

Simulation

Example (1) Monte Carlo Integration

$$I = \int_a^b f(x) dx = (b-a) \int_0^1 f(a+(b-a)u) du \quad \begin{cases} x = a+(b-a)u \\ dx = (b-a)du \end{cases}$$
$$\hat{I}_n = \frac{b-a}{n} \sum_{i=1}^n f(a+(b-a)U_i) \quad \checkmark \quad \underbrace{U_i \sim U(0,1)}$$

$$E(\hat{I}_n) = \frac{b-a}{n} \sum_{i=1}^n \int_0^1 f(a+(b-a)u) du$$
$$= (b-a) \int_0^1 f(a+(b-a)u) du = I \quad \forall n$$

i.e., \hat{I}_n is an unbiased estimator of I .

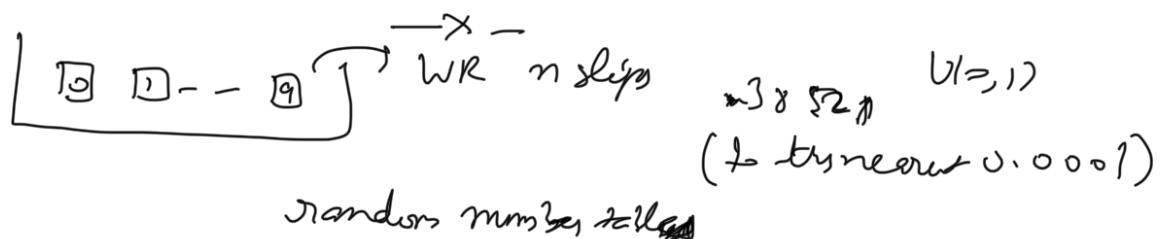
Suppose $I = \int_0^1 (1 + \cos(\pi x)) dx = 1$

$n=4$ $U(0,1)$ sample

U_1, U_2, U_3, U_4
0.419, 0.109, 0.772, 0.893

$$\hat{I}_4 = \frac{1-0}{4} \sum_{i=1}^4 [1 + \cos \pi (0 + (1-0)U_i)] = 0.896$$

which is close to actual answer 1.



digital computers $\xrightarrow{\quad}$ simulated $U(0,1)$

pseudo random number (PRN)

initial seed X_0

$$X_{n+1} = (a X_n + c) \bmod m, \quad n \geq 0 \quad \text{specified integers } a, c \text{ and } m.$$

$$X_n \in \{0, 1, 2, \dots, m-1\}$$

$$U_n = \frac{X_n}{m} \approx U(0,1)$$

Generator

$$X_i = 16807 X_{i-1} \bmod (2^{31}-1)$$

Example (Hines et al)

$$Z_i = (5 Z_{i-1} + 1) \bmod 8, \quad \text{seed } Z_0 = 0$$

$$\begin{array}{l|l} Z_1 = 1 \bmod 8 & 8 \overline{) 160} \\ = 1 & \begin{array}{r} 0 \\ \underline{1} \end{array} \end{array} \quad \begin{array}{l|l} Z_2 = 6 \bmod 8 = 6 & 8 \overline{) 60} \\ & \begin{array}{r} 0 \\ \underline{6} \end{array} \end{array}$$

1, 6, 7, 4, 5, 2, 3, 0

full period or full cycle

$\frac{1}{8}, \frac{6}{8}, \frac{7}{8}, \frac{4}{8}, \frac{5}{8}, \frac{2}{8}, \frac{3}{8}, 0$

generator

$U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8$

PRNs.

Example (Hines et al)

$$Z_i = (3 Z_{i-1} + 1) \bmod 7$$

(i) $Z_0 = 3,$

3, 3, 3, ...

X

$$\begin{array}{r} 7 \overline{) 101} \\ \underline{7} \\ 3 = Z_1 \end{array}$$

(ii) $Z_0 = 0$

1, 4, 6, 5, 2, 0

not full period period

Probability Integral Transform (Inverse transform method):

... Cdf ...

$U \sim U(0,1)$; $X \sim F$, then

$U = F(X)$, i.e., $X = F^{-1}(U)$
Sel F_X $\sim U(0,1)$
 $0 \leq u \leq 1$

$$P(U \leq u) = P(F(X) \leq u)$$

$$= P(X \leq F^{-1}(u))$$

$$= F(F^{-1}(u))$$

$$= u$$

$$F_U(u) = \begin{cases} 1, & 0 \leq u \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$U \sim U(0,1)$$

Example: $F(x) = 1 - e^{-x}$, $x > 0$ $\exp(1)$

$$u = F(x) \Leftrightarrow 1 - e^{-x} = u \Leftrightarrow x = -\log(1-u)$$

$$X = F^{-1}(U) = -\log(1-U) \sim \exp. \text{ with mean } 1.$$

$$(1-U \stackrel{d}{=} U) \underline{E_X}$$

$$\underline{X = -\log U} \sim \exp \text{ with mean } 1$$