3. Method of successive approximation

To solve

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\,dt \longrightarrow (1)$$

In this method,

we take some initial approximation $u_0(x)$ for $u(x)$.

Substitute $u(x) = u_0(x)$ in the r.h.s of (1) and get the next approximation $u_1(x)$.

i.e,

$$u_1(x) = f(x) + \lambda \int_a^b K(x,t)\, u_0(t)\,dt. \longrightarrow (2)$$

Again, substitute $u_1(x)$ in the r.h.s. of (1) and get the next approximation $u_2(x)$.

i.e $$u_2(x) = f(x) + \lambda \int_a^b K(x,t)\, u_1(t)\,dt.$$

$\vdots$

Continuing in this way, we obtain,

$$u_n(x) = f(x) + \lambda \int_a^b K(x,t)\, u_{n-1}(t)\,dt.$$

If 1) $f(x) \neq 0$ is continuous in $[a,b]$.

2) $K(x,t)$ is continuous in $R = \{(x,t) : a \leq x, t \leq b\}$ & $|K(x,t)| \leq M$ $\forall (x,t) \in R$

3) $|\lambda| M (b-a) < 1$

then $u_n(x) \longrightarrow u(x)$ as $n \to \infty$.

i.e as $n \to \infty$, $u_n(x)$ approaches the unique solution of the F.I.E. (1).

Ex-1 Solve by method of successive approximation, the F.I.E

$$u(x) = x + e^x - \int_0^1 xt\, u(t)\, dt,$$

taking $u_0(x) = 0$.

Solu^n: $u(x) = x + e^x - \int_0^1 xt\, u(t)\, dt$

$$u_1(x) = x + e^x - \int_0^1 xt\, u_0(t)\, dt = x + e^x, \quad \because u_0(t) = 0.$$

$$u_2(x) = x + e^x - \int_0^1 xt\, u_1(t)\, dt.$$

$$= x + e^x - \int_0^1 xt(t + e^t)\, dt$$

$$= x + e^x - x\int_0^1 t^2\, dt - x\int_0^1 te^t\, dt.$$

$$= x + e^x - x\left[\frac{t^3}{3}\right]_0^1 - x\left\{\left[te^t\right]_0^1 - \int_0^1 e^t\, dt\right\}$$

$$= x + e^x - \frac{x}{3} - x\left\{1.e - \left[e^t\right]_0^1\right\}$$

$$= x + e^x - \frac{x}{3} - x(\cancel{e} - \cancel{e} + 1) = e^x - \frac{x}{3}.$$

$$u_3(x) = x + e^x - \int_0^1 xt\, u_2(t)\, dt.$$

$$= x + e^x - \int_0^1 xt\left(e^t - \frac{t}{3}\right) dt.$$

$$= x + e^x - x\int_0^1 te^t\, dt + x\int_0^1 \frac{t^2}{3}\, dt.$$

$\therefore \quad u_3(x) = x + e^x - x \cdot 1 + x \cdot \frac{1}{3} \cdot \left[\frac{t^3}{3}\right]_0^1$

$= x + e^x - x + x \cdot \frac{1}{3^2} = e^x + \frac{x}{3^2}.$

$u_4(x) = x + e^x - \int_0^1 x\, t\, u_3(t)\, dt.$

$= x + e^x - \int_0^1 x t \left(e^t + \frac{t}{3^2}\right) dt$

$= x + e^x - x \int_0^1 t e^t\, dt - \frac{x}{3^2} \int_0^1 t^2\, dt.$

$= x + e^x - x \cdot 1 - \frac{x}{3^2} \cdot \frac{1}{3}.$

$= \cancel{x}\; e^x - \frac{x}{3^3}.$

$\vdots$

Continuing in this way,

$u_n(x) = e^x + x \cdot \frac{(-1)^{n-1}}{3^{n-1}}$

Now, $\left| \left(-\frac{1}{3}\right)^{n-1} \right| \to 0$ as $n \to \infty$.

$\therefore \quad u_n(x) \to u(x) = e^x.$

Ans. $u(x) = e^x.$