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Conversion of F.I.E. to B.V.P

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Ex-1 Convert the following F.I.E to BVP.

$$u(x) = 3x^2 + \int_0^1 K(x,t)\,u(t)\,dt \quad \to (1) \quad ; \quad K(x,t) = \begin{cases} t(1-x); & 0 \le t < x \\ x(1-t); & x < t \le 1 \end{cases}$$

Sol.

$$u(x) = 3x^2 + \int_0^x t(1-x)\,u(t)\,dt + \int_x^1 x(1-t)\,u(t)\,dt. \quad \to (2)$$

Take derivative on both sides of (2) w.r.to x

$$\frac{du}{dx} = u'(x) = 6x + \int_0^x \frac{\partial}{\partial x}\left\{t(1-x)\,u(t)\right\}dt + \cancel{1.\,x(1-x)\,u(x)}$$

$$+ \int_x^1 \frac{\partial}{\partial x}\left\{x(1-t)\,u(t)\right\}dt - \cancel{1.\,x(1-x)\,u(x)}$$

$$u'(x) = 6x - \int_0^x t\,u(t)\,dt + \int_x^1 (1-t)\,u(t)\,dt$$

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Taking derivative on both sides of (3), w.r.to x,

$$u''(x) = 6 - \frac{d}{dx}\int_0^x t\,u(t)\,dt + \frac{d}{dx}\int_x^1 (1-t)\,u(t)\,dt$$

$$= 6 - \int_0^x \underbrace{\frac{\partial}{\partial x}\{t\,u(t)\}}_{=0} dt - 1.\,x\,u(x) + \int_x^1 \underbrace{\frac{\partial}{\partial x}\{(1-t)\,u(t)\}}_{=0} dt - 1\,(1-x)\,u(x)$$

or, $$u''(x) = 6 - x\,u(x) - (1-x)\,u(x)$$

$$= 6 - \cancel{x\,u(x)} - u(x) + \cancel{x\,u(x)}$$

$$\boxed{u''(x) + u(x) = 6} \rightarrow \text{required diff. eqn.} \quad 0 < x < 1 \quad (4)$$

together with the boundary conditions, $\boxed{u(0) = 0,\ u(1) = 3}$ (5)

The given F.I.E is converted into a BVP given by the DE (4) & b.c.'s (5).

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Ex2 Convert the following F.I.E. to a BVP.

$$u(x) = e^x + 1 + \int_0^1 K(x,t)\,u(t)\,dt; \quad K(x,t) = \begin{cases} 2t, & 0 \le t \le x \\ 2x, & x \le t \le 1 \end{cases}$$

$$\int_0^x t\,u(t)\,dt + \int_x^1 2x\,u(t)\,dt \longrightarrow (1)$$

$$u'(x) = e^x + \int_x^1 2u(t)\,dt \longrightarrow (2)$$

$$u''(x) = e^x - 2u(x) \Rightarrow u''(x) + 2u(x) = e^x$$

$$u(0) = 2$$

$$u'(1) = e.$$

Put $x=0$ in (1), $u(0) = e^0 + 1 + \cancel{\int_0^0 t\,u(t)\,dt} + \cancel{\int_x^1 2\cdot 0\cdot u(t)\,dt} = 2$

$$u'(1) = e^1 + \int_1^1 2u(t)\,dt$$

$$u'(1) = e.$$

$$u(0) + u'(0) = 1$$

$$u(0) - 2u'(1) = 0$$

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Methods of finding solutions of Fredholm Integral Equations.

A. Method of separable kernel.

Definition: A kernel $K(x,t)$ is said to be separable or degenerate if it can be expressed as,

$$K(x,t) = a_1(x)\,b_1(t) + a_2(x)\,b_2(t) + \cdots + a_n(x)\,b_n(t)$$

Let us consider some F.I.E of the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\,u(t)\,dt \longrightarrow (1)$$

where,

$$K(x,t) = \sum_{i=1}^{n} a_i(x)\,b_i(t) \longrightarrow (2)$$

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Substituting (2) in (1), we get

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} a_i(x) \underbrace{\int_a^b b_i(t)\,u(t)\,dt}_{?}, \quad a \le x \le b \longrightarrow (3)$$

Observe that the integral in (3) gives some unknown constants, say A_i $(i = 1, 2, \ldots, n)$

i.e
$$A_i = \int_a^b b_i(t)\,u(t)\,dt \longrightarrow (4); \quad i = 1, 2, \ldots, n$$

Then (3) is,
$$u(x) = f(x) + \lambda \sum_{i=1}^{n} A_i\, a_i(x) \longrightarrow (5)$$

Putting $u(t)$ from (5) (by replacing x by t), into R.H.S of (4), we get

$$A_i = \int_a^b b_i(t) \left\{ f(t) + \lambda \sum_{j=1}^{n} A_j\, a_j(t) \right\} dt$$

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$$A_i = \int_a^b b_i(t) f(t)\, dt + \lambda \sum_{j=1}^{n} A_j \int_a^b a_j(t)\, b_i(t)\, dt$$

$i = 1, 2, \dots, n.$

$$\Rightarrow \quad A_i - \lambda \sum_{j=1}^{n} A_j \underbrace{\int_a^b a_j(t)\, b_i(t)\, dt}_{\boxed{\mu_{ij} = \int_a^b b_i(t)\, a_j(t)\, dt}} = \int_a^b b_i(t) f(t)\, dt = \delta_i ; \quad i = 1, 2, \dots, n.$$

$i = 1$

$$A_1 - \lambda (A_1 \mu_{11} + A_2 \mu_{12} + A_3 \mu_{13} + \dots + A_n \mu_{1n}) = \delta_1$$

$i = 2$

$$A_2 - \lambda (A_1 \mu_{21} + A_2 \mu_{22} + A_3 \mu_{23} + \dots + A_n \mu_{2n}) = \delta_2$$

$\vdots$

$i = n$

$$A_n - \lambda (A_1 \mu_{n1} + A_2 \mu_{n2} + A_3 \mu_{n3} + \dots + A_n \mu_{nn}) = \delta_n$$

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$$(1-\lambda\mu_{11})A_1 - \lambda\mu_{12}A_2 - \lambda\mu_{13}A_3 \cdots - \lambda\mu_{1n}A_n = \delta_1$$

$$-\lambda\mu_{21}A_1 + (1-\lambda\mu_{22})A_2 - \lambda\mu_{23}A_3 \cdots - \lambda\mu_{2n}A_n = \delta_2$$

$$\vdots$$

$$-\lambda\mu_{n1}A_1 - \lambda\mu_{n2}A_2 - \cdots + (1-\lambda\mu_{nn})A_n = \delta_n$$

$$\begin{bmatrix} 1-\lambda\mu_{11} & -\lambda\mu_{12} & -\lambda\mu_{13} & \cdots & -\lambda\mu_{1n} \\ -\lambda\mu_{21} & 1-\lambda\mu_{22} & & & -\lambda\mu_{2n} \\ & & \ddots & & \\ -\lambda\mu_{n1} & -\lambda\mu_{n2} & & & 1-\lambda\mu_{nn} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix} \longrightarrow (6)$$

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Solving the matrix equation (6), get the unknowns A_i's. Substitute these A_i's into (5) to get the solution $u(x)$ to the given integral equation.

Ex-1 $u(x) = 3x - 2 + \int_0^1 (x+t)u(t)\,dt \longrightarrow (1)$

Kernel $K(x,t) = x + t \Rightarrow$ It is separable.

$x + t = x.1 + t.1 \equiv a_1(x)b_1(t) + a_2(x)b_2(t)$

or, $u(x) = 3x - 2 + x\int_0^1 u(t)\,dt + \int_0^1 t\,u(t)\,dt \longrightarrow (2)$

$A_1 = \int_0^1 u(t)\,dt \longrightarrow (3)$, $A_2 = \int_0^1 t\,u(t)\,dt \longrightarrow (4)$

$a_1(x) = x$, $b_1(t) = 1$

$a_2(x) = 1$, $b_2(t) = t$.

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By virtue of (3) & (4), eq. (2) becomes,

$$u(x) = 3x - 2 + A_1 x + A_2 \rightarrow (5)$$

$$A_1 = \int_0^1 u(t)\,dt = \int_0^1 (3t - 2 + A_1 t + A_2)\,dt$$

$$= \frac{3}{2} - 2 + \frac{A_1}{2} + A_2$$

$$\text{or, } \frac{1}{2}A_1 - A_2 = -\frac{1}{2} \rightarrow (6)$$

$$A_2 = \int_0^1 t\,u(t)\,dt$$

$$= \int_0^1 t\,(3t - 2 + A_1 t + A_2)\,dt$$

$$= 1 - 1 + \frac{A_1}{3} + \frac{A_2}{2}$$

$$\frac{A_1}{3} - \frac{A_2}{2} = 0 \rightarrow (7)$$

Solving (6) & (7), $A_1 = 3$, $A_2 = 2$.

Putting A_1, A_2 into (5), $u(x) = 3x - 2 + 3x + 2 = 6x$.

$$\boxed{u(x) = 6x}$$

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Solve! $y(x) = e^x + \lambda \int_0^1 2 e^x e^t y(t)\, dt$

$A =$

$$y(x) = e^x + \lambda e^x \frac{(e^2-1)}{1-\lambda(e^2-1)} = \frac{e^x}{1-\lambda(e^2-1)}$$

$$\lambda \neq \frac{1}{e^2-1}$$

Corr. hom IE: $y(x) = \lambda \int_0^1 2 e^x e^t y(t)\, dt$