Introduction to Probability
Chapter 8 Joint Distributions

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Outline

1. Joint PMF
2. Joint PDF
3. Independence
4. Covariance
5. Correlation

References

1. Probability and statistics in engineering by Hines et al (2003) Wiley.
2. Mathematical Statistics by Richard J. Rossi (2018) Wiley.
3. Probability and Statistics with reliability, queuing and computer science applications by K. S. Trivedi (1982) Prentice Hall of India Pvt. Ltd.

Joint PMF

- Consider X and Y as discrete random variable. Then (X, Y) is two-dimensional discrete random variable with Joint probability mass function $p(x, y)$ if
  1. $p(x, y) \geq 0, \ \forall x, \ \forall y$, and
  2. $\sum_x \sum_y p(x, y) = 1$
- Marginal density of X is given by $p_X(x) = \sum_y p(x, y), \ \forall x$. Marginal density of Y is given by $p_Y(y) = \sum_x p(x, y), \ \forall x$.
- Conditional density of X given $Y = y$ is given by $p_{X|Y=y}(x) = \frac{p(x,y)}{p_Y(y)}, \ \forall x$. Conditional density of Y given $X = x$ is given by $p_{Y|X=x}(y) = \frac{p(x,y)}{p_X(x)}, \ \forall y$.

Example

A rover (a small vehicle that can move over rough ground, often used on the surface of other planets) is checked for tire wear, and headlight is checked for proper adjustment. Let X be the number of defective tires and Y be the number of headlights that needs adjustment. Consider (X, Y) having joint pmf $p_{XY}(x, y)$ as

$Y = y \downarrow X = x \rightarrow$	0	1	2	
0	$2/15$	$1/15$	$1/15$	4/15
1	$3/15$	$2/15$	$1/15$	6/15
2	$2/15$	$1/15$	$2/15$	5/15
	7/15	4/15	5/15	

Here marginal density of X is

x	0	1	2
$p_X(x)$	$7/15$	$4/15$	$4/15$

$E(X) = 1 \times \frac{4}{15} + 2 \times \frac{4}{15}$

Example

marginal density of Y is

y	0	1	2
$p_Y(y)$	$4/15$	$6/15$	$5/15$

The conditional density of X given $Y = 2$ is

x	0	1	2
$p_{X\|Y=2}(x)$	$2/5$	$1/5$	$2/5$

$$p_{X|Y=2}(x) = P(X=x|Y=2) = \frac{P(X=x, Y=2)}{P(Y=2)} = \frac{p(x,2)}{p_Y(2)}$$

$5/15$

$$E(X|Y=2) = 1 \times \frac{1}{5} + 2 \times \frac{2}{5}.$$

Joint PDF

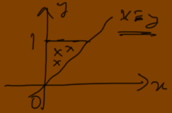
- Consider X and Y as continuous random variables. Then (X, Y) is two-dimensional continuous random variable with Joint probability density function $f_{XY}(x, y)$ if
  1. $f_{XY}(x, y) \geq 0, \ -\infty < x < \infty, \ -\infty < y < \infty$, and
  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dxdy = 1$
- Marginal density of X is given by $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$. Marginal density of Y is given by $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$
- Conditional density of X given $Y = y$ is given by $f_{X|Y=y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)}$. Conditional density of Y given $X = x$ is given by $f_{Y|X=x}(y) = \frac{f_{XY}(x,y)}{f_X(x)}$.

Example

The front tire and back tires of the rover (a small vehicle that can move over rough ground, often used on the surface of other planets) is supposed to be filled to a pressure of 0.4 psi and 0.6 psi, respectively. Suppose the actual air pressure in each tire is a rv, X for the front tire and Y for the back tire, with joint pdf as $f(x, y) = ky$, $0 < x < y < 1$. Then $\int \int f(x, y)\, dx\, dy = 1 \Rightarrow \int_0^1 \int_0^y ky\, dx\, dy = 1 \Rightarrow k = 3$. The marginal density of X is

$k\int_0^1 y^2 dy = 1 \Rightarrow \frac{k}{3} = 1$

$$f_X(x) = \int f(x, y)\, dy = \int_x^1 3y\, dy = \frac{3}{2}(1 - x^2),\ 0 < x < 1.$$



The marginal density of Y is

$$f_Y(y) = \int f(x, y)\, dx = \int_0^y 3y\, dx = 3y^2,\ 0 < y < 1.$$

Example...Contd..

Example

The conditional distribution of Y given $X = x$ is

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{1-x^2}, \; x < y < 1.$$

Independence

- (X, Y) is two-dimensional discrete random variable. Then X and Y are said to be independent if

$$p(x, y) = p_X(x)p_Y(y), \ \forall x, \ \forall y.$$

- (X, Y) is two-dimensional continuous random variable. Then X and Y are said to be independent if

$$f_{XY}(x, y) = f_X(x)f_Y(y).$$

- That is independence implies that the joint density is product of their marginal densities.

Example

A binary message is transmitted, which is either 0 or 1. Assume that the channel has an additive noise and it corrupts the transmission. Let X denote the transmitted message and Y denote the received message by the receiver. Consider the (X, Y) having density $p_{XY}(x, y)$ as

$X = x \downarrow Y = y \rightarrow$	0	1	
0	1/4	1/4	= 2/4
1	1/4	1/4	= 2/4
	2/4	2/4	

Here marginal density of X is

x	0	1
$p_X(x)$	1/2	1/2

$p(0,0) = \frac{1}{4} = p_X(0)p_Y(0)$

$p(0,1) = \frac{1}{4} = p_X(0)p_Y(1)$

⋮

marginal density of Y is

y	0	1
$p_Y(y)$	1/2	1/2

example contd

Example

Here $p_{X,Y}(x,y) = p_X(x)p_Y(y)$, $\forall x$, $\forall y$. Hence X and Y are independent.

Covariance

- Let X and Y are random variable with means μ_X and μ_Y, respectively. Then covariance between X and Y is

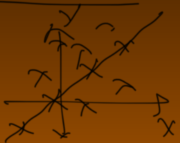
$$Cov(X, Y) = E(X - \mu_X)(Y - \mu_Y) = E(XY) - \mu_X \mu_Y$$

- If X and Y are independent, then $Cov(X, Y) = 0$, i.e., $E(XY) = E(X)E(Y)$. But the converse may not be true.

Correlation

- Let X and Y are random variable with means μ_X and μ_Y and variances σ_X^2 and σ_Y^2, respectively. Then the correlation coefficient between X and Y is

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$



- $-1 \leq \rho_{XY} \leq 1$
- $|\rho_{XY}| = 1$ if and only if $Y = \alpha + \beta X$ for some real numbers α and $\beta \neq 0$.
- X and Y are uncorrelated if $Cov(X,Y) = 0 \Leftrightarrow \rho_{XY} = 0$
- If X and Y are independent, then $\rho_{XY} = 0$. But the converse may not be true.

Example

Consider the (X, Y) having density $p_{XY}(x, y)$ as

$X = x \downarrow Y = y \rightarrow$	-1	0	1
-1	0	1/4	0
0	1/4	0	1/4
1	0	1/4	0

Here marginal density of X is

x	-1	0	1
$p_X(x)$	1/4	1/2	1/4

marginal density of Y is

y	-1	0	1
$p_Y(y)$	1/4	1/2	1/4

Here $p_{X,Y}(-1, -1) \neq p_X(-1)p_Y(-1)$. Hence X and Y are not independent. But $Cov(X, Y) = 0$. Hence X and Y are uncorrelated.

Let X and Y are random variables and $Var(X)$ is finite , then

1. $E\left(E(X|Y)\right) = E(X)$.
2. $E\left(E(g(X)|Y)\right) = E(g(X))$.
3. $Var(X) = E\left(Var(X|Y)\right) + Var\left(E(X|Y)\right)$.

Sol

$$E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y=y}(x)\,dx = \phi(y)$$

$$E(\phi(Y)) = E(E(X|Y)) = \int \phi(y) f_Y(y)\,dy$$

$$= \int\left(\int x f_{X|Y=y}(x)\,dx\right) f_Y(y)\,dy$$

$$= \int x \int f_{X|Y=y}(x) f_Y(y)\,dy\,dx$$

$$= \int x \left(\int f(x,y)\,dy\right) dx$$

$$= \int x f_X(x)\,dx$$

$$= E(X)$$

Summary

In this chapter we presented topics related to jointly distributed random variables. The concept of independence, covariance and correlation were introduced.