# Method 2.: Method of Successive Substitution

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\,u(t)\,dt \longrightarrow (1)$$

Here RHS of $u(x)$ is substituted for $u(t)$ inside the integral repeatedly.

**Step-1**

$$u(x) = f(x) + \lambda \int_a^b K(x,t) \left\{ f(t) + \lambda \int_a^b K(t,t_1)\,u(t_1)\,dt_1 \right\} dt$$

$$= f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1)\,u(t_1)\,dt_1\,dt$$

**Step-2** Again $u(t_1)$ shall be replaced by its expression as given in the RHS of (1). This gives,

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t)\,K(t,t_1) \left\{ f(t_1) + \lambda \int_a^b K(t_1,t_2)\,u(t_2)\,dt_2 \right\} dt_1\,dt$$

$$\therefore u(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1)\,dt_1\,dt + \lambda^3 \int_a^b \int_a^b \int_a^b K(x,t)\,K(t,t_1)\,K(t_1,t_2)\,u(t_2)\,dt_2\,dt_1\,dt$$

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Step n $u(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1)\,dt_1\,dt$

$+ \cdots + \lambda^n \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{n \text{ times}} K(x,t) K(t,t_1) \cdots K(t_{n-2},t_{n-1}) f(t_{n-1})\,dt_{n-1}\,dt_{n-2} \cdots dt_1\,dt$

$+ \lambda^{n+1} \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{(n+1) \text{ times}} K(x,t) K(t,t_1) \cdots K(t_{n-1},t_n)\, u(t_n)\,dt_n\,dt_{n-1} \cdots dt_1\,dt.$

Reminder after $(n+1)$ terms denoted as $R_{n+1}$

If $|\lambda| M(b-a) < 1$ ; $M = \sup\{K(x,t) : a \le x,t \le b\}$,

then $R_{n+1} \longrightarrow 0$.

In that case the solution is given by

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1)\, dt_1 dt$$

$$+ \lambda^3 \int_a^b \int_a^b \int_a^b K(x,t) K(t,t_1) K(t_1,t_2) f(t_2)\, dt_2 dt_1 dt$$

$$+ \cdots\cdots + \lambda^n \underbrace{\int_a^b \int_a^b \cdots\cdots \int_a^b}_{n \text{ times}} K(x,t) K(t,t_1) \cdots\cdots K(t_{n-2}, t_{n-1}) f(t_{n-1})\, dt_{n-1} dt_{n-2} \cdots dt_1 dt.$$

$$+ \cdots\cdots$$

Ex-1 Solve $u(x) = 1 + \frac{1}{2} \int_0^{\pi/4} \sec^2 x \, u(t)\, dt.$

by method of successive substitution.

Note: $f(x) = 1$, $\lambda = \frac{1}{2}$, $K(x,t) = \sec^2 x$.

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The solution to the IE is given by,

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1)\,dt_1\,dt + \cdots$$

$f(x) = 1$, $K(x,t) = \sec^2 x$, $\lambda = \frac{1}{2}$; $a = 0$, $b = \frac{\pi}{4}$.

$$\therefore u(x) = 1 + \frac{1}{2}\int_0^{\pi/4} \sec^2 x\,dt + \left(\frac{1}{2}\right)^2 \int_0^{\pi/4}\int_0^{\pi/4} \sec^2 x \sec^2 t\,dt_1\,dt$$

$$+ \left(\frac{1}{2}\right)^3 \int_0^{\pi/4}\int_0^{\pi/4}\int_0^{\pi/4} \sec^2 x \sec^2 t \sec^2 t_1\,dt_2\,dt_1\,dt + \cdots$$

$$= \left(\frac{1}{2}\right)^2 \left(\int_0^{\pi/4} \sec^2 x\,dt_1\right)\left(\int_0^{\pi/4} \sec^2 t\,dt\right)$$

$$= 1 + \frac{1}{2} \cdot \sec^2 x \cdot \frac{\pi}{4} + \left(\frac{1}{2}\right)^2 \sec^2 x \cdot \frac{\pi}{4} \cdot 1 + \left(\frac{1}{2}\right)^3 \sec^2 x \cdot \frac{\pi}{4} \cdot 1 \cdot 1 + \cdots$$

$$= 1 + \frac{\pi}{4}(\sec^2 x)\left(\frac{1}{2}\right)\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$= 1 + \frac{\pi}{8}\sec^2 x \cdot \frac{1}{1-\frac{1}{2}}$$

$$= 1 + \frac{\pi}{4}\sec^2 x$$

Ex-2 Solve $u(x) = \frac{7}{8}x^2 + \frac{1}{2}\int_0^1 x^2 t\, u(t)\, dt$.

by method of successive substitution.

$$u(x) = \frac{7}{8}x^2 + \frac{1}{2}\int_0^1 x^2 t \cdot \frac{7}{8}t^2\, dt + \left(\frac{1}{2}\right)^2 \int_0^1\int_0^1 x^2 t \cdot t^2 t_1 \cdot \frac{7}{8} t_1^2\, dt_1\, dt$$

$$+ \left(\frac{1}{2}\right)^3 \int_0^1\int_0^1\int_0^1 x^2 t \cdot t^2 t_1 \cdot t_1^2 t_2 \cdot \frac{7}{8} t_2^2\, dt_2\, dt_1\, dt$$

$$= \frac{7}{8}x^2 + \frac{1}{2}\cdot\frac{7}{8}x^2 \int_0^1 t^3\, dt + \left(\frac{1}{2}\right)^2 \cdot \frac{7}{8}x^2 \left(\int_0^1 t^3\, dt\right)\left(\int_0^1 t_1^3\, dt_1\right)$$

$$+ \left(\frac{1}{2}\right)^3 \cdot \frac{7}{8}x^2 \left(\int_0^1 t^3\, dt\right)\left(\int_0^1 t_1^3\, dt_1\right)\left(\int_0^1 t_2^3\, dt_2\right)$$

$$= \frac{7}{8}x^2\left[1 + \frac{1}{2}\cdot\frac{1}{4} + \left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^3\left(\frac{1}{4}\right)^3 + \cdots\right] = \frac{7}{8}x^2 \cdot \frac{1}{1-\frac{1}{8}}$$

$$= x^2.$$

**Theorem** Consider the Fredholm Integral Equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt \longrightarrow (1).$$

If a) $K(x,t)$ is real and continuous in $R : \{(x,t) : a \le x, t \le b\}$ such that $|K(x,t)| \le M \;\forall (x,t) \in R$.

b) $f(x) \not\equiv 0$ in $[a,b]$ & is real & continuous in $[a,b]$.

c) $\lambda$ is a constant such that $|\lambda| < \dfrac{1}{M(b-a)}$

Then method of successive substitution yields one and only one continuous solution in $[a,b]$ for the IE (1).

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<u>Proof.</u> Substitute for $u(t)$ its expression given in R.H.S. of (1).

$$u(x) \overset{1\text{st step}}{=} f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) u(t_1)\,dt_1\,dt$$

$$\overset{2\text{nd step}}{=} f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1)\,dt_1\,dt + \lambda^3 \int_a^b \int_a^b \int_a^b K(x,t) K(t,t_1) K(t_1,t_2) u(t_2)\,dt_2\,dt_1\,dt$$

$$\vdots$$

$$\overset{(n-1)\text{th step}}{=} f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1)\,dt_1\,dt + \cdots + \lambda^{n-1} \int_a^b \int_a^b \cdots \int_a^b K(x,t) K(t,t_1) \cdots K(t_{n-3}, t_{n-2}) f(t_{n-2})\,dt_{n-2}\,dt_{n-1} \cdots dt_1\,dt$$

$$+ \lambda^n \int_a^b \int_a^b \cdots \int_a^b K(x,t) K(t,t_1) \cdots K(t_{n-2}, t_{n-1})\, u(t_{n-1})\,dt_{n-1}\,dt_{n-2} \cdots dt_1\,dt.$$

Let the last term containing $u$, i.e the remainder after $n$ terms be denoted by $R_n$.

If $u(x)$ is assumed to be continuous in $[a,b]$, then $u(x)$ is bounded there. So, $|u(x)| \le N$, say $\forall x \in [a,b]$.

Then $|R_n| = \left| \lambda^n \int_a^b \int_a^b \cdots \int_a^b K(x,t) K(t,t_1) \ldots K(t_{n-2}, t_{n-1}) u(t_{n-1}) \, dt_{n-1} dt_{n-2} \cdots dt_1 dt \right|$

$\le |\lambda|^n \int_a^b \int_a^b \cdots \int_a^b |K(x,t)| |K(t,t_1)| \cdots |K(t_{n-2}, t_{n-1})| . |u(t_{n-1})| \, dt_{n-1} dt_{n-2} \cdots dt_1 dt$

$\le |\lambda|^n M^n N \int_a^b dt \int_a^b dt_1 \cdots \int_a^b dt_{n-1} = |\lambda|^n (b-a)^n M^n . N$

$\because |\lambda| < \frac{1}{M(b-a)}$ $\therefore \underbrace{|\lambda|^n (b-a)^n M^n}$ $N \longrightarrow 0$ as $n \longrightarrow \infty$

This shows that $u(x)$ is given by

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t_1) f(t_1)\,dt_1\,dt + \cdots \longrightarrow (*)$$

$(*)$ represents an infinite series in which every term is continuous in $[a,b]$, since $f(x)$ is cont. in $[a,b]$ & $K(x,t)$ is cont. in $R = \{(x,t) : a \le x, t \le b\}$

This series represents a continuous function in $[a,b]$, provided it converges uniformly in $[a,b]$.

$- \left( \int_a f_1(x) \ldots \right) \ldots \left( \int_a^b f_3(u)\,du \right)$