Let Ω be any nonempty set.

Then $F(\Omega)$ is the space of all K-valued functions defined on Ω with addition and scalar multiplication defined by

$$(f+g)(t) = f(t)+g(t), \quad t\in\Omega$$

$$(\alpha f)(t) = \alpha f(t), \quad \forall t\in\Omega, \ \alpha\in K.$$

is a vector space.

Clear $B(\Omega)$ the space of K-valued bounded functions on Ω is a subspace of $F(\Omega)$.

Ex: For $1\leq p<\infty$, let

$$\ell^p = \left\{ x\in F(\mathbb{N}) \,\middle|\, \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{\frac{1}{p}} < \infty \right\}$$

$= \{$

$= \{x = (x(1), x(2), x(3), \ldots\ldots) \; / \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p} < \infty\}$

Now for $p \in [1, \infty]$, define

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p}, & x \in \ell^p, \; 1 \leq p < \infty, \\ \sup\{|x(i)| \,/\, i \in \mathbb{N}\}, & x \in \ell^\infty, \; p = \infty \end{cases}$$

Then we can clearly see that

$\|\cdot\|_\infty$ is a norm on ℓ^∞

$\|\cdot\|_1$ is a norm on ℓ^1.

Next for $1 < p < \infty$, we show that ~~ℓ^∞ is~~ ℓ^p is a n.l.s.

For this, let $x, y \in \ell^p$.

Now for any $n \in \mathbb{N}$, we know

that

$$\left(\sum_{i=1}^{n} |x(i)+y(i)|^{p}\right)^{1/p}$$

$$\leq \left(\sum_{i=1}^{n} |x(i)|^{p}\right)^{1/p} + \left(\sum_{i=1}^{n} |y(i)|^{p}\right)^{1/p}$$

$$\leq \left(\sum_{i=1}^{\infty} |x(i)|^{p}\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y(i)|^{p}\right)^{1/p}$$

$$= \|x\|_p + \|y\|_p$$

$$\Rightarrow \sum_{i=1}^{n} |x(i)+y(i)|^{p} \leq \left[\|x\|_p + \|y\|_p\right]^{p}$$

letting $n \to \infty$, we get

$$\sum_{i=1}^{\infty} |x(i)+y(i)|^{p} \leq \left(\|x\|_p + \|y\|_p\right]^{p}$$

$$\Rightarrow \|x+y\|_p^p \le \left[\|x\|_p + \|y\|_p\right]^p$$

$$\Rightarrow \|x+y\|_p \le \|x\|_p + \|y\|_p$$

Thus for $x, y \in l^p$,

$$x+y \in l^p$$

and the inequality is satisfied.

Illy we can prove ~~$\alpha x \in l$~~ $\alpha x \in l^p$ and remaining properties of a norm.

Hence l^p is a n.l.s for $1 \le p \le \infty$.

* As in the case of K^n, $n \ge 2$,

We see ~~th~~ that for $0 < p < 1$,

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p}$$ does

not define a norm on ℓ^p.

* $c_{00} = \{ x \in F(\mathbb{N}) \,/\, \exists\, k \in \mathbb{N} \ni x(n) = 0,\ \forall n \geq k \}$

$= \{ x = (x(1), x(2), \ldots x(k), 0, 0, 0 \ldots) \in F(\mathbb{N}) \}$

$c_0 = \{ x \in F(\mathbb{N}) \,/\, x(n) \longrightarrow 0 \text{ as } n \to \infty \}$

$c = \{ x \in F(\mathbb{N}) \,/\, x(n) \text{ cgs as } n \to \infty \}$.

Clearly we have

$$c_{00} \subsetneq l^p \subsetneq c_0 \subsetneq c \subsetneq l^\infty$$

$1 \le p < \infty$

$x = (1, -1, 1, -1, 1, -1, \ldots \ldots) \in l^\infty$

but $x \notin c$

$x = (1, 1, 1, \ldots \ldots) \in c$, $x \notin c_0$

$x = (1, \frac{1}{2}, \frac{1}{3}, \ldots \ldots) \in c_0$,

but $x \notin l^1$

$\because \sum_j \frac{1}{j}$ dgt

$x = (1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots) \in l^1$,

but $x \notin c_{00}$

* Note for $1 \le p < r < \infty$.

we prove $l^p \subset l^r$, $l^p \subset l^\infty$.

Let $1 \le p < r < \infty$.

Let $x \in \ell^p$ with $\|x\|_p \leq 1$,

Then $\left(\sum_{j=1}^{n} |\cancel{x}(j)|^p\right)^{\frac{1}{p}} \leq 1$

$\implies |x(j)| \leq 1$

$\implies |x(j)|^r \leq |x(j)|^p \quad [p<r]$

$\implies \sum_{j=1}^{\infty} |x(j)|^r \leq \sum_{j=1}^{\infty} |x(j)|^p$

$\implies \|x\|_r \leq \|x\|_p \leq 1$

$\implies \|x\|_r \leq 1$

Now for any $0 \neq x \in \ell^p$

Let $y = \frac{x}{\|x\|_p} \Rightarrow \|y\|_p = \frac{\|x\|_p}{\|x\|_p}$

Then $\|y\|_p = 1$

$\Rightarrow \|y\|_r \leq 1$ (by ⊛)

$\Rightarrow \left\| \frac{x}{\|x\|_p} \right\|_r \leq 1$

$\Rightarrow \|x\|_r \leq \|x\|_p$

Thus for $1 \leq p < r < \infty$,

$\|x\|_r \leq \|x\|_p$, this is known as Jensen's inequality.

If $x \in \ell^p \Rightarrow \|x\|_p < \infty$

$\Rightarrow \|x\|_r < \infty$

$\Rightarrow x \in \ell^r$

$\Rightarrow \ell^p \subset \ell^r, \; 1 \leq p < r < \infty$

Also for $p \geq 1, \; x \in \ell^p$

$$\Rightarrow \sum_{j=1}^{\infty} |x(j)|^p < \infty$$

$$\Rightarrow |x(j)| \leq \sum_{j=1}^{\infty} |x(j)|^p < \infty \quad \forall j$$

$$\Rightarrow \max_j \{ |x(j)| \} \leq \sum_{j=1}^{\infty} |x(j)|^p < \infty$$

$$\Rightarrow \underline{\|x\|_\infty \leq \|x\|_p} < \infty$$

$$\Rightarrow x \in \ell^\infty.$$

Thus for $p \geq 1, \; \ell^p \subseteq \ell^\infty$

Also we ~~have~~ have

$$\|x_n - x\|_\infty \leq \|x_n - x\|_p$$

$$\xrightarrow{\text{Then}} 0 \qquad \xrightarrow{\text{if}} 0$$

$$x_n \to x \text{ in } \ell^p \implies x_n \to x \text{ in } \ell^\infty$$

$p \geq 1$

Also $1 \leq p < r < \infty$

$$\|x_n - x\|_r \leq \|x_n - x\|_p$$

$$\xrightarrow{\text{Then}} 0 \qquad \xrightarrow{\text{if}} 0$$

Thus $x_n \to x$ in $\ell^p \implies x_n \to x$ in ℓ^r.

$—//—$