Proof: For $x \in K^n$, let

$$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}$$

Clearly $\|x\|_p = 0 \iff x = 0$

If at least $x = 0$ or $y = 0$

then the Holder inequality is true.

So assume $x \neq 0$ & $y \neq 0$

Let $a = \frac{|x(i)|}{\|x\|_p}$, $b = \frac{|y(i)|}{\|y\|_q}$

Then using

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

1

We have

$$\frac{|x(j)|\,|y(j)|}{\|x\|_p\,\|y\|_q} \le \frac{1}{p}\frac{|x(j)|^p}{\|x\|_p^p} + \frac{1}{q}\frac{|y(j)|^q}{\|y\|_q^q}$$

$$\Rightarrow \sum_{j=1}^{n} \frac{|x(j)y(j)|}{\|x\|_p\|y\|_q} \le \frac{\cancel{\sum_{j=1}^{n}|x(j)|^p}}{p\,\cancel{\|x\|_p^p}} + \frac{\cancel{\sum_{j=1}^{n}|y(j)|^q}}{q\,\cancel{\|y\|_q^q}}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \sum_{j=1}^{n} |x(j)y(j)| \le \|x\|_p \|y\|_q$$

— // —

Theorem: For $1 \le p < \infty$,

let $\|x\|_p = \left(\sum_{j=1}^{n} |x(j)|^p\right)^{\frac{1}{p}}$,

Then $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

Proof: For $p=1$, the theorem is already proved.

So assume $1 < p < \infty$ and $x, y \in K^n$.

Consider

$$\|x+y\|_p^p = \sum_{i=1}^{n} |x(i)+y(i)|^p$$

$$= \sum_{i=1}^{n} \left(|x(i)+y(i)|\right)\left(|x(i)+y(i)|^{p-1}\right)$$

$$\leq \sum_{i=1}^{n} |x(i)|\,|x(i)+y(i)|^{p-1}$$

$$+ \sum_{i=1}^{n} |y(i)|\,|x(i)+y(i)|^{p-1}.$$

Now using ~~Holder~~ Inequality, (*)

We have

$$\sum_{i=1}^{n} |x(i)| \, |x(i)+y(i)|^{p-1}$$

$$\leq \|x\|_p \left[\sum_{i=1}^{n} \left(|x(i)+y(i)|\right)^{q(p-1)} \right]^{1/q}$$

$$= \underline{\|x\|_p \, \|x+y\|_p^{p/q}}$$

$$\left[\because \ \frac{1}{p} + \frac{1}{q} = 1 \right.$$

$$\Rightarrow pq = q+p$$

$$\Rightarrow pq - q = p$$

$$\left. \Rightarrow (p-1)q = p \right]$$

$\underline{|||^{ly}}$ For the second term, we get

$$\sum_{i=1}^{n} |y(i)| \left[|x(i)+y(i)| \right]^{p-1}$$

$$\leq \|y\|_p \, \|x+y\|_p^{p/q}$$

So using these in ⊛, we get

$$\|x+y\|_p^p \leq \left[\|x\|_p + \|y\|_p\right] \|x+y\|_p^{p/q}$$

$$\Rightarrow \|x+y\|_p^{(1-\frac{1}{q})p} \leq \|x\|_p + \|y\|_p$$

$$\Rightarrow \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Other conditions of norm easily follows from

$$\|x\|_p = \left(\sum_{j=1}^{n} |x(j)|^p\right)^{1/p}$$

$\therefore \|\cdot\|_p$ is a norm on K^n

— // —

For $p=2$, $\|\cdot\|_2$ is called Euclidean norm on K^n.

Remark:- For $n>1$, ~~for~~ $0<p<1$,

$$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}$$ is

not a norm on K^n

$\because$ $$\|e_1 + e_2\|_p = 2^{1/p} > 2 = \|e_1\|_p + \|e_2\|_p$$

Ex: Let Ω be any nonempty set. Denote $B(\Omega)$, the set of K-valued bounded functions on Ω, is a vector space w.r.t addition and scalar multiplication defined pointwise. It is denoted by $B(\Omega) = l^{\infty}(\Omega)$

[$(x+y)(t) = x(t) + y(t)$; $(\alpha x)(t) = \alpha x(t)$]

For $x \in B(\Omega)$, define

$$\|x\|_\infty = \sup_{t \in \Omega} |x(t)|$$

Then $\|\cdot\|_\infty$ is a ~~norm~~ norm on $B(\Omega)$.

In particular, if $\Omega = \mathbb{N}$, then norm on $B(\mathbb{N})$ is

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x(n)|, \quad x \in B(\mathbb{N}).$$

Here $B(\mathbb{N}) = \ell^\infty(\mathbb{N})$

Also

$C[a,b]$ and $R[a,b]$ are subspaces of $B([a,b])$

So, $\|\cdot\|_\infty$ is ~~also~~ also a norm on $C[a,b]$ & $R[a,b]$.

Next Consider the p-norm on $C[a,b]$.

Theorem (Holder's Inequality):
Let p and q be positive real numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$.
Then for all $x, y \in C[a,b]$, we have

$$\int_a^b |x(t)\,y(t)|\,dt \le \left(\int_a^b |x(t)|^p\right)^{1/p} \times \left(\int_a^b |y(t)|^q\right)^{1/q}.$$

This can be proved by replacing

Generalization by integrals in the one of the previous theorem.

* Using above ~~inte~~ inequality, we can prove that $C[a,b]$ is a normed linear space with the norm

$$\|x\|_p = \left(\int_a^b |x(t)|^p\right)^{1/p}$$

$1 \le p < \infty$

* $\frac{1}{p} + \frac{1}{q} = 1$, we say p is ~~con~~ conjugate exponent of q and vice versa.