Convex Optimization
C3 Review Test

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Raushan Raj
IIT2018031

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Ques 7. A. Barrier Method :→

In convex optimization, a barrier method is a continuous function, whose value on a point increases to infinity as the point approaches the boundary of the feasible region of an optimization problem.

Mathematically :→

A barrier function for P is any function $b(x) : \mathbb{R}^n \to \mathbb{R}$ which satisfies

→ $b(x) \geq 0 \; \forall x$ that satisfy $g(x) < 0$ and

→ $b(x) \to \infty$ as $\lim_x \max_i \{ g_{(i)}(x) \} \to 0$

The main idea in a barrier method is to dissuade points x from ever approaching the boundary of the feasible region. So now let us consider this

$$B(c) : \text{minimize } f(x) + \frac{1}{c} b(x)$$

$$\text{such that} \quad g(x) < 0$$
$$x \in \mathbb{R}^n$$

For a sequence of $c_k \to +\infty$. Note that the constraints "$g(x) < 0$" are effectively unimportant in $B(c)$, as they are never binding in $B(c)$.

★ Barrier methods solve a sequence of problems

$$\min \quad t f(x) + \phi(x)$$

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Ans 1. B Duality

The term duality, in convex optimization theory is the principle which optimization problems may be viewed from either of two perspectives, the primal problem or the dual problem. Basically, the dual problem's solution provides a lower bound to the solution of the primal problem.

Basically, duality theory applies to general linear programs, that can involve greater than less than and equality constraint

Mathematically :→

$$\text{Maximize } c^T x$$
$$\text{subject to } Ax < b$$
$$x \geq 0$$

$$\text{Maximize } b^T w$$
$$\text{s.t } A^T w \leq c$$
$$w \in R^m$$

$$\text{Minimize } bY$$
$$\text{s.t } AY \geq c$$
$$Y \geq 0$$

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Ques2 Sol^n

(a) $\nabla f = 0$

$$\begin{bmatrix} 4x^3 - 4y \\ 4y - 4x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$y = x$;

$$4x^3 - 4x = 0$$
$$\Rightarrow 4x(x^2 - 1) = 0$$
$$x^2 = 1$$

By solving this, we get three stationary points
(0, 0), (-1, -1), (+1, +1)

at (0, 0)

$$\nabla^2 f = \begin{bmatrix} 0 & -4 \\ -4 & 4 \end{bmatrix}$$ is an indefinite matrix.

$\therefore$ (0, 0) is a saddle point

(-1, 1)

$$\nabla^2 f = \begin{bmatrix} 12 & -4 \\ -4 & 4 \end{bmatrix}$$ is +ve definite

at (1, 1)

$$\nabla^2 f = \begin{bmatrix} 12 & -4 \\ -4 & 4 \end{bmatrix}$$ is +ve definite

(-1, -1) and (+1, +1) are local minima

(b)

$$H = \begin{bmatrix} 12x^2 & -4 \\ -4 & 4 \end{bmatrix}$$

Since It is +ve definite for some $\begin{pmatrix} x \\ y \end{pmatrix}$ and indefinite for some $\begin{pmatrix} x \\ y \end{pmatrix}$.

$\therefore$ It is not a convex function.

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Ques3 Solⁿ

(a) $x_0 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\nabla f = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$

So Methods direction is

$d_0 = \cancel{-\frac{\nabla f}{x_0}}$ $-\frac{\nabla f}{x_0} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$

$x_1 = x_0 + \alpha d_0 = \begin{bmatrix} 1 - 2\alpha \\ -1 + 2\alpha \\ 0 \end{bmatrix}$

$f(\alpha) = (1-2\alpha)^2 + (2\alpha-1)^2 + 0$

$\frac{df}{d\alpha} = 2(1-2\alpha)(-2) + 2(2\alpha-1)(2) = 0$

$\Rightarrow 2\alpha - 1 = 0$

$\alpha = 1/2$

$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ at which $\nabla f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x_1$ is a local minimum

It took one iteration to reach the ontimum.

(b) Yes, Because the condition number 2 The Hessian matrix is 1. $\left(\frac{\lambda_{max}}{\lambda_{min}}\right)$

$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\kappa_{cond} = 1$

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Ques 4 Soln

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$ [Newton's method for optimisation]

$$\Rightarrow x_{k+1} = x_k - \frac{4x_k^3}{12x_k^3}$$

$$\Rightarrow x_{k+1} = \frac{2}{3}x_k$$

If, $x_0 = 1$

$$x_1 = \frac{2}{3}x_0 = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$x_2 = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$$

$$x_3 = \frac{2}{3} \times \frac{4}{9} = \frac{8}{27}$$

⋮

$\therefore$ Convergence rate is linear

$$\boxed{x_{k+1} = \frac{2}{3}x_k}$$

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Ques 5 Soln

First Order KKT Condition

$$\nabla f + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 + \mu_3 \nabla g_3 = 0$$

$$g_1 \leq 0$$

$$g_2 < 0$$

$$g_3 \leq 0$$

$$\mu_1 g_1 = 0, \quad \mu_2 g_2 = 0, \quad \mu_2 g_3 = 0$$

$$\mu_1 \ \mu_2 \ \mu_3 \geq 0$$

$$\nabla f = \begin{bmatrix} -1 \\ 0 \end{bmatrix}; \quad \nabla g_1 = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}; \quad \nabla g_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\nabla g_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

at $(1, 0)$

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$g_1 = 1$ (active)

g_2 is ~~also~~ inactive

g_3 is also active

Since g_2 is inactive, $\mu_2 = 0$, Based on Complementary Slackness Condition.

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 2\mu_1 + 0\mu_3 &= 1 \\ 0\mu_1 - \mu_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} \mu_1 &= 1/2 \\ \mu_3 &= 0 \end{aligned}$$

Since all the condition are satisfied at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore$ ~~It~~ It is a KKT point.

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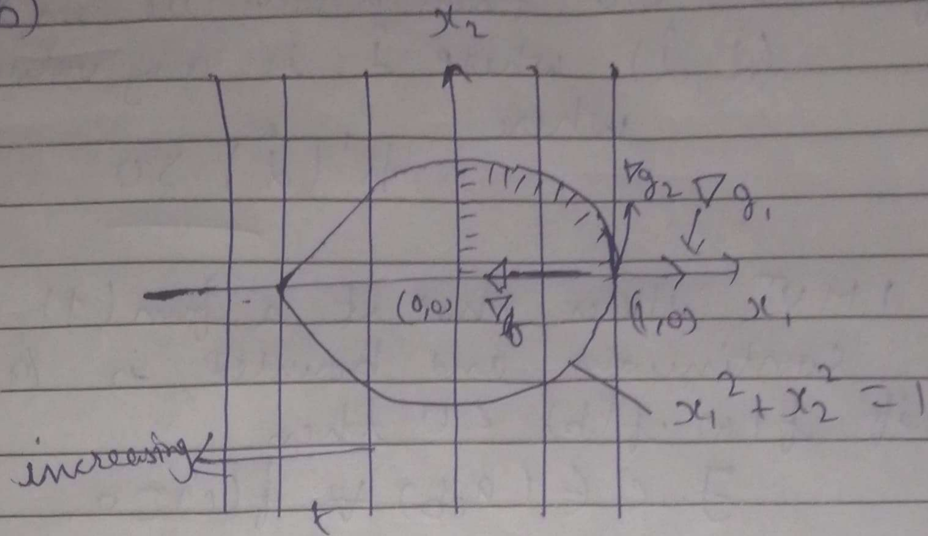
(b)



From graph, it is a unique minimum

Ques 1 Sol^n

Aim → Find the optimal step factor α such that $f(x+\alpha d)$ is ~~maxim~~ minimised.

Suppose,

x be the current iteration and d is the minimal direction i.e

$$f(x+\epsilon d) < f(x) \quad \forall \; \epsilon > 0$$

Assume a function $h(\alpha) = f(x+\alpha d)$

$h(\alpha)$ is a function in α (step factor)

thus our problem now becomes

find α such that $h(\alpha)$ is minimum.

hence $h(\alpha) = 0$

$h'(\alpha) = \nabla f(x+\alpha d)^T \cdot d$

$h'(\bar{\alpha})$ can be one of the 3 rd case.

$h'(\bar{\alpha}) = 0$, then $\bar{\alpha}$ is the optimal one.

$h'(\bar{\alpha}) > 0$, then α lies in the region $(0, \bar{\alpha})$ because, $h'(0) < 0$ and $h'(\alpha)$ is monotonically in as it is a convex function. Hence by intermediate value theorem The function

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If $h'(\hat{\alpha}) < 0$, then α will lie in range $(\bar{\alpha}, \hat{\alpha})$. where $\hat{\alpha}$ is any value where

$$h'(\hat{\alpha}) > 0$$

IMVT states that if a fun (t) is continuous and bounded in (a, b) and if $f(a) \cdot f(b) < 0$ then

$$\exists\, c \in (a, b) \ \forall\ f(c) = 0.$$

Ques 6. Solⁿ

$$\cancel{A^2 f = \begin{bmatrix} 12x^2 & -4 \\ -4 & 4 \end{bmatrix}}$$

~~Maximizing $f(x_1, x_2) = x_1 + x_2$~~
~~subject to $g(x_1, x_2) = -(x_1^2 + x_2^2) \geq 0,\ x_1, x_2 \geq 0$~~

~~(a) The feasible set will contain $(0, 0)$.~~

~~(b) Slater Condition is not satisfied.~~

~~(c) Yes, we have a solution to the problem i.e $(0, 0)$.~~

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Q6 Solⁿ Maximise $f(x_1, x_2) = x_1 + x_2$
subject to $g(x_1, x_2) = -(x_1^2 + x_2^2) \geq 0$
$x_1 \geq 0$
$x_2 \geq 0$

Can be rearranged
$f(x_1, x_2) = x_1 + x_2$
$g(x_1, x_2) = x_1^2 + x_2^2 \leq 0$
$-x_1 \leq 0$
$-x_2 \leq 0$

In general for such conditions,
Lagrangian is

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \mu_i (0 - g_i(x)) + \sum_{j=1}^{k} \lambda_j (0 - h_j(x))$$

So for the given ques
Lagrangian is
$L(x_1, x_2, \mu_1, \mu_2, \mu_3)$
$= x_1 + x_2 + \mu_1(-x_1^2 - x_2^2) + \mu_2 x_2 + \mu_3 x_2$

KKT Conditions are

$$\frac{\partial L}{\partial x_1} = 1 - 2\mu_1 x_1 + \mu_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 1 - 2\mu_2 x_2 + \mu_3 = 0$$

$1 + \mu_2 = 2\mu_1 x_1$
$1 + \mu_3 = 2\mu_1 x_2$

(b) Slater condition is not satisfied.

(c) Yes, we have a solution to this problem