Ex1: Test whether the motion specified by

$$\vec{q} = \frac{K^2(x\hat{j} - y\hat{i})}{x^2+y^2 \ (\neq 0)}$$ is possible or not.

for an incompressible.

Sol^n: $\vec{\nabla}.\vec{q} = 0$? ✓

Ex2: Determine the velocity of an irrotational flow if the velocity potential is $\phi = \frac{1}{2}a(x^2+y^2-2z^2)$.

Sol^n: $\vec{q} = -\nabla\phi = -(ax\hat{i} + ay\hat{j} - 2za\hat{k})$ ✓

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(1)

Circulation: It is a scalar quantity which measures the rotations of a fluid particle along a curve. The circulation C about a closed curve/contour is given by

$$C = \oint_C \vec{q}.d\vec{r}.$$

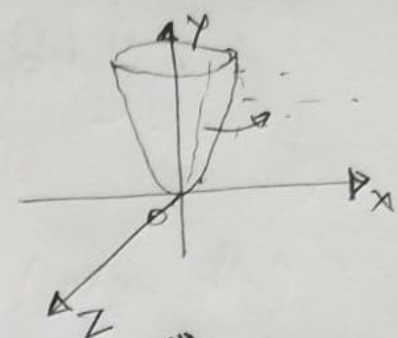
vorticity: $\omega = \vec{\nabla}\times\vec{q}$

Relation between vorticity and circulation:

$$C = \oint_C \vec{q}.d\vec{r} = \iint_S (\vec{\nabla}\times\vec{q}).\hat{n}\,ds$$

$$= \iint_S \vec{\omega}.\hat{n}\,ds.$$

Vortex lines: A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector $\vec{\omega}$.



We are looking for curves st. tangent to it is $\parallel$ to ω.

$\vec{\omega}$ is $\perp^r$ to the plane containing $\vec{q}$.

Let $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$ be the vorticity vector and $\vec{r}$ be the position vector of point P on the vortex line.

$$\vec{\omega} \times d\vec{r} = \vec{0} \Rightarrow (\omega_x, \omega_y, \omega_z) \times (dx, dy, dz) = \vec{0}$$

$$\Rightarrow \underbrace{\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}}_{\text{The curve / vortex lines.}}$$

§ Angular velocity of a rotational flow:

$$\vec{\theta} = \frac{1}{2}\operatorname{curl}\vec{q} = \frac{1}{2}\left[\hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right].$$

$$\Rightarrow \quad \vec{\omega} = 2\vec{\theta}$$

Ex 1: (i) Consider $\vec{q} = (Kx, 0, 0)$, $K \neq 0$.

(ii) " $\vec{q} = (Ky, 0, 0)$, $K \neq 0$.

which one of them is rotational and irrotational.

Solⁿ: (i) is irrotational. (ii) $\vec{\nabla} \times \vec{q} = -K\hat{k} = (0, 0, -K)$

Ex 2: verify the velocity vector

$$\vec{q} = \left(\frac{ax - by}{x^2 + y^2}, \frac{ay + bx}{x^2 + y^2}, 0\right), \; x \neq 0, y \neq 0$$

is for a possible fluid flow of an incompressible, Check if it is irrotational.

Then Determine ϕ.

Solⁿ: $\vec{\nabla} \cdot \vec{q} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \vec{q}$ determines a fluid flow for an incompressible fluid.

$\vec{\nabla} \times \vec{q} = 0 \Rightarrow \vec{q}$ represent an irrotational flow.

We know, $\vec{q} = -\vec{\nabla}\phi \Rightarrow (u, v, w) = -\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \underbrace{\frac{\partial \phi}{\partial z}}_{0}dz$$

$$= -\left[\frac{ax-by}{x^2+y^2}dx + \frac{ay+bx}{x^2+y^2}dy\right].$$

$$\Rightarrow \int d\phi = -\int\left[a\,\frac{x\,dx + y\,dy}{x^2+y^2} + b\,\frac{(x\,dy - y\,dx)}{x^2+y^2}\right]$$

$$\Rightarrow \phi(x,y) = -\frac{a}{2}\log_e|x^2+y^2| - b\tan^{-1}\frac{y}{x} + C$$

§ Irrotational flow in 2D: Let $\vec{q}$ be the velocity vector

then $\quad \vec{\nabla}\times\vec{q} = 0$

$$\Rightarrow i\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\Rightarrow -i\frac{\partial v}{\partial z} + \hat{j}\,\frac{\partial u}{\partial z} + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0.$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0,\quad \frac{\partial v}{\partial z} = 0,\quad \frac{\partial u}{\partial z} = 0.$$

In case of 2D-flow, $\vec{\nabla} \cdot \vec{q} = 0$ from equ^n. of Continuity

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad — (1)$$

Let $\exists$ a function $\psi(x, y, z) = c$ s.t. $u = \frac{\partial \psi}{\partial y}$,

$$v = -\frac{\partial \psi}{\partial x}.$$

This $\psi(x, y, z) = C$ is called the stream function, For every value of C, we get one streamline.

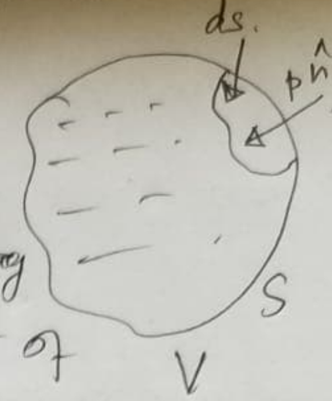
Consider an arbitrary fluid element in 2D. The volume rate of flow across such a line element

$$v\,dx + (-u)\,dy = -\frac{\partial \psi}{\partial x}dx - \frac{\partial \psi}{\partial y}dy = -d\psi$$

$$\Rightarrow \int v\,dx + \int -u\,dy = -\int d\psi = -\psi$$

This shows that the volume rate between a pair of Stream lines is numerically equal to their differces in ψ values.

§ Motion of an inviscous fluid/inviscid fluids.



Consider any arbitrary closed Surface S drawn in the region occupied by an incom-pressible inviscid fluid and S moves ~~along~~ with V so that it Contains same amount of fluid at any instant of time "t". Let p be the pressure, $\vec{q}$ be the fluid velocity, $\hat{n}$ be the outward drawn normal. Then by Newton's 2^{nd} law.

total Force acting on the mass of the fluid
= the rate of change of linear momentum. —(1)

Now, the total force on the fluid is subjected to the following two forces:

(i) The normal pressure (thrusts) on the boundary

(ii) The external force $(\vec{F})$ per unit mass.

Let ρ be the fluid density of the fluid particle P within the closed Surface S and let dv be the volume enclosing P whose surface area is ds.

The mass of fluid element around $P = \rho\, dv$, which is always constant. Then the total linear momentum of the volume V is

$$\vec{M} = \int_V \vec{q}\,\rho\, dv \quad \text{— (i)}.$$

$$\Rightarrow \frac{d\vec{M}}{dt} = \frac{d}{dt}\int_V \vec{q}\,\rho\, dv$$

$$= \int_V \frac{d\vec{q}}{dt}\rho\, dv + \cancel{\vec{q}\,\frac{d(\rho\, dv)}{dt}}_{\to 0}$$

$$= \int_V \rho\,\frac{d\vec{q}}{dt}\, dv \quad \text{— (ii)}$$

If $\vec{F}$ be the external force per unit mass on the particle P, then the total force on the volume $V = \int_V \vec{F}\rho\, dv$ — (iii)

Finally, if p be the pressure acting along the normal on the surface ds, then the total force on the Surface S

$$= \iint_S p(-\hat{n})\, ds = -\iint_S p\hat{n}\, ds = -\iiint_V \nabla p\, dv,$$

(due to GDT)

— (iv)

Now,

total force = rate of change of LM

$$\Rightarrow \int_V \vec{F}\rho\, dv - \iiint_V \nabla p\, dv = \int_V \rho\,\frac{d\vec{q}}{dt}\, dv$$

$$\Rightarrow \int_V \left(\rho \frac{d\vec{q}}{dt} - \rho \vec{F} + \nabla P \right) dv = 0$$

V arbitrary

$$\Rightarrow \quad \rho \frac{d\vec{q}}{dt} - \rho \vec{F} + \vec{\nabla P} = \vec{0}$$

$$\Rightarrow \quad \boxed{\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho} \vec{\nabla P}} \quad \text{— (v)}$$

Equ^n. for an inviscid incompressible fluid flow / Euler's equ^n. of motion.

If $\vec{F}$ is conservative, $\vec{F} = -\nabla\phi$, then (v) become.

$$\frac{d\vec{q}}{dt} = -\vec{\nabla\phi} - \frac{1}{\rho} \vec{\nabla P}.$$

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$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla p} \;\Big|\; = -\vec{\nabla\phi} - \frac{1}{\rho}\vec{\nabla p}$$

$$\Rightarrow \quad \frac{d\vec{q}}{dt} + \vec{\nabla\phi} + \frac{1}{\rho}\vec{\nabla p} = \vec{0}.$$

Euler's equ^n. of motion of an incompressible inviscid ~~Conservat~~ flow under Conservative external forces.

$$W_{AB} = \int_C \vec{F}\cdot d\vec{r}$$

$$TE_A = TE_B$$

$$T_A + V_A = T_B + V_B$$

§ Lamb's Hydrodynamical equ^n.:

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla p}.$$

$$\Rightarrow \quad \frac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla p} \quad \text{--- (I)}$$

From vector analysis,

$$(\vec{q}\cdot\vec{\nabla})\vec{q} = \frac{1}{2}\vec{\nabla}(\vec{q}\cdot\vec{q}) - \vec{q}\times \operatorname{curl}\vec{q}$$

$$\Rightarrow (\vec{q}\cdot\vec{\nabla})\vec{q} = \frac{1}{2}\vec{\nabla}(\vec{q}^{\,2}) - \vec{q}\times\operatorname{curl}\vec{q}. \quad \text{--- (II)}$$

From (i) and (ii), we have

$$\frac{\partial \vec{q}}{\partial t} + \frac{1}{2}\vec{\nabla}(\vec{q}^{\,2}) - \vec{q} \times \operatorname{curl} \vec{q}$$

$$= F - \frac{1}{\rho}\nabla P.$$

$$\Rightarrow \frac{\partial \vec{q}}{\partial t} + \vec{\nabla}\left(\frac{\vec{q}^{\,2}}{2}\right) + \operatorname{curl}\vec{q} \times \vec{q}$$

$$= \vec{F} - \frac{1}{\rho}\vec{\nabla P}$$

$$\Rightarrow \frac{\partial \vec{q}}{\partial t} + \vec{\nabla}\left(\frac{\vec{q}^{\,2}}{2}\right) + \vec{\Omega} \times \vec{q}$$

$$= \vec{F} - \frac{1}{\rho}\vec{\nabla P}, \text{ where } \vec{\Omega} = \operatorname{curl}\vec{q}.$$

§ Bernoulli's equ^n. for a One-dimensional inviscid incompressible flow:

When the velocity potential exists (so that the flow is irrotational) and the external forces are derivable from velocity potential, i.e., $\vec{F} = -\vec{\nabla V}$, where V is a scalar function.

By definition, $\vec{F} = -\nabla\vec{V} \Rightarrow X = -\frac{\partial V}{\partial x}$, $Y = -\frac{\partial V}{\partial y}$,

$Z = -\frac{\partial V}{\partial Z}$ where

$$\vec{F} = (X, Y, Z). \quad \text{— (1)}$$

Since the flow is irrotational, then $\vec{\nabla}\times\vec{q}=\vec{0}$

i.e., $\vec{q}=-\vec{\nabla\phi} \Rightarrow u=-\frac{\partial\phi}{\partial x},\ v=-\frac{\partial\phi}{\partial y},$

$$w=-\frac{\partial\phi}{\partial z} \quad \text{— (ii)}.$$

$\vec{\nabla}.\vec{q}$

$m\times 1 \quad 1\times n$

$$\frac{\partial u}{\partial y}=-\frac{\partial^2\phi}{\partial y\partial x}=-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)=+\frac{\partial v}{\partial x} \quad = [\quad]_{m\times n}$$

$$\frac{\partial v}{\partial z}=\frac{\partial w}{\partial y},\quad \frac{\partial w}{\partial x}=\frac{\partial u}{\partial z} \quad \text{— (iii)}$$

By Euler's equ^n. $\frac{d\vec{q}}{dt}=\vec{F}-\frac{\vec{\nabla p}}{\rho} \Rightarrow \frac{\partial\vec{q}}{\partial t}+(\vec{q}.\vec{\nabla})\vec{q}=\vec{F}-\frac{1}{\rho}\nabla p$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{— (iv)}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{— (v)}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}. \quad \text{— (vi)}$$

equ^n (iv)

$$\Rightarrow -\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x}\right)+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow -\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)=X-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow -\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)dx+\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)dx=X\,dx-\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

— (vii)

$$(v) \Rightarrow -\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)dy + \frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2+\omega^2) = Y dy - \frac{1}{\rho}\frac{\partial p}{\partial y}dy \quad -(viii)$$

$$(vi) \Rightarrow -\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)dz + \frac{1}{2}\frac{\partial}{\partial z}(u^2+v^2+\omega^2) = Z dz - \frac{1}{\rho}\frac{\partial p}{\partial z}dz \quad -(ix)$$

Now

(vii) + (viii) + (ix),

$$-\left[\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)dy + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)dz\right]$$

$$+\frac{1}{2}\left[\frac{\partial}{\partial x}(u^2+v^2+\omega^2)\,dx + \frac{\partial}{\partial y}(\,''\,)\,dy + \frac{\partial}{\partial z}(\,''\,)\,dz\right]$$

$$= X dx + Y dy + Z dz - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right).$$

$$= -\left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right] - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \cdots\right)$$

$$\Rightarrow -\int d\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\int d(u^2+v^2+\omega^2) = -\int dV - \int \frac{1}{\rho}dp$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2+v^2+\omega^2) = -V - \int \frac{dp}{\rho} + \text{Constant}$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{1}{2}\vec{q}^2 + V + \int \frac{dp}{\rho} = C$$

Bernoulli's equ^n. / pressure equ^n.

Remark 1: flow is incompressible, ρ = Constant,

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\vec{q}^2 + V + \frac{p}{\rho} = \text{Const.}$$

Remark 2: If the flow is steady, $\frac{\partial \phi}{\partial t} = 0$ $\quad \frac{\vec{q}^2}{2} + V + \frac{p}{\rho} = \text{Cnt}$