Functional involving higher order derivative

Ex. Extremize $I[y(x)] = \int_0^1 (1 + y''^2)\, dx$.

$y(0) = 0,\ y'(0) = 1,\ y(1) = 1,\ y'(1) = 1$.

Necessary condition for extremizing

$$I[y(x)] = \int_a^b f(x, y, y', y'')\, dx$$

is known as Euler-Poisson eqn. It is of the form.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

If $I[y(x)] = \int_a^b f(x, y, y', y'', \ldots, y^{(n)})\, dx$.

then a nec. condition that $y = y(x)$ extremizes. I is that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y'''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right) = 0.$$

$f = 1 + y''^2$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

$\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial y'} = 0$

$$\frac{d^2}{dx^2}\left(2y''\right) = 0.$$

$$\Rightarrow \frac{d^4 y}{dx^4} = 0 \; ; \quad y(0) = 0, \; y'(0) = 1, \; y(1) = 1, \; y'(1) = 1.$$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

$$\boxed{y = x.}$$

Derivation of Euler-Poisson equation.

<u>Statement</u>: To find $y = y(x)$ which extremizes the functional

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y', y'')\,dx \quad \longrightarrow (1)$$

subject to the conditions $y(x_1) = y_1$, $y(x_2) = y_2$, $y'(x_1) = z_1$, $y'(x_2) = z_2$. $\longrightarrow (2)$

Construct the comparison functions

$$\bar{y}(x) = y(x) + \epsilon\,\eta(x). \longrightarrow (3)$$

where ϵ is a small parameter and

$$\bar{y}(x_1) = y_1,\ \bar{y}(x_2) = y_2,\ \bar{y}'(x_1) = z_1,\ \bar{y}'(x_2) = z_2.$$

Thus, $\eta(x_1) = 0 = \eta(x_2) = \eta'(x_1) = \eta'(x_2) \longrightarrow (4)$

Note, $y = y(x)$ is fixed in the sense that it extremizes the function $I[y(x)]$ defined in (1). Also initially let us keep $\eta(x)$ fixed with the properties (4).

Now, replacing y by $\bar{y}(x)$ in (1) we get

$$I[\bar{y}(x)] = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}', \bar{y}'')\,dx \longrightarrow (5)$$

$\because$ $y(x)$ & $\eta(x)$ are assumed to be fixed, so $\bar{y}(x)$ given in (3) & hence $I[\bar{y}(x)]$ given in (5) is a function of ϵ alone.

Thus, $I(\epsilon) = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}', \bar{y}'')\, dx \rightarrow (6)$ $\quad \left|\ \bar{y}(x) = y(x) + \epsilon\,\eta(x)\right.$

Now I takes its extreme value when $\bar{y}(x) = y(x)$ i.e when $\epsilon = 0$. A necessary condition for this is

$$\frac{dI}{d\epsilon} = 0 \quad \text{when } \epsilon = 0.$$

From (6), $\dfrac{dI}{d\epsilon} = \dfrac{d}{d\epsilon}\displaystyle\int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}', \bar{y}'')\, dx.$

$$= \int_{x_1}^{x_2} \frac{\partial}{\partial \epsilon} f(x, \bar{y}, \bar{y}', \bar{y}'')\, dx.$$

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x} \cdot \underbrace{\cancel{\frac{\partial x}{\partial \epsilon}}}_{=0} + \frac{\partial f}{\partial \bar{y}} \cdot \frac{\partial \bar{y}}{\partial \epsilon} + \frac{\partial f}{\partial \bar{y}'} \cdot \frac{\partial \bar{y}'}{\partial \epsilon} + \frac{\partial f}{\partial \bar{y}''} \cdot \frac{\partial \bar{y}''}{\partial \epsilon} \right] dx. \rightarrow (7)$$

Now, $\bar{y}(x) = y(x) + \epsilon\,\eta(x) \Rightarrow \dfrac{\partial \bar{y}}{\partial \epsilon} = \eta(x)$

$\bar{y}'(x) = y'(x) + \epsilon\,\eta'(x) \Rightarrow \dfrac{\partial \bar{y}'}{\partial \epsilon} = \eta'(x)$

$\bar{y}''(x) = y''(x) + \epsilon\,\eta''(x) \Rightarrow \dfrac{\partial \bar{y}''}{\partial \epsilon} = \eta''(x).$

Substituting the above results into (7) we get.

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial \bar{y}} \cdot \eta(x) + \frac{\partial f}{\partial \bar{y}'} \cdot \eta'(x) + \frac{\partial f}{\partial \bar{y}''} \cdot \eta'' \right] dx \rightarrow (8)$$

Now, $\dfrac{dI}{d\epsilon} = 0$ when $\epsilon = 0$, i.e when $y = \bar{y}$:

Thus from (8), $\displaystyle\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \cdot \eta(x) + \frac{\partial f}{\partial y'} \cdot \eta'(x) + \frac{\partial f}{\partial y''} \cdot \eta'' \right] dx = 0 \rightarrow (9)$

Now, $\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x)\, dx = \left[\eta(x) \cdot \frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$

$$= - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx \quad \because \eta(x_1) = 0 = \eta(x_2) \quad \rightarrow (9a).$$

Also, $\int_{x_1}^{x_2} \frac{\partial f}{\partial y''} \eta''(x)\, dx = \left[\eta'(x) \frac{\partial f}{\partial y''}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta'(x) \cdot \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) dx.$

$$= - \int_{x_1}^{x_2} \eta'(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) dx = - \left[\left(\eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right)_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) dx\right]$$

$$= \int_{x_1}^{x_2} \eta(x) \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) dx \longrightarrow (9b).$$

Substituting (9a) & (9b) into (9), get

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right] \eta(x)\, dx = 0. \longrightarrow (10)$$

So far we've assumed $\eta(x)$ to be fixed with $\eta(x_1) = 0 = \eta(x_2) = \eta'(x_1) = \eta'(x_2)$. Now (10) will hold good for all $\eta(x)$ which satisfy the end point conditions. i.e (10) will hold good for arbitrary $\eta(x)$ which satisfy the end point conditions.

So by the lemma, we've

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

$\rightarrow$ Euler - Poisson equation.

Extension of Euler - Poisson equation.

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y', y'', \ldots, y^{(n)})\,dx$$

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_1) = y_1^1,\ y'(x_2) = y_2^1, \ldots, y^{(n-1)}(x_1) = y_1^{(n-1)}\ \ y^{(n-1)}(x_2) = y_2^{(n-1)}.$$

$$f_y - \frac{d}{dx}(f_{y'}) + \frac{d^2}{dx^2}(f_{y''}) - \frac{d^3}{dx^3}(f_{y'''}) + \cdots + (-1)^n \frac{d^n}{dx^n}(f_{y^{(n)}}) = 0.$$