Wednesday

~~Euler~~ Euler Lagrange eqn. (E-L-E).

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

or, $f_y - \frac{d}{dx} f_{y'} = 0.$

<u>Case I.</u> f does not contain y explicitly.

$$f \equiv f(x, y').$$

$f_y = 0$, ELE: $\frac{d}{dx} f_{y'} = 0 \Rightarrow f_{y'} = \text{const}.$

<u>Ex.</u> $I[y(x)] = \int_{0.1}^{1} y'(1 + x^2 y')\, dx$; $y(0.1) = 19$, $y(1) = 1$.

$$f = y' + x^2 y'^2 \equiv f(x, y').$$

$f_{y'} = \text{const} \Rightarrow 1 + 2x^2 y' = A.$

$\therefore y' = \frac{A-1}{2x^2} = -\frac{c}{x^2}$; $c = -\frac{A-1}{2}.$

$dy = -\frac{c}{x^2} dx$ | $y(0.1) = 19.$

$y = \frac{c}{x} + d.$ | $y(1) = 1.$

$$\left.\begin{aligned} 19 &= 10c + d. \\ 1 &= c + d. \end{aligned}\right\} \quad c = 2,\ d = -1. \quad y = \frac{2}{x} - 1.$$

Case 2. f does not contain x explicitly.

$f = f(y, y')$.

Note, $\frac{d}{dx}(f - y' f_{y'})$

$$= \left(\underbrace{\frac{\partial f}{\partial x}}_{=0} + \frac{\partial f}{\partial y} \cdot y' + \frac{\partial f}{\partial y'} \cdot y''\right) - \frac{d}{dx}(y' f_{y'}).$$

$$= \frac{\partial f}{\partial y} y' + \cancel{\frac{\partial f}{\partial y'} y''} - \cancel{y'' f_{y'}} - y' \frac{d}{dx} f_{y'}.$$

$$= y' \underbrace{\left(\frac{\partial f}{\partial y} - \frac{d}{dx} f_{y'}\right)}_{\text{L.H.S. of E-L-E}} = y' \times 0 = 0.$$

$\therefore \frac{d}{dx}(f - y' f_{y'}) = 0 \Rightarrow f - y' f_{y'} = \text{Const.}$

Ex. $I[y(x)] = \int_0^{\alpha} y(1-y'^2)^{1/2} dx$; $y(0) = 0$, $y(\alpha) = 0$.

$f = y(1-y'^2)^{1/2}$ which does not contain x explicitly.

$\therefore f - y' f_{y'} = A$.

or, $y(1-y'^2)^{1/2} - y' \cdot y \cdot \frac{1}{\cancel{2}} \frac{-\cancel{2}y'}{(1-y'^2)^{1/2}} = A$.

or, $y(1-y'^2)^{1/2} + y \cdot \frac{y'^2}{(1-y'^2)^{1/2}} = A$.

or, $y\left[\frac{1 - \cancel{y'^2} + \cancel{y'^2}}{(1-y'^2)^{1/2}}\right] = A$.

or, $y^2 = A^2(1-y'^2)$.

$$1-y'^2 = \frac{y^2}{A^2} \Rightarrow y'^2 = 1-\frac{y^2}{A^2}.$$

$$y' = \pm\sqrt{1-\frac{y^2}{A^2}}$$

or, $\frac{dy}{dx} = \pm\frac{1}{A}\sqrt{A^2-y^2}.$

or, $\pm\frac{A}{\sqrt{A^2-y^2}}\,dy = dx.$

Either.

$$\frac{A}{\sqrt{A^2-y^2}}\,dy = dx$$

Integrating.

$$A\int\frac{dy}{\sqrt{A^2-y^2}} = \int dx + B.$$

or, $A\sin^{-1}\frac{y}{A} = x + B.$

or, $y = A\sin\left(\frac{x+B}{A}\right).$

$y(0)=0,\ y(\alpha)=0.$

$A\sin\frac{B}{A} = 0. \quad \because A\neq 0,$

$\sin\frac{B}{A} = 0 = \sin 0.$

$\frac{B}{A} = 0 \quad y = A\sin\frac{x}{A}.$

$y(\alpha)=0 \Rightarrow 0 = A\sin\frac{\alpha}{A}.$

$\because A\neq 0,\ \sin\frac{\alpha}{A} = 0 = \sin n\pi.$

$\alpha = An\pi;\quad A = \frac{\alpha}{n\pi}.$

$$\boxed{y = \frac{\alpha}{n\pi}\sin\frac{n\pi x}{\alpha}.}$$ $n = 1, 2, 3,\ldots$

$$-\frac{A}{\sqrt{A^2-y^2}}\,dy = dx$$

Integrating,

$$y = A\cos\left(\frac{x+B}{A}\right)$$

$y(0)=0,\ y(\alpha)=0.$

Case 3. f does not contain y'.

$f \equiv f(x, y)$. ELE: $\frac{\partial f}{\partial y} - \frac{d}{dx} f_{y'} = 0$.

$f_{y'} = 0$. $\therefore$ ELE becomes $\frac{\partial f}{\partial y} = 0$.

$$I[y(x)] = \int_0^{\alpha} y^2\, dx. \quad (\alpha > 0).$$

E-L-E becomes $\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{\partial (y^2)}{\partial y} = 0$.

or, $y = 0$.

If the boundary conditions are of the form $y(0) = 0$ and $y(\alpha) = 0$, then only $y = 0$ extremizes $I[y(x)]$ (minimizes). because, $\int_0^{\alpha} y^2 dx \geq 0$.

& $I[y(x)] = 0$ when $y = 0$.

Case 4. f is a function of y' only i.e $f = f(y')$.

E-L-E: $\frac{\partial f}{\partial y} - \frac{d}{dx} f_{y'} = 0$.

or, $\frac{d}{dx} f_{y'} = 0$ or, $\underbrace{\frac{\partial f_{y'}}{\partial x}}_{=0} + \underbrace{\frac{\partial f_{y'}}{\partial y}}_{=0} . y' + \frac{\partial f_{y'}}{\partial y'} . y'' = 0$.

$\therefore$ $y'' \frac{\partial f_{y'}}{\partial y'} = 0 \Rightarrow$ either $y'' = 0$. or, $\frac{\partial f_{y'}}{\partial y'} = 0$.

$y'' = 0$.
$y' = c$.
$y = cx + d$.

$\frac{\partial f_{y'}}{\partial y'} = 0$.
This gives an algebraic equation in y'. Suppose this eqn. has a root $y' = \alpha$.

$y'^2 - y' = 0$
$y' = 0, y' = 1$.

$\frac{\partial f_{y'}}{\partial y'} = 0$.

$\boxed{f = y'^4 - 2y'^3}$
$f_{y'} = 4y'^3 - 6y'^2$
$\frac{\partial f_{y'}}{\partial y'} = 12y'^2 - 12y'$

Integrating $y' = \alpha$ we get

$$y = \alpha x + \beta.$$

$\therefore$ extremals in this case are always straight lines.

Case 5. f is linear in y'.

Suppose $f \equiv M(x,y) + N(x,y)\, y'$.

E-L-E: $f_y - \frac{d}{dx} f_{y'} = 0$.

$$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial y} \cdot y' - \frac{d}{dx} N(x,y) = 0.$$

or, $$\frac{\partial M}{\partial y} + \cancel{\frac{\partial N}{\partial y} \cdot y'} - \frac{\partial N}{\partial x} - \cancel{\frac{\partial N}{\partial y} \cdot y'} = 0.$$

or, $$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0.$$

→ This will give some identity

→ This will not give any sol. to the variational pro

Ex. $$I[y(x)] = \int_{-1}^{1} \left[(2xy^2 + x^2) + (4y^3 + 2x^2 y) y' \right] dx.$$

$$y(-1) = 2, \quad y(1) = 4.$$

$$f = (2xy^2 + x^2) + (4y^3 + 2x^2 y) y'.$$

$$f_y - \frac{d}{dx} f_{y'} = 0 \Rightarrow 4xy + (12y^2 + 2x^2) y' - \frac{d}{dx}(4y^3 + 2x^2 y) = 0.$$

or, $$\cancel{4xy} + \underbrace{(12y^2 + 2x^2) y'}_{\times} - \left[\underbrace{12y^2 \cdot y'}_{\times} + \cancel{4xy} + \underbrace{2x^2 y'}_{\times} \right] = 0.$$

or, $0 = 0$.

This happens because,

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\, dx.$$

$$= \int_{x_1}^{x_2} \left[M(x,y) + N(x,y)\, y' \right] dx.$$

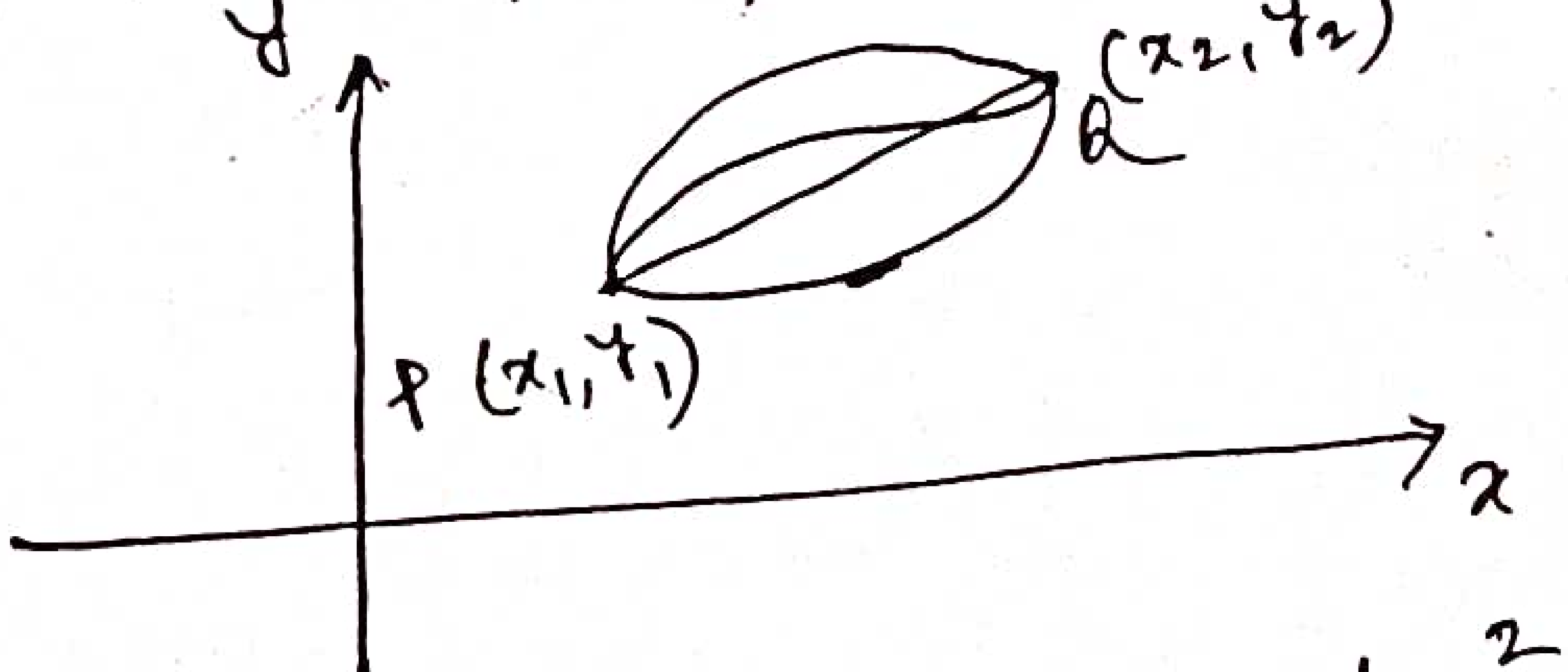
$$= \int_{x_1}^{x_2} M\,dx + N\,dy.$$

$$= \int_{x_1}^{x_2} d f(x,y), \quad \text{since } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$= f(x_2, y_2) - f(x_1, y_1).$$

Applications.

1. Shortest distance between two points lying on a plane (say xy-plane).



Let ds be a small element of the curve $y = y(x)$ joining P & Q.

$$ds^2 = dx^2 + dy^2.$$

or, $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$

$$\therefore\ S = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$