On The Identifiability of Mixture Models from Grouped Samples: Supplemental Material

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Proof of Lemma 1. Because both representations are minimal it follows that $\alpha_i' \neq 0$ for all i and $\mu_i' \neq \mu_j'$ for all $i \neq j$. From this we know $\mathscr{Q}\left(\{\mu_i'\}\right) \neq 0$ for all i. Because $\mathscr{Q}\left(\{\mu_i'\}\right) \neq 0$ for all i it follows that for any i there exists some j such that $\mu_i' = \mu_j$. Let $\psi : [r] \to [r]$ be a function satisfying $\mu_i' = \mu_{\psi(i)}$. Because the elements $\mu_1, \cdots, \mu_r$ are also distinct ψ must be injective and thus a permutation. Again from this distinctness we get that, for all i, $\mathscr{Q}\left(\{\mu_i'\}\right) = \alpha_i' = \alpha_{\psi(i)}$ and we are done. □

Proof of Lemma 2. We will proceed by contradiction. Let $\mathscr{P} = \sum_{i=1}^{l} a_i \delta_{\mu_i}$ be n-identifiable, let $\mathscr{P}' = \sum_{j=1}^{r} b_j \delta_{\nu_j}$ be a different mixture of measures with $r \leq l$ and

$$\sum_{i=1}^{l} a_i \mu_i^{\times q} = \sum_{j=1}^{r} b_j \nu_j^{\times q}$$

for some $q > n$. Let $A \in \mathcal{F}^{\times n}$ be arbitrary. We have

$$\begin{aligned} \sum_{i=1}^{l} a_i \mu_i^{\times q} &= \sum_{j=1}^{r} b_j \nu_j^{\times q} \\ \Rightarrow \sum_{i=1}^{l} a_i \mu_i^{\times q}\left(A \times \Omega^{\times q-n}\right) &= \sum_{j=1}^{r} b_j \nu_j^{\times q}\left(A \times \Omega^{\times q-n}\right) \\ \Rightarrow \sum_{i=1}^{l} a_i \mu_i^{\times n}\left(A\right) &= \sum_{j=1}^{r} b_j \nu_j^{\times n}\left(A\right). \end{aligned}$$

This implies that $\mathscr{P}$ is not n-identifiable, a contradiction. □

Proof of Lemma 3. Let a mixture of measures $\mathscr{P} = \sum_{i=1}^{l} a_i \delta_{\mu_i}$ not be n-identifiable. It follows that there exists a different mixture of measures $\mathscr{P}' = \sum_{j=1}^{r} b_j \delta_{\nu_j}$, with $r \leq l$, such that

$$\sum_{i=1}^{l} a_i \mu_i^{\times n} = \sum_{j=1}^{r} b_j \nu_j^{\times n}.$$

Let $A \in \mathcal{F}^{\times q}$ be arbitrary, we have

$$\begin{aligned} \sum_{i=1}^{l} a_i \mu_i^{\times n}\left(A \times \Omega^{\times n-q}\right) &= \sum_{j=1}^{r} b_j \nu_j^{\times n}\left(A \times \Omega^{\times n-q}\right) \\ \Rightarrow \sum_{i=1}^{l} a_i \mu_i^{\times q}\left(A\right) &= \sum_{j=1}^{r} b_j \nu_j^{\times q}\left(A\right) \end{aligned}$$

and therefore $\mathscr{P}$ is not q-identifiable. □

Proof of Lemma 4. Example 2.6.11 in [2] states that for any two σ-finite measure spaces $(S, \mathscr{S}, m), (S', \mathscr{S}', m')$ there exists a unitary operator $U : L^2(S, \mathscr{S}, m) \otimes L^2(S', \mathscr{S}', m') \to L^2(S \times S', \mathscr{S} \times \mathscr{S}', m \times m')$ such that, for all f, g,

$$U(f \otimes g) = f(\cdot)g(\cdot).$$

Because $(\Psi, \mathcal{G}, \eta)$ is a σ-finite measure space it follows that $(\Psi^{\times m}, \mathcal{G}^{\times m}, \eta^{\times m})$ is a σ-finite measure space for all $m \in \mathbb{N}$. We will now proceed by induction. Clearly the lemma holds for $n = 1$. Suppose the lemma holds for $n - 1$. From the induction hypothesis we know that there exists a unitary transform $U_{n-1} : L^2(\Psi, \mathcal{G}, \eta)^{\otimes n-1} \to L^2\left(\Psi^{\times n-1}, \mathcal{G}^{\times n-1}, \eta^{n-1}\right)$ such that for all simple tensors$f_1 \otimes \cdots \otimes f_{n-1} \mapsto f_1(\cdot) \cdots f_{n-1}(\cdot)$. Combining U_{n-1} with the identity map via Lemma 5 we can construct a unitary operator $T_n : L^2(\Psi, \mathcal{G}, \eta)^{\otimes n-1} \otimes L^2(\Psi, \mathcal{G}, \eta) \to L^2\left(\Psi^{\times n-1}, \mathcal{G}^{\times n-1}, \eta^{n-1}\right) \otimes L^2(\Psi, \mathcal{G}, \eta)$, which maps $f_1 \otimes \cdots \otimes f_{n-1} \otimes f_n \mapsto f_1(\cdot) \cdots f_{n-1}(\cdot) \otimes f_n$

From the aforementioned example there exists a unitary transform $K_n : L^2\left(\Psi^{n-1}, \mathcal{G}^{\times n-1}, \eta^{n-1}\right) \otimes L^2(\Psi, \mathcal{G}, \eta) \to L^2\left(\Psi^{\times n-1} \times \Psi, \mathcal{G}^{\times n-1} \times \mathcal{G}, \eta^{n-1} \times \eta\right)$ which maps $f \otimes f' \mapsto f(\cdot) f'(\cdot)$. Defining $U_n(\cdot) = K_n(T_n(\cdot))$ yields our desired unitary transform. $\square$

Proof of Lemma 5. Proposition 2.6.12 in [2] states that there exists a continuous linear operator $\tilde{U} : H_1 \otimes \cdots \otimes H_n \to H_1' \otimes \cdots \otimes H_n'$ such that $\tilde{U}(h_1 \otimes \cdots \otimes h_n) = U_1(h_1) \otimes \cdots \otimes U_n(h_n)$ for all $h_1 \in H_1, \cdots, h_n \in H_n$. Let $\widehat{H}$ be the set of simple tensors in $H_1 \otimes \cdots \otimes H_n$ and $\widehat{H}'$ be the set of simple tensors in $H_1' \otimes \cdots \otimes H_n'$. Because U_i is surjective for all i, clearly $\tilde{U}(\widehat{H}) = \widehat{H}'$. The linearity of $\tilde{U}$ implies that $\tilde{U}(\operatorname{span}(\widehat{H})) = \operatorname{span}(\widehat{H}')$. Because $\operatorname{span}(\widehat{H}')$ is dense in $H_1' \otimes \cdots \otimes H_n'$ the continuity of $\tilde{U}$ implies that $\tilde{U}(H_1 \otimes \cdots \otimes H_n) = H_1' \otimes \cdots \otimes H_n'$ so $\tilde{U}$ is surjective. All that remains to be shown is that $\tilde{U}$ preserves the inner product. By the continuity of inner product we need only show that $\langle h, g \rangle = \left\langle \tilde{U}(h), \tilde{U}(g) \right\rangle$ for $h, g \in \operatorname{span}(\widehat{H})$. With this in mind let $h_1, \cdots, h_N, g_1, \cdots, g_M \in \widehat{H}$. We have the following

$$\begin{aligned}
\left\langle \tilde{U}\left(\sum_{i=1}^N h_i\right), \tilde{U}\left(\sum_{j=1}^M g_j\right)\right\rangle &= \left\langle \sum_{i=1}^N \tilde{U}(h_i), \sum_{j=1}^M \tilde{U}(g_j)\right\rangle \\
&= \sum_{i=1}^N \sum_{j=1}^M \left\langle \tilde{U}(h_i), \tilde{U}(g_j)\right\rangle \\
&= \sum_{i=1}^N \sum_{j=1}^M \langle h_i, g_j \rangle \\
&= \left\langle \sum_{i=1}^N h_i, \sum_{j=1}^M g_j \right\rangle.
\end{aligned}$$

We have now shown that $\tilde{U}$ is unitary which completes our proof. $\square$

Proof of Lemma 6. We will proceed by induction. For $n = 2$ the lemma clearly holds. Suppose the lemma holds for $n - 1$ and let $h_1, \cdots, h_n$ satisfy the assumptions in the lemma statement. Let $\alpha_1, \cdots, \alpha_n$ satisfy

$$\sum_{i=1}^n h_i^{\otimes n-1} \alpha_i = 0. \tag{1}$$

To finish the proof we will show that α_1 must be zero which can be generalized to any α_i without loss of generality. Let H_1 and H_2 be Hilbert spaces and let $\mathscr{HS}(H_1, H_2)$ be the space of Hilbert-Schmidt operators from H_1 to H_2. Hilbert-Schmidt operators are a closed subspace of bounded linear operators. Proposition 2.6.9 in [2] states that for a pair of Hilbert spaces H_1, H_2 there exists an unitary operator $U : H_1 \otimes H_2 \to \mathscr{HS}(H_1, H_2)$ such that $U(g_1 \otimes g_2) = g_1 \langle g_2, \cdot \rangle$. Applying this operator to (1) we get

$$\sum_{i=1}^n h_i^{\otimes n-2} \langle h_i, \cdot \rangle \alpha_i = 0. \tag{2}$$

Because h_1 and h_n are linearly independent we can choose z such that $\langle h_1, z\rangle \neq 0$ and $z \perp h_n$. Plugging z into (2) yields

$$\sum_{i=1}^{n-1} h_i^{\otimes n-2} \langle h_i, z\rangle \alpha_i = 0$$

and therefore $\alpha_1 = 0$ by the inductive hypothesis. $\square$

Proof of Lemma 7. The fact that f is positive and integrable implies that the map $S \mapsto \int_S f^{\times n} d\gamma^{\times n}$ is a bounded measure on $(\Psi^{\times n}, \mathcal{G}^{\times n})$ (see [1] Exercise 2.12).

Let $R = R_1 \times \ldots \times R_n$ be a rectangle in $\mathcal{G}^{\times n}$. Let $\mathbb{1}_S$ be the indicator function for a set S. Integrating over R and using Tonelli's theorem we get

$$\begin{aligned}
\int_R f^{\times n} d\gamma^{\times n} &= \int \mathbb{1}_R f^{\times n} d\gamma^{\times n} \\
&= \int \mathbb{1}_R f^{\times n} d\gamma^{\times n} \\
&= \int \left(\prod_{i=1}^{n} \mathbb{1}_{R_i}(x_i)\right)\left(\prod_{j=1}^{n} f(x_j)\right) d\gamma^{\times n}(x_1, \cdots, x_n) \\
&= \int \cdots \int \left(\prod_{i=1}^{n} \mathbb{1}_{R_i}(x_i)\right)\left(\prod_{j=1}^{n} f(x_j)\right) d\gamma(x_1)\cdots d\gamma(x_n) \\
&= \int \cdots \int \left(\prod_{i=1}^{n} \mathbb{1}_{R_i}(x_i) f(x_i)\right) d\gamma(x_1)\cdots d\gamma(x_n) \\
&= \prod_{i=1}^{n} \left(\int \mathbb{1}_{R_i}(x_i) f(x_i) d\gamma(x_i)\right) \\
&= \prod_{i=1}^{n} \eta(R_i) \\
&= \eta^{\times n}(R).
\end{aligned}$$

Any product probability measure is uniquely determined by its measure over the rectangles (this is a consequence of Lemma 1.17 in [3] and the definition of product σ-algebra) therefore, for all $B \in \mathcal{G}^n$,

$$\eta^{\times n}(B) = \int_B f^{\times n} d\gamma^{\times n}.$$

$\square$

References

[1] Gerald B. Folland. *Real analysis: modern techniques and their applications*. Pure and applied mathematics. Wiley, 1999.

[2] R.V. Kadison and J.R. Ringrose. *Fundamentals of the theory of operator algebras. V1: Elementary theory*. Pure and Applied Mathematics. Elsevier Science, 1983.

[3] Olav Kallenberg. *Foundations of modern probability*. Probability and its applications. Springer, New York, Berlin,, Paris, 2002. Sur la 4e de couv. : This new edition contains four new chapters as well as numerous improvements throughout the text.