Simulation

Example (1) Monte Carlo Integration

$$I = \int_a^b f(x)\,dx = (b-a)\int_0^1 f(a+(b-a)u)\,du \qquad \left\{ \begin{array}{l} x = a+(b-a)u \\ dx = (b-a)\,du \end{array}\right.$$

$$\hat{I}_n = \frac{b-a}{n}\sum_{i=1}^{n} f(a+(b-a)U_i) \qquad U_i \sim U(0,1)$$

$$E(\hat{I}_n) = \frac{b-a}{n}\sum_{i=1}^{n}\int_0^1 f(a+(b-a)u)\,du$$

$$= (b-a)\int_0^1 f(a+(b-a)u)\,du = I \quad \forall n$$

i.e., $\hat{I}_n$ is an unbiased estimator of I.

Suppose $I = \int_0^1 (1+\cos(\pi x))\,dx = 1$

$n = 4$ $U(0,1)$ sample

U_1, U_2, U_3, U_4: 0.419, 0.709, 0.732, 0.893

$$\hat{I}_4 = \frac{1-0}{4}\sum_{i=1}^{4}\left[1+\cos\pi(0+(1-0)U_i)\right] = 0.896$$

which is close to actual answer 1.

—x—

$\boxed{0}\ \boxed{1} - - \boxed{9}$ WR n slips $U(0,1)$

0.3852 (to the nearest 0.0001)

random number tables

—x—

digital computers simulate $U(0,1)$

pseudo random numbers (PRN)

initial seed X_0

$X_{n+1} = (a X_n + c) \bmod m$, $\quad n \geq 0$, specified integer a, c and m.

$X_n \in \{0, 1, 2, \dots, m-1\}$

$U_n = \frac{X_n}{m} \overset{apx}{\sim} U(0,1)$

Generator

$X_i = 16807 X_{i-1} \bmod (2^{31} - 1)$

Example (Hines et al)

$$Z_i = (5 Z_{i-1} + 1) \bmod 8, \quad \text{seed } Z_0 = 0$$

$Z_1 = 1 \bmod 8 = 1$ $\quad 8 \overline{)1} (0$, 0, $Z_1 = 1$

$Z_2 = 6 \bmod 8 = 6$ $\quad 8 \overline{)6} (0$, 0, $Z_2 = 6$

$1, 6, 7, 4, 5, 2, 3, 0$ $\quad$ full period or full cycle generator

$$\underset{U_1}{\frac{1}{8}}, \underset{U_2}{\frac{6}{8}}, \underset{U_3}{\frac{7}{8}}, \underset{U_4}{\frac{4}{8}}, \underset{U_5}{\frac{5}{8}}, \underset{U_6}{\frac{2}{8}}, \underset{U_7}{\frac{3}{8}}, \underset{U_8}{0}$$

PRN's.

Example (Hines et al)

$$Z_i = (3 Z_{i-1} + 1) \bmod 7$$

(i) $Z_0 = 3$,

$3, 3, 3, \dots$ $\quad \times$

$7 \overline{)10} (1$, 7, $3 = Z_1$

(ii) $Z_0 = 0$

$1, 4, 6, 5, 2, 0$ $\quad$ not full period generator

Probability Integral Transform (Inverse transform method):

cdf

$U \sim U(0,1)$; $\underline{X \sim F}$, then

$$U = F(X) \sim U(0,1), \text{ i.e., } X = F^{-1}(U)$$

Sol For $0 \le u \le 1$

$$P(U \le u) = P(F(X) \le u)$$

$$= P(X \le F^{-1}(u))$$

$$= F F^{-1}(u)$$

$$= u$$

$$f_U(u) = \begin{cases} 1, & 0 \le u \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$U \sim U(0,1)$$

<u>Example</u>: $F(x) = 1 - e^{-x}$, $x > 0$ $\exp(1)$

$$u = F(x) \Leftrightarrow 1 - e^{-x} = u \Leftrightarrow x = -\log(1-u)$$

$$X = F^{-1}(U) = -\log(1-U) \sim \exp. \text{ with mean } 1.$$

$$1 - U \overset{d}{=} U$$ <u>Ex.</u>

$$\underline{X = -\log U} \sim \exp \text{ with mean } 1.$$