Lemma: Let X be a n.l.s and Y be a subspace of X.

(a) For $x \in X$, $y \in Y$ and $k \in K$, we have $\|kx + y\| \geq |k| \operatorname{dist}(x, Y)$.

(b) Let Y be finite dimensional. Then Y is complete. In particular, it is closed in X.

Let $\{y_1, y_2, \ldots y_m\}$ be a basis for Y and $\{x_n\}$ be a sequence in Y.

If $x_n = \sum_{j=1}^{m} k_{nj} y_j$, $n = 1, 2, 3 \ldots$

Then $x_n \to x = \sum_{j=1}^{m} k_j y_j$

iff $k_{nj} \to k_j$, $j = 1, 2, \ldots m$ as $n \to \infty$.

Also $\{x_n\}$ is bounded iff $\{k_{nj}\}$ is bounded for $j = 1, 2, \ldots m$.

Proof:

(a) If $k=0$, the result is true.

If $k \neq 0$, then

$$\|kx+y\| = |k|\,\|x + y/k\|$$
$$= |k|\,\|x-(-y/k)\|$$
$$\geq |k| \operatorname{dist}(x, Y).$$

$$\left[\because \operatorname{dist}(x, Y) = \inf\{\|x-y\| / y \in Y\} \leq \|x-y\|,\ y \in Y \right].$$

(b) We prove this by induction.

Let $\dim Y = 1$, then

$Y = \operatorname{span}\{y\} = \{ky \mid k \in K\}$, $y \neq 0$.

Let $\{x_n\}$ be a Cauchy sequence in Y

Then $x_n = k_n y_1$, $k_n \in K$.

Now

$$\|x_n - x_m\| = \|k_n y - k_m y\|$$
$$= |k_n - k_m| \, \|y\|, \quad y \neq 0.$$

$$\Rightarrow \quad |k_n - k_m| = \frac{\|x_n - x_m\|}{\|y\|} \quad \text{---} \; (*)$$

$\because$ $\{x_n\}$ is a Cauchy sequence, given $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$,

$$\|x_n - x_m\| < \epsilon \|y\|$$

Then from $(*)$, we get

$$|k_n - k_m| < \epsilon, \quad \forall \, n, m \geq n_0$$

$\Rightarrow$ $\{k_n\}$ is a Cauchy sequence in K.

Since K is complete, $k_n \to k \in K$.

$$\Rightarrow x_n = k_n y \longrightarrow x = ky \in Y$$

$4 \| k_n y - k y \| = |k_n - k| \, \|y\| \to 0$]

$\Rightarrow Y$ is Complete.

Now assume that every $n-1$ dimensional subspace of X is Complete.

Let Y be n-dimensional subspace of X and let $\{x_n\}$ be a Cauchy sequence in Y.

Let $\{y_1, y_2, \ldots y_m\}$ be a basis for Y.

Let
$$Z = [y_2, y_3, \ldots y_m]$$
be the $n-1$ dimensional subspace of X. Then by inductive assumption, Z is Complete.

Now we prove Y is Complete.

$\because \{x_n\}$ is a Cauchy sequence in Y, we have

$$x_n = k_n y_1 + z_n,$$

where $k_n \in K$, $z_n \in Z$.

Then by using (a), we have

$$\|x_n - x_p\| = \|(k_n - k_p) y_1 + z_n - z_p\|$$
$$\geq |k_n - k_p| \operatorname{dist}(y_1, Z) \quad (**)$$

$\because y_1 \notin Z$, Z is closed in X, we see that $\operatorname{dist}(y_1, Z) > 0$

So from $(**)$, we see that $\{k_n\}$ is a Cauchy sequence in K, so $k_n \to k \in K$

Also as $z_n = x_n - k_n y_1$, we can see that $\{z_n\}$ is a Cauchy sequence in Z, which is complete. $\therefore z_n \to z \in Z$.

$\therefore x_n = k_n y_1 + z_n \to k y_1 + z \in Y$

$\therefore Y$ is Complete.

In particular, it is closed

Next, let $x_n \in Y$ and $x_n = \sum_{j=1}^{m} k_{nj} y_j$

$n = 1, 2, 3, \ldots$

If $k_{nj} \to k_j$, $j = 1, 2, \ldots m$,
$n \to \infty$

Let $x = \sum_{j=1}^{m} k_j y_j$, then

$$\|x_n - x\| = \left\| \sum_{j=1}^{m} (k_{nj} - k_j) y_j \right\|$$

$$\leq \sum_{j=1}^{m} |k_{nj} - k_j| \, \|y_j\|$$

$$\to 0 \text{ as } n \to \infty$$

$\Rightarrow x_n \to x$.

Conversely, let $x_n \to x = \sum_{j=1}^{m} k_j y_j \in Y$.

Then we have

$$\|x_n - x\| = \left\| \sum_{j=1}^{m} (k_{nj} - k_j) y_j \right\|$$

$\longrightarrow$ (***)

Now, let

$$Y_j = [y_1, y_2, \ldots y_{j-1}, y_{j+1}, \ldots y_m]$$

$j = 1, 2, \ldots m$.

Then $y_j \notin Y_j$ and $\operatorname{dist}(y_j, Y_j) > 0$,

since Y_j is closed.

$\therefore$ from (***) and by (a),

$$\|x_n - x\| \geq |k_{nj} - k_j| \operatorname{dist}(y_j, Y_j)$$

$\Rightarrow k_{nj} \to k_j$, $[\because x_n \to x]$,

$j = 1, 2, \ldots m$.

Finally, if $\{k_{nj}\}$ is bounded

for each $j = 1, 2, \ldots m$, say

$$|k_{nj}| \leq \alpha_j, \quad \forall\ n \in \mathbb{N}.$$

Then

$$\|x_n\| = \| \sum_{j=1}^{m} k_{nj} y_j \|$$

$$\leq \sum_{j=1}^{m} |k_{nj}| \, \|y_j\|$$

$$\leq \sum_{j=1}^{m} \alpha_j \, \|y_j\|$$

$$< \infty$$

$\Rightarrow \{x_n\}$ is bounded

Conversely, let $\{x_n\}$ be bounded.

For each $j = 1, 2, \ldots m$, let

Y_j defined as above. Then

by (a), we have

$$\|x_n\| = \| \sum_{j=1}^{m} k_{nj} y_j \|$$

$$\geq |k_{nj}| \operatorname{dist}(y_j, Y_j)$$

$\Rightarrow \{x_{nj}\}$ is bounded, for $j=1,2,\ldots,m$

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Remark :- An infinite dimensional subspace of a n.l.s X need not be closed in X.

Ex: $X = \ell^{\infty}$

$Y = c_{00}$ is subspace of X.

but it is not closed in X.

$\because\ x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots) \in c_{00}$

But $x_n \to x\,(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots)$

$\notin c_{00}$.

Theorem: Let X be a n.l.s. Then the following are equivalent:

(i) Every closed and bounded

subset of X is compact.

(ii) The subset $\{x \in X \mid \|x\| \leq 1\}$ of X is compact.

(iii) X is finite dimensional.

Proof: (i) $\Rightarrow$ (ii)

$\because$ $\{x \in X \mid \|x\| \leq 1\}$ is closed and bounded, it follows that

(i) $\Rightarrow$ (ii)

Now we prove

(ii) $\Rightarrow$ (iii)

Let, if possible, $\{y_1, y_2, y_3 \cdots\}$ be infinite linearly independent subset of X and consider

$$Z_n = [y_1, y_2, \cdots y_n], \; n = 1, 2, 3 \cdots$$

Then Z_n is finite dimensional,

hence it is closed subspace of $Z_{n+1} = [y_1, y_2 \ldots y_{n+1}]$.

Also $Z_n \neq Z_{n+1}$

Now by Riesz Lemma, there exists $x_n \in Z_{n+1}$ such that

$\|x_n\| = 1$ and $\text{dist}(x_n, Z_n) \geq \frac{1}{2}$

(Take here $r = \frac{1}{2}$ in Riesz Lemma)

$[Z_1, \underline{Z_2}$ $\quad$ Z_2, Z_3 $\quad$ Z_3, Z_4

x_1 $\quad$ $x_2 \in Z_3$ $\quad$ x_3

$\text{dist}(x_1, Z_1) \geq \frac{1}{2}$ $\quad$ $\text{dist}(x_2, Z_2) \geq \frac{1}{2}$ $\quad$ $\text{dist}(x_3, Z_3) \geq \frac{1}{2}$

$\ldots \quad \ldots$

$\{x_1, x_2, x_3 \ldots\} \ni \|x_n\| = 1 \ \forall n$

and $\text{dist}(x_i, x_j) \geq \frac{1}{2}$

$\Rightarrow$ From above, we can see that

$$\|x_n - x_m\| \geq \frac{1}{2} \quad \forall\, n \neq m.$$

So $\{x_n\}$ is a sequence in $\{x \in X \,/\, \|x\| = 1\}$ having no convergent sub-sequence.

$\Rightarrow \{x \in X \,/\, \|x\| \leq 1\}$ is not compact,

which is contradiction.

$\therefore$ X has to be finite dimensional.

Now we prove (ii) $\Rightarrow$ (i).

Suppose X is finite dimentional and E is closed and bounded subset of X.

Claim: E is compact.

let $\{x_n\}$ be a sequence in E.

Let $\{y_1, y_2, \ldots y_m\}$ be a basis for X.

$\because E \subseteq X$, $x_n \in X$, then

$$x_n = \sum_{j=1}^{m} k_{nj} y_j \quad , \quad n = 1, 2, 3, \ldots$$

$\because x_n \in E \Rightarrow \{x_n\}$ is bounded sequence

$\Rightarrow \{k_{nj}\}$ is a bounded sequence in K, $j = 1, 2, \ldots m$.

By Bolzano-weistrass theorem for K and passing to subsequency of subsequency several times, we find $n_1 < n_2 < \ldots$ such that

$\{k_{n_p,j}\}$ converges in K

as $p \to \infty$, $j = 1, 2, \ldots, m$

Then by previous lemma,

the corresponding subsequence

$\{x_{n_p}\}$ converges to some $x \in X$.

Since $x_{n_p} \in E$ and E is closed,

then $x \in E$.

Hence every sequence in E has

a subsequence which converges

in E.

$\therefore$ E is compact

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