Consequences of Hahn-Banach theorem :—
Let X be a n.l.s over K.

(a) Let $0 \neq a \in X$. Then there is some $f \in X'$ such that $f(a) = \|a\|$ and $\|f\| = 1$. Consequently

$$\|a\| = \sup\{ |f(a)| \mid f \in X', \|f\| \leq 1 \}.$$

Proof: Let $Y = \{ ka \mid k \in K \} = \text{Span}\{a\}$.

Then Y is a subspace of X.

Define $g : Y \longrightarrow K$ by

$$g(ka) = k\|a\|, \ \forall k \in K.$$

For $y \in Y$, we have $y = ka$, $k \in K$

$$\therefore g(y) = k\|a\|$$

$$\Rightarrow |g(y)| = |k| \, \|a\| = \|ka\| = \|y\|$$

$$\Rightarrow \|g\| = 1 \quad \text{and } g \in Y'.$$

Hence by Hahn-Banach extension theorem, there exists $f \in X'$ such that

$f|_Y = g$ and $\|f\| = \|g\| = 1$.

Now $f(a) = f(1.a) = g(1.a) = 1.\|a\|$
$= \|a\|$

$\therefore f(a) = \|a\|$ — (1).

Now for $f \in X'$, we have

$|f(a)| \leq \|f\| \, \|a\|, \ \forall f \in X'$

$\Rightarrow \dfrac{|f(a)|}{\|f\|} \leq \|a\|, \ \forall f \in X'$

$\Rightarrow \sup\{ |f(a)| \,/\, f \in X', \|f\| \leq 1\} \leq \|a\|$ — (2)

Combining (1) & (2) we get

$\|a\| = \sup\{ |f(a)| \,/\, f \in X', \|f\| \leq 1\}$.

b) Let Y be a sub-space of a n.l.s X over K and $a \in X$, but $a \notin \overline{Y}$. Then there is some $f \in X'$ such that

$f|_Y = 0$ and $f(a) = \text{dist}(a, \overline{Y})$

and $\|f\| = 1$.

Consequently $x \in \overline{Y} \iff x \in X$

and $f(x) = 0$, whenever $f \in X'$
and $f|_Y = 0$.

Proof: $\overline{Y}$ is a closed subspace of X.

Consider the quotient space $\frac{X}{\overline{Y}}$

and it is a n.l.s with $|||\cdot|||$.

Since $a \notin \overline{Y} \Rightarrow a + \overline{Y} \neq \overline{Y}$

Thus $a + \overline{Y}$ is a nonzero element in the n.l.s $\frac{X}{\overline{Y}}$.

Then by (a), $\exists$ some $\tilde{f} \in \left(\frac{X}{\overline{Y}}\right)'$

such that

$\tilde{f}(a + \overline{Y}) = |||a + \overline{Y}|||$, $\|\tilde{f}\| = 1$.

Now define $f : X \longrightarrow \mathbb{K}$ by

$$f(x) = \tilde{f}(x+\overline{Y}), \quad x \in X.$$

Now for any $x \in \overline{Y}$, we have

$$x + \overline{Y} = \overline{Y}.$$

$$\Rightarrow f(x) = \tilde{f}(x+\overline{Y}) = |||x+\overline{Y}|||$$
$$= \operatorname{dist}(x, \overline{Y})$$
$$= 0$$

$$\Rightarrow f(x) = 0, \ \forall\, x \in \overline{Y}.$$

and

$$f(a) = \tilde{f}(a+\overline{Y}) = |||a+\overline{Y}|||$$
$$= \operatorname{dist}(a, \overline{Y}).$$

Now for any $x \in X$,

$$|f(x)| = |\tilde{f}(x+\overline{Y})|$$
$$\leq \|\tilde{f}\| \, |||x+\overline{Y}|||$$

$$\leq \|\tilde{f}\| \, \|x\| \quad [0 \in \bar{Y}]$$

$$\Rightarrow \|f\| \leq \|\tilde{f}\| \quad \text{—— (1)}$$

Again for any $y \in \bar{Y}$,

$$|\tilde{f}(x+\bar{Y})| = |\tilde{f}(x+y+\bar{Y})|$$

$$= |f(x+y)|$$

$$\leq \|f\| \, \|x+y\|, \quad \forall \, y \in \bar{Y}.$$

$\because$ Above is true for any $y \in \bar{Y}$, we have

$$|\tilde{f}(x+\bar{Y})| \leq \|f\| \, |||x+\bar{Y}|||$$

$$\Rightarrow \|\tilde{f}\| \leq \|f\| \quad \text{—— (2)}.$$

$\therefore$ We have $\|f\| = \|\tilde{f}\| = 1$.

c) Let $\{a_1, a_2, \ldots a_n\}$ be L.I set in a n.l.s X. Then there are

$f_1, f_2, \ldots f_m$ in X' such that

$$f_i(a_j) = \delta_{ij}, \quad \forall\, i,j = 1,2 \ldots m.$$

Proof:- Let $\{a_1, a_2 \ldots a_m\}$ be L.I set in a n.l.s X.

Let

$$Y_j = \text{Span}\{a_1, a_2 \ldots a_{j-1}, a_{j+1}, \ldots a_m\}$$

$$j = 1,2 \ldots m.$$

Then Y_j is a closed subspace of X and $a_j \notin Y_j$, $j = 1,2 \ldots m$.

$$\Rightarrow \text{dist}(a_j, Y_j) > 0$$

Then by (b), $\exists$ $g_j \in X'$ such that $g_j|_{Y_j} = 0$

and

$$g_j(a_j) = \text{dist}(a_j, Y_j) > 0$$

Now Define

$$f_j = \frac{g_j}{\text{diff}(a_j, Y_j)} \quad , j = 1, 2 \cdots m$$

Then

$$f_j(a_j) = \frac{g_j(a_j)}{\text{diff}(a_j, Y_j)} = 1$$

and for $i \neq j$

$$f_j(a_i) = 0$$

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Ex: $X = K^2$ with norm $\|\cdot\|_1$

let $Y = \{ (x(1), x(2)) \mid x(2) = 0 \}$

$$= \text{Span}\{ a = (1, 0) \}.$$

Define $g : Y \to K$ by

$$g(x(1), x(2)) = x(1).$$

Then $g(a) = 1 = \|g\|$.

So $g \in Y'$.

So by Hahn-Banach extension theorem, there exists $f \in X'$ such that

$$f|_Y = g \quad \text{and} \quad \|f\| = 1.$$

If $f : X \to K$ is a linear map, then for any $x = (x(1), x(2)) \in X$, we have

$$x = x(1)(1,0) + x(2)(0,1)$$

$$\Rightarrow f(x) = x(1) f(1,0) + x(2) f(0,1)$$

$$= k_1 x_1 + k_2 x_2, \quad \forall (x(1), x(2)) \in X$$

where $k_1 = f(1,0)$, $k_2 = f(0,1) \in K$.

Now for $x = (x(1), 0) \in Y$, we have

$$f(x(1),0) = k_1 x(1) + k_2 \cdot 0$$

$$= k_1 x(1)$$

$$= g(x(1),0)$$

$$= x(1)$$

Thus $k_1 x(1) = x(1)$

$$\Rightarrow k_1 = 1.$$

[$f \in (\mathbb{R}^2)'$

$f(x(1),x(2)) = k_1 x(1) + k_2 x(2)$

$\because f|_Y = g$

$\Rightarrow f(x(1),0) = g(x(1),0)$

$$\begin{array}{c} f(x(1),0) \\ \parallel \\ k_1 x(1) + k_2 \cdot 0 \end{array} = x(1)$$

$$\Rightarrow k_1 x(1) = x(1)$$

$$\Rightarrow k_1 = 1.$$

$\therefore f(x) = 1 \cdot x(1) + k_2 x(2),$

$\forall\ (x(1), x(2)) \in X.$

Now

$$|f(x)| = |x(1) + k_2 x(2)|$$

$$\leq \max\{1, |k_2|\} \left[|x(1)| + |x(2)|\right]$$

$$= \max\{1, |k_2|\}\ \|x\|_1$$

$$\Rightarrow \|f\| \leq \max\{1, |k_2|\}$$

$$\therefore \|f\| = 1, \quad \text{if } |k_2| \leq 1.$$

$\therefore$ The Hahn–Banach extension is

$$f(x(1), x(2)) = x(1) + k_2 x(2)$$

with $|k_2| \leq 1$.

Ex: $Y = \{(x(1), x(2)) \mid x(1) = x(2)\}$

$= \text{Span}\{ b = (\tfrac{1}{2}, \tfrac{1}{2}) \}$.

Define $h : Y \to K$ by

$$h(x(1), x(2)) = 2x(1)$$

Then $\|h\| = 1$, $h(b) = 1$

$\Rightarrow h \in Y'$

$\therefore$ By Hahn-Banach theorem, there exists $f \in X'$ $\ni$

$$f|_Y = h, \quad \|f\| = \|h\| = 1$$

$$h(b) = 1 = f(b).$$

$\because$ $f(x(1), x(2)) = k_1 x(1) + k_2 x(2)$,

$k_1, k_2 \in K$

$$|f(x(1), x(2))| = \left| \sum_{i=1}^{2} k_i x(i) \right|$$

$$\leq \max\{|k_1|, |k_2|\} \|x\|,$$

$$\Rightarrow |f(x)| \leq \max\{|k_1|, |k_2|\} \|x\|,$$

Now for any $(x(1), x(2)) \in Y$,

$$f(x(1), x(2)) = h(x(1), x(2))$$

$$\Rightarrow k_1 x(1) + k_2 x(2) = 2x(1)$$

$$\Rightarrow (k_1 + k_2)\, x(1) = 2x(1) \quad [\because x(1) = x(2) \text{ in } Y]$$

$$\left.\begin{array}{l} \Rightarrow k_1 + k_2 = 2. \\ \text{and } \max\{|k_1|, |k_2|\} = 1 \end{array}\right\}$$

$$\Rightarrow k_1 = k_2 = 1.$$

$\therefore$ f is given by

$$f(x(1), x(2)) = x(1) + x(2)$$

Unique Hahn-Banach extensions

Theorem: Let X be a n.l.s. For every subspace Y of X and every $g \in Y'$, there is unique Hahn-Banach extension of g to X iff X' is strictly convex, i.e., for $f_1 \neq f_2$ in X', with $\|f_1\| = \|f_2\| = 1$, we have

$$\|f_1 + f_2\| < 2.$$

Proof: See Limaye book.

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