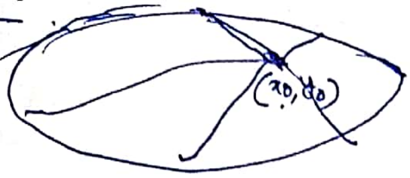
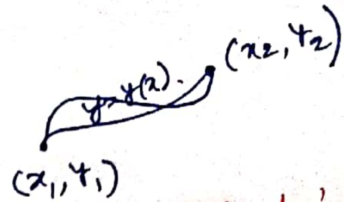
Moving boundary problems.

Case I.

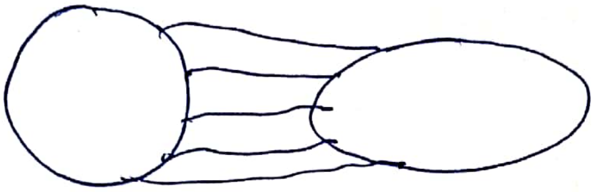


one end is fixed.
other end moving



2 fixed end-points

Case 2.



both ends are moving.

Assumption.

Let $f = f(x, y, y')$ be 3 times differentiable w.r.t. to its arguments.

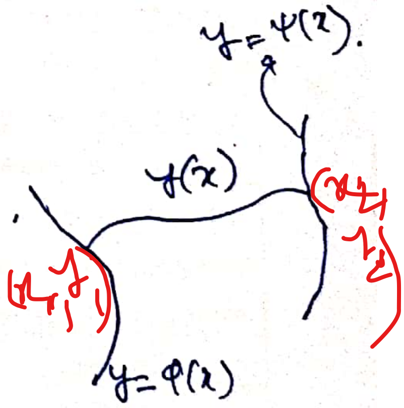
Thm. Let the curve $y = y(x)$ extremizes the functional

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\,dx$$

from all curves joining two arbitrary points of two given curves $y = \phi(x)$ & $y = \psi(x)$. Then the curve $y = y(x)$ satisfies.

(1) the Euler-Lagrange eqn.

(2) the transversality conditions.



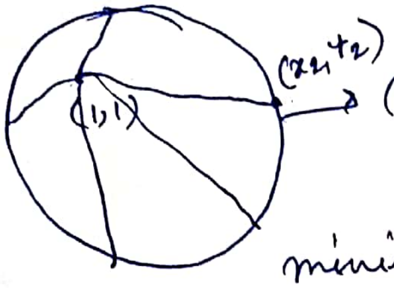
$$\left[f + (\phi' - y') f_{y'}\right]_{x=x_1} = 0 \rightarrow (1)$$

$$\left[f + (\psi' - y') f_{y'}\right]_{x=x_2} = 0 \rightarrow (2)$$

Case I. Left end is fixed. Right end varies along $y = \psi(x)$.
Then $y = y(x)$ satisfies E-L-E and (2).

Case II Right end is fixed. Left end varies along $y = \phi(x)$
Then $y = y(x)$ satisfies E-L-E & (1).

Prob 1. Find the minimum distance of an interior point $(1,1)$ from the curve $(x-2)^2+y^2=9$.



We have to find $y=y(x)$ that minimizes $I[y(x)] = \int_1^{x_2} \sqrt{1+y'^2}\,dx$,

where (x_2, y_2) lies on the circle $(x-2)^2+y^2=9$.

Then $y=y(x)$ will satisfy

$$f_y - \frac{d}{dx} f_{y'} = 0.$$

and $\left[f + (\psi' - y') f_{y'}\right]_{x=x_2} = 0$.

$(x-2)^2+y^2=9$.

$y^2 = 9-(x-2)^2$

$y = \pm\sqrt{9-(x-2)^2}$

$y = \psi(x)$

where $\psi = \sqrt{9-(x-2)^2}$

$y = \psi(x)$

$f = \sqrt{1+y'^2}$ $\quad f_y - \frac{d}{dx} f_{y'} = 0.$

$\because$ f doesn't contain x explicitly,

then, $f - y' f_{y'} = \text{const}$.

$$\sqrt{1+y'^2} - y' \cdot \frac{y'}{\sqrt{1+y'^2}} = c_0$$

or, $\dfrac{1+\cancel{y'^2}-\cancel{y'^2}}{\sqrt{1+y'^2}} = c_0$.

$\therefore$ $1+y'^2 = \frac{1}{c_0^2} = c^2$

$\therefore$ $y'^2 = c^2-1 = a^2$

$y' = \pm a \Rightarrow y = ax+b$.

f doesn't contain y explicitly,

E-L-E becomes

$\frac{d}{dx} f_{y'} = 0$.

$y'' f_{y'y'} = 0$.

Now, $f_{y'y'} = \frac{1}{(1+y'^2)^{3/2}}$

either $y''=0$ or $f_{y'y'}=0$ but, $f_{y'y'} \neq 0$

$y''=0 \Rightarrow y' = a$

$y = ax+b$.

$y = ax + b$ passes through $(1,1)$ & (x_2, y_2)

$\therefore$ $a + b = 1 \longrightarrow (1)$ and $ax_2 + b = y_2 \longrightarrow (2)$.

(x_2, y_2) lies on the circle $y = \sqrt{9-(x-2)^2}$.

$\therefore$ $y_2 = \sqrt{9-(x_2-2)^2} \longrightarrow (3)$

The transversality condition is,

$$\left[f + (\psi' - y') f_{y'}\right]_{x=x_2} = 0.$$

$f = \sqrt{1+y'^2}$, $\psi = \sqrt{9-(x-2)^2}$

$$\therefore \psi' = \frac{-(x-2)}{\sqrt{9-(x-2)^2}}.$$

$$\left\{\sqrt{1+y'^2} + \left[-\frac{x-2}{\sqrt{9-(x-2)^2}} - y'\right]\frac{y'}{\sqrt{1+y'^2}}\right\}_{x=x_2} = 0.$$

or, $$\left[\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}} - \frac{(x-2)}{\sqrt{9-(x-2)^2}} \times \frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_2} = 0$$

or, $$\left[\frac{1}{\sqrt{1+y'^2}} - \frac{(x-2).\,y'}{\sqrt{1+y'^2}\sqrt{9-(x-2)^2}}\right]_{x=x_2} = 0.$$

$$\therefore \sqrt{9-(x_2-2)^2} - (x_2-2)\, y'\Big|_{x_2} = 0.$$

or, $\sqrt{9-(x_2-2)^2} = (x_2-2)\, y' \longrightarrow (4a)$.

$y = ax + b$, $y' = a$.

$\sqrt{9-(x_2-2)^2} = (x_2-2).\,a \longrightarrow (4)$

$a+b=1 \rightarrow (1)$, $ax_2+b=y_2 \rightarrow (2)$,

$y_2=\sqrt{9-(x_2-2)^2} \rightarrow (3)$, $\sqrt{9-(x_2-2)^2}=a(x_2-2) \rightarrow (4)$.

From (3) & (4), $y_2 = ax_2 - 2a . \rightarrow (5)$

From (2) & (5), $ax_2-2a = ax_2+b.$

$\Rightarrow b+2a=0.$

$$\begin{aligned} a+b&=1. \\ 2a+b&=0. \\ \hline -a&=1 \end{aligned}$$

$\therefore a=-1$

$b=1-a=2.$

$y=ax+b. = -x+2$

$(x_2,y_2) = \left(2+\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$,

$(x_2,y_2) = \left(2-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$.

$y=-x+2$

$y'=-1.$

$$I[y(x)] = \int_1^{x_2} \sqrt{1+y'^2}\,dx = \int_1^{x_2} \sqrt{2}\,dx.$$

or, $d = \sqrt{2}(x_2-1)$

$\sqrt{2}+3$

$x_2 = 2+\frac{3}{\sqrt{2}}$, $d=\sqrt{2}\left(2+\frac{3}{\sqrt{2}}-1\right) = \sqrt{2}\left(1+\frac{3}{\sqrt{2}}\right)$

$x_2 = 2-\frac{3}{\sqrt{2}}$ $d=\sqrt{2}\left(2-\frac{\cancel{\sqrt{3}}\,3}{\sqrt{2}}-1\right) = \sqrt{2}\left(1-\frac{3}{\sqrt{2}}\right)$

$\therefore$ Minimum distance is $|\sqrt{2}-3| = \sqrt{2}-3$

$= 3-\sqrt{2}$ units.