CS 250 Course Study Guide

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Notes on Intended Use:

We hope you find this study guide useful, however, we would like to make a few notes on its intended use.

1. This study guide is not intended to be exhaustive. Its intended purpose is for you to ascertain whether or not you have a basic understanding of each topic. *Please use other resources (lecture notes, past exams, etc.) to study for the final.*
2. This resource was made by graduate students as a learning supplement and may not perfectly reflect exam-level difficulty.
3. We advise you to use this to first to gauge your basic understanding of a topic before filling in any gaps in your knowledge with other materials to improve practice exam performance.

Contents

Chapter 1: Sets and Predicate Logic

Q1.1

Solve the following syllogism. Report the answer as a logical implication, as well as a naturally-spoken English sentence.

- The only foods that my doctor recommends are ones that aren't very sweet.
- Nothing that agrees with me isn't good for dinner.
- Cake is always very sweet.
- My doctor recommends all foods that are good for dinner.

Q1.2

a. What values of a and b make the following statement true? Prove using a truth table.

$$(a \vee \neg b) \wedge (b \wedge \neg a)$$

b. What values of a, b, and c make the following statement true? Prove without using a truth table.

$$(a \vee \neg b) \wedge (b \wedge \neg a) \rightarrow (\neg c \wedge (a \vee b))$$

Q1.3

Complete a deductive sequence proof based on the following propositions:

1. $\neg p \land q$
2. $r \to p$
3. $\neg r \to s$
4. $s \to t$

Your answer should either be a conjunction of all of the variables p,q,r,s,t which enumerates their truth values, or you may explicitly state the truth value of each variable.

Q1.4

Prove by contradiction that the sum of a rational number and an irrational number is irrational. Recall that a rational number is one which can be expressed as the quotient of two integers. An irrational number is one which cannot be represented as the quotient of two integers. You may assume that the operations addition, subtraction, and multiplication on rationals are well-defined. In other words, the sum, difference, or product of two rationals is always rational.

Q1.5

Prove that if n is divisible by 6, then $n + 10$ is not divisible by 6.

Additional Resources for Sets and Predicate Logic

1. Translating Statements to Propositional Logic
2. Excellent Discrete Math Playlist
3. Playlist Containing Many Worked Out Deductive Proofs
4. MIT Video of Proof by Contradiction
5. Proof by Contrapositive Intro
6. Rules of Inference Video

Chapter 2: Quantifiers and Relations

Q2.1

Consider the set $A = \mathbb{N}$
Identify whether the following relations are reflexive, antireflexive, symmetric, antisymmetric, or transitive.

- x relates to y if and only if $x = 2y$
- x relates to y if and only if $x\%2 = y\%2$

Q2.2

Let A, B, C, D be nonempty sets.

a) Prove that $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$.

b) What happens to the result in part (a) if any of the sets A, B, C, D is empty?

Q2.3

Determine whether or not each of the following relations is a function. If a relation is a function, find its range.

a) $\{(x, y) \mid x, y \in \mathbb{Z}, y = x^2 + 7\}$, a relation from $\mathbb{Z}$ to $\mathbb{Z}$

b) $\{(x, y) \mid x, y \in \mathbb{R}, y^2 = x\}$, a relation from $\mathbb{R}$ to $\mathbb{R}$

c) $\{(x, y) \mid x, y \in \mathbb{R}, y = 3x + 1\}$, a relation from $\mathbb{R}$ to $\mathbb{R}$

d) $\{(x, y) \mid x, y \in \mathbb{Q}, x^2 + y^2 = 1\}$, a relation from $\mathbb{Q}$ to $\mathbb{Q}$

e) $\mathcal{R}$ is a relation from A to B where $|A| = 5$, $|B| = 6$, and $|\mathcal{R}| = 6$.

Q2.4

Determine whether each of the following statements is true or false. If it is true, give a short proof. If it is false, give an example.

a. If $x \in A$ and $A \in B$ (not a typo), then $x \in B$.

b. If $A \subseteq B$ and $B \in C$, then $A \in C$.

c. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

d. If $A \not\subset B$ and $B \not\subseteq C$, then $A \not\subseteq C$.

e. If $x \in A$ and $A \not\subseteq B$, then $x \in B$.

f. If $A \subseteq B$ and $x \notin B$, then $x \notin A$.

Q2.5

Suppose you have sets $A = \{3, 5, 7, 11, 13, 17\}$ and $B = \{0, 1, 2, 3, 5, 8, 13\}$ and some function $f : A \to B$.

a. How many one-to-one functions f are possible? Give one example.

b. How many onto functions f are possible? Give one example.

Additional Resources for Quantifiers and Relations

1. Set Theory Playlist
2. Equivalence Relations and Equivalence Classes Explained
3. Bijective, Injective, Surjective Explained
4. Existential and Universal Quantifiers Explained
5. Negating Quantifiers
6. Negating Multiple Quantifiers
7. Reflexive, Symmetric, Transitive

Chapter 3: Number Theory

Q3.1

Prove that for some relatively prime a and n, multiplying all numbers in the set $\{1, 2, ..., n-2, n-1\}$ by a will output a permutation of $\{1, 2, ..., n-2, n-1\}$.
Hint: There are two parts to this proof—existence and uniqueness. To show that multiplication by a permutes the elements of the set, you must show that the multiplication always produces an element in the set (existence) and you must show that that element will never be repeatedly produced via multiplication by a (uniqueness). Consider how these relate to modular arithmetic and modular inverses.

Q3.2

Let $a = 473$ and $b = 47$. Run Euclid's Algorithm on a and b to find their GCD. What can you conclude about a and b from your run of Euclid's Algorithm?

Q3.3

Find the smallest integer x such that:

$$x \equiv 3 \pmod{17}$$
$$x \equiv 5 \pmod{19}$$

Additional Resources for Number Theory

1. Euclid's Algorithm
2. Chinese Remainder Theorem

Chapter 4: Induction and Recursion

Q4.1

Explain the error in the following induction proof:
Claim: All students are the same height.
Base case: Consider a set of students of size 1. Any student is the same height as themselves, and so all students in the set are the same height.
Inductive hypothesis: All students in a set of size n have the same height.
Inductive step: Now suppose we have a set of $n+1$ students, labeled s_1 through s_{n+1}: $\{s_1, s_2, s_3, ..., s_{n-1}, s_n, s_{n+1}\}$.
Consider the entire set of $n+1$ students as two individual sets of n students each.
The first set contains students s_1 through s_n: $\{s_1, s_2, s_3, ..., s_{n-1}, s_n\}$
The second set contains students s_2 through s_{n+1}: $\{s_2, s_3, ..., s_{n-1}, s_n, s_{n+1}\}$
Both sets contain n students, and so by the inductive hypothesis, all students in each set must be of the same height. And if all students in each set have the same height, then all students in the full $n+1$ set must have the same height.
In conclusion, all students in any set of size $n \geq 1$ must have the same height.

Q4.2

Suppose we have a recursively-defined function (much like a Fibonacci sequence) $f_n = 5f_{n-1} - 6f_{n-2}$ where $f_0 = 2$ and $f_1 = 5$.
Prove inductively that $f_n = 2^n + 3^n$ for all $n \geq 0$.

Q4.3

Suppose you are assembling trains. There are two types of train cars that you can use: one has a length of 1, the other has a length of 2. Trains are ordered sequences of train cars: for example, if a train has a total length of 3, then it could have been built using a length-2 car, followed by a length 1. Or a length 1 car, followed by a length 2. Or a length 1 car, followed by two more 1's.
Prove that for a train with a total length of $n \geq 1$, there are F_n number of different ways to construct the train, where F_n is the n-th Fibonacci number.
As a reminder, $F_0 = 0$ and $F_1 = 1$ in this class.

Q4.4

Prove by induction that $n^2 - n$ is always even for all $n \in \mathbb{N}$.

Additional Resources for Induction and Recursion

1. Induction Playlist
2. Strong Induction
3. Induction Visualization
4. Recursively Defined Functions

Chapter 9: Graphs and Trees

Q9.1

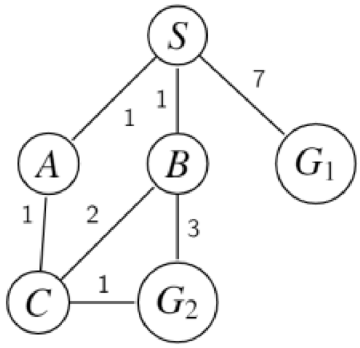
How many different undirected graphs with n nodes are possible? Assume that two graphs with n nodes are "different" if they have different edges and give your answer as an expression of n.

Q9.2

Develop a pseudocode method for a recursive implementation of DFS.

Q9.3

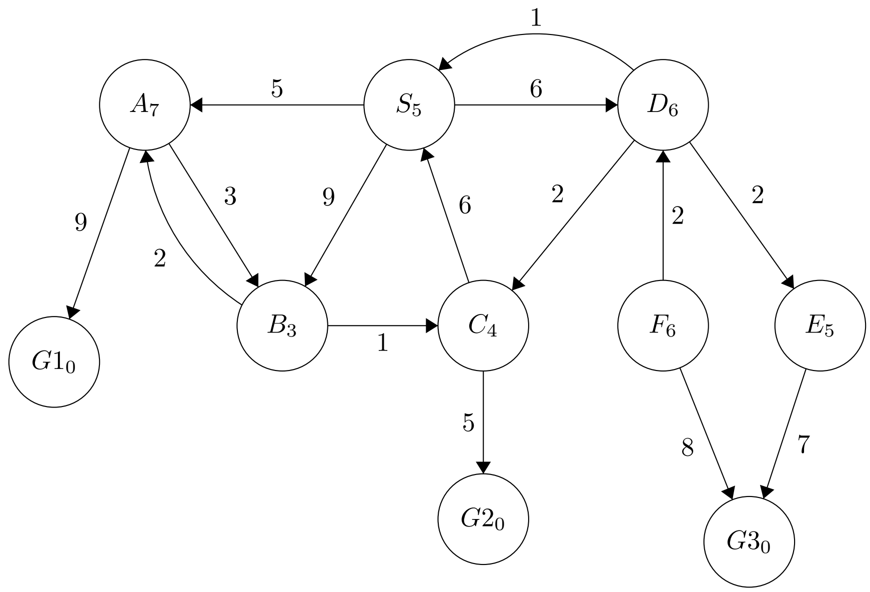
SOURCE: UW Madison https://pages.cs.wisc.edu/ dyer/cs540/exams/exam1-f19-sol.pdf
Consider the following graph.



S is a start state and G1 and G2 are end states. Assume that nodes are searched in alphabetical order. Conduct a BFS and UCS on the graph.
What path does BFS return and what is its cost? What path does UCS return and what is its cost? Produce the overall results of both search methods as a rigorous proof.

9.4

Run A^* search on the following graph, starting at node S and ending at any of the 3 possible goal G nodes. The subscripts on each node are their heuristic values, and you may assume that it is an admissible, consistent heuristic. Show the final search tree and show or explain the state of the priority queue at each step, keeping a closed list of visited nodes.



Chapter 5: Regular Expressions

Q5.1

Assume the alphabet $\sum = \{0, 1\}$. If the English description is given, provide the regular expression. If the regular expression is given, give a short, English description of the accepted set of strings.

a. The set of strings that contain at least one 0 and at least one 1.

b. $(1 + \lambda)(00^*1)^*0^*$

c. The set of strings such that all pairs of adjacent 0's appear before any pairs of adjacent 1's.

Q5.2

Is the Kleene star operation distributive over the + operation? In other words, is $(S+T)^* = S^* + T^*$ true for any languages S and T? Why or why not?
If your answer is no, is this always true, or can you think of specific values for S and T that would make it true?

Chapter 14: Finite Automata

Q14.1

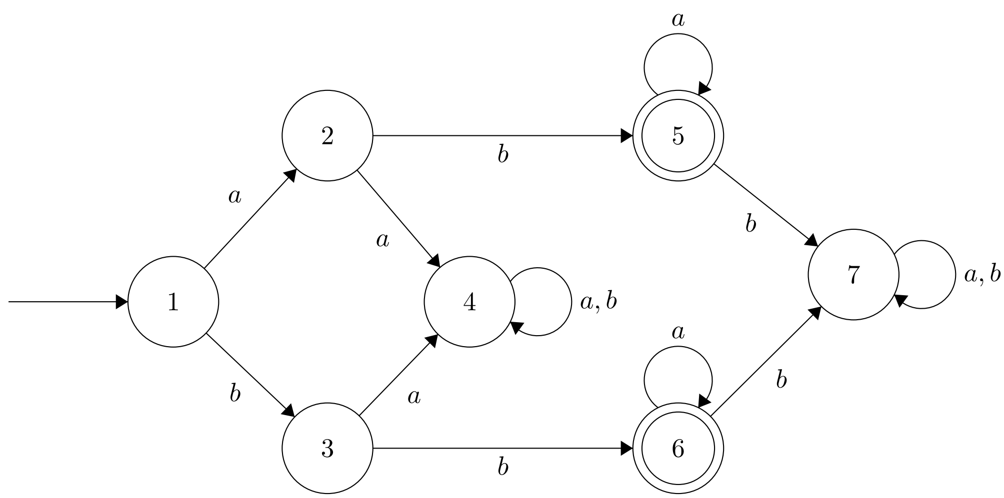
Assume $\sum = \{0, 1\}$. Draw a DFA for the language that accepts all binary strings with either an even number of 0's or contains exactly two 1's.

Q14.2

Construct a DFA that accepts a language consisting of all strings over the alphabet $\Sigma = \{a, b\}$ that either end with the letter 'a' or contain an even number of 'b's.

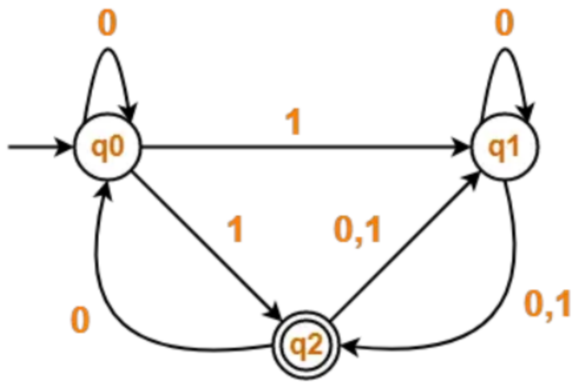
Q14.3

Use the DFA minimization algorithm on the following DFA:



Q14.4

Convert the following NFA to a DFA:



Q14.5

Let L^R be any arbitrary regular language. Let the language $L^R = \{w^R : w \in L\}$ be the reversal of L. Show that L^R is regular.

Q14.6

For $\Sigma = \{a, b, c, d\}$, give a regular expression that captures all strings that use their letters in reverse alphabetical order, but use at most three of the four possible letters.
Note: The strings themselves can be longer than 3 letters long, since letters can repeat. Draw an NFA that captures the regular expression from above.

Additional Resources for Finite Automata

1. Myhill Nerode
2. DFA Minimization
3. Regular Expression to Automaton
4. Lambda/Epsilon NFA to NFA

Chapter 15: Formal Language Theory

Q15.1

Let L be a language that is Turing Recognizable by some Turing Machine. Let L' be the language $\Sigma^* - L$, and let that language also be Turing Recognizable by some other Turing Machine. Prove that L is Turing Decidable. (Hint: Recall that a single-tape Turing Machine can be used to simulate a multi-tape Turing Machine)

Additional Resources for Formal Language Theory

1. Turing Machines Visualized
2. TR / TD
3. Halting Problem
4. Theory of Computation Playlist
5. `https://www.geeksforgeeks.org/conversion-from-nfa-to-dfa/`