Numerical Methods in Engineering

ORDINARY DIFFERENTIAL EQUATIONS-2 (ODEs)

Taylor Series Methods

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Learning Objectives of Lesson 2

- Derive Euler formula using the Taylor series expansion.
- Solve the first order ODEs using Euler method.
- Assess the error level when using Euler method.
- Appreciate different types of errors in the numerical solution of ODEs.
- Improve Euler method using higher-order Taylor Series.

Taylor Series Method

The problem to be solved is a first order ODE:

$$\boxed{\frac{dy(x)}{dx} = f(x, y), \quad y(x_0) = y_0}$$

Estimates of the solution at different base points:

$$y(x_0 + h),\ y(x_0 + 2h),\ y(x_0 + 3h),\ \ldots.$$

are computed using the truncated Taylor series expansions.

Taylor Series Expansion

Truncated Taylor Series Expansion

$$y(x_0 + h) \approx \sum_{k=0}^{n} \frac{h^k}{k!} \left(\left. \frac{d^k y}{dx^k} \right|_{x=x_0,\, y=y_0} \right)$$

$$\approx y(x_0) + h \left. \frac{dy}{dx} \right|_{\substack{x=x_0, \\ y=y_0}} + \frac{h^2}{2!} \left. \frac{d^2 y}{dx^2} \right|_{\substack{x=x_0, \\ y=y_0}} + \ldots + \frac{h^n}{n!} \left. \frac{d^n y}{dx^n} \right|_{\substack{x=x_0, \\ y=y_0}}$$

The n^{th} order Taylor series method uses the n^{th} order Truncated Taylor series expansion.

Euler Method

- First order Taylor series method is known as Euler Method.

- Only the constant term and linear term are used in the Euler method.

- The error due to the use of the truncated Taylor series is of order $O(h^2)$.

First Order Taylor Series Method (Euler Method)

$$y(x_0+h)=y(x_0)+h\left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}}+O(h^2)$$

$$Notation:$$

$$x_n=x_0+nh, \qquad y_n=y(x_n),$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=x_i,\\ y=y_i}}=f(x_i,y_i)$$

$$Euler\ \ Method$$

$$y_{i+1}=y_i+h\ f(x_i,y_i)$$

Euler Method

Problem:

Given the first order ODE: $\dot{y}(x) = f(x, y)$

with the initial condition: $y_0 = y(x_0)$

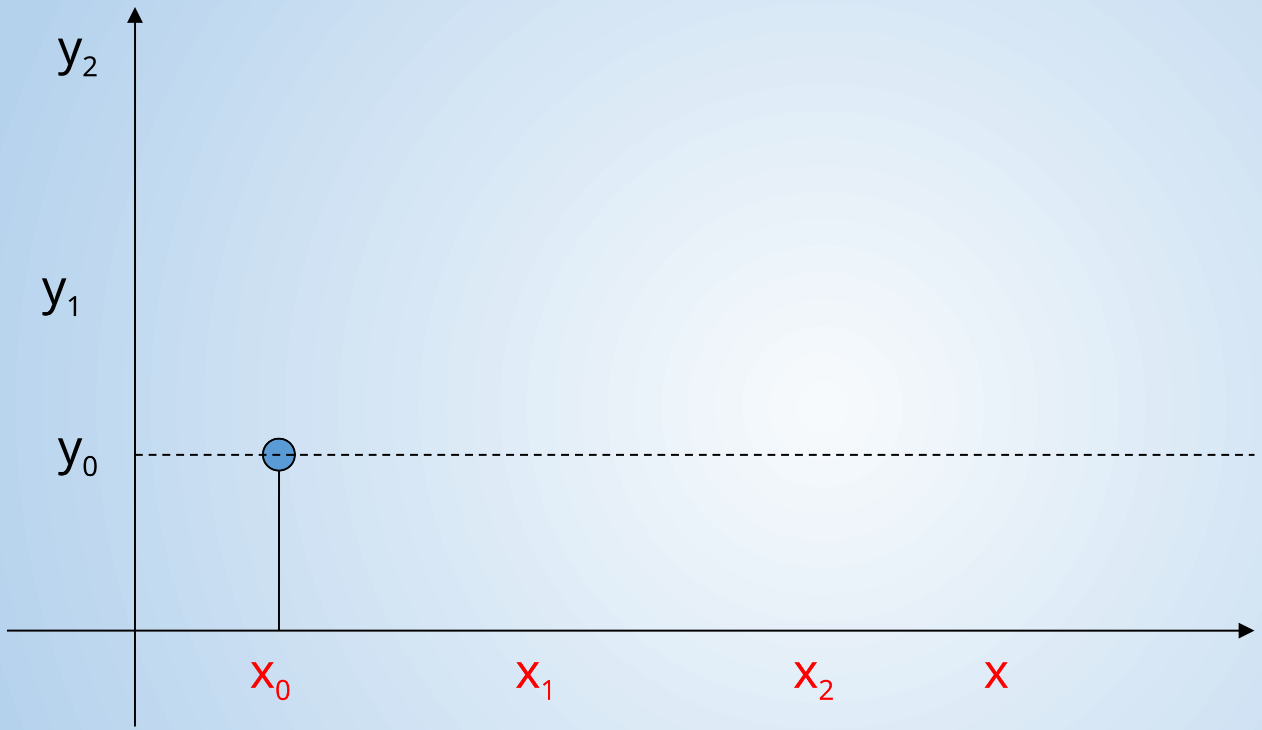
Determine: $y_i = y(x_0 + ih)$ $\quad$ *for* $i = 1, 2, \ldots$

Euler Method:

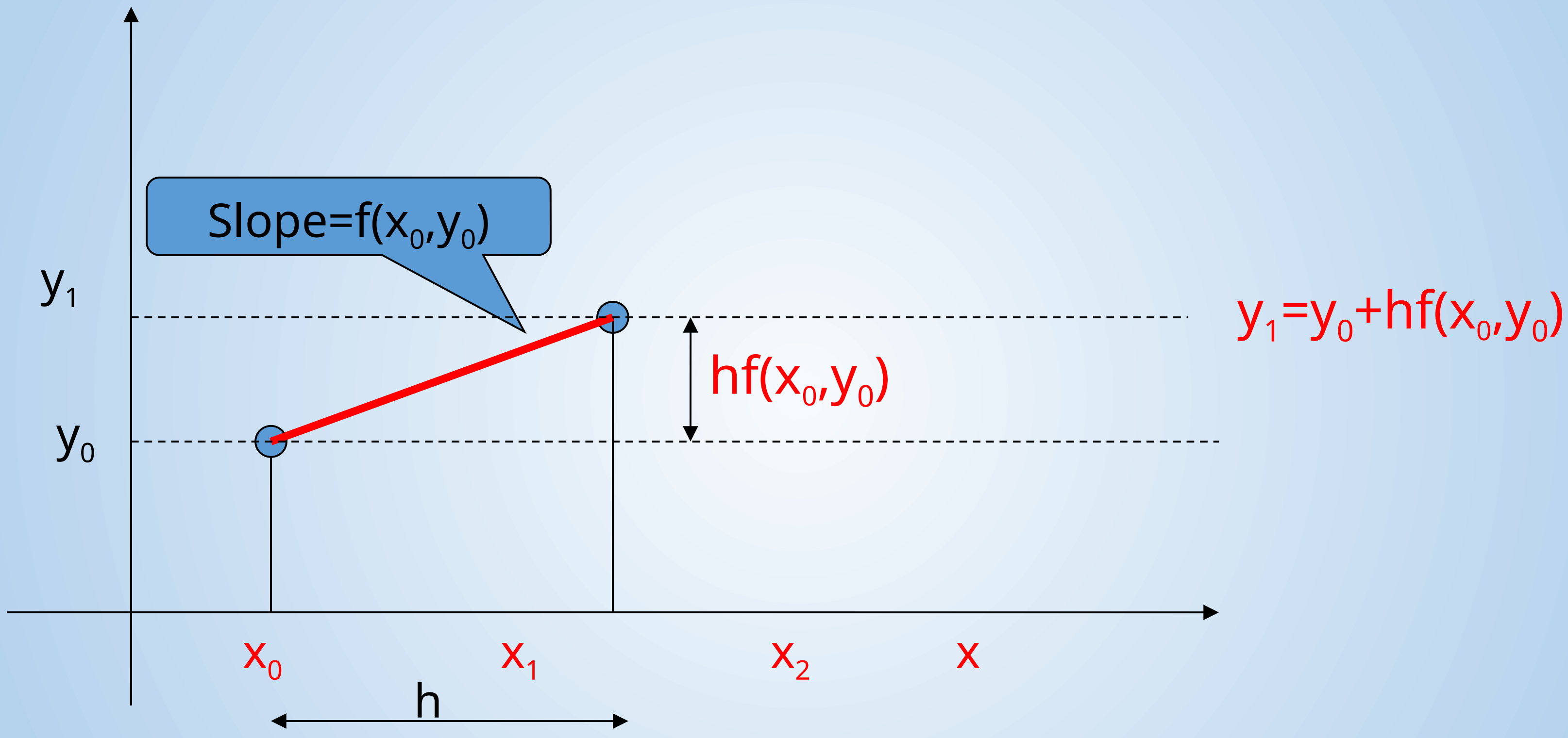
$$y_0 = y(x_0)$$

$$y_{i+1} = y_i + h\, f(x_i, y_i) \qquad \text{for } i = 1, 2, \ldots$$

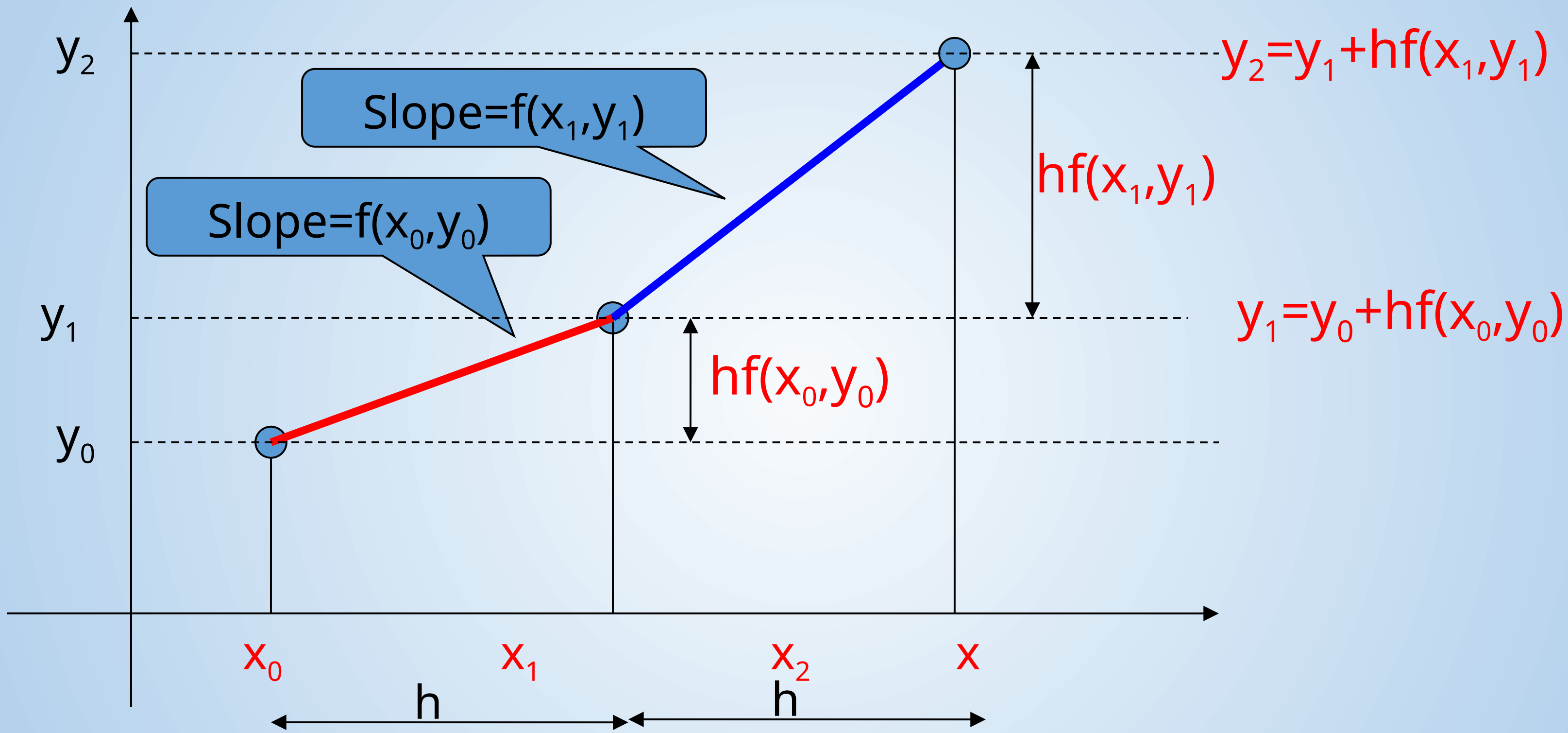
Interpretation of Euler Method



Interpretation of Euler Method



Interpretation of Euler Method



Example 1

Use Euler method to solve the ODE:

$$\frac{dy}{dx} = 1 + x^2, \qquad y(1) = -4$$

to determine y(1.01), y(1.02) and y(1.03).

Example 1

$$f(x,y) = 1 + x^2, \quad x_0 = 1, \quad y_0 = -4, \quad h = 0.01$$

Euler Method

$$y_{i+1} = y_i + h\, f(x_i, y_i)$$

$$Step1: \quad y_1 = y_0 + h\, f(x_0, y_0) = -4 + 0.01(1 + (1)^2) = -3.98$$

$$Step2: \quad y_2 = y_1 + h\, f(x_1, y_1) = -3.98 + 0.01\left(1 + (1.01)^2\right) = -3.9598$$

$$Step3: \quad y_3 = y_2 + h\, f(x_2, y_2) = -3.9598 + 0.01\left(1 + (1.02)^2\right) = -3.9394$$

Example 1

$$f(x,y) = 1 + x^2, \quad x_0 = 1, \quad y_0 = -4, \quad h = 0.01$$

Summary of the result:

i	xi	yi
0	1.00	-4.00
1	1.01	-3.98
2	1.02	-3.9595
3	1.03	-3.9394

Example 1

$$f(x,y) = 1 + x^2, \quad x_0 = 1, \quad y_0 = -4, \quad h = 0.01$$

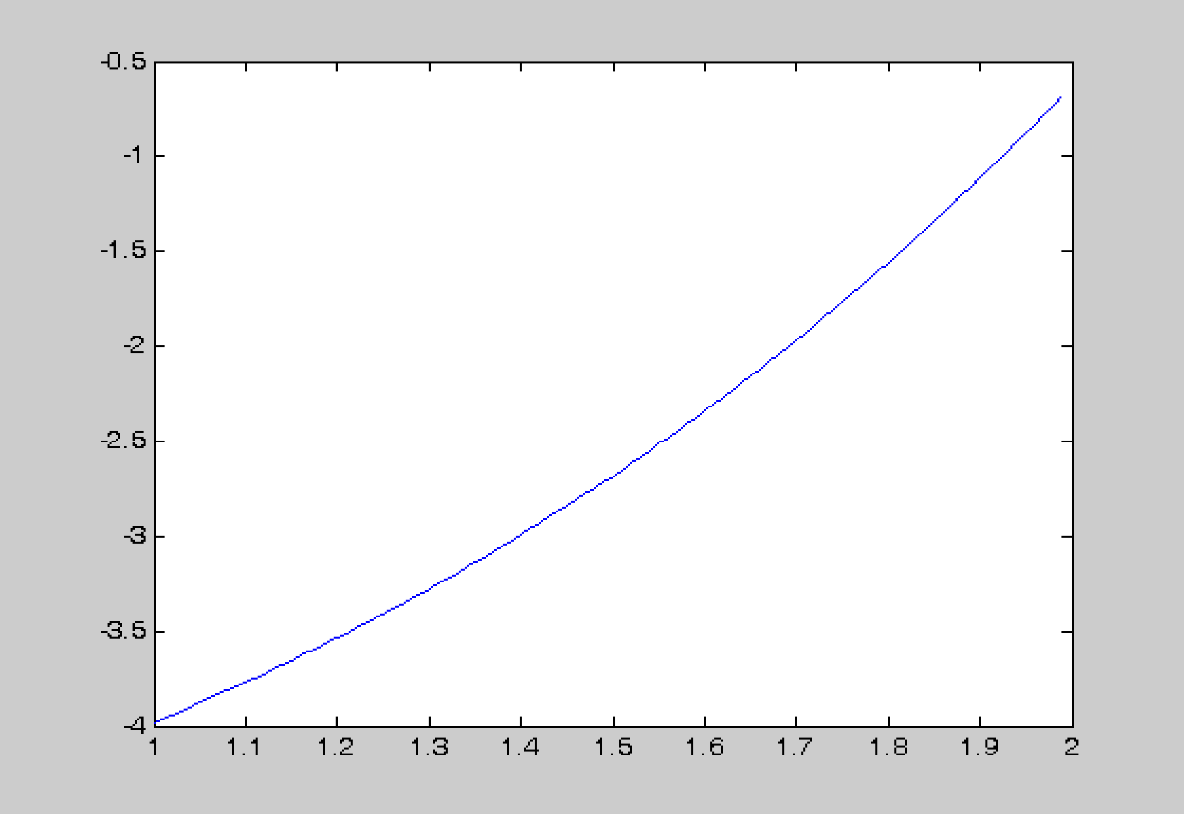
Comparison with true value:

i	xi	yi	True value of yi
0	1.00	-4.00	-4.00
1	1.01	-3.98	-3.97990
2	1.02	-3.9595	-3.95959
3	1.03	-3.9394	-3.93909

Example 1

$$f(x,y) = 1 + x^2, \quad x_0 = 1, \quad y_0 = -4, \quad h = 0.01$$

A graph of the solution of the ODE for 1<x<2



Types of Errors

- Local truncation error:

  Error due to the use of truncated Taylor series to compute x(t+h) in one step.

- Global Truncation error:

  Accumulated truncation over many steps.

- Round off error:

  Error due to finite number of bits used in representation of numbers. This error could be accumulated and magnified in succeeding steps.

Second Order Taylor Series Methods

$$\text{Given} \quad \frac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0$$

Second order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + O(h^3)$$

$\dfrac{d^2y}{dx^2}$ needs to be derived analytically.

Third Order Taylor Series Methods

$$\boxed{Given \quad \frac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0}$$

Third order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + \frac{h^3}{3!}\frac{d^3y}{dx^3} + O(h^4)$$

$\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ need to be derived analytically.

High Order Taylor Series Methods

$$Given\ \frac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0$$

n[th] order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + \ldots. + \frac{h^n}{n!}\frac{d^ny}{dx^n} + O(h^{n+1})$$

$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots, \frac{d^ny}{dx^n}$ need to be derived analytically.

Higher Order Taylor Series Methods

- High order Taylor series methods are more accurate than Euler method.

- But, the 2nd, 3rd, and higher order derivatives need to be derived analytically which may not be easy.

Example 2

Second order Taylor Series Method

Use Second order Taylor Series method to solve:

$$\frac{dx}{dt} + 2x^2 + t = 1, \ x(0) = 1, \qquad use \quad h = 0.01$$

What is: $\dfrac{d^2 x(t)}{dt^2}$?

Example 2

Use Second order Taylor Series method to solve:

$$\frac{dx}{dt} + 2x^2 + t = 1, \; x(0) = 1, \qquad use \quad h = 0.01$$

$$\frac{dx}{dt} = 1 - 2x^2 - t$$

$$\frac{d^2x(t)}{dt^2} = 0 - 4x\frac{dx}{dt} - 1 = -4x(1 - 2x^2 - t) - 1$$

$$\boxed{x_{i+1} = x_i + h(1 - 2x_i^2 - t_i) + \frac{h^2}{2}(-1 - 4x_i(1 - 2x_i^2 - t_i))}$$

Example 2

$$f(t,x) = 1 - 2x^2 - t, \quad t_0 = 0, \quad x_0 = 1, \quad h = 0.01$$

$$x_{i+1} = x_i + h(1 - 2x_i^{\ 2} - t_i) + \frac{h^2}{2}(-1 - 4x_i(1 - 2x_i^{\ 2} - t_i))$$

$Step\,1:$

$$x_1 = 1 + 0.01(1 - 2(1)^2 - 0) + \frac{(0.01)^2}{2}(-1 - 4(1)(1 - 2 - 0)) = 0.9901$$

$Step\,2:$

$$x_2 = 0.9901 + 0.01(1 - 2(0.9901)^2 - 0.01) + \frac{(0.01)^2}{2}(-1 - 4(0.9901)(1 - 2(0.9901)^2 - 0.01)) = 0.9807$$

$Step\,3:$

$$x_3 = 0.9716$$

Example 2

$$f(t,x) = 1 - 2x^2 - t, \quad t_0 = 0, \; x_0 = 1, \; h = 0.01$$

Summary of the results:

i	t_i	x_i
0	0.00	1
1	0.01	0.9901
2	0.02	0.9807
3	0.03	0.9716

Programming Euler Method

Write a MATLAB program to implement Euler method to solve:

$$\frac{dv}{dt} = 1 - 2v^2 - t. \qquad v(0) = 1$$

$$for \quad t_i = 0.01i, \qquad i = 1,2,\ldots,100$$

Programming Euler Method (Matlab)

```
f=inline('1-2*v^2-t','t','v')
h=0.01
t=0
v=1
T(1)=t;
V(1)=v;
for i=1:100
    v=v+h*f(t,v)
    t=t+h;
    T(i+1)=t;
    V(i+1)=v;
end
```

Programming Euler Method

```
f=inline('1-2*v^2-t','t','v')
h=0.01
t=0
v=1
T(1)=t;
V(1)=v;
for i=1:100
    v=v+h*f(t,v)
    t=t+h;
    T(i+1)=t;
    V(i+1)=v;
end
```

Definition of the ODE

Initial condition

Main loop

Euler method

Storing information

Programming Euler Method

Plot of the solution

plot(T,V)