Note. $I[y(x)] = \int_{x_0}^{x_2} A(x,y)\sqrt{1+y'^2}\,dx.$

lower end pt. (x_0, y_0) is fixed.

upper end pt. (x_2, y_2) varies along $y = \psi(x)$.

Transversality condition:

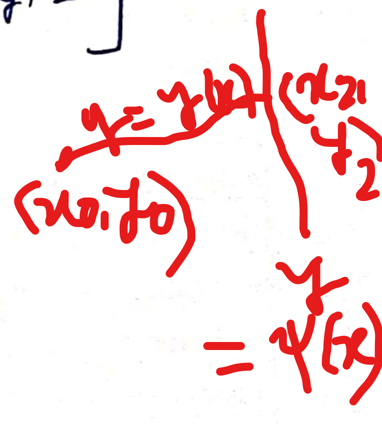
$$\left[f + (\psi' - y') f_{y'}\right]_{x=x_2} = 0.$$

$$f = A(x,y)\sqrt{1+y'^2}$$

$$\left[A(x,y)\sqrt{1+y'^2} + (\psi' - y')\,A(x,y).\frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_2} = 0.$$

Or, $$\left[A(x,y).\sqrt{1+y'^2} - A(x,y).\frac{y'^2}{\sqrt{1+y'^2}} + \frac{\psi' y' A(x,y)}{\sqrt{1+y'^2}}\right]_{x=x_2} = 0$$

Or $$\left[A(x,y).\frac{1+y'^2 - y'^2}{\sqrt{1+y'^2}} + A(x,y).\frac{\psi' y'}{\sqrt{1+y'^2}}\right] = 0.$$



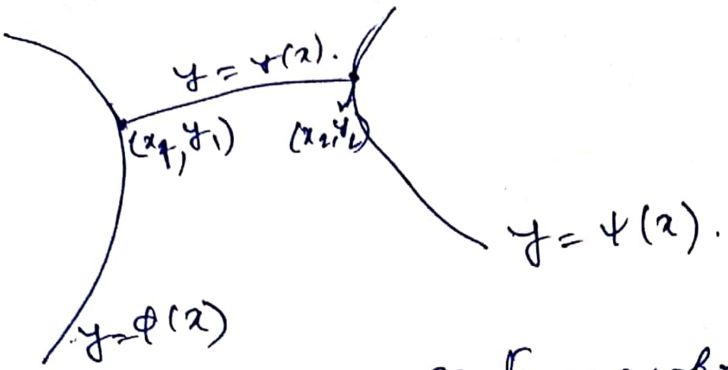
$\therefore$ $$A(x,y).(1 + \psi' y')\Big|_{x=x_2} = 0.$$

But $A(x,y) \neq 0$. $\therefore 1 + \psi' y' = 0.$

$\therefore \psi' y' = -1$ at (x_2, y_2).

$$I[y(x)] = \int_{1}^{x_2} \sqrt{1+y'^2}\,dx \qquad \text{Here } A(x,y) = 1.$$

Two-sided variation.



$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\,dx.$$

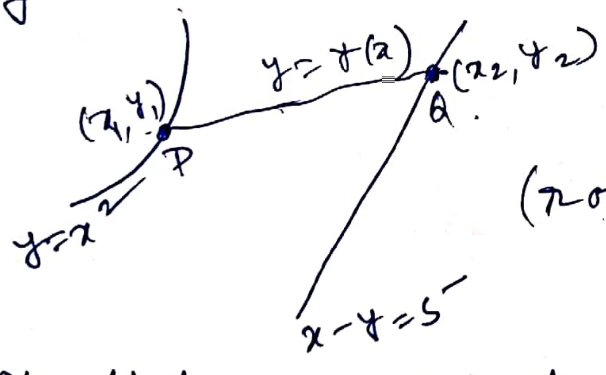
f satisfies E-L-E.

& transversality conditions.

$$\left[f + (\phi' - y') f_{y'}\right]_{x = x_1} = 0.$$

$$\left[f + (\psi' - y') f_{y'}\right]_{x = x_2} = 0.$$

<u>Ex.</u> Find the (minimum) distance between the parabola $y = x^2$ and the straight line $x - y = 5$.



(rough sketch).

To find $y = y(x)$ that will minimize the functional

$$I[y(x)] = \int_P^Q ds = \int_P^Q \sqrt{dx^2 + dy^2}$$

or, $$I[y(x)] = \int_{x = x_1}^{x_2} \sqrt{1 + y'^2}\,dx\,; \quad y' = \frac{dy}{dx}.$$

(x_1, y_1) varies along $y = x^2$
(x_2, y_2) " " $x - y = 5$.

Here $f = \sqrt{1 + y'^2}$

f satisfies E-L-E $\quad f_y - \frac{d}{dx} f_{y'} = 0$

f doesn't contain x explicitly

$$f - y' f_{y'} = C_0$$

$$\sqrt{1+y'^2} - y' \cdot \frac{y'}{\sqrt{1+y'^2}} = C_0$$

$$1 + y'^2 = \frac{1}{C_0^2} = c^2$$

$$y'^2 = \alpha^2$$

$$y' = \alpha.$$

$$y = \alpha x + \beta.$$

f does not contain y explicitly.

$$\frac{d}{dx} f_{y'} = 0.$$

$$y'' f_{y'y'} = 0.$$

$f_{y'y'} \neq 0.$

$\therefore y'' = 0 \rightarrow y = \alpha x + \beta$

Show any 1.

f satisfies $\left[f + (\phi' - y') f_{y'}\right]_{x=x_1} = 0.$

& $\left[f + (\psi' - y') f_{y'}\right]_{x=x_2} = 0.$ $\quad f = \sqrt{1+y'^2}$

$$y = x^2 = \phi(x)$$
$$\phi' = 2x$$

$x - y = 5$

$y = x - 5 = \psi(x).$

$\psi' = 1.$

$$\left[\sqrt{1+y'^2} + (2x - y') \frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_1} = 0$$

$$\left[\frac{1 + \cancel{y'^2} - \cancel{y'^2}}{\sqrt{1+y'^2}} + \frac{2xy'}{\sqrt{1+y'^2}}\right]_{x=x_1} = 0.$$

$$\therefore (1 + 2xy')_{x=x_1} = 0.$$

$y = \alpha x + \beta.$

$y' = \alpha.$

$$\therefore 1 + 2x_1 y' = 0.$$

$$\text{or, } 1 + 2\alpha x_1 = 0 \longrightarrow (1)$$

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$$\left[\sqrt{1+y'^2} + (1-y')\cdot\frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_2} = 0.$$

$\psi(x) = x-5$

$\psi' = 1$

or, $$\left[\frac{1+\cancel{y'^2}-\cancel{y'^2}}{\sqrt{1+y'^2}} + \frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_2} = 0.$$

$\therefore \quad (1+y')_{x=x_2} = 0.$ $\qquad y = \alpha x + \beta.$

$\therefore \quad 1+\alpha = 0. \Rightarrow \boxed{\alpha=-1}$ $\qquad y' = \alpha.$

Putting $\alpha = -1$ in (1) get $1-2x_1=0.$

$\therefore \quad x_1 = \frac{1}{2}$

$(x_1, y_1), (x_2, y_2)$ satisfy $y = \alpha x + \beta.$

$y_1 = \alpha x_1 + \beta \quad , \quad y_2 = \alpha x_2 + \beta.$

(x_1, y_1) lies on $y = x^2$ $\quad \therefore y_1 = x_1^2.$

(x_2, y_2) " " $x - y = 5$ $\quad \therefore y_2 = x_2 - 5$

$y_1 = -\frac{1}{2} + \beta \quad , \quad y_2 = -x_2 + \beta.$

$y_1 = x_1^2 = \frac{1}{4}.$

$\frac{1}{4} = -\frac{1}{2} + \beta \Rightarrow \beta = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$

$y_2 = -x_2 + \frac{3}{4} \qquad y_2 + x_2 = \frac{3}{4}.$

$y_2 = x_2 - 5 \qquad y_2 - x_2 = -5$

$x_2 = \frac{23}{8}.$

$x_1 = \frac{1}{2}$, $x_2 = \frac{23}{8}$, $y' = -1$.

$\therefore \; d = I[y(x)] = \int_{\frac{1}{2}}^{\frac{23}{8}} \sqrt{1+1}\, dx$.

$= \sqrt{2}\left(\frac{23}{8} - \frac{4}{8}\right) = \frac{19\sqrt{2}}{8}$ units.

Note: $I[y(x)] = \int_{x_2}^{x_1} \sqrt{1+y'^2}\, dx = -\frac{19\sqrt{2}}{8}$ units $\times$.

$d = \left| -\frac{19\sqrt{2}}{8} \right| = \frac{19\sqrt{2}}{8}$ units.

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Moving Boundary conditions.

To find $y = y(x)$ which extremizes

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\,dx.$$

where either (x_1, y_1) or (x_2, y_2) or both vary along some curve(s).

If (x_1, y_1) varies along $y = \varphi(x)$, then $y = y(x)$ satisfies the transversality condition

$$\left[f + (\varphi' - y') f_{y'}\right]_{x=x_1} = 0.$$

Natural Boundary condition (N.B.C)

N.B.C arises when one or both end points vary along vertical line(s).

Thus if ~~$y = y(x)$~~ (x_1, y_1) varies along $x = x_1$, then $y = y(x)$ satisfies

$$f_y - \frac{d}{dx} f_{y'} = 0. \longrightarrow (1)$$

and

$$f_{y'}\Big|_{x=x_1} = 0 \longrightarrow (2). \longrightarrow (N.B.C.)$$

If ~~$y = y(x)$~~ (x_2, y_2) varies along $x = x_2$, then $y = y(x)$ satisfies E-L-E (1) and the N.B.C.

$$f_{y'}\Big|_{x=x_2} = 0 \longrightarrow (3)$$

If $(x_1, y_1), (x_2, y_2)$ vary along $x = x_1$ & $x = x_2$, then $y = y(x)$ satisfies E-L-E (1), NBC's (2) & (3).

Ex. Find the extremals for

$$I[y(x)] = \int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right)dx .$$

if the end points vary along $x=0$ and $x=1$.

$y=y(x)$ satisfies

$$f_y - \frac{d}{dx} f_{y'} = 0 .$$

and $f_{y'} = 0$ on $x=0$ and $x=1$.

$$f = \frac{1}{2}y'^2 + yy' + y' + y$$

D. E : $y''=1$. $\Rightarrow y' = x+c_1$, $y = \frac{x^2}{2} + c_1 x + c_2$

solution: $y = \frac{x^2}{2} + c_1 x + c_2$.

$f_{y'} = 0$ on $x=0$. $\quad y' + y + 1 = 0$. on $x=0$.

$f_{y'} = 0$ on $x=1$ $\quad y' + y + 1 = 0$ on $x=1$.

$y' = x + c_1$ $\quad \therefore$ $x + c_1 + y + 1 = 0$ on $x=0$.

& $x + c_1 + y + 1 = 0$ on $x=1$.

$c_1 = -\frac{3}{2}$, $c_2 = \frac{1}{2}$.

$$y = \frac{x^2}{2} - \frac{3}{2}x + \frac{1}{2} .$$

$x + c_1 + \frac{x^2}{2} + c_1 x + c_2 + 1 = 0$

$c_1 + c_2 + 1 = 0$

$c_1 + c_1 + c_2 + 2\frac{1}{2} = 0$.

$2c_1 + \cancel{c_2} = -\frac{5}{2}$

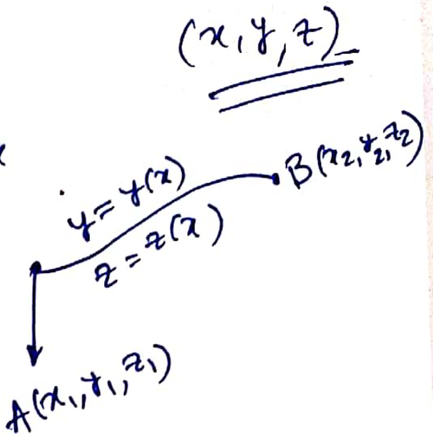
$-c_1 + \cancel{c_2} = -1$

$c_1 = -\frac{3}{2}$, $c_2 = -1 - c_1 = -1 + \frac{3}{2}$.

Case: Functional is of the form.

$$I[y, z] = \int_{x_1}^{x_2} f(x, y, z, y', z')\, dx$$

with $y = y(x)$, $z = z(x)$.



Let $A(x_1, y_1, z_1)$ or, $B(x_2, y_2, z_2)$ or both A & B vary along some curve.

Case I. Suppose $A(x_1, y_1, z_1)$ is fixed. $B(x_2, y_2, z_2)$ varies along the curve $y = \varphi(x)$, $z = \psi(x)$.

Then the extremizing curve $y = y(x)$, $z = z(x)$ satisfies the E-L-E's

$$f_y - \frac{d}{dx} f_{y'} = 0 \longrightarrow (1)$$

$$f_z - \frac{d}{dx} f_{z'} = 0 \longrightarrow (2)$$

and the transversality condition

$$\Big[f + (\varphi' - y') f_{y'} + (\psi' - z') f_{z'} \Big]_{x = x_2} = 0 \longrightarrow (3)$$

Case II. Suppose $B(x_2, y_2, z_2)$ is fixed. $A(x_1, y_1, z_1)$ varies along the curve $y = \hat{\varphi}(x)$, $z = \hat{\psi}(x)$.

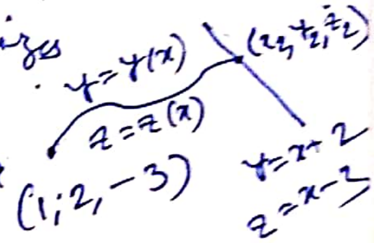
Then $y = y(x)$, $z = z(x)$ satisfies the E-L-E's (1) & (2) and the transversality condition

$$\Big[f + (\hat{\varphi}' - y') f_{y'} + (\hat{\psi}' - z') f_{z'} \Big]_{x = x_1} = 0 \longrightarrow (4)$$

Case III. If $A(x_1, y_1, z_1)$ varies along $y = \hat{\phi}(x), z = \hat{\psi}(x)$
$B(x_2, y_2, z_2)$ " " $y = \phi(x), z = \psi(x)$.
Then $y = y(x)$ satisfies E-L-E's (1) & (2) and the two transversality conditions (3) & (4).

Ex. Find the shortest distance from the point $M(1, 2, -3)$ to the st. line $y = x+2,\ z = x-3$

To find $y = y(x),\ z = z(x)$ which extremizes



Sol. $$I[y(x), z(x)] = \int_{x_1}^{x_2} \sqrt{1 + y'^2 + z'^2}\, dx$$

f satisfies the E-L-E's.

$f = \sqrt{1 + y'^2 + z'^2}$ which does not contain x explicitly.

Then, $f - y' f_{y'} = \text{const}$ & $f - z' f_{z'} = \text{const}$

$$\sqrt{1+y'^2+z'^2} - y' \cdot \frac{y'}{\sqrt{1+y'^2+z'^2}} = c_0, \quad \sqrt{1+y'^2+z'^2} - \frac{z' \cdot z'}{\sqrt{1+y'^2+z'^2}} = d_0.$$

$$\frac{1 + \cancel{y'^2} + z'^2 - \cancel{y'^2}}{\sqrt{1+y'^2+z'^2}} = c_0, \quad \frac{1+y'^2+z'^2 - z'^2}{\sqrt{1+y'^2+z'^2}} = d_0.$$

$$1 + z'^2 = c_0 \sqrt{1+y'^2+z'^2}, \quad 1 + y'^2 = d_0 \sqrt{1+y'^2+z'^2}.$$

$y' = c_1, \quad z' = c_3$ (derive).

$y = c_1 x + c_2, \quad z = c_3 x + c_4.$

$y(x), z(x)$ will satisfy

$$\left[f + (\phi' - y') f_{y'} + (\psi' - z') f_{z'}\right]_{x = x_2} = 0.$$

$y = x + 2, \quad z = x - 3.$

$y = \phi(x) \quad z = \psi(x).$

$\phi(x) = x + 2, \quad \psi(x) = x - 3.$

$\phi'(x) = 1 = \psi'(x). \qquad f = \sqrt{1 + y'^2 + z'^2}.$

$$\left[\sqrt{1+y'^2+z'^2} - \frac{y'.y'}{\sqrt{1+y'^2+z'^2}} - \frac{z'.z'}{\sqrt{1+y'^2+z'^2}} + 1.\frac{y'}{\sqrt{1+y'^2+z'^2}} + 1.\frac{z'}{\sqrt{1+y'^2+z'^2}}\right]_{x=x_2} = 0.$$

$$\therefore \left[1 + \cancel{y'^2} + \cancel{z'^2} - \cancel{y'^2} - \cancel{z'^2} + y' + z'\right]_{x=x_2} = 0.$$

$$(1 + y' + z')_{x = x_2} = 0.$$

$y = c_1 x + c_2, \quad y' = c_1 \qquad \Big| \; z = c_3 x + c_4, \quad z' = c_3.$

$1 + c_1 + c_3 = 0.$

unknowns are $x_2, y_2; z_2, c_1, c_2, c_3, c_4$.

(1) $1 + c_1 + c_3 = 0$.

(2) $2 = c_1 + c_2$

(3) $-3 = c_3 + c_4$.

$\begin{cases} y = c_1 x + c_2 \\ z = c_3 x + c_4 \end{cases}$

pass through.

$M(1, 2, -3)$.

$y = x + 2$

$z = x - 3$.

(4). $y_2 = c_1 x_2 + c_2$

(5) $z_2 = c_3 x_2 + c_4$,

(6) $y_2 = x_2 + 2$

(7) $z_2 = x_2 - 3$.

$c_1 x_2 + c_2 = x_2 + 2$. $\Rightarrow c_1 x_2 + \cancel{2} - c_1 = x_2 + \cancel{2}$

$c_1 x_2 - c_1 = x_2$

$c_3 x_2 + c_4 = x_2 - 3$.

$\Rightarrow c_3 x_2 - 3 - c_3 = x_2 - 3$

$\therefore c_3 x_2 - c_3 = x_2$.

$c_1(x_2 - 1) = x_2$

$c_3(x_2 - 1) = x_2$

$\frac{c_1}{c_3} = 1 \Rightarrow c_1 = c_3$

$1 + c_1 + c_3 = 0$.

$2c_1 + 1 = 0$.

$\therefore c_1 = -\frac{1}{2}$

$c_3 = -\frac{1}{2}$

$y = c_1 x + c_2$

$z = c_3 x + c_4$

$y' = c_1, \ z' = c_3$

$-\frac{1}{2}(x_2 - 1) = x_2$.

$\therefore -\frac{1}{2} x_2 + \frac{1}{2} = x_2$.

$\therefore x_2 \times \frac{3}{2} = \frac{1}{2} \Rightarrow x_2 = \frac{1}{3}$.

with $d = \left| \int_1^{x_2} \sqrt{1 + c_1^2 + c_3^2}\, dx \right|$

$= \frac{2}{3} \times \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{3}$

Ans: $\frac{\sqrt{6}}{3}$