Ex: $X = C[a,b]$ with $\|\cdot\|_\infty$.

and $X_0 = C^1[a,b] \subset X$.

Let $A : X_0 \subseteq X \longrightarrow X$ be the inclusion operator defined by

$$Ax = x.$$

Clearly A is bounded operator.

Since $\overline{X_0} = X$, for $x \in X - X_0$

$\exists$ a sequence $\{x_n\}$ in X_0 such that

$x_n \longrightarrow x \in X$.

$\because x_n \in X_0$, $Ax_n = x_n \longrightarrow x$

$\therefore$ The sequence

$\{(x_n, Ax_n)\}$ is a sequence in

The graph of A, $G(A)$,

and $(x_n, Ax_n) \longrightarrow (x, x) \notin G(A)$

$\therefore$ $G(A)$ is not a closed subspace of $X \times X$.

$\therefore$ A is not a closed operator.

But if the domain of a bounded operator is closed subspace, then it is a closed operator.

Theorem: Let $A : X_0 \subseteq X \longrightarrow Y$ be a bounded operator.

(i) If X_0 is closed in X, then A is a closed operator

(ii) If Y is a Banach space and A is a closed operator, then X_0 is a closed subspace of X.

Proof:

(i) Let X_0 be a closed subspace of X and $A : X_0 \to Y$ be a bounded operator.

Claim: A is a closed operator.

Let $\{x_n\}$ be a sequence in X_0 such that $x_n \to x \in X$ and $Ax_n \to y \in Y$. $\to (1)$

$\because$ X_0 is a closed subspace, and $\{x_n\}$ is a sequence in X_0 such that $x_n \to x \Rightarrow x \in X_0$.

$\because$ A is bounded operator,

$$x_n \to x \Rightarrow Ax_n \to Ax$$
$\longrightarrow (2)$

$\therefore$ from ① & ② we have

$$Ax = y$$

$\therefore$ A is a closed operator.

(ii) Let $A: X_0 \subseteq X \to Y$ be both closed and bounded operator and Y be a Banach space.

Claim: X_0 is a closed subspace

Let $\{x_n\}$ be a sequence in X_0 such that $x_n \to x \in X$.

Since A is a bounded operator, we have

$$\|Ax_n - Ax_m\| \leq \|A\| \, \|x_n - x_m\| \to 0$$

$\Rightarrow \{Ax_n\}$ is a Cauchy sequence in Y. But Y is a Banach space. $\therefore$ $Ax_n \to y \in Y$.

Then we have $\{x_n\}$ is a sequence in X_0 $\ni$ $x_n \to x$ and $Ax_n \to y$, and A is a closed operator.

$\therefore$ $x \in X_0$ and $Ax = y$.

$\Rightarrow$ X_0 is a closed subspace.

Problem: Let $A : X_0 \subseteq X \to Y$ be a closed operator. If Y

is a Banach space and X_0 is not closed in X, then show that A is unbounded operator.

* If every closed operator $A : X_0 \subseteq X \longrightarrow Y$ with closed subspace X_0 and complete Y a bounded operator?

We know that if $A \in BL(X, Y)$, then the null space $N(A)$ is closed subspace of X.

If A is a closed operator, then $N(A)$ is also closed operator. Also if A is $1-1$, then $\overline{A}^{1}: R(A) \to X$ is a closed operator.

Theorem: Suppose $A : X_0 \subseteq X \to Y$ be a closed operator. Then

(i) $N(A)$ is a closed subspace of X

(ii) if A is 1-1, $\overline{A}^{1} : R(A) \subset Y \to X$ is a closed operator.

Proof: (i) let $\{x_n\}$ be a sequence in $N(A)$ $\ni$ $x_n \to x \in X$.

$\because x_n \in N(A) \Rightarrow Ax_n = 0, \forall n.$

$\therefore Ax_n \to 0$ as $n \to \infty$

$\because A$ is a closed operator and $x_n \in N(A)$, $x_n \to x \in X$, $Ax_n \to 0$,

$\Rightarrow \quad Ax = 0 \quad , \; x \in N(A).$

$\therefore$ $N(A)$ is a closed subspace.

(ii) Assume A is $1-1$.

$\therefore$ $A^{-1} : R(A) \subseteq Y \longrightarrow X$ exists

Claim: A^{-1} is a closed operator.

So let $\{y_n\}$ be a sequence in $R(A)$

such that $y_n \to y \in Y$, $A^{-1}y_n \to x \in X$

Let $x_n = A^{-1}y_n \quad \Rightarrow \quad Ax_n = y_n.$

$\therefore \; Ax_n = y_n \longrightarrow y \in Y$

and $x_n \longrightarrow x \in X$

$\because$ A is a closed operator, $Ax = y$, $x \in X_0$.

$\Rightarrow \quad x = A^{-1}y \quad , \; y = Ax \in R(A)$

$\Rightarrow \; A^{-1}$ is a closed operator.

Theorem: Suppose X is a Banach space and $A : X_0 \subseteq X \longrightarrow Y$ be 1–1, closed operator. If $R(A)$ is not closed in Y, then $A^{-1} : R(A) \subseteq Y \longrightarrow X$ is unbounded operator.

Proof: Since $A : X_0 \subseteq X \longrightarrow Y$ is 1–1, closed, by above theorem, $A^{-1} : R(A) \subseteq Y \longrightarrow X$ is a closed operator. If $A^{-1} : R(A) \subseteq Y \to X$ is also bounded, then A^{-1} is both closed and bounded.

Since X is a Banach space, by one of the previous theorem, $R(A)$ the domain of A^{-1} is a closed subspace of Y, which is contradiction to $R(A)$ is not closed.

$\therefore$ A^{-1} is unbounded.

Ex.: $X = C^1[0,1]$ with the

norm $\|x\|_{1,\infty} = \|x\|_\infty + \|x'\|_\infty$

$Y = C^1[0,1]$ with $\|\cdot\|_\infty$

Define $A : X \to Y$ by

$$Ax = x.$$

Then $\|Ax\|_\infty = \|x\|_\infty \le \|x\|_\infty + \|x'\|_\infty$

$$= \|x\|_{1,\infty}$$

$\therefore A : X \to Y$ is bounded.

The inverse of A,

$A^{-1} : R(A) \to X$ is defined

by $A^{-1}y = y, \ \forall y \in R(A)$.

$$\|A^{-1}y\|_{1,\infty} = \|y\|_\infty + \|y'\|_\infty \nleq \|y\|_\infty$$

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$\therefore$ A^{-1} is unbounded.

$$\left[\ A^{-1} : R(A) \subseteq Y \longrightarrow X\right.$$

$$Y = C^1[0,1], \quad \|\cdot\|_\infty$$

$$X = C^1[0,1], \quad \|x\|_{1,\infty} = \|x\|_\infty + \|x'\|_\infty$$

$\forall\, y \in R(A)$,

$$\|A^{-1}y\|_{1,\infty} = \|y\|_{1,\infty}$$

$$= \|y\|_\infty + \|y'\|_\infty.$$

$$\left.\not\leq \|y\|_\infty\ \right]$$

But $A^{-1} : R(A) \subset Y \to X$ is a closed operator

Theorem: Let $A : X \to Y$ be 1-1, bounded operator.
Then $A^{-1} : R(A) \subseteq Y \to X$ is a closed operator.

Proof: Let $\{y_n\}$ be a sequence in $R(A)$ such that

$$y_n \to y \in Y, \quad A^{-1}y_n \to x \in X.$$

Let $x_n = A^{-1}y_n$, $\forall n \in \mathbb{N}$.

$\Rightarrow x_n \to x \in X$, $Ax_n = y_n \to y \in Y$.

$\because$ A is a bounded operator and $x_n \to x$ $\Rightarrow Ax_n \to Ax$

Thus $Ax_n \to y$ & $Ax_n \to Ax$

$\therefore Ax = y \Rightarrow y \in R(A)$ and $x = A^{-1}y \Rightarrow A^{-1}$ is a closed operator.

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Theorem: Let $A_0 : X_0 \subseteq X \to Y$ be a bounded operator, where X_0 is dense in X, and Y is a Banach space. Then there exists a unique $A \in B(X,Y)$ such that A is extension of A_0.

Moreover $\|A\| = \|A_0\|$, and for $x \in X$, $Ax = \lim_{n\to\infty} A x_n$, where $\{x_n\}$ is a sequence in X_0 such that $x_n \to x$ [we prove later].

Closed graph theorem :—

If X and Y are Banach spaces, then every closed operator $A : X \to Y$ is a continuous operator.

Proof: Let X and Y be Banach spaces and $A : X \to Y$ be a closed operator.

Claim: A is a continuous operator.

Let $B_0 = \{x \in X \,/\, \|x\| < 1\}$.

We show that

$$B_0 \subseteq \{x \in X \,/\, \|Ax\| \leq c\}$$

for some $c > 0$, so that A is continuous.

For each $\alpha > 0$, let

$$V_\alpha = \{x \in X / \|Ax\| \leq \alpha\}$$

Then $X = \bigcup_{j=1}^{\infty} V_j$.

Since X is a Banach space, by the Baire-Category theorem, there is some $k > 0$ such that

$$\mathring{\overline{V_k}} \neq \phi$$

Thus there is some $x_0 \in X$ and $r > 0 \ni B(x_0, r) \subseteq \overline{V_k}$.

[i.e., let $x_0 \in \mathring{\overline{V_k}}$].

Now let $x \in B_0$ and

Let $u = x_0 + rx$

$\Rightarrow \|u - x_0\| = \|rx\| = r\|x\| < r$ $\quad [\because \|x\| < 1]$

$\Rightarrow u \in B(x_0, r) \subset \overline{V_K}$.

Now $x_0, u \in B(x_0, r) \subset \overline{V_K}$,

implies there exist sequence

$\{u_n\}$ and $\{v_n\}$ in V_K

such that $u_n \to u$, $v_n \to x_0$

$\because u_n, v_n \in V_K \Rightarrow \|Au_n\| \leq K$

$\|Av_n\| \leq K$.

Thus

$x = \frac{1}{r}(u - x_0)$ $\quad [\because u = x_0 + rx]$

$= \lim_{n \to \infty} \frac{(u_n - v_n)}{r}$, ——(i)

and

$$\left\| A\left(\frac{u_n - v_n}{\gamma}\right) \right\| \leq \frac{1}{\gamma}\left[\|Au_n\| + \|Av_n\|\right] \leq \frac{2k}{\gamma}$$

$$\Rightarrow \frac{u_n - v_n}{\gamma} \in V_{\frac{2k}{\gamma}}$$

by (1)

$$\Rightarrow x \in \overline{V_{\frac{2k}{\gamma}}}.$$

$$\therefore B_0 \subseteq \overline{V_{\frac{2k}{\gamma}}}.$$

let us denote $W = V_{\frac{2k}{\gamma}}$.

let $x \in B_0$, $0 < \epsilon < 1$.

Since $B_0 \subset \overline{W} \Rightarrow x \in \overline{W}$

$\Rightarrow \exists\ x_1 \in W \ni \|x - x_1\| < \epsilon$

$$\Rightarrow \| \epsilon^{-1}(x - x_1) \| < 1$$

$$\Rightarrow \epsilon^{-1}(x - x_1) \in B_0 \subset \overline{W}$$

$$\Rightarrow \exists\, x_2 \in W \ni \| \epsilon^{-1}(x - x_1) - x_2 \| < \epsilon$$

$$\Rightarrow \| x - (x_1 + \epsilon x_2) \| < \epsilon^2$$

$$\Rightarrow \| \epsilon^{-2}[x - (x_1 + \epsilon x_2)] \| < 1$$

$$\Rightarrow \epsilon^{-2}[x - (x_1 + \epsilon x_2)] \in B_0$$

Continuing as above, after obtained $x_1, x_2, x_3 \ldots x_n \in W$ such that

$$\| x - (x_1 + \epsilon x_2 + \epsilon^2 x_3 + \cdots + \epsilon^{n-1} x_n) \| < \epsilon^n$$

$$\Rightarrow \left\| \epsilon^{-n}\left(x-\sum_{j=0}^{n-1}\epsilon^{j}x_{j+1}\right)\right\| < 1$$

$$\Rightarrow \epsilon^{-n}\left(x-\sum_{j=0}^{n-1}\epsilon^{j}x_{j+1}\right) \in B_0\overline{W}$$

$\Rightarrow \exists\, x_{n+1} \in W$ such that

$$\left\| \epsilon^{-n}\left[x-\sum_{j=0}^{n-1}\epsilon^{j}x_{j+1}\right]-x_{n+1}\right\| < \epsilon$$

$\therefore$ By induction, we obtain a sequence $\{x_n\}$ in W such that

$$\left\| x-\sum_{j=0}^{n-1}\epsilon^{j}x_{j+1}\right\| < \epsilon^{n}, \quad (*) \quad \forall\, n \in \mathbb{N}.$$

Set $s_n = \sum_{j=1}^{n}\epsilon^{j-1}x_j$, $\forall\, n \in \mathbb{N}$.

Then $s_n \longrightarrow x_0$ as $n \to \infty$ by (*).

Also since $x_j \in W = V_{\frac{2k}{r}}$

$$\Rightarrow \|Ax_j\| \le \frac{2k}{r}.$$

Hence for $n > m$, we have

$$\|As_n - As_m\| = \left\| \sum_{j=1}^{n} \epsilon^{j-1} Ax_j - \sum_{j=1}^{m} \epsilon^{j-1} Ax_j \right\|$$

$$\le \sum_{j=m+1}^{n} \epsilon^{j-1} \|Ax_j\|$$

$$\le \frac{2k}{r} \sum_{j=m+1}^{n} \epsilon^{j-1}$$

$$\le \frac{2k}{r} \epsilon^{m} [1 + \epsilon + \epsilon^2 \cdots]$$

$$= \frac{2k}{r} \epsilon^{m} \cdot \frac{1}{1-\epsilon} \longrightarrow 0 \text{ as } n, m \to \infty$$

$\Rightarrow \{Af_n\}$ is a Cauchy sequence in Y. Since Y is a Banach space,

$$Af_n \longrightarrow y \in Y.$$

Thus

$$f_n \to x \in X, \quad Af_n \longrightarrow y \in Y$$

and A is closed operator, implies

$$y = Ax = \lim_{n \to \infty} Af_n.$$

Now

$$\|Af_n\| = \left\| A\left(\sum_{j=1}^{n} \epsilon^{j-1} x_j\right)\right\|$$

$$\leq \sum_{j=1}^{n} \epsilon^{j-1} \|Ax_j\| \leq \frac{2k}{r} \sum_{j=1}^{n} \epsilon^{j-1}$$

$$\leq \frac{2k}{r} \sum_{j=1}^{\infty} \epsilon^{j-1}$$

$$= \frac{2k}{r} \frac{1}{1-\epsilon}.$$

$$\therefore \|Ax\| = \|\lim_{n\to\infty} Af_n\|$$

$$= \lim_{n\to\infty} \|Af_n\|$$

$$\leq \lim_{n\to\infty} \frac{2k}{r(1-\epsilon)}$$

$$= \frac{2k}{r(1-\epsilon)}.$$

$\therefore$ for all $x \in B_0$, we have

$$\|Ax\| \leq \frac{2k}{r(1-\epsilon)}$$

$\Rightarrow x \in V_c, \quad c = \frac{2k}{r(r+\epsilon)}$

$\therefore B_0 \subset V_c.$

$\therefore$ A is Continuous.

Continuity of Projection operators: —

A linear map $P : X \to X$

is said to be a projection operator

if $P^2 = P$, i.e., $Px = x, \forall x \in R(P)$

[$P : X \to R(P)$

$x \in R(P) \subset X, \quad Px = x$

$\because$ Now for any $y \in X$

$Py \in R(P)$

$\Rightarrow P(Py) = Py$

$$\Rightarrow P^2 y = Py, \ \forall y \in X$$

$$\Rightarrow P^2 = P.$$

In this case we write

$$X = R(P) + N(P)$$

and $R(P) \cap N(P) = \{0\}$.

Suppose X is n.l.s and $P : X \to X$ be continuous projection operator.

Then $N(P)$ is closed subspace of X.

Here P is called projection onto range $R(P)$ along $N(P)$.

Consider

$$I-P : X \longrightarrow X$$

Now for $x \in R(P)$

$$(I-P)x = x - Px$$
$$= x - x = 0.$$

$\therefore x \in R(P) \Rightarrow x \in N(I-P)$

$$(I-P)^2 = (I-P)(I-P)$$
$$= I - P - (I-P)P$$
$$= I - P - P + P^2$$
$$= I - P - \cancel{P} + \cancel{P} \qquad [P^2 = P]$$
$$= I - P$$

$\therefore I-P : X \longrightarrow R(I-P)$ is

also a projection.

$\because$ $R(P) = N(I-P)$ & $I-P$ is continuous implies $N(I-P)$ is closed implies $R(P)$ is also closed.

Thus if $P : X \to X$ is continuous projection both $R(P)$ and $N(P)$ are closed subspaces of X.

Corollary :- Let X be a Banach space and $P : X \to X$ be a projection operator. If $N(P)$ and $R(P)$ are closed subspaces of X, then P is continuous.

Proof: Suppose $R(P)$ and $N(P)$ are

closed subspaces of X.

To prove $P : X \to X$ is continuous, it is enough to prove that P is a closed operator (by closed graph theorem).

Let $\{x_n\}$ be a sequence in X such that $x_n \to x$, $Px_n \to y$

$\because R(P)$ is closed, it follows that

$$y \in R(P)$$

$$\Rightarrow Py = y \quad [\text{by definition of } P]$$

Also

$$x_n - Px_n \to x - y$$

and $x_n - Px_n = (I-P)x_n \in R(I-P) = N(P)$

$$\left[X = \underset{\parallel}{R(I-P)} \oplus \underset{\parallel}{N(I-P)} \right]$$

$$= N(P) \oplus R(P) \;\Big]$$

$\therefore \quad x - y \in R(I-P) = N(P)$

$\Rightarrow \; P(x-y) = 0$

$\Rightarrow \; Px = Py = y$

Thus $Px = y$

$\therefore$ P is closed operator

$\Rightarrow$ P is continuous by closed graph theorem

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