COMPSCI 250: Fall 2023
Homework 1

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Due Date : Friday, September 22

This assignment has 9 problems. There is also 1 Extra Credit problem. The extra credit is 10 points.

Please submit a single PDF file, with the problems in order (as below), and legible. Look at your PDF before submitting it – it is fine to scan or photograph a handwritten document but it the graders can't read it, they won't grade it.

Please assign pages to problems in Gradescope. Graders will click on the problem number. If no page shows up because it's not assigned, the assumption is you have not solved the problem.

Be sure you are doing Problems in the book and not Exercises: the numbers should start with P rather than E.

For full credit, show your work, explaining your reasoning. This also helps assign partial credit.

You are responsible for following the academic honesty guidelines on the Grading and Requirements page. This means that what you present must be your own work in presentation, and you must acknowledge all sources of aid other than course staff and the textbook. You will get 2 extra points if you typeset your Homework.

(10 points) **Problem 1.1.3**

Let A be the set $\{1, 2, 3\}$. Give explicit descriptions (lists of elements) of each of the following *sets of sets*:

(a) $\{B : B \subseteq A\}$

(b) $\{B : B \subseteq A$ and $|B|$ is even $\}$ (Remember that 0 is an even number.)

(c) $\{B : B \subseteq A$ and $3 \notin B\}$

(d) $\{B : B \subseteq A$ and $A \subseteq B\}$

(e) $\{B : B \subseteq A$ and $B \not\subseteq B\}$

Solution:

(a) $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

(b) $\{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$

(c) $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

(d) $\{\{1, 2, 3\}\}$

(e) $\emptyset$

(10 points) **Problem 1.2.10**

Let Σ be an alphabet with k letters and let n be any natural. How many strings are in the language Σ^n? Justify your answer as best you can, though we won't have formal tools to prove this until Chapter 4.

Solution:

Σ^n contains all the strings on alphabet Σ of length n. Since $|\Sigma| = k$, we have k choices for each letter of a String in Σ^n. k^n is the number of strings in k^n.

(12 points) **Problem 1.4.9**

Choosing variables for the base propositions as needed, translate these English statements into compound propositions.

(a) If you don't eat your meat, you can't have any pudding.

(b) If I'm wearing the antlers, I am dictating, and if I'm not wearing the antlers, I'm not dictating.

(c) If this penguin was from the zoo, it would have "Property of the Zoo" stamped on it, and if penguins molt, this penguin could not have "Property of the Zoo" stamped on it.

(d) It is not true that if I am arguing, then you must have paid.

Solution:

(a) eat your meat (m), have pudding (p)
$\neg m \to \neg p$, or, $m \vee \neg p$

(b) wearing antlers (w), dictating (d)
$(a \to d) \wedge (\neg a \to \neg d)$, or $a \leftrightarrow d$

(c) penguin from zoo (z), have stamp on it (s), penguins molt (m)
$(z \to s) \wedge (m \to \neg s)$

(d) I'm arguing (a), you have paid (p)
$\neg(a \to p)$, or $\neg(\neg p \to \neg a)$, or $a \wedge \neg p$.

(12 points) **Problem 1.5.7**

Any subset statement may be interpreted as saying that some particular set is empty. Let D be a set of dogs, and let B, R, and F be three subsets of D containing the black dogs, the retrievers, and the female dogs respectively. The statement $B \subseteq R$, for example, can be interpreted as "all the black dogs are retrievers", or "there are no black dogs who are not retrievers", that is, $B \cap \overline{R} = \emptyset$. For each of the following subset statements, identify the set that is claimed to be empty, both in English and in symbols:

(a) $B \cap F \subseteq R$

(b) $F \subseteq R \cap B$

(c) $B \subseteq R \cup F$

Solution:

(a) "All the dogs who are black and female are retrievers" or " there is no black female dog who are not retrievers." We can write it as $B \cap F \cap \overline{R} = \emptyset$.

(b) "All the female dogs are black retrievers" or " there is no female dog who is not black retrievers." We can write it as $F \cap \overline{(R \cap B)} = F \cap (\overline{R} \cup \overline{B}) = \emptyset$.

(c) "All the black dogs are either retrievers or female or both", or "there is no black dog who is neither female nor retrievers". We can write it as $B \cap (\overline{R \cup F}) = \emptyset$.

(12 points) **Problem 1.5.10**

Let $X = \{1, 2, 3, \dots, 10\}$ and let I be the set of all intervals in X, that is, subsets of the form $\{a, \dots, b\}$ for some naturals a and b.

(a) How many intervals are in the set I? (Don't forget the empty set.)

(b) How many of the intervals are subsets of $\{2, 3, 4, 5, 6\}$?

(c) How many of the intervals are disjoint from $\{2, 3, 4, 5, 6\}$?

Solution:

(a) $|I| = 56$

10 intervals start with 1
9 intervals start with 2
...
1 interval starts with 10
1 interval for empty set

$|I| = 1 + \sum_{i=1}^{10} i = 56$

(b) The number of intervals that are subsets of $\{2, 3, 4, 5, 6\}$ is equal to the number of intervals over $\{2, 3, 4, 5, 6\}$.
$1 + \sum_{i=1}^{5} i = 16$

(c) Number of intervals disjoint from $\{2, 3, 4, 5, 6\}$ = Number of intervals starting and ending in $\{1\}$ + Number of intervals starting and ending in $\{7, 8, 9, 10\}$ + 1 (empty set).
$= 1 + [\sum_{i=1}^{4} i] + 1 = 12$

(10 points) **Problem 1.7.9**

Here we have another islander problem as in Problem 1.6.9. There are three islanders: A says "If B and I are both telling the truth, then so is C", B says "If C is telling the truth, then A is lying", and C says "It is not the case that all three of us are telling the truth." Using propositional variables a, b, and c to represent the truth of the statements of A, B, and C respectively, we can represent the problem by the three premises $a \leftrightarrow ((a \wedge b) \rightarrow c)$, $b \leftrightarrow (c \rightarrow \neg a)$, and $c \leftrightarrow \neg(a \wedge b \wedge c)$. Determine the conclusion as a conjunction of three literals, and give a deductive sequence proof of this conclusion from the premises.

Solution:

P1: $a \leftrightarrow ((a \wedge b) \rightarrow c)$
P2: $b \leftrightarrow (c \rightarrow \neg a)$
P3: $c \leftrightarrow \neg(a \wedge b \wedge c)$

4. $a \leftrightarrow (\neg(a \wedge b) \vee c)$ Implication Definition of P1
5. $a \leftrightarrow (\neg a \vee \neg b \vee c)$ DeMorgan's of 4
6. $c \leftrightarrow (\neg a \vee \neg b \vee \neg c)$ DeMorgan's of P3
7. $b \leftrightarrow (\neg c \vee \neg a)$ Implication Definition of P2
8. $c \leftrightarrow (\neg c \vee \neg a \vee \neg b)$ commutative law of 6
9. $c \leftrightarrow (b \vee \neg b)$ from 7 and 8
10. $c \leftrightarrow T$ from 9
11. $a \leftrightarrow (a \wedge b) \rightarrow T)$ from P1 and 9
12. $a \leftrightarrow T$
13. $b \leftrightarrow (T \wedge F)$ from P2, 9 and 12.
14. $b \leftrightarrow F$ from 13.
15. $a \wedge \neg b \wedge c$ from 9,12 and 14.

(10 points) **Problem 1.8.3**

Suppose that you have proved 0 from the premise $P \wedge \neg Q$. Show how you can use Proof By Cases and this proof to construct a direct proof of $P \rightarrow Q$.

Solution:

For a direct proof we want to assume P and derive Q.

We can use $\neg Q$ as our new proposition R in the proof by cases. So we will derive Q in the two cases, one where $\neg Q$ is true and one where it is false. (We chose $\neg Q$ so that we could use the proof we are given, that has $P \wedge \neg Q$ as a premise.)

Case 1: (Assuming $\neg Q$.)
Since we have assumed P, we have $P \wedge \neg Q$ by Conjunction. Using the given proof, we derive 0. The implication $0 \rightarrow Q$ is true by the rule of Vacuous Proof. Since we have 0 and $0 \rightarrow Q$, we can derive Q by Modus Ponens.

Case 2: (Assuming $\neg(\neg Q)$)
Without even using the premise of P, we get Q in one step by Double Negation.

Since we have proved Q in both cases we have completed our direct proof.

(12 points) **Problem 1.8.4**

Here you will complete a famous proof, known to the ancient Greeks, that the number $\sqrt{2}$ is **irrational** (that it cannot be expressed as p/q where p and q are naturals). Suppose that $\sqrt{2} = p/q$ and that the fraction p/q is in lowest terms. Then by arithmetic, $p^2 = 2q^2$. Now use Proof By Cases, where the new proposition is "p is even". Derive a contradiction in each case. amd argue using Proof By Contradiction that $\sqrt{2}$ is irrational. (Remember, as we will show formally in Chapter 3, that a natural is even if and only if its square is even.)

Solution:

Case 1: Let's assume "p is even. If p is even, then we can write it as: $p = 2k$ for some natural k. Now we have $(2k)^2 = 2q^2 \rightarrow 4k^2 = 2q^2 \rightarrow 2k^2 = q^2$. Now we have two cases:
Case 1.1: If q is even then we can write it as $2k'$. In this case p and q are both even which contradicts the assumption that $\frac{p}{q}$ is in its simplest form.

Case 1.2: If q is odd then we can write it as $2k' + 1$ where k' is a natural number. In this case $2k^2 = (2k' + 1)^2 \rightarrow 2k^2 = 4k'^2 + 4k' + 1$. In this case an even number is equal to an odd number which is not possible and so it is a contradiction.

Case 2: Now assume p is an odd number so there must exist a natural number like t such that $p = 2t + 1$. Now $(2t + 1)^2 = 2q^2 \rightarrow 4t^2 + 4t + 1 = 2q^2$ which means we have an odd number equal to an even number which is a contradiction.

(12 points) **Problem 1.10.1**

Let $\Sigma = \{a, b\}$. Using the same string predicates defined in Exercises 1.10.2 and 1.10.3 above, express the following sets in set builder notation (that is, determine their predicate form):

(a) $\{aa, bb\}$.

(b) $\{a, aa, aaa, aba, aaaa, aaba, abaa, abba, aaaaa, aaaba, aabaa, aabba, \cdots\}$.

(c) $\{aaaa, aaab, abaa, abab, baba, babb, bbba, bbbb\}$. (You will also need the concatenation operation here.)

Solution:

(a) $\{w : P(w) \wedge Q(w)\}$

(b) $\{w : Q(w) \wedge R(w, \text{“}a\text{”})\}$

(c) $\{w : (w = xy) \wedge R(x, y) \wedge P(x) \wedge P(y)\}$

Extra credit:

(10 points) **Problem 2.3.3**

Let A and B be two types such that A is a proper subset of B (so that B contains all the elements of A plus at least one other element). Let P be a unary predicate on A, and let Q be a unary predicate on B. Suppose that $\forall a : \forall b : P(a) \rightarrow Q(b)$ is true. Which of the following four statements is guaranteed to be true? For each statement, explain why it is always true or give an example where it is false. (**Hint:** Consider the case where A is empty.)

(a) $(\forall a : P(a)) \rightarrow (\forall b : Q(b))$

(b) $(\forall b : Q(b)) \rightarrow (\forall a : P(a))$

(c) $(\exists a : P(a)) \to (\exists b : Q(b))$

(d) $(\exists b : Q(b)) \to (\exists a : P(a))$

Solution:

(a) False. First, if $A = \emptyset$ then $\forall a : P(a)$ is true. However, If B has an element b for which $Q(b)$ is false, then the implication is false.

(b) False. If $A = \{a\}$ and $B = \{a, b\}$ and $Q(a) \wedge Q(b) \wedge \neg P(a)$, then the statement $(\forall b : Q(b)) \to (\forall a : P(a))$ is false, while all the premises remain true.

(c) True. By Instantiation, $P(a)$ is true for some element a of A. Since A is a subset of B, a is an element of B. By Specification on the given statement, $P(a) \to Q(a)$ is true. By Modus Ponens, $Q(a)$ is true. By Existence, $\exists b : Q(b)$ is true.

(d) False. A could be empty and B could have an element b for which $Q(b)$ is true.