(Projection Theorem) :—

Let H be a Hilbert space and F be a non empty closed subspace of H. Then $H = F + F^{\perp}$.

Equivalently, there is an orthogonal projection onto F. Moreover

$$F^{\perp\perp} = F.$$

Proof: If $F = \{0\}$, then $F^{\perp} = H$

then clearly $H = F + F^{\perp}$.

So let $F \neq \{0\}$. Since F is a closed subspace of H, F itself is a Hilbert space.

Let $\{u_\alpha\}$ be an orthonormal basis for F.

Let $x \in H$. Then $\{u_\alpha \,/\, \langle x, u_\alpha\rangle \neq 0\}$

is a countable set say $\{u_1, u_2, u_3 - - -\}$ and the series $\sum_n \langle x, u_n\rangle u_n$ converges to some $y \in H$ such that

$x - y \perp u_\alpha, \forall \alpha.$

$\because$ each $u_n \in F$ and F is closed,

$$\sum_{n=1}^{m} \langle x, u_n\rangle u_n \to y \Rightarrow y \in F$$

Also since $\{u_\alpha\}$ is an orthonormal basis for F, implies $F = \overline{\text{span}\{u_\alpha\}}$.

$\because x - y \perp \{u_\alpha\} \Rightarrow x - y \perp \overline{\text{span}\{u_\alpha\}}$

$$\Rightarrow x - y \perp F$$

$$\Rightarrow x - y \in F^{\perp}$$

Thus every $x \in H$ can be written

a) $x = y + x-y$, with $y \in F$, $x-y \in F^{\perp}$

$= y + z$

Hence $H = F + F^{\perp}$, $F \cap F^{\perp} = \{0\}$.

Hence there exists an orthogonal projection $P : H \to H$

such that $R(P) = F$

and $N(P) = F^{\perp}$

Claim: $F^{\perp\perp} = F$.

Let $x \in F$. Now for any $z \in F^{\perp}$,

we have $\langle x, z \rangle = 0$

$\Rightarrow x \in (F^{\perp})^{\perp} = F^{\perp\perp}$

$\Rightarrow F \subseteq F^{\perp\perp}$ — (1).

Now let $x \in F^{\perp\perp} \Rightarrow x \in H$

Then $\quad x = y + z, \quad y \in F$
$\qquad\qquad\qquad\qquad z \in F^{\perp}$

$\because y \in F \Rightarrow y \in F^{\perp\perp}$ (by (1))

Thus $\quad z = x - y \in F^{\perp\perp}$ $\left\{ \begin{array}{l} \because x \in F^{\perp\perp} \\ y \in F^{\perp\perp} \\ \Rightarrow x - y \in F^{\perp\perp} \end{array} \right]$

$\Rightarrow z \in F^{\perp} \cap (F^{\perp})^{\perp} = \{0\}$

$\Rightarrow z = 0$

$\therefore x - y = 0 \Rightarrow x = y \in F$

$\Rightarrow F^{\perp\perp} \subseteq F$ — (2)

$\therefore$ from (1) & (2) we have

$$F^{\perp\perp} = F.$$

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The projection theorem show that

every Hilbert space H has Complementary subspace property.

That is for every non empty Closed subspace F of H, there is a closed subspace G of H such that

$$H = F + G, \quad F \cap G = \{0\}.$$

~~Here~~ $G = F^{\perp}$, it is a closed subspace of H.

* If H is a Hilbert space, every $x \in H$ can be written as

$$x = y + z, \quad y \in F, \; z \in F^{\perp}$$

Define $P : H \to F$ by

$$P(x) = P(y+z) = y$$

then P is linear map and

$P^2 = P$.

Also $R(P) = F$, $N(P) = F^{\perp}$

$\therefore$ $R(P) \perp N(P)$.

Thus P is a orthogonal projection onto $R(P)$ along $N(P)$.

Continuous linear functionals:—

Let X be an I.P.S over K.

Let $f : X \to K$ be a linear

functional on X. let f be continuous on X.

Then f is continuous at zero and $f(0) = 0$.

Hence $\exists\ \delta > 0 \ni$

$$|f(x)| \leq 1 \quad \forall\ x \in X, \ \|x\| \leq \delta$$

Now for any $x \neq 0$, $y = \frac{\delta x}{\|x\|}$, we see that

$$|f(y)| \leq 1, \quad \|y\| = \delta$$

$$\Rightarrow \ |f(x)| \leq \alpha \|x\|, \ \alpha = \frac{1}{\delta}.$$

Let $X' = B(X, K)$ be the set of all continuous linear functionals on X.

X' is a linear space. And for $f \in X'$, we let

$$\|f\| = \sup\{ |f(x)| \,/\, x \in X, \|x\| \le 1 \}.$$

$$\Rightarrow \quad |f(x)| \le \|f\| \, \|x\|$$

—//—

Lemma :- Let X be an I.P.S and $f \in X'$.

(a) Let $\{u_1, u_2, \ldots \ldots\}$ be an orthonormal set in X. Then $\sum_n |f(u_n)|^2 \le \|f\|^2$.

(b) Let $\{u_\alpha\}$ be an orthonormal set in X and

$$E_f = \{u_\alpha \mid f(u_\alpha) \neq 0\}.$$

Then E_f is countable set say $\{u_1, u_2, \dots\}$.

If E_f denumerable, then $f(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof:

(a) For $m = 1, 2, \dots$, let

$$y_m = \sum_{n=1}^{m} \overline{f(u_n)}\, u_n. \text{ Then}$$

$$\|y_m\|^2 = \langle y_m, y_m \rangle$$

$$= \left\langle \sum_{n=1}^{m} \overline{f(u_n)}\, u_n, \sum_{p=1}^{m} \overline{f(u_p)}\, u_p \right\rangle$$

$$= \sum_{n=1}^{M} |f(u_n)|^2 = \beta_M \text{ (say)}$$

— (1)

Also

$$f(y_m) = f\left(\sum_{n=1}^{M} \overline{f(u_n)}\, u_n\right)$$

$$= \sum_{n=1}^{M} \overline{f(u_n)}\, f(u_n)$$

$$= \sum_{n=1}^{M} |f(u_n)|^2 = \beta_M$$

— (2)

Now since

$$|f(y_m)| \le \|f\|\,\|y_m\|$$

$$= \|f\| \sqrt{\beta_M} \quad (\text{by } (1))$$

— (*)

Now from (*), (2), we have

$$\beta_M \le |f(y_m)| \le \|f\| \sqrt{\beta_M}$$

$$\Rightarrow \beta_M^2 \le \|f\|^2 \beta_M$$

$\Rightarrow \beta_M \leq \|f\|^2$

$\Rightarrow \sum_{n=1}^{M} |f(u_n)|^2 \leq \|f\|^2$

Letting $M \to \infty$, we get

$$\sum_{n=1}^{\infty} |f(u_n)|^2 \leq \|f\|^2.$$

(b) If $f = 0$,

$$E_f = \{u_\alpha \mid f(u_\alpha) \neq 0\} = \varphi.$$

So let $f \neq 0$.

For $j = 1, 2, \ldots$, let

$$E_j = \{u_\alpha \mid \|f\| \leq j\,|f(u_\alpha)|\}$$

Fix j, Suppose E_j contains

distinct elements say $u_{\alpha_1}, u_{\alpha_2}, \ldots u_{\alpha_M}$

then

$$\|f\| \leq j\,|f(u_{\alpha_i})|, \quad i = 1,2,\ldots M$$

Squaring and adding

$$\Rightarrow M\|f\|^2 \leq j^2 \sum_{n=1}^{M} |f(u_{\alpha_i})|^2$$

$$\leq j^2 \|f\|^2$$

(by (a))

$$\Rightarrow M\|f\|^2 \leq j^2 \|f\|^2$$

$$\Rightarrow M \leq j^2$$

Thus E_j contains atmost j^2 elements.

Since $E_f = \bigcup_j E_j$, we see that E_f is countable.

If E_f is denumerable, then

$$\sum_{n=1}^{\infty} |f(u_n)|^2 \leq \|f\|^2 \quad (\text{by } (9))$$

$\sum_{n=1}^{\infty} |f(u_n)|^2$ is cgt series

$\therefore$ nth term $|f(u_n)|^2 \longrightarrow 0$

i.e., $f(u_n) \longrightarrow 0$.

as $n \longrightarrow \infty$.

—//—

Let X be an I.P.S over K.

For a fixed $y \in X$, define

$f : X \longrightarrow K$ by

$f(x) = \langle x, y \rangle, \ \forall x \in X$

Then f is linear

$\because f(ax+bz) = \langle ax+bz, y \rangle$

$= a\langle x,y \rangle + b\langle z,y \rangle$

$= a f(x) + b f(z)$

Also

$$|f(x)| = |\langle x, y \rangle|$$

$$\leq \|x\| \, \|y\| \quad \text{(Schwarz Inequ)}$$

$$\Rightarrow \|f\| \leq \|y\| \quad \text{---(1)}$$

If $y = 0$, then $f = 0$.

So let $y \neq 0$, let $x = \dfrac{y}{\|y\|}$.

Then $\|x\| = 1$.

and

$$f(x) = \langle x, y \rangle$$

$$= \left\langle \frac{y}{\|y\|}, y \right\rangle$$

$$= \frac{1}{\|y\|} \langle y, y \rangle = \frac{\|y\|^2}{\|y\|} = \|y\|$$

$\therefore \quad \|f\| = \|y\|.$

Riesz Representation Theorem :—

Let H be a Hilbert space and $f \in H'$. Then there is a unique $y \in H$ such that

$$f(x) = \langle x, y \rangle, \forall x \in H.$$

In fact, if z is a nonzero element of H such that $z \perp Z(f)$, then $y = \frac{\overline{f(z)}}{\langle z, z \rangle} z$.

Also if $\{u_\alpha\}$ is a orthonormal basis for H and $\{u_\alpha \mid f(u_\alpha) \neq 0\}$

$= \{u_1, u_2, \ldots\}$, then $y = \sum_n \overline{f(u_n)}\, u_n$.

Proof :-

If $f = 0$, then we let $y = 0$

so that $f(x) = 0 = \langle x, y\rangle$

$\forall\, x \in H$.

So let $f \neq 0$.

Then $Z(f)$ null space of f

is a closed subspace of H.

$\therefore$ By projection theorem, we have

$$H = Z(f) + Z(f)^{\perp}$$

As $Z(f) \neq H$, so let

$0 \neq z \in Z(f)^{\perp}$

Since $Z(f)$ is a Hyperspace in H, we have

$$H = Z(f) \cup \text{Span}\{z\}.$$

Let $x \in H$. Then

$$x = w + kz, \quad w \in Z(f)$$

$kz \in \text{Span}\{z\}$.

Then

$$\langle x, z \rangle = \langle w, z \rangle + \langle kz, z \rangle$$

$$\underset{0}{\overset{||}{}} \quad + \; k \langle z, z \rangle \quad \left\{ \begin{array}{l} \because w \in Z(f) \\ z \in Z(f)^\perp \end{array} \right.$$

$$\therefore \; k = \frac{\langle x, z \rangle}{\langle z, z \rangle}.$$

$$\therefore \; x = w + \frac{\langle x, z \rangle}{\langle z, z \rangle} \cdot z$$

Applying f on both side we get

$$f(x) = f(w) + \frac{\langle x, z \rangle}{\langle z, z \rangle} \cdot f(z)$$

$$= 0 + \langle x, z \rangle \frac{f(z)}{\langle z, z \rangle}.$$

$$= \left\langle x, \overline{\frac{f(z)}{\langle z, z \rangle}} \cdot z \right\rangle$$

$$= \langle x, y \rangle,$$

where $y = \dfrac{\overline{f(z)}}{\langle z, z \rangle} \cdot z$

Also $\|f\| = \|y\|$.

Uniqueness of y :—

If $f(x) = \langle x, y_1 \rangle$, $\forall x \in H$

and some $y_1 \in H$ as well.

Then

$f(x) = \langle x, y \rangle = \langle x, y_1 \rangle$, $\forall x \in H$

$\Rightarrow \langle x, y - y_1 \rangle = 0$, $\forall x \in H$.

In particular letting $x = y - y_1$, we get

$$\langle y - y_1, y - y_1 \rangle = 0$$

$$\Rightarrow \|y - y_1\|^2 = 0$$

$$\Rightarrow y = y_1 .$$

Thus for every $f \in H$, $\exists!\, y \in H$ such that $f(x) = \langle x, y \rangle$, $\forall x \in H$.

This y is representer of f.

and it satisfies $\|y\| = \|f\|$

Alternatively, we proceed as follows:-

Let $\{u_\alpha\}$ be an orthonormal basis for H and $\{u_\alpha \mid f(u_\alpha) \neq 0\}$

$= \{u_1, u_2, \dots\}$ is

Countable. Then

$$\sum_n |f(u_n)|^2 \leq \|f\|^2 < \infty.$$

Since H is a Hilbert space, then by Riesz–Fisher theorem, we have $\sum_n \overline{f(u_n)}\, u_n$ converges in H.

Let $y = \sum_n \overline{f(u_n)}\, u_n$.

Claim: $f(x) = \langle x, y \rangle, \ \forall x \in H$.

Let $x \in H$ and $\{u_\alpha \mid \langle x, u_\alpha \rangle \neq 0\}$

$= \{ v_1, v_2, \ldots \ldots \}$ is

orthonormal basis for H.

Then we have Fourier expansion

$$x = \sum_n \langle x, v_n \rangle v_n$$

$$\Rightarrow f(x) = \sum_n \langle x, v_n \rangle f(v_n) \quad \text{--- (1)}$$

On the other hand

$$\langle x, y \rangle = \left\langle \sum_n \langle x, v_n \rangle v_n , y \right\rangle$$

$$= \sum_n \langle x, v_n \rangle \langle v_n, y \rangle \quad \text{--- (2)}$$

We show that from (1) & (2)

$$f(v_n) = \langle v_n, y \rangle$$

Fix n, Then

$$\langle v_m, y\rangle = \langle v_m, \sum_n \overline{f(u_n)}\, u_n\rangle$$

$$= \sum_n f(u_n) \langle v_m, u_n\rangle$$

If $v_m = u_{n_0}$ for some n_0, then

$$\langle v_m, y\rangle = \sum_n f(u_n) \langle u_{n_0}, u_n\rangle$$

$$= f(u_{n_0}).$$

$$= f(v_m)$$

Now let $v_m \neq u_n$ for any n,

Then $f(v_m) = 0$

$$\langle v_m, y\rangle = \langle v_m, \sum_n \overline{f(u_n)}\, u_n\rangle$$

$$= \sum_n f(u_n) \underbrace{\langle v_m, u_n\rangle}_{\substack{0 \\ m \neq n}}$$

$$= 0 \qquad \because v_m \neq u_n \ \forall n$$

$\therefore$ from (1) & (2) & from above we get

$$f(x) = \langle x, y \rangle, \ \forall x \in H.$$

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