$\therefore$ solution to the given IE,

$$u(x) = f(x) + \int_0^x R(x,t;\lambda) f(t)\,dt$$

or, $$u(x) = e^{x^2} + \int_0^x e^{x^2-t^2} \cdot e^{x-t} \cdot e^{t^2}\,dt$$

$$= e^{x^2} + e^{x+x^2} \int_0^x e^{-t}\,dt$$

$$= e^{x^2} + e^{x+x^2} \cdot \left[e^{-t}\right]_x^0 = e^{x^2} + e^{x+x^2}(1-e^{-x})$$

$$= e^{x^2} + e^{x+x^2} - e^{x^2} = e^{x+x^2}.$$

5B : Abel Integral Equation

Abel Integral Equation is of the form

$$f(x) = \int_0^x \frac{u(t)\,dt}{(x-t)^\alpha} \longrightarrow (1);\quad 0<\alpha<1.$$

Note: It is a 1st kind non-homogeneous weakly-singular Volterra Integral Equation.

Abel's theorem: If $f(x)$ is continuous in $[a,b]$, then solution to Abel's IE (1) is given by

$$u(x) = \frac{\sin \pi\alpha}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}.$$

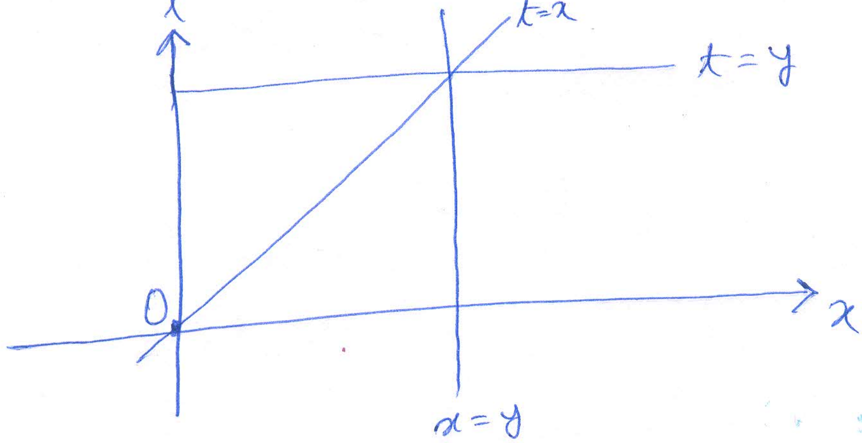
Proof. Multiply both sides of (1) by $(y-x)^{\alpha-1}$ and integrate w. r. to. x between 0 and y.

Then,

$$\int_0^y f(x)\,(y-x)^{\alpha-1}dx = \int_0^y (y-x)^{\alpha-1}dx\left(\int_0^x \frac{u(t)\,dt}{(x-t)^{\alpha}}\right)$$

or,

$$\int_0^y \frac{f(x)\,dx}{(y-x)^{1-\alpha}} = \int_{t=} u(t)\,dt \int_{x=} \frac{dx}{(y-x)^{1-\alpha}(x-t)^{\alpha}}.$$



In order to evaluate the integral

$$I = \int_t^y \frac{dx}{(y-x)^{1-\alpha}(x-t)^{\alpha}}$$

we change the variable x to the variable p such that, when $x=t$, $p=0$, when $x=y$, $p=1$.

$x = t + (y - t)p$

$$y-x = y-t-(y-t)\,p = (y-t)(1-p).$$

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Then, $\displaystyle\int_t^y \frac{dx}{(y-x)^{1-\alpha}(x-t)^{\alpha}}$

$$= \int_0^1 \frac{(y-t)\,dp}{(y-t)^{1-\alpha}(1-p)^{1-\alpha}(y-t)^{\alpha}p^{\alpha}}$$

$$= \int_0^1 p^{-\alpha}(1-p)^{\alpha-1}\,dp$$

$$= B(1-\alpha, \alpha) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)}$$

$$= \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)(=1)} = \frac{\pi}{\sin\pi\alpha},$$

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt.$$

Thus, $\displaystyle\int_0^y \frac{f(x)\,dx}{(y-x)^{1-\alpha}} = \int_0^y \frac{\pi}{\sin\pi\alpha}\,u(t)\,dt.$

$$\therefore \int_0^y u(t)\,dt = \frac{\sin\pi\alpha}{\pi}\int_0^y \frac{f(x)\,dx}{(y-x)^{1-\alpha}}.$$

Take derivative w.r.to y. This gives,

$$u(y) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dy}\int_0^y \frac{f(x)\,dx}{(y-x)^{1-\alpha}}.$$

Changing $y\to x$, $x\to t$,

$$u(x) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}.$$

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§ If f possesses continuous derivatives . . .

In that case,

$$u(x) = \frac{\sin \pi \alpha}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}$$

$$= \frac{\sin \pi \alpha}{\pi} \frac{d}{dx} \int_0^x (x-t)^{\alpha-1} f(t)\,dt.$$

$$= \frac{\sin \pi \alpha}{\pi} \frac{d}{dx} \left[f(t) \cdot \frac{(x-t)^{\alpha}}{\alpha} \Big|_x^0 + \int_0^x \frac{(x-t)^{\alpha}}{\alpha} f'(t)\,dt \right]$$

$$= \frac{\sin \pi \alpha}{\pi} \frac{d}{dx} \left[\frac{x^\alpha}{\alpha} f(0) - 0 + \frac{1}{\alpha} \int_0^x (x-t)^\alpha f'(t)\,dt \right]$$

$$\left[f(x) = \int_0^x \frac{u(t)\,dt}{(x-t)^\alpha} \Rightarrow f(0) = 0 \right]$$

$$\therefore\ u(x) = \frac{\sin \pi \alpha}{\pi \alpha} \frac{d}{dx} \int_0^x (x-t)^\alpha f'(t)\,dt.$$

$$= \frac{\sin \pi \alpha}{\pi \alpha} \int_0^x \frac{\partial}{\partial x} (x-t)^\alpha \cdot f'(t)\,dt.$$

$$= \frac{\sin \pi \alpha}{\pi \alpha} \cdot \alpha \int_0^x (x-t)^{\alpha-1} f'(t)\,dt.$$

$$\therefore\ u(x) = \frac{\sin \pi \alpha}{\pi} \int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}}\,dt.$$

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Example: Solve:

$$\frac{128}{231} x^{\frac{11}{4}} = \int_0^x \frac{u(t)\,dt}{(x-t)^{\frac{1}{4}}} \longrightarrow (1)$$

Solutn. Here $f(x) = \frac{128}{231} x^{\frac{11}{4}}$

$\therefore f'(x) = \frac{128}{231} \times \frac{11}{4} x^{\frac{7}{4}}$ exists in $[0, a]$; $a > 0$

$\therefore$ Solution to (1) is,

$$u(x) = \frac{\sin \pi\alpha}{\pi} \int_0^x \frac{f'(t)\,dt}{(x-t)^{1-\alpha}}$$

Here $\alpha = \frac{1}{4}$ $\therefore 1-\alpha = \frac{3}{4}$

$$\therefore u(x) = \frac{\sin\frac{\pi}{4}}{\pi} \int_0^x \frac{32}{21} \cdot \frac{t^{\frac{7}{4}}\,dt}{(x-t)^{\frac{3}{4}}}$$

Put $t = vx$; when $t = x$, $v = 1$; $t = 0$, $v = 0$.

$$\therefore u(x) = \frac{\sin\frac{\pi}{4}}{\pi} \int_0^1 \frac{32}{21} \cdot \frac{v^{\frac{7}{4}} x^{\frac{7}{4}}\, x\, dv}{x^{\frac{3}{4}} (1-v)^{3/4}}$$

$$\therefore u(x) = \frac{1}{\pi\sqrt{2}} \cdot \frac{32}{21} \cdot x^2 \int_0^1 v^{\frac{7}{4}} (1-v)^{-3/4}\, dv$$

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\,du$$

$$= \frac{1}{\pi\sqrt{2}} \cdot \frac{32x^2}{21} B\left(\frac{11}{4}, \frac{1}{4}\right)$$

$$= \frac{32x^2}{\pi \cdot 21 \cdot \sqrt{2}} \cdot \frac{\Gamma\left(\frac{11}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(3)}$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$\Gamma(n+1) = n\Gamma(n)$

$$\Gamma\left(\frac{11}{4}\right) = \Gamma\left(\frac{7}{4}+1\right) = \frac{7}{4}\Gamma\left(\frac{7}{4}\right) = \frac{7}{4}\cdot\frac{3}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\therefore \frac{\Gamma\left(\frac{11}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(3)} = \frac{\frac{7}{4}\cdot\frac{3}{4}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\,\Gamma(2)}.$$

$\Gamma(2) = 1.\Gamma(1) = 1.$

Also, $\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin \pi\alpha}.$

$$\therefore \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}}.$$

So,
$$u(x) = \frac{32x^2}{21\pi\sqrt{2}}\,\frac{\Gamma\left(\frac{11}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(3)}.$$

$$= \frac{32x^2}{21\cdot\cancel{\pi}\cdot\cancel{\sqrt{2}}}\cdot\frac{\frac{7}{4}\cdot\frac{3}{4}}{2}\cdot\cancel{\sqrt{2}}\,\cancel{\pi} = x^2.$$

Ans $u(x) = x^2.$

<u>Example 2</u>. Solve

$$\sqrt{x} = \int_0^x \frac{u(t)\,dt}{\sqrt{x-t}}.$$

Here $f(x) = \sqrt{x}$ $\therefore$ $f'(x) = \dfrac{1}{2\sqrt{x}}$ is not continuous in $[0,a]$; $a>0$.

So, $$u(x) = \frac{\sin\pi\alpha}{\pi}\,\frac{d}{dx}\left[\int_0^x \frac{\sqrt{t}}{\sqrt{x-t}}\,dt\right]$$