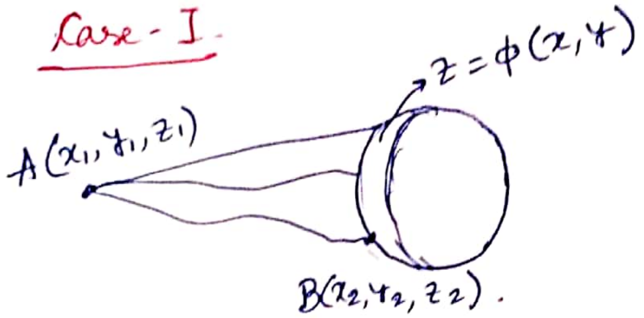
Variation of end points along surfaces

Case - I.



Let $A(x_1, y_1, z_1)$ be fixed and $B(x_2, y_2, z_2)$ vary along a surface $z=\phi(x,y)$. Then, $y=y(x)$, $z=z(x)$ will satisfy the E-L-E's

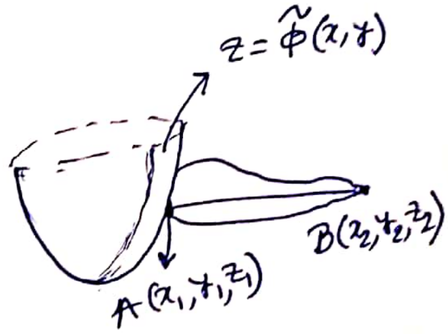
$$f_y - \frac{d}{dx}(f_{y'}) = 0; \quad f_z - \frac{d}{dx}(f_{z'}) = 0. \quad (1) \,\&\, (2).$$

and the transversallity conditions

$$\Big[f - y' f_{y'} + (\phi_x - z') f_{z'}\Big]_{\substack{x=x_2 \\ B}} = 0, \longrightarrow (3)$$

$$\Big[f_{y'} + f_{z'}\,\phi_y\Big]_{\substack{x=x_2 \\ B}} = 0. \longrightarrow (4)$$

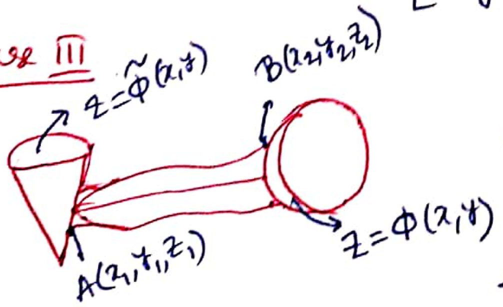
Case - II. Let $B(x_2, y_2, z_2)$ be fixed and $A(x_1, y_1, z_1)$ vary along a surface $z = \tilde{\phi}(x,y)$. Then $y=y(x)$, $z=z(x)$ will satisfy E-L-E's (1) & (2) and the transvarsality conditions.



$$\Big[f - y' f_{y'} + (\tilde{\phi}_x - z') f_{z'}\Big]_{\substack{x=x_1 \\ A}} = 0 \longrightarrow (5)$$

$$\Big[f_{y'} + f_{z'}\,\tilde{\phi}_y\Big]_{\substack{x=x_1 \\ A}} = 0. \longrightarrow (6)$$

Case III



Here $A(x_1, y_1, z_1)$ varies along $z=\tilde{\phi}(x,y)$, $B(x_2, y_2, z_2)$ varies along $z=\phi(x,y)$. Then the extremal $y=y(x)$, $z=z(x)$ satisfy E-L-E's (1) & (2) and T.C.'s (3) - (6).

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Example 1: Find the minimum distance between the point $(1,1,1)$ and the sphere $x^2+y^2+z^2=1$.

Sol. Here $(x_1, y_1, z_1) = (1,1,1)$ is fixed.

(x_2, y_2, z_2) varies along $z = \phi(x,y) = \sqrt{1-x^2-y^2}$.

To find $y = y(x)$, $z = z(x)$ which extremizes

$$I[y(x), z(x)] = \int_{x_1}^{x_2} \sqrt{1+y'^2+z'^2}\, dx = \int_{1}^{x_2} \sqrt{1+y'^2+z'^2}\, dx.$$

$y = y(x)$, $z = z(x)$ will satisfy the E-L-E's

$$f_y - \frac{d}{dx} f_{y'} = 0 \quad \& \quad f_z - \frac{d}{dx} f_{z'} = 0.$$

or equivalently, $f - y' f_{y'} = \text{Const}$, $f - z' f_{z'} = \text{Const}$.

Then show for at least y or z that the extremal is a straight line.

Get $y = c_1 x + c_2 \longrightarrow (1)$, $z = c_3 x + c_4 \longrightarrow (2)$

Thus, $y' = c_1 \longrightarrow (3)$, $z' = c_3 \longrightarrow (4)$

Transversality conditions are,

$$\left[f - y' f_{y'} + (\phi_x - z') f_{z'} \right]_{x=x_2} = 0, \longrightarrow (5)$$

$$\left[f_{y'} + f_{z'} \phi_y \right]_{x=x_2} = 0. \longrightarrow (6).$$

Now $f = \sqrt{1+y'^2+z'^2}$, $\phi = \sqrt{1-x^2-y^2}$

$$\therefore f_{y'} = \frac{y'}{\sqrt{1+y'^2+z'^2}}, \quad f_{z'} = \frac{z'}{\sqrt{1+y'^2+z'^2}}$$

$$\phi_x = -\frac{x}{\sqrt{1-x^2-y^2}}, \quad \phi_y = -\frac{y}{\sqrt{1-x^2-y^2}}.$$

From (5),

$$\left[\sqrt{1+y'^2+z'^2} - \frac{y'^2}{\sqrt{1+y'^2+z'^2}} - \frac{z'^2}{\sqrt{1+y'^2+z'^2}} - \frac{x z'}{\sqrt{1-x^2-y^2}\sqrt{1+y'^2+z'^2}}\right]_{x=x_2} = 0.$$

or, $$\left[\frac{(1+y'^2+z'^2-y'^2-z'^2)\sqrt{1-x^2-y^2} - xz'}{\sqrt{1-x^2-y^2}\sqrt{1+y'^2+z'^2}}\right]_{x=x_2} = 0$$

or, $$\left[\sqrt{1-x^2-y^2} - xz'\right]_{x=x_2} = 0.$$

or, $z_2 - x_2 c_3 = 0$, $\because (x_2, y_2, z_2)$ lies on. $\longrightarrow (7)$ $z = \sqrt{1-x^2-y^2}$ & $z' = c_3$.

Also, from (6),

$$\left[\frac{y'}{\sqrt{1+y'^2+z'^2}} + \frac{z'}{\sqrt{1+y'^2+z'^2}} \times \frac{-y}{\sqrt{1-x^2-y^2}}\right]_{x=x_2} = 0$$

or, $$\left[\sqrt{1-x^2-y^2}\, y' - y z'\right]_{x=x_2} = 0$$

or, $z_2 c_1 - y_2 c_3 = 0$, using (3) & (4).

$\longrightarrow (8)$

Unknowns are, $x_2, y_2, z_2, c_1, c_2, c_3, c_4$.

To find them the equations are the following:

$\overline{(1)}$ $z_2 - x_2 c_3 = 0$ $\rightarrow$ obtained from T.C.

$\overline{(2)}$ $z_2 c_1 - y_2 c_3 = 0$ $\rightarrow$ "

$\overline{(3)}$ $x_2^2 + y_2^2 + z_2^2 = 1$ $\rightarrow$ $\because (x_2, y_2, z_2)$ lies on $z = \sqrt{1-x^2-y^2}$

$$\left.\begin{array}{ll} \overline{(4)} & c_1 + c_2 = 1 \\ \overline{(5)} & c_3 + c_4 = 1 \end{array}\right\}$$ $\because$ $y = y(x) = c_1 x + c_2$, $z = z(x) = c_3 x + c_4$ passes through $(1,1,1)$

$$\left.\begin{array}{ll} \overline{(6)} & c_1 x_2 + c_2 = y_2 \\ \overline{(7)} & c_3 x_2 + c_4 = z_2 \end{array}\right\}$$ $\because$ $y = c_1 x + c_2$, $z = c_3 x + c_4$ passes through (x_2, y_2, z_2).

From $\overline{(1)}$ & $\overline{(7)}$: $z_2 = x_2 c_3 = x_2 c_3 + c_4 \Rightarrow c_4 = 0$.

From $\overline{(5)}$, $c_3 = 1$.

$\overline{(1)} \times c_1$ gives : $z_2 c_1 - x_2 c_3 c_1 = 0 \longrightarrow \overline{(8)}$

comparing $\overline{(2)}$ & $\overline{(8)}$: $y_2 c_3 = x_2 c_3 c_1$

$\Rightarrow y_2 = c_1 x_2 \longrightarrow \overline{(9)}$

By virtue of $\overline{(9)}$, $\overline{(6)}$ yields, $c_2 = 0$

$\therefore$ From $\overline{(4)}$, $c_1 = 1$

★ Thus the minimum distance between (1,1,1) and the sphere is $(\sqrt{3} - 1)$ units.

From $\overline{(1)}$, $\overline{(2)}$ we get after putting $c_1 = c_3 = 1$,

$x_2 = z_2 = y_2 = r$, say

From $\overline{(3)}$, $3r^2 = 1 \Rightarrow r = \pm \frac{1}{\sqrt{3}} \Rightarrow x_2 = y_2 = z_2 = \pm \frac{1}{\sqrt{3}}$

Required distance is,

either $d_1 = \left| \int_{x_2 = \frac{1}{\sqrt{3}}}^{1} \sqrt{1 + y'^2 + z'^2}\, dx \right| = \left| \int_{1/\sqrt{3}}^{1} \sqrt{1 + c_1^2 + c_3^2}\, dx \right| = \sqrt{3} \left| \left(1 - \frac{1}{\sqrt{3}}\right) \right|$

$= (\sqrt{3} - 1)$ units ★

or, $d_2 = \left| \int_{x_2 = -\frac{1}{\sqrt{3}}}^{1} \sqrt{1 + y'^2 + z'^2}\, dx \right| = \left| \int_{-1/\sqrt{3}}^{1} \sqrt{1+1+1}\, dx \right| = \sqrt{3} \left| \left(1 + \frac{1}{\sqrt{3}}\right) \right| = (\sqrt{3} + 1)$ units