[1] Functional Analysis by Balmohan Limaye, New AGE International Publication

[2] Functional Analysis, A First Course. PHI Publications by M. T. Nair

Let the ~~field~~ field $K = \mathbb{R}$ or $\mathbb{C}$.

Let X be a Linear space over the field K. A norm on X is a function $\|\cdot\| : X \longrightarrow \mathbb{R}^+$ such that ~~for~~ for all $x, y \in X$, $\alpha \in K$

(i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|x+y\| \leq \|x\| + \|y\|$

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$.

A normed space X is a linear space with a norm $\|\cdot\|$ on it.

It is denoted by $(X, \|\cdot\|)$.

1) The space

$$K^n = \{ (\alpha_1, \alpha_2, \ldots \alpha_n) \mid \alpha_i \in K, \; i = 1, 2, \ldots n \}$$

of all n-tuples of numbers in K with coordinate wise addition and scalar multiplication, i.e

if $x = (\alpha_1, \alpha_2 \ldots \alpha_n)$

$y = (\beta_1, \beta_2 \ldots \beta_n)$

$$x + y = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots \alpha_n + \beta_n)$$

and for any $\alpha \in K$,

$$\alpha x = (\alpha \alpha_1, \alpha \alpha_2, \ldots \alpha \alpha_n)$$

is a ~~linear~~ linear space.

Now for any $x = (x(1), x(2), \ldots x(n))$

Define $\|x\|_1 = \sum_{i=1}^{n} |x(i)|$

$\|x\|_\infty = \max\{|x(i)| \mid i = 1, \ldots n\}$

- Then $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are ~~Dams~~ norms on X.

$\|\cdot\|_1$:—

$\because |x(i)| \geq 0 \quad \forall i$

$\Rightarrow \sum_{i=1}^{n} |x(i)| \geq 0$

$\Rightarrow \|x\|_1 \geq 0$

and $\|x\|_1 = 0 \Rightarrow \sum_{i=1}^{n} |x(i)| = 0$

$\Rightarrow |x(i)| = 0$
$i = 1, 2 \ldots n$

$\Rightarrow x(i) = 0, \ i = 1, 2 \ldots n$

$$\Rightarrow x = (x(1), x(2), \dots x(n))$$
$$= (0, 0, 0 \dots 0)$$

$\because$ $|\alpha x(i)| = |\alpha|\,|x(i)|$

$$\Rightarrow \sum_{i=1}^{n} |\alpha x(i)| = \sum_{i=1}^{n} |\alpha|\,|x(i)|$$
$$= |\alpha| \sum_{i=1}^{n} |x(i)|$$
$$= |\alpha|\, \|x\|_1$$

$\because$ For $i = 1, 2, \dots n$,

$$|x(i) + y(i)| \le |x(i)| + |y(i)|$$

$$\Rightarrow \sum_{i=1}^{n} (|x(i) + y(i)|) \le \sum_{i=1}^{n} |x(i)| + \sum_{i=1}^{n} |y(i)|$$

$$\Rightarrow \|x+y\|_1 \le \|x\|_1 + \|y\|_1$$

$\therefore$ $(K^n, \|\cdot\|_1)$ is a

normed linear space.

IIIly we can prove

$(K^n, \|\cdot\|_\infty)$ is also a n.l.s.

(2) $X = C[a, b]$

For any $x \in X$, Define

$$\|x\|_1 = \int_a^b |x(t)|\,dt$$

$$\|x\|_\infty = \sup_{t \in [a,b]} |x(t)|.$$

Then $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are ~~have~~ norms on $C[a, b]$.

Theorem: (i) For $x, y \in K^n$,

$$\sum_{i=1}^{n} |x(i)y(i)| \leq \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2} \times \left(\sum_{i=1}^{n} |y(i)|^2\right)^{1/2} \quad (*)$$

(ii) For any $x, y \in C[a,b]$

$$\int_a^b |x(t)y(t)|\,dt \leq \left(\int_a^b |x(t)|^2\right)^{1/2} \times \left(\int_a^b |y(t)|^2\right)^{1/2}$$

Proof :-

(i) For $x = (x(1), x(2), \ldots x(n)) \in K^n$

Let $\|x\|_2 = \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2}$.

For $x = 0$, the inequality $(*)$ is true.

So assume $x \neq 0$, $y \neq 0$

$\Rightarrow \|x\|_2 \neq 0, \quad \|y\|_2 \neq 0$

Also, for any $a, b \in \mathbb{R}$,

$$ab \leq \frac{a^2 + b^2}{2}$$

Letting $a = \frac{|x(i)|}{\|x\|_2}$

$b = \frac{|y(i)|}{\|y\|_2}$

We have

$$\frac{|x(i)|}{\|x\|_2} \frac{|y(i)|}{\|y\|_2} \leq \frac{1}{2} \left[\frac{|x(i)|^2}{\|x\|_2^2} + \frac{|y(i)|^2}{\|y\|_2^2} \right]$$

Taking summation from $i = 1, 2, \ldots n$, we get

$$\sum_{i=1} \frac{|x(i)\,y(i)|}{\|x\|_2\,\|y\|_2} \le \frac{1}{2}\left[\cancel{\frac{\sum_{i=1}^{n}|x(i)|^2}{\|x\|_2^2}} + \cancel{\frac{\sum_{i=1}^{n}|y(i)|^2}{\|y\|_2^2}}\right]$$

$$= \frac{1}{2}[1+1] = 1$$

$$\Rightarrow \underline{\sum_{i=1}^{n} |x(i)\,y(i)|} \le \underline{\|x\|_2}\;\underline{\|y\|_2}.$$

(ii) For $x \in C[a,b]$,

let $\|x\|_2 = \left(\int_a^b |x(t)|^2\right)^{1/2}$

$a = \dfrac{|x(t)|}{\|x\|_2}$, $\quad b = \dfrac{|y(t)|}{\|y\|_2}$, $\quad t \in [a,b]$

and letting

$$ab \leq \frac{1}{2}\left[a^2+b^2\right],$$

We can prove second inequality

—//—

Now we can prove that K^n with $\|x\|_2 = \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2}$ is also a ~~n.l.s.~~ n.l.s.

$\because |x(i)|^2 \geq 0 \quad \forall\, i = 1, 2, \ldots, n$

$$\Rightarrow \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2} \geq 0$$

$$\Rightarrow \|x\|_2 \geq 0$$

and $\|x\|_2 = 0 \Leftrightarrow \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2} = 0$

$\Leftrightarrow |x(i)| = 0, \forall i = 1 \dots n$

$\Leftrightarrow x = (0, 0, 0 \dots 0).$

$$\|\alpha x\|_2^2 = \sum_{i=1}^{n} |\alpha x(i)|^2$$

$$= \sum_{i=1}^{n} |\alpha|^2 |x(i)|^2$$

$$= |\alpha|^2 \sum_{i=1}^{n} |x(i)|^2$$

$$\Rightarrow \|\alpha x\|_2 = |\alpha| \, \|x\|_2 .$$

Now for any $x = (x(1), x(2) \dots x(n))$

$y = (y(1), y(2) \dots y(n)) \in K^n,$

we have

$$\|x+y\|_2^2 = \sum_{i=1}^{n} |x(i) + y(i)|^2$$

$$\leq \sum_{i=1}^{n} [|x(i)| + |y(i)|]^2$$

$$= \sum_{i=1}^{n} [|x(i)|^2 + |y(i)|^2 + 2|x(i)||y(i)|]$$

$$= \sum_{i=1}^{n} |x(i)|^2 + \sum_{i=1}^{n} |y(i)|^2 + 2\sum_{i=1}^{n} |x(i)\, y(i)|$$

$$\leq \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2 \|y\|_2$$

$$= (\|x\|_2 + \|y\|_2)^2$$

$\Rightarrow \quad \|x+y\|_2 \leq \|x\|_2 + \|y\|_2 .$

$\Rightarrow \quad \|\cdot\|_2$ is a norm on K^n.

Illy we can prove that

$C[a, b]$, with

$$\|x\|_2 = \left(\int_a^b |x(t)|^2\right)^{1/2}$$

is also a n.l.s.

Now we will explore about

$$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}$$

on K^n.

We will see that $\|\cdot\|_p$ is ~~also~~ norm on K^n for $1 \le p \le \infty$.

Lemma:- Let p and q be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for every positive real numbers ~~and~~ a and b,

~~there~~ there hold:

$$ab \leq \frac{a^p}{p} + \frac{b^p}{p}.$$

Proof: Note that a function φ is a convex function on an interval J, iff for $\alpha, \beta \in J$ and $\lambda, \mu \in [0,1]$, with $\lambda + \mu = 1$,

$$\varphi(\lambda\alpha + \mu\beta) \leq \lambda\varphi(\alpha) + \mu\varphi(\beta)$$

Letting $\varphi(t) = e^t$, $t > 0$, which is a convex function, we have

$$e^{\lambda\alpha + \mu\beta} \leq \lambda e^{\alpha} + \mu e^{\beta}$$

$$\Rightarrow e^{\lambda\alpha} \cdot e^{\mu\beta} \leq \lambda e^{\alpha} + \mu e^{\beta}$$

Taking $\lambda = \frac{1}{p}$, $\mu = \frac{1}{q}$

and α, β such ~~that~~ that

$$a = e^{\alpha/p} \quad \& \quad b = e^{\beta/q}$$

$$\Rightarrow a^p = e^{\alpha} \quad \text{and} \quad b^q = e^{\beta},$$

we have

$$ab = e^{\alpha/p} e^{\beta/q} \leq \frac{1}{p} a^p + \frac{1}{q} b^q$$

$$\Rightarrow ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q \quad , \frac{1}{p} + \frac{1}{q} = 1$$

$$(\because \lambda + \mu = 1)$$

Holder's Inequality: ___

Inequality.

Let p and q be positive real ~~numbers~~ numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

For $x, y \in K^n$,

$$\sum_{i=1}^{n} |x(i)y(i)| \leq \left(\sum_{i=1}^{n} |x(i)|^p \right)^{1/p} \times \left(\sum_{i=1}^{n} |y(i)|^q \right)^{1/q}.$$