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Let $X = \left(C[a,b], \|\cdot\|_\infty\right)$

For $k(\cdot,\cdot) \in C\left([a,b]\times[a,b]\right)$,

let
$$Ax(s) = \int_a^b k(s,t)\,x(t)\,dt, \quad s\in[a,b]$$
$$\forall x\in C[a,b]$$

We prove $Ax \in C[a,b]$, $\forall x \in C[a,b]$.

Consider for any $s_0, s \in [a,b]$ and $x \in C[a,b]$

$$|Ax(s) - Ax(s_0)| = \left|\int_a^b [k(s,t) - k(s_0,t)]\,x(t)\,dt\right|$$

$$\leq \int_a^b |k(s,t) - k(s_0,t)|\,|x(t)|\,dt$$

$$\leq \sup_{t\in[a,b]} |x(t)| \int_a^b |k(s,t) - k(s_0,t)|\,dt \quad \text{——(1)}$$

Since $k(\cdot,\cdot)$ is continuous on a compact set $[a,b]\times[a,b]$, implies $k(\cdot,\cdot)$ is uniformly continuous.

$\therefore$ given any $\epsilon>0$, $\exists\ \delta=\delta(\epsilon)>0$ such that:

$$|s-s_0|<\delta \implies |k(s,t)-k(s_0,t)|<\epsilon \quad \forall t\in[a,b]$$

$\therefore$ from ① we have

$$|Ax(s)-Ax(s_0)| \leq \|x\|_\infty \cdot \int_a^b |k(s,t)-k(s_0,t)|\,dt$$

$$\leq \|x\|_\infty \cdot \sup_{t\in[a,b]} |k(s,t)-k(s_0,t)| \int_a^b 1\,dt$$

$$\leq \|x\|_\infty\, \epsilon\,(b-a), \quad \forall x\in C[a,b]$$

$\implies Ax\in C[a,b],\ \forall x\in C[a,b]$

$\therefore A: C[a,b]\longrightarrow C[a,b]$

Clearly A is a linear map

$\because$ Consider for any $x, y \in C[a,b], \alpha, \beta \in \Gamma$

$$A(\alpha x+\beta y)(s) = \int_a^b k(s,t)\,[\alpha x + \beta y](t)\,dt$$

$$\vdots$$

$$= \alpha \int_a^b k(s,t)\,x(t)\,dt + \beta \int_a^b k(s,t)\,y(t)\,dt$$

$$= \alpha Ax(s) + \beta Ay(s)$$

$\forall s \in [a,b]$

$$\Rightarrow \quad A(\alpha x+\beta y) = \alpha Ax + \beta Ay.$$

Now Consider for any $x \in C[a,b]$, $s \in [a,b]$

$$|Ax(s)| = \left|\int_a^b k(s,t)\,x(t)\,dt\right|$$

$$\leq \left(\sup_{s\in[a,b]} \int_a^b |k(s,t)|\,dt\right) \|x\|_\infty \quad (*)$$

$\because s \longrightarrow \int_a^b |k(s,t)|\,dt$ is Continuous on a compact interval $[a,b]$,

we have

$$C = \sup_{s\in[a,b]} \int_a^b |k(s,t)|\,dt < \infty$$

$$\left[\because \left| \int_a^b f(t) g(t)\,dt \right| \le \int_a^b |f(t)|\,|g(t)| \le \sup_{t\in[a,b]} |f(t)| \int_a^b g(t)\,dt \right]$$

$$\left[\because \int_a^b |k(s,t)|\,dt - \int_a^b |k(s_0,t)|\,dt\right.$$

$$= \int_a^b \left[|k(s,t)| - |k(s_0,t)| \right] dt$$

$$\le \int_a^b |k(s,t) - k(s_0,t)|\,dt$$

$[\because |a| - |b| < |a-b|]$

$$< \epsilon(b-a) \text{ whenever } |s-s_0| < \delta.$$

$$\left.\therefore s \to \int_a^b |k(s,t)| \text{ is continuous}\right]$$

$\therefore$ from $\circledast$ we have

$$|Ax(t)| \leq c\,\|x\|_\infty \,, \quad \forall t \in [a,b]$$
$$\forall x \in C[a,b]$$

$$\Rightarrow \sup_{t\in[a,b]} |Ax(t)| \leq c\,\|x\|_\infty$$

$$\Rightarrow \|Ax\|_\infty \leq c\,\|x\|_\infty .$$

$\therefore$ $A : C[a,b] \longrightarrow C[a,b]$

is a bounded linear operator.

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We know that a linear map from a n.l.s X to a n.l.s Y is continuous iff it maps bounded sets in X to bounded sets in Y. We call such a map is a bounded linear map.

The set of all bounded linear maps denoted by $BL(X,Y)$ or $B(X,Y)$.

A linear map from n.l.s X to itself is called linear operator on X. We denote set of all bounded linear operators on X by $BL(X,X)$ or $BL(X)$ or $B(X)$.

Also, we write

$X' = BL(X, K)$, is set of all bounded linear functionals on X.

If $F \in BL(X, Y)$, $F \neq 0$, then there exists $\alpha > 0$ $\ni$

$$\|F(x)\| \leq \alpha \|x\|, \quad \forall x \in X.$$

We say a linear map $F : X \longrightarrow Y$ is bounded below if $\exists$ $\beta > 0$ $\ni$

$$\beta \|x\| \leq \|F(x)\|, \quad \forall x \in X.$$

Problem: Show that $BL(X, Y)$ is a linear space under the pointwise operations

For $x \in X$,

$$(F+G)(x) = F(x) + G(x)$$

$$(\alpha F)(x) = \alpha F(x), \quad \alpha \in K$$

Theorem: Let X and Y be n.l.s.

For $F \in BL(X,Y)$, define

$$\|F\| = \sup\{\|F(x)\| \,/\, x \in X, \|x\| \leq 1\}.$$

Then $\|\cdot\|$ is a norm on X called operator norm.

For all $x \in X$, $\|F(x)\| \leq \|F\| \, \|x\|$

In fact

$$\|F\| = \inf\{\alpha \geq 0 \,/\, \|F(x)\| \leq \alpha \|x\|, \ \forall x \in X\}$$

Also if $X \neq \{0\}$,

$$\|F\| = \sup\{\|F(x)\| / x \in X, \|x\| = 1\}$$

$$= \sup\{\|F(x)\| / x \in X, \|x\| < 1\}$$

Proof. If $X = \{0\}$, there is nothing to prove.

So let $X \neq \{0\}$.

$\because \|F(x)\| \geq 0, \forall x \in X$

$\Rightarrow \sup\{\|F(x)\| / x \in X, \|x\| \leq 1\} \geq 0$

$\Rightarrow \|F\| \geq 0$

Also

$\|F\| = 0 \iff \sup\{\|F(x)\| \,/\, x \in X, \|x\| \le 1\}$

$= 0$

$\iff \|F(x)\| = 0, \ \forall x \in X$
$\|x\| \le 1.$

$\iff F(x) = 0, \ \forall x \in X,$
$\|x\| \le 1$

$\iff F(y) = 0 \ \forall y \in X$

$y = \frac{x}{\|x\|}$

$\|y\| = 1$

$\iff F\left(\frac{x}{\|x\|}\right) = 0, \ \forall x \in X$

$\iff F(x) = 0, \ \forall x \in X$

$\iff F := 0$

Now for any scalar $\alpha \in K$,

$\|\alpha F\| = \sup\{\|(\alpha F)(x)\| \,/\, x \in X, \|x\| \le 1\}$

$= \sup\{\|\alpha F(x)\| \,/\, x \in X, \|x\| \le 1$

$$= \sup\{ |\alpha| \, \|F(x)\| \,/\, x \in X, \|x\| \leq \}$$

$$= |\alpha| \sup\{ \|F(x)\| \,/\, x \in X, \|x\| \leq 1\}$$

$$= |\alpha| \, \|F\|$$

Now for any $F_1, F_2 \in BL(X, Y)$,

$$\|F_1 + F_2\| = \sup\{ \|(F_1 + F_2)(x)\| \,/\, x \in X, \|x\| \leq 1\}$$

$$= \sup\{ \|F_1(x) + F_2(x)\| \,/\, \|x\| \leq 1, x \in X\}$$

$$\leq \sup\{ \|F_1(x)\| + \|F_2(x)\| \,/\, x \in X, \|x\| \leq 1\}$$

$$\leq \sup\{ \|F_1(x)\| \,/\, x \in X, \|x\| \leq 1\} + \sup\{ \|F_2(x)\| \,/\, x \in X, \|x\| \leq 1\}$$

$$= \|F_1\| + \|F_2\|$$

$\therefore$ $BL(X, Y)$ is a n.l.s ;

with

$$\|F\| = \sup\{\|F(x)\| \,/\, x \in X, \|x\| \leq 1\}.$$