Theorem: Let X be a normed linear space and Y be a closed subspace of X. Then X is a Banach space iff Y and $\frac{X}{Y}$ are Banach spaces in the induced norms, respectively.

Proof: Let X be a Banach space and Y be a closed subspace of X.

We prove $\frac{X}{Y}$ is a Banach space.

Consider a sequence $\{x_n + Y\}$ in $\frac{X}{Y}$ such that $\sum_{n=1}^{\infty} |||x_n + Y||| < \infty$.

Then by definition of $|||\cdot|||$, $\exists$ a some $y_n \in Y$ such that

$$\|x_n + y_n\| < |||x_n + Y||| + \frac{1}{n^2}$$

(by definition of infimum)

$$\Rightarrow \sum_{n=1}^{\infty} \|x_n+y_n\| < \sum_{n=1}^{\infty} |||x_n+Y||| + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$<\infty \qquad <\infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \|x_n+y_n\| < \infty.$$

Since X is a Banach space, and in a Banach space every absolutely summable series is summable, it follows that

$$\sum_{n=1}^{\infty} (x_n+y_n) = s \in X.$$

Now for $m = 1, 2, 3, \ldots$

$$\left|\left|\left| \sum_{n=1}^{m} (x_n+Y) - (s+Y) \right|\right|\right|$$

$$= \left|\left|\left| \sum_{n=1}^{m} (x_n+y_n+Y) - (s+Y) \right|\right|\right|$$

$$= \left|\left|\left| \sum_{n=1}^{m} (x_n + y_n) - f + Y \right|\right|\right|$$

$$\leq \left\| \sum_{n=1}^{m} (x_n + y_n) - f \right\| \longrightarrow 0 \text{ as } m \to \infty$$

[$\because 0 \in Y$ & then

$$|||x+Y||| = \inf \{ \|x+y\| / y \in Y \}$$

$$\leq \|x-0\| = \|x\|]$$

This implies

$$\left|\left|\left| \sum_{n=1}^{\infty} (x_n + Y) - (f+Y) \right|\right|\right| = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (x_n + Y) = f + Y \in \frac{X}{Y}$$

Thus the absolutely summable series $\sum_{n=1}^{\infty} |||x_n + Y|||$ is summable in $\frac{X}{Y}$. This implies $\frac{X}{Y}$ is a Banach space.

Conversely assume that Y and $\frac{X}{Y}$ are Banach spaces.

We prove X is a Banach space.

Let $\{x_n\}$ be a Cauchy sequence in X. Then $\|x_n - x_m\| \to 0$ as $n, m \to \infty$.

$\because$

$$|||(x_n + Y) - (x_m + Y)|||$$

$$= |||(x_n - x_m) + Y|||$$

$$\leq \|x_n - x_m\| \to 0$$

as $m, n \to \infty$.

$\Rightarrow \{x_n + Y\}$ is a Cauchy sequence in $\frac{X}{Y}$.

Let $x_n + Y \to x + Y \in \frac{X}{Y}$

$\left[\because \frac{X}{Y}\right.$ is Banach space$\left.\right]$

Then there exists a sequence $\{y_n\}$ in Y such that

$$x_n + y_n \longrightarrow x \text{ in } X$$

$$|||x_n + Y - (x+Y)||| \longrightarrow 0$$

$$\Rightarrow |||(x_n - x) + Y||| \longrightarrow 0$$

$$= \inf\{\|x_n - x + y\| \mid y \in Y\} \to$$

Then $\exists$ a sequence $\{y_n\}$ in Y

$\ni$ $$x_n - x + y_n \longrightarrow 0$$

$$x_n + y_n \longrightarrow x \text{ in } X.$$

Since

$$y_n - y_m = y_n + x_n - x - x_n + x_m - x_m - y_m + x$$

$$\Rightarrow \|y_n - y_m\| \le \|y_n + x_n - x\| + \|x_n - x_m\| + \|x_m + y_m - x\|$$

$$\to 0 \text{ as } n, m \to \infty.$$

$\Rightarrow \{y_n\}$ is a Cauchy sequence in Y.

Since Y is a Banach space,

$y_n \to y \in Y$.

Then, we have

$$x_n = (x_n + y_n) - y_n \to x - y \text{ in } X$$

$\Rightarrow X$ is a Banach space

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Riesz lemma :—

Let X be a normed linear space, Y be a closed subspace of X and $Y \neq X$. Let r be a real number such that $0 < r < 1$. Then there exists some $x_r \in X$ such that

$\|x_r\| = 1$ and $r \leq \text{dist}(x_r, Y) \leq 1.$

Proof: Since $Y \neq X$, consider $x \in X$ and $x \notin Y$.

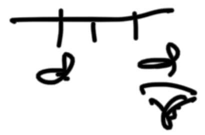
Since Y is closed, $\text{dist}(x, Y) > 0$

$\Rightarrow d = \inf\{\|x - y\| \,/\, y \in Y\} > 0.$

Also, as $r < 1$, $\exists$ some $y_0 \in Y$ such that

$$\|x - y_0\| \leq \frac{\text{dist}(x, Y)}{r} \longleftarrow (*)$$

$[\because r < 1 \Rightarrow \frac{1}{r} > 1 \Rightarrow \frac{d}{r} \geq d]$



Let $x_r = \dfrac{x - y_0}{\|x - y_0\|}$. Then $\|x_r\| = 1$.

Since $0 \in Y$, we see that

$$\text{dist}(x_r, Y) \leq \|x_r - 0\| = \|x_r\| = 1$$

Now consider

$$\operatorname{dist}(x_r, Y) = \inf\{\|x_r - y\| \,/\, y \in Y\}$$

$$= \inf\left\{\left\|\frac{x-y_0}{\|x-y_0\|} - y\right\| \,\middle/\, y \in Y\right\}$$

$$= \frac{1}{\|x-y_0\|}\inf\left\{\|x-(y_0+y\|x-y_0\|)\| \,/\, y \in Y\right\}$$

$$= \frac{1}{\|x-y_0\|}\inf\left\{\|x-z\| \,/\, z \in Y\right\}$$

$$= \frac{1}{\|x-y_0\|}\operatorname{dist}(x, Y)$$

$$\geq r \qquad (\text{by } \circledast).$$

Riesz lemma says that if Y is a proper closed subspace of n.l.s X, then there exists a point on the unit sphere of X whose ~~dist~~

distance from Y is very small.

$$\left[\ \overline{U}(0,1) = \{ x \in X \,/\, \|x\| \leq 1 \} \text{ closed unit ball} \right.$$

$$B(x_0, r) = x \in X \,/\, \|x - x_0\| < r \}$$

$$\left. \overline{B(x_0, r)} = \{ x \in X \,/\, \|x - x_0\| \leq r \} \right]$$