Example 4·1. Check whether any non-zero solution exists for the given IE.

$$u(x) = \lambda \int_0^1 (\sqrt{x}\, t - \sqrt{t}\, x)\, u(t)\, dt.$$

Ans. $$u(x) = \lambda\sqrt{x} \int_0^1 t\, u(t)\, dt - \lambda x \int_0^1 \sqrt{t}\, u(t)\, dt \longrightarrow (1).$$

Let $A = \int_0^1 t\, u(t)\, dt$, $\longrightarrow (2)$ $B = \int_0^1 \sqrt{t}\, u(t)\, dt.$ $\longrightarrow (3)$.

$$\therefore\ u(x) = \lambda\sqrt{x}\, A - \lambda x\, B. \longrightarrow (4).$$

Substitute (4) in (2) & get.

$$A = \int_0^1 t\, (\lambda\sqrt{t}\, A - \lambda t\, B)\, dt$$

$$= \lambda A \int_0^1 t^{3/2}\, dt - \lambda B \int_0^1 t^2\, dt = \frac{2\lambda A}{5} - \frac{\lambda B}{3}.$$

$$B = \int_0^1 \sqrt{t}\, (\lambda\sqrt{t}\, A - \lambda t\, B)\, dt.$$

$$= \frac{\lambda A}{2} - \frac{2\lambda B}{5}.$$

$\therefore$ the system of equations are,

$$\left(1 - \frac{2\lambda}{5}\right) A + \frac{\lambda B}{3} = 0.$$

$$\lambda \frac{A}{2} - \left(1 + \frac{2\lambda}{5}\right) B = 0.$$

The above homogeneous system will have a non-zero solution if

$$\begin{vmatrix} 1-\frac{2\lambda}{5} & \frac{\lambda}{3} \\ \frac{\lambda}{2} & -\left(1+\frac{2\lambda}{5}\right) \end{vmatrix} = 0.$$

$$\text{or, } -\left(1-\frac{4\lambda^2}{25}\right) - \frac{\lambda^2}{6} = 0.$$

$$\text{or, } 1 + \left(\frac{1}{6} - \frac{4}{25}\right)\lambda^2 = 0 \Rightarrow 1 + \frac{\lambda^2}{150} = 0$$

$$\Rightarrow \lambda = \pm i\sqrt{150}.$$

Since λ is purely imaginary, there exist no real e-funcn (non-zero sol.) for the given homogeneous I.E.

Fredholm Alternatives.

Consider the homogeneous IE.

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt \longrightarrow (1).$$

Q1 If $\lambda = \lambda_0$ is an e-value of the IE (1), what happens to the corresponding non-homoge-neous IE

$$u(x) = f(x) + \lambda_0 \int_a^b K(x,t)\, u(t)\, dt \, ? \longrightarrow (2).$$

Q2 If $\lambda = \lambda_0$ is not an e-value of (1), what happens to the non-hom. IE (2)?

Definitⁿ 4·4 Associated kernel $K^*(x,t)$.

It is defined as, $K^*(x,t) = K(t,x)$.

Definitⁿ 4·5. Symmetric kernel $K(x,t)$

$K(x,t)$ is symmetric, if $K(x,t) = K(t,x)$.

Thus, for symmetric kernel, associated kernel is same as the original kernel. Because,

$$K^*(x,t) = K(t,x) = K(x,t).$$

Theorem (Statements of Fredholm Alternatives)

- Let $f(x) \not\equiv 0$ be a continuous function in $I[a,b]$,
- $K(x,t) \not\equiv 0$ be real, continuous in $R = \{(x,t) : a \le x, t \le b\}$.
- Let $D(\lambda)$ denote Fredholm determinant for the kernel $K(x,t)$.

① If λ_0 is not an e-value of the homogeneous IE

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt \longrightarrow (1)$$

then the non-homogeneous equation

$$u(x) = f(x) + \lambda_0 \int_a^b K(x,t)\, u(t)\, dt \longrightarrow (2)$$

has a unique continuous solution in I.

② If λ_0 is an e-value of the homogeneous IE (1), then there exists a continuous solution of the non-homogeneous equation (2), if and only if

$$\int_a^b f(x)\, u^*(x)\, dx = 0.$$

where $u^*(x)$ is the solution to the homogeneous equation $u(x) = \lambda \int_a^b K^*(x,t)\, u(t)\, dt$

where $K^*(x,t) = K(t,x)$ is the associated kernel.

4.12

Associated homogeneous IE

Verification of statements of Fredholm Alternatives

Example 4·2

Consider the non-homogeneous IE

$$u(x) = 2x + \lambda \int_{0^+}^{1} \sin(\ln x)\, u(t)\, dt \longrightarrow (1).$$

The corresponding homogeneous IE is

$$u(x) = \lambda \int_{0^+}^{1} \sin(\ln x)\, u(t)\, dt \longrightarrow (2).$$

We will verify the statements of Fred. Alt.s through this example.

Sol (2) is,

$$u(x) = \lambda \sin(\ln x) \int_{0^+}^{1} u(t)\, dt.$$

or, $u(x) = \lambda \sin(\ln x)\, A$, say $\longrightarrow (3)$.

$$A = \int_{0^+}^{1} u(t)\, dt = \int_{0^+}^{1} \lambda A \sin(\ln t)\, dt.$$

or, $A \left\{ 1 - \lambda \int_{0^+}^{1} \sin(\ln t)\, dt \right\} = 0 \longrightarrow (4)$

$$I = \int_{0^+}^{1} \sin(\ln t)\, dt = \Big[t \cdot \sin(\ln t) \Big]_{0^+}^{1} - \int_{0^+}^{1} t \cos(\ln t) \frac{1}{t}\, dt.$$

$$= -\int_{0^+}^{1} \cos(\ln t)\, dt = -\Big[t \cdot \cos(\ln t) \Big]_{0^+}^{1} - \int_{0^+}^{1} t \cdot \sin(\ln t) \frac{1}{t}\, dt.$$

$$= -1 - I$$

$$\Rightarrow 2I = -1. \Rightarrow I = -\frac{1}{2}.$$

Thus, from (4),

$$\cancel{u(x) =} \quad A\left(1+\frac{\lambda}{2}\right) = 0.$$

For non-zero sol., $A \neq 0$. $\therefore\ 1+\frac{\lambda}{2} = 0$

$$\Rightarrow \lambda = -2.$$

$\therefore\ \lambda = -2$ is the only eigenvalue.

Now, consider the non-hom. eq. (1).

Here, we introduce $A_1 = \int_{0^+}^{1} u(t)\,dt. \rightarrow (5).$

Then (2) becomes,

$$u(x) = 2x + \lambda A_1 \overset{\sin(\ln x)}{\cancel{h}} \longrightarrow (6).$$

By virtue of (6), (5) reduces to,

$$A_1 = \int_{0^+}^{1} \left\{2t + \lambda A_1 \cancel{\}\,dt} \sin(\ln t)\right\} dt.$$

$$= 1 + \lambda A_1 \times -\frac{1}{2}$$

or, $A_1\left(1+\frac{\lambda}{2}\right) = 1. \Rightarrow A_1 = \frac{2}{\lambda+2}.$

From (6),

$$u(x) = 2x + \frac{2\lambda}{\lambda+2} \sin(\ln x) \longrightarrow (7)$$

Thus, the unique solution of (1) is given by (7), provided $\lambda \neq -2$, i.e provided λ is not the eigenvalue of the corresponding homogeneous eq. (2).

In fact, if $\lambda = -2$, there does not exist any continuous sol. for the non-hom. eq. (1).

This is due to the following fact:

Consider the associated homogeneous eq.

$$u^*(x) = \lambda_0 \int_a^l K^*(x,t)\, u^*(t)\, dt .$$

Here $\lambda_0 = -2$, $K(x,t) = \sin(\ln x)$, $a = 0^+$, $b = 1$.

$$K^*(x,t) = K(t,x) = \sin(\ln t) .$$

To solve, $\quad u^*(x) = -2 \int_{0^+}^{1} \sin(\ln t)\, u^*(t)\, dt$

Let $\quad C = \int_{0^+}^{1} \sin(\ln t)\, u^*(t)\, dt$

$\therefore \quad u^*(x) = -2C .$

So, $\quad C = -2C \int_{0^+}^{1} \sin(\ln t)\, dt .$

or, $\quad C \left[1 + 2 \int_{0^+}^{1} \sin(\ln t)\, dt \right] = 0$

or, $\quad C \left(1 + 2 \times -\frac{1}{2} \right) = 0 \quad , \because \int_{0^+}^{1} \sin(\ln t)\, dt = -\frac{1}{2}$

or, $\quad C . 0 = 0 . \Rightarrow C$ is arbitrary $(\neq 0)$.

$\therefore \quad \int_a^l f(x)\, u^*(x)\, dx = \int_{0^+}^{1} 2x . (-2C)\, dx = -2C \neq 0 .$

Hence the corr. non-hom. eq. (1) will have no continuous sol., when $\lambda = -2$ is an e-value of the hom. IE. (1).

Example: Verify Fredholm alternatives for the IE

$$u(x) = \cos 3x + \lambda \int_0^{\pi} \cos(x+t)\, u(t) dt.$$

Step – 1

Find eigenvalues of

$$u(x) = \lambda \int_0^{\pi} \cos(x+t)\, u(t) dt.$$

Turn Over

$$u(x) = x\cos x\, A - \lambda \sin x .\, B .$$

where $\quad A = \int_0^\pi \cos t\, u(t)\, dt, \quad B = \int_0^\pi \sin t\, u(t)\, dt .$

$$A = \int_0^\pi \cos t \left(x \cos t\, A - \lambda \sin t . B \right) dt .$$

$$= x A \int_0^\pi \cos^2 t\, dt - x B \int_0^\pi \cos t \sin t\, dt .$$

$$= \frac{\lambda A}{2} \int_0^\pi (1 + \cos 2t)\, dt - \frac{\lambda B}{2} \int_0^\pi \sin 2t\, dt .$$

$$= \frac{\lambda A}{2} \left[\pi + \left[\frac{\sin 2t}{2} \right]_0^\pi \right] - \frac{\lambda B}{2} \left[- \frac{\cos 2t}{2} \right]_0^\pi$$

$$A = \frac{\lambda \pi A}{2} . \quad \text{or,} \quad A\left(1 - \frac{\lambda \pi}{2}\right) = 0 \longrightarrow (3).$$

$$B = \int_0^\pi \sin t \left(x \cos t . A - \lambda \sin t . B \right) dt .$$

$$= \frac{x A}{2} \int_0^\pi \sin 2t\, dt - \frac{\lambda B}{2} \int_0^\pi (1 - \cos 2t)\, dt$$

$$B = - \frac{\lambda B \pi}{2} . \qquad B\left(1 + \frac{\lambda \pi}{2}\right) = 0 \longrightarrow (4).$$

(3) & (4) can be written together as

$$\begin{pmatrix} 1 - \lambda\frac{\pi}{2} & 0 \\ 0 & 1 + \frac{\lambda \pi}{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Characteristic eqn.: $\begin{vmatrix} 1 - \lambda\frac{\pi}{2} & 0 \\ 0 & 1 + \frac{\lambda \pi}{2} \end{vmatrix} = 0 .$

or, $\left(1-\frac{\lambda\pi}{2}\right)\left(1+\frac{\lambda\pi}{2}\right)=0$.

$\therefore$ $\lambda = \frac{2}{\pi}$ or $-\frac{2}{\pi}$. are eigenvalues of the homogeneous equation (2).

Case I. $\lambda \neq \frac{2}{\pi}$ or $\lambda \neq -\frac{2}{\pi}$.

$$u(x) = \cos 3x + \lambda \int_0^{\pi} \cos(x+t)\, u(t)\, dt.$$

$$u(x) = \cos 3x + \lambda \cos x\, C - \lambda \sin x\, D.$$

$$C = \int_0^{\pi} \cos t\, u(t)\, dt, \quad D = \int_0^{\pi} \sin t\, u(t)\, dt.$$

$$C = \int_0^{\pi} \cos t \left(\cos 3t + \lambda \cos t\, C - \lambda \sin t . D\right) dt.$$

$$= \int_0^{\pi} \cos t \cos 3t\, dt + \lambda C \int_0^{\pi} \cos^2 t\, dt - \lambda D \int_0^{\pi} \sin t \cos t\, dt.$$

$$= \frac{1}{2}\underbrace{\int_0^{\pi} \cos 4t\, dt}_{=0} + \frac{1}{2}\underbrace{\int_0^{\pi} \cos 2t\, dt}_{=0} + \lambda C . \frac{\pi}{2}.$$

$C\left(1-\frac{\lambda\pi}{2}\right)=0$. But $\lambda \neq \frac{2}{\pi}$ $\therefore$ $\underline{C=0}$.

$$D = \int_0^{\pi} \sin t\, u(t)\, dt = \int_0^{\pi} \sin t \left\{\cos 3t + \lambda \cos t\, C - \lambda \sin t\, D\right\} dt.$$

$$= \frac{1}{2}\int_0^{\pi} \underbrace{2 \sin t \cos 3t}_{(\sin 4t - \sin 2t)}\, dt + \frac{\lambda C}{2}\int_0^{\pi} \sin 2t\, dt - \frac{\lambda D}{2}\underbrace{\int_0^{\pi} (1-\cos 2t)\, dt}_{\text{non-0}}.$$

$$= -\frac{\lambda D \pi}{2}$$

$\therefore$ $D\left(1+\frac{\lambda\pi}{2}\right)=0$; $\lambda \neq -\frac{2}{\pi}$ $\therefore$ $\underline{D=0}$.

$\therefore u(x) = \cos 3x + \lambda \cos x . C - \lambda \sin x . D$

$= \cos 3x .$

So $\lambda \neq \frac{2}{\pi}, -\frac{2}{\pi}$ then the non-hom. eqn has a unique sol. $u(x) = \cos 3x$.

<u>Case 2.</u> $\lambda = \frac{2}{\pi}$ or $-\frac{2}{\pi}$.

<u>Case 2a.</u> $\lambda = \frac{2}{\pi}$.

$$u(x) = \cos 3x + \lambda \int_0^{\pi} \cos(x+t)\, u(t)\, dt .$$

$$C = \int_0^{\pi} \cos t\, u(t)\, dt , \quad D = \int_0^{\pi} \sin t\, u(t)\, dt$$

$$u(x) = \cos 3x + \lambda \cos x . C - \lambda \sin x . D .$$

$$C = \int_0^{\pi} \cos t \left(\cos 3t + \lambda \cos t . C - \lambda \sin t . D \right) dt .$$

$$= \lambda C . \frac{\pi}{2} .$$

or, $C\left(1 - \lambda \frac{\pi}{2}\right) = 0$ Now $\lambda = \frac{2}{\pi}$

$\therefore C . 0 = 0 .$

$\Rightarrow C$ is arbitrary.

$$D = \int_0^{\pi} \sin t \left(\cos 3t + \lambda \cos t . C - \lambda \sin t . D \right) dt$$

$= -\lambda D . \frac{\pi}{2}$. or, $D . \left(1 + \lambda \frac{\pi}{2}\right) = 0$.

$\because \lambda = \frac{2}{\pi}$ $\therefore D . (1+1) = 0$. $\Rightarrow D = 0$.

$$u(x) = \cos 3x + \lambda \cos x . C - \lambda \sin x . D .$$

$$= \cos 3x + \frac{2C}{\pi} \cos x$$

Case 2b. $\lambda = -\frac{2}{\pi}$.

Here, $C\left(1 - \lambda\frac{\pi}{2}\right) = 0$.

$\lambda = -\frac{2}{\pi}$ $\therefore C(1+1) = 0$ $\therefore C = 0$.

$D\cdot\left(1 + \lambda\frac{\pi}{2}\right) = 0$ $\therefore$ when $\lambda = -\frac{2}{\pi}$,

then, $D(1-1) = 0 \Rightarrow D.0 = 0$.

$\Rightarrow$ D is arbitrary.

$u(x) = \cos 3x + \lambda \cos x . C - \lambda \sin x . D$.

$= \cos 3x + \frac{2D}{\pi}\sin x$.

To prove.

$$\int_a^b f(x)\, u^*(x)\, dx = 0.$$

$u^*(x)$ is a solution of

$$u(x) = \lambda_0 \int_a^b K^*(x,t)\, u(t)\, dt.$$

$K(x,t) = \cos(x+t)$.

$K^*(x,t) = K(t,x) = \cos(t+x) = K(x,t)$.

To find solution of

$$u(x) = \frac{2}{\pi}\int_0^\pi \cos(x+t)\, u(t)\, dt.$$

and

$$u(x) = -\frac{2}{\pi}\int_0^\pi \cos(x+t)\, u(t)\, dt.$$

$$u(x) = \frac{2}{\pi}\cos x \int_0^{\pi} \cos t\, u(t)\, dt - \frac{2}{\pi}\sin x \int_0^{\pi} \sin t\, u(t)\, dt$$

$$u(x) = \frac{2}{\pi}\cos x\, A - \frac{2}{\pi}\sin x . B.$$

$$A = \int_0^{\pi} \cos t\, u(t)\, dt = \int_0^{\pi} \cos t \left(\frac{2}{\pi}\cos t\, A - \frac{2}{\pi}\sin t\, B\right) dt$$

$$= \frac{2A}{\pi}\int_0^{\pi} \cos^2 t\, dt - \frac{1}{\pi}\int_0^{\pi} \underbrace{\sin 2t\, dt}_{0}.$$

$$= \frac{A}{\pi}\int_0^{\pi} (1 + \cos 2t)\, dt = A.$$

$\underline{A = A}$ $\qquad A \cdot 0 = 0$.

$$B = \int_0^{\pi} \sin t\, u(t)\, dt = \int_0^{\pi} \sin t \left(\frac{2}{\pi}\cos t\, A - \frac{2}{\pi}\sin t\, B\right) dt$$

$$= \frac{A}{\pi}\int_0^{\pi} \sin 2t\, dt - \frac{B}{\pi}\int_0^{\pi} (1 - \cos 2t)\, dt.$$

$$= -B. \qquad \therefore 2B = 0 \Rightarrow B = 0.$$

$$\therefore u(x) = \frac{2}{\pi}\cos x . A = u^*(x).$$

$$\int_a^b f(x)\, u^*(x)\, dx = \int_0^{\pi} \cos 3x . \frac{2}{\pi}\cos x . A\, dx.$$

$$= \frac{A}{\pi}\int_0^{\pi} (\cos 4x + \cos 2x)\, dx = 0.$$

Similarly for $\lambda = -\frac{2}{\pi}$, it can be proved that $\int_a^b f(x)\, u^*(x)\, dx = 0$ where $u^*(x)$ is a solution of

$$u(x) = -\frac{2}{\pi}\int_0^{\pi} K^*(x,t)\, u(t)\, dt$$