3. Method of resolvent kernel

3.1 Resolvent kernel

Definition 1 Fredholm determinant $D(\lambda)$

$$D(\lambda) = 1 - \lambda A_1 + \frac{\lambda^2}{2!} A_2 - \frac{\lambda^3}{3!} A_3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^n A_n \quad \text{with } A_0 = 1.$$

A_n's are given by,

$$A_n = \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{n \text{ times}} K\begin{pmatrix} t_1 & t_2 & \cdots & t_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} dt_1\, dt_2 \cdots dt_n.$$

Here,

$$K\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \cdots & K(x_1, y_n) \\ K(x_2, y_1) & K(x_2, y_2) & \cdots & K(x_2, y_n) \\ \vdots & & & \\ K(x_n, y_1) & K(x_n, y_2) & \cdots & K(x_n, y_n) \end{vmatrix}$$

Definition 2 Fredholm minor $D(x, y; \lambda)$.

$$D(x, y; \lambda) = \lambda B_0(x, y) - \lambda^2 B_1(x, y) + \frac{\lambda^3}{2!} B_2(x, y) - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^{n+1} B_n(x, y)$$

$$B_0(x, y) = K(x, y)$$

$$B_n(x, y) = \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{n \text{ times}} K\begin{pmatrix} x & t_1 & t_2 & \cdots & t_n \\ y & t_1 & t_2 & \cdots & t_n \end{pmatrix} dt_1\, dt_2 \cdots dt_n$$

Definitn 3: The resolvent kernel $R(x,t;\lambda)$ is defined by, $R(x,t;\lambda) = \dfrac{D(x,t;\lambda)}{D(\lambda)}$.

Solution to the F.I.E.

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt$$

is given by,

$$u(x) = f(x) + \int_a^b R(x,t;\lambda) f(t)\,dt$$

Example 1 Using the concept of Fredholm determinant find the resolvent kernel corresponding to ~~K(x,t)~~

$K(x,t) = xe^t; \quad 0 \le x, t \le 1.$

Hence solve the IE:

$$u(x) = e^{-x} - 2\int_0^1 xe^t u(t)\,dt.$$

Solution: Resolvent kernel $R(x,t;\lambda)$ is given by,

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)};$$

$$D(\lambda) = 1 - \lambda A_1 + \frac{\lambda^2}{2!}A_2 - \frac{\lambda^3}{3!}A_3 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\lambda^n A_n.$$

$$A_0 = 1, \quad A_n = \int_a^b\int_a^b \cdots \int_a^b K\begin{pmatrix} t_1 & \cdots & t_n \\ t_1 & \cdots & t_n\end{pmatrix} dt_1\, dt_2 \cdots dt_n$$

$$K\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n\end{pmatrix} = \begin{vmatrix} K(x_1,y_1) & K(x_1,y_2) & \cdots & K(x_1,y_n) \\ K(x_2,y_1) & K(x_2,y_2) & \cdots & K(x_2,y_n) \\ \vdots & & & \\ K(x_n,y_1) & K(x_n,y_2) & \cdots & K(x_n,y_n)\end{vmatrix}$$

$$A_1 = \int_a^b K\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} dt_1 \; ; \quad K\begin{pmatrix} t_1 \\ t_1 \end{pmatrix} = K(t_1, t_1) = t_1 e^{t_1}$$

$$K(x,t) = x e^t.$$

$$A_1 = \int_0^1 t_1 e^{t_1} dt_1 = \left[t_1 e^{t_1}\right]_0^1 - \int_0^1 e^{t_1} dt_1$$

$$= e - \left[e^{t_1}\right]_0^1 = e - e + 1 = 1$$

$$A_2 = \int_a^b \int_a^b K\begin{pmatrix} t_1 & t_2 \\ t_1 & t_2 \end{pmatrix} dt_1\, dt_2$$

$$K\begin{pmatrix} t_1 & t_2 \\ t_1 & t_2 \end{pmatrix} = \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) \\ K(t_2,t_1) & K(t_2,t_2) \end{vmatrix} = \begin{vmatrix} t_1 e^{t_1} & t_1 e^{t_2} \\ t_2 e^{t_1} & t_2 e^{t_2} \end{vmatrix}$$

$$= t_1 t_2 e^{t_1+t_2} - t_1 t_2 e^{t_1+t_2} = 0.$$

$\therefore A_2 = 0.$

$$A_3 = \int_a^b \int_a^b \int_a^b K\begin{pmatrix} t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 \end{pmatrix} dt_1\, dt_2\, dt_3.$$

$$K\begin{pmatrix} t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 \end{pmatrix} = \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) & K(t_1,t_3) \\ K(t_2,t_1) & K(t_2,t_2) & K(t_2,t_3) \\ K(t_3,t_1) & K(t_3,t_2) & K(t_3,t_3) \end{vmatrix} = 0$$

It can be shown that,

$$A_n = 0 \text{ for } n \geq 2.$$

$$\therefore D(\lambda) = 1 - \lambda A_1 = 1 - \lambda.$$

$$D(x,y;\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^{n+1} B_n(x,y) \ ; \qquad K(x,y) = x e^y$$

$$B_0(x,y) = K(x,y)$$

$$B_1(x,y) = \int_a^b K\begin{pmatrix} x & t_1 \\ y & t_1 \end{pmatrix} dt_1 = \int_0^1 K\begin{pmatrix} x & t_1 \\ y & t_1 \end{pmatrix} dt_1 .$$

$$K\begin{pmatrix} x & t_1 \\ y & t_1 \end{pmatrix} = \begin{vmatrix} K(x,y) & K(x,t_1) \\ K(t_1,y) & K(t_1,t_1) \end{vmatrix} = \begin{vmatrix} xe^y & xe^{t_1} \\ t_1 e^y & t_1 e^{t_1} \end{vmatrix}$$

$$= x t_1 e^{y+t_1} - x t_1 e^{y+t_1}$$

$$= 0 .$$

$\therefore B_1(x,y) = 0$

We can show, $B_n(x,y) = 0 \quad \forall\, n \geq 1$.

$\therefore D(x,y;\lambda) = \lambda B_0(x,y) = \lambda x e^y$.

$\therefore D(x,t;\lambda) = \lambda x e^t$.

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} = \frac{\lambda x e^t}{1-\lambda} .$$

$$R(x,t;-2) = -\frac{2}{3} x e^t$$

Solution to the given IE is,

$$u(x) = f(x) + \int_a^b R(x,t;\lambda) f(t)\, dt$$

$$\text{or, } u(x) = e^{-x} + \int_0^1 -\frac{2}{3} x e^t \cdot e^{-t} dt = e^{-x} - \frac{2x}{3} .$$

3.2 Recurrence Relations

$$A_n = \int_a^b B_{n-1}(x,x)\,dx \;;\quad n = 1,2,3,\ldots$$

$$B_n(x,t) = A_n K(x,t) - n\int_a^b K(x,s)\,B_{n-1}(s,t)\,ds \;;\quad n=1,2,3\ldots$$

Let us now solve the IE.

$$u(x) = e^{-x} - 2\int_0^1 x e^t u(t)\,dt$$

Here $K(x,t) = xe^t$; $\lambda = -2$

$$\boxed{\begin{aligned}&\int_0^1 xe^x\,dx\\ &= \left[xe^x\right]_0^1 - \int_0^1 e^x\,dx\\ &= 1.e^1 - \left[e^x\right]_0^1\\ &= e - e + 1 = 1\end{aligned}}$$

$A_0 = 1$ (always)

$$A_1 = \int_0^1 B_0(x,x)\,dx = \int_0^1 K(x,x)\,dx = \int_0^1 xe^x\,dx = 1.$$

$$\begin{aligned}B_1(x,t) &= A_1 K(x,t) - 1.\int_0^1 K(x,s)\,B_0(s,t)\,ds.\\ &= 1\times xe^t - \int_0^1 K(x,s)\,K(s,t)\,ds\\ &= xe^t - \int_0^1 xe^s s e^t\,ds\\ &= xe^t - xe^t\int_0^1 se^s\,ds = xe^t - xe^t.1 = 0.\end{aligned}$$

$$A_2 = \int_0^1 B_1(x,x)\,dx = \int_0^1 0.\,dx = 0.$$

$$B_2 = \underbrace{A_2}_{=0} K(x,t) - 2\int_0^1 K(x,s).\underbrace{B_1(s,t)}_{=0}\,ds = 0.$$

$\therefore A_n = 0$, $n \geq 2$; $B_n = 0$; $n \geq 1$.

$$\therefore\ D(\lambda) = 1 - \lambda A_1 + \frac{\lambda^2}{2!} A_2 - \frac{\lambda^3}{3!} A_3 + \cdots$$

$$= 1 - \lambda A_1 = 1 + 2 \cdot 1 = 3.$$

$$D(x,t;\lambda) = \lambda B_0(x,t) - \lambda^2 B_1(x,t) + \frac{\lambda^3}{2!} B_2(x,t) - \cdots$$

$$= \lambda K(x,t) = -2 \cdot x e^t$$

Thus, $R(x,t;\lambda) = \dfrac{D(x,t;\lambda)}{D(\lambda)} = -\dfrac{2}{3} x e^t.$

$\therefore$ Solu^n to $u(x) = e^{-x} - 2\int_0^1 x e^t u(t)\,dt$

is given by, $u(x) = f(x) + \int_a^b R(x,t;\lambda) f(t)\,dt$

$\because\ f(x) = e^{-x},\ a = 0,\ b = 1,\ R(x,t;\lambda) = -\frac{2}{3} x e^t,$

$$\therefore\ u(x) = e^{-x} + \int_0^1 \left(-\frac{2}{3}x\right) e^t \cdot e^{-t}\,dt = e^{-x} - \frac{2x}{3}. //$$

<u>Exercise</u>: Find the resolvent kernel corresponding to $K(x,t) = e^{x-t};\ 0 \le x \le 1.$

Hence solve the IE:

$$u(x) = e^x - \int_0^1 e^{x-t} u(t)\,dt$$

Solu^n $A_0 = 1$, $B_0(x,t) = K(x,t)$

Here $f(x) = e^x$, $\lambda = -1$, $K(x,t) = e^{x-t}$.

$$A_1 = \int_0^1 B_0(x,x)\,dx = \int_0^1 K(x,x)\,dx = \int_0^1 e^{x-x}dx = \int_0^1 1.\,dx = 1$$

$$B_1(x,t) = A_1 \cdot K(x,t) - 1.\int_0^1 K(x,s)\,B_0(s,t)\,ds$$

$$= 1.\,e^{x-t} - 1\int_0^1 e^{x-s} \cdot e^{s-t}\,ds$$

$$= e^{x-t} - \int_0^1 e^{x-t}\,ds = e^{x-t} - e^{x-t} = 0.$$

$$A_2 = \int_0^1 B_1(x,x)\,dx = 0.$$

$$B_2(x,t) = \underbrace{A_2}_{=0} K(x,t) - 2\int_0^1 K(x,s)\,\underbrace{B_1(s,t)}_{=0}\,ds = 0$$

$\therefore$ $A_n = 0$, $n \geq 2$, $B_n(x,t) = 0$; $n \geq 1$

$$\therefore\ D(\lambda) = 1 - \lambda A_1 + \frac{\lambda^2}{2!}A_2 - \frac{\lambda^3}{3!}A_3 + \cdots = 1 - \lambda A_1 = 1 + 1.1 = 2.$$

$$D(x,t;\lambda) = \lambda B_0(x,t) - \lambda^2 B_1(x,t) + \frac{\lambda^3}{2!}B_2(x,t) - \cdots$$

$$= \lambda K(x,t) = -e^{x-t}$$

$$\therefore\ R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} = -\frac{1}{2}.e^{x-t}.$$

$\therefore$ Sol. to $u(x) = e^x - \int_0^1 e^{x-t}u(t)\,dt$ is,

$$u(x) = e^x + \int_0^1 \left(-\frac{1}{2}e^{x-t}\right)e^t\,dt = e^x - \frac{1}{2}e^x = \frac{e^x}{2}.\ //$$

3.3 Alternative method of finding resolvent kernel:

Iterated kernels

Resolvent kernel $R(x,t;\lambda)$ can also be expressed as,

$$R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^n K_n(x,t)$$

$$= \lambda K_1(x,t) + \lambda^2 K_2(x,t) + \lambda^3 K_3(x,t) + \cdots$$

Here $K_n(x,t)$ $(n=1,2,3,\ldots)$ are iterated kernels given as,

$$K_1(x,t) = K(x,t)$$

$$K_r(x,t) = \int_a^b K(x,s)\, K_{r-1}(s,t)\, ds \,; \quad r = 2,3,4,\ldots$$

$$\therefore\; K_2(x,t) = \int_a^b K(x,s)\, K_1(s,t)\, ds = \int_a^b K(x,s)\, K(s,t)\, ds$$

$$K_3(x,t) = \int_a^b K(x,s)\, K_2(s,t)\, ds = \int_a^b K(x,s)\left(\int_a^b K(s,t_1)\, K(t_1,t)\, dt_1\right) ds$$

$$= \int_a^b\int_a^b K(x,s)\, K(s,t_1)\, K(t_1,t)\, dt_1\, ds.$$

$$K_4(x,t) = \int_a^b K(x,s)\, K_3(s,t)\, ds = \int_a^b K(x,s)\left(\int_a^b\int_a^b K(s,t_1)\, K(t_1,t_2)\, K(t_2,t)\, dt_2\, dt_1\right) ds$$

$$= \int_a^b\int_a^b\int_a^b K(x,s)\, K(s,t_1)\, K(t_1,t_2)\, K(t_2,t)\, dt_2\, dt_1\, ds$$

$\vdots$

$$K_n(x,t) = \underbrace{\int_a^b\int_a^b \cdots \int_a^b}_{(n-1)\text{ times}} K(x,s)\, K(s,t_1)\, K(t_1,t_2) \cdots K(t_{n-2},t)\; dt_{n-2}\, dt_{n-3} \cdots dt_2\, dt_1\, ds.$$

It can be shown that,

$$K_{m+n}(x,t) = \int_a^b K_m(x,s) K_n(s,t)\,ds = \int_a^b K_n(x,s) K_m(s,t)\,ds.$$

Example: Find the iterated kernels for the kernel

$K(x,t) = x-t$, if $a=0, b=1$.

Solut^n: $K_1(x,t) = K(x,t) = x-t$.

$$K_2(x,t) = \int_a^b K(x,s) K_1(s,t)\,ds = \int_0^1 (x-s)(s-t)\,ds$$

$$= \int_0^1 \left[s(x+t) - s^2 - xt\right] ds = \frac{x+t}{2} - \frac{1}{3} - xt.$$

$$K_3(x,t) = \int_0^1 K(x,s) K_2(s,t)\,ds = \int_0^1 (x-s)\left(\frac{s+t}{2} - \frac{1}{3} - st\right) ds$$

$$= \int_0^1 (x-s)\left\{s\left(\frac{1}{2}-t\right) + \frac{t}{2} - \frac{1}{3}\right\} ds$$

$$= \int_0^1 \left[s\left\{x\left(\frac{1}{2}-t\right) - \left(\frac{t}{2}-\frac{1}{3}\right)\right\} - s^2\left(\frac{1}{2}-t\right) + x\left(\frac{t}{2}-\frac{1}{3}\right)\right] ds$$

$$= \frac{1}{2}\left\{x\left(\frac{1}{2}-t\right) - \frac{t}{2} + \frac{1}{3}\right\} - \frac{1}{3}\left(\frac{1}{2}-t\right) + x\left(\frac{t}{2}-\frac{1}{3}\right)$$

$$= \frac{t-x}{12} = -\frac{1}{12}(x-t).$$

$$K_4(x,t) = \int_0^1 K(x,s) K_3(s,t)\,ds \quad ; \quad K(x,t) = x-t$$

$$= \int_0^1 (x-s) \times -\frac{1}{12}(s-t)\,ds = -\frac{1}{12}\int_0^1 (x-s)(s-t)\,ds$$

$$= -\frac{1}{12} \cdot K_2(x,t)$$

$$= -\frac{1}{12}\left(\frac{x+t}{2} - \frac{1}{3} - xt\right).$$

$$K_5(x,t) = \int_0^1 K(x,s)\, K_4(s,t)\, ds.$$

$$= \int_0^1 (x-s) \times -\frac{1}{12}\left(\frac{s+t}{2} - \frac{1}{3} - st\right) ds.$$

$$= -\frac{1}{12} \times K_3(x,t) = -\frac{1}{12} \times -\frac{1}{12}(x-t)$$

$$\therefore \quad K_5(x,t) = \left(-\frac{1}{12}\right)^2 (x-t).$$

Thus,

$$K_1(x,t) = x - t \qquad\qquad K_2(x,t) = \frac{x+t}{2} - \frac{1}{3} - xt$$

$$K_3(x,t) = -\frac{1}{12}(x-t) \qquad\qquad K_4(x,t) = \left(-\frac{1}{12}\right)\left(\frac{x+t}{2} - \frac{1}{3} - xt\right).$$

check

$$K_5(x,t) = \left(-\frac{1}{12}\right)^2 (x-t) \qquad\qquad K_6(x,t) = \left(-\frac{1}{12}\right)^2 \left(\frac{x+t}{2} - \frac{1}{3} - xt\right).$$

Iterated kernels are given by.

$$K_{2n-1}(x,t) = \left(-\frac{1}{12}\right)^{n-1} (x-t), \qquad n = 1, 2, 3 \ldots$$

$$K_{2n}(x,t) = \left(-\frac{1}{12}\right)^{n-1} \left(\frac{x+t}{2} - \frac{1}{3} - xt\right); \quad n = 1, 2, 3 \ldots$$