Ex: $X = C[a,b]$ with $\|\cdot\|_\infty$.
and $X_0 = C'[a,b] \subset X$.

Let $A : X_0 \subseteq X \to X$ be the inclusion operator defined by

$$Ax = x.$$

Clearly A is bounded operator.

Since $\overline{X_0} = X$, for $x \in X - X_0$
$\exists$ a sequence $\{x_n\}$ in X_0 such that

$x_n \to x \in X$.

$\because x_n \in X_0$, $Ax_n = x_n \to x$

$\therefore$ The sequence
$\{(x_n, Ax_n)\}$ is a sequence in

the graph of A, $G(A)$,

and $(x_n, Ax_n) \longrightarrow (x, x) \notin G(A)$

$\therefore$ $G(A)$ is not a closed subspace of $X \times X$.

$\therefore$ A is not a closed operator.

But if the domain of a bounded operator is closed subspace, then it is a closed operator.

Theorem: Let $A : X_0 \subseteq X \longrightarrow Y$ be a bounded operator.

(i) If X_0 is closed in X, then A is a closed operator

(ii) If Y is a Banach space and A is a closed operator, then X_0 is a closed subspace of X.

Proof:

(i) Let X_0 be a closed subspace of X and $A : X_0 \longrightarrow Y$ be a bounded operator.

Claim: A is a closed operator.

Let $\{x_n\}$ be a sequence in X_0 such that $x_n \longrightarrow x \in X$ and $Ax_n \longrightarrow y \in Y$. $\longrightarrow (1)$

$\because$ X_0 is a closed subspace, and $\{x_n\}$ is a sequence in X_0 such that $x_n \longrightarrow x \Rightarrow x \in X_0$.

$\because$ A is bounded operator,

$$x_n \to x \Rightarrow Ax_n \to Ax \quad \text{---(2)}$$

$\therefore$ from (1) & (2) we have

$$Ax = y$$

$\therefore$ A is a closed operator.

(ii) Let $A : X_0 \subseteq X \to Y$ be both closed and bounded operator and Y be a Banach space.

Claim: X_0 is a closed subspace

Let $\{x_n\}$ be a sequence in X_0 such that $x_n \to x \in X$.

Since A is a bounded operator, we have

$$\|Ax_n - Ax_m\| \leq \|A\| \, \|x_n - x_m\| \to 0$$

$\Rightarrow \{Ax_n\}$ is a Cauchy sequence in Y. But Y is a Banach space. $\therefore$ $Ax_n \to y \in Y$.

Then we have $\{x_n\}$ is a sequence in X_0 $\ni$ $x_n \to x$ and $Ax_n \to y$, and A is a closed operator.

$\therefore$ $x \in X_0$ and $Ax = y$.

$\Rightarrow$ X_0 is a closed subspace.

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Problem: Let $A : X_0 \subseteq X \to Y$ be a closed operator. If Y

is a Banach space and X_0 is not closed in X, then show that A is unbounded operator.

* If every closed operator $A : X_0 \subseteq X \longrightarrow Y$ with closed subspace X_0 and complete Y a bounded operator?

We know that if $A \in BL(X, Y)$, then the null space $N(A)$ is closed subspace of X.

If A is a closed operator, then $N(A)$ is also closed operator. Also if A is 1-1, Then $\vec{A}^{1}: R(A) \to X$ is a closed operator.

Theorem: Suppose $A : X_0 \subseteq X \to Y$ be a closed operator. Then

(i) $N(A)$ is a closed subspace of X

(ii) if A is $1-1$, $\vec{A}^{1} : R(A) \subset Y \to X$ is a closed operator.

Proof: (i) Let $\{x_n\}$ be a sequence in $N(A)$ & $x_n \to x \in X$.

$\because x_n \in N(A) \Rightarrow Ax_n = 0, \forall n.$

$\therefore Ax_n \to 0$ as $n \to \infty$

$\because A$ is a closed operator and $x_n \in N(A)$, $x_n \to x \in X$, $Ax_n \to 0$,

$\Rightarrow \quad Ax = 0 \quad , \quad x \in N(A).$

$\therefore$ $N(A)$ is a closed subspace.

(ii) Assume A is $1-1$.

$\therefore$ $A^{-1} : R(A) \subseteq Y \longrightarrow X$ exists

Claim: A^{-1} is a closed operator.

So let $\{y_n\}$ be a sequence in $R(A)$

such that $y_n \to y \in Y$, $A^{-1}y_n \to x \in X$

Let $x_n = A^{-1}y_n \Rightarrow Ax_n = y_n.$

$\therefore$ $Ax_n = y_n \to y \in Y$

and $x_n \longrightarrow x \in X$

$\because$ A is a closed operator, $Ax = y$, $x \in X_0$.

$\Rightarrow x = A^{-1}y \,, \; y = Ax \in R(A)$

$\Rightarrow A^{-1}$ is a closed operator.

Theorem: Suppose X is a Banach space and $A : X_0 \subseteq X \to Y$ be 1-1, closed operator. If $R(A)$ is not closed in Y, then $A^{-1} : R(A) \subseteq Y \to X$ is unbounded operator.

Proof: Since $A : X_0 \subseteq X \to Y$ is 1-1, closed, by above theorem, $A^{-1} : R(A) \subseteq Y \to X$ is a closed operator. If $A^{-1}: R(A) \subseteq Y \to X$ is also bounded, then A^{-1} is both closed and bounded.

Since X is a Banach space, by one of the previous theorem, $R(A)$ the domain of A^{-1} is a closed subspace of Y, which is contradiction to $R(A)$ is not closed.

$\therefore A^{-1}$ is unbounded.

Ex: $X = C^1[0,1]$ with the

norm $\|x\|_{1,\infty} = \|x\|_\infty + \|x'\|_\infty$.

$Y = C^1[0,1]$ with $\|\cdot\|_\infty$.

Define $A : X \to Y$ by

$$Ax = x.$$

Then $\|Ax\|_\infty = \|x\|_\infty \leq \|x\|_\infty + \|x'\|_\infty$
$= \|x\|_{1,\infty}$

$\therefore A : X \to Y$ is bounded.

The inverse of A,

$A^{-1} : R(A) \to X$ is defined

by $A^{-1}y = y$, $\forall y \in R(A)$.

$$\|A^{-1}y\|_{1,\infty} = \|y\|_\infty + \|y'\|_\infty$$

$$\nleq \|y\|_\infty$$

$\therefore\ A^{-1}$ is unbounded.

$\Big[\ A^{-1} : R(A) \subseteq Y \longrightarrow X$

$Y = C^1[0,1],\ \|\cdot\|_\infty$

$X = C^1[0,1],\ \|x\|_{1,\infty} = \|x\|_\infty + \|x'\|_\infty$

$\forall\, y \in R(A),$

$$\|A^{-1}y\|_{1,\infty} = \|y\|_{1,\infty}$$
$$= \|y\|_\infty + \|y'\|_\infty.$$
$$\not\leq \|y\|_\infty \Big]$$

But $A^{-1} : R(A) \subseteq Y \longrightarrow X$ is a closed operator.

Theorem: let $A : X \to Y$ be 1-1, bounded operator. Then $A^{-1} : R(A) \subseteq Y \longrightarrow X$ is a closed operator.

Proof: let $\{y_n\}$ be a sequence in $R(A)$ such that

$$y_n \to y \in Y, \quad A^{-1}y_n \to x \in X.$$

Let $x_n = A^{-1}y_n$, $\forall n \in \mathbb{N}$.

$\Rightarrow x_n \to x \in X$, $Ax_n = y_n \to y \in Y$.

$\because A$ is a bounded operator and $x_n \to x \Rightarrow Ax_n \to Ax$

Thus $Ax_n \to y$ & $Ax_n \to Ax$

$\therefore Ax = y \Rightarrow y \in R(A)$

and $x = A^{-1}y$

$\Rightarrow \bar{A}$ is a closed operator.

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Theorem: let $A_0 : X_0 \subseteq X \to Y$ be a bounded operator, where X_0 is dense in X, and Y is a Banach space. Then there exists a unique $A \in B(X,Y)$ such that A is extension of A_0.

Moreover $\|A\| = \|A_0\|$, and for $x \in X$, $Ax = \lim_{n\to\infty} A x_n$, where $\{x_n\}$ is a sequence in X_0 such that $x_n \to x$ [we prove later].

Closed graph theorem :—

If X and Y are Banach spaces, then every closed operator $A : X \to Y$ is a continuous operator.

Proof: Let X and Y be Banach spaces and $A : X \to Y$ be a closed operator.

Claim: A is a continuous operator.

Let $B_0 = \{x \in X / \|x\| < 1\}$.

We show that

$$B_0 \subseteq \{x \in X / \|Ax\| \leq c\}$$

for some $c > 0$, so that A is continuous.

For each $\alpha > 0$, let

$$V_\alpha = \{x \in X / \|Ax\| \leq \alpha\}$$

Then $X = \bigcup_{j=1}^{\infty} V_j$.

Since X is a Banach space, by the Baire-Category theorem, there is some $k > 0$ such that

$$\overset{\circ}{\overline{V_k}} \neq \phi$$

Thus there is some $x_0 \in X$ and $r > 0 \ni B(x_0, r) \subseteq \overline{V_k}$.

[i.e., let $x_0 \in \overset{\circ}{\overline{V_k}}$].

Now let $x \in B_0$ and set

$$u = x_0 + rx$$

$$\Rightarrow \|u - x_0\| = \|rx\| = r\|x\| < r$$

$[\because \|x\|<1]$

$$\Rightarrow u \in B(x_0, r) \subset \overline{V_K}.$$

Now $x_0, u \in B(x_0, r) \subset \overline{V_K}$, implies there exist sequence $\{u_n\}$ and $\{v_n\}$ in V_K such that $u_n \to u$, $v_n \to x_0$

$$\because u_n, v_n \in V_K \Rightarrow \|Au_n\| \le K$$
$$\|Av_n\| \le K.$$

Thus

$[\because u = x_0 + rx]$

$$x = \frac{1}{r}(u - x_0)$$

$$= \lim_{n\to\infty} \frac{(u_n - v_n)}{r},$$

and

$$\left\|A\left(\frac{u_n - v_n}{r}\right)\right\| \le \frac{1}{r}\left[\|Au_n\| + \|Av_n\|\right]$$

$$\leq \frac{2k}{r}$$

$$\Rightarrow \frac{u_n - v_n}{r} \in V_{\frac{2k}{r}}$$

$$\Rightarrow x \in \overline{V_{\frac{2k}{r}}}.$$

$$\therefore B_0 \subseteq \overline{V_{2k}}.$$