Convex Optimization
C2 Review Test

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Answers

Ques 1 Solⁿ

The equation of dual problem are as follows:→

$$2y_1 - 3y_2 - n_1 = 2$$
$$y_1 + y_2 - n_2 = 16$$
$$-y_1 + 2y_2 - n_3 = 2$$

Now, given the values of x_1, x_2 and x_3, Here, first of all we calculate the value of the slack variables s_1 and s_2. The solution needs to be feasible in order to optimal. In this case that means the decision and slack variables, all should be greater than or equal to zero. Next, it, it is optimal whenever an x_j or s_i is non-zero.

Now complementary slackness says the corresponding n_j or y_i must be 0. We see that what's left of the equation of the dual when those variables are set ~~of 0.~~ to 0. If we can find a solution that ~~it~~ is feasible, then we conclude the alleged solution is optimal. If there's not a feasible solution, it isn't.

We shall discuss with case.

☆☆ For solution to be feasible all decision

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1st case → with $x_1=6, x_2=0$ and $x_3=12$ and $s_1=3$. This is not feasible therefore it is not optimal.

2nd case → with $x_1=0, x_2=2, x_3=5, s_1=0$ and $s_2=0$

⇒ the solution of the primal is feasible. From complementary slackness, $n_2=0$ and $n_3=0$. So, the equation of dual become

$$2y_1-3y_2-n_1=2 \quad -(I)$$
$$y_1+y_2=16 \quad -(II)$$
$$-y_1+2y_2=2 \quad -(III)$$

Now from the eqn 2nd and 3rd, $y_1=10, y_2=6$ and then from the first $n_1=0$.

⇒ All ≥0. Hence the solution of the dual is feasible.

⇒ Case 2 is feasible and optimal as well.

Case 3 with $x_1=0, x_2=0, x_3=6, S_1=3$ and $s_2=0$. Then the solution of primes is feasible.

From complementary slackness, $n_3=0$ and $y_1=0$. So the equation of dual become as follows:-

$$-3y_2-n_1=2 \quad -(IV)$$
$$+y_2-n_2=16 \quad -(V)$$
$$+2y_2=2 \quad -(VI)$$

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The (vi)th and the last equation says $y_2 = 1$ then from the (iv)th equation $n_1 = -5$.
The solution of the dual is not feasible
Since solution is not feasible, so Case 3 is not optimal.

In the last part, we again take $x_1 = 0$ $x_2 = 0$ and $x_3 = 6$, so $s_1 = 3$ and $s_2 = 0$ but c_2 and c_3 are unspecified ($\because c_1$ is kept at 21. With complementary slackness telling us $n_3 = 0$ and $y_1 = 0$. the eqn of dual becomes

$$-3y_2 - n_1 = 2 \quad -(VII)$$
$$+y_2 - n_2 = c_2 \quad -(VIII)$$
$$+2y_2 = c_3 \quad -(IX)$$

But here, if $y_2 \geq 0$ and $n_2 \geq 0$. the left side of the (VII)th eqn ≤ 0. So This is impossible. There are no such values of c_2 and c_3.

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Ques1 $f(x) = (x_1 + 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2$

given $x^1 = [9, 8]^T$

Sol^n

$$g(x) = \begin{pmatrix} 2(x_1+2x_2-7) + 2(2x_1+x_2-5)(2) \\ 2(x_1+2x_2-7)(2) + 2(2x_1+x_2-5) \end{pmatrix}$$

$$H(x) = \begin{bmatrix} 2+8 & 4+4 \\ 4+4 & 8+2 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}$$

As we see that $H(x)$ is always positive definite, So we have no issues that will occur and hence we can continue.

~~$H^{-1}(x) = \frac{1}{18}\begin{bmatrix} 5 & -4 \\ -4 & \end{bmatrix}$~~

$$H^{-1}(x) = \frac{1}{18}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$ [$\because$ here $g^k = g(x^k)$]

$\therefore$ $d^1 = -(H^2)^{-1} g$

$$d^1 = \frac{-1}{18}\begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}\begin{bmatrix} 120 \\ 114 \end{bmatrix} = \frac{-1}{18}\begin{bmatrix} 114 \\ 90 \end{bmatrix}$$

$a = 1$

$\therefore$ $x^2 = x^1 + \alpha^1 d^1$

$$= \begin{pmatrix} 9 \\ 8 \end{pmatrix} + 1\left(-\frac{1}{18}\right)\begin{pmatrix} 114 \\ 90 \end{pmatrix}$$

$$= \frac{9}{8} - \begin{pmatrix} \frac{16}{3} \\ 5 \end{pmatrix}$$

As $g'(x) = 0$, for only one value that is $x_1 = 1$ and $y_1 = 3$. So we have to do one more iteration.

So, Here we go.

$$d_2 = -(H_x)^{-1} g^2$$

$$\Rightarrow d_2 = -\frac{1}{18}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} \frac{50}{3} \\ 40/3 \end{bmatrix} = -\frac{1}{18}\begin{bmatrix} 30 \\ 0 \end{bmatrix}$$

$$a_2 = 1$$

$$\therefore x^3 = x^2 + a_2 d_2$$

$$= \begin{bmatrix} \frac{8}{3} \\ 3 \end{bmatrix} + \frac{-1}{18}\begin{bmatrix} 30 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{3} - \frac{30}{18} \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} - \frac{5}{3} \\ 3 \end{bmatrix} = \frac{1}{3}$$

$\therefore$ We got $x^3 = \frac{1}{3}$

Also, $g(x^3) = 0$

and $H(x^3)$ is positive definite.

Hence x^3 is local minima

(ii) After 2 iteration using newton's method, my function returns the optimal value (minimum)

(iii) Yes, the minimum is the global minimum as $g(x)$ is a linear equation that can only have one root.

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Ques3

KKT Condition

Given general problem

$\min_{x \in R^n} f(x)$

subject to $h_i(x) \leq 0 \; ; i = 1 \ldots n$

$l_j(x) = 0 \; ; j = 1 \ldots r$

x, KKT Conditions are

$\Rightarrow 0 \in \partial f(x) + \sum_{i=1}^{m} \partial h_i(x) + \sum_{i=1}^{r} v_j \, \partial l_j(x)$

(Stationarity)

$\Rightarrow \mu_i h_i(x) = 0 \;\; \forall i$ (complementary slackness)

$\Rightarrow h_i(x) \leq 0, \; l_j(x) = 0 \;\; \forall i, j$ (primal feasibility)

$\Rightarrow \mu_i \geq 0 \;\; \forall i$ (dual feasibility)

Necessity

If x^* and μ^*, v^* are primal and dual and with 0 duality gap then $x^*, u^* \; v^*$ satisfy under KKT condition.

If x^* and μ^*, v^* satisfy the KKT condition then x^*, μ^*, v^* are primal and dual solution.

In summary, KKT condition remains -

(I) always sufficient

(II) ~~nece~~ necessary under strong duality

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Ques 3 Soln

As we see in newton step for $\min_{x \in R^n}$ $f(x)$ subject to $Ax = b$

This is convex problem, so no inequality constraints. So by KKT condition x is a soln if and only if

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -c \\ 0 \end{bmatrix}$$

For some u. Linear system combines ~~stationary~~ stationarity, primal feasibility.

→ Stationary condition can be obtained by
$\nabla_x L(x, v) = 0$
Using derivative properties

$$\boxed{Qx + b + A^T v = 0}$$

→ Primal feasibility: $\boxed{Ax = 0}$

Stationarity condition and primal feasibility can be combined as →

$$\begin{bmatrix} Q & AT \\ A & 0 \end{bmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{bmatrix} -b \\ v \end{bmatrix}$$

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Ques2 Sol^n If Q has negative eigen value
$f^* = -\infty$
if $Q > 0 \Rightarrow$ unique global solution

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

To find $\alpha_k \Rightarrow$ to minimize $f(x_k+1)$
Let $g(\alpha) = f(x_k - \alpha \nabla f(x_k))$

$$= \frac{1}{2}\left[(x_k - \alpha \nabla f(x_k))^T Q (x_k - \alpha \nabla f(x_k)) - b^T (x_k - \nabla f(x_k))\right]$$

we can see $g(\alpha)$ is quadoratic and concave.

$$g(\alpha) = a\alpha^2 + d\alpha + c$$

$\Rightarrow$ min $(g(\alpha))$ will be at $\frac{-d}{2a}$

$$a = \frac{1}{2} \nabla f^T(x_k) Q \nabla f(x_k)$$

$$d = (b^T - x_k^T Q) \nabla f^T(x_k)$$

$$= -\nabla f^T(x_k) \nabla f(x_k)$$

$$\Rightarrow \alpha_k = \frac{\nabla f^T(x_k) \nabla f(x_k)}{\nabla f^T(x_k) Q \nabla f(x_k)}$$

$$\Rightarrow x_{k+1} = x_k - \frac{\nabla f^T(x_k) \nabla f(x_k)}{\nabla f^T(x_k) Q \nabla f(x_k)} \nabla f(x_k)$$

where $\nabla f(x_k) = Q x_k - b$