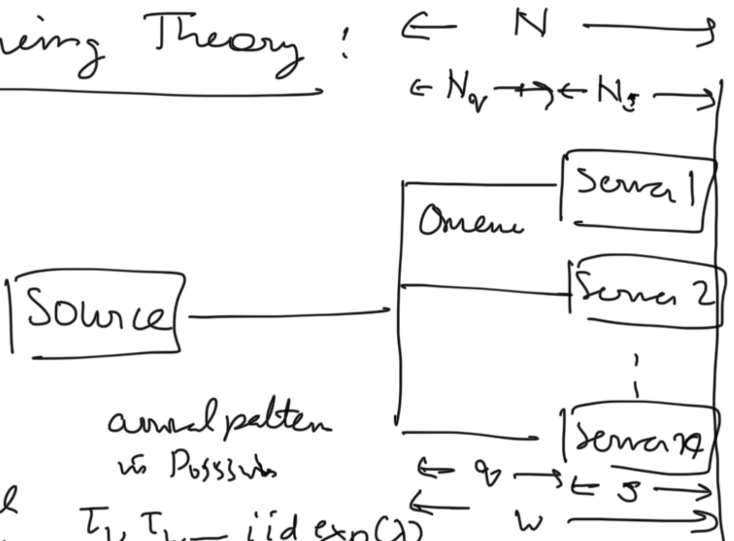
Queueing Theory :



service time dist.

$S \sim \exp(\mu)$

$E(S) = \frac{1}{\mu} = W_s$

T_i: remaining service time for server i, $i = 1, \ldots, n$

$T_i \sim \exp(\mu)$

n identical servers, let n servers are busy

$T = \min(T_1, \ldots, T_n) \sim \exp(n\mu)$

time until the next service completion

arrival pattern is Poisson

interarrival times $T_1, T_2, \ldots$ iid $\exp(\lambda)$

r.v. $N(t)$ = # of customers in the queueing system at time t

$N_q(t)$ = " " " " queue in the " " ... t.

$N_s(t)$ = " " " receiving service " ... t.

$$N(t) = N_q(t) + N_s(t)$$

Steady state

r.v. N = steady state # of customers in the queueing system

N_q = " " " " " " in queue ...

N_s = " " " " receiving service ...

$$N = N_q + N_s$$

$$E(N) = E(N_q) + E(N_s)$$

$$L = L_q + L_s$$

Relationship btw time

r.v. w = total time a customer spends in queueing system

q = time spent in line waiting for service

s = service time

$$W = q + S$$

$$E(W) = E(q) + E(S)$$

$$W = W_q + W_s$$

Notation

$A / B / c / K$

c → # of servers

K → system capacity

A → interarrival time dist

B → service time dist

$$A, B = \begin{cases} GI & \text{general indep interarrival times} \\ G & \text{general service times} \\ M & \text{expo. interarrival or service time dist} \\ U & \text{uniform " " " " "} \\ D & \text{deterministic " " " " "} \end{cases}$$

eg $M/M/1$, $M/M/c$, $M/M/1/K$, $M/M/c/c$, $M/M/\infty$.

λ_a → av. arrival rate of entering customers

$N(t)$ → # of customer arrival by time t

$$\lambda_a = \lim_{t \to \infty} \frac{N(t)}{t} = \text{rate of renewal process.}$$

<u>Basic cost identity:</u>

Imagine that the entering customers are forced to pay money (according to some rule) to the system.

$$\begin{bmatrix} \text{av. rate at which} \\ \text{the system earns} \end{bmatrix} = \lambda_a \times \begin{bmatrix} \text{av. amt an entering} \\ \text{customers pays} \end{bmatrix}$$

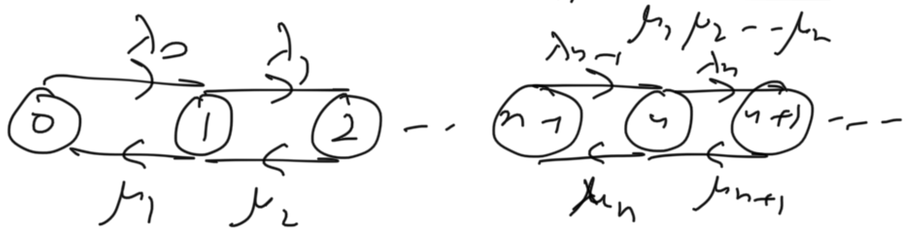
<u>Little's law</u> :

$$L = \lambda_a W$$
$$L_q = \lambda_a W_q$$
$$L_s = \lambda_a W_s$$

<u>B & D process</u> :

$N(t) = n \qquad \lambda_n \qquad \mu_n$

$S = 1 + C_1 + C_2 + \cdots < \infty \qquad C_n = \dfrac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$



$$P_0 = P(N=0) = \frac{1}{S}$$

$$P_n = P(N=n) = C_n p_0, \quad n = 1, 2, 3, \ldots$$

<u>M/M/1</u> : <u>random (Poisson) arrival pattern</u>

" (expo) service time dist

P(arrival in a time interval of length $h > 0$)

$$= e^{-\lambda h} \lambda h = \lambda h \left(1 - \lambda h + \frac{(\lambda h)^2}{2!} - \cdots\right) = \lambda h + o(h)$$

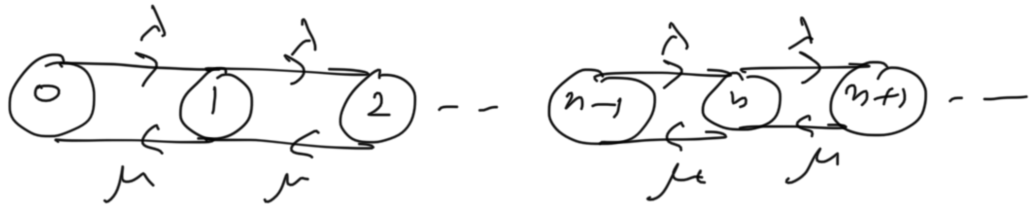
$$\therefore \lambda_n = \lambda, \quad n = 0, 1, 2, \ldots$$

$$P(\text{service completion}) = P(T \le t+h \mid T > t) = P(T \le h)$$

$$= 1 - e^{-\mu h} = \mu h + o(h)$$

$$\mu_n = \mu, \quad n = 1, 2, \ldots$$

State transition diagram for M/M/1



let $\frac{\lambda}{\mu} = \rho$; $C_n = \left(\frac{\lambda}{\mu}\right)^n = \rho^n$

↳ arrival/service ratio

$$S = 1 + C_1 + C_2 + \cdots = 1 + \rho + \rho^2 + \cdots = \frac{1}{1-\rho}$$

$$P_0 = \frac{1}{S} = 1 - \rho$$

If we assume $\rho < 1$, so steady state sol^n does exist.

$$P_n = P(N=n) = C_n P_0 = \underline{\underline{\rho^n(1-\rho)}}, \quad n = 0, 1, 2, \ldots$$

$$N \sim Geo(1-\rho)$$

$$L = E(N) = \frac{\rho}{1-\rho}, \quad \sigma_N^2 = \frac{\rho}{(1-\rho)^2} \quad ; \underline{\rho = \frac{\lambda}{\mu} = \lambda W_s}$$

$$L = \lambda W \Rightarrow W = \frac{L}{\lambda} = \frac{\rho}{\lambda(1-\rho)} = \frac{W_s}{1-\rho} \quad \Big| \quad \lambda_a = \lambda \times 1 = \lambda$$

$$W_s = \frac{1}{\mu}$$

$$W_q = W - W_s = \left(\frac{1}{1-\rho} - 1\right) W_s = \frac{\rho}{1-\rho} W_s$$

$$L = L_q + L_s$$

$$L_q = \lambda W_q = \frac{\lambda \rho W_s}{1-\rho} = \frac{\rho^2}{1-\rho}$$

Performance of queueing system

W, W_q, L, L_q as well as P_n

$$P(\text{server busy}) = 1 - P(N=0) = 1 - (1-\rho) = \rho = \underline{\lambda W_s}$$

→ fraction of time server is busy / server

... / server utilization.

Example: Suppose that customers arrive at a Poisson rate of one per every 12 min, and that the service time is exp. at a rate of one service per 8 min, parameters M/M/1.

Sol
$$\lambda = \frac{1}{12}, \quad \mu = \frac{1}{8}$$

$$\rho = \frac{\lambda}{\mu} = \frac{8}{12} = \frac{2}{3} < 1 \qquad \lambda_a = \lambda$$

$$L = \frac{\rho}{1-\rho} = \frac{2/3}{1/3} = 2 \qquad W = \frac{L}{\lambda} = 2 \times 12 = 24$$

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(2/3)^2}{1/3} = \frac{4}{3} \qquad W_q = \frac{L_q}{\lambda} = \frac{4}{3} \times 12 = 16$$

$$L_s = 2 - \frac{4}{3} = \frac{2}{3} \qquad W_s = W - W_q = 8$$

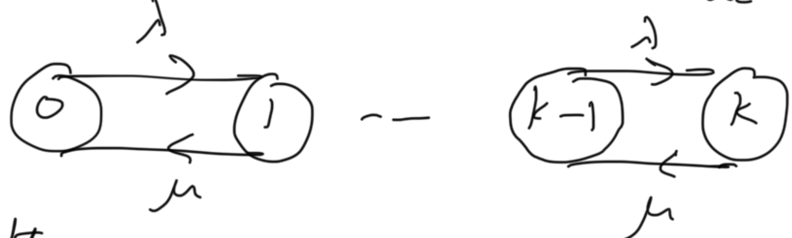
$$N \sim Geo\left(\frac{1}{3}\right)$$

—x—

M/M/1/K

1 service, $k-1$ queue

if k customers system, arriving customers are turned away.



arrival/births rate $\lambda_n = \begin{cases} \lambda & , n = 0, 1, \ldots k-1 \end{cases}$

departure/death/service rate $\mu_n = \begin{cases} \mu, & n = 1, 2, \ldots k \\ 0, & n = k+1, \ldots \end{cases}$

$W_s = \frac{1}{\mu}$

let $a = \frac{\lambda}{\mu} = \lambda W_s$

$$P_n = C_n P_0 = \left(\frac{\lambda}{\mu}\right)^n P_0 = a^n P_0 \quad , n = 0, 1, \ldots, k$$

$$1 = P_0 + P_1 + \cdots + P_k = \underbrace{P_0 \sum_{n=0}^{k} a^n} = \begin{cases} \frac{1-a^{k+1}}{1-a} P_0 & , a \neq 1 \\ (k+1) P_0 & , a = 1 \end{cases}$$

$$P_0 = \begin{cases} \frac{1-a}{1-a^{k+1}} & , a \neq 1 \\ \frac{1}{k+1} & , a = 1 \end{cases}$$

$\underline{a = 1 \Leftrightarrow \lambda = \mu}$; $P_n = \frac{1}{k+1}$, $n = 0, 1, \ldots k$

$\underline{a \neq 1} \Leftrightarrow \lambda \neq \mu$;

$$P_n = \frac{(1-a)a^n}{1-a^{k+1}} \quad , n = 0, 1, \ldots, k$$

$\left[k \to \infty \right.$, $\underline{\lambda < \mu \Leftrightarrow a < 1}$; $P_n = a^n(1-a)$, $\left. M/M/1 \right]$

$a \neq 1$

$$L = \sum_{n=0,1}^{k} n P_n = \frac{1-a}{1-a^{k+1}} \sum_{n=1}^{k} n a^n = \frac{(1-a)a}{1-a^{k+1}} \sum_{n=1}^{k} n a^{n-1}$$

$$= \frac{(1-a)a}{1-a^{k+1}} \frac{d}{da} \sum_{n=1}^{k} a^n$$

$\frac{d}{da} a^n$

$$= \frac{(1-a)a}{1-a^{k+1}} \frac{d}{da} \left(\frac{1-a^{k+1}}{1-a} \right)$$

$$= \frac{(1-a)a}{1-a^{k+1}} \left[\frac{-(1-a)(k+1)a^k + (1-a^{k+1})}{(1-a)^2} \right]$$

$$= \frac{a}{1-a} - \frac{(k+1)\,a^{k+1}}{1-a^{k+1}} \; ; \; a \neq 1$$

$$\xrightarrow{a=1} \; ; \; L = \sum_{n=0,1}^{k} n P_n = \frac{1+2+\cdots+k}{k+1} = \frac{k}{2} \; ; \; a=1$$

$$L_s = E(N_s) = \underbrace{P(N=0)}\,\underbrace{E(N_s \mid N=0)} + P(N>0) \cdot \underbrace{E(N_s \mid N>0)}$$

$$= P_0 \times 0 + (1-P_0) \times 1 = 1-P_0$$

$\because \underline{M/M/1/k}$

$$L_q = L - L_s = L - (1-P_0)$$

$$\because E(N_s \mid N>0) = 1 \cdot \underbrace{P(N_s=1 \mid N>0)}_{1} + \sum_{i \neq 1} i \, \underbrace{P(N_s=i \mid N>0)}_{0} = 1$$

$$P_{blocking} = P_K$$

$$\lambda_a = \lambda \times (1 - P_{blocking})$$

$$= \lambda \times (1-P_k)$$

$$W = \frac{L}{\lambda_a} \; , \; W_q = \frac{L_q}{\lambda_a} \; ; \; W_s = \frac{1}{\mu}$$

time server utilization

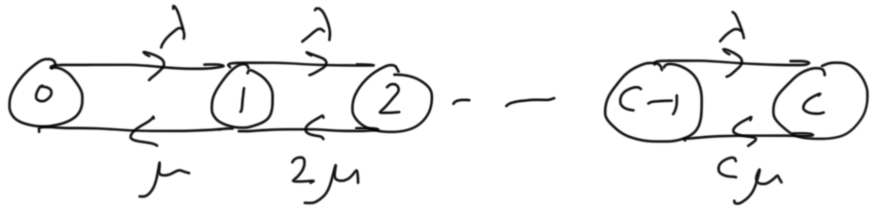
$$\rho = \lambda_a W_s = \lambda (1-P_k) W_s = (1-P_k) a .$$

— X —

<u>M/M/c/c</u> (or M/M/c loss system)

[illegible]

If all servers are busy, customers will not allowed to wait for service and thus are lost to the system



$$C_n = \frac{a^n}{n!}, \quad n = 1, 2, \ldots, c \qquad , \quad a = \frac{\lambda}{\mu} = \lambda W_s$$

$$S = \frac{1}{P_0} = 1 + a + \frac{a^2}{2!} + \cdots + \frac{a^c}{c!} \quad \Bigg| \quad W_s = \frac{1}{\mu}$$

$$P_n = C_n P_0 = \frac{a^n/n!}{1 + a + \frac{a^2}{2!} + \cdots + \frac{a^c}{c!}}, \quad n = 0, 1, \ldots, c$$

$$P_{blocking} = P_c$$

$$\lambda_a = \lambda \times (1 - P_c)$$

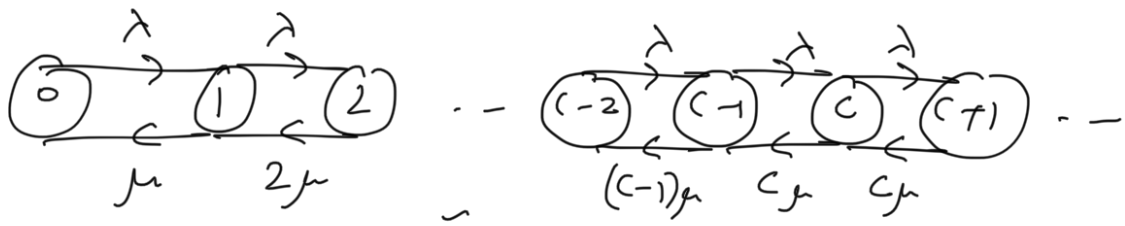
Since no customers are allowed to wait for service

$$L_q = 0 \qquad ; \quad W_q = 0$$

$$L = E(N) = \sum_{n=\cancel{0}\,1}^{c} n P_n = P_0 \sum_{n=1}^{c} \frac{n\, a^n}{n!}$$

— x —

M/M/c



$$\lambda_n = \lambda, \quad n = 0, 1, 2, \ldots$$

$$\mu_n = \{ n\mu, \quad n = 1, 2, \ldots$$

$$\mu_n = \begin{cases} n\mu, & n = 1,2,\ldots,c \\ c\mu, & n = c+1, c+2, \ldots \end{cases}$$

let $a = \frac{\lambda}{\mu}$

$\rho = \frac{a}{c}$

$$C_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} = \begin{cases} \frac{a^n}{n!}, & n = 1,2,\ldots,c \\ \frac{a^c \rho^{n-c}}{c!} & n = c+1, c+2, \ldots \end{cases}$$

$\rho < 1$ steady state sol^n exist (see CTMC notes)

$$S = \frac{1}{P_0} = 1 + C_1 + C_2 + \cdots$$

$$= 1 + a + \frac{a^2}{2!} + \cdots + \frac{a^{c-1}}{(c-1)!} + \frac{a^c}{c!}(1 + \rho + \rho^2 + \cdots)$$

$$= \sum_{n=0}^{c-1} \frac{a^n}{n!} + \frac{a^c}{c!(1-\rho)}$$

$$\Rightarrow P_0 = \frac{1}{S}$$

$$P_n = C_n P_0$$

M/M/c

$$L_q = \sum_{n=c}^{\infty} (n-c) P_n = P_0 \frac{a^c}{c!} \sum_{n=c}^{\infty} (n-c) \rho^{n-c}$$

$$= \frac{P_0 a^c}{c!} \sum_{k=0}^{\infty} k \rho^k = P_0 \frac{a^c}{c!} \rho \sum_{k=0}^{\infty} k\rho^{k-1}$$

$$\frac{d}{d\rho} \rho^k$$

$$= P_0 \frac{a^c}{c!} \rho \frac{d}{d\rho} \left(\frac{1}{1-\rho} \right)$$

$$= \frac{P_0 a^c \rho}{c! (1-\rho)^2}$$

$$\lambda_e = \lambda \times 1$$

$W_s = \frac{1}{\mu}$

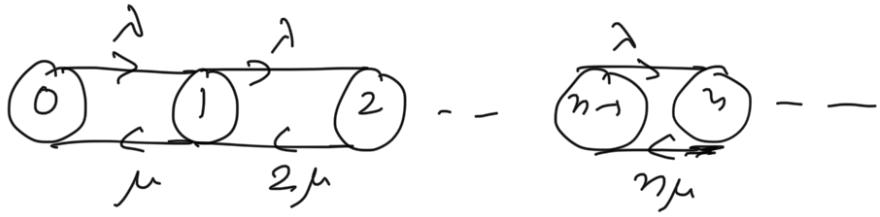
$$W_q = \frac{L_q}{\lambda_a} = \frac{L_q}{\lambda} \qquad \rho = \frac{a}{c} = \frac{\lambda}{\mu c} = \frac{\lambda}{c} W_s$$

$$W = W_q + W_s \quad , \quad L = \lambda W$$

—x—

M/M/∞

No real life queueing system can have an infinite # of servers, what is meant, here, is that a server is immediately provided for each arriving customer.



$$C_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} = \frac{a^n}{n!}, \; n = 1, 2, \ldots \quad \text{where } a = \frac{\lambda}{\mu}$$

$$S = \frac{1}{P_0} = 1 + C_1 + C_2 + \cdots = \sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$$

$$P(N=n) = P_n = C_n P_0 = \frac{e^{-a} a^n}{n!}, \; n = 0, 1, 2, \ldots$$

$$N \sim \text{Poiss}(a)$$

$$L = E(N) = a$$

$$\sigma_N^2 = a$$

av. # of busy servers

$$L_q = 0 \qquad L = L_s = a$$

$$W_q = 0 \qquad W_s = \frac{1}{\mu} = W$$

M/M/∞ model can be used to estimate the # of lines in use in a large communication network or as a gross estimate of values in an M/M/c or M/M/c/c queueing system for large values of c.

Example: Calls in a telephone system arrive randomly at an exchange at the Poisson rate of 140 per hr. If there is very large # of lines available to handle the calls, that last according to expo. dist an average of 3 min, what is the av. # of lines in use?

Sol

$$\lambda = 140 \text{ per hr} = \frac{140}{60} \text{ per min} \qquad \underline{M/M/\infty}$$

$$\mu = \frac{1}{3} \text{ per min}$$

$$a = \frac{\lambda}{\mu} = \frac{140}{60} \times 3 = 7$$

$$L = L_s = a = 7$$

av. # of lines in use is 7.

—x—