DEPARTMENT OF MATHEMATICS, IIT KHARAGPUR

Integral Equations and Variational Methods
Spring 2021 * Exam 3/3 * Date: 09.04.2021 * FM = 20
Time-duration: 65 minutes * Mode: Online

Instructions:

- Show each step of calculations. Without showing proper step, no marks will be awarded.
- Give your file name as **rollno_exam3_09042021,** only PDF files will be accepted.
- Write your **name, roll number and serial number** on the top of every page.
- Make sure that there is no shadow in your photo that will be uploaded. If a student fails to upload clear picture of each page, his/ her exam will be considered as cancelled. NO equivalent exam will be taken.
- Your video shall be ON and audio shall be OFF. Focus your camera on the paper on which you will write. Failure of this will lead to cancellation of your exam.
- NO EMAIL/ MOODLE SUBMISSION – link will be provided to you where you have to submit.
- Time-duration is 65 minutes, 60 minutes for writing the paper and 5 minutes for taking photo and uploading.

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1. (a) Find the eigenvalues λ_1 and λ_2 and the corresponding eigenfunctions of the homogenous integral equation

$$u(x) = \lambda \int_1^2 [xt + x^2t^2]\, u(t)\mathrm{d}t.$$

(b) Hence applying the concept of Fredholm alternatives comment on the existence of the solutions to the following non-homogenous integral equations: (Give proper justification in support of your answer.)

(i) $u(x) = x^2 + 1 + \lambda_1 \int_1^2 [xt + x^2t^2]\, u(t)\mathrm{d}t,$

(ii) $u(x) = x^2 + 1 + \lambda_2 \int_1^2 [xt + x^2t^2]\, u(t)\mathrm{d}t.$

(6m)

2. If the Green's function for the operator $L \equiv \frac{d}{dx}\left(x\frac{d}{dx}\right) - \frac{n^2}{x}$ associated with a second order linear differential equation with prescribed boundary conditions is given by

$$g(x,t) = \begin{cases} \dfrac{x^n}{2nt^n}(1-t^{2n})\,;0 \le x < t \\ \dfrac{t^n}{2nx^n}(1-x^{2n})\,;\ t < x \le 1 \end{cases}$$

and if the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (3x^2 - n^2)y = 0$ is reduced to an integral equation $y(x) = f(x) + \int_0^1 K(x,t)y(t)dt$, find $f(x)$ and $K(x,t)$.

(2m)

3. Find the curve $y = y(x)$ that extremizes the functional

$$I(y) = \int_1^4 (y')^2 \mathrm{d}x$$

subject to the condition

$$J(y) = \int_1^4 y \mathrm{d}x = 36$$

and that passes through the points P(1, 3) and Q(4, 24).

(6m)

4. Find the curve $y = y(x)$ that extremizes the functional

$$I(y) = \int_1^e \left\{\frac{1}{2}x^2(y')^2 - \frac{1}{8}y^2\right\} \mathrm{d}x \ ;\ y(1) = 1, y(e) \text{ is free.}$$

(6m)

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