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$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\vec{q}^2 + V + \int \frac{dp}{\rho} = \text{Constant} = f(t). \quad \text{--- } (1)$$

Bernoulli's equ^n of motion.

<u>Bernoulli's theorem:</u> "When the fluid flow is steady and the velocity potential does not exist, then we have

$$\frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = \text{Constant}, \text{ where } q = |\vec{q}|$$

and V is the force potential from which external forces are derivable."

<u>Proof:</u> From (1) for steady flow $\frac{\partial \phi}{\partial t} = 0$. Take $q = |\vec{q}|$

$$\frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = \text{Constant}.$$

§ <u>Principal of Conservation of Energy:</u>

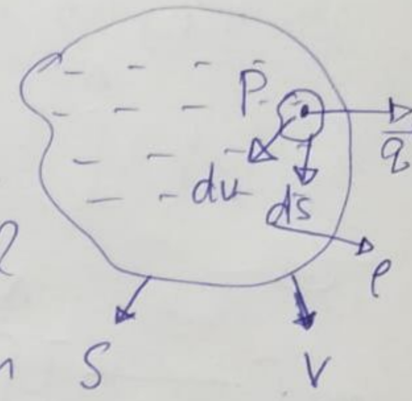
KE : The KE is due to the motion of the fluid and it is given by $\frac{1}{2}mq^2$

PE : It is the energy held by an object because of its position relative to other object.

IE (internal energy): It is the total energy Contained within the system

Statement: The rate of change of total energy (i.e, sum of KE, PE and IE) of any portion of compressible inviscid fluid as it moves about is equal to the rate at which the work done by the pressure on the boundary.

Sol^n: Consider any arb. closed surface drawn in the region occupied by the inviscid fluid and let V be its volume. Let ρ be the density and $\vec{q}$ be the velocity of fluid. Let us consider an infinitesimal small volume dv at a point P in the fluid whose surface area is ds. By Euler's equ^n,



$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla p}, \quad \text{where } \vec{F} \text{ is the external force and } p \text{ is the pressure.} \quad \text{—(I)}$$

Now, let external force be conservative so that there exists a force potential ϕ_1 which is independent of time and

$$\vec{F} = -\vec{\nabla}\phi_1 \quad \text{and} \quad \frac{\partial\phi_1}{\partial t} = 0 \quad \text{—(II)}$$

Multiply (I) by $\vec{q}\rho$, then,

$$\rho\vec{q}\cdot\frac{d\vec{q}}{dt} = \rho\vec{q}\cdot\vec{F} - \frac{1}{\rho}\rho\vec{q}\cdot\vec{\nabla p}$$

$$\Rightarrow \rho\left[\frac{1}{2}\frac{d}{dt}(\vec{q}^{\,2}) + \vec{q}\cdot\vec{\nabla}\phi_1\right] = -\vec{q}\cdot\vec{\nabla p}$$

—(III)

Again, $\frac{d\phi_1}{dt} = \underset{\overset{\|}{0}}{\frac{\partial \phi_1}{\partial t}} + \vec{q}\cdot\nabla\vec{\phi_1} = 0 + \vec{q}\cdot\vec{\nabla\phi_1} = \vec{q}\cdot\vec{\nabla\phi_1}$ — (iv)

From (iii) & (iv),

$$\rho\left[\frac{d}{dt}\left(\frac{1}{2}\vec{q}^{\,2}\right) + \frac{d\phi_1}{dt}\right] = -\vec{q}\cdot\nabla p. \quad \text{— (v)}$$

Now, let I be the internal energy per unit mass.

$$\vec{q}\cdot\vec{\nabla p} = \vec{\nabla}\cdot(p\vec{q}) - p\,\vec{\nabla}\cdot\vec{q} \quad \text{— (vi)}$$

The r.h.s. of (v) $= -\vec{q}\cdot\vec{\nabla p} = -\vec{\nabla}\cdot(p\vec{q}) + p\,\vec{\nabla}\cdot\vec{q}$.

Then (v) reduces to

$$\int_V \rho\frac{d}{dt}\left[\frac{1}{2}\vec{q}^{\,2} + \phi_1\right]dv = \int_V\left[\vec{\nabla}\cdot(p\vec{q}) + p\,\vec{\nabla}\cdot\vec{q}\right]dv.$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\int_V \rho\vec{q}^{\,2}\,dv + \int_V \rho\phi_1\,dv\right] = \int_V[\text{———}]\,dv. \quad \text{— (vii)}$$

Let $T = \frac{1}{2}\int_V \rho\vec{q}^{\,2}\,dv$ be the KE and $W = \int_V \rho\phi_1\,dv$ be the potential energy, $E = \int_V \rho I\,dv$ be the total internal energy.

The r.h.s. of (vii) $= -\int_V \nabla\cdot(p\vec{q})\,dv + \int_V p\,\vec{\nabla}\cdot\vec{q}\,dv$

$\downarrow$ (Gauss div. th^m.)

$$= -\int_S p\,\vec{q}\cdot\vec{n}\,ds + \int_V p\,\vec{\nabla}\cdot\vec{q}\,dv \quad \text{— (viii)}$$

where $\hat{n}$ is the outward drawn normal. We will show that

$$\int_V p\,\vec{\nabla}\cdot\vec{q}\,dv = \frac{dE}{dt}. \quad \text{— (ix)}$$

Now I_1 be the work done by the unit mass of the fluid against external pressure p from actual state to the some standard state where p_0 and ρ_0 be the values of pressure and density at the standard state.

$$I_1 = \int_V^{V_0} p\,dv, \quad \text{and} \quad m = \rho V = 1 \Rightarrow V = \frac{1}{\rho}$$

$$= \int_\rho^{\rho_0} p\; d\left(\frac{1}{\rho}\right)$$

$$= \int_\rho^{\rho_0} p \times -\frac{1}{\rho^2}\,d\rho = \int_{\rho_0}^{\rho} \frac{p}{\rho^2}\,d\rho \quad \text{— (x)}$$

Now,
$$\frac{dI_1}{d\rho} = \frac{d}{d\rho}\left[\int_{\rho_0}^{\rho} \frac{p}{\rho^2}\,d\rho\right] = \frac{p}{\rho^2}$$

$$\Rightarrow \frac{dI_1}{dt} = \frac{dI_1}{d\rho}\cdot\frac{d\rho}{dt} = \frac{p}{\rho^2}\cdot\frac{d\rho}{dt}$$

$$\Rightarrow \quad \frac{dI_1}{dt}\rho\,dv = \frac{p}{\rho^2}\frac{d\rho}{dt}\cdot \rho\,dv$$

$$\Rightarrow \quad \frac{dI_1}{dt}\rho\,dv = \frac{p}{\rho}\frac{d\rho}{dt}\,dv. \quad -\text{(XI)}$$

But $$\frac{d}{dt}(I_1\rho\,dv) = \frac{dI_1}{dt}\rho\,dv + I_1\frac{d}{dt}(\rho\,dv) \quad -\text{(XII)}$$

$$\Rightarrow \quad \frac{d}{dt}(I_1\rho\,dv) - \underbrace{I_1\frac{d}{dt}(\rho\,dv)}_{=0} = \frac{p}{\rho}\frac{d\rho}{dt}\,dv$$

$$\Rightarrow \quad \frac{d}{dt}(I_1\rho\,dv) = \frac{p}{\rho}\frac{d\rho}{dt}\,dv.$$

$$= \frac{p}{\rho}\left[-\rho\,\vec{\nabla}\cdot\vec{q}\right]dv, \text{ since}$$

from equ^n. of continuity $\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{q} = 0.$

$$\Rightarrow \quad \frac{d}{dt}(I_1\rho\,dv) = -p\,\vec{\nabla}\cdot\vec{q}\,dv.$$

$$\Rightarrow \quad \frac{d}{dt}\int_V I_1\rho\,dv = -\int_V p\,\vec{\nabla}\cdot\vec{q}\,dv.$$

$$\Rightarrow \quad \frac{dE}{dt} = -\int_V p\,\vec{\nabla}\cdot\vec{q}. \quad -\text{(XIII)}$$

Combining (xiii) with (vii) and (viii),

$$\frac{d}{dt}[T+W] = -\frac{dE}{dt} - \int_S p\,\vec{q}\cdot\vec{n}\,ds.$$

$$\Rightarrow \frac{d}{dt}[T+W+E] = -\int_S p\,\vec{q}\cdot\vec{n}\,ds. \quad \text{— (XIV)}$$

Again the work done by the pressure on an infinitesimal surface element δS is $= -p\cdot\delta S\,\vec{q}\cdot\vec{n}$. The total work done by the fluid region V with surface area S

$$= \int_S -p\,\vec{q}\cdot\vec{n}\,ds = R \quad \text{— (XV)}$$

We will obtain,

$$\boxed{\frac{d}{dt}[T+W+E] = R}$$

δM $dM = \text{const}$ $\rho dv = dm$

$\rho v = M.$

§ Motion in two-dimension:

Stream function: Let $\vec{q} = (u, v)$ be the velocity component in a 2D motion. Then, the lines of flow or streamlines are given by

$$\vec{q} \times d\vec{r} = 0$$

$$\Rightarrow \frac{dx}{u} = \frac{dy}{v} \Rightarrow v\,dx - u\,dy = 0. \quad \text{—(i)}$$

Also assume that the flow is incompressible, by eqn. of contn.

$$\vec{\nabla} \cdot \vec{q} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\partial (-u)}{\partial x} \quad \text{—(ii)}$$

(i) is exact ODE by (ii), $\exists$ a $\psi(x, y)$ s.t.

$$v\,dx - u\,dy = d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$\Rightarrow v = \frac{\partial \psi}{\partial x}, \quad u = -\frac{\partial \psi}{\partial y}.$$

The function ψ is called Streamfunction. ✓

From (1), we obtain $d\psi = 0 \Rightarrow \psi(x, y) = \text{Const.} = \underline{C}$.

Remark 1: Let the flow be irrotational incompressible 2D flow. Then $\exists$ a ϕ s.t. $\vec{q} = -\vec{\nabla\phi}$

$$\Rightarrow \quad u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}. \checkmark \quad \text{— (a)}$$

Then $\exists$ a ψ (stream func^n.) s.t.

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \quad \text{— (b)}.$$

From (a) and (b).

$$\underbrace{\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}}_{\text{Cauchy Riemann equ}^n\text{. (CR equ}^n\text{.)}} \quad \text{— (c)}$$

Therefore, $\omega = \phi + i\psi$, $i = \sqrt{-1}$ will be the complex potential associated with a 2D irrotational incompressible flow.

$$\omega = f(z) = \phi(x, y) + i\psi(x, y), \quad z = x + iy.$$

From relation (1), we also obtain

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}.$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\Rightarrow \nabla^2 \phi = \Delta \phi = 0 \quad \& \quad \nabla^2 \psi = \Delta \psi = 0$$

Remark 2: Let $\omega = f(z) = \phi + i\psi$, $z = x+iy = re^{i\theta}$

$$\Rightarrow \phi(r,\theta) + i\psi(r,\theta) = f(re^{i\theta}).$$

$$\Rightarrow \frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r} = f'(re^{i\theta})\, e^{i\theta}$$

$$\frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = f'(re^{i\theta})\, ire^{i\theta}.$$

From these two eqn's.

$$ir\left(\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r}\right) = \frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta}$$

$$\Rightarrow \frac{\partial \phi}{\partial \theta} = -r\frac{\partial \psi}{\partial r} \quad \text{and} \quad \frac{\partial \psi}{\partial \theta} = r\frac{\partial \phi}{\partial r}$$

Therefore
$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}.$$

CR equ^n in polar form.

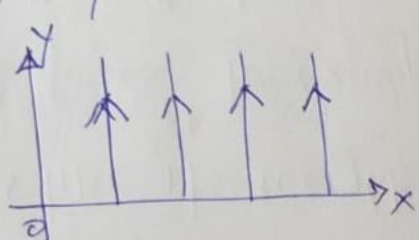
Ex 1: Complex potential for some uniform flows:

(i) Consider $w = ikz$, $z = x+iy$.

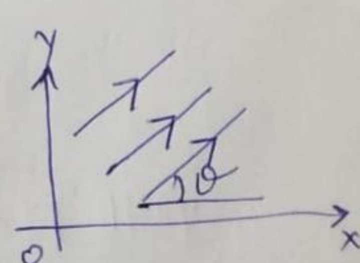
(For the time being $\frac{dw}{dz} = -u+iv \Rightarrow u^2+v^2 = \left|\frac{dw}{dz}\right|^2$
$\Rightarrow \left|\frac{dw}{dz}\right| = |\vec{q}|$)

$$\frac{dw}{dz} = ik \Rightarrow -u+iv = ik \Rightarrow u=0, \; v=k$$

This clearly implies that flow is || to y-axis.



(ii) $w = -ke^{-i\theta} z$.



$$\frac{dw}{dz} = -ke^{-i\theta}\frac{d}{dz}(z) = -ke^{-i\theta}$$

$$\Rightarrow -u+iv = -ke^{-i\theta} = -k(\cos\theta - i\sin\theta)$$

$$\Rightarrow u = k\cos\theta, \; v = k\sin\theta.$$

ϕ and $\psi \Rightarrow w = \phi + i\psi$ 04.09.21

$$\frac{1}{2}\frac{dw}{dz} = -u + iv \Rightarrow q = |\vec{q}| = \frac{1}{2}\left|\frac{dw}{dz}\right| = \frac{1}{2}\sqrt{u^2+v^2},$$

$$\vec{q} = (u, v).$$

Small calculation:

$$w = \phi + i\psi \qquad z = x + iy$$

$$\Rightarrow \frac{dx}{dz} = 1, \ \frac{dy}{dz} = -i$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$$

$$\Rightarrow \frac{dw}{dz} = \frac{\partial w}{\partial x}\frac{dx}{dz} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dz} \checkmark$$

$\vec{q} = -\nabla\phi$, $u = -\frac{\partial \phi}{\partial x}$, $v = -\partial\phi/\partial y$

$$= \frac{1}{2}\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right) + \left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right)\left(\frac{1}{2i}\right)$$

$x = \frac{z+\bar{z}}{2}$, $\frac{dx}{dz} = \frac{1}{2}$, $y = \frac{z-\bar{z}}{2i}$, $\frac{dy}{dz} = \frac{1}{2i}$

$$= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}i + \frac{\partial \psi}{\partial y}$$

$$= \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} - i\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}$$

(CR-equ^ns.)

$$= 2\frac{\partial \phi}{\partial x} - 2i\frac{\partial \phi}{\partial y}.$$

$$\cancel{2}\frac{dw}{dz} = \cancel{2}\left[-u + iv\right].$$

$$\Rightarrow 2\cdot\frac{1}{2}\frac{dw}{dz} = -u + iv \Rightarrow \frac{dw}{dz} = -u + iv$$

Ex1: The velocity potential for a 2D flow is $\phi(x,y) = x(2y-1)$. Then determine the velocity and the stream function at the point P(4, 5).

Soln: $\phi(x,y) = x(2y-1)$. The velocity components u and v of $\vec{q}$ (velocity) is given by

$$u = -\frac{\partial \phi}{\partial x} = (1-2y) \Rightarrow u\Big|_{(4,5)} = -9 \qquad \Rightarrow \vec{q} = (-9, -8)$$

$$v = -\frac{\partial \phi}{\partial y} = -2x \Rightarrow v\Big|_{(4,5)} = -8$$

Let $\psi(x,y)$ be the stream function. Then,

$$u = -\frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x}, \; w =$$

$$\Rightarrow \frac{\partial \psi}{\partial y} = -u = 2y - 1$$

$$\Rightarrow \psi(x,y) = y^2 - y + C(x) \quad \text{--- (i)}$$

Again,

$$v = \frac{\partial \psi}{\partial x} \Rightarrow \frac{\partial \psi}{\partial x} = -2x$$

$$\Rightarrow \psi(x,y) = -x^2 + d(y) \quad \text{--- (ii)}$$

Alternative:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$

$$= -2x\,dx + (2y-1)\,dy$$

$$\Rightarrow \psi(x,y) = -x^2 + y^2 - y + C.$$

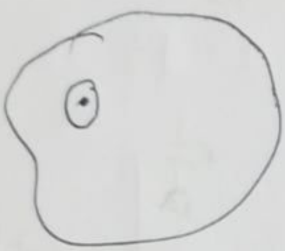
For $\psi = 0$ at origin we have

$$0 = \psi(0,0) = 0 - 0 + 0 + C \Rightarrow C = 0$$

The required stream function is

$$\psi(x,y) = -x^2 + y^2 - y$$

$$\Rightarrow \psi(4,5) = -16 + 25 - 5 = 4.$$



$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow u\,dy - v\,dx = 0.$$

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

$$\Rightarrow v\,dx - u\,dy = 0. \qquad w\,dy - v\,dz = 0.$$

$$u\,dz - w\,dx = 0.$$

$$f(z) = u + iv = u(x,y) + iv(x,y)$$

$$\downarrow$$

CR equ^n, analy. $\qquad u(x,y,z) + iv(x,y,z)$

Ex 2: The streamlines are represented by $\psi(x,y) = x^2+y^2$. Then determine the velocity and the direction at $(2,2)$. Also sketch the stream lines.

Solⁿ: Give that $\psi(x,y) = x^2+y^2$. By 2D flow we have

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

$$\Rightarrow u(x,y) = 2y \text{ and } v(x,y) = -2x$$

$$\Rightarrow \vec{q} = (u,v) = 2(y,-x) \Rightarrow \vec{q}\big|_{(2,2)} = 4(1,-1).$$

The magnitude of the velocity is: $|\vec{q}| = \sqrt{4^2+4^2} = 4\sqrt{2}$ units.

The slope of the velocity, $\tan\theta = \frac{v}{u} \Rightarrow \theta = \tan^{-1}\left(-\frac{4}{4}\right)$

$$= \tan^{-1}(-1)$$

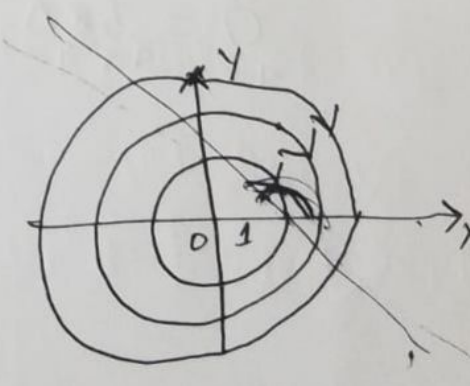
$$= 135^\circ \text{ or } \frac{3\pi}{4}$$

The stream function is given by

$$\psi(x,y) = C$$

$$\Rightarrow x^2+y^2 = C \quad \text{—(1)}$$

$\forall C > 0$ equⁿ (1) represents family of concentric circles.



$$u = -\frac{\partial \phi}{\partial x} \Rightarrow \frac{\partial \phi}{\partial x} = -2y$$

$$v = -\frac{\partial \phi}{\partial y} \Rightarrow \frac{\partial \phi}{\partial y} = 2x.$$

$$\phi(x,y) = -2xy + c$$

$$\Rightarrow \phi(x,y) = -2xy + c(y) = 2xy + c$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = -2x + c'(y)$$

$$\Rightarrow 2x = -2x + c'(y)$$

$$\Rightarrow c(y) = 4xy + c$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$= -2y\, dx + 2x\, dy.$$

Curl $\vec{q} \neq 0 \Rightarrow \vec{q} \neq -\vec{\nabla}\phi$

$\Rightarrow \vec{\nabla} \times (\vec{\nabla}\phi) \neq \vec{0}$

$\vec{q} = (u, v) = (2y, -2x) \Rightarrow \vec{\nabla} \times \vec{q} = \vec{0}$

$$\nabla \times q = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(-1-1).$$

$$= -2\hat{k} \neq 0$$

Ex 3: Let the streamfunction $\psi(x,y) = x^3 - 3xy^2$. Then determine whether the flow is rotational or irrotational. Find ϕ.

Sol^n: Give $\psi(x,y) = x^3 - 3xy^2$. We know $u = -\frac{\partial \psi}{\partial y}$, $v = +\frac{\partial \psi}{\partial x}$.

$\Rightarrow u(x,y) = 6xy$ and $v(x,y) = 3x^2 - 3y^2$.

Method 1: $\vec{q} = (u,v)$, $\vec{\nabla} \times \vec{q} = \vec{0}$

" 2: $\Omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = (6x - 6x) = 0$

A 2D-flow in xy-plane will be irrotational if the vorticity vector component Ω_z in the z-direction is zero.

$\vec{q}$ is irrotational. $\Rightarrow \exists\ \phi$ s.t. $\vec{q} = -\vec{\nabla}\phi$

$$\Rightarrow 6xy = -\frac{\partial \phi}{\partial x} \text{ and } 3x^2 - 3y^2 = -\frac{\partial \phi}{\partial y}.$$

We know,

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$\vec{q} = -\vec{\nabla}\phi$

$\vec{\nabla} \times \vec{q} = \vec{0}$

$\vec{q} = \vec{\nabla}\phi$

$$= -6xy\,dx - (3x^2 - 3y^2)\,dy$$

$$= -3x^2 dy - 6xy\,dx + 3y^2 dy.$$

$$= -[3x^2 dy + 6xy\,dx] + 3y^2 dy$$

$$= -d(3x^2 y) + 3y^2 dy$$

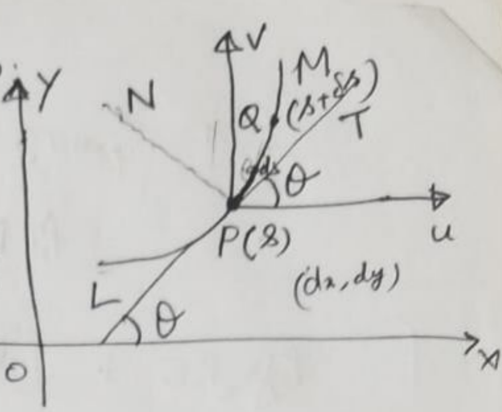
$$\Rightarrow \phi(x,y) = y^3 - 3x^2 y + \text{Const.}$$

$\phi(x,y) = -y^3 + 3x^2y = -\phi$

$\Rightarrow \phi_1(x,y) = y^3 - 3x^2y$

$\nabla^2\phi = 0$ and $\nabla^2\psi = 0$??

§ Physical Significance of stream function:



Let LM be any arb. curve in xy-plane and let P and Q be any two nhbing points s.t. $\widehat{LP} = s$ and $\widehat{LQ} = s+\delta s$.

Let ψ_1 and ψ_2 be two stream functions at the points L and M, respectively. Let the tangent T makes an angle θ with +ve direction of x-axis. Let $\vec{q} = (u, v)$ be the velocity vector at P.

Then, we know

$$u = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x} \quad — (1)$$

Also, from diff. calculus,

$$\cos\theta = \frac{dx}{ds} \quad \text{and} \quad \sin\theta = \frac{dy}{ds} \quad — (2)$$

The velocity at P along the inward drawn normal PN

$$= v\cos\theta - u\sin\theta$$

Total flux across the curve LM $= \int_{LM} (v\cos\theta - u\sin\theta)\, ds$

$$= \int_{LM} \left(\frac{\partial \psi}{\partial x} \frac{dx}{ds} + \frac{\partial \psi}{\partial y} \frac{dy}{ds} \right) ds$$

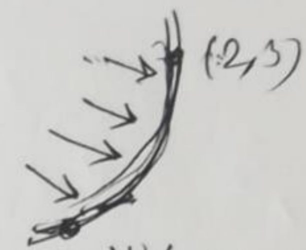
$$= \int_{LM} \left(\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \right)$$

$$= \int d\psi$$

$$= \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

$\Rightarrow$ ~~The stream~~

$\Rightarrow$ The difference of the values of the streamfunction at any two points is equal to the total flux across the curve adjoining two points.

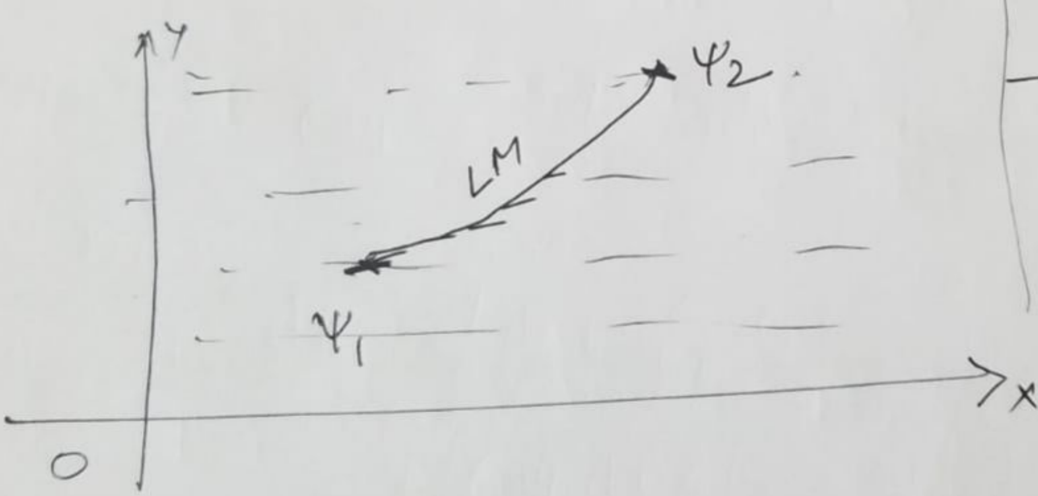


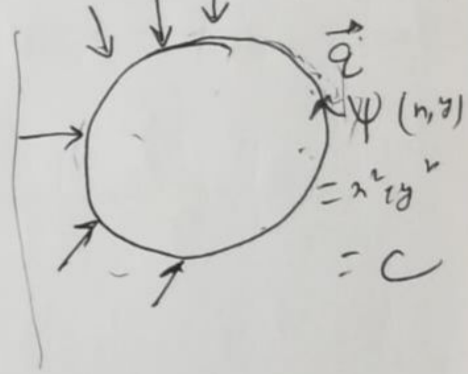
~~For~~ Flux $= MT^{-1}x^{-1}$ (Kg/s/L)

$= MT^{-1}L^{-2}$

(1,1) $\psi(x,y) = x^2+y^2$

$= C$





$$\psi(x,y) = C$$

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow v\,dx - u\,dy = 0$$

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$0 = v\,dx - u\,dy = d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \Rightarrow u = -\frac{\partial \psi}{\partial y},\ v = \frac{\partial \psi}{\partial x}$$

$\Rightarrow \psi(x,y) = C =$ Constant.

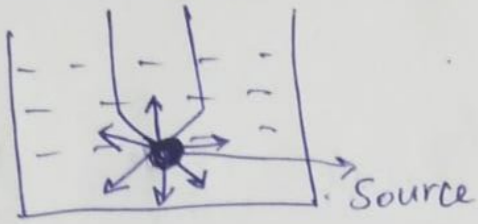
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow u = \frac{\partial \varphi}{\partial y}, \quad v = -\frac{\partial \varphi}{\partial x}$$

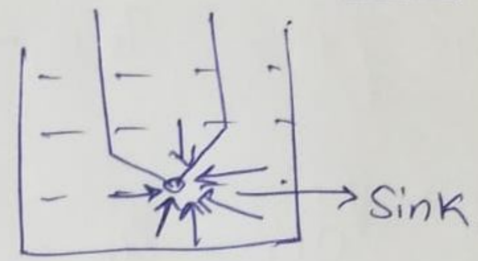
$$v\,dx - u\,dy = 0 \qquad u = -\frac{\partial \varphi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial x}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} = 0 = -\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y \partial x}$$

08.09.21

§ Sources and Sinks :





If the motion of the fluid consists of symmetrical radial flow in all directions proceeding from a point, then the point is called a simple source. However, if the flow is such that the fluid is directed inwards to a point from all directions in a symmetrical manner, then the point is called sink.

Consider a source at the origin. Then the mass "m" of the fluid coming out from the origin in a unit time is known as strength of the source. Similarly the amount of fluid going out of the fluid region in a unit time is called the strength of sink.

Sources and Sinks in two dimension: [Sink is ~~are~~ also considered to be a source of strength $-m$.]

In a 2D- flow a source of strength m is considered, then the flow across any small curve surrounding it is $2\pi m$.

Now consider a circle of radius r with source at its center. Then for a two dimensional stream function $\psi(r,\theta)$ exists and.

compressible

$$q_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad — (1).$$

Or, $q_r = -\frac{\partial \phi}{\partial r}$ since $\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$.

— (2)

Now the flow across the circle $= 2\pi r q_r$. Hence we have

$$\underline{\underline{2\pi m}} = 2\pi r q_r \Rightarrow r q_r = m \quad — (3).$$

From (2),

$$q_r = \frac{m}{r} = -\frac{\partial \phi}{\partial r}$$

$$\Rightarrow -m\frac{\partial r}{r} = \partial \phi$$

$$\Rightarrow \phi = -m \log r.$$

$$\checkmark\ \phi(r,\theta) = -m \log r. \quad — (4).$$

From (2),

$$\frac{\partial \psi}{\partial \theta} = r\frac{\partial \phi}{\partial r} = \cancel{r} \times -\frac{m}{\cancel{r}} = -m$$

$$\Rightarrow \checkmark\ \psi(r,\theta) = -m\theta \quad — (5)$$

Now, the complex potential w is given by

$$w = \phi(r,\theta) + i\psi(r,\theta) = -m\log r - im\theta$$

$$= -m[\log r + i\theta].$$

$= -m\,[\log_e r + \theta \log_e e^{i\theta}]$.

$= -m\,[\log_e re^{i\theta}] = -m\log_e z, \quad z = re^{i\theta}$

$\Rightarrow \quad \omega = -m\log z,$ ✓

i.e, the complex potential due to a source of strength m and located at origin is $-m\log z$.

In case if there is a sink of strength $-m$ located at the origin then,

$$\omega = -(-m)\log z = m\log_e z. \checkmark$$

Suppose the source is located at z_1, then

$$\omega(z) = -m\log(z - z_1) \qquad \underline{z_0 = z - z_1}$$

" " sink " " z_1, then

$$\omega(z) = m\log(z - z_1).$$

Suppose there are N sources of strength $m_1, m_2, \ldots m_N$ ($z_1, z_2, \ldots z_N$)

then

$$\omega(z) = \sum_{i=1}^{N} -m_i \log(z - z_i)$$

$$= -m_1\log(z - z_1) - m_2\log(z - z_2) - \cdots - m_N\log(z - z_N).$$

$$\phi(r,\theta) = -m_1 \log r_1 - m_2 \log r_2 \cdots - m_N \log r_N.$$

$$\psi(r,\theta) = -m_1 \theta_1 - m_2 \theta_2 \cdots - m_N \theta_N,$$

$$r_i = |z - z_i|, \quad \theta_i = \arg(z - z_i),$$

$$\forall i = 1, 2, \ldots, N.$$

$$z = re^{i\theta}$$

Suppose both sources and sinks are present of strengths m_1 and m_2 then
(z_1) (z_2)

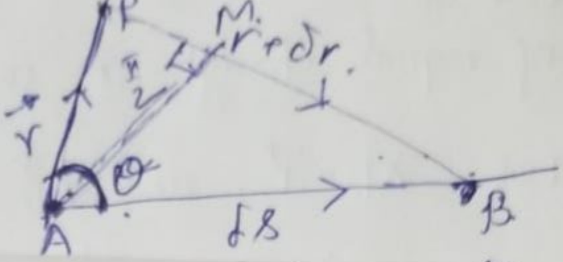
$$w(z) = -m_1 \log(z - z_1) + m_2 \log(z - z_2)$$

~~When $m_1 = m_2$ then the Combinati~~

A combination of a source of strength m and a sink of strength $-m$ at a small distance δs apart, where in the limit m is taken infinitely large and δs is infinitely small so that the product $m\,\delta s$ remains finite and is equal to a constant μ (>0). Then the combination is called a doublet or a dipole.

Complex potential due to a doublet in 2D:

Let A and B positions of the Sink and Source and P be any point. Let $AP = r$ and $BP = r + \delta r$ and $\angle PAB = \theta$



$\overline{|u|}$

$h u_1 \frac{\partial f}{\partial x_1} + h x_2 \frac{\partial f}{\partial x_2}$

$\frac{h \cdot \nabla f}{}$

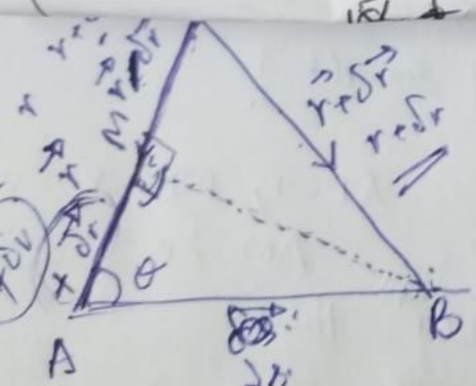
$$\left| \nabla_u f - \frac{f(u + h\vec{n}) - f(u)}{h} \right| \to 0 \text{ as } h \to 0$$

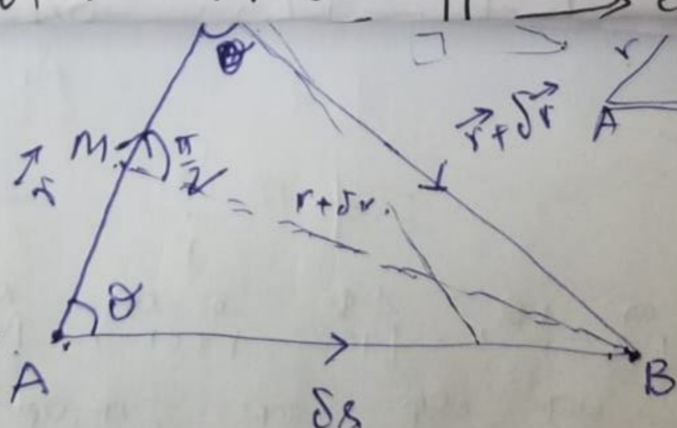
$T: X \to Y$

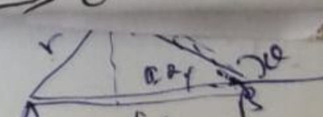
$h t \, DF = h D_t f$

DF

$-F(x + t) + F(u) \quad || \to 0$







BM is $\perp^r$ to AP. the $\perp^r$ BM is taken as of similar length as PB, i.e., s.t. ~~BM = r+δr~~.

$MP = r - \delta r$, $AM = \delta r$

§ Complex potential due to doublet

Let A and B be the position of the ~~Source~~ ~~Sink~~ Source and Sink ~~Source~~ of equal strength m and P be any arb. point within the flow.

Let $AP = \vec{r}$, and $\vec{BP} = \vec{r} + \delta\vec{r}$ and PB makes an angle θ with axis of the doublet. Then, the complex potential is given by

$$w(z) = \pm m \log z \mp m \log \cancel{(r+\delta r)}(z+\delta z)$$

From here,

$$\phi(r) = \mp m \log r \pm m \log(r+\delta r), \quad r = |\vec{r}| = \sqrt{x^2+y^2} = |z|$$

$$= +m \log_e \frac{r+\delta r}{r} \quad \cancel{=} \; -m \log\left(1+\frac{\delta r}{r}\right)$$

$-\delta\vec{r} = -$

$$= +m\left[\frac{\delta r}{r} - \text{neglect higher order terms as } \delta r \text{ is very small}\right].$$

$-\delta r = -$

$$= +m\,\frac{\delta r}{r}. \quad \text{——(i)}. \qquad AP = r,\ MP = r - \delta r,\ AM = \delta r$$

~~Let $\vec{PA} = \vec{r}$, $\vec{PB} = \vec{r} + \delta\vec{r}$, $\vec{MP} \approx \vec{r} + \delta\vec{r}$ then.~~

~~$\vec{AM} = \vec{AP} - \vec{MP} = \vec{r} - (\vec{r} + \delta\vec{r}) = -\delta\vec{r}$.~~ (δr)

Since $\angle PAB = \theta$, $AM = +\delta r$ and $AB = \delta s$ then

$$\cos\theta = \frac{AM}{AB} = \frac{+\delta r}{\delta s} \Rightarrow \pm\delta r = \delta s \cos\theta \quad \text{—(ii)}$$

(i) & (ii):

$$\phi(r) = \frac{m\,\delta s \cos\theta}{r} = \frac{\mu \cos\theta}{r}, \quad \mu = m\,\delta s = \text{strength of doublet}.$$

$$\Rightarrow \phi(r) = \frac{\mu\cos\theta}{r}.$$

$$\Rightarrow \frac{\partial\phi}{\partial r} = -\frac{\mu}{r^2}\cos\theta \qquad — (2).$$

By CR equ^n.: $\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} = -\frac{\mu}{r^2}\cos\theta$

$$\Rightarrow \frac{\partial\psi}{\partial\theta} = -\frac{\mu}{r}\cos\theta$$

$$\Rightarrow \psi(r,\theta) = -\frac{\mu}{r}\sin\theta + f(r).$$

$$— (3)$$

From (3):

$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

$$\Rightarrow +\frac{\mu}{r^2}\sin\theta = -\frac{1}{r}\times -\frac{\mu\sin\theta}{r} + f'(r)$$

$$\Rightarrow f'(r) = 0. \Rightarrow f(r) \equiv \text{const.}$$

Since $f(r)$ is a const. we omit it.

From (3): $\psi(r,\theta) = -\frac{\mu}{r}\sin\theta.. \quad — (4)$

From (2) and (4).

$$\omega = \phi + i\psi = \frac{\mu\cos\theta}{r} - i\frac{\mu}{r}\sin\theta$$

$$= +\frac{\mu}{r}(\cos\theta - i\sin\theta) = \frac{\mu e^{-i\theta}}{r} = \frac{\mu}{re^{i\theta}}$$

$$= \mu/z$$

$$\Rightarrow \omega(z) = \mu/z.$$

The streamlines are given by

$$\psi(r,\theta) = \text{const.} = c$$

$$\Rightarrow \quad -\frac{\mu \sin\theta}{r} = c$$

$$\Rightarrow \quad -\mu y = c(x^2+y^2)$$

$$-\frac{\mu \sin\theta}{r} + d = c$$
$$= c-d$$
$$\underline{= e}$$

$$\Rightarrow \quad x^2 + y^2 + c'y = 0, \quad c' = \mu/c$$

$$\Rightarrow \quad x^2 + \left(y - \frac{c'}{2}\right)^2 + \frac{c'^2}{4} = 0$$

$$z = x+iy$$

The equi-potentials are given by

$$x = \frac{z+\bar{z}}{2}$$

$$\phi(r,\theta) = c$$

$$\Rightarrow \quad \frac{\mu \cos\theta}{r} = c$$

$$\Rightarrow \quad \mu x = c(x^2+y^2).$$

$$\Rightarrow \quad x^2+y^2 - cx = 0.$$

<u>Remark 1</u>: within the flow we have n doublets of strengths $\mu_1, \mu_2, \ldots, \mu_n$ located at $z_1, z_2, \ldots, z_n$ then.

$$w(z) = \frac{\mu_1}{z-z_1} + \frac{\mu_2}{z-z_2} + \cdots + \frac{\mu_n}{z-z_n}.$$

10.0921

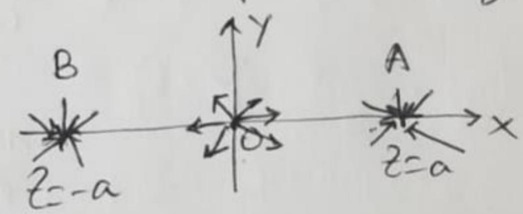
Ex 1: What arrangements of sources and sinks will give rise to the complex potential $w = \log\left(z - \frac{a^2}{z}\right)$, $a > 0$. Draw the stream lines.

Solⁿ: Given that $w(z) = \log\left(z - \frac{a^2}{z}\right) = \log_e \frac{z^2 - a^2}{z}$

$$\Rightarrow w(z) = \log(z^2 - a^2) - \log z$$

$$= \log(z-a) + \log(z+a) - \log z \quad \text{—(1)}$$

There are two sinks each located at $z = a$ and $z = -a$ of unit strength and there is one source of unit strength located $z = 0$.



From (1): $z = x + iy$

$$w(z) = \log(x+iy-a) + \log(x+iy+a) - \log(x+iy)$$

$$= \log\{(x-a)+iy\} + \log\{(x+a)+iy\} - \log(x+iy)$$

$$\Rightarrow \phi + i\psi = \frac{1}{2}\left[\log\{(x-a)^2+y^2\} + \log\{(x+a)^2+y^2\} - \log(x^2+y^2)\right] + i\left[\tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}\right].$$

$$\left(\log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}\right) \checkmark$$

For streamlines $\psi(x, y) = c$

$$\Rightarrow \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x} = c$$

$$\Rightarrow \tan^{-1}\frac{\frac{y}{x-a} + \frac{y}{x+a}}{1 - \frac{y^2}{x^2-a^2}} - \tan^{-1}\frac{y}{x} = c$$

$$\Rightarrow \tan^{-1}\frac{\frac{2xy}{x^2-a^2}}{\frac{x^2-a^2-y^2}{x^2-a^2}} - \tan^{-1}\frac{y}{x} = c$$

$$\Rightarrow \tan^{-1}\frac{\frac{2xy}{x^2-a^2-y^2} - \frac{y}{x}}{1 + \frac{2xy}{x^2-y^2-a^2} \cdot \frac{y}{x}} = c$$

$$\Rightarrow \frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = \underbrace{\tan^{-1} c}_{\text{Constant}} = c'$$

$$\Rightarrow y(x^2+y^2+a^2) = c_1 x(x^2+y^2-a^2) \quad -②$$

This is the required equ^n. of the streamlines.

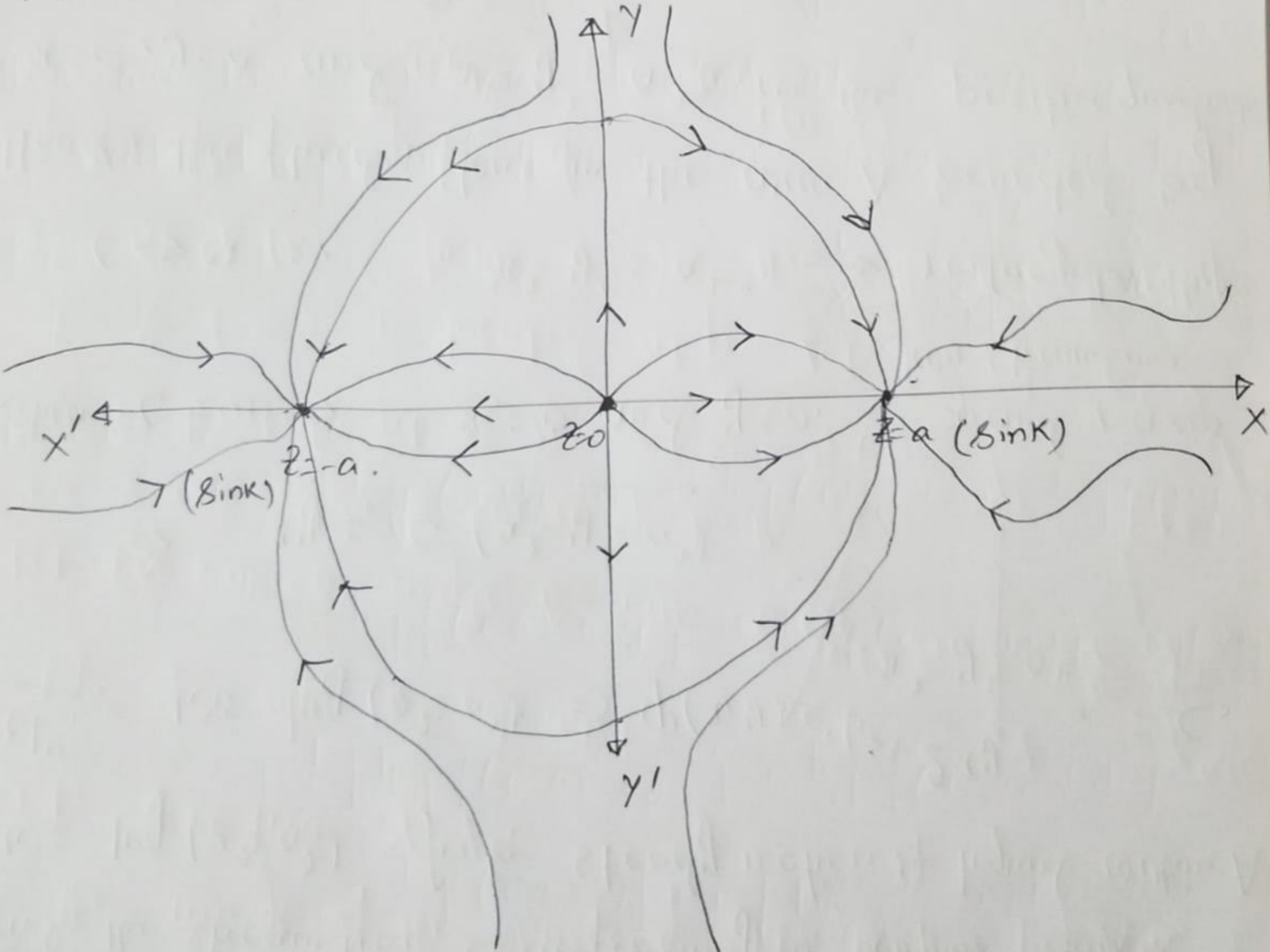
Case I: When $c = 0$, $y = 0 \to$ streamlines, x-axis is the streamline.

Case II: When $c \to \infty$, then from (2),

$$x(x^2 + y^2 - a^2) = 0$$

$\Rightarrow$ $x = 0$ or $x^2 + y^2 = a^2$

$\Rightarrow$ y-axis and the circle centered at origin and radius "a" are the streamlines.



Ex! Let A be area bounded by x-axis for which $x > a > 0$, and by the branch of $x^2 - y^2 = a^2$ in the positive quadrant. Find the stream line corresponding to the complex potential $\omega = \log(z^2 - a^2)$ for a steady motion of liquid within A.

Sol^n:

$$\omega = \log(z^2 - a^2) \Rightarrow \psi(x, y) = \frac{2xy}{x^2 - y^2 - a^2} = C$$

$$\Rightarrow \quad xy = c\,(x^2 - y^2 - a^2)$$

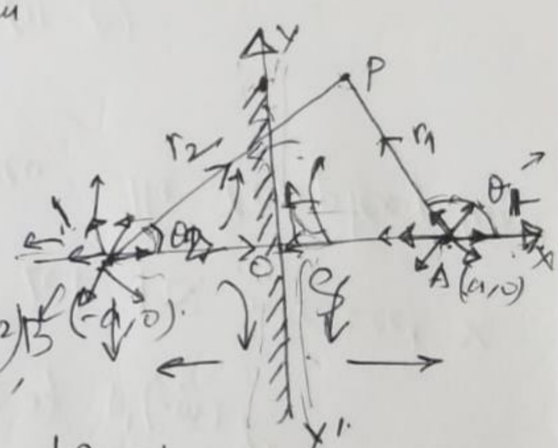
when $c = 0 \Rightarrow \quad x = 0$ and $y = 0$. $\Rightarrow$ x and y axes are streamlines

$c \Rightarrow \infty \Rightarrow \quad x^2 - y^2 = a^2 \longrightarrow$ rectangular Hyper..

Hence, the fluid flows in the area A bounded by $x = 0, y = 0$ and $x^2 - y^2 = a^2$, in the positive quadrant.

§ Images in a 2D Flow: If a surface S can be drawn in a moving fluid in such a way that there is no transport of fluid accross that surface then any system of sources, sinks and doublets on one side of the surface is said to be the ~~system~~ image system of sources, sinks and doublets on the other side with regard to the surface S. The fluid flows tangentially to the surface.

§ Image of a source with regard to a plane: Consider two sources of equal strength $+m$ at $A(a,0)$ and $B(-a,0)$ at an equidistant from the axis OY. Then the complex potential is



$$w(z) = -m\log(z-a) - m\log(z+a)$$

$$= -m\log(r_1 e^{i\theta_1}) - m\log(r_2 e^{i\theta_2}),$$

$$|z-a| = r_1$$

$$|z+a| = r_2$$

$$= -m\log(r_1 r_2 e^{i(\theta_1+\theta_2)})$$

$$\Rightarrow \phi + i\psi = -m\left[\log r_1 r_2 + i(\theta_1+\theta_2)\right]$$

$$\Rightarrow \phi = -m\log_e(r_1 r_2) \quad \text{and} \quad \psi = -m(\theta_1+\theta_2)$$

The stream lines are given by $\psi(r,\theta) = c \Rightarrow -m(\theta_1+\theta_2) = c$

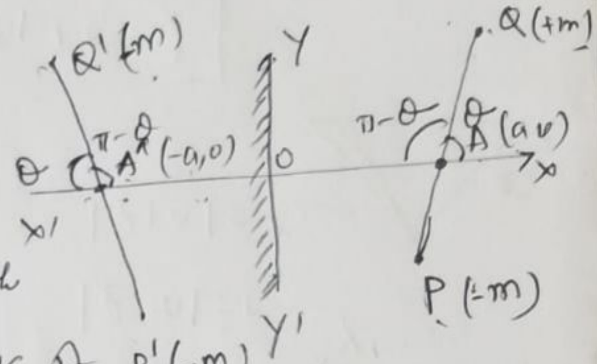
$$\Rightarrow \theta_1 + \theta_2 = c$$

Let P be on Y-axis which is a streamlines then $\theta_1 + \theta_2 = \pi$

There is no flow across y-axis i.e. the line YOY'.

Therefore the Source of Strength $+m$ at $B(-a,0)$ is the image of the source of strength $+m$ of $A(a,0)$ w.r.t. the line YOY'.

§ Equ^n of image of a doublet:



Let PQ makes an angle θ with +ve x-axis where a source of strength $+m$ is located at Q and a sink of strength $(-m)$ is located at P. Let PQ intersect X axis at $A(a,0)$. (Write......). Then the complex potential of the flow is given by

$$w = \frac{\mu e^{i\theta}}{z-a} + \frac{\mu e^{i(\pi-\theta)}}{z+a}$$

$$= \frac{\mu e^{i\theta}}{z-a} - \frac{\mu e^{-i\theta}}{z+a} \quad ✓$$

§ The Circle Theorem: Let $f(z)$ be a complex potential of a fluid motion of a 2D ~~flow~~ irrotational and incompressible flow. with no rigid boundary and $f(z)$ has no singularities within the circle $|z|=a$, $a>0$. If a circular cylinder, classifie by its cross section, the circle $|z|=a$, be introduced into the flow, then the complex potential ~~is~~ becomes

$$\hat{\omega}(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right), \quad |z| \geq a$$

where $\bar{f}$ is the conjugate of f.

$\omega = f(z) = \phi + i\psi$

$\omega(z) = m(z+a)$

$$\hat{\omega}(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

$$= m(z+a) + m\left(\frac{a^2}{z} + a\right), \quad |z| = a$$

$=$