Note:- Since a basis of a n.l.s depends only on the linear structure of the space, not on the norm, from ~~above~~ the last theorem, we can say that if a linear space is a Banach space w.r.t some norm on it, then it cannot have a denumerable basis.

Ex: $X = C[a,b]$ with $\|\cdot\|_1$ is not a Banach space but $(C[a,b], \|\cdot\|_\infty)$ is a Banach space. So $C[a,b]$ cannot have a denumerable basis.

(ii) If a linear space X has a denumerable basis, then no norm on X makes it ~~complete~~ Banach space.

That is a Banach space X has either finite basis or uncountable basis.

Ex (1): $X = P$, the linear space of all polynomials with coefficients in the field $\mathbb{K}$.

$\text{Lin} \{ u_j(t) = t^j, \; j = 0, 1, 2, 3, \ldots \}$ is a denumerable basis for P. So X is not a Banach space w.r.t. to any norm on it.

(2) $X = c_{00}$, is not a Banach space w.r.to any norm on it, since $\{e_1, e_2, e_3, \ldots\}$ with $e_i(j) = \delta_{ij}$, is a denumerable basis for c_{00}.

Now let us relax the requirement of a basis that every element of a linear space must be finite linear combinations of basis elements, and admit ~~de~~ denumerable linear ~~combinat~~ combinations of elements of a Banach space. This leads to the following definition.

Schauder basis :

A countable subset $\{x_1, x_2, x_3, \ldots\ldots\}$ of a Banach space X is called a Schauder basis if $\|x_n\| = 1,\ \forall n \in \mathbb{N}$ and if every $x \in X$, there are unique scalars $k_1, k_2, k_3 \ldots\ldots$ in K such that

$$x = \sum_{j=1}^{\infty} k_j x_j .$$

Ex: Suppose $(X\ \|\cdot\|)$ is a

finite dimensional normed linear space, and $\{x_1, x_2, \ldots x_n\}$ be a basis for X. Then X is a Banach space and

$$\left\{ \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \ldots \frac{x_n}{\|x_n\|} \right\}$$

is a Schauder basis for X.

In particular, if $X = K^n$, then standard basis $\{e_1, e_2, \ldots e_n\}$ with $e_i(j) = \delta_{ij}$ is a Schauder basis for $(K^n, \|\cdot\|_p)$, $p = 1, 2, \infty$.

[*] $e_i(j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$

(iii) $X = \ell^p$, $p = 1, 2$,

$\{e_1, e_2, e_3 \ldots\}$ is not a basis for ℓ^p, but it is a Schauder basis for ℓ^p.

$\because$ $x = (x(1), x(2), x(3) \ldots) \in \ell^p$ and $m \in \mathbb{N}$, let

$$f_m(x) = (x(1), x(2), \ldots x(m), 0, 0, \ldots)$$
$$= \sum_{n=1}^{m} x(n) e_n$$

Then

$$\|f_m(x) - x\|_p^p = \sum_{n=m+1}^{\infty} |x(n)|^p \to 0 \text{ as } n \to \infty$$

Thus $x = \lim_{n\to\infty} S_n(x) = \sum_{j=1}^{\infty} x(j) e_j$.

The Schauder basis $\{e_1, e_2, \dots\}$ is known as standard basis for ℓ^p.

Ex: A Schauder basis for $C[0,1]$ can be constructed as follows:

For $t \in \mathbb{R}$, let $y_0(t) = t$

$y_1(t) = 1-t$

$$y_2(t) = \begin{cases} 2t, & 0 \le t \le \frac{1}{2} \\ 2-2t, & \frac{1}{2} < t \le 1 \\ 0, & \text{if } t<0 \text{ or } t>1 \end{cases}$$

and define

... then

$$y_{2^n+j}(t) = y_2\left(2^n t - j + 1\right)$$

$n \in \mathbb{N}$
$j = 1, 2, 3, \ldots 2^n.$

Let $x_n = y_n \Big|_{[0,1]}$, $n = 0, 1, 2, 3, \ldots$

Then $\{x_1, x_2, x_3, \ldots\}$ is a Schauder basis for $C[0,1]$.

Equivalent norms

Let X be a n.l.s with norm $\|\cdot\|$ and $\|\cdot\|_*$

These norms are said to be equivalent if there exists $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|x\| \le \|x\|_* \le c_2\|x\|, \quad \forall x \in X.$$

Theorem: Suppose $\|\cdot\|$ and $\|\cdot\|_*$ be two equivalent norms on a n.l.s X. Then X is a Banach space w.r.t $\|\cdot\|$ iff X is a Banach space w.r.t $\|\cdot\|_*$.

Def:- Two norms $\|\cdot\|$ and $\|\cdot\|'$ are comparable if one of them is stronger than the other, i.e., if $\|x_n\| \to 0 \Rightarrow \|x_n\|' \to 0$ as $n \to \infty$.

Then we have $\|\cdot\|$ is

... say $\|\cdot\|$ is
stronger than $\|\cdot\|'$
and we say $\|\cdot\|'$ is
weaker than $\|\cdot\|$.

Theorem: Let $\|\cdot\|$ and $\|\cdot\|'$
be norms on a linear space X.

Then $\|\cdot\|$ is stronger ~~that~~
than $\|\cdot\|'$ iff there exists
$\alpha > 0 \ni \|x\|' \leq \alpha \|x\|, \ \forall x \in X$.

Further $\|\cdot\|$ and $\|\cdot\|'$ equivalent
iff there exists $\alpha, \beta > 0 \ni$
$\beta \|x\| \leq \|x\|' \leq \alpha \|x\|, \ \forall x \in X$

Proof: Suppose $\|x\|' \leq \alpha \|x\|, \forall x \in X$
Let $\{x_n\}$ be a sequence in X
such that $\|x_n\| \to 0$ as $n \to \infty$.

Then by the above inequality

$$\|x_n\|' \leq \alpha \|x_n\| \longrightarrow 0$$

$$\Rightarrow \|x_n\|' \longrightarrow 0 \text{ as } n \to \infty$$

$\therefore$ $\|\cdot\|$ is stronger than $\|\cdot\|'$.

Conversely, suppose that $\|\cdot\|$ is stronger than $\|\cdot\|'$.

Claim: $\exists$ $\alpha > 0$ $\ni$

$$\|x\|' \leq \alpha \|x\|, \forall x \in X.$$

Suppose there is no $\alpha > 0$ $\ni$ $\|x\|' \leq \alpha \|x\|$, $\forall x \in X$

Then for every $n \in \mathbb{N}$, $\exists$ a non zero element $x_n \in X$ such that

$$\|x_n\|' > n\|x_n\|.$$

Let $y_n = \dfrac{x_n}{n\|x_n\|}$, $n = 1, 2, 3 \ldots$

Then $\|y_n\| = \dfrac{1}{n} \to 0$ as $n \to \infty$

$$\|y_n\|' = \frac{\|x_n\|'}{n\|x_n\|} > 1 \not\to 0 \text{ as } n \to \infty.$$

which is contradiction to $\|\cdot\|$ is stronger than $\|\cdot\|'$.

$\therefore$ our assumption is wrong.

Hence $\exists$ $\alpha > 0$ $\ni$

$$\|x\|' \leq \alpha\|x\|, \ \forall x \in X.$$

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Ex: $\|\cdot\|_1$, $\|\cdot\|_2$, & $\|\cdot\|_\infty$ are equivalent on K^n.

$$\because \quad \|x\|_1 = \sum_{i=1}^{n} |x(i)| = \sum_{i=1}^{n} 1 \cdot |x(i)|$$

$$\leq \left(\sum_{i=1}^{n} 1^2\right)^{1/2} \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2}$$

$$= \sqrt{n}\, \|x\|_2 .$$

$$\|x\|_2 = \left[\sum_{i=1}^{n} |x(i)|^2\right]^{1/2}$$

$$\leq \left[\left(\sum_{i=1}^{n} |x(i)|\right)^2\right]^{1/2}$$

$$= \sum_{i=1}^{n} |x(i)| = \|x\|_1$$

Also

$$|x(i)| \leq \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2} = \|x\|_2$$

$$\Rightarrow \max_{i=1 \ldots n} |x(i)| \leq \|x\|_2$$

$$\Rightarrow \|x\|_\infty \leq \|x\|_2$$

$\Rightarrow \|x\|_\infty \le \|x\|_2$.

|||ly we can prove

$$\|x\|_\infty \le \|x\|_1$$

$$\therefore \|x\|_1 \le \sqrt{n}\,\|x\|_2 \le \sqrt{n}\,\|x\|_1$$

$\Rightarrow \|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent.

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$$

$$\left[\because \|x\|_2 = \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2} \le \sup_i |x(i)| \left(\sum_{i=1}^{n} 1\right)^{\frac{1}{2}} = \|x\|_\infty \cdot \sqrt{n}\right].$$

$\Rightarrow \|\cdot\|_\infty$ & $\|\cdot\|_2$ are equivalent.

Also $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$

$$\|x\|_1 \le \sqrt{n}\,\|x\|_2 \le n\,\|x\|_\infty$$

$\because$ $\|x\|_\infty \le \|x\|_1 \le n\,\|x\|_\infty$.

$\therefore$ $\|\cdot\|_\infty$ & $\|\cdot\|_1$ are equivalent.

* Any two norms on $\mathbb{K}^n$ are equivalent.

$\boxed{{}^* \|\cdot\|_p \; [1 \le p \le \infty]}$

Ex: On ℓ^1, the norm $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$.

$\because$ $x = (x(1), x(2), \dots) \in \ell^1$, we have

$$\|x\|_2 = \left(\sum_{i=1}^{\infty} |x(i)|^2\right)^{1/2}$$

$$\leq \left[\left(\sum_{i=1}^{\infty} |x(i)|\right)^2\right]^{\frac{1}{2}}$$

$$= \sum_{i=1}^{\infty} |x(i)| = \|x\|_1$$

$\Rightarrow \|x\|_2 \leq \|x\|_1$

$\Rightarrow \|\cdot\|_1$ is stronger than $\|\cdot\|_2$.

But these two are not equivalent.

Since $x_n = (\underbrace{\tfrac{1}{n}, \tfrac{1}{n}, \ldots \tfrac{1}{n}}_{n \text{ times}}, 0, 0 \ldots) \in \ell^1$.

$$\therefore \|x_n\|_1 = \sum_{i=1}^{\infty} |x_n(i)| = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

$$\|x_n\|_2 = \left(\sum_{i=1}^{\infty} |x_n(i)|^2\right)^{1/2}$$

$$= \left(\sum_{i=1}^{n} \frac{1}{n^2}\right)^{1/2}$$

$$\leq \frac{1}{n}\left(\sum_{i=1}^{n} 1\right)^{1/2}$$

$$= \frac{1}{\sqrt{n}}$$

$\therefore \|x_n\|_1 = 1 \not\leq \frac{1}{\sqrt{n}} = \|x_n\|_2.$

|||ly on ℓ^2, $\|\cdot\|_2$ is stronger than $\|\cdot\|_\infty$

$\because \|x\|_\infty \leq \|x\|_2$, on ℓ^2.

But these two are not equivalent.

$$\therefore \|\sqrt{n}\, x\|_2 = \left[\sum_{i=1}^{\infty} [\sqrt{n}\, |x_n(i)|]^2\right]^{1/2}$$

$$= \sqrt{n}\left(\sum_{i=1}^{n} \frac{1}{n^2}\right)^{1/2}$$

$$= \sqrt{n}\cdot \frac{1}{n}\left(\sum_{i=1}^{n} 1\right)^{1/2}$$

$$= 1$$

and $\|\sqrt{n}\, x_n\|_\infty = \frac{1}{\sqrt{n}}, \ \forall n \in \mathbb{N}$

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