$\therefore \ \frac{\partial g}{\partial x}(t+0, t) - \frac{\partial g}{\partial x}(t-0, t)$

$= t - (t-1) = t - t + 1 = 1.$

This can be expressed as,

$$\frac{\partial g}{\partial x}(t+0, t) - \frac{\partial g}{\partial x}(t-0, t) = -\frac{1}{p(t)}.$$

Self-adjoint ordinary differential equations of order 2

Consider

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = r(x) \longrightarrow (1).$$

$a \le x \le b.$

a_0, a_1, a_2, r are known continuous fns. in $[a,b]$.

The corresponding adjoint equation is,

$$a_0(x)\frac{d^2y}{dx^2} + (2a_0' - a_1)\frac{dy}{dx} + (a_0' - a_1 + a_2)y = r(x) \longrightarrow (2).$$

A necessary and sufficient condition for a differential equation to be self-adjoint is that, $\boxed{\frac{da_0}{dx} = a_1(x)}$; $a \le x \le b.$

In the diff. equation,

$$-\left(p(x)\frac{dy}{dx}\right)' + q(x)y = r(x) \longrightarrow (3)$$

or, $-p(x)\frac{d^2y}{dx^2} - p'(x)\frac{dy}{dx} + q(x)y = r(x)$

$$\frac{da_0}{dx} = \frac{d}{dx}\{-p(x)\} = -p'(x) = a_1(x)$$

Thus (3) is self-adjoint.

E.L. Ince – ODE

Finding Green's function for a self-adjoint differential equation with homogeneous boundary condition:

Consider the self-adjoint DE

$$-\left(p(x)\frac{dy}{dx}\right)' + q(x)\,y = f(x)\,;\quad a \le x \le b \longrightarrow (1).$$

with $B_1 y = 0,\quad B_2 y = 0.$

[B.C.'s may look like:

$\left.\begin{aligned} y(a)&=0\\ y(b)&=0\end{aligned}\right\}$ or, $\left.\begin{aligned} y'(a)&=0\\ y'(b)&=0\end{aligned}\right\}$ or $\left.\begin{aligned} y(a)&=0\\ y'(b)&=0\end{aligned}\right\}$ $\left.\begin{aligned} \mu_1 y(a) + \lambda_1 y'(b) &= 0\\ \mu_2 y(a) + \lambda_2 y'(b) &= 0\end{aligned}\right\}$]

Equation (1) may be expressed as,

$$Ly = f(x)\,;\quad a \le x \le b. \longrightarrow (1a)$$

where the operator L is self-adjoint.

The solution to (1a) is given by,

$$y = L^{-1} f(x) = \int_a^b g(x,t)\, f(t)\, dt.$$

- <u>How to find $g(x,t)$?</u>

First get $$g(x,t) = \begin{cases} A_1(t)\, y_1(x) + A_2(t)\, y_2(x)\,; & x < t\\ B_1(t)\, y_1(x) + B_2(t)\, y_2(x) & x > t.\end{cases}$$

by solving $L\,g(x,t) = \delta(x-t)$

[In practice one has to solve $L g(x,t) = 0$].

The 4 unknowns A_1, A_2, B_1, B_2 are determined from.

1. $g(t+0,t) = g(t-0,t)$ (continuity of g at $x=t$)

2. $\frac{\partial g}{\partial x}(t+0,t) - \frac{\partial g}{\partial x}(t-0,t) = -\frac{1}{p(t)}$ (jump of g' at $x=t$)

3. $B_1 g = 0$.

4. $B_2 g = 0$.

<u>Example</u>: Find the Green's funct $g(x,t)$ corresponding to the BVP:
$$-\frac{d^2u}{dx^2} = \lambda u\,; \quad 0 \le x \le l\,;$$
$$u'(0) = 0 = u'(l)$$

<u>Solution</u> The given DE may be expressed as,

$$Lu = 0.$$

where $L = -\frac{d^2}{dx^2} - \lambda$

$g(x,t)$ satisfies $Lg(x,t) = \delta(x-t)$.

We need to solve, $-\frac{d^2g}{dx^2} - \lambda g = 0$.

or, $\frac{d^2g}{dx^2} + \lambda g = 0. \longrightarrow (1)$

auxiliary eq: $m^2 + \lambda = 0 \Rightarrow m = \pm\sqrt{\lambda}\, i$

Thus the two linearly independent solutions to (1) are $\cos\sqrt{\lambda}\,x$ and $\sin\sqrt{\lambda}\,x$.

Thus, $g(x,t)$ can be expressed as,
$$g(x,t) = \begin{cases} A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x\,; & 0 \le x < t \\ C\cos\sqrt{\lambda}\,x + D\sin\sqrt{\lambda}\,x & t < x \le l \end{cases}$$

To find A, B, C, D :

1. $g'(0,t)=0$ 2) $g'(l,t)=0$.

3. $g(t+0,t) = g(t-0,t)$.

4. $\frac{\partial g}{\partial x}(t+0,t) - \frac{\partial g}{\partial x}(t-0,t) = -\frac{1}{p(t)}$

$g'(0,t)=0$.

$$\Rightarrow \sqrt{\lambda}\left(-A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x\right)_{x=0} = 0$$

or, $\sqrt{\lambda}.B=0 \Rightarrow B=0$.

$g'(l,t)=0$

$$\Rightarrow \sqrt{\lambda}\left(-C\sin\sqrt{\lambda}x + D\cos\sqrt{\lambda}x\right)_{x=l} = 0$$

or, $-C\sin\sqrt{\lambda}l + D\cos\sqrt{\lambda}l = 0$.

$$\therefore \quad D = \frac{C\sin\sqrt{\lambda}l}{\cos\sqrt{\lambda}l}.$$

$$\therefore g(x,t) = \begin{cases} A\cos\sqrt{\lambda}x; & 0 \le x < t \\ C\left[\cos\sqrt{\lambda}x + \frac{\sin\sqrt{\lambda}l}{\cos\sqrt{\lambda}l}\sin\sqrt{\lambda}x\right] & \\ = \frac{C}{\cos\sqrt{\lambda}l}\cos\sqrt{\lambda}(l-x); & t \le x \le l. \end{cases}$$

(*)

$g(t+0,t) = g(t-0,t)$ gives,

$$\frac{C}{\cos\sqrt{\lambda}l}.\cos\sqrt{\lambda}(l-t) = A\cos\sqrt{\lambda}t. \longrightarrow (2)$$

Also, $\frac{\partial g}{\partial x}(t+0,t) - \frac{\partial g}{\partial x}(t-0,t) = -\frac{1}{p(t)}$

where $p(x)$ is the coeff. of $y'(x)$ in the DE

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + qy = f(x)$$

Comparing the given DE $-\frac{d^2u}{dx^2} = \lambda u$

(which can be re-written as $-\frac{d}{dx}\left(1 \cdot \frac{du}{dx}\right) - \lambda u = 0$)

with the ~~sta~~ above standard form, we see,

here $p(x) = 1$.

$$\therefore \frac{\partial g}{\partial x}(t+0, t) - \frac{\partial g}{\partial x}(t-0, t) = -\frac{1}{p(t)}$$

or, $\sqrt{\lambda}\left(-C\sin\sqrt{\lambda}\,t + D\cos\sqrt{\lambda}\,t\right)$

$$-\sqrt{\lambda}\left(-A\sin\sqrt{\lambda}\,t + B\cos\sqrt{\lambda}\,t\right) = -1$$

or, $\sqrt{\lambda}\,C\left(\frac{\sin\sqrt{\lambda}\,l}{\cos\sqrt{\lambda}\,l} \cdot \cos\sqrt{\lambda}\,t - \sin\sqrt{\lambda}\,t\right) + \sqrt{\lambda}\,A\sin\sqrt{\lambda}\,t = -1$

or, $\sqrt{\lambda}\,C\,\frac{\sin\sqrt{\lambda}(l-t)}{\cos\sqrt{\lambda}\,l} + \sqrt{\lambda}\,A\sin\sqrt{\lambda}\,t = -1 \longrightarrow (3)$

Now from (2), $\frac{C}{\cos\sqrt{\lambda}\,l}\cos\sqrt{\lambda}(l-t) = A\cos\sqrt{\lambda}\,t$

So that $A = C \cdot \frac{\cos\sqrt{\lambda}(l-t)}{\cos\sqrt{\lambda}\,l\,\cos\sqrt{\lambda}\,t} \longrightarrow (4)$

Substituting A from (4) into (3), we have

$$C\left[\frac{\sin\sqrt{\lambda}(l-t)}{\cos\sqrt{\lambda}\,l} + \frac{\cos\sqrt{\lambda}(l-t)\sin\sqrt{\lambda}\,t}{\cos\sqrt{\lambda}\,l\,\cos\sqrt{\lambda}\,t}\right] = -\frac{1}{\sqrt{\lambda}}$$

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or, $$\frac{C}{\cos\sqrt{\lambda}\,l} \cdot \frac{\sin\sqrt{\lambda}\,(l-t+t)}{\cos\sqrt{\lambda}\,t} = -\frac{1}{\sqrt{\lambda}}.$$

$$\therefore\quad C = -\frac{1}{\sqrt{\lambda}} \cdot \frac{\cos\sqrt{\lambda}\,t\,\cos\sqrt{\lambda}\,l}{\sin\sqrt{\lambda}\,l}$$

$$A = C \cdot \frac{\cos\sqrt{\lambda}\,(l-t)}{\cos\sqrt{\lambda}\,l\,\cos\sqrt{\lambda}\,t}$$

$$= -\frac{1}{\sqrt{\lambda}} \cdot \frac{\cos\sqrt{\lambda}\,t\,\cos\sqrt{\lambda}\,l}{\sin\sqrt{\lambda}\,l} \cdot \frac{\cos\sqrt{\lambda}\,(l-t)}{\cos\sqrt{\lambda}\,l\,\cos\sqrt{\lambda}\,t}$$

$$\therefore\quad A = -\frac{1}{\sqrt{\lambda}} \cdot \frac{\cos\sqrt{\lambda}\,(l-t)}{\sin\sqrt{\lambda}\,l}.$$

Thus, from ( )

$$g(x,t) = \begin{cases} -\dfrac{1}{\sqrt{\lambda}} \cdot \dfrac{\cos\sqrt{\lambda}\,(l-t)\,\cos\sqrt{\lambda}\,x}{\sin\sqrt{\lambda}\,l}\,; & 0 \le x < t \\[2ex] -\dfrac{1}{\sqrt{\lambda}}\,\dfrac{\cos\sqrt{\lambda}\,(l-x).\,\cos\sqrt{\lambda}\,t}{\sin\sqrt{\lambda}\,l}\,; & t < x \le l. \end{cases}$$