Case 1.

$$I[z(x_1, x_2, \ldots, x_n)]$$

$$= \iint_{D_n} \cdots \int f(x_1, x_2, \ldots, x_n, z, z_{x_1}, z_{x_2}, \ldots, z_{x_n}) \, dx_1 \, dx_2 \ldots dx_n .$$

$$D_n \subset \mathbb{R}^n .$$

Then a nec. condition that
$z = z(x_1, \ldots, x_n)$ will extremize. I is that.

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial p_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial p_2}\right) - \cdots - \frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial p_n}\right) = 0 .$$

where $p_i = \frac{\partial z}{\partial x_i}$; $i = 1, 2, \ldots, n$.

Case 2. $$I[z(x, y)] = \iint_D f(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) \, dx \, dy .$$

Then a nec. condition that $z = z(x, y)$ will extremize I is that

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial r}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial f}{\partial s}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial t}\right) = 0 .$$

$r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.

Ex1. Obtain the Euler–Ostrogradsky eq. for,

$$I[z(x,y)] = \iint_D \left\{ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2z\,h(x,y) \right\} dx\,dy.$$

$$f(x,y,z,p,q) = p^2 + q^2 + 2z\,h(x,y).$$

E-O. eq. is, $f_z - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q}\right) = 0.$

or, $2h(x,y) - \frac{\partial}{\partial x}(2p) - \frac{\partial}{\partial y}(2q) = 0.$

or, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = h(x,y)$

or, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = h(x,y) \rightarrow$ Poisson's eqn.

Ex2. $I[z(x,y)] = \iint_D \left\{ \left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \right\} dx\,dy.$

$f = r^2 + t^2 + 2s^2.$

A nec. condition that $z = z(x,y)$ will extremize the functional I is that.

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial t}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial f}{\partial s}\right) = 0.$$

or, $0 - 0 - 0 + \frac{\partial^2}{\partial x^2}(2r) + \frac{\partial^2}{\partial y^2}(2t) + \frac{\partial^2}{\partial x \partial y}(4s) = 0.$

or, $\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 z}{\partial x^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 z}{\partial y^2}\right) + 2\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 z}{\partial x \partial y}\right) = 0.$

or, $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} = 0$

or, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 z = 0$

or, $\nabla^4 z = 0 \longrightarrow$ biharmonic equation