* Uniformly bounded implies pointwise bounded.

If $\mathcal{A} = \{A \mid A \in BL(X,Y)\}$ is uniformly bounded implies, there exists $M > 0 \ni \|A\| \leq M, \forall A \in \mathcal{A}$.

Now for any $x \in X$, we have

$$\|Ax\| \leq \|A\| \|x\|, \forall A \in \mathcal{A}$$
$$\leq M \|x\|$$

$\Rightarrow \mathcal{A}$ is pointwise bounded.

But converse need not be true.

Ex: $X = c_{00}$, $\|\cdot\|_\infty$

For $x = (x(1), x(2), \ldots) \in c_{00}$,

define $f_n(x) = \sum_{j=1}^{n} x(j)$.

Then $\|f_n\| = n$

$\left[x_n = (\underbrace{1,1,1,\ldots 1}_{n \text{ times}}, 0, 0, 0 \cdots) \right]$

$$f_n(x) = \sum_{j=1}^{n} x(j) \longrightarrow \sum_{j=1}^{\infty} x(j) = f(x).$$

Then

$f_n(x_n) = n, \ \forall n \in \mathbb{N}.$

but $\{f(x_n)\}$ is unbounded $\to \infty$.

$\Rightarrow \{\|f_n\|\}$ not uniformly bounded

$$f(x) = \lim_{n \to \infty} f_n(x)$$

$$\|fx\| \leq \liminf_{n \to \infty} \|f_n\| \, \|x\|$$

$$\leq n \|x\|_\infty \longrightarrow \infty$$

— $*$

if $\{\|f_n\|\}$ is uniformly bounded, then $\|f_n x\| \le M\|x\|$, which is not true by $\cancel{x}$.

If X is finite dimensional, then pointwise bounded implies uniformly bounded.

Let $\{u_1, u_2 \ldots u_n\}$ be a basis for X and let $\{f_1, f_2 \ldots f_n\}$ be a dual basis for X.

Then $f_i(u_j) = \delta_{ij} \quad \forall\, i,j$

$\because$ Every $x \in X$ can be written uniquely as

$$x = \sum_{i=1}^{n} \alpha_i u_i$$

$$\Rightarrow f_j(x) = \sum_{i=1}^{n} \alpha_i \underbrace{f_j(u_i)}_{\delta_{ij}}$$

$$= \alpha_j$$

$$\therefore \quad x = \sum_{i=1}^{n} f_i(x) u_j .$$

Now let $\mathcal{A} = \{ A \,/\, A \in BL(X, Y) \}$ be a family of operators, which are pointwise bounded.

Then for any $A \in \mathcal{A}$, we have

$$Ax = \sum_{i=1}^{n} f_i(x) A u_j$$

$$\Rightarrow \|Ax\| = \left\| \sum_{i=1}^{n} f_i(x) A u_j \right\|$$

$$\leq \sum_{i=1}^{n} |f_i(x)| \, \|A u_j\|$$

$$\leq \sum_{i=1}^{n} \|f_i\| \, \|x\| \, \|A u_j\| \quad (*)$$

$\because$ $\mathcal{A}$ is pointwise bounded,

$\exists\ \beta_1, \beta_2 \dots \beta_n \geq 0 \ \ni$

$\|Ax_j\| \leq \beta_j \quad , j = 1, 2 \ldots n$

Let $\beta = \max\{\beta_1, \beta_2 \ldots \beta_n\}$.

Let $\alpha = \sum_{i=1}^{n} \|f_i\|$

$\therefore$ from $(*)$ we have for any $A \in \mathcal{A}$

$$\|Ax\| \leq \left(\sum_{i=1}^{n} \|f_i\|\right) \beta_j \|x\|$$

$$\leq \alpha \beta \|x\|, \quad \forall A \in \mathcal{A}$$

$$\Rightarrow \quad \|A\| \leq \alpha \beta, \quad \forall A \in \mathcal{A}$$

$\Rightarrow \mathcal{A}$ is uniformly bounded.

Uniform boundedness Principle

Let X be a Banach space and Y be a n.l.s, and $\mathcal{A} \subseteq BL(X,Y)$.
If $\mathcal{A}$ is pointwise bounded, then $\mathcal{A}$ is uniformly bounded.

Proof: —

Suppose $\mathcal{A}$ is pointwise bounded.
Then for each $x \in X$, $\exists\, M_x > 0 \ni$

$$\|Ax\| \leq M_x \|x\|, \ \forall A \in \mathcal{A}.$$

For each $n \in \mathbb{N}$, let

$$E_n = \{x \in X \mid \|Ax\| \leq n, \ \forall A \in \mathcal{A}\}$$

claim: E_n is closed.

let $x \in \overline{E_n}$

Then there exists a sequence $\{x_k\}$ in E_n such that $x_k \to x$.

$\because$ each $x_k \in E_n$, we have

$$\|Ax_k\| \leq n, \quad \forall k$$
$$\forall A \in \mathcal{A}.$$

$\because$ $x_k \to x$, $A \in \mathcal{A} \subset BL(X,Y)$

$\Rightarrow$ $Ax_k \to Ax$, $\forall A \in \mathcal{A}$.

$\Rightarrow$ $\|Ax_k\| \to \|Ax\|$.

$$\therefore \|Ax\| = \lim_{k\to\infty} \|Ax_k\|$$
$$\leq \left(\lim_{k\to\infty} n\right)$$
$$= n$$

$\Rightarrow$ $\|Ax\| \leq n$, $\forall A \in \mathcal{A}$

$\Rightarrow$ $x \in E_n$

$$\therefore \overline{E_n} \subseteq E_n \subseteq \overline{E_n}$$

$\Rightarrow \bar{E}_n = E_n$

$\therefore$ E_n is a closed set.

Claim: $X = \bigcup_{n=1}^{\infty} E_n$.

If $X \neq \bigcup_{n=1}^{\infty} E_n$, then there exists $x \in X$ and $x \notin \bigcup_{n=1}^{\infty} E_n$.

$\Rightarrow x \notin E_n, \ \forall n$

$\Rightarrow \|Ax\| \geq n, \ \forall A \in \mathcal{A}$
$\forall n.$

Which is contradiction to $\mathcal{A}$ is a family of bounded operators.

$\therefore \ X = \bigcup_{n=1}^{\infty} E_n$.

[Baire-Catagory theorem: Let X be a complete metric space. If $\{X_n\}$

is a sequence of subsets of X such that $X = \bigcup_{n=1}^{\infty} X_n$, then there exists some j $\ni$ interior of $\overline{X_j} \neq \phi$.]

$\therefore$ By Baire-category theorem, there exists $k \in \mathbb{N}$ such that interior of $E_k \neq \phi$ $\quad$ [$\because \overline{E_k} = E_k$]

So let $u \in E_k$ and $r > 0$ such that

$$B(u, r) \subset E_k$$

Also $\|Au\| \leq k$ $\quad \forall A \in \mathcal{A}$.

Then for any $x \in X$, we have

$$\left\| u + \frac{rx}{2\|x\|} - u \right\| = \left\| \frac{rx}{2\|x\|} \right\|$$

$$= \frac{r}{2} < r$$

$$\Rightarrow \quad u + \frac{rx}{2\|x\|} \in B(u, r) \subset E_k$$

$$\Rightarrow \quad \left\| A\left(u + \frac{rx}{2\|x\|}\right) \right\| \leq k \quad \forall A \in \mathcal{A}.$$

Now

$$\left\| A\left(\frac{rx}{2\|x\|}\right) \right\| = \left\| A\left(\frac{rx}{2\|x\|} + u\right) - Au \right\|$$

$$\leq \left\| A\left(\frac{rx}{2\|x\|} + u\right) \right\| + \|Au\|$$

$$\leq k + k = 2k$$

$$\Rightarrow \quad \|Ax\| \leq \frac{4k}{r} \|x\|, \quad \forall A \in \mathcal{A}$$

$$\Rightarrow \quad \|A\| \leq \frac{4k}{r}, \quad \forall A \in \mathcal{A}.$$

$\Rightarrow$ $\mathcal{A}$ is uniformly bounded.

—— // ——

$\left[\because \|A\| = \sup\{\|Ax\| \,/\, x \in X, \|x\| \le 1\}\right]$

Corollary (Banach-Steinhaus theorem)

Let X be a Banach space, Y be a n.l.s and $\{A_n\}$ be a sequence in $BL(X, Y)$ such that for every $x \in X$, $\{A_n x\}$ converges in Y. Let $A : X \to Y$ be defined by

$$Ax = \lim_{n \to \infty} A_n x, \quad x \in X.$$

Then $\{\|A_n\|\}$ is bounded and $A \in BL(X, Y)$.

Proof: For each $x \in X$, $\{A_n x\}$ converges in Y.

$\therefore \{A_n x \mid n = 1, 2, \dots\}$ is bounded

$\{A_n \mid n \in \mathbb{N}\}$ is pointwise bounded.

$\because$ X is a Banach space, by Uniform boundedness Principle,

$\{\|A_n\|\}$ is bounded.

$\Rightarrow \|A_n\| \leq C, \ \forall n, \ C > 0$

Also, for $x \in X$,

$$Ax = \lim_{n \to \infty} A_n x$$

$$\|Ax\| \leq \left(\limsup_{n} \|A_n\|\right) \|x\|$$

$$\leq C\|x\|, \ \forall x \in X.$$

$\Rightarrow A \in BL(X, Y)$

—||—

Theorem: Let X and Y be Banach spaces and $\{A_n\}$ be a sequence in $BL(X,Y)$.
Then $\{A_n x\}$ converges in Y for every $x \in X$ iff $\{\|A_n\|\}$ is bounded and there exists a dense subset D of X $\ni$ $\{A_n u\}$ converges for every $u \in D$.

Proof: Suppose $\{\|A_n\|\}$ is bounded and $\{A_n u\}$ converges for every $u \in D$.

Claim: $\{A_n x\}$ converges for every $x \in X$.

Since $x \in X = \overline{D}$, $\exists\, u \in D$ such that $\|x - u\| < \epsilon$. —(1)

Now for $m, n \in \mathbb{N}$,

$$\|A_n x - A_m x\| \leq \|A_n x - A_n u\| + \|A_n u - A_m u\| + \|A_m u - A_m x\|.$$

$$\leq \|A_n\| \|x-u\| + \|A_n u - A_m u\| + \|A_m\| \|x-u\|$$

$$\leq (\|A_n\| + \|A_m\|) \|x-u\| + \|A_n u - A_m u\|. \quad -(2)$$

$\because$ $\{A_n u\}$ converges for every $u \in D$

implies $\{A_n u\}$ is a Cauchy sequence.

$\therefore$ for $\epsilon > 0$ $\exists$ $n_0 \in N$ $\ni$

$\|A_n u - A_m u\| < \epsilon$, $\forall$ $n, m \geq n_0$. $\quad -(3)$

Using (1) & (3) in (2) we get

$$\|A_n x - A_m x\| \leq (\|A_n\| + \|A_m\|) \epsilon + \epsilon$$

$$\leq (2C+1)\,\epsilon \quad \left[\because \|A_n\| \leq C \ \forall n\right]$$

$$\longrightarrow 0$$

$\therefore \{A_n x\}$ is Cauchy sequence in a Banach space Y.

$\therefore \{A_n x\}$ converges in Y.

Conversely suppose that $\{A_n x\}$ converges in Y for every $x \in X$.

$\Rightarrow \{A_n x \mid n = 1, 2, 3 \ldots\}$ is pointwise bounded.

$\because X$ is a Banach space,

$\{A_n \mid n = 1, 2 \ldots\}$ is uniformly bounded

i.e., $\{\|A_n\|\}$ is bounded.

Since $\{A_n x\}$ Cgs for every $x \in X$

and $D \subseteq X$, implies

$\{A_n x\}$ also Cgs for every $x \in D$.

— // —

Corollary :— Let X be a Banach space, Y be a n.l.s and $\{A_n\}$ be a sequence in $BL(X, Y)$ such that $\{A_n x\}$ Converges for every $x \in X$. Let $A : X \to Y$ be defined by $Ax = \lim_{n \to \infty} A_n x$, $x \in X$.

Then for every totally bounded subset $S \subseteq X$,

$$\sup_{x \in S} \|A_n x - Ax\| \longrightarrow 0$$

as $n \longrightarrow \infty$.

Proof: