Theorem: Let X and Y be n.l.s.

(a) If Z is a closed subspace of X, then the quotient map $Q : X \to \frac{X}{Z}$ is continuous and open.

(b) Let $F : X \to Y$ is a linear map such that the null space $Z(F)$ is closed in X. Define $\tilde{F} : \frac{X}{Z(F)} \to Y$ by $\tilde{F}(x + Z(F)) = F(x), \forall x \in X$.

Then F is open map iff $\tilde{F}$ is an open map.

Proof

(a) The map $Q : X \to \frac{X}{Z}$ is linear and surjective.

Here $Q(x) = x + Z$.

Now $|||Q(x)||| = |||x+Z|||$

$$= \inf\{\|x+z\| \,/\, z \in Z\}$$

$$\leq \|x\|, \quad [\because z = 0 \in Z]$$

$\therefore$ Q is Continuous.

Claim: Q is open map.

To prove this we use the previous class theorem.

Consider any $\epsilon > 0$. let $x+Z \in \frac{X}{Z}$

Then

$$\inf\{\|x+z\| \,/\, z \in Z\}$$

$$= |||x+Z||| < (1+\epsilon)|||x+Z|||$$

So there exists $z_0 \in Z$ such that

$$\|x+z_0\| < (1+\epsilon)\,|||x+Z|||$$

Also

$$Q(x+z_0) = x+z_0+Z$$
$$= x+Z \quad \left[\because z_0 \in Z \Rightarrow z_0+Z = Z\right]$$

$\therefore$ By taking $\delta = 1+\epsilon$, in the last class theorem, we see that Q is a open map.

(b) Since $Z(F)$ is closed in X, the quotient space $\frac{X}{Z(F)}$ is closed in the quotient norm.

Let $Q : X \longrightarrow \frac{X}{Z(F)}$ be the quotient map $Q(x) = x+Z(F)$, $\forall x \in X$.

Let $E \subseteq X$ be an open set

$$F(x) = \tilde{F}(x + Z(F))$$
$$= \tilde{F}(Q(x)), \quad \forall x \in X.$$
$$\Rightarrow F = \tilde{F} \circ Q$$
$$\Rightarrow F(E) = \tilde{F}(Q(E)).$$

By (a), we know that Q is open map, so it follows that F is open map whenever $\tilde{F}$ is open map.

Now assume F is open map. We prove $\tilde{F}$ is an open map.

Let $\tilde{E} \subset \dfrac{X}{Z(F)}$ be an open set.

Since $F = \tilde{F} \circ Q$

$$\Rightarrow \tilde{F}(\tilde{E}) = F(Q^{-1}(\tilde{E})) \longrightarrow (*)$$

$\because Q : X \to \dfrac{X}{Z(F)}$ is continuous,

$\tilde{E}$ is open in $\dfrac{X}{Z(F)}$, implies

$Q^{-1}(\tilde{E})$ is open in X.

Hence it follows that

$\tilde{F}(\tilde{E})$ is open iff $F(Q^{-1}(\tilde{E}))$ is open by $\circledast$.

$\Rightarrow \tilde{F}$ is an map iff F is open map.

$(*)$ Also $\tilde{F}$ defined above is bounded linear map and

$$\|F\| = \|\tilde{F}\|.$$

Open mapping Theorem :—

Let X and Y be Banach spaces and $F : X \to Y$ be a linear map which is closed and surjective. Then F is continuous and open map.

Proof: By closed graph theorem, it follows that F is continuous.

$\Rightarrow Z(F)$ is closed in X.

$\therefore \tilde{F} : \frac{X}{Z(F)} \longrightarrow Y$ defined by

$$\tilde{F}(x + Z(F)) = F(x), \ \forall x \in X$$

is continuous and bijective.

$$[\because \ \tilde{F}(x_1 + Z(F)) = \tilde{F}(x_2 + Z(F))$$
$$\Leftrightarrow \quad F(x_1) = F(x_2)$$
$$\Leftrightarrow \quad F(x_1 - x_2) = 0$$
$$\Leftrightarrow \quad x_1 - x_2 \in Z(F)$$
$$\Leftrightarrow \quad x_1 + Z(F) = x_2 + Z(F).$$

Since F is onto, we have

$Y = R(F) \Rightarrow$ given $y \in Y$

$\exists\, x \in X \ni F(x) = y$

$\Rightarrow \tilde{F}(x + Z(F)) = F(x) = y$

Then for every $y \in Y$,

$\exists\, x + Z(F) \in \dfrac{X}{Z(F)}$ such that

$\tilde{F}(x + Z(F)) = y$

$\therefore \tilde{F}$ is onto]

Since $\tilde{F} : \dfrac{X}{Z(F)} \to Y$ is

and bounded linear operator.

Then $\tilde{F}^{-1} : Y \to \dfrac{X}{Z(F)}$ is bijective.

Closed operator.

[If $A : X \to Y$ is 1–1, bounded

operator, $A^{-1} : R(A) \to X$ is a closed operator]

$\because$ Y and $\frac{X}{Z(F)}$ are Banach spaces and $\tilde{F}^{-1} : Y \to \frac{X}{Z(F)}$ is closed linear map, so by Closed Graph theorem $\tilde{F}^{-1}$ is a continuous linear map

let $\tilde{E}$ be any open set in $\frac{X}{Z(F)}$

$\Rightarrow (\tilde{F}^{-1})^{-1}(\tilde{E})$ is open in Y

$\Rightarrow \tilde{F}(\tilde{E})$ is open in Y

$\Rightarrow \tilde{F}$ is open map

$\Rightarrow$ F is open map

(by previous theorem).

—//—

Theorem (Bounded inverse theorem).

Let X and Y be Banach spaces and $F : X \to Y$ be bounded, bijective linear map. Then $F^{-1} \in BL(X, Y)$.

[Hint: Use $A : X \to Y$ is 1-1, bounded operator, then $A^{-1} : R(A) \to X$ is closed operator and use Closed graph theorem].

—//—

Partial order :— A binary relation "R" on a set S is called partial order if it is

(a) Reflexive : $x R x$, $\forall x \in S$

(b) Anti symmetric : $x R y$ and $y R x$ $\Rightarrow x = y$ $\forall x, y \in S$.

(c) Transitive :—

$x R y$, $y R z \Rightarrow x R z$, $\forall x, y, z \in S$.

A set S together with partial order is called partially ordered set (POSET).

$S = \{A_1, A_2, A_3 \ldots \ldots\}$

the class of subsets of real

numbers.

Let the relation $R = \subseteq$, set inclusion.

Then $(S, \subseteq)$ is a POSET.

Totally ordered set :—

A subset T of a POSET S with the partial order relation R is called Totally ordered set if for all $x, y \in T$, either xRy or yRx, i.e., any two elements of T are comparable.

Suppose S is a POSET with the partial order relation R or "$\leq$".

An element $x \in S$ is called maximal element of S if for every $y \in S$, xRy

Zorn's lemma :-

Let X be a non empty POSET with a partial order relation "$\leq$" such that every totally ordered subset of X has an upper bound, then X has a maximal element.

—//—