Ex: $X = P[a,b]$ of all polynomials ~~on~~ on $[a,b]$ is not a Banach space w.r.t $\|\cdot\|_\infty$.

We know that by Weierstrass approximation theorem, for every $x \in C[a,b]$, there exist a sequence ~~$\{x_n\}$~~ $\{x_n\}$ ~~in~~ of polynomials such that

$$\|x_n - x\|_\infty \longrightarrow 0$$

as $n \to \infty$

$\therefore P[a,b]$ is dense in $C[a,b]$ w.r.t $\|\cdot\|_\infty$.

[Then $P[a,b]$ is not closed, otherwise

$$P[a,b] = \overline{P[a,b]} = C[a,b]]$$

$\therefore P[a,b]$ is not a Banach space.

Also $P[a,b]$ is not a Banach space w.r.t $\|\cdot\|_p$, $1 \leq p < \infty$.

To see this, let $x \in C[a,b] \setminus P[a,b]$

let $\{x_n\}$ be a sequence in $P[a,b]$

$\ni$ $\|x_n - x\|_\infty \to 0$ as $n \to \infty$.

Then

$$\|x_n - x\|_p^p = \int_a^b |x_n(t) - x(t)|^p \, dt$$

$$\leq \|x_n - x\|_\infty^p \left(\int_a^b dt \right)$$

$$\therefore \|x_n - x\|_p \leq (b-a)^{\frac{1}{p}} \|x_n - x\|_\infty$$

$1 \leq p < \infty$.

Thus $P[a,b]$ dense in $C[a,b]$

w.r.t $\|\cdot\|_p$, $1 \leq p < \infty$

Also $P[a,b]$ is dense in $P[a,b]$

$\Rightarrow$ $P[a,b]$ is not closed in $C[a,b]$ w.r.t $\|\cdot\|_p$, $1 \leq p < \infty$.

$\therefore$ $P[a,b]$ is not a Banach space w.r.t $\|\cdot\|_p$ $1 \leq p < \infty$.

$$\boxed{\begin{array}{c}\overline{P[a,b]} = C[a,b] \\ \{x_n\} \longrightarrow x\end{array}}$$

Ex: $X = C_{00}$ with $\|\cdot\|_p$, $1 \leq p \leq \infty$ is not a Banach space.

Let $x \in C_0$ and $n \in \mathbb{N}$, let

$$x_n = (x(1), x(2), \ldots x(n), 0, 0, \ldots) \in C_{00}$$

$x = (x(1), x(2), x(3), \ldots) \in C_0$

Then

$$\|x_n - x\|_\infty = \sup\{|x(j)| \mid j > n\} \longrightarrow 0 \text{ as } n \to \infty.$$

$\Rightarrow \overline{C_{00}} = C_0$, w.r.t $\|\cdot\|_\infty$.

Now for $1 \leq p < \infty$, Consider for any $x \in \ell^p$

$$\|x_n - x\|_p = \sum_{j=n+1}^{\infty} |x(j)|^p \longrightarrow 0$$

as $n \to \infty$

$\Rightarrow \overline{c_{00}} = \ell^p, \ 1 \leq p < \infty.$

But c_{00} is not closed,

Otherwise

$$c_{00} = \overline{c_{00}} = \ell^p$$

$$c_{00} = \overline{c_{00}} = c_0$$

which is not true.

$\therefore$ c_{00} is not a Banach space for $1 \leq p \leq \infty$.

$$\left[x_n - x = (0, 0, \ldots 0, -x(n+1), -x(n+2), \ldots \ldots) \right]$$

$$x_{n+1} - x = (0, 0, \ldots 0, 0, -x(n+2), \ldots)$$

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Baire Category Theorem: —

If X is a Complete metric space and $\{X_n\}$ is a sequence of subsets of X such that $X = \bigcup_{n=1}^{\infty} X_n$, then there exists some $j \in N$ such that interior of $\overline{X_j}$ is non empty, [i.e., $\overline{X_j}^{\,o} \neq \phi$].

Lemma :— The interior of a proper subspace of a normed linear space X is empty.

Proof: Suppose W is a proper subspace of a n.l.s X

such that $W^\circ \neq \phi$.

$\left[B(x_0, r) = \{x \in X \,/\, \|x - x_0\| < r\} \right.$ is a Ball around x_0 with radius r $\left. \right]$

$\because W^\circ \neq \phi$, let $x_0 \in W$ be interior point of W.

Then there exist $r > 0$ such that

$$B(x_0, r) \subset W.$$

Also for any $0 \neq x \in X$, we have

$$u = x_0 + \frac{r}{2\|x\|} x \in B(x_0, r)$$

$\because \|u - x_0\| = \left\| \frac{r}{2\|x\|} \cdot x \right\| = \frac{r}{2} < r$

$\Rightarrow u \in B(x_0, r)$.

$\Rightarrow x = \frac{2\|x\|}{r}(u - x_0) \in W$

[$\because$ W is a subspace of X]

$\Rightarrow X \subseteq W \subseteq X$

$\therefore X = W$, which is contradiction to W is a proper subspace of X.

$\therefore W^{\circ} = \phi$

i.e., W cannot have any interior points.

—//—

Theorem: A Banach space cannot have denumerable basis.

Proof: Suppose X is a Banach space with denumerable basis $\{u_1, u_2, u_3, \ldots u_n, \ldots \ldots\}$.

Let $X_n = \operatorname{Span}\{u_1, u_2, u_3 \ldots u_n\}$

$\forall\, n = 1, 2, 3, \ldots$

Then

$$X = \bigcup_{n=1}^{\infty} X_n$$ ~~are~~ and

each X_n is a proper closed subspace of X.

Then by above lemma interior of each X_n is empty.

$\therefore$ By Baire-Category theorem X is not a Banach space.

$\therefore$ X cannot have a denumerable basis.

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