Green's function for higher order ordinary differential operator.

Consider the diff. equation

$$p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_n(x)y = f(x);$$

$$a \le x \le b \rightarrow (1)$$

The B.C.s are,

$$B_1 y = 0,\ B_2 y = 0,\ \ldots,\ B_n y = 0 \longrightarrow (2)$$

Let $g(x,t)$ be the Green's fn. corresponding to

$$Ly = f(x); \quad a \le x \le b.$$

where $\quad L \equiv p_0(x)\frac{d^n}{dx^n} + p_1(x)\cdot\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x).$

Solution of (1) in terms of $g(x,t)$ is,

$$y(x) = \int_a^b g(x,t)\, f(t)\, dt$$

where $\quad Lg(x,t) = \delta(x-t); \quad a \le x, t \le b.$

We actually solve $Lg(x,t)=0$. & get

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

$c_1, c_2, \ldots, c_n$ are arbitrary constants.
$y_1, y_2, \ldots, y_n$ are n l.i. solutions of (1).

~~Thus~~,

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Thus, $g(x,t)$ will be of the form,

$$g(x,t) = \begin{cases} A_1 y_1(x) + A_2 y_2(x) + \cdots + A_n y_n(x); & a \le x < t \\ B_1 y_1(x) + B_2 y_2(x) + \cdots + B_n y_n(x); & t < x \le b. \end{cases}$$

g will satisfy the boundary conditions

$B_1 g = 0, \quad B_2 g = 0, \; \cdots, \; B_n g = 0.$

Continuity of $g, g', g'', \cdots, g^{(n-2)}$ at $x = t$ gives,

$g(t+0,t) = g(t-0,t); \quad g'(t+0,t) = g'(t-0,t),$

$g''(t+0,t) = g''(t-0,t); \; \cdots, \; g^{(n-2)}(t+0,t) = g^{(n-2)}(t+0,t).$

Also, $g(x,t)$ satisfies,

$$\frac{\partial^{(n-1)} g(t+0,t)}{\partial x^{(n-1)}} - \frac{\partial^{(n-1)} g(t-0,t)}{\partial x^{(n-1)}} = \frac{1}{p_0(t)}$$

<u>Example</u> Solve the following BVP using Green's funct.

$$\frac{d^4 y}{dx^4} = 1, \quad y(0) = y'(0) = 0, \quad y''(1) = y'''(1) = 0.$$

<u>Sol.</u> To solve $\dfrac{d^4 g(x,t)}{dx^4} = 0$

$$g(x,t) = \begin{cases} a_0 + a_1 x + a_2 x^2 + a_3 x^3; & 0 \le x < t \\ b_0 + b_1 x + b_2 x^2 + b_3 x^3; & t < x \le 1. \end{cases}$$

a_i, b_i are determined from

$g(0,t) = 0 = g'(0,t); \quad g''(1,t) = 0 = g'''(1,t).$

Also, $g(t+0,t) = g(t-0,t)$

$$g'(t+0,t) = g'(t-0,t)$$

$$g''(t+0,t) = g''(t-0,t).$$

and, $\dfrac{\partial^3 g}{\partial x^3}(t+0,t) - \dfrac{\partial^3 g}{\partial x^3}(t-0,t) = \dfrac{1}{p_0(t)} = \dfrac{1}{1}.$

$g(0,t) = 0 \Rightarrow \boxed{a_0 = 0}$

$g'(0,t) = 0 \Rightarrow \boxed{a_1 = 0}$

When $t < x \leq 1$,

$$g(x,t) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$g'(x,t) = b_1 + 2b_2 x + 3b_3 x^2$$

$$g''(x,t) = 2b_2 + 6b_3 x$$

$$g'''(x,t) = 6b_3.$$

$g''(1,t) = 2b_2 + 6b_3 = 0$; $\quad g'''(1,t) = 6b_3 = 0.$

$\Rightarrow \boxed{b_3 = 0}$

$\therefore \boxed{b_2 = 0}$

$$g(x,t) = \begin{cases} a_2 x^2 + a_3 x^3 & ; 0 \leq x < t \\ b_0 + b_1 x & ; t < x \leq 1. \end{cases}$$

$g(t+0,t) = g(t-0,t) \Rightarrow a_2 t^2 + a_3 t^3 = b_0 + b_1 t \longrightarrow (1)$

$g'(t+0,t) = g'(t-0,t) \Rightarrow b_1 = 2a_2 t + 3a_3 t^2 \longrightarrow (2)$

$g''(t+0,t) = g''(t-0,t) \Rightarrow 0 = 2a_2 + 6a_3 t \longrightarrow (3)$

$g'''(t+0,t) - g'''(t-0,t) = 1 \Rightarrow 0 - 6a_3 = 1$

$\Rightarrow \boxed{a_3 = -\frac{1}{6}}$

$\therefore$ from (3),

$2a_2 - t = 0 \Rightarrow \boxed{a_2 = \frac{t}{2}}$

From (2),

$$t^2 - \frac{t^2}{2} = b_1 \Rightarrow \boxed{b_1 = \frac{t^2}{2}}$$

From (1),

$$\frac{t}{2} \cdot t^2 - \frac{1}{6} t^3 = b_0 + \frac{t^3}{2} \Rightarrow \boxed{b_0 = -\frac{t^3}{6}}$$

$$g(x,t) = \begin{cases} \frac{x^2 t}{2} - \frac{1}{6} x^3 ; & 0 \le x < t \\ \frac{x t^2}{2} - \frac{1}{6} t^3 ; & t < x \le 1. \end{cases}$$

$$y(x) = \int_0^1 g(x,t) f(t)\, dt$$

$$= \int_0^x \left(\frac{x t^2}{2} - \frac{1}{6} t^3 \right) dt + \int_x^1 \left(\frac{x^2 t}{2} - \frac{x^3}{6} \right) dt$$

$$= \left[\frac{x t^3}{6} - \frac{t^4}{24} \right]_0^x + \left[\frac{x^2 t^2}{4} - \frac{x^3 t}{6} \right]_x^1$$

$$= x^4 \left(\frac{1}{6} - \frac{1}{24} \right) + \left(\frac{x^2}{4} - \frac{x^3}{6} - \frac{x^4}{4} + \frac{x^4}{6} \right)$$

$$= \frac{x^2}{4} - \frac{x^3}{6} + x^4 \left(\frac{2}{6} - \frac{1}{4} - \frac{1}{24} \right)$$

$$= \frac{x^2}{4} - \frac{x^3}{6} + x^4 \cdot \frac{8 - 6 - 1}{24}$$

$$= \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{24}.$$

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Reduction of BVP to an integral equation using Green's fn.

Consider the BVP

$$Ly = \lambda r(x)\, y + f(x) \; ; \quad a \le x \le b \longrightarrow (1)$$

$$B_j y = 0, \quad j = 1, 2, \ldots, n \longrightarrow (2)$$

$L \to$ differential operator of the form

$$Ly = p_0 \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y$$

λ is some real parameter.

$f(x)$ is a continuous funct. of x in $[a, b]$.

Then the BVP given in (1) & (2) can be reduced to the following integral equation

$$y(x) = \lambda \int_a^b g(x,t)\, r(t)\, y(t)\, dt + \int_a^b g(x,t)\, f(t)\, dt.$$

Example: Reduce the BVP

$$y'' + xy = 1, \quad y(0) = 0 = y(1) \quad \rightarrow 0 \le x \le 1$$

to an IE using Green's funct.

Solut.: The ODE is,

$$y'' = -xy + 1. \quad 0 \le x \le 1$$

Compare with $Ly = \lambda r(x) y + f(x)$.

Then, $L \equiv \dfrac{d^2}{dx^2}$, $\lambda = -1$, $r(x) = x$; $f(x) = 1$.

The Green's funct^n $g(x,t)$ will satisfy

$$L\,g(x,t) = \delta(x-t).$$

Solving $L\,g(x,t)=0$ i.e. $\frac{d^2}{dx^2}g(x,t)=0$, get

$$g(x,t) = \begin{cases} a_1 + a_2 x; & 0 \le x < t \\ b_1 + b_2 x; & t < x \le 1 \end{cases} \quad (1)$$

g satisfies, $g(0,t) = 0 = g(1,t)$,

$$g(t+0,t) = g(t-0,t)$$

$$\frac{\partial g}{\partial x}(t+0,t) - \frac{\partial g}{\partial x}(t-0,t) = -\frac{1}{p(t)}.$$

Writing $\frac{d^2}{dx^2}$ as, $-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)$, we find $p(x) = -1$.

$$\therefore \quad \frac{\partial g}{\partial x}(t+0,t) - \frac{\partial g}{\partial x}(t-0,t) = 1.$$

$g(0,t)=0 \Rightarrow a_1 = 0.$

$g(1,t)=0 \Rightarrow b_1 + b_2 = 0.$

$$\therefore \; g(x,t) = \begin{cases} a_2 x & ; \; 0 \le x < t \\ b_1 - b_1 x & ; \; t < x \le 1 \end{cases} \quad (2)$$

$$g(t+0,t) = g(t-0,t)$$

$$\Rightarrow \; b_1 - b_1 t = a_2 t \;\Rightarrow\; b_1(1-t) = a_2 t. \quad (3)$$

$$g_x(t+0,t) - g_x(t-0,t) = 1$$

$$\Rightarrow \; -b_1 - a_2 = 1 \longrightarrow (4)$$

From (3), $b_1 = a_2 \cdot \frac{t}{1-t}.$

Substituting (4) into (3) we get,

$$-\frac{a_2 t}{1-t} - a_2 = 1$$

or, $-a_2(t + 1 - t) = 1 - t$

$$\Rightarrow a_2 = t - 1$$

$$b_1 = \frac{a_2 t}{1-t} = \frac{(t-1)t}{1-t} = -t.$$

So, from (2),

$$g(x,t) = \begin{cases} (t-1)x & ; \; 0 \le x < t \\ (x-1)t & ; \; t < x \le 1. \end{cases}$$

The IE is of the form,

$$y(x) = \lambda \int_a^b g(x,t)\, r(t)\, y(t)\, dt + \int_a^b f(t)\, g(x,t)\, dt$$

$$= -\int_0^1 g(x,t).\, t\, y(t)\, dt + \int_0^1 g(x,t)\, dt$$

$$= -\int_0^1 t\, g(x,t)\, y(t)\, dt + \int_0^x (x-1)\, t\, dt + \int_x^1 (t-1)\, x\, dt$$

$$= (x-1)\left[\frac{t^2}{2}\right]_0^x + x\left[\frac{(t-1)^2}{2}\right]_x^1 - \int_0^1 t\, g(x,t)\, y(t)\, dt.$$

$$\therefore\; y(x) = \frac{x^2(x-1)}{2} - \frac{x(x-1)^2}{2} - \int_0^1 t\, g(x,t)\, y(t)\, dt.$$

$$y(x) = \frac{x(x-1)}{2} - \int_0^1 \underbrace{t\, g(x,t)}_{K(x,t)}\, y(t)\, dt.$$