1.3.1

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} a^*a + b^*b & c^*a + d^*b \\ a^*c + b^*d & c^*c + d^*d \end{pmatrix} = \mathbb{1}$$

$$a^*a + b^*b = 1 = c^*c + d^*d$$

$$a^*c + b^*d = 0 = c^*a + d^*b$$

$$a^*c = -b^*d \qquad c^*a = -d^*b$$

$$|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$$

1.3.2

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow U = \begin{pmatrix} a & -e^{i\beta}c^* \\ c & e^{i\beta}a^* \end{pmatrix}$$

$$b = -\frac{c^*a}{d^*} \qquad d = \frac{a^*c}{-b^*}$$

Hint : det and Inverse

$$UU^\dagger = \mathbb{1} = UU^{-1}$$

Eq. 1 $$U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \frac{1}{\det(U)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$b = -\det(U)\, c^*$$

$$a = \det(U)\, d^*$$

$$\det(\underbrace{UU^\dagger}_{\text{remember this gives the } \mathbb{1} \text{ matrix}}) = 1 = \overbrace{\det(U)\det(U^\dagger)}^{\text{property of determinants}}$$

Some other determinant properties:

$$\det(UU^\dagger) = \det(U)\det(U^\dagger)$$

$$\det(U^{-1}) = 1/\det(U)$$

$$\det(U^\dagger) = \overline{\det(U)}$$

With this knowledge, we now know that $\det(U) = e^{i\beta}$ and with $\det(U^\dagger) = \overline{\det(U)} = e^{-i\beta}$ we can satisfy

U

$$1 = e^{i\beta} e^{-i\beta} = \det(UU^\dagger)$$

U can be rewritten as $\begin{pmatrix} a & -e^{i\beta}c^* \\ c & e^{i\beta}a^* \end{pmatrix}$

1.3.3

From Eq 1 $\det(U)$

$$a = \det(U)\, d^*$$

$$b = -\det(U)\, c^*$$

$$c = -\det(U)\, b^*$$

$$d = \det(U)\, a^*$$

$$\begin{pmatrix} e^{i\beta} d^* & -e^{i\beta} c^* \\ -e^{i\beta} b^* & e^{i\beta} a^* \end{pmatrix}$$

$$\beta = (\phi + \omega)/2$$

Hint 1 $\rightarrow$ start by expressing a and c is polar form to uncover the θ dependence.

then, find a way to re-express the phases involving ϕ, ω to match the result

Starting with:

$$\begin{pmatrix} a & -e^{i\beta} c^* \\ c & e^{i\beta} a^* \end{pmatrix}$$

polar form of complex numbers:

$a = |a| e^{i\theta}$ $\qquad$ $c = |c| e^{i\psi}$

$a^* = |a| e^{-i\theta}$ $\qquad$ $c^* = |c| e^{-i\psi}$

Hint 2: In quant computing U and $e^{i\theta} U$ are effectively the same operation, you can thus factor out complex numbers with modulus 1 from any unitary matrix.

Im Still Stumped!

following the solution:

In Exersice 1.3.1 we learnt U has orthonormal rows and columns i.e. its rows and columns are orthogonal and normalized.

$UU^\dagger$ gave us $|a|^2 + |b|^2 = 1$ [row is normalized]

similiarly $U^\dagger U$ would give us $|a|^2 + |c|^2 = 1$ [column is normalized]

Quick Sanity Check:

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} a^* \\ b^* \end{pmatrix} + c\begin{pmatrix} c^* \\ d^* \end{pmatrix} & b\begin{pmatrix} a^* \\ b^* \end{pmatrix} + d\begin{pmatrix} c^* \\ d^* \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} aa^* + cc^* & ba^* + dc^* \\ ab^* + cd^* & bb^* + dd^* \end{pmatrix}$$

$$= \begin{pmatrix} |a|^2 + |c|^2 & \cdots \\ \cdots & |b|^2 + |d|^2 \end{pmatrix}$$

$|a|^2 + |c|^2 = |b|^2 + |d|^2 = 1$

$a = re^{i\alpha}$ $\quad c = se^{i\gamma}$ [polar form]

$r^2 + s^2 = 1$

we can set $r = \cos(\theta/2)$ $\quad s = \sin(\theta/2)$

$$\begin{pmatrix} a & -e^{i\beta}c^* \\ c & e^{i\beta}a^* \end{pmatrix} = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & -e^{i(\beta-\gamma)}\sin(\theta/2) \\ e^{i\gamma}\sin(\theta/2) & e^{i(\beta-\alpha)}\cos(\theta/2) \end{pmatrix}$$

We can remove a factor $e^{i\beta/2}$ (as per hint 2)

$$U = e^{i\beta/2} \begin{pmatrix} e^{i(\alpha-\beta/2)}\, c(\theta/2) & -e^{i(\beta/2-\gamma)}\, s(\theta/2) \\ e^{i(\gamma-\beta/2)}\, s(\theta/2) & e^{i(\beta/2-\alpha)}\, c(\theta/2) \end{pmatrix}$$

adjust signs to match expression

$$U = \begin{pmatrix} e^{-i(\beta/2-\alpha)}\, c(\theta/2) & -e^{i(\beta/2-\gamma)}\, s(\theta/2) \\ e^{-i(\beta/2-\gamma)}\, s(\theta/2) & e^{i(\beta/2-\alpha)}\, c(\theta/2) \end{pmatrix}$$

$$\begin{aligned} & \beta/2-\alpha = \phi+\omega \\ + \quad & \beta/2-\gamma = \phi-\omega \\ & \beta-\alpha-\gamma = 2\phi \\ & \boxed{\phi = \beta/2 - \alpha/2 - \gamma/2} \end{aligned}$$

$$\begin{aligned} & \beta/2-\alpha = \phi+\omega \\ - \quad & \underline{\beta/2-\gamma = \phi-\omega} \\ & -\alpha+\gamma = 2\omega \\ & \boxed{\omega = \gamma/2 - \alpha/2} \end{aligned}$$

$$U = \begin{pmatrix} e^{-i(\phi+\omega)} \cos(\theta/2) & -e^{i(\phi-\omega)} \sin(\theta/2) \\ e^{-i(\phi-\omega)} \sin(\theta/2) & e^{i(\phi+\omega)} \cos(\theta/2) \end{pmatrix}$$

Most general parameterization of a unitary matrix