Example 4·1. Find the characteristic constants and the ~~functions~~ ch. funct$^{\underline{n}}$ of the IE

$$u(x) = \lambda \int_0^{\pi} (\cos^2 x \cos 2t + \cos 3x \cos^3 t)\, u(t)\, dt. \longrightarrow (1)$$

or, $u(x) = \lambda \cos^2 x \int_0^{\pi} \cos 2t\, u(t)\, dt + \lambda \cos 3x \int_0^{\pi} \cos^3 t\, u(t)\, dt$

or, $u(x) = \lambda \cos^2 x\, A + \lambda \cos 3x \cdot B. \longrightarrow (2)$

where, $A = \int_0^{\pi} \cos 2t\, u(t)\, dt \longrightarrow (3)$, $B = \int_0^{\pi} \cos^3 t\, u(t)\, dt. \longrightarrow (4)$.

Substitute (2) into (3) & get,

$$A = \int_0^{\pi} \cos 2t \left\{ \lambda \cos^2 t\, A + \lambda \cos 3t\, B \right\} dt$$

or, $\left\{ 1 - \lambda \int_0^{\pi} \cos 2t \cos^2 t\, dt \right\} A - \lambda \left(\int_0^{\pi} \cos 2t \cos 3t \right) B = 0$

Substitute (2) into (4) & get,

$$B = \int_0^{\pi} \cos^3 t \left\{ \lambda \cos^2 t\, A + \lambda \cos 3t\, B \right\} dt$$

or, $-\lambda \int_0^{\pi} \cos^3 t \cos^2 t\, dt \cdot A + \left(1 - \lambda \int_0^{\pi} \cos^3 t \cos 3t\, dt \right) B = 0.$

Thus we have,

$$\left. \begin{aligned} a_{11} A + a_{12} B &= 0 \\ a_{21} A + a_{22} B &= 0. \end{aligned} \right\} \longrightarrow (5)$$

Thus (A, B) will have a non-trivial solution, if and only if,
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.$$

Now,
$$a_{11} = 1 - \lambda \int_0^{\pi} \cos 2t \cos^2 t \, dt$$
$$= 1 - \frac{\lambda}{2} \int_0^{\pi} \cos 2t \,(1 + \cos 2t)\, dt$$
$$= 1 - \frac{\lambda}{2} \int_0^{\pi} \cos 2t \, dt - \frac{\lambda}{4} \int_0^{\pi} (1 + \cos 4t)\, dt.$$
$$= 1 - \frac{\lambda \pi}{4}.$$

$$a_{12} = -\lambda \int_0^{\pi} \cos 2t \cos 3t \, dt$$
$$= -\frac{\lambda}{2} \int_0^{\pi} (\cos t + \cos 5t)\, dt = 0.$$

$$a_{21} = -\lambda \int_0^{\pi} \cos^3 t \cos^2 t \, dt.$$
$$= -\lambda \int_0^{\pi} (1 - \sin^2 t)^2 . \cos t \, dt.$$
$$= -\lambda \left[\int_0^{\pi} (1 - 2\sin^2 t + \sin^4 t) \cos t \, dt \right]$$
$$= -\lambda \left\{ \left[\sin t\right]_0^{\pi} - \frac{2}{3}\left[\sin^3 t\right]_0^{\pi} + \frac{1}{5}\left[\sin^5 t\right]_0^{\pi} \right\}$$
$$= 0.$$

$$a_{22} = 1 - \lambda \int_0^{\pi} \cos^3 t \cos 3t \, dt \; ; \quad \cos^3 t = \frac{1}{4}(\cos 3t + 3\cos t)$$

$$= 1 - \frac{\lambda}{4} \int_0^{\pi} (\cos 3t + 3\cos t)\, dt \times \cos 3t$$

$$= 1 - \frac{\lambda}{4} \int_0^{\pi} \cos^2 3t \, dt - \frac{3\lambda}{4} \int_0^{\pi} \cos t \cos 3t \, dt$$

$$= 1 - \frac{\lambda}{8} \int_0^{\pi} (1 + \cos 6t)\, dt - \frac{3\lambda}{8} \int_0^{\pi} \{\cos(4t) + \cos(2t)\}\, dt$$

$$= 1 - \frac{\lambda\pi}{8}.$$

Thus,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 - \frac{\lambda\pi}{4} & 0 \\ 0 & 1 - \frac{\lambda\pi}{8} \end{vmatrix} = 0$$

or, $\left(1 - \frac{\lambda\pi}{4}\right)\left(1 - \frac{\lambda\pi}{8}\right) = 0 \quad \therefore \lambda = \frac{4}{\pi}, \frac{8}{\pi}.$

<u>Case - I</u>: $\lambda = \frac{4}{\pi}$.

$$u(x) = \lambda \cos^2 x \, A + \lambda \cos 3x \, B.$$

From (5), $a_{11} A + a_{12} B = 0.$

$$a_{21} A + a_{22} B = 0$$

Taking ~~the 2nd~~ eq., $a_{21} A + \cancel{a_{22}} B = 0.$

or, $\overset{0}{\cancel{\left(1 - \frac{\lambda\pi}{4}\right)}} A + \cancel{0 . B = 0}.$

Since now $\lambda = \frac{4}{\pi}$, $\therefore$ 0. A i.e $\left(1-\frac{\lambda\pi}{4}\right)A + 0 \cdot B = 0.$

$0 \cdot A + \left(1-\frac{\lambda\pi}{8}\right)B = 0.$

or, $\left(1-\frac{\lambda\pi}{4}\right)A = 0$. and $\left(1-\frac{\lambda\pi}{8}\right) \cdot B = 0$. $\therefore \lambda = \frac{4}{\pi}$

$\Rightarrow \frac{1}{2} B = 0 \Rightarrow B = 0$

$\because \lambda = \frac{4}{\pi}$ for this case, so,

$\left(1-\frac{\lambda\pi}{4}\right)A = 0 \Rightarrow 0 \cdot A = 0 \Rightarrow A$ is arbitrary $\neq 0$

$\therefore$ from (2), $u(x) = \frac{4A}{\pi}\cos^2 x = A_1 \cos^2 x \rightarrow$ e-functions corr. to $\lambda = \frac{4}{\pi}$.

Case II : $\lambda = \frac{8}{\pi}$.

From (5), $\left(1-\frac{\lambda\pi}{4}\right)A + 0 \cdot B = 0.$

$0 \cdot A + \left(1-\frac{\lambda\pi}{8}\right)B = 0,$

or, $\cancel{-\frac{\lambda\pi}{8}A = 0}$, $-A = 0 \Rightarrow A = 0$, $0 \cdot B = 0$.

$\Rightarrow A = 0$ & B arbitrary $\neq 0$

$\therefore$ from (2), $u(x) = \frac{8B}{\pi}\cos 3x = B_1 \cos 3x.$

$\rightarrow$ e-functions corr. to $\lambda = \frac{8}{\pi}$.

Exercise: Find e-value(s) & e-functions for the IE : $u(x) = \lambda \int_0^1 x^2 t\, u(t)\, dt$.

Ans: $\lambda = 4$, $u(x) = A_1 x^2$; $A_1 = 4A$.