MA41031 Stochastic Processes in Finance

Wed 10–11, Thus 9–10, Fri 11–12, 12–1

CT-1 10/9 Fri ; CT-2 8/10 Fri ; CT-3 18/11 Thurs

Books
1) Probability Models SM Ross
2) Sto. Process SM Ross
3) Prob., Stat. and Queueing Theory by Arnold O. Allen
4) Intro. to Stochastic Modelling by Pinsky, Karlin
5) Stochastic Calculus in Finance I, II by Shreve

Review: Introduction to probability

Stochastic Process: (S.P.)

Defn A S.P. is a family of r.v's $\{X(t), t \in T\}$, defined on a given probability space, indexed by the parameter t, where $t \in T$ → parameter/time space

→ The values assumed by r.v $X(t)$ are called states and set of all possible values form the state space (S) of the process.

—x—

1) discrete state, discrete parameter S.P.
2) " ", continuous " S.P.
3) C " " S.P.

3) Cont. " , " " SP

4) " " , discrete " S.P.

— × —

Example: Consider a queueing system with jobs arriving at random point in time, queueing for service and departing from the system after service completion.

a) $X(t)$ # of jobs in the system at time t

$$S = \{0, 1, 2, \ldots\} \; ; \; T = \{t \mid 0 \le t < \infty\}$$

$\{X(t)\}$ discrete state, continuous parameter S.P.

b) W_k time that the k^{th} customer has to wait in the system before receiving service.

$$S = \{x \mid 0 \le x < \infty\} , \; T = \{1, 2, 3, \ldots\}$$

$\{W_k\}$ continuous state , discrete parameter S.P.

c) $Y(t)$ cummulative service requirement (experience) of all jobs in the system at time t.

$$S = [0, \infty) \; , \; T = [0, \infty)$$

$\{Y(t)\}$ cont state, cont. parameter S.P.

d) N_k # of jobs in the system at the time of the departure of the k^{th} customer (after service complition)

$S = \{0, 1, 2, \ldots\}$, $T = \{1, 2, 3, \ldots\}$

$\{N_k\}$ discrete state, discrete parameter/time S.P.

— x —

Discrete Time Markov Chain (DTMC):

discrete state, discrete time/parameter S.P.

$\{X_n, n = 0, 1, 2, \ldots\}$

State space $\rightarrow S = \{x_0, x_1, x_2, \ldots\}$
$= \{0, 1, 2, \ldots\}$

$X_n = i \iff$ process is in state i at time/step n.

$\{i, j, i_{n-1}, \ldots, i_0\} \in S$

$\{X_n\}$ is DTMC iff

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$$

$$= P(X_{n+1} = j \mid X_n = i)$$

$$= p_{ij}^{(1)}(n) \rightarrow \text{transition probability}$$

$$= p_{ij}^{(1)} \rightarrow \text{stationary transition probability (Homogeneous M.C.)}$$

$$= p_{ij}$$

$= P(X_1 = j \mid X_0 = i)$
$= P(X_{m+1} = j \mid X_m = i)$

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

final state, initial state, $i, j \in S$

$$0 \le p_{ij} \le 1, \quad \forall i, \forall j$$

For fixed i; $\displaystyle\sum_{j=0}^{\infty} p_{ij} = 1$

j

0 1 2 ---

$$i\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array}\begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ & & & \end{pmatrix} = P = P^{(1)} \checkmark$$

Transition Probability Matrix (TPM or tpm)

<u>Example</u> Consider a game of ladder climbing. There are 5 levels in the game, level 1 is lowest (bottom) and level 5 is the highest (top). A player starts at the bottom. Each time, <u>a fair</u> coin is tossed. If it turns up heads, the player moves up one rung. If tails, the player moves down to the very bottom. Once at the top level, the player moves to the very bottom if tails turns up and stays at the top if head turns up.

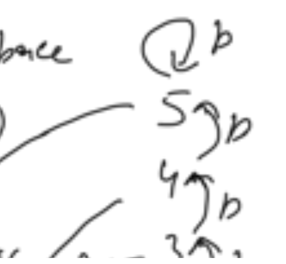
Let X_n be the level of the game in the n^{th} step/transition.

$$X_n \in \{1,2,3,4,5\} = S$$

State space

$$P(X_{n+1}=\underline{j} \mid X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0)$$

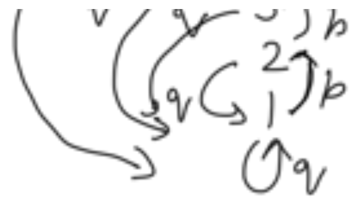
$$= P(X_{n+1}=j \mid X_n=i) = p_{ij}$$



$\ldots = p_{ij}$

$\{X_n\}$ DTMC $\quad i, j \in S$

$X_{n+1} = j$



$p + q = 1$

Tpm

$$((p_{ij})) = P = X_n = i \; \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & q & p & 0 & 0 & 0 \\ 2 & q & 0 & p & 0 & 0 \\ 3 & q & 0 & 0 & p & 0 \\ 4 & q & 0 & 0 & 0 & p \\ 5 & q & 0 & 0 & 0 & p \end{array}$$

$p = q = \frac{1}{2}$ fair coin

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Example(2) Let $\{X_n\}_{n=0,1,2,\ldots}$ be a sequence of i.i.d. discrete r.v. with $\underbrace{P(X_1 = j)}_{p_j} = \left(\frac{1}{2}\right)^{j+1} \; \forall j = 0, 1, 2, \ldots$

Determine whether each of the following chain is Markovian or not. If so find its corresponding state space (S) and tpm

(i) $\{S_n\}_{n=0,1,2,\ldots}$ where $S_n = \sum_{i=1}^{n} X_i$

(ii) $\{M_n\}\ldots$

$\dots, n = 0, 1, 2, \dots$ where $M_n = \max\{X_1, X_2, \dots, X_n\}$

Sol (i) $S_n = \sum_{i=1}^{n} X_i$

$\{S_n\}$ is MC with state space $S = \{0, 1, 2, \dots\}$

$$p_{ij} = P(S_{n+1} = j \mid S_n = i), \quad \forall i, j \in S$$

$$P = ((p_{ij})) = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} & \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & p_0 & p_1 & p_2 & \cdots \\ 0 & 0 & p_0 & p_1 & \cdots \\ 0 & 0 & 0 & p_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Example (3) (Transformation of a process into M.C.)

Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Suppose that if it has rained for the past two days, then it will rain tomorrow with prob (wp) 0.7; if it has rained today but not yesterday, then it will rain tomorrow w/p 0.5; if it has rained yesterday but not today, then it will rain tomorrow w/p 0.4; if it has not rained in the past two days, then it will rain tomorrow w/p 0.2.

Y_n weather condition of that day

(Y_n) not a M.C

X_n : state at any time is determined by the weather conditions during both that day and the previous day.

(X_n) M.C.

State X_n	Rained yesterday	Rained today
0	✓	✓
1	X	✓
2	✓	X
3	X	X

$$S = \{0,1,2,3\} \equiv \{✓✓, X✓, ✓X, XX\}$$

tpm

yesterday today ↓ ↓ ; today tomorrow ↓ ↓ ; $X_{n+1} = j$

$$P = \begin{array}{c|cccc} X_n = i & ✓✓ & X✓ & ✓X & XX \\ \hline ✓✓ & 0.7 & 0 & 0.3 & 0 \\ X✓ & 0.5 & 0 & 0.5 & 0 \\ ✓X & 0 & 0.4 & 0 & 0.6 \\ XX & 0 & 0.2 & 0 & 0.8 \end{array}$$

$$p_{ij} = P(X_{n+1} = j \mid X_n = i), \quad \forall i,j \in S$$

n-step transition probability: $\{X_n\}$ DTMC, $S = \{0, 1, 2, \ldots\}$ State space

$$p_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i)$$
$$= P(X_n = j \mid X_0 = i) \quad , i, j \in S$$

$0 \le p_{ij}^{(n)} \le 1$

For fixed i

$$\sum_j p_{ij}^{(n)} = 1$$

n-step tpm

$$P^{(n)} = \left(\left(p_{ij}^{(n)}\right)\right) = \begin{pmatrix} p_{00}^{(n)} & p_{01}^{(n)} & \cdots \\ p_{10}^{(n)} & p_{11}^{(n)} & \cdots \\ \cdots & \cdots & \\ \cdots & \cdots & \end{pmatrix}$$

Chapman kolomgorov equation: $\{X_n\}$ M.C

$$p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)} = \sum_k p_{ik}^{(n)} p_{kj}^{(m)}$$

Sol $(i,j)^{th}$ element of $P^{(m+n)}$ is

$$p_{ij}^{(m+n)} = P(X_{m+n} = j \mid X_0 = i)$$

$$= \sum_k P(X_{m+n} = j, X_n = k \mid X_0 = i)$$

[using thm of total prob

$$= \sum_k P(X_{m+n} = j \mid X_n = k, X_0 = i) \cdot P(X_n = k \mid X_0 = i)$$

$\because P(AB|C) = \dfrac{P(ABC)}{P(BC)} \times \dfrac{P(BC)}{P(C)}$

$= P(A|BC) \cdot P(B|C)$

$$= \sum_k P(X_{m+n}=j \mid X_n=k) \cdot P(X_n=k \mid X_0=i)$$

$\because (X_n)$ is MC.

$$= \sum_k p_{kj}^{(m)} \, p_{ik}^{(n)}$$

$$= \sum_k p_{ik}^{(n)} \, p_{kj}^{(m)}$$

— x —

$$p_{ij}^{(m+n)} = \sum_k p_{ik}^{(n)} \, p_{kj}^{(m)}$$

$$\begin{array}{cc} & P^{(n)} \\ \text{ith row } i & \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ p_{i0}^{(n)} & p_{ij}^{(n)} & - & - \\ - & - & - & \end{pmatrix} \end{array} \quad \begin{array}{c} P^{(m)} \\ \begin{array}{ccccc} & & j & & \end{array} \\ \begin{pmatrix} - & - & p_{0j}^{(m)} & - & - \\ - & - & p_{1j}^{(m)} & - & - \\ & & \vdots & & \\ - & - & \vdots & - & - \end{pmatrix} \end{array}$$

$$P^{(m+n)} = P^{(n)} P^{(m)}$$

We know $P^{(1)} = P$

$m=1, n=1 \quad P^{(2)} = P^{(1)} P^{(1)} = P.P = P^2$

$\vdots$

$$\underline{P^{(n)} = P^n} \qquad \underline{P^n = P^{(n)}} = ((p_{ij}^{(n)}))$$

— x —

(X_n) DTMC $\quad S = \{0, 1, 2, \ldots\} \quad P$ ↑ tpm

$X_0 \qquad P(X_0 = i) = p_i^{(0)} \qquad i \in S$

pmf $p^{(0)} = (p_0^{(0)}, p_1^{(0)}, p_2^{(0)} \quad)$ initial state

$\sim$ ... $, p_1, p_2, \ldots)$ ... some prob. distn

X_n $\qquad P(X_n = i) = p_i^{(n)} \qquad i \in S$

pmf $\quad (p_0^{(n)}, p_1^{(n)}, p_2^{(n)}, \ldots) = \underset{\sim}{p}^{(n)}$

✓ $\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(0)} P^n$

✓ $\boxed{\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(n-1)} P}$ ✓

$$\left(p_0^{(n)}, p_1^{(n)}, \ldots, p_i^{(n)}, \ldots\right) = \left(p_0^{(n-1)}, p_1^{(n-1)}, \ldots\right) \begin{array}{c} \quad\quad\quad\quad\ \downarrow i \\ \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0i} & \cdots \\ p_{10} & p_{11} & \cdots & p_{1i} & \cdots \\ \cdots & \cdots & & & \\ & \cdots & & \cdots & \end{pmatrix} \end{array}$$

$$\boxed{p_i^{(n)} = \sum_k p_k^{(n-1)} p_{ki}}$$

Sol $\quad p_i^{(n)} = P(X_n = i)$

$$= \sum_k P(X_n = i, X_{n-1} = k)$$

| using thm of total prob

$$= \sum_k P(X_n = i \mid X_{n-1} = k)\, P(X_{n-1} = k)$$

$$= \sum_k p_{ki}^{(1)}\, p_k^{(n-1)}$$

$$= \sum_k p_{ki}\, p_k^{(n-1)}$$

Example: Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with prob. α; and if it does not rain today, then it will rain tomorrow with prob. β.

Let X_n: weather condition on nth day

$\{X_n\}$ mc

$S = \{0, 1\}$ — 0: rain, 1: not raining

tpm

$$P = \begin{array}{cc} & X_{n+1} = j \\ & \begin{array}{cc} 0 & 1 \end{array} \\ X_n = i \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix} \end{array}$$

$\alpha = 0.7$, $\beta = 0.3$

$$P = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

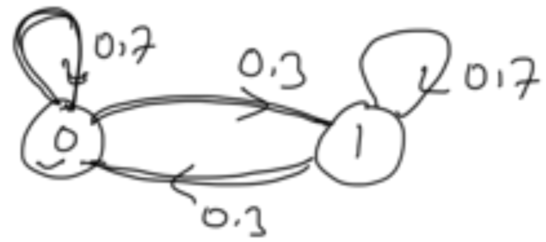
$$p_{00}^{(2)} = P(X_2 = 0 \mid X_0 = 0)$$

$$P^{(2)} = P^2 = P.P = \begin{pmatrix} 0.7 & 0.3 \\ 0.7 & 0.7 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \boxed{0.58} & 0.42 \\ 0.42 & 0.58 \end{pmatrix} \end{array} = \begin{pmatrix} p_{00}^{(2)} & p_{01}^{(2)} \\ p_{10}^{(2)} & p_{11}^{(2)} \end{pmatrix}$$

$$p_{00}^{(2)} = 0.58$$

$$p_{10}^{(2)} = \sum_{k=0}^{1} p_{1k}^{(1)} p_{k0}^{(1)} = p_{10} p_{00} + p_{11} p_{10} = 0.42$$



$$p_{00}^{(2)} = 0.7 \times 0.7 + 0.3 \times 0.3 = 0.58$$

<u>Example</u>: $\underline{\{X_n\}\ \text{DTMC}}$ $\quad S = \{1,2,3\}$ $\quad$ tpm

$$\text{tpm } P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.1 & 0.5 & 0.4 \\ 2 & 0.6 & 0.2 & 0.2 \\ 3 & 0.3 & 0.4 & 0.3 \end{array}$$

$\underline{X_0 \text{ pmf}}$ $\quad (0.7, 0.2, 0.1) = (p_1^{(0)}, p_2^{(0)}, p_3^{(0)}) = p^{(0)}$

$\qquad\quad 1 \quad \cancel{2} \quad 3$

$\qquad\quad P(X_0=2) \qquad\qquad P(X_0 = i) = p_i^{(0)}$

a) $P(X_1=3, X_2=3, X_0=2, X_3=2)$

$= P(X_3=2, X_2=3, X_1=3, X_0=2)$

$\qquad | \ P(ABCD) = P(A|BCD)\,P(B|CD)\,P(C|D)\,P(D)$

$= P(X_3=2 | X_2=3, X_1=3, X_0=2) \cdot P(X_2=3 | X_1=3, X_0=2)$

$\qquad \cdot P(X_1=3 | X_0=2) \cdot P(X_0=2)$

$= P(X_3=2 | X_2=3) \cdot P(X_2=3 | X_1=3) \cdot P(X_1=3 | X_0=2)$

$\qquad \cdot P(X_0=2)$

$\qquad | \because (X_n)$ MC

$= p_{32}\ p_{33}\ p_{23}\ P(X_0=2)$

$= 0.4 \times 0.3 \times 0.2 \times 0.2 = 0.0048$

(b) $P(X_2=3, X_1=3 \mid X_0=2)$

$= P(X_2=3 \mid X_1=3, \cancel{X_0=2}) \cdot P(X_1=3 \mid X_0=2)$

$$P(AB|C) = \frac{P(ABC)}{P(BC)} \times \frac{P(BC)}{P(C)}$$
$$= P(A|BC)\; P(B|C)$$

$= P(X_2=3 \mid X_1=3) \cdot P(X_1=3 \mid X_0=2)$

$= p_{33}^{(1)}\; p_{23}^{(1)} = 0.3 \times 0.2 = 0.06$

(c) $P(X_3=2, X_0=2, X_1=3)$

$= P(X_3=2, X_1=3, X_0=2)$

$= P(\underline{X_3=2} \mid \underline{X_1=3}, \cancel{X_0=2}) \cdot P(X_1=3 \mid X_0=2) \cdot P(X_0=2)$

$= P(X_3=2 \mid X_1=3) \cdot P(X_1=3 \mid X_0=2) \cdot P(X_0=2)$

$= \underbrace{p_{32}^{(2)}}\; \underbrace{p_{23}^{(1)}}\; P(X_0=2) = 0.35 \times 0.2 \times 0.2$
$= 0.0048,$

where

$$p_{32}^{(2)} = \sum_{k\in\{1,2,3\}} p_{3k}^{(1)} p_{k2}^{(1)} = p_{31}p_{12} + p_{32}p_{22} + p_{33}p_{32}$$
$$= 0.3\times0.5 + 0.4\times0.2 + 0.3\times0.4$$
$$= 0.35$$

(d) $p_3^{(2)} = P(X_2=3)$

X_2 pmf $\underset{\sim}{p}^{(2)} = (p_1^{(2)}, p_2^{(2)}, \underline{\underline{p_3^{(2)}}})$

X_0 pmf $\underset{\sim}{p}^{(0)} = (0.7, 0.2, 0.1)$

X_1 $\quad \underset{\sim}{p}^{(1)} = \underset{\sim}{p}^{(0)} P = (0.7, 0.2, 0.1) \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$

$$= (0.22, 0.43, 0.35)$$

X_2 $\quad \underset{\sim}{p}^{(2)} = \underset{\sim}{p}^{(1)} P = (0.22, 0.43, 0.35) \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$

$$= (0.385, 0.336, 0.279)$$

$$P(X_2 = 3) = 0.279 = p_3^{(2)}$$

—x—

Classification of States:

(X_n) MC $\quad S = \{0, 1, 2, \dots\}$, tpm $P = ((p_{ij}))$

state $i, j \in S$

Def $i \to j$, state j is accesible from state i if $p_{ij}^{(n)} > 0$ for some n.

Def $i \leftrightarrow j$, state i and j communicate with each other if $i \to j$ and $j \to i$

Remark $\quad i \leftrightarrow j, \; j \leftrightarrow k \Rightarrow i \leftrightarrow k$

Sol $\exists n, m$ $\quad p_{ij}^{(n)} > 0 \quad p_{jk}^{(m)} > 0 \qquad p_{ik}^{(m+n)} = \sum_{l} p_{il}^{(n)} p_{lk}^{(m)}$

$$\geq p_{ij}^{(n)} p_{jk}^{(m)}$$

>0

<u>Defⁿ</u> M.C is irreducible/conctable if every state communicate with every other state otherwise M.C. reducible.

<u>Defⁿ</u> period of state i : $d(i)$ }

$d(i) = \gcd \mathbb{I}^+ = \{1,2,\ldots\}$ n s.t. $p_{ii}^{(n)} > 0$

(If $p_{ii}^{(n)} = 0 \ \forall n \geq 1$, define $d(i) = 0$)

If $d(i) = 1$ $\quad i \to$ aperiodic

<u>Example</u>: ① $\{X_n\}$ DTMC $\quad S = \{0,1,2\}$ tpm

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 2 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}$$

Eg ①

$p_{11}^{(n)} = 0 \ \forall n \geq 1$

$d(1) = 0$

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}$$

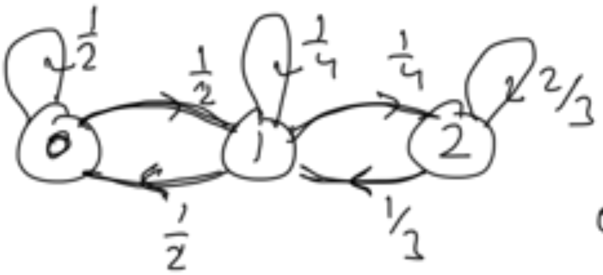
$f_0 = 1$

$f_0^{(1)} =$

$f_0^{(2)} =$

$f_0^{(3)} =$

$\vdots$



0, 1, 2 recurrent

$0 \leftrightarrow 1 \leftrightarrow 2$ $\quad$ M.C. irreducible/conctable

Class $\{0,1,2\}$

$d(0) = \gcd\{1,2,3,\ldots\} = 1 = d(1) = d(2)$

②
$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$
$S = \{0,1\}$

$p^{(1)}$ $\quad$ 1

$f_0 = 0$ — —

$f_0^{(2)} = 1$ $\quad 0 \leftrightarrow 1$

$f_0^{(3)} = 0$ Class $\{0,1\}$

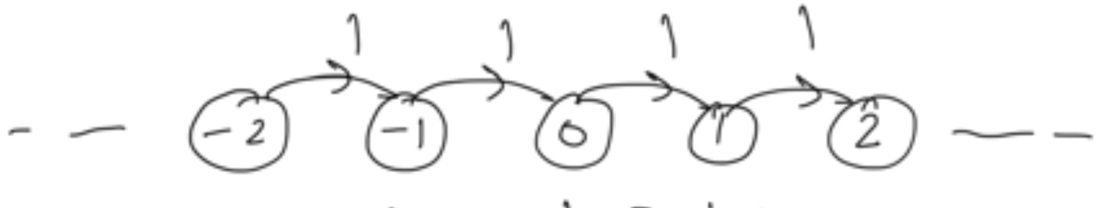
1 $\quad$ 0, 1 <u>recurrent</u>

$f_0^{(4)} = 0$ $\quad p_{00}^{(\infty)} > 0$ $\quad$ MC - irreducible

$\vdots$

$\overline{f_0 = 1}$ $\quad d(0) = \gcd\{2, 4, 6, \ldots\} = 2 = d(1)$

Eg (3)



$$P = \begin{array}{c|c} & \cdots \; -1 \; 0 \; 1 \; \cdots \\ \hline \vdots & \\ -1 & \cdots \; 0 \; 1 \; 0 \; \cdots \\ 0 & \cdots \; 0 \; 0 \; 1 \; \cdots \\ 1 & \cdots \\ \vdots & \end{array}$$

<u>Reducible M.C.</u>

$p_{00}^{(n)} = 0, \ \forall n \geq 1 \quad d(0) = 0$

$d(i) = 0$

X_n DTMC, $P = ((p_{ij}))$ tpm, S $\quad p_{ij} = P(X_{n+1} = j \mid X_n = i)$

<u>For state i</u>

$$f_{ii}^{(n)} \text{ or } f_i^{(n)} = P(X_n = i, X_k \neq i, k = 1, 2, \ldots, n-1 \mid X_0 = i)$$

prob of <u>first</u> visit to state i in n steps /transitions starting from state i

or recurrence time prob

$$f_{ii}^{(0)} \text{ or } f_i^{(0)} = 1$$

$$f_{ii} \text{ or } f_i = f_i^{(1)} + f_i^{(2)} + f_i^{(3)} + \cdots$$

prob of ever visiting state i

... of ... visiting state i

$f_i = 1 \Leftrightarrow i$ recurrent state $\Leftrightarrow$ return to state i is certain

$f_i < 1 \Leftrightarrow i$ transient state $\Leftrightarrow$ return to state i is uncertain

Let $$I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}$$

$\sum_{n=1}^{\infty} I_n \rightarrow$ # of time period, the process is in state i

$$E\left(\sum_{n=1}^{\infty} I_n \,\middle|\, X_0 = i \right) = \sum_{n=1}^{\infty} E(I_n | X_0 = i)$$

$$= \sum_{n=1}^{\infty} \left(1 \cdot P(X_n = i | X_0 = i) + 0 \cdot P(X_n \neq i | X_0 = i)\right)$$

$$= \sum_{n=1}^{\infty} p_{ii}^{(n)}$$

i recurrent $\Leftrightarrow f_i = 1 \Leftrightarrow \sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ ✓

---

i transient $\Leftrightarrow f_i < 1 \Leftrightarrow \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$

— x —

Def^n i recurrent

$m_i = \sum_{n=1}^{\infty} n f_i^{(n)}$ mean recurrance time / expected time the process return to state i

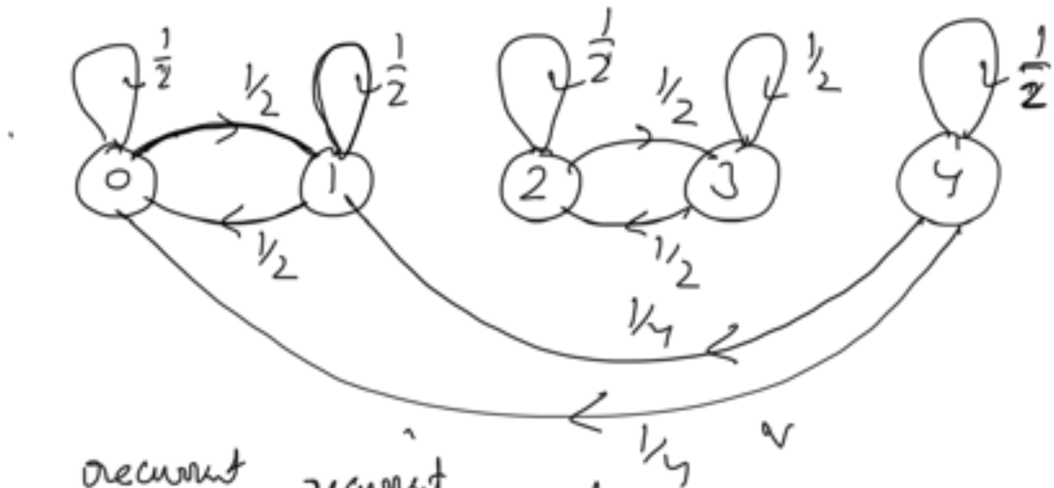
$m_i = \infty$, i null recurrent

$m_i < \infty$, i non-null recurrent / positive recurrent.

— x —

Example: M.C. having $S = \{0,1,2,3,4\}$ and tpm

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & & & & & \\ 1 & & & & & \\ 2 & & & & & \\ 3 & & & & & \\ 4 & & & & & \end{array} \quad \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \end{pmatrix}$$



Classes $\underbrace{C_1 = \{0,1\}}$ recurrent, $C_2 = \{2,3\}$ recurrent, $\{4\} = C_3$ → transient

$d(0) = 1 = d(1) = d(2) = d(3) = d(4)$ aperiodic

$$\underline{\underline{f_0}} = f_0^{(1)} + f_0^{(2)} + f_0^{(3)} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$= \frac{1}{2} \times \frac{1}{1 - \frac{1}{2}} = 1.$$

0 → recurrent state.

P1 $i \leftrightarrow j$, i recurrent $\Rightarrow j$ recurrent

Sol

Given $\begin{cases} \exists\ n, m \\ p_{ij}^{(n)} > 0 , \ p_{ji}^{(m)} > 0 , \ \sum_{\nu=1}^{\infty} p_{ii}^{(\nu)} = \infty \end{cases}$

$$p_{jj}^{(m+n+\nu)} \geq p_{ji}^{(m)}\, p_{ii}^{(\nu)}\, p_{ij}^{(n)} \quad \text{using CK eqn}$$

$$\sum_{\nu} p_{jj}^{(m+n+\nu)} \geq p_{ji}^{(m)}\, p_{ij}^{(n)} \underbrace{\left(\sum_{\nu} p_{ii}^{(\nu)}\right)}_{=\infty}$$

$$\Rightarrow \sum_{n} p_{jj}^{(n)} = \infty$$

j recurrent state.

P2 $i \leftrightarrow j$, i transient $\Rightarrow j$ transient.

Sol On contrary suppose j recurrent, since $i \leftrightarrow j$

$\Rightarrow i$ recurrent (using P1) # a contradiction

P3 In a finite state M.C. all states can not be transient.

P4 In a finite state, irreducible M.C. all states are recurrent.

P5 In a irreducible M.C., all states are recurrent or transient.

$\rightarrow$ $x \leftrightarrow y \Rightarrow d(x) = d(y)$

Sol

$d(x) = \gcd\{ n \geq 1 : p_{xx}^{(n)} > 0 \}$

$x \leftrightarrow y$

$\exists\, m, n \quad p_{x,y}^{(m)} > 0 \;,\; p_{y,x}^{(n)} > 0$

$p_{xx}^{(s)} > 0$

$p_{yy}^{(n+m)} \geq p_{yx}^{(n)}\, p_{x,y}^{(m)} > 0$

$p_{yy}^{(n+s+m)} \geq p_{yx}^{(n)}\, p_{xx}^{(s)}\, p_{x,y}^{(m)} > 0$

$d(y)$ divides both $n+m$ and $n+s+m$

$\Rightarrow d(y)$ divides every s with $p_{xx}^{(s)} > 0$

$\Rightarrow d(y)$ divides gcd of such s

$\Rightarrow d(y)$ divides $d(x)$

Repeat by changing the role of x and y

$d(x)$ divides $d(y)$

$d(x) = d(y)$

Example M.C. $S = \{1,2,3,4\}$ tpm

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \end{array}$$

$1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ Irreducible

Class $\{1,2,3,4\}$ $S = \{1,2,3,4\}$

All states are recurrent. (using P4)

<u>Example</u> (One-dimensional random walk) $\{X_n\}$

$$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

$p_{i,i+1} = p$; $p_{i,i-1} = q = 1-p$; $p_{i,j} = 0$, $j \neq i-1, i+1$

$0 < p < 1$

Irreducible M.C

$$\sum_n p_{ii}^{(n)} = \sum_m a_m$$

$$p_{ii}^{(n)} = P(X_n = i \mid X_0 = i)$$

$$= \begin{cases} \underbrace{\binom{2m}{m} p^m q^m}, & n = \underline{2m} \\ 0, & n = 2m+1 \end{cases}, \quad m = 1, 2, 3, \ldots$$

$$= \begin{cases} a_m, & n = 2m \\ 0, & n = 2m+1 \end{cases}$$

$$\frac{a_{m+1}}{a_m} = \frac{\binom{2m+2}{m+1} p^{m+1} q^{m+1}}{\binom{2m}{m} p^m q^m}$$

$$= \frac{(2m+2)(2m+1)}{(m+1)(m+1)} pq$$

$$\lim_{m \to \infty} \frac{a_{m+1}}{a_m} = 4pq$$

$$= \begin{cases} 1 & \underline{p = \frac{1}{2}} \\ < 1 & \underline{p \neq \frac{1}{2}} \end{cases}, p \in (0,1)$$

Ratio test

$$\lim_{m \to \infty} \frac{a_{m+1}}{a_m} \begin{cases} \underline{<1}, & \underline{\sum a_m \text{ converges}} \\ >1, & \sum a_m \text{ diverges} \end{cases}$$

If $p \neq \frac{1}{2}$, $\sum_m a_m = \sum_n p_{ii}^{(n)} < \infty \Rightarrow i$ transient state

Stirling approx $n! \sim n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}$

where $a_n \sim b_n$ when $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$

$$p_{ii}^{(2n)} \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}} = \frac{1}{\sqrt{\pi n}} \quad \text{if} \quad p = \frac{1}{2}$$

$$\underline{\sum p_{ii}^{(2n)}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi} n^{1/2}} \underline{= \infty} \quad i \text{ recurrent}$$

if $p = \frac{1}{2}$

$$\sum_{n=1} \frac{1}{n^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p \le 1 \end{cases}$$

All states are recurrent if $p=\frac{1}{2}$

" " " transient if $p \neq \frac{1}{2}$.

—x—

Gambler's Ruin Problem: $i = \underline{0}, 1, \dots, N$

initial capital Rs i aim Rs N

Z_i i^{th} bet / step / transition / time

$$P(Z_i = 1) = p, \quad P(Z_i = -1) = q = 1-p$$

$Z_1, Z_2, \dots$ are independent

X_n : Fortune of the gambler after n steps

$$X_n = i + Z_1 + Z_2 + \dots + Z_n \qquad X_{n+1}$$

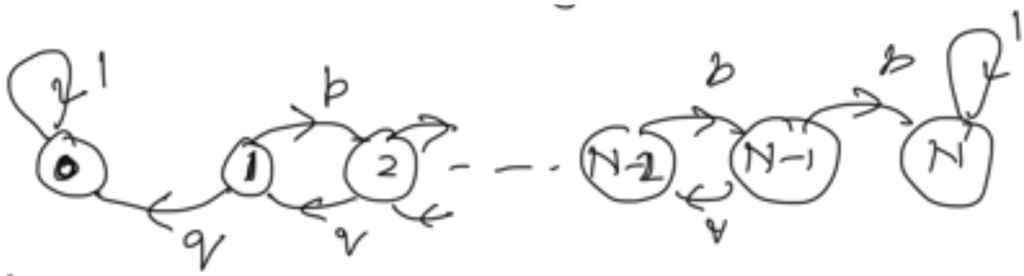
$$X_n \in \underline{\{0, 1, \dots, N\} = S} \qquad \{X_n\} \text{ is a M.C.}$$

$$p_{ij} = P(X_{n+1} = j \mid X_n = i) \qquad i, j \in S$$

$$\begin{cases} p_{00} = 1 = p_{NN} \\ p_{i,i+1} = p, \quad p_{i,i-1} = q, \quad i = 1, \dots, N-1 \end{cases}$$

tpm

$$P = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \\ N \end{array} & \begin{bmatrix} 1 & 0 & 0 & \cdots & & 0 \\ q & 0 & p & 0 & \cdots & 0 \\ 0 & q & 0 & p & \cdots & 0 \\ - & - & - & & & \\ - & - & - & & & \\ 0 & 0 & 0 & \cdots & & 1 \end{bmatrix} \end{array}$$



Class

$\{0\}$ $\{1, 2, \ldots, N-1\}$ $\{N\}$

recurrent or absorbing — transient — recurrent or absorbing

T_0 = time he broke = $\inf\{n : X_n = 0\}$

T_N = time he has Rs N = $\inf\{n : X_n = N\}$

$P_i = P(T_N < T_0)$ Prob. starting with i units, the gambler's fortune will reach N before reaching 0.

$$P_i = P(T_N < T_0 \mid Z_1 = 1) \cdot P(Z_1 = 1) + P(T_N < T_0 \mid Z_1 = -1)\, P(Z_1 = -1)$$

$$P_i = P_{i+1}\, p + P_{i-1}\, q$$

$$\Rightarrow \quad p P_i + q P_i = P_{i+1}\, p + P_{i-1}\, q$$

$$\Rightarrow \quad (P_{i+1} - P_i)\, p = q\, (P_i - P_{i-1})$$

$$\Rightarrow \quad P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1})$$

$P_0 = 0$

$P_N = 1$

$i=1 \quad P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p} P_1$

$i=2 \quad P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$

$$P_i - P_{i-1} = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$P_i - P_1 = \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right] P_1$$

$$P_i = \left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right] P_1$$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}} P_1 , & \frac{q}{p} \neq 1 \\ i P_1 , & \frac{q}{p} = 1 \end{cases} \quad \text{— (1)}$$

$P_N = 1$

$\underline{i = N}$ in (1)

$$P_1 = \begin{cases} \dfrac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} , & \underline{\frac{q}{p} \neq 1} \\ \dfrac{1}{N} , & \frac{q}{p} = 1 \end{cases}$$

$$\Rightarrow P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{if } \frac{q}{p} \neq 1 \Leftrightarrow p \neq \frac{1}{2} \\ \dfrac{i}{N} & \text{if } \frac{q}{p} = 1 \Leftrightarrow \underline{p = \frac{1}{2}} \end{cases}$$

— x —

$N \to \infty$

$$P_i = \begin{cases} 1-\left(\frac{q}{p}\right)^i & \text{if } \frac{q}{p} < 1 \Leftrightarrow p > \frac{1}{2} \\ 0 & \text{if } \frac{q}{p} \geq 1 \Leftrightarrow p \leq \frac{1}{2} \end{cases}$$

Example A rat is put into the linear maze as shown below

0	1	Rat 2	3	4	5
Shock					Food

At each step the rat moves to the right with prob $3/4$ and to the left with prob. $\frac{1}{4}$.

What is the prob. that the rat finds the food before getting shocked?

Sol. Gambler's ruin problem

$$i = 2, \quad N = 5, \quad p = \frac{3}{4}, \quad q = \frac{1}{4}$$

$$\frac{q}{p} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3} \neq 1$$

$$P_2 = \frac{1-\left(\frac{q}{p}\right)^2}{1-\left(\frac{q}{p}\right)^5} = \frac{1-\left(\frac{1}{3}\right)^2}{1-\left(\frac{1}{3}\right)^5} = 0.892$$

Prob. that rat will reach 0 before 5 $= 1 - P_2$

$= 1 - 0.892$

2. The prob. of the thrower winning in the dice game called "Craps" is $p = 0.49$. Suppose Player A is the thrower and begins the game with \$5, and Player B, his opponent, begins with \$10. What is the probability that player A goes bankrupt

before Player B? Assume that the bet is \$1 per round.

Sol $i=5,\ N=15,\ p=0.49,\ q=0.51,\ \frac{q}{p}\neq 1$

$$1-P_5 = 1-\frac{1-\left(\frac{q}{p}\right)^5}{1-\left(\frac{q}{p}\right)^{15}}$$

$\underline{i=5, N=15}$

$=$

A B

\$5 \$10

\$15 $p^* = q = \underline{\underline{0.51}}$

$p = q^* = 0.49$

$$P^*_{10} = \frac{1-\left(\frac{q^*}{p^*}\right)^{10}}{1-\left(\frac{q^*}{p^*}\right)^{15}}$$

$\frac{q^*}{p^*}\neq 1$

<u>Example</u> (1)

0, 3 → absorbing

1, 2 → transient

$f_0 = f_{0,0} = \underline{\underline{\sum_{n=1}^{\infty} f_0^{(n)}}}$

$f_{0,1} = 0 \leftarrow$

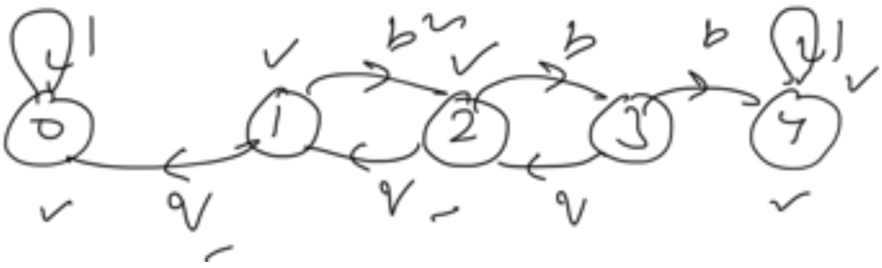
recurrent → transient

$f_{0,0} = 1$

$$f_{1,0} = f_{1,0}^{(1)} + f_{1,0}^{(2)} + f_{1,0}^{(3)} + \cdots$$

transient recurrent $= q + 0 + pq^2 + 0 + (pq)^2 q + \cdots$

(2)



$$f_{2,1} = q + pq^2 + p^2q^3 + p^3q^4 + \cdots$$

$$= q + pq\underline{(q + pq^2 + \sim)}$$

$$= q + pq\, f_{2,1}$$

$$\Rightarrow f_{2,1} = \frac{q}{1-pq}$$

If $p = 0.4$

$$\underline{f_{2,1}} = \frac{0.6}{1 - 0.6\times 0.4} = \frac{0.6}{0.76} = 0.78$$

using Gambler's ruin problem

$f_{2,1}^{*} \rightarrow$ start in 2, visit $\underline{1}$ before 4

$\rightarrow$ Start in 1 goes broke before reaching 3 *

$$= 1 - \frac{1 - \left(\frac{0.6}{0.4}\right)^1}{1 - \left(\frac{0.6}{0.4}\right)^3} = 0.78$$

— x —

Mean time spent in transient states:

Finite state (S) M.C. $\{X_n\}$

$T = \{1, 2, \ldots, t\}$ set of transient states

$$P_T = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1t} \\ \vdots & \vdots & \cdots & | \\ p_{t1} & p_{t2} & \cdots & p_{tt} \end{bmatrix}$$

sum of rows < 1

$i, j \in T$

s_{ij} : expected # of time periods that the M.C. is in state j, given that it starts in state i.

Let $\delta_{ij} = \begin{cases} 1 & \text{when } i=j \\ 0 & \text{o.w.} \end{cases}$; $I_{n,j} = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{o.w.} \end{cases}$

$$s_{ij} = \delta_{ij} + E\left(\sum_{n=1}^{\infty} I_{n,j} \mid X_0 = i \right)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} E(I_{n,j} \mid X_0 = i)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} P(X_n = j \mid X_0 = i)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} p_{ij}^{(n)} \longrightarrow (*)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} \sum_{k} p_{ik} \, p_{kj}^{(n-1)}$$

$$= \delta_{ij} + \sum_{k} p_{ik} \sum_{n=1}^{\infty} p_{kj}^{(n-1)}$$

$$= \delta_{ij} + \sum_{k} p_{ik} \left[\delta_{kj} + \sum_{n=2}^{\infty} p_{kj}^{(n-1)} \right]$$

$$= \delta_{ij} + \sum_{k} p_{ik} \left(\delta_{kj} + \sum_{n=1}^{\infty} p_{kj}^{(n)} \right)$$

$$= \delta_{ij} + \sum_{k} p_{ik} \, s_{kj} \qquad \text{/ using } (*)$$

$$= \delta_{ij} + \sum_{k=1}^{t} p_{ik} \, s_{kj} \, ,$$

since it is impossible to go from a recurrent to transient state $\Rightarrow$ $s_{kj} = 0$ when k is a recurrent state

$$\boxed{s_{ij} = \delta_{ij} + \sum_{k=1}^{t} p_{ik} s_{kj}} \quad , i,j \in T$$

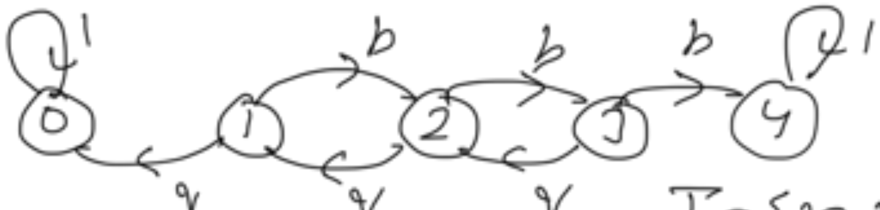
$$S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1t} \\ - & - & - & \\ s_{t1} & s_{t2} & \cdots & s_{tt} \end{bmatrix} \quad , P_T$$

$$S = I + P_T S$$

$$\Rightarrow (I - P_T) S = I$$

$$\Rightarrow \boxed{S = (I - P_T)^{-1}}$$

Example: (A) Gambler's ruin problem $p = 0.4$, $N = 4$



$T = \{1, 2, 3\}$

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0.6 & 0 & 0.4 & 0 & 0 \\ 2 & 0 & 0.6 & 0 & 0.4 & 0 \\ 3 & 0 & 0 & 0.6 & 0 & 0.4 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array} \rightarrow P_T$$

$$I - P_T = \begin{pmatrix} 1 & -0.4 & 0 \\ -0.6 & 1 & -0.4 \\ 0 & -0.6 & 1 \end{pmatrix}$$

$$S = (I - P_T)^{-1} \; s \; \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1.46 & 0.76 & 0.31 \\ 2 & \boxed{1.15} & 1.92 & [illegible] \end{array}$$

$$\begin{array}{c} \\ 3 \end{array}\begin{pmatrix} \boxed{\ } & & \boxed{0.76} \\ 0.69 & 1.15 & 1.46 \end{pmatrix}$$

→ Starting with 2 units, determine the expected amt of time the gambler has 3 units

$$s_{23} = 0.76$$

→ " " 2 " , " " " " " " " " " 1 unit

$$s_{21} = \underline{1.15}$$

$i, j \in T$

f_{ij} : prob. that the M.C ever makes a transition into state j given that it start in state i.

$$s_{ij} = E(\text{time in } j \mid \text{start in } i)$$

$$= \underbrace{E(\text{time in } j \mid \text{start in } i, \text{ ever transit to } j)} \cdot f_{ij}$$

$$+ E(\text{time in } j \mid \text{start in } i, \text{ never transit to } j) \cdot (1 - f_{ij})$$

$$= (\underbrace{\delta_{ij}} + s_{jj}) \cdot f_{ij} + \delta_{ij} \cdot (1 - f_{ij})$$

$$= \delta_{ij} + f_{ij}\, s_{jj}$$

$$\Rightarrow \quad f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}} \quad ✓$$

Example (A) (contd) Starting with 2 units, what is the prob. that the gambler ever has a fortune of 1?

$$f_{21} = \frac{s_{21} - \delta_{21}}{\ldots} = \frac{1.15}{\ldots} \ldots$$

$$r_{2,1} = \frac{\quad\quad}{s_{11}} = \frac{1.15 - 0}{1.46} = 0.78$$

—x—

$\{X_n\}$ M.C $S = \{0,1,2,\dots\}$

<u>Def^n</u> M.C is said to have stationary prob. dist^n $\underset{\sim}{\pi} = (\pi_0, \pi_1, \dots)$ if

$$\underset{\sim}{\pi} = \underset{\sim}{\pi} P \qquad P \to TPM$$

$$\sum_{i=0}^{\infty} \pi_i = 1$$

<u>Def^n</u> M.C have limiting probability

$$\underline{\lim_{n\to\infty} p_j^{(n)} = \lim_{n\to\infty} P(X_n = j) = \pi_j} \quad \forall j$$

<u>Example</u> $S = \{0,1\}$

$$P = \underbrace{\begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}}$$

Stationary prob. dist^n?

$\underset{\sim}{\pi} = (\pi_0, \pi_1)$

$$\begin{array}{l} \underset{\sim}{\pi} = \underset{\sim}{\pi} P \\ \pi_0 + \pi_1 = 1 \end{array} \Rightarrow \begin{array}{l} \pi_0 = \alpha\pi_0 + \beta\pi_1 \\ \pi_0 + \pi_1 = 1 \end{array}$$

$$\pi_0 = \frac{\beta}{1+\beta-\alpha}, \quad \pi_1 = 1 - \pi_0 = \frac{1-\alpha}{1+\beta-\alpha}$$

<u>Example</u>: $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S = \{1,2\}$

(i) Find stationary prob. dist

$\underset{\sim}{\Pi} = (\Pi_1, \Pi_2)$

$\underset{\sim}{\Pi} = \underset{\sim}{\Pi} P$, $\Pi_1 + \Pi_2 = 1$ $\Rightarrow$ $\Pi_1 = \Pi_2$, $\Pi_1 + \Pi_2 = 1$ $\Rightarrow$ $\underset{\sim}{\Pi} = \left(\frac{1}{2}, \frac{1}{2}\right)$

(ii) $\underset{\sim}{p}^{(0)} = (\alpha, 1-\alpha)$

$\underset{\sim}{p}^{(1)} = \underset{\sim}{p}^{(0)} P = (\alpha, 1-\alpha)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (1-\alpha, \alpha)$

$\underset{\sim}{p}^{(2)} = \underset{\sim}{p}^{(1)} P = (\alpha, 1-\alpha)$

Here limiting prob dist does not exist.

—x—

<u>Ergodic M.C.</u> M.C. irreducible, aperiodic (period 1) and all states are positive recurrent.

→ Finite state M.C. (X_n) that is irreducible and aperiodic is ergodic

→ For ergodic M.C., the limiting prob. dist and stationary prob. dist are same and

$$\Pi_j = \frac{1}{m_j} \quad \forall j \quad , \text{where } m_j = \sum_{n=1}^{\infty} n f_j^{(n)}.$$

—x—

<u>Example</u> NCD (No Claim Discount) systems

NCD class	E_0	E_1	E_2
% discount	0	20	40
annual premium	100 ✓	80	60

Movement in the system is determined by the rule whereby one steps back one discount level (or stays in E_0) with one claim in a year, and returns to a level of no discount if more than one claim is made. A claim-free year results in a step up to a higher discount level (or one remains in E_2 if already there).

If we suppose that for anyone in this scheme the prob. of one claim in a year is 0.2 while the prob. of two or more claims is 0.1. Find

(i) tpm of system

(ii) In long run, what proportion of time is the process in each of the three discount classes

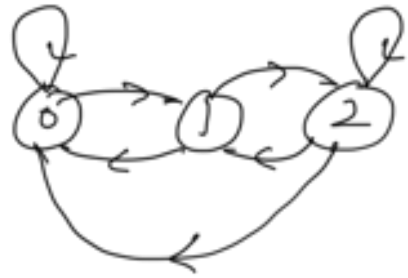
(iii) Find the av. annual premium paid.

$E_i \to i$ state, $i = 0, 1, 2$

tpm

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.3 & 0.7 & 0 \\ 1 & 0.3 & 0 & 0.7 \end{array}$$

$$\begin{array}{c} 1 \\ 2 \end{array}\begin{pmatrix} 0.3 & 0 & 0.7 \\ 0.1 & \underline{0.2} & 0.7 \end{pmatrix}$$



$S = \{0,1,2\}$, irreducible, aperiodic M.C.

$\Rightarrow$ ergodic M.C..

$\underset{\sim}{\pi} = (\pi_0, \pi_1, \pi_2)$

$$\left.\begin{array}{l} \underset{\sim}{\pi} = \underset{\sim}{\pi} P \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{array}\right] \Rightarrow \begin{array}{l} 0.3\pi_0 + 0.3\pi_1 + 0.1\pi_2 = \pi_0 \\ 0.7\pi_0 \qquad + 0.2\pi_2 = \pi_1 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{array}$$

$$\Rightarrow \underset{\sim}{\pi} = (0.1860, 0.2442, 0.5698)$$

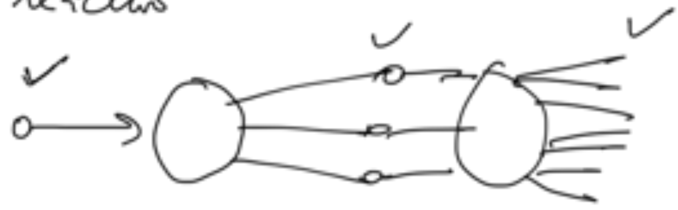
av. annual premium $= 100 \times 0.1860 + 80 \times 0.2442 + 60 \times 0.5698 = 72.324$

—x—

Branching Process:

electron multipliers

neutron chain reaction



survival of family name

by the end of its lifetime

○ each individual produce j new offsprings

j wp $p_j < 1 \ \forall j \geq 0$ (let independently of number produced by other individual

X_0 size of zeroth generation

X_1 all offsprings of zeroth generations or first generation

$\vdots$

X_n size of n^{th} generation

$\{X_n : n \geq 0\}$ M.C. $\quad S = \{0, 1, 2, \ldots\}$

$p_0 > 0$

State 0 recurrent

all other states are transient

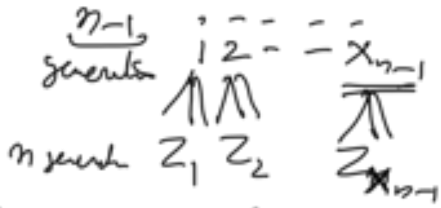
If $p_0 > 0$, then the popn will either die out or its size will converge to ∞.

mean # of offspring of a single individual $\mu = \sum_{j=0}^{\infty} j p_j$

var " " ------ $\sigma^2 = \sum_{j=0}^{\infty} (j-\mu)^2 p_j$

Let $X_0 = 1$

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$



Z_i # of offsprings of ith individual of the $(n-1)^{st}$ generation ...

... generations St. $E(Z_i)=\mu,\ V(Z_i)=\sigma^2$

$$E(X_n) = E\big(E(X_n \mid X_{n-1})\big)$$

$$= E\Big(\underbrace{E\Big(\sum_{i=1}^{X_{n-1}} Z_i \mid X_{n-1}\Big)}_{\to X_{n-1}\mu}\Big)$$

$$\left[E\Big(\sum_{i=1}^{X_{n-1}} Z_i \mid X_{n-1}=y\Big) = E\Big(\sum_{i=1}^{y} Z_i\Big) = y\mu\right.$$

$$= E(X_{n-1}\mu) = \mu E(X_{n-1})$$

$$= \mu^n E(X_0) = \mu^n$$

$$V(X_n) = E\big(\underbrace{V(X_n \mid X_{n-1})}_{\sigma^2 X_{n-1}}\big) + V\big(\underbrace{E(X_n \mid X_{n-1})}_{\mu X_{n-1}}\big)$$

$$= \sigma^2 E(X_{n-1}) + \mu^2 V(X_{n-1})$$

$$= \sigma^2\mu^{n-1} + \mu^2 V(X_{n-1})$$

$$= \sigma^2\mu^{n-1} + \mu^2[\sigma^2\mu^{n-2} + \mu^2 V(X_{n-2})]$$

$$= \sigma^2(\mu^{n-1}+\mu^n) + \mu^4 \underbrace{V(X_{n-2})}_{\sigma^2\mu^{n-3}+\mu^2 V(X_{n-3})}$$

$$= \sigma^2(\mu^{n-1}+\mu^n+\mu^{n+1}) + \mu^6 V(X_{n-3})$$

$$= \sigma^2(\mu^{n-1}+\mu^n+\cdots+\mu^{2n-2}) + \mu^{2n} \underbrace{V(X_0)}_{\to 0}$$

$$= \sigma^2 \mu^{n-1}(1+\mu+\cdots\mu^{n-1})$$

$$V(X_n) = \begin{cases} \sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

$$\underline{u_{n+1} = P(X_{n+1}=0)} = \sum_j P(X_{n+1}=0 \mid X_1=j)\, p_j$$

$$= \sum_j \left(P(X_n=0)\right)^j p_j$$

$$= \sum_j u_n^j\, p_j$$

Let Π_0 denote the prob. that popl^n will eventually die out (under the assumption that $X_0=1$)

, i.e., prob. of ultimate extinction

$$\Pi_0 = \lim_{n\to\infty} P(X_n=0 \mid X_0=1)$$

Note that

$\rightarrow$ $\underline{\Pi_0 = 1 \text{ if } \mu<1}$, since

$$\mu^n = E(X_n) = \sum_{j=\cancel{0}\,1}^{\infty} j\,P(X_n=j) \geq \sum_{j=1}^{\infty} 1\cdot P(X_n=j)$$

$$= P(X_n \geq 1)$$

Since $\mu^n \to 0$ if $\mu<1$, $n\to\infty$

as $n\to\infty$

$\therefore$ $P(X_n \geq 1) \leq \mu^n \to P(X_n \geq 1) \to 0$

$\ldots_n > j) \to 0 \Rightarrow P(X_n = j) = 0$

$$\therefore P(X_n = 0) \to 1$$

$$\pi_0 = 1$$

→ It can be shown that $\pi_0 = 1$ even when $\mu = 1$

→ When $\mu > 1$, it turns out $\pi_0 < 1$

$$\pi_0 = P(\text{popl}^n \text{ dies out}) = \sum_{j=0}^{\infty} \underbrace{P(\text{popl}^n \text{ dies out} \mid X_1 = j)}\, p_j$$

thm of total prob

$$\boxed{\pi_0 = \sum_{j=0}^{\infty} \pi_0^j\, p_j} \quad \text{—— (1)}$$

It can be shown that π_0 is the smallest +ve number satisfying equation (1).

Example (1) If $p_0 = \frac{1}{2}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{4}$ $X_0 = 1$

π_0 ?

$$\mu = \frac{3}{4} \leq 1 \Rightarrow \pi_0 = 1 \qquad \pi_0^n = 1 \quad X_0 = n$$

(2) If $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$, $X_0 = 1$

$$\mu = \frac{1}{4} + 1 = \frac{5}{4} > 1$$

$$\pi = \sum_{j=0}^{\infty} \pi^j p_j$$

$$\pi = \frac{1}{4} + \frac{\pi}{4} + \frac{\pi^2}{2}$$

$4 \quad 2$

$$\Rightarrow 2\pi^2 - 3\pi + 1 = 0$$

$$\Rightarrow \pi = \frac{1}{2}, 1$$

$$\boxed{\pi_0 = \frac{1}{2}}$$

—x—

Transformed M.C

Example A pensioner receives ₹ 2 (0,000) at the beginning of each month. The amt of money he need to spend during a month is independent of the amt he has and is equal to i with prob p_i, $i = 1, 2, 3, 4$, $\sum_{i=1}^{4} p_i = 1$. If the pensioner has more than 3 at the end of a month, he gives the amt greater than 3 to his son. If, after receiving his payment at the beginning of a month, the pensioner has a capital of 5, what is the prob. that his capital is ever 1 or less at any time within four months?

Sol.

X_n amt the pensioner has at the end of month

$$S = \{1^*, 2^*, 3^*\}$$

1^*: less than or equal to 1 $(0, 1)$

2^*: 2

3^*: greater than or equal to 3

transformed tpm ✓ $\{X_n\}$ is M.C.

$$Q = \begin{array}{c|ccc} & 1^* & 2^* & 3^* \\ 1^* & 1 & 0 & 0 \\ 2^* & p_3+p_4 & p_2 & p_1 \\ 3^* & p_4 & p_3 & p_1+p_2 \end{array}$$

make absorbing state (requirement of Qn)

$\underline{2^*}$ $\underline{2}+\underline{2}$ (4,3)

1^* (0,1)

2^* 2+2 (2)

2^*

(2)

$3^+2=5$, 3,2

3^* 3^* (3,4,5 —)

$\underline{3^*}$ $\underline{3}$ $\cancel{+2}$ 4

1^* (0,1)

$Q^4 = Q^{(4)}$

Ans $\rightarrow$ $Q^{(4)}_{3^*,1^*} = Q^4_{3^*,1^*}$

— x —

Example Suppose that a production process changes states in accordance with a M.C. having transition probabilities p_{ij} ; $i,j = 1, -, n$; and suppose that certain of the states are considered acceptable (A) and the remaining unacceptable (A^c)

The production process is said to be "up" when in an A and "down" when in in A^c. Determine

1. the rate of breakdowns
2. the av. ...

... av. length of time the process remains down when it goes down; and

U " " " " " — — — — up — — — goes up.

Sol $\underline{\pi_k}$, $k = 1, \ldots, n$ $\underline{\text{limiting prob.}}$

$\underline{i \in A}$, $\underline{j \in A^c}$

rate of breakdown $= \sum_{j \in A^c} \sum_{i \in A} \pi_i p_{ij}$

$\bar{U}$, $\bar{D}$ → av. time the process remains down when it goes down

$\bar{U}$ ↳ av. time the process remains up when it goes up

Single breakdown every $\bar{U} + \bar{D}$ time

rate at which breakdown occur $= \dfrac{1}{\bar{U} + \bar{D}}$

$\therefore \dfrac{1}{\quad} = \sum \sum \pi_i p_{ij}$ ...