# Isoperimetric Problems

Originally, isoperimetric problem is the problem of finding the curve enclosing the greatest area, among all closed curve of a given length.

isoperimetric → with the same perimeter

(Ancient Greeks proposed the problem).

In calculus of variation, the term isoperimetric is extended to the cases where the admissible curves which extremizes a given integral, in addition to satisfy given boundary conditions, also satisfy some other conditions called subsidiary conditions / constraints.

**Ex-1.** Find an admissible curve $y = y(x)$ which extremizes the functional

$$I[y] = \int_{x_1}^{x_2} F(x, y, y')\,dx \longrightarrow (1)$$

satisfying the boundary conditions

$$y(x_1) = y_1 \,,\quad y(x_2) = y_2 \longrightarrow (2)$$

and the subsidiary condition

$$J[y] = \int_{x_1}^{x_2} G(x, y, y')\,dx = l \text{ (given)} \longrightarrow (3)$$

Scanned by CamScanner

Solution: Consider the two-parameter families of comparison curves

$$\bar{y}(x) = y(x) + \epsilon_1 \eta_1(x) + \epsilon_2 \eta_2(x) \quad (*) \longrightarrow (4)$$

where $\eta_1(x)$ and $\eta_2(x)$ are arbitrary twice-differentiable functions on $[x_1, x_2]$ such that

$$\eta_1(x_1) = \eta_1(x_2) = 0 \text{ and } \eta_2(x_1) = \eta_2(x_2) = 0 \longrightarrow (5)$$

Thus, $\bar{y}(x_1) = y(x_1) = y_1$ & $\bar{y}(x_2) = y(x_2) = y_2$.

Again, for every $\eta_1(x)$ and $\eta_2(x)$, $\bar{y}(x) = y(x)$ if $\epsilon_1 = \epsilon_2 = 0$. From (4),

$$\bar{y}'(x) = y'(x) + \epsilon_1 \eta_1'(x) + \epsilon_2 \eta_2'(x). \longrightarrow (6).$$

Replacing $y$ and $y'$ by $\bar{y}(x)$ and $\bar{y}'(x)$ respectively in (1) & (2), we get

$$I(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx \longrightarrow (7)$$

$$\text{and } J(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} G(x, \bar{y}, \bar{y}')\,dx \longrightarrow (8)$$

Since a functional relation

$$J(\epsilon_1, \epsilon_2) = l \longrightarrow (9)$$

exists between $\epsilon_1$ & $\epsilon_2$, these parameters are not independent.

(*) any change in value of a single parameter would change the value of the integral in ~~(2)~~ 3, whose value must remain fixed.

Now, by definition, $y(x)$ extremizes $I$ subject to the constraint (2). Hence when $\epsilon_1 = \epsilon_2 = 0$, $I(\epsilon_1, \epsilon_2)$ has an extremum relative to values of $\epsilon_1$ and $\epsilon_2$ which satisfy relation 9. Our given isoperimetric problem thus reduces to finding the conditions which $y(x)$ must satisfy when it is known that, for $\epsilon_1 = \epsilon_2 = 0$, the ordinary function $I(\epsilon_1, \epsilon_2)$ of two variables has an extremum subject to condition (9). Such problem is generally solved by the method of Lagrange multipliers. Accordingly, we multiply (2) by $\lambda$ and then add it to (1). Thus we get,

$$K(\epsilon_1, \epsilon_2) = I(\epsilon_1, \epsilon_2) + \lambda J(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} H(x, \bar{y}, \bar{y}')\, dx \longrightarrow (10)$$

where $\lambda$ is the Lagrange's multiplier and $H(x, \bar{y}, \bar{y}')$ is defined by

$$H(x, \bar{y}, \bar{y}') = F(x, \bar{y}, \bar{y}') + \lambda G(x, \bar{y}, \bar{y}') \longrightarrow (11)$$

Now, if $H(\epsilon_1, \epsilon_2)$ has an extremum for $\epsilon_1 = 0 = \epsilon_2$, necessary conditions are then by calculus, given by $\frac{\partial H}{\partial \epsilon_1} = \frac{\partial H}{\partial \epsilon_2} = 0$ when $\epsilon_1 = 0 = \epsilon_2$ $\longrightarrow$ (12).

computing these derivatives with the help of (4), (6) and (10), we get -

$$\frac{\partial K}{\partial \cancel{\alpha}_i \, \epsilon_i} = \int_{x_1}^{x_2} \left( \frac{\partial H}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \cancel{\alpha}_i \, \epsilon_i} + \frac{\partial H}{\partial \bar{y}'} \cdot \frac{\partial \bar{y}'}{\partial \cancel{\alpha}_i \, \epsilon_i} \right) dx = \int_{x_1}^{x_2} \left( \frac{\partial H}{\partial \bar{y}} \cdot \eta_i + \frac{\partial H}{\partial \bar{y}'} \cdot \eta_i' \right) dx \longrightarrow (13)$$

Since $\bar{y}$ and $\bar{y}'$ reduce to the extremizing function $y$ and its derivatives $y'$, when $\cancel{\alpha}_1 \epsilon_1 = \cancel{\alpha}_2 \epsilon_2 = 0$, we have from (12) and (13),

$$\left[ \frac{\partial K}{\partial \cancel{\alpha}_i \, \epsilon_i} \right]_{\cancel{\alpha}_1 \epsilon_1 = 0, \cancel{\alpha}_2 \epsilon_2 = 0} = \int_{x_1}^{x_2} \left( \frac{\partial H}{\partial y} \eta_i + \frac{\partial H}{\partial y'} \cdot \eta_i' \right) dx = 0; \quad i = 1, 2. \longrightarrow (14)$$

Using (12) and integrating by parts the second term on the R.H.S of (14), we obtain

$$\int_{x_1}^{x_2} \frac{\partial H}{\partial y} \eta_i(x)\, dx + \left[ \frac{\partial H}{\partial y'} \cdot \eta_i(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left( \frac{\partial F}{\partial y'} \right) \eta_i(x)\, dx = 0; \quad i = 1, 2.$$

or, $$\int_{x_1}^{x_2} \eta_i(x) \left\{ \frac{\partial H}{\partial y} - \frac{d}{dx}\left( \frac{\partial H}{\partial y'} \right) \right\} dx = 0$$ using (5), $(i = 1, 2) \longrightarrow (15)$

Since $\eta_1(x)$ and $\eta_2(x)$ are arbitrary, we have,

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left( \frac{\partial H}{\partial y'} \right) = 0. \longrightarrow (15)$$

This is the Euler's equation which is satisfied by an admissible curve $y = y(x)$ which extremizes $I$ subject to the integral constraint (2).

Form $H = F + \lambda G$. Then $H$ satisfies the ELE

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{H}{\partial y'}\right) = 0 \longrightarrow (*)$$

Note 1. Eq. $(\cancel{*})$ $(*)$ is a 2nd order DE. So its solution will involve two arbitrary constants. These two constants and $\lambda$ are determined by the two boundary conditions (2) and the subsidiary condit. (3)

Note 2. Suppose If we're looking for an extremum of the funct.

$$I[y_1, \ldots, y_n] = \int_{a}^{b} F(x; y_1, \ldots, y_n, y_1', \ldots, y_n')\,dx \longrightarrow (1)$$

subject to the conditions

$$y_i(a) = A_i,\quad y_i(b) = B_i \quad (i = 1, 2, \ldots, n) \longrightarrow (2)$$

&

$$\int_a^b G_j(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\,dx = l_j;$$

$$j = 1, \ldots, k. \longrightarrow (3)$$

In this case a necessary condit. for an extremum is that

$$\frac{\partial}{\partial y_i}\left(F + \sum_{j=1}^{k} \lambda_j G_j\right) - \frac{d}{dx}\left\{\frac{\partial}{\partial y_i'}\left(F + \sum_{j=1}^{k} \lambda_j G_j\right)\right\} = 0,$$

$$i = 1, 2, \ldots, n \longrightarrow (4)$$

The $2n$ arbitrary constants and the $k$ Lagrange's multipliers $\lambda_j$ $(j = 1, 2, \ldots, k)$ are determined from $2n$ boundary condits (2) and the $k$ subsidiary conditions given in (3).

Q. Find the solution to the isoperimetric problem.

$$I[y(x)] = \int_0^1 y'^2 \, dx$$

subject to $y(0)=0$, $y(1)=1$. and $\int_0^1 y \, dx = 2$.

Sol. form $h = y'^2 + \lambda y$.

$$\frac{\partial h}{\partial y} - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = 0 \qquad \text{or, } \lambda - 2y'' = 0.$$

or, $\lambda - \frac{d}{dx}(2y') = 0$. $\quad \therefore y'' = \frac{\lambda}{2}$.

Integrate, $y' = \frac{\lambda x}{2} + c_1$

Again integrating $y(x) = \frac{\lambda x^2}{4} + c_1 x + c_2$.

$y(0)=0 \Rightarrow 0 = c_2$. $\Rightarrow y(x) = \frac{\lambda x^2}{4} + c_1 x$.

$y(1)=1 \Rightarrow 1 = \frac{\lambda}{4} + c_1$ $\qquad \therefore y(x) = -9x^2 + 10x$

$\int_0^1 y \, dx = 2 \Rightarrow \int_0^1 \left(\frac{\lambda x^2}{4} + c_1 x\right) dx = 2$.

or, $\frac{\lambda}{12} + \frac{c_1}{2} = 2 \to \times 3 \Rightarrow \frac{\lambda}{4} + \frac{3c_1}{2} = 6$.

$\frac{\lambda}{4} + c_1 = 1$ $\qquad\qquad \frac{\lambda}{4} + c_1 = 1$

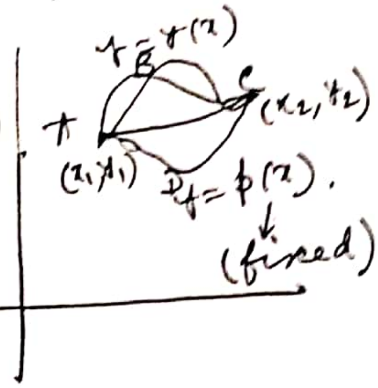
$\boxed{\frac{\lambda}{4} = 1 - 10 = -9}$. $\qquad \frac{c_1}{2} = 5 \Rightarrow \boxed{c_1 = 10}$

53

# Isoperimetric Problem.

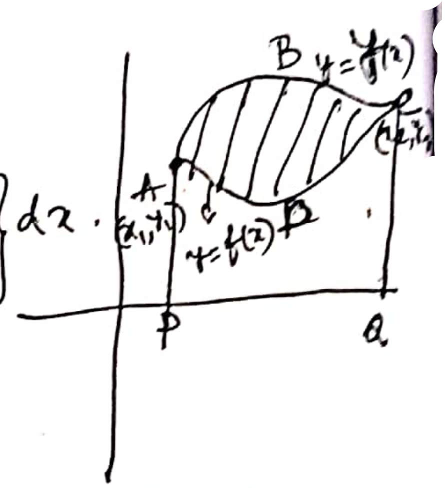
**Problem.**

ABCDA is max. for which $y=y(x)$ together with $y=f(x)$ (fixed).




Given length of $y=y(x)$ between A & C is const. say $l$.

Maximize,




$$I[y(x)] = \int_{x_1}^{x_2} \{y(x)-f(x)\}dx.$$

subject to the conditions that

$y(x_1)=y_1, \quad y(x_2)=y_2$

and $J[y(x)] = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = l$.

area ABCPQ. $= \int_{x_1}^{x_2} y(x)\,dx$

Construct

$$\boxed{[I+\lambda J]}(y(x)) = \int_{x_1}^{x_2} (\underbrace{F+\lambda G}_{H})\,dx$$

area ADCQP $= \int_{x_1}^{x_2} f(x)\,dx$

$F = y(x)-f(x)$

$G = \sqrt{1+y'^2}$.

$$= \int_{x_1}^{x_2} H(x,y,y')\,dx.$$

$\therefore$ area ABCDA.

$=$ area ABCPQ.

$-$ area ADCQP.

$$= \int_{x_1}^{x_2} \{y(x)-f(x)\}dx$$

$y=y(x)$ is determined by solving Euler-Lagrange eqn.

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0.$$

$H = F + \lambda G$

$= y(x) - f(x) + \lambda \sqrt{1+y'^2}$

$H_y = 1$ , $H_{y'} = \lambda . \frac{y'}{\sqrt{1+y'^2}}$ , $\frac{d}{dx} H_{y'} = \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right)$.

Then from E-L-E,

$1 - \lambda \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$ ; $\frac{d}{dx}\left(x - \frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$

Integrate to get

$x - \frac{\lambda y'}{\sqrt{1+y'^2}} = c$ . or, $x - c = \frac{\lambda y'}{\sqrt{1+y'^2}}$.

or, $(x-c)^2 (1+y'^2) = \lambda^2 y'^2$

or, $y'^2 \left\{ \lambda^2 - (x-c)^2 \right\} = (x-c)^2$

or, $\frac{dy}{dx} = y' = \frac{x-c}{\sqrt{\lambda^2 - (x-c)^2}}$

$dy = \frac{x-c}{\sqrt{\lambda^2 - (x-c)^2}} dx$ .

Integrating, $\int dy = \int \frac{x-c}{\sqrt{\lambda^2 - (x-c)^2}} dx + d$ .

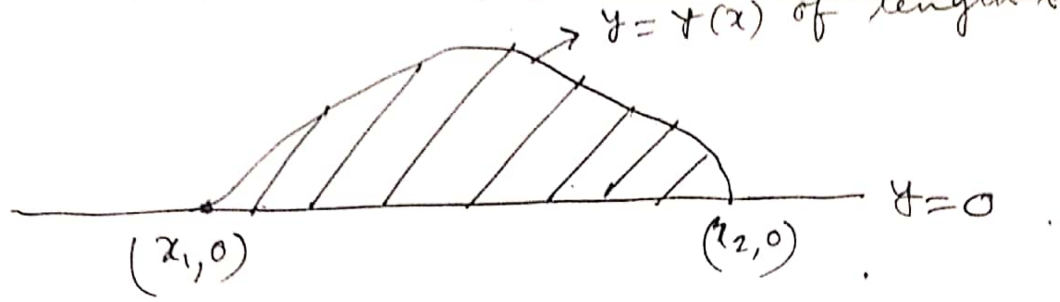
or, $y - d = -\sqrt{\lambda^2 - (x-c)^2}$

or, $(y-d)^2 = \lambda^2 - (x-c)^2$

or, $(x-c)^2 + (y-d)^2 = \lambda^2 \rightarrow \therefore y = y(x)$ between $(x_1, y_1)$, $(x_2, y_2)$ is an arc of a circle.

# Isoperimetric problem. (Dido's problem)

$y = f(x) = 0$.




Extremize

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\, dx.$$

subject to. $y(x_1) = y_1$, $y(x_2) = y_2$.

& $$J(y(x)) = \int_{x_1}^{x} g(x, y, y')\, dx = l.$$

$h = f + \lambda g$. $\lambda \to$ arbitrary parameter.

Method of Lagrange's multiplier.

Maximize / Minimize the function $f(x, y)$.

subject to the condition $g(x, y) = l$.

Construct $h(x, y) = f(x, y) + \lambda g(x, y)$.

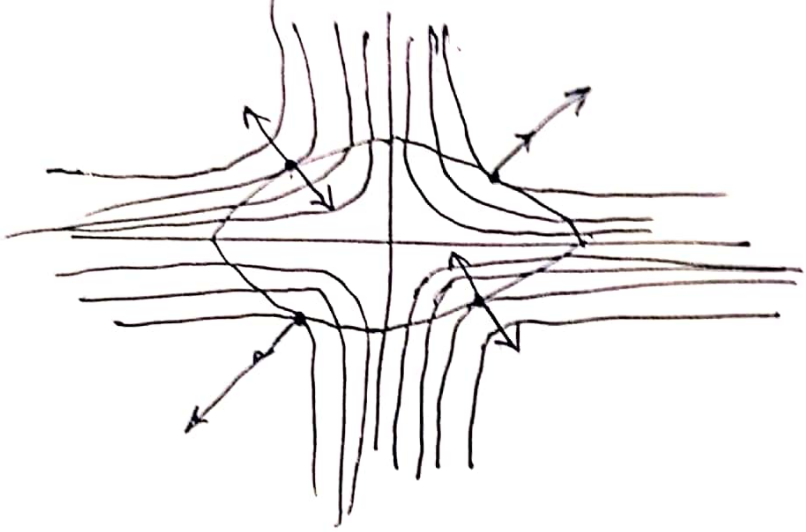
Take $\frac{\partial h}{\partial x} = 0$, $\frac{\partial h}{\partial y} = 0$

$$\frac{\partial (f + \lambda g)}{\partial x} = 0, \quad \frac{\partial}{\partial y}(f + \lambda g) = 0.$$

$$\frac{\partial f}{\partial x}\hat{i} = -\lambda \frac{\partial g}{\partial x} \times \hat{i} \qquad f_x \hat{i} + f_y \hat{j} = -\lambda (g_x \hat{i} + g_y \hat{j})$$

$$\frac{\partial f}{\partial y}\hat{j} = -\lambda \frac{\partial g}{\partial y} \times \hat{j} \qquad \vec{\nabla} f = -\lambda \vec{\nabla} g.$$

Find the extreme value of $f = xy$ that it takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.



$(\pm 2, 1)$, $(\pm 2, -1)$.