Martingales :

Probability spaces $\quad \xi$ modelled by $(\Omega, \mathcal{F}, P)$ prob. space

random expt

sample pts $\omega \in \Omega$

prob. on $\mathcal{F}$ (prob. is defined for events, not sample pts)

n coin $\quad |\Omega| = \underbrace{2 \times 2 \times \dots \times 2}_{n \text{ times}} = 2^n$

roll a dice $\quad \Omega = \{1, 2, \dots, 6\} \quad (|\Omega| = 6)$

two dice $\quad \Omega = \{(1,1), (1,2), \dots, (6,6)\} \quad ; |\Omega| = 6 \times 6 = 36$ points

product spaces $\quad \Omega_1, \dots, \Omega_n$ spaces

$\Omega = \Omega_1 \times \dots \times \Omega_n = \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \Omega_i, i = 1, \dots, n\}$

$|\Omega| = |\Omega_1| \times \dots \times |\Omega_n|$

if $\Omega_i = \Omega_1 \forall i \; ; \; |\Omega| = |\Omega_1|^n$

— x —

$(\Omega, \mathcal{F})$ $\quad \mathcal{F}$ is σ-field (or σ-algebra)

1. $\Omega \in \mathcal{F}$ 2. If $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$. 3. If $A_i \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, $i = 1, 2, \dots$

eg (i) $\Omega = \{H, T\}$, $\mathcal{F}_1 = \{\phi, \Omega\}$ $\quad$ (ii) $\mathbb{R}, \mathcal{B}$

$\mathcal{F}_2 = \{\phi, \{H\}, \{T\}, \Omega\}$

— x —

→ **Toss a coin infinitely many times**

$\Omega = \{(HH \dots), (HT \dots), (TT \dots), (TH \dots)\}$

$\mathcal{F}_0 = \{\phi, \Omega\}$

$\Omega_1 = \{A_H, A_T\}$

$\mathcal{F}_1 = \{\phi, A_H, A_T, \Omega_1\}$ contains the information based by observing the first toss

$X = 0, 1, 2$

$\Omega_2 = \{A_{HH}, A_{TH}, A_{HT}, A_{TT}\}$ $A_H = A_{HH} \cup A_{HT}$

$\mathcal{F}_2 = \{\phi, A_{HH}, A_{TH}, A_{HT}, A_{TT}, \{A_{HH}, A_{TH}\}, \ldots, \Omega_2\}$

contain the information learned by observing the first two consecutive tosses

$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots$

$\mathcal{F}_k$ represent all the information available by the time k.

Such increasing families of σ-fields are called flows of σ-algebra or filtration.

—x—

$(\Omega, \mathcal{F}), P$

(i) $A \in \mathcal{F}$, $P(A) \in [0,1]$ (ii) $P(\Omega) = 1$

(iii) $A_1, A_2, \ldots$ seq of disjoint events; $\forall A_i \in \mathcal{F}$, $A_i \cap A_j = \phi$, $i \neq j$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \rightarrow \text{countable additivity or } \sigma\text{-additivity}$$

—x—

$\mu(\cdot) \geq 0$ with $\mu(\phi) = 0$ defined on a σ-field and having σ-additivity is called a measure.

eg (i) counting measure on set of integers $\mathbb{Z}$.

$$\mu(A) = |A| \ (\# \text{ of pts in } A), \ A \subset \mathbb{Z}$$

$(\mathbb{R}, \mathcal{B})$

$\mu(B) = \#\{\text{integers } n \in B\}, \; B \subset \mathbb{R}.$

(ii) Lebesgue measure on real line (the "length")

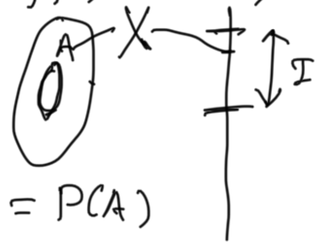
μ on $(\mathbb{R}, \mathcal{B})$

$\mu([a,b]) = b-a$ for all $a<b$ s.t. $\mu([0,1])=1$

E

—x—

$\Omega, \mathcal{F}, P \qquad \mathbb{R}, \mathcal{B}, P_X$



$E(X) = \int_x x \, dP_X(x)$

$= \int_x x \, dF_X(x)$ $\quad P_X(I) = P(A)$

(Lebesgue-Stieltjes integral) ↳ integral w.r.t. function not measure $\Leftarrow$ $\quad X^{-1}(B) \in \mathcal{F}, \; B \in \mathcal{B}$

$X^{-1}(-\infty, x]$ is event $\Leftrightarrow$ $X^{-1}(-\infty, x] \in \mathcal{F}$ $\quad\Updownarrow$

X is r.v.

cdf $F_X(x) = P_X((-\infty, x]) = P(\omega : X(\omega) \le x)$

$P_X(B) = P(\{\omega : X(\omega) \in B\}), B \in \mathcal{B}$ $\quad = P(X \le x)$

—x—

<u>Martingale (MG):</u>

$\{X_t\}_{t \in \underset{\sim}{T}}$ on $(\Omega, \mathcal{F}, P)$, $\underset{\sim}{T} = \{0, 1, 2, \ldots\}$, $[0, \infty)$ or $\{\ldots, -1, 0, 1, \ldots\}$

→ given t, $\mathcal{F}_t$ collection of events "observable" by that time.

→ filtration $\underset{\sim}{\mathcal{F}}$ is ↑ seq of sub σ-fields $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$

→ SP $\{X_t\}_{t \ge 0}$ is adapted to filtration $\underset{\sim}{\mathcal{F}}$ if for any $t = 0, 1, 2, \ldots$ the r.v. X_t is $\mathcal{F}_t$-measurable.

, i.e., $\underline{\{X_t \in B\}} \in \mathcal{F}_t$ for any $B \in \mathcal{B}$.

$(\Omega, \mathcal{F}, \underset{\sim}{F}, P)$ filtered prob. space or stochastic basis

eg $\{S_t\}$ is adapted to filtration $\underset{\wedge}{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$

$\mathcal{F}_t$ $\to$ information of the past (upto time t) of the price process

$\mathcal{F}_t$ generated by $(S_0, S_1, \dots S_t)$

$\underset{\wedge}{F}$ natural filtration of the process $\{S_t\}$

$\{S_t^2\}$ is also adapted to $\underset{\sim}{F}$?

$Y_t = S_{t+1} - S_t$? not.

Defn $(\Omega, \mathcal{F}, \underset{\sim}{F}, P)$ SP $\{X_t\}_{t \geq 0}$ adapted to filtration $\underset{-}{F}$.

$\{X_t\}$ is MG if for any $t = 0, 1, 2, \dots$

$\underline{E|X_t| < \infty}$ and $\underline{E(X_{t+1} | \mathcal{F}_t)} = \underline{X_t}$

$$\mathcal{F}_t := \underbrace{\sigma(X_1, \dots, X_t)}$$

$$E(X_{t+1} \mid X_0, X_2, \dots, X_t) = X_t$$

Let $\underset{\sim}{X_t \text{ MG}}$; for any $s \geq 1$

$$E(X_{t+s} | \mathcal{F}_t) = E\Big(\underbrace{E(X_{t+s} \mid \mathcal{F}_{t+s-1})}_{X_{t+s-1}} \Big| \mathcal{F}_t\Big) \quad \because X_t \text{ is MG}$$

$$= E(X_{t+s-1} | \mathcal{F}_t) = \dots = E(X_t | \mathcal{F}_t) = X_t$$

$E(X_{t+s-1} \mid \mathcal{F}_t) = \cdots = E(X_{t+1} \mid \mathcal{F}_t) = X_t$

—x—

$\Rightarrow E(X_{t+s}) = E\left(E(X_{t+s} \mid \mathcal{F}_t)\right) = E(X_t)$

—x—

→ A cont. time MG is defined as an adapted

$\left(\{\mathcal{F}_t\}_{t\geq 0},\ s,t\geq 0,\ \mathcal{F}_t \subset \mathcal{F}_{t+s} \subset \mathcal{F}\right)$ process

$\{X_t\}$ s.t. $\quad s,t \geq 0$

$$E(X_{t+s} \mid \mathcal{F}_t) = X_t.$$

—x—

notion of a game is fair

eg (1) Fairness in gambling, X_n → player's fortune after n^{th} play of game

fair game iff player's fortune neither ↑ nor ↓ at each play.

$\{X_n\}$ MG, requires player's fortune after the next play to equal, on av. his current fortune and not be otherwise affected by the previous history

(2) (Stock price in a perfect market)

X_n closing price at the end of day n of a certain publicly traded security such as a share of stock.

While daily prices may fluctuate, in a perfect market, then price seq. should be martingale.

x. MC

not possible to predict whether future price X_{n+1} will be higher or lower than current price X_n.

—x—

Example (1) Let $X_1, X_2, \ldots$ be indep r.v's with zero mean.

Let $Z_n = \sum_{i=1}^{n} X_i$. $\{Z_n\}$ MG?

$$Z_{n+1} = \underbrace{\sum_{i=1}^{n} X_i}_{Z_n} + X_{n+1}$$

Sol

$$E(Z_{n+1} \mid Z_1, \ldots, Z_n) = E(Z_n + X_{n+1} \mid Z_1, \ldots, Z_n)$$

$$= E(Z_n \mid Z_1, \ldots, Z_n) + E(X_{n+1} \mid Z_1, \ldots, Z_n)$$

$$= Z_n + E(X_{n+1}) = Z_n + 0 = Z_n$$

$\{Z_n\}$ MG.

(2) $X_1, X_2, \ldots$ are indep r.v's with $E(X_i) = 1$

$Z_n = \prod_{i=1}^{n} X_i$; Q. $\{Z_n\}_{n \geq 1}$ MG ?

Sol

$$E(Z_{n+1} \mid Z_1, \ldots, Z_n) = E(Z_n X_{n+1} \mid Z_1, \ldots, Z_n)$$

$$= Z_n \, E(X_{n+1} \mid Z_1, \ldots, Z_n)$$

$$= Z_n \, E(X_{n+1}) = Z_n$$

$\{Z_n\}$ MG.

(3) Consider a Branching process and let X_n denote the size of n^{th} generation. If m is the mean # of offsprings per individual, then $\{U_n, n \geq 1\}$ MG, when $U_n = \frac{X_n}{\ldots}$

m^n

Sol

$E(U_{n+1} \mid U_1, \ldots, U_n)$

$= \frac{1}{m^{n+1}} E(X_{n+1} \mid U_1, \ldots, U_n)$

$X_{n+1} = \sum_{i=1}^{X_n} Z_i$,

where Z_i # of offspring of i^{th} individual of n^{th} generation

$E(Z_i) = m$

$= \frac{1}{m^{n+1}} E\left(\sum_{i=1}^{X_n} Z_i \mid X_1, \ldots, X_n \right)$

$= \frac{1}{m^{n+1}} \times m X_n = \frac{X_n}{m^n} = U_n$

$\{U_n\}$ is MG.

—x—

(4) Random walk

$X_0 := 0$; $X_n := Y_1 + \cdots + Y_n$, $n \geq 1$,

Y_j IID r.v's with $E(|Y_1|) < \infty$

When is the SP $\{X_n\}_{n \geq 0}$ MG?

Sol natural filtration $\{\mathcal{F}_n\}_{n \geq 0}$ of $\{X_n\}_{n \geq 0}$

$E|X_n| = E|Y_1 + \cdots + Y_n| \leq E|Y_1| + \cdots + E|Y_n| = n E|Y_1| < \infty$

Now

$E(X_{n+1} \mid \mathcal{F}_n) = E(X_n + Y_{n+1} \mid \mathcal{F}_n) = E(X_n \mid \mathcal{F}_n) + E(Y_{n+1} \mid \mathcal{F}_n)$

$= X_n + E(Y_1)$

$\{X_n\}$ is MG if $E(Y_1) = 0$.

—x—

Ex Geometric Random walk

$$X_n := X_0 e^{Y_1 + \dots + Y_n}, n \geq 1$$

where $X_0 :=$ const. > 0, Y_j's are IID r.v.

Q When is $(X_n)_{n\geq 0}$ MG w.r.t filtration $\mathcal{F}_t = \sigma(Y_1, \dots, Y_n)$ ($\mathcal{F}_0$ being trivial)?

<u>Ans</u> $\underline{\phi_Y(1) = 1}$. where $\phi_Y(u) = E(e^{uY})$

—x—

$(\Omega, \mathcal{F}, \mathbb{F}, P)$ filtered prob. space

A r.v. τ "stopping time" (ST) if $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t = 0, 1, 2, \dots$.

—x—

For ST τ

$$\{\tau = t\} = \underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_t} \cap \underbrace{\{\tau \leq t-1\}^c}_{\in \mathcal{F}_{t-1} \subset \mathcal{F}_t} \in \mathcal{F}_t \quad \text{for each } t = 0, 1, 2, \dots$$

Let τ (random) time when we decide to stop doing something (stop gambling or to sell a block of shares at a stock exchange).

$\tau = t$, you act on the basis of what you already know by that time

$\{\tau = t\} \in \mathcal{F}_t$ → represent all the info available to us at time t.

<u>Example</u> (First hitting time)

adapted process (X_t), a (boundary) fn u_t, $t = 0, 1, 2, \dots$

Show that the first hitting (or crossing) time

$\tau := \inf\{t \geq 0 : X_t \geq u_t\}$ is a ST.

Sol For any $t = 0, 1, 2, \ldots$

$$\{\tau \leq t\} = \bigcup_{s=0}^{t} \underbrace{\{X_s \geq u_s\}}_{\in \mathcal{F}_s \subset \mathcal{F}_t} \in \mathcal{F}_t$$

τ is a ST.

<u>Thm:</u> $\underline{\{X_t\}_{t\geq 0}\ MG}$, τ ST on common filtered prob. space $a \wedge b = \min(a,b)$

$Z_t := \underline{X_{t\wedge\tau}}$, $t = 0, 1, 2, \ldots$ is a MG on that space.

pf. $Z_0 = X_0$

$t \geq 0$

$$Z_{t+1} = \underbrace{\sum_{k=0}^{t} X_k 1_{\{\tau = k\}}}_{|\cdot| \leq |X_k|} + \underbrace{X_{t+1} 1_{\{\tau > t\}}}_{|\cdot| \leq |X_{t+1}|} \quad — (*)$$

$$\Rightarrow E|Z_{t+1}| \leq E\left(\sum_{k=0}^{t+1} |X_k|\right) = \sum_{k=0}^{t+1} \underbrace{E|X_k|}_{<\infty} < \infty \quad \because X_t\ MG$$

$$E(Z_{t+1} | \mathcal{F}_t) = E\left(\sum_{k=0}^{t} X_k 1_{\{\tau=k\}} + X_{t+1} 1_{\{\tau>t\}} \,\middle|\, \mathcal{F}_t\right)$$

$$= \sum_{k=0}^{t} \underline{X_k 1_{\{\tau=k\}}} + E\left(X_{t+1} \underline{1_{\{\tau>t\}}} | \mathcal{F}_t\right)$$

$$\downarrow$$

$$\underline{1_{\{\tau>t\}} E(X_{t+1} | \mathcal{F}_t)}$$

$$\downarrow$$

$$\underline{1_{\{\tau>t\}}} X_t \quad \because X_t\ MG$$

$$= Z_t \quad — (*)$$

$$E(Z_{t+1} \mid \mathcal{F}_t) = Z_t \qquad \therefore \{Z_t\} \text{ MG.}$$

The optimal stopping thm:

$\{X_t\}_{t \geq 0}$ MG, τ bdd ST (i.e., for a constt $c < \infty$ one has $\tau < c$ a.s.).

Then $E(X_\tau) = E(X_0)$ —— (1)

(Thus, in a fair game, one cannot invent a rule for quitting the game that would "beat the system" : the game will remain fair)

Pf $Z_t = X_{t \wedge \tau}$ is MG | Using previous results

$\Rightarrow E(X_{t \wedge \tau}) = E(X_0)$ | $\because$ MG has constt mean

Setting $t := c$ yields (1).

—x—

Markov Inequality: $X \geq 0$, +ve constt λ

$$E(X) \geq \lambda P(X \geq \lambda)$$

Sol Using law of total prob.

$$E(X) = E(X 1_{[\lambda,\infty)}(X)) + E(X 1_{(-\infty,\lambda)}(X))$$

$$\geq E(X 1_{[\lambda,\infty)}(X))$$

$$1_{[a,b)}(u) = \begin{cases} 1 & \text{if } a \leq u < b \\ 0 & \text{o.w.} \end{cases}$$

$$\geq \lambda P(X \geq \lambda)$$

eg if $E(X) = 1$ then $P(X \geq 4) \leq \frac{1}{4}$, no matter what the actual distn of X is.

— x —

$(X_n) \geq 0$ MG, using Markov inequality

$$P(X_n \geq \lambda) \leq \frac{E(X_0)}{\lambda}, \quad \lambda > 0.$$

— x —

Maximal Inequality for non-negative martingales:

Let $X_0, X_1, \ldots$ be a MG with non-negative values, i.e., $P(X_n \geq 0) = 1$ for $n = 0, 1, \ldots$. For any $\lambda > 0$

$$P\left(\max_{0 \leq n \leq m} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \quad \text{for } 0 \leq n \leq m \quad — (1)$$

(m)

and $$P\left(\max_{n \geq 0} X_n > \lambda\right) \leq \frac{E(X_0)}{\lambda}, \quad \forall n \quad — (2)$$

Sol $\{X_0, \ldots, X_m\}$ seq. rises above λ for first time at some index n or it remains always below λ.

$$E(X_m) = \sum_{n=0}^{m} E(X_m 1\{X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda\})$$

$$+ E(X_m 1\{X_0 < \lambda, \ldots, X_m < \lambda\})$$

$$\geq \sum_{n=0}^{m} \underline{E(X_m 1\{X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda\})}$$

$$E(g(X) h(Y)) = E(h(Y) E(g(X) \mid Y))$$

Let $X = X_m$, $Y = (X_0, \ldots, X_n)$

(*) $\therefore E(X_m 1\{X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda\})$

$$= E(1\{X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda\} E(X_m \mid X_0, \ldots, X_n))$$

$$= E(X_n 1\{X_0 < \lambda, \ldots, X_{n-1} < \lambda, X_n \geq \lambda\})$$

$$\leq \sum_{n=0}^{m} E(X_n 1\{X_0<\lambda, \ldots, X_{n-1}<\lambda, X_n\geq\lambda\}) \quad \left| \begin{array}{l} \because X_n \text{ ms} \\ \text{large} \\ (2) \end{array} \right.$$

$$\geq \lambda \sum_{n=0}^{m} P(X_0<\lambda, \ldots, X_{n-1}<\lambda, X_n\geq\lambda)$$

$$= \lambda\, P\left(\bigcup_{n=0}^{m} \{X_0<\lambda, \_\_\_, X_{n-1}<\lambda, X_n\geq\lambda\}\right)$$

$$= \lambda\, P\left(\max_{0\leq n\leq m} X_n \geq \lambda\right)$$

—x—

Rk Instead of limiting the prob. of a large value for a single obs. X_n, the maximal inequality (1) limits the prob. of observing a large value anywhere in the time interval $0, \ldots, m$, and since the RHS of (1) does not depend on the length of interval, the maximal inequality limits the prob. of observing a large value at any time in the infinite future of the martingale.

—x—

Example: Gambler faces a series of indep fair games

$X_0 = 1$, bets the amt the amt in proportion of p, $0<p<1$

$$X_1 = \begin{cases} X_0 + pX_0 & \text{wp } \frac{1}{2} \end{cases}$$

$$1X_0 - pX_0 \quad wp\ \tfrac{1}{2}$$

$$= \begin{cases} 1+p & wp\ \tfrac{1}{2} \\ 1-p & wp\ \tfrac{1}{2} \end{cases}$$

$$X_2 = \begin{cases} X_1 + pX_1 & wp\ \tfrac{1}{2} \\ X_1 - pX_1 & wp\ \tfrac{1}{2} \end{cases} \equiv \begin{cases} (1+p)X_1 & wp\ \tfrac{1}{2} \\ (1-p)X_1 & wp\ \tfrac{1}{2} \end{cases}$$

<u>after n^{th} play</u>

current fortune of gambler X_n. <u>Wagers</u> pX_n (amt of money risked)

$$X_{n+1} = \begin{cases} (1+p)X_n & wp\ \tfrac{1}{2} \\ (1-p)X_n & wp\ \tfrac{1}{2} \end{cases}$$

$$E(X_{n+1} \mid X_0, \ldots, X_n) = X_n$$

$\{X_n\}, \geq 0$, MG

maximal inequality (1) with $\lambda = 2$.

$\because E(X_{n+1} \mid X_0 = x_0, \ldots X_n = x_n)$

$= (1+p)x_n \times \frac{1}{2} + (1-p)x_n \times \frac{1}{2}$

$= x_n$

the prob. that the gambler ever doubles his money is less than or equal to $\frac{1}{2}$, and this holds no matter what the game is, as long as it is fair, and no matter what fraction p of his fortune is wagered at each play. Indeed, the fraction wagered may vary from play to play, ...

[illegible], as long as it is chosen without knowledge of the next outcome.

— x —