm(s)\,dm(t) < \infty.$$

If $\beta_2 < \infty$, this follows since α^2 is at most equal to

$$\left(\int_a^b \int_a^b |k(s,t)|^2 dm(s)\,dm(t)\right)\left(\int_a^b |x(t)|^2\,dm(t)\right)\left(\int_a^b |y(s)|^2\,dm(s)\right).$$

If $\alpha_1 < \infty$ and $\alpha_\infty < \infty$, this follows since

$$|k(s,t)x(t)\overline{y(s)}| = \left(|k(s,t)|^{1/2}|x(t)|\right)\left(|k(s,t)|^{1/2}|y(s)|\right)$$

and α^2 is at most equal to the product of

$$\int_a^b |x(t)|^2\left(\int_a^b |k(s,t)|\,dm(s)\right)dm(t)$$

and

$$\int_a^b |y(s)|^2\left(\int_a^b |k(s,t)|\,dm(t)\right)dm(s).$$

We now relate the adjoint A^* of $A \in BL(H)$ to the transpose A' of A defined in Section 13. Recall that the dual H' of H is defined to be the set of all $f : H \to \mathbf{K}$ which are linear and continuous. Further, $A' : H' \to H'$ is defined by

$$A'(f)(x) = f(A(x)), \quad f \in H', x \in H.$$

For $f \in H'$, let y_f be the representer of f given by the Riesz representation theorem (24.3). Then

$$A'(f)(x) = \langle A(x), y_f \rangle = \langle x, A^*(y_f) \rangle$$

for all $x \in X$. Thus $A^*(y_f) \in H$ is the representer of $A'(f) \in H'$. Let $T : H' \to H$ be the map which sends $f \in H'$ to its representer $y_f \in H$. Then we have $T(A'(f)) = A^*(T(f))$ for every $f \in H'$. As we have seen in 24.5, T is a conjugate-linear surjective isometry. Thus

$$A^* = TA'T^{-1}$$

for every $A \in BL(H)$.

Using this relation, several results involving A and A^* can be deduced from the results involving A and A' given in 13.7 and 13.9. However, we choose to prove them independently, since direct proofs are simpler and since we can strengthen some of the earlier results.

We say that a linear map $A : H \to H$ is **bounded below** if $\beta\|x\| \le \|A(x)\|$ for all $x \in H$ and some $\beta > 0$.

25.5 Theorem

Let H be a Hilbert space and $A \in BL(H)$.

(a) $Z(A) = R(A^*)^\perp$ and $Z(A^*) = R(A)^\perp$.

A is injective if and only if $R(A^*)$ is dense in H, and A^* is injective if and only if $R(A)$ is dense in H.

(b) The closure of $R(A)$ equals $Z(A^*)^\perp$, and the closure of $R(A^*)$ equals $Z(A)^\perp$.

(c) $R(A) = H$ if and only if A^* is bounded below, and $R(A^*) = H$ if and only if A is bounded below.

Proof:

We observe that the above-mentioned results are symmetric in A and A^*, since $A^{**} = A$. Hence it is enough to prove only a half of them.

(a) Let $x \in H$. Then $x \in Z(A)$, that is, $A(x) = 0$ if and only if $\langle x, A^*(y)\rangle = \langle A(x), y\rangle = 0$ for all $y \in H$, that is, $x \in R(A^*)^\perp$. Hence A is injective, that is, $Z(A) = \{0\}$ if and only if $R(A^*)^\perp = \{0\}$, that is, $R(A^*)$ is dense in H.

(b) Let F denote the closure of $R(A)$, and note that $F^\perp = R(A)^\perp$. Since F is a closed subspace of H, 24.1 and part (a) above show that

$$F = F^{\perp\perp} = [R(A)^\perp]^\perp = Z(A^*)^\perp.$$

(c) Assume that $R(A) = H$. Suppose for a moment that A^* is not bounded below. Then there is a sequence (x_n) in H such that $\|A^*(x_n)\| < \|x_n\|/n$ for $n = 1, 2, \ldots$. Let $y_n = nx_n/\|x_n\|$, so that $\|A^*(y_n)\| < 1$. We show that the sequence (y_n) is weak bounded in H. Consider $y \in H$. Since A is surjective, there is some $x \in H$ such that $A(x) = y$. Then

$$|\langle y_n, y\rangle| = |\langle y_n, A(x)\rangle| = |\langle A^*(y_n), x\rangle| \leq \|A^*(y_n)\|\,\|x\| < \|x\|.$$

By 24.8, the weak bounded sequence (y_n) must be bounded in H. But $\|y_n\| = n \to \infty$. This contradiction proves that A^* is bounded below.

Conversely, assume that A^* is bounded below: $\beta\|x\| \leq \|A^*(x)\|$ for all $x \in H$ and some $\beta > 0$. We first prove that $R(A^*)$ is closed in H. If $A^*(x_n) \to y$ in H, then the inequality

$$\beta\|x_n - x_m\| \leq \|A^*(x_n) - A^*(x_m)\|, \quad n, m = 1, 2 \ldots,$$

shows that (x_n) is Cauchy in H. If $x_n \to x$ in H, then $A^*(x_n) \to A^*(x)$. Hence $y = A^*(x) \in R(A^*)$, as desired. Thus $R(A^*)$ is a Hilbert space.

Let $y \in H$. Define $g : R(A^*) \to \mathbf{K}$ by

$$g(A^*(w)) = \langle w, y \rangle, \quad w \in H.$$

The functional g is well-defined since A^* is bounded below and hence injective. Clearly, g is linear. Also,

$$|g(z)| \leq \|w\| \|y\| \leq \frac{1}{\beta} \|z\| \|y\|$$

for all $z \in R(A^*)$, so that g is continuous on $R(A^*)$.

By the Riesz representation theorem (24.3) for $R(A^*)$, there is some $x \in R(A^*)$ such that $g(z) = \langle z, x \rangle$ for all $z \in R(A^*)$. Hence for every $w \in H$, we have

$$\langle w, y \rangle = g(A^*(w)) = \langle A^*(w), x \rangle = \langle w, A(x) \rangle.$$

Thus $y = A(x) \in R(A)$. Hence $R(A) = H$ □

We shall have an occasion to use some of the results proved above in Section 27.

Problems

X denotes an inner product space and H denotes a Hilbert space over $\mathbf{K}$, unless otherwise stated.

25-1 Let G be a subspace of H and $B \in BL(G)$. Then there is some A in $BL(H)$ such that $A_{|G} = B$ and $\|A\| = \|B\|$. Such an operator A is not unique unless G is dense in H or $B = 0$. (Hint: 24.1)

25-2 Let E be a measurable subset of $\mathbf{R}$ and $H = L^2(E)$. For $x, y \in H$, let

$$\langle x, y \rangle = \int_E x\overline{y}\, dm.$$

Then H is a Hilbert space. Fix $z \in L^\infty(E)$ and define $A(x) = zx, x \in H$. Then $A \in BL(H)$ and $\|A\| = \|z\|_\infty$.

25-3 Let $\{u_1, u_2 \ldots\}$ be an orthonormal set in $H, A \in BL(H)$ and $k_{i,j} = \langle A(u_j), u_i\rangle, i, j = 1, 2, \ldots$. Then the matrix $(k_{i,j})$ defines a bounded operator B on H with respect to $u_1, u_2, \ldots$. In fact, $B = PAP$, where $P(x) = \sum_j \langle x, u_j\rangle u_j, x \in H$. If $u_1, u_2 \ldots$ constitute an orthonormal basis for H, then $B = A$.

25-4 Let $\{u_1, u_2, \ldots\}$ be an orthonormal set in H. Suppose that a matrix $(a_{i,j})$ (resp., $(b_{i,j})$) defines a bounded operator A (resp., B) on H with respect to $u_1, u_2, \ldots$. Then the bounded operator AB on H is defined by the matrix $(c_{i,j})$ (with respect to $u_1, u_2 \ldots$), where $c_{i,j} = \sum_k a_{i,k} b_{k,j}$ for $i, j \ldots$.

25-5 Let A and B denote Fredholm integral operators on $H = L^2([a,b])$ with kernels $k(.,.)$ and $h(.,.)$ in $L^2([a,b]\times[a,b])$, respectively. Then $A = B$ if and only if $k(.,.)$ and $h(.,.)$ are equal almost everywhere on $[a,b]\times[a,b]$. Further, AB is a Fredholm integral operator with kernel

$$k \circ h(s,t) = \int_a^b k(s,u)h(u,t)\,dm(u), \quad (s,t) \in [a,b]\times[a,b],$$

and $\|AB\| \le \|k \circ h\|_2 \le \|k\|_2\|h\|_2$. (Hint: If $A = 0$, then $k(.,.)$ is orthogonal to every degenerate kernel. Use part 4 of 20.4. Compare 12.4.)

25-6 Let $\{u_1, u_2, \ldots\}$ be an orthonormal set in H and $k_1, k_2, \ldots$ be scalars such that $\alpha = \sup\{|k_n| : n = 1, 2, \ldots\} < \infty$. If

$$A(x) = \sum_{n=1}^{\infty} k_n \langle x, u_n\rangle u_n, \quad x \in H,$$

then $A \in BL(H)$ and $\|A\| = \alpha$. In this case, A is called a **diagonal operator** with respect to $\{u_1, u_2, \ldots\}$.

25-7 (a) Let $a_1, a_2, \ldots, b_1, b_2, \ldots, c_2, c_3, \cdots$ be scalars such that

$$\begin{aligned} \alpha_1 &= \sup\{|b_{n-1}| + |a_n| + |c_{n+1}| : n = 1, 2, \ldots\} < \infty, \\ \alpha_\infty &= \sup\{|c_n| + |a_n| + |b_n| : n = 1, 2, \ldots\} < \infty \end{aligned}$$

with $b_0 = 0 = c_1$. Let $\{u_1, u_2, \ldots\}$ be an orthonormal set in H and

$$A(x) = \sum_{n=1}^{\infty} \langle x, u_n\rangle (b_{n-1}u_n + a_n u_n + c_{n+1}u_{n+1}), \quad x \in H.$$

Then $A \in BL(H)$ and $\|A\| \le \sqrt{\alpha_1 \alpha_\infty}$. In this case, A is called a **tridiagonal operator** with respect to $\{u_1, u_2, \ldots\}$.

(b) Let $H = L^2([0,1])$ and

$$k(s,t) = \begin{cases} 0, & \text{if } 0 \le s \le t \le 1 \\ \dfrac{1}{\sqrt{s-t}}, & \text{if } 0 \le t < s \le 1. \end{cases}$$

Then the Fredholm integral operator A on H with kernel $k(.,.)$ is bounded and $\|A\| \le 2$. (Hint: $\alpha_1, \alpha_\infty \le 2$)

25-8 **(Generalized Schur test)** Let $(k_{i,j})$ be an infinite matrix such that for some positive numbers $p_1, p_2, \ldots, q_1, q_2, \ldots, \alpha_1, \alpha_\infty$, we have

$$\sum_{i=1}^{\infty} |k_{i,j}| p_i \le \alpha_1 q_j \quad \text{and} \quad \sum_{j=1}^{\infty} |k_{i,j}| q_j \le \alpha_\infty p_i, \quad i, j = 1, 2, \ldots.$$

Then $(k_{i,j})$ defines a bounded operator A on ℓ^2 with respect to the standard basis for ℓ^2 and $\|A\| \le \sqrt{\alpha_1 \alpha_\infty}$.

In particular, the **Hilbert matrix** $\left(\dfrac{1}{i+j-1}\right), i, j = 1, 2, \ldots$, defines a bounded operator A on ℓ^2 and $\|A\| \le \pi$. (Hint: Let $p_i = q_i = \dfrac{1}{\sqrt{i - 1/2}}, i, j = 1, 2, \ldots$ and use $\displaystyle\int_0^\infty \frac{dt}{t^2+1} = \frac{\pi}{2}$.)

25-9 Let $u_1, u_2, \ldots$ constitute an orthonormal basis for H and $k_{i,j} \in \mathbf{K}$ for $i, j = 1, 2, \ldots$. Then a necessary and sufficient condition that there is some $A \in BL(H)$ with $\langle A(u_j), u_i \rangle = k_{i,j}$ for all $i, j = 1, 2, \ldots$ is that for every pair of scalar sequence (s_i) and (t_j), and every $n = 1, 2, \ldots$,

$$|\sum_{i=1}^{n} \sum_{j=1}^{n} \bar{t}_j k_{i,j} s_i|^2 \le \alpha \left(\sum_{i=1}^{n} |s_i|^2\right) \left(\sum_{j=1}^{n} |t_j|^2\right)$$

for some $\alpha > 0$.

25-10 Let $P : X \to X$ be a **projection**, that is, P is linear and $P^2 = P$. If $0 \ne P \ne I$, then $\|P\| = 1/\sqrt{(1+\alpha)(1-\alpha)}$, where

$$\alpha = \sup\{\operatorname{Re}\langle y, z\rangle : y \in R(P), z \in Z(P), \|y\| = 1 = \|z\|\}.$$

Thus $\|P\| = \csc\theta$, where θ is the supremum of the angles between a unit vector in $R(P)$ and a unit vector in $Z(P)$. In particular, $\|I - P\| = \|P\|$.

(Compare Problem 6.9. Hint: $\|P\| = \sup\{\|P(\beta y + \gamma z)\|/\|\beta y + \gamma z\| : y \in R(P), z \in Z(P), \|y\| = 1 = \|z\|, \beta \geq 0, \gamma \geq 0, \beta + \gamma > 0\}$)

25-11 Let $\{u_1, u_2, \ldots\}$ be an infinite orthonormal set in H. For $x \in H$ and $n = 1, 2, \ldots$, let

$$A_n(x) = \sum_{j=1}^{n} \langle x, u_j \rangle u_j, \quad B_n(x) = \sum_{j=1}^{n} \frac{\langle x, u_j \rangle}{j} u_j,$$

$$A(x) = x, \quad B(x) = \sum_{j=1}^{\infty} \frac{\langle x, u_j \rangle}{j} u_j$$

Then $\|A_n - A\| \not\to 0$, but $\|B_n - B\| \to 0$ as $n \to \infty$.

25-12 **(Invariant subspace) Let $A \in BL(H)$ and F be a closed subspace** of H. Then $A(F) \subset F$ if and only if $A^*(F^\perp) \subset F^\perp$, and in that case $(A_{|F})^* = PA^*{}_{|F}$, where P is the orthogonal projection from H onto F.

25-13 Let $H = \mathbf{K}^2$ and $A(x(1), x(2)) = (x(2), x(1))$ for $(x(1), x(2))$ in H. Then $A^* = A$. If $y_1 = (1, 1)$ and $y_2 = (0, 1)$, then $\{y_1, y_2\}$ is a basis for H, which is not orthonormal. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ defines A as well as A^* with respect to the ordered basis y_1, y_2 of H in the sense of 13.4(a), but it does not equal its conjugate transpose.

25-14 Let $H = \mathbf{C}^2$ and $A(x(1), x(2)) = (2x(1), x(2))$ for $(x(1), x(2))$ in H. Then $A^* = A$. Let $u_1 = (1, 0), u_2 = (0, 1), v_1 = (i/2, -i\sqrt{3}/2), v_2 = (\sqrt{3}/2, 1/2)$. Then both $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are orthonormal bases for H. The matrices $M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $N = \dfrac{1}{4}\begin{bmatrix} 5 & -i\sqrt{3} \\ i\sqrt{3} & 7 \end{bmatrix}$ define A with respect to $\{u_1, u_2\}$ and $\{v_1, v_2\}$, respectively. The transpose of M, also defines A with respect to $\{u_1, u_2\}$, but the transpose of N does not define A with respect to $\{v_1, v_2\}$.

25-15 Let H be a Hilbert space over $\mathbf{R}$. Then $H_{\mathbf{C}} = \{(x, y) : x, y \in H\}$ is a Hilbert space over $\mathbf{C}$ if we let $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, $(s+it)(x, y) = (sx - ty, tx + sy)$ and $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle - i\langle x_1, y_2 \rangle + i\langle y_1, x_2 \rangle$. It is called the **complexification** of H. For A in $BL(H)$, define $A_{\mathbf{C}}((x, y)) = (A(x), A(y))$ for $(x, y) \in H_{\mathbf{C}}$. Then $A_{\mathbf{C}}$ belongs to $BL(H_{\mathbf{C}})$, $\|A_{\mathbf{C}}\| = \|A\|$ and $(A_{\mathbf{C}})^* = (A^*)_{\mathbf{C}}$. Also, $AB = BA$ if and only if $A_{\mathbf{C}}B_{\mathbf{C}} = B_{\mathbf{C}}A_{\mathbf{C}}$.

25-16 Let A and B be operators on H such that $\langle A(x), y\rangle = \langle x, B(y)\rangle$ for all $x, y \in H$. Then $A \in BL(H)$ and $B = A^*$. (Hint: Use 24 8 for the subset $\{x \in H : \|x\| \leq 1\}$.)

25-17 Let H_1 and H_2 be Hilbert spaces with inner products $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, respectively. If $A : H_1 \to H_2$ is a continuous linear map, then there is a unique continuous linear map $A^* : H_2 \to H_1$ such that

$$\langle A(x), y\rangle_2 = \langle x, A^*(y)\rangle_1, \quad x \in H_1, y \in H_2.$$

Also, $\|A^*\| = \|A\|$, $Z(A) = R(A^*)^{\perp}$, the closure of $R(A)$ equals $Z(A^*)^{\perp}$, and $R(A) = H_2$ if and only if A^* is bounded below. (Compare 25.2 and 25.5).

25-18 Let F be a closed subspace of H. For $y \in F$, let $A(y) = y$. Then A is a continuous linear map from F to H, and for $x \in H$, we have $A^*(x) = y$, where $x = y + z$, $y \in F$ and $x \in F^{\perp}$.

25-19 **(Bounded inverse theorem)** If $A \in BL(H)$ is bijective, then A is invertible. (Hint: A^* is invertible by 25.5(a) and (c).) In fact, if H_1 and H_2 are Hilbert spaces and a map $A : H_1 \to H_2$ is linear, continuous and bijective, then A^{-1} is continuous. (Hint: Problem 25-17)

25-20 Let H_1 and H_2 be Hilbert spaces and $A : H_1 \to H_2$ be linear.

(a) **(Closed graph theorem)** A is continuous if and only if the set $G(A) = \{(x, A(x)) : x \in H_1\}$ is closed in the Hilbert space $H_1 \times H_2$ with the inner product $\langle (x_1, x_2), (y_1, y_2)\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle$. (Hint: Define $B : G(A) \to H_1$ by $B(x, A(x)) = x$ and use Problem 25-19)

(b) **(Open mapping theorem)** If A is continuous and surjective, then A maps open sets in H_1 onto open sets in H_2. (Hint: Consider $H_1/Z(A)$ and use Problem 25-19)

25-21 Let $A \in BL(H)$. Then $R(A)$ is closed in H if and only if $\beta\|x\| \leq \|A(x)\|$ for all $x \in Z(A)^{\perp}$ and some $\beta > 0$. Also, the following statements are equivalent: (i) $R(A^*)$ is closed in H (ii) $R(A) = Z(A^*)^{\perp}$ (iii) $R(A)$ is closed in H (iv) $R(A^*) = Z(A)^{\perp}$. (Hint: Proof of 25.5(c))

25-22 Let $A \in BL(H)$. Then $Z(A^*A) = Z(A)$ and the closure of $R(A^*A)$ equals the closure of $R(A^*)$. If $R(A^*)$ is closed, then $R(A^*A) = R(A^*)$.

26 Normal, Unitary and Self-Adjoint Operators

In this section we study some bounded operators on a Hilbert space H which are 'well-behaved' with regard to the adjoint operation. Since commutativity of elements of $BL(H)$ is rare to come by, a bounded operator on H which commutes with its own adjoint calls for a special treatment!

Let $A \in BL(H)$. Then A is called **normal** if $A^*A = AA^*$, **unitary** if $A^*A = I = AA^*$, that is, $A^* = A^{-1}$, and **self-adjoint** if $A^* = A$.

Clearly, if A is unitary or self-adjoint, then A is normal. However, a normal operator need be neither unitary nor self-adjoint. For example, let $H = \mathbf{K}^2$ and for $x = (x(1), x(2))$ in H,

$$A(x) = (x(1) - x(2), x(1) + x(2)).$$

Then it can be easily seen that for $x \in H$,

$$A^*(x) = (x(1) + x(2), -x(1) + x(2)),$$

$$A^*A(x) = 2(x(1), x(2)) = AA^*(x).$$

Hence $A^*A = AA^*$, but $A^*A \neq I$ and $A^* \neq A$.

It is easy to see that if B is a normal operator and a C is a bounded operator such that $C^*C = I$, then the operator $A = CBC^*$ is normal. Note that a bounded operator C may satisfy $C^*C = I$ without being a unitary operator. For example, let $H = \ell^2$ and for $x = (x(1), x(2), \ldots)$ in H, let

$$C(x) = (0, x(1), x(2), \ldots).$$

Then $C^*(x) = (x(2), x(3), \ldots)$ for $x \in H$, as we have seen in 25.4(a). Hence

$$C^*C(x) = C^*((0, x(1), x(2), \ldots)) = (x(1), x(2) \ldots),$$

$$CC^*(x) = C((x(2), x(3), \ldots)) = (0, x(2), x(3), \ldots)$$

for all $x \in H$, so that $C^*C = I$ but $CC^* \neq I$.

Let $A \in BL(H)$. Since $\langle A(x), y\rangle = \langle x, A^*(y)\rangle$ for all $x, y \in H$, we immediately see that A is

normal if and only if $\langle A(x), A(y)\rangle = \langle A^*(x), A^*(y)\rangle$,

unitary if and only if $\langle A(x), A(y)\rangle = \langle x, y\rangle = \langle A^*(x), A^*(y)\rangle$,

self-adjoint if and only if $\langle A(x), y\rangle = \langle x, A(y)\rangle$

for all $x, y \in H$. We have seen in Sections 21 and 22 that the inner product $\langle\ ,\ \rangle$ on H characterizes the geometry of H. Hence an operator A is normal if A and A^* transform the geometry of H in the same fashion, and A is unitary if neither A nor A^* change the geometry of H. For this reason, a unitary operator is known as a **Hilbert space isomorphism**.

To get a clear idea of these operators, we consider a number of examples.

26.1 Examples

(a) Let E be a measurable subset of $\mathbf{R}$ and $H = L^2(E)$. Fix z in $L^\infty(E)$ and define

$$A(x) = zx, \quad x \in H.$$

Then A is an operator on H. Since

$$\|A(x)\|_2^2 = \int_E |zx|^2 \, dm \leq \|z\|_\infty^2 \|x\|_2^2, \quad x \in H,$$

we see that A is bounded. (See Problem 25-2.) For $x, y \in H$,

$$\langle y, A^*(x)\rangle = \langle A(y), x\rangle = \int_E zy\overline{x} \, dm = \langle y, \overline{z}x\rangle.$$

Thus $A^*(x) = \overline{z}x$ for all $x \in H$. Since

$$A^*A(x) = |z|^2 x = AA^*(x), \quad x \in H,$$

we see that A is normal. Further, A is unitary if and only if $|z| = 1$ a.e. on $[a, b]$, and A is self-adjoint if and only if $z(t) \in \mathbf{R}$ for almost all $t \in [a, b]$.

This is a typical example of a normal operator on a Hilbert space in the following sense. If $A \in BL(H)$ is normal, then there exist a measure space (E, μ), a surjective linear isometry $V : H \to L^2(E, \mu)$ and some $z \in L^\infty(E, \mu)$ such that $V(A(x)) = zV(x)$ for all $x \in H$. We refer the interested reader to [25] for this result which is, in fact, a version of the spectral theorem for normal operators.

(b) Let H be a separable Hilbert space and $u_1, u_2, \ldots$ constitute orthonormal basis for H. Let (k_n) be a bounded sequence of scalars and

$$A(x) = \sum_n k_n \langle x, u_n \rangle u_n, \quad x \in H.$$

Then it follows from 25.1 and 25.4(a) that $A \in BL(H)$ and

$$A^*(x) = \sum_n \overline{k}_n \langle x, u_n \rangle u_n, \quad x \in H.$$

Since for all $x \in H$,

$$\begin{aligned} A^*(A(x)) &= \sum_n \overline{k}_n \langle A(x), u_n \rangle u_n = \sum_n |k_n|^2 \langle x, u_n \rangle u_n, \\ A(A^*(x)) &= \sum_n k_n \langle A^*(x), u_n \rangle u_n = \sum_n |k_n|^2 \langle x, u_n \rangle u_n, \end{aligned}$$

we see that A is normal. Further, A is unitary if and only if $|k_n| = 1$ for each n and A is self-adjoint if and only if $k_n \in \mathbf{R}$ for each n.

More generally, let $A \in BL(H)$ be defined by the matrix $M = (k_{i,j})$ with respect to the orthonormal basis $u_1, u_2, \ldots$. Then $k_{i,j} = \langle A(u_j), u_i \rangle$, $i = 1, 2, \ldots$ We have seen in 25.4(a) that $A^* \in BL(H)$ is defined by the matrix $\overline{M}^t = (\overline{k}_{j,i})$ with respect to $u_1, u_2, \ldots$. Hence $A(x) = \sum_i (\sum_j k_{i,j} \langle x, u_j \rangle) u_i$ and $A^*(x) = \sum_i (\sum_j \overline{k}_{j,i} \langle x, u_j \rangle) u_i$ for all $x \in H$. Thus we see that A is self-adjoint if and only if $\overline{M}^t = M$, that is, the matrix M is conjugate-symmetric. Next, $A^*A \in BL(X)$ is defined by the matrix $\overline{M}^t M = (\sum_n \overline{k}_{n,i} k_{n,j})$ with respect to $u_1, u_2, \ldots,$

since

$$\begin{aligned}\langle A^*A(u_j),\, u_i\rangle &= \langle A(u_j),\, A(u_i)\rangle \\ &= \langle \sum_n \langle A(u_j),\, u_n\rangle u_n,\, \sum_m \langle A(u_i),\, u_m\rangle u_m\rangle \\ &= \sum_n \langle A(u_j),\, u_n\rangle \overline{\langle A(u_i),\, u_n\rangle} \\ &= \sum_n k_{n,j}\overline{k}_{n,i}\end{aligned}$$

for all $i, j = 1, 2, \ldots$. Similarly, $AA^* \in BL(H)$ is defined by the matrix $M\overline{M}^t = (\sum_n k_{i,n}\overline{k}_{j,n})$ with respect to $u_1, u_2, \ldots$. Hence we see that A is unitary if and only if $\overline{M}^t M = I = M\overline{M}^t$, that is,

$$\sum_n \overline{k}_{n,i}k_{n,j} = \delta_{i,j} = \sum_n k_{i,n}\overline{k}_{j,n}$$

for all $i, j = 1, 2, \ldots$. This says that the columns of M form an orthonormal set in ℓ^2, and so do its rows. Next, A is normal if and only if $\overline{M}^t M = M\overline{M}^t$. This is certainly the case if M is a diagonal matrix. See Problems 26-1 and 27-5(b) for connections between the normality of A and the representability of A by a diagonal matrix.

Let $H = \mathbf{K}^2$ and $A \in BL(H)$ be given by

$$A(x(1), x(2)) = (ax(1) + bx(2), cx(1) + dx(2)),$$

where a, b, c and d are fixed scalars. Problem 26-2 gives precise conditions on a, b, c, d for A to be self-adjoint, unitary or normal.

We now consider some important properties of self-adjoint, unitary and normal operators.

26.2 Theorem

Let H be a Hilbert space and $A \in BL(H)$.

(a) Let A be self-adjoint. Then

$$\|A\| = \sup\{|\langle A(x),\, x\rangle| : x \in H, \|x\| \leq 1\}.$$

In particular, $A = 0$ if and only if $\langle A(x),\, x\rangle = 0$ for all $x \in H$.

(b) A is unitary if and only if $\|A(x)\| = \|x\|$ for all $x \in H$ and A is surjective. In that case, $\|A^{-1}(x)\| = \|x\|$ for all $x \in H$ and

$$\|A\| = 1 = \|A^{-1}\|.$$

(c) A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$. In that case,

$$\|A^2\| = \|A^*A\| = \|A\|^2.$$

Proof:

(a) We have seen in Section 25 that for every $A \in BL(H)$,

$$\|A\| = \sup\{|\langle A(x), y\rangle| : x, y \in H, \|x\| \leq 1, \|y\| \leq 1\}.$$

Let $\alpha = \sup\{|\langle A(x), x\rangle| : x \in H, \|x\| \leq 1\}$. Clearly, $\alpha \leq \|A\|$. To prove $\|A\| \leq \alpha$, we note that for $x, y \in H$,

$$\begin{aligned}\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle &= 2\langle A(x), y\rangle + 2\langle A(y), x\rangle \\ &= 4\mathrm{Re}\,\langle A(x), y\rangle,\end{aligned}$$

since A is self-adjoint. Hence

$$4\mathrm{Re}\,\langle A(x), y\rangle \leq \alpha(\|x+y\|^2 + \|x-y\|^2) = 2\alpha(\|x\|^2 + \|y\|^2)$$

by the parallelogram law (21.2(b)). Let $\|x\| \leq 1$ and $\|y\| \leq 1$. Then it follows that $\mathrm{Re}\,\langle A(x), y\rangle \leq \alpha$. If $\langle A(x), y\rangle = re^{i\theta}$ for $r \geq 0$ and $\theta \in \mathbf{R}$, then let $x_0 = e^{-i\theta}x$, so that $\|x_0\| = \|x\| \leq 1$ and

$$|\langle A(x), y\rangle| = r = \langle A(x_0), y\rangle = \mathrm{Re}\,\langle A(x_0), y\rangle \leq \alpha.$$

Taking supremum over all $x, y \in H$ with $\|x\| \leq 1, \|y\| \leq 1$, we obtain $\|A\| \leq \alpha$, as desired.

In particular, if $\langle A(x), x\rangle = 0$ for all $x \in H$, then $\|A\| = \alpha = 0$, that is, $A = 0$. The converse is obvious.

(b) For $x \in H$, we have

$$\begin{aligned} \|A(x)\|^2 - \|x\|^2 &= \langle A(x), A(x)\rangle - \langle x, x\rangle \\ &= \langle A^*A(x), x\rangle - \langle x, x\rangle \\ &= \langle (A^*A - I)x, x\rangle. \end{aligned}$$

Since $A^*A - I$ is self-adjoint, it follows from part (a) that $A^*A = I$ if and only if $\|A(x)\| = \|x\|$ for all $x \in H$.

Thus if $\|A(x)\| = \|x\|$ for all $x \in H$ and A is surjective, then $A^*A = I$ and A is bijective, so that

$$AA^* = (AA^*)(AA^{-1}) = A(A^*A)A^{-1} = AA^{-1} = I,$$

that is, A is unitary.

Conversely, if A is unitary, then $A^*A = I$ and $A^{-1} = A^*$, so that $\|A(x)\| = \|x\|$ for all $x \in H$ and A is surjective. In that case, it follows that $\|A^{-1}(x)\| = \|x\|$ for all $x \in H$. Taking supremum over all $x \in H$ with $\|x\| \leq 1$, we obtain $\|A\| = 1 = \|A^{-1}\|$.

(c) For $x \in H$, we have

$$\begin{aligned} \|A(x)\|^2 - \|A^*(x)\|^2 &= \langle A(x), A(x)\rangle - \langle A^*(x), A^*(x)\rangle \\ &= \langle A^*A(x), x\rangle - \langle AA^*(x), x\rangle \\ &= \langle (A^*A - AA^*)(x), x\rangle. \end{aligned}$$

Since $B = A^*A - AA^*$ is self-adjoint, it follows from part (a) that $B = 0$ (that is, A is normal) if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$. In that case, it follows that

$$\|A^2(x)\| = \|A(A(x))\| = \|A^*A(x)\|$$

for all $x \in H$. Hence $\|A^2\| = \|A^*A\| = \|A\|^2$ by 25.3(b). □

We now investigate whether sums, compositions and limits of self-adjoint, unitary and normal operators are respectively self-adjoint, unitary and normal.

26.3 Theorem

Let H be a Hilbert space.

(a) Let A and B be self-adjoint. Then $A+B$ is self-adjoint. Also, AB is self-adjoint if and only if A and B commute.

(b) Let A and B be unitary. Then AB is unitary. Also, $A+B$ is unitary if and only if it is surjective and $\mathrm{Re}\,\langle A(x), B(x)\rangle = -1/2$ for every $x \in H$ with $\|x\| = 1$.

(c) Let A and B be normal. If A commutes with B^* and B commutes with A^*, then $A+B$ and AB are normal.

(d) Let (A_n) be a sequence in $BL(H)$, and $A \in BL(H)$ be such that $\|A_n - A\| \to 0$ as $n \to \infty$. If each A_n is self-adjoint, unitary or normal, then A is self-adjoint, unitary or normal, respectively.

Proof:

(a) Since $(A+B)^* = A^* + B^* = A + B$, we see that $A+B$ is self-adjoint. Also, since $(AB)^* = B^*A^* = BA$, we see that AB is self-adjoint if and only if $BA = AB$.

(b) Since $(AB)^*AB = B^*A^*AB = B^*B = I$ and $AB(AB)^* = ABB^*A^* = I$, we see that AB is unitary. Also, for all $x \in H$, we have

$$\begin{aligned}
\langle (A+B)(x), (A+B)(x)\rangle &= \langle A(x), A(x)\rangle + \langle B(x), B(x)\rangle \\
&\qquad + \langle A(x), B(x)\rangle + \langle B(x), A(x)\rangle \\
&= 2\langle x, x\rangle + 2\mathrm{Re}\,\langle A(x), B(x)\rangle.
\end{aligned}$$

Hence 26.2(b) implies that $A+B$ is unitary if and only if it is surjective and $\langle x, x\rangle + 2\mathrm{Re}\,\langle A(x), B(x)\rangle = 0$ for all $x \in H$.

(c) Let $AB^* = B^*A$ and $A^*B = BA^*$. Then

$$\begin{aligned}
(A+B)^*(A+B) &= A^*A + B^*B + A^*B + B^*A \\
&= AA^* + BB^* + BA^* + AB^* \\
&= (A+B)(A+B)^*,
\end{aligned}$$

$$
\begin{aligned}
(AB)^*AB &= B^*A^*AB = B^*AA^*B \\
&= AB^*BA^* = ABB^*A^* \\
&= AB(AB)^*.
\end{aligned}
$$

Hence $A+B$ and AB are normal.

(d) By 25.3(b), $\|A_n^* - A^*\| = \|A_n - A\| \to 0$. If each A_n is self-adjoint, that is, $A_n^* = A_n$, then (A_n) converges to both A and A^* in $BL(H)$, so that $A^* = A$, that is, A is self-adjoint. Similar reasoning holds if each A_n is unitary or if each A_n is normal. Note that
$\|A_n^*A_n - A^*A\| \to 0$ and $\|A_nA_n^* - AA^*\| \to 0$. $\square$

Part (b) of Theorem 26.3 says that if A and B are unitary, then $A+B$ is unitary if and only if it is surjective and the angle θ_x between $A(x)$ and $B(x)$ is $2\pi/3$ for every nonzero $x \in H$. This follows by noting that for every nonzero $x \in H$, the angle θ_x is given by

$$\arccos \frac{\operatorname{Re}\langle A(x), B(x)\rangle}{\|A(x)\|\,\|B(x)\|} = \arccos \frac{\operatorname{Re}\langle A(x), B(x)\rangle}{\langle x, x\rangle}$$

and by observing that $\arccos(-1/2) = 2\pi/3$. (See Problems 21-3 and 26-3.) It can be proved that if A and B are unitary operators and $\mathbf{K} = \mathbf{C}$, then $A+B$ unitary if and only if

$$B = [e^{2\pi i/3}P + e^{-2\pi i/3}(I-P)]A$$

for some orthogonal projection P.

In view of Theorem 26.3, let us ask the following question. If A and B are normal operators and A commutes with B, then must $A+B$ and AB be normal? By 26.3(c), we are led to find out whether A commutes with B^*, and B commutes with A^*. This is indeed the case. In fact, Fuglede (1950) proved that if $\mathbf{K} = \mathbf{C}$, A is a normal operator on a complex Hilbert space H and A commutes with a bounded operator B on H, then A commutes with B^* as well. We refer the reader to an elegant proof of this result given by Rosenblum

([49], 1958). If $\mathbf{K} = \mathbf{R}$, we can use Problem 25-15. [See Problem 27-20 for the case when H is finite dimensional.]

As the reader may have guessed by now, there seems to be an analogy between the complex numbers and the bounded operators on a complex Hilbert space H, with the adjoint operation playing the role of complex conjugation. Self-adjoint operators correspond to real numbers and unitary operators correspond to complex numbers of absolute value 1. Since a normal operator commutes (at least) with its adjoint, it seems appropriate to let normal operators correspond to complex numbers. This is made precise in the following result.

26.4 Theorem

Let $\mathbf{K} = \mathbf{C}$ and $A \in BL(H)$. Then there are unique self-adjoint operators B and C on H such that

$$A = B + iC$$

Further, A is normal if and only if $BC = CB$, A is unitary if and only if $BC = CB$ and $B^2 + C^2 = I$, and A is self-adjoint if and only if $C = 0$.

Proof:

Let

$$B = \frac{A + A^*}{2} \quad \text{and} \quad C = \frac{A - A^*}{2i}.$$

Then B and C are self-adjoint operators and $A = B + iC$. If we also have $A = B_1 + iC_1$, where B_1 and C_1 are self-adjoint, then $A^* = B_1 - iC_1$, so that

$$B_1 = \frac{A + A^*}{2} = B \quad \text{and} \quad C_1 = \frac{A - A^*}{2i} = C.$$

Thus B and C are unique.

Now A is normal if and only if $(B - iC)(B + iC) = A^*A = AA^* = (B + iC)(B - iC)$, that is, $BC = CB$. Similarly, A is unitary if and only if $(B - iC)(B + iC) = I = (B + iC)(B - iC)$, that is, $(B^2+C^2)+i(BC-CB)=I=(B^2+C^2)-i(BC-CB)$. It can be easily seen that this is equivalent to $B^2+C^2=I$ and $(BC-CB)=0$. Finally, A is self-adjoint if and only if $B-iC = B + iC$, that is, $C = 0$.

□

The analogy between normal operators and complex numbers is well brought out by Example 26.1(a): If $\mathbf{K} = \mathbf{C}$, $H = L^2(E)$ and z is an essentially bounded complex-valued measurable function on E, then the operator A given by

$$A(x) = zx, \quad x \in H,$$

is always normal. Also, A is unitary if $|z| = 1$ almost everywhere on E and A is self-adjoint if z is real almost everywhere on E.

Positive Operators

There is an important subclass of self-adjoint operators to which we now turn. A self-adjoint operator A on a Hilbert space H over $\mathbf{K}$ is said to be **positive** if $\langle A(x), x\rangle \geq 0$ for all $x \in H$ and then we write $A \geq 0$. If A and B are self-adjoint operators and $A - B \geq 0$, then we write $A \geq B$ or $B \leq A$.

The relation $\geq$ on the set of all self-adjoint operators on H is a partial order. Clearly, it is reflexive since $A \geq A$, and transitive since $A \geq B, B \geq C$ imply that $A \geq C$. Also, it is antisymmetric, since $A \geq B$ and $B \geq A$ imply that $A = B$ by 26.2(a).

Let A and B be positive operators on H. It is clear that $A + B$ is a positive operator. In general, the composition operator AB may not be a positive operator. For example, let $H = \mathbf{K}^2$ and

$$A(x(1), x(2)) = (x(1) + x(2), x(1) + 2x(2)),$$
$$B(x(1), x(2)) = (x(1) + x(2), x(1) + x(2)).$$

Then

$$AB(x(1), x(2)) = (2x(1) + 2x(2), 3x(1) + 3x(2))$$

for all $(x(1), x(2)) \in \mathbf{K}^2$. Note that A and B are positive operators. But AB is not a positive operator, since it is not self-adjoint and if $x = (-4, 3)$, then $\langle AB(x), x\rangle = -1$. By showing that a positive operator has a unique positive square root, one can conclude that if A and B are positive operators and $AB = BA$, then AB is a positive operator. See, for example, [48], p. 265.

Each orthogonal projection is a positive operator. To see this, let Y be a closed subspace of H and let P denote the orthogonal projection onto Y. For $j = 1, 2$, consider $x_j \in H, x_j = y_j + z_j$ with $y_j \in Y$ and $z_j \in Y^\perp$, so that $P(x_j) = y_j$. Then

$$\langle P(x_1),\, x_2\rangle = \langle y_1,\, y_2 + z_2\rangle = \langle y_1,\, y_2\rangle = \langle y_1 + z_1,\, y_2\rangle = \langle x_1,\, P(x_2)\rangle,$$

so that P is self-adjoint. Since $\langle P(x_1), x_1\rangle = \langle y_1, y_1\rangle \geq 0$ for all $x_1 \in H$, P is a positive operator.

To get some idea of what a positive operator is like, let us recall the examples given in 26.1.

Let $H = L^2(E)$ and $z \in L^\infty(E)$ be real-valued. Then the operator A on H defined by $A(x) = zx, x \in H$, is self-adjoint (26.1(a)). Now for $x \in H$,

$$\langle A(x),\, x\rangle = \int_E z|x|^2\, dm.$$

If $z(t) \geq 0$ for almost all $t \in E$, then it is clear that $\langle A(x), x\rangle \geq 0$ for all $x \in H$, that is, A is a positive operator. Conversely, let A be a positive operator. For $k, n = 1, 2, \ldots$, let $E_{k,n} = \{t \in E : |t| \leq k, z(t) \leq -1/n\}$ and $x_{k,n}$ denote the characteristic function of the set $E_{k,n}$. Then $x_{k,n} \in L^2(E)$. Since

$$0 \leq \langle A(x_{k,n}),\, x_{k,n}\rangle = \int_{E_{k,n}} z\, dm \leq -\frac{m(E_{k,n})}{n},$$

it follows that $m(E_{k,n}) = 0$ for all k, n and hence $z(t) \geq 0$ for almost all $t \in E$.

Next, let H be a separable Hilbert space and $u_1, u_2, \ldots$ constitute an orthonormal basis for H. If (k_n) is a bounded sequence in $\mathbf{R}$, then

$$A(x) = \sum_{n=1}^{\infty} k_n \langle x,\, u_n\rangle u_n, \quad x \in H,$$

is a self-adjoint operator on H (26.1(b)). Now for $x \in H$,

$$\langle A(x),\, x\rangle = \sum_{n=1}^{\infty} k_n |\langle x,\, u_n\rangle|^2.$$

By considering $x = u_n, n = 1, 2, \ldots$, we see that A is a positive operator if and only if each k_n is nonnegative.

More generally, let $A \in BL(H)$ be defined by the matrix $M = (k_{i,j})$ with respect to the orthonormal basis $u_1, u_2, \ldots$ of H. We have seen in 26.1(b) that A is self-adjoint if and only if $\overline{k}_{j,i} = k_{i,j}, i, j = 1, 2, \ldots$. For $n = 1, 2, \ldots$, let

$$P_n(x) = \sum_{j=1}^{n} \langle x,\, u_j\rangle u_j, \quad x \in H.$$

Then each P_n is an orthogonal projection and $P_n(x) \to x$ as $n \to \infty$ for all $x \in H$ by 22.7(ii). Consider $A_n = P_n A P_n, n = 1, 2, \ldots$. It is easy to see that A is positive if and only if each A_n is positive. Now for $n = 1, 2, \ldots$ and $x \in H$,

$$\begin{aligned} \langle A_n(x),\, x\rangle &= \langle \sum_{i=1}^{n} (\sum_{j=1}^{n} k_{i,j} \langle x,\, u_j\rangle) u_i,\, \sum_{i} \langle x,\, u_i\rangle u_i\rangle \\ &= \sum_{i,j=1}^{n} k_{i,j} \langle x,\, u_j\rangle \overline{\langle x,\, u_i\rangle}. \end{aligned}$$

Hence A_n is positive if and only if it is self-adjoint and the quadratic form

$$q_n(c_1, \ldots, c_n) = \sum_{i,j=1}^{n} k_{i,j} c_j \overline{c_i}, \quad (c_1, \ldots, c_n) \in \mathbf{K}^n,$$

is nonnegative. Let us assume that A_n is self-adjoint, that is, $\overline{k}_{j,i} = k_{i,j}, i, j = 1, \ldots, n$. Then q_n is nonnegative if and only if every principal minor of the $n \times n$ matrix $(k_{i,j}), i, j = 1, \ldots, n$, is nonnegative. (See, for example, an exercise on page 405, Chapter 7 of [30].) It follows that the operator A is positive if and only if $\overline{k}_{j,i} = k_{i,j}$ for $i, j = 1, 2, \ldots$ and every principal minor of the (possibly infinite) matrix $(k_{i,j}), i, j = 1, 2, \ldots$, is nonnegative. See Problem 26-2 for a simple example.

For a positive operator A on a Hilbert space H and $y_0 \in H$, the solution of the operator equation $A(x) = y_0$ can be reduced to the minimization of the quadratic functional

$$q(x) = \langle A(x),\, x\rangle - 2\,\mathrm{Re}\,\langle x,\, y_0\rangle, \quad x \in H.$$

See Problem 26-16. This is the basic principle behind the so-called **variational methods**, also known as **energy methods**, in mathematical physics.

We now prove an important result, which will be useful in 27.5(a).

26.5 Lemma (Generalized Schwarz inequality)
Let $A \in BL(H)$ be self-adjoint. Then A or $-A$ is a positive operator if and only if

$$|\langle A(x),\, y\rangle|^2 \leq \langle A(x),\, x\rangle\langle A(y),\, y\rangle$$

for all $x, y \in H$.

Proof:
Suppose A is a positive operator, that is, $\langle A(x),\, x\rangle \geq 0$ for all $x \in H$. For $x, y \in H$, define

$$\langle x,\, y\rangle_A = \langle A(x),\, y\rangle.$$

Note that $\langle x,\, x\rangle_A \geq 0$ for all $x \in H$ and the function $\langle\ ,\ \rangle_A$ from $H \times H$ to $\mathbf{K}$ is linear in the first variable. Also, it is conjugate-symmetric since A is self-adjoint.

We must prove that for all $x, y \in H$,

$$|\langle x,\, y\rangle_A|^2 \leq \langle x,\, x\rangle_A\langle y,\, y\rangle_A.$$

This follows exactly as in the proof of the Schwarz inequality (21.1(c)), provided $\langle y,\, y\rangle_A \neq 0$. If $\langle y,\, y\rangle_A = 0$, but $\langle x,\, x\rangle_A \neq 0$, then we can interchange x and y and obtain the desired result. Finally, assume that $\langle x, x\rangle_A = 0 = \langle y, y\rangle_A$. Then

$$\langle x+y, x+y\rangle_A + \langle x-y, x-y\rangle_A = 2\langle x, x\rangle_A + 2\langle y, y\rangle_A = 0,$$

so that $\langle x+y, x+y\rangle_A = 0 = \langle x-y, x-y\rangle_A$. Replacing y by iy, we see that $\langle x+iy, x+iy\rangle_A = 0 = \langle x-iy, x-iy\rangle_A$. Hence

$$\begin{aligned} 4\langle x, y\rangle_A &= \langle x+y, x+y\rangle_A - \langle x-y, x-y\rangle_A \\ &\quad +i\langle x+iy, x+iy\rangle_A - i\langle x-iy, x-iy\rangle_A \\ &= 0. \end{aligned}$$

Thus $|\langle x, y\rangle_A|^2 \le \langle x, x\rangle_A\langle y, y\rangle_A$ for all $x, y \in H$, provided A is positive. In case $-A$ is positive, then

$$\begin{aligned} |\langle A(x), y\rangle|^2 &= |\langle -A(x), y\rangle|^2 \\ &\le \langle -A(x), x\rangle\langle -A(y), y\rangle = \langle A(x), x\rangle\langle A(y), y\rangle \end{aligned}$$

for all $x, y \in H$ by what we have just proved.

Conversely, assume that $|\langle A(x), y\rangle|^2 \le \langle A(x), x\rangle\langle A(y), y\rangle$ for all $x, y \in H$. Then $\langle A(x), x\rangle \ge 0$ for all $x \in H$, or $\langle A(x), x\rangle \le 0$ for all $x \in H$, that is, A or $-A$ is a positive operator. □

A self-adjoint operator A on a Hilbert space H is said to be **positive-definite** if $\langle A(x), x\rangle > 0$ for every nonzero $x \in H$. Note that this condition is, in general, weaker than the condition '$\langle A(x), x\rangle \ge \delta\|x\|^2$ for every $x \in X$ and some $\delta > 0$', considered in Problem 26-17.

If A is a positive-definite operator on H, then equality holds in the generalized Schwarz inequality given in 26.5 if and only if the set $\{x, y\}$ is linearly dependent. This follows by noting that

$$\langle x, y\rangle_A = \langle A(x), y\rangle, \quad x, y \in H,$$

defines an inner product on H, and hence the result 21.1(c) is applicable for this inner product.

Fourier-Plancherel Transform

In this subsection we consider a very important example of unitary operators and interpret it in terms of signal analysis. As we have mentioned earlier, a unitary operator keeps the geometry of a Hilbert space in tact, but it allows us to view the Hilbert space from a different angle. The following consideration, which extends the concept of a Fourier integral of integrable functions on $\mathbf{R}$ to square-integrable functions on $\mathbf{R}$, is a case in point.

In Section 4, we have defined the Fourier integral of $x \in L^1(\mathbf{R})$ at $u \in \mathbf{R}$ by

$$\hat{x}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(s)e^{-ius}\, dm(s), \quad u \in \mathbf{R}.$$

Let, now, $x \in L^2(\mathbf{R})$. As we have remarked in Section 5, x may not belong to $L^1(\mathbf{R})$, and the integrals $\int_{-\infty}^{\infty} x(s)e^{-ius}\, dm(s), u \in \mathbf{R}$, may not exist. We shall, therefore, modify the definition of a Fourier integral in such a way that the two definitions agree if x belongs to $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$.

An ingenious approach in this situation is as follows. Instead of employing the kernel e^{-ius} in the definition of $\hat{x}(u)$, we consider the kernel

$$k(u,s) = \int_0^u e^{-ist}\, dt = \frac{1 - e^{-ius}}{is}, \quad u \in \mathbf{R}, 0 \neq s \in \mathbf{R},$$

and then differentiate the resulting integral with respect to u. Let us work out the details.

26.6 Theorem (Plancherel, 1933)
Let $\mathbf{K} = \mathbf{C}$ and $H = L^2(\mathbf{R})$. For $x \in H$, consider

$$U(x)(u) = \frac{1}{\sqrt{2\pi}} \frac{d}{du} \int_{-\infty}^{\infty} x(s)k(u,s)\, dm(s), \quad u \in \mathbf{R}.$$

Then $U(x)(u)$ is well-defined for almost all $u \in \mathbf{R}$, and $U(x)$ belongs to H. The map $U : H \to H$ is a unitary operator and its inverse

$V : H \to H$ is given by

$$V(y)(u) = \frac{1}{\sqrt{2\pi}} \frac{d}{du} \int_{-\infty}^{\infty} y(s)k(u, -s)\, dm(s), \quad y \in H.$$

Further, if $x \in L^1(R) \cap L^2(R)$, then $U(x)(u) = \hat{x}(u)$ for almost all $u \in \mathbf{R}$.

Proof:

Fix $u \in \mathbf{R}$. Noting that $|1 - e^{-ius}| \le \min\{2, |us|\}$ for all $u, s \in \mathbf{R}$, we obtain

$$|k(u,s)| \le \begin{cases} |u|, & \text{if } 0 < |s| \le 1 \\ 2/|s|, & \text{if } |s| \ge 1. \end{cases}$$

Hence

$$\int_{-\infty}^{\infty} |k(u,s)|^2\, ds \le \int_{|s|\le 1} |u|^2\, ds + 2\int_{|s|\ge 1} \frac{ds}{|s|} \le 2|u|^2 + 4,$$

so that $k(u, .) \in L^2(\mathbf{R})$ for every fixed $u \in \mathbf{R}$.

Now fix $a, b \in \mathbf{R}$. By the Schwarz inequality (21.1(c)), we see that $k(a, .)k(b, .) \in L^1(\mathbf{R})$ and

$$\int_{-\infty}^{\infty} k(a,s)\overline{k(b,s)}\, ds = \int_{-\infty}^{\infty} \frac{(1 - e^{-ias})(1 - e^{ibs})}{s^2}\, ds.$$

Since

$$\operatorname{Im} \frac{(1 - e^{-ias})(1 - e^{ibs})}{s^2} = \frac{1}{s^2}[\sin as - \sin bs - \sin(a-b)s]$$

is an odd integrable function on $\mathbf{R}$, its integral over $\mathbf{R}$ is zero. On the other hand,

$$\begin{aligned} \operatorname{Re} \frac{(1 - e^{-ias})(1 - e^{ibs})}{s^2} &= \frac{1}{s^2}[1 + \cos(a-b)s - \cos as - \cos bs] \\ &= \frac{2}{s^2}\left[\sin^2 \frac{as}{2} + \sin^2 \frac{bs}{2} - \sin^2 \frac{(a-b)s}{2}\right]. \end{aligned}$$

Now $\int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi$, as can be seen by integrating the function $(1 - e^{2iz})/z^2$ around a semicircular contour in the upper half-plane dented at 0. Hence

$$\int_{-\infty}^{\infty} k(a,s)\overline{k(b,s)}\, ds = \pi(|a| + |b| - |a-b|).$$

Let $a \in \mathbf{R}$. If $a \geq 0$, let z_a denote the characteristic function of $[0, a]$, and if $a < 0$, let z_a denote the negative of the characteristic function of $[a, 0]$. Then it can be verified that for all $a, b \in \mathbf{R}$,

$$\int_{-\infty}^{\infty} z_a(s) z_b(s)\, ds = \frac{|a| + |b| - |a - b|}{2}.$$

Now we shall define two operators U and V, and show that they have the representations stated in the theorem. Let

$$U(z_a) = \frac{k(a, \cdot)}{\sqrt{2\pi}} \quad \text{and} \quad V(z_a) = \frac{\overline{k(a, \cdot)}}{\sqrt{2\pi}}.$$

Then $U(z_a)$ and $V(z_a)$ belong to $H = L^2(\mathbf{R})$ and

$$\langle U(z_a),\, U(z_b)\rangle = \langle z_a,\, z_b\rangle = \langle V(z_a),\, V(z_b)\rangle$$

for all $a, b \in \mathbf{R}$. Also,

$$\langle U(z_a),\, z_b\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(a, s)\overline{z_b(s)}\, ds = \frac{1}{\sqrt{2\pi}} \int_0^b k(a, s)\, ds.$$

If a and b are nonzero, then the change of variables $t = as/b$ shows that $\int_0^b k(a, s)\, ds = \int_0^a k(b, t)\, dt$, since $k(a, bt/a) = ak(b, t)/b$ for all $t \neq 0$. It follows that

$$\langle U(z_a),\, z_b\rangle = \frac{1}{\sqrt{2\pi}} \int_0^a k(b, t)\, dt = \langle z_a,\, V(z_b)\rangle.$$

Let G denote the subspace of H consisting of all step functions which vanish outside a finite interval and $E = \{z_a : a \in \mathbf{R}\}$. Since E is a linearly independent set and its span is G, we can linearly extend U and V to G. As the inner product is linear in the first variable and conjugate-linear in the second, we obtain

$$\langle U(x),\, U(y)\rangle = \langle x,\, y\rangle = \langle V(x),\, V(y)\rangle,$$

$$\langle U(x),\, y\rangle = \langle x,\, V(y)\rangle$$

for all $x, y \in G$. Now by 4.7(c), the subspace G is dense in H. For $x \in H$, let (x_n) be a sequence in G such that $x_n \to x$ in H. Since

$$\|U(x_n) - U(x_m)\|_2 = \|U(x_n - x_m)\|_2 = \|x_n - x_m\|_2$$

for all $n, m = 1, 2, \dots$, and since H is complete, let $U(x_n) \to y$ in H. Define $U(x) = y$. It is easy to see that this definition does not depend on the choice of the sequence (x_n) in G converging to x. Similarly, we extend the operator V to H. The continuity of the inner product in both the variables (21.2(a)) implies that the relations $(*)$ hold for all $x, y \in H$. Hence U and V are unitary operators and $U^{-1} = U^* = V$.

To obtain the representations of U and V stated in the theorem, consider $x, y \in H$. Then for each fixed $u \in \mathbf{R}$,

$$\langle U(x),\, z_u\rangle = \langle x,\, V(z_u)\rangle,$$

that is,

$$\int_0^u U(x)(s)\, dm(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(s)k(u, s)\, dm(s).$$

The fundamental theorem for Lebesgue integration (4.3) shows that

$$U(x)(u) = \frac{1}{\sqrt{2\pi}} \frac{d}{du} \int_{-\infty}^{\infty} x(s)k(u, s)\, dm(s)$$

for almost all $u \in \mathbf{R}$. Similarly, for each fixed $u \in \mathbf{R}$, $\langle V(y),\, z_u\rangle = \langle y,\, U(x_u)\rangle$, that is,

$$\int_0^u V(y)(s)\, dm(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(s)\overline{k(u, s)}\, dm(s),$$

so that

$$V(y)(u) = \frac{1}{\sqrt{2\pi}} \frac{d}{du} \int_{-\infty}^{\infty} y(s)k(u, -s)\, dm(s)$$

for almost all $u \in \mathbf{R}$.

Finally, let $x \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Since

$$\int_{-\infty}^{\infty} |x(s)e^{-ius}|\, dm \le \int_{-\infty}^{\infty} |x(s)|\, dm(s) < \infty,$$

the integral $\int_{-\infty}^{\infty} x(s)e^{-ius}\, dm(s)$ converges uniformly (and absolutely) for $u \in \mathbf{R}$. Hence differentiating under the integral sign we obtain

$$U(x)(u) \;=\; \frac{1}{\sqrt{2\pi}} \frac{d}{du} \int_{-\infty}^{\infty} x(s)k(u, s)\, dm(s)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(s) \frac{\partial k(u, s)}{\partial u} \, dm(s) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(s) e^{-ius} \, dm(s) \\
&= \hat{x}(u)
\end{aligned}
$$

for almost all $u \in \mathbf{R}$. □

For $x \in L^2(\mathbf{R}), U(x)$ is known as the **Fourier-Plancherel transform** and $V(x)$ is known as the **inverse Fourier-Plancherel transform** of x

Plancherel's theorem has a significance for **signal analysis**. A signal is represented in the time domain by a square-integrable function x on $\mathbf{R}$, the signal at time t being determined by the value $x(t)$. The **total energy** of the signal is given by

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dm(t).$$

The Fourier-Plancherel transform $U(x)$ of x gives a representation of the signal in the frequency domain, the signal at frequency u being determined by the value $U(x)(u)$. For signals x and y, the relation $\langle x, y \rangle = \langle U(x), U(y) \rangle$, that is,

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)} \, dm(t) = \int_{-\infty}^{\infty} U(x)(u)\overline{U(y)(u)} \, dm(u),$$

is known as the **time-frequency equivalence**. If the signals x and y are also integrable on $\mathbf{R}$, then this identity can be written as

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)} \, dm(t) = \int_{-\infty}^{\infty} \hat{x}(u)\overline{\hat{y}(u)} \, dm(u).$$

It can be compared with the Parseval identity given in Problem 22-11(b). If $y = x$, we obtain the so-called **energy principle**

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dm(t) = \int_{-\infty}^{\infty} |U(x)(u)|^2 \, dm(u).$$

A signal x is said to be **causal** if it originates at time $t = 0$, that is, $x(t) = 0$ for all $t < 0$. Thus $L^2([0,\infty))$ is the space of all causal signals. These signals can be represented in the frequency domain by the cosine and the sine transforms given in Problem 26-19. A signal x is said to be **time-limited** if it dies at some time T, that is, $x(t) = 0$ for all $t > T$. Thus $L^2([0,1])$ is the space of all causal time-limited signals which die at time 1. Problem 24-19 shows that if (x_n) is a sequence of such signals, then $x_n \xrightarrow{w} 0$ if and if only (x_n) is a bounded sequence and the energy of x_n in the frequency band $[-\beta, \beta]$ tends to 0 for each $\beta > 0$.

Problems

X (resp., H) denotes an inner product space (resp., a Hilbert space), unless otherwise stated.

26-1 Let $u_1, u_2, \ldots$ constitute an orthonormal basis for H. Suppose that $A \in BL(H)$ is defined by a matrix M with respect to $u_1, u_2, \ldots$.

(a) Assume that M is triangular. Then A is normal if and only if M is diagonal.

(b) Assume that $M = ND\overline{N}^t$, where the matrices N and D define bounded operators on H, the columns of N form an orthonormal set in ℓ^2 and D is diagonal. Then A is normal. [The converse holds if $\mathbf{K} = \mathbf{C}$ and H is finite dimensional. Compare 27.9(b).]

26-2 Let $H = \mathbf{K}^2$ and $A(x(1), x(2)) = (ax(1) + b(x(2), cx(1) + dx(2))$ for $(x(1), x(2)) \in H$, and fixed a, b, c, d in $\mathbf{K}$. If $\mathbf{K} = \mathbf{C}$ (resp., $\mathbf{R}$), then A is normal if and only if $|b|^2 = |c|^2$ and $(a-d)\bar{c} = (\bar{a} - \bar{d})b$ (resp., either $b = c$, or $b = -c$ and $a = d$), A is unitary if and only if $|a|^2 + |b|^2 = 1 = |c|^2 + |d|^2$ and $a\bar{c} + b\bar{d} = 0$ (resp., either $a = \cos\theta = d$ and $b = -\sin\theta = -c$, or $a = \cos\theta = -d$ and $b = \sin\theta = c$ for some $0 \le \theta < 2\pi$), A is self-adjoint if and only if $a, d \in \mathbf{R}$ and $c = \bar{b}$, A is positive if and only if $a \ge 0, d \ge 0, c = \bar{b}$ and $ad \ge |b|^2$. (Hint: 26.5)

26-3 Let M_1 and M_2 be 2×2 matrices with real entries. Then M_1, M_2 and $M_1 + M_2$ define unitary operators on $\mathbf{R}^2$ with respect to the standard basis if and only if

$$M_j = \begin{bmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\alpha_j & \sin\alpha_j \\ \sin\alpha_j & -\cos\alpha_j \end{bmatrix},$$

for $j = 1, 2$, where $0 \leq \alpha_1 < 2\pi$ and $\alpha_2 = \alpha_1 \pm 2\pi/3$.

26-4 Let A be a **rigid motion** on X, that is, A is a (not necessarily linear) map from X to X such that $\|A(x) - A(y)\| = \|x - y\|$ for all $x, y \in X$.

(a) Let $\mathbf{K} = \mathbf{R}$ and X be finite dimensional. Then there is some $x_0 \in X$ such that $A(x) = x_0 + A_0(x)$ for all $x \in X$, where A_0 is a unitary operator on X. (Hint: If $A(0) = 0$ and $u_1, \ldots, u_n$ is an orthonormal basis for X, then $A(u_1), \ldots, A(u_n)$ is also an orthonormal basis for X.) The result does not hold if $\mathbf{K} = \mathbf{C}$.

(b) Let $\mathbf{K} = \mathbf{R}$, $X = \mathbf{R}^2$ and $A(0) = 0$. Then either $A = A_1$ or $A = A_1A_2$, where A_1 is a rotation and A_2 is a reflection on $\mathbf{R}^2$. (Hint: Problem 26-2)

26-5 Let A denote a Fredholm integral operator on $H = L^2([a, b])$ with kernel $k(.,.) \in L^2([a, b]) \times [a, b])$. Then A is self-adjoint if and only if $\overline{k(t, s)} = k(s, t)$ for almost all (s, t) in $[a, b] \times [a, b]$. Further, A is normal if and only if $\int_a^b \overline{k(u, s)} k(u, t)\, dm(u) = \int_a^b k(s, u) \overline{k(t, u)}\, dm(u)$ for almost all (s, t) in $[a, b] \times [a, b]$. (Hint: Problem 25-5)

26-6 $A \in BL(H)$ is called **hyponormal** if $A^*A - AA^*$ is a positive operator on H. [A right shift operator on a separable Hilbert space is hyponormal, but not normal.] A is hyponormal if and only if $\|A(x)\| \geq \|A^*(x)\|$ for all $x \in H$. A is normal if and only if A and A^* are hyponormal.

26-7 Let H denote the Hilbert space of all doubly infinite square-summable scalar sequences $x = (x(j)), j = \ldots, -2, -1, 0, 1, 2, \ldots$. For x in H, let $A(x)(j) = x(j - 1)$ for all j. Then A is a unitary operator on H. (Compare this with the right shift operator discussed in 25.4(a) and 26.1(b).)

26-8 Let $A, B \in BL(X)$ with A self-adjoint. Then $AB = 0$ if and only if $R(A) \perp R(B)$.

26-9 Let $A : X \to X$ be linear. If $\langle A^2(x), x\rangle \geq 0$ and $\langle A(x), x\rangle = 0$ for all $x \in X$, then $A = 0$. (Compare 26.2(a).)

26-10 Let $\mathbf{K} = \mathbf{C}$ and $A : X \to X$ be linear. If $\langle A(x), x\rangle = 0$ for all $x \in X$, then $A = 0$. If $\langle A(x), x\rangle \in \mathbf{R}$ for all $x \in X$, then $\langle A(x), y\rangle = \langle x, A(y)\rangle$ for all $x, y \in X$. These results do not hold if $\mathbf{K} = \mathbf{R}$. (Compare Problem 26-9.)

26-11 Let $A \in BL(H)$. If A is unitary, then for every orthonormal basis $\{u_\alpha\}$ of $H, \{A(u_\alpha)\}$ and $\{A^*(u_\alpha)\}$ are both orthonormal bases for H. Conversely, if for some orthonormal basis $\{u_\alpha\}$ for H, $\{A(u_\alpha)\}$ and $\{A^*(u_\alpha)\}$ are orthonormal sets, then A is unitary.

26-12 A self-adjoint operator on H is also known as an **hermitian** operator. Also, $A \in BL(H)$ is called **skew-hermitian** if $A^* = -A$. For every A in $BL(H)$, there are unique B and C in $BL(H)$ such that B is hermitian and C is skew-hermitian. Further, A is normal if and only if $BC = CB$, A is unitary if and only if $BC = CB$ and $B^2 - C^2 = I$, A is hermitian if and only if $C = 0$ and A is skew-hermitian if and only if $B = 0$. (Compare 26.4.)

26-13 The set of all self-adjoint operators on H is a real-linear subspace of $BL(H)$ and the set of all positive operators on H is a **cone** in this subspace (that is, if $A \geq 0, B \geq 0, t \geq 0$, then $A+B \geq 0, tA \geq 0$ and if $A \geq 0, -A \geq 0$, then $A = 0$). However, $0 \leq A \leq B$ may not imply that $A^2 \leq B^2$. Further, if $A_n, A \in BL(X), A_n(x) \xrightarrow{w} A(x)$ for each $x \in H$ and if every A_n is self-adjoint (resp., positive), then A is self-adjoint (resp., positive).

26-14 Let $A \in BL(H)$ be self-adjoint. Then $A^2 \geq 0$ and $A \leq \|A\|I$. If $A^2 \leq A$, then $0 \leq A \leq I$.

26-15 Let (A_n) be a sequence of self-adjoint operators in $BL(H)$

(a) If $A_n \leq A_{n+1} \leq I$ for all n, then there is a self-adjoint operator A on H such that $A \leq I$ and $A_n(x) \to A(x)$ for every $x \in H$.

(b) If $0 \leq A_{n+1} \leq A_n$ for all n, then there is a positive operator A on H such that $A_n(x) \to A(x)$ for every $x \in H$.

26-16 Let A be a self-adjoint operator on H. Fix $y_0 \in H$ and consider the quadratic functional

$$q(x) = \langle A(x), x\rangle - 2\,\mathrm{Re}\,\langle x, y_0\rangle, \quad x \in H.$$

Then A is a positive operator and $A(x_0) = y_0$ if and only if q attains its minimum at x_0. (Hint: $q(x+y) = q(x) + \langle A(y), y\rangle + 2\,\mathrm{Re}\,\langle A(x) - y_0, y\rangle$ for all $x, y \in H$)

26-17 **(Bubnov-Galerkin method)** Let $A \in BL(H)$ be self-adjoint and satisfy $\langle A(x), x\rangle \geq \delta\|x\|^2$ for all $x \in H$ and some $\delta > 0$. Then A is bijective. Consider $y_0 \in H$ and let $x_0 \in H$ be such that $A(x_0) = y_0$.

Let $E = \{x_1, x_2, \ldots\}$ be a subset of H and for $m = 1, 2, \ldots$, let $F_m = \mathrm{span}\,\{x_1, \ldots, x_m\}$. Then there is a unique $z_m \in F_m$ such that $A(z_m) - y_0$ is orthogonal to F_m. We have the *a posteriori* estimate

$$\delta\|z_m - x_0\| \leq \|A(z_m) - y_0\|.$$

If E is dense in H, then $z_m \to x_0$ as $m \to \infty$. Each z_m can be considered as an approximate solution. (Hint: 25.5(c). Consider $\langle x, y\rangle_A = \langle A(x), y\rangle$ for $x, y \in H$ and use 23.2(a), 23.3.)

26-18 Let H be a Hilbert space over $\mathbf{R}$ and $H_{\mathbf{C}}$ denote its complexification given in Problem 25-15. Then $A \in BL(H)$ is normal, unitary, self-adjoint or positive if and only if $A_{\mathbf{C}} \in BL(H_{\mathbf{C}})$ is respectively so.

26-19 Let $\mathbf{K} = \mathbf{R}$ and $H = L^2([0, \infty))$. For $x \in H$ and $u \in [0, \infty)$, consider

$$U_1(x)(u) = \sqrt{\frac{2}{\pi}}\frac{d}{du}\int_0^\infty \frac{\sin us}{s} x(s)\,dm(s),$$

$$U_2(x)(u) = \sqrt{\frac{2}{\pi}}\frac{d}{du}\int_0^\infty \frac{1-\cos us}{s} x(s)\,dm(s).$$

Then $U_1(x)(u)$ and $U_2(x)(u)$ are well-defined for almost all $u \in [0, \infty)$, and $U_1(x), U_2(x) \in H$. The maps U_1 and U_2 are bounded operators on H, which are self-adjoint as well as unitary, so that $U_1^{-1} = U_1$ and $U_2^{-1} = U_2$. Further, if $x \in H$ and $x(t) = 0$ for all $t > T$, then

$$U_1(x)(u) = \sqrt{\frac{2}{\pi}}\int_0^T x(s)\cos us\,ds \quad \text{and} \quad U_2(x)(s) = \sqrt{\frac{2}{\pi}}\int_0^T x(s)\sin us\,ds$$

for almost all $u \in [0, \infty)$.

26-20 Let $x \in L^2(\mathbf{R})$. For $n = 1, 2, \ldots$ and $u \in \mathbf{R}$, let

$$y_n(u) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} x(s)e^{-ius}\,dm(s) \quad \text{and} \quad z_n(u) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} x(s)e^{ius}\,dm(s).$$

Then $y_n \to U(x)$ and $z_n \to V(x)$ in $L^2(\mathbf{R})$, where $U(x)$ is the Fourier-Plancherel transform of x and $V(x)$ is the inverse Fourier-Plancherel transform of x.

27 Spectrum and Numerical Range

In Section 12 we had introduced the **spectrum** of a bounded operator on a normed space. We give here a brief but independent treatment of the spectrum of a bounded operator A on a Hilbert space H by utilizing its inner product structure and also the adjoint considerations of Sections 25 and 26. [An interested reader can use Section 12 as a supplement.] We give a description of the spectrum of a normal operator, especially when the Hilbert space is finite dimensional. We also study a subset of scalars which is closely related to the spectrum of A and is known as the **numerical range** of A.

A scalar k is called an **eigenvalue** of A if there is a nonzero $x \in H$ such that $A(x) = kx$. [We emphasize the requirement that x be nonzero. Note that $A(0) = k0$ for every scalar k.] In that case, such an element x is called an **eigenvector** of A corresponding to k. The set of all eigenvalues of A constitutes the **eigenspectrum** of A. This nomenclature arises from the following considerations. Physical quantities like position, momentum and energy can be represented by operators defined on subspaces of a Hilbert space. If such a physical quantity is measured in an experiment, then the result of the measurement is one of the eigenvalues of the operator representing that quantity. In an atomic quantum mechanical system, if an operator A represents the energy of an atom, then the differences of various eigenvalues of A are, in fact, the amounts of energy emitted by the atom as it undergoes a transition from one state to another. These amounts can be seen in the form of electromagnetic waves which constitute the optical spectrum of that atom.

We shall denote the eigenspectrum of A by $\sigma_e(A)$. If x is an eigenvector of A corresponding to an eigenvalue k of A, then the same holds for the unit vector $x/\|x\|$. Hence

$$\sigma_e(A) = \{k \in \mathbf{K} : A(x) = kx \text{ for some } x \in H \text{ with } \|x\| = 1\}.$$

Note that a scalar k is an eigenvalue of A if and only if the operator $A - kI$ is not injective. In that case, the closed subspace $Z(A - kI)$ of X is nonzero. It is called the **eigenspace** of A corresponding to the eigenvalue k.

A scalar k is called an **approximate eigenvalue** of A if the operator $A - kI$ is not bounded below, that is, if for every $\beta > 0$, there is some $x \in H$ with $\|x\| = 1$ and $\|A(x) - kx\| < \beta$. The set of all approximate eigenvalues of A constitutes the **approximate eigenspectrum** of A. We shall denote it by $\sigma_a(A)$. It is then clear that

$$\sigma_a(A) = \{k \in \mathbf{K} : A(x_n) - kx_n \to 0 \text{ for some } (x_n) \text{ in } H$$
$$\text{with } \|x_n\| = 1 \text{ for each } n\}.$$

A scalar k is called a **spectral value** of A if the bounded operator $A - kI$ is not invertible (in $BL(H)$). The set of all spectral values of A constitutes the **spectrum** of A. We shall denote it by $\sigma(A)$. It follows from Problem 25-19 that

$$\sigma(A) = \{k \in \mathbf{K} : A - kI \text{ is either not injective or not surjective}\}.$$

We shall not, however, use this result in the sequel. Instead, we base our study on the following theorem.

27.1 Theorem

Let H be a Hilbert space and $A \in BL(H)$. Then

(a) $k \in \sigma(A)$ if and only if $\overline{k} \in \sigma(A^*)$

(b) $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \bigcup \{k : \overline{k} \in \sigma_e(A^*)\}$.

Proof:

(a) Let $k \in \mathbf{K}$. By 25.3(a), $A - kI$ is invertible if and only if $(A - kI)^* = A^* - \overline{k}I$ is invertible. Thus $\sigma(A^*) = \{\overline{k} \in \mathbf{K} : k \in \sigma(A)\}$.

(b) If k is an eigenvalue of A and x is a corresponding eigenvector of norm 1, then letting $x_n = x$ for each n, we see that $A(x_n) - kx_n = A(x) - kx = 0$, so that k is an approximate eigenvalue of A. Thus $\sigma_e(A) \subset \sigma_a(A)$.

Assume now that $k \notin \sigma(A)$, that is, $(A - kI)^{-1} \in BL(H)$. Then

$$\|x\| = \|(A - kI)^{-1}(A - kI)(x)\| \leq \|(A - kI)^{-1}\| \, \|A(x) - kx\|$$

for all $x \in H$. If we let $\beta = 1/\|(A-kI)^{-1}\|$, we see that $\beta \leq \|A(x)-kx\|$ for all $x \in H$ with $\|x\| = 1$. Hence $A - kI$ is bounded below, that is, $k \notin \sigma_a(A)$. Thus $\sigma_a(A) \subset \sigma(A)$.

Further, if $\overline{k} \in \sigma_e(A^*)$, then $\overline{k} \in \sigma(A^*)$, that is, $k \in \sigma(A)$ by (a). Thus $\{k : \overline{k} \in \sigma_e(A)\} \subset \sigma(A)$.

Conversely, assume that $k \notin \sigma_a(A)$ and $\overline{k} \notin \sigma_e(A^*)$. Then the operator $A - kI$ is bounded below and the operator $A^* - \overline{k}I$ is injective. The proof of 25.5(c) (with A^* replaced by $A - kI$)) and the result 25.5(a) show that the range of $A - kI$ is closed as well as dense in H. Hence the operator $A - kI$ is surjective. Since $k \notin \sigma_a(A)$, there is some $\beta > 0$ such that

$$\beta\|x\| \leq \|(A - kI)(x)\|$$

for all $x \in H$. Clearly, the operator $A - kI$ is injective. Now the map $(A - kI)^{-1} : H \to H$ is linear by 2.4(a). If $y \in H$, then we let $x = (A - kI)^{-1}(y)$ and obtain

$$\beta\|(A - kI)^{-1}(y)\| \leq \|(A - kI)(A - kI)^{-1}(y)\| = \|y\|,$$

so that $\|(A - kI)^{-1}\| \leq 1/\beta$. Thus $(A - kI)^{-1} \in BL(H)$, that is, $k \notin \sigma(A)$. □

If $A \in BL(H)$ is normal, then the eigenvalues, the eigenvectors and the spectral values of A have special properties.

27.2 Theorem

Let $A \in BL(H)$ be normal.

(a) If $k \in \sigma_e(A)$, then $\overline{k} \in \sigma_e(A^*)$ and an eigenvector of A corresponding to k is also an eigenvector of A^* corresponding to $\overline{k}$.

(b) If $(A - kI)^2(x) = 0$ and $x \neq 0$, then $k \in \sigma_e(A)$ and x is an eigenvector of A corresponding to k.

(c) If x_1 and x_2 are eigenvectors of A corresponding to distinct eigenvalues, then $x_1 \perp x_2$.

(d) Every spectral value of A is an approximate eigenvalue of A.

Proof:

(a) Let $A(x) = kx$ with $x \neq 0$. Then by 26.2(c),

$$\|A^*(x) - \overline{k}x\| = \|(A - kI)^*(x)\| = \|(A - kI)(x)\| = 0,$$

that is, $A^*(x) = \overline{k}x$.

(b) By the Schwarz inequality (21.1(c)) and by 26.2(c) again,

$$\begin{aligned}
\|(A - kI)(x)\|^2 &= \langle (A - kI)(x),\, (A - kI)(x)\rangle \\
&= \langle (A^* - \overline{k}I)(A - kI)(x),\, x\rangle \\
&\leq \|(A^* - \overline{k}I)(A - kI)(x)\|\, \|x\| \\
&= \|(A - kI)^2(x)\|\, \|x\|.
\end{aligned}$$

Hence if $(A - kI)^2(x) = 0$, then $A(x) = kx$.

(c) Let $A(x_1) = k_1 x_1, A(x_2) = k_2 x_2, x_1 \neq 0$ and $x_2 \neq 0$. Then $A^*(x_2) = \overline{k}_2 x_2$ by (a) and

$$\begin{aligned} k_1 \langle x_1, x_2 \rangle &= \langle k_1 x_1, x_2 \rangle = \langle A(x_1), x_2 \rangle \\ &= \langle x_1, A^*(x_2) \rangle = \langle x_1, \overline{k}_2 x_2 \rangle = k_2 \langle x_1, x_2 \rangle. \end{aligned}$$

Hence if $k_1 \neq k_2$, then $\langle x_1, x_2 \rangle = 0$, that is, $x_1 \perp x_2$.

(d) Let $k \in \sigma(A)$. Then by 27.1(b), $k \in \sigma_a(A)$ or $\overline{k} \in \sigma_e(A^*)$. But if $\overline{k} \in \sigma_e(A^*)$, then $k \in \sigma_e(A)$ by (a). Since $\sigma_e(A) \subset \sigma_a(A)$, we see that $k \in \sigma_a(A)$ in any case. □

27.3 Examples

(a) If $k \in \sigma(A)$, then $\overline{k} \in \sigma(A^*)$, as we have seen in 27.1(a). On the other hand, if $k \in \sigma_e(A)$, it does not follow that $\overline{k} \in \sigma_e(A^*)$. For example, let $H = \ell^2$ and

$$A(x(1), x(2), \ldots) = (x(2), x(3), \ldots)$$

for $x = (x(1), x(2), \ldots) \in H$. Then $A((1, 0, \ldots)) = 0$, so that 0 belongs to $\sigma_e(A)$. However,

$$A^*(x(1), x(2), \ldots) = (0, x(1), x(2), \ldots)$$

for all $x \in H$. Hence it follows that 0 does not belong to $\sigma_e(A^*)$.

If A is normal, then an eigenvector of A corresponding to an eigenvalue k is also an eigenvector of A^* corresponding its eigenvalue $\overline{k}$, as we have seen in 27.2(a). In general, however, this may not hold. For example, let $H = \mathbf{C}^2$ and an operator A on H be defined by the matrix $\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$ with respect to the standard orthonormal basis e_1, e_2 of H. Then A^* is defined by the matrix $\begin{bmatrix} -i & 0 \\ 1 & -i \end{bmatrix}$. It can be easily seen that $A(x(1), x(2)) = i(x(1), x(2))$ if and only if $x(2) = 0$, while $A^*(x(1), x(2)) = -i(x(1), x(2))$ if and only if $x(1) = 0$. Hence an eigenvector of A corresponding to i can never be an eigenvector A^* corresponding to $-i$.

(b) If $(A-kI)^2(x)=0$ for some nonzero x, then k is an eigenvalue of A. Letting $y=(A-kI)(x)$, we see that if $y\neq 0$, then y is an eigenvector of A and if $y=0$, then x is an eigenvector of A corresponding to k. If A is normal, then the second alternative holds, as we have seen in 27.2(b). In general, however, the first alternative can hold. For example, let $H=\mathbf{K}^2$, and an operator A on H be defined by the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2=0$, so that $A^2((0,1))=(0,0)$ but $A((0,1))=(1,0)\neq(0,0)$.

(c) Let $A(x_1)=k_1x_1$ and $A(x_2)=k_2x_2$ with $k_1\neq k_2, x_1\neq 0$ and $x_2\neq 0$. Then the set $\{x_1,x_2\}$ is linearly independent. For if $k_1'x_1+k_2'x_2=0$ for some $k_1',k_2'\in\mathbf{K}$, then

$$k_1k_1'x_1=A(k_1'x_1)=A(-k_2'x_2)=-k_2'k_2x_2=-k_2(-k_1'x_1)=k_2k_1'x_1.$$

Since $k_1\neq k_2$, we see that $k_1'=0$ and, in turn, $k_2'=0$. Hence $\{x_1,x_2\}$ is a linearly independent set. If A is normal, then, in fact, it is an orthogonal set, as we have seen in 27.2(c). This may not hold if A is not normal. For example, let $H=\mathbf{K}^2$ and an operator A on H be defined by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Then $A((1,0))=(1,0)$ and $A((1,1))=2(1,1)$. However, (1,0) is not orthogonal to (1,1).

(d) We have seen in 27.1(b) that $\sigma_e(A)\subset\sigma_a(A)\subset\sigma(A)$. In general, these inclusions are proper. For example, let $H=\ell^2$. If

$$A(x(1),x(2),\ldots)=(x(1),\frac{x(2)}{2},\ldots)$$

for $x=(x(1),x(2),\ldots)\in H$, then it follows that $0\notin\sigma_e(A)$. However, $0\in\sigma_a(A)$ since $\|A(e_n)\|=\|e_n\|/n\to 0$ as $n\to\infty$, where $\|e_n\|=\|(0,\ldots,0,1,0,0,\ldots)\|=1$. (Note that A is a self-adjoint operator.) Similarly, if

$$B(x(1),x(2),\ldots)=(0,x(1),x(2),\ldots)$$

for $x = (x(1), x(2), \ldots) \in H$, then $\|B(x)\| = \|x\|$ for every $x \in H$, so that $0 \notin \sigma_a(B)$. However, $0 \in \sigma(B)$ since B is clearly not surjective.

Of course, if A is a normal operator then $\sigma_a(A) = \sigma(A)$, as we have seen in 27.2(d).

We remark that $\sigma_e(A)$ may be empty even for a self-adjoint operator A. For example, let $H = L^2([a,b])$ and for $x \in H$, let

$$A(x)(t) = tx(t), \quad t \in [a,b].$$

If $k \in \mathbf{K}$ and $A(x) = kx$ for some $x \in H$, then $(t-k)x(t) = 0$ for almost all $t \in [a,b]$, so that $x = 0$. Hence $\sigma_e(A) = \emptyset$.

We now consider a subset of scalars which is closely related to the spectrum of $A \in BL(H)$. The set

$$\omega(A) = \{\langle A(x), x\rangle : x \in H, \|x\| = 1\}$$

is known as the **numerical range** of A. Since $|\langle A(x), x\rangle| \le \|A\|\,\|x\|^2$ for every $x \in H$, we see that $|k| \le \|A\|$ for all $k \in \omega(A)$. In particular, $\omega(A)$ is a bounded subset of $\mathbf{K}$. It, however, may not be a closed subset of $\mathbf{K}$. For example, if $H = \ell^2$ and $A \in BL(H)$ with $A(e_n) = e_n/n, n = 1, 2, \ldots$, then $\langle A(e_n), e_n\rangle = 1/n \in \omega(A)$ for each n, but $\langle A(e_n), e_n\rangle \to 0 \notin \omega(A)$.

A peculiar property of $\omega(A)$ is that it is a convex subset of $\mathbf{K}$. To see this, consider $x_j \in H$ with $\|x_j\| = 1$ for $j = 1, 2$ and $t \in (0,1)$. It is enough to find $x_0 = k_1x_1 + k_2x_2 \in H$ with k_1, k_2 in K $\|x_0\| = 1$ and

$$\langle A(x_0), x_0\rangle = t\langle A(x_1), x_1\rangle + (1-t)\langle A(x_2), x_2\rangle.$$

The problem can thus be reduced to two dimensional Hilbert spaces. We refer the interested reader to [46] for a simple proof.

27.4 Theorem

Let $A \in BL(H)$.

(a) $k \in \omega(A)$ if and only if $\overline{k} \in \omega(A^*)$

(b) $\sigma_e(A) \subset \omega(A)$ and $\sigma(A)$ is contained in the closure of $\omega(A)$.

Proof:

(a) Let $x \in H$ with $\|x\| = 1$. Since

$$\overline{\langle A(x),\, x\rangle} = \langle x,\, A(x)\rangle = \langle A^*(x),\, x\rangle.$$

We see that $\langle A(x),\, x\rangle \in \omega(A)$ if and only if $\overline{\langle A(x),\, x\rangle} \in \omega(A^*)$.

(b) Let $k \in \sigma_e(A)$. Then $A(x) = kx$ for some $x \in X$ with $\|x\| = 1$. Since

$$\langle A(x),\, x\rangle = \langle kx,\, x\rangle = k\|x\|^2 = k,$$

we see that $k \in \omega(A)$.

Next, let $k \in \sigma(A)$. By 27.1(b), $k \in \sigma_a(A)$ or $\overline{k} \in \sigma_e(A^*)$. If k is in $\sigma_a(A)$, then there is a sequence (x_n) in H such that $\|x_n\| = 1$ and $A(x_n) - kx_n \to 0$ as $n \to \infty$. Since

$$\begin{aligned} |\langle A(x_n),\, x_n\rangle - k| &= |\langle (A - kI)(x_n),\, x_n\rangle| \\ &\le \|A(x_n) - kx_n\|\,\|x_n\| \to 0 \end{aligned}$$

as $n \to \infty$, we see that $k = \lim_{n\to\infty}\langle A(x_n),\, x_n\rangle$, so that k belongs to the closure of $\omega(A)$. Also, if $\overline{k} \in \sigma_e(A^*)$, then $\overline{k} \in \omega(A^*)$ as we have just seen, and hence $k \in \omega(A)$ by (a) above. This completes the proof that $\sigma(A)$ is contained in the closure of $\omega(A)$. □

We mention that neither $\sigma(A)$ nor $\omega(A)$ is contained in the other in general. For example, if $H = \ell^2$ and $A \in BL(H)$ with $A(e_n) = e_n/n$, then $0 \in \sigma_a(A) \subset \sigma(A)$, but $0 \notin \omega(A)$ since

$$\langle A(x),\, x\rangle = \langle (x(1), \frac{x(2)}{2}, \ldots), (x(1), x(2), \ldots)\rangle = \sum_{j=1}^{\infty} \frac{|x(j)|^2}{j} \ne 0$$

for every $x \in H$. On the other hand, let $H = \mathbf{K}^2$ and an operator A on H be defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $\sigma(A) = \{1, -1\}$, while $\omega(A) = [-1, 1]$ since

$$\langle A(x),\, x\rangle = \langle (x(1), -x(2)), (x(1), x(2))\rangle = |x(1)|^2 - |x(2)|^2$$

for every $x \in H$.

If $x \in H$ and $\|x\| = 1$, then the scalar $q(x) = \langle A(x), x\rangle$ is known as the **Rayleigh quotient** of A at x. It has a minimum residual property stated in Problem 27-9. Note that the set of all Rayleigh quotients of A constitutes the numerical range of A.

Let us now consider a self-adjoint operator A on H. Since

$$\overline{\langle A(x),\, x\rangle} = \langle x,\, A(x)\rangle = \langle A^*(x),\, x\rangle = \langle A(x),\, x\rangle,$$

we see that $\langle A(x),\, x\rangle \in \mathbf{R}$ for all $x \in H$. We define

$$m_A = \inf_{x\in H, \|x\|=1} \langle A(x),\, x\rangle \quad \text{and} \quad M_A = \sup_{x\in H, \|x\|=1} \langle A(x),\, x\rangle.$$

27.5 Theorem

Let $H \neq \{0\}$ and $A \in BL(H)$ be self-adjoint. Then

(a) $\{m_A, M_A\} \subset \sigma_a(A) = \sigma(A) \subset [m_A, M_A]$.

(b) (Ritz method) Consider $x_1, x_2, \ldots,$ in H. For $n = 1, 2, \ldots,$ let $F_n = \operatorname{span}\{x_1, \ldots, x_n\}$,

$$\alpha_n = \inf_{x\in F_n, \|x\|=1} \langle A(x),\, x\rangle \quad \text{and} \quad \beta_n = \sup_{x\in F_n, \|x\|=1} \langle A(x),\, x\rangle.$$

Then

$$m_A \le \alpha_{n+1} \le \alpha_n \le \cdots \le \alpha_1 \le \beta_1 \le \beta_2 \le \cdots \le \beta_n \le \beta_{n+1} \le M_A.$$

If $\operatorname{span}\{x_1, x_2, \ldots\}$ is dense in H, then

$$m_A = \lim_{n\to\infty} \alpha_n \quad \text{and} \quad \lim_{n\to\infty} \beta_n = M_A$$

Proof:

(a) By the definition of m_A, there is a sequence (x_n) in H such that $\|x_n\| = 1$ and $\langle A(x_n),\, x_n\rangle \to m_A$ as $n \to \infty$. We show that $A(x_n) - m_A x_n \to 0$ as $n \to \infty$. Let $B = A - m_A I$ and observe that B

is a positive operator. By the generalized Schwarz inequality (26.5), we have

$$|\langle B(x),\, y\rangle|^2 \le \langle B(x),\, x\rangle\langle B(y),\, y\rangle$$

for all $x, y \in H$. Letting $x = x_n$ and $y = B(x_n)$ we see that

$$\begin{aligned} \|B(x_n)\|^4 &= |\langle B(x_n),\, B(x_n)\rangle|^2 \\ &\le \langle B(x_n),\, x_n\rangle\langle B^2(x_n),\, B(x_n)\rangle \\ &\le \langle B(x_n),\, x_n\rangle\|B\|^3. \end{aligned}$$

Since $\langle B(x_n),\, x_n\rangle = \langle A(x_n),\, x_n\rangle - m_A \to 0$, we have $\|A(x_n) - m_A x_n\| = \|L(x_n)\| \to 0$ as $n \to \infty$. Hence $m_A \in \sigma_a(A)$. Similarly, we obtain $M_A \in \sigma_a(A)$ by considering $C = A - M_A I$ and noting that $-C$ is a positive operator.

Since A is self-adjoint, it is normal. Hence $\sigma_a(A) = \sigma(A)$ by 27.2(d). Also, 27.4(b) shows that $\sigma(A)$ is contained in the closure of $\omega(A)$. As $\omega(A) \subset [m_A, M_A]$, it follows that $\sigma(A) \subset [m_A, M_A]$ as well.

(b) It is easy to see that (α_n) is a nonincreasing sequence which is bounded below by m_A. Hence $\alpha_n \to m_0$, say. Clearly, we have $m_A \le m_0$.

Assume now that span$\{x_1, x_2, \ldots\}$ is dense in H. Let, if possible, $m_A < m_0$. By the definition of m_A, there is some $x \in H$ with $\|x\| = 1$ and $\langle A(x),\, x\rangle < m_0$. There is a sequence (y_n) in span$\{x_1, x_2, \ldots\}$ such that $y_n \to x$ as $n \to \infty$. Noting that $\|y_n\| \to \|x\| = 1$ and letting $z_n = y_n/\|y_n\|$ for all large n, we find that $z_n \to x$, $\|z_n\| = 1$ and there is an integer j_n with $z_n \in F_{j_n} = \text{span}\{x_1, \ldots, x_{j_n}\}$. Letting $n \to \infty$, we find that

$$m_0 \le \alpha_{j_n} \le \langle A(z_n),\, z_n\rangle \to \langle A(x),\, x\rangle < m_0.$$

This contradiction shows that $m_A = m_0 = \lim_n \alpha_n$.

In exactly the same way, it follows that (β_n) is a nondecreasing sequence and $\lim_n \beta_n = M_A$. Since $\alpha_n \le \beta_n$ for each n, the proof is complete. □

27.6 Example

Let $H = L^2([0,1])$ and A denote the Fredholm integral operator on H with kernel $k(.,.)$ given by

$$k(s,t) = \begin{cases} s(1-t), & \text{if } 0 \le s \le t \le 1 \\ (1-s)t, & \text{if } 0 \le t \le s \le 1. \end{cases}$$

We have seen in 18.6(b) that

$$\sigma(A) = \{0\} \bigcup \{\frac{1}{n^2\pi^2} : n = 1, 2, \ldots\}.$$

Hence $m_A = 0$ and $M_A = 1/\pi^2$. Let us illustrate Ritz's method for finding approximations of m_A and M_A.

Consider $x_j(t) = t^{j-1}, j = 1, 2, \ldots$. Then span $\{x_1, x_2, \ldots\}$ equals the set $P([0,1])$ of all polynomial functions on $[0,1]$. Since $P([0,1])$ is dense in $C([0,1])$ with the sup norm by 3.12 and $C([0,1])$ is dense in $L^2([0,1])$ by 4.7(b), we see that span $\{x_1, x_2 \ldots\}$ is dense in H.

For $x \in X$, we have

$$\langle A(x),\, x\rangle = \int_0^1 A(x)(s)\overline{x(s)}\, dm(s),$$

where $A(x)(s) = (1-s)\displaystyle\int_0^s t\, x(t)\, dm(t) + s\int_s^1 (1-t)x(t)\, dm(t)$.

If $x \in F_1 = \text{span}\,\{x_1\}$ and $x = ax_1$ with $a \in \mathbf{K}$, then

$$\begin{aligned} \langle A(x),\, x\rangle &= |a|^2 \int_0^1 (1-s)(\frac{s^2}{2}) + s\frac{(1-s)^2}{2}\, ds \\ &= \frac{|a|^2}{2}\int_0^1 s(1-s)\, ds = \frac{|a|^2}{12}, \end{aligned}$$

while

$$\langle x,\, x\rangle = \int_0^1 |a|^2\, dm(s) = |a|^2.$$

Hence

$$\alpha_1 = \inf_{x\in F_1, \|x\|=1} \langle A(x),\, x\rangle = \frac{1}{12} = \sup_{x\in F_1, \|x\|=1} \langle A(x),\, x\rangle = \beta_1,$$

Next, if $x \in F_2 = \operatorname{span}\{x_1, x_2\}$ and $x = ax_1 + bx_2$ with $a, b \in \mathbf{K}$, then

$$\begin{aligned}\langle A(x),\, x\rangle &= \int_0^1 (1-s)\left[a\frac{s^2}{2} + b\frac{s^3}{3}\right](\overline{a} + \overline{b}s) \\ &\quad + s\left[a\frac{(1-s)^2}{2} + b\Big(\frac{1}{6} - \frac{s^2}{2} + \frac{s^3}{3}\Big)\right](\overline{a} + \overline{b}s)ds \\ &= \frac{|a|^2}{12} + \frac{\operatorname{Re}\overline{a}b}{12} + \frac{|b|^2}{45},\end{aligned}$$

while

$$\langle x,\, x\rangle = \int_0^1 (a + bs)(\overline{a} + \overline{b}s)\, ds = |a|^2 + \operatorname{Re}\overline{a}b + \frac{|b|^2}{3}.$$

Thus

$$\begin{aligned}&\{\langle A(x),\, x\rangle : x \in F_2, \|x\| = 1\} \\ &\quad = \{\frac{|a|^2}{12} + \frac{\operatorname{Re}\overline{a}b}{12} + \frac{|b|^2}{45} : a, b \in \mathbf{K}, |a|^2 + \operatorname{Re}\overline{a}b + \frac{|b|^2}{3} = 1\}.\end{aligned}$$

Let $b \in \mathbf{K}$. Then by using the theory of quadratic equations, it can be seen that there is some $a \in \mathbf{K}$ with $|a|^2 + \operatorname{Re}\overline{a}b + |b^2|/3 = 1$ if and only if $|b|^2 \leq 12$. In that case,

$$\frac{|a|^2}{12} + \frac{\operatorname{Re}\overline{a}b}{12} + \frac{|b|^2}{45} = \frac{1}{12}\left(1 - \frac{|b|^2}{3}\right) + \frac{|b|^2}{45} = \frac{1}{12} - \frac{|b|^2}{180},$$

so that

$$\{\langle A(x),\, x\rangle : x \in F_2, \|x\| = 1\} = \{\frac{1}{12} - \frac{|b|^2}{180} : b \in \mathbf{K}, |b|^2 \leq 12\}.$$

Hence

$$\begin{aligned}\alpha_2 &= \inf_{x \in F_2, \|x\|=1} \langle A(x),\, x\rangle = \frac{1}{12} - \frac{12}{180} = \frac{1}{60}, \\ \beta_2 &= \sup_{x \in F_2, \|x\|=1} \langle A(x),\, x\rangle = \frac{1}{12}\end{aligned}$$

In a like manner, by considering $F_3 = \operatorname{span}\{x_1, x_2, x_3\}$ and using the Lagrange multipliers for constrained minimization and maximization, one can obtain

$$\alpha_3 = \inf_{x \in F_3, \|x\|=1} \langle A(x),\, x\rangle = \frac{1}{56}\left(3 - \sqrt{\frac{107}{15}}\right),$$

$$\beta_3 = \sup_{x \in F_3, \|x\|=1} \langle A(x), x \rangle = \frac{1}{56}\left(3 + \sqrt{\frac{107}{15}}\right).$$

Note that $\alpha_3 = 0.00587\ldots$ agrees with $m_A = 0$ in the first two decimal places, while $\beta_3 = 0.10126\ldots$ agrees with $M_A = 1/\pi^2 = 0.10132\ldots$ in this first three decimal places.

An alternative method for finding the α_n's and β_n's is outlined in Problem 27-15.

27.7 Corollary

Let $H \neq \{0\}$.

(a) Let $A \in BL(H)$ be self-adjoint. Then

$$\|A\| = \max\{|m_A|, |M_A|\} = \sup\{|k| : k \in \sigma(A)\}.$$

(b) Let $A \in BL(H)$. Then

$$\|A\| = \sup\{\sqrt{|k|} : k \in \sigma(A^*A)\}.$$

Proof:

(a) By 26.2(a),

$$\|A\| = \sup\{|\langle A(x), x\rangle| : x \in H, \|x\| = 1\}.$$

Since A is self-adjoint, $\langle A(x), x\rangle \in \mathbf{R}$ for all $x \in H$. The definitions of m_A and M_A, therefore, show that $\|A\| = \max\{|m_A|, |M_A|\}$. Also, it follows from 27.5(a) that $\max\{|m_A|, |M_A|\} = \sup\{|k| : k \in \sigma(A)\}$.

(b) By 25.3(b), we have $\|A\|^2 = \|A^*A\|$. But since A^*A is self-adjoint, it follows from (a) above that

$$\|A^*A\| = \sup\{|k| : k \in \sigma(A^*A)\}.$$

Hence $\|A\| = \|A^*A\|^{1/2} = \sup\{\sqrt{|k|} : k \in \sigma(A^*A)\}.$ □

While the expression for $\|A\|$ given in 27.7(b) is of considerable theoretical importance, it is hardly of any use in calculating $\|A\|$, in general. Problem 27-17 indicates the difficulties involved even in the simplest cases. It also shows that $\|A\|$ can be strictly greater than $\sup\{|k| : k \in \sigma(A)\}$. If, however, A is a normal operator on a Hilbert space over $\mathbf{C}$, then an analog of 27.7(a) holds, as indicated in Problem 27-11(b).

Finite Dimensional Operators

Consider a finite dimensional Hilbert space H with an orthonormal basis $u_1, \ldots, u_n$. Let A be a linear operator on H. Then for every $x \in H$, we have

$$x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j \quad \text{and} \quad A(x) = \sum_{j=1}^{n} \langle x, u_j \rangle A(u_j).$$

The continuity of the inner product in the first variable and of the operations of scalar multiplication and addition shows that A is a continuous map, that is, $A \in BL(H)$. Hence A is invertible (in $BL(H)$) if and only if A is bijective. We note that A is bijective if and only if A is injective since H is finite dimensional (2.4(b)).

As in 25.1(a), we have

$$A(x) = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \langle x, u_j \rangle \langle A(u_j), u_i \rangle \right] u_i, \quad x \in H.$$

Hence $A(x) = 0$ if and only if $\sum_{j=1}^{n} \langle x, u_j \rangle \langle A(u_j), u_i \rangle = 0$ for each $i = 1, \ldots, n$, while $x = 0$ if and only if $\langle x, u_j \rangle = 0$ for each $i = 1, \ldots, n$. This shows that A is injective if and only if the system

$$\begin{array}{ccccc} \langle A(u_1), u_1 \rangle k_1 & + \cdots + & \langle A(u_n), u_1 \rangle k_n & = & 0 \\ \vdots & & \vdots & & \vdots \\ \langle A(u_1), u_n \rangle k_1 & + \cdots + & \langle A(u_n), u_n \rangle k_n & = & 0 \end{array}$$

of n equations in the n unknowns $k_1, \ldots, k_n$ has $k_1 = \cdots = k_n = 0$ as the only solution. Thus if M denotes the $n \times n$ matrix $(\langle A(u_j), u_i \rangle)$, $i, j = 1, \ldots, n$, then we conclude that

$$\sigma(A) = \sigma_e(A) = \{k \in \mathbf{K} : \det(M - kI) = 0\}.$$

Recall from 25.1(a) that the matrix M is said to define the operator A with respect to $u_1, \ldots, u_n$. Hence if $k \in \sigma_e(A)$, then x is an eigenvector of A corresponding to k if and only if $x = a_1 u_1 + \cdots + a_n u_n$, where

$$(M - kI) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

and not all the scalars $a_1, \ldots a_n$ are zero.

We have seen in 2.6(a) that the characteristic polynomial $p_A(k) = \det(M - kI)$ of A does not depend on the choice of the orthonormal basis $u_1, \ldots, u_n$ for H. [In fact, if $\tilde{u}_1, \ldots, \tilde{u}_n$ is another orthonormal basis for H and $\tilde{M} = (\langle A(\tilde{u}_j), \tilde{u}_i \rangle)$, then $\tilde{M} = P^{-1} M P$, where $P = (\langle \tilde{u}_j, u_i \rangle)$.]

Must A have an eigenvalue? If so, how many eigenvalues can A have? Since the degree of the polynomial p_A is n, it has at most n distinct roots in $\mathbf{K}$. Hence A has at most n distinct eigenvalues. If $\mathbf{K} = \mathbf{C}$, then p_A must have at least one root in $\mathbf{C}$, that is, A must have at least one eigenvalue. If $\mathbf{K} = \mathbf{R}$, this may not be the case as shown by the following simple example. Let $H = \mathbf{R}^2, \mathbf{K} = \mathbf{R}$, and

$$A(x(1)), x(2)) = (-x(2), x(1)), \quad (x(1), x(2)) \in H.$$

Then the matrix $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ defines the operator A with respect to the basis e_1, e_2. Since $p_A(k) = k^2 + 1$ for $k \in \mathbf{R}$, p_A has no root in $\mathbf{R}$, and consequently A has no eigenvalue. If, however, the operator A is self-adjoint, then this situation does not arise, as following result shows.

27.8 Lemma

Let A be a self-adjoint operator on a finite dimensional Hilbert space H. Then every root of the characteristic polynomial of A is real. In particular, A has an eigenvalue.

Proof:

Let $s+it$ be a root of the characteristic polynomial p_A of A. Consider the operator $B = (A - sI)^2 + t^2 I$. If a matrix M defines the operator A, then B is defined by the matrix $(M - sI)^2 + t^2 I$. Since

$$\det((M - sI)^2 + t^2 I) = \det(M - (s + it)I)\det(M - (s - it)I) = 0,$$

there is some nonzero $x \in H$ such that $B(x) = 0$, that is, $(A - sI)^2(x) = -t^2 x$. Since A is self-adjoint, we have

$$\begin{aligned} 0 \le \langle (A - sI)(x), (A - sI)(x)\rangle &= \langle (A^* - sI)(A - sI)(x), x\rangle \\ &= \langle (A - sI)^2(x), x\rangle \\ &= -t^2\langle x, x\rangle. \end{aligned}$$

As $x \neq 0$, we obtain $t = 0$. Hence $s + it = s \in \mathbf{R}$ and $s \in \sigma_e(A)$. $\square$

We now prove the most important result of this subsection. It is known as the finite dimensional **spectral theorem** for self-adjoint or normal operators.

27.9 Theorem

Let H be a finite dimensional Hilbert space over $\mathbf{K}$ and $A \in BL(H)$.

(a) Suppose that there is an orthonormal basis for H consisting of eigenvectors of A. Then A is a normal operator. If $\mathbf{K} = \mathbf{R}$, then A is, in fact, a self-adjoint operator.

(b) Suppose that $\mathbf{K} = \mathbf{C}$ and A is normal, or that $\mathbf{K} = \mathbf{R}$ and A is self-adjoint. Then there is an orthonormal basis for H consisting of eigenvectors of A. If $u_1, \ldots, u_n$ is such a basis and $A(u_j) = k_j u_j$ for

$j = 1, \ldots, n$, then

$$A(x) = \sum_{j=1}^{n} k_j \langle x, u_j \rangle u_j, \quad x \in H.$$

Proof:

(a) Let $u_1, \ldots, u_n$ be an orthonormal basis for H such that $A(u_j) = k_j u_j$ for some $k_j \in \mathbf{K}, j = 1, \ldots, n$. Then for all $x \in H$, we have

$$x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j, \quad A(x) = \sum_{j=1}^{n} \langle x, u_j \rangle k_j u_j \quad A^*(x) = \sum_{j=1}^{n} \langle x, u_j \rangle \overline{k}_j u_j.$$

It follows, as in 26.1(b), that A is normal. If $\mathbf{K} = \mathbf{R}$, then $k_1, \ldots, k_n$ are all real and hence $A^* = A$.

(b) Suppose that $\mathbf{K} = \mathbf{C}$ and A is normal, or that $\mathbf{K} = \mathbf{R}$ and A is self-adjoint. Let $k_1, \ldots, k_m$ be the distinct eigenvalues of A. (Recall Lemma 27.8 if $\mathbf{K} = \mathbf{R}$.) For $j = 1, \ldots, m$, let $F_j = Z(A - k_j I)$ and $F = F_1 + \cdots + F_m$. We claim that $F = H$.

First we prove that $A(F^\perp) \subset F^\perp$. Consider $x \in F^\perp$ and $y \in F$. Then $y = y_1 + \cdots + y_m$ for some $y_j \in F_j$, so that $A(y_j) = k_j y_j$ for $j = 1, \ldots, m$. Since A is normal in any case, it follows from 27.2(a) that $A^*(y_j) = \overline{k}_j y_j$ for $j = 1, \ldots, m$. Now

$$\begin{aligned}
\langle A(x), y \rangle &= \langle x, A^*(y) \rangle = \langle x, A^*(y_1 + \cdots + y_m) \rangle \\
&= \langle x, \overline{k}_1 y_1 + \cdots + \overline{k}_m y_m \rangle \\
&= k_1 \langle x, y_1 \rangle + \cdots + k_m \langle x, y_m \rangle \\
&= 0,
\end{aligned}$$

since $x \perp y_j$ for each j. Thus $A(x) \in F^\perp$. Hence $A(F^\perp) \subset F^\perp$.

Next we prove that $F^\perp = \{0\}$. Were $F^\perp \neq \{0\}$, $A_{|F^\perp}$ will have an eigenvalue k. (Note that if $\mathbf{K} = \mathbf{R}$, then $A_{|F^\perp}$ is a self-adjoint operator on $F^\perp$ and hence Lemma 27.8 applies.) Let x be a nonzero element of $F^\perp$ with $A(x) = kx$. Since k is thus an eigenvalue of A, $k = k_j$ for some $j, 1 \leq j \leq m$. Then $x \in Z(A - k_j I) = F_j \subset F$. But this is impossible since $x \neq 0, x \in F \cap F^\perp$ and $F \cap F^\perp = \{0\}$. Thus $F^\perp = \{0\}$, so that $H = F + F^\perp = F + \{0\} = F$ by 24.1.

Let $\{u_{j,1}, \ldots, u_{j,n_j}\}$ be an orthonormal basis for $F_j, j = 1, \ldots, m$. Since A is normal, $F_i \perp F_j$ for $i \neq j$ by 27.2(c). It follows that $\{u_{1,1}, \ldots, u_{1,n_1}, \ldots, u_{m,1}, u_{m,n_m}\}$ is an orthonormal basis for $H = F = F_1 + \cdots + F_m$, where each $u_{j,k}$ is an eigenvector of A.

The last statement of the theorem is obvious. □

We remark that if $\mathbf{K} = \mathbf{R}$, then a normal operator on a finite dimensional space may not have any eigenvectors. For example, let $H = \mathbf{R}^2$ and let A be defined by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

with respect to the standard orthonormal basis e_1, e_2. If $\theta \in (0, \pi)$ or $\theta \in (\pi, 2\pi)$, then it can be seen that A is normal but has no eigenvectors, since $A(x)$ is obtained by rotating x through an angle θ. [We have earlier considered the case $\theta = \pi/2$.]

Problems

H denotes a Hilbert space over $\mathbf{K}$, unless otherwise stated.

27-1 Let $A \in BL(H)$. Then $\sigma(A)$ is a closed and bounded subset of $\mathbf{K}$. (Hint: $\sigma_a(A)$ is a closed subset of $\{k \in \mathbf{K} : |k| \leq \|A\|\}$. Use 27.1(b).)

27-2 $\sigma(A) = \sigma_e(A) \bigcup \{k : \bar{k} \in \sigma_a(A^*)\}$. (Hint: 27.1)

27-3 Let $A \in BL(H)$ and $k_1 \neq k_2$ in $\mathbf{K}$. If $A(x_1) = k_1 x_1$ and $A^*(x_2) = \bar{k}_2 x_2$ for x_1, x_2 in X, then $x_1 \perp x_2$. (Compare 27.2(c).)

27-4 Let $A \in BL(H)$ be normal.

(a) Let E be a set of eigenvectors of A and F denote the closure of the span of E. Then F and $F^\perp$ are closed invariant subspaces for A. (Hint:

27.2(a))

(b) There is a unique closed subspace F of H such that F and $F^{\perp}$ are invariant for A, the span of all the eigenvectors of $A_{|F}$ is dense in F, while $A_{|F^{\perp}}$ has no eigenvectors.

27-5 Let H be a separable Hilbert space and $A \in BL(H)$.

(a) If A is normal, then $\sigma_e(A)$ is countable. (Hint: 27.2(c) and 22.9)

(b) A is normal and the closure of the span of all the eigenvectors of A is dense in H if and only if there is an orthonormal basis $u_1, u_2, \ldots$ for H such that

$$A(x) = \sum_n k_n \langle x, u_n \rangle u_n, \quad x \in H,$$

for some $k_1, k_2, \ldots$ in $\mathbf{K}$. In that case, $\sigma_e(A) = \{k_1, k_2, \ldots\}$ and $\sigma(A)$ is the closure of $\{k_1, k_2, \ldots\}$. (Compare Problem 25-6. Hint: 22.7, 26.1(b) and 27.2(d))

27-6 Let E be a measurable subset of $\mathbf{R}$ and $z \in L^{\infty}(E)$. The set

$$\{k \in \mathbf{K} : m(\{t \in E : |z(t) - k| < \epsilon\}) > 0 \text{ for every } \epsilon > 0)\}$$

is called the essential range of z. Let $H = L^2(E)$ and $A \in BL(H)$ be defined by $A(x) = zx, x \in H$. Then

$$\sigma_e(A) = \{k \in \mathbf{K} : m(z^{-1}(\{k\})) > 0\}$$

and $\sigma_a(A) = \sigma(A)$ equals the essential range of z. (Compare Problem 25-2 and 26.1(a). Hint: 27.2(d))

27-7 Let $A \in BL(H)$ and $k \in \mathbf{K}$.

(a) If $d = \text{dist}(k, \omega(A)) > 0$, then $\|(A - kI)^{-1}\| \le 1/d$. (Hint: 27.4(b))

(b) If $\gamma\|x\|^2 \le |\langle A(x), x\rangle|$ for some $\gamma > 0$ and all $x \in H$, then A is invertible and $\|A^{-1}\| \le 1/\gamma$. The converse does not hold.

(c) If A is self-adjoint and $k \notin \mathbf{R}$, then $\|(A - kI)^{-1}\| \le 1/|\text{Im}\, k|$.

(d) If A is unitary and $|k| \ne 1$, then $\|(A - kI)^{-1}\| \le 1/|1 - |k||$. (Hint: If $0 < |k| < 1$, then $(A - kI)^{-1} = -(kA)^{-1}(A^{-1} - I/k)^{-1}$.)

27-8 If $A \in BL(H)$ is self-adjoint, then $\sigma(A) \subset \mathbf{R}$ and if A is unitary, then $\sigma(A) \subset \{k \in \mathbf{K} : |k| = 1\}$. On the other hand, if H is infinite dimensional and E is a nonempty closed and bounded subset of $\mathbf{K}$, then there is a normal operator $A \in BL(H)$ such that $\sigma(A) = E$. If the dimension of H is n and E is a nonempty subset of $\mathbf{K}$ having m elements, then there is a normal operator A on H with $\sigma(A) = E$ if and only if $m \leq n$. (Hint: If $\{u_n\} \bigcup \{v_\alpha\}$ is an orthonormal basis for H and $\{k_n\}$ is a countable dense subset of E, define $A(x) = k_1 x + \sum_n (k_n - k_1)(x, u_n)u_n$, $x \in H$.)

27-9 **(Minimum residual property)** For $A \in BL(H)$ and $x \in H$ with $\|x\| = 1$, let $r(x) = A(x) - (A(x), x)x$. Then $r(x) \perp x$ and for all $k \in \mathbf{K}$, $\|r(x)\| \leq \|A(x) - kx\|$. (Hint: 22.1(a))

27-10 Let $A \in BL(H)$ and $k \in \mathbf{K}$ satisfy $|k| = \|A\|$.

(a) If $k \in \omega(A)$, then there is some nonzero $x \in H$ such that $A(x) = kx$ and $A^*(x) = \bar{k}x$, so that $k \in \sigma_e(A)$ and $\bar{k} \in \sigma_e(A^*)$.

(b) If k belongs to the closure of $\omega(A)$, then there is sequence (x_n) in H such that $\|x_n\| = 1$, $\|A(x_n) - kx_n\| \to 0$ and $\|A^*(x_n) - \bar{k}x_n\| \to 0$, so that $k \in \sigma_a(A)$ and $\bar{k} \in \sigma_a(A^*)$. (Compare 27.4.)

27-11 Let $A \in BL(H)$. The **spectral radius** of A is defined by

$$r_\sigma(A) = \sup\{|k| : k \in \sigma(A)\}$$

and the **numerical radius** of A is defined by

$$r_\omega(A) = \sup\{|k| : k \in \omega(A)\}.$$

(a) $r_\sigma(A) \leq r_\omega(A) \leq \|A\|$. Further, $r_\sigma(A) = \|A\|$ if and only if $r_\omega(A) = \|A\|$. In that case, there is some $k \in \sigma_a(A)$ such that $|k| = \|A\|$. (Hint: 27.4(b) and Problem 27-10(b))

(b) Let $\mathbf{K} = \mathbf{C}$. Then $\|A\| \leq 2r_\omega(A)$. (Hint: Problem 21-2.) If A is normal, then $\|A\| = r_\omega(A) = r_\sigma(A)$. (Hint: $\|A^n\| = \|A\|^n$ if $n = 2^j, j = 1, 2, \ldots$ by 26.2(c) and $r_\sigma(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$ by 12.8(b).)

27-12 Let $\mathbf{K} = \mathbf{C}$. If A is self-adjoint, then its **Cayley transform** $T(A) = (A - iI)(A + iI)^{-1}$ is unitary and $1 \notin \sigma(T(A))$. Conversely, if B is unitary and $1 \notin \sigma(B)$, then its **inverse Cayley transform** $S(B) = i(I + B)(I - B)^{-1}$ self-adjoint. Further, $S(T(A)) = A$ and $T(S(B)) = B$.

27-13 Let $A \in BL(H)$ be self-adjoint. Then A is a positive operator if and only if every spectral value of A is a nonnegative real number. In that case, $0 \in \omega(A)$ if and only if $0 \in \sigma_e(A)$. (Hint: 27.5(a) and 26.5)

27-14 Let $A \in BL(H)$ be self-adjoint.

(a) m_A and M_A need not be eigenvalues of A. (Hint: For $x \in \ell^2$, let $A(x)(j) = x(j)/j$ if j is even and $jx(j)/(j+1)$ if j is odd.)

(b) Let $k = m_A$ or M_A. If $k \in \omega(A)$, then, in fact, $k \in \sigma_e(A)$. (Hint: Proof of 27.5(a))

27-15 Let A, F_n, α_n and β_n be as in 27.5(b). Let P_n denote the orthogonal projection onto F_n and $A_n = P_n A_{|F_n}$. Then $A_n \in BL(F_n)$, A_n is self-adjoint and α_n (resp., β_n) is the smallest (resp., largest) eigenvalue of A_n. If the set $\{x_1, x_2, \ldots\}$ is linearly independent and we construct $u_1, u_2, \ldots$ as in 22.2, then α_n (resp., β_n) is the smallest (resp., largest) root of $\det(M - kI) = 0$, where $M = (\langle A(u_j), u_i\rangle), i, j = 1, \ldots, n$.

27-16 Let $H = L^2([0,1])$ and for $x \in H$, let

$$A(x)(s) = \int_0^1 k(s,t)x(t)\,dm(t), \quad 0 \le s \le 1,$$

where $k(s,t) = \min\{s,t\}$ for $0 \le s,t \le 1$. Let $x_1(t) = 1$ and $x_2(t) = t$ for $0 \le t \le 1$. Then the Ritz method (27.5(b)) gives $\alpha_2 = (13 - 2\sqrt{31})/60 = 0.31\ldots$ and $\beta_2 = (13 + 2\sqrt{31})/60 \doteq 0.402\ldots$. [Note that $m_A = 0$ and $M_A = 4/\pi^2 = 0.405\ldots$, as in Problem 18-9.]

27-17 Fix $k \in \mathbf{K}$. For $(x(1), x(2)) \in \mathbf{K}^2$, define

$$A(x(1), x(2)) = (kx(1) + x(2), kx(2)).$$

Then

$$\|A\| = \left[\frac{2|k|^2 + 1 + \sqrt{4|k|^2 + 1}}{2}\right]^{1/2}$$

while $\sigma(A) = \{k\}$, so that $r_\sigma(A) = |k| < \|A\|$. (Hint: 27.7)

27-18 Let $A \in BL(H)$. If $\mathbf{K} = \mathbf{C}$, assume that A is normal and if $\mathbf{K} = \mathbf{R}$, assume that A is self-adjoint. Then

(a) $\|(A - kI)^{-1}\| = 1/\text{dist}(k, \sigma(A))$ for every $k \notin \sigma(A)$.

(b) (Krylov-Weinstein) If $k \in \mathbf{K}$, then there is some $k_0 \in \delta_a(A)$ such that $|k - k_0| \le \|A(x) - kx\|$ for all $x \in H$ with $\|x\| = 1$.

27-19 Let H be finite dimensional A be a normal operator on H.

(a) Let $\mathbf{K} = \mathbf{C}$. Then $AB = BA$ if and only if every eigenspace of A is invariant for B. This result does not hold if $\mathbf{K} = \mathbf{R}$.

(b) $AB = BA$ if and only if $A^*B = BA^*$. (Hint: If $\mathbf{K} = \mathbf{C}$, use (a) above and 27.2(a). If $\mathbf{K} = \mathbf{R}$, use Problem 25-15.)

27-20 Let $\mathbf{K} = \mathbf{C}$, H be finite dimensional and $A \in BL(H)$ be normal. Then $A^* = p(A)$ for some polynomial p with complex coefficients. In particular, if $B \in BL(H)$ and $AB = BA$, then $A^*B = BA^*$.

28 Compact Self-Adjoint Operators

In this section we consider certain bounded operators which provide a natural generalization of finite dimensional operators and include some well-known integral operators. Although we have studied these operators in Chapter V, we give here an independent treatment in the framework of a Hilbert space. We also investigate the spectrum of a self-adjoint operator belonging to this class.

An operator A on a Hilbert space H over $\mathbf{K}$ is said to be **compact** if for every bounded sequence (x_n) in H, the sequence $(A(x_n))$ contains a subsequence which converges in H. (This definition is equivalent to the one given earlier. See 17.1(a).)

If A is a compact operator on H, then there is some $\alpha > 0$ such that $\|A(x)\| \leq \alpha$ for all $x \in H$ with $\|x\| \leq 1$. For otherwise, we can find some $x_n \in H$ with $\|x_n\| \leq 1$ and $\|A(x_n)\| > 1 + \max\{\|A(x_1)\|, \ldots, \|A(x_{n-1})\|\}$ for $n = 1, 2, \ldots$ with $x_0 = 0$. Then the sequence $(A(x_n))$ has no convergent subsequence although (x_n) is a bounded sequence. This shows that every compact operator is bounded. However, the converse is not true. For example, if H is infinite dimensional, then the identity operator I is clearly bounded, but it is not compact. For if (u_n) is an infinite orthonormal sequence in H, then $\|u_n\| = 1$ but $\|u_n - u_m\| = \sqrt{2}$ for all $n \neq m$, so that the bounded sequence $(I(u_n))$ has no convergent subsequence.

We now give some sufficient conditions for a bounded operator to be compact and show that the adjoint of a compact operator is compact.

28.1 Theorem

Let $A \in BL(H)$.

(a) If $R(A)$ is finite dimensional, then A is compact.

(b) If each A_n is a compact operator on H and $\|A_n - A\| \to 0$, then A is compact.

(c) If A is a compact, then so is A^*.

Proof:

(a) Let the range $R(A)$ of A be finite dimensional. Then it is spanned by a finite orthonormal set $\{u_1, \ldots, u_m\}$. (See 22.2.) The proof of 22.9 shows that the map

$$F(x) = (\langle x, u_1\rangle, \ldots, \langle x, u_m\rangle), \quad x \in R(A),$$

is a linear isometry of $R(A)$ onto $\mathbf{K}^m$ with the norm $\| \ \|_2$. Let (x_n) be a bounded sequence in H. Since A is a bounded operator, $(A(x_n))$ is a bounded sequence in $R(A)$ and hence $(F(A(x_n)))$ is a bounded sequence in $\mathbf{K}^m$. By the classical Bolzano-Weierstrass theorem, it contains a convergent subsequence. In turn, this holds for the sequence $(A(x_n))$. Thus A is compact.

(b) For $n = 1, 2, \ldots$, let A_n be a compact operator on H and assume that $\|A_n - A\| \to 0$. Let (x_j) be a sequence in H such that $\|x_j\| \leq \alpha$ for all $j = 1, 2, \ldots$ and some $\alpha > 0$. Since A_1 is compact,

$(A_1(x_j))$ contains a convergent subsequence $(A_1(x_{j,1}))$. Again, since $(x_{j,1})$ is a bounded sequence and A_2 is compact, $(A_2(x_{j,1}))$ contains a convergent subsequence $(A_2(x_{j,2}))$. We continue in this fashion and consider the diagonal sequence $(x_{j,j})$, which is a subsequence of (x_j). Then for each $n = 1, 2, \ldots$, the sequence $(A_n(x_{j,j}))$ converges in H. For $j, k = 1, 2, \ldots$, we have

$$\begin{aligned}
\|A(x_{j,j}) - A(x_{k,k})\| &\leq \|A(x_{j,j}) - A_n(x_{j,j})\| + \|A_n(x_{j,j}) - A_n(x_{k,k})\| \\
&\qquad + \|A_n(x_{k,k}) - A(x_{k,k})\| \\
&\leq \|A - A_n\| \, \|x_{j,j}\| + \|A_n(x_{j,j}) - A_n(x_{k,k})\| \\
&\qquad + \|A_n - A\| \, \|x_{k,k}\| \\
&\leq 2\alpha \|A - A_n\| + \|A_n(x_{j,j}) - A_n(x_{k,k})\|.
\end{aligned}$$

Hence we see that $(A(x_{j,j}))$ is a Cauchy sequence in H. Since H is complete, it converges in H. Thus A is compact.

(c) Let A be compact. To show that A^* is compact, consider a sequence (x_n) in H such that $\|x_n\| \leq \alpha$ for all n and some $\alpha > 0$. Let $y_n = A^*(x_n), n = 1, 2, \ldots$. Since A^* is a bounded operator, (y_n) is a bounded sequence in H and since A is a compact operator, there is a subsequence (y_{n_j}) such that $(A(y_{n_j}))$ converges in H. For $j, k = 1, 2, \ldots$, we have

$$\begin{aligned}
\|A^*(x_{n_j}) - A^*(x_{n_k})\|^2 &= \langle A^*(x_{n_j} - x_{n_k}),\, A^*(x_{n_j} - x_{n_k})\rangle \\
&= \langle AA^*(x_{n_j} - x_{n_k}),\, x_{n_j} - x_{n_k}\rangle \\
&= \langle A(y_{n_j}) - A(y_{n_k}),\, x_{n_j} - x_{n_k}\rangle \\
&\leq 2\alpha \|A(y_{n_j}) - A(y_{n_k})\|.
\end{aligned}$$

This shows that $(A^*(x_{n_j}))$ is a Cauchy sequence in H. Since H is complete, it converges in H. Thus A^* is compact. □

We now consider an important kind of compact operators on a separable Hilbert space H. A bounded operator A on H is called

Hilbert-Schmidt if $\sum_j \|A(v_j)\|^2 < \infty$ for some orthonormal basis $\{v_j\}$ for H.

28.2 Theorem
Let $A \in BL(H)$ be a Hilbert-Schmidt operator. Then

(a) A is compact.

(b) A^* is a Hilbert-Schmidt operator.

Proof:
Let $\{v_j\}$ be an orthonormal basis for H such that $\sum_j \|A(v_j)\|^2 < \infty$.

(a) For $x \in H$, consider its Fourier expansion (22.7(ii))

$$x = \sum_j \langle x, v_j \rangle v_j.$$

Since A is continuous and linear, we have

$$A(x) = \sum_j \langle x, v_j \rangle A(v_j).$$

For $n = 1, 2, \ldots$, define

$$A_n(x) = \sum_{j=1}^{n} \langle x, v_j \rangle A(v_j), \quad x \in H.$$

As $A_n(x) \in \operatorname{span}\{A(v_1), \ldots, A(v_n)\}$ for all $x \in H, R(A_n)$ is finite dimensional. It follows from 28.1(a) that A_n is compact. Now for all $x \in H$,

$$\begin{aligned}
\|A(x) - A_n(x)\|^2 &= \|\sum_{j>n} \langle x, v_j \rangle A(v_j)\|^2 \\
&\le \left(\sum_{j>n} |\langle x, v_j \rangle| \, \|A(v_j)\|\right)^2 \\
&\le \left(\sum_{j>n} |\langle x, v_j \rangle|^2\right) \left(\sum_{j>n} \|A(v_j)\|^2\right) \\
&\le \|x\|^2 \sum_{j>n} \|A(v_j)\|^2
\end{aligned}$$

by Bessel's inequality (22.4). Since $\sum_{j>n}\|A(v_j)\|^2 \to 0$ as $n \to \infty$, we see that $\|A - A_n\| \to 0$. Hence A is compact by 28.1(b).

(**b**) By Parseval's formula (22.7(iii)), we have

$$\begin{aligned}\sum_i \|A^*(v_i)\|^2 &= \sum_i \sum_j |\langle A^*(v_i),\, v_j\rangle|^2 \\ &= \sum_j \sum_i |\langle v_i,\, A(v_j)\rangle|^2 = \sum_j \|A(v_j)\|^2 < \infty.\end{aligned}$$

Hence A^* is a Hilbert-Schmidt operator. □

28.3 Examples

(**a**) Let $H = \ell^2$ and $A \in BL(H)$. Let A be defined by the matrix $(k_{i,j}), i, j = 1, 2, \ldots$ with respect to the standard orthonormal basis $\{e_j\}$ for ℓ^2. Then

$$\sum_{j=1}^{\infty} \|A(e_j)\|^2 = \sum_{j=1}^{\infty} \|\sum_{i=1}^{\infty} k_{i,j} e_i\|^2 = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |k_{i,j}|^2.$$

Hence A is a Hilbert-Schmidt operator if $\Sigma_j \Sigma_i |k_{i,j}|^2 < \infty$. In that case, A is compact by 28.2(a). As we have seen in 25.4(a), the operator A^* is defined by the matrix $(\overline{k_{j,i}}), i, j = 1, 2, \ldots$. Hence we see that A^* is also a Hilbert-Schmidt operator. This also follows from 28.2(b).

(**b**) Let $H = L^2([a, b])$ and $A \in BL(H)$. We show that A is a Hilbert-Schmidt operator if and only if A is a Fredholm integral operator with kernel $k(.,.) \in L^2([a, b] \times [a, b])$.

Let $\{v_j\}$ be an orthonormal basis for H and for $i, j = 1, 2, \ldots$, let

$$w_{i,j}(s,t) = v_i(s)\overline{v_j(t)}, \quad s, t \in [a, b].$$

As in 22.8(d), $\{w_{i,j}\}$ is an orthonormal basis for $L^2([a, b] \times [a, b])$. Let A be a Fredholm integral operator with kernel $k(.,.) \in L^2([a, b] \times [a, b])$. Then by Fubini's theorem (4.4), we have for $i, j = 1, 2, \ldots$,

$$\langle\langle k,\, w_{i,j}\rangle\rangle = \iint_{[a,b]\times[a,b]} k(s,t)\overline{v_i(s)}v_j(t)\,d(m\times m)(s,t)$$

$$\begin{aligned} &= \int_a^b \left[\int_a^b k(s,t)v_j(t)\,dm(t)\right]\overline{v_i(s)}\,dm(s) \\ &= \langle A(v_j),\, v_i\rangle. \end{aligned}$$

By Parseval's formula (22.7(iii)),

$$\sum_j \|A(v_j)\|^2 = \sum_j \sum_i |\langle A(v_j),\, v_i\rangle|^2 = \sum_{i,j} |\langle\langle k,\, w_{i,j}\rangle\rangle|^2 = \|k\|_2^2 < \infty.$$

Hence A is a Hilbert-Schmidt operator.

Conversely, let A be a Hilbert-Schmidt operator. Then

$$\sum_j \sum_i |\langle A(v_j),\, v_i\rangle|^2 = \sum_j \|A(v_j)\|^2 < \infty.$$

By the Riesz-Fischer theorem (22.5(b)), the series $\sum_{i,j}\langle A(v_j),\, v_i\rangle w_{i,j}$ converges to some $k(.,.) \in L^2([a,b]\times[a,b])$ and by 22.5(a),

$$\langle A(v_j),\, v_i\rangle = \langle\langle k,\, w_{i,j}\rangle\rangle, \quad i,j = 1,2,\ldots.$$

If B denotes the Fredholm integral operator with kernel $k(.,.)$, then

$$\langle\langle k,\, w_{i,j}\rangle\rangle = \langle B(v_j),\, v_i\rangle, \quad i,j = 1,2,\ldots,$$

as we have seen earlier. Hence $A(v_j) = B(v_j)$ for each $j = 1,2,\ldots$, and consequently $A = B$, as desired.

By 28.2(a), A is a compact operator. As we have seen in 25.4(b), A^* is a Fredholm integral operator with kernel h given by $h(s,t) = \overline{k(t,s)}$, $s,t \in [a,b]$. Since $h \in L^2([a,b]\times[a,b])$, we see that A^* is also a Hilbert-Schmidt operator. This also follows from 28.2(b).

We now prove an important result concerning the eigenvalues of a compact operator.

28.4 Lemma

Let A be a compact operator on $H \neq \{0\}$.

(a) Every nonzero approximate eigenvalue of A is, in fact, an eigenvalue of A and the corresponding eigenspace is finite dimensional.

(b) If A is self-adjoint, then $\|A\|$ or $-\|A\|$ is an eigenvalue of A.

Proof:

(a) Let $k \neq 0$ be an approximate eigenvalue of A. Then there is some $x_n \in H$ be such that $\|x_n\| = 1$ and $\|A(x_n) - kx_n\| \to 0$ as $n \to 0$. Since A is compact, there is a subsequence (x_{n_j}) of (x_n) such that $(A(x_{n_j}))$ converges to some x in H. It follows that (kx_{n_j}) also converges to x in H. Since $k \neq 0$, and $\|x_{n_j}\| = 1$ for each j, we see that $x \neq 0$. Also,

$$A(x) = A(\lim_{j\to\infty} kx_{n_j}) = k \lim_{j\to\infty} A(x_{n_j}) = kx.$$

Thus k is an eigenvalue of A.

Were the eigenspace $Z(A-kI)$ infinite dimensional, we could find an (infinite) orthonormal sequence (u_n) in $Z(A-kI)$ by the Gram-Schmidt orthonormalization (22.2). Since $A(u_n) = ku_n$ for each n, we see that

$$\|A(u_n) - A(u_m)\|^2 = |k|^2\|u_n - u_m\|^2 = 2|k|^2$$

for all $n \neq m$ by the Pythagoras theorem (22.1(a)). Then (u_n) is a bounded sequence, but $(A(u_n))$ has no convergent subsequence, contrary to the compactness of A. Thus $Z(A-kI)$ is finite dimensional.

(b) Assume that A is self-adjoint. If $A = 0$, then $\pm\|A\| = 0$, which is clearly an eigenvalue of A. Let $A \neq 0$. By 27.5(a), both m_A and M_A are approximate eigenvalues of A. By 27.7(a),

$$\|A\| = \max\{|m_A|, |M_A|\} \neq 0.$$

If $M_A + m_A \geq 0$, then $M_A > 0$ and $\|A\| = |M_A| = M_A$, so that $\|A\|$ is an eigenvalue of A by (a) above. On the other hand, if $M_A + m_A < 0$, then $m_A < 0$ and $\|A\| = |m_A| = -m_A$, so that $-\|A\|$ is an eigenvalue of A, again by (a) above. □

We are now in a position to prove the following basic result about the spectrum of a compact self-adjoint operator. It is known as the **spectral theorem** for compact self-adjoint operators.

28.5 Theorem

Let A be a nonzero compact self-adjoint operator on a Hilbert space H over $\mathbf{K}$.

(a) There exist a finite or infinite sequence (s_n) of nonzero real numbers with $|s_1| \geq |s_2| \geq \cdots$ and an orthonormal set $\{u_1, u_2, \ldots\}$ in H such that

$$A(x) = \sum_n s_n \langle x,\, u_n \rangle u_n, \quad x \in H.$$

Further, if the set $\{u_1, u_2, \ldots\}$ is infinite, then $s_n \to 0$ as $n \to \infty$.

(b) Any real number occurs at most a finite number of times in the sequence (s_n). Also,

$$\{k : k \in \sigma(A), k \neq 0\} = \{s_1, s_2, \ldots\} = \{k : k \in \sigma_e(A), k \neq 0\}.$$

(c) Each u_n is an eigenvector of A and the set $\{u_1, u_2, \ldots\}$ is an orthonormal basis for $Z(A)^\perp$. If $\{v_\alpha\}$ is an orthonormal basis for $Z(A)$, then $\{u_1, u_2, \ldots\} \bigcup \{v_\alpha\}$ is an orthonormal basis for H, consisting of eigenvectors of A.

Proof:

(a) By 28.4(b), $\|A\|$ or $-\|A\|$ is a nonzero eigenvalue of A, that is, A has an eigenvalue s_1 with $|s_1| = \|A\| > 0$. Let u_1 be an eigenvector of A of norm 1 corresponding to s_1 and $F_1 = \operatorname{span}\{u_1\}$. Let $x \in F_1$. Then

$$x = \langle x,\, u_1 \rangle u_1 \quad \text{and} \quad A(x) = \langle x,\, u_1 \rangle A(u_1) = s_1 \langle x,\, u_1 \rangle u_1.$$

For $x \in F_1^\perp$, on the other hand, we have

$$\langle A(x),\, u_1 \rangle = \langle x,\, A(u_1) \rangle = \langle x,\, s_1 u_1 \rangle = s_1 \langle x,\, u_1 \rangle = 0,$$

so that $A(x) \in F_1^{\perp}$. Thus A maps $F_1^{\perp}$ into itself. Now $H_1 = F_1^{\perp}$ is a Hilbert space and $A_1 = A_{|H_1}$ is a compact self-adjoint operator on H_1.

Suppose that $A_1 = 0$. Let $x \in H$. Then $x = y + z$ for some $y \in F_1$ and $z \in H_1$ by the projection theorem (24.1). Hence

$$\begin{aligned} A(x) &= A(y) + A(z) = s_1\langle y, u_1\rangle u_1 + 0 \\ &= s_1\langle y + z, u_1\rangle u_1 = s_1\langle x, u_1\rangle u_1, \end{aligned}$$

which gives a desired representation of A.

On the other hand, if $A_1 \neq 0$, then again by 28.4(b), A_1 has an eigenvalue s_2 such that $|s_2| = \|A_1\| > 0$. Clearly,

$$|s_1| = \|A\| \geq \|A_1\| = |s_2|.$$

Let $u_2 \in H_1$ be an eigenvector of A_1 (and hence of A) of norm 1 corresponding to s_2 and $F_2 = \operatorname{span}\{u_1, u_2\}$. It follows that $u_1 \perp u_2$, A maps $F_2^{\perp}$ into itself, $H_2 = F_2^{\perp}$ is a Hilbert space and $A_2 = A_{|H_2}$ is a compact self-adjoint operator on H_2. If $A_2 = 0$, then it can be seen, as earlier, that

$$A(x) = s_1\langle x, u_1\rangle u_1 + s_2\langle x, u_2\rangle u_2$$

for all $x \in H$. On the other hand, if $A_2 \neq 0$, we can continue this process. If $A_n = A_{|H_n} = 0$ for some positive integer n, then

$$A(x) = s_1\langle x, u_1\rangle u_1 + \cdots + s_n\langle x, u_n\rangle u_n$$

for all $x \in H$, where $|s_1| \geq \cdots \geq |s_n| > 0$ and the set $\{u_1, \ldots, u_n\}$ is orthonormal.

On the other hand, if $A_n \neq 0$ for each $n = 1, 2, \ldots$, then we find a sequence (s_n) of nonzero real numbers such that $|s_1| \geq |s_2| \geq \cdots$ and an infinite orthonormal set $\{u_1, u_2, \ldots\}$ with $A(u_n) = s_n u_n$ for each $n = 1, 2, \ldots$. In this case, we show that $s_n \to 0$ as $n \to \infty$. Assume for a moment that there is some $\delta > 0$ such that $|s_{n_j}| \geq \delta$ for $j = 1, 2, \ldots$ with $n_1 < n_2 < \ldots$. If $j \neq k$, then

$$\|A(u_{n_j}) - A(u_{n_k})\|^2 = \|s_{n_j} u_{n_j} - s_{n_k} u_{n_k}\|^2 = |s_{n_j}|^2 + |s_{n_k}|^2 \geq 2\delta^2$$

by the Pythagoras theorem (22.1(a)). But this is not possible, since (u_{n_j}) is a bounded sequence and A is a compact operator, so that $(A(u_{n_j}))$ must have a convergent subsequence. Thus $s_n \to 0$ as $n \to \infty$.

Let F_∞ denote the closure of the linear span of $\{u_1, u_2, \ldots\}$ and $H_\infty = F_\infty^\perp$. Then H_∞ is a Hilbert space, and it can be seen that A maps H_∞ into itself. Let $A_\infty = A|_{H_\infty}$. Then it can be seen that A_∞ is a compact self-adjoint operator on H_∞. We show that $A_\infty = 0$. Let $x \in H_\infty = F_\infty^\perp$. Since $F_n \subset F_\infty$, we see that $x \in F_n^\perp = H_n$ for each $n = 1, 2, \ldots$. By the Schwarz inequality (21.1(c)),

$$|\langle A_\infty(x), x\rangle| = |\langle A_n(x), x\rangle| \leq \|A_{n|H_\infty}\| \, \|x\|^2 \leq \|A_n\| \, \|x\|^2 = |s_n| \, \|x\|^2.$$

Since $s_n \to 0$ as $n \to \infty$, it follows that $\langle A_\infty(x), x\rangle = 0$. By 26.2(a), $A_\infty = 0$.

Let $x \in H$. Then $x = y + z$ for some $y \in F_\infty$ and $z \in H_\infty$. By 22.7(iv), $\{u_1, u_2, \ldots\}$ is an orthonormal basis for F_∞ and

$$y = \sum_{n=1}^{\infty} \langle y, u_n\rangle u_n.$$

Hence

$$\begin{aligned} A(x) &= A(y) + A(z) = A\Big(\sum_{n=1}^{\infty} \langle y, u_n\rangle u_n\Big) + 0 \\ &= \sum_{n=1}^{\infty} s_n \langle y, u_n\rangle u_n = \sum_{n=1}^{\infty} s_n \langle y + z, u_n\rangle u_n \\ &= \sum_{n=1}^{\infty} s_n \langle x, u_n\rangle u_n, \end{aligned}$$

as desired.

In fact, for $m = 1, 2, \ldots$,

$$\begin{aligned}
\|A(x) - \sum_{n=1}^{m} s_n \langle x,\, u_n\rangle u_n\|^2 &= \|\sum_{n=m+1}^{\infty} s_n \langle x,\, u_n\rangle u_n\|^2 \\
&= \sum_{n=m+1}^{\infty} |s_n|^2 |\langle x,\, u_n\rangle|^2 \\
&\leq |s_{m+1}|^2 \sum_{n=m+1}^{\infty} |\langle x,\, u_n\rangle|^2 \\
&\leq |s_{m+1}|^2 \|x\|^2
\end{aligned}$$

by Bessel's inequality (22.4).

(b) Let $s \in \mathbf{R}$. If $s = 0$, then s does not occur in the sequence (s_n) since each $s_n \neq 0$. Let $s \neq 0$. Then s occurs at most a finite number of times in the sequence (s_n) since $s_n \to 0$ as $n \to \infty$, if the orthonormal set $\{u_1, u_2, \ldots\}$ is infinite.

Let s be a nonzero spectral value of A. Since A is self-adjoint, s is a nonzero approximate eigenvalue of A by 27.5(a) and since A is compact s is a nonzero eigenvalue of A by 28.4(a). Let u be an eigenvector of A corresponding to s. Then by (a) above,

$$su = A(u) = \sum_n s_n \langle u,\, u_n\rangle u_n.$$

If $s \neq s_n$ for any n, then $u \perp u_n$ for all n by 27.2(c) and hence $su = 0$. But this is not possible, since $s \neq 0$ and $u \neq 0$. Thus $s = s_n$ for some n, so that $\{s_1, s_2, \ldots\}$ consists of all nonzero spectral values of A.

(c) Let $F = Z(A)^{\perp}$. By 27.2(c), $u_n \in F$ for all $n = 1, 2, \ldots$. Consider $x \in F$ and assume that $\langle x,\, u_n\rangle = 0$ for all $n = 1, 2, \ldots$. Then $A(x) = \sum_n s_n \langle x,\, u_n\rangle u_n = 0$. Hence $x \in Z(A) = F^{\perp}$. Since $x \in F \bigcap F^{\perp}$, we see that $x = 0$. By 22.7(v), the orthonormal set $\{u_1, u_2, \ldots\}$ is, in fact, an orthonormal basis for F.

Now $H = F + F^{\perp}$, where $F^{\perp} = Z(A)$. Hence, if $\{v_\alpha\}$ is an orthonormal basis for $Z(A)$, it follows that $\{u_1, u_2, \ldots\} \bigcup \{v_\alpha\}$ is an orthonormal basis for H, again by appealing to 22.7(v). □

The preceding theorem is a structure theorem for nonzero compact self-adjoint operators in the sense that it describes such an operator completely in terms of its nonzero eigenvalues and corresponding eigenvectors. [It should be compared with a similar result for finite dimensional operators given in 27.9(b).]

The proofs of 28.4 and 28.5 give a procedure for finding the eigenvalues and eigenvectors of a compact self-adjoint operator A. Let us recount it. Since

$$\|A\| = \sup\{|\langle A(x),\, x\rangle| : x \in H, \|x\| = 1\},$$

there is a sequence (x_n) in H such that $\|x_n\| = 1$ and $\langle A(x_n),\, x_n\rangle$ tends to $\|A\|$ or to $-\|A\|$. Then $s_1 = \lim\limits_{n\to\infty} \langle A(x_n),\, x_n\rangle$ is an eigenvalue of A and if (x_{n_j}) is a convergent subsequence of (x_n), then $u_1 = \lim_{j\to\infty} x_{n_j}$ is a corresponding eigenvector of A. This process is repeated by restricting A to the orthogonal complement of span $\{u_1\}$. Having found nonzero eigenvalues $s_1, \ldots, s_n$ and corresponding eigenvectors $u_1, \ldots, u_n$ of A, we restrict A to the orthogonal complement of span $\{u_1, \ldots, u_n\}$, and look for additional nonzero eigenvalues. All nonzero eigenvalues and corresponding eigenvectors of A are found in this way.

28.6 Corollary

Let A be a nonzero self-adjoint Hilbert-Schmidt operator on H. If (s_n) is the sequence of the nonzero eigenvalues of A as given in 28.5(a), then $\sum_n |s_n|^2 < \infty$.

Proof :

By 28.2(a), A is a compact operator. Let

$$A(x) = \sum_n s_n \langle x,\, u_n\rangle u_n, \quad x \in H,$$

as given in 28.5(a). Since $A(u_n) = s_n u_n$, $n = 1, 2, \ldots$, we have

$$\sum_n |s_n|^2 = \sum_n \|A(u_n)\|^2.$$

Let $\{v_j\}$ be an orthonormal basis for H such that $\sum_j \|A(v_j)\|^2 < \infty$. As $\{u_1, u_2, \ldots\}$ is an orthonormal set in H, Bessel's inequality (22.4) shows that

$$\begin{aligned}\sum_n \|A(u_n)\|^2 &= \sum_n \sum_j |\langle A(u_n),\, v_j\rangle|^2 = \sum_j \sum_n |\langle u_n,\, A(v_j)\rangle|^2 \\ &\leq \sum_j \|A(v_j)\|^2 < \infty.\end{aligned}$$

Hence $\sum_n |s_n|^2 < \infty$. □

We shall now use Theorem 28.5 to obtain explicit solutions of some operator equations. Parts (a) and (b) of Theorem 28.7 give an explicit version of the **Fredholm alternative** considered in 19.1(a).

28.7 Theorem

Let A be a nonzero compact self-adjoint operator on a Hilbert space H over $\mathbf{K}$. Let A have the representation

$$A(x) = \sum_n s_n \langle x,\, u_n\rangle u_n, \quad x \in H,$$

as given in 28.5(a). Fix $\mu \in \mathbf{K}, \mu \neq 0$.

(a) If $\mu \notin \{1/s_1, 1/s_2, \ldots\}$, then for every $y \in H$, there is a unique $x \in H$ such that $x - \mu A(x) = y$. In fact,

$$x = y + \mu \sum_n \frac{s_n \langle y,\, u_n\rangle}{1 - \mu s_n} u_n.$$

Also, there is some $\alpha > 0$ (independent of y) such that $\|x\| \leq \alpha \|y\|$, so that the solution x depends continuously on the right side y.

(b) If $\mu \in \{1/s_1, 1/s_2, \ldots\}$ and $1/\mu$ is repeated exactly m times in the sequence (s_n), say $\mu = 1/s_{j_1}, \ldots, 1/s_{j_m}$, then for a given $y \in H$, there is some $x \in H$ such that $x - \mu A(x) = y$ if and only if y is orthogonal to $u_{j_1}, \ldots u_{j_m}$ and then

$$x = y + k_{j_1} u_{j_1} + \cdots + k_{j_m} u_{j_m} + \mu \sum_{n \neq j_1, \ldots, j_m} \frac{s_n \langle y,\, u_n\rangle}{1 - \mu s_n} u_n,$$

where $k_{j_1}, \ldots, k_{j_m}$ are scalars.

Proof:
If $x, y \in H$ satisfy $x - \mu A(x) = y$, then

$$x = y + z, \quad \text{where } z = \mu A(x) = \mu \sum_n s_n \langle x, u_n \rangle u_n,$$

so that $\langle x, u_n \rangle = \langle y, u_n \rangle + \mu s_n \langle x, u_n \rangle$, that is,

$$(1 - \mu s_n)\langle x, u_n \rangle = \langle y, u_n \rangle, \quad n = 1, 2, \ldots.$$

(a) Assume that $\mu \notin \{1/s_1, 1/s_2, \ldots\}$. Then $1/\mu \notin \sigma(A)$ by 28.5(b). Hence for every $y \in H$, there is a unique $x \in H$ such that $(A - \frac{I}{\mu})(x) = -\frac{y}{\mu}$, that is, $x - \mu A(x) = y$. In fact, since $1 - \mu s_n \neq 0$ for each n, we have $\langle x, u_n \rangle = \langle y, u_n \rangle / (1 - \mu s_n)$ and hence

$$z = \mu \sum_n \frac{s_n \langle y, u_n \rangle}{1 - \mu s_n} u_n.$$

Now there is some $\beta > 0$ such that

$$\left| \frac{\mu s_n}{1 - \mu s_n} \right| \leq \beta, \quad n = 1, 2, \ldots,$$

because $s_n \to 0$ whenever the sequence (s_n) is infinite. Hence Bessel's inequality (22.4) shows that

$$\|z\|^2 \leq \beta^2 \sum_n |\langle y, u_n \rangle|^2 \leq \beta^2 \|y\|^2$$

Therefore

$$\begin{aligned} \|x\|^2 &= \|y\|^2 + 2\mathrm{Re}\,\langle y, z \rangle + \|z\|^2 \\ &\leq \|y\|^2 + 2\beta\|y\|^2 + \beta^2\|y\|^2 = (1+\beta)^2\|y\|^2, \end{aligned}$$

that is, $\|x\| \leq (1 + \beta)\|y\|$, where $1 + \beta$ is independent of y. If $x_n - \mu A(x_n) = y_n$, then $x - x_n - \mu A(x - x_n) = y - y_n$ and $\|x - x_n\| \leq (1 + \beta)\|y - y_n\|$. Hence if $y_n \to y$ in H, then $x_n \to x$ in H, that is, the solution x depends continuously on the right side y.

(b) Assume that $\mu = 1/s_{j_1} = \cdots = 1/s_{j_m}$, and $\mu \neq 1/s_n$ for $n \notin \{j_1, \cdots, j_m\}$. Fix $y \in H$.

If $x \in H$ satisfies $x - \mu A(x) = y$ and if $n \in \{j_1, \ldots, j_m\}$, then $1 - \mu s_n = 0$, so that $\langle y, u_n \rangle = (1 - \mu s_n)\langle x, u_n \rangle = 0$, that is, y is orthogonal to $u_{j_1}, \ldots, u_{j_m}$.

Conversely, let $y \in H$ be orthogonal to $u_{j_1}, \ldots, u_{j_m}$. As in (a) above, there is some $\beta > 0$ such that $|\frac{\mu s_n}{1 - \mu s_n}| \leq \beta$ for each $n \neq j_1, \ldots, j_m$. Hence

$$\sum_{n \neq j_1, \ldots, j_m} |\frac{\mu s_n}{1 - \mu s_n}|^2 |\langle y, u_n \rangle|^2 \leq \beta^2 \sum_n |\langle y, u_n \rangle|^2 \leq \beta^2 \|y\|^2$$

and by the Riesz-Fischer theorem (22.5(a)),

$$\sum_{n \neq j_1, \ldots, j_m} \frac{\mu s_n \langle y, u_n \rangle}{1 - \mu s_n} u_n = z_0 \in H.$$

Let $x_0 = y + z_0$. Since $\langle y, u_{j_1} \rangle = \cdots = \langle y, u_{j_m} \rangle = 0$, we see that

$$\begin{aligned} A(x_0) &= A(y) + A(z_0) = \sum_n s_n \langle y, u_n \rangle u_n + \mu \sum_{n \neq j_1, \ldots, j_m} \frac{s_n \langle y, u_n \rangle}{1 - \mu s_n} s_n u_n \\ &= \sum_{n \neq j_1, \ldots, j_m} s_n \langle y, u_n \rangle \left(1 + \frac{\mu s_n}{1 - \mu s_n}\right) u_n \\ &= \sum_{n \neq j_1, \ldots, j_m} \frac{s_n \langle y, u_n \rangle}{1 - \mu s_n} u_n. \end{aligned}$$

Thus $\mu A(x_0) = z_0$ and hence $x_0 - \mu A(x_0) = x_0 - z_0 = y$, that is, x_0 is a solution of $x - \mu A(x) = y$. Let now x be any solution. If $n \neq j_1, \ldots, j_m$, then $1 - \mu s_n \neq 0$ and hence $\langle x, u_n \rangle = \langle y, u_n \rangle / (1 - \mu s_n)$, while if $n \in \{j_1, \ldots, j_m\}$, then $1 - \mu s_n = 0 = \langle y, u_n \rangle$, so that $\langle x, u_n \rangle$ remains arbitrary. Thus

$$\begin{aligned} x &= y + \mu A(x) \\ &= y + \mu \sum_n s_n \langle x, u_n \rangle u_n \\ &= y + k_{j_1} u_{j_1} + \cdots + k_{j_m} u_{j_m} + z_0 \end{aligned}$$

for some scalars $j_1, \ldots, j_m$. $\square$

28.8 Examples

(a) Let $H = \ell^2$. Consider $k_{i,j} \in \mathbf{K}$ for $i, j = 1, 2, \ldots$ such that $\overline{k_{j,i}} = k_{i,j}$ and $0 < \sum_j \sum_i |k_{i,j}|^2 < \infty$. We have seen in 28.3(a) that the matrix $M = (k_{i,j})$ defines a Hilbert-Schmidt operator A on H. Further, A is self-adjoint since $\overline{M}^t = M$ (26.1(b)). Also, $A \neq 0$ since $\langle A(e_j),\, e_i\rangle = k_{i,j} \neq 0$ for some i, j. Hence Theorems 28.5 and 28.7 are applicable. In particular, if the sequence (s_n) of nonzero eigenvalues of A is infinite, then $s_n \to 0$ as $n \to \infty$. In fact, 28.6 shows that $\sum_n |s_n|^2 < \infty$.

(b) Let $H = L^2([a, b])$ and $k(\cdot, \cdot)$ be a measurable function on $[a, b] \times [a, b]$ with $\int_a^b \int_a^b |k(s, t)|^2 \, dm(t)\, dm(s) < \infty$, $\overline{k(t, s)} = k(s, t)$ for all $s, t \in [a, b]$ and the set $\{(s, t) : k(s, t) \neq 0\}$ has positive Lebesgue measure. We have seen in 28.3(b) that the Fredholm integral operator A with kernel $k(\cdot, \cdot)$ is a Hilbert-Schmidt operator on H. Further, it can be seen that A is self-adjoint. (Compare Problem 26-5.) Also, the set $\{(s, t) : |k(s, t)| \geq 1/n\}$ has positive Lebesgue measure for some $n = 1, 2, \ldots$. Therefore

$$0 < \int_a^b \int_a^b |k(s, t)|^2 \, dm(t)\, dm(s) = \sum_j \|A(v_j)\|^2,$$

where $\{v_j\}$ is an orthonormal basis for $L^2([a, b])$. (See 28.3(b).) Thus $A \neq 0$. Hence Theorems 28.5 and 28.7 are applicable. Thus we obtain explicit solutions of the **Fredholm integral equation of the second kind**

$$x(s) - \mu \int_a^b k(s, t) x(t)\, dm(t) = y(s), \quad s \in [a, b],$$

If (s_n) is the sequence of nonzero eigenvalues of A, then $\sum_n |s_n|^2 < \infty$ by 28.6. Also, if (u_n) is the corresponding orthonormal sequence of eigenvectors of A, then

$$A(x) = \sum_n s_n \langle x,\, u_n\rangle u_n, \quad x \in H,$$

that is, $\|A(x) - A_m(x)\|_2 \to 0$, as $m \to \infty$, where

$$A_m(x) = \sum_{n=1}^{m} s_n \langle x, u_n \rangle u_n, \quad x \in H, \quad m = 1, 2 \ldots.$$

Let us ask whether for a given $s \in [a, b]$, we have $A_m(x)(s) \to A(x)(s)$ as $m \to \infty$. In this regard, we shall show that if the kernel $k(.,.)$ satisfies

$$\int_a^b |k(s,t)|^2 \, dm(t) \leq \alpha$$

for all $s \in [a, b]$ and some $\alpha > 0$, then $A_m(x)(s) \to A(x)(s)$ uniformly for $s \in [a, b]$ and the series $\sum_n s_n \langle x, u_n \rangle u_n(s)$ converges absolutely. First note that if (x_m) is a sequence in H and $\|x_m - x\|_2 \to 0$, then $A(x_m)(s) \to A(x)(s)$ uniformly for $s \in [a, b]$. This follows by observing that

$$|A(x)(s) - A(x_m)(s)|^2 = \left| \int_a^b k(s,t)[x(t) - x_m(t)] \, dt \right|^2,$$

which is less than or equal to

$$\left[\int_a^b |k(s,t)|^2 \, dm(t) \right] \left[\int_a^b |x(t) - x_m(t)|^2 \, dm(t) \right] \leq \alpha \|x - x_m\|^2.$$

Now let $x \in H$. If we let

$$z = x - \sum_n \langle x, u_n \rangle u_n,$$

then

$$A(z) = A(x) - \sum_n \langle x, u_n \rangle A(u_n) = A(x) - \sum_n s_n \langle x, u_n \rangle u_n = 0.$$

Thus

$$x = \sum_n \langle x, u_n \rangle u_n + z$$

where $A(z) = 0$. For $m = 1, 2, \ldots$, define

$$x_m = \sum_{n=1}^{m} \langle x, u_n \rangle u_n + z.$$

Then $\|x_m - x\|_2 \to 0$, so that $A(x_m)(s) \to A(x)(s)$ uniformly for $s \in [a, b]$. But

$$\begin{aligned} A(x_m)(s) &= \sum_{n=1}^{m} \langle x, u_n\rangle A(u_n)(s) + 0 \\ &= \sum_{n=1}^{n} \langle x, u_n\rangle s_n u_n(s) \\ &= A_m(x)(s). \end{aligned}$$

Hence $A_m(x)(s) \to A(x)(s)$ uniformly for $s \in [a, b]$. To conclude that the series $\sum_n s_n\langle x, u_n\rangle u_n(s)$ converges absolutely for each s in $[a, b]$, we show that every rearrangement of the series also converges. (See 3.54 of [51].) Consider a rearrangement $\sum_n s_{k_n}\langle x, u_{k_n}\rangle u_{k_n}$. Now the series $\sum_n \langle x, u_{k_n}\rangle u_{k_n}$ converges in H since $\sum_n |\langle x, u_{k_n}\rangle|^2 \le \|x\|^2 < \infty$. (See 22.5(a).) Hence the series $\sum_n s_{k_n}\langle x, u_{k_n}\rangle u_{k_n}(s)$ converges to $A(x)(s)$ for every $s \in [a, b]$, as before.

The result proved above implies that the convergence of the series in the solution of the Fredholm integral equation given in 28.8(b) is uniform and absolute if $\int_a^b |k(s,t)|^2\,dm(t) \le \alpha$ for all $s \in [a, b]$ and some $\alpha > 0$. We remark that this condition is satisfied if $k(.,.)$ is a bounded measurable function (in particular, a continuous function) on $[a, b]\times[a, b]$.

Let us consider a specific example. Let

$$k(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1 \\ s(1-t), & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Then $k(.,.)$ is a real-valued continuous function on $[0,1]\times[0,1]$ and $k(t,s) = k(s,t)$ for all $s, t \in [0,1]$. Also, $k(s,t) \neq 0$ for almost all $s, t \in [0,1]$. In 18.7(c) we have seen that the nonzero eigenvalues of the Fredholm integral operator A with this kernel are given by $s_n = 1/n^2\pi^2, n = 1, 2, \ldots$ and the eigenspace of A corresponding to the eigenvalue s_n equals span $\{u_n\}$, where $u_n(s) = \sqrt{2}\sin n\pi s, s \in [0,1]$. Hence for every $x \in L^2([0,1])$ and $s \in [0,1]$, we have

$$A(x)(s) = \sum_n s_n\langle x, u_n\rangle u_n(s),$$

that is,

$$(1-s)\int_0^s t\,x(t)\,dm(t) + s\int_s^1 (1-t)x(t)\,dm(t)$$
$$= \frac{2}{n^2\pi^2}\sum_{n=1}^{\infty}\left[\int_0^1 x(t)\sin n\pi t\,dm(t)\right]\sin n\pi s,$$

where the series converges absolutely and uniformly for $s \in [0,1]$.

For $0 \neq \mu \in \mathbf{K}$ and $y \in L^2([0,1])$, consider the integral equation

$$x(s) - \mu\left[(1-s)\int_0^s t\,x(t)\,dm(t) + s\int_s^1 (1-t)x(t)\,dm(t)\right] = y(s)$$

for $s \in [0,1]$.

(i) If $\mu \neq n^2\pi^2$ for any $n = 1, 2, \ldots$, then the unique solution of the integral equation is given by

$$\begin{aligned} x(s) &= y(s) + \sum_{n=1}^{\infty}\frac{\mu s_n}{1-\mu s_n}\langle y,\, u_n\rangle u_n(s) \\ &= y(s) + \sum_{n=1}^{\infty}\frac{2\mu}{n^2\pi^2-\mu}\left[\int_0^1 y(t)\sin n\pi t\,dm(t)\right]\sin n\pi s \end{aligned}$$

for almost all $s \in [0,1]$.

(ii) If $\mu = n_0^2\pi^2$ for some $n_0 = 1, 2, \ldots$, then the integral equation has a solution if and only if

$$\langle y,\, u_{n_0}\rangle = \sqrt{2}\int_0^1 y(t)\sin n_0\pi t\,dm(t) = 0.$$

In that case, the general solution is given by

$$x(s) = x_0(s) + c\sin n_0\pi s,$$

where $c \in \mathbf{K}$ and

$$x_0(s) = y(s) + \sum_{n\neq n_0}\frac{2n_0^2}{n^2-n_0^2}\left[\int_0^1 y(t)\sin n\pi t\,dm(t)\right]\sin n\pi s$$

for almost all $s \in [0,1]$. All the series in these solutions converge absolutely and uniformly on $[0,1]$. These explicit results can be contrasted with the existence and uniqueness results given in 19.5.

Of course, we have merely obtained the explicit form of a solution. To be able to compute such a solution exactly, we need to know the entire set of nonzero eigenvalues of the given self-adjoint compact operator A and also the corresponding eigenvectors. This is, in general, a formidable task. Note that in the solution

$$x = y + \mu \sum_n \frac{s_n \langle y,\, u_n\rangle}{1 - \mu s_n} u_n,$$

a major contribution comes from the terms $s_n\langle y,\, u_n\rangle/(1-\mu s_n)$ whose denominator is small, that is, when s_n is close to $1/\mu$ (but $s_n \neq 1/\mu$). Hence one can often ignore those eigenvalues of A which are far from $1/\mu$ and attempt to compute only the ones that are near $1/\mu$.

Finally, we mention that the well-known regular Sturm-Liouville problems in the theory of ordinary differential equations can be reduced to finding solutions of integral equations considered in the present section. This aspect is treated in Appendix C.

Problems

H denotes a Hilbert space over $\mathbf{K}$, unless otherwise stated.

28-1 Let A be a compact operator on H. If (u_n) is an infinite orthonormal sequence in H, then $A(u_n) \to 0$ as $n \to \infty$.

In particular, if a matrix $(k_{i,j})$ defines a compact operator on ℓ^2,

$$\gamma_j = \sum_{i=1}^{\infty} |k_{i,j}|^2 \quad \text{and} \quad \delta_i = \sum_{j=1}^{\infty} |k_{i,j}|^2,$$

then $\gamma_j \to 0$ as $j \to \infty$ and $\delta_i \to 0$ as $i \to \infty$.

28-2 The tridiagonal operator considered in Problem 25-7 is compact if and only if the sequences $(a_n), (b_n)$ and (c_n) tend to zero. In particular, a diagonal operator on ℓ^2 is compact if and only if the diagonal sequence tends to zero, and a weighted (left or right) shift operator on ℓ^2 is compact if and only if the sequence of weights tends to zero.(See Problem 17-10.)

28-3 Let A be a Fredholm integral operator on $L^2([0,1])$ with kernel $k(.,.)$ given by

$$k(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1 \\ s(1-t), & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Then $\|A\| = 1/\pi^2 = 0.10132\ldots$, which is less than $\|k(.,.)\|_2 = 1/3\sqrt{10} = 0.10541\ldots$.

28-4 Let A be a compact operator on an infinite dimensional Hilbert space H and let $p(s,t)$ be a polynomial in the variables s and t. Then $p(A, A^*)$ is a compact operator on H if and only if $p(0,0) = 0$.

28-5 Let $A \in BL(H)$. Then A is compact if and only if A^*A is compact. (Hint: If $(A^*A(x_n))$ is a Cauchy sequence in H, then so is $(A(x_n))$.)

28-6 Consider $\phi \in L^1([-\pi,\pi])$ and extend it to $\mathbf{R}$ so as to have period 2π. Let $H = L^2([-\pi,\pi])$ and $x \in H$. Define the **convolution**

$$(\phi * x)(s) = \int_{-\pi}^{\pi} \phi(s-t)x(t)\,dm(t), \quad -\pi \le s \le \pi.$$

Then $\|\phi * x\|_2 \le \|\phi\|_1 \|x\|_2$. If we let $A(x) = \phi * x$ for $x \in H$, then A is a compact operator on H. (Compare Problem 17-15.) If $\phi \in H$, then A is, in fact, Hilbert-Schmidt.

If $\mathbf{K} = \mathbf{C}$, then $\widehat{\phi * x}(n) = 2\pi\hat{\phi}(n)\hat{x}(n)$ for $n = 0, \pm1, \pm2, \ldots$ and if $e_n(s) = e^{ins}$ for $s \in [-\pi,\pi]$, then

$$A(x) = 2\pi \sum_{n=-\infty}^{\infty} \hat{\phi}(n)\hat{x}(n)e_n, \quad x \in H.$$

In particular, $k \in \mathbf{C}$ is an eigenvalue of A if and only if $k = 2\pi\hat{\phi}(n)$ for some n and then the corresponding eigenspace equals the closure of the span of $\{e_n : 2\pi\hat{\phi}(n) = k, n = 0, \pm1, \pm2, \ldots\}$. (Hint: 17.4(b), 28.3(b) and 22.8(b))

28-7 Let A be a compact self-adjoint operator on $H \ne \{0\}$. Then there is some $x_0 \in H$ with $\|x_0\| = 1$ such that

$$|\langle A(x_0), x_0\rangle| = \sup\{|\langle A(x), x\rangle| : x \in H, \|x\| = 1\}.$$

In fact, x_0 is an eigenvector of A of norm 1 corresponding to an eigenvalue of absolute value $\|A\|$. (Compare Problem 27-11(a).)

28-8 Let H be a Hilbert space over $\mathbf{K}$ and A be a compact operator on H. Let $\|\ \|_0$ be a norm on H and

$$\|A\|_0 = \sup\{\|A(x)\|_0 : x \in H, \|x\|_0 \le 1\}.$$

Then $\|A\| \le \|A\|_0^{1/2} \|A^*\|_0^{1/2}$. In particular, if $\|\ \|_0$ is a norm on $\mathbf{K}^n$ and M is an $n \times n$ matrix, then $\|M\|_2 \le \|M\|_0^{1/2} \|\overline{M}^t\|_0^{1/2}$. (Hint: First consider the case when A is self-adjoint and then consider $B = A^* A$.)

28-9 H is separable if and only if there is a compact self-adjoint operator A on H such that $Z(A)$ is separable.

28-10 Let A be a compact self-adjoint operator on H. Then A is a positive operator if and only if every eigenvalue of A is nonnegative.

28-11 Let A be the Fredholm integral operator on $L^2([0,1])$ with kernel $k(s,t) = \min\{s,t\}, 0 \le s,t \le 1$. For $x \in L^2([0,1])$ and $n = 1, 2, \ldots$, let

$$k_n(x) = \int_0^1 x(t) \sin(2n-1)\frac{\pi t}{2}\, dm(t).$$

Then for every $x \in L^2([0,1])$ and $s \in [0,1]$,

$$A(x)(s) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} k_n(x) \sin(2n-1)\frac{\pi s}{2}, \quad s \in [0,1].$$

Let $0 \ne \mu \in \mathbf{K}$ and $y \in L^2([0,1])$.

(a) If $\mu \ne (2n-1)^2\pi^2/4$ for any $n = 1, 2, \ldots$, then there is a unique $x \in L^2([0,1])$ satisfying $x - \mu A(x) = y$. In fact,

$$x(s) = y(s) + \sum_{n=1}^{\infty} \frac{8\mu k_n(y)}{(2n-1)^2\pi^2 - 4\mu} \sin(2n-1)\frac{\pi s}{2}, \quad s \in [0,1].$$

(b) If $\mu = (2n_0 - 1)^2\pi^2/4$ for some $n_0 = 1, 2, \ldots$, then there is x in $L^2([0,1])$ satisfying $x - \mu A(x) = y$ if and only if $k_{n_0}(y) = 0$, and in that case

$$x(s) = x_0(s) + y(s) + \frac{(2n_0-1)^2}{2} \sum_{n \ne n_0} \frac{k_n(y)}{(n-n_0)(n+n_0-1)} \sin(2n-1)\frac{\pi s}{2},$$

where $x_0(s) = c \sin(2n_0 - 1)\pi s/2$ for all $s \in [0,1]$ and some $c \in \mathbf{K}$.

All the series appearing above converge absolutely and uniformly for $s \in [0,1]$. (Hint: Problem 19-6, 28.8(b))

28-12 Let $k(.,.) \in L^2([a,b]\times[a,b])$ satisfy $\overline{k(t,s)} = k(s,t)$ for almost all $(s,t) \in [a,b]\times[a,b]$ and assume that the set $\{(s,t) : k(s,t) \neq 0\}$ has positive Lebesgue measure. Consider the Fredholm integral operator A on $L^2([a,b])$ with kernel $k(.,.)$. If (s_n) is the sequence of nonzero (repeated) eigenvalues of A and (u_n) is the corresponding orthonormal sequence of eigenvectors, then

$$k(s,t) = \sum_n s_n u_n(s)\overline{u_n(t)}, \quad s,t \in [0,1],$$

where the series on the right converges to $k(.,.)$ in $L^2([a,b]\times[a,b])$. In particular,

$$\int_a^b \int_a^b |k(s,t)|^2 \, dm(s)\, dm(t) = \sum_n |s_n|^2$$

and for $N = 1, 2, \ldots,$

$$\int_a^b \int_a^b |k(s,t) - \sum_{n=1}^{N} s_n u_n(s)\overline{u_n(t)}|^2 \, dm(s)\, dm(t) = \sum_{n=N+1}^{\infty} |s_n|^2.$$

Consequently, A has only a finite number of distinct nonzero eigenvalues if and only if the kernel $k(.,.)$ is degenerate, and in that case

$$k(s,t) = \sum_{n=1}^{N} s_n u_n(s)\overline{u_n(t)}, \quad s,t \in [0,1]$$

for some positive integer N.

28-13 Let $\mathbf{K} = \mathbf{C}$ and A be a compact normal operator on H. Then Theorem 28.5 holds if we replace 'real number(s)' by 'complex number(s)'. (Hint: Proof of 28.5, 27.2(a), (c), (d) and Problem 27-11(b))

28-14 Let A be a compact operator on H. Then there is an orthonormal basis for $Z(A)^\perp$ consisting of eigenvectors of A if and only if A is normal (resp., self-adjoint), provided $\mathbf{K} = \mathbf{C}$ (resp., $\mathbf{R}$). (Compare 27.9. Hint: 28.5, Problem 28-13)

28-15 Let A be a compact operator on H. Then there exist a finite or infinite nonincreasing sequence (α_n) of positive numbers and orthonormal sets $\{u_1, u_2, \ldots\}$ and $\{v_1, v_2, \ldots\}$ in H such that

$$A(x) = \sum_n \alpha_n \langle x, u_n \rangle v_n, \quad x \in H.$$

For $m = 1, 2, \ldots$, let $A_m(x) = \sum_{n=1}^{m} \alpha_n \langle x, u_n \rangle v_n, x \in H$. Then $\|A - A_m\| \leq \alpha_{m+1}$, which tends to 0 as $m \to \infty$ whenever the sequence (α_n) is infinite. For $x \in H$, we have

$$A^*(x) = \sum_n \alpha_n \langle x, v_n \rangle u_n, A^*A(x) = \sum_n \alpha_n^2 \langle x, u_n \rangle u_n, AA^*(x) = \sum_n \alpha_n^2 \langle x, v_n \rangle v_n.$$

The sequence (α_n) is called the sequence of (repeated) nonzero **singular values** of A. (Hint: Use 28.5(a) for A^*A.)

28-16 Let A be a compact operator on H and (α_n) be the sequence of its (repeated) nonzero singular values.

(a) If $\{v_1, v_2, \ldots\}$ is an orthonormal basis for H, then

$$\sum_n \alpha_n^2 = \sum_n \|A(v_n)\|^2.$$

In particular, A is Hilbert-Schmidt if and only if $\sum_n \alpha_n^2 < \infty$.

(b) A is said to be of **trace class** if $\sum_n \alpha_n < \infty$. If A is of trace class, then A is Hilbert-Schmidt, but not conversely. In fact, A is Hilbert-Schmidt if and only if A^*A is of trace class. If A and B are Hilbert-Schmidt, then AB is of trace class. [The converse also holds.]

28-17 Let $u_1, u_2, \ldots$ be a denumerable orthonormal basis for $H, (k_n)$ be a bounded sequence of scalars and

$$A(x) = \sum_n k_n \langle x, u_n \rangle u_n, \quad x \in H.$$

Then A is a compact if and only if $k_n \to 0$ as $n \to \infty$, A is Hilbert-Schmidt if and only if $\sum_n |k_n|^2 < \infty$ and A is of trace class if and only if $\sum_n |k_n| < \infty$. [As examples, consider $k_n = 1/\sqrt{n}, 1/n$ and $1/n^2$.]

28-18 Let A be a compact operator on H. If (x_n) is an infinite sequence of linearly independent elements in H and $A(x_n) = k_n x_n$ for some $k_n \in \mathbf{K}$, then $k_n \to 0$ as $n \to \infty$. As a consequence, A has only a countable number of distinct eigenvalues $k_1, k_2, \ldots$, and if they are infinite, then $k_n \to 0$ as $n \to \infty$. Also, for each nonzero eigenvalue k of A, the eigenspace $Z(A - kI)$ is finite dimensional. (Hint: 22.2, Problem 28-1, 18.4(a))

Appendix A

Fixed Points

A **fixed point** of a function F from a set S to itself is a point x in S such that $F(x) = x$. Fixed points arise naturally while solving operator equations. For example, let us consider the integral equation

$$x(s) = x_0(s) + \int_a^b f(s, u, x(u))\, du, \quad s \in [a, b],$$

where x_0 is a given continuous real-valued function on $[a, b]$. Let S denote the set of all real-valued continuous functions on $[a, b]$ and

$$F(x)(s) = x_0(s) + \int_a^b f(s, u, x(u))\, du, \quad x \in S,\ s \in [a, b].$$

If for each $s \in [a, b]$, the function $f(s, u, x(u))$ is an integrable function of u on $[a, b]$ and the function $\int_a^b f(s, u, x(u))\, du$ is a continuous function of s on $[a, b]$, then $F(x) \in S$. Clearly, x is a solution of this integral equation if and only if it is a fixed point of the function F.

We begin our study of fixed points with a well-known classical result about the existence and uniqueness of a fixed point of a **contraction**, that is, a function F from a metric space X to itself satisfying

$$d(F(x), F(y)) \leq r\, d(x, y)$$

for all x, y in X and some r with $0 \leq r < 1$. We shall then give an application of this result to the solution of an initial value problem by converting it to an integral equation of the type considered above.

A1 Theorem (Contraction mapping principle, Banach, 1922) Let X be a complete metric space and $F : X \to X$ such that for some positive integer k, the function F^k is a contraction. Then F has a unique fixed point in X.

Proof:
Consider any $x_0 \in X$ and define

$$x_n = F(x_{n-1}), \quad n = 1, 2, \ldots .$$

First assume that F is itself a contraction. We claim that (x_n) is a Cauchy sequence in X. Note that $x_n = F^n(x_0)$ for $n = 1, 2, \ldots$. If $1 \le m < n$, then

$$\begin{aligned} d(x_m, x_n) &= d(F^m(x_0), F^n(x_0)) \\ &\le r^m d(x_0, F^{n-m}(x_0)) \\ &\le r^m[d(x_0, F(x_0)) + d(F(x_0), F^2(x_0)) + \cdots \\ &\qquad + d(F^{n-m-1}(x_0), F^{n-m}(x_0))] \\ &\le r^m d(x_0, F(x_0))[1 + r + \cdots + r^{n-m-1}] \\ &\le \frac{r^m}{1-r} d(x_0, F(x_0)). \end{aligned}$$

Since $r < 1, r^m \to 0$ as $m \to \infty$. Hence (x_n) is a Cauchy sequence. As X is a complete metric space, let $x_n \to x$ in X. Then

$$F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = x,$$

because F is continuous. Thus x is a fixed point of F.

To show the uniqueness of a fixed point of F, let $F(y) = y$ for some $y \in X$. Then

$$d(x, y) = d(F(x), F(y)) \le r\, d(x, y).$$

Again since $r < 1$, we obtain $d(x, y) = 0$, that is, $y = x$.

Next assume that F^k is a contraction for some integer $k \ge 2$. By what we have just proved, the function $G = F^k$ has a unique fixed point x in X. Since $G(F(x)) = F(G(x)) = F(x)$, it follows that $F(x)$ is also a fixed point of G. The uniqueness of the fixed point of G shows that $F(x) = x$, that is, x is a fixed point of F itself. Also, if y is a fixed point of F, then clearly it is a fixed point of G. Again, the uniqueness of the fixed point of G shows that $y = x$. Thus F has a unique fixed point in X. $\square$

The remarkable thing about the contraction mapping principle is that the desired unique fixed point is obtained by starting with an arbitrary point x_0 and employing a simple iterative process of applying the given contraction mapping F. If x is the unique fixed point, then the successive approximations $x_n = F^n(x_0)$ satisfy

$$d(x, x_n) = d(F(x), F(x_{n-1})) \leq r\, d(x, x_{n-1}) \leq \cdots \leq r^n d(x, x_0).$$

Since

$$\begin{aligned} d(x, x_n) &\leq d(x, x_{n+1}) + d(x_{n+1}, x_n) \\ &= d(F(x), F(x_n)) + d(x_{n+1}, x_n) \\ &\leq r\, d(x, x_n) + d(x_{n+1}, x_n), \end{aligned}$$

it follows that $d(x, x_n) \leq \dfrac{d(x_{n+1}, x_n)}{1 - r}$. Thus if we want $d(x, x_n) < \epsilon$, we only need to make sure that $d(x_{n+1}, x_n) < \epsilon(1 - r)$.

Suppose that X is a complete metric space and $F : X \to X$ such that for some positive integer k, $d(F^k(x), F^k(y)) < d(x, y)$ for all x and y in X. Then F^k may not be a contraction and F may not have a fixed point in X, as the example $F(x) = x + 1/x$ for $x \in [1, \infty)$ shows (with $k = 1$). However, it follows from the proof of Theorem A1 that such a function can have at most one fixed point.

We now give an application of the contraction mapping principle.

Theorem A2 (Picard, 1890)
Let $a > 0, b > 0, (s_0, t_0) \in \mathbf{R}^2$ and $E = [s_0 - a, s_0 + a] \times [t_0 - b, t_0 + b]$. Let $f : E \to \mathbf{R}$ be continuous and $|f(s, t)| \leq \alpha$ for all $(s, t) \in E$. Let $\delta = \min\{a, b/\alpha\}$. Then there is a unique continuously differentiable function x on $[s_0 - \delta, s_0 + \delta]$ such that for all $s \in [s_0 - \delta, s_0 + \delta]$,

$$x'(s) = f(s, x(s)) \quad \text{and} \quad x(s_0) = t_0,$$

provided f satisfies a **Lipschitz condition** in the second variable:

$$|f(s, t_1) - f(s, t_2)| \leq L|t_1 - t_2|$$

for all (s,t_1) and (s,t_2) in E and some $L \geq 0$.

Proof:
Note that for $s \in [s_0-\delta, s_0+\delta]$, $x'(s) = f(s,x(s))$ and $x(s_0) = t_0$ if and only if

$$x(s) = t_0 + \int_{s_0}^{s} f(u, x(u))\,du$$

by the fundamental theorem for Riemann integration (4.2). Consider

$$S = \{x \in C([s_0-\delta, s_0+\delta]) : |x(s) - t_0| \leq b \text{ for all } s \in [s_0-\delta, s_0+\delta]\}$$

and define

$$F(x)(s) = t_0 + \int_{s_0}^{s} f(u, x(u))\,du, \quad x \in S, s \in [s_0-\delta, s_0+\delta].$$

Since for all $x \in C([a,b])$ and $s \in [s_0-\delta, s_0+\delta]$,

$$|F(x)(s) - t_0| \leq \int_{s_0}^{s} |f(u,x(u))|\,du \leq \alpha|s-s_0| \leq \alpha\delta \leq b,$$

it follows that F maps S into S. Note that S is a closed subset of $C([s_0-\delta, s_0+\delta])$ and hence it is complete in the metric $d(x,y) = \|x-y\|_\infty$. Also, for all $x, y \in S$ and $s \in [s_0-\delta, s_0+\delta]$, we have

$$\begin{aligned}
|F(x)(s) - F(y)(s)| &= \left|\int_{s_0}^{s} [f(u,x(u)) - f(u,y(u))]\,du\right| \\
&\leq \left|\int_{s_0}^{s} L|x(u)-y(u)|\,du\right| \\
&\leq L|s-s_0|\,\|x-y\|_\infty,
\end{aligned}$$

so that $\|F(x)-F(y)\|_\infty \leq L\delta\|x-y\|_\infty$. If $L\delta < 1$, then F is itself a contraction. Otherwise, consider the map F^2. Now for all $x, y \in S$ and all $s \in [s_0-\delta, s_0+\delta]$, we have

$$\begin{aligned}
|F^2(x)(s) - F^2(y)(s)| &= \left|\int_{s_0}^{s} [f(u,F(x)(u)) - f(u,F(y)(u))]\,du\right| \\
&\leq \left|\int_{s_0}^{s} L|F(x)(u) - F(y)(u)|\,du\right| \\
&\leq L\left|\int_{s_0}^{s} |u-s_0|\,\|x-y\|_\infty\,du\right| \\
&= L\frac{|s-s_0|^2}{2}\|x-y\|_\infty,
\end{aligned}$$

so that $\|F^2(x) - F^2(y)\|_\infty \le \dfrac{(L\delta)^2}{2}\|x - y\|_\infty$.

By induction, it can be seen that for $n = 1, 2, \ldots,$

$$\|F^n(x) - F^n(y)\|_\infty \le \frac{(L\delta)^n}{n!}\|x - y\|_\infty$$

for all $x, y \in S$. Since $(L\delta)^n/n! \to 0$ as $n \to \infty$, choose k so that $(L\delta)^k/k! < 1$, so that F^k is a contraction. By A1 Theorem, F has a unique fixed point x in S, that is, there is a unique $x \in C([s_0-\delta, s_0+\delta])$ such that

$$x(s) = F(x)(s) = t_0 + \int_{s_0}^{s} f(u, x(u))\, du,$$

for all $s \in [s_0 - \delta, s_0 + \delta]$. Hence there is a unique continuously differentiable function x on $[s_0 - \delta, s_0 + \delta]$ such that $x'(s) = f(s, x(s))$ for all $s \in [s_0 - \delta, s_0 + \delta]$ and $x(s_0) = t_0$. □

The function f in Theorem A2 was required to satisfy a Lipschitz condition because we invoked the contraction mapping principle. We shall now investigate fixed points of continuous functions which may not be contractions. This will enable us, in particular, to drop the requirement of a Lipschitz condition if we are interested merely in the existence (and not the uniqueness) of a solution of the initial value problem.

We restrict ourselves to continuous functions defined on subsets of a normed space. To begin with a very simple example, we observe that every continuous function F from $[-1, 1]$ to itself has a fixed point. This follows by noting that the function $G(t) = t - F(t), t \in [-1, 1]$, satisfies $G(-1) \le 0 \le G(1)$ and the intermediate value property of the continuous function G implies that $G(t) = 0$ (that is, $F(t) = t$) for some $t \in [-1, 1]$. Which properties of the set $[-1, 1]$ are used in this result ?

(i) The set $[-1, 1]$ cannot be replaced by the set $[-1, 1)$, as the example $F(t) = (t + 1)/2, -1 \le t < 1$, shows. Note that $[-1, 1]$ is **closed**, but $[-1, 1)$ is not.

(ii) The set $[-1, 1]$ cannot be replaced by the set $[-1, \infty)$, as the example $F(t) = t + 1, -1 \leq t < \infty$, shows. Note that $[-1, 1]$ is **bounded**, but $[-1, \infty)$ is not.

(iii) The set $[-1, 1]$ cannot be replaced by the set $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$, as the example $F(t) = -t, \frac{1}{2} \leq |t| \leq 1$, shows. Note that the set $[-1, 1]$ is **convex**, but $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ is not.

We are thus led to consider nonempty closed bounded convex subsets of a normed space. To tackle the finite dimensional case, the following well-known result is useful.

Theorem A3 (Brouwer, 1910)

Let $n \geq 1$. Every continuous function from the closed unit ball $\{(x(1), \ldots, x(n)) \in \mathbf{R}^n : x(1)^2 + \cdots + x(n)^2 \leq 1\}$ of $\mathbf{R}^n$ to itself has a fixed point.

We have just proved this result when $n = 1$. A simple but slightly involved proof for the case $n = 2$ can be found in [14], pp. 251-255. All known proofs for the cases $n \geq 3$ are far from trivial. Most of them are based on techniques of algebraic topology. An elementary proof is given in [33], pp. 38-40, while an analytic proof can be found in [19], pp. 467-470.

Corollary A4

Let X be a finite dimensional normed space over $\mathbf{K}$ and S be a nonempty closed bounded convex subset of X. Then every continuous function from S to itself has a fixed point.

Proof:

By 6.3(b), there is a $\mathbf{R}$-linear homeomorphism from X onto $\mathbf{R}^n$ (with the norm $\| \ \|_2$) for some positive integer n. (Note that $\mathbf{C}^m$ is $\mathbf{R}$-linearly homeomorphic to $\mathbf{R}^{2m}$.) Hence we let $X = \mathbf{R}^n$ (with the norm $\| \ \|_2$). Since S is bounded, we can assume that it is a subset of the closed unit ball U of $\mathbf{R}^n$, without loss of generality.

For every $x \in \mathbf{R}^n$, there is a unique $G(x) \in S$ such that $\|x - G(x)\|_2 = \operatorname{dist}(x, S)$. [See, for example, 23.5.] Also, the function $G : X \to S$ is continuous. (In fact, $\|G(x) - G(y)\|_2 \le \|x - y\|_2$ for all $x, y \in X$. See, for example, Problem 23-14.) Also, note that if $x \in S$, then $G(x) = x$.

Let $F : S \to S$ be a continuous function. By Theorem A3, the continuous function $F \circ G : U \to S \subset U$ has a fixed point $x \in U$. Since $x = F(G(x)) \in S$, we have $G(x) = x$. Thus x is a fixed point of F. □

This result does not hold, in general, for infinite dimensional normed spaces. For example, let $X = \ell^2$ and S denote the closed unit ball of X. For $x = (x(1), x(2), \ldots) \in S$, let

$$F(x) = (1 - \|x\|_2, x(1), x(2), \ldots).$$

Then $\|F(x)\|_2^2 = (1 - \|x\|_2)^2 + \|x\|_2^2 \le 1 - \|x\|_2 + \|x\|_2 = 1$ for all $x \in S$, so that $F(x) \in S$. Also, F is easily seen to be a continuous function. However, $F(x) = x$ for some $x \in S$ implies that

$$x = (1 - \|x\|_2, 1 - \|x\|_2, \ldots),$$

which is impossible. Thus the continuous function F from the closed bounded convex subset S of ℓ^2 to itself has no fixed point. Note, however, that every continuous function F from the **Hilbert cube** $C = \{x \in \ell^2 : |x(j)| \le 1/j, j = 1, 2, \ldots\}$ to itself has a fixed point. To see this, let for $n = 1, 2, \ldots,$

$$C_n = \{x \in C : x(j) = 0 \text{ for } j \ge n + 1\}.$$

Then C_n is homeomorphic to a nonempty closed bounded convex subset of $\mathbf{R}^n$. Let $P_n(x) = (x(1), \ldots, x(n), 0, 0, \ldots)$ for $x \in C$. Then $P_n \circ F$ is a continuous function from C_n to C_n. By Corollary A4, there is some $x_n \in C_n$ such that $P_n \circ F(x_n) = x_n$. Since C is compact (Problem 5-15(b)), there is a subsequence (x_{n_k}) such that $x_{n_k} \to x$ in C.

Now

$$\|F(x_n) - P_n \circ F(x_n)\|_2^2 \leq \sum_{j=n+1}^{\infty} \frac{1}{j^2} \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$F(x) = \lim_{k\to\infty} F(x_{n_k}) = \lim_{k\to\infty} P_{n_k} \circ F(x_{n_k}) = \lim_{k\to\infty} x_{n_k} = x.$$

This example indicates that for proving fixed point results in infinite dimensional normed spaces, we ought to consider **compact** convex sets instead of closed bounded convex sets. The first success in this direction was achieved by Birkhoff and Kellog in 1922, when they proved that every continuous function from a nonempty compact convex subset of $C([0,1])$ or $L^2([0,1])$ to itself has a fixed point. (See [3].) This result was generalized later by Schauder to arbitrary normed spaces. We remark, however, that in infinite dimensional normed spaces, compact sets are hard to come by. Hence one needs to modify these results from the point of view of applications.

Theorem A5 (Schauder, 1930)
Let S be a nonempty closed convex subset of a normed space X. Then every continuous function from S into a compact subset of S has a fixed point.

We refer the reader to [29], pp. 101-102 for a proof which reduces the problem to the case where X is a separable strictly convex normed space and S is bounded. (Compare Problem 7-18(b).) The proof then employs Corollary A4 in a crucial manner. While these arguments are within the scope of this book, we refrain ourselves from giving them here.

As an application of Theorem A5, we note that the existence part of Theorem A2 remains valid without the requirement of a Lipschitz condition on f. To see this we merely observe that the set S considered in the proof of Theorem A2 is a nonempty closed convex subset of the normed space $C([s_0 - \delta, s + \delta])$ and that F is a continuous function from S to a compact subset of S by Ascoli's theorem (3.10(a)). Hence a fixed point of F, guaranteed by Theorem A5, gives a solution of the initial value problem. This result is known as **Peano's theorem**.

Here is another application of Theorem A5.

Theorem A6

Let X be a normed space and $F : X \to X$ be a continuous function which maps each bounded subset of X into a compact subset of X. Let

$$\limsup_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|} < 1.$$

Then for every $y \in X$, there is some $x \in X$ such that $x - F(x) = y$.

Proof:

Let $y \in X$ and $G(x) = F(x) + y$ for $x \in X$. Then $G : X \to X$ is a continuous function which maps each bounded subset of X into a compact subset of X. Also,

$$\limsup_{\|x\| \to \infty} \frac{\|G(x)\|}{\|x\|} \le \limsup_{\|x\| \to \infty} \left(\frac{\|F(x)\|}{\|x\|} + \frac{\|y\|}{\|x\|} \right) = \limsup_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|} < 1.$$

For $n = 1, 2, \ldots$, let $S_n = \{x \in X : \|x\| \le n\}$. We claim that $G(S_n)$ is contained in S_n for some n. If not, there is some $x_n \in S_n$ with $\|G(x_n)\| > n$ for each $n = 1, 2, \ldots$. But then $\|x_n\| \to \infty$, since otherwise G will map the bounded set $\{x_1, x_2, \ldots\}$ into a compact (and hence bounded) set. Since $\|G(x_n)\| > n \ge \|x_n\|$, we must have

$$\limsup_{\|x\| \to \infty} \frac{\|G(x)\|}{\|x\|} \ge \limsup_{n \to \infty} \frac{\|G(x_n)\|}{\|x_n\|} \ge 1,$$

which is a contradiction. Thus $G(S_n) \subset S_n$ for some n. Now S_n is a nonempty closed bounded convex subset of X and G maps S_n into a compact subset of S_n. By Theorem A5, there is some $x \in S_n$ with $G(x) = x$, that is, $x - F(x) = y$. $\square$

Recall that if X is a **complete** normed space and $F : X \to X$ is a continuous **linear** map with $\|F^p\| < 1$ for some positive integer p, then for every $y \in X$, there is a unique $x \in X$ with $x - F(x) = y$ by 12.3. Note that Theorem A6 does not require either X to be complete or F to be linear.

Schauder's result can be generalized to set-valued functions as follows.

Theorem A7 (Fan-Glicksberg, 1952)
Let S be a nonempty compact convex subset of a normed space X. For each $x \in S$, let $F(x)$ denote a nonempty closed convex subset of S. Assume that if $x \in S$ and V is an open subset of S containing $F(x)$, then there is an open subset U of S containing x such that $F(y) \subset V$ for all $y \in U$. Then there is some $x \in S$ with $x \in F(x)$.

For a proof of this result, see [22,24]. Set-valued functions arise in many practical situations such as differential games, control of various systems in engineering and economics. Fixed point results for such functions then turn out be useful.

Common fixed points

Consider a family $\mathcal{F}$ of functions from a set S to itself. Even if each $F \in \mathcal{F}$ has a fixed point in S, no point of S may be a fixed point for all $F \in \mathcal{F}$. Example: Let $S = [-1, 1]$, $F(t) = (t+1)/2, G(t) = (t-1)/2$ for $t \in [-1, 1]$. Since 1 is the only fixed point of F and -1 is the only fixed point of G, it follows that F and G do not have a common fixed point. Thus we are led to look for conditions on the family $\mathcal{F}$ which will guarantee a common fixed point of all members of $\mathcal{F}$. One such condition is that $\mathcal{F}$ be a **commuting family**, that is, $F \circ G = G \circ F$ for all F, G in $\mathcal{F}$. In 1967, Boyce and Huneke independently found functions F and G from $[-1,1]$ to itself which commute but do not have a common fixed point. (See [5] and [31].) Thus we need to consider special kinds of commuting continuous functions.

Let S be a subset of a linear space X. A function $F : S \to X$ is said to be **affine** if $F(rx+(1-r)y) = rF(x)+(1-r)F(y)$ whenever $x, y \in S, 0 < r < 1$ and $rx+(1-r)y \in S$.

Theorem A8 (Markov, 1936 and Kakutani, 1938)
Let S be a nonempty compact convex subset of a normed space X. Then every commuting family of continuous affine functions from S to itself has a common fixed point.

Proof:
Let $\mathcal{F}$ be such a family, $F \in \mathcal{F}$ and $n = 1, 2, \ldots$. Consider

$$F_n = \frac{I + F + \cdots + F^{n-1}}{n},$$

where I denotes the identity function from S to itself. Since S is a convex set, we see that $F_n(S) \subset S$. Since S is compact and F is continuous, it follows that $F_n(S)$ is compact, and hence it is closed in S. We claim that any finite subcollection of the collection $\{F_n(S) : F \in \mathcal{F}, n = 1, 2, \ldots\}$ has a nonempty intersection. Let $F, G \in \mathcal{F}$ and $m, n = 1, 2, \ldots$. For $x \in S$, we have

$$F_n(G_m(x)) \in F_n(S) \quad \text{and} \quad G_m(F_n(x)) \in G_n(S).$$

Since $F \circ G = G \circ F$, we see that $F_n(G_m(x)) = G_m(F_n(x))$ for all $x \in S$. Thus $F_n \circ G_m(S) \subset F_n(S) \cap G_m(S)$. Hence our claim is justified. Since $\{F_n(S) : F \in \mathcal{F}, n = 1, 2, \ldots\}$ is a collection of closed subsets of the compact set S having the finite intersection property, there is some $x \in S$ such that $x \in F_n(S)$ for all $F \in \mathcal{F}$ and $n = 1, 2, \ldots$, as we have noted in Section 4.

We show that $F(x) = x$ for all $F \in \mathcal{F}$. Since $x \in F_n(S)$, there is some $x_n \in S$ such that

$$x = F_n(x_n) = \frac{1}{n}\left[x_n + F(x_n) + \cdots + F^{n-1}(x_n)\right].$$

Since F is an affine function,

$$F(x) = \frac{1}{n}\left[F(x_n) + F^2(x_n) + \cdots + F^n(x_n)\right].$$

As S is a bounded set, let $\|x\| \le \alpha$ for all $x \in S$ and some $\alpha > 0$. Then

$$\|F(x) - x\| = \frac{1}{n}\|F^n(x_n) - x_n\| \le \frac{2\alpha}{n},$$

which tends to zero as $n \to \infty$. Hence $F(x) = x$. □

Finally, we consider a commuting family of continuous functions F on a subset S of a normed space X which are **nonexpansive**, that is, $\|F(x) - F(y)\| \le \|x - y\|$ for all x, y in S. Note that each contraction is nonexpansive, while an isometry is nonexpansive but not contractive.

Theorem A9 (Browder, 1965)
Let X be a uniformly convex Banach space and S be a nonempty closed bounded convex subset of X. Then every commuting family of nonexpansive functions from S to itself has a common fixed point.

We refer the reader to [7] for a proof, which is beyond the scope of this book as it involves the concept of 'weak compactness'. We note that uniform convexity cannot be dropped from Browder's theorem even when the commuting family of nonexpansive functions consists of a single function. For example, let $X = c_0$, the linear space of all scalar sequences which tend to 0 (with the norm $\| \ \|_\infty$) and let S denote the closed unit ball of c_0. For $x = (x(1), x(2), \ldots) \in S$, let

$$F(x) = (1, x(1), x(2), \ldots).$$

Then $\|F(x) - F(y)\| = \|x - y\|$ for all $x, y \in S$, so that F is a nonexpansive function from S to itself. (It is even affine !) However, $F(x) = x$ for some $x \in S$ will imply that $x = (1, 1, \ldots)$, which does not belong to c_0! Whether the condition of uniform convexity of a Banach space can be weakened to the strict convexity and reflexivity of a normed space is an open question.

Since Browder's theorem does not require the functions to be contractive (unlike A1 Theorem) or affine (unlike Theorem A8) and since the ranges of the functions need not be contained in compact sets (unlike Theorem A5), it has proved to be of remarkable value in applications. In [6] Browder himself had employed a special case of the theorem to show the existence of periodic solutions of a very general class of nonlinear equations of evolution in infinite dimensional spaces.

Appendix B

Extreme Points

Given a real-valued function h defined on a set S, one is often interested in finding points of S at which h assumes its supremum or infimum. As this problem is very difficult in general, we consider a special situation. Let S be a subset of a linear space and h be an **affine** function on S, that is, $h(rx + (1-r)y) = rh(x) + (1-r)h(y)$ whenever $x, y \in S$, $0 < r < 1$ and $rx + (1-r)y \in S$. It is easy to see that if h assumes its supremum on S at $rx + (1-r)y$ for some $x, y \in S$ and $0 < r < 1$, then h assumes its supremum on S at x as well as at y. A similar statement holds for the infimum of h. This peculiar property turns out to be crucial in the search for points at which a real-valued function assumes its extreme values.

Let S be a subset of a linear space X over $\mathbf{K}$ and $E \subset S$. We say that E is an S-**extremal subset** of X or E is S-**extreme** in X when the following holds: If $x, y \in S$ and $rx + (1-r)y \in E$ for some r satisfying $0 < r < 1$, then $x, y \in E$, that is, no point of E lies on an open segment having its end points in the complement of E. If an S-extreme set consists of a single point, then that point is called an **extreme point** of S. Let the set of all extreme points of S be denoted by $\text{ext}\, S$.

If $x \in S$ and the complement of $\{x\}$ in S is convex, it is clear that $x \in \text{ext}\, S$. Conversely, if S is convex and $x \in \text{ext}\, S$, then it is easy to see that the complement of $\{x\}$ in S is convex.

We remark that if X is a linear space over $\mathbf{C}$ and $X_{\mathbf{R}}$ denotes the set X regarded as a linear space over $\mathbf{R}$, then for $E \subset S \subset X$, E is S-extreme in X if and only if E is S-extreme in $X_{\mathbf{R}}$.

If $E \subset T \subset S$, T is S-extreme and E is T-extreme, then it follows that E is S-extreme. In particular, an extreme point of an S-extreme set is an extreme point of S itself. Thus to determine extreme points of S, it is natural to seek S-extreme sets. We now give a general construction of S-extreme sets.

A real-valued function h on a subset S of a linear space X is said to be **convex** on S if

$$h(rx + (1 - r)y) \leq rh(x) + (1 - r)h(y)$$

whenever $x, y \in S, 0 < r < 1$ and $rx+(1-r)y \in S$. Also, h is said to be **concave** on S if the inequality $\leq$ above is replaced by the inequality $\geq$. Note that a real-valued function h on S is affine if and only if it is convex as well as concave.

Theorem B1

Let h be a real-valued function on a subset S of a linear space X and

$$E^h = \{x \in S : h(x) = \sup_{y \in S} h(y)\}, \quad E_h = \{x \in S : h(x) = \inf_{y \in S} h(y)\}.$$

If h is a convex function, then E^h is an S-extreme set and if h is a concave function, then E_h is an S-extreme set.

Proof:

Let h be convex on S. Consider $x, y \in S$ and $0 < r < 1$ such that $rx + (1 - r)y \in E^h$. Then $h(rx + (1 - r)y) = \sup_{z \in S} h(z) = \alpha$, say. Were $h(x) < \alpha$ or $h(y) < \alpha$, we would have

$$\alpha = h(rx + (1 - r)y) \leq rh(x) + (1 - r)h(y) < r\alpha + (1 - r)\alpha = \alpha,$$

which is impossible. Hence $h(x) = \alpha = h(y)$, that is, $x \in E^h$. Thus E^h is an S-extreme set.

Similarly, if h is concave on S, then E_h is an S-extreme set. $\square$

We give some simple examples of extreme points and extreme sets. Let S be a subset of $\mathbf{R}$. If S is bounded above, then the least upper bound of S is an extreme point of S. Similarly, if S is bounded below, then the greatest lower bound of S is an extreme point of S. Next, let S denote a closed triangular region in $\mathbf{R}^2$ with vertices x, y and z. If $[a,b]$ denotes the segment $\{ta+(1-t)b : 0 \le t \le 1\}$ from a to b, then the S-extreme sets are

$$\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}, [x,y],$$
$$[x,y]\cup[y,z], [x,y]\cup[z,x], [y,z], [y,z]\cup[z,x], [x,y]\cup[y,z]\cup[z,x], S.$$

Now let $S = \{x \in \mathbf{R}^2 : \|x\|_p \le 1\}$, with $1 \le p \le \infty$. (See Figure 2 in Section 5.) If $p = 1$, then the extreme points of S are $(1,0), (-1,0), (0,1)$ and $(0,-1)$. If $p = \infty$, then the extreme points of S are $(1,1), (1,-1), (-1,1)$ and $(-1,-1)$. On the other hand, if $1 < p < \infty$, then every $x \in \mathbf{R}^2$ with $\|x\|_p = 1$ is an extreme point of S. This is a special case of the following result.

Theorem B2

Let X be a normed space and S denote the closed unit ball in X. Then $\operatorname{ext} S = \{x \in X : \|x\| = 1\}$ if and only if X is strictly convex.

Proof:

Let $\operatorname{ext} S = \{x \in X : \|x\| = 1\}$. Consider $x, y \in X$ such that $x \ne y$ and $\|x\| = 1 = \|y\|$. Were $\|(x+y)/2\| = 1$, $(x+y)/2$ will be an extreme point of S. But this is not possible since $x \ne y$. Hence $\|(x+y)/2\| < 1$. Thus X is strictly convex.

Conversely, assume that X is strictly convex. It is clear that $\operatorname{ext} S \subset \{x \in X : \|x\| = 1\}$. To prove the other inclusion, let $x \in X$ with $\|x\| = 1$. Consider $y, z \in S$ with $x = ry + (1-r)z$ for some r satisfying $0 < r < 1$. It follows that $\|y\| = 1 = \|z\|$. If $r = 1/2$, then the strict convexity of X shows that $y = z = x$. Let $0 < r < 1/2$. Were $\|y+z\| < 2$, we would have

$$\begin{aligned} 1 = \|x\| = \|ry + (1-r)z\| &= \|r(y+z) + (1-2r)z\| \\ &\le r\|y+z\| + (1-2r)\|z\| \\ &< 2r + (1-2r) = 1, \end{aligned}$$

which is impossible. Hence $\|y+z\| = 2$. Again, the strict convexity of X shows that $y = z = x$. The case $1/2 < r < 1$ is similar. (Compare Problem 5-1.) Hence $x \in \operatorname{ext} S$. □

Corollary B3

Let n be a positive integer and S be a nonempty closed bounded subset of $\mathbf{R}^n$ with a given norm. Then S has an extreme point.

Proof:

Since all norms on $\mathbf{R}^n$ are equivalent to each other by 6.3(b), we can assume that S is closed and bounded with respect to the Euclidean norm $\| \ \|_2$. Let $r = \sup\{\|x\|_2 : x \in S\}$. Since the function $x \mapsto \|x\|_2$ is continuous on the compact subset S of $\mathbf{R}^n$, there is some $x_0 \in S$ such that $\|x_0\|_2 = r$. Let

$$S_r = \{x \in \mathbf{R}^n : \|x\|_2 \leq r\}.$$

Since $\mathbf{R}^n$ with the norm $\| \ \|_2$ is strictly convex, we see from Theorem B2 that $x_0 \in \operatorname{ext} S_r$. As $x_0 \in S$ and $S \subset S_r$, we have $x_0 \in \operatorname{ext} S$. □

Let X be a linear space. If $x_1, \ldots, x_k \in X$, $r_1 \geq 0, \ldots, r_k \geq 0$ and $r_1 + \cdots + r_k = 1$, then $r_1x_1 + \cdots + r_kx_k$ is called a **convex combination** of $x_1, \ldots, x_k$. Recalling the definition of the convex hull $\operatorname{co}(E)$ of $E \subset X$ given in Section 2, we note that $\operatorname{co}(E)$ is, in fact, the set of all convex combinations of points in E. We shall now consider a nonempty closed bounded convex subset S of $\mathbf{R}^n$ and show that each point of S is a convex combination of at most $n+1$ points in $\operatorname{ext} S$. For this purpose, we need the following result which is of independent interest as well.

Lemma B4

Let n be a positive integer and S be a nonempty convex subset of $\mathbf{R}^n$. Then either S is contained in a hyperplane of $\mathbf{R}^n$ or $S^o \neq \emptyset$.

Proof:

We assume without loss of generality that $0 \in S$. If $S = \{0\}$, then we are through. If $S \neq \{0\}$, that is, there is some nonzero x_1 in S, then either $S \subset \operatorname{span}\{x_1\}$, or there is some $x_2 \in S$ such that $x_2 \notin \operatorname{span}\{x_1\}$. Continuing in this fashion, we find linearly independent elements $x_1, \ldots, x_m$ in S such that $S \subset \operatorname{span}\{x_1, \ldots, x_m\}$ for some positive integer $m \leq n$. If $m \leq n-1$, then it is clear that S is contained in a hyperplane of $\mathbf{R}^n$. Now suppose that $m = n$. For $x = (r_1, \ldots, r_n) \in \mathbf{R}^n$, define

$$F(x) = r_1 x_1 + \cdots + r_n x_n.$$

Then F is a linear map from $\mathbf{R}^n$ to $\mathbf{R}^n$. Since $\{x_1, \ldots, x_n\}$ is a (Hamel) basis for $\mathbf{R}^n$ (2.3), F is surjective. Hence F sends each open subset of $\mathbf{R}^n$ onto an open subset of $\mathbf{R}^n$. (See 10.6 or Problem 10-20.) Consider

$$E = \{(r_1, \ldots, r_n) \in \mathbf{R}^n : r_1, \ldots, r_n > 0 \text{ and } r_1 + \cdots + r_n < 1\}.$$

Since E is an open subset of $\mathbf{R}^n$, $F(E)$ is open in $\mathbf{R}^n$ and since S is a convex subset of $\mathbf{R}^n$, $F(E) \subset S$. Thus $F(E) \subset S^o$, the interior of S. In particular, $(x_1 + \ldots + x_n)/n$ belongs to S^o. □

Theorem B5 (Minkowski, 1911)
Let n be a positive integer and S be a nonempty closed bounded convex subset of $\mathbf{R}^n$. Then every boundary point of S is a convex combination of at most n points in $\operatorname{ext} S$ and every interior point of S is a convex combination of at most $n+1$ points in $\operatorname{ext} S$.

Proof:

Without loss of generality, assume that $0 \in S$. We proceed by induction on n. If $n = 1$, then $S = [a, b]$ for some $a, b \in \mathbf{R}$ and the result is obvious. Assume now that the result holds for all nonempty closed bounded convex subsets of $\mathbf{R}^{n-1}$. Let S be a nonempty closed bounded convex subset of $\mathbf{R}^n$. By Lemma B4, either S is contained in a hyperplane of $\mathbf{R}^n$ or $S^o \neq \emptyset$. In the former case, S is contained in a subspace of $\mathbf{R}^n$ of dimension $n-1$ and the result follows by the inductive assumption.

Assume now that $S^\circ \neq \emptyset$. Consider a boundary point b of S. By 7.6(b), there exists a nonzero continuous linear functional h on $\mathbf{R}^n$ such that $h(x) \leq h(b)$ for all $x \in S$. Then

$$E^h = \{x \in S : h(x) = h(b)\}$$

can be thought of as a nonempty closed convex set contained in a subspace of $\mathbf{R}^n$ of dimension $n-1$. By the inductive assumption, the point b of E^h is a convex combination of at most $(n-1)+1 = n$ extreme points of E^h. But by Theorem B1, the set E^h is S-extreme, so that these are, in fact, extreme points of S.

Let now c be an interior point of S. By Corollary B3, S has an extreme point d. Consider the points $tc + (1-t)d$ for $t \geq 0$. There is some $t_0 > 1$ such that $b = t_0c + (1-t_0)d$ is a boundary point of S. Thus $c \in (b, d)$. Since b must be a convex combination of at most n extreme points of S and d is itself an extreme point of S, we see that c is in a convex combination of at most $n+1$ extreme points of S. □

Corollary B6

Let X be a finite dimensional normed space and S be a nonempty closed bounded subset of X. Then S has an extreme point.

If, in addition, S is a convex subset, then $S = \operatorname{co}(\operatorname{ext} S)$.

Proof:

By 6.3(b), X is $\mathbf{R}$-linearly homeomorphic to $\mathbf{R}^n$ for some positive integer n. (Note that $\mathbf{C}^m$ is $\mathbf{R}$-linearly homeomorphic to $\mathbf{R}^{2m}$.) Hence S has an extreme point by Corollary B3.

If, in addition, S is convex, then every point of S is a convex combination of points in $\operatorname{ext} S$ by Theorem B5. □

We remark that Corollary B6 does not hold, in general, for an infinite dimensional normed space X. For example, let $X = c_0$, the linear space of all scalar-sequences which tend to zero with the norm $\| \|_\infty$ and let S denote the closed unit ball in c_0. Then S is a nonempty closed bounded convex subset of X. However S has no extreme point. To see this, consider $x \in S$. Since $x(n) \to 0$ as $n \to \infty$, there is a positive integer m such that $|x(m)| \le 1/2$. Construct $y, z \in c_0$ by letting $y(n) = z(n) = x(n)$ if $n \neq m$ and $y(m) = x(m) + 1/2, z(m) = x(m) - 1/2$. Then it is clear that $y, z \in S$ and $x = (y+z)/2$, but $y \neq z$. Hence $x \notin \operatorname{ext} S$.

On the other hand, let $X = c_0$ and

$$C = \{x \in X : |x(j)| \le 1/j \ \text{ for } \ j = 1, 2, \ldots\}.$$

Then C is a nonempty compact convex subset of c_0. (Compare Problem 5-15(b).) Let

$$x = (1, \frac{1}{2}, \frac{1}{3} \cdots).$$

We show that $x \in \operatorname{ext} C$. Consider $y, z \in C$, $0 < r < 1$ and $x = ry + (1-r)z$. For $j = 1, 2, \ldots$, we have

$$ry(j) + (1-r)z(j) = x(j) = \frac{1}{j},$$

where $|y(j)| \le 1/j$ and $|z(j)| \le 1/j$. Since the absolute value is a strictly convex norm on $\mathbf{K}$, $y(j) = 1/j = z(j)$ for $j = 1, 2, \ldots$, so that $y = x = z$.

These examples indicate that we may consider a compact (convex) set rather than a closed bounded (convex) set while looking for extreme points of subsets of an infinite dimensional normed space.

Theorem B7 (Krein-Milman, 1940)
Let C be a nonempty compact subset of a normed space X. Then C has an extreme point. In fact, C is contained in the closure of $\operatorname{co}(\operatorname{ext} C)$.

If, in addition, C is a convex subset, then C equals the closure of $\operatorname{co}(\operatorname{ext} C)$.

Proof:

We proceed by transfinite induction. Consider the family

$$\mathcal{E} = \{E \subset C : E \text{ nonempty, closed and } C\text{-extreme }\}.$$

Since $C \in \mathcal{E}$, the family $\mathcal{E}$ is nonempty. For E_1 and E_2 in $\mathcal{E}$, define $E_1 \leq E_2$ if $E_2 \subset E_1$. Let $\mathcal{F}$ be a totally ordered subfamily of $\mathcal{E}$. Then $E = \cap\mathcal{F}$ is a closed subset of C, since each member of $\mathcal{F}$ is a closed subset of C. Also, E is C-extreme, since each member of $\mathcal{F}$ is C-extreme. Since the subfamily $\mathcal{F}$ is totally ordered, it has the finite intersection property. As C is compact, it follows that E is nonempty. Thus E is an upper bound for $\mathcal{F}$. By Zorn's lemma , the family $\mathcal{E}$ has a maximal element E_0. In particular, $E_0 \neq \emptyset$. We show that E_0 consists of a single point.

Let, if possible, $x_0, x_1 \in E_0$ with $x_1 \neq x_0$. By the Hahn-Banach separation theorem (7.5), there is a continuous linear functional f on X such that $\operatorname{Re} f(x_1) < \operatorname{Re} f(x_0)$. Let $h = \operatorname{Re} f_{|E_0}$. Then h is a convex function on E_0. Since h is continuous on the compact set E_0, we see that $\sup\limits_{x\in E_0} h(x) = \alpha < \infty$. Hence the closed set

$$E_0^h = \{x \in E_0 : h(x) = \alpha\}$$

is nonempty. By Theorem B1, it is C-extreme. Moreover, $E_0^h \subset E_0$. But $x_1 \in E_0$ while $x_1 \notin E_0^h$ since $h(x_1) < h(x_0) \leq \alpha$. Thus E_0^h is a member of the family $\mathcal{E}$ and is strictly contained in E_0. This contradicts the minimality of E_0 as a member of the family $\mathcal{E}$. Hence $E_0 = \{x_0\}$, that is, x_0 is an extreme point of the set C.

Let $\tilde{C}$ denote the closure of $\operatorname{co}(\operatorname{ext} C)$. Let, if possible, $c \in C$ but $c \notin \tilde{C}$. By 7.5, there is a continuous linear functional f on X such that

$$\operatorname{Re} f(x) < \operatorname{Re} f(c) \quad \text{for all } x \in \tilde{C}.$$

Let $g = \operatorname{Re} f_{|C}$. Then g is a convex function on C. Since g is continuous on the compact set C, we see that $\sup\limits_{x\in C} g(x) = \beta < \infty$. Thus the set

$$C^g = \{x \in C : g(x) = \beta\}$$

is nonempty. Also, it is compact. By what we have just proved, C^g has an extreme point x. Since the set C^g is C-extreme by Theorem B1, x is, in fact, an extreme point of C. Now since $x \in C^g$, we have $g(x) = \beta$, while since $x \in \operatorname{ext} C \subset \tilde{C}$, we have $g(x) = \operatorname{Re} f(x) < \operatorname{Re} f(c) \le \beta$, which is a contradiction. This shows that $C \subset \tilde{C}$.

Finally, assume that C is, in addition, a convex set. Since $\operatorname{ext} C$ is contained in C and $\tilde{C}$ is the smallest closed convex set containing $\operatorname{ext} C$, we see that $\tilde{C} \subset C$. Thus $\tilde{C} = C$. □

We remark that if X is a strictly convex normed space, then the fact that a nonempty compact subset of X has an extreme point can be proved rather simply as in Corollary B3. The same holds if X is a separable normed space since there is an equivalent strictly convex norm on such a space. (See Problem 7-18(b).)

In 1947, Milman proved the following partial converse of Theorem B7. Let E be a nonempty compact subset of a normed space X. Assume that the closure C of $\operatorname{co}(E)$ is compact. Then $\operatorname{ext} C \subset E$. We refer the reader to 10.1.3 of [20] and 3.25 of [53] for proofs of these results. Note that if X is a Banach space, then the closure of the convex hull of a compact subset of X is compact. (See 3.20(c) of [53].)

A nonempty compact convex subset C of an infinite dimensional normed space need not equal $\operatorname{co}(\operatorname{ext} C)$, unlike the finite dimensional case treated in Corollary B6. For example, let $X = c_0$ and $E = \{0, e_1, \frac{e_2}{2}, \ldots\}$, where $e_k(j) = \delta_{k,j}, k, j = 1, 2, \ldots$. Then E is a compact subset of the Banach space X. Let C denote the closure of $\operatorname{co}(E)$. By Milman's result quoted earlier, we see that $\operatorname{ext} C \subset E$. For $n = 1, 2, \ldots$, let

$$x_n = \sum_{k=1}^{n} \frac{e_k}{k2^k} \quad \text{and} \quad x = \sum_{k=1}^{\infty} \frac{e_k}{k2^k}.$$

It is easy to see that $x_n \in \operatorname{co}(E)$ for each n and $x_n \to x$ in X, so that $x \in C$. On the other hand, it is clear that $x \notin \operatorname{co}(E)$, so that $x \notin \operatorname{co}(\operatorname{ext} C)$.

The remarkable thing about Theorem B7 is that it establishes the existence of an algebraically defined entity (an extreme point) from a purely topological assumption (compactness). The content of Theorem B7 is that each point of a compact subset of a normed space can be approximated by convex combinations of its extreme points.

We shall now show that certain real-valued functions defined on a compact set attain their suprema or infima at extreme points of that set. A real-valued function h on a metric space S is said to be **upper semicontinuous** if $h^{-1}((-\infty, a))$ is an open subset of S for every a in $\mathbf{R}$. It is said to be **lower semicontinuous** if $h^{-1}((a, \infty))$ is an open subset of S for every $a \in \mathbf{R}$.

Theorem B8

Let C be a nonempty compact subset of a normed space X. If h is a convex upper semicontinuous function on C, then h attains its supremum on C at an extreme point of C. Similarly, if h is a concave lower semicontinuous function on C, then h attains its infimum at an extreme point of C.

Proof:

Let $h : C \to \mathbf{R}$ be upper semicontinuous. Considering the open cover $\{V_a\}$ of C, where $a \in \mathbf{R}$ and $V_a = \{x \in C : h(x) < a\}$, it follows that h is bounded above on C. Let $\alpha = \sup\{h(x) : x \in C\}$ and (x_n) be a sequence in C such that $h(x_n) \to \alpha$. Since C is compact, there is a subsequence (x_{n_k}) which converges to some $x \in C$. Then we see that $h(x) = \alpha$. Thus h attains its supremum on C at a point of C. Consider then the nonempty set

$$E^h = \{x \in C : h(x) = \alpha\}.$$

Since E^h is the complement of $\{x \in C : h(x) < \alpha\}$ in C and h is upper semicontinuous on C, it follows that E^h is a closed subset of C. Since C is compact, E^h is itself compact. By Theorem B7, E^h has an extreme point x_0.

Let h be convex. Then by Theorem B1, the set E^h is C-extreme. Hence x_0 is an extreme point of C and since $x_0 \in E^h, h(x_0) = \alpha = \sup\{h(x) : x \in C\}$.

Similar proof holds if h is concave and lower semicontinuous. $\square$

Before concluding this appendix, we consider the following two questions of practical importance.

1. Which nonempty closed convex subsets of $\mathbf{R}^n$ have extreme points?

Let a subset S of $\mathbf{R}^n$ have an extreme point x_0. Suppose that $x + S = S$ for some $x \in \mathbf{R}^n$. Then $x + x_0 = y$ and $x + z = x_0$ for some $y, z \in S$. Thus $x_0 = \frac{1}{2}(x_0 + x) + \frac{1}{2}(x_0 - x)$ for some $x_0 + x$ and $x_0 - x$ in S, so that $x_0 + x = x_0 = x_0 - x$, that is, $x = 0$.

We say that a nonempty subset S of $\mathbf{R}^n$ is **line-free** if $x + S = S$ holds only for $x = 0$. We have just seen that if S has an extreme point, then S is line-free. It turns out that the converse also holds if S is a nonempty closed convex subset of $\mathbf{R}^n$. See [29], p. 33 for a proof.

2. If a convex (resp., concave) function on a nonempty closed convex line-free subset S of $\mathbf{R}^n$ attains its supremum (resp., infimum) at some point of S, then will it do so at an extreme point of S?

The answer to this question is affirmative. Again, we refer the reader to [29], p. 37 for a proof with the following comments: This proof needs to be modified since the set of all points where a convex (resp., concave) function attains its supremum (resp., infimum) need not be a convex subset, as claimed by Holmes! Also, although Holmes requires the convex (resp., concave) function to be upper semicontinuous (resp., lower semicontinuous), this requirement can be dropped.

In fact, the following converse also holds. Let S be a nonempty closed convex line-free subset of $\mathbf{R}^n$, and $h : S \to \mathbf{R}$ be convex and bounded above. If there is some $x_0 \in \operatorname{ext} S$ with $h(x_0) = \sup\{h(x) : x \in \operatorname{ext} S\}$, then, in fact, $h(x_0) = \sup\{h(x) : x \in S\}$. A similar statement holds if h is concave and bounded below. See [29], p. 37.

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Let h be convex. Then by Theorem B1, the set E^h is C-extreme. Hence x_0 is an extreme point of C and since $x_0 \in E^h, h(x_0) = \alpha = \sup\{h(x) : x \in C\}$.

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These results are extremely(!) useful in abstract mathematical programming. A **convex programme** (S, h) consists of a set S in a linear space X and a real-valued convex function h on S, called the **objective functional**. Consider the optimization problem of finding the supremum of h on S. Any $x_0 \in S$ with $h(x_0) = \sup\{h(x) : x \in S\}$ is called a **solution** of the convex programme (S, h).

Let S be a closed convex line-free subset of $\mathbf{R}^n$ and h be a real-valued convex function on S. The preceding results say that if the convex programme (S, h) has a solution in S, then it has a solution in ext S as well. If, in addition, the function h is bounded above on S, then every solution of (ext S, h) is also a solution of (S, h). Hence we need to compare the values of h at only the extreme points of S in order to find a solution of the convex programme (S, h). In particular, if ext S is a finite set, then our search of a solution of the convex programme (S, h) is bound to be successful. For example, if S is as in Figure 10(a), then x_0 has to be a solution and if S is as in Figure 10(b), then a solution must occur either at x_1 or at x_2.

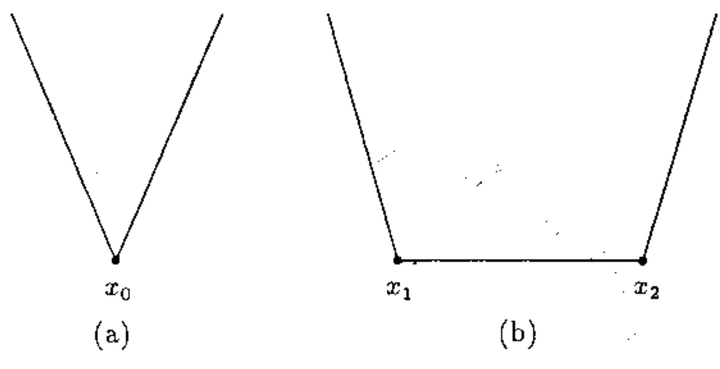

Figure 10

Convex programmes are used in maximizing a convex profit function on a set which represents certain constraints. Similarly, **concave programmes** are used in minimizing a concave cost function.

Appendix C

Sturm-Liouville Problems

We study some **boundary value problems** involving ordinary linear differential equations of second order which occur in the study of several physical entities like vibrating strings, transmission lines, resonances in a cavity etc. They also appear in optimal control theory and quantum mechanics. The corresponding differential operators are not, in general, bounded, but the inverse integral operators are, in fact, compact and self-adjoint. Then the spectral theory of such operators given in Section 28 becomes applicable.

Let us first consider the following differential equation:

$$-x''(t) + q(t)x(t) - \mu x(t) = y(t), \quad t \in [a, b],$$

where q is a real-valued continuous function on a closed bounded interval $[a, b]$, μ is a complex number and $y \in C([a, b])$, the set of all complex-valued continuous functions on $[a, b]$. Let $C^2([a, b])$ denote the set of all complex-valued functions on $[a, b]$ whose first two derivatives are continuous on $[a, b]$. By a **solution** of the differential equation we mean a function $x \in C^2([a, b])$ which satisfies the equation stated above for all $t \in [a, b]$. For $x \in C^2([a, b])$, let

$$S(x) = -x'' + qx.$$

Consider the following **boundary conditions**:

$$\alpha x(a) + \alpha_1 x'(a) = 0 = \beta x(b) + \beta_1 x'(b),$$

where $\alpha, \alpha_1, \beta, \beta_1$ are real numbers such that neither both α and α_1 nor both β and β_1 are zero. For $x \in C^2([a, b])$, let

$$B_a(x) = \alpha x(a) + \alpha_1 x'(a), \quad B_b(x) = \beta x(b) + \beta_1 x'(b)$$

and

$$D = \{x \in C^2([a, b]) : B_a(x) = 0 = B_b(x)\}.$$

Let $y \in C([a, b])$ and $\mu \in \mathbf{C}$. The **Sturm-Liouville problem** consists of finding $x \in D$ such that $S(x) - \mu x = y$. The (linear) operator S defined on D with range in $C([a, b])$ is known as the **Sturm-Liouville operator** . For $x, y \in C([a, b])$, we shall use the notation

$$\langle x, y \rangle = \int_a^b x(t)\overline{y}(t)\, dt = \int_a^b x\overline{y}\, dm.$$

Before attempting to solve the Sturm-Liouville problem, we state a result regarding the solution of an **initial value problem** : Let q be a real-valued continuous function on $[a, b]$, $s_0 \in [a, b]$ and $r, r_1 \in \mathbf{R}$. Then there is a unique real-valued function $x \in C^2([a, b])$ such that

$$-x'' + qx = 0, \quad x(s_0) = \cdot \text{ and } x'(s_0) = r_1.$$

For the existence of such a function, we refer the reader to Theorem 8 in Chapter 6 of [11]. Here a constructive proof is given which is based on Picard's method applied to the system of first order differential equations

$$[\mathbf{x}(t)]' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}_0 = (r, r_1),$$

with $\mathbf{x} = (x_1, x_2), x_1' = x_2, x_2' = qx_1, \mathbf{f} = (f_1, f_2), f_1(t, \mathbf{x}) = x_2$, and $f_2(t, \mathbf{x}) = q(t)x_1(t)$. For the uniqueness of such a solution, we refer the reader to Ziebur's proof given in [41], p. A77.

Now let α and α_1 be real numbers not both zero. Then there is a nonzero real-valued function $u \in C^2([a, b])$ satisfying

$$-u'' + qu = 0 \quad \text{and} \quad \alpha u(s_0) + \alpha_1 u'(s_0) = 0.$$

This follows by considering the **initial conditions** $x(s_0) = r = \alpha_1$ and $x'(s_0) = r_1 = -\alpha$ Further, if $v \in C^2([a, b])$ satisfies

$$-v'' + qv = 0 \quad \text{and} \quad \alpha v(s_0) + \alpha_1 v'(s_0) = 0,$$

then v is a constant multiple of u. To see this, we recall the following.

If x and y are complex-valued differentiable functions on $[a, b]$, then the **Wronskian** of x and y is defined by

$$W(x, y) = xy' - x'y.$$

If x and y are twice differentiable on $[a, b]$, then it follows that

$$[W(x, y)]' = xy'' - x''y,$$

and if x and y are in $C^2([a, b])$, then

$$\int_a^b (xy'' - x''y)\, dm = \int_a^b [W(x, y)]'\, dm = W(x, y)(b) - W(x, y)(a).$$

This is known as **Green's formula**. It remains valid if x and y are twice continuously differentiable on the open interval (a, b), x, y, x', y' have finite limits as $s \to a$ or $s \to b$ and x'', y'' remain bounded on (a, b). Of course, then one must substitute the appropriate limits in place of the corresponding values of functions in the formula.

Now suppose that $x, y \in C^2([a, b])$ satisfy

$$-x'' + qx = 0 = -y'' + qy.$$

Then $[W(x, y)]' = x(qy) - (qx)y = 0$, so that the function $W(x, y)$ is constant on $[a, b]$. If x or y is a constant multiple of the other, then it is easy to see that $W(x, y) = 0$. Conversely, assume that $W(x, y) = 0$. Let $s_0 \in [a, b]$ and consider the homogeneous system of linear equations

$$\begin{aligned} k_1 x(s_0) + k_2 y(s_0) &= 0 \\ k_1 x'(s_0) + k_2 y'(s_0) &= 0 \end{aligned}$$

in the unknowns k_1 and k_2. Since the determinant $W(x, y)(s_0)$ of this system equals 0, it has a nonzero solution (k_1, k_2). Let $u = k_1 x + k_2 y$. Then $u \in C^2([a, b]), -u'' + qu = 0, u(s_0) = 0 = u'(s_0)$. The uniqueness of the solution of the initial value problem shows that $u = 0$, that is, the set $\{x, y\}$ is linearly dependent. Thus we have proved that if $x, y \in C^2([a, b])$ satisfy $-x'' + qx = 0 = -y'' + qy$, then they are linearly independent if and only if $W(x, y) \neq 0$.

Coming back to the situation $-u'' + qu = 0 = -v'' + qv$ and

$$\begin{aligned} \alpha u(s_0) + \alpha_1 u'(s_0) &= 0 \\ \alpha v(s_0) + \alpha_1 v'(s_0) &= 0, \end{aligned}$$

where $u \neq 0$ and not both α and α_1 are zero, we note that the determinant $W(u, v)(s_0)$ of this system must be zero and hence the set $\{u, v\}$ is linearly dependent. Since $u \neq 0$, we see that v must be a constant multiple of u, as desired.

We now take up the homogeneous Sturm-Liouville problem of finding $x \in D$ with $S(x) - \mu x = 0$. If this problem has a nonzero solution $x \in D$, then we say that μ is an **eigenvalue** of the Sturm-Liouville operator S and that x is a corresponding **eigenfunction** of S.

Theorem C1

Let $S : D \to C([a, b])$ denote the Sturm-Liouville operator $S(x) = -x'' + qx$. Then

(a) S is symmetric : $\langle S(x),\, y\rangle = \langle x,\, S(y)\rangle$ for all $x, y \in D$.

(b) The eigenvalues of S are all real. Also, if μ is an eigenvalue of S, then there is a corresponding real-valued eigenfunction u such that $\langle u,\, u\rangle = 1$ and any other eigenfunction corresponding to μ is a constant multiple of u.

(c) If μ and ν are distinct eigenvalues of S with u and v corresponding eigenfunctions in D, then $\langle u,\, v\rangle = 0$.

(d) The eigenvalues of S are countable.

Proof:

(a) Let $x, y \in D$. Since q is real-valued, we have

$$\begin{aligned} \langle S(x),\, y\rangle - \langle x,\, S(y)\rangle &= \int_a^b [(-x'' + qx)\overline{y} - x(-\overline{y}'' + q\overline{y})]\, dm \\ &= \int_a^b (x\overline{y}'' - x''\overline{y})\, dm \\ &= W(x, \overline{y})(b) - W(x, \overline{y})(a) \end{aligned}$$

Green's formula. Since α and α_1 are real, we have

$$\begin{aligned}\alpha x(a) + \alpha_1 x'(a) &= B_a(x) = 0\\ \alpha\overline{y}(a) + \alpha_1\overline{y}'(a) &= B_a(\overline{y}) = \overline{B_a(y)} = 0,\end{aligned}$$

ıere not both α and α_1 are zero. Hence the determinant $W(x,\overline{y})(a)$ this system is zero. Similarly, $W(x,\overline{y})(b) = 0$. Hence $\langle S(x),\, y\rangle =$ $, S(y)\rangle$.

(b) Let μ be an eigenvalue of S and $S(x) = \mu x$ with $x \neq 0$. Then

$$\mu\langle x,\, x\rangle = \langle \mu x,\, x\rangle = \langle Sx,\, x\rangle = \langle x,\, Sx\rangle = \langle x,\, \mu x\rangle = \overline{\mu}\langle x,\, x\rangle.$$

ince $x \neq 0$, we see that $\langle x,\, x\rangle \neq 0$ and hence $\mu = \overline{\mu}$, that is, μ is real.

Since $\operatorname{Re} x + i \operatorname{Im} x = x \neq 0$, we see that either $\operatorname{Re} x \neq 0$ or m $x \neq 0$. Considering ix in place of x, if necessary, we can assume hat $\operatorname{Re} x \neq 0$. As $q, \alpha, \alpha_1, \beta, \beta_1$ and μ are real, we see that

$$\begin{aligned}&S(\operatorname{Re} x) = \operatorname{Re} S(x) = \operatorname{Re} \mu x = \mu \operatorname{Re} x,\\ &B_a(\operatorname{Re} x) = \operatorname{Re} B_a(x) = 0 = \operatorname{Re} B_b(x) = B_b(\operatorname{Re} x).\end{aligned}$$

Ience $\operatorname{Re} x$ is an eigenfunction of S corresponding to μ. Let $u =$ ٤e $x/\langle \operatorname{Re} x,\, \operatorname{Re} x\rangle^{1/2}$. Then u is a real-valued eigenfunction of S cor-:esponding to μ and $\langle u,\, u\rangle = 1$.

Let v be another eigenfunction of S corresponding to μ. Then the system of linear equations

$$\begin{aligned}B_a(u) &= \alpha u(a) + \alpha_1 u'(a) = 0\\ B_a(v) &= \alpha v(a) + \alpha_1 v'(a) = 0\end{aligned}$$

has a nonzero solution (α, α_1). Hence the determinant $W(u,v)(a)$ of the system is zero, so that the set $\{u, v\}$ is linearly dependent. As $u \neq 0$, we see that v must be a constant multiple of u.

(c) Let μ and ν be distinct (real) eigenvalues of S with u and v corresponding eigenfunctions. Then

$$\mu\langle u,\, v\rangle = \langle \mu u,\, v\rangle = \langle S(u),\, v\rangle = \langle u,\, S(v)\rangle = \langle u,\, \nu v\rangle = \nu\langle u,\, v\rangle,$$

since S is symmetric and ν is real. But $\mu \neq \nu$, so that $\langle u, v\rangle = 0$.

(d) Since D is a subspace of the separable Hilbert space $L^2([a,b])$, it cannot contain an uncountable orthonormal set (22.9). Hence part (c) shows that the set of eigenvalues of S is countable. □

We remark that there is a real number r such that every eigenvalue of the Sturm-Liouville operator S is greater than r. We refer the reader to 11.7.4 of [17] for a proof.

Assume that 0 is not an eigenvalue of S, that is, the conditions

$$x \in C^2([a,b]), \quad B_a(x) = 0 = B_b(x) \quad \text{and} \quad S(x) = 0$$

imply that $x = 0$. In this case , we show that the inverse of the operator S exists and it is a Fredholm integral operator with a symmetric continuous kernel.

From our discussion before Theorem C1, it follows that there are nonzero real-valued functions u_a and u_b in $C^2([a,b])$ such that

$$\begin{aligned} -u_a'' + qu_a = 0, &\quad B_a(u_a) = \alpha u_a(a) + \alpha_1 u_a'(a) = 0, \\ -u_b'' + qu_b = 0, &\quad B_b(u_b) = \beta u_b(b) + \beta_1 u_b'(b) = 0. \end{aligned}$$

These functions u_a and u_b are linearly independent, since otherwise $u_a = \gamma u_b$ for some constant γ and

$$B_a(u_a) = 0, B_b(u_a) = B_b(\gamma u_b) = \gamma B_b(u_b) = 0,$$

so that $u_a \in D$ and $S(u_a) = 0$. Then 0 would be an eigenvalue of S, contrary to our assumption.

As $u_a, u_b \in C^2([a, b])$ and $u_a'' = qu_a$, $u_b'' = qu_b$, the function $W(u_a, u_b)$ is constant on $[a,b]$ and since u_a and u_b are linearly independent, this constant is not zero. Let $\omega = W(u_a, u_b)$ and define

$$k(s,t) = \begin{cases} -\dfrac{u_a(s)u_b(t)}{\omega}, & \text{if } a \leq s \leq t \leq b \\[2ex] -\dfrac{u_a(t)u_b(s)}{\omega}, & \text{if } a \leq t \leq s \leq b. \end{cases}$$

We remark that the function $k(.,.)$ does not depend on our choice of the nonzero functions u_a and u_b in $C^2([a,b])$ with $u_a'' = qu_a$, $u_b'' = qu_b$ $B_a(u_a)=0=B_b(u_b)$. For if some functions v_a and v_b also satisfy these conditions, then we have earlier shown that v_a is a constant multiple of u_a and v_b is a constant multiple of u_b. The function $k(.,.)$ is known as the **Green function** for the Sturm-Liouville operator S (when 0 is not an eigenvalue of S).

Lemma C2

Assume that 0 is not an eigenvalue of the Sturm-Liouville operator S. Let $k(.,.)$ be the Green function for S. Then

(a) $k(.,.)$ is a real-valued continuous function on $[a,b]\times[a,b]$.

(b) $k(s,t) = k(t,s)$ for all $s,t \in [a,b]$.

(c) For a fixed $s \in [a,b]$, define

$$k_s(t) = k(s,t), \quad t \in [a,b].$$

Then on $[a,s)$ and $(s,b]$, k_s is continuously twice differentiable, k_s'' is bounded and $-k_s'' + qk_s = 0$. Also, $B_a(k_s) = 0 = B_b(k_s)$.

(d) For each fixed $s \in (a,b)$, we have

$$\lim_{t\to s^-} \frac{dk_s}{dt} - \lim_{t\to s^+} \frac{dk_s}{dt} = 1.$$

Proof:
Parts (a), (b) and (c) are obvious. Fix $s \in (a,b)$. Then,

$$\lim_{t\to s^-} \frac{dk_s}{dt} = -\frac{1}{\omega}u_b(s) \lim_{t\to s^-} u_a'(t) = -\frac{1}{\omega}u_b(s)u_a'(s),$$

$$\lim_{t\to s^+} \frac{dk_s}{dt} = -\frac{1}{\omega}u_a(s) \lim_{t\to s^+} u_b'(t) = -\frac{1}{\omega}u_a(s)u_b'(s).$$

Since $u_a(s)u_b'(s) - u_a'(s)u_b(s) = W(u_a,u_b)(s) = \omega$ for all $s \in [a,b]$, we obtain the desired result. □

Theorem C3

Assume that 0 is not an eigenvalue of the Sturm-Liouville operator S. Let $k(.,.)$ be the Green function for S. Define $A : L^2([a,b]) \to L^2([a,b])$ by

$$A(y)(s) = \int_a^b k(s,t)y(t)\,dm(t), \quad y \in L^2([a,b]), s \in [a,b].$$

(**a**) Let $y \in C([a,b])$ and $\mu \in \mathbf{C}$. Then $x \in D$ and $S(x) - \mu x = y$ if and only if $x \in L^2([a,b])$ and $x - \mu A(x) = A(y)$.

(**b**) Let μ be a nonzero real number. Then μ is an eigenvalue of S if and only if $1/\mu$ is an eigenvalue of A. In that case, the corresponding eigenfunctions are the same.

Proof:

(**a**) It is enough to prove that $x \in D$ and $S(x) = y$ if and only if $x \in L^2([a,b])$ and $x = A(y)$. For then $x \in D$ and $S(x) = \mu x + y$ if and only if $x \in L^2([a,b])$ and $x = A(\mu x + y) = \mu A(x) + A(y)$, as desired.

Assume that $x \in D$ and $S(x) = y$. Fix $s \in (a,b)$. Applying Green's formula to the functions x and k_s (as defined in part (c) of Lemma C2) on the intervals $[a,s)$ and $(s,b]$, we have

$$\begin{aligned}\int_a^{s^-} (xk_s'' - x''k_s)\,dm &+ \int_{s^+}^b (xk_s'' - x''k_s)\,dm \\ &= [W(x,k_s)(s^-) - W(x,k_s)(a)] + [W(x,k_s)(b) - W(x,k_s()(s^+)].\end{aligned}$$

Let us first consider the left side. On $[a,s)$ and $(s,b]$, we have $-x'' + qx = y$ and $-k_s'' + qk_s = 0$ by part (c) of Lemma C2, so that

$$xk_s'' - x''k_s = xqk_s - (qx - y)k_s = yk_s.$$

Hence the left side equals

$$\int_a^b yk_s\,dm = \int_a^b y(t)k(s,t)\,dt = A(y)(s).$$

Turning now to the right side, we see that $W(x,k_s)(a) = x(a)k_s'(a) - x'(a)k_s(a) = 0$, since the linear system

$$\begin{aligned}\alpha x(a) + \alpha_1 x'(a) &= B_a(x) = 0 \\ \alpha k_s(a) + \alpha_1 k_s(a) &= B_a(k_s) = 0\end{aligned}$$

has a nonzero solution (α, α_1). Similarly, $W(x, k_s)(b) = 0$. Hence the right side equals

$$
\begin{aligned}
&W(x, k_s)(s^-) - W(x, k_s)(s^+) \\
&\quad = x(s^-)k_s'(s^-) - x'(s^-)k_s(s^-) - x(s^+)k_s'(s^+) + x'(s^+)k_s(s^+) \\
&\quad = x(s)k_s'(s^-) - x'(s)k_s(s) - x(s)k_s'(s^+) + x'(s)k_s(s) \\
&\quad = x(s)[k_s'(s^-) - k_s'(s^+)] \\
&\quad = x(s)
\end{aligned}
$$

by part (d) of Lemma C2. Thus we see that

$$A(y)(s) = x(s) \quad \text{for all } s \in (a, b).$$

By the continuity of x and $A(y)$, it follows by $A(y) = x$.

Conversely, let $x \in L^2([a, b])$ and for all $s \in [a, b]$,

$$
\begin{aligned}
x(s) = A(y)(s) &= \int_a^b k(s, t)y(t)\, dt \\
&= -\frac{u_b(s)}{\omega} \int_a^s u_a(t)y(t)\, dt - \frac{u_a(s)}{\omega} \int_s^b u_b(t)y(t)\, dt.
\end{aligned}
$$

Since u_a, y and u_b are continuous on $[a, b]$, the fundamental theorem for Riemann integration (4.2) shows that for all $s \in [a, b]$,

$$
\begin{aligned}
x'(s) &= -\frac{1}{\omega}\Big(u_b'(s) \int_a^s u_a(t)y(t)\, dt + u_b(s)u_a(s)y(s) \\
&\qquad + u_a'(s) \int_s^b u_b(t)y(t)\, dt - u_a(s)u_b(s)y(s)\Big) \\
&= -\frac{1}{\omega}\left(u_b'(s) \int_a^s u_a(t)y(t)\, dt + u_a'(s) \int_s^b u_b(t)y(t)\, dt\right)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
x''(s) &= -\frac{1}{\omega}\Big(u_b''(s) \int_a^s u_a(t)y(t)\, dt + u_b'(s)u_a(s)y(s) \\
&\qquad + u_a''(s) \int_s^b u_b(t)y(t)\, dt - u_a'(s)u_b(s)y(s)\Big) \\
&= -\frac{1}{\omega}\left(u_b''(s) \int_a^s u_a(t)y(t)\, dt + u_a''(s) \int_s^b u_b(t)y(t)\, dt\right) - y(s),
\end{aligned}
$$

since $u_a(s)u_b'(s) - u_a'(s)u_b(s) = W(u_a, u_b)(s) = \omega$. This shows that $x'' \in C([a,b])$. Using $u_b'' = qu_b$ and $u_a'' = qu_a$, it can be seen that

$$S(x)(s) = -x''(s) + q(s)x(s) = y(s).$$

Also,

$$\begin{aligned} B_a(x) &= \alpha x(a) + \alpha_1 x'(a) \\ &= -\frac{\alpha u_a(a)}{\omega}\int_a^b u_b(t)y(t)\,dt - \frac{\alpha_1 u_a'(a)}{\omega}\int_a^b u_b(t)y(t)\,dt \\ &= -\frac{1}{\omega}\int_a^b u_b(t)y(t)\,dt\,\left(\alpha u_a(a) + \alpha_1 u_a'(a)\right) \\ &= 0, \end{aligned}$$

since $\alpha u_a(a) + \alpha_1 u_a'(a) = B_a(u_a) = 0$. Similarly, $B_b(x) = 0$. Thus $x \in D$ and $S(x) = y$.

(b) Letting $y = 0$ in part (a), we obtain the desired results. □

When 0 is not an eigenvalue of S, part (b) of Theorem C3 reduces the homogeneous Sturm-Liouville problem of finding $x \in D$ with $S(x) - \mu x = 0$ (that is, of finding eigenvalues and eigenfunctions of S) to finding eigenvalues and eigenfunctions of the Fredholm integral operator A with the Green function of S as its kernel.

Let us now consider the nonhomogeneous Sturm-Liouville problem. In view of Theorem C3, we have the following result.

Theorem C4

Assume that 0 is not an eigenvalue of the Sturm-Liouville operator S. As in parts (b), (c) and (d) of Theorem C1, let the distinct real eigenvalues of S be arranged in a sequence (μ_n) satisfying $0 < |\mu_1| \leqslant |\mu_2| \leqslant \cdots$ and let (u_n) be a corresponding sequence of real-valued eigenfunction of S satisfying $\langle u_n, u_m\rangle = \delta_{n,m}, n, m = 1, 2, \ldots$.

(a) Let $\mu \in \mathbb{C}$ with $\mu \neq \mu_n$ for $n = 1, 2, \ldots$. For every $y \in C([a,b])$,

there is a unique $x \in D$ such that $S(x) - \mu x = y$. In fact,

$$x = \sum_n \frac{\langle y, u_n \rangle}{\mu_n - \mu} u_n.$$

(b) Let $\mu \in \mathbf{C}$ with $\mu = \mu_j$ for some positive integer j. For a given $y \in C([a, b])$, there is some $x \in D$ such that $S(x) - \mu x = y$ if and only if $\langle y, u_j \rangle = 0$ and then

$$x = cu_j + \sum_{n \neq j} \frac{\langle y, u_n \rangle u_n}{\mu_n - \mu},$$

where $c \in \mathbf{C}$.

The series occurring in the expression for x in (a) and (b) converge absolutely and uniformly on $[a, b]$.

Proof:
Let $A : L^2([a, b]) \to L^2([a, b])$ denote the Fredholm integral operator with the Green function $k(.,.)$ for S as its kernel. Since $k(.,.)$ is a real-valued continuous function on $[a, b] \times [a, b]$ and $k(s, t) = k(t, s)$ for all $s, t \in [a, b]$ by parts (a) and (b) of Lemma C2, the result 28.8(b) shows that A is a compact self-adjoint operator on $L^2([a, b])$.

Let $s_n = 1/\mu_n$ for $n = 1, 2, \ldots$. By part (b) of Theorem C3, (s_n) is a sequence consisting of all nonzero eigenvalues of A with $|s_1| > |s_2| > \cdots$ and $\{u_1, u_2, \ldots\}$ is an orthonormal set in $L^2([a, b])$ with $A(u_n) = s_n u_n, n = 1, 2, \ldots$. Then 28.5(a) shows that

$$A(x) = \sum_n s_n \langle x, u_n \rangle u_n$$

for all $x \in L^2([a, b])$.

Let $\mu \in \mathbf{C}$ and $y \in C([a, b])$. By part (a) of Theorem C3, we have $x \in D$ and $S(x) - \mu x = y$ if and only if $x \in L^2([a, b])$ and $x - uA(x) = A(y)$. Since $A(y) = \sum_n s_n \langle y, u_n \rangle u_n$, it follows that

$$\langle A(y), u_n \rangle = s_n \langle y, u_n \rangle, \quad n = 1, 2, \ldots.$$

(a) Let $\mu \neq \mu_n, n = 1, 2, \ldots$. Since $\mu \notin \{1/s_1, 1/s_2, \ldots\}$, 28.7(a shows that the unique $x \in L^2([a, b])$ with $x - \mu A(x) = A(y)$ is

$$\begin{aligned} x &= A(y) + \mu \sum_n \frac{s_n \langle A(y), u_n\rangle}{1 - \mu s_n} u_n. \\ &= \sum_n s_n \langle y, u_n\rangle u_n + \mu \sum_n \frac{s_n^2 \langle y, u_n\rangle}{1 - \mu s_n} u_n \\ &= \sum_n \frac{s_n \langle y, u_n\rangle}{1 - \mu s_n} u_n \\ &= \sum_n \frac{\langle y, u_n\rangle}{\mu_n - \mu} u_n. \end{aligned}$$

(b) Let $\mu = \mu_j$ for some positive integer j. Then $\mu = 1/s_j$ and 28.7(b) shows that for a given $y \in C([a, b])$, there is some $x \in L^2([a, b])$ with $x - \mu A(x) = A(y)$ if and only if $\langle A(y), u_j\rangle = 0$, that is, $\langle y, u_j\rangle = 0$ since $\langle A(y), u_j\rangle = s_j \langle y, u_j\rangle$ and $s_j \neq 0$. Also, any such x is given by

$$x = A(y) + cu_j + \mu \sum_{n \neq j} \frac{s_n \langle A(y), u_n\rangle}{1 - \mu s_n} u_n = cu_j + \sum_{n \neq j} \frac{\langle y, u_n\rangle}{\mu_n - \mu} u_n,$$

as before, where $c \in \mathbf{C}$.

Since the kernel $k(.,.)$ of the Fredholm integral operator A satisfies

$$\int_a^b |k(s,t)|^2 \, dm(t) \leq (b - a) \|k\|_\infty^2 < \infty$$

for all $s \in [a, b]$, our discussion in 28.8(b) implies that the series occurring in the expressions for x in parts (a) and (b) converge absolutely and uniformly on $[a, b]$. □

We make several important remarks. Consider $u \in D$, that is, $u \in C^2([a, b])$ with $B_a(u) = 0 = B_b(u)$. If $S(u) = v$, then letting $\mu = 0$ in part (a) of Theorem C4, we have

$$u = A(v) = \sum_n \frac{\langle v, u_n\rangle}{\mu_n} u_n.$$

It follows that $\langle u, u_n\rangle = \langle v, u_n\rangle/\mu_n$ for $n = 1, 2, \dots$. Hence

$$u = \sum_n \langle u, u_n\rangle u_n.$$

Thus the orthonormal set $\{u_1, u_2, \dots\}$ is dense in D with respect to the norm $\| \ \|_2$. By employing a modification of Weierstrass' theorem (3.12), we can show that the set $\{x \in C^2([a,b]) : x(a) = 0 = x(b)\}$ is dense in $L^2([a,b])$. Therefore the orthonormal set $\{u_1, u_2, \dots\}$ is dense $L^2([a,b])$. Thus it is, in fact, an orthonormal basis for $L^2([a,b])$ by 22.7. Since $L^2([a,b])$ is an infinite dimensional linear space, the set $\{u_1, u_2, \dots\}$ and, in turn, the set $\{\mu_1, \mu_2, \dots\}$ of eigenvalues of S are infinite. Also, by 28.3(b), A is a Hilbert-Schmidt operator and hence Corollary 28.6 shows that

$$\sum_{n=1}^{\infty} \frac{1}{|\mu_n|^2} = \sum_{n=1}^{\infty} |s_n|^2 < \infty.$$

So far we have treated the case when 0 is not an eigenvalue of the Sturm-Liouville operator S. Suppose now that 0 is an eigenvalue of S. By part (d) of Theorem C1 (or by our remark after its proof), there is some real number r which is not an eigenvalue of S. Let $S_r = S - rI$. Then 0 is not an eigenvalue of the Sturm-Liouville operator S_r. The earlier arguments, therefore, go through if we replace the function q by the function $q - r$ and note that $S_r(x) - \mu x = y$ if and only if $S(x) - (\mu + r)x = y$ for $x \in D$, $\mu \in \mathbf{C}$ and $y \in C([a,b])$.

Examples C5

Let $a = 0, b = 1$ and $q(t) = 0$ for $t \in [0,1]$. Then $S(x) = -x''$ with an appropriate domain D depending on the boundary conditions. These conditions arise naturally in considering longitudinal vibrations of an elastic string of unit length. In all these cases, the eigenvalues and the eigenfunctions of S can be found out by considering the general solution of $-x'' - \mu x = 0$.

(a) **(Rigidly fixed ends)** $x(0) = 0 = x(1)$. The eigenvalues of

S are $\mu_n = n^2\pi^2, n = 1, 2, \ldots$, with the corresponding eigenfunctions $u_n(t) = \sqrt{2}\sin n\pi t, t \in [0, 1]$.

To find the Green function $k\,(.,.)$ for S, we take $u_a(t) = t, u_b(t) = 1 - t$ for $t \in [0, 1]$ and obtain

$$k(s,t) := \begin{cases} (1-t)s, & \text{if } 0 \leq s \leq t \leq 1, \\ (1-s)t, & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

We remark that we have solved the corresponding Fredholm integral equation in 28.8(b).

If $\mu \neq n^2\pi^2$ for $n = 1, 2, \ldots$ and $y \in C([0, 1])$, then the unique solution of

$$-x'' - \mu x = y, \quad x(0) = 0 = x(1)$$

is given by

$$x(s) = 2\sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2 - \mu}\int_0^1 y(t)\sin n\pi t\,dt\right)\sin n\pi s, \quad s \in [0, 1]$$

where the series converges absolutely and uniformly on $[0, 1]$.

If $\mu = j^2\pi^2$ for some $j = 1, 2, \ldots$, then the problem

$$-x'' - \mu x = y, \quad x(0) = 0 = x(1)$$

has a solution if and only if $\int_0^1 y(t)\sin j\pi t\,dt = 0$. In that case, the solutions are given by

$$x(s) = c\sin j\pi s + 2\sum_{n \neq j}\frac{1}{\pi^2(n^2 - j^2)}\left(\int_0^1 y(t)\sin n\pi t\,dt\right)\sin n\pi s,$$

where $c \in \mathbf{C}$.

(b) (**Free ends**) $x'(0) = 0 = x'(1)$. The eigenvalues of S are $\mu_n = n^2\pi^2, n = 0, 1, 2, \ldots$, with the corresponding eigenfunctions $u_n(t) = \sqrt{2}\cos n\pi t$ for $t \in [0, 1]$.

Since 0 is an eigenvalue of S, there is no Green function for S. Yet, the Sturm-Liouville problem can be solved by considering $S - r\,I$, where $r \in \mathbf{R}$ and $r \neq n^2\pi^2, n = 0, 1, \ldots,$ as we have remarked after the proof of Theorem C4.

If $\mu \neq n^2\pi^2$ for $n = 0, 1, 2, \ldots$ and $y \in C([0,1])$, then the unique solution of

$$-x'' - \mu x = y, \quad x'(0) = 0 = x'(1)$$

is given by

$$x(s) = 2\sum_{n=0}^{\infty}\left(\frac{1}{n^2\pi^2 - \mu}\int_0^1 y(t)\cos n\pi t\,dt\right)\cos n\pi s, \quad s \in [0,1].$$

If $\mu = 0$, then the problem

$$-x' = y, \quad x'(0) = 0 = x'(1)$$

has a solution if and only if $\int_0^1 y(t)\,dt = 0$. In that case, the solutions are given by

$$x(s) = c + 2\sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2}\int_0^1 y(t)\cos n\pi t\,dt\right)\cos n\pi s, \quad s \in [0,1],$$

where $c \in \mathbf{C}$.

(c) (One end fixed, one end free) $x(0) = 0 = x'(1)$. The eigenvalues of S are $\mu_n = (n+\frac{1}{2})^2\pi^2, n = 0, 1, 2, \ldots,$ with the corresponding eigenfunctions $u_n(t) = \sqrt{2}\sin(n + \frac{1}{2}\pi)t, t \in [0,1]$.

To find the Green function $k(.,.)$ for S, we take $u_a(t) = t, u_b(t) = 1$ for $t \in [0,1]$ and obtain

$$k(s,t) = \begin{cases} s, & \text{if } 0 \leq s \leq t \leq 1, \\ t, & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

For a solution of the corresponding Fredholm integral equation, see Problem 28-11.

(d) (Elastically held ends) $x(0) = x'(0), x(1) = -x'(1)$. The eigenvalues of S are $\mu_n = \theta_n^2$, where $\theta_n > 0$ is a root of $\tan\theta = -2\theta/(1-\theta^2)$, with the corresponding eigenfunctions $u_n(t) = \theta_n\cos\theta_n t$

$+\sin\theta_n t, t \in [0,1]$. To find the Green function for S, we take $u_a(t) = t+1, u_b(t) = t-2$ for $t \in [0,1]$ and obtain

$$k(s,t) = \begin{cases} \frac{1}{3}(1+s)(2-t), & \text{if } 0 \le s \le t \le 1, \\ \frac{1}{3}(1+t)(2-s), & \text{if } 0 \le t \le s \le 1. \end{cases}$$

(e) Consider the boundary conditions

$$x(0) = -x'(0), \quad x(1) = x'(1).$$

In this case, there is one negative eigenvalue $\mu_0 = -\theta_0^2$, where θ_0 is the only root of $e^\theta = (\theta+1)/(\theta-1)$ which is greater than 1 ($\mu_0 = -2.382$ approximately) with the corresponding eigenfunction $(\theta_0 - 1)e^{\theta_0 t} + (\theta_0+1)e^{-\theta_0 t}, t \in [0,1]$. The other eigenvalues are $\mu_n = \theta_n^2$, where $\theta_n > 0$ is a root of $\tan\theta = 2\theta/(1-\theta^2)$ with the corresponding eigenfunctions $u_n(t) = k_n(-\theta_n \cos\theta_n t + \sin\theta_n t), t \in [0,1]$ for some k_n.

To find the Green function for S, we take $u_a(t) = 1-t$ and $u_b(t) = t$ for $t \in [0,1]$ and obtain

$$k(s,t) = \begin{cases} (s-1)t, & \text{if } 0 \le s \le t \le 1 \\ (t-1)s, & \text{if } 0 \le t \le s \le 1. \end{cases}$$

Notice that $k(s,t) \le 0$ for all $s,t \in [0,1]$.

The method which we have developed can be used to solve the following kinds of boundary value problems:

$$\begin{aligned} &\frac{1}{w(t)}\left[-(p(t)x'(t))' + q(t)x(t)\right] - \mu x(t) = y(t), \quad t \in [a,b], \\ &\alpha x(a) + \alpha_1 x'(a) = 0 = \beta x(b) + \beta_1 x'(b), \end{aligned}$$

where $[a,b]$ is a finite closed interval, $p \in C^1([a,b])$, w and q are real-valued continuous functions on $[a,b]$, p and w are positive functions, $\alpha, \alpha_1, \beta, \beta_1$ are real numbers as before and $\mu \in \mathbf{C}$. Such a problem is known as a **regular Sturm-Liouville problem** .

If any one of the conditions mentioned above is violated, then our procedure may not work. For example, the interval under consideration may be infinite, or one of the functions w and p may vanish at one of the end-points of the interval, or w, p, q may fail to be continuous at some points in the interval. In these situations, the Sturm-Liouville problem is said to be **singular**. We shall not go into a discussion of such a problem, but only mention that in many cases, it is still possible to obtain the inverse of the Sturm-Liouville operator as a compact integral operator. We shall now state some of these cases, which are of considerable importance in the discussion of the Schrödinger equation in quantum mechanics. (See Appendix D.) They also arise while solving some classical partial differential equations by separating the variables. Note the occurrence of orthonormal polynomials (considered in 22.3) as the eigenfunctions of some of these singular Sturm-Liouville operators.

1) $a = -1, b = 1, w(t) = 1$, $p(t) = 1 - t^2$, $q(t) = 0$ for $t \in [-1, 1]$ so that the homogeneous differential equation is

$$((1-t^2)x'(t))' + \mu x(t) = 0 \quad \text{or} \quad (1-t^2)x''(t) - 2tx'(t) + \mu x(t) = 0.$$

As boundary conditions, we require that $\lim\limits_{t \to -1} x(t)$ and $\lim\limits_{t \to +1} x(t)$ exist and are finite.

Then the eigenvalues are $\mu_n = n(n+1), n = 0, 1, 2, \ldots$ and the corresponding eigenfunctions are the **Legendre polynomials**.

2) $a = -1, b = 1, w(t) = (1-t^2)^{-\frac{1}{2}}, p(t) = (1-t^2)^{\frac{1}{2}}, q(t) = 0$ for $t \in [-1, 1]$, so that the homogeneous differential equation is

$$((1-t^2)^{\frac{1}{2}}x'(t))' + \mu(1-t^2)^{-\frac{1}{2}}x(t) = 0 \text{ or } (1-t^2)x''(t) - tx'(t) + \mu x(t) = 0$$

The boundary conditions are the same as in 1) above.

Then the eigenvalues are $\mu_n = n(n-1), n = 1, 2, \ldots$ and the corresponding eigenfunctions are the **Tchebychev polynomials of the first kind** .

3) $a = 0, b = \infty, w(t) = e^{-t},\ p(t) = te^{-t},\ q(t) = 0$ for $t \in [0, \infty)$, so that the homogeneous differential equation is

$$(te^{-t}x'(t))' + \mu e^{-t}x(t) = 0 \quad \text{or} \quad tx''(t) + (1-t)x'(t) + \mu x(t) = 0.$$

As boundary conditions, we require that $\lim_{t\to 0} x(t)$ exists and $|x(t)/t^m|$ is bounded as $t \to \infty$ for some nonnegative integer m.

Then the eigenvalues are $\mu_n = n, n = 1, 2, \ldots$ and the corresponding eigenfunctions are the **Laguerre polynomials** .

4) $a = -\infty, b = \infty, w(t) = e^{-t^2} = p(t), q(t) = 0$ for $t \in (-\infty, \infty)$, so that the homogeneous differential equation is

$$(e^{-t^2}x'(t))' + \mu e^{-t^2}x(t) = 0 \quad \text{or} \quad x''(t) - 2tx'(t) + \mu x(t) = 0.$$

As boundary conditions, we require that $|x(t)/t^m|$ is bounded as $t \to \pm\infty$ for some nonnegative integer m.

Then the eigenvalues are $\mu_n = 2n, n = 0, 1, 2, \ldots$ and the corresponding eigenfunctions are the **Hermite polynomials** .

Letting $\psi(s) = \exp(-s^2/2)\, x(s)$, we obtain

$$\psi''(s) - s^2\psi(s) + (\mu + 1)\psi(s) = 0, \quad s \in (-\infty, \infty),$$

which is the time-independent Schrödinger equation of an harmonic oscillator after a change of units and with μ replaced by $\mu + 1$. (See Appendix D.)

For a detailed discussion of singular Sturm-Liouville problems, we refer the reader to [60].

Appendix D

Unbounded Operators and Quantum Mechanics

Solutions of linear differential equations can be studied systematically by considering appropriate differential operators defined on suitable subspaces of a Hilbert space. Such an operator is rarely defined on the whole Hilbert space and is almost never bounded. As the simplest example, consider the linear space $C^1([0,1])$ of all continuously differentiable scalar-valued functions on $[0,1]$ as a subspace of the Hilbert space $L^2([0,1])$ and for $x \in C^1([0,1])$, let $T(x) = x'$, the derivative of x. Then the operator $T : C^1([0,1]) \to L^2([0,1])$ is unbounded, since by letting $x_n(t) = t^n, t \in [0,1]$ and $n = 1, 2, \ldots$, we see that

$$\|x_n\|_2^2 = \int_0^1 t^{2n}\, dt = \frac{1}{2n+1}, \quad \|T(x_n)\|_2^2 = \int_0^1 n^2 t^{2n-2}\, dt = \frac{n^2}{2n-1}.$$

In this appendix, we shall extend some of the results for bounded operators on a Hilbert space given in Chapter VII to possibly unbounded operators defined on subspaces of a Hilbert space. [If an operator T defined on a subspace of a Hilbert space H is bounded, then T has a unique bounded linear extension $\bar{T}$ to the closure F of that subspace and if for $x \in H$ with $x = y + z, y \in F, z \in F^{\perp}$, we let $\tilde{T}(x) = \bar{T}(y)$, then $\tilde{T}$ is a bounded linear extension of T to H. In this case no new treatment is necessary.] Such operators occur in quantum mechanics. In fact, quantum mechanics provided on impetus to the development of operator theory in a Hilbert space setting during the late 1920's and the early 1930's. We shall consider a simple quantum mechanical system to illustrate this point later in this appendix.

Throughout this appendix, H will denote a Hilbert space over the scalars $\mathbf{K}$. A linear map T from a subspace of H to H is called an **operator in** H and the subspace, denoted by D_T, is called the **domain** of T. Let S be another operator in H. We say that S extends T if $D_T \subset D_S$ and $S(x) = T(x)$ for all $x \in D_T$. We write $T = S$ if T and S extend each other.

We note that $S + T$ and ST are operators in H with

$$D_{S+T} = D_S \cap D_T \quad \text{and} \quad D_{ST} = \{x \in D_T : T(x) \in D_S\}.$$

Although, in general, differential operators are not continuous, many of them possess a related property. We say that an operator T in H is **closed** if the following holds: Whenever $x_n \in D_T$, $x_n \to x$ in H and $T(x_n) \to y$ in H, we have $x \in D_T$ and $T(x) = y$. [This definition agrees with the definition of a closed operator on H given in Section 10.]

We show that the operator $T(x) = x'$ with $D_T = C^1([0,1])$, which is a subspace of $L^2([0,1])$, is not closed, but it can be extended to a closed operator $\tilde{T}$ by enlarging D_T to

$$D_{\tilde{T}} = \{x \in C([0,1]) : x \text{ is aboslutely continuous and } x' \in L^2([0,1])\}.$$

Consider

$$y(t) = \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

and

$$y_n(t) = \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ n(t - \frac{1}{2} + \frac{1}{n}), & \text{if } \frac{1}{2} - \frac{1}{n} < t \le \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

for $n = 2, 3, \ldots$ Then it is clear that $y_n \to y$ in $L^2([0,1])$. For $t \in [0,1]$, define

$$x(t) = \int_0^t y(s)\,ds \quad \text{and} \quad x_n(t) = \int_0^t y_n(s)\,ds.$$

Then $x'(t) = y(t)$ for all $t \in [0,1]$ except when $t = 1/2$ and $x_n'(t) = y_n(t)$ for all $t \in [0,1]$. Also,

$$|x_n(t) - x(t)| \le \int_0^t |y_n(s) - y(s)|\,ds \le \|y_n - y\|_1 \le \|y_n - y\|_2.$$

Hence (x_n) converges to x uniformly on $[0,1]$. Thus $x_n \in D_T, x_n \to x$ and $x_n' = y_n \to y$ in $L^2([0,1])$. It can be easily seen, however, that x is not differentiable at $1/2$. Hence $x \notin D_T$. This shows that the operator T is not closed. On the other hand, if $x_n \in D_{\tilde{T}}, x_n \to x$ in $L^2([0,1])$ and $\tilde{T}(x_n) = x_n' \to y$ in $L^2([0,1])$, then it can be shown that $(x_n(0))$ is a Cauchy sequence and if $x_n(0) \to k$ as $n \to \infty$, then

$$x(t) = k + \int_0^t y(s)\, dm(s), \quad t \in [0,1].$$

(See Example 21.3(d) and Problem 21-13.) Thus $x \in D_{\tilde{T}}$ and $\tilde{T}(x) = x' = y$ in $L^2([0,1])$. This shows that $\tilde{T}$ is a closed operator.

We remark that if T is a closed operator in H and D_T is a closed (and hence a complete) subspace of H, then T is continuous by the closed graph theorem. (See 10.2 or Problem 25-20(a).)

We now pass on to adjoint considerations. Given an operator T in H, we wish to define an operator T^* in H such that $\langle T(x),\, y\rangle = \langle x,\, T^*(y)\rangle$ for all $x \in D_T$ and all y in the largest possible subspace of H. In order that the element $T^*(y)$ satisfying $\langle T(x),\, y\rangle = \langle x,\, T^*(y)\rangle$ for all $x \in D_T$ be uniquely determined by T, it is necessary and sufficient that D_T be dense in H. Then the largest possible subset of H on which T^* can be so defined is

$$D_{T^*} = \{y \in H : \langle T(x),\, y\rangle = \langle x,\, z\rangle \text{ for all } x \in D_T \text{ and some } z \in H\}.$$

Note that D_{T^*} is a subspace of H. Let $y \in D_{T^*}$. Since D_T is dense in H, there is a unique $z \in H$ with $\langle T(x),\, y\rangle = \langle x,\, z\rangle$ for all $x \in D_T$. We define $T^*(y) = z$. Thus for a **densely defined** operator T in H, the adjoint operator T^* is defined on the subspace D_{T^*} of H and we have

$$\langle T(x),\, y\rangle = \langle x,\, T^*(y)\rangle \quad \text{for all } x \in D_T, y \in D_{T^*}.$$

Let us give some illustrative examples. Consider $\mathbf{K} = \mathbf{C}, H = L^2([0,1])$ and $T_1(x) = T_2(x) = T_3(x) = ix'$ with

$$D_{T_1} = \{x \in C([0,1]) : x \text{ absolutely continuous and } x' \in L^2([0,1])\},$$

$$D_{T_2} = \{x \in D_{T_1} : x(0) = x(1)\}, \quad D_{T_3} = \{x \in D_{T_1} : x(0) = 0 = x(1)\}.$$

Note that T_1 extends T_2 and T_2 extends T_3. Since polynomials are dense in $C([0,1])$ by 3.12 and since $C([0,1])$ is dense in $L^2([0,1])$ by 4.7(b), it follows that D_{T_1} is dense in $L^2([0,1])$. In fact, by requiring the approximating polynomials to be zero at 0 and 1, we can show that D_{T_3} is dense in $L^2([0,1])$. Thus T_1, T_2 and T_3 are densely defined. For $x, y \in D_{T_1}$, we have

$$\begin{aligned} \langle ix', y\rangle &= i\int_0^1 x'\overline{y}\,dm = i\left[x(1)\overline{y}(1) - x(0)\overline{y}(0) - \int_0^1 x\overline{y}'\,dm\right] \\ &= i\,[x(1)\overline{y}(1) - x(0)\overline{y}(0)] + \langle x,\, iy'\rangle. \end{aligned}$$

This relation allows us to conclude that $T_1{}^*$ extends $T_3, T_2{}^*$ extends T_2 and $T_3{}^*$ extends T_1. On the other hand, if $y \in D_{T_j{}^*}, j = 1, 2, 3$, and we let

$$z(t) = \int_0^t T^*(y)(s)\,dm(s),$$

then for $j = 1, 2, 3$ and all $x \in D_{T_j}$, we have

$$\begin{aligned} \langle T_j(x),\, y\rangle &= \langle x,\, T_j{}^*(y)\rangle = \int_0^1 x\,\overline{T_j{}^*(y)}\,dm \\ &= \int_0^1 x\overline{z}'\,dm = x(1)\overline{z}(1) - x(0)\overline{z}(0) - \int_0^1 x'\overline{z}\,dm. \end{aligned}$$

This relation allows us to conclude that T_3 extends $T_1{}^*, T_2$ extends $T_2{}^*$ and T_1 extends $T_3{}^*$. Thus we have

$$T_1{}^* = T_3, \quad T_2{}^* = T_2 \quad \text{and} \quad T_3{}^* = T_1.$$

This also exhibits how D_{T^*} is sensitive to D_T.

Let T be a densely defined operator in H. Then T^* is always a closed operator. For if $y_n \in D_{T^*}, y_n \to y$ in H and $T^*(y_n) \to z$ in H, then for all $x \in D_T$,

$$\langle T(x),\, y\rangle = \lim_{n\to\infty}\langle T(x),\, y_n\rangle = \lim_{n\to\infty}\langle x,\, T^*(y_n)\rangle = \langle x,\, z\rangle,$$

so that $y \in D_{T^*}$ and $T^*(y) = z$. In particular, we see that the operators T_1, T_2 and T_3 considered above are closed since they are adjoints of some operators in H.

Note, however, that T^* need not be densely defined. In fact, D_{T^*} can reduce to zero. For example, let $H = \ell^2$. Consider a double indexing $e_{k,j}, , k, j = 1, 2, \ldots$ of the standard orthonormal basis $\{e_1, e_2, \ldots\}$ of ℓ^2. Let $T(e_{k,j}) = e_k$ for $k, j = 1, 2, \ldots$ and extend T linearly to the span of $\{e_{k,j} : k, j = 1, 2, \ldots\} = D_T$. If $y \in D_{T^*}$, then for each $k = 1, 2, \ldots,$

$$y(k) = \langle y, e_k \rangle = \langle y, T(e_{k,j}) \rangle = \langle T^*(y), e_{k,j} \rangle,$$

which tends to zero as $j \to \infty$ by 22.6, so that $y = 0$.

On the other hand, if T is a closed densely defined operator, then T^* is also a closed densely defined operator. This allows us to prove the existence and uniqueness of solutions of several operator equations involving T and T^*.

Theorem D1

Let T be a closed densely defined operator in H. Then

(**a**) D_{T^*} is dense in H and $T^{**} = T$.

(**b**) Given z and w in H, there exist unique $x \in D_T$ and $y \in D_{T^*}$ such that

$$T(x) + y = z \quad \text{and} \quad x - T^*(y) = w.$$

(**c**) Given $w \in H$, there is a unique $x \in D_{T^*T}$ such that

$$x + T^*T(x) = w.$$

Proof:

Consider the Hilbert space $H \times H$ with the inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$$

for $x_1, x_2, y_1, y_2 \in H$. Since T is a closed operator, the set

$$F = \{(x, T(x)) : x \in D_T\}$$

is a closed subspace of $H \times H$. By the projection theorem (24.1), we have $H \times H = F + F^{\perp}$ with $F \cap F^{\perp} = \{0\}$. Now $(u, y) \in F^{\perp}$ if and only if for all $x \in D_T$,

$$0 = \langle (x, T(x)), (u, y) \rangle = \langle x, u \rangle + \langle T(x), y \rangle,$$

that is, $\langle T(x), y \rangle = \langle x, -u \rangle$, or $y \in D_{T^*}$ and $u = -T^*(y)$. Thus

$$F^{\perp} = \{(-T^*(y), y) : y \in D_{T^*}\},$$

so that

$$H \times H = \{(x, T(x)) : x \in D_T\} + \{(-T^*(y), y) : y \in D_{T^*}\}.$$

(**a**) Let $u \perp D_{T^*}$. Then for all $y \in D_{T^*}$, we have

$$\langle (0, u), (-T^*(y), y) \rangle = \langle 0, -T^*(y) \rangle + \langle u, y \rangle = 0 + 0 = 0.$$

Hence $(0, u) \in (F^{\perp})^{\perp}$, which equals F by 24.1. Thus $u = T(0) = 0$. This proves that D_{T^*} is dense in H.

Let $x \in D_T$. Then for every $y \in D_{T^*}$, we have $\langle T^*(y), x \rangle = \langle y, T(x) \rangle$. This shows that $x \in D_{T^{**}}$ and $T^{**}(x) = T(x)$. Thus T^{**} extends T. On the other hand, let $z \in D_{T^{**}}$. Then it is easy to see that $(z, T^{**}(z)) \perp (-T^*(y), y)$ for every $y \in D_{T^*}$, that is, $(z, T^{**}(z))$ is in $F^{\perp\perp} = F$. Hence $z \in D_T$ and $T(z) = T^{**}(z)$. Thus $T^{**} = T$.

(**b**) Since $(w, z) \in H \times H = F + F^{\perp}$, there are unique $x \in D_T$ and $y \in D_{T^*}$ such that

$$(w, z) = (x, T(x)) + (-T^*(y), y),$$

that is, $w = x - T^*(y)$ and $z = T(x) + y$, as desired.

(c) Letting $z = 0$ in (b) above, we see that there exist unique x in D_T and y in D_{T^*} such that $T(x) = -y$ and $w = x - T^*(y) = x + T^*T(x)$, as desired. □

Noting that the operators T_1, T_2 and T_3 considered earlier are closed and densely defined, we deduce the following result from part (b) of Theorem D1. Given z and w in $L^2([0,1])$, there are unique absolutely continuous functions x and y on $[0,1]$ with x' and y' in $L^2([0,1]), y(0) = 0 = y(1)$ and $ix' + y = z, x - iy' = w$. The conditions $y(0) = 0 = y(1)$ can be replaced either by the conditions $x(0) = x(1), y(0) = y(1)$, or by the conditions $x(0) = 0 = x(1)$. Similarly, part (c) of Theorem D1 yields the following result. Given $w \in L^2([0,1])$, there is a unique absolutely continuous function x on $[0,1]$ with x' absolutely continuous on $[0,1], x'' \in L^2([0,1]), x'(0) = 0 = x'(1)$ and $-x'' + x = w$. Again, the conditions $x'(0) = 0 = x'(1)$ can be replaced either by the conditions $x(0) = x(1), x'(0) = x'(1)$, or by the conditions $x(0) = 0 = x(1)$.

We now consider a special kind of operators in H. An operator T in H is called **symmetric** if for all $x, y \in D_T$, we have

$$\langle T(x),\, y\rangle = \langle x,\, T(y)\rangle.$$

It is clear that if T is symmetric, then $\langle T(x),\, x\rangle \in \mathbf{R}$ for every $x \in D_T$. Also, it follows that a densely defined operator T is symmetric if and only if T^* extends T. If T is symmetric and $D_T = H$, then T is, in fact, a bounded operator on H. This can be seen as follows. Let $E = \{T(x) : x \in H, \|x\| \leq 1\}$. For a fixed $y \in H$, we have

$$|\langle T(x),\, y\rangle| = |\langle x,\, T(y)\rangle| \leq \|x\|\, \|T(y)\| \leq \|T(y)\|$$

for all $x \in H$ with $\|x\| \leq 1$. The uniform boundedness principle (9.1 or 24.8) then shows that E is a bounded subset of H, that is, T is a bounded operator on H. This result is known as the **Hellinger-Toeplitz theorem**. It was proved in 1910.

Let T be a densely defined operator in H. Then T is called **self-adjoint** if $T^* = T$. It is clear that a self-adjoint operator is always closed. As examples, we note that the operator T_2 is self-adjoint, the operator T_3 is symmetric but not self-adjoint, while the operator T_1 is not even symmetric.

It is often of interest to know when the sum of two self-adjoint operators is self-adjoint. In this regard, we state the following result. Let T be a self-adjoint operator and S be a symmetric operator in H with $D_T \subset D_S$. Suppose that

$$\|S(x)\| \leq \alpha\|x\| + \beta\|T(x)\|$$

for all $x \in D_T$ and some nonnegative constants α and β with $\beta < 1$. Then $T + S$ is self-adjoint. This result is known as the **Kato-Rellich theorem**. See Theorem X.12 of [47] for a proof. Choosing a self-adjoint operator A with $D_A \neq H$ and $B = -A$, we easily see that the result does not hold if $\beta = 1$. For nontrivial examples when $\beta > 1$, see p. 172 of [47]. We shall later consider an application of the Kato-Rellich theorem in quantum mechanics.

We now turn to spectral considerations of operators in H. An operator T in H is said to **invertible** if T is a bijective map from D_T to H and its inverse (denoted by T^{-1}) is a bounded linear operator on H. As in Section 12, the set

$$\rho(T) = \{k \in \mathbf{K} : T - kI \text{ is invertible }\}$$

is called the **resolvent set** of T. The complement $\sigma(T)$ of $\rho(T)$ in $\mathbf{K}$ is called the **spectrum** of T.

If T is an invertible operator in H, then it can be seen that

$$\sigma(T) = \{\frac{1}{k} : k \in \sigma(T^{-1}), k \neq 0\}.$$

(Compare Problem 12-1.) Let us consider an example. In Appendix C, we have discussed the Sturm-Liouville operator S given by $S(x) = -x'' + qx$ for $x \in D_S = \{x \in C^2([a,b]) : B_a(x) = 0 = B_b(x)\}$. Let $D_T = \{x \in C^1([a,b]) : x'$ absolutely continuous on $[a,b]$, x'' in $L^2([a,b]), B_a(x) = 0 = B_b(x)\}$ and $T(x) = -x'' + qx$ for $x \in D_T$.

Note that T extends S. Assume that 0 is not an eigenvalue of S. As in the proof of part (a) of Theorem C3, it can be shown that for each y in $L^2([a,b])$, $T(x) = y$ if and only if $A(y) = x$, where A is the Fredholm integral operator with the Green function for S as its kernel. Then it follows that T is an invertible operator in $L^2([a,b])$. As A is a nonzero compact self-adjoint operator on $L^2([a,b])$, $\{k : k \in \sigma(A), k \neq 0\} = \{s_1, s_2, \ldots\}$ by 28.5(b), so that $\sigma(T) = \{1/s_1, 1/s_2, \ldots\}$.

Consider an invertible operator T in H. Since $U = T^{-1}$ is a bounded linear operator on H, it follows from 12.3 that $I - kU$ is an invertible operator on H, provided $|k| < 1/\|U\|$. Using $TU = I$ and $UT(x) = x$ for all $x \in D_T$, we easily see that $(T - kI)U(I - kU)^{-1} = I$ and $U(I - kU)^{-1}(T - kI)(x) = x$ for all $x \in D_T$. Thus for all $|k| < 1/\|U\|$, $T - kI$ is a bijective linear map from $D_{T-kI} = D_T$ to H and its inverse $U(I - kU)^{-1}$ is a bounded linear operator on H, that is, $k \in \rho(T)$. This implies that $\rho(T)$ is an open subset and $\sigma(T)$ is a closed subset of $\mathbf{K}$. (Compare 12.6(b) and Problem 27-1.)

Theorem D2

Let T be a closed operator in H. Then $k \in \rho(T)$ if and only if $T - kI$ is bounded below (that is, $\beta\|x\| \leq \|(T - kI)(x)\|$ for some $\beta > 0$ and all $x \in D_T$) and the range of $T - kI$ is dense in H.

Proof:

Since H is complete and T is a closed operator, the proof of 12.1(b) holds, so that T is invertible if and only if T is bounded below and the range of T is dense in H. The desired result follows by noting that $T - kI$ is a closed operator for every $k \in \mathbf{K}$. □

Corollary D3

Let T be a self-adjoint operator in H. Then $\sigma(T) \subset \mathbf{R}$. In fact, if $k \in \mathbf{K}$ and $\operatorname{Im} k \neq 0$, then

$$\|(T - kI)^{-1}\| \leq \frac{1}{|\operatorname{Im} k|}.$$

Proof:
Let $k \in \mathbf{K}$ and $\operatorname{Im} k \neq 0$. We prove that $k \in \rho(T)$. Since a self-adjoint operator is closed, it is enough to show that

$$|\operatorname{Im} k| \, \|x\| \leq \|(T - kI)x\|$$

for all $x \in D_T$, that the range of $T - kI$ is dense in H and apply Theorem D2.

Now for all $x \in D_T$,

$$\|(T - kI)(x)\|^2 = \|(T - \operatorname{Re} k \, I)x - i \operatorname{Im} kx\|^2.$$

If we let $S = T - \operatorname{Re} k \, I$, then S is a symmetric operator and hence

$$\begin{aligned} \|S(x) - i\operatorname{Im} kx\|^2 &= \|S(x)\|^2 + i \operatorname{Im} k\langle S(x), x\rangle \\ &\qquad - i \operatorname{Im} k\langle x, S(x)\rangle + |\operatorname{Im} k|^2 \|x\|^2 \\ &= \|S(x)\|^2 + |\operatorname{Im} k|^2 \|x\|^2. \end{aligned}$$

This shows that

$$\|(T - kI)x\|^2 \geq |\operatorname{Im} k|^2 \|x\|^2,$$

as desired. To show that the range of $T - kI$ is dense in H, consider $y \in H$ such that $\langle (T - kI)x, y\rangle = 0$ for all $x \in D_T$. Since

$$\langle T(x), y\rangle = \langle kx, y\rangle = \langle x, \overline{k}y\rangle$$

for all $x \in D_T$, it follows that $y \in D_{T^*}$ and $T^*(y) = \overline{k}y$. As T is self-adjoint, we have $T(y) = T^*(y) = \overline{k}y$. But then

$$\overline{k}\langle y, y\rangle = \langle \overline{k}y, y\rangle = \langle T(y), y\rangle \in \mathbf{R},$$

which is impossible, unless $\langle y, y\rangle = 0$. (Note $\operatorname{Im} k \neq 0$.) Thus $y = 0$, showing that the range of $T - kI$ must be dense in H. □

We remark that the spectrum of a closed densely defined symmetric operator in H may not be contained in $\mathbf{R}$. For example, the operator T_3 considered earlier is closed, densely defined and symmetric. However, $\sigma(T_3) = \mathbf{C}$, since $T_3 - kI$ is not surjective for every $k \in \mathbf{C}$. This can be seen as follows. If y belongs to the range of $T_3 - kI$, then there is some absolutely continuous function x on $[0,1]$ such that $x' \in L^2([0,1])$ and

$$ix' - kx = y, \quad x(0) = 0 = x(1).$$

Solving the differential equation, we see that for all $t \in [0,1]$

$$x(t) = ce^{-ikt} - ie^{-ikt} \int_0^t e^{iks} y(s)\, dm(s)$$

for some $c \in \mathbf{C}$. Now the condition $x(0) = 0$ implies that $c = 0$ and the condition $x(1) = 0$ implies that

$$\int_0^1 e^{iks} y(s)\, dm(s) = 0.$$

In particular, if we let $y(s) = e^{-iks}$ for $s \in [0,1]$, then y does not belong to the range of $T_3 - kI$.

The preceding example also shows that the spectrum of an operator in H may not be bounded. (Compare 12.6(b) and Problem 27-1.) This holds even for a self-adjoint operator in H. For example, the operator T_2 considered earlier is self-adjoint, but $T_2 - 2n\pi I$ is not injective for each $n = 0, \pm 1, \pm 2, \ldots$. This follows by noting that $T_2(x_n) = 2n\pi x_n$, where $x_n(t) = e^{-2n\pi i t}$ for $t \in [0,1]$.

We also point out that even when $\mathbf{K} = \mathbf{C}$, the spectrum of an operator in H may be empty. (Compare 12.8(a).) For example, let $H = L^2([0,1]$ and $T_4(x) = ix'$ for $x \in D_{T_4} = \{x \in D_{T_1} : x(0) = 0\}$. Consider $k \in \mathbf{C}$. Then for every $y \in L^2([0,1])$, there is a unique $x \in D_{T_4}$ such that

$$T_4(x) - kx = ix' - kx = y,$$

namely,

$$x(t) = -ie^{-ikt} \int_0^t e^{iks} y(s)\, dm(s), \quad t \in [0,1].$$

Also, this x depends continuously on $y \in L^2([0,1])$. Hence $T_4 - kI$ is invertible. Thus $\sigma(T_4) = \emptyset$.

We now study two important operators which will enter our discussion of quantum mechanics. Let $H = L^2(\mathbf{R})$ and consider the **multiplication operator** $S(x)(t) = tx(t)$ and the **differentiation operator** $T(x) = ix'$ with

$$D_S = \{x \in H : S(x) \in H\} \quad \text{and} \quad D_T = \{x \in H : T(x) \in H\}$$

Note that $x \in D_T$ if and only if x is absolutely continuous on every finite subinterval of $\mathbf{R}$ and $x' \in L^2(\mathbf{R})$ by the fundamental theorem for Lebesgue integration (4.3). There is a strong connection between the operators S and T. It allows us to obtain information for either one of them from the other.

Theorem D4

Let S and T denote the multiplication and the differentiation operators, respectively.

(a) For $x \in L^2(\mathbf{R})$, let $U(x) \in L^2(\mathbf{R})$ denote Fourier-Plancherel transform of x considered in 26.6. Then $S = U^{-1}TU$.

(b) S and T are self-adjoint operators in $L^2(\mathbf{R})$.

(c) $\sigma(S) = \mathbf{R} = \sigma(T)$. In fact, for every $k \in \mathbf{R}$, both $S - kI$ and $T - kI$ are not bounded below although they are injective.

Proof:

(a) We refer the reader to [48], pp. 293-295 for a technical proof.

(b) For a subset E of $\mathbf{R}$ let c_E denote the characteristic function of E. To see that S is an unbounded operator, let $x_n = c_{[n,n+1]}$, $n = 1, 2, \ldots$. Then $\|x_n\|_2 = 1$, but

$$\|S(x_n)\|_2^2 = \int_n^{n+1} t^2\, dt = n^2 + n + \frac{1}{3} \to \infty \quad \text{as } n \to \infty.$$

To show that S is densely defined, consider $x \in L^2(\mathbf{R})$. If we let $x_n = c_{[-n,n]}x$ for $n = 1, 2, \ldots$, then $x_n \in D_S$ and $\|x_n - x\|_2 \to 0$ as $n \to \infty$ by Lebesgue's dominated convergence theorem (1.1(b)). Next, for x and y in D_S, we have

$$\langle S(x),\, y\rangle = \int_{\mathbf{R}} tx(t)\overline{y(t)}\,dm(t) = \int_{\mathbf{R}} x(t)\overline{ty(t)}\,dm(t) = \langle x,\, S(y)\rangle.$$

Thus S is symmetric and hence S^* extends S. Finally, let $y \in D_{S^*}$. By the definition of D_{S^*}, there is $z \in L^2(\mathbf{R})$ such that $\langle S(x),\, y\rangle = \langle x,\, z\rangle$ for all $x \in D_S$, that is,

$$\int_{\mathbf{R}} x(t)[\overline{ty(t)} - \overline{z(t)}]\,dm(t) = 0$$

for all $x \in D_S$. If we let $x_n = (S(y) - z)c_{[-n,n]}$ for $n = 1, 2, \ldots$, then it follows that $x_n \in D_S$ and

$$\int_{-n}^{n} |x_n|^2\,dm = \int_{-n}^{n} |ty(t) - z(t)|^2\,dm(t) = 0.$$

Hence $x_n(t) = 0$, that is, $ty(t) = z(t)$ for almost all $t \in [-n, n]$. Since this is true for each $n = 1, 2, \ldots$, we see that $ty(t) = z(t)$ for almost all $t \in \mathbf{R}$. Therefore $S(y) = z \in L^2(\mathbf{R})$, showing that $D_{S^*} \subset D_S$. Thus $S^* = S$, that is, S is self-adjoint.

By 26.6, U is a unitary operator on $L^2(\mathbf{R})$, that is, U is a bounded linear operator with $U^* = U^{-1}$. As $T = USU^{-1}$ by (a) above, we conclude that T is also an unbounded self-adjoint operator in $L^2(\mathbf{R})$.

(c) Again, it is enough to consider the operator S. Since S is self-adjoint, Corollary D3 shows that $\sigma(S) \subset \mathbf{R}$. On the other hand, consider $k \in \mathbf{R}$. Let $0 < t_n \to 0$ as $n \to \infty$ and $x_n = c_{[k-t_n, k+t_n]}$ for $n = 1, 2, \ldots$. Then

$$\|x_n\|^2 = \int_{k-t_n}^{k+t_n} dt = 2t_n$$

and

$$\|(S - kI)(x_n)\|^2 = \int_{\mathbf{R}} (t-k)^2 |x_n(t)|^2\,dt = \int_{k-t_n}^{k+t_n} (t-k)^2\,dt = \frac{2t_n^3}{3},$$

so that $\|(S-kI)(x_n)\| = t_n\|x_n\|/\sqrt{3}$. Since $t_n \to 0$, we see that $S-kI$ is noboundedbelow. Hence it is not invertible by Theorem D2. Thus $\sigma(S) = \mathbf{R}$.

Finally, let $k \in \mathbf{R}, x \in D_S$ and $(S-kI)(x) = 0$. Then

$$\int_{\mathbf{R}} (t-k)^2 |x(t)|^2 \, dm(t) = \|(S-kI)(x)\|^2 = 0.$$

Hence $x(t) = 0$ for almost all $t \in \mathbf{R}$. Thus $S-kI$ is injective. □

Quantum Mechanics

We indicate how unbounded operators arise in quantum mechanics. To illustrate the basic ideas, we consider a simple physical system consisting of a single particle of mass $m = 1$ which is constrained to move in one dimension, that is, it moves in $\mathbf{R}$. We first consider this system at an arbitrary but fixed time.

In classical mechanics, this system is completely specified by two real numbers which give the position and the momentum of the particle. Instead of this deterministic specification, quantum mechanics offers a probabilistic description of the system. Consider a complex-valued measurable function ψ on $\mathbf{R}$ and let $|\psi(s)|^2, s \in \mathbf{R}$, play the role of the density of a probability distribution on $\mathbf{R}$, that is, if E is a measurable subset of $\mathbf{R}$, then let

$$\int_E |\psi(s)|^2 \, dm(s)$$

be the probability that the particle lies in E. Since the particle must always be somewhere in $\mathbf{R}$, we must have $\int_{\mathbf{R}} |\psi(s)|^2 \, dm(s) = 1$. In particular, $\psi \in L^2(\mathbf{R})$. Such a function ψ is called a **pure state** of the system under consideration. Sometimes ψ is also referred to as a **wave function** .

If the system is in state ψ, then the average position of the particle, that is, the **mean value** of the corresponding probability **distri-**

bution, is given by

$$\mu_\psi = \int_{\mathbf{R}} s|\psi(s)|^2\, dm(s)$$

and the spread of its positions about the average position, that is, the **variance** of the corresponding probability distribution, is given by

$$\mathrm{var}_\psi = \int_{\mathbf{R}} (s-\mu_\psi)^2|\psi(s)|^2\, dm(s).$$

Let $Q(\psi)(s) = s\psi(s)$ for $s \in \mathbf{R}$, $D_Q = \{\psi \in L^2(\mathbf{R}) : Q(\psi) \text{ in } L^2(\mathbf{R})\}$. Then Q is the multiplication operator considered earlier and

$$\mu_\psi = \langle Q(\psi), \psi\rangle \quad \text{and} \quad \mathrm{var}_\psi = \langle (Q-\mu_\psi I)^2(\psi), \psi\rangle$$

for all $\psi \in D_Q$. Thus it is most natural to call Q the **position operator**. We have already seen in part (b) of Theorem D4 that Q is a self-adjoint operator in $L^2(\mathbf{R})$. Position is one of the important 'observable' quantities about a physical system. It expresses one of the major aspects of the system that can be measured experimentally.

With this in mind, we define an **observable** is to be a densely defined possibly unbounded self-adjoint operator in $L^2(\mathbf{R})$. The **mean value** and the **variance** of an observable T in state ψ are defined by

$$\mu_\psi(T) = \langle T(\psi), \psi\rangle \quad \text{and} \quad \mathrm{var}_\psi(T) = \langle (T-\mu_\psi I)^2(\psi), \psi\rangle.$$

Some other important observable quantities about a physical system are its momentum and energy. A physical argument, which we do not give here, motivates us to define the **momentum operator** by

$$P(\psi)(s) = \frac{h}{2\pi i}\psi'(s), \quad s \in \mathbf{R},$$

with $D_P = \{\psi \in L^2(\mathbf{R}) : P(\psi) \in L^2(\mathbf{R})\}$, where h is **Planck's constant**. Let p denote the momentum of the particle. Since its mass equals 1, the kinetic energy of the system is $p^2/2$. Hence we define the **kinetic energy operator** E_0 by

$$E_0(\psi)(s) = \frac{P^2}{2}(\psi)(s) = -\frac{h^2}{8\pi^2}\psi''(s)$$

with $D_{E_0} = \{\psi \in D_P : P(\psi) \in D_P\} = \{\psi \in L^2(\mathbf{R}) : \psi', \psi'' \in L^2(\mathbf{R})\}$. Since $P = -hT/2\pi$, where T is the differentiation operator, we see by Theorem D4 that P is self-adjoint and

$$E_0 = \frac{P^2}{2} = \frac{h^2}{8\pi^3}T^2 = \frac{h^2}{8\pi^2}UQ^2U^{-1},$$

where U is the unitary operator on $L^2(\mathbf{R})$ given by the Fourier-Plancherel transform.

As in the proof of part (b) of Theorem D4c, we see that Q^2 is a self-adjoint operator. This, in turn, shows that E_0 is self-adjoint. Thus P and E_0 are observables.

We shall now consider some properties of these observables. First notice that if $\psi \in D(PQ) \cap D(QP)$, then for $s \in \mathbf{R}$,

$$(PQ - QP)(\psi)(s) = \frac{h}{2\pi i}\left[(s\psi(s))' - s\psi'(s)\right] = \frac{h}{2\pi i}\psi(s),$$

that is, $PQ - QP$ equals a scalar multiple of the identity operator on the domain $D(PQ) \cap D(QP)$. This domain is dense in $L^2(\mathbf{R})$ since it includes the functions $\psi_n(s) = s^n e^{-s^2/2}, s \in \mathbf{R}, n = 0, 1, 2, \ldots$ (See [38], p. 45.) This remarkable fact raises an interesting question. Is it possible to have $AB - BA = I$ for some bounded operators A and B on a Hilbert space H? This question was answered negatively by Wintner in 1947. We outlines here a proof given by Wielandt in 1949. Suppose A and B are bounded operators on H with $AB - BA = I$. Then it follows by mathematical induction that

$$nB^{n-1} = AB^n - B^nA$$

for $n = 1, 2, \ldots$, so that

$$n\|B^{n-1}\| = \|AB^n - B^nA\| \leq 2\|A\|\,\|B^n\| \leq 2\|A\|\,\|B\|\,\|B^{n-1}\|.$$

If $\|B^{n-1}\| \neq 0$, then $n \leq 2\|A\|\,\|B\|$. This shows that $\|B^m\| = 0$ for some positive integer m. But then $mB^{m-1} = 0$ by above, which in turn shows that $(m-1)B^{m-2} = 0$. Proceeding in this fashion, we have $B^2 = 0, B = 0$ and $B^0 = 0$, that is, $I = 0$, which is a contradiction.

There is a nice result for the commutator $ST - TS$ of symmetric operators S and T in H. If $x \in D_{ST} \cap D_{TS}$, then

$$|\langle (ST - TS)(x), x\rangle| \leq 2\|S(x) - \langle S(x), x\rangle x\| \, \|T(x) - \langle T(x), x\rangle x\|.$$

This can be proved as follows.

$$\begin{aligned}
\langle (ST - TS)(x), x\rangle &= \langle ST(x), x\rangle - \langle TS(x), x\rangle \\
&= \langle ST(x), x\rangle - \langle S(x), T(x)\rangle \\
&= \langle ST(x), x\rangle - \langle x, ST(x)\rangle = 2i \operatorname{Im} \langle ST(x), x\rangle.
\end{aligned}$$

Let $t = \langle T(x), x\rangle$ and $s = \langle S(x), x\rangle$. Then $t, s \in \mathbf{R}$, since S and T are symmetric. Now

$$ST - TS = (S - sI)(T - tI) - (T - tI)(S - sI).$$

Replacing S and T by $S - sI$ and $T - tI$, we have

$$\begin{aligned}
|\langle (ST - TS)(x), x\rangle| &= 2|\operatorname{Im} \langle (S - sI)(T - tI)(x), x\rangle| \\
&= 2|\operatorname{Im} \langle (T - tI)(x), (S - sI)(x)\rangle| \\
&\leq 2\|(S - sI)(x)\| \, \|(T - tI)(x)\|,
\end{aligned}$$

as desired. Letting $S = P$ and $T = Q$, we obtain

$$\begin{aligned}
|\langle \frac{h}{2\pi i}\psi, \psi\rangle| &= |\langle (PQ - QP)(\psi), \psi\rangle| \\
&\leq 2\|P(\psi) - \langle P(\psi), \psi\rangle\psi\| \, \|Q(\psi) - \langle Q(\psi), \psi\rangle\psi\|
\end{aligned}$$

for $\psi \in D_{PQ} \cap D_{QP}$. If $\langle \psi, \psi\rangle = 1$, it follows that

$$\frac{h}{4\pi} \leq (\operatorname{var}_\psi(P))^{1/2}(\operatorname{var}_\psi(Q))^{1/2}.$$

This is the famous **Heisenberg uncertainty principle** . Since the nonnegative square root of the variance gives the standard deviation of the probability distribution, this principle can be interpreted to say that the position and the momentum of the particle cannot be simultaneously determined with arbitrarily small error in the measurements. Thus the amount of possible precision is limited *a priori*.

In classical mechanics, the total energy of the system under consideration is given by its **Hamiltonian function** $H = H(q,p)$, which depends on the position coordinate q and the momentum coordinate p of the particle. In the quantum mechanical description, we replace q and p by the position operator Q and the momentum operator P respectively and obtain the **Hamiltonian operator** $H = H(Q,P)$. Since Q and P do not commute with each other, this replacement process is not, in general, unique. The equation

$$H(\psi)(s) = \mu\psi(s), \quad s \in \mathbf{R},$$

where $\psi \in D_H$ and μ is a scalar, is known as the **time-independent Schrödinger equation** . An eigenvalue of the Hamiltonian operator H, that is, a scalar μ for which this equation has a nonzero solution $\psi \in D_H$, represents a **quantized energy level** of the system under consideration.

In many cases of physical interest, the Hamiltonian function has the form

$$H(p,q) = \frac{p^2}{2} + V(q),$$

because the total energy of the system is the sum of the kinetic energy and the potential energy. The real-valued function V is called the **potential function**. Here the replacement of q and p by Q and P can be done uniquely and the Hamiltonian operator is given by

$$H(Q,P) = \frac{P^2}{2} + V(Q),$$

that is,

$$H(\psi)(s) = -\frac{h^2}{8\pi^2}\psi''(s) + V(s)\psi(s), \quad s \in \mathbf{R},$$

with an appropriate domain D_H. Let us consider some examples. In the case of a **free particle**, we have $V = 0$ and the time-independent Schrödinger equation is given by

$$\psi''(s) + \frac{8\pi^2}{h^2}\mu\psi(s) = 0, \quad s \in \mathbf{R}.$$

It can be easily seen that for each $\mu \in \mathbf{R}$, this equation admits no square-integrable solution on $\mathbf{R}$ except for the identically zero solution, that is, the Hamiltonian operator has no eigenvalues.

Next, in the case of a **harmonic oscillator**, we have $V(s) = \lambda_0 s^2$, where $\lambda_0 > 0$ is a physical constant and the time-independent Schrödinger equation is given by

$$\psi''(s) - \frac{8\pi^2}{h^2}\lambda_0 s^2 \psi(s) + \mu \frac{8\pi^2}{h^2}\psi(s) = 0, \quad s \in \mathbf{R}.$$

The eigenvalues of the corresponding Hamiltonian operator are

$$\mu_n = (2n+1)\sqrt{\frac{\lambda_0}{2}}\frac{h}{2\pi}, \quad n = 0, 1, 2, \ldots.$$

An eigenfunction corresponding to μ_n is given by $\exp(-s^2/2)\, h_n(s)$, where h_n is the nth Hermite polynomial. (See [38], pp. 213-214.)

It is important to know whether an Hamiltonian operator H is self-adjoint. Let $H = \dfrac{P^2}{2} + V(Q)$. We know that $E_0 = P^2/2$ is self-adjoint. Since H is obtained by perturbing E_0 by $V(Q)$, one wants to find conditions on the potential function V so that the $E_0 + V(Q)$ remains self-adjoint. One such condition can be deduced from the Kato-Rellich theorem. First note that every ψ in D_{E_0} is a bounded continuous function on $\mathbf{R}$. Also, for every $\epsilon > 0$, there is some $\delta > 0$ such that

$$\|\psi\|_\infty \le \delta\|\psi\|_2 + \epsilon\|E_0(\psi)\|_2, \quad \psi \in D_{E_0}.$$

See Theorem IX.28(a) of [47] for a proof.

Let $V = V_1 + V_2$, where $V_1 \in L^2(\mathbf{R})$ and $V_2 \in L^\infty(\mathbf{R})$. We have

$$\|V\psi\|_2 \le \|V_1\|_2\,\|\psi\|_\infty + \|V_2\|_\infty\,\|\psi\|_2, \quad \psi \in D_{E_0},$$

so that $D_{E_0} \subset D_V$. Choose ϵ such that $0 < \epsilon < 1/\|V_1\|_2$. Then

$$\|V\psi\|_2 \le (\delta\|V_1\|_2 + \|V_2\|_\infty)\|\psi\|_2 + \epsilon\|V_1\|_2\,\|E_0(\psi)\|_2, \quad \psi \in D_{E_0}.$$

Letting $\alpha = \delta\|V_1\|_2 + \|V_2\|_\infty$ and $\beta = \epsilon\|V_1\|_2$ in the Kato-Rellich theorem, we conclude that $H = E_0 + V(Q)$ is self-adjoint. This consideration applies to the **square-well potential**

$$V(s) = \begin{cases} V_0, & \text{if } |s| \leq s_0, \\ 0, & \text{if } |s| > s_0, \end{cases}$$

where $V_0 < 0$ and $s_0 > 0$, and also to the **modified Coulomb potential** $V(s) = 1/(|s| + a)$, $s \in \mathbf{R}$, where $a > 0$.

Before concluding this appendix, we mention the dynamics or the time-evolution of our quantum mechanical system.

For time $t \geq 0$, let $\psi_t \in L^2(\mathbf{R})$. We define

$$\frac{d\psi_t}{dt} = \lim_{\delta \to 0} \frac{\psi_{t+\delta} - \psi_t}{\delta},$$

whenever the limit exists in $L^2(\mathbf{R})$. The equation

$$H(\psi_t) = -\frac{h}{2\pi i}\frac{d\psi_t}{dt}$$

together with a given initial state $\psi_0 \in D_H$ is known as the **quantum mechanical equation of motion** or as the **time-dependent Schrödinger equation**. Its role is parallel to that of Hamilton's equations of motion in classical mechanics, namely

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

In case the initial state ψ_0 is an eigenvector of the Hamiltonian operator H corresponding to an eigenvalue μ_0, then we can easily verify that

$$\psi_t = \psi_0 \exp\left(-\frac{2\pi i}{h} t\mu_0\right), \quad t \in \mathbf{R},$$

is a solution of the quantum mechanical equation of motion. If the initial state ψ_0 is not an eigenvector of H, the solution ψ_t of the quantum mechanical equation of motion can be expressed as $\exp\left(-\frac{2\pi i}{h} tH\right)(\psi_0)$. However, this is beyond the scope of this book. (See [54], p. 104.)

Bibliography

[1] Alexander, F. E. (1974). Compact and finite rank operators on subspaces of ℓ_p, *Bull. London Math. Soc.* **6**, pp. 341-342.

[2] Banach, S. (1932), *Théorie des Opérations Linéaires*, Monografje Matematyczne, Warsaw.

[3] Birkhoff, G. D. and O. D. Kellog (1922), Invariant points in function space, *Trans. Amer. Math. Soc.* **23**, pp. 96-115.

[4] Bishop, E. (1967), *Foundations of Constructive Analysis*, McGraw-Hill, New York.

[5] Boyce, W. M. (1969), Commuting functions with no common fixed point, *Trans. Amer. Math. Soc.* **137**, pp. 77-92.

[6] Browder, F. E. (1953), Existence of periodic solutions for nonlinear equations of evolution, *Proc. Nat. Acad. Sci. U.S.A.*, **53**, 1100-1103.

[7] Browder, F. E. (1954), Nonexpansive nonlinear operators in a Banach space, *Proc. Nat. Acad. Sci. U.S.A.* **54**, pp. 1041-1044.

[8] Carleson, A. (1966), On the Convergence and growth of partial sums of Fourier series, *Acta Math.* **116**, pp. 135-157.

[9] Chatelin, F. (1983), *Spectral Approximation of Linear Operators*, Academic Press, New York.

[10] Clarkson, J. A. (1936), Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40**, pp. 396-414.

[11] Coddington, E. A. (1961), *An Introduction to Ordinary Differential Equations*, Prentice-Hall, Englewood Cliffs, N. J.

[12] Cohen, P. J. (1966), *Set Theory and the Continuum Hypothesis*, Benjamin, New York.

[13] Courant, R. and D. Hilbert (1953), *Methods of Mathematical Physics*, vol. I, Interscience, New York.

[14] Courant, R. and H. Robbins (1968), *What is Mathematics? An Elementary Approach to Ideas and Methods*, Oxford Univ. Press, London.

[15] Day, M. M. (1941), Reflexive Banach spaces not isomorphic to uniformly convex spaces, *Bull. Amer. Math. Soc.* **47**, pp. 313-317.

[16] Diestel, J. (1984), *Sequences and Series in Banach Spaces*, Springer-Verlag, New York.

[17] Dieudonné, J. (1969), *Foundations of Modern Analysis*, Academic Press, New York.

[18] Dixmier, J. (1953), Sur les bases orthonormales dans les espaces préhilbertiens, *Acta Math. Szeged* **15**, pp. 29-30.

[19] Dunford, N. and J. T. Schwartz (1958), *Linear Operators*, Part I, Interscience, New York.

[20] Edwards, R. E. (1965), *Functional Analysis*, Holt, Rinehart and Winston, New York.

[21] Enflo, P. (1973), A counter-example to the approximation problem in Banach spaces, *Acta Math.* **103**, pp. 309-317.

[22] Fan, K. (1952), Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. U.S.A* **38**, pp. 121-126.

[23] Fichtenholtz, G. and L. Kantorowitch (1934), Sur les opérations linéaires l'espace des fonctions bornées, *Studia Math.* **5**, pp. 69-98.

[24] Glicksberg, I. L. (1952), A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points, *Proc. Amer. Math. Soc.* **3**, pp. 170-174.

[25] Halmos, P. R. (1963), What does the spectral theorem say? *Amer. Math. Monthly* **70**, pp. 241-247.

[26] Halmos, P. R. and V. S. Sunder (1978), *Bounded Integral Operators on L^p spaces*, Springer Verlag, New York.

[27] Hardy, G. H. (1949), *Divergent Series*, Oxford Univ. Press, London.

[28] Hoffman, K. (1962), *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N. J.

[29] Holmes, R. B. (1975), *Geometric Functional Analysis and its Applications*, Springer-Verlag, New York.

[30] Horn, R. A. and C. R. Johnson (1985), *Matrix Analysis*, Cambridge Univ. Press, Cambridge.

[31] Huneke, J. P. (1967), Two counterexamples to a conjecture on commuting continuous functions of the closed interval, Abstract 67T-231, *Notices Amer. Math. Soc.* **14**, p. 284.

[32] Hunt, R. A. (1968), *On the Convergence of Fourier Series, Orthogonal Expansions and Their Continuous Analogs*, Southern Illinois Univ. Press, Carbondale, Ill.

[33] Hurewicz, W. and H. Wallman (1948), *Dimension Theory*, Princeton Univ. Press, Princeton.

[34] Jacobson, N. (1984), *Basic Algebra*, vol. 2, Hindustan Publishing Corporation (India) Delhi.

[35] James, R. C. (1951), A nonreflexive Banach space isomorphic with its second conjugate, *Proc. Nat. Acad. Sci., U.S.A.* **37**, pp. 174-177.

[36] James, R. C. (1971), A counterexample for a sup theorem in normed spaces, *Israel J. Math.* **9**, pp. 511-512.

[37] James, R. C. (1972), Reflexivity and the sup of linear functionals, *Israel J. Math.*, **13**, pp. 289-300.

[38] Jauch, J. M. (1968), *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Mass.

[39] Katznelson, Y. (1976), *An Introduction to Harmonic Analysis*, Dover, New York.

[40] Kolmogorov, A. (1926), Une sèrie de Fourier-Lebesgue divergente partout, *C. R.* **183**, pp. 1327-1328.

[41] Kreyszig, E. (1993), *Advanced Engineering Mathematics*, 7th ed., Wiley, New York.

[42] Krylov, V. I. (1962), *Approximate Calculation of Integrals*, MacMillan, New York.

[43] Limaye, B. V. (1987), *Spectral Perturbation and Approximation with Numerical Experiments*, vol. 13, Proceedings of the Centre for Math. Anal., Australian Nat. Univ., Canberra.

[44] Lindenstrauss, J. and L. Tzafriri (1971), On the complemented subspaces problem, *Israel J Math.* **9**, pp. 263-269.

[45] Löwig, H. (1934), Über die Dimension linearer Räume, *Studia Math.* **5**, pp. 18-23.

[46] Raghavendran, R. (1969), Toeplitz-Hausdorff theorem on numerical ranges, *Proc. Amer. Math. Soc.* **20**, pp. 284-285.

[47] Reed, M. and B. Simon (1975), *Methods of Mathematical Physics*, vol. II, Academic Press, New York.

[48] Riesz, F. and B. Sz.-Nagy (1955), *Functional Analysis*, Frederick Ungar, New York.

[49] Rosenblum, M. (1958), On a theorem of Fuglede and Putnam, *J. London Math. Soc.* **33**, pp. 376-377.

[50] Royden, H. L. (1968), *Real Analysis*, 2nd ed., Macmillan, New York.

[51] Rudin, W. (1976), *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York.

[52] Rudin, W. (1986), *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York.

[53] Rudin, W. (1991), *Functional Analysis*, 2nd ed., McGraw-Hill, New York.

[54] Schechter, M. (1981), *Operator Methods in Quantum Mechanics*, North-Holland, New York.

[55] Szegö, G. (1959), *Orthogonal Polynomials*, vol. XXIII, Amer. Math. Soc. Colloq. Publications, New York.

[56] Watkins, D. S. (1991), *Fundamentals of Matrix Computations*, Wiley, New York.

[57] Whitley, R. (1966), Projecting m onto c_0, *Amer. Math. Monthly* **73**, pp. 285-286.

[58] Whitley, R. (1986), The stability of finite rank methods with applications to integral equations, *SIAM J. Numer. Anal.* **23**, pp. 118-134.

[59] Widder, D. V. (1946), *The Laplace Transform*, Princeton Univ. Press, Princeton.

[60] Yoshida, K. (1960), *Lectures on Differential and Integral Equations*, Interscience, New York.

List of Symbols

$\in$	belongs to 1
$E \subset F$	E is a subset of F 2
$E \cup F$	union of subsets E and F 2
$E \cap F$	intersection of subsets E and F 2
E^c	complement of a subset E 2
$\emptyset$	empty set 2
$F : X_1 \to X_2$	F is a function from X_1 to X_2 2
$F_{\|Y}$	restriction of a function F to a subset Y 2
$F^{-1}(Y)$	inverse image of Y under a function F 2
F^{-1}	inverse of an injective function or operator F 3
$G \circ F$	composition of functions F and G 3
$\mathbf{K}$	scalar field of real numbers or complex numbers 5
$\mathbf{R}$	field of real numbers 5
$\mathbf{C}$	field of complex numbers 5
$E + F$	sum of subsets E and F of a linear space 6
kE	scalar multiple of a subset E of a linear space 6
$\mathrm{co}(E)$	convex hull of a subset E of a linear space 6
$\mathrm{span}\, E$	linear span of a subset E of a linear space 6
$\dim X$	dimension of a linear space X 9
X/Y	quotient space of a linear space X by a subspace Y
$X_1 \times \cdots \times X_n$	product space of linear spaces $X_1, \ldots, X_n$ 10
$\mathbf{K}^n$	product space of $\mathbf{K}, \ldots, \mathbf{K}$ (n times) 10
$R(F)$	range space of a linear map F 11
$Z(F)$	zero space of a linear map F 11
$(k_{i,j})$	matrix whose (i,j)th element is $k_{i,j}$ 14
$d(x,y)$	distance between x and y 19
$d_p(x,y)$	p-distance between x and y in $\mathbf{K}^n, \ell^p$ or $L^p(E)$, $1 \le p < \infty$ 19, 22
$d_\infty(x,y)$	∞-distance between x and y in $\mathbf{K}^n$, ℓ^∞, $L^\infty(E)$ or $B(T)$ 19, 22, 23

E^a	annihilator of a subset E of a normed space 233
c_t	characteristic function of $(a, t]$ 236
$BV([a, b])$	space of scalar-valued functions of bounded variation on $[a, b]$ 242
$NBV([a, b])$	space of scalar-valued normalized functions of bounded variation on $[a, b]$ 243
$z(t^+)$	limit of $z(s)$ as $s \to t, s > t$ 243
$\triangle$	forward difference operator 249
$x_n \xrightarrow{w} x$	(x_n) is weak convergent to x 260, 432
$x_n' \xrightarrow{w*} x'$	(x_n') is weak* convergent to x' 266
$CL(X, Y)$	space of compact linear maps from a normed space X to a normed space Y 304
$CL(X)$	space of compact operators on a normed space X 304
$\langle x, y \rangle$	inner product of x with y 368
H	Hilbert space 373
$x \perp y$	x is orthogonal to y 381
$E \perp F$	E is orthogonal to F 381
$\sum\limits_n$	summation over $n = 1, 2, \ldots$ (finite or infinite) 385
$E^\perp$	elements orthogonal to a subset E 420
H'	dual space of a Hilbert H 424
A^*	adjoint of a bounded operator A 449
$\overline{M}^t$	conjugate transpose of a matrix M 451
$A \geq 0$	A is a positive operator 469
$\omega(A)$	numerical range of an operator A 489
m_A	infimum of the numerical range of a self-adjoint operator A 491
M_A	supremum of the numerical range of a self-adjoint operator A 491
$r_\omega(A)$	numerical radius of an opeartor A 502
$\operatorname{ext} S$	extreme points of a subset S of a linear space 542
D_T	domain of an operator T 572
T^*	adjoint of a densely defined operator T 573

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