# COMPSCI 250: Introduction to Computation

Lecture #32: The Myhill-Nerode Theorem
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# The Myhill-Nerode Theorem

- Review: L-Distinguishable Strings
- The Language Prime has no DFA
- The Relation of L-Equivalence
- More Than k Classes Means More Than k States
- Constructing a DFA From the Relation
- Completing the Proof
- The Minimal DFA and Minimizing DFA’s

# Review: L-Distinguishable Strings

- Let $L \subseteq \Sigma^*$ be any language. Two strings u and v are **L-distinguishable** (or **L-inequivalent**) if there exists a string w such that $uw \in L \oplus vw \in L$.
- They are **L-equivalent** if for every string w, $uw \in L \leftrightarrow vw \in L$ (we write this as $u \equiv_L v$).
- We proved last time that if a DFA takes two L-distinguishable strings to the same state, it cannot have L as its language.

# Using Distinguishable Strings

- If we know that two strings u and v are L-distinguishable, this means that there is a w such that one of uw and vw is in L, and the other is not.
- If a DFA takes both u and v to the same state, that is, if $\delta^*(i, u) = \delta^*(i, v)$, then that DFA must also take uw and vw to the same state.
- This state would have to be both final (to accept the string in L) and non-final (to reject the other one). So the DFA is wrong for L.

# Clicker Question #1

- Let $\Sigma = \{a, b\}$ and X be the language $(b+ab^*a)^*$, which contains all strings with an even number of a's. Which one of these pairs of strings *is* X-distinguishable?
- (a) abb and abaababb
- (b) aabbabbbab and $\lambda$
- (c) bbbaaba and bababa
- (d) ba and baabaaba

# Not the Answer

# Clicker Question #1

- Let $\Sigma = \{a, b\}$ and X be the language $(b+ab^*a)^*$, which contains all strings with an even number of a's. Which one of these pairs of strings *is* X-distinguishable? any odd vs. even or even vs. odd
- *(a) abb and abaababb* 1 vs. 4
- (b) aabbabbbab and $\lambda$ 4 vs. 0
- (c) bbbaaba and bababa 3 vs. 3
- (d) ba and baabaaba 1 vs. 5

# L-Distinguishable Strings

- We use this fact to prove a lower bound on the number of states in a DFA for L. Suppose we can find a set S of k strings that are *pairwise* L-distinguishable. Then it is impossible for a DFA with *fewer than k* states to have L as its language.
- If S is an *infinite* set of pairwise L-distinguishable strings, no correct DFA for L can exist at all.

# The Paren Language

- For example, the language Paren $\subseteq \{L, R\}^*$ has such a set, $\{L^i : i \geq 0\}$, because if $i \neq j$ then $L^iR^i$ is in Paren but $L^jR^i$ is not.
- So any two distinct strings in the set are L-distinguishable.
- No DFA for Paren exists, and thus Paren is not a regular language.

# Another Non-DFA Language

- Another example of a language that has no DFA is EQ = $\{a^n b^n : n \geq 0\} = \{\lambda, ab, aabb, aaabbb, \ldots\}$.
- Just as with Paren, we can find an infinite set of strings that are pairwise EQ-distinguishable: $\{\lambda, a, aa, aaa, \ldots\}$.
- If $i \neq j$, the strings $a^i$ and $a^j$ are distinguished by $b^i$, since $a^i b^i$ is in EQ and $a^j b^i$ is not.

# Prime Has No DFA

- Let Prime be the language $\{a^n: n \text{ is a prime number}\}$. It doesn't seem likely that any DFA could decide Prime, but this is a little tricky to prove.
- Let $i$ and $j$ be two naturals with $i > j$. We'd like to show that $a^i$ and $a^j$ are Prime-distinguishable, by finding a string $a^k$ such that $a^i a^k \in \text{Prime}$ and $a^j a^k \notin \text{Prime}$ (or vice versa).
- We need to find a natural $k$ such that $i + k$ is prime and $j + k$ not, or vice versa.

# Prime Has No DFA

- Pick a prime $p$ bigger than both $i$ and $j$ (since there are infinitely many primes).
- Does $k = p - j$ work?
  It depends on whether $i + (p - j)$ is prime
  -- if it isn't we win because $j + (p - j)$ is prime.
  If it is prime, look at $k = p + i - 2j$.
  Now $j + k$ is the prime $p + (i - j)$,
  so if $i + k = p + 2(i - j)$ is not prime we win.
- We find a value of $k$ that works unless all the numbers $p$, $p + (i - j)$, $p + 2(i - j)$,..., $p + r(i - j)$,... are prime.
  But $p + p(i - j)$ is not prime as it is divisible by $p$.

# The Relation of L-Equivalence

- The relation of L-equivalence is aptly named because we can easily prove that it is an equivalence relation.
- Clearly $\forall w$: $uw \in L \leftrightarrow uw \in L$, so it is reflexive.
- If we have that $\forall w$: $uw \in L \leftrightarrow vw \in L$, we may conclude that $\forall w$: $vw \in L \leftrightarrow uw \in L$, and thus it is symmetric.
- Transitivity is equally simple to prove.

# Clicker Question #2

- Again let $\Sigma = \{a, b\}$, and now X be the language of strings with even numbers of *both* a's and b's. Three of these sets of strings are pairwise X-inequivalent, and thus contains exactly one element of each X-equivalence class.
  Which one *is not* pairwise X-inequivalent?
- (a) {λ, aabaa, ba, baaba}
- (b) {λ, ab, ba, aabaa}
- (c) {λ, aabaa, bbabb, ababab}
- (d) {λ, ba, aba, bab}

# Not the Answer

# Clicker Question #2

- Again let $\Sigma = \{a, b\}$, and now X be the language of strings with even numbers of *both* a's and b's. Three of these sets of strings are pairwise X-inequivalent, and thus contains exactly one element of each X-equivalence class. Which one *is not* pairwise X-inequivalent?
- (a) $\{\lambda$, aabaa, ba, baaba$\}$ {ee, eo, oo, oe}
- *(b) $\{\lambda$, ab, ba, aabaa$\}$* {ee, oo, oo, eo}
- (c) $\{\lambda$, aabaa, bbabb, ababab$\}$ {ee, eo, oe, oo}
- (d) $\{\lambda$, ba, aba, bab$\}$ {ee, oo, eo, oe}

# The Myhill-Nerode Theorem

- We know that any equivalence relation partitions its base set into equivalence classes.
- The **Myhill-Nerode Theorem** says that for any language L, there exists a DFA for L with k or fewer states if and only if the L-equivalence relation's partition has k or fewer classes.

# The Myhill-Nerode Theorem

- That is, if the number of classes is a natural k then there is a minimal DFA with k states.
- If the number of classes is infinite then there is no DFA at all.
- It may be easiest to view the theorem in this form: "k or fewer states $\leftrightarrow$ k or fewer classes".

# ($\leq$ k States) $\rightarrow$ ($\leq$ k Classes)

- We've essentially already proved half of this theorem. We can take the contrapositive of "k or fewer states $\rightarrow$ k or fewer classes", to get "more than k classes $\rightarrow$ more than k states".
- Let L be an arbitrary language and assume that the L-equivalence relation has more than k (non-empty) equivalence classes. Let $x_1, \ldots, x_{k+1}$ be one string from each of the first $k + 1$ classes.
- Since any two distinct strings in this set are in different classes, by definition they are not L-equivalent, and thus they are L-distinguishable.

# $(\leq k \text{ States}) \rightarrow (\leq k \text{ Classes})$

- By our result from last lecture, since there exists a set of $k + 1$ pairwise L-distinguishable strings, no DFA with k or fewer states can have L as its language.
- This proves the first half of the Myhill-Nerode Theorem.
- The second half is a bit more complicated.

# Making a DFA From the Relation

- Now to prove the other half,
“k or fewer classes → k or fewer states”.
- In fact we will prove that if there are exactly k classes, we can build a DFA with exactly k states.
- This DFA will necessarily be the smallest possible for the language, because a smaller one would contradict the first half of the theorem, which we have just proved.

# Making a DFA From the Relation

- Let L be an arbitrary language and assume that the classes of the relation are $C_1, \ldots, C_k$. We will build a DFA with states $q_1, \ldots, q_k$, each state corresponding to one of the classes.
- The initial state will be the state for the class containing $\lambda$. The final states will be any states that contain strings that are in L. The transition function is defined as follows. To compute $\delta(q_i, a)$, where $a \in \Sigma$, let w be any string in the class $C_i$ and define $\delta(q_i, a)$ to be the state for the class containing the string wa.

# Making a DFA From the Relation

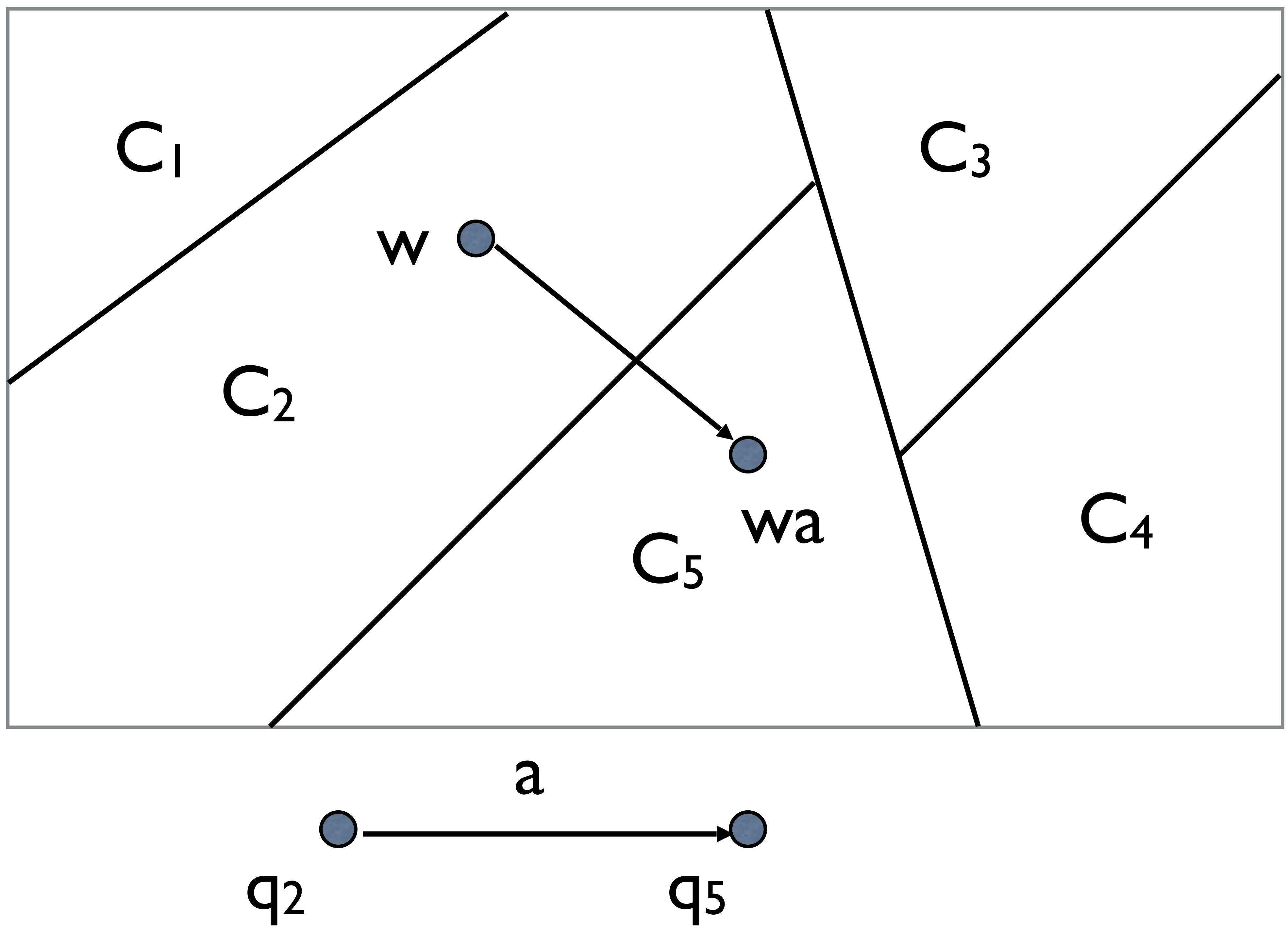
- It's not obvious that this $\delta$ function is well-defined, since its definition contains an arbitrary choice. We must show that any choice yields the same result.
- Let u and v be two strings in the class $C_i$. We need to show that ua and va are in the same class as each other.
- That is, for any u, v, and a, we must show that $(u \equiv_L v) \to (ua \equiv_L va)$.

# The δ Function is Well-Defined

- Assume that $\forall w$: $uw \in L \leftrightarrow vw \in L$.
- Let z be an arbitrary string.
- Then $uaz \in L \leftrightarrow vaz \in L$, because we can specialize the statement we have to az.
- We have proved that $\forall z$: $uaz \in L \leftrightarrow vaz \in L$, which by definition means that $ua \equiv_L va$.

# Defining the Transition Function

- $\delta(q_2, a)$ = state for class of wa = $q_5$



# Completing the Proof

- Now we prove that for this new DFA and for any string w, $\delta^*(i, w) = q_j \leftrightarrow w \in C_j$.
  (Here "i" is the initial state of the DFA.)
- We prove this by induction on w.
  Clearly $\delta^*(i, \lambda) = i$, which matches the class of $\lambda$.
- Assume as IH that $\delta^*(i, w) = x$ matches the class of w. Then for any a, $\delta^*(i, wa)$ is defined as $\delta(x, a)$, which matches the class of wa by the definition, which is what we want.

# Completing the Proof

- If two strings are in the same class, either both are in L or both are not in L.
- So L is the union of the classes corresponding to our final states.
- Since the DFA takes a string to the state for its class, $\delta^*(i, w) \in F \leftrightarrow w \in L$.
- Thus this DFA decides the language L.

# Clicker Question #3

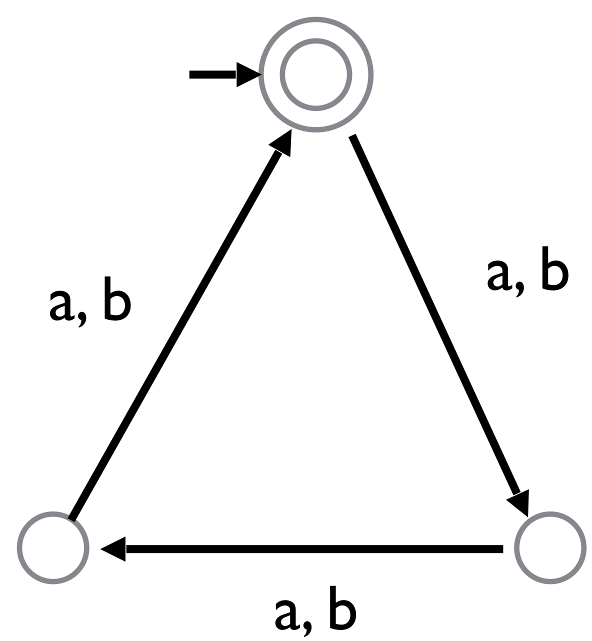
- Let $\Sigma = \{a, b\}$ and let $X = (\Sigma^3)^*$. There are three X-equivalence classes (string lengths modulo 3), so the MN theorem gives us a DFA for X with three states. Which statement about this DFA is *false*?
- (a) The a-arrow and b-arrow from a given state s always go to the same state.
- (b) There is one final state and two non-final states.
- (c) There are two distinct states x and y both with a-arrows to the same state z.
- (d) The initial state is for the class of bababa.

# Not the Answer

# Clicker Question #3

- Let $\Sigma = \{a, b\}$ and let $X = (\Sigma^3)^*$. There are three X-equivalence classes (string lengths modulo 3), so the MN theorem gives us a DFA for X with three states. Which statement about this DFA is *false*?
- (a) The a-arrow and b-arrow from a given state s always go to the same state.
- (b) There is one final state and two non-final states.
- *(c) There are two distinct states x and y both with a-arrows to the same state z.*
- (d) The initial state is for the class of bababa.

# The DFA for $(\Sigma^3)^*$



# The Minimal DFA

- Let X be a regular language and let M be *any* DFA such that L(M) = X.
- We will show that the minimal DFA, constructed from the classes of the L-equivalence relation,
  is **contained within** M.
- We begin by eliminating any unreachable states of M, which does not change M's language.

# The Minimal DFA

- Remember that a correct DFA cannot take two L-distinguishable strings to the same state.
- So for any state p of M, the strings w such that $\delta(i, w) = p$ are all L-equivalent to one another.
- Each state of M is thus associated with one of the classes of the L-equivalence relation.

# Minimizing a DFA

- The states of M are thus partitioned into classes themselves.
- If we combine each class into a single state, we get the minimal DFA.
- In section 14.3 of the book there is an example of an algorithm that gives you these classes, and thus gives you the minimal DFA. We'll put an example of this on the homework and in Discussion #10.