Quotient Space :—

Let X be a n.l.s, Y be a closed subspace of X. The coset of an element $x \in X$ w.r.t Y is defined as

$$x + Y = \{ x + y \mid y \in Y \}$$

Any two cosets are either disjoint or identical and distinct cosets form the partition of X.

$$\frac{X}{Y} = \{ x + Y \,/\, x \in X \}$$

define linear operations on $\frac{X}{Y}$

$$(x_1+Y)+(x_2+Y) = x_1+x_2+Y$$

$$k(x+Y) = kx+Y$$

$\forall x, x_1, x_2 \in X$
$k \in K.$

Then $\frac{X}{Y}$ is a linear space w.r.t above operations.

Y is the additive identity

and $-(x+Y) = -x+Y$ is the additive inverse of $x+Y$ in $\frac{X}{Y}$.

Now Define a function $|||\cdot|||$ on $\frac{X}{Y}$

by $|||x+Y||| = \inf\{ \|x+y\| \,/\, y \in Y\}$

claim: $\frac{X}{Y}$ is a n. l. s.

$\because$ $\|x+y\| \geq 0$, $\forall\, y \in Y$, $x \in X$

$\Rightarrow \inf\{ \|x+y\| \,/\, y \in Y\} \geq 0$ $\forall x \in X$

$\Rightarrow |||x+Y||| \geq 0$, $\forall\, x \in X$.

Next let $|||x+Y||| = 0$, ~~then~~

$\Rightarrow \inf\{ \|x+y\| / y \in Y\} = 0$

$\Rightarrow \exists$ a sequence $\{y_n\}$ in Y

such that $\|x+y_n\| \longrightarrow 0$

$$\Rightarrow y_n + x \to 0 \text{ as } n \to \infty$$

$$\Rightarrow y_n \to -x \in Y$$

[$\because Y$ is closed]

$$\Rightarrow x \in Y \quad [\because Y \text{ is a subspace}]$$

$$\Rightarrow x + Y = Y$$

Thus

$$|||x+Y||| = 0 \Rightarrow x+Y = Y.$$

$$|||k(x+Y)||| = \inf\{\|kx+y\| \mid y \in Y\}$$

$$= |k| \inf\{\|x + y/k\| \mid y \in Y\}$$

$$= |k|\, |||x+Y|||.$$

$$|||(x_1+Y) + (x_2+Y)|||$$

$$= |||x_1+x_2+Y|||$$

$$= \inf\{\|x_1+x_2+y\| \mid y \in Y\}$$

$$= \inf \{ \| x_1 + x_2 + y_1 + y_2 \| / y = y_1 + y_2 \in Y \}$$

$$= \inf \{ \| (x_1 + y_1) + (x_2 + y_2) \| / y_1, y_2 \in Y \}$$

$$\leq \inf \{ \| x_1 + y_1 \| / y_1 \in Y \} + \inf \{ \| x_2 + y_2 \| / y_2 \in Y \}$$

$$= |||x_1 + Y||| + |||x_2 + Y|||.$$

$\therefore \dfrac{X}{Y}$ is a normed linear space.

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Def :- Let X be a n. l. s. A series $\sum_{n=1}^{\infty} x_n$ in X is said to be absolutely

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summable if $\sum_{n=1} \|x_n\| < \infty$.

Def:- A series $\sum_{n=1}^{\infty} x_n$ is said to be summable if

$$s_n = \sum_{j=1}^{n} x_j \longrightarrow x \in X.$$

Theorem: A normed linear space X is a Banach space iff every absolutely summable series is summable in X.

Proof: Let X be a Banach space.

Suppose $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

We prove $\sum_{n=1}^{\infty} x_n$ is summable.

So let $s_n = \sum_{j=1}^{n} x_j$, the

for $m >> n$, we have

$$\|s_m - s_n\| = \left\| \sum_{j=n+1}^{m} x_j \right\|$$

$$\leq \sum_{j=n+1}^{m} \|x_j\|$$

$\because$ [define $\alpha_n = \sum_{j=1}^{n} \|x_j\|$, $m >> n$,

if $\{\alpha_n\}$ is cgt, $\exists\, n_0 \in \mathbb{N}$

$\ni$ $\forall\, n, m \geq n_0$ $|\alpha_n - \alpha_m| < \epsilon$]

$$\therefore \|s_n - s_m\| \leq \sum_{j=n+1}^{m} \|x_j\|$$

$$\leq |\alpha_n - \alpha_m| < \epsilon$$

$\Rightarrow$ $\{s_n\}$ is a Cauchy sequence in X.

$\Rightarrow$ $s_n \to x \in X$ [$\because X$ is a Banach space]

Conversely assume that every absolutely summable series in X is summable in X.

Claim: X is a Banach space.

Let $\{f_n\}$ be a Cauchy sequence in X.

Then $\exists\ m_1 \in \mathbb{N}$ $\ni$

$$\|f_m - f_{m_1}\| < 1, \quad \forall\ m \geq m_1$$

Choose $m_2 > m_1$ $\ni$

$$\|f_m - f_{m_2}\| < \frac{1}{2^2}, \quad \forall\ m > m_2$$

$\vdots$

Choose $m_n > m_{n-1}$, such that

$$\|f_m - f_{m_n}\| < \frac{1}{n^2}, \quad \forall\ m \geq m_n$$

Also for $m_{n+1} > m_n$, let

$$x_n = f_{m_{n+1}} - f_{m_n}, \quad n = 1, 2, 3, \ldots$$

Then $\|x_n\| = \|f_{m_{n+1}} - f_{m_n}\| < \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \|x_n\| = \sum_{n=1}^{\infty} \|f_{m_{n+1}} - f_{m_n}\|$

$$< \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Hence by the assumption,

$\sum_{n=1}^{\infty} x_n$ is summable in X.

Hence $\sum_{n=1}^{\infty} x_n \to x \in X$.

Since $f_{m_n} = f_{m_1} + \sum_{j=1}^{n-1} x_j$, it follows that subsequence $\{x_{n_k}\}$ of a Cauchy sequence $\{x_n\}$ is convergent.

Hence $\{x_n\}$ is itself convergent.

$\Rightarrow$ X is a Banach space

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Theorem: Let X be a normed linear space and Y be a closed subspace of X.

Then X is a Banach space iff Y and $\frac{X}{Y}$ are Banach spaces in the induced norms, respectively