Continuous Time Markov Chain (CTMC)

Defⁿ 1 Let $X(t)$: state at time t.

$\{X(t), t \geq 0\}$ is CTMC if $\forall\, s, t \geq 0$ and for non-negative integers i, j, $x(u)$, $0 \leq u < s$,

$$P(X(t+s)=j \mid X(s)=i, X(u)=x(u), 0 \leq u < s)$$

$$= P(X(t+s)=j \mid X(s)=i) = P(X(t)=j \mid X(0)=i)$$

$$:= P_{ij}(t) \quad \text{transition probability function} \quad (*)$$

In addition if $P(X(t+s)=j \mid X(s)=i)$ is indep. of s, then CTMC is said to have stationary or homogeneous transition probabilities.

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Let T_i denote the sojourn time or the amt of time the process stays in state i before making a transition into a different state, then using $(*)$, we have

$$P(T_i > t+s \mid T_i > s) = P(T_i > t), \ \forall\, s, t \geq 0$$

$$\therefore \quad T_i \sim \text{exponential}(\nu_i) \qquad \nu_i = q_i$$

Defⁿ 2 SP having the property that each time the process enters state i (i) the amt of time it spends in that state before making a transition into different state is exp. distributed with mean $\frac{1}{v_i}$.

and (ii) When the process leaves state i, it next enters state j with some probability, say, P_{ij}

$$P_{ii} = 0, \forall i, \quad \sum_j P_{ij} = 1$$

Example: (1) Birth & death process (B&D process)

Given $X(t) = n$

(i) new arrival enters the system at an exp. rate λ_n.

and (ii) people leave " " " " " " μ_n.

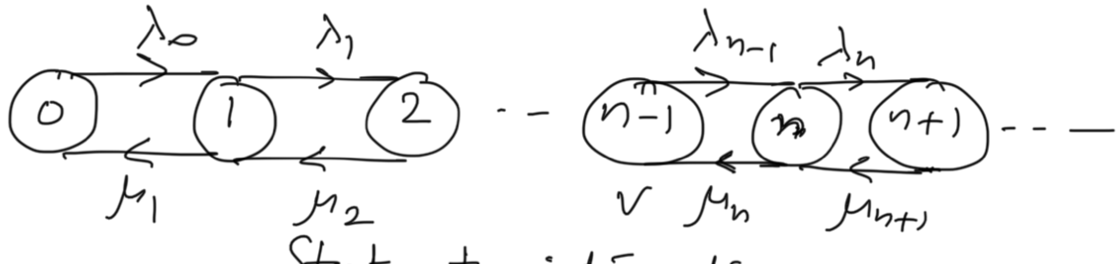
$h \to$ small

$$P(X(t+h) = n+1 \mid X(t) = n) = \lambda_n h + o(h)$$

$$P(X(t+h) = n-1 \mid X(t) = n) = \mu_n h + o(h)$$

$\{\lambda_n\}_{n=0}^{\infty}$ arrival / birth rate

$\{\mu_n\}_{n=1}^{\infty}$ departure / death rate



State transition diagram

$v_0 = \lambda_0$

$v_n = \lambda_n + \mu_n$

$X(t) = n$

indep. $X \sim exp(\lambda_n)$, $Y \sim exp(\mu_n)$

$T_n = \min(X, Y) \sim exp(\lambda_n + \mu_n)$

$P(\min(X,Y) > t) = P(X > t, Y > t)$

$$= P(X>t)\, P(Y>t)$$
$$= e^{-\lambda_n t}\, e^{-\mu_n t}$$
$$= e^{-(\lambda_n+\mu_n)t}$$

$P_{01} = 1$

$$\underline{P_{n,n+1}} = P(X<Y) = \int_0^\infty P(Y>t)\, \lambda_n e^{-\lambda_n t}\, dt$$

$$= \int_0^\infty e^{-\mu_n t} \lambda_n e^{-\lambda_n t}\, dt = \frac{\lambda_n}{\lambda_n+\mu_n}$$

$$P_{n,n-1} = \frac{\mu_n}{\lambda_n+\mu_n}.$$

(2) <u>Poisson Process</u> P.P. (λ)

B&D process $\lambda_n = \lambda$, $\mu_n = 0$ $\forall n \geq 0$

(3) <u>Pure Birth process</u>

B&D process $\mu_n = 0$, $\forall n$.

(4) <u>Yule process or a birth process with linear birth rate</u>

Pure Birth process with $\lambda_n = n\lambda$, $n = 0, 1, 2, \ldots$

Here each individual in the popl^n is assumed to give birth at an expo. rate λ

$$n \xrightarrow{\lambda_n} n+1$$

$\underbrace{X_1, \ldots, X_n}_{n}$ $X_i \sim \exp(\lambda)$

$\to n+1$

$\min(X_1, \ldots, X_n)$

$\sim \exp(n\lambda)$

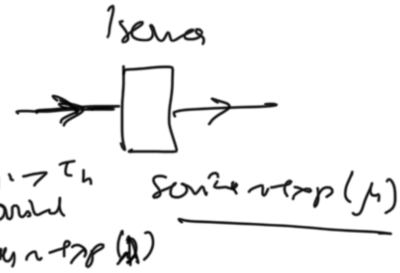
(5) A linear growth model with immigration:

$$\mu_n = n\mu \quad , n = 1, 2, \ldots$$

$$\lambda_n = n\lambda + \theta \quad , n = 0, 1, 2, \ldots$$

Here each individual in the pop^n is assumed to give birth at an exponential rate λ, in addition there is an exponential rate of increase θ of the pop^n due to an external source such as immigration.

(6) M/M/1 queueing system

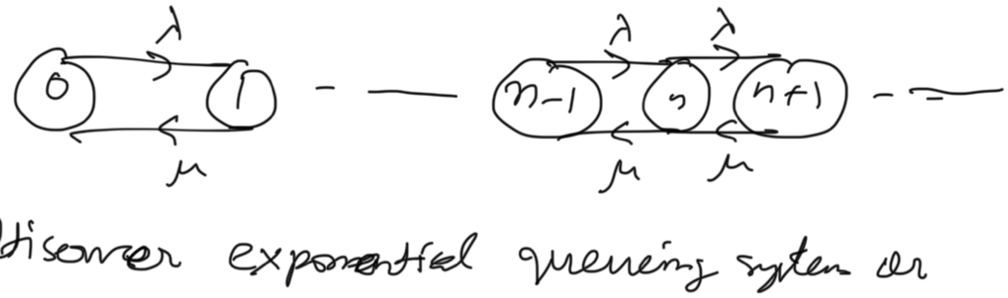


B & D process

$$\lambda_n = \lambda \ , n = 0, 1, 2, \ldots$$

$$\mu_n = \mu \ , n = 1, 2, \ldots$$

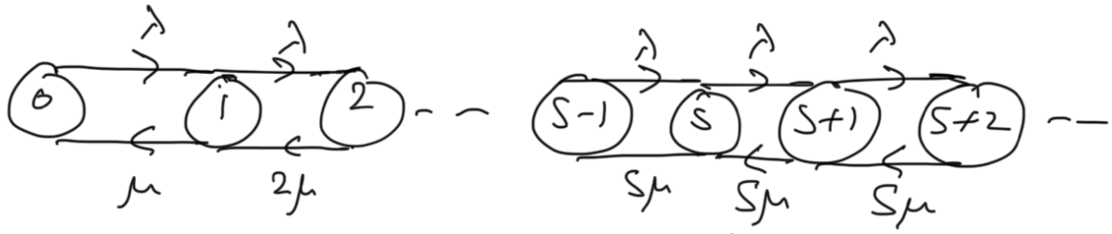
$X(t)$: # of customers in queueing system at time t.



(7) Multiserver exponential queueing system or M/M/s queueing system

s = servers $\lambda_n = \lambda \quad , n = 0, 1, 2, \ldots$

and $\mu_n = \begin{cases} n\mu, & n = 1, 2, \dots, s \\ s\mu, & n = s+1, s+2, \dots \end{cases}$



indep $\begin{cases} Y_1 \sim exp(\mu) \\ Y_2 \sim exp(\mu) \\ \vdots \\ Y_s \end{cases}$ $\min(Y_1, Y_2) \sim exp(2\mu)$

Expected time to go from state i to state j in B&D process

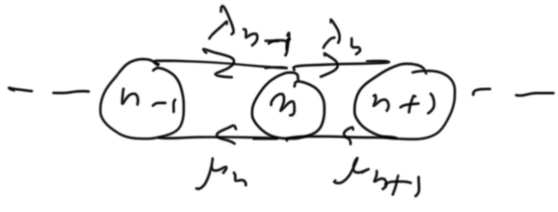
B&D process birth rates $\{\lambda_n\}_{n=0}^{\infty}$
death rates $\{\mu_n\}_{n=1}^{\infty}$

Let T_i time, starting from state i, it takes for the process to enter state $i+1$, $i \geq 0$.

$$T_0 \sim exp(\lambda_0) \qquad E(T_0) = \frac{1}{\lambda_0}$$

For $i > 0$, let

$$I_i = \begin{cases} 1 & \text{if the first transition from } i \text{ to } i+1 \\ 0 & \text{,, ,, ,, ,, ,, } i \text{ to } i-1 \end{cases}$$



$E(T_i \mid I_i = 1) = \dfrac{1}{\lambda_i + \mu_i}$, since independent of whether the first transition is from a birth or death, the time it occurs is exp. with rate $\lambda_i + \mu_i$.

$$E(T_i \mid I_i=0) = \frac{1}{\lambda_i+\mu_i} + E(T_{i-1}) + E(T_i)$$

Now $E(T_i) = E(T_i \mid I_i=1)\,P(I_i=1) + E(T_i \mid I_i=0)\,P(I_i=0)$

$$= \frac{1}{\lambda_i+\mu_i} \times \frac{\lambda_i}{\lambda_i+\mu_i} + \left[\frac{1}{\lambda_i+\mu_i} + E(T_{i-1}) + E(T_i)\right] \times \frac{\mu_i}{\lambda_i+\mu_i}$$

$$\Rightarrow E(T_i) = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E(T_{i-1}) \quad , i \geq 1$$

with $E(T_0) = \frac{1}{\lambda_0}$

$\therefore$ Expected time to go from state i to state j

$= E(T_i) + E(T_{i+1}) + \cdots + E(T_{j-1})$

where $i<j$.

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indep $\left\langle \begin{array}{l} X_1 \sim \exp(\lambda) \\ X_2 \sim \ \text{"} \ (\lambda) \end{array} \right.$ $X_1 + X_2 \sim \text{Gamma}(2, \lambda)$

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Hypoexponential dist^n^

Let $X_1, X_2, \ldots, X_n$ be indep. r.v's s.t.

$X_i \sim \exp(\lambda_i)$, $i = 1, \ldots, n$, then

$\sum_{i=1}^{n} X_i$ hypoexponential r.v.

pdf of $X_1 + X_2$

$$f_{X_1+X_2}(t) = \int_0^t f_{X_1}(s)\, f_{X_2}(t-s)\, ds$$

$$= \int_0^t \lambda_1 e^{-\lambda_1 s}\, \lambda_2 e^{-\lambda_2 (t-s)}\, ds$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 t} \int_0^t e^{-(\lambda_1-\lambda_2)s} ds$$

$$= \frac{\lambda_1 \lambda_2 e^{-\lambda_2 t}}{\lambda_1-\lambda_2} \left(1-e^{-(\lambda_1-\lambda_2)t}\right)$$

$$= \frac{\lambda_1}{\lambda_1-\lambda_2} \lambda_2 e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2-\lambda_1} \lambda_1 e^{-\lambda_1 t}$$

Similarly $f_{X_1+X_2+X_3}(t) = \sum_{i=1}^{3} \lambda_i e^{-\lambda_i t} \left(\prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \right)$

$$f_{\sum_{i=1}^{n} X_i}(t) = \sum_{i=1}^{n} C_{i,n} \lambda_i e^{-\lambda_i t}, \text{ where } C_{i,n} = \prod_{j\neq i} \frac{\lambda_j}{\lambda_j-\lambda_i}$$

$$\bar{F}_{\sum_{i=1}^{n} X_i}(t) = P\left(\sum_{i=1}^{n} X_i > t\right) = \int_t^\infty f_{\sum_{i=1}^{n} X_i}(u)\, du$$

$$= \sum_{i=1}^{n} C_{i,n} e^{-\lambda_i t}$$

<u>Pure Birth process</u>

$$0 \xrightarrow{\lambda_0} 1 \xrightarrow{\lambda_1} 2 \; \cdots$$

$X(t)$ state at time t

$$P_{ij}(t) = P(X(t)=j \mid X(0)=i)$$

Let X_k time pure birth process spends in state k before making a transition into state $k+1$; $k \geq 1$

$X_k \sim \exp(\lambda_k)$ $\qquad X_1, X_2, \ldots$ indep

$X(t)<j \mid X(0)=i \iff X_i + X_{i+1} + \cdots + X_{j-1} > t$

$$P(X(t)<j \mid X(0)=i) = P\Big(\underbrace{\sum_{k=i}^{j-1} X_k}_{\sim \text{Hypo exp}} > t\Big)$$

$$= \sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{r\neq k, r=i}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}$$

$\therefore$ $P_{ij}(t) = P(X(t)=j \mid X(0)=i)$

$$= P(X(t)<j+1 \mid X(0)=i) - P(X(t)<j \mid X(0)=i)$$

$$= \sum_{k=i}^{j} e^{-\lambda_k t} \prod_{r\neq k, r=i}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k} - \sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{r\neq k, r=i}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}$$

Also $P_{ii}(t) = P(X_i > t) = e^{-\lambda_i t}$, $i<j$.

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For CTMC

For any pair of states i and j, let

v_i rate at which the process makes a transition when in state i

P_{ij} prob. that this transition is into state j

q_{ij} rate, when in state i, at which the process makes a transition into state j

Note that $q_{ij} = v_i P_{ij}$

generator of CTMC
$Q = ((q_{ij}))$
$q_{ij} = \begin{cases} -v_i, & i=j \end{cases}$

$$\sum_j q_{ij} = \nu_i \sum_j P_{ij} = \nu_i = q_i$$

$q_{ij},\ i \neq j$

$Q = \begin{bmatrix} \ & \ & 0 \\ \ & \ & 0 \\ \ & \ & 0 \end{bmatrix}$

Lemma (*) (a) $\displaystyle\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = \nu_i$

(b) $\displaystyle\lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}$, when $i \neq j$

Sol (a) Note that $1 - P_{ii}(h) = \nu_i h + o(h)$

$$\therefore \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = \nu_i$$

(b) Note that $P_{ij}(h) = q_{ij} h + o(h)$

$$\lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}$$

Lemma (A) For all $s \geq 0$, $t \geq 0$

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

Chapman-Kolmogorov equations

$\underset{\sim}{P}(t+s) = \underset{\sim}{P}(t)\, \underset{\sim}{P}(s)$

$\underset{\sim}{P}(t) = ((P_{ij}(t)))$

Sol $P_{ij}(t+s) = P(X(t+s) = j \mid X(0) = i)$

$$= \sum_{k=0}^{\infty} P(X(t+s) = j,\ X(t) = k \mid X(0) = i)$$

$$= \sum_{k=0}^{\infty} P(X(t+s) = j \mid X(t) = k,\ \cancel{X(0) = i}) \cdot P(X(t) = k \mid X(0) = i)$$

$$= \sum_{k=0}^{\infty} P(X(t+s) = j \mid X(t) = k) \cdot P(X(t) = k \mid X(0) = i)$$

$$\sum^{\infty} \ldots$$

$$= \sum_{k=0} P_{kj}(s) \cdot P_{ik}(t)$$

Kolmogorov's Backward equation: For all states i, j and time $t \geq 0$,

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$

$$\underset{\sim}{P}'(t) = Q \underset{\sim}{P}(t)$$

Sol

$$P_{ij}(h+t) - P_{ij}(t)$$

$$= \sum_{k=0}^{\infty} P_{ik}(h) P_{kj}(t) - P_{ij}(t) \quad \text{(Using CK eqn's}$$

$$= \sum_{k \neq i} P_{ik}(h) P_{kj}(t) - (1 - P_{ii}(h)) P_{ij}(t)$$

$$\lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k \neq i} \left(\lim_{h \to 0} \frac{P_{ik}(h)}{h} \right) P_{kj}(t) - \left(\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} \right) P_{ij}(t)$$

$$\Rightarrow P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t) \quad \text{(Using Lemma } \circledast$$

$$\text{———— (1)}'$$

Example (1) Backward equations for B & D process

$\nu_0 = \lambda_0$, $\nu_i = \lambda_i + \mu_i$, $P_{0,1} = 1$, $P_{i,i+1} = \dfrac{\lambda_i}{\lambda_i + \mu_i}$, $P_{i,i-1} = \dfrac{\mu_i}{\lambda_i + \mu_i}$

$q_{ij} = \nu_i P_{ij}$; $q_{i,i+1} = \lambda_i$, $q_{i,i-1} = \mu_i$, $q_{i,j} = 0, j \neq i-1, i+1$

From (1)'

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t), \quad i > 0$$

$$P'_{0j}(t) = \lambda_0 P_{1j}(t) - \lambda_0 P_{0j}(t)$$

(2) <u>Backward equation for Pure Birth process</u>

B & D process $\mu_i = 0 \ \forall i$

$$P_{ij}'(t) = \lambda_i P_{i+1,j}(t) - \lambda_i P_{ij}(t)$$

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<u>Kolmogorov's forward equations</u> Under suitable regularity conditions

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) \qquad -(2)'$$

$\underset{\sim}{P}'(t) = \underset{\sim}{P}(t) Q$

Pf

$P_{ij}(t+h) - P_{ij}(t)$

$$= \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(h) - P_{ij}(t) \qquad | \text{Using CK eqns}$$

$$= \sum_{k \neq j} P_{ik}(t) P_{kj}(h) - (1 - P_{jj}(h)) P_{ij}(t)$$

divide by h and take $\lim_{h \to 0}$

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) \qquad -(2)'$$

| Using Lemma (*)

<u>Example</u> Forward equations for B & D process

$v_0 = \lambda_0$, $v_i = \lambda_i + \mu_i$, $q_{i,i+1} = \lambda_i$, $q_{i,i-1} = \mu_i$

$q_{ij} = 0$, $j \neq i-1, i+1$

$$P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t) - (\lambda_j + \mu_j) P_{ij}(t)$$

$$P_{i0}(t) = \mu_1 P_{i1}(t) - \lambda_0 P_{i0}(t)$$

Limiting probabilities / Steady state prob./ stationary state prob.:

The prob. that a CTMC will be in state j at time t often converges to a limiting value which is indep of the initial state, i.e.,

$$(\pi_j \text{ or}) \; P_j = \lim_{t\to\infty} P_{ij}(t)$$

forward eq'n

$$P_{ij}'(t) = \sum_{k\neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

$$\lim_{t\to\infty} P_{ij}'(t) = \sum_{k\neq j} q_{kj}\left(\lim_{t\to\infty} P_{ik}(t)\right) - v_j\left(\lim_{t\to\infty} P_{ij}(t)\right)$$

$$0 = \sum_{k\neq j} q_{kj} P_k - v_j P_j$$

$P_{ij}(t)$ bdded fn

$P_{ij}(t) \to P_j$ as $t\to\infty$

hence $P_{ij}'(t) \to 0$ as $t\to\infty$

$$\Rightarrow v_j P_j = \sum q_{kj} P_k \quad \forall \text{ states } j$$

$P_j = \pi_j$

$$k \neq j$$

$$\sum_j P_j = 1$$

$\pi Q = 0$

$\sum \pi_i = 1$

$\pi_j = P_j$ long run proportion of time that the process is in state j.

Example For B&D process

State	$v_n P_n = \sum_{k \neq n} q_{kn} P_k$
n	$(\lambda_n + \mu_n) P_n = \mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1}$
0	$\lambda_0 P_0 = \mu_1 P_1$

$$\Rightarrow P_1 = \frac{\lambda_0}{\mu_1} P_0 \quad ; \quad (\lambda_1 + \cancel{\mu_1}) P_1 = \mu_2 P_2 + \cancel{\lambda_0 P_0}$$

$$\Rightarrow \mu_2 P_2 = \lambda_1 P_1 \Rightarrow P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0$$

$$P_n = \underbrace{\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}}_{C_n} P_0 = C_n P_0 ,$$

$$\sum_{n=0}^{\infty} P_n = 1 \Rightarrow (1 + C_1 + C_2 + \cdots) P_0 = 1$$

$$S P_0 = 1 \quad , \text{where } S = 1 + C_1 + C_2 + \cdots$$

$$\Rightarrow P_0 = \frac{1}{S}$$

A necessary condition for these limiting probabilities to exist is $S < \infty$. —— $*_1$

Example (1) M/M/s queueing system

$$\lambda_n = \lambda, \; n = 0, 1, 2, \ldots \quad ; \quad \mu_n = \begin{cases} n\mu, & n = 1, 2, \ldots, s \\ s\mu, & n = s+1, s+2, \ldots \end{cases}$$

$*_1$ reduces to $\sum_{n=1}^{\infty} \frac{\lambda^n}{(s\mu)^n} < \infty \equiv \frac{\lambda}{s\mu} < 1$

(2) Linear growth model with immigration

$$\lambda_n = n\lambda + \theta \quad , n \geq 0$$

$$\mu_n = n\mu \quad , n \geq 1$$

$*_1$ reduces to $$\sum_{n=1}^{\infty} \underbrace{\frac{\theta(\theta+\lambda)\cdots(\theta+(n-1)\lambda)}{n!\,\mu^n}}_{T_n \text{ (say)}} < \infty$$

Using ratio test, this converges when

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} < 1 \iff \lim_{n\to\infty} \frac{\theta(\theta+\lambda)\cdots(\theta+n\lambda)}{(n+1)!\,\mu^{n+1}} \times \frac{n!\,\mu^n}{\theta(\theta+\lambda)\cdots(\theta+(n-1)\lambda)} < 1$$

$$\iff \lim_{n\to\infty} \frac{\theta+n\lambda}{(n+1)\mu} < 1$$

$$\iff \frac{\lambda}{\mu} < 1 \iff \lambda < \mu.$$

Example (1) (Assgn CTMC Q.13) A small barbershop operated by a single barber, has room for atmost two customers. Potential customers arrive at a Poisson rate of three per hr, and the successive service times are independent expo. r.v's with mean $\frac{1}{4}$ hr.

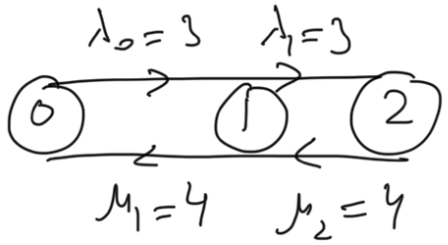
(i) What is the av. # of customers in the shop?

(1) What is the proportion of potential customers that enter the shop?

Sol $\quad X(t)$: # of customers in the shop at time t

$$X(t) \in \{0, 1, 2\}$$

CTMC



single barber

B&D process

$S < \infty$

Steady state sol^n exist

$$S = 1 + C_1 + C_2 + \cdots$$
$$= 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + 0 + 0 + \cdots$$
$$= \frac{37}{16}$$
$$P_0 = \frac{1}{S} = \frac{16}{37}$$
$$P_1 = C_1 P_0 = \frac{3}{4} \times \frac{16}{37} = \frac{12}{37}$$
$$P_2 = C_2 P_0 = \left(\frac{3}{4}\right)^2 \times \frac{16}{37} = \frac{9}{37}$$

(i) $\sum_n n P_n = 1 \times \frac{12}{37} + 2 \times \frac{9}{37} = \frac{30}{37}$

(ii) $P_0 + P_1 = \frac{28}{37}$.

(2) A job shop consists of three machines and two repairmen. The amt. of time a machine works before breaking down is expo. distributed with mean 10. If the amt of time it takes a single

repairman to fix a machine is exp. dist with mean 8, then

(a) What is the av. # of machines not in use?

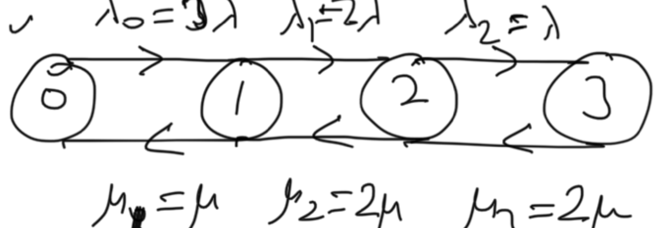
(b) What proportion of time both repairmen busy?

Sol

$X(t) = $ # of machines that are down at time t

$X(t) \in \{0, 1, 2, 3\}$ CTMC

$\lambda = \frac{1}{10}, \mu = \frac{1}{8}$



$\frac{\lambda}{\mu} = \frac{8}{10}$

$S = 1 + C_1 + C_2 + \cdots$

$= 1 + 3 \times \frac{8}{10} + \frac{6}{2} \times \left(\frac{8}{10}\right)^2 + \frac{6}{4} \times \left(\frac{8}{10}\right)^3 + 0 + \cdots$

$= \frac{1522}{250} < \infty$

$P_0 = \frac{1}{S} = \frac{250}{1522}$

$P_1 = C_1 P_0$

$P_2 = C_2 P_0$

$P_3 = C_3 P_0$

(i) $\sum_n n P_n = P_1 + 2P_2 + 3P_3$

(ii) $P_2 + P_3$