Lemma :- Let X be a linear space, Y be a n.l.s with norm $\|\cdot\|_Y$, and $T : X \to Y$ be a 1-1 linear map. Then $\|x\|_X = \|Tx\|_Y$ defines a ~~norm~~ norm on X.

Proof: $\because \|Tx\|_Y \geq 0$

$$\Rightarrow \|x\|_X \geq 0$$

and $\|x\|_X = 0 \Rightarrow \|Tx\|_Y = 0$

$$\Rightarrow Tx = 0 = T0$$

$$\Rightarrow x = 0 \quad (\because T \text{ is } 1\text{-}1)$$

$$\|x+y\|_X = \|T(x+y)\|_Y$$

$$= \|Tx + Ty\|_Y$$

$$\leq \|Tx\|_Y + \|Ty\|_Y$$

$$= \|x\|_X + \|y\|_X$$

and for any $\alpha \in K$, $x \in X$

$$\|\alpha x\|_X = \|T(\alpha x)\|_Y$$

$$= \|\alpha T x\|_Y$$

$$= |\alpha| \, \|Tx\|_Y$$

$$= |\alpha| \, \|x\|.$$

$\therefore \|\cdot\|_X$ is a norm on X.

Ex: Let X be a finite dimensional linear space and $E = \{u_1, u_2, \ldots u_n\}$ be an ordered basis for X.

Let $\|\cdot\|$ be any norm on K^n.

Define $T : X \longrightarrow K^n$ by

$$T(x) = T\left(\sum_{i=1}^{n} \alpha_i u_i\right)$$

$= (\alpha_1, \alpha_2 \ldots \alpha_n)$

where $(\alpha_1, \alpha_2 \ldots \alpha_n) \in K^n$

$\ni \quad x = \sum_{i=1}^{n} \alpha_i u_i$

Then T is a linear map and it is $1-1$.

$\because T(x) = 0 \Rightarrow (\alpha_1, \alpha_2 \ldots \alpha_n)$

$= (0, \ldots 0)$

$\Rightarrow \alpha_i = 0 \;\forall i = 1 \ldots n.$

$\Rightarrow x = \sum_{i=1}^{n} \alpha_i u_i = 0.$

$\therefore$ By previous lemma we have

$\|x\|_X = \|Tx\|_Y$ is a

norm on X

— // —

Banach Spaces

A normed linear space X over a

field K is called a Banach space, if X is complete ~~in the~~ in the metric $d(x,y) = \|x-y\|$ induced by the norm.

i.e., A complete n.l.s is called Banach space.

[Complete metric space :—

$\{a_n\}$ is a sequence in R

$\ni |a_n - a_m| \to 0$ as $n, m \to \infty$

$d(a_n, a_m)$

Then $\{a_n\}$ is said to be Cauchy sequence.

cgt sequence :—

$|a_n - l| \to 0$ as $n \to \infty$.

If every Cauchy sequence $\{a_n\}$ is also a cgt sequence, then we ~~say~~ say ~~$\{a_n\}$~~ $\mathbb{R}$ is complete.

* We say $\{x_n\}$ is a Cauchy sequence in a n.l.s $(X, \|\cdot\|)$ if given $\epsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ $\ni$

$$\|x_n - x_m\| < \epsilon, \ \forall \ n, m > n_0$$

i.e.,

$$\|x_n - x_m\| \longrightarrow 0 \text{ as } n, m \longrightarrow \infty.$$

If every Cauchy sequence $\{x_n\}$ in a n.l.s X, is also a convergent sequence in X,

i.e., $\|x_n - x\| \to 0$ as

$n \to \infty$, $x \in X$

i.e., $x_n \to x \in X$.

[$\because$ $\|x_n - x_m\| \to 0 \Rightarrow \|x_n - x\| \to 0$ for some $x \in X$]

* Since a subset Y of a complete metric space X is complete iff it is closed in X, it follows that

"A subspace Y of a Banach space X is a Banach space iff Y is a closed subspace of X".

Ex 1. For each $n \in \mathbb{N}$, $(K^n, \|\cdot\|_p)$, $1 \le p \le \infty$ is a Banach space.

Sol: For $p=\infty$, $\|x\|_\infty = \max\{|x(i)| \mid i=1,2,\ldots n\}$

We know that $(K^n, \|\cdot\|_\infty)$ is a n.l.s.

Now, Let $\{x_n\}$ be a Cauchy sequence in K^n w.r.t the norm $\|\cdot\|_\infty$.

Then for every $\epsilon>0$, $\exists\, n_0 \in \mathbb{N}$ such that

$$\|x_n - x_m\|_\infty < \epsilon, \quad \forall\, n,m \geq n_0$$

$$\Rightarrow \max\{|x_n(i) - x_m(i)| \mid i=1,2,\ldots n\} < \epsilon$$

$\Rightarrow |x_n(1)-x_m(1)|<\epsilon,\ |x_n(2)-x_m(2)|<\epsilon, \ldots |x_n(n)-x_m(n)|<\epsilon$ $\forall\, n,m \geq n_0$

$\Rightarrow \{x_n(i)\}$ is a Cauchy sequence in K, $i=1,2,\ldots n$.

Since K is a Complete,

$$x_n(i) \longrightarrow x(i) \in K$$

$\Rightarrow |x_n(i) - x(i)| \to 0, \quad i = 1, 2, \ldots n.$

Let $x = (x(1), x(2), \ldots x(n)) \in K^n$.

Then

$$\|x_n - x\|_\infty = \max\{|x_n(i) - x(i)| \,/\, i = 1, 2, \ldots n\} \longrightarrow 0$$

$\Rightarrow \{x_n\}$ is a Cgt Sequence in K^n w.r.t $\|\cdot\|_\infty$.

$\therefore (K^n, \|\cdot\|_\infty)$ is a Banach Space.

Note that for $1 \le p < \infty$, and $x = (x(1), x(2), \ldots, x(n)) \in K^n$, we have

$$|x(i)| \le \|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{\frac{1}{p}}$$

$\forall\ i = 1, 2, \ldots n$

$$\Rightarrow \max_{i=1,2,\dots,n} |x(i)| \leq \|x\|_p$$

$$\Rightarrow \|x\|_\infty \leq \|x\|_p \quad \text{--- (1)}$$

Also

$$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}$$

$$\leq \max_{i \in \{1,2,\dots,n\}} |x(i)| \left(\sum_{i=1}^{n} 1\right)^{1/p}$$

$$= \|x\|_\infty \cdot n^{1/p} \quad \text{--- (2)}$$

$\therefore$ From (1) & (2), we have

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty$$

$\forall x \in K^n$,

$1 \leq p < \infty$

Thus for any sequence $\{x_n\}$ in K^n, we have

(i) $\|x_\ell - x_m\|_\infty \leq \|x_\ell - x_m\|_p$

$\leq \ {}^{1/p} \|x \ \ x$

$- \ n \ \ \ e \ \ m\|_\infty$

(ii) $\|x_n - x\|_\infty \leq \|x_n - x\|_p \leq n^{1/p} \|x_n - x\|_\infty$

Thus

$\{x_n\}$ is Cauchy w.r.t $\|\cdot\|_\infty$

iff $\{x_n\}$ is Cauchy w.r.t $\|\cdot\|_p$

$1 \leq p < \infty$

and

$x_n \to x \in K^n$ w.r.t $\|\cdot\|_\infty$

iff $x_n \to x \in K^n$ w.r.t $\|\cdot\|_p$

$1 \leq p < \infty$

$\therefore$ $(K^n, \|\cdot\|_p)$, $1 \leq p \leq \infty$

is a Banach space.

Ex: $B(\Omega)$ is a Banach space, where Ω is a nonempty set.

Let $\{x_n\}$ be a Cauchy-sequence in $B(\Omega)$ and $\epsilon > 0$ be given.

Then $\exists\ n_0 \in \mathbb{N} \ni$

$$\|x_n - x_m\|_\infty < \epsilon, \quad \forall\ n, m \geq n_0$$

In particular, for each $t \in \Omega$, we have

$$|x_n(t) - x_m(t)| \leq \|x_n - x_m\|_\infty < \epsilon \quad \forall\ n, m \geq n_0$$

$\Rightarrow \{x_n(t)\}$ is a Cauchy sequence in K, for each $t \in \Omega$.

$\because K$ is Complete, $\exists\ \alpha_t \in K$

$$\ni x_n(t) \longrightarrow \alpha_t \in K.$$

Define $x(t) = \alpha_t, \quad t \in \Omega,$

Then

$$|x_n(t) - x(t)| = \lim_{m \to \infty} \|x_n(t) - x_m(t)\|$$

Since this is true $\forall t \in \Omega$, $\forall n \geq n_0$

we have

$$\|x_n - x\|_\infty \to 0 \text{ as } n \to \infty.$$

$\therefore$

$$\|x\|_\infty \leq \underbrace{\|x - x_M\|_\infty}_{\to 0} + \underbrace{\|x_M\|_\infty}_{< \infty}$$

$\Rightarrow \|x\|_\infty < \infty$

$\Rightarrow x \in B(\Omega)$.

$\therefore$ $B(\Omega)$ is a Banach space.

In particular, if $\Omega = N$,

$$B(\Omega) = l^\infty(N) \text{ is}$$

a Banach space.

Ex: Let Ω be a metric space. Then $C(\Omega)$, the space of K-valued continuous functions defined on Ω

is a subspace of $F(\Omega)$

Denote

$$C_b(\Omega) = C(\Omega) \cap B(\Omega).$$

the space of all K- ~~value~~ $\mathbb{C}$ valued continuous and bounded functions on Ω.

we prove $C_b(\Omega)$ is a closed subspace of $B(\Omega)$.

Let $\{x_n\}$ be a sequence in $C_b(\Omega)$

$\ni \quad \|x_n - x\|_\infty \longrightarrow 0$, for some $x \in B(\Omega)$

Claim: $x \in C_b(\Omega)$.

i.e., x is continuous at every $\tau \in \Omega$.

Let $\epsilon > 0$ be given, then $\exists$ ~~$k_0 \in \mathbb{N}$~~

$n_0 \in \mathbb{N} \quad \ni$

$$\|x_n - x\|_\infty < \epsilon \quad \forall\, n \geq n_0$$

Hence for each $t \in \Omega$, $n \geq n_0$,

$$|x(t) - x(\tau)| \leq |x(t) - x_n(t)| + |x_n(t) - x_n(\tau)| + |x_n(\tau) - x(\tau)|$$

$$< \epsilon + |x_n(t) - x_n(\tau)| + \epsilon$$

$\forall\, n \geq n_0$.

In particular for $n = n_0$, we have

$$|x(t) - x(\tau)| < \epsilon + \underline{|x_{n_0}(t) - x_{n_0}(\tau)|} + \epsilon$$

$\because x_{n_0} \in C_b(\Omega)$, there exists an open set $G \subseteq \Omega$ containing τ such that

$$|x_{n_0}(t) - x_{n_0}(\tau)| < \epsilon, \ \forall\, t \in G$$

Thus for every $\epsilon > 0$ $\exists$ an open set G containing τ

such that

$$|x(t) - x(\tau)| < 3\epsilon \quad \forall \ t \in G$$

$\Rightarrow x \in C_b(\Omega)$

Thus $C_b(\Omega)$ is a closed subspace of a Banach space $B(\Omega)$. This implies $C_b(\Omega)$ is also a Banach space

— // —