Method of successive substitution

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt \longrightarrow (1)$$

Here R.H.S of $u(x)$ is substituted for $u(t)$ inside the integral repeatedly.

Step 1: $u(x) = f(x) + \lambda \int_a^b K(x,t) \left\{ f(t) + \lambda \int_a^b K(t,t_1) u(t_1) dt_1 \right\} dt$

$\therefore$

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1)\, u(t_1)\, dt_1\, dt$$

Step 2: Substitute $u(t_1)$ from R.H.S of (1) into the double integral.

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) \left\{ f(t_1) + \lambda \int_a^b K(t_1,t_2) u(t_2) dt_2 \right\} dt_1\, dt$$

$\therefore$

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1)\, dt_1\, dt$$
$$+ \lambda^3 \int_a^b \int_a^b \int_a^b K(x,t) K(t,t_1) K(t_1,t_2)\, u(t_2)\, dt_2\, dt_1\, dt,$$

<u>Theorem</u>: Consider the integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt \longrightarrow (1)$$

If (i) $K(x,t)$ is real and continuous in $R: \{a \le x, t \le b\}$ such that $|K(x,t)| \le M \ \forall (x,t) \in R$.

(ii) $f(x) \neq 0$ is real and continuous in $I: [a,b]$

(iii) λ is a constant : $|\lambda| < \dfrac{1}{M(b-a)}$,

then, method of successive substitution yields one and only one continuous solution in I for the IE (1).

<u>Proof</u>. Substitute for $u(t)$ its expression given in r.h.s. of (1).

$$u(x) = f(x) + \lambda \int_a^b K(x,t) \left\{ f(t) + \lambda \int_a^b K(t,t_1)\, u(t_1)\, dt_1 \right\} dt$$

$$= f(x) + \lambda \int_a^b K(x,t)\, f(t)\, dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1)\, u(t_1)\, dt_1\, dt$$

$$= f(x) + \lambda \int_a^b K(x,t)\, f(t)\, dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1)\, f(t_1)\, dt_1\, dt$$
$$+ \lambda^3 \int_a^b \int_a^b \int_a^b K(x,t) K(t,t_1) K(t_1,t_2)\, u(t_2)\, dt_2\, dt_1\, dt$$

$$= f(x) + \lambda \int_a^b K(x,t)\, f(t)\, dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1)\, f(t_1)\, dt_1\, dt$$
$$+ \cdots + \lambda^{n-1} \int_a^b \int_a^b \cdots \int_a^b K(x,t) K(t,t_1) \cdots K(t_{n-3}, t_{n-2})\, f(t_{n-2})\, dt_{n-2} \cdots dt_1\, dt$$
$$+ \lambda^n \int_a^b \int_a^b \cdots \int_a^b K(x,t) K(t,t_1) \cdots K(t_{n-2}, t_{n-1})\, u(t_{n-1})\, dt_{n-1} \cdots dt_1\, dt \longrightarrow (2)$$

Let the last term containing the funct. u be denoted by R_n. Let us assume $u(x)$ to be continuous in $[a,b]$. $\therefore |u(x)| \le N \;\forall x \in [a,b]$.

Then,
$$|R_n| = \left| \lambda^n \int_a^b \int_a^b \cdots \int_a^b K(x,t)\,K(t,t_1) \cdots K(t_{n-2})K(t_{n-1})\,u(t_{n-1})\,dt_{n-1} \cdots dt_1\,dt \right|$$

$$= \left| \lambda^n \int_a^b K(x,t) \int_a^b K(t,t_1) \cdots \int_a^b K(t_{n-2}, t_{n-1})\,u(t_{n-1})\,dt_{n-1} \cdots dt_1\,dt \right|$$

$$\le |\lambda|^n \int_a^b |K(x,t)| \int_a^b |K(t,t_1)| \cdots \int_a^b |K(t_{n-2}, t_{n-1})|\,|u(t_{n-1})|\,dt_{n-1} \cdots dt_1\,dt.$$

$$\le |\lambda|^n M^n N \int_a^b \int_a^b \cdots \int_a^b dt_{n-1} \cdots dt_1\,dt.$$

$$= |\lambda|^n M^n (b-a)^n N.$$

Given, $|\lambda| < \dfrac{1}{M(b-a)}$ $\quad\therefore |\lambda| M (b-a) < 1$

Thus, as $n \to \infty$, $|\lambda|^n M^n (b-a)^n N \to 0$.

So, $|R_n| \to 0$ as $n \to \infty$.

This shows that $u(x)$ is given by [from (2)]

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1)\,dt_1\,dt$$
$$+ \cdots + \lambda^{n-1} \int_a^b K(x,t) \cdots \int_a^b K(t_{n-3}, t_{n-2}) f(t_{n-2})\,dt_{n-2}\,dt_1 \cdots dt$$
$$+ \cdots \quad \to (3).$$

The r.h.s. of eq (3) represents an infinite series in which every term is continuous in I, since $K(x,t)$ is continuous in R and $f(x)$ is continuous in I. Thus, the series represents a continuous funct. in I, provided it converges uniformly in I.

To show the r.h.s of (3) is indeed a solut. to the F.I.E. (1), note that $u(x)$ can also be written as,

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1)\,dt_1\,dt$$

$$+ \cdots + \lambda^{n-1} \int_a^b K(x,t) \cdots \int_a^b K(t_{n-3}, t_{n-2}) f(t_{n-2})\,dt_{n-2} \cdots dt$$

$$+ \lambda^n \int_a^b \int_a^b \cdots \int_a^b K(x,t) \cdots K(t_{n-2}, t_{n-1}) f(t_{n-1})\,dt_{n-1}\,dt_{n-2} \cdots dt_1\,dt \quad \rightarrow (4)$$

$$+ \cdots$$

Now, in (3) replace x by t, t by t_1, t_1 by t_2, ...

$$u(t) = f(t) + \lambda \int_a^b K(t,t_1) f(t_1)\,dt_1 + \lambda^2 \int_a^b \int_a^b K(t,t_1) K(t_1,t_2) f(t_2)\,dt_2\,dt_1$$

$$+ \cdots + \lambda^{n-1} \int_a^b \int_a^b \cdots \int_a^b K(t,t_1) \cdots K(t_{n-2},t_{n-1}) f(t_{n-1})\,dt_{n-1} \cdots dt_1$$

$$+ \cdots \quad \rightarrow (5)$$

Multiply both sides of (5) with $\lambda K(x,t)$ and integrate between a and b w.r.to t. This gives,

$$\lambda \int_a^b K(x,t) u(t)\,dt = \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1)\,dt_1\,dt$$

$$+ \cdots + \lambda^{n-1} \int_a^b \cdots \int_a^b K(x,t) \cdots K(t_{n-3},t_{n-2}) f(t_{n-2})\,dt_{n-2} \cdots dt$$

$$+ \lambda^n \int_a^b \int_a^b \cdots \int_a^b K(x,t) \cdots K(t_{n-2}, t_{n-1}) f(t_{n-1})\,dt_{n-1}\,dt_{n-2} \cdots dt_1\,dt \quad \rightarrow (6)$$

Comparing equations (4) & (6) (look at r.h.s.), we get,

$$u(x) - f(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt.$$

or, $$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt.$$

Thus, $u(x)$ as given by the r.h.s of (3), is indeed a solutⁿ of (1).

<u>Uniqueness:</u> Let $u(x)$ and $\hat{u}(x)$ be two continuous solutions of

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt \longrightarrow (A1)$$

So, $$\hat{u}(x) = f(x) + \lambda \int_a^b K(x,t)\, \hat{u}(t)\, dt \longrightarrow (A2)$$

Define, $$U(x) = |u(x) - \hat{u}(x)|.$$

From (A1) & (A2),

$$U(x) = \left| \lambda \int_a^b K(x,t) \left\{ u(t) - \hat{u}(t) \right\} dt \right|$$

$$\leq |\lambda| \int_a^b |K(x,t)|\, U(t)\, dt.$$

$$\leq |\lambda| M \int_a^b U(t)\, dt, \quad \text{since } |K(x,t)| \leq M \longrightarrow (A3)$$

Since $u(x)$ & $\hat{u}(x)$ are continuous in $[a,b]$, ~~...~~, then $U(x)$ is also cont. in $[a,b]$ and therefore bounded in $[a,b]$.

Let $$U(x) \leq B \quad \forall x \in [a,b].$$

From (A3),

$$U(x) \leq |\lambda| MB \int_a^b dt = |\lambda| M B(b-a).$$

$$\therefore \quad U(x) \leq |\lambda| M B(b-a) \longrightarrow (A4)$$

On the r.h.s of A3), if we apply (A4), we get,

$$U(x) \leq |\lambda| M \int_a^b U(t)\, dt$$

$$\leq |\lambda| M \int_a^b |\lambda| M B(b-a)\, dt = |\lambda|^2 M^2 (b-a)^2 B$$

$$\therefore \quad U(x) \leq |\lambda|^2 M^2 (b-a)^2 B \longrightarrow (A5).$$

Again, by virtue of (A5), (A3) becomes,

$$U(x) \leq |\lambda|^3 M^3 (b-a)^3 B.$$

Continuing in this way, we will have,

$$U(x) \leq |\lambda|^n M^n (b-a)^n B.$$

Now, $|\lambda| < \dfrac{1}{M(b-a)}$, so that $|\lambda| M(b-a) < 1$.

$$|\lambda|^n M^n (b-a)^n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

$$\therefore \quad |\lambda|^n M^n (b-a)^n B \longrightarrow 0$$

$\therefore$ As $n \longrightarrow \infty$, $U(x) \leq |\lambda|^n M^n (b-a)^n B$

$$\Rightarrow \quad u(x) = \hat{u}(x) \quad \forall x \in [a,b].$$