<u>Example</u> Solve the following BVP using Green's function technique:

$$u'' - u = x; \quad u(0) = 0 = u(1).$$

$Lu = x$

<u>Solution</u>: $L = \frac{d^2}{dx^2} - 1 = -\frac{d}{dx}\left(-1 \cdot \frac{d}{dx}\right) - 1.$

Comparing with $L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$,

we find, $\quad p(x) = -1, \quad q(x) = -1.$

To solve, $\quad Lg = 0.$

subject to, $\quad g(0,t) = 0, \quad g(1,t) = 0.$

$$g(t+0,t) = g(t-0,t); \quad \frac{\partial g}{\partial x}(t+0,t) - \frac{\partial g}{\partial x}(t-0,t) = -\frac{1}{p(t)}.$$

$Lg = 0 \Rightarrow g'' - g = 0$, a.e: $m^2 - 1 = 0 \Rightarrow m = \pm 1.$

$$g(x,t) = \begin{cases} Ae^{x} + Be^{-x} & ; \ 0 \le x < t \\ Ce^{x} + De^{-x} & ; \ t < x \le 1 \end{cases} \longrightarrow (1)$$

From the boundary conditions,

$g(0,t) = 0 \Rightarrow A + B = 0$

$g(1,t) = 0 \Rightarrow Ce + De^{-1} = 0.$

$$\therefore \ g(x,t) = \begin{cases} A(e^{x} - e^{-x}) & ; \ 0 \le x < t \\ C(e^{x} - e^{2}.e^{-x}) & ; \ t < x \le 1 \end{cases} \longrightarrow (2)$$

We know, $g(t+0, t) = g(t-0, t)$.

$$C\left(e^t - e^{2-t}\right) = A\left(e^t - e^{-t}\right).$$

or, $$A = \frac{C \cdot e\left(e^{t-1} - e^{1-t}\right)}{e^t - e^{-t}}. \longrightarrow (3)$$

Also, $$\frac{\partial g}{\partial x}(t+0, t) - \frac{\partial g}{\partial x}(t-0, t) = -\frac{1}{p(t)}.$$

$$C\left(e^t + e^2 . e^{-t}\right) - A\left(e^t + e^{-t}\right) = 1$$

or, using (3),

$$Ce\left(e^{t-1} + e^{1-t}\right) - \frac{Ce\left(e^{t-1} - e^{1-t}\right)}{e^t - e^{-t}} . \left(e^t + e^{-t}\right) = 1$$

or, $$Ce\left\{\left(e^{t-1} + e^{1-t}\right)\left(e^t - e^{-t}\right) - \left(e^{t-1} - e^{1-t}\right)\left(e^t + e^{-t}\right)\right\} = e^t - e^{-t}.$$

or, $$Ce\left\{\cancel{e^{2t-1}} + e - e^{-1} - \cancel{e^{1-2t}} - \cancel{e^{2t-1}} + e - e^{-1} + \cancel{e^{1-2t}}\right\} = e^t - e^{-t}$$

or, $$2Ce\left(e - e^{-1}\right) = e^t - e^{-t}$$

or, $$C = \frac{e^t - e^{-t}}{2e\left(e - e^{-1}\right)} \longrightarrow (4)$$

Using (4) in (3),

$$A = \frac{\cancel{\left(e^t - e^{-t}\right)}}{2\cancel{e}\left(e - e^{-1}\right)} . \cancel{e} . \frac{e^{t-1} - e^{1-t}}{\cancel{\left(e^t - e^{-t}\right)}} = \frac{e^{t-1} - e^{1-t}}{2\left(e - e^{-1}\right)}.$$

$\longrightarrow (5)$.

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Substituting (4) & (5) into (2),

$$g(x,t) = \begin{cases} \dfrac{\left(e^{t-1} - e^{1-t}\right)}{2\left(e - e^{-1}\right)} \cdot \left(e^{x} - e^{-x}\right) ; & 0 \le x < t \\[2ex] \dfrac{e^{t} - e^{-t}}{2e\left(e - e^{-1}\right)} \cdot \left(e^{x} - e^{2} \cdot e^{-x}\right) ; & t < x \le 1. \end{cases}$$

Or, $$g(x,t) = \begin{cases} \dfrac{\sinh x \ \sinh(t-1)}{\sinh 1} ; & 0 \le x < t \\[2ex] \dfrac{\sinh t \cdot \sinh(x-1)}{\sinh 1} ; & t < x \le 1. \end{cases}$$

The solution to the given BVP is,

$$u(x) = \int_0^1 g(x,t)\, f(t)\, dt$$

$$= \int_0^x \frac{\sinh t \cdot \sinh(x-1)}{\sinh 1}\, t\, dt + \int_x^1 \frac{\sinh x \ \sinh(t-1)}{\sinh 1}\, t\, dt$$

$$= \frac{\sinh(x-1)}{\sinh 1} \int_0^x t \sinh t\, dt + \frac{\sinh x}{\sinh 1} \int_x^1 \left\{\sinh(t-1)\right\} t\, dt$$

$$= \frac{\sinh(x-1)}{\sinh 1} \left\{ \left[\cosh t \cdot t\right]_0^x - \int_0^x \cosh t\, dt \right\}.$$

$$+ \frac{\sinh x}{\sinh 1} \left\{ \left[\cosh(t-1) \cdot t\right]_x^1 - \int_x^1 \cosh(t-1)\, dt \right\}.$$

$$\therefore\ u(x) = \frac{\sinh(x-1)}{\sinh 1}\left(x\cosh x - \sinh x\right)$$

$$+ \frac{\sinh x}{\sinh 1}\left\{x\cosh(x-1) + 1 - \left[\sinh(x-1)\right]'_x\right\}$$

$$= \frac{x\sinh(x-1)\cosh x}{\text{sh}\,1}\checkmark - \frac{\text{sh}(x-1)\,\cancel{\text{sh}\,x}}{\cancel{\text{sh}\,1}}$$

$$- \frac{x\,\text{sh}\,x\,\text{ch}(x-1)}{\text{sh}\,1}\checkmark + \frac{\text{sh}\,x}{\text{sh}\,1} + \frac{\text{sh}\,x\,\cancel{\text{sh}}(x-1)}{\cancel{\text{sh}}\,1}$$

$$= \cancel{\frac{x - \text{sh}\,x}{\text{sh}\,1}} - \cancel{\frac{2\,\text{sh}\,x\,\text{sh}(1-x)}{\text{sh}\,1}}$$

$$= \frac{x}{\text{sh}\,1}\left\{\text{sh}(x-1)\,\text{ch}\,x - \text{sh}\,x\,\text{ch}(x-1)\right\} + \frac{\text{sh}\,x}{\text{sh}\,1}.$$

= (x/sinh1)sinh(x-1-x) + sinh$x$/sinh1

= (x/sinh*1*)sinh(*-1*) + sinh$x$/sinh1

= (-$x$sinh1)/sinh1 + sinh$x$/sinh1

= -x + sinh$x$/sinh1

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Our Procedure.

$$Lu = f(x); \quad B_1 u = 0, \quad B_2 u = 0; \quad a \le x \le b.$$

$$L \equiv -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q(x).$$

If $g(x,t)$ be the G.F. corr. to L, then,

$$u(x) = \int_a^b g(x,t) f(t)\,dt$$

is the solution to the BVP with the understanding that,

$$Lg(x,t) = \delta(x-t),$$

$$\frac{\partial g}{\partial x}(t+0,t) - \frac{\partial g}{\partial x}(t-0,t) = -\frac{1}{p(t)}.$$

Alt. Procedure.

$$Lu = f(x); \quad B_1 u = 0, \quad B_2 u = 0; \quad a \le x \le b.$$

$$L \equiv \frac{d}{dx}\left(p\frac{d}{dx}\right) + q(x).$$

$g(x,t) \to$ G. F. corr. to L.

$$u(x) = -\int_a^b g(x,t) f(t)\,dt.$$

where, $$Lg(x,t) = -\delta(x-t)$$

$$\frac{\partial g}{\partial x}(t+0,t) - \frac{\partial g}{\partial x}(t-0,t) = -\frac{1}{p(t)}.$$