Renewal Theory:

→ Interarrival times of a P.P. are i.i.d. expo r.v's.

→ Generalizing, counting process, for which interarrival times are i.i.d. with an arbitrary dist^n, called renewal process.

Let $\{X_n;\ n=1,2,\ldots\}$, $X_n \geq 0$, be indep r.v's s.t. $X_i \overset{CDF}{\sim} F(\cdot)$, suppose that

$$F(0) = P(X_n = 0) < 1$$

X_n : interarrival times, i.e., time between $(n-1)^{st}$ and n^{th} event/renewal.

Since $X_n \geq 0$, $F(0) < 1 \Rightarrow 0 < \mu < \infty$; $\mu = E(X_n)$

Let $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$.

S_n : time for n^{th} event/renewal.

$N(t)$: number of event by time t.

$$N(t) = \sup\{n;\ S_n \leq t\}$$

→ We say n^{th} renewal occur at time S_n.

→ Since interarrival times are i.i.d., it follows that at each renewal the process probabilistically starts over.

Q: Whether an infinite number of

... renewals can occur in finite time

Ans it cannot.

SLLN wp1 $\frac{S_n}{n} \to \mu$ as $n \to \infty$

Since $\mu > 0$ $\therefore$ $S_n \to \infty$ as $n \to \infty$

$\therefore$ $S_n \le t$ for atmost a finite number of values of n.

$N(t) = \sup\{n : S_n \le t\} < \infty$

$\therefore$ $N(t) = \max\{n ; S_n \le t\}$

— x —

$S_n > t \equiv N(t) \le n-1$

$P(S_n > t) = P(N(t) \le n-1)$

$\Leftrightarrow$ $P(S_n \le t) = P(N(t) \ge n)$

$S_n \le t \equiv N(t) \ge n.$

— x —

X_n i.i.d. cdf $F(\cdot)$; $\underline{S_n = \sum_{i=1}^{n} X_i \sim F_n ,}$

where F_n n fold convolution of F with itself.

$m(t) = E(N(t)) \to$ renewal fn or mean value fn.

Prop 2 $m(t) = \sum_{n=1}^{\infty} F_n(t)$

pf Let $I_n = \begin{cases} 1 & \text{if } n^{th} \text{ renewal occured in } [0,t] \\ 0 & \text{O.W.} \end{cases}$

$N(t) = \sum_{n=1}^{\infty} I_n$

$$E[N(t)] = E\left(\sum_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} E(I_n) = \sum_{n=1}^{\infty} P(S_n \le t)$$

$$= \sum_{n=1}^{\infty} F_n(t)$$

<u>Prop II</u> $m(t) < \infty \;\forall\; 0 \le t < \infty$ (, i.e., $N(t)$ has finite expectation).

<u>Example:</u> $m(t) = \lambda t$, $t \geq 0$. What is the distn of the renewal occurring by <u>time 10</u>?

Sol $m(t) = E(N(t)) = \lambda t$

$N(t) \sim P.P.(\lambda)$

F is exponential with mean $1/\lambda$.

Renewal process is a P.P. with rate λ and hence

$$P(N(10) = n) = \frac{e^{-10\lambda}(10\lambda)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

— x —

Assume interarrival distn F is continuous with density fn f

$$m(t) = E(N(t)) = E\left[E[N(t) \mid X_1]\right]$$

$$= \int_0^{\infty} E(N(t) \mid X_1 = x) f(x)\, dx$$

$$\begin{cases} E(N(t) \mid X_1 = x) = 1 + E(N(t-x)) & \text{if } x < t \\ E(N(t) \mid X_1 = x) = 0 & \text{if } x > t \end{cases}$$

$$m(t) = \int_0^t \left(1 + m(t-x)\right) f(x)\,dx$$

$$\boxed{m(t) = F(t) + \int_0^t m(t-x)\, f(x)\,dx}$$

Fundamental renewal equation

<u>Example:</u> Let interarrival distn

$$X_i \sim U(0,1)$$

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$F(x) = x$$

For $t \leq 1$

$$m(t) = F(t) + \int_0^t m(t-x)\,dx$$

$$m(t) = t + \int_0^t m(y)\,dy$$

$t - x = y$

$-dx = dy$

$$\Rightarrow \quad m'(t) = 1 + m(t) = h(t) \text{ (say)} \qquad \text{——— (1)}$$

$$\therefore \quad h'(t) = m'(t) = h(t) \Rightarrow \frac{h'(t)}{h(t)} = 1$$

$$\Rightarrow \log h(t) = t + c$$

$$\therefore h(t) = k e^t$$

<u>From (1)</u> $\quad m(t) = k e^t - 1$

$$N(0) = 0 \quad \Rightarrow \quad m(0) = 0 \Rightarrow k = 1$$

$$\therefore m(t) = e^t - 1 \qquad 0 \leq t \leq 1$$

$\therefore\ m(t) = e^t - 1 \ , \quad 0 \le t \le 1 .$

—x—

We know $N(t) < \infty \ , \ \forall t$

$N(\infty) = \lim_{t \to \infty} N(t)$: the total number of renewals that occur

<u>Prop:</u> $N(\infty) = \infty$ with prob 1.

Sol

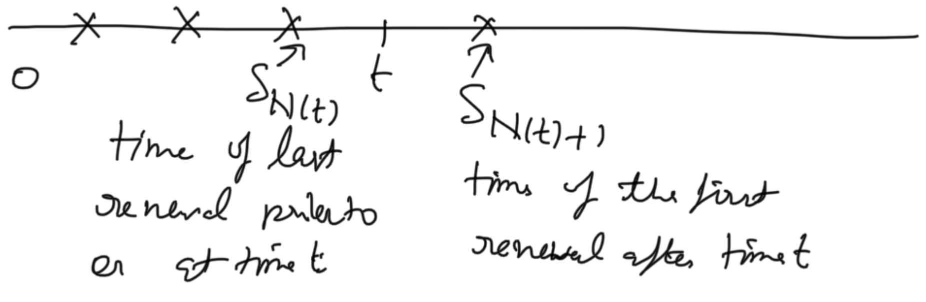
$$P(N(\infty) < \infty) = P(X_n = \infty \text{ for some } n)$$
$$= P\left(\bigcup_{n=1}^{\infty} \{X_n = \infty\} \right)$$
$$\le \sum_{n=1}^{\infty} P(X_n = \infty)$$
$$= 0$$

$P(X_1 = \infty)$
$= P(\text{no renewal occurs})$
$= 0$
$\because F(0) < 1$
some renewal will occur

$\therefore$ $N(t) \to \infty$ as $t \to \infty$.

However, it would be nice to know the rate at which $N(t)$ goes to ∞, i.e.,

$$\lim_{t \to \infty} \frac{N(t)}{t} \quad : \text{ rate of renewal process}$$



<u>Prop!</u> With prob. 1

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty$$

Sol

$$S_{N(t)} \le t < S_{N(t)+1}$$

$$\Rightarrow \quad \frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} \quad \text{——} \ \text{✡}$$

$$\frac{S_{N(t)}}{N(t)} = \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \longrightarrow \mu \quad \text{as } N(t) \to \infty$$

But $N(t) \to \infty$ as $t \to \infty$

Also $$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \times \frac{N(t)+1}{N(t)} \to \mu \quad \text{as } t \to \infty$$

Using ✡ $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$ rate of renewal process

Hence with prob 1, the long-run rate at which renewal occur will equal $\frac{1}{\mu}$

→ When the mean time btw renewals is infinite in that case $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} = 0$

Example: 'A' has a radio that works on a single battery. As soon as the battery in use fails, 'A' immediately replaces it with a new battery. If the lifetime of a battery (in hrs) is Uniformly (30,60), then at what rate does 'A' have to change batteries?

Sol X : life time of a battery ...

$\sim U(30, 60)$

pdf

$$f(x) = \begin{cases} \frac{1}{30}, & 30 < x < 60 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu = \int_{30}^{60} x \times \frac{1}{30}\, dx = \frac{60+30}{2} = 45$$

$N(t)$ # of batteries that have failed by time t

$\therefore$ rate at which 'A' replaces batteries is given by

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{45}$$

, i.e., in long run, 'A' will have to replace one battery every 45 hours.

— x —