

COMPSCI 250: Introduction to Computation

Lecture #38: Turing Machine Semantics
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Turing Machine Semantics

- Review: A Turing Machine Example
- Turing Recognizable Languages
- Turing Decidable Languages
- The TR/TD Theorem
- Running Two Machines in Parallel
- Multitape Turing Machines
- Nondeterministic Turing Machines

A Turing Machine Example

- Here is a machine that solves a problem that a DFA cannot. When started in configuration $i \sqsubset w_1 w_2 \dots w_n$, it will halt if and only if w is in the language $\{a^n b^n : n \geq 0\}$ -- otherwise it will hang.
- With input $aabb$ we get $i \sqsubset aabb$, $\sqsubset paabb$, $\sqsubset \sqsubset qabb$, $\sqsubset \sqsubset aqbb$, $\sqsubset \sqsubset abqb$, $\sqsubset \sqsubset abbq \sqsubset$, $\sqsubset \sqsubset abrb$, $\sqsubset \sqsubset asb$, $\sqsubset \sqsubset sab$, $\sqsubset s \sqsubset ab$, $\sqsubset \sqsubset pab$, $\sqsubset \sqsubset \sqsubset qb$, $\sqsubset \sqsubset \sqsubset bq \sqsubset$, $\sqsubset \sqsubset \sqsubset rb$, $\sqsubset \sqsubset s \sqsubset$, $\sqsubset \sqsubset \sqsubset p \sqsubset$, $\sqsubset \sqsubset \sqsubset h \sqsubset$. The string $aabb$ is accepted.

A Turing Machine Example

In i: Move R and go to p.
In p: On \square , go to h.
On b, move L and go to z.
On a, print \square , move R, and go to q.
In q: On a or b, move R and stay in q.
On \square , move L and go to r.
In r: On a or \square , move L and go to z.
On b, print \square , move L, and go to s.
In s: On a or b, move L and stay in s.
On \square , move R and go to p.
In h: Halt (final state).
In z: Move left and stay in z.

The Church-Turing Thesis

- The **Church-Turing Thesis** says that any “reasonable” general-purpose model of computation will be able to compute exactly the same functions from strings to strings as Turing machines or the lambda calculus.
- (More precisely, they compute the same set of **partial functions**, because a general computation always has the possibility of not returning an output.)

The Church-Turing Thesis

- We can't mathematically prove this thesis, only amass evidence for it. In fact it actually serves as an implicit definition of "reasonable".
- Serious people have argued against the thesis -- for example physicist Roger Penrose argues that quantum effects in the brain compute in ways that a Turing machine could not. (Dave, at least, doesn't buy it.)
- For more on this see Turing's article *On Minds and Machines* or almost anything by Douglas Hofstadter.

The Church-Turing Thesis

- You probably believe that we could simulate a Turing machine in Java, given unlimited memory.
Could a Turing machine simulate any Java program?
- We know that Java can be compiled into machine language, so we would have to believe that any machine language program could be simulated by a TM.

Turing Recognizable Languages

- We've seen that a Turing machine, when started on a string, may or may not ever reach a final state. We define the **language of the machine M** , called $L(M)$, to be the set of strings on which M eventually halts.
- If a language X is equal to $L(M)$ for some Turing machine M , we say that X is **Turing recognizable**. The idea is that the machine “recognizes” strings in the language by halting, but gives no output on strings not in the language.

Turing Recognizable Languages

- In the example earlier, the language of our TM is $\{a^n b^n : n \geq 0\}$, a language that we now know is not regular.
- A recognizer for a language is not all that useful in practice, because we would like to know whether or not the string is in the language, and the recognizer makes us wait forever on the “no” strings. We can’t always recognize that it is in a loop, because it might just use more and more tape.

Turing Decidable Languages

- Suppose, though, that the machine always halts, but halts in an **accepting state** when the input is in the language and in a **rejecting state** when it is not.
- We redefine $L(M)$ to be the set of strings that are accepted. In this case we say that the language $L(M)$ is **Turing decidable**.
- Remembering the Church-Turing thesis, a decidable language is one where the decision problem can be solved by an algorithm, as long as we give the algorithm enough time and memory.

Turing Decidable Languages

- Every Turing decidable language is also Turing recognizable. If we have a decider M for a language X , we can modify M to make a recognizer for X .
- We replace the rejecting final state of M with a state in which the machine always moves left, like the state “ z ” of our example. Now this new machine M' halts on input w if and only if M accepts w , which is true if and only if $w \in X$.
- We'll show next time that there exist TR languages that are not TD.

Clicker Question #1

- Let X and Y be two languages. Suppose we have a TM M such that on any input string w ,
 M accepts $\Leftrightarrow w$ is in X but not in Y , and
 M rejects $\Leftrightarrow w$ is in Y but not in X .
Which statement follows from the existence of M ?
- (a) Both X and Y are Turing recognizable
- (b) Both X and Y are Turing decidable
- (c) $X \Delta Y$ is Turing decidable
- (d) Trick question: None of these three follows

Not the Answer

Clicker Answer #1

- Let X and Y be two languages. Suppose we have a TM M such that on any input string w ,
 M accepts $\Leftrightarrow w$ is in X but not in Y , and
 M rejects $\Leftrightarrow w$ is in Y but not in X .

Which statement follows from the existence of M ?

- (a) Both X and Y are Turing recognizable
- (b) Both X and Y are Turing decidable
- (c) $X \Delta Y$ is Turing decidable we can't tell about strings that are in both X and Y
- (d) *Trick question: None of these three follows*

The TR/TD Theorem

- The **TR/TD Theorem** says that a language X is Turing decidable if and only if both X and its complement \bar{X} are Turing recognizable.
- One half of the proof is easy. If X is decidable, so is \bar{X} (by exchanging the accepting and rejecting states, as we did for DFA's). By the proof on an earlier slide, both X and \bar{X} must also be recognizable.

The TR/TD Theorem

- For the other half, we assume that both X and \bar{X} are recognizable.

So we have two Turing machines M and N , with $X = L(M)$ and $\bar{X} = L(N)$. We need to build a decider for X .

- But if we have an arbitrary string w , we can't run either M or N on it and be confident of getting an answer. Strings in X will cause N to run forever, and strings not in X will cause M to run forever. Connecting M and N in **series** will not work.

Running Two TM's in Parallel

- The trick is to run both M and N on w in **parallel**. When our decider starts in configuration $i \sqsubset w$, the first thing it does is to make a new copy of w to the right of the original one, so that the tape contents become $@ \sqsubset w \# @ \sqsubset w$.
- The left copy of w represents the initial configuration of M , and the right copy is the initial configuration of N . The two $@$ symbols mark the head positions.

Running Two TM's in Parallel

- We now want the decider to run M and N for one step each on their respective tapes.
- We make a sweep of the tape. When the machine finds the first @, it changes the next letter and moves the @ left or right, based on what M should do in its current state seeing that letter.
- Then it continues to the right and does the same thing for the other @, based on what N should do in its current state seeing that letter.

Running Two TM's in Parallel

- The new TM must remember both an M-state and an N-state in its own state. Thus its state set is the direct product of M's state set and N's state set.
- One slight complication -- if the character to the right of the first @ is #, the machine must make space for a new blank by shifting everything to the right.

Clicker Question #2

- We've just seen how to build a single TM that runs two given Turing machines in parallel. Which of the following statements about combining languages *is not* necessarily true?
- (a) If $X \cup Y$ is Turing recognizable, then both X and Y are both Turing recognizable.
- (b) If X and Y are Turing decidable, then $X \cap Y$ is Turing decidable.
- (c) If X and \bar{X} are Turing recognizable, then \bar{X} is Turing decidable.
- (d) Trick question: All statements are true.

Not the Answer

Clicker Answer #2

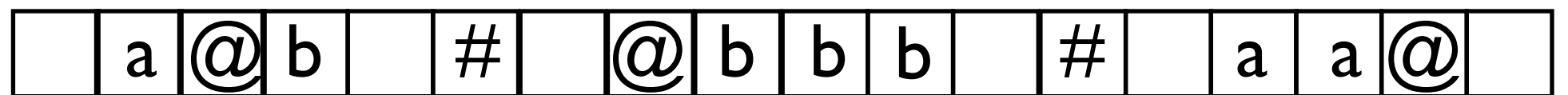
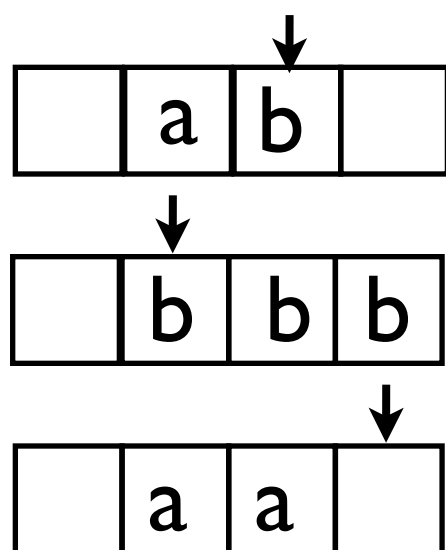
- We've just seen how to build a single TM that runs two given Turing machines in parallel. Which of the following statements about combining languages *is not* necessarily true?
- (a) *If $X \cup Y$ is Turing recognizable, then both X and Y are both Turing recognizable.* No way to tell which you have recognized
- (b) If X and Y are Turing decidable, then $X \cap Y$ is Turing decidable. Test for both, you get both answers
- (c) If X and \bar{X} are Turing recognizable, then \bar{X} is Turing decidable. Use the TD/TR theorem for \bar{X}
- (d) Trick question: All statements are true.

Multitape Turing Machines

- We can use a similar idea to show that a multitape Turing machine can be simulated by an ordinary one. A Turing machine with k tapes has a transition function from $Q \times \Gamma^k$ to $Q \times (\Gamma \times \{L, R\})^k$.
- On any step, based on its current state and the letters it sees at the head on each tape, it writes a letter and moves left or right on each tape.

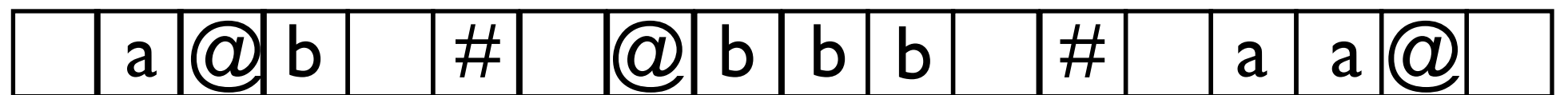
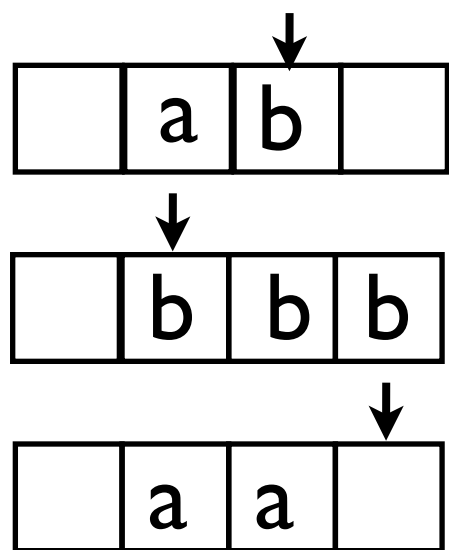
Multitape Turing Machines

- To simulate this multitape machine by an ordinary single-tape TM, we use our single tape to store the k different tapes in series, with $\#$ symbols between every pair of adjacent tape contents and $@$ symbols to mark each head position.



Multitape Turing Machines

- To simulate a move of the multitape machine, our ordinary machine sweeps the tape left to right, remembers the character after each @, then goes back, implements the write and move operations for each head, and updates its state.



Other Data Structures

- We can imagine versions of the Turing machine with more sophisticated access to memory.
- We might have tape cells arranged in two dimensions, or even in registers with numbers as in our real-life computers.
- With enough effort, each of these types of machines can be simulated by a one-tape machine.

Nondeterministic TM's

- A nondeterministic Turing machine or NDTM is like an ordinary deterministic Turing machine (or DTM) except that for a given state and letter seen, it may have no options, one option, or more than one option as to the character it writes and the direction it moves.
- As with NFA's, we say that a string w is in the language $L(N)$ if *it is possible* for N to accept w .

Nondeterministic TM's

- Nondeterminism did not make NFA's able to decide languages that DFA's could not, though the smallest NFA for a language might have many fewer states than the smallest DFA.
- In the case of Turing machines, we will see that nondeterminism again does not help *qualitatively* though it might help *quantitatively*.
- Every NDTM has an equivalent DTM with the same language, but the NDTM might take much less time to recognize a given string.

Clicker Question #3

- Which of these statements, about a k -state nondeterministic Turing machine M and a k -state NFA D , is not necessarily true?
- (a) $L(D)$ is regular but $L(M)$ may not be.
- (b) M can be converted into a DFA, but it may have many more states than D .
- (c) $L(D)$ is Turing decidable but $L(M)$ may not be.
- (d) Trick question: All three are true.

Not the Answer

Clicker Answer #3

- Which of these statements, about a k -state nondeterministic Turing machine M and a k -state NFA D , is not necessarily true?
- (a) $L(D)$ is regular but $L(M)$ may not be.
- (b) *M can be converted into a DFA, but it may have many more states than D .* Turing machines can have non-regular languages!
- (c) $L(D)$ is Turing decidable but $L(M)$ may not be.
- (d) Trick question: All three are true.

Simulating NDTM's with DTM's

- To simulate an NDTM with a DTM, we first build a DTM with three tapes.
- The first tape will store the input string w and will never change.
- The second will be a work tape to exactly simulate a particular computation of the NDTM.
- The third tape will hold a **choice sequence**, which is a string of symbols telling the NDTM which of its options to take on each of its moves.

Simulating NDTM's with DTM's

- The input w is in the language of the NDTM N if and only if there exists a choice sequence that causes N to halt, starting from $i \sqsubseteq w$.
- So our simulation tests all possible choice sequences, starting with the one of length 0, then all the ones of length 1, then length 2, and so forth.
- For each choice sequence, the DTM clears its work tape, copies w onto it, then runs N using the sequence. If the simulated N ever halts, the DTM accepts w .

Simulating NDTM's with DTM's

- So if $w \in L(N)$, the DTM will eventually reach a good choice sequence and will accept w .
- If $w \notin L(N)$, the DTM will run forever because it will keep trying longer and longer choice sequences.
- Thus the DTM accepts exactly those strings in $L(N)$, and so simulates N .
- If the DTM accepts, it takes exponentially longer to do so than N did on that input.