Chapter 6 : Green's Function

Observation :

$$\frac{d}{dx}(x^2) = 2x.$$

and $\quad x^2 = \int 2x\,dx.$

Thus ~~∃~~ if $D \equiv \frac{d}{dx}$ be the differential operator and I be the integral operator,

then, $\quad Dx^2 = 2x$

$\Rightarrow \quad x^2 = I(2x)$, neglecting constant.

Now, take any differential equation.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \sin x; \quad a \le x \le b.$$

or, $\quad D^2y - 4Dy + 3y = \sin x.$

or, $\quad (D^2 - 4D + 3I)y = \sin x$ $\qquad$ Identity operator

or, $\quad Ly = \sin x$

where, $\quad L \equiv D^2 - 4D + 3I.$

i.e. L is some differential operator.

Thus $y(x)$ can be obtained by operating L^{-1} on $\sin x$, where it is expected that L^{-1} will be some integral operator.

Thus, any linear diff. equation can be expressed as $\quad Ly = f(x) \longrightarrow (1)$

where $f(x)$ is known function.

Then $y(x)$ can be obtained as,

$$y(x) = L^{-1} f(x) \longrightarrow (2)$$

and it can be shown that,

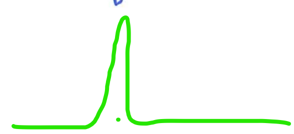
$$y(x) = \int_a^b g(x,t) f(t)\, dt. \longrightarrow (3)$$

Here $g(x,t)$ is the Green's funct. for the operator L, with the understanding that

$$L g(x,t) = \delta(x-t) \longrightarrow (4)$$

where $\delta(x-t)$ is the Dirac Delta funct.

$$\delta(x-t) = \begin{cases} 0, & x \neq t \\ \infty, & x = t. \end{cases}$$



Also, $\int_{-\infty}^{\infty} \delta(x-t)\, dt = 1$, $\int_{-\infty}^{\infty} \delta(x-t) f(t)\, dt = f(x)$,

$\int_{-\infty}^{\infty} \delta(x-t) f(x)\, dx = f(t)$; $f(t)$ is a smooth function in $(-\infty,\infty)$.

Note: $y(x) = \int_a^b g(x,t) f(t)\, dt$.

$$\therefore Ly = L \int_a^b g(x,t) f(t)\, dt = \int_a^b L g(x,t) f(t)\, dt.$$

$$= \int_a^b \delta(x-t) f(t)\, dt$$

Let $F(t) = \begin{cases} f(t); & a \le t \le b. \\ 0; & \text{otherwise}. \end{cases}$

Then, $Ly = \int_{-\infty}^{\infty} \delta(x-t) F(t)\, dt = F(x)$

$$= f(x); \quad a \le x \le b.$$

Example. Find the Green's function for the BVP

$$\frac{d^2u}{dx^2} = f(x); \quad 0 \le x \le 1; \quad u(0)=0, \; u(1)=0.$$

($f(x)$ is known).

Solu^{n}: Here $Lu = f(x)$; $L \equiv \frac{d^2}{dx^2}$.

$$u(x) = \int_0^1 g(x,t) f(t)\, dt.$$

To find $g(x,t)$.

We should have, $Lg(x,t) = \delta(x-t)$.

or, $\frac{d^2}{dx^2} g(x,t) = \delta(x-t)$.

Integrating w.r.to x, we get

$$\frac{d}{dx} g(x,t) = \int \delta(x-t)\, dx + \alpha(t) \longrightarrow (5).$$

~~Again, integrate w.r.to x~~

Now, we use the property

$$\frac{d}{dx} H(x-t) = \delta(x-t)$$

where $H(x)$ is the Heaviside's fn. given by,

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases}$$

Thus, from (5),

$$\frac{d}{dx} g(x,t) = \int H'(x-t)\, dx + \alpha(t)$$

$$\therefore \frac{d}{dx} g(x,t) = H(x-t) + \alpha(t).$$

Integrate again w.r.to. x and get

$$g(x,t) = \int \underset{\substack{\downarrow \\ d(x-t)}}{H(x-t)\,dx} + \int \alpha(t)\,dx + \beta(t).$$

$$= \left\{H(x-t)\right\}(x-t) - \int H'(x-t)(x-t)\,dx + x\alpha(t) + \beta(t)$$

or, $$g(x,t) = (x-t)H(x-t) - \int \delta(x-t)(x-t)\,dx + x\alpha(t) + \beta(t).$$

Now, $$\int_{-\infty}^{\infty} \delta(x-t)\,\phi(x)\,dx = \phi(t).$$

Here $\phi(x) = x-t$ so that $\phi(t) = t-t = 0$.

$$\therefore\ g(x,t) = (x-t)H(x-t) + x\alpha(t) + \beta(t) \longrightarrow (6)$$

$\alpha(t)$, $\beta(t)$ are unknown functions. These are to be determined with the help of the bounda-ry conditions.

The solution to $Lu = f$ is,

$$u = \int_0^1 g(x,t)\,f(t)\,dt.$$

or, $$u(x) = \int_0^1 \left\{(x-t)H(x-t) + x\alpha(t) + \beta(t)\right\} f(t)\,dt.$$

Since $$H(x-t) = \begin{cases} 1, & x > t \\ 0, & x < t \end{cases}$$

so, $u(x)$ becomes

$$u(x) = \int_0^x (x-t)\,f(t)\,dt + \int_0^1 \left\{x\alpha(t) + \beta(t)\right\} f(t)\,dt.$$

Now, the b.c. $u(0)=0$ gives,

$$0 = \int_0^1 \beta(t) f(t)\, dt.$$

Since $f(t) \not\equiv 0$ in $[0,1]$, $\beta(t) \equiv 0$ in $[0,1]$

Also, $u(1) = 0$

$$\Rightarrow \int_0^1 (1-t) f(t)\, dt + \int_0^1 f(t)\alpha(t)\, dt = 0.$$

or, $$\int_0^1 \{(1-t) + \alpha(t)\} f(t)\, dt = 0.$$

Since $(1-t) + \alpha(t) = 0$,

$$\alpha(t) = t-1$$

Thus, from (6),

$$g(x,t) = (x-t)H(x-t) + x(t-1)$$

$$= \begin{cases} (x-t)\cdot 1 + x(t-1) ; & x > t \\ (x-t)\cdot 0 + x(t-1) ; & x < t. \end{cases}$$

$$= \begin{cases} \cancel{x} - t + xt - \cancel{x} ; & x > t \\ x(t-1) ; & x < t \end{cases}$$

$$\therefore\ g(x,t) = \begin{cases} t(x-1) ; & x > t \\ x(t-1) ; & x < t. \end{cases} \longrightarrow (7)$$

From (7), $$g(x, x^+) = x(x-1) = x^2 - x$$
$$g(x, x^-) = x(x-1) = x^2 - x$$

Thus, $g(x,t)$ is continuous at $t = x$.

Note, (7) can be re-written as,

$$g(x,t) = \begin{cases} x(t-1) \; ; 0 \le x \le t \\ t(x-1) \; ; t < x \le 1. \end{cases} \longrightarrow (8)$$

$$g(t,x) = \begin{cases} t\,(x-1) & t < x \\ x.(t-1) & t > x \end{cases} \quad \begin{bmatrix} \text{changing roles of} \\ x \,\&\, t \end{bmatrix}.$$

$$\longrightarrow (9)$$

From (8) & (9), $g(x,t) = g(t,x)$

Thus, $g(x,t)$ is symmetric in x & t.

Also, $g(0,t) = 0.t - 0 = 0$, from (8a)

$g(1,t) = t.(1-1) = 0$, from (8b)

Thus, $g(x,t)$ satisfies the boundary conditions.

Note that the given diff. equation

$$\frac{d^2u}{dx^2} = f(x)$$

can be expressed as, $-\left(-\frac{du}{dx}\right)' + 0.y = f(x)$.

which is of the form $-\left(p(x)\frac{du}{dx}\right)' + q(x).y = f(x)$

with $p(x) = -1$, $q(x) = 0$.

Next let us compute $\frac{\partial g}{\partial x}(t+0,t)$ & $\frac{\partial g}{\partial x}(t-0,t)$.

For this we compute $\frac{\partial g}{\partial x}(x,t)$.

$$\frac{\partial g}{\partial x}(x,t) = \begin{cases} t-1 \; ; & 0 \le x < t \\ t \; ; & t < x \le 1. \end{cases}$$

$$\therefore \frac{\partial g}{\partial x}(t+0, t) - \frac{\partial g}{\partial x}(t-0, t)$$

$$= t - (t-1) = t - t + 1 = 1.$$

This can be expressed as,

$$\frac{\partial g}{\partial x}(t+0, t) - \frac{\partial g}{\partial x}(t-0, t) = -\frac{1}{p(t)}.$$