Practice it to improve your reasoning. Don't be upset if you can not solve it.
You need not submit anything. Please don't ask for solution set.

1 If x, y, z is a basis for $\mathbb{R}^3$, which of the following are also bases for $\mathbb{R}^3$? (i) $x+2y$, $y+3z$, $x+2z$. (ii) $x+y-2z$, $x-2y+z$, $-2x+y+z$. (iii) x, y, $x+y+z$.

2 Let A be a square matrix with all row sums equal to 1. If $AA' = A'A$, then show that the column sums of A are also equal to 1.

3 Let A be a square matrix. Prove that the following conditions are equivalent: (i) $A = A'$. (ii) $A^2 = AA'$. (iii) trace $A^2 =$ trace AA'. (iv) $A^2 = A'A$. (v) trace $A^2 =$ trace $A'A$.

4 Let

$$\boldsymbol{X} = \begin{bmatrix} 1 & .2 & 0 \\ 1 & .4 & 0 \\ 1 & .6 & 0 \\ 1 & .8 & 0 \\ 1 & .2 & .1 \\ 1 & .4 & .1 \\ 1 & .6 & .1 \\ 1 & .8 & .1 \end{bmatrix} \qquad \boldsymbol{Y} = \begin{pmatrix} 242 \\ 240 \\ 236 \\ 230 \\ 239 \\ 238 \\ 231 \\ 226 \end{pmatrix}.$$

(a) Compute $\boldsymbol{X}'\boldsymbol{X}$ and $\boldsymbol{X}'\boldsymbol{Y}$. Verify by separate calculations that the $(i,j) = (2,2)$ element in $\boldsymbol{X}'\boldsymbol{X}$ is the sum of squares of column 2 in $\boldsymbol{X}$. Verify that the $(2,3)$ element is the sum of products between columns 2 and 3 of $\boldsymbol{X}$. Identify the elements in $\boldsymbol{X}'\boldsymbol{Y}$ in terms of sums of squares or products of the columns of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

(b) Is $\boldsymbol{X}$ of full column rank? What is the rank of $\boldsymbol{X}'\boldsymbol{X}$?

(c) Obtain $(\boldsymbol{X}'\boldsymbol{X})^{-1}$. What is the rank of $(\boldsymbol{X}'\boldsymbol{X})^{-1}$? Verify by matrix multiplication that $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X} = \boldsymbol{I}$.

(d) Compute $\boldsymbol{P} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ and verify by matrix multiplication that $\boldsymbol{P}$ is idempotent. Compute the trace $\mathrm{tr}(\boldsymbol{P})$. What is $r(\boldsymbol{P})$?

5 Find the inverse of the following matrix,

$$\boldsymbol{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 2 \\ 0 & 2 & 3 \end{bmatrix}.$$

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6 Given the following eigenvalues with their corresponding eigenvectors, and knowing that the original matrix was square and symmetric, reconstruct the original matrix.

$$\begin{aligned} \lambda_1 &= 6 \quad z_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \lambda_2 &= 2 \quad z_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

7 If $Y_1, Y_2, \ldots, Y_n$ is a random sample from $N(\mu, \sigma^2)$, prove that $\overline{Y}$ is independent of $\sum_{i=1}^{n-1}(Y_i - Y_{i+1})^2$.

8 Let $\mathbf{Y} \sim N_n(\mathbf{0}, \mathbf{I}_n)$, and put $\mathbf{X} = \mathbf{AY}$, $\mathbf{U} = \mathbf{BY}$ and $\mathbf{V} = \mathbf{CY}$. Suppose that $\text{Cov}[\mathbf{X}, \mathbf{U}] = \mathbf{0}$ and $\text{Cov}[\mathbf{X}, \mathbf{V}] = \mathbf{0}$. Show that $\mathbf{X}$ is independent of $\mathbf{U} + \mathbf{V}$.

9 Given $\mathbf{Y} \sim N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\Sigma} = \sigma^2 \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix},$$

for what value(s) of ρ are $Y_1 + Y_2 + Y_3$ and $Y_1 - Y_2 - Y_3$ statistically independent?

10 Let X_1, X_2, and X_3 be i.i.d. $N(0,1)$. Let

$$\begin{aligned} Y_1 &= (X_1 + X_2 + X_3)/\sqrt{3}, \\ Y_2 &= (X_1 - X_2)/\sqrt{2}, \\ Y_3 &= (X_1 + X_2 - 2X_3)/\sqrt{6}. \end{aligned}$$

Show that Y_1, Y_2 and Y_3 are i.i.d. $N(0,1)$. (The transformation above is a special case of the so-called *Helmert transformation.*)

11 Suppose that $Y_1, Y_2, \ldots, Y_n$ are independently distributed as $N(0,1)$. Calculate the m.g.f. of the random vector

$$(\overline{Y}, Y_1 - \overline{Y}, Y_2 - \overline{Y}, \ldots, Y_n - \overline{Y})$$

and hence show that $\overline{Y}$ is independent of $\sum_i (Y_i - \overline{Y})^2$.

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12 Suppose that $\mathbf{Y} \sim N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the joint distribution of $Z_1 = Y_1 + Y_2 + Y_3$ and $Z_2 = Y_1 - Y_2$.

13 The random variables $X_1, X_2, \ldots, X_n$ have a common nonzero mean μ, a common variance σ^2, and the correlation between any pair of random variables is ρ.

(a) Find $\text{var}[\overline{X}]$ and hence prove that $-1/(n-1) \le \rho \le 1$.

(b) If

$$Q = a\sum_{i=1}^{n} X_i^2 + b\left(\sum_{i=1}^{n} X_i\right)^2$$

is an unbiased estimate of σ^2, find a and b. Hence show that, in this case,

$$Q = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{(1-\rho)(n-1)}.$$

14 If $X_1, X_2, \ldots, X_n$ are independent random variables with common mean μ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, prove that $\sum_i (X_i - \overline{X})^2/[n(n-1)]$ is an unbiased estimate of $\text{var}[\overline{X}]$.

15 If X is a random variable with a density function symmetric about zero and having zero mean, prove that $\text{cov}[X, X^2] = 0$.

16 If X and Y are random variables with the same variance, prove that $\text{cov}[X+Y, X-Y] = 0$. Give a counterexample which shows that zero covariance does not necessarily imply independence.

17 Let X and Y be discrete random variables taking values 0 or 1 only, and let $\text{pr}(X = i, Y = j) = p_{ij}$ $(i = 1, 0; j = 1, 0)$. Prove that X and Y are independent if and only if $\text{cov}[X, Y] = 0$.

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18	If $X_1, \ldots, X_n$ are independently and identically distributed as $N(0, \sigma^2)$, and $\mathbf{A}$ and $\mathbf{B}$ are any $n \times n$ symmetric matrices, prove that $$\text{Cov}[\mathbf{X}'\mathbf{A}\mathbf{X}, \mathbf{X}'\mathbf{B}\mathbf{X}] = 2\sigma^4 \operatorname{tr}(\mathbf{AB}).$$
19	If X and Y are random variables, prove that $$\text{var}[X] = E_Y\{\text{var}[X\|Y]\} + \text{var}_Y\{E[X\|Y]\}.$$
20	Let $\mathbf{X} = (X_1, X_2, X_3)'$ with $$\text{Var}[\mathbf{X}] = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix}.$$ (a) Find the variance of $X_1 - 2X_2 + X_3$. (b) Find the variance matrix of $\mathbf{Y} = (Y_1, Y_2)'$, where $Y_1 = X_1 + X_2$ and $Y_2 = X_1 + X_2 + X_3$.
21	2.2. Find the rank of each of the following matrices. Which matrices are of full rank? $$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$ $$\boldsymbol{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}.$$ 2.3. Use $\boldsymbol{B}$ in Exercise 2.2 to compute $\boldsymbol{D} = \boldsymbol{B}(\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'$. Determine whether $\boldsymbol{D}$ is idempotent. What is the rank of $\boldsymbol{D}$?

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22 Suppose that X_1, X_2, and X_3 are random variables with common mean μ and variance matrix

$$\mathrm{Var}[\mathbf{X}] = \sigma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} \\ 0 & \frac{1}{4} & 1 \end{pmatrix}.$$

Find $E[X_1^2 + 2X_1X_2 - 4X_2X_3 + X_3^2]$.