COMPSCI 250 Discussion #3: Practicing Proofs
Individual Handout

David Mix Barrington and Ghazaleh Parvini
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In this discussion we will apply our proof methods, for both the propositional and predicate calculi, to prove some statements about functions and relations.

The predicate proof rules will be important: Generalization and Specification to work with universal quantifiers, and Existence and Instantiation for existential quantifiers.

The key definitions will be properties of functions and relations. These were introduced in Lecture 9. We'll work a lot with them in Lectures 10 and 11, but for now you can just work with them as quantified statements, using these definitions.

If R is a relation from A to B:

- R is **total** if $\forall a : \exists b : R(a, b)$.
- R is **well-defined** if $\neg\exists b_1 : \exists b_2 : \exists a : R(a, b_1) \wedge R(a, b_2) \wedge (b_1 \neq b_2)$.
- R is **onto** if $\forall b : \exists a : R(a, b)$.
- R is **one-to-one** if $\neg\exists a_1 : \exists a_2 : \exists b : R(a_1, b) \wedge R(a_2, b) \wedge (a_1 \neq a_2)$.

And if R is a binary relation on a set A:

- R is **reflexive** if $\forall x : R(x, x)$.
- R is **antireflexive** if $\forall x : \neg R(x, x)$.
- R is **symmetric** if $\forall x : \forall y : R(x, y) \rightarrow R(y, x)$.
- R is **antisymmetric** if $\forall x : \forall y : (R(x, y) \wedge R(y, x)) \rightarrow (x = y)$.
- R is **transitive** if $\forall x : \forall y : \forall z : (R(x, y) \wedge R(y, z)) \rightarrow R(x, z)$.

Here's a warmup from Problem 2.3.3 in the text, which uses all four of the quantifier rules:

In this problem you were given sets A and B, where $A \subseteq B$ and $(B \setminus A) \neq \emptyset$, unary predicates P on A and Q on B, and a premise $\forall a : \forall b : P(a) \rightarrow Q(b)$.

- **Theorem 1** (part of part (a)): If $A \neq \emptyset$, prove $(\forall a : P(a)) \rightarrow (\forall b : Q(b))$.
- **Proof of Theorem 1**:
  1. To start the direct proof, assume $\forall a : P(a)$
  2. By Instantiation, let x be an element of a.
  3. By Specification on $\forall a : P(a)$, derive $P(x)$.
  4. Let y be an arbitrary element of B.
  5. Specify a to x and b to y in the premise $\forall a : \forall b : P(a) \rightarrow Q(b)$ to get $P(x) \rightarrow Q(b)$.
  6. By Modus Ponens on lines (3) and (5), derive $Q(b)$.
  7. By Generalization, since y was arbitrary, derive $\forall b : Q(b)$.
- Note that without the assumption $A \neq \emptyset$, we can't carry out this proof, so the answer to part (a) is "false".
- **Theorem 2** (part (c)) $(\exists a : P(a)) \rightarrow (\exists b : Q(b))$
- **Proof of Theorem 2**:
  1. Assume $\exists a : P(a)$.
  2. By Instantiation, let x be an element of A such that $P(x)$ is true.
  3. Note that x is also an element of B since $A \subseteq B$.
  4. Specify a to x and b to x in $\forall a : \forall b : P(a) \rightarrow Q(b)$, to get $P(x) \rightarrow Q(x)$.
  5. Using Modus Ponens on lines (2) and (4), derive $Q(x)$.
  6. By Existence, derive $\exists b : Q(b)$.

Writing Exercise:

Here are four practice statements to prove. For this exercise, you should go slightly overboard in justifying your steps. In the first example you should use variables x of type X, y of type Y, and z of type Z. In the other three examples all the variables are of type A.

1. Let X, Y, and Z be three sets, and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions (both total and well-defined). Assume that f and g are each onto functions, and let $h : X \rightarrow Z$ be defined by the rule $h(x) = g(f(x))$. **Prove that h is onto**.
2. Let R be a binary relation on A. Assume that R is transitive and symmetric. Also assume $\forall x : \exists y : R(x, y)$. **Prove that R is reflexive**.
3. Let R and S be two binary relations on the same set A. Assume that both R and S are antisymmetric. Define T to be the intersection of R and S, so that $\forall x : \forall y : T(x, y) \leftrightarrow (R(x, y) \wedge S(x, y))$. **Prove that T is antisymmetric**.