Hilbert Space :— An inner product space, which is complete in the norm induced by the inner product is called Hilbert space. We use the letter H to denote the Hilbert space.

Ex: $H = K^n$,

$x = (x(1), x(2), \ldots x(n)) \in H$

$y = (y(1), y(2), \ldots y(n)) \in H$,

Let

$$\langle x, y \rangle = \sum_{i=1}^{n} x(i)\overline{y(i)}.$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on H.

and

$$\|x\|_2 = \sqrt{\langle x, x \rangle}$$

$$= \sqrt{\sum_{i=1}^{n} |x(i)|^2}.$$

Then H is also n.l.s.

Let $\{x_n\}$ be a Cauchy Sequence in H.

Then

$$\|x_n - x_m\|_2 \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \quad \text{---(1)}$$

$$\Rightarrow \sqrt{\sum_{i=1}^{n} |x_n(i) - x_m(i)|^2} \longrightarrow 0$$

$\Rightarrow \{x_n(i)\}_{i=1}^{n}$ is a Cauchy Sequence in $\mathbb{R}$. But $\mathbb{R}$ is Complete.

Let $x_n(i) \longrightarrow x_0(i)$, $i = 1, 2 \ldots n$

$$\Rightarrow [x_n(i) - x(i)]^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

$$i = 1, 2 \ldots n.$$

Fixing n and letting $m \longrightarrow \infty$ in (1)

We get

$$\|x_n - x_0\|_2 = \lim_{m \to \infty} \|x_n - x_0\|_2 \longrightarrow 0.$$

$$= \lim_{m \to \infty} \sqrt{\sum_{i=1}^{n} |x_n(i) - x_m(i)|^2} \longrightarrow 0$$

$\therefore$ $H = \mathbb{K}^n$ is complete w.r.t norm $\|\cdot\|_2$. $\therefore$ H is a Hilbert space.

Note:- Among all the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$ on $\mathbb{K}^n$ $(n \geq 2)$, only the norm $\|\cdot\|_2$ is induced by the inner product, because,

if $p \neq 2$, and $x = (1, 0, 0 \ldots 0)$
$y = (0, 1, 0 \ldots 0)$

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2^{2/p} + 2^{2/p}$$
$$= 2 \cdot 2^{2/p}$$
$$= 2^{1+2/p}.$$

$$\|x\|_p^2 = 1$$

$$\|y\|_p^2 = 1.$$

Now $4 = 2\left(\|x\|_p^2 + \|y\|_p^2\right) \neq 2^{1+\frac{1}{p}}$

$$= \|x+y\|_p^2 + \|x-y\|_p^2$$

Hence parallelogram does not hold.

Ex: Consider the space ℓ^p, $1 \le p < \infty$ with the norm given by

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p},$$

$$x = (x(1), x(2) \ldots) \in \ell^p.$$

For $p = 2$, ℓ^2.

$x, y \in \ell^2$, define

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x(i)\overline{y(i)},$$

where $x = (x(1), x(2) \cdot \cdot \cdot) \in \ell^2$
$y = (y(1), y(2), \cdot \cdot) \in \ell^2$.

Then $\langle \cdot, \cdot \rangle$ is an inner product on ℓ^2.

Define $\|x\|_2 = \sqrt{\langle x, x\rangle}$

$$= \sqrt{\sum_{i=1}^{\infty} |x(i)|^2}.$$

Then ℓ^2 is also n.l.s.

Let $\{x_n\}$ be a Cauchy Sequence in ℓ^2

where $x_n = (x_n(1), x_n(2), \ldots)$.

Then given $\epsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ $\ni$

$\|x_n - x_m\|_2 < \epsilon$, $\forall$ $n, m \geq n_0$.

$$\Rightarrow \|x_n - x_m\|_2 = \left(\sum_{i=1}^{\infty} |x_n(i) - x_m(i)|^2\right)^{1/2} < \epsilon \quad (2)$$

$\Rightarrow$ $\{x_n(i)\}$ is Cauchy Sequence in $\mathbb{R}$, $i = 1, 2, 3, \ldots$

$\therefore$ $x_n(i) \longrightarrow x(i)$, as $n \to \infty$.

Fixing n & letting $m \to \infty$, in ⊛, we get

$$\|x_n - x\|_2 = \lim_{m\to\infty} \|x_n - x_m\|_2$$

$$= \lim_{m\to\infty} \sqrt{\sum_{i=1}^{\infty} |x_n(i) - x_m(i)|^2}$$

$$\longrightarrow 0$$

$\therefore$ l^2 is a Hilbert space w.r.t norm $\|\cdot\|_2$ induced by the inner product $\|x\|_2 = \sqrt{\langle x, x\rangle}$.

Note:— For $p \neq 2$, $1 \leq p \leq \infty$, l^p is not a inner product space

$$x = (-1, -1, 0, 0 \cdots) \in l^p$$

$$y = (-1, 1, 0, \cdots) \in l^p$$

$$\|x\|_p = 2^{1/p}, \quad \|y\|_p = 2^{1/p}$$

$\|x+y\|_p = 2 \qquad \|x-y\|_p = 2$

$\therefore \|x+y\|_p^2 + \|x-y\|_p^2 = 2^2+2^2 = 8$

& $2\left(\|x\|_p^2 + \|y\|_p^2\right) = 2\left(2^{2/p}+2^{2/p}\right)$
$= 4 \cdot 4^{1/p}.$

Thus in ℓ^p, $p \neq 2$, $\|\cdot\|_p$ does not satisfy parallelogram law.

$\therefore \ell^p$, $p \neq 2$ is not an I.P.S and hence it is not an Hilbert space.

Ex: $X = c_{00}$

For $x, y \in X$, define

$$\langle x, y\rangle = \sum_{j=1}^{\infty} x(j)\overline{y(j)}$$

Then $\langle \cdot, \cdot\rangle$ is an I.P on c_{00}.

and the induced norm

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \left(\sum_{j=1}^{\infty} |x(j)|^2\right)^{1/2}$$

Let $x \in \ell^2$, $x = (x(1), x(2), x(3) \ldots)$

and $x_n = (x(1), x(2), \ldots x(n), 0, 0 \ldots) \in c_{00}$

and

$$\|x - x_n\|_2^2 = \sum_{j=n+1}^{\infty} |x(j)|^2 \longrightarrow 0$$

as $n \to \infty$

$\Rightarrow$ c_{00} is dense in ℓ^2.

i.e., $\overline{c_{00}} = \ell^2$

However $c_{00} \neq \ell^2$

$\because$ $(1, \frac{1}{2}, \frac{1}{3} \ldots \ldots) \in \ell^2$

$\because$ $\sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$.

But $(1, \frac{1}{2}, \frac{1}{3}, \ldots) \notin c_{00}$

$\therefore$ $X = c_{00}$ cannot be closed in $H = \ell^2$

Hence c_{00} is incomplete I.P.S.

Ex: Let $X = C[a,b]$.

for $x, y \in X$, define

$$\langle x, y \rangle = \int_a^b x(t)\overline{y(t)}\,dt.$$

then $\langle \cdot, \cdot \rangle$ is an inner product on $C[a,b]$ and the induced norm is

$$\|x\|_2 = \langle x, x \rangle^{1/2}$$

$$= \left(\int_a^b |x(t)|^2 dt \right)^{1/2}, \quad x \in C[a,b]$$

Orthonormal set :—

Let X be an I.P.S over the field K. For any $x, y \in X$, we say x and y are orthogonal if $\langle x, y\rangle = 0$, $x \neq y$ and we write $x \perp y$.

Let E and F be any two subsets of an I.P.S X. We say E and F are orthogonal if

$\langle x, y\rangle = 0$, $\forall\, x \in E$ and $y \in F$.

In this case we write $E \perp F$.

We say a subset E of X is an orthonormal set if $\langle x, y\rangle = 0$, $x \neq y$

$\forall\, x, y \in E$ and $\|x\| = 1 \ \forall\, x \in E$.

Theorem: Let X be an I.P.S.

(a) (Pythagoras) Let $\{x_1, x_2, \dots x_n\}$ be an orthogonal set in X.

Then

$$\|x_1 + x_2 + \dots + x_n\|^2 = \sum_{i=1}^{n} \|x_i\|^2.$$

(b) Let E be an orthogonal subset of X and $0 \notin E$. Then E is L.I. If, in fact, E is orthonormal, then $\|x-y\| = \sqrt{2}$, $\forall\, x, y \in E$, $x \neq y$.

Proof.

(a) Given that $\langle x_i, x_j \rangle = 0 \ \forall\, i \neq j$
$i, j = 1, 2, \dots n.$

Consider

$$\|x_1+x_2+\cdots+x_n\|^2 = \left\langle \sum_{i=1}^{n} x_i , \sum_{j=1}^{n} x_j \right\rangle$$

$$= \sum_{i=1}^{n} \left\langle x_i , \sum_{j=1}^{n} x_j \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x_i , x_j \rangle$$

$$= \sum_{i=1}^{n} \langle x_i , x_i \rangle$$

$$= \sum_{i=1}^{n} \|x_i\|^2 .$$

b) Let $x_1, x_2, \ldots x_n \in E$ and $k_1, k_2 \ldots k_n \in K$ such that

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n = 0$$

Then taking inner product on both side with x_j, $j = 1, 2 \ldots k$, we get

$$\left\langle \sum_{i=1}^{n} k_i x_i , x_j \right\rangle = \langle 0, x_j \rangle$$

$$\Rightarrow \sum_{i=1}^{n} k_i \langle x_i , x_j \rangle = 0$$

$$\Rightarrow \quad k_j \langle x_j , x_j \rangle = 0$$

$$\Rightarrow \quad k_j = 0 \qquad [\because x_j \neq 0 \Rightarrow \langle x_j, x_j \rangle \neq 0]$$

$$j = 1, 2, \ldots n.$$

$\Rightarrow \{x_1, x_2 \ldots x_n\}$ is L.I

$\Rightarrow E$ is L.I.

If E is orthonormal set,
then for any $x, y \in E$, $x \neq y$,
we have $\langle x, y \rangle = 0$, and

$\|x\| = \|y\| = 1$.

$\therefore \|x-y\|^2 = \langle x-y, x-y \rangle$

$$= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$$

$$= 1 - 0 - 0 + 1$$

$$= 2.$$

$\therefore \|x-y\| = \sqrt{2}, \quad x \neq y.$

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