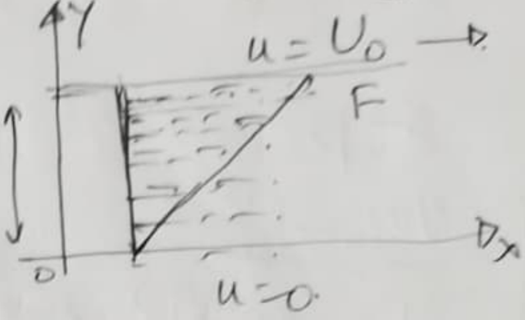
General theory of stresses and strain.

20.10.21

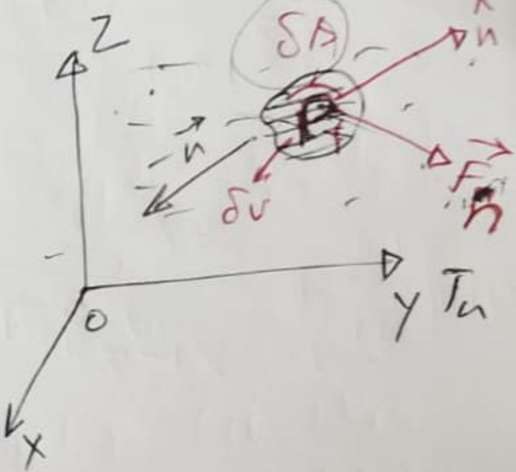
Newton's law of viscosity:



$$\tau = \mu \frac{du}{dy}$$

μ is a constant called co-efficient of viscosity and τ is the shear stress/tangential stress.

§ Stress at a point,



Consider a small area δs (within the given fluid ~~at~~ regions S of volume V) and it is at the point P in the fluid region. Let (x, y, z) be the co-ordinates of P referred to OX, OY and OZ as the co-ordinate axes. Let $\hat{n}$ be the unit outward drawn normal to δs on its RHS. Let the area δs be acted upon by the external forces at various pts. in various direction. Let δF be the Combined external forces acting at P and let δC be the couple about some axis. Let $\delta s \to 0$ in such a way as to always include or shrinks to the point P and δC

appears to be zero.

We know that if the fluid is inviscid fluid. then δF will act only along the direction of $\hat{n}$ so there is only normal Stress. On the other hand we have viscous fluid so together with normal Stress we also have tangential or shear Stress. The normal Stress and the shear stress are given by:

$$\text{The normal Stress} = \lim_{\delta s \to 0} \frac{\delta F_{nn}}{\delta S}$$

$$\text{" Shear "} = \lim_{\delta s \to 0} \frac{\delta F_{ns}}{\delta s}.$$

Clearly, $\frac{\delta F}{\delta S}$ is a definite number as $\delta s \to 0$. This number will depend not only on the point P but also on the orientation of the δs. It makes sense to put $\vec{F_n}$ instead $\vec{F}$. Therefore, this $\vec{F_n}$ is called the Stress vector or Stress traction at P corresponding to the direction $\hat{n}$ of the area δs.

$$\vec{T}_{(n)} = \vec{F}_{(n)} = \lim_{\delta s \to 0} \frac{\delta \vec{F}}{\delta S}.$$

Let σ_{nx}, σ_{ny} and σ_{nz} be the Cartesian components of F along $\hat{i}$, $\hat{j}$ and $\hat{k}$ direction.

$$\vec{T}_{(n)} = \sigma_{nx} i + \sigma_{ny} j + \sigma_{nz} \hat{k}.$$

Let $\hat{n}$ be parallel to x-axis, then,

$$\vec{T}_{(x)} = \sigma_{xx} \hat{i} + \sigma_{xy} \hat{j} + \sigma_{xz} \hat{k}$$

Similarly, $\vec{T}_{(y)} = \sigma_{yx} \hat{i} + \sigma_{yy} \hat{j} + \sigma_{yz} \hat{k}$

$$\vec{T}_{(z)} = \sigma_{zx} \hat{i} + \sigma_{zy} \hat{j} + \sigma_{zz} \hat{k}.$$

Therefore, $\vec{T}^{i}_{(n)} = \sigma \cdot \hat{n} = \sum_{j=1}^{3} \sigma_{ij} n_j$, $i=1,2,3$; $j=1,2,3$.

where

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \qquad 3^2 = 9$$

is called Cauchy stress tensor (of rank 2).

$$\vec{T}_{(n)} = \sigma \cdot n \Rightarrow T^{i}_{(n)} = \sum_{j=1}^{3} \sigma_{ij} n_j$$

$$\sigma = (\sigma_{ij})_{\substack{1 \le i \le 3 \\ 1 \le j \le 3}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$ ✓

Stress at a pt.: The stress at point is completely known if the nine components of the stress tensor at the pt. are known.

~~Wh~~ Here σ_{11}, σ_{22} and σ_{33} are called principal or normal stresses. and σ_{21}, σ_{12}, σ_{23}, σ_{32}, σ_{13} and σ_{31} are called shearing stresses.

$$\underline{\underline{T}}^{(n)} = \sigma \cdot \underline{\underline{\vec{n}}}$$

Ex: The stress tensor at a pt. P is given by

$$\sigma = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

Find the stress vector on the plane at a pt. P whose outward unit normal is $\hat{n} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

Solⁿ: By Formula, $\vec{T}_n = (\sigma_{nx}, \sigma_{ny}, \sigma_{nz}) = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$

$\Rightarrow \vec{T}(n) = \left(4, -\frac{10}{3}, 0\right)$

§ Levi-Civita Symbol: In 2D, the LC Symbol is given by.

$$\varepsilon_{ij} = 1 \quad \text{if } (i,j) = (1,2).$$

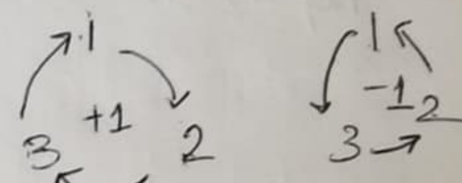
$1 \underset{-1}{\overset{+1}{\rightleftarrows}} 2$

$$= -1 \quad \text{if } (i,j) = (2,1)$$

$$= 0 \quad \text{if } i = j$$

In 3D, LC Symbol ~~by~~ is given by

$$\varepsilon_{ijk} = 1 \quad \text{if } (i,j,k) \text{ is } (1,2,3),\ (2,3,1) \text{ or } (3,1,2)$$



$$= -1 \quad \text{if } (i,j,k) \text{ is } (2,1,3),\ (1,3,2) \text{ or } (3,2,1).$$

$$= 0. \quad \text{if } i = j \text{ or, } j = k \text{ or } k = i$$

In n-dimension.

$$\varepsilon_{i_1 i_2 i_3 \dots i_n} = (-1)^p \varepsilon_{123\dots n},$$

where p is called the parity of permutation. which is the pairwise interchanges of indices to unscramble $i_1 i_2 \dots i_n$ into the order $1, 2, \dots n$.

* Cauchy postulate: "The stress vector ~~$\vec{T}_{(n)}$~~ $\vec{T}_{(n)}$ remains unchanged for all surfaces passing through the point P and having the same normal $\hat{n}$ at the point P; i.e., the stress vector $\vec{T}_{(n)}$ ~~is~~ depends only on the normal $\hat{n}$, i.e., $\vec{T}_{(n)}$ is a function of $\hat{n}$ only."

* Cauchy's fundamental lemma/Cauchy Reciprocal Theorem:

"The stress vector $\vec{T}_{(n)}$ acting on opposite sides of the same surface are equal in magnitude and opposite in direction. Cauchy's fundamental lemma is equivalent to Newton's third law which means to every action there is an equal and opposite reaction.

$$\vec{T}_{(-\hat{n})} = -\vec{T}_{(n)} \quad \text{i.e., } |\vec{T}_{(-n)}| = |\vec{T}_{(n)}|$$

(Infinitesimal stress tensor: $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^t)$, where $u: \Omega \to \mathbb{R}$ is the ~~deformation~~ displacement).

Gurtin, Continuum Mechanics

§ Derivation of equilibrium equⁿ: Consider a continuum body occupying a volume V, having surface area S; with the stress vector / surface traction as $\left(\vec{T}_i^{(n)}\right)$ per unit area acting on every point of the body surface. Let $\vec{F}_i$ be the body forces per unit volume on every point within the volume V. If the body is in equilibrium the resultant force acting on the volume must be zero, i.e.,

$$\underbrace{\int_S T_i^{(n)}\, ds}_{\text{ith comp. of total surface force}} + \underbrace{\int_V F_i\, dv}_{\text{i-th comp. of total body force}} = 0.$$

$T_i^{(n)} = \sigma \cdot n$

$\Rightarrow T_i^{(n)} = \sigma_{ij} n_j$

$= f_{,i}$

$= f_{,n}$

$$\Rightarrow \int_S \sigma_{ij} n_j\, ds + \int_V F_i\, dv = 0.$$

$$\Rightarrow \int_V \sigma_{ij,j}\, dv + \int_V F_i\, dv = 0.$$

$$\Rightarrow \int_V (\sigma_{ij,j} + F_i)\, dv = 0$$

V arbitrary. $\quad \sigma_{ij,j} + F_i = 0$:

This is the required Cauchy's equilibrium equ^n.

$$\left[\ \vec{T}_n = \sigma \cdot n = \underset{3\times 3}{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}} = \begin{bmatrix} \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\ \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{bmatrix}\right.$$

$$= \begin{bmatrix} \sum_{j=1}^{3} \sigma_{1j} n_j \\ \sum_{j=1}^{3} \sigma_{2j} n_j \\ \sum_{j=1}^{3} \sigma_{3j} n_j \end{bmatrix} \Rightarrow T^1_{(n)} = \sum \sigma_{1j} n_j = \sigma_{1j} n_j$$

$$T^2_{(n)} = \sigma_{2j} n_j$$

$$\left. T^i_{(n)} = \sigma_{ij} n_j \ \right]$$

Summation Convention
Repeated index.

(Derivative of tensors.)

$$S_n = \sum_{i=1}^{n} S_i = S_i$$

§ Principal Stresses: The formula to calculate principal stresses is

$$|\sigma_{ij} - \lambda \delta_{ij}| = 0$$

$$\Rightarrow \det(\sigma_{ij} - \lambda \delta_{ij}) = 0$$

$$\Rightarrow \begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda = \lambda_1, \lambda_2, \lambda_3 \rightarrow$ principal stresses.

$\sigma_1 = \max(\lambda_1, \lambda_2, \lambda_3) \rightarrow$ first principal stress

$\sigma_3 = \min(\lambda_1, \lambda_2, \lambda_3)$

$$\sigma_2 = I_1 - \sigma_1 - \sigma_3 = (\sigma_{11} + \sigma_{22} + \sigma_{33}) - (\sigma_1 - \sigma_3)$$

$= \checkmark$

Ex 1: Find the principal stresses of

$$\sigma = \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Solⁿ: $\lambda = \cancel{9,3,3} \cdot 3, 8, 9.$

$$\sigma_1 = \max.(3,8,9) = 9$$

$$\sigma_3 = \min(3,8,9) = 3$$

$$\sigma_2 = I - \sigma_1 - \sigma_3 = 8$$

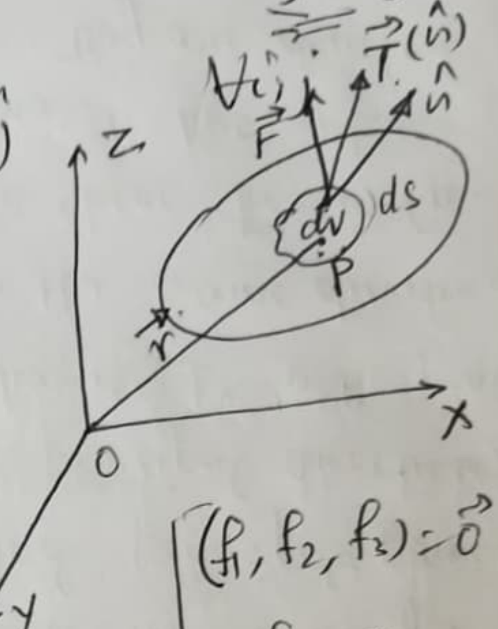
$$\sigma_1 = 9, \ \sigma_2 = 8, \ \sigma_3 = 3$$

22.10.21

§ Principal stresses: At every point in a stressed body, there are at least three planes, called principal planes, with normal vector $\hat{n}$, called the principal direction, where the corresponding stress vector is $\perp^r$ to the plane i.e., $\parallel$ to the normal $\hat{n}$ or in the same direction as in $\hat{n}$ and where there is no shear stresses τ_n. The three stress components normal to these planes are called principal stresses. They are denoted by σ_{nn} or σ_n, where $n = 1, 2, 3$ or, x, y, z.

§ The stress tensor σ is Symmetric. $\sigma_{ij} = \sigma_{ji}$ $\forall i,j$

Sol^n: we Know that (by equilibrium condition)



$$T_i^{(n)}\, ds + F_i\, dv = 0$$

$$\Rightarrow \vec{T}^{(n)}\, ds + \vec{F}\, dv = \vec{0}$$

$(f_1, f_2, f_3) = \vec{0}$

$f_1 = 0$

$f_2 = 0$

$f_3 = 0$

$$\Rightarrow \vec{r} \times \vec{T}^{(n)}\, ds + \vec{r} \times \vec{F}\, dv = \vec{r} \times \vec{0} = \vec{0}$$

$$\Rightarrow \vec{r} \times \underbrace{(\vec{T}^{(n)}\, ds + \vec{F}\, dv)}_{\text{moment of the total force / resultant torque}} = \vec{0}$$

The resultant moment of the force equals to zero implies that

$$\int_S \vec{r} \times \vec{T}^{(n)}\, ds + \int_V \vec{r} \times \vec{F}\, dv = \vec{0} \qquad -(1)$$

we Know $\vec{r} = x_j \hat{e}_j$, $\vec{F} = F_j \hat{e}_j$,

$$\vec{T}^{(n)} = \sigma \cdot n = \sigma_{mk}\, n_m$$

Then from ①.

$$\int_S \varepsilon_{ijk}\, x_j\, \sigma_{mk}\, n_m\, ds + \int_V \varepsilon_{ijk}\, x_j\, F_k\, dv = 0.$$

[Side calculation: $\int_V \vec{r} \times \vec{F}\, dv$

$$x_{j,m} = \frac{\partial x_j}{\partial x_m} = \delta_{jm}$$

$$= \int_V \{\hat{i}\,(x_2 F_3 - x_3 F_2) + \hat{j}\,(x_3 F_1 - x_1 F_3) + \hat{k}\,(x_1 F_2 - x_2 F_1)\}\, dv.$$

$$= \int_V [\hat{i}\,\{\varepsilon_{123}\, x_2 F_3 + \varepsilon_{132}\, x_3 F_2\} + \cdots$$

$$= \int_V \sum_{i,j,k=1}^{3} [\{\varepsilon_{ijk}\, x_j F_k\}\, \hat{e}_k\,]$$

By Gauss Div. Th^m,

$$\int_V (\varepsilon_{ijk}\, x_j\, \sigma_{mk})_{,m}\, dv + \int_V \varepsilon_{ijk}\, x_j\, F_k\, dv = 0$$

$$\Rightarrow \int_V \varepsilon_{ijk}\,(x_{j,m}\sigma_{mk} + x_j\, \sigma_{mk,m})\, dv + \text{''} = 0$$

$$\Rightarrow \int_V \varepsilon_{ijk}\,(\delta_{jm}\sigma_{mk} + x_j\, \sigma_{mk,m})\, dv + \int_V \cdots = 0$$

$$\Rightarrow \int_V \varepsilon_{ijk}\, \delta_{jm}\, \sigma_{mk}\, dv + \int_V \varepsilon_{ijk}\, x_j\, \underbrace{(\sigma_{mk,m} + F_k)}_{=0}\, dv = 0.$$

(By Cauchy's law for equilibrium eqn. $\sigma_{mk,m} + F_k = 0$

$$\Rightarrow \int_V \varepsilon_{ijk}\, \delta_{jm}\, \sigma_{mk}\, dv = 0.$$

arbitrary, which implies.

$$\varepsilon_{ijk}\, \delta_{jm}\, \sigma_{mk} = 0 \quad \forall\, v$$

$i, j, k, m = 1, 2, 3$

~~Let i=1~~, $\varepsilon_{ijk}\, \delta_{jm}\, \sigma_{mk} = 0$

$\cancel{j=2 \quad \varepsilon_{i2k}\, \delta_{2m}\, \sigma_{mk} = 0.}$ $\Rightarrow \varepsilon_{ijk}\, \delta_{jj}\, \sigma_{jk} = 0$

$\cancel{m=2 \Rightarrow \varepsilon_{12k}\, \delta_{22}\, \sigma_{2k} = 0}$ $\Rightarrow \varepsilon_{ijk}\, \sigma_{jk} = 0$

$\cancel{\Rightarrow \varepsilon_{12k}\, \sigma_{2k} = 0}$ $\Rightarrow \sigma_{23} - \sigma_{32} = 0$

$\Rightarrow \sigma_{23} = \sigma_{32}$

Similarly, $\sigma_{13} = \sigma_{31}$ and $\sigma_{12} = \sigma_{21}$ /

Ex 1: The Stress matrix σ at a point P is given by

$$\sigma = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 1 & 2 \\ -3 & 2 & 1 \end{pmatrix}.$$

Find the stress vector on the plane passing through P and ∥ to the plane whose unit normal is $\left(\frac{3}{7}, \frac{6}{7}, \frac{2}{7}\right)$. Also determine principal stresses. Verify if σ is symmetric.

Solⁿ: $\vec{T}^{(n)} = \left(\frac{6}{7}, \frac{13}{7}, \frac{5}{7}\right)$, σ is symmetric ✓

$\sigma_1 = 4.63897$, $\sigma_2 =$ ✓ , $\sigma_3 =$ ✓.

§ Navier-Stokes Equn: In general the NS equn. is given by

$$\frac{d\vec{q}}{dt} = \vec{F} - \vec{\nabla}\int \frac{dp}{\rho} + \frac{1}{3}\nu\,\vec{\nabla}(\vec{\nabla}\cdot\vec{q})$$

$$+ \nu\,\nabla^2\vec{q},$$

where $\vec{q}$ is the fluid velocity, ρ is the density and $\nu = \frac{\mu}{\rho}$, μ is the co-efficient of viscosity.

§ Stress in a fluid at rest:

$$\sigma = \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} = \mathrm{diag}(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$$

$$\sigma_{ij} = 0 \quad \forall\, i \neq j$$ ✓

Fluid at rest are normally in a state of compression and it is therefore convenient to write the stress tensor σ in a fluid at rest as: $\sigma_n = -p\hat{n}$ for all directions of $\hat{n}$. The corresponding stress tensor will then by

$$\sigma = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

where the parameter p (+ve number) is the static fluid pressure and it may be a function of position.

§ Stress on a fluid in motion: