Numerical Methods in Engineering

NUMERICAL INTEGRATION

Lecture 24-27

Read Chapter 21, Section 1
Read Chapter 22, Sections 2-3

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Lecture 24

INTRODUCTION TO NUMERICAL INTEGRATION

- Definitions
- Upper and Lower Sums
- Trapezoid Method (Newton-Cotes Methods)
- Romberg Method
- Gauss Quadrature
- Examples

Integration

Indefinite Integrals

$$\int x\,dx = \frac{x^2}{2} + c$$

Indefinite Integrals of a function are functions that differ from each other by a constant.

Definite Integrals

$$\int_0^1 x dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

Definite Integrals are numbers.

Fundamental Theorem of Calculus

If f is continuous on an interval $[a,b]$,

F is antiderivative of f (i.e., $F'(x) = f(x)$)

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

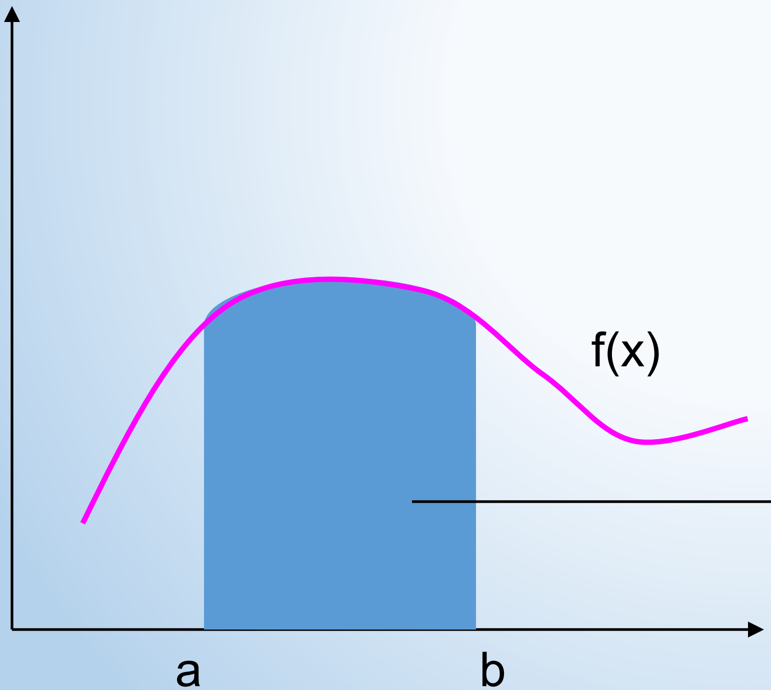
There is no antiderivative for: e^{x^2}

No closed form solution for: $\int_a^b e^{x^2} dx$

The Area Under the Curve

One interpretation of the definite integral is:

<u>Integral = area under the curve</u>



$$Area = \int_a^b f(x)dx$$

Upper and Lower Sums

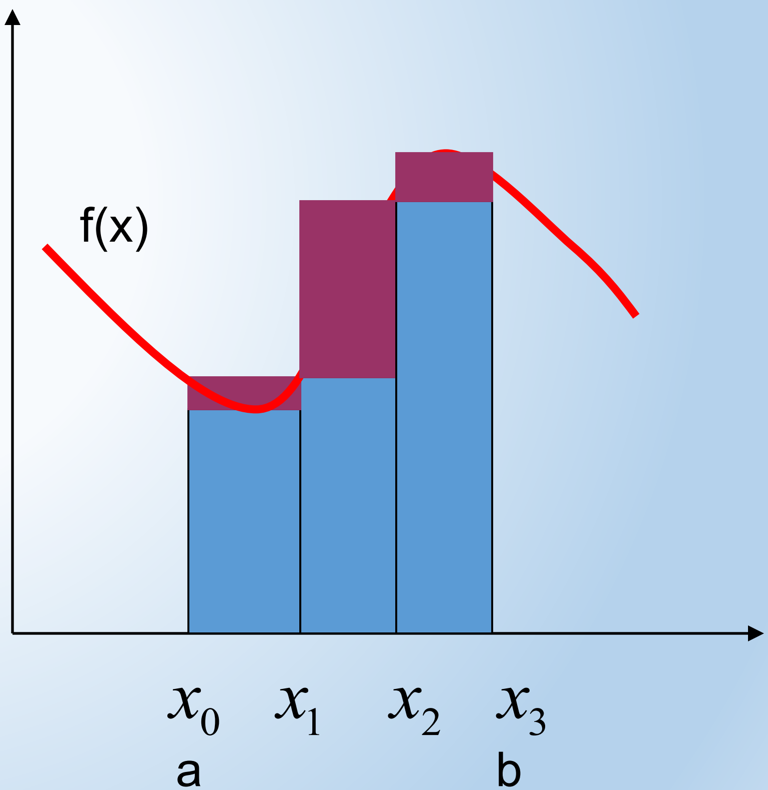
The interval is divided into subintervals.

$$Partition\ P = \{a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b\}$$

Define

$$m_i = \min\{f(x) : x_i \le x \le x_{i+1}\}$$

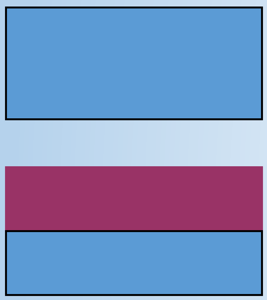
$$M_i = \max\{f(x) : x_i \le x \le x_{i+1}\}$$



Lower sum $\quad L(f,P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$

Upper sum $\quad U(f,P) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$

Upper and Lower Sums

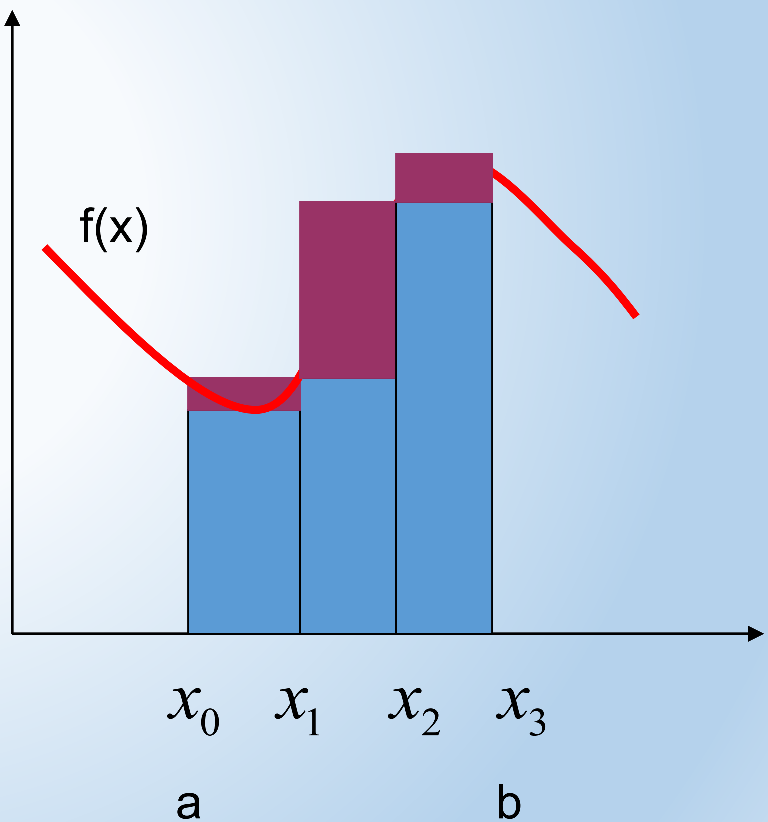


$$L(f,P)=\sum_{i=0}^{n-1} m_i\left(x_{i+1}-x_i\right)$$

$$U(f,P)=\sum_{i=0}^{n-1} M_i\left(x_{i+1}-x_i\right)$$

$$\text{Estimate of the integral} = \frac{L+U}{2}$$

$$Error \le \frac{U-L}{2}$$



Example

$$\int_0^1 x^2 \, dx$$

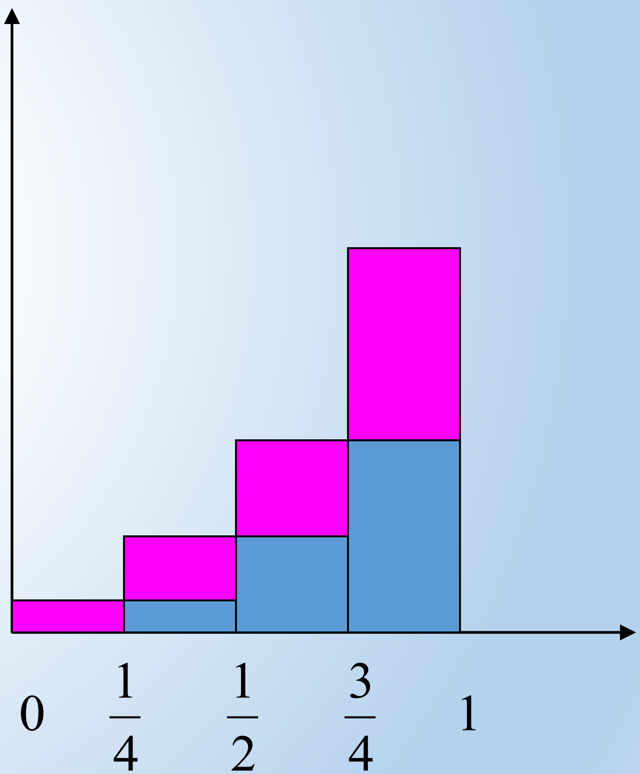
$$Partition : P = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$$

$$n = 4 \text{ (four equal intervals)}$$

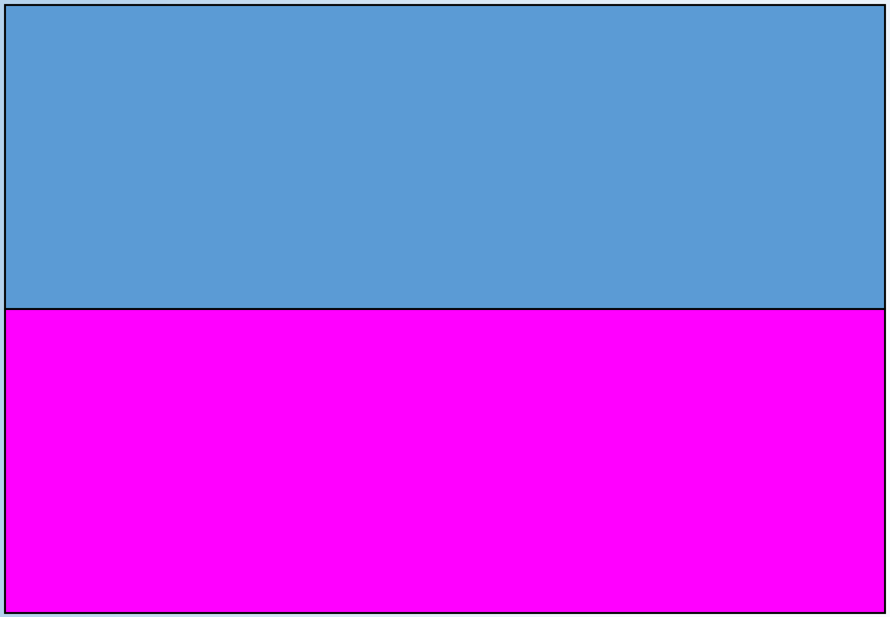
$$m_0 = 0, \quad m_1 = \frac{1}{16}, \quad m_2 = \frac{1}{4}, \quad m_3 = \frac{9}{16}$$

$$M_0 = \frac{1}{16}, \quad M_1 = \frac{1}{4}, \quad M_2 = \frac{9}{16}, \quad M_3 = 1$$

$$x_{i+1} - x_i = \frac{1}{4} \qquad for \quad i = 0,1,2,3$$

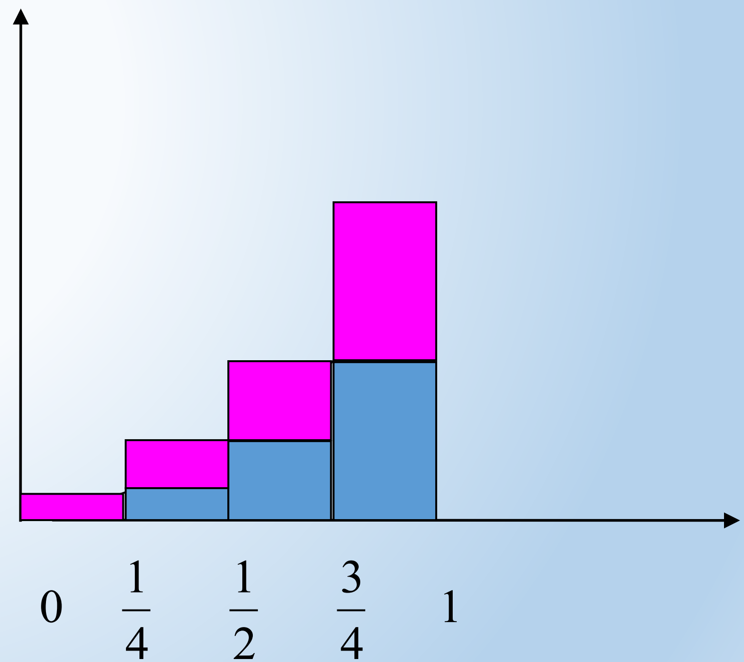


Example



$$\text{Estimate of the integral} = \frac{1}{2}\left(\frac{30}{64} + \frac{14}{64}\right) = \frac{11}{32}$$

$$Error < \frac{1}{2}\left(\frac{30}{64} - \frac{14}{64}\right) = \frac{1}{8}$$



Upper and Lower Sums

- Estimates based on Upper and Lower Sums are easy to obtain for **monotonic** functions (always increasing or always decreasing).
- For non-monotonic functions, finding maximum and minimum of the function can be difficult and other methods can be more attractive.

Newton-Cotes Methods

- In Newton-Cote Methods, the function is approximated by a polynomial of order *n*.
- Computing the integral of a polynomial is easy.

$$\int_a^b f(x)dx \approx \int_a^b \left(a_0 + a_1 x + \ldots + a_n x^n\right)dx$$

$$\int_a^b f(x)dx \approx a_0(b-a) + a_1 \frac{(b^2-a^2)}{2} + \ldots + a_n \frac{(b^{n+1}-a^{n+1})}{n+1}$$

Newton-Cotes Methods

- Trapezoid Method (First Order Polynomials are used)

$$\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1 x)dx$$

- Simpson 1/3 Rule (Second Order Polynomials are used)

$$\int_a^b f(x)dx \approx \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

Lecture 25

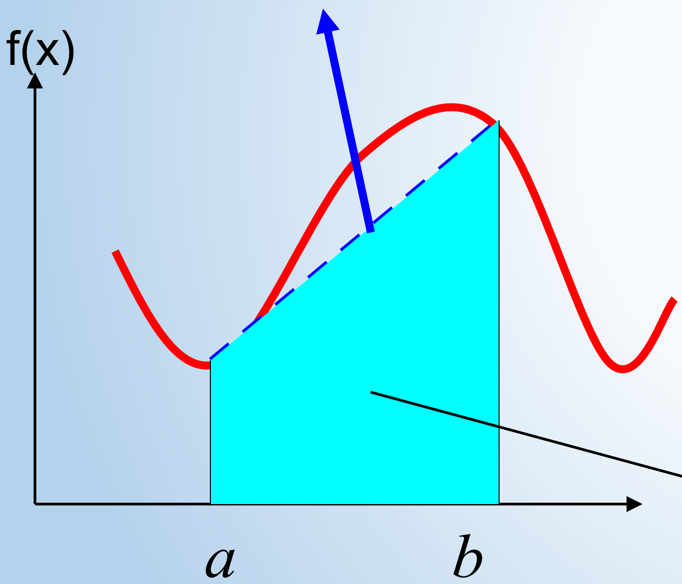
TRAPEZOID METHOD

- Derivation-One Interval
- Multiple Application Rule
- Estimating the Error
- Recursive Trapezoid Method

Read 21.1

Trapezoid Method

$$f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$$



$$I=\int_a^b f(x)dx$$

$$I\approx\int_a^b\left(f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\right)dx$$

$$=\left(f(a)-a\frac{f(b)-f(a)}{b-a}\right)x\Big|_a^b$$

$$+\frac{f(b)-f(a)}{b-a}\frac{x^2}{2}\Big|_a^b$$

$$\boxed{=(b-a)\frac{f(b)+f(a)}{2}}$$

Trapezoid Method Derivation-One Interval

$$I = \int_a^b f(x)dx \approx \int_a^b \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right) dx$$

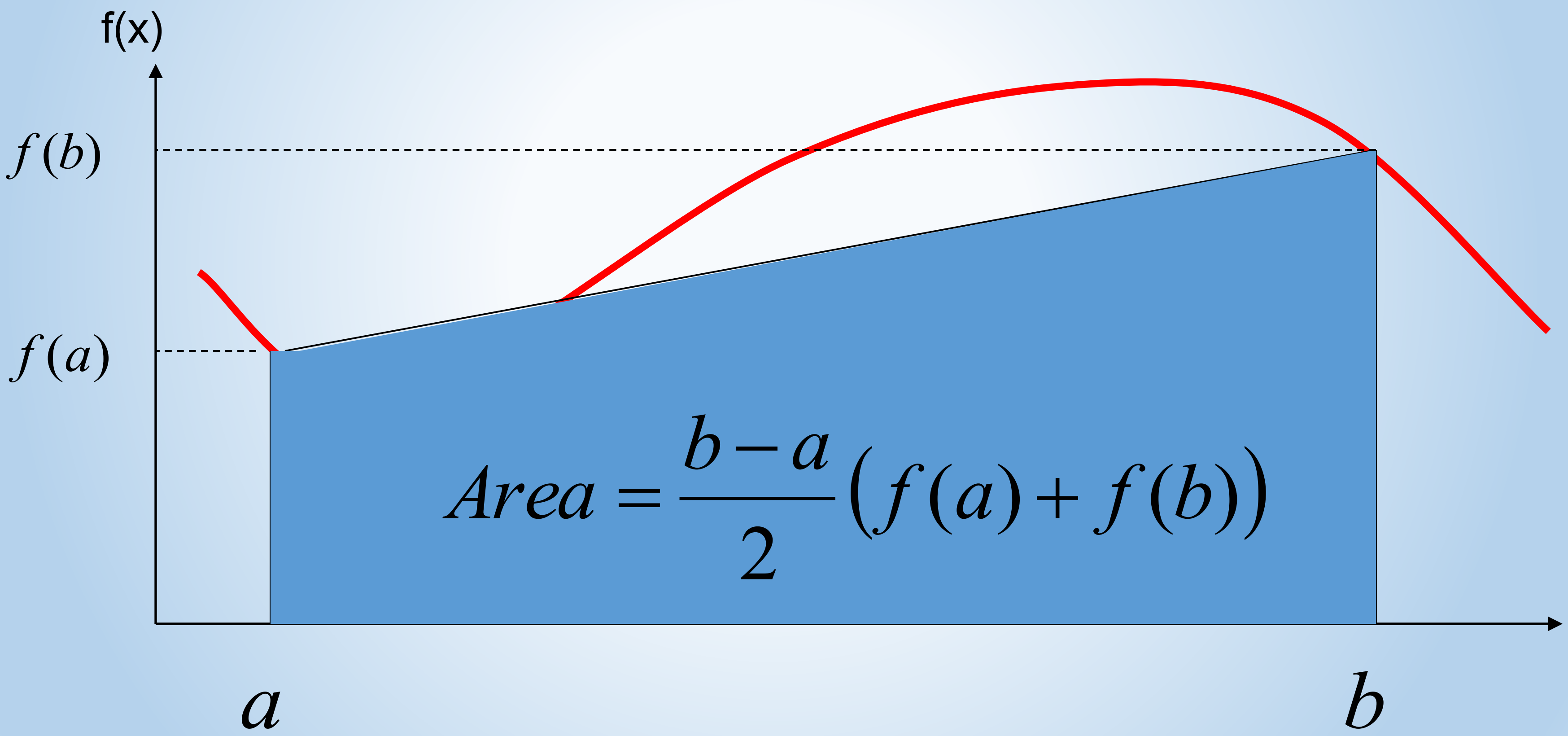
$$I \approx \int_a^b \left(f(a) - a\frac{f(b) - f(a)}{b - a} + \frac{f(b) - f(a)}{b - a}x \right) dx$$

$$= \left(f(a) - a\frac{f(b) - f(a)}{b - a} \right) x \Big|_a^b + \frac{f(b) - f(a)}{b - a}\frac{x^2}{2}\Big|_a^b$$

$$= \left(f(a) - a\frac{f(b) - f(a)}{b - a} \right) (b - a) + \frac{f(b) - f(a)}{2(b - a)}(b^2 - a^2)$$

$$= (b - a)\frac{f(b) + f(a)}{2}$$

Trapezoid Method



$$Area = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

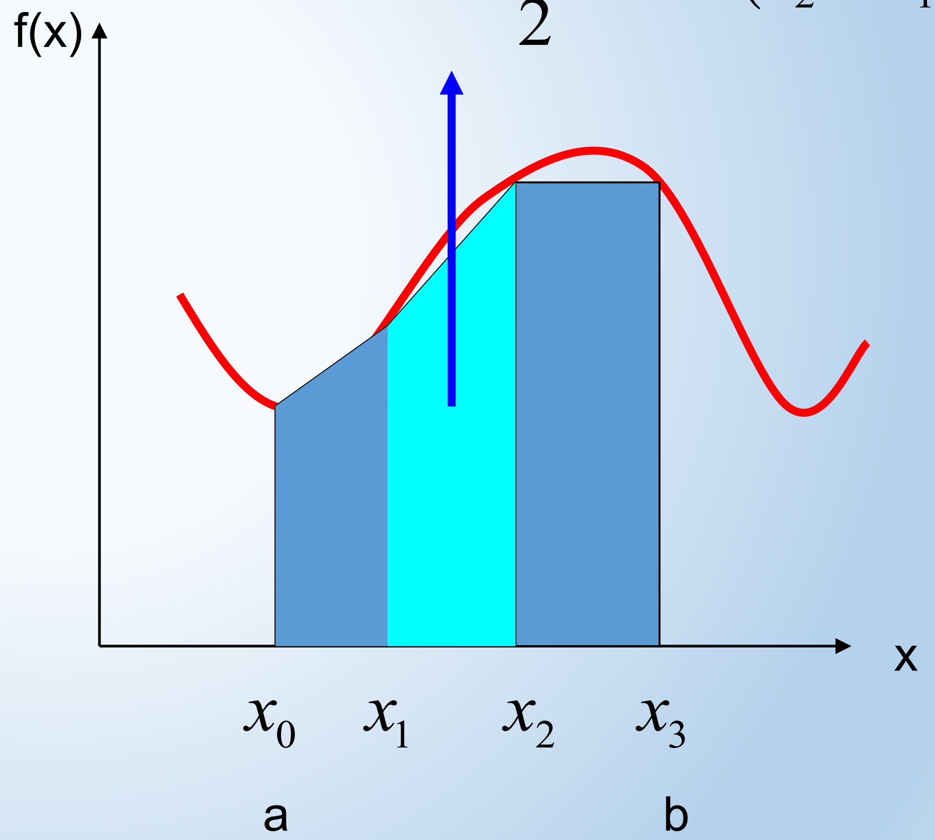
Trapezoid Method
Multiple Application Rule

$$Area = \frac{f(x_2)+f(x_1)}{2}(x_2 - x_1)$$

The interval $[a, b]$ is partitioned into n segments

$$a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b$$

$$\int_a^b f(x)dx = \text{sum of the areas of the trapezoids}$$



Trapezoid Method
General Formula and Special Case

If the interval is divided into n segments (not necessarily equal)

$$a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b$$

$$\boxed{\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \frac{1}{2}\left(x_{i+1} - x_i\right)\left(f(x_{i+1}) + f(x_i)\right)}$$

Special Case (Equaliy spaced base points)

$$x_{i+1} - x_i = h \quad \text{for all} \quad i$$

$$\int_a^b f(x)dx \approx h\left[\frac{1}{2}\left[f(x_0) + f(x_n)\right] + \sum_{i=1}^{n-1} f(x_i)\right]$$

Example

Given a tabulated values of the velocity of an object.

Time (s)	0.0	1.0	2.0	3.0
Velocity (m/s)	0.0	10	12	14

Obtain an estimate of the distance traveled in the interval [0,3].

Distance = integral of the velocity

$$\text{Distance} = \int_0^3 V(t)\, dt$$

Example 1

The interval is divided into 3 subintervals

Base points are $\{0,1,2,3\}$

Time (s)	0.0	1.0	2.0	3.0
Velocity (m/s)	0.0	10	12	14

Trapezoid Method

$$h = x_{i+1} - x_i = 1$$

$$T = h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\left(f(x_0) + f(x_n)\right)\right]$$

$$\text{Distance} = 1\left[(10+12) + \frac{1}{2}(0+14)\right] = 29$$

Error in estimating the integral

Theorem

Assumption: $f''(x)$ is continuous on $[a,b]$

Equal intervals $(\text{width} = h)$

Theorem : If Trapezoid Method is used to

approximate $\int_a^b f(x)dx$ then

$$Error = -\frac{b-a}{12} h^2 f''(\xi) \quad where\ \xi \in [a,b]$$

$$|Error| \leq \frac{b-a}{12} h^2 \max_{x\in[a,b]} |f''(x)|$$

Estimating the Error for Trapezoid Method

How many equally spaced intervals are needed to compute $\int_0^{\pi} \sin(x)dx$ to 5 decimal digit accuracy ?

Example

$$\int_0^{\pi} \sin(x)dx, \quad \mathit{find}\ h\ \mathit{so\ that}\ |\text{error}| \le \frac{1}{2}\times 10^{-5}$$

$$|Error| \le \frac{b-a}{12}\, h^2 \max_{x\in[a,b]} |f''(x)|$$

$$b=\pi;\ a=0;\quad f'(x)=\cos(x);\quad f''(x)=-\sin(x)$$

$$|f''(x)| \le 1 \quad \Rightarrow |Error| \le \frac{\pi}{12}h^2 \le \frac{1}{2}\times 10^{-5}$$

$$\Rightarrow\ h^2 \le \frac{6}{\pi}\times 10^{-5} \quad \Rightarrow\ h \le 0.00437$$

$$\Rightarrow\ n \ge \frac{(b-a)}{h} = \frac{\pi}{0.00437} = 719 \text{ intervals}$$

Example

x	1.0	1.5	2.0	2.5	3.0
f(x)	2.1	3.2	3.4	2.8	2.7

Use Trapezoid method to compute: $\int_1^3 f(x)dx$

$$Trapezoid \quad T(f,P) = \sum_{i=0}^{n-1} \frac{1}{2}\left(x_{i+1} - x_i\right)\left(f(x_{i+1}) + f(x_i)\right)$$

$$Special\ Case: \quad h = x_{i+1} - x_i \quad \text{for all} \quad i,$$

$$T(f,P) = h\left[\sum_{i=1}^{n-1} f(x_i) \; + \frac{1}{2}\left(f(x_0) + f(x_n)\right)\right]$$

Example

x	1.0	1.5	2.0	2.5	3.0
f(x)	2.1	3.2	3.4	2.8	2.7

$$\int_1^3 f(x)dx \approx h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\big(f(x_0) + f(x_n)\big)\right]$$

$$= 0.5\left[\ 3.2 + 3.4 + 2.8 + \frac{1}{2}(2.1 + 2.7)\right]$$

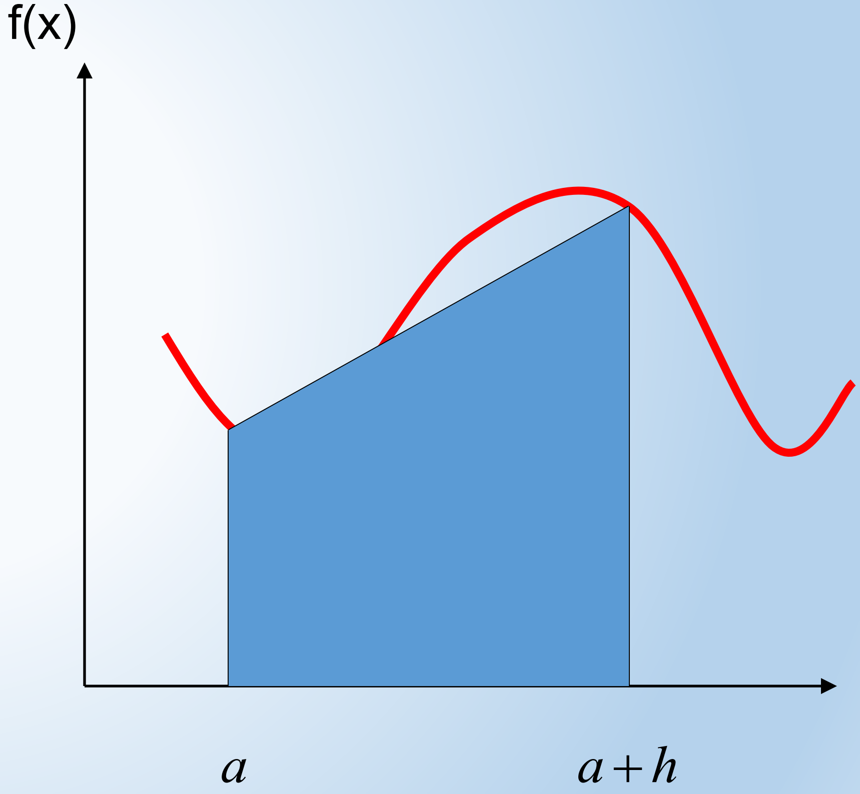
$$= 5.9$$

Recursive Trapezoid Method

Estimate based on one interval:

$$h = b - a$$

$$\mathrm{R}(0,0) = \frac{b-a}{2}\left(f(a) + f(b)\right)$$



Recursive Trapezoid Method

Estimate based on 2 intervals :

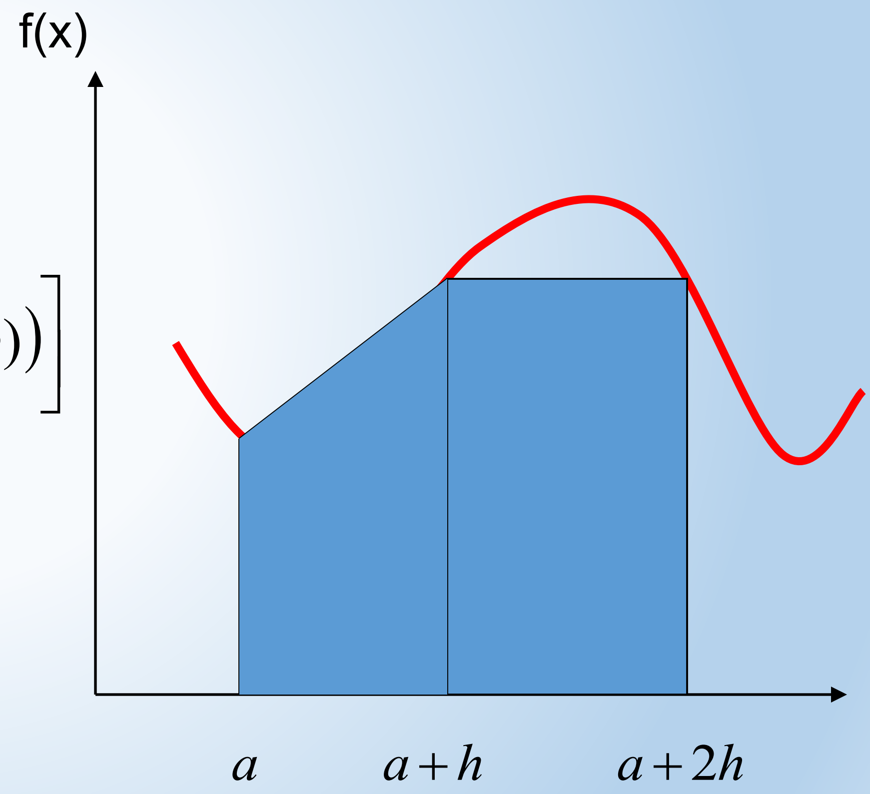
$$h=\frac{b-a}{2}$$

$$\mathrm{R}(1{,}0)=\frac{b-a}{2}\left[f(a+h)+\frac{1}{2}\big(f(a)+f(b)\big)\right]$$

$$R(1{,}0)=\frac{1}{2}R(0{,}0)+h\big[f(a+h)\big]$$

Based on previous estimate

Based on new point



Recursive Trapezoid Method

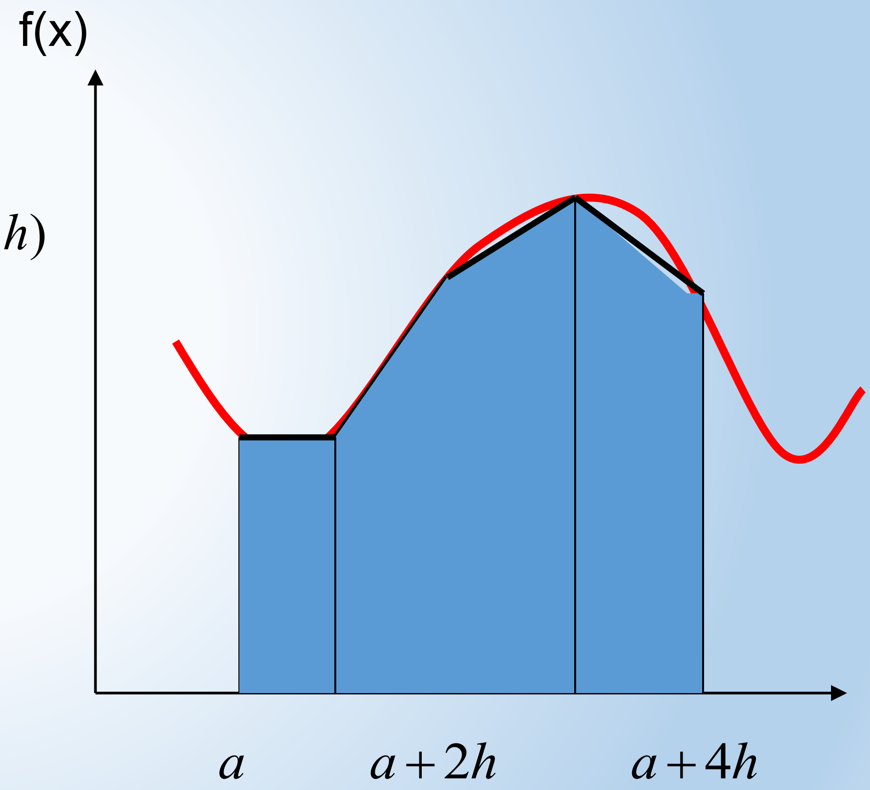
$$h=\frac{b-a}{4}$$

$$\mathrm{R}(2,0)=\frac{b-a}{4}\left[f(a+h)+f(a+2h)+f(a+3h)\right.$$
$$\left.+\frac{1}{2}\big(f(a)+f(b)\big)\right]$$

$$R(2,0)=\frac{1}{2}R(1,0)+h\left[f(a+h)+f(a+3h)\right]$$

Based on previous estimate

Based on new points



Recursive Trapezoid Method Formulas

$$\mathrm{R}(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\big(a + (2k-1)h\big)\right]$$

$$h = \frac{b-a}{2^n}$$

Recursive Trapezoid Method

$$h = b - a, \qquad \mathrm{R}(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$h = \frac{b-a}{2}, \qquad R(1,0) = \frac{1}{2}R(0,0) + h\left[\sum_{k=1}^{1} f\big(a + (2k-1)h\big)\right]$$

$$h = \frac{b-a}{2^2}, \qquad R(2,0) = \frac{1}{2}R(1,0) + h\left[\sum_{k=1}^{2} f\big(a + (2k-1)h\big)\right]$$

$$h = \frac{b-a}{2^3}, \qquad R(3,0) = \frac{1}{2}R(2,0) + h\left[\sum_{k=1}^{2^2} f\big(a + (2k-1)h\big)\right]$$

....

$$h = \frac{b-a}{2^n}, \qquad R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\big(a + (2k-1)h\big)\right]$$

Example on Recursive Trapezoid

Use Recursive Trapezoid method to estimate:

$$\int_0^{\pi/2} \sin(x)dx \text{ by computing R(3,0) then estimate the error}$$

n	h	R(n,0)
0	(b-a)=π/2	(π/4)[sin(0) + sin(π/2)]=0.785398
1	(b-a)/2=π/4	R(0,0)/2 + (π/4) sin(π/4) = 0.948059
2	(b-a)/4=π/8	R(1,0)/2 + (π/8)[sin(π/8)+sin(3π/8)] = 0.987116
3	(b-a)/8=π/16	R(2,0)/2 + (π/16)[sin(π/16)+sin(3π/16)+sin(5π/16)+ sin(7π/16)] = 0.996785

Estimated Error = |R(3,0) – R(2,0)| = 0.009669

Advantages of Recursive Trapezoid

Recursive Trapezoid:

- Gives the same answer as the standard Trapezoid method.
- Makes use of the available information to reduce the computation time.
- Useful if the number of iterations is not known in advance.

Lecture 26

ROMBERG METHOD

- Motivation
- Derivation of Romberg Method
- Romberg Method
- Example
- When to stop?

Read 22.2

Motivation for Romberg Method

- Trapezoid formula with a sub-interval h gives an error of the order $O(h^2)$.

- *We can combine two Trapezoid estimates with intervals h and $h/2$ to get a better estimate.*

Romberg Method

Estimates using Trapezoid method intervals of size h, h/2, h/4, h/8 ...

are combined to improve the approximation of $\int_a^b f(x)\,dx$

First column is obtained using Trapezoid Method

R(0,0)			
R(1,0)	R(1,1)		
R(2,0)	R(2,1)	R(2,2)	
R(3,0)	R(3,1)	R(3,2)	R(3,3)

The other elements are obtained using the Romberg Method

First Column
Recursive Trapezoid Method

$$\mathrm{R}(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\big(a + (2k-1)h\big)\right]$$

$$h = \frac{b-a}{2^n}$$

Derivation of Romberg Method

$$\int_a^b f(x)dx = R(n-1,0) + O(h^2) \quad \text{Trapezoid method} \quad with \ h = \frac{b-a}{2^{n-1}}$$

$$\int_a^b f(x)dx = R(n-1,0) + a_2h^2 + a_4h^4 + a_6h^6 + \ldots \qquad (eq1)$$

More accurate estimate is obtained by R(n,0)

$$\int_a^b f(x)dx = R(n,0) + \frac{1}{4}a_2h^2 + \frac{1}{16}a_4h^4 + \frac{1}{64}a_6h^6 + \ldots \qquad (eq2)$$

$eq1 - 4 * eq2 \qquad gives$

$$\int_a^b f(x)dx = \frac{1}{3}\left[4 \times R(n,0) - R(n-1,0)\right] + b_4h^4 + b_6h^6 + \ldots$$

Romberg Method

R(0,0)			
R(1,0)	R(1,1)		
R(2,0)	R(2,1)	R(2,2)	
R(3,0)	R(3,1)	R(3,2)	R(3,3)

$$\mathrm{R}(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$h = \frac{b-a}{2^n},$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\big(a + (2k-1)h\big)\right]$$

$$R(n,m) = \frac{1}{4^m - 1}\left[4^m \times R(n,m-1) - R(n-1,m-1)\right] \; n \geq 1, \quad m \geq 1$$

Property of Romberg Method

Theorem

$$\int_a^b f(x)dx = R(n,m) + O(h^{2m+2})$$

R(0,0)			
R(1,0)	R(1,1)		
R(2,0)	R(2,1)	R(2,2)	
R(3,0)	R(3,1)	R(3,2)	R(3,3)

Error Level $O(h^2)$ $O(h^4)$ $O(h^6)$ $O(h^8)$

Example

$$\text{Compute} \int_0^1 x^2 dx$$

0.5	
3/8	1/3

$$h = 1,\ \ R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] = \frac{1}{2}\left[0+1\right] = 0.5$$

$$h = \frac{1}{2},\ R(1,0) = \frac{1}{2}R(0,0) + h(f(a+h)) = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{3}{8}$$

$$R(n,m) = \frac{1}{4^m - 1}\left[4^m \times R(n,m-1) - R(n-1,m-1)\right] \ \ for \ \ n \geq 1, \quad m \geq 1$$

$$R(1,1) = \frac{1}{4^1 - 1}\left[4 \times R(1,0) - R(0,0)\right] = \frac{1}{3}\left[4 \times \frac{3}{8} - \frac{1}{2}\right] = \frac{1}{3}$$

Example (cont.)

0.5		
3/8	1/3	
11/32	1/3	1/3

$$h = \frac{1}{4},\ R(2,0) = \frac{1}{2}R(1,0) + h(f(a+h) + f(a+3h)) = \frac{1}{2}\left(\frac{3}{8}\right) + \frac{1}{4}\left(\frac{1}{16} + \frac{9}{16}\right) = \frac{11}{32}$$

$$R(n,m) = \frac{1}{4^m - 1}\left[4^m \times R(n,m-1) - R(n-1,m-1)\right]$$

$$R(2,1) = \frac{1}{3}\left[4 \times R(2,0) - R(1,0)\right] = \frac{1}{3}\left[4 \times \frac{11}{32} - \frac{3}{8}\right] = \frac{1}{3}$$

$$R(2,2) = \frac{1}{4^2 - 1}\left[4^2 \times R(2,1) - R(1,1)\right] = \frac{1}{15}\left[\frac{16}{3} - \frac{1}{3}\right] = \frac{1}{3}$$

When do we stop?

$$STOP \;\; if$$

$$\left| R(n,n) - R(n,n-1) \right| \leq \varepsilon$$

or

After a given number of steps,

for example, STOP at R(4,4)

Lecture 27

GAUSS QUADRATURE

- Motivation
- General integration formula

Read 22.3

Motivation

Trapezoid Method:

$$\int_a^b f(x)dx \approx h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\big(f(x_0) + f(x_n)\big)\right]$$

It can be expressed as:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i \, f(x_i)$$

$$where \quad c_i = \begin{cases} h & i = 1,2,\ldots,n-1 \\ 0.5h & i = 0 \;\; and \;\; n \end{cases}$$

General Integration Formula

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i \; f(x_i)$$

$c_i : Weights \qquad x_i : Nodes$

Problem :

How do we select c_i *and* x_i so that the formula gives a good approximation of the integral?

Lagrange Interpolation

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx$$

where $P_n(x)$ is a polynomial that interpolates f(x) at the nodes: $x_0, x_1, \ldots, x_n$

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx = \int_a^b \left(\sum_{i=0}^{n} \ell_i(x) f(x_i) \right) dx$$

$$\Rightarrow \int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i \; f(x_i) \quad \textit{where} \;\; c_i = \int_a^b \ell_i(x)\, dx$$

Example

- Determine the Gauss Quadrature Formula of

$$\int_{-2}^{2} f(x)dx$$

  If the nodes are given as (-1, 0 , 1)

- Solution: First we need to find $l_0(x)$, $l_1(x)$, $l_2(x)$

- Then compute:

$$c_0 = \int_{-2}^{2} l_0(x)dx, \quad c_1 = \int_{-2}^{2} l_1(x)dx, \quad c_2 = \int_{-2}^{2} l_2(x)dx$$

Solution

$$l_0(x) = \frac{(x-x1)(x-x2)}{(x0-x1)(x0-x2)} = \frac{x(x-1)}{2}$$

$$l_1(x) = \frac{(x-x0)(x-x2)}{(x1-x0)(x1-x2)} = -(x+1)(x-1)$$

$$l_2(x) = \frac{(x-x0)(x-x1)}{(x2-x0)(x2-x1)} = \frac{x(x+1)}{2}$$

$$c_0 = \int_{-2}^{2} \frac{x(x-1)}{2}\,dx = \frac{8}{3}, \quad c_1 = \int_{-2}^{2} -(x+1)(x-1)\,dx = -\frac{4}{3}, \quad c_2 = \int_{-2}^{2} \frac{x(x+1)}{2}\,dx = \frac{8}{3}$$

$$\text{The Gauss Quadrature Formula for } \int_{-2}^{2} f(x)\,dx = \frac{8}{3} f(-1) - \frac{4}{3} f(0) + \frac{8}{3} f(1)$$

Using the Gauss Quadrature Formula

Case 1 : Let $f(x) = x^2$

The exact value for $\int_{-2}^{2} f(x)dx = \int_{-2}^{2} x^2 dx = \frac{16}{3}$

The Gauss Quadrature Formula $= \frac{8}{3} f(-1) - \frac{4}{3} f(0) + \frac{8}{3} f(1)$

$= \frac{8}{3}(-1)^2 - \frac{4}{3}(0)^2 + \frac{8}{3}(1)^2 = \frac{16}{3}$, which is the same exact answer

Using the Gauss Quadrature Formula

$\text{Case 2 : Let } f(x) = x^3$

$$\text{The exact value for } \int_{-2}^{2} f(x)dx = \int_{-2}^{2} x^3 dx = 0$$

$$\text{The Gauss Quadrature Formula} = \frac{8}{3} f(-1) - \frac{4}{3} f(0) + \frac{8}{3} f(1)$$

$$= \frac{8}{3}(-1)^3 - \frac{4}{3}(0)^3 + \frac{8}{3}(1)^3 = 0, \text{ which is the same exact answer}$$

Improper Integrals

Methods discussed earlier cannot be used directly to approximate improper integrals (one of the limits is ∞ or $-\infty$)

$\Rightarrow$ Use a transformation like the following

$$\int_{a}^{b} f(x)dx = \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{1}{t^2} f\left(\frac{1}{t}\right)dt, \quad \text{(assuming } ab > 0\text{)}$$

and apply the method on the new function.

$$Example: \quad \int_{1}^{\infty} \frac{1}{x^2}\, dx = \int_{0}^{1} \frac{1}{t^2}\left(\frac{1}{\left(\frac{1}{t}\right)^2}\right) dt$$