This happens because,

$$J[y(x)] = \int_{x_1}^{x_2} f(x,y,y')\,dx.$$

$$= \int_{x_1}^{x_2} \left[M(x,y) + N(x,y)\,y'\right] dx.$$

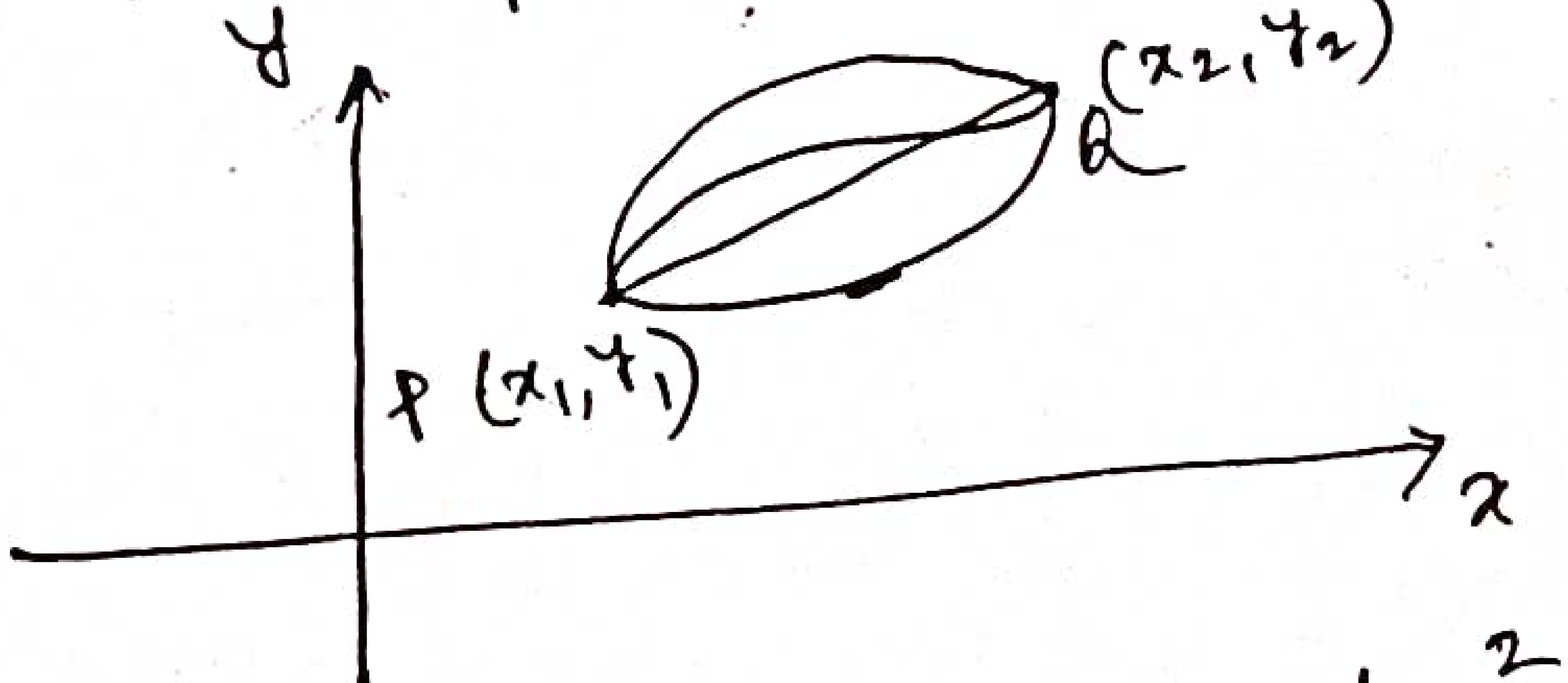
$$= \int_{x_1}^{x_2} M\,dx + N\,dy.$$

$$= \int_{x_1}^{x_2} d\,f(x,y), \quad \text{since } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$= f(x_2,y_2) - f(x_1,y_1).$$

Applications.

1. Shortest distance between two points lying on a plane (say xy-plane).



Let ds be a small element of the curve $y = y(x)$ joining P & Q.

$$ds^2 = dx^2 + dy^2.$$

$$\text{or, } ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.dx.$$

$$\therefore S = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

or, $S[y(x)] = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$.

$f - y' f_{y'} = A$ $\because f = \sqrt{1+y'^2}$ does not contain ~~x~~ explicitly.

or, $y'' \cdot \frac{\partial f_{y'}}{\partial y'} = 0$.

$$\sqrt{1+y'^2} - \frac{y' \cdot \cancel{2}y'}{\cancel{2}\sqrt{1+y'^2}} = A.$$

or, $1+y'^2 - y'^2 = A\sqrt{1+y'^2}$.

$1+y'^2 = B^2$; $B = \frac{1}{A}$.

$y'^2 = B^2 - 1$.

$\therefore$ $y' = \alpha$. $\Rightarrow$ $y = \alpha x + \beta$.

$\rightarrow$ a st. line.

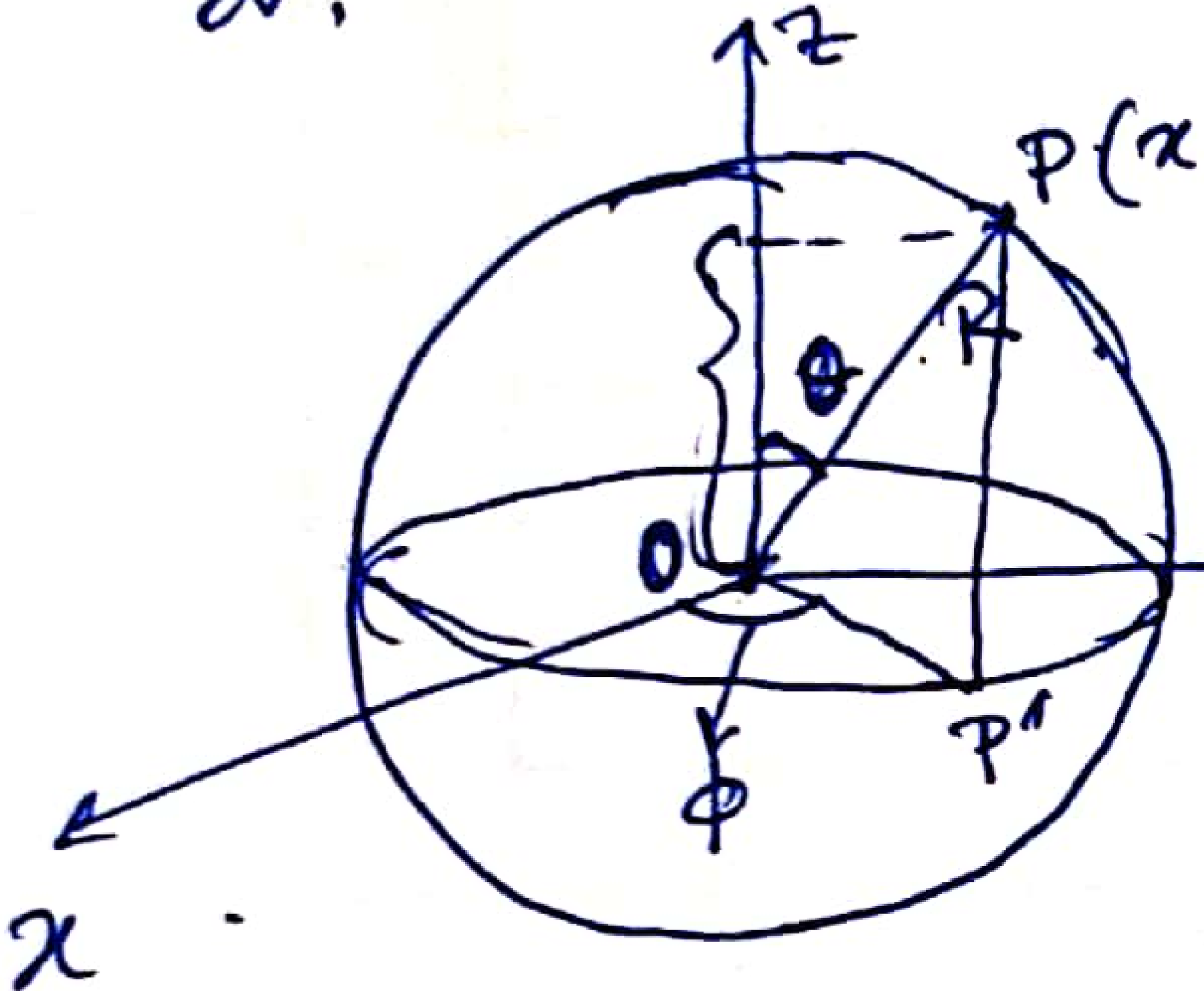
Conditions are $y(x_1) = y_1$, $y(x_2) = y_2$.

Substituting, $y - y_1 = \frac{(y_2 - y_1)(x - x_1)}{(x_2 - x_1)}$. (verify)

$\rightarrow$ the curve that extremizes (minimizes) the distance between P & Q is a st. line

<u>Definitn</u>. A curve lying on a surface that minimizes the distance between two points on the surface is called a geodesic.

2. Geodesic on the surface of a sphere (of radius R).



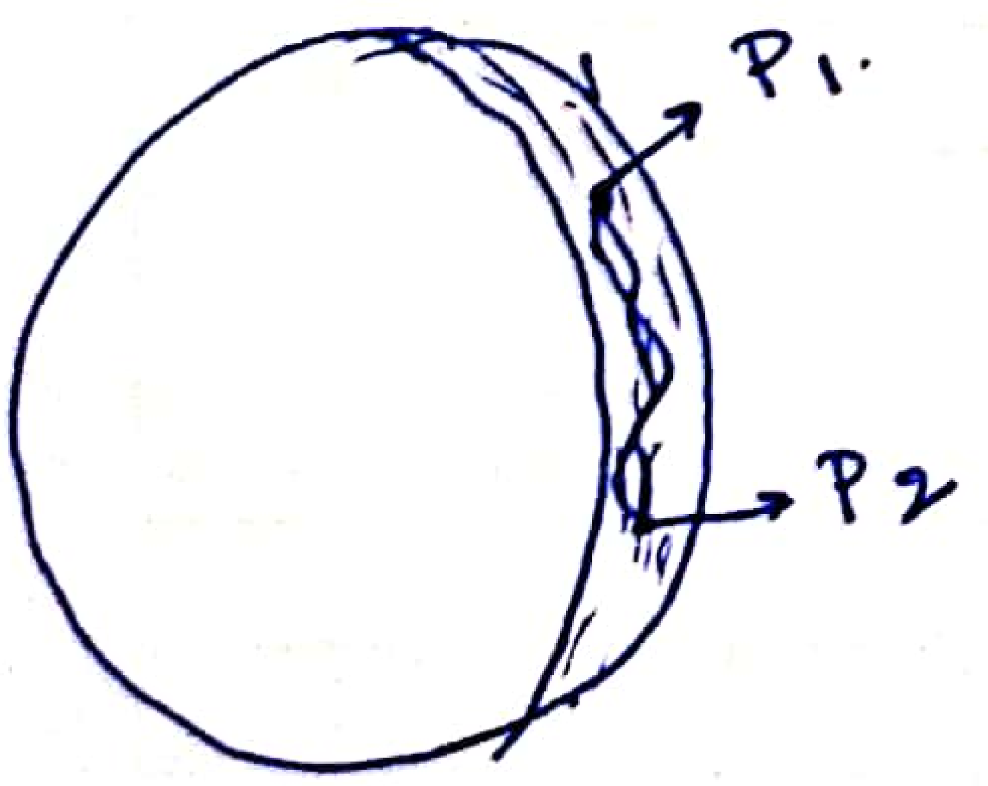
$(x, y, z) \to$ Cartesian coordinates of P.

$(\cancel{r}, \theta, \phi) \to$ spherical polar coordinates of P.
$r \to R$.

$z = R\cos\theta$, $OP' = R\cos\left(\frac{\pi}{2} - \theta\right) = R\sin\theta$.

$x = OP'\cos\phi$, $y = OP'\sin\phi$.

$\therefore x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\theta$.



$ds \to$ line element of P_1P_2.

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$dx = R\cos\theta\cos\phi\, d\theta - R\sin\theta\sin\phi\, d\phi.$$

$$dy = R\cos\theta\sin\phi\, d\theta + R\sin\theta\cos\phi\, d\phi.$$

$$dz = -R\sin\theta\, d\theta.$$

$$dx^2 + dy^2 + dz^2 = R^2\cos^2\theta\, d\theta^2 + R^2\sin^2\theta\, d\phi^2 + R^2\sin^2\theta\, d\theta^2$$

or, $$ds^2 = R^2 d\theta^2 + R^2\sin^2\theta\, d\phi^2.$$

$$\therefore ds = R\sqrt{\sin^2\theta\, d\phi^2 + d\theta^2}$$
$$= R\sqrt{\sin^2\theta + \left(\frac{d\theta}{d\phi}\right)^2}\, d\phi.$$

$$S = \int_{P_1}^{P_2} ds = R \int_{\phi=\phi_1}^{\phi_2} \sqrt{\sin^2\theta + \theta'^2}\, d\phi ,$$

$$\theta' = \frac{d\theta}{d\phi}.$$

$S \downarrow$

$S[\theta(\phi)]$

$I[y(x)]$

$= \int_{x_1}^{x_2} f(x,y,y')\,dx.$

$y' = \frac{dy}{dx}$

$f - y' f_{y'} = \text{Const}$

Here $f = \sqrt{\sin^2\theta + \theta'^2} = f(\theta, \theta')$.

$\because \phi$ (independent variable) does not appear explicitly in f, so E-L-E becomes

$$f - \theta' . f_{\theta'} = A. \quad (A \to \text{const}).$$

or, $\sqrt{\sin^2\theta + \theta'^2} - \theta' . \frac{\cancel{2}\theta'}{\cancel{2}\sqrt{\sin^2\theta + \theta'^2}} = A.$

or, $\frac{\sin^2\theta + \cancel{\theta'^2} - \cancel{\theta'^2}}{\sqrt{\sin^2\theta + \theta'^2}} = A$

or. $\sin^4\theta = A^2(\sin^2\theta + \theta'^2)$.

or, $\sin^2\theta(\sin^2\theta - A^2) = A^2\theta'^2$.

or, $\theta'^2 = \frac{1}{A^2} . \sin^2\theta(\sin^2\theta - A^2)$

$\therefore \theta' = \pm \frac{1}{A} . \sin\theta\,(\sin^2\theta - A^2)^{1/2}$

$\therefore \frac{\pm A\, d\theta}{\sin\theta\,(\sin^2\theta - A^2)^{1/2}} = d\phi.$

Let us take '+' sign.

$$A \int \frac{d\theta}{\sin\theta\,(\sin^2\theta - A^2)^{1/2}} = \phi + B.$$

$$\text{or, } A \int \frac{d\theta}{\sin\theta\,(\sin^2\theta - A^2\sin^2\theta - A^2\cos^2\theta)^{1/2}} = \phi + B.$$

$$\text{or, } A \int \frac{d\theta}{\sin\theta\,\{\sin^2\theta\,(1-A^2) - A^2\cos^2\theta\}^{1/2}} = \phi + B.$$

$$\text{or, } A \int \frac{d\theta}{\sin^2\theta\,\{(1-A^2) - \cot^2\theta . A^2\}^{1/2}} = \phi + B.$$

$$\text{or, } \cancel{A} \int \frac{\operatorname{cosec}^2\theta\, d\theta}{\cancel{A}\left\{\frac{1-A^2}{A^2} - \cot^2\theta\right\}^{1/2}} = \phi + B.$$

$$\frac{1-A^2}{A^2} = c^2$$

$$\text{or, } \int \frac{\operatorname{cosec}^2\theta\, d\theta}{\{c^2 - \cot^2\theta\}^{1/2}} = \phi + B.$$

$$\cot\theta = c\,t$$

$$-\operatorname{cosec}^2\theta\, d\theta = c\,dt$$

$$\text{or, } \int \frac{-\cancel{c}\,dt}{\cancel{c}\sqrt{1-t^2}} = \phi + B.$$

$$\text{or, } \cos^{-1} t = \phi + B.$$

$$\therefore \frac{\cot\theta}{c} = \cos(\phi + B).$$

$$\therefore \cot\theta = c\cos(\phi + B).$$

or, $\cot\theta = c\cos\phi\cos B - c\sin\phi\sin B$.

$= A_1\cos\phi + B_1\sin\phi$; $A_1 = c\cos B$; $B_1 = -c\sin B$,

$x = R\sin\theta\cos\phi$, $\frac{\cos\phi}{x} = (R\sin\theta)^{-1}$

$y = R\sin\theta\sin\phi$. $\frac{\sin\phi}{y} = (R\sin\theta)^{-1}$

$z = R\cos\theta$

$= A_2\cos\phi + B_2\sin\phi$

$\therefore \frac{\cos\phi}{x} = \frac{\sin\phi}{y} = \frac{1}{\sqrt{x^2+y^2}}$.

$\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\phi = \frac{y}{\sqrt{x^2+y^2}}$

$R^2\sin^2\theta = x^2+y^2$,

$\sin\theta = \frac{\sqrt{x^2+y^2}}{R}$.

$\cos\theta = \frac{z}{R}$

$\therefore \cot\theta = \frac{z}{\sqrt{x^2+y^2}}$

$\therefore \cot\theta = \frac{A_1 x}{\sqrt{x^2+y^2}} + \frac{B_1 y}{\sqrt{x^2+y^2}}$

$\therefore A_1 x + B_1 y = \sqrt{x^2+y^2}\cot\theta$.

or, $A_1 x + B_1 y = z$.

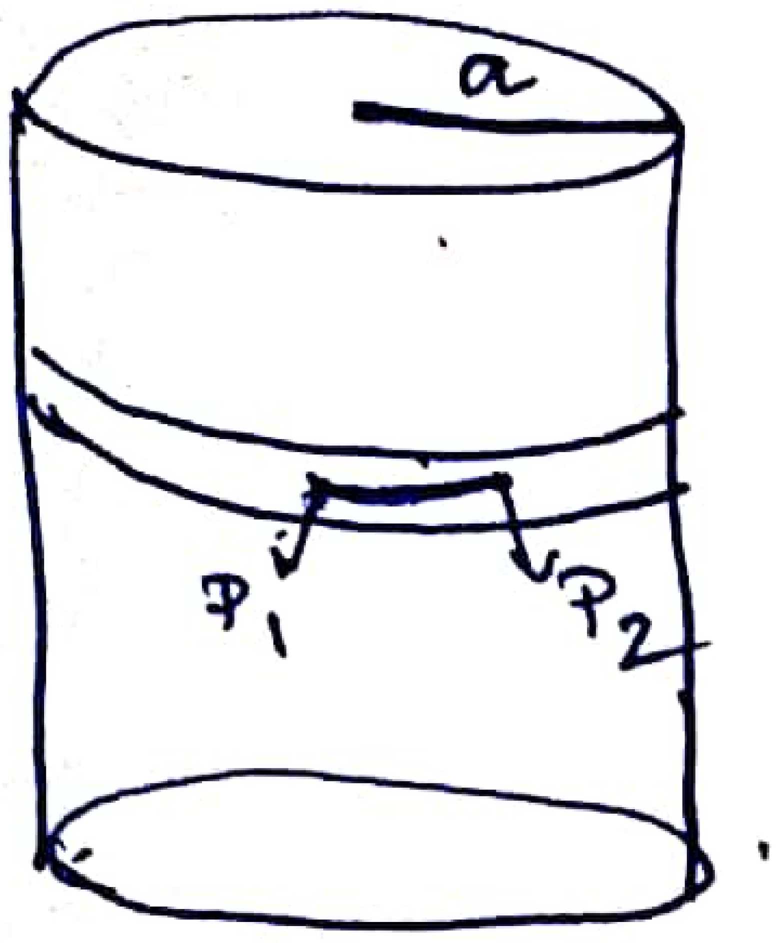
$\therefore A_1 x + B_1 y - z = 0$.

$\rightarrow$ plane passing through centre of the sphere $(0,0,0)$.

$\therefore$ the geodesic is the intersection of the surface of the sphere and the central plane. i.e it is an arc of a great circle.

Any pt. (x, y, z) on the surface of the cylinder can be expressed as.
$x = a\cos\theta, \quad y = a\sin\theta, \quad z = z.$

Geodesic on the surface of a right circular cylinder



$$ds^2 = dx^2 + dy^2 + dz^2.$$
$$= (-a\sin\theta\, d\theta)^2 + (a\cos\theta\, d\theta)^2 + dz^2$$
$$= a^2\sin^2\theta\, d\theta^2 + a^2\cos^2\theta\, d\theta^2 + dz^2$$
$$= a^2 d\theta^2 + dz^2.$$

Let $z = z(\theta)$.

Then, $$S[z(\theta)] = \int_{\theta=\theta_1}^{\theta_2} \sqrt{a^2 d\theta^2 + dz^2} = \int_{\theta_1}^{\theta_2} \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\, . d\theta.$$

$$S[z(\theta)] = \int_{\theta_1}^{\theta_2} \sqrt{a^2 + z'^2}\, d\theta. \quad ; \quad z' = \frac{dz}{d\theta}.$$

E-L-E. in this case, $f - z' f_{z'} = \text{Const}.$

or, $$\sqrt{a^2 + z'^2} - z' . \frac{\cancel{2}z'}{\cancel{2}\sqrt{a^2 + z'^2}} = A.$$

Or, $$\frac{a^2 + \cancel{z'^2} - \cancel{z'^2}}{\sqrt{a^2 + z'^2}} = A \quad \text{or,} \quad a^2 + z'^2 = \frac{a^4}{A^2}$$

$$\therefore \quad z'^2 = B^2 \quad ; \quad B^2 = \frac{a^4}{A^2} - a^2.$$

$$\therefore \quad z' = B.$$

$$\therefore \quad z = B\theta + z_0 \quad . \quad \text{or,} \quad z - z_0 = B\theta.$$

$\longrightarrow$. equation of the curve that minimizes the distance $P_1 P_2$.

Since the curve lies on the surface of the cylinder, $\therefore$ any pt. (x,y,z) on the curve can be expressed as

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z - z_0 = B\theta.$$

$\rightarrow$ equation of a circular helix.

$\therefore$ the geodesic on the surface of a circular cylinder is an arc of a circular helix.

Integrating, $\theta = mz + b$, where $m = \frac{(1-c^2)^{1/2}}{ac}$.

This shows that the required geodesics are circular helix.

Geodesics on a right-circular cone of semi-vertical angle α.

Spherical polar coordinates: (r, θ, ϕ)

$$ds^2 = dr^2 + (r\,d\theta)^2 + (r\sin\theta\, d\phi)^2$$

For the cone, semi-vertical angle α. $\therefore \theta = \alpha$

$\therefore d\theta = 0$. So, $ds^2 = dr^2 + (r\sin\alpha\, d\phi)^2$

$$\therefore ds = (r^2\sin^2\alpha + r'^2)^{1/2} d\phi \; ; \quad r' = \frac{dr}{d\phi}.$$

$\therefore$ Arc length s between $P_1(r_1, \alpha, \phi_1)$ and $P_2(r_2, \alpha, \phi_2)$ is given by

$$s = \int_{P_1}^{P_2} ds = \int_{\phi_1}^{\phi_2} (r^2\sin^2\alpha + r'^2)^{1/2} d\phi.$$

Here $F(\phi, r, r') = (r^2\sin^2\alpha + r'^2)^{1/2}$.

Since F is independent of ϕ, we have

$$F - r'\left(\frac{\partial F}{\partial r'}\right) = \text{constant}$$

or, $r^4\sin^4\alpha = c^2(r^2\sin^2\alpha + r'^2)$

$$\therefore \frac{dr}{d\phi} = \frac{r\sin\alpha}{c}(r^2\sin^2\alpha - c^2)^{1/2}$$

or, $d\phi = \frac{c}{\sin\alpha} \cdot \frac{dr}{r(r^2\sin^2\alpha - c^2)^{1/2}}$

Integrating,

$$\phi = \frac{c}{\sin\alpha}\int \frac{-\frac{1}{u^2}\,du}{\frac{1}{u}\left\{\frac{1}{u^2}\sin^2\alpha - c^2\right\}^{1/2}} \quad ; \quad r = \frac{1}{u}$$

$$\therefore \phi = -\frac{c}{\sin\alpha}\int \frac{du}{(\sin^2\alpha - u^2c^2)^{1/2}} = \frac{1}{\sin\alpha}\cos^{-1}\left(\frac{cu}{\sin\alpha}\right) - \frac{l}{\sin\alpha}$$

$$\therefore \phi\sin\alpha + l = \cos^{-1}\left(\frac{c}{r\sin\alpha}\right).$$

or, $r = a\sec(\phi\sin\alpha + l)$; a, l are arbit. constants.

Hence the required geodesics are given by the two parameter family of the above curves.

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3. Minimum Surface of Revolution:

To find the curve joining the points (x_1, y_1) and (x_2, y_2) that yields a surface of revolution of minimum area when revolved about the x-axis, we must minimize

$$I = \int_{x_1}^{x_2} 2\pi y \sqrt{1+(y')^2}\, dx. \longrightarrow (1)$$

Here x does not appear explicitly in $f = 2\pi y\sqrt{1+(y')^2}$. So, the E-L equatn becomes,

$$f - y' f_{y'} = \text{const.}$$

~~which simplifies to~~ or,

$$\frac{y(y')^2}{\sqrt{1+(y')^2}} - y\sqrt{1+(y')^2} = c_1$$

which simplifies to

$$c_1 y' = \sqrt{y^2 - c_1^2}.$$

On separating variables and integrating, we get

$$x = c_1 \int \frac{dy}{\sqrt{y^2 - c_1^2}} = c_1 \log\left(\frac{y + \sqrt{y^2 - c_1^2}}{c_1}\right) + c_2$$

and solving for y gives

$$y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right).$$

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$$x = c_1 \int \frac{dy}{\sqrt{y^2 - c_1^2}} = c_1 \log \left| \frac{y + \sqrt{y^2 - c_1^2}}{c_1} \right| + c_2$$

$$\therefore \ln \left| \frac{y + \sqrt{y^2 - c_1^2}}{c_1} \right| = \frac{x - c_2}{c_1}$$

$$\therefore y + \sqrt{y^2 - c_1^2} = c_1 e^{\frac{x - c_2}{c_1}}$$

$$y^2 - (y^2 - c_1^2) = c_1^2$$

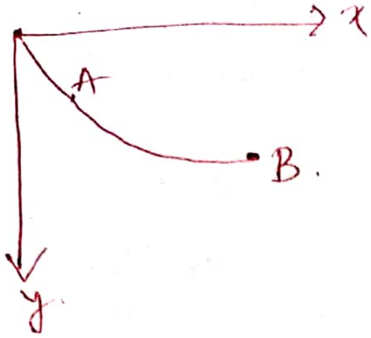
or, $\left(y + \sqrt{y^2 - c_1^2}\right)\left(y - \sqrt{y^2 - c_1^2}\right) = c_1^2$

$$\therefore y - \sqrt{y^2 - c_1^2} = \frac{c_1^{\cancel{2}}}{\cancel{c_1}} e^{-\frac{x - c_2}{c_1}}$$

$$\therefore 2y = c_1 (e^{+} + e^{-})$$

$$y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right)$$

1. The Brachistochrone Problem: (G-F)



Let A and B be two fixed points. Then the time it takes a particle to slide under the influence of gravity along some path joining A & B depends on the choice of the path (curve), and hence is a functional. The curve such that the particle takes the least time to go from A to B is called the brachistochrone. The Brachistochrone problem was posed by Johan Bernoulli in 1696, and played an important part in the development of the calculus of variations. The problem was solved by John Bernoulli, James Bernoulli, Newton and L'Hospital. The Brachistochrone turns out to be a cycloid, lying in the vertical plane and passing through A and B.

Proof: We take the origin at A, x-axis horizontal and y-axis vertically downwards. If v be the velocity of the massive particle

Greek: brachistos - shortest
chronos - time

at time t, when it is at $P(x, y)$, then we have by principle of conservation of energy

$\frac{1}{2} m v^2 - m g y = \text{const.}$

Assuming the particle to start from rest at A,

$$v^2 = 2gy, \quad \frac{ds}{dt} = v = \sqrt{2gy},$$

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx$$

T = total time of descent

$$= \frac{1}{\sqrt{2g}} \int \sqrt{\frac{1+y'^2}{y}} dx$$

So, next problem is to minimize T.

Here $F = \sqrt{\frac{1+y'^2}{y}}$, which does not contain x explicitly. So the E-L equatn. reduces to

$$F - y' F_{y'} = \text{const.}$$

or, $y(1+y'^2) = \text{const.} = c_1$, say

To solve the D.E., let us put $y' = \cot\theta$

$\therefore \; y = \frac{c_1}{1+y'^2} = c_1 \sin^2\theta = \frac{1}{2} c_1 (1 - \cos 2\theta)$

Again, $dx = \frac{dx}{dy} \cdot dy = \frac{dy}{y'} = \frac{c_1 \sin 2\theta \cdot}{\cot\theta} d\theta = c_1 (1 - \cos 2\theta) d\theta$

Integrating, $x = \frac{c_1}{2}(2\theta - \sin 2\theta) + c_2.$

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Since $x = y = 0$ at A, it is clear that $c_2 = 0$ and consequently the parametric equations of the curve of quickest descent are

$$x = \frac{1}{2} c_1 (2\theta - \sin 2\theta), \quad y = \frac{1}{2} c_1 (1 - \cos 2\theta);$$

which can be put in the form

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

which represent a cycloid. Hence the curve of quickest descent is a cycloid.