Example 2.

Find the iterated kernel $K_2(x,t)$ corresponding to the kernel

$$K(x,t) = \begin{cases} e^x; & x<t \\ e^t; & x>t. \end{cases} \qquad 0 \le x, t \le 1.$$

Soln:

$$K_2(x,t) = \int_0^1 K(x,s)\,K(s,t)\,ds.$$

Case 1 $\quad t < x$.

$$K_2(x,t) = \int_0^t \cdots + \int_t^x \cdots + \int_x^1 \cdots$$

In $(0,t)$, $t<x$, $\therefore s<t<x$

In (t,x), $t<s<x$

In $(x,1)$, $t<x<s$.

$$\boxed{K(x,s) = \begin{cases} e^x; & x<s \\ e^s; & x>s \end{cases}} \qquad \boxed{K(s,t) = \begin{cases} e^s; & s<t \\ e^t; & s>t \end{cases}}$$

$$K_2(x,t) = \int_0^t K(x,s)\,K(s,t)\,ds + \int_t^x K(x,s)\,K(s,t)\,ds + \int_x^1 K(x,s)\,K(s,t)\,ds$$

$s<t<x$ $\qquad$ $t<s<x$ $\qquad$ $t<x<s$

$$= \int_0^t e^s e^s\,ds + \int_t^x e^s e^t\,ds + \int_x^1 e^x e^t\,ds$$

$$= \left[\frac{e^{2s}}{2}\right]_0^t + e^t\left[e^s\right]_t^x + e^{x+t}\int_x^1 ds$$

$$= \frac{e^{2t}-1}{2} + e^t\left(e^x - e^t\right) + e^{x+t}(1-x)$$

$$= -\frac{1}{2}e^{2t} + 2e^{x+t} - x e^{x+t} - \frac{1}{2}$$

$$= (2-x)e^{x+t} - \frac{1}{2}e^{2t} - \frac{1}{2}$$

Case 2: $t > x$

.

Similarly it can be shown that

$$K_2(x,t) = (2-t)\, e^{x+t} - \frac{1}{2} e^{2x} - \frac{1}{2} \quad \text{when } t > x$$

$$\therefore \; K_2(x,t) = \{2 - \max(x,t)\}\, e^{x+t} - \frac{1}{2}$$
$$- \frac{1}{2} e^{2\min(x,t)}.$$

<u>Relation between resolvent and iterated kernels.</u>

Let $R(x,t;\lambda)$ and $K_n(x,t)$ be the resolvent and iterated kernels respectively.

Then, $$R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^n K_n(x,t)$$

Solution of
$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt$$
is, $$u(x) = f(x) + \int_a^b R(x,t;\lambda)\, f(t)\, dt.$$

<u>Example:</u> Find the iterated kernels corresponding to the kernel $K(x,t) = x^2 t$; $a=0$, $b=1$. Hence find the resolvent kernel $R(x,t;\lambda)$ and solve the IE

$$u(x) = \frac{7}{8} x^2 + \frac{1}{2} \int_0^1 x^2 t\, u(t)\, dt.$$

Solution:

$$K_1(x,t) = K(x,t) = x^2 t.$$

$$K_2(x,t) = \int_0^1 K(x,s)\, K_1(s,t)\, ds = \int_0^1 x^2 s \cdot s^2 t \, ds$$

$$= x^2 t \int_0^1 s^3 ds = x^2 t \cdot \left[\frac{s^4}{4}\right]_0^1 = \frac{x^2 t}{4}.$$

$$K_3(x,t) = \int_0^1 K(x,s)\, K_2(s,t)\, ds = \int_0^1 x^2 s \cdot \frac{s^2 t}{4} ds.$$

$$= \frac{x^2 t}{4} \cdot \int_0^1 s^3 ds = \frac{x^2 t}{4^2}.$$

$$K_4(x,t) = \int_0^1 K(x,s)\, K_3(s,t)\, ds = \int_0^1 x^2 s \cdot \frac{s^2 t}{4^2} ds$$

$$= \frac{x^2 t}{4^2} \int_0^1 s^3 ds = \frac{x^2 t}{4^3}.$$

$$\therefore K_n(x,t) = \frac{x^2 t}{4^{n-1}}.$$

$$\therefore R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^n K_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{1}{4}\right)^{n-1} x^2 t$$

$$= \frac{1}{2} \cdot x^2 t \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^{n-1} = \frac{1}{2} x^2 t \cdot \frac{1}{1-\frac{1}{8}} = \frac{4x^2 t}{7}$$

$$\therefore u(x) = f(x) + \int_a^b R(x,t;\lambda)\, f(t)\, dt$$

$$= \frac{7}{8}x^2 + \int_0^1 \frac{\cancel{4}}{\cancel{7}} x^2 t \cdot \frac{\cancel{7}}{\cancel{8}_2} t^2 dt = \frac{7}{8}x^2 + \frac{x^2}{2}\int_0^1 t^3 dt$$

$$= \frac{7}{8}x^2 + \frac{x^2}{8} = x^2.$$