Conversion of BVP to FIE

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BVP: Boundary value Problem

FIE: Fredholm Integral Equation

Let us consider the BVP given by,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = r(x); \quad a \le x \le b \longrightarrow (1)$$

and the boundary conditions

$$y(a) = c_0, \quad y(b) = c_1; \longrightarrow (2)$$

c_0, c_1 are known constants.

$p(x), q(x), r(x)$ are known functions.

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<u>Procedure:</u> Assume: $\frac{d^2y}{dx^2} = u(x) \longrightarrow (3)$

Integrate (3) w.r. to x between a and x

$$\int_a^x \frac{d^2y}{dx^2}\,dx = \int_a^x u(x)\,dx = \int_a^x u(t)\,dt$$

Assume $\boxed{y'(a) = \mu}$

or, $y'(x) - y'(a) = \int_a^x u(t)\,dt$

$$\Rightarrow \quad y'(x) = \mu + \int_a^x u(t)\,dt \longrightarrow (4)$$

Integrating (4) w.r. to x between a and x, we get

$$\int_a^x y'(x)\,dx = \mu \int_a^x dx + \int_a^x u(t)\,dt^2$$

or, $y(x) - y(a) = \mu(x-a) + \int_a^x (x-t)\,u(t)\,dt$

$$y(x) = C_0 + \mu(x-a) + \int_a^x (x-t)\,u(t)\,dt \longrightarrow (5)$$

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We have, $y(b) = C_1$. Thus putting $x = b$ on both sides of (5),

$$y(b) = C_1 = C_0 + \mu(b-a) + \int_a^b (b-t)\,u(t)\,dt$$

$$\Rightarrow \boxed{\mu = \left\{ C_1 - C_0 - \int_a^b (b-t)\,u(t)\,dt \right\} (b-a)^{-1}} \longrightarrow (\star)$$

Putting $y''(x)$, $y'(x)$, $y(x)$ from (3), (4), (5) into (1)

we obtain, $$u(x) + p(x)\left\{ \mu + \int_a^x u(t)\,dt \right\} + q(x)\left\{ C_0 + \mu(x-a) + \int_a^x (x-t)\,u(t)\,dt \right\} = r(x)$$

or, $$u(x) = r(x) - C_0\,q(x) - \mu\left\{ p(x) + (x-a)\,q(x) \right\} - \int_a^x \left\{ p(x) + (x-t)\,q(x) \right\} u(t)\,dt \longrightarrow (6)$$

Subst. value of μ from $(*)$, get

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$$u(x) = r(x) - c_0\, q(x) - \left\{ p(x) + (x-a)\, q(x) \right\} \left\{ \frac{c_1 - c_0}{b-a} - \underbrace{\int_a^b}_{\int_a^x + \int_x^b} \frac{b-t}{b-a}\, u(t)\, dt \right\}$$

$$- \int_a^x \left\{ p(x) + (x-t)\, q(x) \right\} u(t)\, dt$$

$$= r(x) - c_0\, q(x) - \frac{c_1 - c_0}{b-a} \left\{ p(x) + (x-a)\, q(x) \right\} + \int_x^b \left\{ p(x) + (x-a)\, q(x) \right\} \frac{b-t}{b-a} \times u(t)\, dt$$

$$+ \int_a^x \left[\frac{(b-t)}{b-a} \left\{ p(x) + (x-a)\, q(x) \right\} - \left\{ p(x) + (x-t)\, q(x) \right\} \right] u(t)\, dt$$

$$= \text{1st line} + \int_a^x \left\{ \frac{a-t}{b-a}\, p(x) + \frac{(t-a)(b-x)}{b-a}\, q(x) \right\} u(t)\, dt$$

$$= \quad " \quad + \int_a^x \frac{(t-a)}{b-a} \left\{ -p(x) + (b-x)\, q(x) \right\} u(t)\, dt$$

$$(b-t)(x-a) - (x-t)(b-a)$$

$$= \cancel{bx} - xt - ab + \cancel{at} - \cancel{bx} + bt + ax - \cancel{at}$$

$$= t(b-x) - a(b-x) = (t-a)(b-x)$$

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$$u(x) = r(x) - c_0\, q(x) - \frac{c_1 - c_0}{b-a}\left\{\phi(x) + (x-a)\, q(x)\right\}$$

$$+ \int_a^x \frac{t-a}{b-a}\left\{-\phi(x) + (b-x)\, q(x)\right\} u(t)\, dt + \int_x^b \left\{\phi(x) + (x-a)\, q(x)\right\} \frac{b-t}{b-a}\, u(t)\, dt$$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt$$

$$f(x) = r(x) - c_0\, q(x) - \frac{c_1 - c_0}{b-a}\left\{\phi(x) + (x-a)\, q(x)\right\}; \quad \lambda = \frac{1}{b-a}$$

$$K(x,t) = \begin{cases} \left\{-\phi(x) + (b-x)\, q(x)\right\}(t-a); & a \le t < x \\ \left\{\phi(x) + (x-a)\, q(x)\right\}(b-t); & x < t \le b \end{cases}$$

$$K(x,x^-) = (x-a)\left\{-\phi(x) + (b-x)\, q(x)\right\}, \quad K(x,x^+) = (b-x)\left\{\phi(x) + (x-a)\, q(x)\right\} \ne K(x,x^-)$$

Example Reduce the BVP $y'' + 2xy = 1$, $0 < x < 1$ $\longrightarrow (1)$

$y(0) = 0 = y(1)$ $\longrightarrow (2)$ to a FIE

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Sol. Let $\boxed{y''(x) = u(x)} \longrightarrow (3)$

Int. (3) between 0 and x

$$\int_0^x y''(x)\,dx = \int_0^x u(x)\,dx = \int_0^x u(t)\,dt$$

or, $y'(x) - y'(0) = \int_0^x u(t)\,dt$

Take $\boxed{\mu = y'(0)}$

$$\Rightarrow \boxed{y'(x) = \mu + \int_0^x u(t)\,dt} \longrightarrow (4)$$

Int. (4) between 0 and x,

$$\int_0^x y'(x)\,dx = \mu\int_0^x 1.\,dx + \int_0^x u(t)\,dt^2$$

$$y(x) - y(0) = \mu x + \int_0^x (x-t)\,u(t)\,dt$$

$$\therefore \boxed{y(x) = \mu x + \int_0^x (x-t)\,u(t)\,dt} \longrightarrow (5)$$

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Put $x=1$ on both sides of (4) & use (2).

This gives, $0 = \mu + \int_0^1 (1-t)\, u(t)\, dt$

$$\Rightarrow \quad \mu = \int_0^1 (t-1)\, u(t)\, dt \longrightarrow (6)$$

Substituting $y''(x)$, ~~$y'(x)$~~, $y(x)$ from (3), ~~(4)~~, (5) into (1), get,

$$u(x) + 2x \left\{ \mu x + \int_0^x (x-t)\, u(t)\, dt \right\} = 1 \rightarrow (7).$$

Putting μ from (6) into (7), we obtain,

$$u(x) + 2x \left\{ x \int_0^1 (t-1)\, u(t)\, dt + \int_0^x (x-t)\, u(t)\, dt \right\} = 1$$

$$\int_0^1 \rightarrow \int_0^x + \int_x^1$$

or, $$u(x) + 2x \left[\int_0^x \underbrace{\{x(t-1) + x - t\}}_{= xt - t = t(x-1)} u(t)\, dt + \int_x^1 x(t-1)\, u(t)\, dt \right] = 1$$

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$$u(x) = 1 - 2\int_0^x xt(x-1)\,u(t)\,dt - 2\int_x^1 x^2(t-1)\,u(t)\,dt$$

$$u(x) = f(x) + \lambda\int_0^1 K(x,t)\,u(t)\,dt$$

$$\lambda = -2, \quad K(x,t) = \begin{cases} xt(x-1); & 0 \le t < x \\ x^2(t-1); & x < t \le 1 \end{cases}$$

cont. at $t = x$.

$$\boxed{K(x,x^-) = x^2(x-1) = K(x,x^+)}$$

$$K(t,x) = \begin{cases} tx(t-1) & x < t \\ t^2(x-1) & x > t. \end{cases}$$

$$K(x,t) \neq K(t,x).$$

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~~Solve:~~ convert $y'' + 4y = x$; $0 < x < 1$,

$y(0) = 1$, $y'(1) = 0$

to a FIE.

Sol: $u(x) = f(x) + \lambda \int_0^1 K(x,t)\, u(t)\, dt$

$f(x) = x - 4$, $\lambda = 4$, $K(x,t) = \begin{cases} t; & 0 \le t < x \\ x; & x < t \le 1. \end{cases}$

Regularity conditions for $K(x,t)$.

$K(x,t)$ is said to be regular in the square $\{(x,t) : a \le x, t \le b\}$ when

(1) $\int_a^b \int_a^b |K(x,t)|^2 dx\, dt < \infty$, (2) $\int_a^b |K(x,t)|^2 dt < \infty$

(3) $\int_a^b |K(x,t)|^2 dx < \infty$.