ROJECT REPORT

Swallowtail Catastrophe Model

"Introduction to Singularity Theory"

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Introduction

A dynamical system can be defined as a set of interacting variables with properties of continuity, determination, and infinite duration. The mechanisms of a dynamical system include: bifurcations, transformation of a system from one type to another. Bifurcation theory, as proposed by [1], was further developed through the application of the work of mathematician Rene Thom. [2] provided the complete proof for the existence of seven elementary catastrophes, which he termed, in order of complexity, the fold, cusp, swallowtail, butterfly, elliptic umbilic, hyperbolic umbilic, and parabolic umbilic. Catastrophe theory is derived from topology, a field of mathematics that studies the properties of surfaces in numerous dimensions. According to Thom's theory, catastrophe models are the key points to analyze the catastrophe characteristics of system. However, how to transform the mathematical models into catastrophe models of a system is a worth studying problem. For this purpose, this report discus the transformation methods to change the mathematical models into swallowtail catastrophe models of a nonlinear system.

The Swallowtail Catastrophe Modeling

Next, we derive the catastrophe control parameters as function of parameters. Analyzing the degenerate critical points of catastrophe potential function is proposed to determine the qualitative properties of potential function at those points. The potential function of Swallowtail catastrophe model is defined by [3]. Here the unfolding is

$$V_{abc}(x) = x^5 + ax^3 + bx^2 + cx \tag{1}$$

Where a, b, and c are control parameters and x is the state variable. Equilibrium points are obtained by taking the first derivative of (1) with respect to x equal to 0; and the catastrophe manifold is given by

$$0 = \frac{d}{dx} V_{abc}(x) = 5x^4 + 3ax^2 + 2bx + c \tag{2}$$

Next, considering (2), which describes a balanced curved surface, we use Taylor expansion of the nonlinear system which is approximated to fourth power, as expansion (3) shows:

$$F(t) = a_0 + \ a_1 t + a_2 t^2 + a_3 x^3 + a_4 t^4 \tag{3}$$

Through Tschirnhaus transformation [6], let $t = x - \frac{a_3}{4a_4}$, we get

$$F(x) = a_4 x^4 + \left(a_2 - \frac{3a_3^2}{8a_4}\right)x^2 + \left(-\frac{a_2 a_3}{2a_4} + \frac{a_3^2}{8a_4} + a_1\right)x + a_0 - \frac{a_1 a_3}{4a_4} - \frac{3a_3^4}{256a_4^3} + \frac{a_2 a_3^2}{16a_4^2} \tag{4}$$

The corresponding potential function of expression (4) is

$$Y(x) = \frac{1}{5}a_4 x^5 + \left(\frac{1}{3}a_2 - \frac{a_3^2}{8a_4}\right)x^3 + \left(-\frac{a_2 a_3}{4a_4^2} + \frac{a_3^2}{16a_4^3} + a_1\right)x^2 + \left(a_0 - \frac{a_1 a_3}{4a_4^2} - \frac{3a_3^4}{256a_4^3} + \frac{a_2 a_3^2}{16a_4^2}\right)x \tag{5}$$

Change the coefficient of the highest order item to be 1,

$$V(x) = x^5 + \left(\frac{5}{3a_4}a_2 - \frac{5a_3^2}{8a_4^2}\right)x^3 + \left(-\frac{5a_2 a_3}{4a_4^2} + \frac{5a_3^2}{16a_4^3} + a_1\right)x^2 + \left(\frac{5a_0}{a_4} - \frac{5a_1 a_3}{4a_4^2} - \frac{15a_3^4}{256a_4^4} + \frac{5a_2 a_3^2}{16a_4^3}\right)x \tag{6}$$

$$Let \begin{cases} u = \dfrac{5}{3a_4}a_2 - \dfrac{5a_3^2}{8a_4^2} \\ v = -\dfrac{5a_2 a_3}{4a_4^2} + \dfrac{5a_3^2}{16a_4^3} + a_1 \\ w = \dfrac{5a_0}{a_4} - \dfrac{5a_1 a_3}{4a_4^2} - \dfrac{15a_3^4}{256a_4^4} + \dfrac{5a_2 a_3^2}{16a_4^3} \end{cases}$$

We get,

$$V(x) = x^5 + ux^3 + vx^2 + wx \tag{7}$$

Expression (7) is the normal potential function form of swallowtail catastrophe model (1), and the corresponding catastrophe manifold is by taking the first derivative of (7) with respect to x equal to 0,

$$M: 5x^4 + 3ux^2 + 2vx + w = 0 \tag{8}$$

And we can obtain the singularity set by vanishing the second derivative of (7) with respect to x,

$$20x^3 + 6ux + 2v = 0 \tag{9}$$

Eliminate x from expression (8) and (9), and then bifunction set can be obtained as follows,

$$\begin{cases} x(u,v) = w(uv^2 + 3v^4) \\ y(u,v) = w(-2uv - 4v^3) \\ z(u,v) = wu \end{cases} \tag{10}$$

Because the bifurcation set looks like a swallowtail, we call it swallowtail catastrophe. In the next section, we will show you how we visualize swallowtail catastrophe in term of 2D graph and 3D graph.

Visualization

We conducted our experiment using Wolfram Mathematica 13.0. The full project files have been posted on my GitHub (https://github.com/rauzansumara/swallowtail-catastrophe-modeling). A catastrophe which can occur for three control factors and one behavior axis. The swallowtail catastrophe is the universal unfolding of singularity with codimension 3. In three unfolding parameters, the parametric equations (10) will be used. The surface graph uses $u \in [-2,2]$, $v \in [-0.8,0.8]$, and $w \in [1.0,2.0]$.

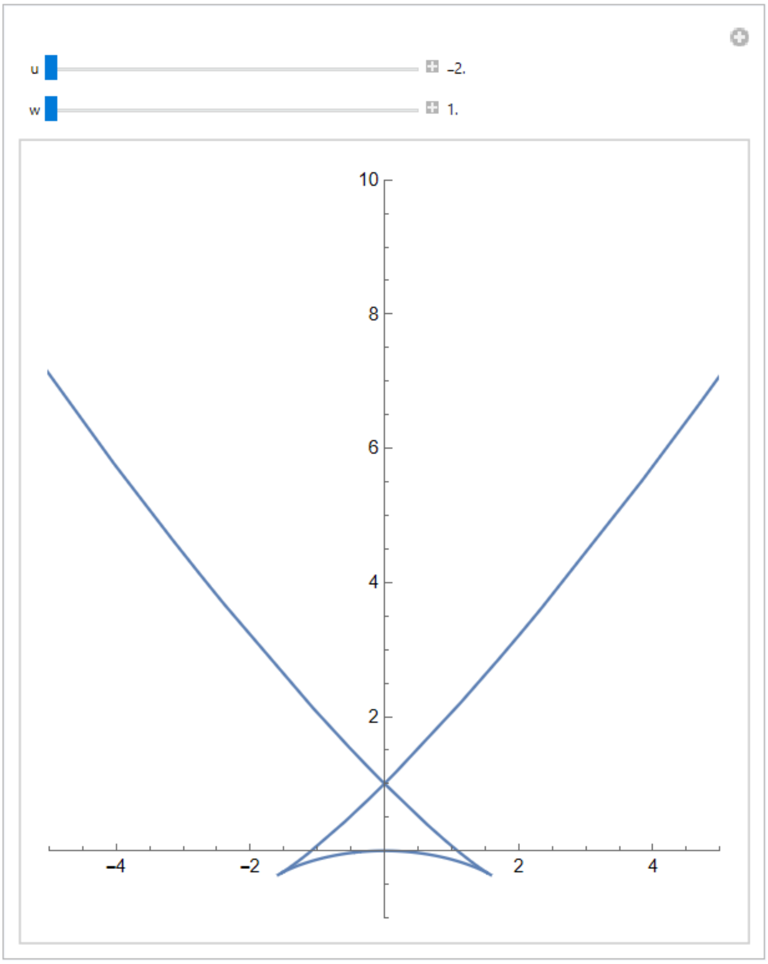

Figure 1. The set of Swallowtail ramification points for u < 0

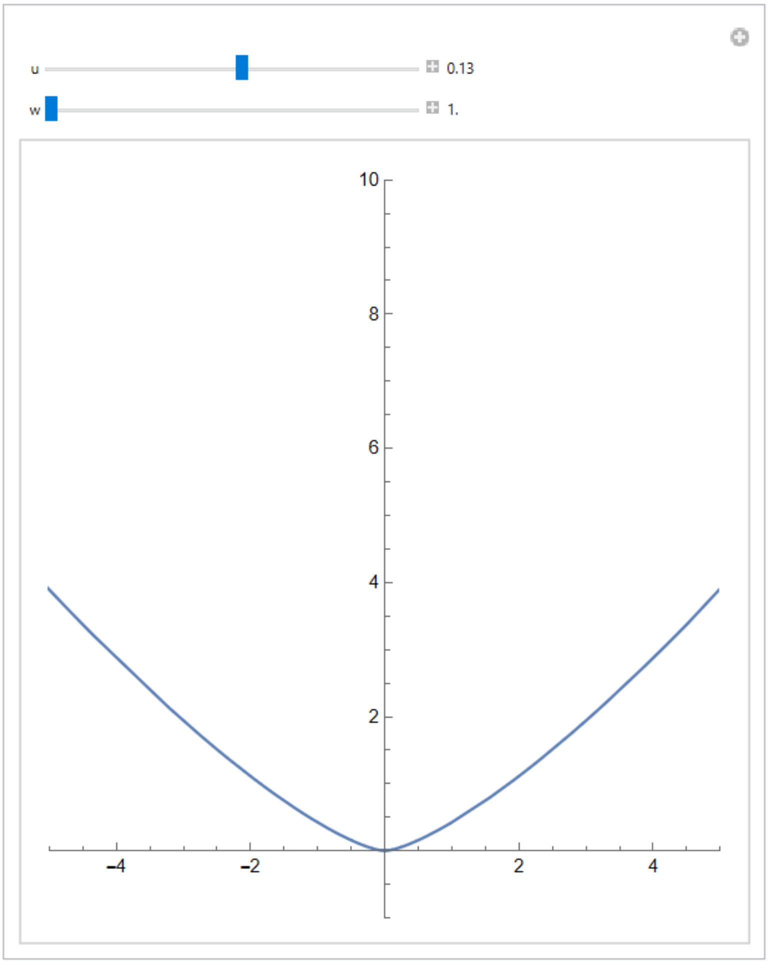

Figure 2. The set of Swallowtail ramification points for u > 0

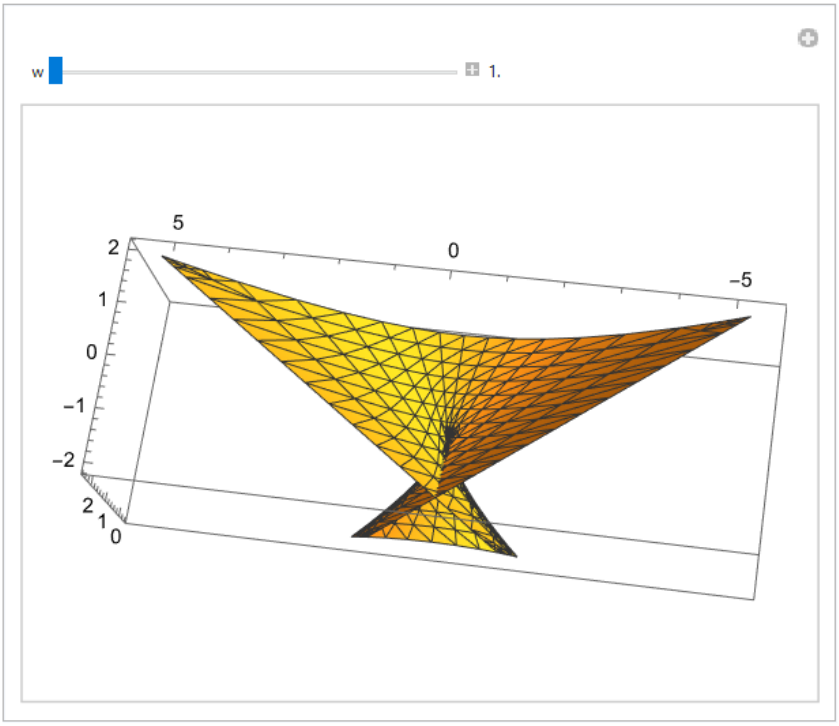

Figure 3. The 3D Graph of Parametric Equations for w = 1

References

[1] Raschevsky, N. (1968). Looking at history through mathematics. Cambridge: MIT Press

[2] Thom, R. (1972). Stabilite structurelle et m orphogenese (Structural stability and morphogenesis'). (H. Fowler, Trans.). Reading: Benjamin

[3] R. Gilmore. (1981). Catastrophe Theory for Scientists and Engineers, John Wiley & Sons, New York, USA

[4] Mathworld (2022). Swallowtail Catastrophe. [Article is accessed on January 12, 2022] https://mathworld.wolfram.com/SwallowtailCatastrophe.html