COMPSCI 250: Introduction to Computation

Lecture #27: Games and Adversary Search
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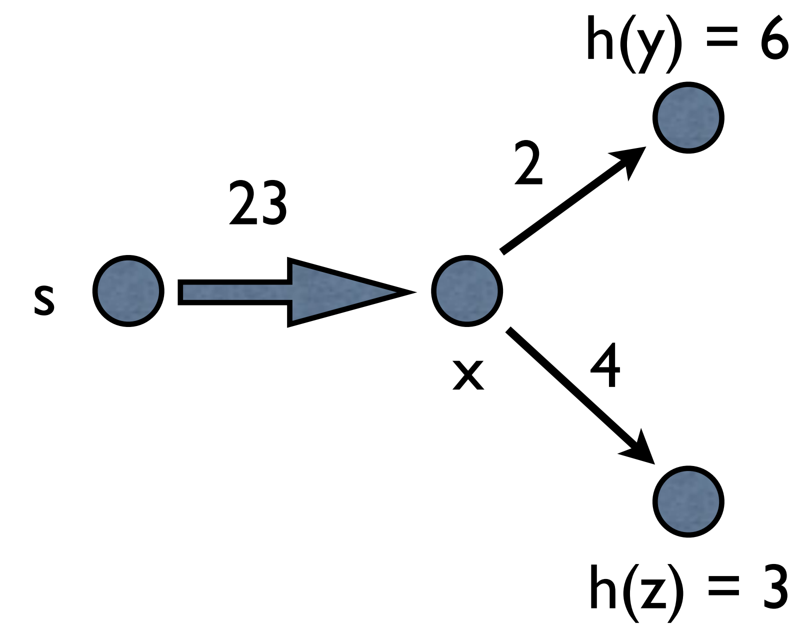
Games and Adversary Search

- Review: A* Search
- Modeling Two-Player Games
- When There is a Game Tree
- The Determinacy Theorem
- Searching a Game Tree
- Examples of Games

Review: A^* Search

- The A^* Search depends on a **heuristic function**, which is a **lower bound** on the distance to the goal.
- If x is a node, and g is the nearest goal node to x, the **admissibility condition** on h is that $0 \leq h(x) \leq d(x, g)$.

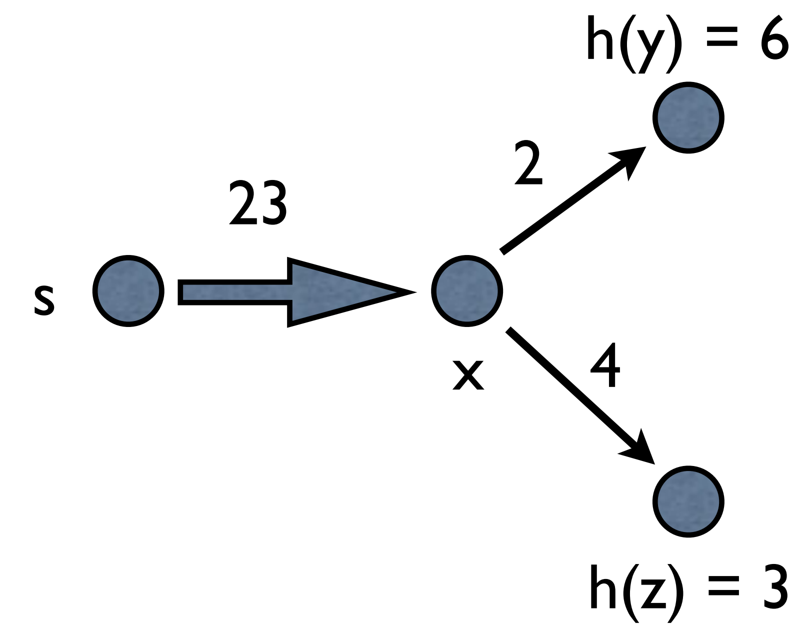


p(y) = 23 + 2 + 6 = 31

p(z) = 23 + 4 + 3 = 30

Review: A^* Search

- Suppose we consider taking y off of the open list. The best-path distance from the start s to the goal g through y is d(s, y) + d(y, g), and this cannot be less than d(s, y) + h(y).
- Thus when we find a path from s to y of length k, we put y onto the open list with priority k + h(y). We still record the distance d(s, y) when we take y off the open list.



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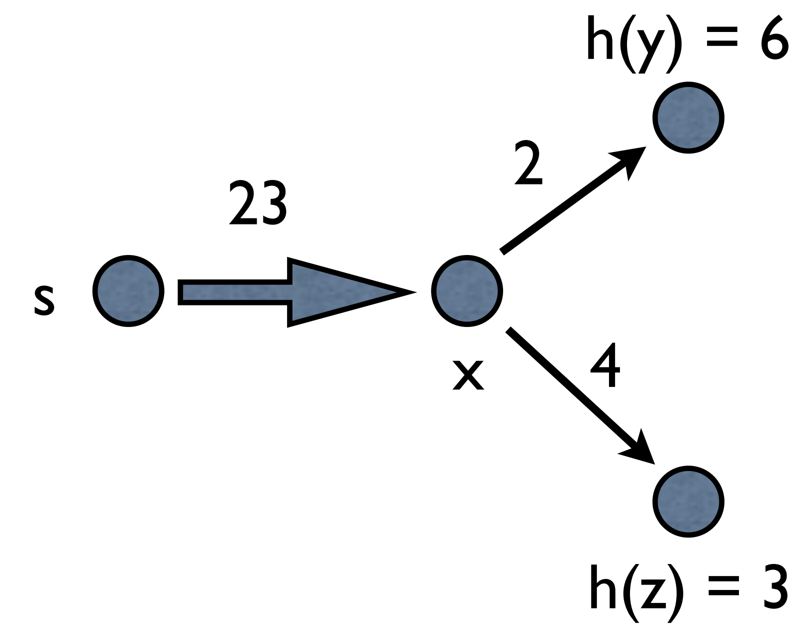
Review: A^* Search

- The advantage of A^* over uniform-cost search is that we do not consider entries x in the closed list for which $d(s, x) + h(x)$ is greater than the actual best-path distance from s to g.

- This is because when we find the best path to g with length $d(s, g)$, we put g on the open list with priority $d(s, g) + h(g) = d(s, g)$ and it will come off before any node with higher priority value.



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p(z) = 23 + 4 + 3 = 30

The 15 Puzzle

- The 15-puzzle is a 4×4 grid of pieces with one missing, and the goal is to put them in a certain arrangement by repeatedly sliding a piece into the hole.
- We can imagine a graph where nodes are positions and edges represent legal moves.

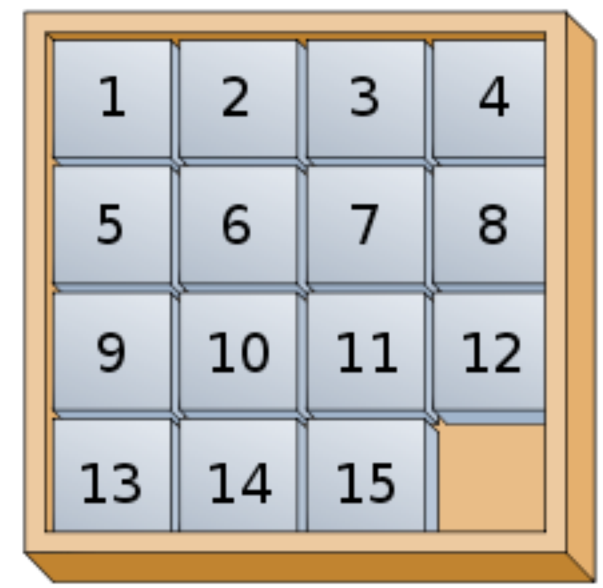


Figure from
en.wikipedia.org
“Fifteen puzzle”

The 15 Puzzle

- In order to move from a given position to the goal, each piece must move at least the Manhattan distance from its current position to its goal position.
- The sum of all these Manhattan distances gives us an admissible, consistent heuristic for the actual minimum number of moves to reach the goal. So an A* search will be faster than a uniform-cost search.

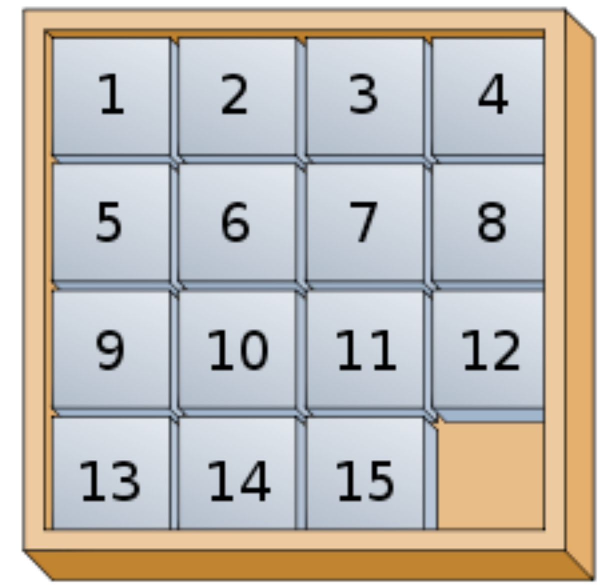


Figure from en.wikipedia.org "Fifteen puzzle"

Clicker Question #1

- Suppose we construct the directed graph where nodes are 15-puzzle positions and there is a directed edge labeled “1” for every legal move. Which search *is* equivalent to uniform-cost search from x?
- (a) DFS with start node x
- (b) A^* search from x, with h = distance to goal.
- (c) BFS with start node x
- (d) Trick question, none of these are equivalent to UCS.

Not the Answer

Clicker Answer #1

- Suppose we construct the directed graph where nodes are 15-puzzle positions and there is a directed edge labeled “1” for every legal move. Which search *is* equivalent to uniform-cost search from x?
- (a) DFS with start node x Not similar to UCS at all
- (b) A^* search from x, with h = distance to goal A^* will get the same answer, but will be faster
- (c) BFS with start node x
- (d) Trick question, none of these are equivalent to UCS.

Modeling Two-Player Games

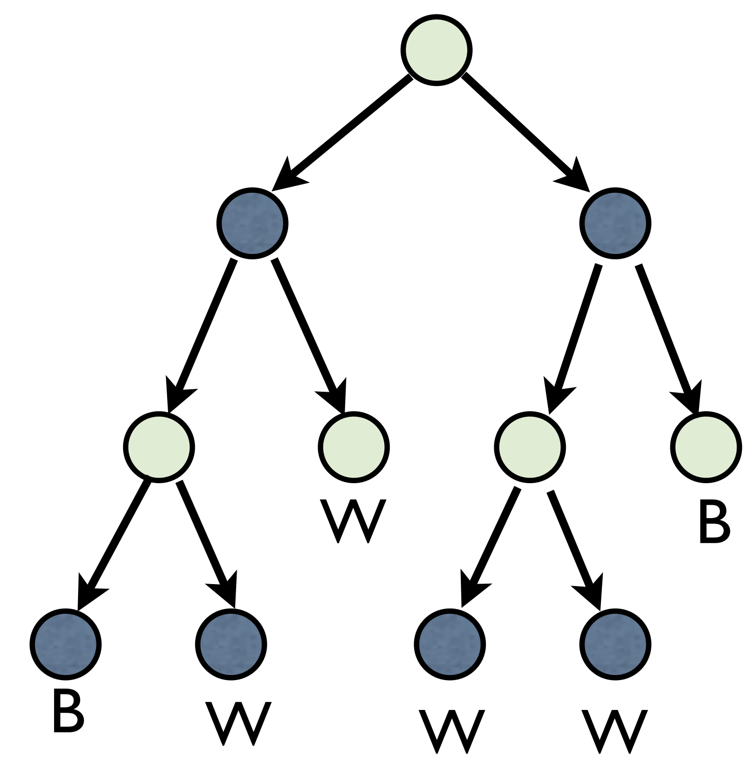
- There are many kinds of games, and we are now going to look at a theory which will let us model and analyze some of them.
- You probably know that the game of **tic-tac-toe** is not very interesting to play, because if both players are familiar with the game the result is always a draw.
- There is a strategy for the first player, X, that allows her to always win or draw. There is also a strategy for O, the second player, letting him win or draw.

Modeling Two-Player Games

- Any game that shares certain particular features of tic-tac-toe is **determined** in the same way.
- We must have **sequential moves**, **two players**, a **deterministic** game with no randomness, a **zero-sum** game, and **perfect information**.
- In these cases we can model the game by a **game tree**.

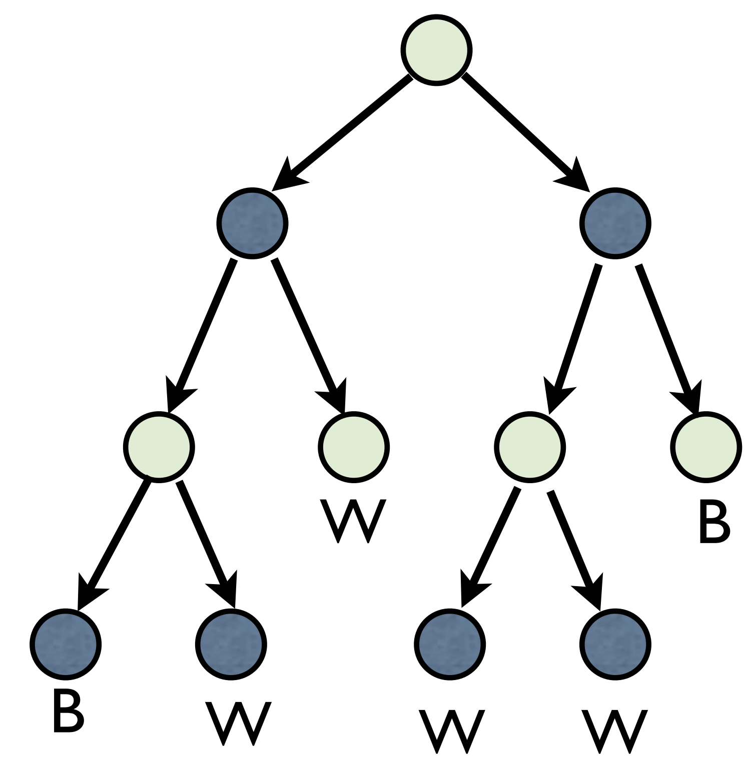
Game Trees

- A game tree has a node for every possible **state** or **position** of the game. The root node represents the **start position**.
- A node y is a child of a node x if it is possible, according to the rules of the game, to get to y from x in one move.



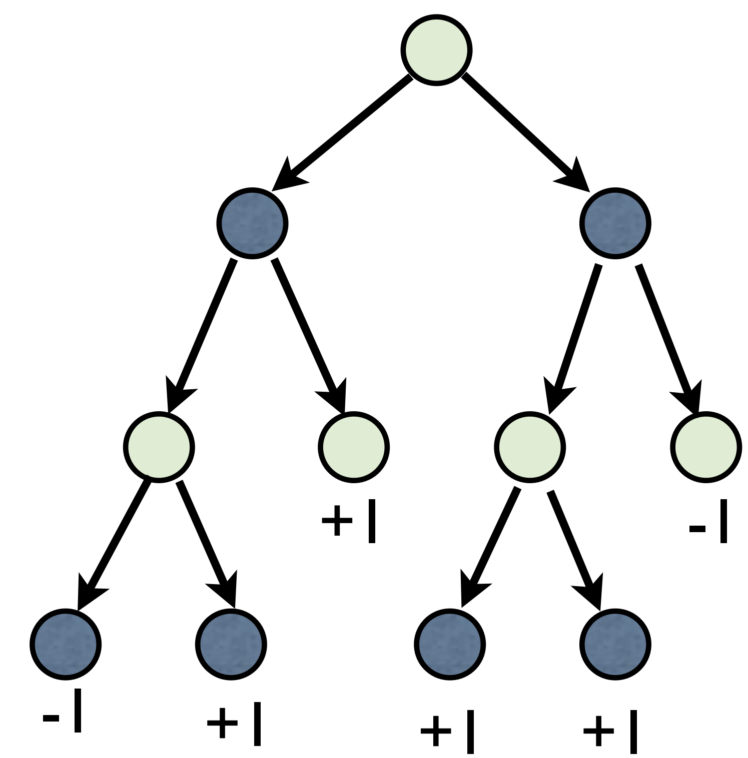
Game Trees

- Every node is labeled by **whose turn** it is. (Leaves can be colored any way.)
- Usually the two players alternate moves, so we can call them the **first** and **second** player (White and Black), but our analysis will not change if one player can make several moves in a row.



Game Trees

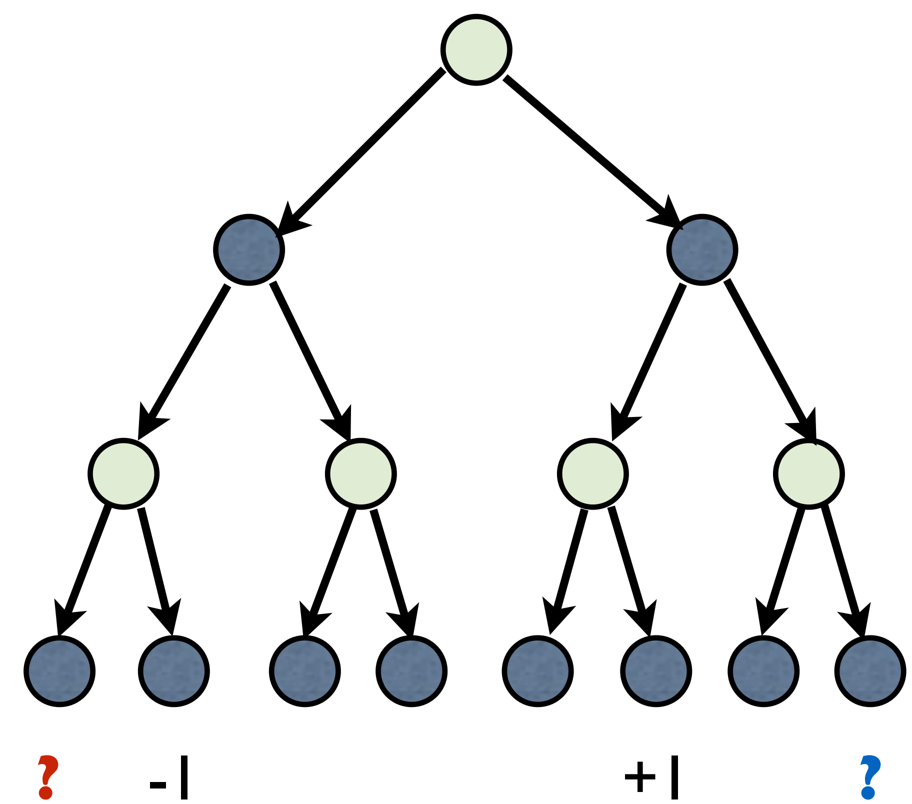
- The **leaves** of the tree represent positions where the **result** of the game is known.
- We label leaves with a real number indicating how much White is paid by Black, typically 1 for a White win, 0 for a draw, and -1 for a Black win, but any real number values are possible.



Clicker Question #2

For the given game tree, assume there are no draws. Which of the following is true, if both play optimally?

- (a) Black cannot win if ? = +1
- (b) Black wins if ? = -1
- (c) White wins if ? = +1
- (d) White wins whatever the other values are.



Not the Answer

Clicker Answer #2

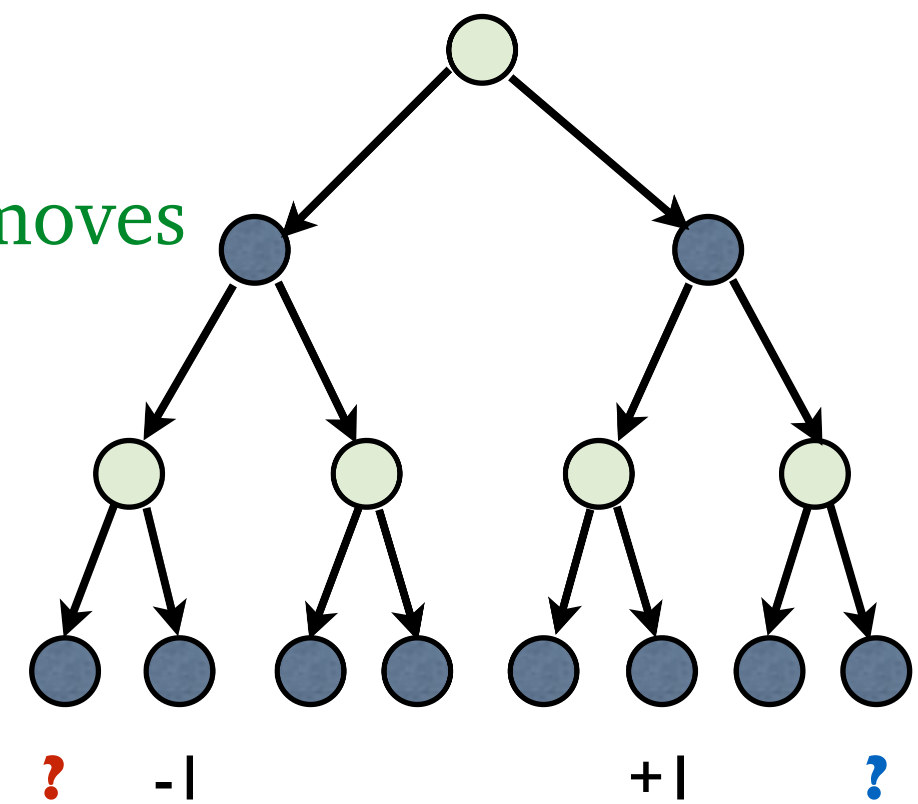
For the given game tree, assume there are no draws. Which of the following is true, if both play optimally?

- (a) Black cannot win if ? = +1

  White moves right on both her moves
- (b) Black wins if ? = -1

  White could win by going right
- (c) White wins if ? = +1

  Black might go right on his first move
- (d) White wins whatever the other values are.



Black can win if both ?, ?, and the leaf before ? are -1

When We Have a Game Tree

- To be represented by such a tree the game must be **discrete**, **deterministic**, **zero-sum**, and have **perfect information**.
- The tree is **finite** if there are only finitely many sequences of moves that can ever occur. We could have a finite **game graph** where nodes can be reached in more than one way or even revisited, but we won't analyze these here.

The Determinacy Theorem

- Each leaf has a **game value**, the real number we defined above. We can inductively assign a game value to *every node* of the tree, by the following rules.
- The value val(s) of a final position is its label.
- If White is to move in position s, val(s) is the *maximum* value of any child of s.
- If Black is to move in position s, val(s) is the *minimum* value of any child of s.

The Determinacy Theorem

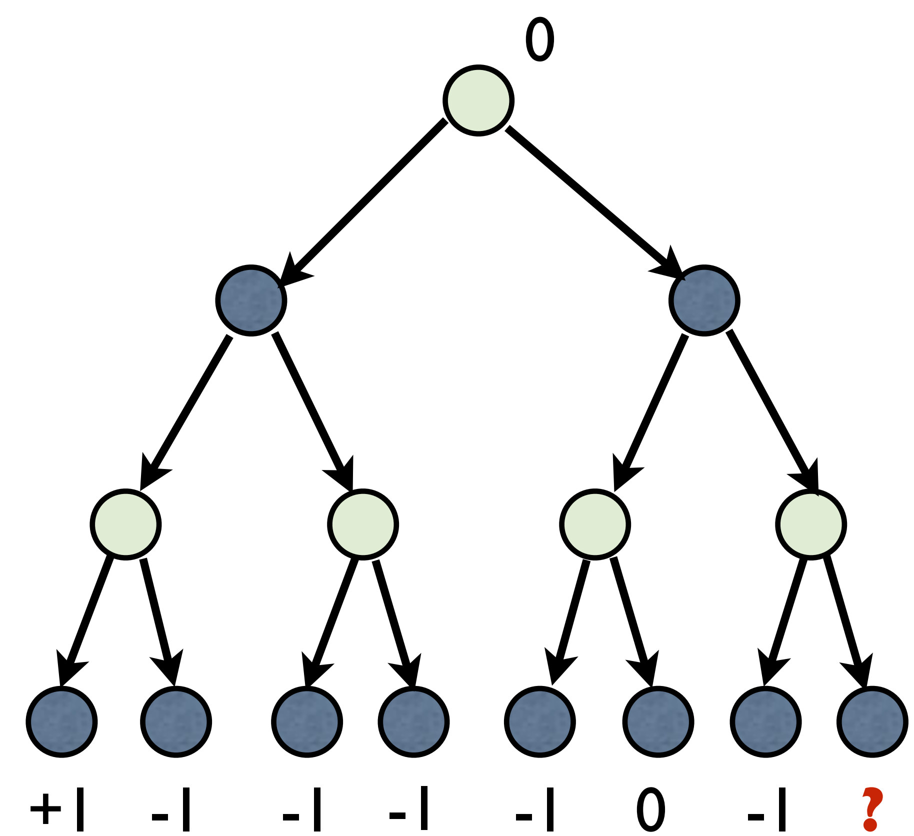
- The **Determinacy Theorem** says that:
- (1) any game given by a finite tree has a game value v (the value of the root given by the definition above),
- (2) White has a strategy that guarantees her a result of *at least* v, and
- (3) Black has a strategy that guarantees him that the result will be *at most* v. Thus v is the result if both players play *optimally*.

Proving Determinacy

- We prove that for each node x in the tree, each player has a strategy that gets them either a result of val(x) or a result that is even better for them.
- If x is a leaf of the tree this is obvious.
- If it is White's move she can move to the child with value val(x), and by the IH get at least this result.
- It's just the same if Black is to move.

Clicker Question #3

- The value of this game is 0. What is the value of ?
- (a) must be -1
- (b) must be 0
- (c) must be 1
- (d) could be 0 or 1



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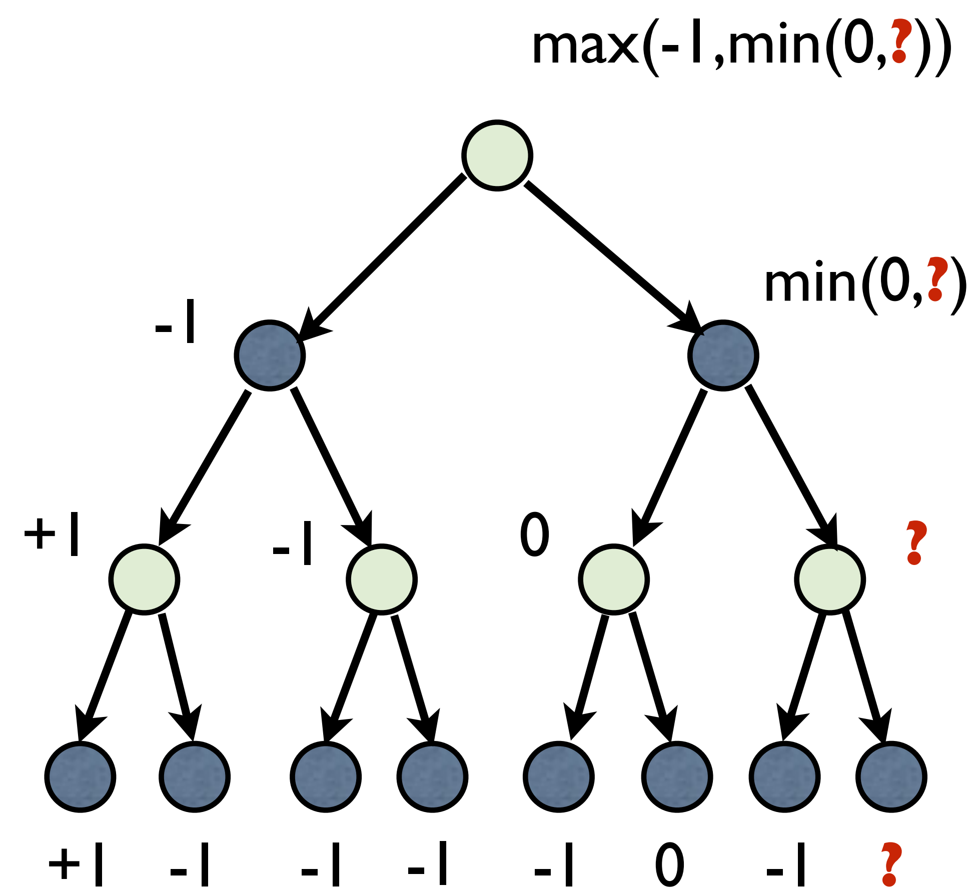
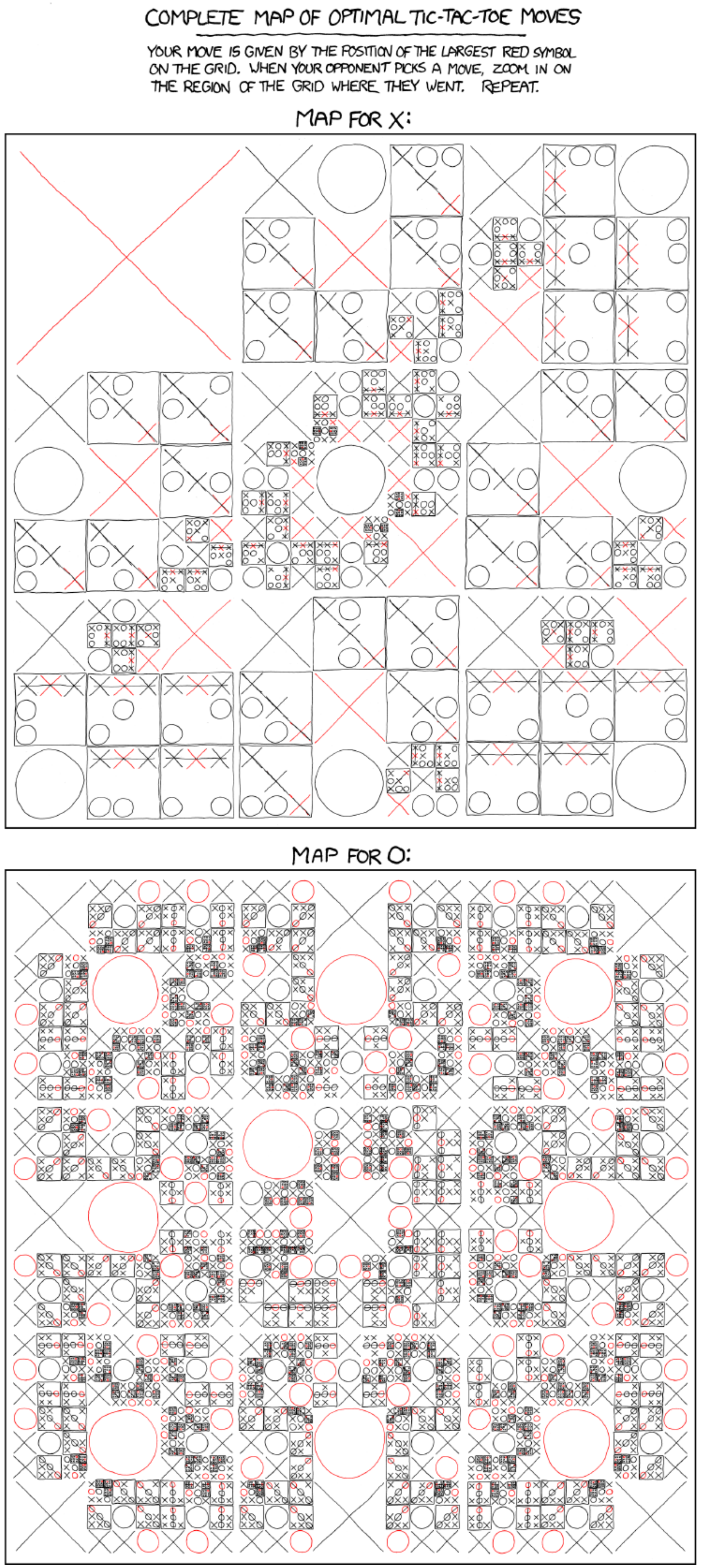


Winning Tic-Tac-Toe

- The chart to the right, if it were big enough to read, would tell you **complete strategies** for each player guaranteeing a result of 0 (a draw) or better.



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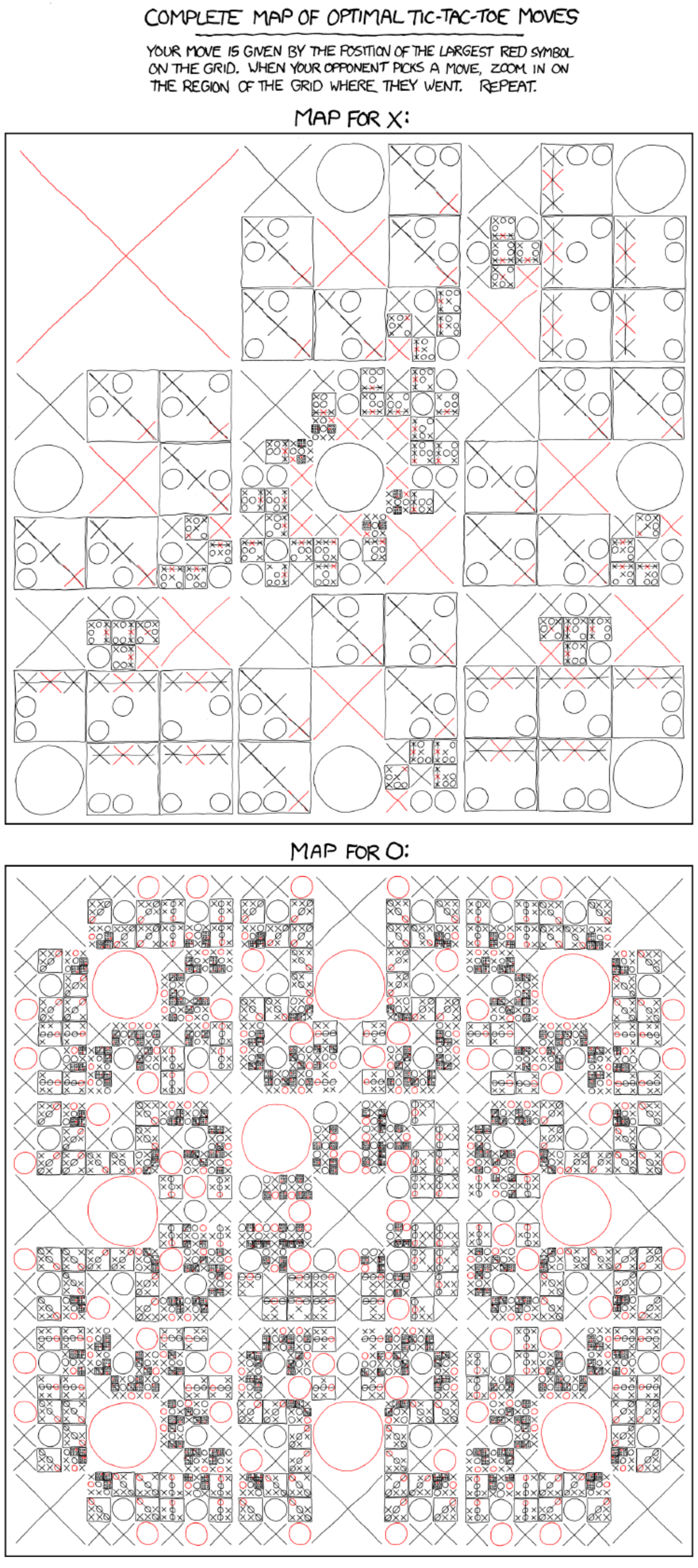
Winning Tic-Tac-Toe

- The X strategy starts with moving to the top left, then has a reply to each of the eight O moves that could follow, then a reply to each of the six possible O responses to that move, and so on.
- The desired moves are in red.



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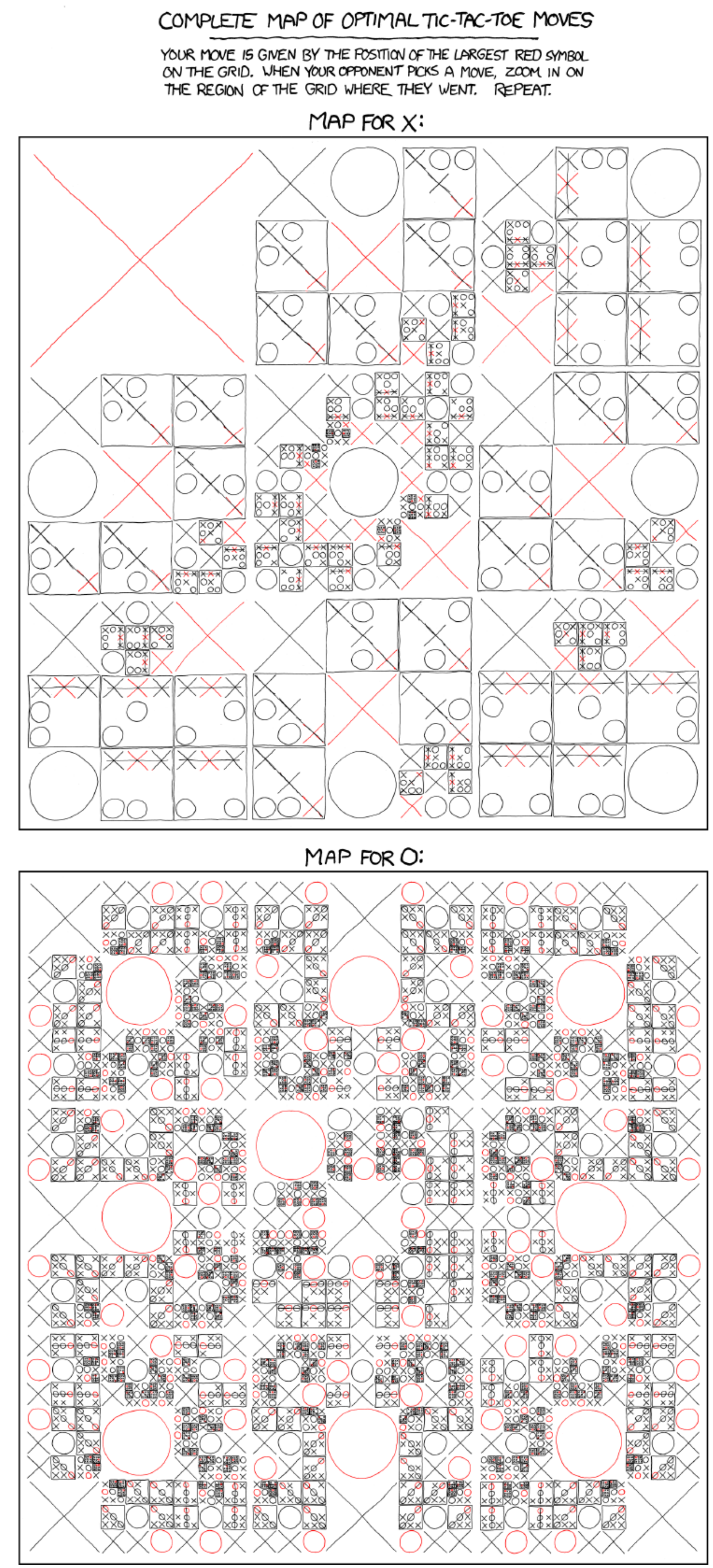
Winning Tic-Tac-Toe

- The O strategy must have responses to all nine initial X moves, then to all seven X responses to each of those moves, and so on.
- The messiest parts of the chart is where the game goes for all nine moves, since each board is 1/9 the area of the last.



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Searching a Game Tree

- The Determinacy Theorem only tells us that these optimal strategies exist, not that they are possible to implement.
- If it is possible to **calculate the game value** of any node, then choosing the right move is easy. And we have a recursive algorithm to compute the game value, so what is the problem?
- The tree could be *really really big*.

Adversary Search

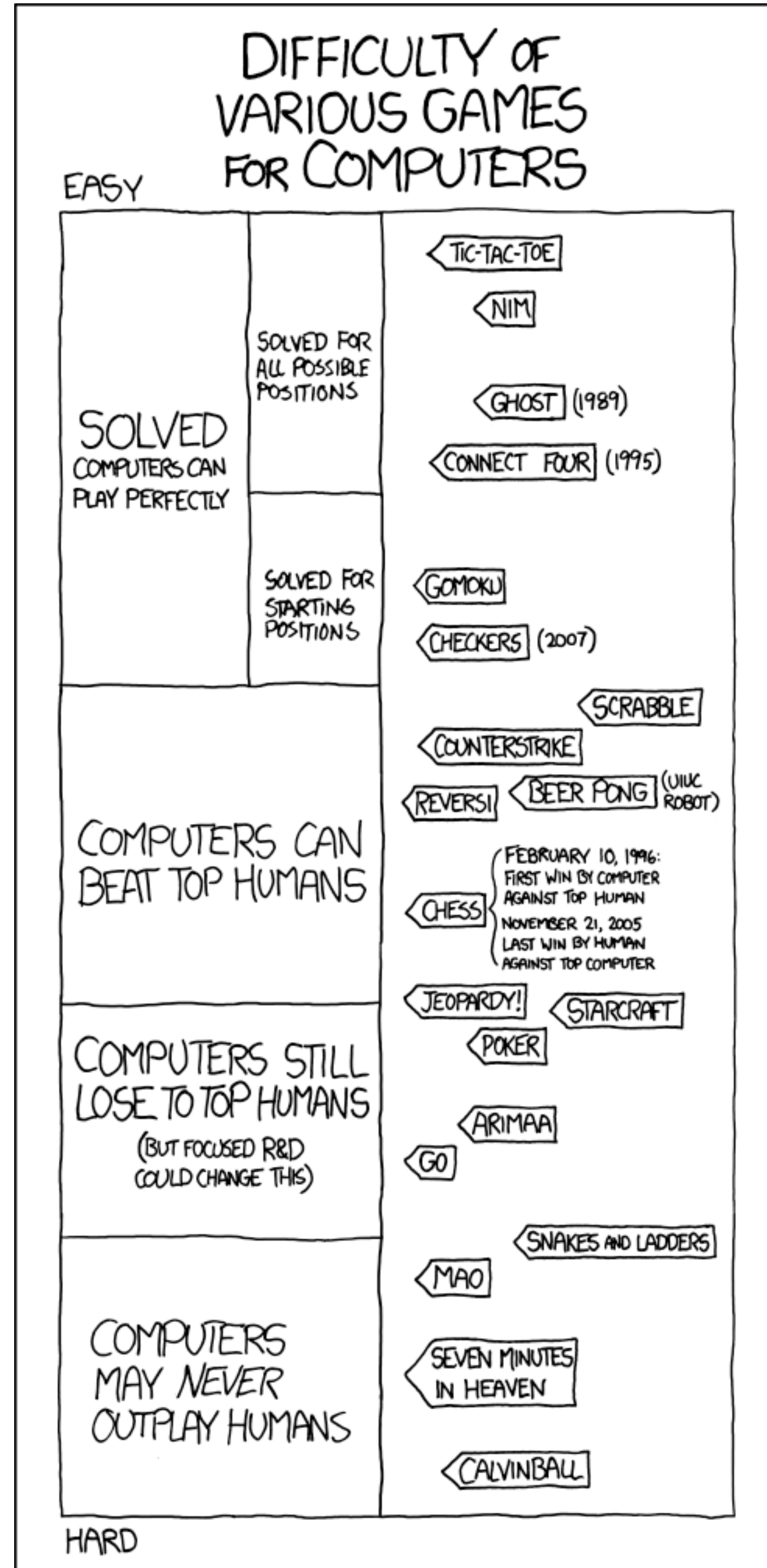
- An exhaustive adversary search computes the exact value.
- If we can’t do that, we need an **estimate** of the game value.
- In Chess, for example, we can evaluate material and some positional facts to get a good idea whether one position is better than another.

Adversary Search

- We can then use **finite lookahead**, playing a game that ends in k moves, where the payoff is the estimated value of the position at the end of those k moves.
- **Alpha-beta pruning**, which we won't do in this course, is a way to improve the search. But the required time is still usually exponential in the number of moves to go.

Examples of Games

- In 1965, Dave' father's M.S. thesis was to build a tic-tac-toe program that learned from its mistakes.
- By 1992, and probably earlier, students in CMPSCI 187 could build an optimal program that exhaustively searched the game tree *on every move*.



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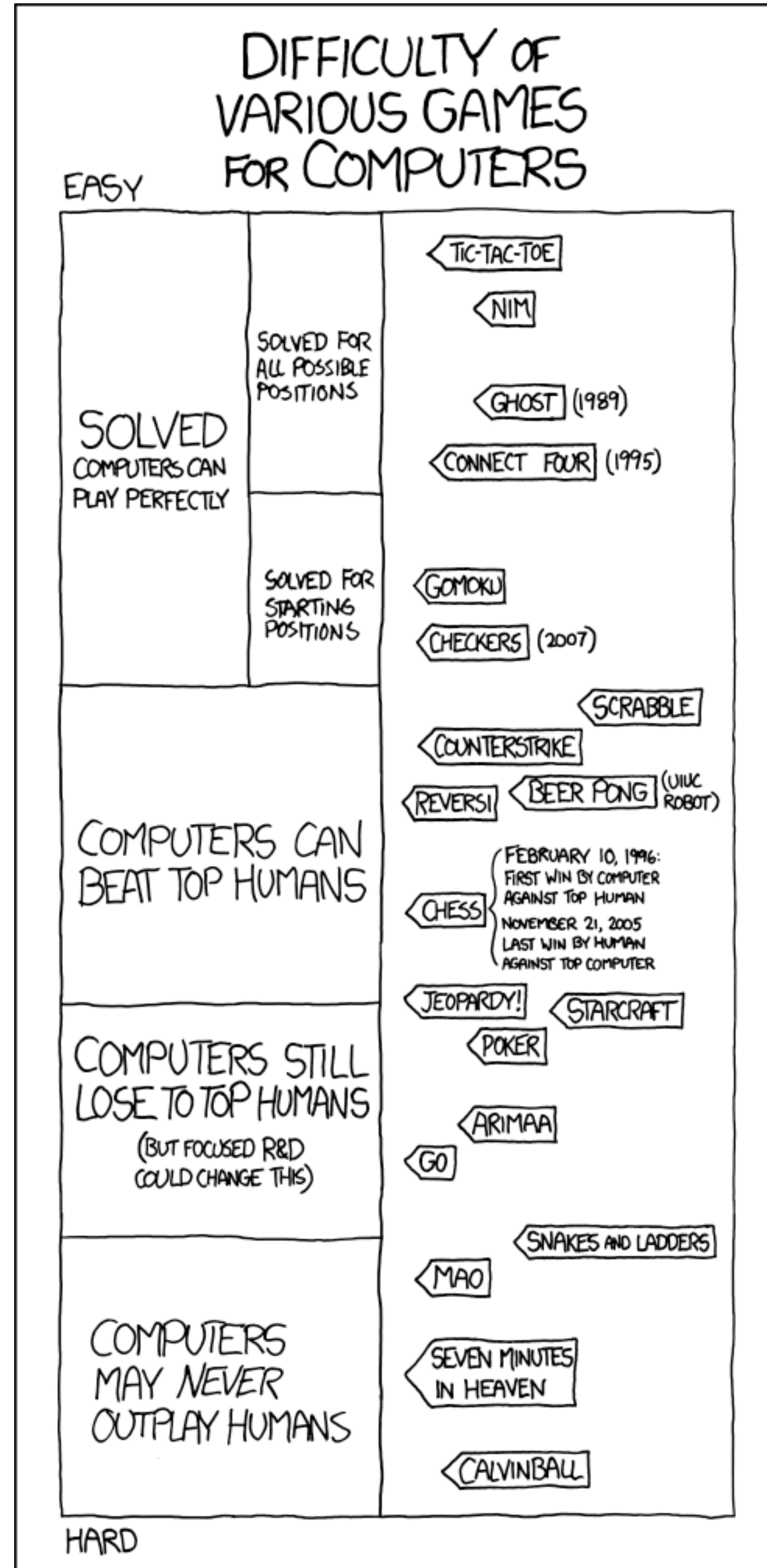
Examples of Games

- There is either a winning strategy for White in Chess, or a drawing strategy for Black. But no one knows which is true.
- Current Chess programs succeed by doing a better job of searching and evaluating positions.



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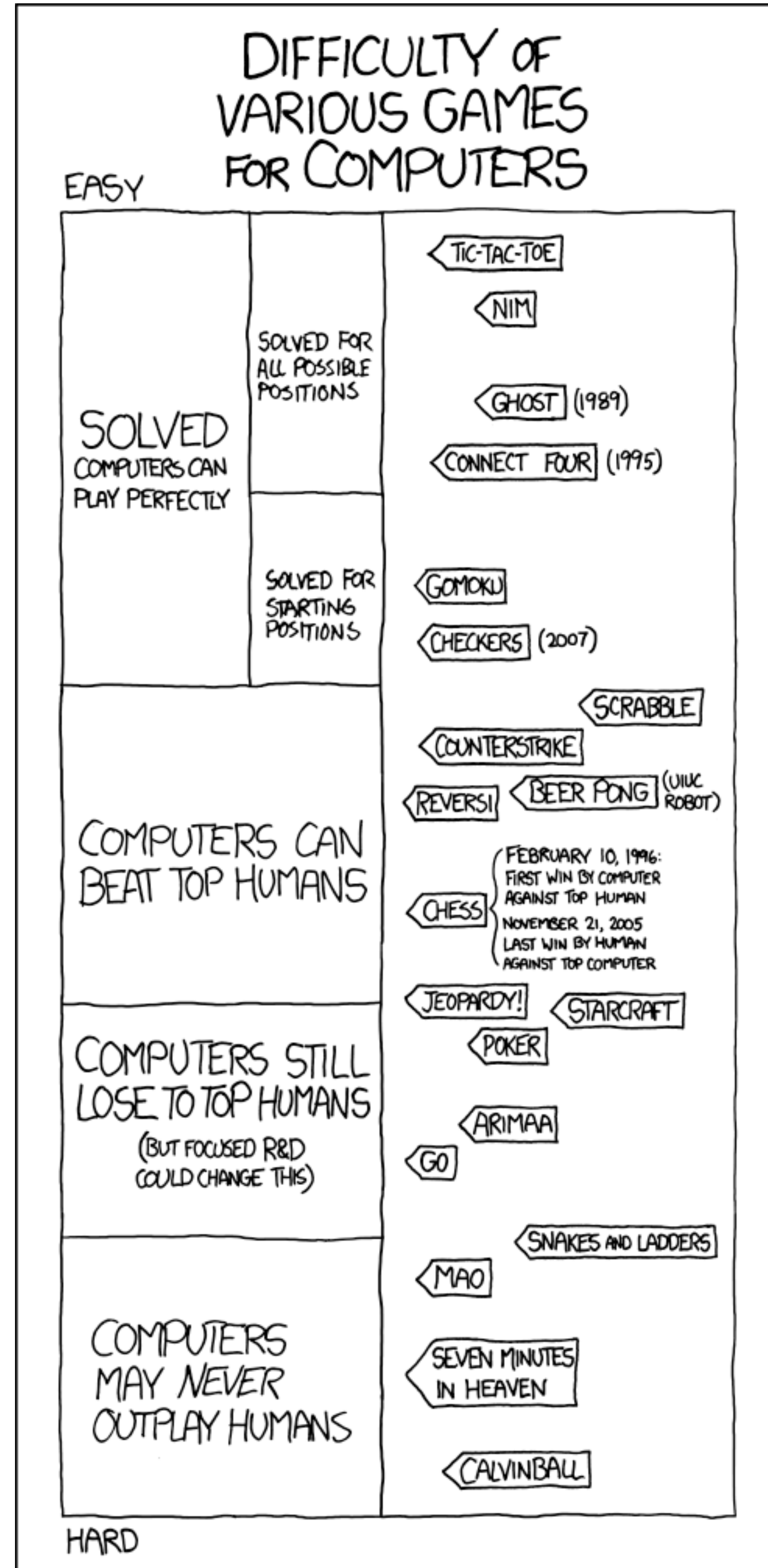
Examples of Games

- Computers don't approach chess the way good human players do. We can use games as benchmarks for AI achievement.
- Checkers is easier than Chess, and Go is harder.
- Calvinball (from *Calvin and Hobbes*) allows rules to be changed at will.



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