Calculus of Variations

Introduction:

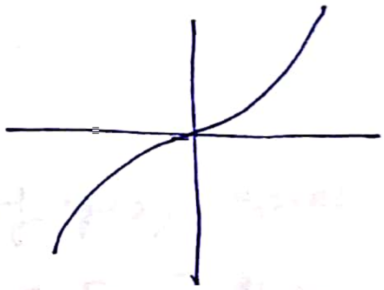
Let $y = f(x)$ be a cont. function defined on $[a, b]$. Then for max/min., we have.

$$f'(x_0) = 0.$$

Now if $f''(x_0) > 0$ $f(x)$ is min. at $x = x_0$

$f''(x_0) < 0$ $f(x)$ is max at $x = x_0$.

Without ~~one~~, either of the above two conditions it is not possible to say whether the funct. is max/min at $x = x_0$. or $x = x_0$ is a pt. of inflexion.



[Def.: Pt. of inflexion is a pt. at which the curvature of a curve changes sign. At that pt. $f'(x_0) = 0$.

$f''(x_0) = 0 \rightarrow$ nec. condit.

$f''(x_0 - \epsilon)$ & $f''(x_0 + \epsilon)$ have opposite signs in the nbd. of x_0.

1. Weinstock, Robert. Calculus of variations : with applications to physics and engineering.
2. Gel'fand, I. M., and S. V. Fomin. Calculus of Variations. Englewood Cliffs, NJ: Prentice Hall, 1963.
3. M.L. Krasnov et.al., Problems and exercises in the Calculus of Variations, (Mir publishers, 1975).
4. Elsgolc, Calculus of variations.

Definition of a functional:

Suppose we have a certain class M of functions $y(x)$. If to each function $y(x) \in M$, there is associated, by some law, a definite number I, then we say that a functional I is defined in the class M and we write $I = I[y(x)]$.

The class M of functions $y(x)$ on which the functional $I[y(x)]$ is defined is called the domain of definition of the functional.

Example 1. Let $M = C[0,1]$ be the collection of all continuous functions $y(x)$ specified on the interval $[0,1]$ and let

$$I[y(x)] = \int_0^1 y(x)\,dx.$$

Then $I[y(x)]$ is a functional of $y(x)$ such that to every function $y(x) \in C[0,1]$ there is associated a definite value of $I[y]$.

Let $y(x) = 1$ $\quad I[1] = \int_0^1 1\,dx = 1$

Let $y(x) = e^x$ $\quad I[e^x] = e - 1$

$y(x) = \sin \pi x$ $\quad I[\sin \pi x] = \left. \frac{\cos \pi x}{\pi} \right|_1^0 = \frac{2}{\pi}.$

Ex-2 Let $M = C^1[1,3]$ be the class of functions $y(x)$ that have continuous derivatives on the interval $[1,3]$ and let

$$I[y(x)] = y'(x_0); \quad x_0 \in [1,3].$$

Choose $x_0 = 2$. & $y(x) = x^2$.

$$I[y(x)] = 2x\big|_{x=2} = 4.$$

Choose $x_0 = 2$, $y(x) = \ln(1+x)$.

$$I[y(x)] = \frac{1}{1+x}\Big|_{x=2} = \frac{1}{3}.$$

Ex-3. $M = C[-1,1]$ be the class of all continuous functions defind in $[-1,1]$. Let $\phi(x,y)$ be a given function defined and continuous for all $-1 \le x \le 1$ and for all real y. Then

$$I[y(x)] = \int_{-1}^{1} \phi[x, y(x)]\,dx$$

is a functional defined on the class M.

Let $\phi(x,y) = \dfrac{x}{1+y^2}$

Then for $y(x) = x$, $I[y(x)] = \displaystyle\int_{-1}^{1} \frac{x}{1+x^2} = 0.$

" $y(x) = 1+x$, $I[y(x)] = \displaystyle\int_{-1}^{1} \frac{x}{1+(1+x)^2} = \ln\sqrt{5} - \tan^{-1} 2.$

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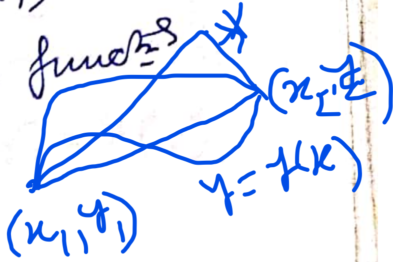
Aim of Calculus of Variation:

Let P and Q have coordinates (x_1, y_1) and (x_2, y_2), and consider the family of func^ns



$$y = y(x) \longrightarrow (1)$$

that satisfy the boundary condi^ns $y(x_1) = y_1$, $y(x_2) = y_2$ i.e the graph of (1) must join P & Q. Then we wish to find the func^n in this family that ~~minimizes~~ extremizes the integral of the form

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')\,dx$$

The func^ns $y = y(x)$ which satisfy the end pt. condi^ns are called admissible func^n or curve.

Assumptions:

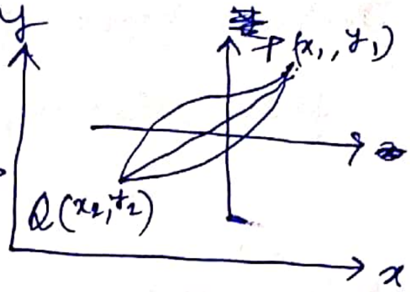
1. The func^nl $f(x, y, y')$ has continuous partial derivatives of the 2nd order w.r.to x, y and y'.

2. The func^ns $y(x)$ have continuous second derivatives and satisfy the given boundary conditions $y(x_1) = y_1$ & $y(x_2) = y_2$.

[Func^ns of this kind will be called admissible. We can imagine a competition in which only admissible func^ns are allowed to enter, and the problem is to select from this family the func^n or func-tions that yield the ~~smallest~~ extreme value for I.]

§1. Euler - Lagrange Equation

Introduction



Consider a family of curves $y = y(x)$ that passes through $P(x_1, y_1)$ and $Q(x_2, y_2)$. i.e $y_1 = y(x_1)$, $y_2 = y(x_2)$.

Then,

1) the length of the curve between P and Q is given by

$$I(y) = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

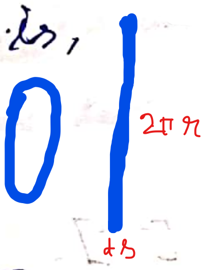
$ds^2 = dx^2 + dy^2$

$ds = \sqrt{dx^2+dy^2}$

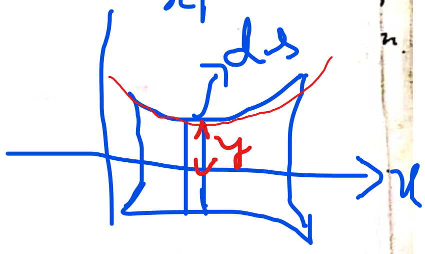
$= dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$s = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$

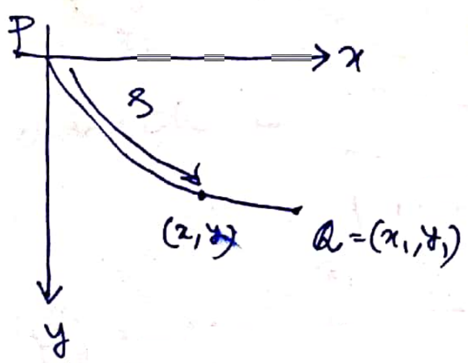
2) the area of the surface of revolution obtained by revolving it about the x-axis is,



$$I(y) = \int_{x_1}^{x_2} 2\pi y \sqrt{1+(y')^2}\, dx.$$



3) In case of the curve of quickest descent, it is convenient to invert the coordinate system and take the pt. P at the origin.



Speed $v = \frac{ds}{dt}$ is given by

$v = \sqrt{2gy}$. $\frac{ds}{dt} = \sqrt{2gy} \Rightarrow dt = \frac{ds}{\sqrt{2gy}}$

$\therefore$ t = total time of descent

$$= \int_{0}^{x_1} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx$$

$t = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx$