Functionals involving several dependent variables

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Let $x_1(t), x_2(t), \ldots, x_n(t)$ be twice differentiable functions of the independent variable t. Let these functions extremize the integral

$$I = \int_{t_1}^{t_2} F(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n, t)\,dt. \longrightarrow (1)$$

Here $\dot{x}_j = \frac{dx_j}{dt}$; $j = 1, 2, \ldots, n$.

Let ϵ be a small parameter. Let us form the comparison functions

$$X_1(t) = x_1(t) + \epsilon\,\xi_1(t),\quad X_2(t) = x_2(t) + \epsilon\,\xi_2(t), \ldots,$$
$$X_n(t) = x_n(t) + \epsilon\,\xi_n(t). \longrightarrow (2)$$

Here ξ_j $(j = 1, 2, \ldots, n)$ are arbit. differentiable functions for which

$$\xi_j(t_1) = \xi_j(t_2) = 0\ ; \quad j = 1, 2, \ldots, n. \longrightarrow (3)$$

Suppose the boundary conditions $x_j(t_1) = x_j^1$ & $x_j(t_2) = x_j^2$ are given. Then substituting $t = t_1$ & $t = t_2$ in turn, into (2), we get after using (3),

$$X_j(t_1) = x_j(t_1) = x_j^1 \text{ and } X_j(t_2) = x_j(t_2) = x_j^2. \longrightarrow (4)$$

Thus the functions $X_j(t)$ satisfy the given boundary conditions.

Next, consider the function $I(\epsilon)$ of ϵ defined by

$$I(\epsilon) = \int_{t_1}^{t_2} F(X_1, \ldots, X_n, \dot{X}_1, \ldots, \dot{X}_n, t)\,dt \longrightarrow (5)$$

Observe that

$$I(0) = \int_{t_1}^{t_2} F(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, t)\, dt. \longrightarrow (6)$$

Thus $I(\epsilon)$ takes its extreme value when $\epsilon = 0$, since the functions $x_j(t)$ extremizes the integral given in (1) which is same as the above integral.

Now for extreme value of $I(\epsilon)$ at $\epsilon = 0$,

we must have $\dfrac{dI}{d\epsilon} = 0$ at $\epsilon = 0$. $\longrightarrow$ (7)

Now from (2) $\dot{X}_j(t) = \dot{x}_j(t) + \epsilon \dot{\xi}_j(t)$; $j = 1, 2, \ldots n$.
$\longrightarrow$ (8)

Taking derivative of (5), we obtain

$$\frac{dI}{d\epsilon} = \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial X_1} \cdot \xi_1 + \frac{\partial F}{\partial \dot{X}_1} \cdot \dot{\xi}_1 + \cdots + \frac{\partial F}{\partial X_n} \cdot \xi_n + \frac{\partial F}{\partial \dot{X}_n} \cdot \dot{\xi}_n \right) dt.$$
$\longrightarrow$ (9)

Observe that (from (2) & (8)) if we set $\epsilon = 0$, then X_j & $\dot{X}_j$'s can be replaced by x_j & $\dot{x}_j$'s.

$$\text{So, } \frac{dI}{d\epsilon}(0) = \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x_1} \xi_1 + \frac{\partial F}{\partial \dot{x}_1} \cdot \dot{\xi}_1 + \cdots + \frac{\partial F}{\partial x_n} \cdot \xi_n + \frac{\partial F}{\partial \dot{x}_n} \cdot \dot{\xi}_n \right) dt.$$
$$= 0. \longrightarrow (10)$$

The above relation must hold for all choices of the functions $\xi_j(t)$ $(j = 1, 2, \ldots n)$. In particular, it holds for the special choice in which $\xi_2, \ldots \xi_n$ are identically zero, but for which $\xi_1(t)$ is still arbitrary, ~~constant~~ satisfying $\xi_1(t_1) = 0 = \xi_1(t_2)$.

With this choice, we have

$$\int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x_1} \xi_1 + \frac{\partial F}{\partial \dot{x}_1} \cdot \dot{\xi}_1 \right) dt = 0 \longrightarrow (11).$$

Now, $\displaystyle\int_{t_1}^{t_2} \frac{\partial F}{\partial \dot{x}_1} \cdot \dot{\xi}_1 \, dt$

$$= \left. \frac{\partial F}{\partial \dot{x}_1} \cdot \xi_1 \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_1} \cdot \xi_1 \, dt = - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}_1} \right) \cdot \xi_1 \, dt.$$

So, (11) reduces to

$$\int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x_1} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}_1} \right) \right) \xi_1 \, dt = 0 \longrightarrow (12).$$

Since (12) holds for any arbitrary ξ_1, we have

$$\frac{\partial F}{\partial x_1} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}_1} \right) = 0$$

Next, taking $\xi_1 = 0 = \xi_3 = \xi_4 = \cdots = \xi_n$ and letting ξ_2 to be arbitrary we will obtain

$$\frac{\partial F}{\partial x_2} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}_2} \right) = 0.$$

Thus for functionals involving n dependent variables of a single variable t, we will have a system of n Euler-Lagrange equations:

$$\boxed{\frac{\partial F}{\partial x_j} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}_j} \right) = 0 \; ; \; j = 1, 2, \ldots, n.}$$

These equations must be satisfied by the functions $x_j(t)$ which make the integral in (1) an extremum.

$$I[x_1, x_2, \dots, x_n] = \int_{t_1}^{t_2} F(x_1, \dots, x_n; \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n, t)\,dt$$

Ex Find the extremals of the functional

$$I[y(x), z(x)] = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz)\,dx,$$

$y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 1, \quad z(0) = 0, \quad z\left(\frac{\pi}{2}\right) = -1.$

Ans. ~~The system of Euler equations is~~

Here $F = y'^2 + z'^2 + 2yz$

The 2 Euler Lagrange equations are: $F_y - (d/dx)(F_{y'}) = 0$ i.e. $2z - (d/dx)(2y') = 0$

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$$y'' - z = 0, \quad \dots(1) \qquad z'' - y = 0. \quad \dots(2)$$

Eliminating z from (1) and (2), we get

$$\Rightarrow \quad y^{(iv)} - y = 0 \qquad \frac{d^4}{dx^4}$$

$$\Rightarrow \quad y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

From (1), $z = y''$. Thus,

$$z = c_1 e^{x} + c_2 e^{-x} - c_3 \cos x - c_4 \sin x.$$

$$y(0) = 0 \Rightarrow c_1 + c_2 + c_3 = 0$$

$$y\left(\tfrac{\pi}{2}\right) = 1 \Rightarrow c_1 e^{\frac{\pi}{2}} + c_2 e^{-\frac{\pi}{2}} + c_4 = 1$$

$$z(0) = 0 \Rightarrow c_1 + c_2 - c_3 = 0$$

$$z\left(\tfrac{\pi}{2}\right) = -1 \Rightarrow c_1 e^{\frac{\pi}{2}} + c_2 e^{-\frac{\pi}{2}} - c_4 = -1$$

$c_3 = 0$

$c_4 = 1$

$$c_1 = 0 = c_2 = c_3; \quad c_4 = 1.$$

$$\therefore \quad y = \sin x, \quad z = -\sin x.$$

P.T.O

§ Functional involving one dependent variable and more than one independent variable: Euler-Ostrogradski eqn.

To find the surface $z = z(x,y)$ that extremizes the functional

$$I(z(x,y)) = \iint_D f(x, y, z, z_x, z_y)\, dx\, dy \longrightarrow (1)$$

subject to the condition $z = z_0$ on Γ; Γ is the boundary of the domain D.

Lemma - Let $V(x,y)$ be a continuous funct. in the domain D of xy-plane.

Suppose $\iint_D V(x,y)\,\eta(x,y)\, dx\, dy = 0$.

for all continuously differentiable functions $\eta(x,y)$ which satify $\eta(x,y) = 0$ on Γ, Γ is the boundary of the domain D.

Then $V(x,y) \equiv 0 \;\; \forall (x,y) \in D$.

To derive the necessary condition that $z = z(x,y)$ extremizes the functional I given in (1).

Let us form the comparison functions

$$Z(x,y) = z(x,y) + \epsilon\, \eta(x,y) \longrightarrow (2).$$

such that $Z(x,y) = z_0(x,y)$ on Γ. $\therefore \eta(x,y) = 0$ on Γ.
$z_0 \to$ (known).

Let us assume that $\eta(x,y)$ is fixed.

Also, $z(x,y)$ is fixed because it extremizes $I(z(x,y))$. Then changing z by Z in (1) we get

$$I[Z(x,y)] = \iint_D f(x,y,Z,Z_x,Z_y)\,dx\,dy.$$

or, $$I(\epsilon) = \iint_D f(x,y,Z,Z_x,Z_y)\,dx\,dy.$$

$$\frac{dI}{d\epsilon} = \iint_D \frac{\partial f}{\partial \epsilon}(x,y,Z,Z_x,Z_y)\,dx\,dy.$$

$$= \iint_D \left[\frac{\partial f}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial \epsilon}}_{=0} + \frac{\partial f}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial \epsilon}}_{=0} + \frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial \epsilon} + \frac{\partial f}{\partial Z_x}\cdot\frac{\partial Z_x}{\partial \epsilon} + \frac{\partial f}{\partial Z_y}\cdot\frac{\partial Z_y}{\partial \epsilon}\right]dx\,dy \longrightarrow (3)$$

From (2), $$Z_x(x,y) = z_x(x,y) + \epsilon\,\eta_x(x,y) \longrightarrow (4)$$

$$Z_y(x,y) = z_y(x,y) + \epsilon\,\eta_y(x,y) \longrightarrow (5).$$

Take partial derivative of (2), (4), (5) w.r.to ϵ.

$$\frac{\partial Z}{\partial \epsilon} = \eta, \quad \frac{\partial Z_x}{\partial \epsilon} = \eta_x, \quad \frac{\partial Z_y}{\partial \epsilon} = \eta_y.$$

Substituting into (3), we get –

$$\frac{dI}{d\epsilon} = \iint_D \left(\frac{\partial f}{\partial Z}\cdot\eta + \frac{\partial f}{\partial Z_x}\cdot\eta_x + \frac{\partial f}{\partial Z_y}\cdot\eta_y\right)dx\,dy \longrightarrow (6).$$

Now, $\frac{dI}{d\epsilon} = 0$ when $\epsilon = 0$. because $z = z(x,y)$ extremizes $I[z(x,y)]$ and $Z = z$ when $\epsilon = 0$.

From (6),

$$0 = \iint_D \left(\frac{\partial f}{\partial z} \cdot \eta + \frac{\partial f}{\partial z_x} \cdot \eta_x + \frac{\partial f}{\partial z_y} \cdot \eta_y \right) dx\, dy \longrightarrow (7).$$

Note; $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x} \cdot \eta \right) = \eta_x \cdot \frac{\partial f}{\partial z_x} + \eta \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x} \right)$.

$$\Rightarrow \eta_x \frac{\partial f}{\partial z_x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x} \cdot \eta \right) - \eta \cdot \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x} \right).$$

Also, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y} \cdot \eta \right) = \eta_y \frac{\partial f}{\partial z_y} + \eta \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y} \right)$.

$$\Rightarrow \eta_y \cdot \frac{\partial f}{\partial z_y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y} \cdot \eta \right) - \eta \cdot \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y} \right).$$

Substituting into eqn. (7), we get

$$0 = \iint_D \Big[\frac{\partial f}{\partial z} \eta - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x} \right) \cdot \eta - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y} \right) \cdot \eta + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x} \cdot \eta \right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y} \cdot \eta \right) \Big] dx\, dy \longrightarrow (8).$$

Now, $\iint_D \Big\{ \frac{\partial}{\partial x} \underbrace{\left(\frac{\partial f}{\partial z_x} \cdot \eta \right)}_{Q} + \frac{\partial}{\partial y} \underbrace{\left(\frac{\partial f}{\partial z_y} \cdot \eta \right)}_{-P} \Big\} dx\, dy$

[Compare with $\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \oint (P\,dx + Q\,dy)$]

$$= \oint_\Gamma \left(\frac{\partial f}{\partial z_x} dy - \frac{\partial f}{\partial z_y} dx \right) \eta = 0.$$

$\because \eta = 0$ on Γ.

By virtue of the above result (8) becomes

$$\iint_D \left\{ \frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y} \right) \right\} \eta \, dx\, dy = 0$$

$\longrightarrow$ The above result holds for all such η which vanishes on Γ.

So by the lemma, $\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p} \right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q} \right) = 0$.

$p = z_x$, $q = z_y$ Euler-Ostrogradsky eqn. $\forall (x,y) \in D$ $\forall y$

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