Change of variables.

Let the coordinates (x, y) be transformed to (u, v) coordinates. Then what would happen to,

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\, dx \longrightarrow (1)$$

subject to conditions $y(x_1) = y_1,\ y(x_2) = y_2$?

Observe, $$dx = \left(x_u + x_v \frac{dv}{du}\right) du.$$

$$\frac{dy}{dx} = \frac{y_u\, du + y_v\, dv}{x_u\, du + x_v\, dv}.$$

$$\Rightarrow y' = \frac{y_u + y_v v'}{x_u + x_v v'}$$; prime $\rightarrow$ differentiation w.r. to independent variable.

Then, $$f(x, y, y')\, dx$$

$$= f\left(x(u,v),\ y(u,v),\ y'(u,v,v')\right)\left(x_u + x_v v'\right) du$$

$$= F(u, v, v')\, du$$

Thus, (1) reduces to,

$$J[v(u)] = \int_{u_1}^{u_2} F(u, v, v')\, du.$$

In the new-coordinates (u, v), it can be shown that $F(u, v, v')$ also satisfies the E-L-E

$$\frac{\partial F}{\partial v} - \frac{d}{du}\left(\frac{\partial F}{\partial v'}\right) = 0.$$

Example: Find the extremals of the functional.

$$I[y(x)] = \int_0^{\ln 2} (e^{-x} y'^2 - e^x y^2)\, dx \longrightarrow (1)$$

by transforming (x, y) to (u, v) plane where $u = e^x$, $y = v$.

Sol. $u = e^x \Rightarrow e^x dx = du \Rightarrow \boxed{dx = e^{-x} du = \frac{du}{u}.}$

$y = v \Rightarrow \boxed{dy = dv}$

So, $y' = \frac{dy}{dx} = u \frac{dv}{du} = u v'$.

Thus, eq. (1) reduces to,

$$J[v(u)] = \int_1^2 \left(\frac{1}{u} \cdot u^2 v'^2 - u v^2\right) \frac{du}{u}$$

$$= \int_1^2 (v'^2 - v^2)\, du$$

Note: $F = v'^2 - v^2$. It does not contain the independent variable. But since it does not contain square root, we will apply original form of ELE.

$$\therefore \quad F_v - \frac{d}{du} F_{v'} = 0.$$

$$\Rightarrow -2v - 2v'' = 0 \quad \Rightarrow \quad v'' + v = 0.$$

$$\therefore \quad v = A\cos u + B\sin u$$

$$y = A\cos(e^x) + B\sin(e^x).$$