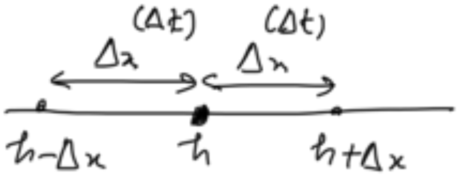
Brownian Motion (BM) process:



$X(t)$: position of particle at time t.

$$X_i = \begin{cases} +1 & \text{if the } i^{th} \text{ step of length } \Delta x \text{ is to the right} \\ -1 & \text{'' '' '' '' '' '' '' '' left} \end{cases}$$

$$X(t) = \Delta x \left(\underline{X_1 + X_2} + \cdots + X_{\left[\frac{t}{\Delta t}\right]} \right)$$

Where $[\cdot]$: greatest integer less than or equal to the number eg $[4.4] = 4$

X_i's are indep.

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$$

$$\underline{E(X_i) = 0} \quad ; V(X_i) = \underline{E(X_i^2) = 1}$$

$$E(X(t)) = 0$$

$$V(X(t)) = (\Delta x)^2 \sum_{i=1}^{[t/\Delta t]} V(X_i) \rightarrow 1$$

$$= (\Delta x)^2 \left[\frac{t}{\Delta t}\right]$$

Let $\Delta x \to 0, \Delta t \to 0$

(i) if $\Delta x = \Delta t \to 0$; $E(X(t)) = 0$, $V(X(t)) = 0$
trivial result

(ii) if we let $\underline{\Delta x = \sigma \sqrt{\Delta t}}$, $\sigma > 0$
as $\Delta t \to 0$

$$E(X(t)) = 0 \quad , \quad V(X(t)) \to \sigma^2 t$$

$-2-$

Def^n A SP $\{X(t), t \geq 0\}$ BM process if

(i) $X(0) = 0$ (ii) $\{X(t), t \geq 0\}$ has stationary indep. increments

(iii) $\forall t > 0$, $X(t) \sim N(0, \sigma^2 t)$

$-x-$

If $\underline{\sigma = 1}$ Standard BM / Wiener process

$\{X(t)\}$ BM, then $W(t) = \frac{X(t)}{\sigma} \sim N(0, t)$ SBM / Wiener process

$\rightarrow W(t) \sim N(0, t)$

density $f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$, $-\infty < x < \infty$.

Note that

$W(t_1) = x_1, \ldots, W(t_n) = x_n \equiv W(t_1) = x_1, W(t_2) - W(t_1) = x_2 - x_1,$
$\ldots, W(t_n) - W(t_{n-1}) = x_n - x_{n-1}$

Also $\underline{W(t_1), W(t_2) - W(t_1)}, \ldots, W(t_n) - W(t_{n-1})$ are

indep and has stationary increments

$\underline{W(t_k) - W(t_{k-1})} \overset{d}{=} W(t_k - t_{k-1})$
$\sim N(0, t_k - t_{k-1})$.

Joint density of $\underline{W(t_1), \ldots, W(t_n)}$ is

$$\underline{f(x_1, \ldots, x_n)} = f_{t_1}(x_1)\, f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right]\right\}}{(2\pi)^{n/2}\left[t_1(t_2 - t_1) \cdots (t_n - t_{n-1})\right]^{1/2}} \checkmark$$

Conditional $[W(s) \mid W(t) = B]$, $s < t$
density

density $f_{s|t}(x|B) = f_s(x)\, f_{t-s}(B - x)$

$$\ldots_{s|t}\ldots = \frac{\ldots}{f_t(B)}$$

$$= k_1 \exp\left\{ -\frac{x^2}{2s} - \frac{(B-x)^2}{2(t-s)} \right\}$$

$$= k_2 \exp\left\{ -x^2\left(\frac{1}{2s} + \frac{1}{2(t-s)} \right) + \frac{Bx}{t-s} \right\}$$

$$= k_2 \exp\left\{ -\frac{t}{2s(t-s)} \left(x^2 - \frac{2sB}{t}x \right) \right\}$$

$$= k_3 \exp\left\{ -\frac{(x - Bs/t)^2}{2s(t-s)/t} \right\}$$

For $s<t$ — k_1, k_2, k_3 indep of x

$$[W(s) \mid W(t) = B] \sim N\left(\frac{s}{t}B,\ \frac{s(t-s)}{t} \right)$$

$E(W(s) \mid W(t)=B]$ $\quad$ $V(W(s) \mid W(t)=B)$

Example: In a bicycle race btw two competitors, let $Y(t)$: the amt of time (in secs) by which the racer that started in the inside position is ahead when $100t$% of the race has been completed, and suppose that $\{Y(t)\}_{0\le t\le 1}$ can be effectively modeled as BM process with variance parameter σ^2.

(a) If the inside racer is leading by σ sec's at the midpoint of the race, what is the prob. that she is the winner?

$Y(t) \sim N(0, \sigma^2 t)$

Sol $P(Y(1) > 0 \mid Y(\tfrac{1}{2}) = \sigma)$

$$= P(\underbrace{Y(1) - Y(\tfrac{1}{2})} > 0 - \sigma \mid \underbrace{Y(\tfrac{1}{2}) = \sigma})$$

$$= P(Y(1) - Y(\tfrac{1}{2}) > -\sigma)$$ | ∵ indep increment of $Y(t)$

$$= P(Y(\tfrac{1}{2}) > -\sigma)$$ | Stationary increment of $Y(t)$

$$= P\left(\frac{Y(\frac{1}{2})}{\sigma/\sqrt{2}} > \frac{-\sigma}{\sigma/\sqrt{2}}\right) = P(Z > -\sqrt{2})$$

$Z \sim N(0,1)$

$$= 1 - \Phi(-\sqrt{2})$$ | $\Phi(x) = P(Z \leq x)$

$$= \Phi(\sqrt{2}) \simeq 0.9213$$ | $\Phi(x) + \Phi(-x) = 1$

(b) If the inside racer wins the race by a margin of σ sec's, what is the prob that she was ahead at the midpoint?

Sol

$$P(Y(\tfrac{1}{2}) > 0 \mid Y(1) = \sigma) = ?$$

$$\sigma W(t) = Y(t) \sim N(0, \sigma^2 t)$$ $W(t) \sim N(0, t)$

For $s < t$

$$[W(s) \mid W(t) = c] \sim N\left(\frac{s}{t}c, \frac{s}{t}(t-s)\right)$$

$$[W(s) \mid Y(t) = c] \equiv [W(s) \mid W(t) = \tfrac{c}{\sigma}] \sim N\left(\frac{sc}{t\sigma}, \frac{s}{t}(t-s)\right)$$

$$[Y(s) \mid Y(t) = c] \equiv [\sigma W(s) \mid Y(t) = c] \sim N\left(\frac{sc}{t}, \frac{\sigma^2 s(t-s)}{t}\right)$$

$$[Y(\tfrac{1}{2}) \mid Y(1) = \sigma] \sim N\left(\frac{\sigma}{2}, \frac{\sigma^2}{4}\right)$$ $s = \frac{1}{2}, t = 1, c = \sigma$

$$P(Y(\tfrac{1}{2}) > 0 \mid Y(1) = \sigma) = P\left(Z > \frac{0 - \sigma/2}{}\right)$$

$$\sigma/2 \,)$$

$$= P(Z > -1) = 1 - P(Z \le -1) = 1 - \Phi(-1)$$

$$= \Phi(1) \simeq 0.8413$$

— x —

→ $W(t)$ is MG. ?

Sol For $s<t$
$E(W(t) \mid W(u), 0 \le u \le s)$

$$= E(W(t) - W(s) + W(s) \mid W(u), 0 \le u \le s)$$

$$= E(W(t) - W(s) \mid W(u), 0 \le u \le s) + E(W(s) \mid W(u), 0 \le u \le s)$$

$$= \underbrace{E(W(t) - W(s))}_{\text{by independent increment}} + W(s)$$

$W(t) - N(0,t)$

$$= 0 + W(s) = W(s)$$

$W(t)$ MG.

— x —

Martingale Stopping thm: An important property of MG $Y(t)$ is that if you continually observe the process and stop at some time T, then, subject to some technical condition $E(Y(T)) = E(Y(0))$

T → stopping time for MG.
expected value of the stopped MG is equal to its fixed time expectation.

— x —

Eg. Let $T = \min\{t : W(t) = 2 - 4t\}$, i.e., T is the first time that $W(t)$ hits the line $2-4t$. $E(T)=?$

Sol Using MG stopping thm

$$\underline{E}(W(T)) = E(W(0)) = 0$$

$$W(T) = 2-4T \Rightarrow E(W(T)) = 2-4E(T)$$

$$\Rightarrow 2-4E(T)=0 \Rightarrow E(T) = 1/2$$

—x—

Q. Let $Y(t) = W^2(t) - t$.

$Y(t)$ MG ? $\quad E(Y(t)) = ?$ $\quad$ Ex.

—x—

<u>Geometric BM</u>

$Y(t)$ BM drift coeff μ, var-parameter σ^2

$$Y(t) \sim N(\mu t, \sigma^2 t),$$

$$X(t) = e^{Y(t)}$$

$\{X(t), t \geq 0\}$ Geometric BM.

For $s<t$

$$E(X(t) \mid X(u), 0 \leq u \leq s)$$

$$= E(e^{Y(t)} \mid Y(u), 0 \leq u \leq s)$$

$$= E(e^{Y(s)+Y(t)-Y(s)} \mid Y(u), 0 \leq u \leq s)$$

$$= e^{Y(s)} E(e^{Y(t)-Y(s)} \mid Y(u), 0 \leq u \leq s)$$

$$= X(s)\, E(\underline{e^{Y(t)-Y(s)}}) \quad \Big| \text{ indep. increments}$$

$$= X(s)\, e^{\mu(t-s) + (t-s)\sigma^2/2} \quad \Big| \; Y(t)-Y(s) \overset{d}{=} Y(t-s) \text{ stationary increments}$$

$$= X(s)\, e^{(t-s)(\mu+\sigma^2/2)}$$

$\sim N(\mu(t-s), (t-s)\sigma^2)$

$W \sim N(\cdot,\cdot)$

$E(e^{aW}) = e^{aE(W) + \frac{1}{2}a^2 \mathrm{Var}(W)}$

$a=1$

$$\Rightarrow E(X(t) \mid X(u),\ 0 \le u \le s) = X(s)\, e^{(t-s)(\mu+\sigma^2/2)} \quad *_u$$

$$\Rightarrow E(X(t)) = E(X(s))\, e^{(t-s)(\mu+\sigma^2/2)}$$

Geo. BM is useful in modeling of stock prices over time when you feel the %age changes are IID.

eg Let X_n : price of some stock at time n.

It might be reasonable to suppose that

$\frac{X_n}{X_{n-1}}$, $n \ge 1$, are I.I.D.

Let $Y_n = \frac{X_n}{X_{n-1}} \Rightarrow X_n = Y_n X_{n-1}$

$$= Y_n Y_{n-1} X_{n-2}$$

$$\vdots$$

$$= Y_n Y_{n-1} \cdots Y_1 X_0$$

$$\Rightarrow \log X_n = \sum_{i=1}^{n} \log(Y_i) + \log(X_0)$$

$\log(Y_i)$, $i \ge 1$, IID, $\{\log(X_n)\}$ will, when suitably normalized, approx. BM with a drift, so.

$e^{\log X_n} = X_n$, $\{X_n\}$ approx. Geo. BM.

—x—

$X(t)$ BM with drift wrt μ: $\quad X(t) \sim N(\mu t, t)$

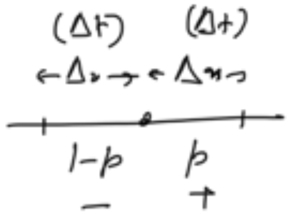
$; W(t) \sim N(0, t)$

$X(t) = W(t) + \mu t$

BM also be defined as a limit of random walks

Seal

$X(t)$: position at time t



$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ step in +ve direction} \\ -1 & \text{o.w.} \end{cases}$$

X_i's indep.

$$X(t) = \Delta x \left(X_1 + \cdots + X_{[t/\Delta t]} \right)$$

$$E(X(t)) = \Delta x \left[\frac{t}{\Delta t}\right](2p-1)$$

$$V(X(t)) = (\Delta x)^2 \left[\frac{t}{\Delta t}\right]\left(1-(2p-1)^2\right)$$

$$E(X_i) = 1 \cdot p + (-1)(1-p) = 2p-1$$

$$V(X_i) = E(X_i^2) - (E(X_i))^2 = 1-(2p-1)^2$$

If we let $\underline{\Delta x = \sqrt{\Delta t}}$, $\underline{p = \frac{1}{2}(1+\mu\sqrt{\Delta t})}$

and let $\Delta t \to 0$

$$E(X(t)) \to \mu t$$

$$V(X(t)) \to t$$

$$\cancel{\sqrt{\Delta t}} \times \frac{t}{\cancel{\Delta t}} \times \mu \cancel{\sqrt{\Delta t}} \to \mu t$$

$$\cancel{\Delta t} \times \frac{t}{\cancel{\Delta t}} \times (1-\underline{\mu^2 \Delta t}) \to t$$

—x—

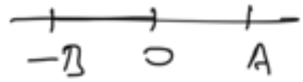
Probability that the process will hit A before $-B$; $A, B > 0$

Let $P(0) = P(X(t) \text{ hits } A \text{ before } -B \mid X(0)=0)$, $-B < 0 < A$

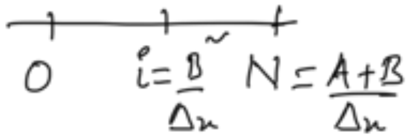
where $P(x)$ in the prob. that process will hit A before B given that we are now at x.

Boundary condition $P(A) = 1$, $P(-B) = 0$

Δx



Gambler's ruin problem



$$p = \frac{1}{2}(1 + \mu \Delta x)$$

$$\lim_{\Delta x \to 0} \left(\frac{1-p}{p}\right)^{1/\Delta x} = \lim_{\Delta x \to 0} \left(\frac{1-\mu \Delta x}{1+\mu \Delta x}\right)^{1/\Delta x}$$

$$= \frac{e^{-\mu}}{e^{\mu}} = e^{-2\mu}$$

$$\frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}$$

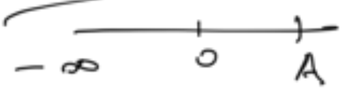
Letting $\Delta x \to 0$, we see that

$$P(\text{up } A \text{ before down } B) = \frac{1-e^{-2B\mu}}{1-e^{-2\mu(A+B)}} \quad \text{--- (1)}$$

$$= \frac{e^{2B\mu}-1}{e^{2\mu(A+B)}-1} \; e^{2\mu A}$$

Case I If $\mu < 0$ by letting $B \to \infty$ (2)

$$P(\text{process ever goes to } A) = e^{2\mu A}$$



In this case, the process drift off to $-\infty$ and its max is an expo. r.v. with rate (-2μ).

Case II let $\mu \to 0$ in (1)

$$P(\text{BM goes up } A \text{ before down } B) = \frac{B}{A+B}.$$

In general

$$P(x) = \frac{1 - e^{-2\mu(x+B)}}{1 - e^{-2\mu(A+B)}}$$

—x—

<u>Example</u> (Exercising a Stock Option)

Suppose we have the option of buying, at some time in the future, <u>one unit</u> of a stock at <u>a fixed price A</u>, indep. of current market price. The current market price of the stock is taken to be 0, and we suppose that it changes in accordance with a BM having a negative drift coefficient $-d$, where $d > 0$. The question is, when, if ever, should we exercise our option?

Sol policy exercise the option when market price is x

<u>expected gain</u> $= (x-A)\,P(x)$,

where $P(x)$ prob. that the process will ever reach x,

$\mu = -d < 0$, $d > 0$

From (2) $\quad P(x) = e^{-2dx}$

Optimal value of x is one max. $(x-A)\,e^{-2dx} = f(x)$

$$f'(x) = (x-A)\,e^{-2dx}(-2d) + e^{-2dx} = 0$$

$$\Rightarrow x = A + \frac{1}{2d}$$

$$f''(x)\Big|_{x = A + \frac{1}{2d}} > 0$$

$$\boxed{x = A + \frac{1}{2d}}$$

eg if $A = 100$, $d = 2$

$$x = 100 + \frac{1}{4} = 100.25$$

— x —

Pricing Stock Option:

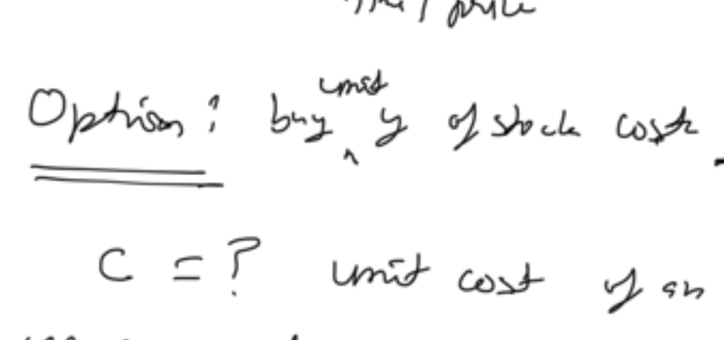
example in option pricing:

amt $\underline{\underline{v}}$ time '0' value present value — v at t is $ve^{-\alpha t}$

$\alpha \to$ discount factor

$e^{-\alpha t}$ discount fn

Option — share of stock at a future time at a fixed price.



Option: buy unit of stock cost $\underline{150}$ per share (time 0 price)

$C = ?$ unit cost of an option

We will show unless $C = \frac{50}{3}$, there will be a combination of purchases that will always result in a +ve gain.

suppose Time 0, we

$\begin{cases} \text{buy } x \text{ unit of stock} \\ \text{buy } y \text{ " " option} \end{cases}$ sell means $-ve$

value of our holding at time 1

$$\text{value} = \begin{cases} 200x + 50y & \text{if price is } 200 \\ 50x & \text{if price is } 50 \end{cases}$$

let we choose y s.t.

$$200x + 50y = 50x \Rightarrow \boxed{y = -3x} \checkmark$$

the value of holding $= 50x$

Orignal cost $= 100x + cy$

x unit stock + $-3x$ unit of option $= 100x - 3xc$

$$\text{gain} = 50x - (100x - 3xc) = x(3c - 50)$$

$$\begin{cases} \boxed{0 \quad \text{if } 3c = 50} \\ +ve \quad \text{if } 3c > 50 \\ -ve \quad \text{if } 3c < 50 \end{cases}$$

only option cost c that does not remeet in an arbitrage is $c = 50/3$

A sure win betting scheme is called an arbitrage

eg (1) $c = 20$ (unit cost per option)

$x = 1$, $y = -3$ $\qquad y = -3x$

initially cost us $= 100 - 60 = 40$

value of holding at time 1 ($\uparrow 200$, $\downarrow 50$)

$= 50$

guaranteed profit $= 50 - 40 = 10$ is attained

(2) $c = 15$

$x = -1$, $y = 3$

initial gain $= 100 - 45 = 55$

Value of holding at time 1 is $= -50$

guaranteed profit $= 55 - 50 = 5$

Arbitrage thm: expt. whose set of possible outcomes $\underline{S = \{1, 2, \ldots, m\}}$.

$\underline{n \text{ wagers}}$

amt x is bet on wager i, then return $x\, r_i(j)$ is earned if the outcome of expt. is j, $j \in S = \{1, 2, \ldots, m\}$, $i \in \{1, 2, \ldots, n\}$

$r_i(\cdot) \to$ return fn for unit bet on wager i.

Betting scheme $\underset{\sim}{x} = (x_1, \ldots, x_n)$

Outcome of expt is j, then return from $\underline{x} = \sum_{i=1}^{n} x_i r_i(j)$

$\exists$ $\underset{\sim}{p} = (p_1, \ldots, p_m)$ on $S = \{1, \ldots, m\}$ under which each wager has expected return 0, or else there is a betting scheme that guarantees a positive win.

Arbitrage thm: Exactly one of the following is true:

Either

(i) $\exists$ a $\underset{\sim}{p} = (p_1, \ldots, p_m)$ for which $\sum_{i=1}^{n} p_i r_i(j) = 0$ $\forall j = 1, \ldots, m$

or

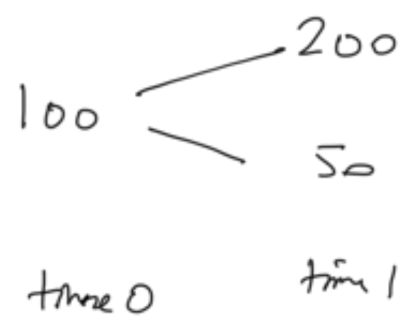
(ii) $\exists$ a betting scheme $\underset{\sim}{x} = (x_1, \ldots, x_n)$ for which

$$\sum_{i=1}^{n} x_i \, r_i(j) > 0 \quad \forall j = 1, \ldots, m.$$

In other words, if X is the outcome of the expt, then arbitrage thm states that either there is a prob. vector $\underset{\sim}{p}$ for X s.t. $\quad E_{\underset{\sim}{p}}(r_i(X)) = 0 \quad \forall i = 1, \ldots, n$

or else there is a betting scheme that leads to a sure win.

Example (contd)



C $\qquad$ Option: to buy at time 1 at present value of 150 per share

s.t. no sure win is possible

two outcomes

two wagers $\rightarrow$ $\begin{cases} \text{buy / sell stock} \\ \text{buy / sell option} \end{cases}$

no sure wins if $\quad \underset{\sim}{p} = (p, 1-p) \qquad \underline{E(\text{return}) = 0}.$

returns from purchasing 1 unit of ... $= \begin{cases} 200 - 100 = 100, \text{ if price } 200 \text{ at time } 1 \end{cases}$

time of stock $\quad \Big| \; 50-100=-50$ if ... 50 ... 1.

if $p \to$ prob. that price is $\underline{200}$ at time 1, then

$E(\text{return}) = 100p - 50(1-p)$

$\underline{E(\text{return})=0} \Rightarrow \underline{100 = 200p + 50(1-p)}$

$\Rightarrow p = \frac{1}{3}$

$(p, 1-p) = \left(\frac{1}{3}, \frac{2}{3}\right)$ for which wager yields an expected return 0.

return from purchasing one share of option $= \begin{cases} 50 - c & \text{if price is } 200 \\ -c & \text{" " " } 50 \end{cases}$

expected return when $p = \frac{1}{3}$ is

$E(\text{return}) = (50-c) \times \frac{1}{3} - c \times \frac{2}{3} = \frac{50}{3} - c$

$E(\text{return}) = 0 \Rightarrow c = \frac{50}{3}$

arbitrage then only value of c for which there will not be a sure win is $c = 50/3$.

—x—

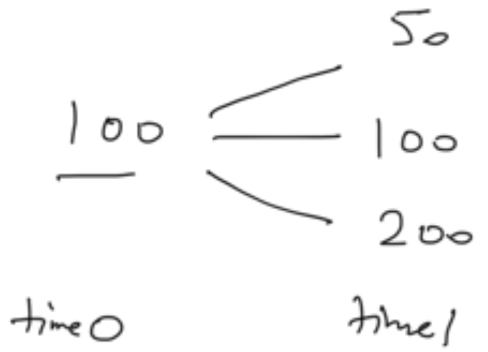
Example 1 The present price of a stock is $\underline{100}$. The price at time 1 will be either $\underline{50, 100 \text{ or } 200}$. An option to purchase y shares of the stock at time 1 for the (present value) price ky cost cy.

(a) If $\underline{k = 120}$, show that an arbitrage opportunity occurs iff $c > 80/3$

occurs iff $c >$ [illegible]

(b) If $k = 80$, show that there is not an arbitrage opportunity iff $20 \le c \le 40$.

Sol



Stock will be a fair bet under probabilities $(p_1, p_2, 1-p_1-p_2)$ on $(50, 100, 200)$

C　　　　K

$$100 = 50p_1 + 100p_2 + 200(1-p_1-p_2)$$

$$\Rightarrow 2 = p_1 + 2p_2 + 4(1-p_1-p_2)$$

$$\Rightarrow 3p_1 + 2p_2 = 2 \quad —(1)$$

(a) $k = 120$　　　　(c)

option

$$\text{return} = \begin{cases} -c & \text{if } 50 \text{ wp } p_1 \\ -c & \text{if } 100 \text{ wp } p_2 \\ 80-c & \text{if } 200 \text{ wp } 1-p_1-p_2 \end{cases}$$

$$E(\text{return}) = 0 \Rightarrow -cp_1 - cp_2 + (80-c)(1-p_1-p_2) = 0$$

$$\Rightarrow c = 80(1-p_1-p_2) \quad —(2)$$

$\Rightarrow$ solving (1) & (2)

$$p_1 = \frac{c}{40}, \quad p_2 = \frac{80-3c}{80}$$

$1 - p_1 - p_2 = c$ [illegible]

$p_2 = \frac{\quad}{80}$

$\Rightarrow \quad \underline{\underline{80 \geq 3c}} \iff$ no arbitrage

$c > \frac{80}{3} \Rightarrow$ arbitrage opportunity occurs.

(b) $\underline{k = 80}$

$$\underline{C = 20p_2 + 120(1 - p_1 - p_2)} \quad \text{— (2)}$$

$0 \le p_i \le 1$

Solve (1) & (2)

$$p_1 = \frac{c-20}{30}, \quad p_2 = \frac{40-c}{20}$$

$$1 - p_1 - p_2 = \frac{c-20}{60}$$

iff $\underline{\underline{20 \le c \le 40}}$

no arbitrage

— x —

Black – Scholes Option pricing formula:

present price of stock $X(0) = x_0$; $X(t)$ stock's price at time t

Stock $[0, T]$

$\alpha \to$ discount factor

$e^{-\alpha t} X(t)$

Observe $X(t)$ upto $\underline{s < t}$ buy (or sell) stock at price $X(s)$, then sell (buy) these shares at time t for price $X(t)$

purchase N different options at time 0

Option i cost $\underline{c_i}$ per share option purchase share of stock at time t_i for fixed price K_i per share $i = 1, \ldots, N$.

$\underline{c_i}$? for which there

... is no betting strategy that lead to a sure win.

<u>Use Arbitrage theorem</u>

$\underset{\sim}{P}$ prob. measure on the set of outcomes

wager $s \quad t$

$$\underline{e^{-\alpha s}X(s)} \qquad e^{-\alpha t}X(t)$$

$\underset{\sim}{P}$ on $X(t)$, $0 \le t \le T$

$$E_{\underset{\sim}{P}}\left[e^{-\alpha t}X(t) \mid X(u),\ 0 \le u \le s\right] = e^{-\alpha s}X(s) \quad \text{— (1)}$$

$X(t)$ MG

(k)

$$\text{worth of option at time } t = \begin{cases} X(t) - k & \text{if } X(t) \ge k \\ 0 & \text{if } X(t) < k \end{cases}$$

present value of worth of option $= e^{-\alpha t}(X(t)-k)^+$

$$E_{\underset{\sim}{P}}\left(e^{-\alpha t}(X(t)-k)^+\right) = C \quad \text{— (2)}$$

$$C_i = E_{\underset{\sim}{P}}\left[e^{-\alpha t_i}(X(t_i)-k_i)^+\right], \quad i = 1, 2, \ldots, N \quad \text{— (2)'}$$

By arbitrage thm if we can find a prob measure $\underset{\sim}{P}$ on the set of outcomes that satisfy (1), then if c, the cost of an option to purchase one share at time t at the fixed price k, is given by (2), then no arbitrage is possible. On the other hand, if for given prices C_i, $i = 1, \ldots, N$, there is no prob. measure

$\underset{\sim}{P}$ that satisfy both ① and equality ②'
then a sure win is possible.

$\underset{\sim}{P}$ $\quad X(t)\,,\quad 0\le t\le T$

Let $\quad \underline{X(t) = x_0\, e^{Y(t)} \sim \text{Geo BM}}$

$Y(t) \sim \text{BM } \mu\,,\ \sigma^2$

using $\mathcal{F}_u$, for $s<t$

$$E(X(t)\mid X(u),\ 0\le u\le s) = X(s)\, e^{(t-s)\underline{(\mu+\sigma^2/2)}}$$

choose $\boxed{\mu + \frac{\sigma^2}{2} = \alpha}$

$$\Rightarrow\ E(e^{-\alpha t}X(t)\mid X(u),\ 0\le u\le s) = e^{-\alpha s}X(s)$$

$\therefore$ Eq (1) is satisfied.

Eq (2) $\Leftrightarrow$ satisfied $\quad C = \underline{E_p}(e^{-\alpha t}(\underline{X(t)}-k)^+)$, then no arbitrage is possible

Since $\quad X(t) = \underline{x_0\, e^{Y(t)}}\quad , Y(t)\sim N(\mu t, \sigma^2 t)$

$$\Rightarrow C e^{\alpha t} = \int_{-\infty}^{\infty}(x_0 e^{y}-k)^{+}\frac{1}{\sqrt{2\pi t\sigma^2}}\, e^{-(y-\mu t)^2/2t\sigma^2}\,dy$$

$$C e^{\alpha t} = \int_{\log(k/x_0)}^{\infty}(x_0 e^{y}-k)\frac{1}{\sqrt{2\pi t\sigma^2}}\, e^{-(y-\mu t)^2/2t\sigma^2}\,dy \qquad \left|\ \begin{array}{l} x_0 e^{y}\ge k \\ \Leftrightarrow y \ge \log(k/x_0)\end{array}\right.$$

$$C e^{\alpha t} = \underline{x_0 e^{\mu t}\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{w\sigma\sqrt{t}}\, e^{-w^2/2}\,dw} - k\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{-w^2/2}\,dw\,, \qquad \left|\ \begin{array}{l} w = \frac{y-\mu t}{\sigma\sqrt{t}} \\ dw = \frac{dy}{\sigma\sqrt{t}}\end{array}\right.$$

where $a = \log(k/x_0) - \mu t \qquad$ —— (3)

$$\overline{\qquad\qquad}$$
$$\sigma\sqrt{t}$$

Now $\frac{1}{\sqrt{2\pi}}\int_a^\infty e^{w\sigma\sqrt{t}}e^{-w^2/2}dw = \frac{e^{t\sigma^2/2}}{\sqrt{2\pi}}\underline{\int_a^\infty e^{-(w-\sigma\sqrt{t})^2/2}dw}$

$$= e^{t\sigma^2/2}\,P(N(\sigma\sqrt{t},1)\geq a)$$

$$= e^{t\sigma^2/2}\,P(N(0,1)\geq a-\sigma\sqrt{t})$$

$$= e^{t\sigma^2/2}\,(1-\Phi(a-\sigma\sqrt{t}))$$

$$= \underline{e^{t\sigma^2/2}\,\Phi(\sigma\sqrt{t}-a)} \qquad \Big|\ \Phi(x)+\Phi(-x)=1$$

$\therefore$ (3) $\quad c\, e^{\alpha t} = x_0\, \underline{e^{\mu t+\frac{1}{2}\sigma^2 t}}\,\Phi(\sigma\sqrt{t}-a) - k\,\Phi(-a)$

Since $\quad \mu+\underline{\frac{\sigma^2}{2}}=\alpha$, let $b=-a$

$$\underline{C} = x_0\ \Phi(\sigma\sqrt{t}+b) - k\,e^{-\alpha t}\,\Phi(b) \quad \text{—} \ (4)$$

, where $b=-a=\frac{\mu t-\log(k/x_0)}{\sigma\sqrt{t}} = \frac{\alpha t-\sigma^2 t/2-\log(k/x_0)}{\sigma\sqrt{t}}$.

Option price formula (4) depends on x_0, t, k, α, σ^2

$\hookrightarrow$ then no arbitrage is possible.

(4) Black – Scholes Option cost valuation.

(it does not depend on μ (drift) but only on σ^2 (var-parameter))

Example The current price of a stock is 100. Suppose that the logarithm of the price of the stock changes according to a BM process with drift coeff $\mu = 2$ and var parameter $\sigma^2 = 1$. Give the Black-Scholes cost of an option to buy the stock at time 10 for a cost of 100 per unit.

Sol

$$\mu = 2, \ \sigma^2 = 1, \quad \alpha = \mu + \frac{\sigma^2}{2} = 2 + \frac{1}{2}$$

$$x_0 = 100, \ t = 10, \ k = 100$$

$$b = \frac{\alpha t - \sigma^2 t/2 - \log(k/x_0)}{\sigma\sqrt{t}} =$$

$$C = x_0\, \Phi(\sigma\sqrt{t} + b) - k e^{-\alpha t}\, \Phi(b) =$$

Ex.

— x —

White Noise :

$W(t) \sim N(0,t)$

$\{W(t), t \geq 0\}$ SBM, f has cont derivative on $[a,b]$

Sto. integral

Itô's integral correspond

$$\int_a^b f(t)\, dW(t) \equiv \lim_{\substack{n\to\infty \\ \max(t_i - t_{i-1}) \to 0}} \sum_{i=1}^{n} f(t_{i-1}) [W(t_i) - W(t_{i-1})] \quad \text{——} (1)^*$$

where $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of $[a,b]$

$$\sum_{i=1}^{n} f(t_{i-1}) [W(t_i) - W(t_{i-1})]$$

$$= f(b) W(b) - f(a) W(a) - \sum_{i=1}^{n} W(t_i) [f(t_i) - f(t_{i-1})]$$

Observing this

$$\cancel{f(t_n)}W(t_n) - f(t_0)W(t_0) - W(t_1)[f(t_1)-f(t_0)] - W(t_2)[f(t_2)-f(t_1)]$$
$$- \cdots - W(t_n)[\cancel{f(t_n)} - f(t_{n-1})]$$

$$\boxed{\int_a^b f(t)\,dW(t) = f(b)W(b) - f(a)W(a) - \int_a^b W(t)\,df(t)} \quad \underline{\underline{\text{deps}}} \quad \text{---}(2)^*$$

Assume interchangeability btw exp and limit

$$E\left(\int_a^b f(t)\,dW(t)\right) = 0$$

$$V\left(\sum_{i=1}^n f(t_{i-1})\left(W(t_i)-W(t_{i-1})\right)\right)$$

$$= \sum_{i=1}^n f^2(t_{i-1})\,\mathrm{Var}\left(W(t_i)-W(t_{i-1})\right)$$

$$\hookrightarrow t_i - t_{i-1}$$

$$= \sum_{i=1}^n f^2(t_{i-1})\,(t_i - t_{i-1})$$

$$V\left(\int_a^b f(t)\,dW(t)\right) = \int_a^b f^2(t)\,dt = E\left(\int_a^b f(t)\,dW(t)\right)^2$$

(Itô's isometry)

$$\int_a^b f(t)\,\underbrace{dW(t)}_{\text{White noise}} \quad \text{white noise transformation}$$

f travel through white noise medium to yield the output (at time b) $\int^b f(t)\,dW(t)$.

** integrand f_t is non-random

$W_t = W(t)$ $\quad$ — × — $\quad$ (*) $\int_a^b f(t)\,dW(t) \sim N\left(0, \int_a^b f^2(t)\,dt\right)$

Eg (1) $E\left(\int_0^T W_t\,dW_t\right) = E\left(\sum_j W_{t_j}(W_{t_{j+1}} - W_{t_j})\right)$

$$= \sum_j E(W_{t_j})\,E(W_{t_{j+1}} - W_{t_j}) = 0$$

W(t) has indep. increments

$E\left(\int_0^t W_s\,dW_s\right) = E\left(\frac{W_t^2}{2} - \frac{t}{2}\right) = \frac{t}{2} - \frac{t}{2} = 0$

— × —

$\cancel{\int_0^t \frac{1}{2} d(W_s)^2 = \frac{1}{2} W_t^2}$ wrong

W_t is nowhere differentiable so usual calculus cannot be applied

Example (2) Itô's integral

$$\int_0^t W_s\,dW_s \approx \sum_j W_{t_j}(W_{t_{j+1}} - W_{t_j})$$

$$= \frac{W_t^2}{2} - \frac{1}{2}\underbrace{\sum_j (W_{t_{j+1}} - W_{t_j})^2}_{\to t} \longrightarrow \frac{W_t^2}{2} - \frac{t}{2} \quad \blacksquare$$

$$\left(\text{Ex. } \sum_{k=1}^n a_{k-1}(a_k - a_{k-1}) = \frac{1}{2} a_n^2 - \frac{1}{2}\sum_{k=1}^n (a_k - a_{k-1})^2 \right)$$

$\sqrt{h}\,Z \overset{d}{=} (W_{t+h} - W_t) \sim N(0,h)$ $\qquad Z \sim N(0,1)$

$W_t \sim N(0,t)$

Quadratic variation

$$\sum_{j=1}^n \left(W_{tj/n} - W_{t(j-1)/n}\right)^2 \overset{d}{=} \sum_{j=1}^n \left(\sqrt{\frac{t}{n}}\,Z_j\right)^2$$

For BM trajectories

$$= t \times \underbrace{\frac{1}{n}\sum_{j=1}^n Z_j^2}_{\to E(Z^2) = 1} \to t$$

Thus the paths of the BM are very irregular (nowhere differentiable) randomly that nevertheless have a very special property that their quadratic variation in $\{g(t)\}$ ↑ exactly as the deterministic part..

$$dW_t\, dW_t = (dW_t)^2 = dt$$

Similarly $dW_t\, dt = dt\, dW_t = 0$ | $**$

$$dt \cdot dt = (dt)^2 = 0$$

Example (3) Consider a particle of unit mass that is suspended in a liquid and suppose that, due to the liquid, there is a viscous force that retards the velocity of the particle at a rate proportional to its present velocity. In addition, let us suppose that the velocity instantaneously changes according to a constant multiple of white noise, i.e., if $V(t)$: particle's vel. at t, suppose that

$$V'(t) = -\beta V(t) + \alpha W'(t),$$

where $\{W(t), t \geq 0\}$ SBM

$$\Rightarrow \quad e^{\beta t}\left[V'(t) + \beta V(t)\right] = \alpha e^{\beta t} W'(t)$$

$$\frac{d}{dt}\left[e^{\beta t} V(t)\right] = \alpha e^{\beta t} W'(t)$$

$$\Rightarrow \quad e^{\beta t} V(t) = V(0) + \alpha \int_0^t e^{\beta s} W'(s)\, ds$$

$$V(t) = V(0)\, e^{-\beta t} \ldots \int^t \ldots e^{-\beta(t-s)}$$

$$V(t) = V(0)e^{-\beta t} + \alpha \int_0^t \underline{e^{-\beta(t-s)}}\, dW(s)$$

using (2)*

$$V(t) = V(0)e^{-\beta t} + \alpha\left[W(t) - \int_0^t W(s)\,\beta\, e^{-\beta(t-s)}\, ds \right]$$

$| W(0) \equiv 0$

(Ornstein – Uhlenbeck process)

It has been proposed as model for describing the vel. of a particle immersed in a liquid or gas; and as such is useful in statistical mechanics.

—x—

Multivariate normal distn (MVN):

$$Z_i \sim NID(0,1) \quad , i = 1, \dots, n$$

constt a_{ij} , $1 \le i \le m$, $1 \le j \le n$ and μ_i , $1 \le i \le m$

If $X_i = a_{i1}Z_1 + \dots + a_{in}Z_n + \mu_i$, $i = 1, \dots, m$,

then the rvs $X_1, \dots, X_m$ are said to have a MVN.

$$E(X_i) = \mu_i \quad ; \quad V(X_i) = \sum_{j=1}^{n} a_{ij}^2$$

$$E\left(\sum_{i=1}^{m} t_i X_i\right) = \sum_{i=1}^{m} t_i \mu_i$$

$$V\left(\sum_{i=1}^{m} t_i X_i\right) = Cov\left(\sum_{i=1}^{m} t_i X_i , \sum_{j=1}^{m} t_j X_j\right)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} t_i t_j \, Cov(X_i, X_j)$$

$$\sum_{i=1}^{m} t_i X_i \sim N(\cdot\, , \cdot)$$

joint mgf of $X_1, \to X_m$

$$\phi(t_1, \to t_m) = E\left(e^{\sum_{i=1}^{m} t_i X_i}\right)$$

$$= \exp\left\{ \sum_{i=1}^{n} t_i \mu_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} t_i t_j \operatorname{Cov}(X_i, X_j) \right\}$$

which shows that the joint distn of $X_1, \to X_m$ is completely determined from a knowledge of values of $E(X_i)$ and $\operatorname{Cov}(X_i, X_j)$, $i, j = 1, \to m$. ⊛

—x—

Gaussian Process

Defn S.P. $X(t), t \geq 0$ is Gaussian or a normal process if $X(t_1), \ldots, X(t_n)$ has MVN $\forall t_1, \to t_n$.

—x—

If $\{B(t), t \geq 0\}$ BM process, then

$B(t_1), B(t_2), \to B(t_n)$ can be expressed as l.c of indep. $N(\cdot, \cdot)$ r.vs $\overbrace{B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})}^{\text{indep}}$.

$$\left.\begin{aligned} &B(t_1) \\ &B(t_2) = B(t_2) - B(t_1) + B(t_1) \\ &\vdots \end{aligned}\right]$$

$\Rightarrow \{B(t), t \geq 0\}$ is a Gaussian process.

Using ⊛, SBM could also be defined as a Gaussian process.

having $E(W(t)) = 0$, and for $s < t$

$$\operatorname{Cov}(W(s), W(t)) = \operatorname{Cov}(W(s), W(s) + W(t) - W(s))$$

$$= \underbrace{Cov(W(s), W(s))}_{\searrow \, Var(W(s)) \, \searrow s} + \underbrace{Cov(W(s), W(t)-W(s))}_{\searrow 0}$$

| $\because$ indep increment

| $W(t) \sim N(0, t)$

$$= s \; = \; s \wedge t$$

$\wedge \to$ min

| A Gaussian SP $\{X(t)\}$ is a SBM iff

| $m_X(t) := E(X(t)) = 0$ and $r_X(s,t) := Cov(X(s), X(t)) = s \wedge t$, $s, t \geq 0$.

— x —

Example (4) Find the distr of $X := \int_0^t s\, dW_s$

As integrand is non-random

$$X \sim N\left(0, \frac{t^3}{3}\right) \text{, where}$$

$$V\left(\int_0^t s\, dW_s\right) = E\left(\int_0^t s\, dW_s\right)^2 = \int_0^t s^2 ds = \frac{t^3}{3}$$

Itô Formula :

$\{X_t\}_{t \in [0,T]}$ Itô process (on $(\Omega, \mathcal{F}, F, P)$ with

BM $\{W_t\}_{t \in [0,T]}$ on it) if

$$X_t = X_0 + \int_0^t \underbrace{a_s}_{\downarrow \text{ drift coeff}} ds + \int_0^t \underbrace{b_s}_{\downarrow \text{ diffusion coeff}} dW_s \quad \text{———} (I1)$$

(I1) has on $[0,T]$ stochastic differential

$$dX_t = a_t\, dt + b_t\, dW_t \quad \text{———} (I2)$$

(One-dimensional Itô formula)

$(X_t)_{t\in[0,T]}$ Itô process with (I 2), $f(\cdot)$ twice cont. differentiable fn. Then

$Y_t := f(X_t)$, $t \in [0,T]$ is also Itô process with stochastic differential

$$dY_t \equiv df(X_t) = \left(f'(X_t)a_t + \frac{1}{2}f''(X_t)b_t^2\right)dt + f'(X_t)b_t dW_t \quad \text{—— (I 3)}$$

Sol multiplication table **

Taylor's formula for f with two terms

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \quad \text{—— (I 4)}$$

Using (I 2)

$$dX_t = a_t dt + b_t dW_t$$

$$(dX_t)^2 = dX_t \cdot dX_t = (a_t dt + b_t dW_t)(a_t dt + b_t dW_t)$$

$$= a_t^2 \underbrace{(dt)^2}_{=0} + 2a_t b_t \underbrace{dt\,dW_t}_{=0} + b_t^2 \underbrace{(dW_t)^2}_{=dt}$$

$$= b_t^2 dt$$

using this in (I 4) we get (I 3).

Example (5) $f(x) = \frac{1}{2}x^2$, $X_t = W_t$

$$f'(x) = x, \quad f''(x) = 1$$

$$Y_t = df(X_t) = df(W_t) = d\left(\frac{1}{2}W_t^2\right)$$

using (I 4)

$$dY_t = d\left(\frac{1}{2}W_t^2\right) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2$$

$$= W_t dW_t + \frac{1}{2}dt$$

$$\Rightarrow \frac{d(W_t^2)}{2} - \frac{1}{2}dt = W_t dW_t$$

Which is in perfect agreement with result of Example (2).

Example (6) $f(x) = e^x$, $X_t = W_t$

$f'(x) = f''(x) = e^x$ $\qquad Y_t = f(X_t) = f(W_t) = e^{W_t}$

Using (I4)

$$dY_t = d(e^{W_t}) = e^{W_t} dW_t + \frac{1}{2} e^{W_t} (dW_t)^2$$

$$= Y_t \, dW_t + \frac{1}{2} Y_t \, dt$$

(univariate Itô process and time variable)

Statement: $f(t,x)$ cont. $\partial_t f = \frac{\partial f}{\partial t}$ and be twice cont. differentiable in x, $\partial_x := \frac{\partial}{\partial x}$, $\partial_{xx} := \frac{\partial^2}{\partial x^2}$;

let $\{X_t\}$ Itô process (I2).

Then $Y_t := f(t, X_t)$ is also an Itô process with sto. differential

$$dY_t = \underline{\partial_t f(t, X_t)\, dt} + \underline{\partial_x f(t, X_t)\, dX_t} + \underline{\tfrac{1}{2} \partial_{xx} f(t, X_t) (dX_t)^2} \longrightarrow (I5)$$

$$= \left[\partial_t f(t, X_t) + a_t \partial_x f(t, X_t) + \frac{1}{2} b_t^2 \partial_{xx} f(t, X_t)\right] dt$$

$$+ b_t \partial_x f(t, X_t)\, dW_t$$

Example: GBM

$$Z_t = e^{X_t} = Z_0 \, e^{\mu t + \sigma W_t}, \quad t \geq 0; \; Z_0 = e^{X_0}$$

$\mu, \sigma \in \mathbb{R}$

let $Z_0 = 1$

$$Z_t = f(t, W_t) \quad \text{with } f(t,x) = e^{\mu t + \sigma x}$$

$$\partial_t f = \mu f \ , \ \partial_x f = \sigma f \ , \ \partial_{xx} f = \sigma^2 f$$

Using (I5)

$$dZ_t = \mu f(t, W_t) dt + \sigma f(t, W_t) dW_t + \frac{\sigma^2}{2} f(t, W_t) \underbrace{(dW_t)^2}_{dt}$$

$$= \left(\mu + \frac{\sigma^2}{2}\right) Z_t \, dt + \sigma Z_t \, dW_t$$

Result: (The product rule of Itô calculus):

(X_t), (Y_t) two Itô processes on a common filtered prob. space satisfying

$$dX_t = a_t \, dt + b_t \, dW_t$$

$$dY_t = \tilde{a}_t \, dt + \tilde{b}_t \, dW_t$$

with common BM (W_t).

$Z_t := X_t Y_t$ is also an Itô process with

$$dZ_t \equiv d(X_t Y_t) = Y_t \, dX_t + X_t \, dY_t + dX_t \, dY_t \quad \text{— (I6)}$$

$$= Y_t (a_t \, dt + b_t \, dW_t) + X_t (\tilde{a}_t \, dt + \tilde{b}_t \, dW_t)$$

$$+ (a_t \, dt + b_t \, dW_t)(\tilde{a}_t \, dt + \tilde{b}_t \, dW_t)$$

$$= (Y_t a_t + X_t \tilde{a}_t + b_t \tilde{b}_t) dt$$

$$+ (Y_t b_t + X_t \tilde{b}_t) dW_t$$

$dt \, dW_t = 0 = dW_t \, dt$
$(dt)^2 = 0$
$(dW_t)^2 = dt$

Sol $\quad d(X_t Y_t) = (X_t + dX_t)(Y_t + dY_t) - X_t Y_t$

now expand

Example: $\quad Z_t = W_t e^{W_t} = X_t Y_t$

let $X_t := W_t$, $Y_t := e^{W_t}$

Using (E6)

$$dZ_t = e^{W_t} dW_t + W_t \, \underbrace{de^{W_t}} + dW_t \, de^{W_t}$$

$$= e^{W_t} dW_t + W_t \left(e^{W_t} dW_t + \tfrac{1}{2} e^{W_t} dt \right) + dW_t \cdot \left(e^{W_t} dW_t + \tfrac{1}{2} e^{W_t} dt \right)$$

using example (6)

$$= \left(1 + \tfrac{1}{2} W_t\right) e^{W_t} dt + (1 + W_t) e^{W_t} dW_t$$

| using multiplication table

Stochastic Differential Equations: (SDE)

SDE one means an equation involving stochastic differentials

we here deal SDE's with form

$$dX_t = a(t, X_t) + b(t, X_t) dW_t \quad (S1), \quad t \in [0,T], \quad X_0 = x_0$$

non-random initial condition

$a(t,x)$, $b(t,x)$, $t \geq 0$, $x \in \mathbb{R}$

non-random fn

(W_t) SBM on $(\Omega, \mathcal{F}, F, P)$

An Itô process $(X_t)_{t \in [0,T]}$ on $(\Omega, \mathcal{F}, F, P)$ is said to be a sol^n to SDE (S1) if

$$X_t = x_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s, \quad t \in [0,T]$$

└ (S2)

(S1) will have a unique solⁿ provided that the fⁿ a, b are regular enough.

SDE rarely admit solⁿ of an analytic closed form (one exceptional class being linear SDEs e.g. a special case Ornstein-Uhlenbeck process)

$\rightarrow$ $\boxed{\text{Mathematical Modeling and Computation in Finance by Oosterlee \& Grzelak}}$ $\leftarrow$

<u>Example</u> (Ornstein-Uhlenbeck process) is the solⁿ of <u>linear SDE</u> with additive noise

$$dX_t = -r X_t \, dt + \sigma \, dW_t \;, \quad X_t\big|_{t=0} = x_0$$

$$r, \sigma > 0.$$

If we use this to model dynamics of a particle, then drift term represents a restoring force that prevent the particle from getting too far away from the origin. We will see that the distrⁿ of the particle position has a limit as time goes to ∞.

This is a very different behavior from that of the BM.

Sol $\quad e^{rt} dX_t + r e^{rt} X_t \, dt = \sigma e^{rt} dW_t$

$$\Rightarrow \underline{d} \left(e^{rt} X_t \right) = \sigma e^{rt} dW_t$$

$$dt \qquad\qquad dW_t$$

$$\Rightarrow \quad e^{rt} X_t - X_0 = \int_0^t \sigma e^{rs}\, dW_s$$

$$\Rightarrow \quad X_t = e^{-rt} X_0 + \sigma \int_0^t e^{-r(t-s)}\, dW_s$$

is a sol^n

Let $\Theta_t := \int_0^t e^{-r(t-s)}\, dW_s$

Θ_t is a Gaussian process with (using $\otimes_2$)

$$E(\Theta_t) = 0$$

$$E(\Theta_t^2) = E\left(\int_0^t e^{-r(t-s)}\, dW_s \right)^2$$

$$= \int_0^t e^{-2r(t-s)}\, ds$$

| using Itô's isometry

$$= e^{-2rt} \int_0^t e^{+2rs}\, ds$$

$$= e^{-2rt} \left(\frac{e^{2rs}}{2r} \right)_{s=0}^{t}$$

$$= \frac{e^{-2rt}}{2r} \left(e^{2rt} - 1 \right) = \frac{1}{2r} \left(1 - e^{-2rt} \right)$$

$$\underline{X_t \sim N\left(e^{-rt} x_0 ,\ \sigma^2_{r,t} \right)} \ ,$$

where $\sigma^2_{r,t} = \frac{\sigma^2}{2r}(1-e^{-2rt})$

if X_0 Gaussian

$$X_t \to N\left(0, \frac{\sigma^2}{2r}\right) \quad \text{as } t \to \infty$$

Vasicek interest rate model:

"mean-reverting property" of Ornstein-Uhlenbeck process (i.e., the tendency of SP's trajectory to keep returning to its "historic av. value") made it a candidate for a mathematical model of the interest rate dynamics.

In this model spot interest rate r_t is assumed to satisfy the SDE

$$dr_t = a(b-r_t)dt + \sigma dW_t, \quad t>0; \; a, b, \sigma, r_0 > 0$$

Putting $X_t := r_t - b$; $dX_t = dr_t$

$$dX_t = -aX_t dt + \sigma dW_t, \quad t > 0$$

$$r_t - b \sim N\left(b + e^{-at}(r_0 - b), \sigma^2_{a,t}\right)$$

using Ornstein-Uhlenbeck process

This model has an obvious deficiency with +ve prob., the interest rate r_t can assume -ve values, which is undesirable.

Th...

This is fixed in the model in next example

Cox – Ingersell – Ross Interest rate model:

assume

$$dr_t = a(b - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t \quad , t > 0$$

$$a, b, \sigma, r_0 > 0$$

"freezes" the random oscillation as $r_t \to 0$ and so the +ve drift term becomes dominating, Hence the model will never produce negative interest rate values. Moreover r_t will never turn into zero provided that $2ab \geq \sigma^2$.