Alternative form of Isoperimetric Problem

To extremize

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\,dx.$$

subject to $y(x_1) = y_1$, $y(x_2) = y_2$

and the conditions

$$\phi_i(x, y) = 0 \;;\; i = 1, 2, \dots, m$$

Form

$$h = f + \sum_{i=1}^{m} \lambda_i \phi_i \;;$$ λ_i $(i = 1, 2, \dots, m)$ are unknown parameters.

Then h will satisfy the E-L-E:

$$h_y - \frac{d}{dx} h_{y'} = 0.$$

Note: The geodesic problem can also be cast into an isoperimetric problem.

Ex: Find the shortest distance between the points $P(1,-1,0)$ and $Q(2,1,-1)$ lying on the surface $15x - 7y + z - 22 = 0$.

$s = s[y(x), z(x)] = \int ds \qquad ds^2 = dx^2 + dy^2 + dz^2$

Soln: The dist. between P & Q are given by,

$$PQ = \int_1^2 \sqrt{1 + y'^2 + z'^2}\, dx \;; \quad y = y(x), \; z = z(x) \longrightarrow (1)$$

We have to find minimum of PQ provided $\phi(x, y, z) = 0$, where $\phi(x,y,z) = 15x - 7y + z - 22$.

Let us form the auxiliary functional

$$K[y] = \int_1^2 \Big[\underbrace{\sqrt{1 + y'^2 + z'^2} + \lambda(x)\{15x - 7y + z - 22\}}_{h} \Big] dx \longrightarrow (2)$$

The ELE's are

$h_y - \frac{d}{dx}(h_{y'}) = 0$

$$\lambda(x)(-7) - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0 \longrightarrow (3)$$

$h_z - \frac{d}{dx}(h_{z'}) = 0$

$$\lambda(x) \cdot 1 - \frac{d}{dx}\left(\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0 \longrightarrow (4)$$

$$7 \times (4) + (3) \Rightarrow \frac{d}{dx}\left(\frac{y' + 7z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0.$$

$$\Rightarrow \frac{y' + 7z'}{\sqrt{1 + y'^2 + z'^2}} = C_1 \longrightarrow (5)$$

We have,

$$15x - 7y + z - 22 = 0 \longrightarrow (6).$$

$$\Rightarrow z' = 7y' - 15 \longrightarrow (7).$$

subst. this value into (5), we get

$$\frac{y' + 7(7y'-15)}{\sqrt{1+y'^2+(7y'-15)^2}} = c_1$$

$$\Rightarrow \frac{50y' - 105}{\sqrt{50y'^2 - 210y' + 226}} = c_1$$

$$\Rightarrow (50y' - 105)^2 = c_1^2(50y'^2 - 210y' + 226).$$

$$\Rightarrow (50^2 - 50c_1^2)y'^2 + (210c_1 - 2\times 105\times 50)y' + (105)^2 - 226c_1^2 = 0$$

$$\Rightarrow 50(50 - c_1^2)y'^2 + 210(c_1 - 50)y' + (105)^2 - 226c_1^2 = 0.$$

$$y' = \frac{-210(c_1-50) \pm \sqrt{(210)^2(c_1-50)^2 - 4\times 50(50-c_1^2)[105^2 - 226]}}{2\times 50(50-c_1^2)}.$$

$$y' = \tilde{c}_1$$

$$\Rightarrow y = \tilde{c}_1 x + \tilde{c}_2.$$

$$y(1) = -1, \quad y(2) = 1, \quad z(1) = 0, \quad z(2) = -1.$$

$$2c_1 + c_2 = 1$$
$$c_1 + c_2 = -1$$
$$c_1 = 2$$
$$c_2 = -3.$$

$$y = 2x - 3$$

$$z' = -1 \Rightarrow z = -x + p \Rightarrow z = 1 - x.$$

$$\therefore PQ = \sqrt{6}.$$

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