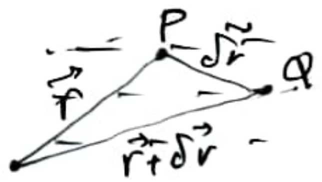
Stream Surface: A stream surface is a surface made by the stream lines passing through an arbitrary line in the fluid region at any instant of time "t".

Streamtube

§ **Velocity of a fluid particle at a point:** Consider P and Q be the positions of the fluid particle at any instant of time t and $t+\delta t$ from a fixed origin O s.t.

$$\vec{OP} = \vec{r} \quad \text{and} \quad \vec{OQ} = \vec{r} + \delta\vec{r}$$



Let $\vec{q}$ be the velocity at the point P, then

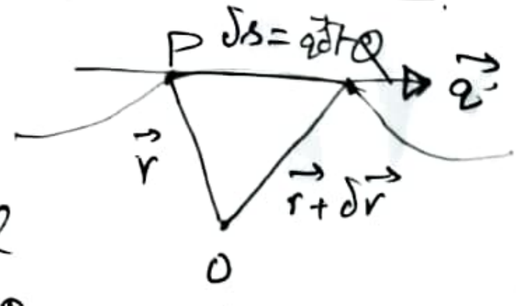
$$\vec{q} = \lim_{\delta t \to 0} \frac{(\vec{r}+\delta\vec{r}) - \vec{r}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

Here $\vec{q}$ depends both on $\vec{r}$ and t.

§ **Local, Convective and Material (Spatial) derivatives:**

Consider $\vec{F}(\vec{r},t)$ be some fluid property related to the flow that



s $\vec{F}$ is some fluid property related

the fluid element at the point P s.t. $\vec{OP} = \vec{r}$.

Let the element moves through a distance $\delta s \ (= \vec{q}\,\delta t)$ in an interval δt, where $\delta t = t+\delta t - t$. Then $Q(\vec{r}+\delta\vec{r})$ will be the new position of the fluid element at an instant $t+\delta t$ and the fluid property at Q will become $\vec{F}(\vec{r}+\delta\vec{r}, t+\delta t)$. The rate of change in the fluid property in time δt is :

$$\frac{d\vec{F}}{dt} = \lim_{\delta t \to 0} \frac{\delta \vec{F}}{\delta t} = \lim_{\delta t \to 0} \frac{F(\vec{r}+\delta\vec{r}, t+\delta t) - F(\vec{r}, t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\vec{F}(\vec{r}+\vec{q}\,\delta t, t+\delta t) - F(\vec{r}, t)}{\delta t}, \quad \vec{q} = \frac{\delta \vec{r}}{\delta t} \text{ as } \delta t \to 0$$

(assuming the smoothness of the func^n. F / cont^n. diff.)

$$= \lim_{\delta t \to 0} \frac{1}{\delta t}\Big[\vec{F}(\vec{r}, t) + \frac{\partial F}{\partial t}\delta t + \frac{\partial^2 F}{\partial t^2}(\delta t)^2 + \cdots$$

$$+ \frac{\partial F}{\partial r}\vec{q}\,\delta t + \frac{\partial^2 F}{\partial \vec{r}^2}(\vec{q}\,\delta t)^2 + \cdots\Big] - \vec{F}(r, t)$$

$$= \lim_{\delta t \to 0} \frac{1}{\delta t}\left[\frac{\partial F}{\partial t}\delta t + \frac{\partial \vec{F}}{\partial r}\vec{q}\,\delta t\right]$$

$$= \lim_{\delta t \to 0} \left[\frac{\partial F}{\partial t} + \vec{q}\,\frac{\partial \vec{F}}{\partial r}\right].$$

$$f(x+h, y+K) = f(x,y) + h\frac{\partial f}{\partial x} + K\frac{\partial f}{\partial y} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{K^2}{2}\frac{\partial^2 f}{\partial y^2}$$

$$+ \cdots$$

$$\Rightarrow \frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} + \underline{\vec{q} \cdot \frac{\partial \vec{F}}{\partial \vec{r}}} \quad \text{—(I)}$$

Spatial derivative / Material derivative

local derivative

~~Spatial~~ Convective derivative / advective derivative

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{F} = \vec{F}(\vec{r}, t) = \vec{F}(\vec{r}(x,y,z), t)$

$$\frac{\partial \vec{F}}{\partial \vec{r}} = \frac{\partial \vec{F}}{\partial x} \cdot \frac{\partial x}{\partial \vec{r}} \hat{i} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \vec{r}} \hat{j} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial r} \hat{k}$$

$$= \hat{i} \frac{\partial \vec{F}}{\partial x} + \hat{j} \frac{\partial F}{\partial y} + \hat{k} \frac{\partial \vec{F}}{\partial z} = \underline{\vec{\nabla} F} \quad \text{—(II)}$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{F} = F(r, t) = F(r(x,y,z), t)$

$$\frac{\partial F}{\partial r} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial r} \hat{i} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial r} \hat{j} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial r} \hat{k}$$

From (I) and (II)

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \vec{q} \cdot (\vec{\nabla} F)$$

$$\Rightarrow \frac{d}{dt} \equiv \underline{\frac{\partial}{\partial t}} + \vec{q} \cdot \underline{\vec{\nabla}}$$

material   local   convective

F is Scalar.

t · P $F(x,y,z,t)$. $t+\delta t$ · Q $F(x+\delta x, y+\delta y, z+\delta z, t+\delta t)$

Change in ~~for~~ F,

$$\cancel{\frac{df}{dt} = \frac{\partial f}{\partial x}}$$

$$dF = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial t}dt$$

$$\Rightarrow \quad \frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} + \frac{\partial F}{\partial t}$$

$$= u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t}$$

$$= \vec{q}\cdot\vec{\nabla}.F + \frac{\partial F}{\partial t}$$

$$\Rightarrow \quad \frac{dF}{dt} = \vec{q}\cdot\vec{\nabla}F + \frac{\partial F}{\partial t} \quad \Rightarrow \quad \frac{d}{dt} = \vec{q}\cdot\nabla + \frac{\partial}{\partial t}$$

$$\vec{q}\cdot\frac{\partial F}{\partial F}\delta t$$

Steady and Non-uniform.

$$\frac{\partial F}{\partial t}\delta t$$

uniform and unsteady

+

non-uniform and unsteady

Ex 1: Determine the acc^n at $P(2,1,3)$ at $t=1$ sec if $u = yz+t$, $v = xz-t$ and $\omega = xy$.

Sol^n: Here $\vec{q} = (yz+t)\hat{i} + (xz-t)\hat{j} + xy\hat{k}$. Now by formula,

$$\vec{a} = \frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \vec{q}\cdot(\vec{\nabla}\vec{q})$$

$$= \frac{\partial \vec{q}}{\partial t} + u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + \omega\frac{\partial \vec{q}}{\partial z}$$

$$= (\hat{i}-\hat{j}) + (yz+t)(z\hat{j}+y\hat{k}) + (xz-t)(z\hat{i}+x\hat{k})$$
$$+ xy(y\hat{i}+x\hat{j})$$

At the point $P(2,1,3)$, the value $\vec{a}$ is

$$\vec{a}\Big|_{P(2,1,3)} = (\hat{i}-\hat{j}) + (3+t)(3\hat{j}+\hat{k})$$
$$+ (6-t)(3\hat{i}+2\hat{k}) + 2(\hat{i}+2\hat{j})$$

$$\vec{a}(t=1) = (\hat{i}-\hat{j}) + 4(3\hat{j}+\hat{k}) + 5(3\hat{i}+2\hat{k})$$
$$+ (2\hat{i}+4\hat{j})$$

$$= \checkmark\ 18\hat{i} + 13\hat{j} + 6\hat{k}\ (?)$$

Find the path lines and streamlines of the flow 6

$$u = \frac{x}{1+t}, \quad v = \frac{y}{1+t}, \quad w = \frac{z}{1+t}.$$

Sol^n: The equ^n for st. lines: $\dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w}$.

$$\Rightarrow \quad x = c_1 y \text{ and } y = c_2 z.$$

The equ^n for path lines:

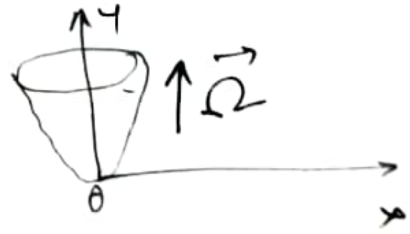
$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w.$$

$$\Rightarrow \quad x = c_1(1+t), \; y = c_2(1+t), \; z = c_3(1+t)$$

Conservation Laws: mass $\longrightarrow$ equ^n of continuity.

Rotational Flow: A fluid flow is said to be rotational if $\text{curl}\, \vec{q} \neq 0$. If $\text{curl}\, \vec{q} = 0$ then the flow is irrotational.

Vorticity: Let $\vec{q}$ be the fluid velocity associated with a fluid flow. Then the vorticity vector is defined as $\vec{\Omega} = \text{curl}\, \vec{q}$.



If $\vec{\Omega} = 0$, then the flow is

If the flow is irrotational, $\text{Curl}\,\vec{q} = 0, \Rightarrow \vec{q} = -\nabla\phi$, ϕ is the velocity potential. If the flow is incompressible and irrotational.

$$\vec{\nabla}\cdot\vec{q} = 0 \quad \& \quad \vec{\nabla}\times\vec{q} = 0$$

$$\Rightarrow \quad \vec{\nabla}\cdot(\vec{\nabla}\phi) = 0$$

$$\Rightarrow \quad \nabla^2\phi = 0.$$