Chapter 5 : Volterra Integral Equations

5A. Methods of solutions of VIE

5A.1 Successive approximation method.

Consider

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt.$$

In this method, take some initial approximation $u_0(x)$ and define successive iterates as,

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)\, u_{n-1}(t)\, dt, \quad n = 1, 2, 3, \ldots$$

Then, $u_n(x) \longrightarrow u(x)$ as $n \to \infty$,

if $|\lambda| < \dfrac{1}{M(b-a)}$, $M = \sup\limits_{(x,t)\in R} |K(x,t)|$.

$R : \{ (x,t);\ a \le x, t \le b \}$.

Example 1: Solve $u(x) = 1 + x - \int_0^x (x-t)\, u(t)\, dt$ by method of successive approximation, taking $u_0(x) = 1$.

Soln: $u_1(x) = f(x) + \lambda \int_0^x K(x,t)\, u_0(t)\, dt.$

Here $f(x) = 1 + x$, $\lambda = -1$, $K(x,t) = x - t$.

$$\therefore\ u_1(x) = 1 + x - \int_0^x (x-t) \cdot 1\, dt$$

$$= 1 + x - \frac{(x-t)^2}{2} \Big|_x^0 = 1 + x - \frac{x^2}{2}.$$

$$u_2(x) = 1+x - \int_0^x (x-t)\left(1+t-\frac{t^2}{2}\right)dt$$

$$= 1+x - \int_0^x (x-t)\,dt - \int_0^x (x-t)\,t\,dt + \int_0^x (x-t)\frac{t^2}{2}\,dt.$$

Now, $I_1 = \int_0^x (x-t)\,dt = \frac{x^2}{2}.$

$$I_2 = \int_0^x (x-t)\,t\,dt = \left[\frac{(x-t)^2}{2}\,t\right]_x^0 + \int_0^x \frac{(x-t)^2}{2}\,dt$$

$$= 0 + \frac{(x-t)^3}{2.3}\Bigg|_{t=x}^{0} = \frac{x^3}{3!}.$$

$$I_3 = \int_0^x (x-t)\frac{t^2}{2}\,dt = \left[\frac{(x-t)^2}{2}\cdot\frac{t^2}{2}\right]_x^0 + \int_0^x \frac{(x-t)^2}{2}\cdot t\,dt$$

$$= 0 + \left[\frac{(x-t)^3}{3!}\,t\right]_x^0 + \int_0^x \frac{(x-t)^3}{3!}\,dt = \frac{(x-t)^4}{4!}\Bigg|_x^0$$

$$= \frac{x^4}{4!}$$

$$\therefore\ u_2(x) = 1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}.$$

$$u_3(x) = 1+x - \int_0^x (x-t)\left(1+t-\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}\right)dt$$

$$= u_2(x) + \int_0^x (x-t)\frac{t^3}{3!}\,dt - \int_0^x (x-t)\frac{t^4}{4!}\,dt$$

$$= 1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}-\frac{x^6}{6!}$$

$$\therefore \quad u(x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)$$

$$+ \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos x + \sin x.$$

5A.2 Method of Resolvent kernel.

- Iterated kernels

$$K_1(x,t) = K(x,t)$$

$$K_n(x,t) = \int_t^x K(x,s)\, K_{n-1}(s,t)\, ds \;; \quad n = 2,3,4 \ldots$$

- Resolvent kernel $R(x,t;\lambda)$

$$R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^n K_n(x,t)$$

Solution to the VIE

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt$$

is given by,

$$u(x) = f(x) + \int_0^x R(x,t;\lambda)\, f(t)\, dt.$$

Note: Another way to write resolvent kernel,

$$R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,t).$$

In that case

$$u(x) = f(x) + \lambda \int_0^x R(x,t;\lambda)\, f(t)\, dt.$$

Example: Find the resolvent kernel corresponding to $K(x,t) = e^{x^2-t^2}$. Hence solve,

$$u(x) = e^{x^2} + \int_0^x e^{x^2-t^2} u(t)\,dt.$$

Soln: $K_1(x,t) = K(x,t) = e^{x^2-t^2}$.

$$K_2(x,t) = \int_t^x K(x,s) K_1(s,t)\,ds$$

$$= \int_t^x e^{x^2-s^2} e^{s^2-t^2}\,ds = \left[e^{x^2-t^2}\right]\int_t^x ds$$

$$= (x-t) \cdot e^{x^2-t^2}.$$

$$K_3(x,t) = \int_t^x K(x,s)\, K_2(s,t)\,ds.$$

$$= \int_t^x e^{x^2-s^2}(s-t)e^{s^2-t^2}\,ds = e^{x^2-t^2}\int_t^x (s-t)\,ds.$$

$$= e^{x^2-t^2}\left[\frac{(s-t)^2}{2}\right]_t^x = \frac{(x-t)^2}{2!} e^{x^2-t^2}.$$

$\vdots$

$$K_n(x,t) = \underline{(x-t)}\, e^{x^2-t^2}$$

$$\therefore R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^n K_n(x,t) \qquad \begin{matrix}\text{Here} \\ (\lambda = 1)\end{matrix}$$

$$= K_1(x,t) + K_2(x,t) + K_3(x,t) + \cdots$$

$$= e^{x^2-t^2}\left\{1 + (x-t) + \frac{(x-t)^2}{2!} + \cdots\right\}$$

$$= e^{x^2-t^2} \cdot e^{x-t}.$$

$\therefore$ solution to the given IE,

$$u(x) = f(x) + \int_0^x R(x,t;\lambda) f(t)\,dt.$$

$$\text{or, } u(x) = e^{x^2} + \int_0^x e^{x^2-t^2} \cdot e^{x-t} \cdot e^{t^2}\,dt.$$

$$= e^{x^2} + e^{x+x^2} \int_0^x e^{-t}\,dt.$$

$$= e^{x^2} + e^{x+x^2} \cdot \left[e^{-t}\right]_x^0 = e^{x^2} + e^{x+x^2}(1-e^{-x}).$$

$$= e^{x^2} + e^{x+x^2} - e^{x^2} = e^{x+x^2}.$$

5B : Abel Integral Equation

Abel Integral Equation is of the form

$$f(x) = \int_0^x \frac{u(t)\,dt}{(x-t)^\alpha} \longrightarrow (1); \quad 0 < \alpha < 1.$$

Note: It is a 1st kind non-homogeneous weakly-singular Volterra Integral Equation.

Abel's theorem: If $f(x)$ is continuous in $[a,b]$, then solution to Abel's IE (1) is given by,

$$u(x) = \frac{\sin \pi\alpha}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}.$$