CMPSCI 250: Introduction to Computation

Lecture #11: Partial Orders

David Mix Barrington and Ghazaleh Parvini

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Partial Orders

- Review of Binary Relation Properties
- Definition of Partial Orders and Total Orders
- The Division Relation and Other Examples
- Hasse Diagrams
- The Hasse Diagram Theorem

Definition of a Partial Order

- A **partial order** is a particular kind of binary relation on a set. Remember that R is a **binary relation** on a set A if $R \subseteq A \times A$, that is, if R is a set of ordered pairs where both elements of every pair are from A.
- In Lecture #9 we used quantifiers to define four particular properties that a binary relation on a set might have.
- A relation is a partial order if and only if it is reflexive, antisymmetric, and transitive.

Properties of a Partial Order

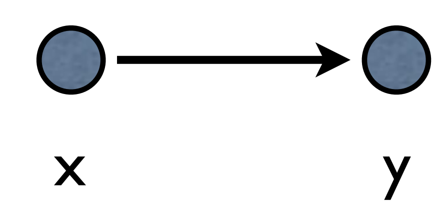
- A relation R is **reflexive** if every element is related to itself -- in symbols, $\forall x: R(x, x)$.
- It is **antisymmetric** if we can't reverse the order of elements in any pair unless they are the same — in symbols, $\forall x: \forall y: (R(x, y) \wedge R(y, x)) \rightarrow (x = y)$.
- Finally, R is **transitive** if
  $\forall x: \forall y: \forall z: (R(x, y) \wedge R(y, z)) \rightarrow R(x, z)$.
  This says that a chain of pairs in the relation must be accompanied by a single pair whose elements are the start and end of the chain.

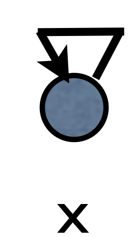
Equivalence Relations and P.O.'s

- Last lecture we defined **equivalence relations**, which are binary relations on a set that are reflexive, **symmetric**, and transitive.
- An equivalence relation divides its set into equivalence classes: If x is an element, [x] is the set of elements equivalent to x.
- Partial orders are different because they are **antisymmetric**. This does not mean "not symmetric" but is a different property.

Diagrams of Binary Relations

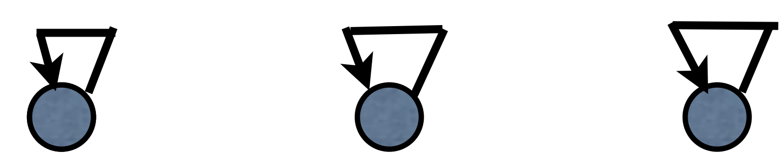
- If A is a finite set and R is a binary relation on A, we can draw R in a diagram called a graph.
- We make a dot for each element of A, and draw an arrow from the dot for x to the dot for y whenever R(x, y) is true.
If R(x, x), we draw a loop from the dot for x to itself.





Seeing the Properties

- The properties are perhaps easier to see in one of these diagrams.

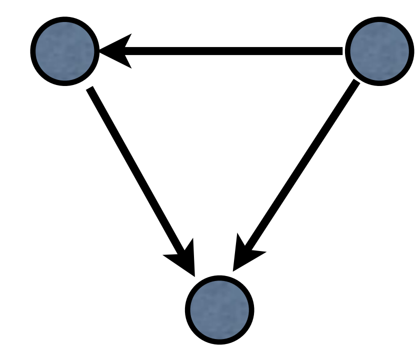


- A relation is reflexive if its diagram has a loop at every dot.

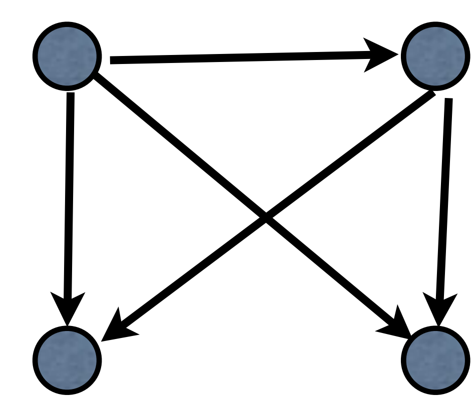
- It is symmetric if every arrow (except loops) has a matching opposite arrow.

Seeing the Properties

- It is antisymmetric if there are never two arrows in opposite directions between two different nodes.

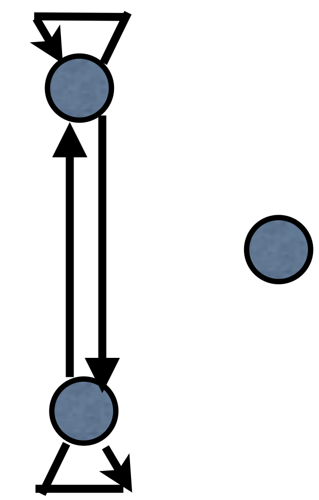


- It is transitive if for every path of arrows (a chain where the start of each arrow is the end of the previous one) there is a single arrow from the start of the chain to the end.



Clicker Question #1

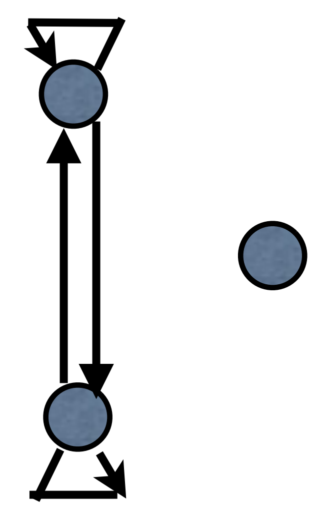
- Which properties does the diagrammed relation have?
- (a) reflexive, antisymmetric, transitive
- (b) not reflexive, symmetric, transitive
- (c) not reflexive, symmetric, not transitive
- (d) not reflexive, not symmetric, not transitive



Not the Answer

Clicker Answer #1

- Which properties does the diagrammed relation have?
- (a) reflexive, antisymmetric, transitive
- *(b) not reflexive, symmetric, transitive*
- (c) not reflexive, symmetric, not transitive
- (d) not reflexive, not symmetric, not transitive



Total Orders

- When we studied **sorting** in a data structures course, we assumed that the elements to be sorted came from a type with a defined comparison operation.
- Given any two elements in the set, we can determine which is "smaller" according to the definition. (In Java the type would have a `compareTo` method or have an associated `Comparator` object.)

Total Orders

- The “smaller” relation is not normally reflexive, but the related “smaller or equal to” relation is.
- Both these relations are normally antisymmetric, unless it is possible for the comparison relations to have ties between different elements.
- And both relations are transitive, just as $\leq$ is on numbers.

Total Orders

- But ordered sets have an additional property called being **total**, which we write in symbols as $\forall x: \forall y: R(x, y) \vee R(y, x)$.
- In general a partial order need not have this property -- two distinct elements could be **incomparable**.
- For example, the equality relation E, defined by $E(x, y) \leftrightarrow (x = y)$, is reflexive, antisymmetric, and transitive, but any two distinct elements are incomparable.

The Division Relation

- Here's another example of a partial order that is not total.
- Our base set will be the natural numbers $\{0, 1, 2, 3,\ldots\}$, and we define the **division relation** D so that D(x, y) means "x divides into y without remainder".
- In symbols, D(x, y) means $\exists z: x \cdot z = y$. (Here we use the dot operator $\cdot$ for multiplication.)

The Division Relation

- Any natural divides 0, but 0 divides only itself. D(1, y) is always true. D(2, y) is true for even y's (including 0) but not for odd y's. D(100, x) is true if and only if the decimal for x ends in at least two 0's.
- In Excursion 3.2 (not a Discussion this term) the text looks at some tricks to determine whether D(k, y) is true for some particular small values of k.

Division is a Partial Order

- It's easy to prove that D is a partial order.
- $D(x, x)$ is always true because we can take z to be 1 and $x \cdot 1 = x$.
- If $D(x, y)$ and $D(y, x)$ are both true, x must equal y because $D(x, y)$ implies that $x \leq y$ (unless x or y is 0).
- And if $D(x, y)$ and $D(y, z)$, then there exist naturals u and v such that $x \cdot u = y$ and $y \cdot v = z$, and then we see that $x \cdot (u \cdot v) = z$.

Clicker Question #2

- Recall that D(x, y) means $\exists z: xz = y$, where all variables are of type "natural". For *how many* values of x is D(x, 60) true? (Hint: $60 = 2^2 \cdot 3 \cdot 5$).
- a) 8
- b) 9
- c) 12
- d) 15

Not the Answer

Clicker Answer #2

- Recall that D(x, y) means $\exists z: xz = y$, where all variables are of type "natural".
  For *how many* values of x is D(x, 60) true?
  (Hint: $60 = 2^2 \cdot 3^1 \cdot 5^1$).
- a) 8
- b) 9
- *c) 12* *(2+1)(1+1)(1+1), or*
  d =1,2,3,4,5,6 and corresponding 60/d
- d) 15

More Partial Order Examples

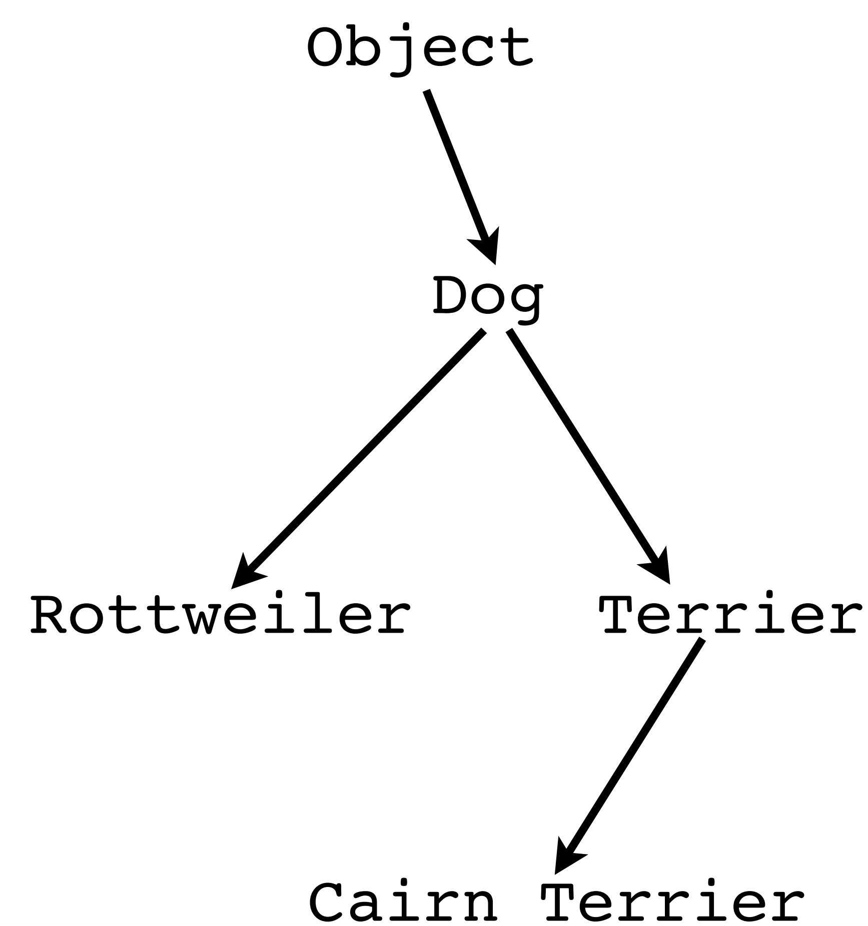
- There are several easily defined partial orders on strings.
- We say that u is a **prefix** of v if $\exists w: uw = v$. (Here we write concatenation as algebraic multiplication.)
  We say u is a **suffix** of v if $\exists w: wu = v$.
  And u is a **substring** of v if $\exists w: \exists z: wuz = v$.
- It's easy to check that each of these relations is reflexive, antisymmetric, and transitive.

More Partial Order Examples

- **Inclusion** on sets is another partial order, as $X \subseteq X$, $X \subseteq Y$ and $Y \subseteq X$ imply $X = Y$, and $X \subseteq Y$ and $Y \subseteq Z$ imply $X \subseteq Z$.
- The **subclass** relation on Java classes is a partial order, since every class is a subclass of itself, two different classes can never each be subclasses of the other, and a subclass of a subclass is a subclass.

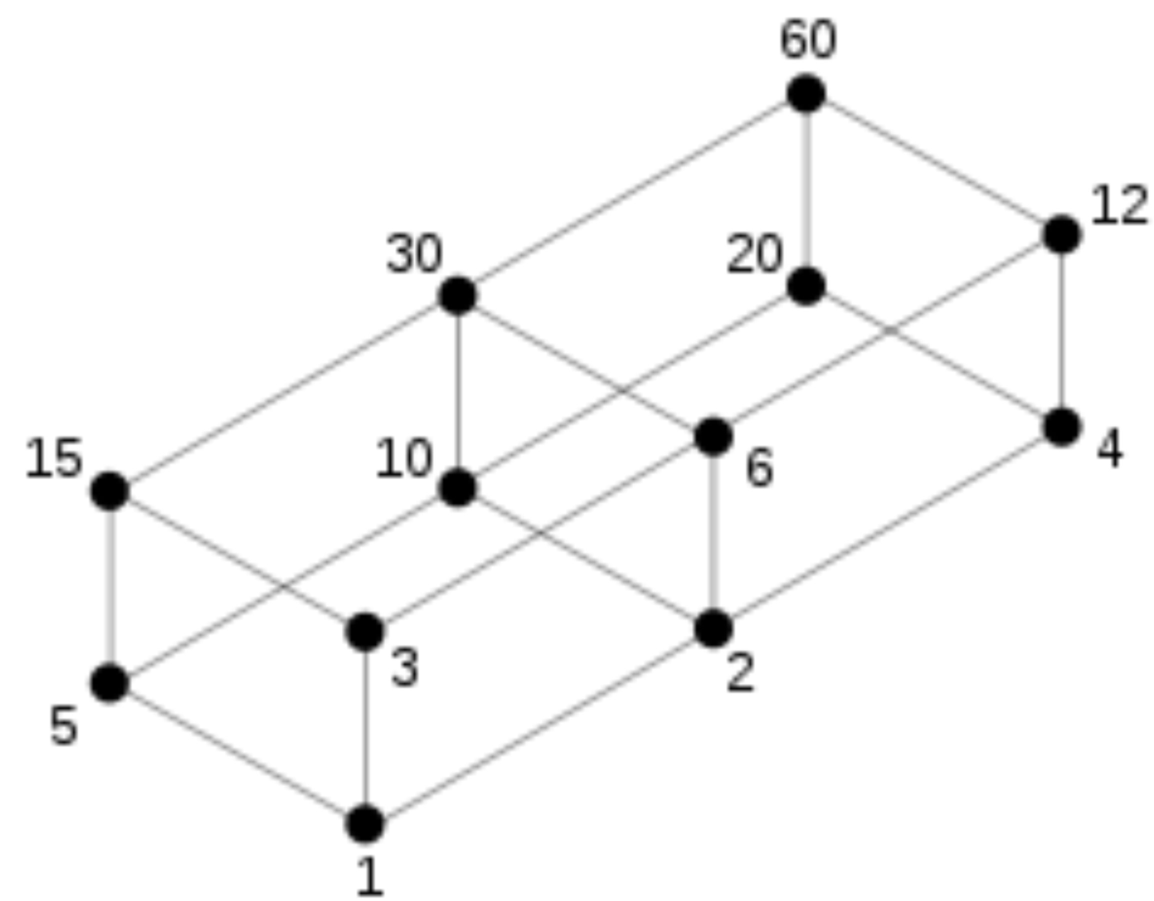
More Partial Order Examples

- We represent this relation by an object hierarchy diagram in the form of a **tree**.
- One class is a subclass of another if we can trace a path of `extends` relationships in the diagram from the subclass up to the superclass.



Hasse Diagrams

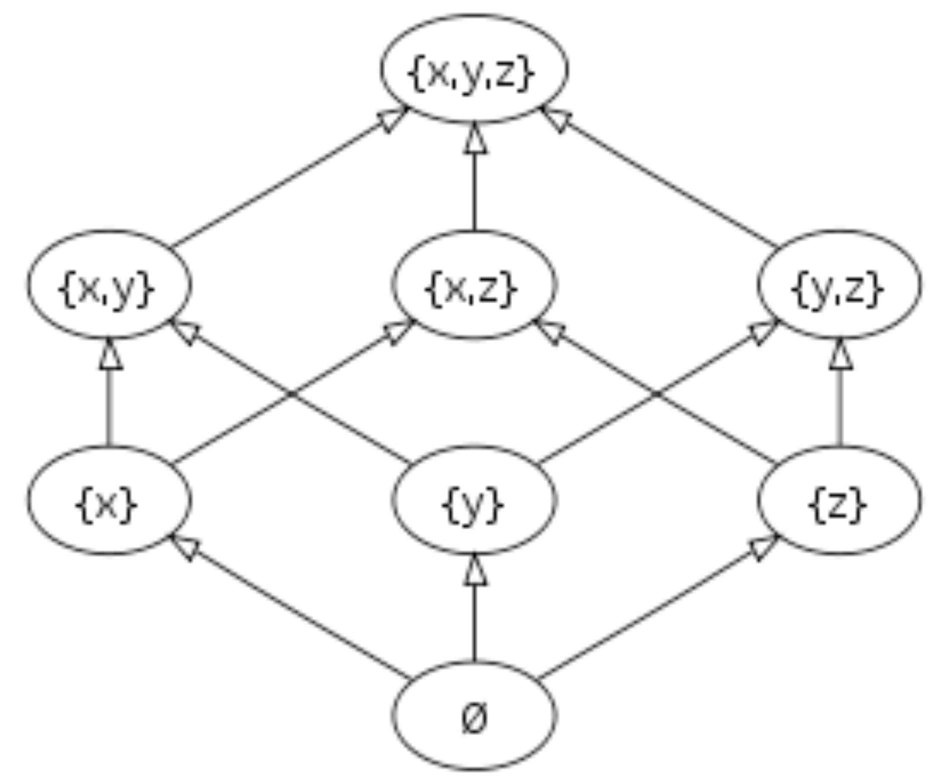
- We make a Hasse diagram by making a dot for each element of the set, and making lines so that R(x, y) is true if and only if there is a path from x up to y.
- (Relative position of points in a graph usually doesn't matter, but here it does.)



Relation D on Divisors of 60
(wikipedia.org)

Hasse Diagram

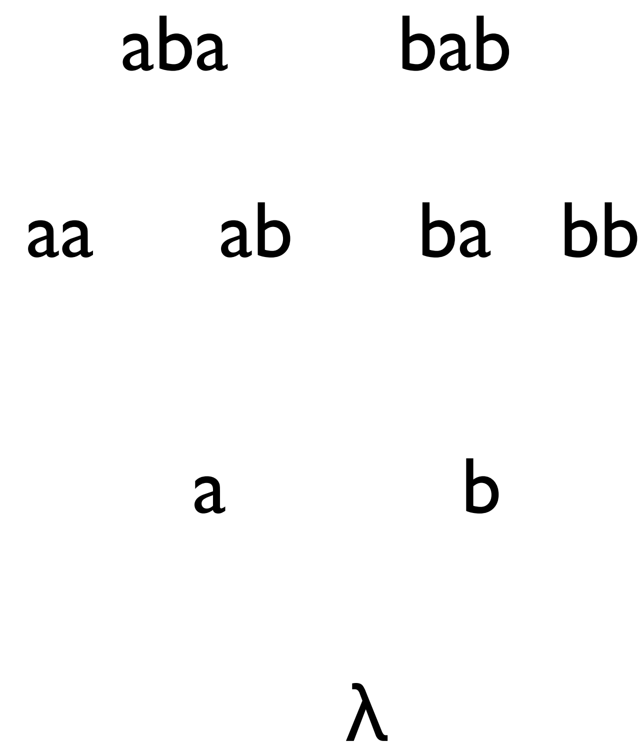
- Starting from the graph of a partial order, we make a Hasse diagram as follows.
- We first delete the loops.
- We then position the dots so the all arrows go upward.
- Finally we delete arrows that are implied by transitivity from other arrows.



Inclusion on Sets
(wikipedia.org)

Clicker Question #3

- Let X be the set of strings {λ, a, b, aa, ab, ba, bb, aba, bab}. Let R be the *substring* relation on X. How many lines do we need to add at right to make a Hasse diagram for R?

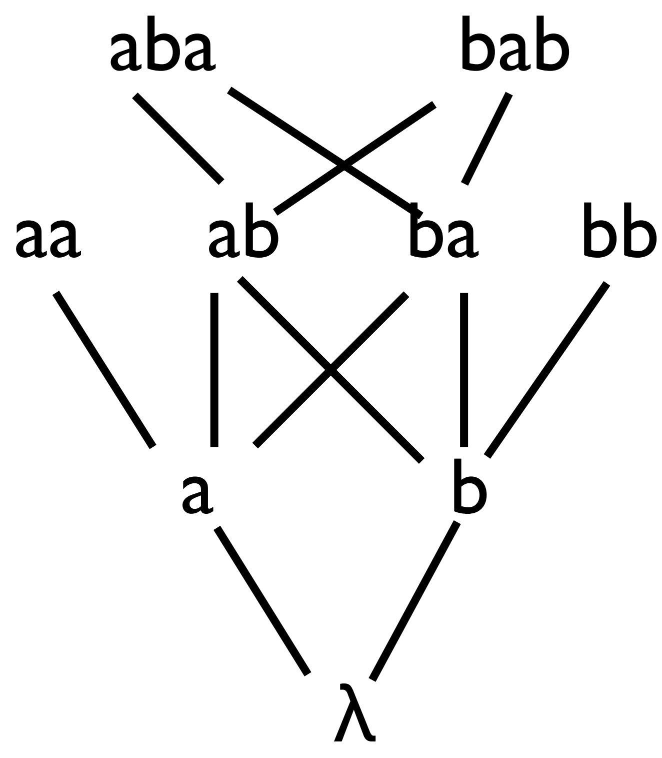


- (a) 7 (b) 8
- (c) 10 (d) 12

Not the Answer

Clicker Answer #3

- Let X be the set of strings {λ, a, b, aa, ab, ba, bb, aba, bab}. Let R be the *substring* relation on X. How many lines do we need to add at right to make a Hasse diagram for R?



- (a) 7 (b) 8
- (c) 10 *(d) 12*

The Hasse Diagram Theorem

- A Hasse diagram is a convenient way to represent a partial order if we can make one.
- But if I am just given R and told that it is a partial order, can I always make a Hasse diagram for it?
- The potential problem comes with the rule that the points must be arranged so that every arrow goes upward.

The Hasse Diagram Theorem

- The **Hasse Diagram Theorem** says that any finite partial order is the “path-below” relation of some Hasse diagram, and the “path-below” relation of any Hasse diagram is a partial order.
- The second of these two statements is easy to prove -- we just have to check that the path-below relation is reflexive, antisymmetric, and transitive.
- The text proves the first statement -- we’ll prove it later using mathematical induction.

Idea of the Proof

- Suppose we have a partial order on a finite set X. We'll outline a way to draw a Hasse diagram for X.
- We first show that X must have a **minimal element**, which is some element a such that $\forall b: R(b, a) \rightarrow (a = b)$ is true.
- Then we recursively make a Hasse diagram for the partial order we get by removing a from X.

Idea of the Proof

- This recursion is grounded (as we recall from COMPSCI 187) because the recursive call is always to a smaller set.
- The base case is when the set is empty.
- Given the Hasse diagram for $X \setminus \{a\}$, we add a point for element a back in at the bottom.
- We draw new lines up from a to any element b such that no element c satisfies $R(a, c)$ and $R(c, b)$ except for $c = a$ or $c = b$.