Numerical Methods in Engineering

ORDINARY DIFFERENTIAL EQUATIONS (ODEs)

Lecture 28-36

Read 25.1-25.4, 26-2, 27-1

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Lecture 28

Lesson 1: Introduction to ODEs

Learning Objectives of Lesson 1

- Recall basic definitions of ODEs:
  - Order
  - Linearity
  - Initial conditions
  - Solution
- Classify ODEs based on:
  - Order, linearity, and conditions.
- Classify the solution methods.

Derivatives

Derivatives

Ordinary Derivatives

$$\frac{dv}{dt}$$

v is a function of one independent variable

Partial Derivatives

$$\frac{\partial u}{\partial y}$$

u is a function of more than one independent variable

Differential Equations

Differential Equations

Ordinary Differential Equations

$$\frac{d^2v}{dt^2} + 6tv = 1$$

involve one or more

Ordinary derivatives of

unknown functions

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more

partial derivatives of

unknown functions

Ordinary Differential Equations

Ordinary Differential Equations (ODEs) involve one or more ordinary derivatives of unknown functions with respect to one independent variable

Examples :

$$\frac{dv(t)}{dt} - v(t) = e^t$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

x(t): unknown function

t: independent variable

Example of ODE:
Model of Falling Parachutist

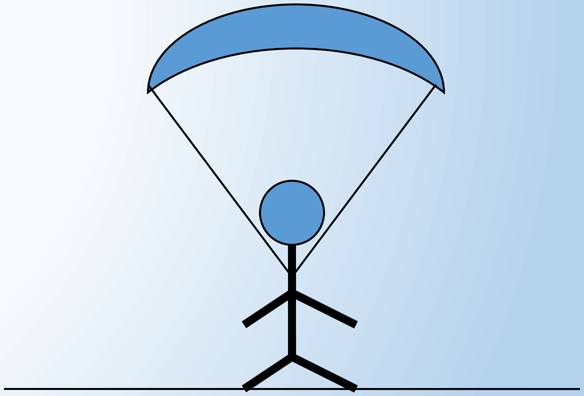
The velocity of a falling parachutist is given by:

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

$M : mass$

$c : drag\ coefficient$

$v : velocity$



Definitions

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

Ordinary differential equation

Definitions (Cont.)

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

(Dependent variable) unknown function to be determined

Definitions (Cont.)

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

(independent variable)
the variable with respect to which other variables are differentiated

Order of a Differential Equation

The **order** of an ordinary differential equations is the order of the highest order derivative.

Examples :

$$\frac{dx(t)}{dt} - x(t) = e^{t}$$

First order ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

Second order ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + 2x^4(t) = 1$$

Second order ODE

Solution of a Differential Equation

A **solution** to a differential equation is a function that satisfies the equation.

$$Example:$$

$$\frac{dx(t)}{dt} + x(t) = 0$$

Linear ODE

An ODE is linear if

The unknown function and its derivatives appear to power one

No product of the unknown function and/or its derivatives

Examples :

$$\frac{dx(t)}{dt} - x(t) = e^t$$

Linear ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2t^2 x(t) = \cos(t)$$

Linear ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + \sqrt{x(t)} = 1$$

Non-linear ODE

Nonlinear ODE

An ODE is linear if

The unknown function and its derivatives appear to power one

No product of the unknown function and/or its derivatives

Examples of nonlinear ODE :

$$\frac{dx(t)}{dt} - \cos(x(t)) = 1$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt}x(t) = 2$$

$$\frac{d^2x(t)}{dt^2} - \left|\frac{dx(t)}{dt}\right| + x(t) = 1$$

Solutions of Ordinary Differential Equations

$$x(t) = \cos(2t)$$

is a solution to the ODE

$$\frac{d^2 x(t)}{dt^2} + 4x(t) = 0$$

Is it unique?

All functions of the form $x(t) = \cos(2t + c)$ (where c is a real constant) are solutions.

Uniqueness of a Solution

In order to uniquely specify a solution to an n^{th} order differential equation we need n conditions.

$$\frac{d^2 y(x)}{dt^2} + 4y(x) = 0$$

Second order ODE

$$\boxed{\begin{aligned} y(0) &= a \\ \dot{y}(0) &= b \end{aligned}}$$

Two conditions are needed to uniquely specify the solution

Auxiliary Conditions

Auxiliary Conditions

Initial Conditions

- All conditions are at one point of the independent variable

Boundary Conditions

- The conditions are not at one point of the independent variable

Boundary-Value and Initial value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

Classification of ODEs

ODEs can be classified in different ways:

- Order
  - First order ODE
  - Second order ODE
  - N^{th} order ODE
- Linearity
  - Linear ODE
  - Nonlinear ODE
- Auxiliary conditions
  - Initial value problems
  - Boundary value problems

Analytical Solutions

- Analytical Solutions to ODEs are available for linear ODEs and special classes of nonlinear differential equations.

Numerical Solutions

- Numerical methods are used to obtain a graph or a table of the unknown function.
- Most of the Numerical methods used to solve ODEs are based directly (or indirectly) on the truncated Taylor series expansion.

Classification of the Methods

Numerical Methods
for Solving ODE

Single-Step Methods

Estimates of the solution at a particular step are entirely based on information on the previous step

Multiple-Step Methods

Estimates of the solution at a particular step are based on information on more than one step