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§ Reduction of an initial value problem (IVP) to a Volterra Integral Equation (VIE)

Consider the ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y(x) = r(x), \quad a \le x \le b \longrightarrow (1)$$

Initial conditions: $y(a) = c_0$, $y'(a) = c_1$

Let $\boxed{\frac{d^2y}{dx^2} = u(x) \longrightarrow (2)}$ Integrating eq (2) w.r.to x between a and x, we get $\int_a^x \frac{d^2y}{dx^2}\,dx = \int_a^x u(t)\,dt$

or, $\frac{dy}{dx} - \frac{dy}{dx}\Big|_{x=a} = \int_a^x u(t)\,dt$ or, $y'(x) - c_1 = \int_a^x u(t)\,dt$

or, $\boxed{y'(x) = c_1 + \int_a^x u(t)\,dt \longrightarrow (3)}$ Int. eq. (3) w.r.to x bet. a & x, we get

$$y(x) - y(a) = c_1\int_a^x dx + \int_a^x u(t)\,dt^2 = c_1(x-a) + \int_a^x (x-t)\,u(t)\,dt$$

$$\boxed{y(x) = c_0 + c_1(x-a) + \int_a^x (x-t)\,u(t)\,dt} \longrightarrow (4)$$

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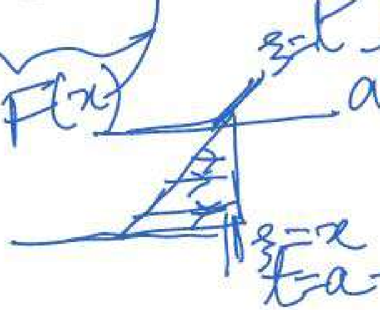
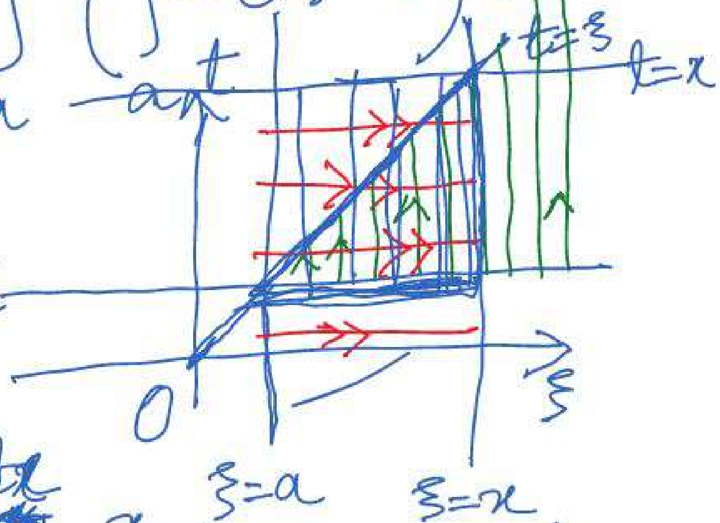
Result: $$\int_a^x u(t)\,dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\,u(t)\,dt$$

Let $n=2$

$$\int_a^x u(t)\,dt^2 = \int_a^x \Bigg(\underbrace{\int_a^x u(t)\,dt}_{F(x)}\Bigg)dx = \int_a^x \Bigg(\overbrace{\int_a^\xi u(t)\,dt}^{F(\xi)}\Bigg)d\xi$$

$$= \int_{t=a}^x \left(\int_t^x d\xi\right) u(t)\,dt = \int_a^x (x-t)\,u(t)\,dt$$





$n=3$

$$\int_a^x u(t)\,dt^3 = \int_a^x \left(\int_a^x u(t)\,dt^2\right) = \int_a^x \Bigg(\overbrace{\int_a^x (x-t)\,u(t)\,dt}^{G(x)}\Bigg)dx = \int_a^x \Bigg(\underbrace{\int_a^\xi (\xi-t)\,u(t)\,dt}_{G(\xi)}\Bigg)d\xi$$

$$= \int_{t=a}^x \left(\int_t^x (\xi-t)\,d\xi\right) u(t)\,dt = \int_a^t \frac{(x-t)^2}{2}\,u(t)\,dt.$$

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$$u(x) + p(x)\left\{c_1 + \int_a^x u(t)\,dt\right\} + q(x)\left\{c_0 + c_1(x-a) + \int_a^x (x-t)\,u(t)\,dt\right\} = r(x)$$

or,
$$u(x) = r(x) - c_1 p(x) - q(x)\{c_0 + c_1(x-a)\} - \int_a^x \{p(x) + q(x)(x-t)\}\,u(t)\,dt$$

This is in the form
$$u(x) = f(x) + \lambda \int_a^x K(x,t)\,u(t)\,dt \longrightarrow (5)$$

where
$$f(x) = r(x) - c_1 p(x) - q(x)\{c_0 + c_1(x-a)\}$$
$$\lambda = -1, \quad K(x,t) = p(x) + q(x)(x-t)$$

$\therefore$ The given IVP is equivalent to the VIE in (5)

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1. Convert the IVP $y''' - 3y'' - 6y' + 5y = 0$; IC's: $y(0) = y'(0) = y''(0) = 1$

to an equivalent VIE.

Sol. Let $\dfrac{d^3y}{dx^3} = u(x) \longrightarrow (1)$

Int (1) w.r.to x bet. 0 and x we get $\dfrac{d^2y}{dx^2} - \left.\dfrac{d^2y}{dx^2}\right|_{x=0} = \int_0^x u(t)\,dt$

or, $y''(x) = 1 + \int_0^x u(t)\,dt \longrightarrow (2)$

Int (2) w.r.to x bet. 0 and x we get $y'(x) - y'(0) = \int_0^x 1.\,dx + \int_0^x \left(\int_0^x u(t)\,dt\right) dx$

$= \int_0^x u(t)\,dt^2$

$\therefore y'(x) = 1 + x + \int_0^x (x-t)\,u(t)\,dt \longrightarrow (3)$

Int (3) w.r.to x bet. 0 & x,

$y(x) - y(0) = \int_0^x (1+\xi)\,d\xi + \int_0^x u(t)\,dt^3 = \int_0^x (x-t)^2 u(t)\,dt$

$\Rightarrow y(x) = 1 + x + \dfrac{x^2}{2} + \int_0^x \dfrac{(x-t)^2}{2} u(t)\,dt \longrightarrow (4)$

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$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt$$

$$\lambda = -1$$

$$f(x) = 4 + x - \frac{5}{2}x^2$$

$$K(x,t) = \frac{5}{2}(x-t)^2 - 6(x-t) - 3.$$

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Conversion of VIE to IVP

Leibnitz rule: (Differentiation under the integral sign)

Suppose $f(x,t)$ and the partial derivative $\frac{\partial f}{\partial t}(x,t)$ are continuous on rectangle $a \le x \le b$, $c \le t \le d$. Then

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx \longrightarrow (1)$$

Now if $a \equiv \alpha(t)$, $b \equiv \beta(t)$, then the generalized Leibnitz rule is,

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}(x,t)\,dx + \beta'(t)\cdot f(\beta(t),t) - \alpha'(t)\, f(\alpha(t),t)$$

provided $\alpha(t)$, $\beta(t)$ are continuous in $c \le t \le d$..

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Example: Reduce the VIE to an IVP. Hence ~~solve~~ solve it. Verify that the derived sol. is indeed the solution of the given IE.

$$u(x) = 1 - 2x - 4x^2 + \int_0^x \underbrace{\{3 + 6(x-t) - 4(x-t)^2\}}_{f(x,t)} u(t)\,dt \longrightarrow (1)$$

$u(0) = 1$

Sol Differentiate both sides w.r.t. x:

$$u'(x) = -2 - 8x + \frac{d}{dx}\int_0^x \{3 + 6(x-t) - 4(x-t)^2\} u(t)\,dt$$

$$u'(x) = -2 - 8x + \int_0^x \frac{\partial}{\partial x}\{3 + 6(x-t) - 4(x-t)^2\} u(t)\,dt + 1.f(x,x)$$

$f(x,x) = 3u(x)$

$$= -2 - 8x + \int_0^x \{6 - 8(x-t)\} u(t)\,dt + 3u(x) \longrightarrow (2)$$

$u'(0) = -2 + 3u(0) = -2 + 3.1 = 1$

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Differentiating (2) w.r.to x, we get

$$u''(x) = -8 - \int_0^x 8u(t)\,dt + 1.\{6 - 8(x-x)\}u(x) + 3u'(x)$$

$$u''(x) = -8 - 8\int_0^x u(t)\,dt + 6u(x) + 3u'(x) \longrightarrow (3)$$

$$u''(0) = -8 + 6u(0) + 3u'(0) = 1$$

$$\frac{d}{dx}\int_0^x u(t)\,dt + 1.u(x) = \int_0^x \cancel{\frac{\partial}{\partial x}u(t)\,dt}_{\,0} + u(x)$$

Diff. (3) w.r.to x, we get

$$u'''(x) = \overset{0}{\cancel{0}} \overset{0+}{-} 8u(x) + 6u'(x) + 3u''(x)$$

$$\boxed{\begin{array}{l} u'''(x) - 3u''(x) - 6u'(x) + 8u(x) = 0 \\ u(0) = u'(0) = u''(0) = 1 \longrightarrow (4) \end{array}} \longrightarrow \text{IVP}$$

$m^3 - 3m^2 - 6m + 8 = 0$ $m = 1, 4, -2$ $u(x) = c_1 e^{x} + c_2 e^{4x} + c_3 e^{-2x}$

The IC's (4) give $c_1 = 1$, $c_2 = c_3 = 0$. $u(x) = e^{x}$