Lemma: If $x_1$ and $x_2 (> x_1)$ are two fixed constants and $G(x)$ is a continuous funct$^n$ in $[x_1, x_2]$ and if

$$\int_{x_1}^{x_2} \eta(x)\, G(x)\, dx = 0 \longrightarrow (1).$$

for every choice of the continuously differentiable funct$^n$ $\eta(x)$ for which $\eta(x_1) = \eta(x_2) = 0$, then $G(x) = 0$ for all $x$ in $[x_1, x_2]$.

For proof look at Weinstock

## Euler's equation for an extremal:

Consider the functional $I(y)$ given by

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')\,dx. \longrightarrow (1)$$

Here $f(x,y,y')$ and $y(x)$ satisfy the assumptions 1 an 2 (of previous page).

We look for a function $y(x)$ that ~~maximizes~~ extremizes the functional $I(y)$ given in (1).

Euler's equation is the differential equation for $y(x)$ which is obtained by comparing the values of $I$ that correspond to neighboring admissible functions. The central idea is that since $y(x)$ gives an ~~minimum~~ extreme value to $I$, $I$ will ~~increase~~ change, if we disturbe $y(x)$ slightly. These disturbed functions are constructed as follows:

Let $\eta(x)$ be any function with the property that $\eta''(x)$ is continuous and

$$\eta(x_1) = \eta(x_2) = 0. \longrightarrow (2)$$

If $\alpha$ is a small parameter, then

$$\bar{y}(x) = y(x) + \alpha\,\eta(x) \longrightarrow (3)$$

represents a one-parameter family of admissible functions. The vertical deviation of a curve in this family from the ~~minimizing~~ extremizing curve $y(x)$ is $\alpha\eta(x)$, as shown in fig. 1. The significance of (3), lies in the fact that for each family of this type, i.e for each choice of the funct. $\eta(x)$, the ~~minimizing~~ extremizing funct. $y(x)$ belongs to the family and corresponds to the value of the parameter $\alpha = 0$.

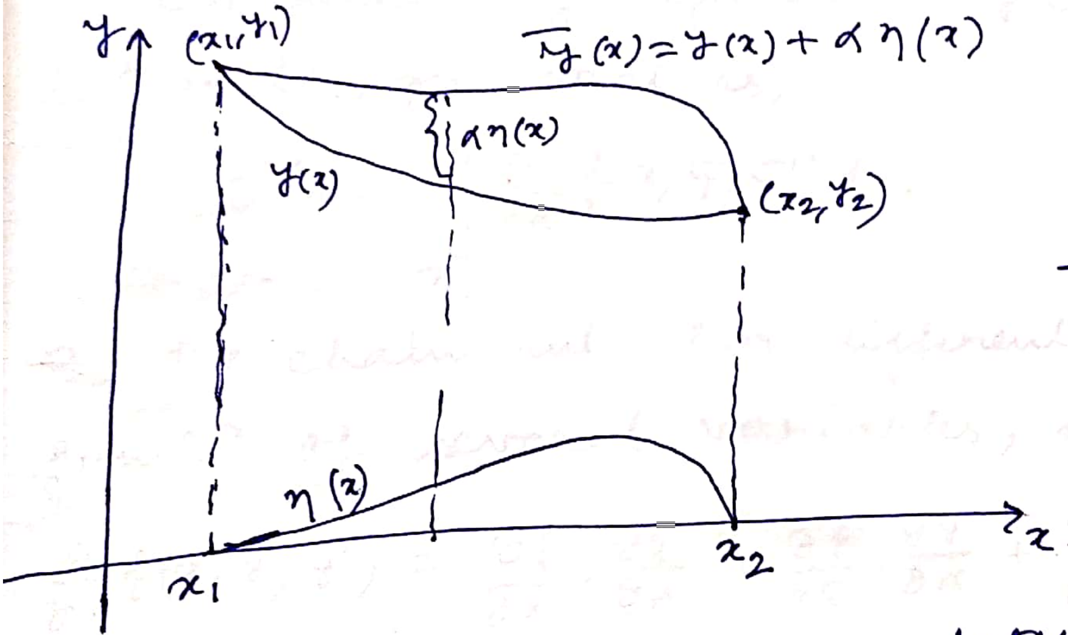


Fig. 1

Now, with $\eta(x)$ fixed, we substitute

$\bar{y}(x) = y(x) + \alpha\eta(x)$ and $\bar{y}'(x) = y'(x) + \alpha\eta'(x)$

into the integral (1), and get a funct. of $\alpha$,

$$I(\alpha) = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\,dx$$

$$= \int_{x_1}^{x_2} f\left[x, y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x)\right] dx. \longrightarrow (4)$$

When $\alpha = 0$, formula (3) yields $\bar{y}(x) = y(x)$; and since $y(x)$ ~~minimizes~~ extremizes the integral, we know that $I(\alpha)$ must have a minimum/maximum when $\alpha = 0$. By elementary calculus, a necessary condition for this is the vanishing of the derivative $I'(\alpha)$ when $\alpha = 0$ : $I'(0) = 0$. The derivative $I'(\alpha)$ can be computed by differentiating (4) under the integral sign, that is,

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}')\, dx. \longrightarrow (5)$$

By the chain rule for differentiating functions of several variables, we have

$$\frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}') = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial \bar{y}} \cdot \frac{\partial \bar{y}}{\partial \alpha} + \frac{\partial f}{\partial \bar{y}'} \cdot \frac{\partial \bar{y}'}{\partial \alpha}$$

$$= \frac{\partial f}{\partial \bar{y}} \cdot \eta(x) + \frac{\partial f}{\partial \bar{y}'} \cdot \eta'(x).$$

So, (5) can be written as

$$I'(\alpha) = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial \bar{y}} \eta(x) + \frac{\partial f}{\partial \bar{y}'} \cdot \eta'(x) \right] dx. \longrightarrow (6)$$

Now, $I'(0) = 0$, so putting $\alpha = 0$ in (6) yields

$$\int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx = 0 \longrightarrow (7)$$

In this equatⁿ the derivative $\eta'(x)$ appears along with the funcⁿ $\eta(x)$. We can eliminate $\eta'(x)$ by integrating the second term by parts, which gives,

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x)\,dx = \left[\eta(x)\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

$$= -\int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx \text{ , by virtue of (2).}$$

We can therefore write (7) in the form,

$$\int_{x_1}^{x_2} \eta(x)\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]dx = 0. \longrightarrow (8)$$

Our reasoning up to this point is based on a fixed choice of the funcⁿ $\eta(x)$. However, since the integral in (8) must vanish for <u>every</u> such funcⁿ, we at once conclude that the expressions in bracket must also vanish. This yields,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0,$$

which is Euler's equatⁿ, or Euler-Lagrange (E-L) equatⁿ.

Note 1. The difference $\bar{y} - y = \alpha\eta$ is called the variation of the funct. $y$ and is usually denoted by $\delta y$.

Note 2. Alternative form of Euler-Lagrange equat.

The E-L equat. is, $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$.

Now, $\frac{\partial f}{\partial y'}$ is a funct. of $x$ explicitly, and also implicitly through $y$ and $y'$.

So,
$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)\cdot\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)\cdot\frac{dy'}{dx}.$$

$\frac{d}{dx}(y') = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$

$\therefore$ E-L equat. is

$$\frac{\partial^2 f}{\partial y'^2}\cdot\frac{d^2y}{dx^2} + \frac{\partial^2 f}{\partial y\,\partial y'}\cdot\frac{dy}{dx} + \frac{\partial^2 f}{\partial x\,\partial y'} - \frac{\partial f}{\partial y} = 0.$$

Say, $f = y'^2 y + x^2 y$

This is a second-order non-linear differential equat. So, the extremals, its solutions - constitute a two-parameter family of curves.

Note: The extremum of the funct$^{nl}$

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\,dx$$

can be realized only on those ~~extrem~~ curves which satisfy $y(x_1) = y_1$, $y(x_2) = y_2$

The B V P

$$f_y - \frac{d}{dx} f_{y'} = 0$$

$$y(x_1) = y_1, \quad y(x_2) = y_2$$

does not always have a solution, and if a solution exists, it may not be unique.

1). $I[y(x)] = \int_0^{\pi/2} (y'^2 - y^2)\,dx$ ; $y(0)=0$ ; $y(\frac{\pi}{2})=1$

$y(0)=0$, $y(\frac{\pi}{2})=1$

$f = y'^2 - y^2$.

E-L-E: $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ ; $\frac{\partial f}{\partial y} = -2y$, $\frac{\partial f}{\partial y'} = 2y'$

$\therefore -2y - \frac{d}{dx}(2y') = 0$.

or, $-2y - 2y'' = 0 \Rightarrow y'' + y = 0$ ; $y(0)=0$, $y(\frac{\pi}{2})=1$

$y = A\cos x + B\sin x$. $y(0)=0 \Rightarrow 0 = A$

$\therefore y = B\sin x$.

$y(\frac{\pi}{2}) = 1$

$\therefore 1 = B$. $\therefore y = \sin x \rightarrow$ unique sol.

2). $I[y(x)] = \int_0^1 (y^2 + x^2 y')\,dx$ ; $y(0)=0$, $y(1)=2$.

E-L-E: $f_y - \frac{d}{dx} f_{y'} = 0$ ; $f = y^2 + x^2 y'$.

$f_y = 2y$, $f_{y'} = x^2$

or, $2y - \frac{d}{dx}(x^2) = 0$.

or, $2y - 2x = 0$, $y = x$.

$y = x$ satisfies $y(0)=0$, but doesn't satisfy $y(1)=2$.

$\Rightarrow$ the variational problem does not have any solution.

§ $y = x \rightarrow$ is an extremal but it is not an extremum / extremizing curve

$y = y(x)$ is an extremal when it satisfies E-L-E only.

$y = y(x)$ is an extremum / extremizing curve when it satisfies E-L-E as well as the boundary conditions.

3. $I[y(x)] = \int_0^{2\pi} (y'^2 - y^2)\, dx$; $y(0) = 1$, $y(2\pi) = 1$.

E-L-E: $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(f_{y'}\right) = 0$. $\quad f = y'^2 - y^2$.

$f_y = -2y$.

$\therefore -2y - \frac{d}{dx}(2y') = 0$. $\quad f_{y'} = 2y'$.

or, $-2y - 2y'' = 0$.

$\therefore y'' + y = 0$.

$y(x) = A\cos x + B\sin x$.

$y(0) = 1$

$\Rightarrow 1 = A$

$\therefore y(x) = \cos x + B\sin x$.

$y(2\pi) = 1$, $\Rightarrow 1 = 1$

$\therefore y(x) = \cos x + B\sin x$;

$B \to$ arbitrary

Hence the given variational problem has infinitely many solutions.