Chapter-4: Homogeneous Integral Equations

Consider the Homogeneous Fredholm Integral Equatn.

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt. \longrightarrow (1).$$

Note 4.1 $u(x) = 0$ is a trivial solution.

Definitn 4.1 λ is called an eigenvalue of the IE (1), ~~and~~ if (1) has a non-trivial solution $u(x)$. $u(x)$ is called an e-functn corresponding to λ.

Note 4.2 Writing $\mu = \frac{1}{\lambda}$ and defining $Iu = \int_a^b K(x,t)u(t)dt$, (1) can be expressed as

$$Iu = \mu u \longrightarrow (2)$$

which is similar to an e-value problem,

$$T\underset{\sim}{v} = \lambda \underset{\sim}{v}. \longrightarrow (3).$$

Note 4.3 In (3), λ may be zero.

But, in (1), λ cannot be zero.

Because, if $\lambda = 0$, $u(x) = 0$ so that $u(x)$ can't be an eigenfunction.

Case 1. Separable kernel.

Here the kernel $K(x,t)$ can be expressed as,

$$K(x,t) = \sum_{i=1}^{n} a_i(x) b_i(t) \longrightarrow (4)$$

Substitute (4) in (1) & get

$$u(x) = \lambda \int_a^b \sum_{i=1}^{n} a_i(x) b_i(t) u(t) dt$$

Let $\quad A_i = \int_a^b b_i(t) u(t) dt \longrightarrow (5)$

Then,

$$u(x) = \lambda \sum_{i=1}^{n} a_i(x) A_i = \lambda \sum_{j=1} a_j(x) A_j \longrightarrow (6)$$

Substituting $u(x)$ from (6) into R.H.S. of (5) we obtain

$$A_i = \int_a^b b_i(t) \left(\lambda \sum_{j=1}^{n} a_j(t) A_j \right) dt.$$

or, $\quad A_i = \lambda \sum_{j=1}^{n} A_j \int_a^b b_i(t) a_j(t) dt \longrightarrow (7)$

$; \; i = 1, 2, \ldots, n.$

Let $\quad C_{ij} = \int_a^b b_i(t) a_j(t) dt.$

Then (7) becomes, $\quad A_i = \lambda \sum_{j=1}^{n} C_{ij} A_j \longrightarrow (8)$

Bringing A_i's on the same side and putting $i = 1, 2, 3, \ldots, n$ we get,

$$(1-\lambda C_{11})A_1 - \lambda C_{12}A_2 - \cdots - \lambda C_{1n}A_n = 0$$

$$-\lambda C_{21}A_1 + (1-\lambda C_{22})A_2 - \cdots - \lambda C_{2n}A_n = 0$$

$$\vdots$$

$$-\lambda C_{n1}A_1 - \lambda C_{n2}A_2 - \cdots + (1-\lambda C_{nn})A_n = 0.$$

or,
$$\begin{pmatrix} 1-\lambda C_{11} & -\lambda C_{12} & \cdots & -\lambda C_{1n} \\ -\lambda C_{21} & 1-\lambda C_{22} & & -\lambda C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ +\lambda C_{n1} & +\lambda C_{n2} & & 1-\lambda C_{nn} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This is a homogeneous system of n equations in n unknowns $A_1, A_2, \ldots, A_n$. Thus the system will have a non-zero solution $(A_1, A_2, \ldots, A_n)$ if and only if,

$$\begin{vmatrix} 1-\lambda C_{11} & -\lambda C_{12} & \cdots & -\lambda C_{1n} \\ -\lambda C_{21} & 1-\lambda C_{22} & \cdots & -\lambda C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda C_{n1} & -C_{n2} & \cdots & 1-\lambda C_{nn} \end{vmatrix} = 0. \quad \rightarrow (9)$$

Definition 4.2 The above equation is known as the characteristic equation corresponding to the IE (1) where the kernel $K(x,t)$ is separable as given in (4).

Note 4.4 The equation (9) is a polynomial equation in λ where the degree of the polynomial is n. Solving this equation we get the eigenvalues.

$\boxed{\text{Case 2}}$ General kernel.

Eigenvalues are the roots of $D(\lambda) = 0$.
Here $D(\lambda)$ is the Fredholm's determinant.

Definition 4.3: The index q of an e-value λ is defined as the number of linearly independent eigen functions corresponding to an e-value.

Note 4.5 There may exist no eigenvalue, one e-value, more than one eigenvalue, no real eigenvalues.

Note 4.6 If λ is complex, then there does not exist any real non-trivial solution $u(x)$ of the homogeneous equation.