CMPSCI 250: Introduction to Computation

Lecture #9: Properties of Functions and Relations

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25 September 2023

Relations and Functions

- Review of the Dog Example
- Defining Functions With Quantifiers
- Total and Well-Defined Relations
- One-to-One and Onto Functions
- Bijections
- Composition and Inverse Functions
- Properties of Binary Relations on a Set
- Examples of Binary Relations on a Set

The Dog Example

- We have a set of dogs D, and predicates R(x) "x is a Rottweiler", T(x) "x is a terrier", S(x, y) "x is smaller than y", W(x) "x likes to go for walks".
- Our desired conclusion is as follows:
  "There exists a Rottweiler that is larger than some terrier who likes walks", which we may write as
  $\exists x: \exists y: R(x) \wedge S(y, x) \wedge T(y) \wedge W(y)$.
- We will work from five premises on the next slide.

Dog Example Premises

- (1) All dogs like to go for walks: $\forall x: W(x)$
- (2) Duncan is a terrier: $T(d)$
- (3) Cardie is smaller than some Rottweiler:
  $\exists x: R(x) \wedge S(c, x)$
- (4) All terriers are smaller than Cardie:
  $\forall x: T(x) \rightarrow S(x, c)$
- (5) S is transitive:
  $\forall x: \forall y: \forall z: (S(x, y) \wedge S(y, z)) \rightarrow S(x, z)$

Dog Example Strategy

- Recall the goal: There exists a Rottweiler that is larger than some terrier who likes walks ($\exists x$: $\exists y$: $R(x) \wedge S(y, x) \wedge T(y) \wedge W(y)$).
- Overall strategy: Figure out which dogs x and y ought to be -- maybe constants, maybe dogs forced to exist by the premises.
  In this case y should be Duncan, and x should be the Rottweiler provided by premise (3).

More of the Dog Example

- We use Instantiation on (3) to get a dog r such that $R(r) \wedge S(c, r)$.
- We need four facts about d and r: We have $R(r)$, and we need $W(d)$, $T(d)$, and $S(d, r)$.
- We have $T(d)$ by (2), and we get $W(d)$ by Specification on (1).

Finishing the Dog Example

- To get $S(d, r)$, we use Specification on (4) to get $T(d) \rightarrow S(d, c)$, Modus Ponens to get $S(d, c)$ since we have $T(d)$, and finally Specification on (5) to get $(S(d, c) \wedge S(c, r)) \rightarrow S(d, r)$ and Conjunction and Modus Ponens to get $S(d, r)$.
- Once we have these four facts we use Existence twice to get our desired conclusion $\exists x: \exists y: R(x) \wedge S(y, x) \wedge T(y) \wedge W(y)$.

Relations and Direct Products

- Recall that when A and B are two sets, a **relation** from A to B is any set of ordered pairs, where the first element of each pair is from A and the second is from B.
- We say that the relation R is a subset of the **direct product** $A \times B$, which is the set of all such ordered pairs.

Functions

- A **function** in ordinary computing usage is an entity that gives an **output** of a given type (the **range** or **codomain**) whenever it is called with an input of a given type (the **domain**).
- A function **from A to B** takes input from A and gives output from B.
- A relation from A to B may or may not define a function from A to B.

Relations and Functions

- We say that the relation is a function if for each input, there is *exactly one* possible output.
- That is, for every element x of A, there is exactly one element y of B such that the pair (x, y) is in the relation.
- We can put this definition into formal terms using predicates and quantifiers.

When a Relation is a Function

- Let R be a relation from A to B. We'll write "$(x, y) \in R$" as "$R(x, y)$", identifying the relation with its corresponding predicate. What does it mean for R to be a function?

- Part of the answer is that each x must have at least one y such that $R(x, y)$ is true.
  In symbols, we say $\forall x: \exists y: R(x, y)$. This property of a relation is called being **total**.

When a Relation is a Function

- We also require that each x may have at most one y such that R(x, y) is true -- this is the property of being **well-defined**.

- We can write that no x has more than one y, by saying $\forall x: \forall y: \forall z: (R(x, y) \wedge R(x, z)) \rightarrow (y = z)$. Another way to say this is $\neg\exists x: \exists y: \exists z: R(x, y) \wedge R(x, z) \wedge (y \neq z)$.

- A relation that is well-defined, but not necessarily total, is called a **partial function**. A non-void Java method computes a partial function since it may not terminate for all possible inputs.

Clicker Question #1

- Here are four binary relations on $\mathbb{N}$, the set of natural numbers. Which one *is not a function* from $\mathbb{N}$ to $\mathbb{N}$? Remember that a function must be both total and well-defined.
- (a) $A(x, y) = \{(x, y): y - x = 3\}$
- (b) $B(x, y) = \{(x, y): y = x^2 - 3x + 3\}$
- (c) $C(x, y) = \{(x, y): y = (x - 1)^2 - 2(x - 1) - 2\}$
- (d) $D(x, y) = \{(x, y): x^2 = y^2\}$

Not the Answer

Clicker Answer #1

- Here are four binary relations on $\mathbb{N}$, the set of natural numbers. Which one *is not a function* from $\mathbb{N}$ to $\mathbb{N}$? Remember that a function must be both total and well-defined.
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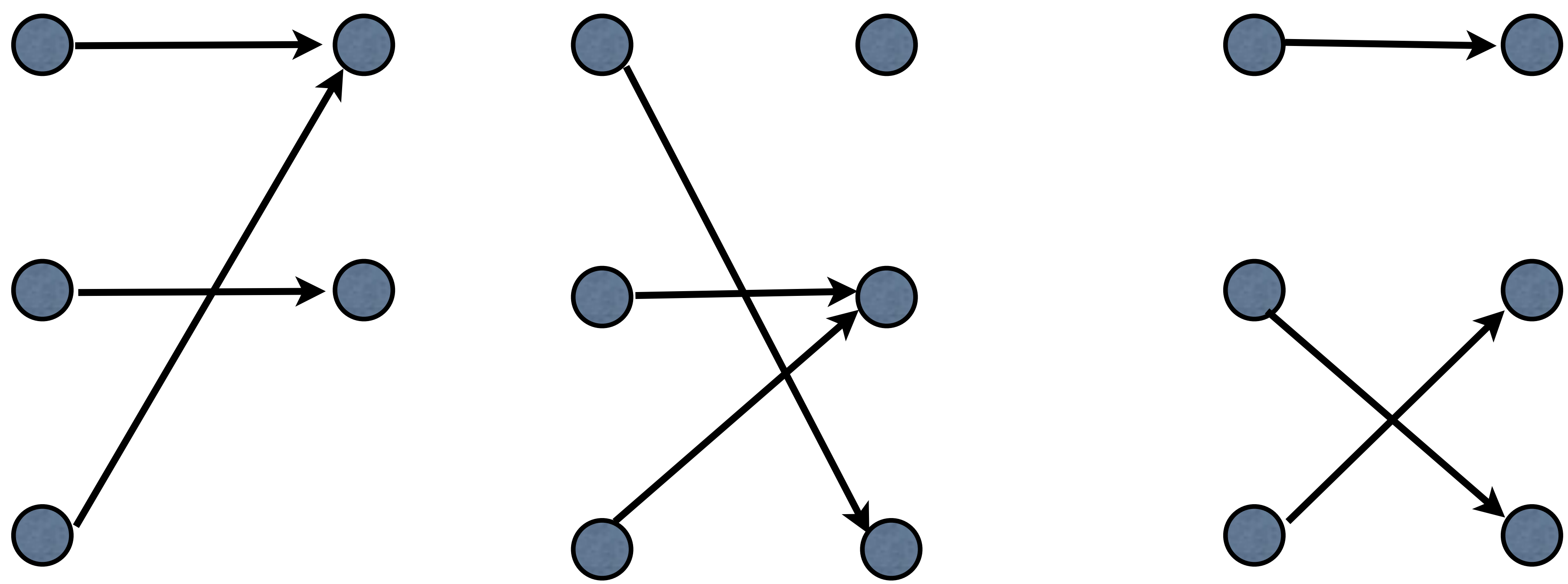
- (b) $B(x, y) = \{(x, y): y = x^2 - 3x + 3\}$
- (c) $C(x, y) = \{(x, y): y = (x - 1)^2 - 2(x - 1) - 2\}$
  $y = (x - 2)^2 - 3$, we don't have $C(2, y)$ for $y \in \mathbb{N}$
- (d) $D(x, y) = \{(x, y): x^2 = y^2\}$

Onto Functions (Surjections)

- We can also use quantifiers to define two important properties of functions.
- A function is **onto** (also called a **surjection**) if every element of the range is the output for at least one input, in symbols $\forall y: \exists x: R(x, y)$. Note that this is not the same as the definition of "total" because the x and y are switched -- it is the **dual** property of being total.

Onto Functions (Surjections)

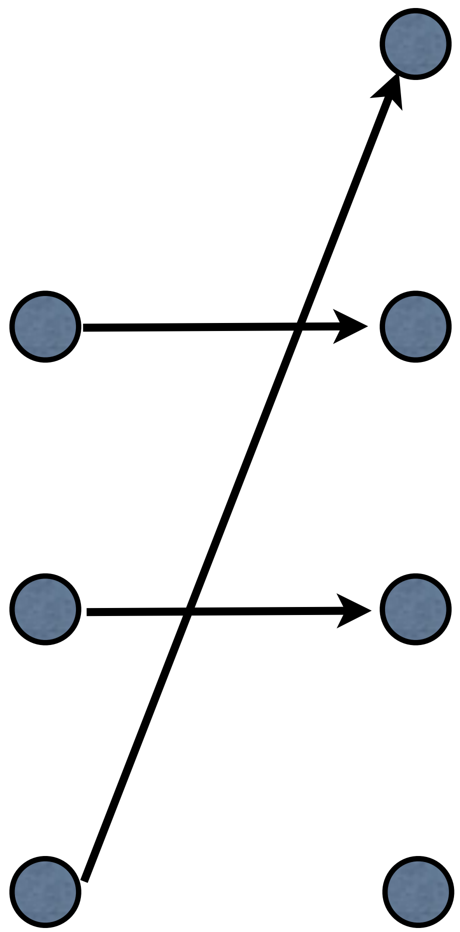


Onto, not 1-1 | Not Onto | Onto and 1-1

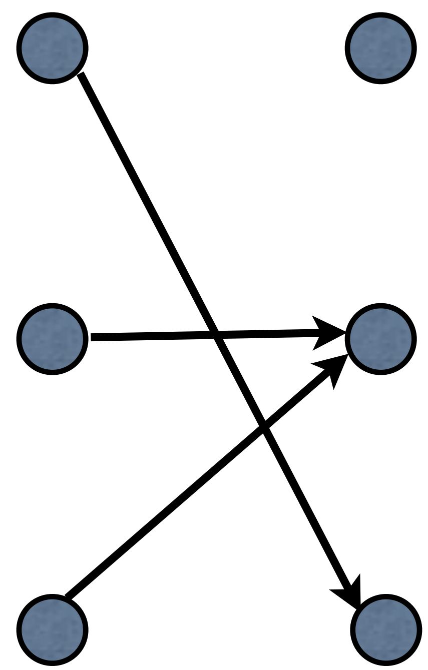
One-to-One Functions

- A function is **one-to-one** (an **injection**) if it is never true that two different inputs map to the same output.
- We can write this as
$\forall w\colon \forall x\colon \forall y\colon (R(w, y) \wedge R(x, y)) \rightarrow (w = x)$,
or equivalently
$\neg\exists w\colon \exists x\colon \exists y\colon R(w, y) \wedge R(x, y) \wedge (w \neq x)$.
This is obtained from the well-defined property by switching domain and range.

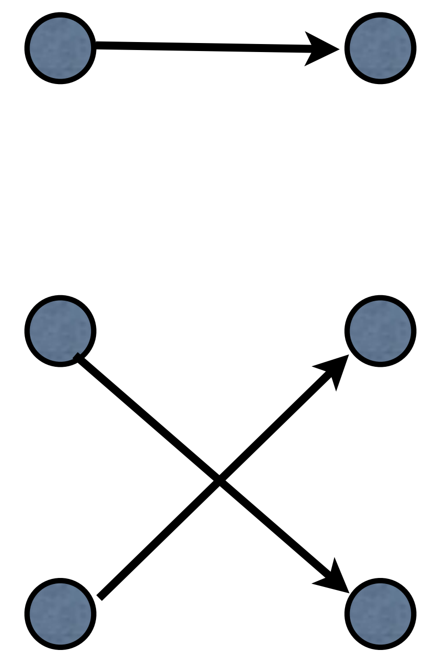
One-to-one Functions (Injections)



1-1, not onto



Not 1-1



Onto and 1-1

Functions and Sizes of Sets

- These properties are important in **combinatorics** -- if A and B are finite sets, we can have a surjection from A to B if and only if $|A| \geq |B|$.
- Similarly, we can have an injection from A to B if and only if $|A| \leq |B|$.
- (Here "$|A|$" denotes the number of elements in A, and "$|B|$" the number in B.)

Bijections

- It is possible for a function to be both onto and one-to-one. We call such a function a **bijection** (also sometimes a **one-to-one correspondence** or a **matching**).
- From what we just said about the sizes of finite sets in a surjection or injection, we can see that a bijection from A to B is possible if and only if $|A| = |B|$.

Clicker Question #2

- Suppose that A and B are two nonempty finite sets, and that f is a function from A to B. If A is *no larger than* B, that is, $|A| \leq |B|$, which one of these statements *must be true?*
- (a) f is not onto
- (b) f is one-to-one
- (c) f can't be both one-to-one and onto
- (d) if f is onto, then it is one-to-one

Not the Answer

Clicker Answer #2

- Suppose that A and B are two nonempty finite sets, and that f is a function from A to B. If A is *no larger than* B, that is, $|A| \leq |B|$, which one of these statements *must be true?*
- (a) f is not onto
- (b) f is one-to-one
- (c) f can't be both one-to-one and onto
- *(d) if f is onto, it is one-to-one* *yes, since then* $|A| = |B|$

Bijections

- There is an interesting theory, which we don’t have time for in this course, about the sizes of **infinite** sets, where we define two sets to have the same “size” if there is a bijection from one to the other.
- A bijection from a set to itself is also called a **permutation**. The problem of sorting is to find a permutation of a set that puts it in some desired order.

Composition of Functions

- If f is a function from A to B, and g is a function from B to C, we can define a function h from A to C by the rule $h(x) = g(f(x))$. We map x by f to some element y of B, then map y by g to an element of C. This new function is called the **composition** of f and g, and is written "$g \circ f$".
- The notation $g \circ f$ is chosen so that $(g \circ f)(x) = g(f(x))$, that is, the order of f and g remains the same in these two ways of writing it.

Inverse Functions

- With quantifiers, we can define $(g \circ f)(x) = z$ to mean $\exists y$: $(f(x) = y) \wedge (g(y) = z)$.
- If A and C are the same set, it is possible that the function g *undoes* the function f, so that g(f(x)) is always equal to x and f(g(y)) is always equal to y. This can only happen when f is a bijection -- in this case A and B have the same size, and g must also be a bijection. We say that f and g are **inverse functions** for one another.

Properties of Binary Relations

- Binary relations from a set to itself (called **relations on a set**) may or may not have certain properties that we also define with quantifiers.
- A relation R is **reflexive** if $\forall x$: $R(x, x)$ is true, and **antireflexive** if $\forall x$: $\neg R(x, x)$. Note that "antireflexive" is not the same thing as "not reflexive".

More Properties

- R is **symmetric** if $\forall x: \forall y: R(x, y) \rightarrow R(y, x)$, or equivalently $\forall x: \forall y: R(x, y) \leftrightarrow R(y, x)$.
- R is **antisymmetric** if $\forall x: \forall y: (R(x, y) \wedge R(y, x)) \rightarrow (x = y)$. Again "antisymmetric" is a different property from "not symmetric".
- R is **transitive** if $\forall x: \forall y: \forall z: (R(x, y) \wedge R(y, z)) \rightarrow R(x, z)$. We saw this property in the last lecture with the "smaller than" property for dogs.

Examples of Binary Relations

- The **equality relation** E is defined so that E(x, y) is true if and only if x = y.
- This relation is reflexive, symmetric, and transitive.
- We'll soon see that any relation with these three properties, called an **equivalence relation**, acts in many ways like equality.

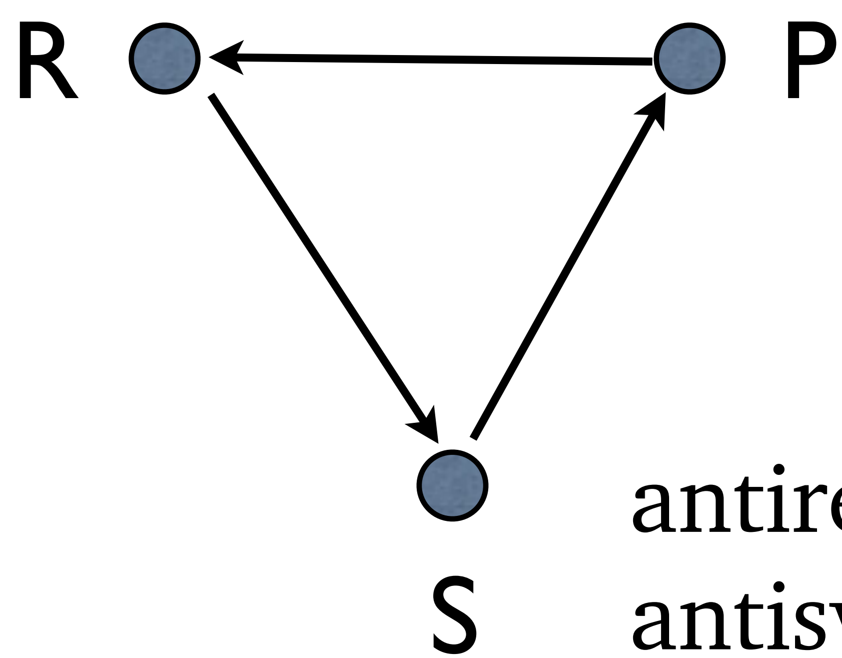
Examples of Binary Relations

- On numbers, for example, we can define LE(x, y) to mean $x \leq y$, and LT(x, y) to mean $x < y$.
- LE is reflexive, antisymmetric, and transitive, and relations with those three properties are called **partial orders**.
- LT, on the other hand, is antireflexive, antisymmetric, and transitive.

Examples of Binary Relations

- In the game of rock-paper-scissors, we can define a "beats" relation so that B(x, y) means "x beats y in the game".
- So B(r, s), B(s, p), and B(p, r) are true and the other six possible atomic statements are false.
- This relation is antireflexive, antisymmetric, and *not* transitive.

Rock-Paper-Scissors



antireflexive -- no self-loops
antisymmetric -- no two-way arrows
not transitive -- two-step paths have no shortcuts

Clicker Question #3

- Let the binary relation R on **Z** be defined so that $R(x, y)$ is $\{(x, y): x + y \leq 3\}$.
  This relation is:
- (a) reflexive and symmetric and transitive
- (b) symmetric, not transitive, and not reflexive
- (c) antireflexive, antisymmetric, transitive
- (d) neither reflexive, symmetric, nor transitive

Not the Answer

Clicker Answer #3

- Let the binary relation R on **Z** be defined so that R(x, y) is $\{(x, y): x + 3 \leq 3\}$.
  This relation is:
- (a) reflexive and symmetric and transitive
- *(b) symmetric, not transitive, and not reflexive*
- (c) antireflexive, antisymmetric, transitive
- (d) neither reflexive, symmetric, nor transitive

$\{(x, y): x + y \leq 3\}$

- Not reflexive because $R(2, 2)$ is false
- Not antireflexive because $R(0, 0)$ is true
- Symmetric because $x + y \leq 3$ is true if and only if $y + x \leq 3$
- Not transitive, because $R(2, 0)$ and $R(0, 2)$ are both true but $R(2, 2)$ is false.