# COMPSCI 250: Introduction to Computation

Lecture #39: The Halting Problem and Unsolvability
David Mix Barrington and Ghazaleh Parvini
17 May 2023

# Halting and Unsolvability

- Simulating an NDTM With a DTM
- Proving Something to be Impossible
- Representing TM's By Strings
- The Universal Turing Machine
- The Barber of Seville Language
- Undecidable, Non-Recognizable Languages
- Getting More Undecidable Languages
- Turing Complete Languages

# Simulating NDTM's with DTM's

- To simulate an NDTM with a DTM, we first build a DTM with three tapes.
- The first tape will store the input string w and will never change.
- The second will be a work tape to exactly simulate a particular computation of the NDTM.
- The third tape will hold a **choice sequence**, which is a string of symbols telling the NDTM which of its options to take on each of its moves.

# Simulating NDTM's with DTM's

- The input w is in the language of the NDTM N if and only if there exists a choice sequence that causes N to halt, starting from i□w.
- So our simulation tests all possible choice sequences, starting with the one of length 0, then all the ones of length 1, then length 2, and so forth.
- For each choice sequence, the DTM clears its work tape, copies w onto it, then runs N using the sequence.
  If the simulated N ever halts, the DTM accepts w.

# Simulating NDTM's with DTM's

- So if $w \in L(N)$, the DTM will eventually reach a good choice sequence and will accept w.
- If $w \notin L(N)$, the DTM will run forever because it will keep trying longer and longer choice sequences.
- Thus the DTM accepts exactly those strings in L(N), and so simulates N.
- If the DTM accepts, it takes exponentially longer to do so than N did on that input.

# Proving Things to Be Impossible

- When a problem can be solved with a particular set of resources, we can prove this to be the case by showing how to do it.
- But what about when a problem *can't* be solved with those resources? We can't just show algorithms that don't work, because these don't rule out the existence of other algorithms that do.

# Proving Things to Be Impossible

- We have one example in this course -- if a language cannot be decided by a DFA, the Myhill-Nerode Theorem can be used to prove it. This also shows that the language has no regular expression.

- Gödel proved in 1931 that there is a true statement of number theory that can't be proved (or a false statement that can be proved). The idea is that the statement can be interpreted as "I am not provable".

# Clicker Question #1

- Suppose that Statement n means “there is no proof of Statement n in the system”. Which one of these statements *could* be true?
- (a) Statement n is true.
- (b) Statement n is false and the system has no proof for any false statement.
- (c) Statement n is true and the system has a proof for every true statement.
- (d) Statement n is both true and false.

# Not the Answer

# Clicker Answer #1

- Suppose that Statement n means "there is no proof of Statement n in the system". Which one of these statements *could* be true?
- *(a) Statement n is true.*
- (b) Statement n is false and the system has no proof for any false statement. If it's false, there is a proof for Statement n, which is false.
- (c) Statement n is true and the system has a proof for every true statement. But not n?
- (d) Statement n is both true and false. Can't be.

# Proving Limits on TM's

- By the Church-Turing Thesis, if we prove that no Turing machine can decide a particular language, that means that no algorithm can decide it.
- Deciding a language means solving a general class of problems, not just a single instance.
- The basic idea is called **diagonalization**, for reasons we won't be able to go into here.

# Proving Limits on TM's

- Like the Gödel argument, we get a contradiction out of applying a hypothetical Turing machine to *itself*. The assumption that our target problem is decidable leads to this contradiction, so it is false and the problem is not decidable.
- To formulate this argument, we will have to say more about Turing machines that take *other Turing machines* as input.

# The Universal Turing Machine

- We could, with some effort, formalize a scheme for representing Turing machines by strings. We would need the string to code the number of states, the number of letters in the input alphabet and in the tape alphabet, the special states, and the transition function.
- It doesn't really matter how this information is stored, as long as it's possible for an algorithm (and therefore a Turing machine) to answer questions about the states and transition function.

# The Universal Turing Machine

- Once this is done, it is possible to build a **universal Turing machine**.
- This machine takes two inputs, a Turing machine M and a string w over M's input alphabet.
- It simulates the action of M on w, accepting or rejecting if and only if M would accept or reject w.
- Now we have a Turing machine that *acts on Turing machines*.

# The Barber of Seville Language

- *The Barber of Seville shaves exactly those men of Seville who do not shave themselves.*
- Bertrand Russell proposed this statement as a logical paradox.
- If the barber is a man of Seville who does not shave himself, the rule obligates him to shave himself.
- And if he is a man of Seville who *does* shave himself, the rule forbids him from shaving himself.

# The Barber of Seville Language

- The only solution is that he is not from Seville, or that she is (or they are) not a man.
- Define the **Barber of Seville language** to be the set of TM's that do not accept themselves. Formally, $L_{BS}$ is the set $\{M: M \notin L(M)\}$ or $\{M: (M, M) \notin L(U)\}$ where U is the universal TM.
- A **Barber of Seville Turing machine** would be a TM $M_{BS}$ such that $L(M_{BS}) = L_{BS}$, a TM that accepts exactly those TM's that don't accept themselves.

# Non-TD and Non-TR Languages

- Just as the Barber can't be a man of Seville, the machine $M_{BS}$ cannot exist. If it did, $M_{BS}$ either would accept $M_{BS}$ or it wouldn't. If it does, by definition it doesn't, and if it doesn't, by definition it does.
- This tells us that the language $L_{BS}$ is *not Turing recognizable* because it is not the language of any Turing machine. Since all decidable languages are also recognizable, $L_{BS}$ is *not decidable* either.

# Clicker Question #2

- We can define the set of machines X that accept their own descriptions: $\{X: \text{“X”} \in L(X)\}$for various kinds of machines. Which of these sets are *not* Turing recognizable?
- (a) All DFAs that accept their own descriptions.
- (b) All NFAs that accept their own descriptions.
- (c) All nondeterministic Turing machines that recognize their own descriptions.
- (d) Trick question: all are Turing recognizable.

# Not the Answer

# Clicker Answer #2

- We can define the set of machines X that accept their own descriptions: {X: “X” ∈ L(X)}for various kinds of machines. Which of these sets are *not* Turing recognizable?
- (a) All DFAs that accept their own descriptions.
- (b) All NFAs that accept their own descriptions.
- (c) All nondeterministic Turing machines that recognize their own descriptions.
- *(d) Trick question: all are Turing recognizable.*
  Run U on (“X”, X), accept if it accepts.

# Non-TD and Non-TR Languages

- But note that the language $L_{BS}$-bar *is* recognizable. It is the union of the set of strings that don't code Turing machines at all, and the set of TM's that *do* accept themselves.
- We can recognize the latter set by taking any machine M and feeding the pair (M, M) to the universal TM. The former set is decidable, assuming that we have defined our coding scheme unambiguously.
- So we have an example of a language that is recognizable but not decidable.

# Getting More Non-TD Languages

- Of course it would be much more interesting to have an undecidable language that we actually might have wanted to decide.
- We can do this by the **method of reduction**. Given a language X, we prove that we could decide $L_{BS}$ *if* we had a decider for X. Then since the decider for $L_{BS}$ cannot exist, the decider for X cannot exist either.

# More Non-TD Languages

- For example, let $L_{halt}$ be the set of all pairs (M, w) such that M is a TM that eventually halts (either accepts or rejects) on the input string w. Suppose I had a decider for $L_{halt}$.
- Given any Turing machine M, I can now decide whether M is in $L_{BS}$ by forming the pair (M, M) and feeding it to the $L_{halt}$ decider. If the decider says that M will not halt on M, then M *is* in $L_{BS}$. If it will halt, I can then run M on M and see whether it accepts, knowing that this computation will not run forever.

# Clicker Question #3

- Let $Loop_{BT}$ be the set of TM's that fail to halt on a blank input tape. Suppose I show how, given any M, I can make a Turing machine f(M) so that $M \in L_{BS} \leftrightarrow f(M) \in Loop_{BT}$. What could I derive from here?
- (a) The language $Loop_{BT}$ is Turing recognizable .
- (b) The language $Loop_{BT}$ is Turing recognizable but not Turing decidable.
- (c) The language $Loop_{BT}$ is neither Turing decidable nor Turing recognizable.
- (d) $Loop_{BT}$ and $L_{BS}$ are the same language.

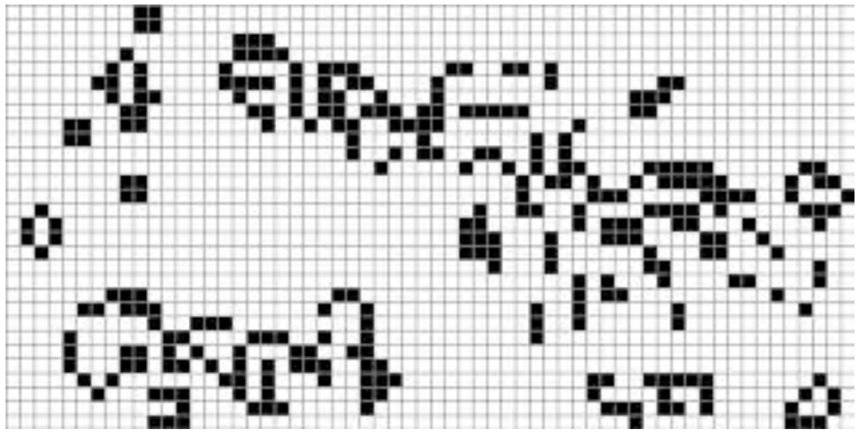
Not the Answer

# Clicker Question #3

- Let $Loop_{BT}$ be the set of TM's that fail to halt on a blank input tape. Suppose I show how, given any M, I can make a Turing machine f(M) so that $M \in L_{BS} \leftrightarrow f(M) \in Loop_{BT}$. What could I derive from here?
- (a) The language $Loop_{BT}$ is Turing recognizable .
- (b) The language $Loop_{BT}$ is Turing recognizable but not Turing decidable.
- *(c) The language $Loop_{BT}$ is neither Turing decidable nor Turing recognizable.* Same properties as $L_{BS}$.
- (d) $Loop_{BT}$ and $L_{BS}$ are the same language.

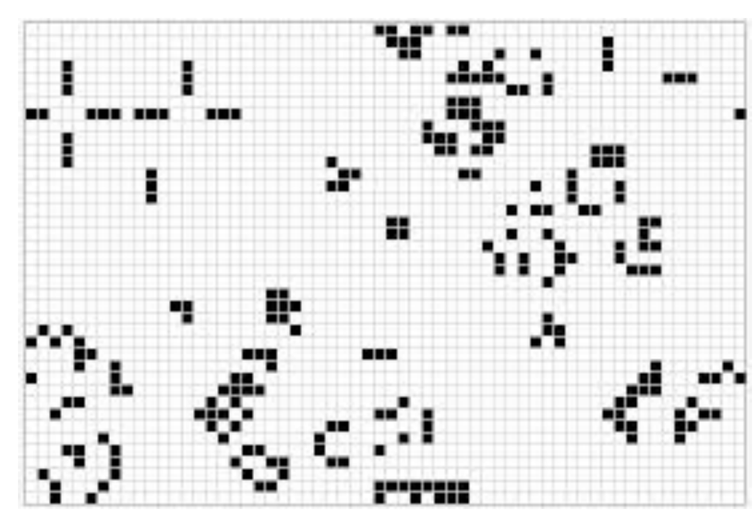
# More Non-TD Languages

- As we build up a library of undecidable languages, we can use any of them in place of $L_{BS}$ in this kind of argument.
- For example, **Conway's Game of Life** is a set of rules that lets patterns of pixels "evolve" over time. It's *undecidable* whether a pattern will stay bounded, or ever include a given pixel.



# More Non-TD Languages

- The proof of this is remarkable -- someone designed a way to emulate an arbitrary Turing machine with a pixel pattern.
- So a decider for the Game of Life questions could be used to answer undecidable questions about TM's.



# Turing Completeness

- In COMPSCI 311 and 501, you will spend a lot of time with the concept of **complete languages** for a class.
- $L_{halt}$ turns out to be **Turing complete**, or complete for the set of recognizable languages. We can take any recognizable language X, and any string w, and transform w into a string f(w) such that w is in X if and only if f(w) is in $L_{halt}$.

# Turing Completeness

- This means that the language $L_{halt}$ captures every possible Turing recognizable computation.
- If we could decide $L_{halt}$, then, we could decide *every* TR language. But we know that there are TR languages that are not TD. So $L_{halt}$ is not TD.
- In the same way, we see that no Turing-complete language is TD.

# NP-Completeness

- The class **NP** or nondeterministic polynomial time is the set of languages that are recognized by nondeterministic Turing machines in polynomial time.
- If we prove a language to be **NP-complete** by showing that any NP language can be reduced to it, we are pretty sure that it is not actually decidable in polynomial time by a deterministic TM.
- This is because if it were the classes P and NP would be the same, and we are pretty sure that they are not.