4th January 2022

IE: $y(x) = f(x) + \lambda \int_a^b k(x,t)\, y(t)\, dt$ To find $y(x)$

$f(x)$ → known; $y(t)$ → unknown; $k(x,t)$ → kernel

Classifications

Fredholm, Volterra, 1st kind, 2nd kind, homogenous, non-homogenous. linear, non-linear, singular, non-singular

Fredholm: a and b are constants

Volterra: at least 1 limit is function of x.

1st kind: if $y(x)$ appears under integral sign only

2nd kind: if $y(x)$ appears under as well as outside integral sign.

Homogenous: if $f(x) = 0$, here $y(x) = 0$ is trivial solution

non-homogenous: $f(x) \neq 0$

linear IE: of the form $y(x) = f(x) + \lambda \int_a^{\alpha(x)} k(x,t)\, y(t)\, dt$

note 1. power of $y = 1$

2. no product of $y(x)\, y(t)$ appears.

3. no transcendental fn of $y(x)$ appears.

$$y(x) - \lambda \int_a^{\alpha(x)} k(x,t)\, y(t)\, dt = f(x)$$

define $I[y(x)] = y(x) - \lambda \int_a^{\alpha(x)} k(x,t)\, y(t)\, dt$

$$I[y(x)] = f(x)$$

$$I[c_1 y_1(x) + c_2 y_2(x)] = c_1 I[y_1(x)] + c_2 I[y_2(x)]$$

Take $I[y(x)] = y(x) - \int_a^b k(x,t)\, y^2(t)\, dt$

$$y(x) = f(x) + \int_a^{\phi(x)} k(x,t)\, y(t)\, dt \quad \Big| \quad y(x) = f(x) + \lambda \int_a^b k(x,t)\, y(t)\, dt$$

if $k(x,t)$ has an infinite discontinuity in $[a,b]$ or if a or b become ∞, then IE is singular

Eg: $$y(x) = x + \int_0^1 \frac{y(t)\, dt}{(2t-1)^2} \qquad 0 \le x \le 1$$

$$y(x) = \sin x + \int_0^\infty e^{-(t+x)}\, y(t)\, dt$$

suppose

$k(x,t)$ becomes unbounded at a point in $[a,b]$ (range of integration)

strong

$$k(x,t) = \frac{m(x,t)}{(x-t)^\alpha}$$

$\alpha \geq 1$

$m(x,t)$ is a regular/cts. fn. defined on domain where $k(x,t)$ is defined

weak / integrable singularity

$$k(x,t) = \frac{N(x,t)}{(x-t)^\alpha} \qquad 0 < \alpha < 1$$

or $N(x,t) \log|x-t|$

where $N(x,t)$ is regular

$t \to 0 \leftrightarrow \infty \qquad v \leftrightarrow -1$ to 1

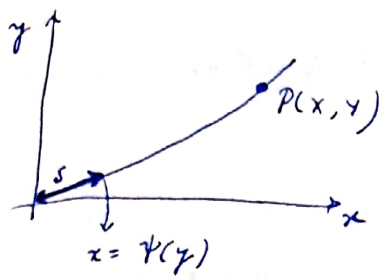
$$\int_0^\infty e^{-(x+t)} y(t)\, dt \to \int_{-1}^1 e^{-x} e^{-\left(\frac{v+1}{v-1}\right)} y\left(\frac{v+1}{v-1}\right) \times \frac{-2}{(v-1)^2}\, dv$$

Solution of an I.E.

Ex: verify whether $\phi(x) = 1-x$ is a soln. of $x = \int_0^x e^{x-t} \phi(t)\, dt$

Then, $$\int_0^x e^{x-t}(1-t)\, dt = \int_0^x e^x$$

Abel's Problem



suppose a particle is falling along the curve $x = \psi(y)$. It starts from rest at the point $P(X, Y)$. It slides down freely under gravity along the curve $x = \psi(y)$. It reaches the lowermost point 0.

using conservation of energy.

$$\frac{1}{2}mv^2 + mgy = 0 + mgY \Rightarrow v^2 = 2g(Y-y)$$

$$v = \frac{ds}{dt} = \pm\sqrt{2g(Y-y)} \Rightarrow \qquad v = \pm\sqrt{2g(Y-y)}$$

Observe $ds^2 = dx^2 + dy^2$

$$\Rightarrow \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$v^2 = 2g(Y-y)$

$$\Rightarrow v = \pm\sqrt{2g(Y-y)} \qquad \Rightarrow \frac{ds}{dt} = \pm\sqrt{2g(Y-y)}$$

but $\frac{ds}{dt} = -\sqrt{2g(Y-y)}$ (as s is ↓)

$x = \psi(y)$

$$\frac{ds}{dy} = \sqrt{1 + \{\psi'(y)\}^2}$$

Note $$\frac{dy}{dt} = \frac{\frac{ds}{dt}}{\frac{ds}{dy}} = \frac{-\sqrt{2g(Y-y)}}{\sqrt{1+(\psi'(y))^2}}$$

$$dt = -\frac{\sqrt{1+\{\psi'(y)\}^2}}{\sqrt{2g(Y-y)}}\,dy \quad \text{---}(*)$$

let the time of fall be T. Integrating, we get

$$\int_0^T dt = -\int_Y^0 \frac{\sqrt{1+\{\psi'(y)\}^2}}{\sqrt{2g(Y-y)}}\,dy$$

let $\phi(y) = \frac{1}{\sqrt{2g}}\sqrt{1+\{\psi'(y)\}^2}$

$$T = \int_0^Y \frac{\phi(y)}{(Y-y)^{1/2}}\,dy \quad \leftarrow \text{Abel's integral equation}$$

If T is known and we need to find shape of the curve $x = \psi(y)$, then it becomes a problem of integral equations

10th January 2022

§ Reduction of an IVP to a Volterra Integral Equation

Consider the ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y(x) = r(x) \quad \dots (1) \qquad a \le x \le b$$

Initial conditions $y(a) = c_0, \quad y'(a) = c_1$

let $$\frac{d^2y}{dx^2} = u(x) \quad \dots (2)$$

Result :

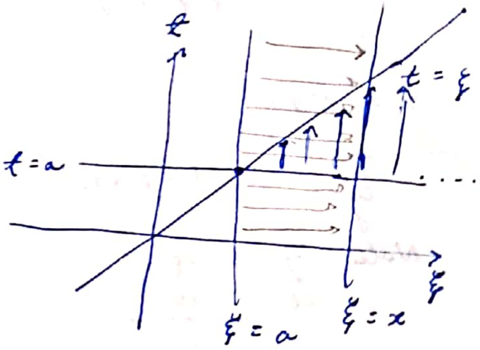
$$\int_a^x u(t)\,dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\,u(t)\,dt$$

let $n = 2$, then

$$\int_a^x u(t)\,dt^2 = \int_a^x \left(\int_a^x u(t)\,dt\right) dx$$

$$= \int_a^x \left(\int_a^\xi u(t)\,dt\right) d\xi$$

we want to change order of integration.



$$= \int_a^x \left(\int_t^x u(t)\,d\xi\right) dt$$

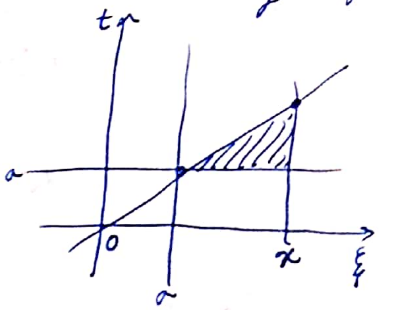
$$= \int_{t=a}^x \left(\int_t^x d\xi\right) u(t)\,dt$$

$$= \int_a^x (x-t)\,u(t)\,dt$$

$$\int_a^x \int_t^x d\xi\,dt$$

⇓

resulting region



$n = 3$

$$\int_a^x u(t)\,dt^3 = \int_a^x \left(\int_a^x u(t)\,dt^2\right) dt$$

$$= \int_a^x \left(\int_a^x (x-t)\,u(t)\,dt\right) dx = \int_a^x \left(\int_a^\xi (\xi - t)\,u(t)\,dt\right) d\xi$$

$$= \int_{t=a}^x \int_t^x (\xi - t)\,d\xi \Big)\, u(t)\,dt \qquad = \int_a^t \frac{(x-t)^2}{2}\,u(t)\,dt$$

Integrating eqn (2) wrt x between a and x

$$\int_a^x \frac{d^2y}{dx^2}\,dx = \int_a^x u(t)\,dt$$

or $$\frac{dy}{dx} - \frac{dy}{dx}\Big|_{x=a} = \int_a^x u(t)dt \quad \text{or} \quad y'(x) - c_1 = \int_a^x u(t)\,dt$$

or $$y'(x) = c_1 + \int_a^x u(t)dt \quad \text{--- (3)}$$

Integrating eqn (2) wrt x between a and x, we get

$$y(x) - y(a) = c_1 \int_a^x dx + \int_a^x u(t)\,dt^2 = c_1(x-a) + \int_a^x (x-t)\,u(t)\,dt$$

$$y(x) = c_0 + c_1(x-a) + \int_a^x (x-t)\,u(t)\,dt \quad \text{--- (4)}$$

$$u(x) + p(x)\left\{c_1 + \int_a^x u(t)dt\right\} + q(x)\left\{c_0 + c_1(x-a) + \int_a^x (x-t)\,u(t)\,dt\right\} = r(x)$$

or

$$u(x) = r(x) - c_1 p(x) - q(x)\{c_0 + c_1(x-a)\} - \int_a^x \{p(x) + q(x)(x-t)\}\,u(t)\,dt$$

This is in the form

$$u(x) = f(x) + \lambda \int_a^x k(x,t)\,u(t)\,dt$$

where $f(x) = r(x) - c_1 p(x) - q(x)\{c_0 + c_1(x-a)\}$

$\lambda = -1$, $k(x,t) = p(x) + q(x)(x-t)$

$\therefore$ The given IVP is equivalent to the VIE in (5)

Ques: Convert the IVP $y''' - 3y'' - 6y' + 5y = 0$, IC's $y(0), y'(0), y''(0) = 1$ to an equivalent volterra Integral Equation.

let $$\frac{d^3y}{dx^3} = u(x) \quad \text{— (1)}$$

Integrating 1. wrt x between 0 and x, we get

$$\frac{d^2y}{dx^2} - \frac{d^2y}{dx^2}\Big|_{x=0} = \int_0^x u(t)dt$$

or $$y''(x) - 1 = \int_0^x u(t)\,dt \quad \text{— (2)}$$

Integrating (2) wrt x between 0 and x

we get
$$\frac{dy}{dx} - y'(0) = \int_0^x 1\,dt + \int_0^x \left(\int_0^x u(t)\,dt \right) dx$$
$$= \int_0^x dt + \int_0^x u(t)\,dt^2$$
$$= \int_0^x dt + \int_0^x (x-t)\,u(t)\,dt$$
$$\therefore y'(x) = 1 + \int_0^x dt + \int_0^x (x-t)\,u(t)\,dt$$
$$\therefore y'(x) = 1 + x + \int_0^x (x-t)\,u(t)\,dt \quad \text{--- (3)}$$
Integrating (3), we get
$$y(x) - y(0) = \int_0^x (1+\xi)\,d\xi + \int_0^x u(t)\,dt^3$$
$$\Rightarrow y(x) = 1 + x + \frac{x^2}{2} + \int_0^x \frac{(x-t)^2}{2} u(t)\,dt \quad \text{--- (4)}$$
$$y(x) = f(x) + \lambda \int_0^x k(x,t)\,u(t)\,dt$$
$$f(x) = 4 + x - \frac{5}{2}x^2$$
$$k(x,t) = \frac{5}{2}(x-t)^2 - 6(x-t) - 3$$

Conversion of VIE to IVP

Leibnitz Rule (diff. under integral sign)

suppose $f(x,t)$ and the partial derivative $\frac{\partial f}{\partial t}(x,t)$ are continuous on rectangle $a \le x \le b$, $c \le t \le d$, then
$$\frac{d}{dt} \int_a^b f(x,t)\,dx = \int_a^b \frac{\partial}{\partial t} f(x,t)\,dx$$
now if $a = \alpha(t)$

$b = \beta(t)$ then the generalized Leibniz rule is
$$\frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x,t)}{\partial t}\,dx + \beta'(t)\, f(\beta(t), t) - \alpha'(t)\, f(\alpha(t), t)$$
provided $\alpha(t)$, $\beta(t)$ are continuous in $c \le t \le d$.

Example: Reduce the VIE to an IVP. Hence solve it verify that the derived soln. is indeed the soln of given IE.

$$u(x) = 1 - 2x - 4x^2 + \int_0^x \{3 + 6(x-t) - 4(x-t)^2\} u(t)\,dt$$

Soln:

$$u'(x) = -2 - 8x + \frac{d}{dx}\int_0^x 3 + 6(x-t) - 4(x-t)^2\} u(t)\,dt$$

$$u'(x) = -2 - 8x + \int_0^x \frac{\partial}{\partial x}\{3 + 6(x-t) - 4(x-t)^2\} u(t)\,dt + 1\cdot f(x,x)$$

$$= -2 - 8x + \int_0^x \{6 - 8(x-t)\} u(t)\,dt + 3u(x)$$

Differentiating (2) wrt x

$$u''(x) = -8 - \int_0^x 8u(t)\,dt + 1\{6 - 8(x-x)\} u(x) + 3u'(x)$$

$$u''(x) = -8 - 8\int_0^x u(t)\,dt + 6u(x) + 3u'(x)$$

differentiating we get

$$u'''(x) = -\cancel{8} - 8u(x) + 6u'(x) + 3u''(x)$$

dts

$$u'''(x) - 3u''(x) - 6u'(x) + 8(u(x)) = 0$$

$$u(0) = u'(0) = u''(0) = 1$$

$$u(x) = c_1 e^{x} + c_2 e^{4x} + c_3 e^{-2x}$$

11th January 2022

Conversion of BVP to Fredholm Integral Equation

Consider the BVP given by

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x) \quad \text{---- (1)} \qquad a \leq x \leq b$$

and the boundary conditions $y(a) = c_0$, $y(b) = c_1$

c_0 and c_1 are known constants.

$p(x)$, $q(x)$, $r(x)$ are known functions

Procedure: Assume $\frac{d^2y}{dx^2} = u(x)$ --- (3)

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integrate (3) wrt x between a and x

$$\int_a^x \frac{d^2y}{dx^2}\,dx = \int_a^x u(x)\,dx$$

or

$$y'(x) - y'(a) = \int_a^x u(t)\,dt \qquad \text{assume } y'(a) = \mu$$

$$\Rightarrow \quad y'(x) = \mu + \int_a^x u(t)\,dt \quad \ldots (4)$$

integrating (4) wrt x between a and x

$$\int_a^x y'(x)\,dx = \mu\int_a^x dx + \int_a^x u(t)\,dt^2$$

$$y(x) - y(a) = \mu(x-a) + \int_a^x (x-t)\,u(t)\,dt$$

$$y(x) = c_0 + \mu(x-a) + \int_a^x (x-t)\,u(t)\,dt \quad \ldots (5a)$$

we have $y(b) = c_1$

Thus putting $x = b$ on both sides of 5a, we obtain

$$y(b) = c_1 = c_0 + \mu(b-a) + \int_a^b (b-t)\,u(t)\,dt$$

$$\Rightarrow \quad \mu = \left\{c_1 - c_0 - \int_a^b (b-t)\,u(t)\,dt\right\}(b-a)^{-1} \quad \ldots (*)$$

putting $y''(x)$, $y'(x)$, $y(x)$ from eqns (3), (4) and (5a) into 1, we obtain.

$$u(x) + p(x)\left\{\mu + \int_a^x u(t)\,dt\right\} + q(x)\left\{c_0 + \mu(x-a) + \int_a^x (x-t)\,\mu(t)\,dt\right\} = r(x)$$

$$u(x) = r(x) - c_0\,q(x) - \mu\{p(x) + (x-a)\,q(x)\} - \int_a^x \{p(x) + (x-t)\,q(x)\}\,u(t)\,dt \quad \ldots (6)$$

substituting value of μ from $(*)$, we get

$$u(x) = r(x) - c_0\,q(x) - \{p(x) + (x-a)\,q(x)\}\left\{\frac{c_1 - c_0}{b-a} - \underbrace{\int_a^b}_{\int_a^x + \int_x^b} \frac{b-t}{b-a}\,u(t)\,dt\right\} - \int_a^x \{p(x) + (x-t)\,q(x)\}\,u(t)\,dt$$

$$= r(x) - c_0 q(x) - \frac{c_1 - c_0}{b-a}\{p(x) + (x-a)q(x)\} + \int_x^b \{p(x) + (x-a) q(x)\} \frac{b-t}{b-a} \times u(t)\,dt$$

$$+ \int_a^x \left[\frac{b-t}{b-a}\{p(x) + (x-a)q(x)\} - \{p(x) + (x-t)q(x)\}\right] u(t)\,dt$$

$$= 1^{st} \text{ line} + \int_a^x \left\{ \frac{a-t}{b-a} p(x) + \frac{(t-a)(b-x)}{b-a} q(x) \right\} u(t)\,dt$$

$$= 1^{st} \text{ line} + \int_a^x \frac{(t-a)}{b-a}\left\{-p(x) + \frac{b-x}{b-a} q(x)\right\} u(t)\,dt$$

$$r(x) = r(x) - c_0 q(x) - \frac{c_1 - c_0}{b-a}\{p(x) + (x-a)q(x)\}$$
$$+ \int_a^x \frac{t-a}{b-a}\{-p(x) + (b-x)q(x)\} u(t)\,dt + \int_x^b \{p(x) + (x-a)q(x)\} \frac{b-t}{b-a} \cdot u(t)\,dt$$

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\,dt$$

$$f(x) = r(x) - c_0 q(x) - \frac{c_1 - c_0}{b-a}\{p(x) + (x-a)q(x)\}, \quad \lambda = \frac{1}{b-a}$$

$$k(x,t) = \begin{cases} \{-p(x) + (b-x)q(x)\}(t-a) & a \le t < x \\ \{p(x) + (x-a)q(x)\}(b-t) & x < t \le b \end{cases}$$

Example: Reduce the BVP $y'' + 2xy = 1 \quad 0 < x < 1$ — (1)

$y(0) = 0 = y(1)$ to a FIE — (2)

Soln: let $y''(x) = u(x) \longrightarrow (3)$

Integrating between 0 and x.

$$\int_0^x y''(x)\,dx = \int_0^x u(x)\,dx = \int_0^x u(t)\,dt$$

or $y'(x) - y'(0) = \int_0^x u(t)\,dt$ Take $\mu = y'(0)$

$$\Rightarrow y'(x) = \mu + \int_0^x u(t)\,dt \longrightarrow (4)$$

Integrate (4)

$$\int_0^x y'(x)\,dx = \mu \int_0^x 1 \cdot dx + \int_0^x u(t)\,dt^2$$

$$y(x) - y(0) = \mu x + \int_0^x (x-t)u(t)\,dt$$

$$\therefore y(x) = \mu x + \int_0^x (x-t)u(t)\,dt \quad \cdots (5)$$

put $x=1$ on both sides of (4)

This gives

$$0 = \mu + \int_0^1 (1-t)\,u(t)\,dt$$

$$\Rightarrow \quad \mu = \int_0^1 (t-1)\,u(t)\,dt$$

Substituting $y''(x)$, $y'(x)$, $y(x)$ from 3, 4, 5 into 1

$$u(x) + 2x\left\{\mu x + \int_0^x (x-t)\,u(t)\,dt\right\} = 1 \quad \longrightarrow 7$$

Putting μ from 6 into 7, we obtain

$$u(x) + 2x\left\{x\int_0^1 (t-1)\,u(t)\,dt + \int_0^x (x-t)\,u(t)\,dt\right\} = 1$$

or

$$u(x) + 2x\left[\int_0^x \{x(t-1) + x - t\}\,u(t)\,dt + \int_1^x x(t-1)\,u(t)\,dt\right] = 1$$

$$u(x) = 1 - 2\int_0^x xt(x-1)\,u(t)\,dt - 2\int_x^1 x^2(t-1)\,u(t)\,dt$$

$$u(x) = f(x) + \lambda\int_0^1 k(x,t)\,u(t)\,dt$$

$\lambda = -2$

$$k(x,t) = \begin{cases} xt(x-1) & 0 \le t < x \\ x^2(t-1) & x < t \le 1 \end{cases}$$

$$k(x,x^-) = x^2(x-1) = k(x,x^+)$$

$$k(t,x) = \begin{cases} tx(t-1) & x < t \\ t^2(x-1) & x > t \end{cases} \qquad k(x,t) \neq k(t,x)$$

<u>Regularity conditions for $k(x,t)$</u>.

$k(x,t)$ is said to be regular in the square $\{(x,t) : a \le x, t \le b\}$ when

(1) $\displaystyle\int_a^b\int_a^b |k(x,t)|^2\,dx\,dt < \infty$

(2) $\displaystyle\int_a^b |k(x,t)|^2\,dt < \infty$

(3) $\displaystyle\int_a^b |k(x,t)|^2\,dx < \infty$