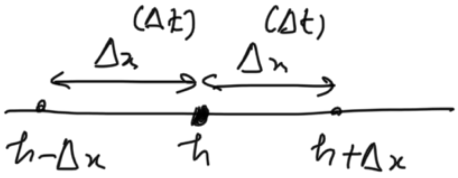
Brownian Motion (BM) process:



$X(t)$: position of particle at time t.

$$X_i = \begin{cases} +1 & \text{if the } i^{th} \text{ step of length } \Delta x \text{ is to the right} \\ -1 & \text{" " " " " " " " " left} \end{cases}$$

$$X(t) = \Delta x \left(\underline{X_1 + X_2} + \cdots + X_{\left[\frac{t}{\Delta t}\right]} \right)$$

where $[\cdot]$: greatest integer less than or equal to the number eg $[4.4] = 4$

X_i's are indep.

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$$

$$\underline{E(X_i) = 0} \quad ; \quad V(X_i) = \underline{E(X_i^2) = 1}$$

$$E(X(t)) = 0$$

$$V(X(t)) = (\Delta x)^2 \sum_{i=1}^{[t/\Delta t]} V(X_i) \rightarrow 1$$

$$= (\Delta x)^2 \left[\frac{t}{\Delta t} \right]$$

Let $\Delta x \to 0$, $\Delta t \to 0$

(i) if $\Delta x = \Delta t \to 0$; $E(X(t)) = 0$, $V(X(t)) = 0$

trivial result

(ii) if we let $\underline{\Delta x = \sigma \sqrt{\Delta t}}$, $\sigma > 0$

as $\Delta t \to 0$

$$E(X(t)) = 0 \quad , \quad V(X(t)) \to \sigma^2 t$$

Defⁿ A SP $\{X(t), t\geq 0\}$ BM process if

(i) $X(0)=0$ (ii) $\{X(t), t\geq 0\}$ has stationary indep. increments

(iii) $\forall t>0$, $X(t) \sim N(0, \sigma^2 t)$

—x—

If $\sigma = 1$ Standard BM / Wiener process

$\{X(t)\}$ BM, then $W(t) = \dfrac{X(t)}{\sigma} \sim N(0,t)$ SBM / Wiener process

$\rightarrow W(t) \sim N(0,t)$

density $f_t(x) = \dfrac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$, $-\infty < x < \infty$.

Note that

$W(t_1)=x_1, \ldots, W(t_n)=x_n \equiv W(t_1)=x_1, W(t_2)-W(t_1)=x_2-x_1, \ldots, W(t_n)-W(t_{n-1})=x_n-x_{n-1}$

Also $\underline{W(t_1), W(t_2)-W(t_1)}, \ldots, W(t_n)-W(t_{n-1})$ are indep and has stationary increments

$\underline{W(t_k)-W(t_{k-1})} \stackrel{d}{=} W(t_k-t_{k-1}) \sim N(0, t_k-t_{k-1})$.

Joint density of $\underline{W(t_1), \ldots, W(t_n)}$ is

$$\underline{f(x_1,\ldots,x_n)} = f_{t_1}(x_1)\, f_{t_2-t_1}(x_2-x_1) \cdots f_{t_n-t_{n-1}}(x_n-x_{n-1})$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \cdots + \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\right]\right\}}{(2\pi)^{n/2}\left[t_1(t_2-t_1)\cdots(t_n-t_{n-1})\right]^{1/2}} \quad ✓$$

Conditional density of $[W(s) \mid W(t)=B]$, $s<t$

density $f\ (x|B) = f_s(x)\, f_{t-s}(B-x)$

$$f_{s|t}(x|B) = \frac{\quad\quad}{f_t(B)}$$

$$= k_1 \exp\left\{-\frac{x^2}{2s} - \frac{(B-x)^2}{2(t-s)}\right\}$$

$$= k_2 \exp\left\{-x^2\left(\frac{1}{2s} + \frac{1}{2(t-s)}\right) + \frac{Bx}{t-s}\right\}$$

$$= k_2 \exp\left\{-\frac{t}{2s(t-s)}\left(x^2 - \frac{2sB}{t}x\right)\right\}$$

$$= k_3 \exp\left\{-\frac{(x - Bs/t)^2}{2s(t-s)/t}\right\}$$

For $s<t$ $\quad$ k_1, k_2, k_3 indep of x

$$[W(s) \mid W(t)=B] \sim N\left(\frac{s}{t}B,\ \frac{s(t-s)}{t}\right)$$

$E(W(s)|W(t)=B]$ $\quad$ $V(W(s)|W(t)=B)$

Example: In a bicycle race btw two competitors, let $Y(t)$: the amt of time (in secs) by which the racer that started in the inside position is ahead when $100t\%$ of the race has been completed, and suppose that $\{Y(t)\}_{0\le t\le 1}$ can be effectively modeled as BM process with variance parameter σ^2.

(a) If the inside racer is leading by σ sec's at the midpoint of the race, what is the prob. that she is the winner?

$Y(t) \sim N(0, \sigma^2 t)$

Sol $\quad P(Y(1) > 0 \mid Y(\tfrac{1}{2}) = \sigma)$

$$= P\big(\underbrace{Y(1) - Y(\tfrac{1}{2})}\, > 0 - \sigma \,\big|\, \underbrace{Y(\tfrac{1}{2}) = \sigma}\big)$$

$$= P\big(Y(1) - Y(\tfrac{1}{2}) > -\sigma\big) \quad \Big| \because \text{indep increment of } Y(t)$$

$$= P\big(Y(\tfrac{1}{2}) > -\sigma\big) \quad \Big| \text{ stationary increment of } Y(t)$$

$$= P\left(\frac{Y(\frac{1}{2})}{\sigma/\sqrt{2}} > \frac{-\sigma}{\sigma/\sqrt{2}}\right) = P(Z > -\sqrt{2})$$

$Z \sim N(0,1)$

$$= 1 - \Phi(-\sqrt{2}) \quad \Big| \; \Phi(x) = P(Z \leq x)$$

$$= \Phi(\sqrt{2}) \simeq 0.9213 \quad \Big| \; \Phi(x) + \Phi(-x) = 1$$

(b) If the inside racer wins the race by a margin of σ sec's, what is the prob that she was ahead at the midpoint?

Sol

$$P\big(Y(\tfrac{1}{2}) > 0 \,\big|\, Y(1) = \sigma\big) = ?$$

$$\sigma W(t) = Y(t) \sim N(0, \sigma^2 t) \qquad W(t) \sim N(0,t)$$

For $s < t$

$$[W(s) \mid W(t) = c] \sim N\left(\frac{s}{t}c,\ \frac{s}{t}(t-s)\right)$$

$$[W(s) \mid Y(t) = c] \equiv \left[W(s) \,\middle|\, W(t) = \frac{c}{\sigma}\right] \sim N\left(\frac{sc}{t\sigma},\ \frac{s}{t}(t-s)\right)$$

$$[Y(s) \mid Y(t) = c] \equiv [\sigma W(s) \mid Y(t) = c] \sim N\left(\frac{sc}{t},\ \frac{\sigma^2 s(t-s)}{t}\right)$$

$$[Y(\tfrac{1}{2}) \mid Y(1) = \sigma] \sim N\left(\frac{\sigma}{2}, \frac{\sigma^2}{4}\right)$$

$s = \frac{1}{2}, t = 1, c = \sigma$

$$P\big(Y(\tfrac{1}{2}) > 0 \mid Y(1) = \sigma\big) = P\big(Z > \frac{0 - \sigma/2}{\ldots}\big)$$

$$P(Y(2) \ldots X(1)=0) = P\left(Z > \frac{\ldots}{\sigma/2}\right)$$

$$= P(Z > -1) = 1 - P(Z \le -1) = 1 - \Phi(-1)$$

$$= \Phi(1) \simeq 0.8413$$

— x —

→ $W(t)$ is MG. ?

Sol For $s<t$

$$E(W(t) \mid W(u),\ 0 \le u \le s)$$

$$= E(W(t) - W(s) + W(s) \mid W(u),\ 0 \le u \le s)$$

$$= E(W(t) - W(s) \mid W(u),\ 0 \le u \le s) + E(W(s) \mid W(u),\ 0 \le u \le s)$$

$$= \underbrace{E(W(t) - W(s))}_{\text{by independent increment}} + W(s)$$

$$= 0 + W(s) = W(s) \qquad \Big|\ W(t) \sim N(0,t)$$

$W(t)$ MG.

— x —

<u>Martingale</u> Stopping thm: An important property of MG $Y(t)$ is that if you continually observe the process and stop at some time T, then, subject to some technical condition

$$E(Y(T)) = E(Y(0))$$

T → stopping time for MG.

expected value of the stopped MG is equal to its fixed time expectation.

— x —

Eg. Let $T = \min\{t : \underline{W(t) = 2 - 4t}\}$, i.e., T is the first time that SBM hits the line $2-4t$. $E(T)=?$

Sol Using MG stopping thm

$$\underline{E}(W(T)) = E(W(0)) = 0$$

$$W(T) = 2-4T \Rightarrow E(W(T)) = 2-4E(T)$$

$$\Rightarrow 2-4E(T)=0 \Rightarrow E(T) = 1/2$$

—x—

Q. Let $Y(t) = W^2(t) - t$.

$Y(t)$ MG ? $E(Y(t)) = ?$ Ex.

—x—

<u>Geometric BM</u>

$Y(t)$ BM drift coeff μ, var-parameter σ^2

$$Y(t) \sim N(\mu t, \sigma^2 t),$$

$$X(t) = e^{Y(t)}$$

$\{X(t), t \geq 0\}$ Geometric BM.

For $s<t$

$$E(X(t) \mid X(u), 0 \leq u \leq s)$$

$$= E(e^{Y(t)} \mid Y(u), 0 \leq u \leq s)$$

$$= E(e^{Y(s)+Y(t)-Y(s)} \mid Y(u), 0 \leq u \leq s)$$

$$= e^{Y(s)} E(e^{Y(t)-Y(s)} \mid Y(u), 0 \leq u \leq s)$$

$$= X(s)\, E(\underline{e^{Y(t)-Y(s)}})$$ | indep. increments

$$= X(s)\, e^{\mu(t-s) + (t-s)\sigma^2/2}$$ | stationary increments

$$= X(s)\, e^{(t-s)(\mu+\sigma^2/2)}$$

$Y(t)-Y(s) \stackrel{d}{=} Y(t-s)$

$\sim N(\mu(t-s), (t-s)\sigma^2)$

$W \sim N(\cdot,\cdot)$

$E(e^{aW}) = e^{aE(W)+\frac{1}{2}a^2 Var(W)}$

$a=1$

$$\Rightarrow E(X(t) \mid X(u),\ 0 \le u \le s) = X(s)\, e^{(t-s)(\mu+\sigma^2/2)}$$

$$\Rightarrow E(X(t)) = E(X(s))\, e^{(t-s)(\mu+\sigma^2/2)}$$

Geo. BM is useful in modeling of stock prices over time when you feel the %age changes are IID.

eg Let X_n : price of some stock at time n.

It might be reasonable to suppose that

$\frac{X_n}{X_{n-1}}$, $n \ge 1$, are I.I.D.

Let $Y_n = \frac{X_n}{X_{n-1}} \Rightarrow X_n = Y_n X_{n-1}$

$= Y_n Y_{n-1} X_{n-2}$

$\vdots$

$= Y_n Y_{n-1} \cdots Y_1 X_0$

$$\Rightarrow \log X_n = \sum_{i=1}^{n} \log(Y_i) + \log(X_0)$$

$\log(Y_i)$, $i \ge 1$, IID, $\{\log(X_n)\}$ will, when suitably normalized, approx. BM with a drift, so.

$e^{\log X_n} = X_n$, $\{X_n\}$ approx. Geo. BM.

—x—

$X(t)$ BM with drift wrt μ : $X(t) \sim N(\mu t, t)$

, with $N(0,t)$

$X(t) = W(t) + \mu t$

BM also be defined as a limit of random walks

Sol

$X(t)$: position at time t

(Δt) (Δt)
$\leftarrow \Delta x \rightarrow \leftarrow \Delta x \rightarrow$

$1-p$ p
$-$ $+$

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ step in +ve direction} \\ -1 & \text{o.w.} \end{cases}$$

X_i's indep.

$X(t) = \Delta x \, (X_1 + \cdots + X_{[t/\Delta t]})$

$E(X(t)) = \Delta x \left[\frac{t}{\Delta t}\right](2p-1)$

$V(X(t)) = (\Delta x)^2 \left[\frac{t}{\Delta t}\right]\left(1-(2p-1)^2\right)$

$E(X_i) = 1 \cdot p + (-1)(1-p)$
$= 2p-1$

$V(X_i) = E(X_i^2) - (E(X_i))^2$
$= 1-(2p-1)^2$

If we let $\underline{\Delta x = \sqrt{\Delta t}}$, $\underline{p = \frac{1}{2}(1+\mu\sqrt{\Delta t})}$

and let $\Delta t \to 0$

$E(X(t)) \longrightarrow \mu t$

$V(X(t)) \to t$

$\cancel{\sqrt{\Delta t}} \times \frac{t}{\cancel{\Delta t}} \times \mu \cancel{\sqrt{\Delta t}}$
$\to \mu t$

$\cancel{\Delta t} \times \frac{t}{\cancel{\Delta t}} \times (1-\underline{\mu^2 \Delta t})$
$\to t$

---x---

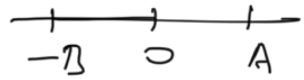
Probability that the process will hit A before $-B$; $A, B > 0$

Let $P(0) = P(X(t) \text{ hits } A \text{ before } -B \mid X(0)=0)$, $-B<0<A$

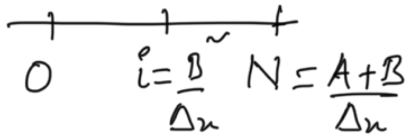
where $P(x)$ is the prob. that process will hit A before B given that we are now at x.

$x-B$ x $x+A$

Boundary condition $P(A)=1$, $P(-B)=0$ (Δx)



Gambler's ruin problem



$$p = \frac{1}{2}(1+\mu\Delta x)$$

$$\lim_{\Delta x \to 0}\left(\frac{1-p}{p}\right)^{1/\Delta x} = \lim_{\Delta x \to 0}\left(\frac{1-\mu\Delta x}{1+\mu\Delta x}\right)^{1/\Delta x} = \frac{e^{-\mu}}{e^{\mu}} = e^{-2\mu}$$

$$\frac{1-\left(\frac{q}{p}\right)^{i}}{1-\left(\frac{q}{p}\right)^{N}}$$

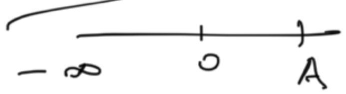
Letting $\Delta x \to 0$, we see that

$$P(\text{up } A \text{ before down } B) = \frac{1-e^{-2B\mu}}{1-e^{-2\mu(A+B)}} \quad \text{—} \quad (1)$$

$$= \frac{e^{2B\mu}-1}{e^{2\mu(A+B)}-1}\, e^{2\mu A}$$

Case I If $\mu < 0$ by letting $B \to \infty$ (2)

$$P(\text{process ever goes to } A) = e^{2\mu A}$$



In this case, the process drift off to $-\infty$ and its max is an expo. r.v. with rate (-2μ).

Case II let $\mu \to 0$ in (1)

$$P(\text{BM goes up } A \text{ before down } B) = \frac{B}{A+B}.$$

In general

$$P(x) = \frac{1 - e^{-2\mu(x+B)}}{1 - e^{-2\mu(A+B)}}$$

—x—

<u>Example</u> (Exercising a Stock Option)

Suppose we have the option of buying, at some time in the future, <u>one unit</u> of a stock at a <u>fixed price A</u>, indep. of current market price. The current market price of the stock is taken to be 0, and we suppose that it changes in accordance with a BM having a negative drift coefficient $-d$, where $d > 0$. The question is, when, if ever, should we exercise our option?

Sol policy exercise the option when market price is x

↳ <u>expected gain</u> $= (x-A)\,P(x)$,

where $P(x)$ prob. that the process will ever reach x,

$\mu = -d < 0$, $d > 0$

From (2) $\quad P(x) = e^{-2dx}$

Optimal value of x is one max. $(x-A)\,e^{-2dx} = f(x)$

$$f'(x) = (x-A)\,e^{-2dx}(-2d) + e^{-2dx} = 0$$

$$\Rightarrow \quad x = A + \frac{1}{2d}$$

$$f''(x)\Big|_{x=A+\frac{1}{2d}} > 0$$

$$\ldots = A + \frac{1}{2d}$$

$$\boxed{x = A + \frac{1}{2d}}$$

eg if $A = 100$, $d = 2$

$$x = 100 + \frac{1}{4} = 100.25$$

— x —

Pricing Stock Option:

example in option pricing:

amt $\underline{\underline{v}}$ — v at t is $ve^{-\alpha t}$

time '0' value present value

$\alpha \to$ discount factor

$e^{-\alpha t}$ discount fn

Option — share of stock at a future time at a fixed price.

$$100 \begin{cases} 200 \ (200e^{-\alpha}) \\ 50 \ (50e^{-\alpha}) \end{cases}$$

time 0 price — time 1 price

Option: buy unit of stock cost 150 per share (time 0 price)

$C = ?$ unit cost of an option

We will show unless $C = \frac{50}{3}$, there will be a combination of purchases that will always result in a +ve gain.

Suppose time 0, we

$$\begin{cases} \text{buy } x \text{ unit of stock} \\ \text{buy } y \text{ " " option} \end{cases}$$

sell means $-x$

value of our holding at time 1

$$\text{value} = \begin{cases} 200x + 50y & \text{if price is } 200 \\ 50x & \text{if price is } 50 \end{cases}$$

let we choose y s.t.

$$200x + 50y = 50x \Rightarrow \boxed{y = -3x}$$

the value of holding $= \underline{50x}$

Origned cost $= 100x + cy$

x unit stock & $-3x$ unit of option $= 100x - 3xc$

gain $= 50x - (100x - 3xc)$

$= x(3c - 50)$

$$\begin{cases} \boxed{0 \quad \text{if } 3c = 50} \\ +ve \quad \text{if } 3c > 50 \\ -ve \quad \text{if } 3c < 50 \end{cases}$$

$\rightarrow$ only option cost c that does not result in an arbitrage is $\underline{c = 50/3}$

A sure win betting scheme is called an arbitrage

eg (1) $\underline{c = 20}$ (unit cost per option)

$\underline{x = 1}, \underline{y = -3}$ $\qquad y = -3x$

initially cost us $= 100 - 60 = 40$

value of holding at time 1 (↑200, ↓50)

$= 50$

guaranteed profit $= 50 - 40 = 10$ is attained

(2) $C = 15$

$x = -1, \; y = 3$

initial gain $= 100 - 45 = 55$

Value of holding at time 1 is $= -50$

guaranteed profit $= 55 - 50 = 5$

Arbitrage thm: expt. whose set of possible outcomes $S = \{1, 2, \ldots, m\}$.

n wagers

amt x is bet on wager i, then return $x\, r_i(j)$ is earned if the outcome of expt is j, $j \in S = \{1, 2, \ldots, m\}$

$i \in \{1, 2, \ldots, n\}$

$r_i(\cdot) \to$ return fn for unit bet on wager i.

Betting scheme $\underset{\sim}{x} = (x_1, \ldots, x_n)$

Outcome of expt is j, then return from $\underset{\sim}{x} = \sum_{i=1}^{n} x_i r_i(j)$

$\exists$ $\underset{\sim}{p} = (p_1, \ldots, p_m)$ on $S = \{1, \ldots, m\}$ under which each wager has expected return 0, or else there is a betting scheme that guarantees a positive win.

Arbitrage thm: Exactly one of the following is true:

Either

(i) $\exists$ a $\underset{\sim}{p} = (p_1, \ldots, p_m)$ for which $\sum_{j=1}^{m} p_j r_i(j) = 0$ ...

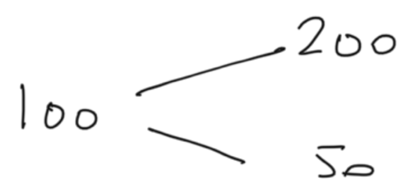
ver $\qquad \forall j = 1, \dots, m$

(ii) $\exists$ a betting scheme $\underset{\sim}{x} = (x_1, \dots, x_n)$ for which
$$\sum_{i=1}^{n} x_i \, r_i(j) > 0 \quad \forall j = 1, \dots, m.$$

In other words, if X is the outcome of the expt, then arbitrage thm states that either there is a prob. vector $\underset{\sim}{p}$ for X s.t. $E_{\underset{\sim}{p}}(r_i(X)) = 0 \quad \forall i = 1, \dots, n$

or else there is a betting scheme that leads to a sure win.

Example (contd)



time 0 $\qquad$ time 1

C $\qquad$ Option: to buy at time 1 at present value of 150 per share

Set no sure win is possible

two outcomes

two wagers $\to \begin{cases} \text{buy / sell stock} \\ \text{buy / sell option} \end{cases}$

no sure wins if $\underset{\sim}{p} = (p, 1-p)$ $\qquad \underline{E(\text{return}) = 0}$.

returns from purchase $= \{ \; 200 - 100 = 100$ if price 200

1 unit of stock $\begin{cases} & \text{at time 1} \\ 50-100 = -50 & \text{if} \;\text{-- 50 --}\; 1. \end{cases}$

if $p \to$ prob. that price is 200 at time 1, then

$$E(\text{return}) = 100p - 50(1-p)$$

$$\underline{E(\text{return}) = 0} \Rightarrow \underline{100 = 200p + 50(1-p)}$$

$$\Rightarrow p = \frac{1}{3}$$

$(p, 1-p) = \left(\frac{1}{3}, \frac{2}{3}\right)$ for which wager yields an expected return 0.

$$\text{return from purchasing one share of option} = \begin{cases} 50 - c & \text{if price is } 200 \\ -c & \text{if price is } 50 \end{cases}$$

expected return when $p = \frac{1}{3}$ is

$$E(\text{return}) = (50-c) \times \frac{1}{3} - c \times \frac{2}{3} = \frac{50}{3} - c$$

$$E(\text{return}) = 0 \Rightarrow c = \frac{50}{3}$$

arbitrage thm only value of c for which there will not be a sure win is $c = 50/3$.

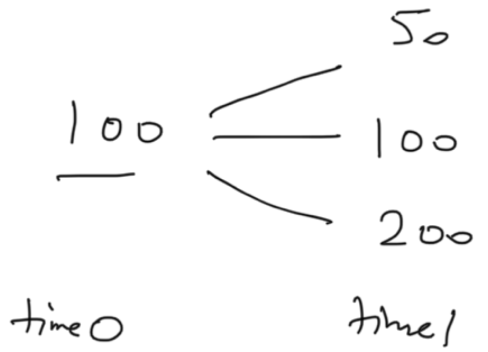
—x—

Example 1 The present price of a stock is 100. The price at time 1 will be either 50, 100 or 200. An option to purchase y shares of the stock at time 1 for the (present value) price ky costs cy.

(a) If $k = 120$, show that an arbitrage opportunity occurs iff $c > 80/3$

(b) If $k = 80$, show that there is not an arbitrage opportunity iff $20 \le c \le 40$.

Sol



Stock will be a fair bet under probabilities $(p_1, p_2, 1-p_1-p_2)$ on $(50, 100, 200)$

(c) (k)

$$100 = 50p_1 + 100p_2 + 200(1-p_1-p_2)$$

$$\Rightarrow 2 = p_1 + 2p_2 + 4(1-p_1-p_2)$$

$$\Rightarrow 3p_1 + 2p_2 = 2 \quad \text{— (1)}$$

(a) $k = 120$

option

$$\text{return} = \begin{cases} -c & \text{if } 50 \text{ wp } p_1 \\ -c & \text{if } 100 \text{ wp } p_2 \\ 80-c & \text{if } 200 \text{ wp } 1-p_1-p_2 \end{cases}$$

$$E(\text{return}) = 0 \Rightarrow -cp_1 - cp_2 + (80-c)(1-p_1-p_2) = 0$$

$$\Rightarrow c = 80(1-p_1-p_2) \quad \text{— (2)}$$

$\Rightarrow$ solving (1) & (2)

$$p_1 = \frac{c}{40}, \quad p_2 = \frac{80-3c}{80}$$

$$1 - p_1 - p_2 = \frac{C}{80}$$

$\Rightarrow \quad \underline{\underline{80 \geq 3C}} \iff$ no arbitrage

$C > \frac{80}{3} \iff$ arbitrage opportunity occurs.

(b) $\underline{K = 80}$

$$\underline{C = 20p_2 + 120(1 - p_1 - p_2)} \quad — (2)$$

Solve (1) & (2)

$0 \leq p_i \leq 1$

$$\left.\begin{aligned} p_1 &= \frac{C - 20}{30}, \quad p_2 = \frac{40 - C}{20} \\ 1 - p_1 - p_2 &= \frac{C - 20}{60} \end{aligned}\right\} \text{ iff } \underline{20 \leq C \leq 40}$$

no arbitrage

— x —

Black – Scholes Option pricing formula:

present price of stock $X(0) = x_0$; $X(t)$ stocks price at time t

Stock $[0, T]$

$\alpha \to$ discount factor

$$e^{-\alpha t} X(t)$$

Observe $X(t)$ upto $\underline{s < t}$ buy (or sell) stock at price $X(s)$, then sell (buy) these shares at time t for price $X(t)$

purchase N different options at time 0

Option i cost C_i per share option purchase share of stock at time t_i for fixed price K_i

c_i for which there is no betting strategy that lead to a sure win. per share $i=1,\ldots,N$.

Use Arbitrage thm

$\underset{\sim}{P}$ prob. measure on the set of outcomes

wager s t

$$\underline{e^{-\alpha s}X(s)} \qquad e^{-\alpha t}X(t)$$

$\underset{\sim}{P}$ on $X(t)$, $0 \le t \le T$

$$E_{\underset{\sim}{P}}\left[e^{-\alpha t}X(t) \mid X(u),\ 0 \le u \le s\right] = e^{-\alpha s}X(s) \quad \text{— } (1)$$

$X(t)$ MG

k

$$\text{worth of option at time } t = \begin{cases} X(t)-k & \text{if } X(t) \ge k \\ 0 & \text{if } X(t) < k \end{cases}$$

present value of worth of option $= e^{-\alpha t}(X(t)-k)^+$

$$E_{\underset{\sim}{P}}\left(e^{-\alpha t}(X(t)-k)^+\right) = c \quad \text{— } (2)$$

$$c_i = E_{\underset{\sim}{P}}\left[e^{-\alpha t_i}(X(t_i)-k_i)^+\right], \quad i=1,2,\ldots,N \quad \text{— } (2)'$$

By arbitrage thm if we can find a prob. measure $\underset{\sim}{P}$ on the set of outcomes that satisfy (1), then if c, the cost of an option to purchase one share at time t at the fixed price k, is given by (2), then no arbitrage is possible. On the other hand, if for

... prices c_i, $1 \le i \le N$, then is no prob. measure $\underset{\sim}{P}$ that satisfy both (1) and equality (2)'