Poisson Process and Related distributions

eg ① $N(t)$ # of persons who enters a store by time t

→ indep increment reasonable

→ stationary increment is reasonable if there were no times of day at which people were more likely to enter the store.

② soccer player $N(t)$ # of goals by the player $(0, t]$

Stationary assumption do not seem to be reasonable

$\boxed{25-30 \text{ age}}$ $\quad$ $35-40$ age

—x—

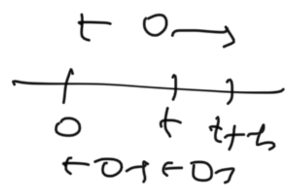
Statement

Under the assumptions 1, 2, 3,

$$P_n(t) = P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

Sol h - small

$$P_0(t+h) = P(N(0, t+h] = 0)$$



$$= P(N(0,t] = 0,\ N(t, t+h] = 0)$$

$$= P(N(0,t] = 0) \cdot P(N(t, t+h] = 0)$$

| Using indep increments

$$= P(N(t) = 0) \cdot P(N(h) = 0)$$

| Using stationary increment

$$= P_0(t)\,.\,(1-\lambda h+o(h)) \qquad \Big|\text{Using assum. (3)}$$

$$\Rightarrow \frac{P_0(t+h)-P_0(t)}{h} = -\lambda P_0(t) + P_0(t)\frac{o(h)}{h}$$

Limit $h \to 0$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

$$P_0(t) = Ce^{-\lambda t}$$

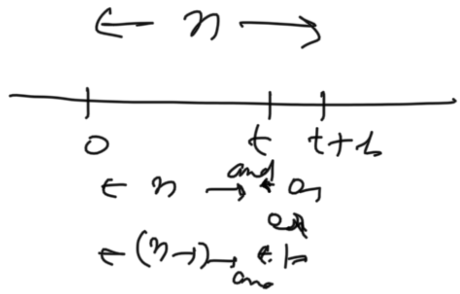
$$P_0(0) = 1 \quad \Rightarrow 1 = C$$

$$P_0(t) = e^{-\lambda t}$$

$$P_n(t+h) = P(N(0,t+h]=n)$$

$$= P_n(t)\,.\,(1-\lambda h+o(h))$$

$$+ P_{n-1}(t)\,.\,(\lambda h+o(h))$$



$$\Rightarrow \frac{P_n(t+h)-P_n(t)}{h} = -\lambda\,(P_n(t)-P_{n-1}(t)) + \frac{o(h)}{h}$$

Lt $h \to 0$

$$\frac{dP_n(t)}{dt} = -\lambda\,(P_n(t)-P_{n-1}(t))$$

$$P_0(t) = e^{-\lambda t} \checkmark$$

Claim $P_n(t) = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n = 0, 1, 2, \ldots$

$n=0$ true

assume → $n-1$ true

$$\frac{dP_n(t)}{dt} = -\lambda\left(P_n(t) - \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\right)$$

$$e^{\lambda t}\frac{dP_n(t)}{dt} + \lambda e^{\lambda t}P_n(t) = \frac{\lambda^n}{(n-1)!}t^{n-1}$$

$$\frac{d}{dt}\left(e^{\lambda t}P_n(t)\right) = \frac{\lambda^n}{(n-1)!}t^{n-1}$$

$$e^{\lambda t}P_n(t) = \frac{\lambda^n}{(n-1)!}\times\frac{t^n}{n}$$

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad \text{holds } \forall n.$$

—x—

PI mgf of $N(t)$

$$M_{N(t)}(u) = e^{\lambda t(e^u-1)}$$

indep $\begin{cases} N_1(t) \sim P.P.(\lambda_1) \\ N_2(t) \sim P.P.(\lambda_2) \end{cases}$

$$M_{N_1(t)+N_2(t)}(u) = M_{N_1(t)}(u)\,M_{N_2(t)}(u)$$

$$= e^{(\lambda_1+\lambda_2)t(e^u-1)}$$

$$N_1(t)+N_2(t) \sim P.P.(\lambda_1+\lambda_2)$$

—x—

PII

$N_1(t)$ is p

$N_2(t)$ is q $\boxed{N(t) \sim P.P.(\lambda)}$

$$N(t) = N_1(t) + N_2(t)$$

$\Rightarrow$ indep. $\begin{cases} N_1(t) \sim P.P.(\lambda p) \\ N_2(t) \sim PP(\lambda q) \end{cases}$

Sol

$$P(N_1(t)=n, N_2(t)=m) = \sum_{i=0}^{\infty} P(N_1(t)=n, N_2(t)=m \mid N(t)=i)\, P(N(t)=i)$$

$$= P(N_1(t)=n, N_2(t)=m \mid N(t)=n+m)\ P(N(t)=n+m)$$

$$= {}^{n+m}C_n\, p^n q^m \quad \frac{e^{-\lambda t} (\lambda t)^{n+m}}{(n+m)!}$$

$$= \frac{\cancel{(n+m)!}}{n!\, m!} \quad \frac{e^{-\lambda p t} e^{-\lambda q t} (\lambda p t)^n (\lambda q t)^m}{\cancel{(n+m)!}}$$

$$= \frac{e^{-\lambda p t} (\lambda p t)^n}{n!} \quad \frac{e^{-\lambda q t} (\lambda q t)^m}{m!}$$

$$= P(N_1(t)=n) \cdot P(N_2(t)=m)$$

indep. $\begin{cases} N_1(t) \sim P.P.(\lambda p) \\ N_2(t) \sim PP(\lambda q) \end{cases}$

— x —

$N(t) \sim P.P.(\lambda)$

$$E(N(t)) = \lambda t$$

$$V(N(t)) = \lambda t$$

— x —

$N(t) \sim PP(\lambda)$

T_i IID $\exp(\lambda)$

$\longleftarrow S_n \longrightarrow$

$\leftarrow T_1 \rightarrow \leftarrow T_2 \rightarrow \quad \leftarrow T_n \rightarrow$

event 1 2 --- n-1 n

$T_1 \sim \exp(\lambda)$

<u>Interarrival and waiting time dist^n:</u>

$N(t)$ # of event $(0, t] \sim PP(\lambda)$

T_1 time for first event

T_n time elapsed b/w $(n-1)^{st}$ and n^{th} event

$$T_1 > t \equiv N(t) = 0$$

$$P(T_1 > t) = P(N(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$T_1 \sim \exp(\lambda)$$

For exp. dist^n pdf

$f(t) = \lambda e^{-\lambda t}, t > 0, \lambda > 0$

CDF

$F(t) = P(X \le t)$

$= 1 - e^{-\lambda t}$

Reliability fn

$\bar{F}(t) = P(X > t) = e^{-\lambda t}$

Note that

$$P(T_2 > t \mid T_1 = s) = P(N(s, s+t] = 0)$$

$$= P(N(t) = 0)$$

$$= e^{-\lambda t}$$

$s \quad s+t$

$$P(T_2 > t \mid T_1) = e^{-\lambda t}$$

$$P(T_2 > t) = E(P(T_2 > t \mid T_1)) = e^{-\lambda t}$$

$\therefore T_2 \sim \exp(\lambda)$

$E(P(T_2 > t \mid T_1))$

$\int P(T_2 > t \mid T_1 = t_1) f_{T_1}(t_1) dt_1$

$= \int \left(\int_t^\infty f_{T_2 \mid T_1 = t_1}(t_2) dt_2 \right) f_{T_1}(t_1) dt_1$

$= \int^\infty (\int f$ ...

$$\int_t^{\infty} \underbrace{\left(\int f_{T_1,T_2}(t_1,t_2)\,dt_1\right)} dt_2$$

$$= \underbrace{\int_t^{\infty} f_{T_2}(t_2)\,dt_2}$$

$$= P(T_2 > t)$$

$N(t) \sim P.P.(\lambda)$

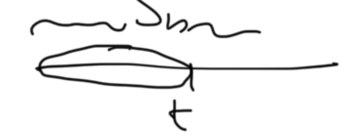
interarrival times $T_1, T_2, \ldots$ are i.i.d exp. r.v's having common mean $\frac{1}{\lambda}$

$S_n = \sum_{i=1}^{n} T_i$ arrival time of n^{th} event, i.e, waiting time until the n^{th} event

mgf $\quad M_{T_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}, \; i = 1, 2, \ldots$

$$M_{S_n}(t) = \prod_{i=1}^{n} M_{T_i}(t) = \underline{\left(1 - \frac{t}{\lambda}\right)^{-n}}$$

$S_n \sim \text{Gamma}(n, \lambda)$ $\qquad E(S_n) = \frac{n}{\lambda}, \; V(S_n) = \frac{n}{\lambda^2}$



Alter

$$\underline{S_n > t} \equiv N(t) \le n-1$$

$$P(S_n > t) = P(N(t) \le n-1) = \sum_{i=0}^{n-1} \underline{\frac{e^{-\lambda t} (\lambda t)^i}{i!}}$$

CDF of S_n

$F_{S_n}(t) = P(S_n > t) = 1 - P(S_n > t)$

pdf of S_n

$$f_{S_n}(t) = \frac{d}{dt} F_{S_n}(t) = \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1}, \; t > 0, \; \lambda > 0$$

<u>Example</u> Suppose that people migrate into a territory at Poisson rate $\lambda = 1$ per day (a) What is the expected time until the tenth immigration arrives?

(b) What is the prob. that the elapsed time b/w the tenth and the eleventh arrivals exceeds two days?

Sol (a) $N(t) \sim P.P.(\lambda)$ $\lambda = 1$ $S_n \sim \text{Gamma}(n, \lambda)$

$$E(S_{10}) = \frac{10}{1} = 10$$

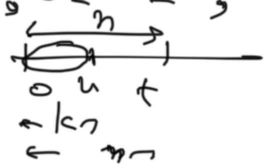
(b) $$P(T_{11} > 2) = e^{-1 \times 2} = e^{-2} \simeq 0.133$$

—x—

<u>Binomial dist also arises in the context of P.P.</u>

If $N(t) \sim P.P.(\lambda)$, then for $0 < u < t$, $0 \le k \le n$,

$$[N(u) \mid N(t) = n] \sim \text{Bin}\left(n, \frac{u}{t}\right)$$



Sol

$$P(N(u) = k \mid N(t) = n) = \frac{P(N(0,u] = k, N(0,t] = n)}{P(N(t) = n)}$$

$$= \frac{P(N(0,u] = k, N(u,t] = n-k)}{P(N(t) = n)}$$

$$= \frac{P(N(u) = k) \cdot P(N(t-u) = n-k)}{P(N(t) = n)}$$

using stat. & indep. increments of pp

$$= \frac{\dfrac{e^{-\lambda u} (\lambda u)^k}{k!} \times \dfrac{e^{-\lambda(t-u)} (\lambda(t-u))^{n-k}}{(n-k)!}}{\dfrac{e^{-\lambda t} (\lambda t)^n}{n!}}$$

$$= \frac{n!}{\quad} \quad \frac{u^k (t-u)^{n-k}}{\quad}$$

$$\overline{k!(n-k)!} \qquad \overline{t^n}$$

$$= \binom{n}{k}\left(\frac{u}{t}\right)^k\left(1-\frac{u}{t}\right)^{n-k}, \quad k=0,1,\ldots,n$$

—x—

P.P. & Beta dist

Ex indep $\begin{cases} X \sim \text{Gamma}(\alpha,\lambda) \\ Y \sim \text{Gamma}(\beta,\lambda) \end{cases}$ ✓ $\underline{U = \dfrac{X}{X+Y}}$, $V = X+Y$

then U, and V are indep $U \sim \text{Beta}(\alpha,\beta)$,
$V \sim \text{Gamma}(\alpha+\beta,\lambda)$

pdf $$f_U(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\,\Gamma\beta}\,u^{\alpha-1}(1-u)^{\beta-1}, \quad 0<u<1$$

$N(t) \sim \text{P.P.}(\lambda)$

$T_1, T_2, \ldots$ $\underline{\text{i.i.d } \exp(\lambda)}$

$n>m$

$$S_m = \sum_{i=1}^{m} T_i, \quad S_n = S_m + (S_n - S_m)$$

$$= \underbrace{\sum_{i=1}^{m} T_i}_{*} + \underbrace{\sum_{i=m+1}^{n} T_i}_{*}$$

indep $\begin{cases} \overset{\nearrow X}{S_m} \sim \text{Gamma}(m,\lambda) \\ \overset{\nearrow Y}{S_n - S_m} \sim \text{Gamma}(n-m,\lambda) \end{cases}$

$$\underline{U = \frac{S_m}{S_n}} \sim \text{Beta}(\underline{m}, \underline{n-m}) \quad \Big| \text{ using Ex.}$$

Also $\underline{U}$ and $\underline{S_n}$ are indep.

$$f_U(u) = \frac{\Gamma n}{\Gamma m\,\Gamma(n-m)}\,u^{m-1}(1-u)^{n-m-1}, \quad 0<u<1.$$

—x—

Example: Suppose customers stream into a drug store at the constant Poisson arr. rate λ of 15 per hr. The pharmacy opens its door at 8 AM and closes at 8 pm. Given that the 100^{th} customer on a particular day walked in at 2 pm, we want to know what is the prob. that the 50^{th} customer came before noon.

Sol

$$N(t) \sim P.P.(\lambda)$$

S_j arrival time of the j^{th} customer on that day.

$$\underset{8+12}{n=100}, \quad \underset{8+2}{m=50}$$

$$P(S_m < 4 \mid S_n = 6) = P\left(\frac{S_m}{S_n} < \frac{4}{6} \mid S_n = 6\right)$$

$$= P\left(\frac{S_m}{S_n} < \frac{4}{6}\right) \quad \because U = \frac{S_m}{S_n} \text{ \& } S_n \text{ are indep}$$

$\sim$ Beta$(\overset{m}{50}, \overset{n-m}{50})$

$$= \int_0^{4/6} \frac{\Gamma 100}{(\Gamma 50)^2} u^{49} (1-u)^{49} du \quad \text{exact}$$

If $U \sim$ Beta(α, β)

$$\text{mean} = \frac{\alpha}{\alpha+\beta}$$

$$\text{Var} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$E(U) = \frac{1}{2}$$

$$V(U) = \frac{50 \times 50}{(100)^2 \times (101)} = 0.0025$$

$$\overset{CLT}{=} P\left(Z < \frac{\frac{4}{6} - \frac{1}{2}}{\sqrt{0.0025}}\right)$$

$$= P(Z < 3.33)$$

$$= \Phi(3.33)$$

$= 0.9997$ | using standard normal dist table

—x—

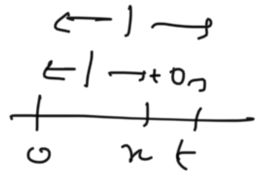
P.P. & uniform dist^n

$N(t) \sim PP$ and one event take place in interval from 0 to t. Let Y r.v. describing the time of occurrence of this Poisson event, has continuous uniform dist $(0, t]$.

Sol $\qquad N(t) \sim PP(\lambda)$

$$Y = [T_1 \mid N(t) = 1]$$

For $0 < x \le t$



$$P(Y \le x) = P(T_1 \le x \mid N(t) = 1)$$

$$= \frac{P(T_1 \le x, N(t) = 1)}{P(N(t) = 1)}$$

$$= \frac{P(N(0,x] = 1, N(x,t] = 0)}{P(N(t) = 1)}$$

$$= \frac{P(N(x) = 1)\, P(N(t-x) = 0)}{P(N(t) = 1)}$$

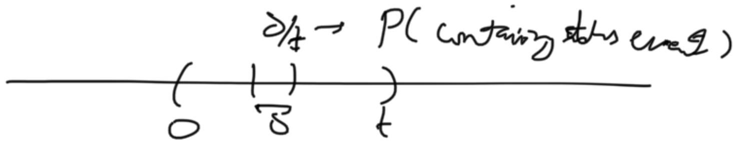
$$= \frac{e^{-\lambda x} \lambda x \cdot e^{-\lambda(t-x)}}{e^{-\lambda t} \lambda t}$$

$$= \frac{x}{t}$$

$\therefore Y = [T_1 \mid N(t) = 1] \sim U(0, t)$

pdf
$$\begin{cases} \frac{1}{t}, & 0 < x < t \\ 0 & \text{o.w.} \end{cases}$$

This means, if $0<\delta<t$, any subinterval of $(0,t)$ of length δ has prob. of $\frac{\delta}{t}$ of containing the time of occurance of the event.



— x —

Compound P.P.

A S.P. $\{X(t), t\geq 0\}$ is a compound P.P. if

$$X(t) = \sum_{i=1}^{N(t)} Y_i \quad , t\geq 0 \text{ , where}$$

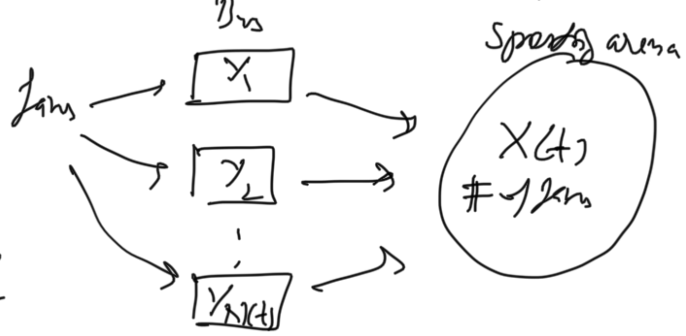
$\{N(t), t\geq 0\}$ is a P.P. and $\{Y_i, i\geq 1\}$ be a family of i.i.d. r.v's that is also indep of $\{N(t), t\geq 0\}$

Example (1) If $Y_i \equiv 1$, then $X(t) = N(t)$
usual P.P.

(2)

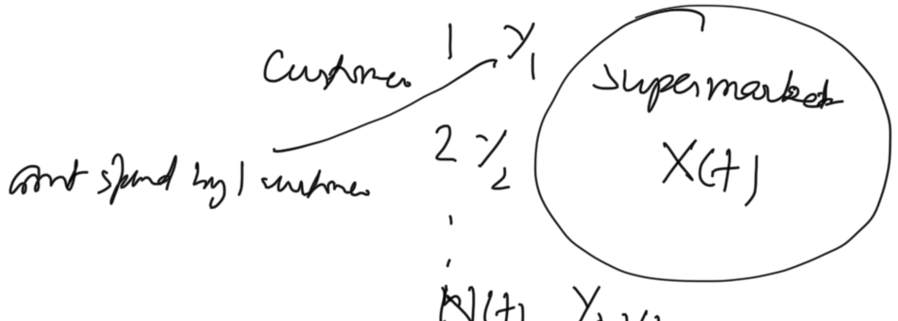
$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

compound P.P.



$N(t) = \#$ of bus $\sim$ P.P. (λ)

(3)



$N(t)$ PP(λ)

$N(t) \sim P.O.(\lambda)$

earning of supermarket $X(t) = \sum_{i=1}^{N(t)} Y_i$ Compound PP

(y) i.i.d. Y_i, $N(t)$

Y_1
$\vdots$
$Y_{N(t)}$

ATM
$X(t)$

$$E(X(t)) = E\left(\sum_{i=1}^{N(t)} Y_i\right) \qquad E(X) = E(E(X|Y))$$

$$= E\Big(\underbrace{E\Big(\sum_{i=1}^{N(t)} Y_i \,\Big|\, N(t)\Big)}_{N(t)\,E(Y_1)}\Big)$$

$$= E(Y_1)\, E(N(t))$$

$$= \lambda t\, E(Y_1)$$

$\because$
$$E\left(\sum_{i=1}^{N} Y_i \,\Big|\, N=n\right)$$
$$= E\left(\sum_{i=1}^{n} Y_i\right)$$
$$= n\,E(Y_1)$$

$$V(X(t)) = E\big(\underbrace{V(X(t)|N)}_{N\,V(Y_1)}\big) + V\big(\underbrace{E(X(t)|N)}_{N\,E(Y_1)}\big)$$

$$= V(Y_1)\, E(N) + (E(Y_1))^2\, V(N)$$

$$= \lambda t\, V(Y_1) + \lambda t\,(E(Y_1))^2$$

$$= \lambda t\left[E(Y_1^2) - (E(Y_1))^2\right] + \lambda t\,(E(Y_1))^2$$

$= \lambda t \, E(Y_1^2)$.

Example Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. If the # of people in each family is indep. and takes on the values 1, 2, 3, 4 with resp. prob. $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$, then what is the expected value and var. of the individuals migrating to this area during a fixed five-week peroid?

Sol $\quad X(t) = \sum_{i=1}^{N(t)} Y_i \sim$ Compound P.P.

Y_i # of people in the ith family

$N(t)$ # of families migrating (at t) $\lambda = 2$

$$E(Y_1) = 1 \times \frac{1}{6} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} \times 4 \times \frac{1}{6} = \frac{5}{2}$$

$$E(Y_1^2) = 1 \times \frac{1}{6} + 4 \times \frac{1}{3} + 9 \times \frac{1}{3} + 16 \times \frac{1}{6} = \frac{43}{6}$$

$$E(X(5)) = \lambda t \, E(Y_1) = 2 \times 5 \times \frac{5}{2} = 25$$

$$V(X(5)) = \lambda t \, E(Y_1^2) = 2 \times 5 \times \frac{43}{6} = \frac{215}{3}.$$

(Contd) In previous example, find the approximate probability that atleast 240 people migrate to the area within next 50 weeks.

$$E(X(50)) = 2 \times 50 \times \frac{5}{2} = 250$$

$$V(X(50)) = 2 \times 50 \times \frac{43}{6} = \frac{4300}{6}$$

$$P(X(50) \geq 240) = P(X(50) > 239)$$

$$\stackrel{CLT}{\simeq} P\left(\frac{X(50) - 250}{\sqrt{4300/6}} \geq \frac{239.5 - 250}{\sqrt{4300/6}} \right)$$

$$= 1 - \Phi(-0.3922)$$

$$= \Phi(0.3922)$$

$$= 0.6517$$

<u>Example</u> Customers arrive at the ATM in accordance with a P.P. with rate 12 per hr. The amt of money withdrawn on each transaction is a r.v. with mean \$30 and sd \$50 (A negative withdrawal means that money was deposited). The machine is in use for 15 hr daily. Approximate the prob. that the total daily withdrawal is less than \$6000.

Sol

$N(t) \sim P.P.(\lambda)$, $\lambda = 12$ per hr

Y_i $\quad E(Y_i) = 30$, $V(Y_i) = (50)^2 \rightarrow E(Y_i^2) - (E(Y_i))^2$

$$X(t) = \sum^{N(t)} Y_i \sim \text{compound P.P.}$$

$$E(X(15)) = 12 \times 15 \times 30 = 5400 \; ; \; V(X(15)) = 12 \times 15 \times \left((50)^2 + (30)^2\right) = 612000$$

$$P(X(15) \leq 6000)$$

$$\overset{CLT}{=} P\left(Z \leq \frac{6000 - 5400}{\sqrt{612000}}\right)$$

$$= P(Z \leq 0{,}767)$$

$$= 0{,}78$$

—x—