Qualifying quiz application for Maths Beyond Limits

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Contents

1 Solution to the first problem

The statement of the problem was:

> **1.** Let n $\geq$ 2 be an integer. We roll m identical dice at the same time; each die has n sides, numbered from 1 to n. For which m is the chance of getting 1 exactly one time the highest?

In order to get an idea about how to solve the problem, let's start with a classic 6-faced dice. Let's call c_m the chance of getting 1 exactly one time if i threw m dice. Clearly:

$$c_1 = \frac{1}{6}$$

If m=2, the chance would be the chance of getting 1 with the first dice and another number with the second dice, plus the chance of getting another number with the first dice and 1 with the second dice. Then:

$$c_2 = \frac{1}{6} * \frac{5}{6} + \frac{5}{6} * \frac{1}{6}$$

Using a similar idea, we can get for m=3:

$$c_3 = \frac{1}{6} * \frac{5}{6} * \frac{5}{6} + \frac{5}{6} * \frac{1}{6} * \frac{5}{6} + \frac{5}{6} * \frac{5}{6} * \frac{1}{6}$$

We can generalise that idea to:

$$c_m = m\frac{1}{n}\left(\frac{n-1}{n}\right)^{m-1} = \frac{m}{n}\left(\frac{n-1}{n}\right)^{m-1}$$

That's because the chance c_m is m multiplied by the chance of getting 1 with a specific dice, $\frac{1}{n}$, and other number with the other dice, $\left(\frac{n-1}{n}\right)^{m-1}$. Now we need to find the highest possible value of c_m. Again, let's make some tests in order to get a better understanding. Let's suppose the dice is two-sided, like a coin, and let's see for which value of m we get the best chance. (It may be more handy to use a programmable calculator or a python script to speed up calculations)

$$c_1 = \frac{1}{2} \qquad c_2 = \frac{1}{2} \qquad c_3 = \frac{3}{8}$$

Considering that after the third step $\left(\frac{n-1}{n}\right)^{m-1}$ will get smaller and smaller very quickly, let's stop there and suppose the best values of m for n=2 are 1 or 2. Let's repeat the same thing for 3.

$$c_1 = \frac{1}{3} \qquad c_2 = \frac{4}{9} \qquad c_3 = \frac{4}{9} \qquad c_4 = \frac{32}{81}$$

Again, we stop because the number won't stop decreasing. Let's try with 4.

$$c_1 = \frac{1}{4} \qquad c_2 = \frac{3}{8} \qquad c_3 = \frac{27}{64} \qquad c_4 = \frac{27}{64} \qquad c_5 = \frac{405}{1024}$$

Again, we stop because the number won't stop decreasing. It looks like that $c_n = c_{n-1}$ and these are the highest values. Let's try to proof that. First of all $c_n = c_{n-1}$ because:

$$c_{n-1} = \frac{n-1}{n}\left(\frac{n-1}{n}\right)^{n-2} = \left(\frac{n-1}{n}\right)^{n-1} = \frac{n}{n}\left(\frac{n-1}{n}\right)^{n-1} = c_n \tag{1}$$

Now let's proof that $\left(\frac{n-1}{n}\right)^{n-1}$ is the highest possible value for any c_m. In order to do so, let's define k as $n+k = m$. We can write down that:

$$\frac{n+k}{n}\left(\frac{n-1}{n}\right)^{n+k-1} \leq \left(\frac{n-1}{n}\right)^{n-1} \implies \frac{n+k}{n}\left(\frac{n-1}{n}\right)^{k} \leq 1$$

We can easily proof that inductively. First of all, let's consider the subcase $k \geq 0$. If $k = 0$:

$$\frac{n}{n} \leq 1$$

Now let's suppose that for k and let's proof that for $k+1$. Clearly:

$$\frac{n+k+1}{n}\left(\frac{n-1}{n}\right)^{k+1} = \frac{n+k}{n}\left(\frac{n-1}{n}\right)^{k}\frac{n+k+1}{n+k}\left(\frac{n-1}{n}\right)$$

We already know that:

$$\frac{n+k}{n}\left(\frac{n-1}{n}\right)^k \leq 1$$

We need to show that:

$$\frac{n+k+1}{n+k}\left(\frac{n-1}{n}\right) \leq 1$$

However, making some calculations we discover that:

$$\frac{n+k+1}{n+k}\left(\frac{n-1}{n}\right) = 1 - \frac{k+1}{n^2+nk}$$

Then, considering that $\frac{k+1}{n^2+nk} > 0$, we have proofed it for $k \geq 0$. Let's consider the subcase $-n < k < 0$. Again, we will proof that inductively. If $k = -1$:

$$\frac{n-1}{n} \cdot \frac{n}{n-1} \leq 1$$

Now let's suppose that for k and proof it for $k-1$.

$$\frac{n+k-1}{n}\left(\frac{n-1}{n}\right)^{k-1} = \frac{n+k}{n}\left(\frac{n-1}{n}\right)^k \frac{n+k-1}{n+k}\left(\frac{n}{n-1}\right)$$

We need to show that:

$$\frac{n+k-1}{n+k}\left(\frac{n}{n-1}\right) \leq 1$$

Making some calculations we get that:

$$\frac{n+k-1}{n+k}\left(\frac{n}{n-1}\right) = 1 + \frac{k}{(n+k)(n-1)}$$

Now, n+k $>$ 0 and n-1 $>$ 0, while k $<$ 0. So we add to 1 a negative number. Therefore, we have proofed the $-n < k < 0$ subcase too. So we have proofed that for every $k : -n < k$, the maximum possible value for c_{n+k} is:

$$\left(\frac{n-1}{n}\right)^{n-1}$$

Because of (1) we can get that for $m = n$ or $m = n-1$. **QED**

2 Solution to the second problem

The problem statement was:

2. Let a, b, c be positive integers satisfying $a^2 = bc + 1$. Prove that $2a + b + c$ is a composite number.

If $a \equiv 0 \pmod 2$ or $a \equiv 0 \pmod 3$ this can be easily done with modular arithmetic, yet there is no easy way to deal with $a \equiv \pm 1 \pmod 6$. In order to simplify the problem, we could write that:

$$a^2 = bc + 1 \implies b = \frac{a^2 - 1}{c}$$

We could then substitute that in $2a + b + c$ getting:

$$2a + \frac{a^2 - 1}{c} + c = \frac{a^2 + 2ac + c^2 - 1}{c} = \frac{(a + c + 1)(a + c - 1)}{c}$$

Now, $c|(a+c+1)(a+c-1)$, because $c|a^2-1$ and $c|2ac+c^2$. Both of them are greater of c if $a > 1$, so, if c divided just one of them, WLOG[1] $cq = (a + c + 1)$, we would get $q(a + c - 1)$ with $q \geq 2$ and $a + c - 1 \geq 2$, which is a composite number. If c divided their product but not them, we would get $c = c_1 c_2$ and $\frac{c_1 p \cdot c_2 q}{c} = pq$ with $p, q \geq 2$, which is a composite number. Now let's consider the case $a = 1$. We can clearly see that in this case we would get:

$$1^2 = bc + 1 \implies bc = 0 \implies b = 0 \lor c = 0$$

Which violates the assumption a, b, c > 0. This proofs that if a, b, c are positive integers satisfying $a^2 = bc + 1$, $2a + b + c$ is a composite number. **QED**

[1] without loss of generality

3 Solution to the third problem

The problem statement was:

> **3.** Szymon coloured squares of a 2024×2024 chessboard in n colours. Then he placed a queen on one of the squares. He wondered if it is possible for the queen to reach all other squares of this colour via a sequence of moves, each ending on a square of this colour. It happened to be impossible, for any starting position of the queen. What is the smallest possible value of n?

This problem asks to divide the chessboard into n colours, such that each colour can be further divided in at least two colours c and c’ such that there is no possible way to travel from c to c’ without changing colour. We need to find the smallest possible value of n for a 2024x2024 chessboard. Clearly the number 2024 has nothing special, so we can scale this down a bit. I’ve managed to colour a 10x10 board with three colours in the same way Szymon did, and this method can be easily generalised for any (2n)x(2n) board just by following the same pattern of 2, 1’ and 3.

3’	2	2	2	2	2	2	2	2	1
1’	1’	1’	1’	1’	1’	1’	1’	2	3
1’	1’	1’	1’	1’	1’	1’	2	1’	3
1’	1’	1’	1’	1’	1’	2	1’	1’	3
1’	1’	1’	1’	1’	2	1’	1’	1’	3
1’	1’	1’	1’	2	1’	1’	1’	1’	3
1’	1’	1’	2	1’	1’	1’	1’	1’	3
1’	1’	2	1’	1’	1’	1’	1’	1’	3
1’	2	1’	1’	1’	1’	1’	1’	1’	3
3’	1’	1’	1’	1’	1’	1’	1’	1’	2’

3’	…	2	2	2	2	2	2	2	1
1’	…	1’	1’	1’	1’	1’	1’	2	3
1’	…	1’	1’	1’	1’	1’	2	1’	3
1’	…	1’	1’	1’	1’	2	1’	1’	3
1’	…	1’	1’	1’	2	1’	1’	1’	3
1’	…	1’	1’	2	1’	1’	1’	1’	3
1’	…	1’	2	1’	1’	1’	1’	1’	3
1’	…	2	1’	1’	1’	1’	1’	1’	3
1’	…	…	…	…	…	…	…	…	…
3’	1’	1’	1’	1’	1’	1’	1’	1’	2’

We can be sure that we can’t go from 1 to 1’ because there is just one 1, and the 1’ are exactly all the squares which 1 can’t reach. There is no way to go from 2’ to 2 because there is only a 2’ which is in the lower-right corner: in the last row and in the right column there are no 2’, and there is just one diagonal which will never encounter anything but 1’ and 3’(This works just if the side of the square is even. That’s because, if this was a normal chessboard, the diagonal from the upper-right corner to the lower-left corner would be completely black and the lower-right corner white: there is no way to go from a white square to a black square moving diagonally in chess!). We can be sure that there is no way to go from 3 to 3’ for a reason similar to the one for 2 and 2’. Now we need to proof that there is no possible way to do that using just two colours(proofing that there is no possible way with just one colour is obvious: wherever the queen went, it would be on the same starting colour, and it would be able to reach every square!). In order to do that, let’s suppose such a board existed. In this case, the first column would be made of two colours (it could not be made of only one colour, otherwise every square would be reachable), let’s call them WLOG[2] 1 and 2. Now, let’s switch to the second column. If this column contained just 1 and 2, let’s switch to the next one: sooner or later we will find 1’ and 2’. When we find a line containing 1’ or 2’, we will be in the situation of the following table. Let’s assume WLOG we have finally found a line containing 2’. There will surely be a square(called x) confining with a 2 and a 1 in the previous column, otherwise the last column should have been completely made of 1 or 2, and from there all the squares would be reachable. x could not contain 2, otherwise in the current row we would have both a 2 and 2’, or 1’, because it’s nearby to a 1. Therefore x must contain 1. This means that this new column is made of 2’ and 1. Because of that, every square bordering with a 2 in the last row, should be 1(because it could not be 2’). However, this means that every row of the chessboard contains a 1: impossible! **QED**

[2]without loss of generality

1	
1	
1	
1	
2	
1	
1	
2	
2	
2	
1	
1	2’
1	
2	
2	

1	
1	
1	
1	
2	
1	
1	
2	1
2	
2	
1	
1	2’
1	
2	
2	

1	
1	
1	
1	
2	1
1	
1	
2	1
2	1
2	1
1	
1	2’
1	
2	1
2	1

Table 1: On every row there would be at least one 1

4 Solution to the sixth problem

The problem statement was:

> **6.** Due to a chain of peculiar events, you became a plumber. You were recruited to fix an old pipe in the city park. Unfortunately, the city council lost the plans, and they do not know where the pipe is anymore. However, in some documents they found out that the pipe goes through the park in a straight line, and that it is closer than 100 meters from both fountains in the park (points A and B in the picture next page). You need to dig a ditch through the park, hoping that you will find the pipe that way. The city council obviously wants the ditch to be as short as possible, not destroying too much of the lawn. How short can you make it, in order to make sure you find the pipe? You should assume the distance between the fountains is
>
> - |AB| = 100 meters,
> - |AB| = 50 meters,
> - |AB| = 150 meters.

There are two types of lines which are closer than 100 meters from both the fountains. They could have a point which is closer than 100 meters from both the fountains, or they could have no such point:

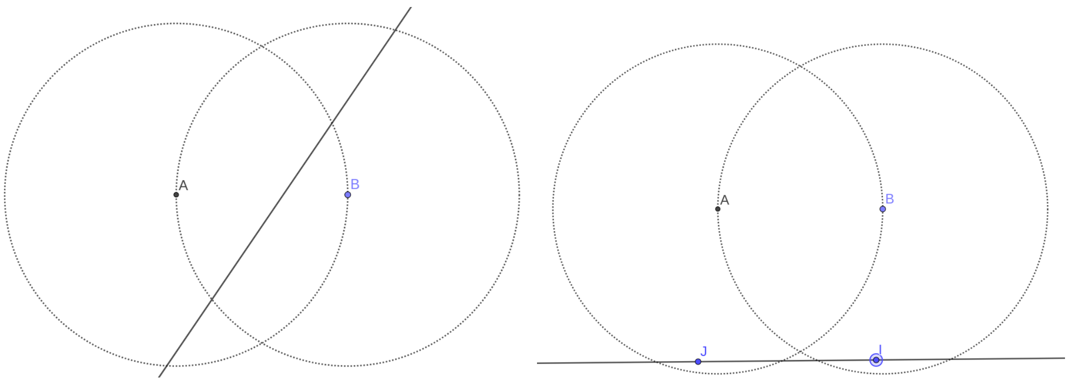


Figure 1: The first and the second type of possible pipes

Clearly, the second type of pipe must intersect the perpendicular bisector of AB between the two grey lines(the tangents to both circles) showed in the following picture, otherwise they could not intersect both circles. Because of this any ditch which follows the perpendicular bisector of AB from E to F, should intercept all these lines.

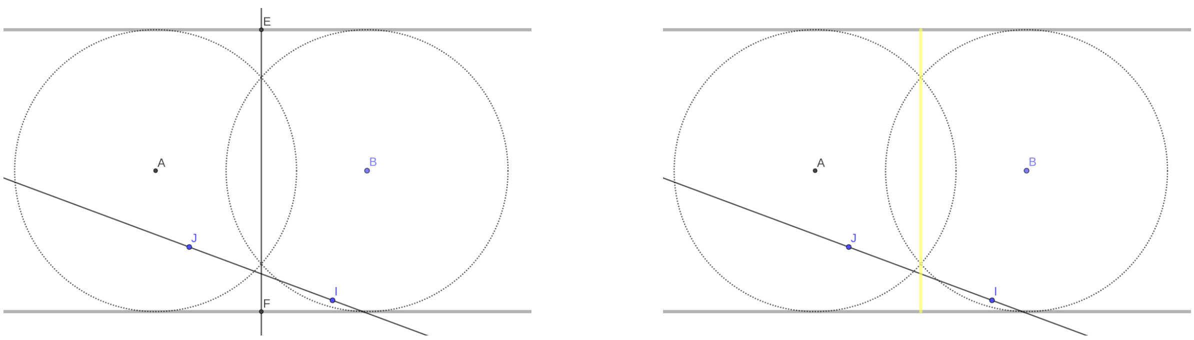


Figure 2: The ditch to intercept the lines of the first type

Now we have to intercept all the lines which have at least one point closer than 100m from both the fountains. This could be done trivially by bordering the intersection of the two circles with a ditch. In this case, we could remove the part of ditch previously dug to intersect the lines of the first type (the one in Figure 2). However, this is far from optimal, especially if AB=50 the intersection border is very long. So we need to find better ways to do that. A nice idea could be to draw from F or E the tangents to the intersection of the two circles, and see where it intersects the line AB. Let's call these points G and H. The segment GH, with the segment EF, would intercept all the pipes of the first type, as can be seen in the following picture (this time, i decided to mark the pipe in blue because we are starting to have too many lines). This works because the line is straight and therefore it can't get near to the intersection without being intercepted by the yellow ditch.

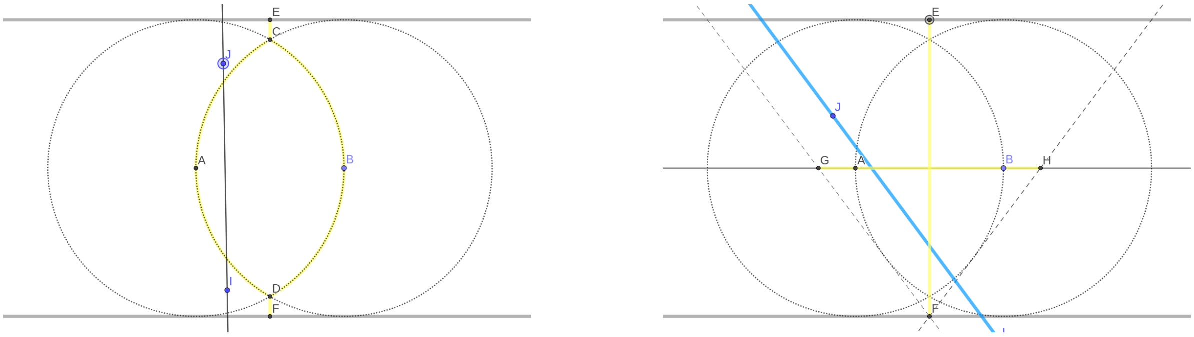
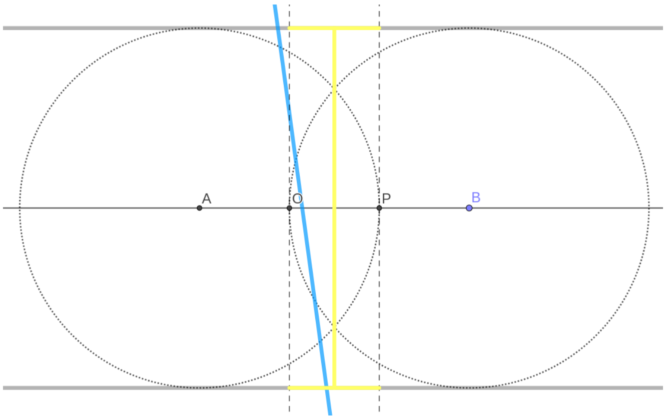


Figure 3: Two different ways to intercept lines of type 1

Another similar way may be to consider the intersection of the segment AB with the two circles and their projections on the grey lines. If we dug the ditch from a projection on one line to the other projection on the same line, we would intercept all the pipes of the first type. This is exceptionally efficient especially for AB=50.



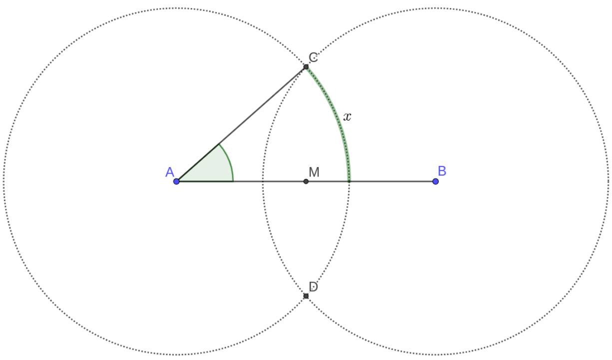
Now that we have implemented these three ways, we need to choose the best one. In order to do so, we need to be able to calculate the cost of each method.

4.1 Cost of the first method

The cost of the first method could be calculated by obtaining with acos the angle CAB:

$$CAB = \cos^{-1}\left(\frac{AB}{2 * 100m}\right)$$

Then $360 : CAB = 2\pi 100m : x$, where $4 * x$ is the length of the border of the intersection of the two circles. Then whe have to add EF(=200m) and subtract the segment CD (now it's useless): its length is two times the length of CM (M is the centre of AB), which could be calculated using Pythagoras's theorem.



The possible costs are

$$AB = 100m \implies c \cong 345m$$
$$AB = 50m \implies c \cong 533m$$
$$AB = 150m \implies c \cong 357m$$

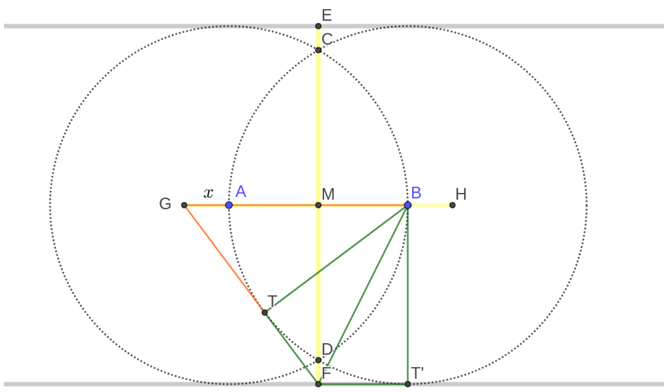
4.2 Cost of the second method

The hard part of this method is to calculate the length of GH. First of all we could say that AB/2=BM=FT'=FT. Then we could say that BTG and GFM is a triangle rectangle. So:

$$GT^2 = (x + AB)^2 - (100m)^2$$
$$TF = AB/2$$
$$(GT + AB/2)^2 = (100m)^2 + (x + AB/2)^2$$
$$\implies GT^2 + \frac{AB^2}{4} + GT * AB = (100m)^2 + (x + AB/2)^2$$
$$\implies (x + AB)^2 - (100m)^2 + \frac{AB^2}{4} + AB * \sqrt{(x + AB)^2 - (100m)^2} = (100m)^2 + (x + AB/2)^2$$
$$\implies (x + 150)^2 - (100)^2 + \frac{150^2}{4} + 150 * \sqrt{(x + 150)^2 - (100)^2} = (100)^2 + (x + 75)^2$$

Now we just need to substitute AB with the appropriate number to get x. The total cost is 2x + AB + 200m.

$$AB = 100m \implies c = 350m$$
$$AB = 50m \implies c = 575m$$
$$AB = 150m \implies c = (258 + \frac{1}{3})m$$



4.3 Cost of the third method

The cost of the third method is the easiest to compute. It is equal to 200m + 2*(200m - AB).

$$AB = 100m \implies c = 400m$$
$$AB = 50m \implies c = 500m$$
$$AB = 150m \implies c = 300m$$

4.4 Conclusion

The length if AB = 100m is 345m and we get that with the first method, the length if AB = 50 is 500m and we get that with the third method, the length if AB=150 is $(258 + \frac{1}{3})m$ and we get that with the second method.

5 Solution to the seventh problem

The problem statement was:

> Let p, q be coprime positive integers. We say an expression $N = a_1 + a_2 + ... + a_m$ is a captivating decomposition of N , if numbers a_i on the right hand side are pairwise distinct numbers of the form $p^k q^l$, with k, l non-negative integers. We say a positive integer N is captivating, if it has a captivating decomposition. The goal of this problem is to prove that every large enough number is captivating.
>
> a) Show that there are infinitely many numbers N which have at least two captivating decompositions.
>
> b) Show that there is a positive integer C, such that for any positive integer n, there is at least one captivating integer in the interval [n, n + C].
>
> c) Assume we can find positive integers R, k, l such that for every positive integer M the number M $p^k q^l$ + R is captivating. Prove that then every integer large enough is captivating, i.e. we can find a constant D such that every integer n D is captivating.
>
> d) Prove the assumption in (c), i.e. that we can find R, k, l such that for every M the number M $p^k q^l$ + R is captivating.

5.1 Attempt proof to *a)*

It is interesting to note that, if one number with at least 2 captivating decompositions existed, we could multiply it for p again and again and get infinitely many numbers with 2 captivating decompositions. It is also interesting to note that, if a number has a captivating decomposition using just p, it can be represented as a number completely made of zeros and ones in base p and vice versa. Therefore, if we proofed that there certainly is a number which can be represented using just zeros and ones in base p and in base q for any lcd(p, q)=1, we would have proofed a). However, i noticed that unless q | p or q + 1 = p, this number either doesn't exist, or is very high(for p=5, q=7, I've not been able to find such a number).

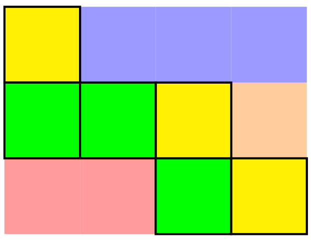
I've also tried to proof that using the pigeonhole principle. I tried to proof that there surely is a number n > p, q such that all the numbers less than n represented only by zero and ones in base p are more than those less than n not represented only by zero and ones in base q or vice-versa, but this didn't work too.

6 Solution to the eighth problem

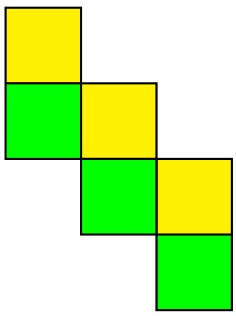
The statement of the problem was:

> MBL recently acquired a puzzle consisting of an m × n rectangular board ($m, n \geq 1$), and mn colourful tiles in the shape of a unit square, which come in k colours ($1 \leq k \leq mn$; there might be a different number of tiles of different colours). The goal of the puzzle is to arrange the tiles on the board in such a way that:
>
> - whenever a tile of colour A is directly over a tile of colour B, you like B more than A;
> - whenever a tile of colour A is directly to the right from a tile of colour B, either you like B more than A, or A = B (i.e. the tiles have the same colour).
>
> Justyna and Daniela both have a very strong sense of aesthetics, in particular for any two colours A, B, they will strictly prefer one over the other. However, their tastes may differ. Prove that the number of ways in which Justyna can arrange the tiles is the same as the number of ways in which Daniela can do it. Can you generalise this result to other board shapes?

In this problem both Justyna and Daniela have a list of colors in their preference order. For instance, in the given example Justyna has: (red, green, yellow, orange, blue) and Daniela has: (orange, yellow, blue, red, green). If we proved that, however chosen two colours adjacent in the preference list, we can swap them without changing the number of possible arrangements, we are done. In fact, it would be enough to swap a few times to go from Justyna preference list to Daniela's, using a Bubble sort. So we will try to proof that: however chosen two colours adjacent in the preference list, we can swap them without changing the number of possible arrangements. First of all, it's interesting to notice that every pair of colours which are consecutive in the preference list, form a ”stair” in the puzzle, going from the top-left corner to the bottom-right corner. That's because each player first starts by putting his/her favourite colour in the lower left corner, then puts the next on it, and so on, without ever putting the same colour above itself, and always putting it as down and as left as possible. Let's analyse this ”stair” properties. First of all, a ”stair” can't be more than 2 blocks tall in any point. In fact, if it was more than two blocks tall, a colour would appear at least two times on a column, which is impossible because of the first rule. Moreover, if a ”stair” is two blocks tall in any point, we can be sure that in that point A is under B (if A > B and A and B are the two colours). Because of that we could say that in every ”stair” 2 blocks tall in every point, there is just one possible combination, and therefore we can swap A and B freely. Also, there would be only one combination if the stair was flat, because it would be enough to flip it horizontally to swap A and B.



(a) Every pair of consecutive elements form a ”stair”



(b) A stair tall two blocks in any point

Now, let's consider the general case in the Picture 2. In the general case, the stair is made of parts two blocks tall and parts one block tall. The parts two blocks tall are fixed because of what said earlier, while the parts one block tall are divided into multiple "chunks". In the Figure 2, for example, there are three "chunks" a, b and c. Now, we need to fill the "chunks" with the remaining tiles (let's suppose, like in the example the colours are green and yellow and green > yellow). Clearly there are multiple ways to do that, yet if we put n green tiles in a "chunk", we are obliged to fill the rest of the chunk with m yellow tiles. Moreover, there is just one way to fill the chunk with n green tiles and m yellow tiles, because we have to put all the green tiles left and all the yellow tiles right. This means that all the possible ways to fill the chunks are equal to all the possible ways to distribute the green tiles between the chunks. Yet, this is equivalent to all the possible ways to distribute (free_tiles - green_tiles) = yellow_tiles. However, this is equivalent to all the possible ways to distribute m yellow tiles between all the chunks, which is equivalent to all the possible combinations if we swapped green and yellow! **QED**

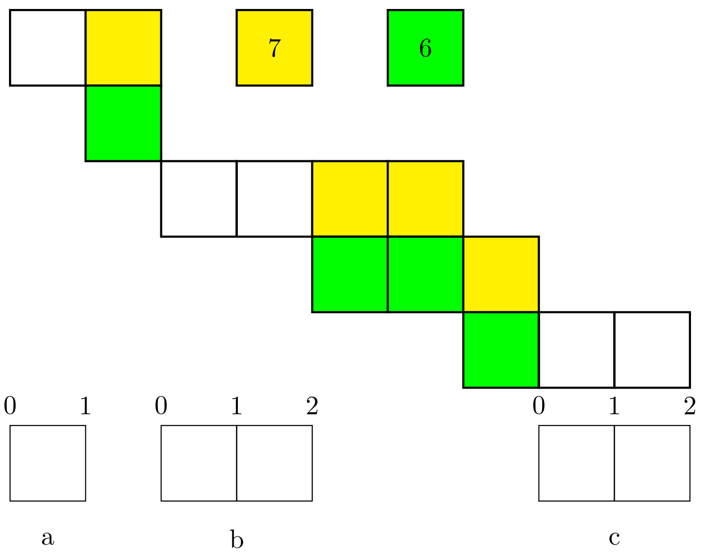


Figure 5: Every pair of consecutive elements form a "stair"