Corollary :- Let X be a Banach space, Y be a n.l.s and $\{A_n\}$ be a sequence in $BL(X,Y)$ such that $\{A_n x\}$ converges for every $x \in X$. Let $A : X \to Y$ be defined by

$$Ax = \lim_{n \to \infty} A_n x, \quad \forall x \in X.$$

Then for every totally bounded subset $S \subseteq X$, $\sup_{x \in S} \|A_n x - Ax\| \to 0$ as $n \to \infty$.

Proof:

[we say S is totally bounded set if for all $\epsilon > 0$,

$$S \subseteq \bigcup_{i=1}^{n} B(x_i, \epsilon)$$

where $x_1, x_2, \dots x_n \in S$.]

Let S be a totally bounded subset of X and $\epsilon > 0$ be given.

Then there exist $x_1, x_2, \ldots x_k$ in S

such that $S \subseteq \bigcup_{i=1}^{k} B(x_i, \epsilon)$

$$= \bigcup_{i=1}^{k} \{x \in X \mid \|x - x_i\| < \epsilon\}$$

Let $x \in S \Rightarrow \exists\ j \in \{1, 2 - k\}$

such that $x \in B(x_j, \epsilon)$

$$\Rightarrow \|x - x_j\| < \epsilon .$$

$\because \{A_n x\}$ converges for every $x \in X$,

$\{A_n x_j\}$ also converges for $x_j \in S \subseteq X$.

$\therefore\ Ax_j = \lim_{n \to \infty} A_n x_j$

$\therefore$ There exists $n_0 \in N$ such that

$$\|A_n x_j - Ax_j\| < \epsilon, \forall\ n \geq n_0$$

Now consider

$$\|A_n x - Ax\| \leq \|A_n x - A_n x_j\| + \|A_n x_j - Ax_j\| + \|Ax_j - Ax\|$$

$$\leq \|A_n\| \, \|x - x_j\| + \|A_n x_j - Ax_j\| + \|A\| \, \|x - x_j\| \quad \text{--- (1)}$$

$\because$ X is a Banach space, By uniform boundedness principle, $\{\|A_n\| / n = 1, 2 \ldots\}$ is bounded.

$\therefore$ $\|A_n\| \leq C \quad \forall n.$

$\therefore$ from (1), we have

$$\|A_n x - Ax\| \leq C\epsilon + \epsilon + \|A\|\epsilon$$

$$= \underbrace{(C + \|A\| + 1)}_{< \infty} \epsilon \quad \forall n \geq n_0.$$

$\therefore$

$$\sup_{x \in S} \|A_n x - Ax\| < (C + \|A\| + 1)\epsilon \longrightarrow 0$$

$$\Rightarrow \|(A_n - A)|_S\| \longrightarrow 0 \text{ as } n \to \infty$$

Closed operator :-

Let X and Y be n.l.s and X_0 be a subspace of X. A linear operator $A : X \to Y$ is said to be closed operator if for every sequence $\{x_n\}$ in X_0 such that $x_n \to x \in X$ and $Ax_n \to y \in Y$,

Then $x \in X_0$ and $Ax = y$.

Ex: $X = Y = C[0,1]$ are n.l.s w.r.t $\|\cdot\|_\infty$.

Let $X_0 = C^1[0,1] \subset C[0,1]$.

Define $A : C^1[0,1] \subset C[0,1] \to C[0,1]$

by $Ax = x'$, $\forall x \in C^1[0,1]$.

Let $\{x_n\}$ be a sequence in $C^1[0,1]$

such that $x_n \to x \in C[0,1]$,

and $Ax_n \to y \in Y = C[0,1]$.

Then for each $t \in [0,1]$, we have

$$\int_0^t y(\tau)\,d\tau = \lim_{n\to\infty} \int_0^t x_n'(\tau)\,d\tau$$

$$= \lim_{n\to\infty} [x_n(t) - x_n(0)]$$

$$= x(t) - x(0)$$

$\therefore$ for each $t \in [0,1]$, we have

$$x(t) = x(0) + \int_0^t y(\tau)\, d\tau$$

$\Rightarrow x'(t) = y(t), \quad t \in [0,1].$

$\Rightarrow Ax = y, \quad x \in C^1[0,1]$

$\Rightarrow$ A is a closed operator.

But A is not bounded operator since $x_n(t) = t^n, \quad t \in [0,1]$

$$Ax_n(t) = n t^{n-1}$$

$\Rightarrow \|Ax_n\|_\infty = n, \quad \|x_n\|_\infty = 1$

$\therefore$ A is unbounded.

* A closed operator need not be a bounded operator.

Is every bounded operator, a closed operator?

Let $A : X_0 \subseteq X \to Y$ be a linear map, where X and Y are n.l.s. Then

$$G(A) = \{ (x, Ax) \mid x \in X_0 \}$$

is called graph of A.

Then $G(A)$ is a subspace of of product space $X \times Y$.

The norm on $X \times Y$ is given by $\|(x,y)\| = \|x\|_X + \|y\|_Y$, $\forall (x,y) \in X \times Y$.

Theorem: Let X and Y be n.l.s and X_0 be a subspace of X. A linear operator $A: X_0 \to Y$ is a closed linear operator iff its graph $G(A) = \{(x, Ax) \mid x \in X_0\}$ is a closed subspace of $X \times Y$.

Proof: Suppose $A: X_0 \to Y$ be a closed operator.

Claim: $G(A)$ is a closed subspace of $X \times Y$.

Let (x, y) belongs to closure of $G(A)$.

Then there exists a sequence $\{(x_n, y_n)\}$ in $G(A)$ such that

$(x_n, y_n) \longrightarrow (x, y)$, where $y_n = Ax_n$

$\Rightarrow (x_n - x, y_n - y) \longrightarrow 0$ in $X \times Y$

$\|(x_n - x, y_n - y)\| \longrightarrow 0$

$\Rightarrow \|x_n - x\|_X + \|y_n - y\|_Y \longrightarrow 0.$

$\therefore x_n \to x$ in X and $y_n \to y$ in Y

$\because Ax_n = y_n \to y$ and $x_n \to x \in X$
and A is a closed operator,
implies $x \in X_0$ & $Ax = y$

$\therefore (x, y) = (x, Ax) \in G(A)$

$\because x \in X_0$

$\therefore G(A)$ is a closed subspace of $X \times Y$.

Conversely suppose that $G(A)$ is a closed subspace of $X \times Y$.

Claim: A is a closed operator.

So let $\{x_n\}$ be a sequence in X_0

$\ni x_n \to x \in X$ and $Ax_n \to y \in Y$.

$\Rightarrow \|x_n - x\|_X \to 0$, $\|Ax_n - y\|_Y \to 0$

$\Rightarrow \|x_n - x\|_X + \|Ax_n - y\|_Y \to 0$

$\Rightarrow \|(x_n, Ax_n) - (x, y)\| \to 0$.

$(x_n, Ax_n) \to (x, y) \quad \text{—} \; * $

$\therefore \{(x_n, Ax_n)\}$ is a sequence in $G(A)$ and $G(A)$ is closed

by ⊛ it follows that $(x, y) \in G(A)$.

$\therefore (x, y) = (x, Ax)$, i.e., $y = Ax$, $x \in X_0$.

$\therefore A$ is a closed operator.

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