COMPSCI 250: Introduction to Computation

Lecture #33: NFA’s and the Subset Construction
David Mix Barrington and Ghazaleh Parvini
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Nondeterministic Finite Automata

- Kleene's Theorem: What and Why?
- Nondeterministic Finite Automata
- The Language of an NFA
- The Model of λ-NFA's
- The Subset Construction: NFA's to DFA's
- Applying the Construction to No-aba
- The Validity of the Construction

The Minimal DFA

- Let X be a regular language and let M be *any* DFA such that L(M) = X.
- We'll show that the minimal DFA, constructed from the classes of the L-equivalence relation, is **contained within** M.
- We begin by eliminating any unreachable states of M, which does not change M’s language.

The Minimal DFA

- Remember that a correct DFA cannot take two L-distinguishable strings to the same state.
- So for any state p of M, the strings w such that $\delta(i, w) = p$ are all L-equivalent to one another.
- Each state of M is thus associated with one of the classes of the L-equivalence relation.

Minimizing a DFA

- The states of M are thus partitioned into classes themselves.
- If we combine each class into a single state, we get the minimal DFA.
- In section 14.3 of the book there is an example of an algorithm that gives you these classes, and thus gives you the minimal DFA. We'll have an example of this on the homework.

Kleene's Theorem: What and Why?

- We have now defined two classes of formal languages -- **regular** languages that are denoted by regular expressions, and what we will call **recognizable** languages that are decided by a DFA.
- **Kleene's Theorem**, the subject of the next several lectures, says that these two classes are the same.

Kleene’s Theorem

- Mathematically, it’s interesting that two classes with such different definitions should turn out to coincide -- it suggests that the class is important.
- But the theorem also has practical consequences.
- A class of languages is **closed** under an operation if applying the operation to elements of the class results in another element.

Kleene's Theorem

- It's easy to see that the regular languages are closed under union, concatenation, and star, and that the recognizable languages are closed under complement and intersection.
- The theorem tells us that *both* classes have *all* these closure properties.
- The efficient way to test whether a string is in a regular language is to create the DFA for the language and run it on the string.

Nondeterminism

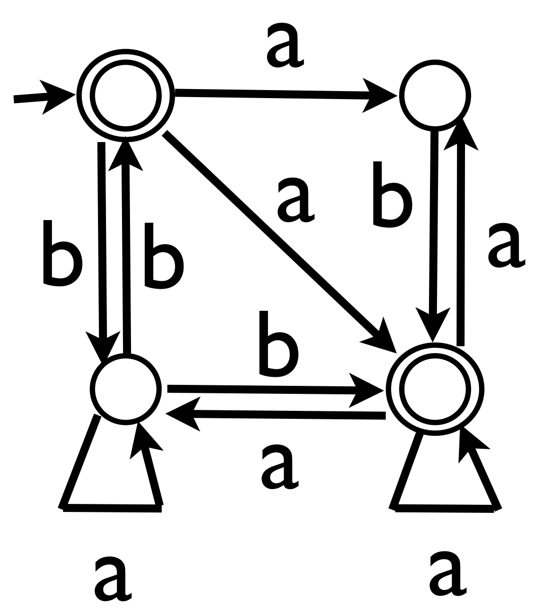
- DFA's are **deterministic** in that the same input always leads to the same output.
- Some algorithms are not deterministic because they are randomized, but here we will consider "algorithms" that are not deterministic because they are **underdefined** -- given a single input, more than one output is possible.
- We had an example of such an algorithm with our generic search, which didn't say *which* element came off the open list when we needed a new one.

Nondeterministic Finite Automata

- Formally, a **nondeterministic finite automaton** or **NFA** has an alphabet, state set, start state, and final state just like a DFA.
- But instead of the transition function δ, it has a **transition relation** $\Delta \subseteq Q \times \Sigma \times Q$. If $(p, a, q) \in \Delta$, the NFA *may* move to state q if it sees the letter a while in state p.

Drawing an NFA

- We draw an NFA like a DFA, with an a-arrow from p to q whenever (p, a, q) ∈ Δ.
- The NFA no longer has the rule that there must be exactly one arrow for each letter out of each state -- there may be more than one, exactly one, or none.

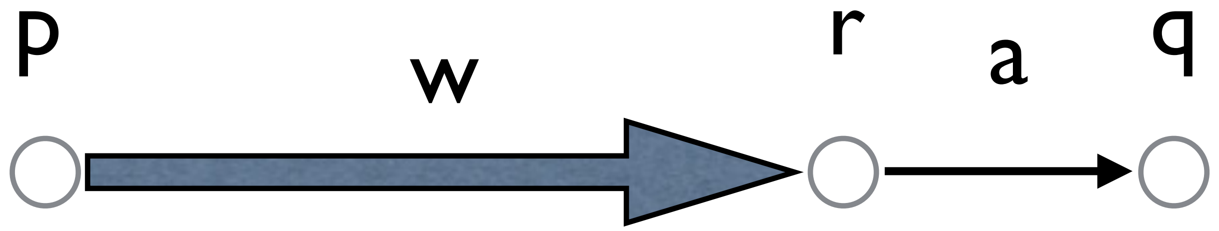


The Language of an NFA

- We can no longer say what the NFA *will* do when reading a string, only what it *might* do. The language of an NFA N is defined to be the set {w: w might be accepted by N}.
- More formally, we define a relation $\Delta^* \subseteq Q \times \Sigma^* \times Q$ so that the triple (p, w, q) is in Δ^* if and only if N *might* go from p to q while reading w.
- Then $w \in L(N) \leftrightarrow (i, w, f) \in \Delta^*$ for some final state $f \in F$.

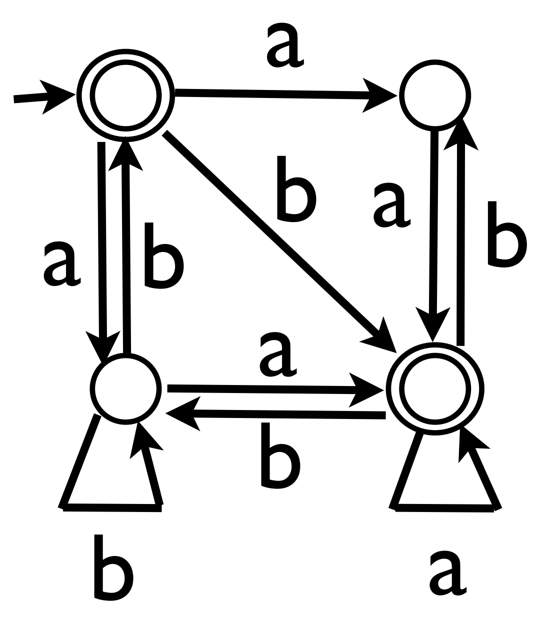
Defining Δ^*

- We can define the relation Δ^* inductively on the string in the middle.
- The triple (p, λ, q) is in Δ^* if and only if $p = q$.
- The triple (p, wa, q), where w is a string and a is a letter, is in Δ^* if and only if there is some state r such that (p, w, r) is in Δ^* and (r, a, q) is in Δ.



Clicker Question #1

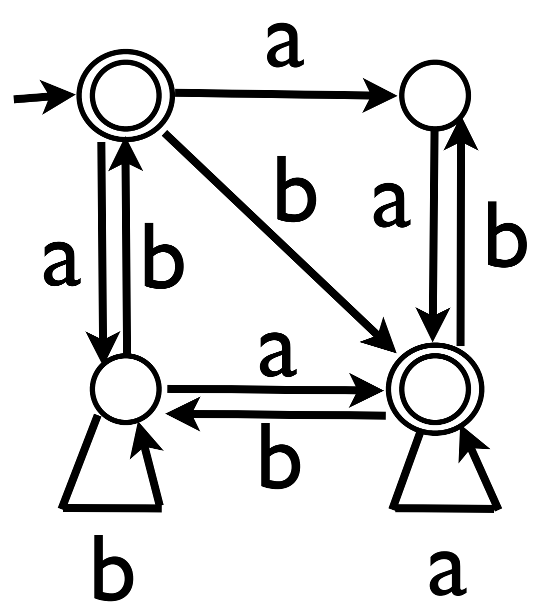
- A string w is in the language of this NFA if it is *possible* to follow a path with the letters of w from the start state to a final state. Which string *is* in L(N)?
- (a) aaab
- (b) aabb
- (c) baab
- (d) bbab



Not the Answer

Clicker Answer #1

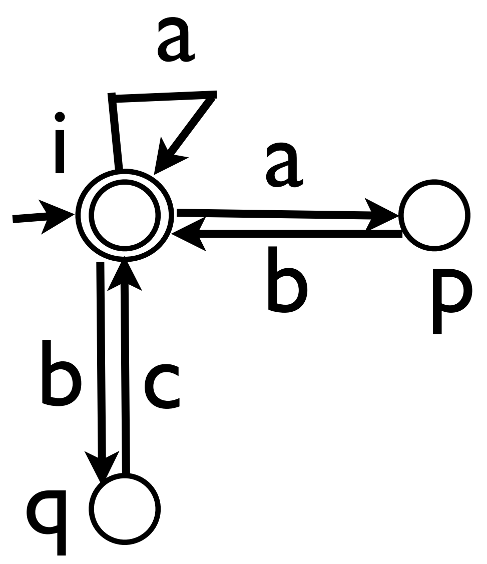
- A string w is in the language of this NFA if it is *possible* to follow a path with the letters of w from the start state to a final state. Which string *is* in L(N)?



- (a) aaab aaa to SE, b to NE/SW
- (b) *aabb* can go SW-SE-SW-NW
- (c) baab b to SE, aa stays, b SW/NE
- (d) bbab bb to SW/NE, a to SE, b back to SW/NE

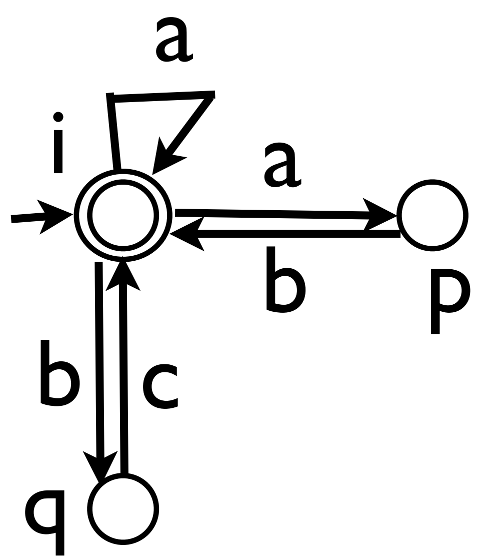
An NFA Example

- Consider the NFA N with alphabet {a, b, c}, state set {i, p, q}, start state i, final state set {i}, and $\Delta = \{(i, a, i), (i, a, p), (p, b, i), (i, b, q), (q, c, i)\}$.
- This is nondeterministic because there are two a-moves out of i, and several situations with no move at all.



An NFA Example

- Here L(N) is the regular language (a + ab+ bc)*, because any path from i to itself must consist of pieces labeled a, ab, or bc.
- It is not immediately clear how, for a larger NFA, we could determine whether a particular string was in L(N). Our method will be to turn N into a DFA.



Interpretations of Nondeterminism

- Because we can’t speak clearly of “what happens when we run N on w”, we need other ways to think of the action of an NFA.
- In our proofs, we will just replace “$w \in L(N)$” by “$\exists f: (i, w, f) \in \Delta^*$” and argue about the possible w-paths in the graph of N.

Interpretations of Nondeterminism

- Suppose the NFA makes a choice uniformly at random whenever it has more than one option. This makes it a **Markov process** in the language of COMPSCI 240.
- In this case w $\in$ L(N) if and only if the probability that N goes to a final state on w is *positive*. If there is a path, there is a nonzero probability of N taking it, and if there is no path, of course it cannot possibly reach a final state.

Interpretations of Nondeterminism

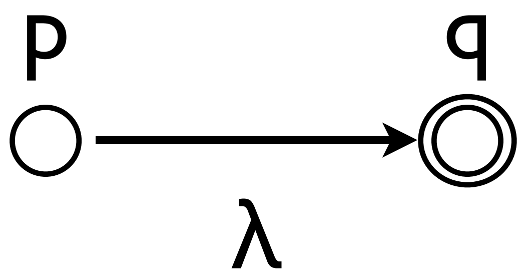
- Another interpretation has us **fork a process** whenever N is faced with a choice. One process takes each choice, and if *any* of the processes reaches a final state when it is done reading w, then w $\in$ L(N).
- *"When you come to a fork in the road... take it." (Y. Berra)*

The Model of λ-NFA's

- The main reason to use NFA's is that they are easier to design in many situations when we have some other definition of the language.
- Often we will find it convenient to give the NFA the option to jump from one state to another *without reading a letter*.
- A **λ-move** is a transition (p, λ, q) that allows a **λ-NFA** to do just that.

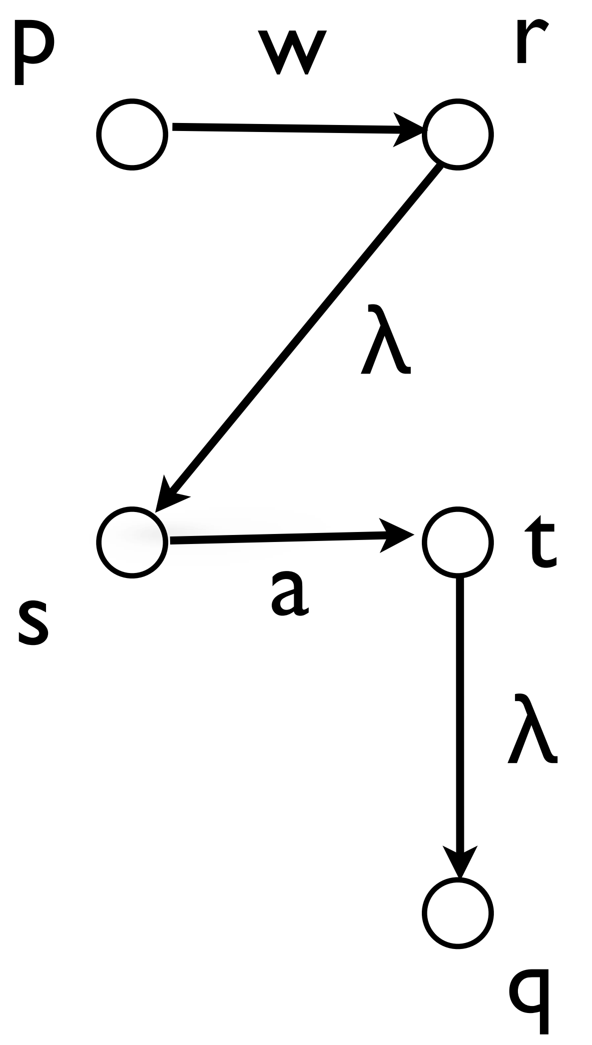
The Model of λ-NFA's

- We need to redefine the type of Δ, so that it is a subset of $Q \times (\Sigma \cup \{\lambda\}) \times Q$.
- In the diagram, this transition is represented by an arrow from p to q labeled with λ.



Paths in a λ-NFA

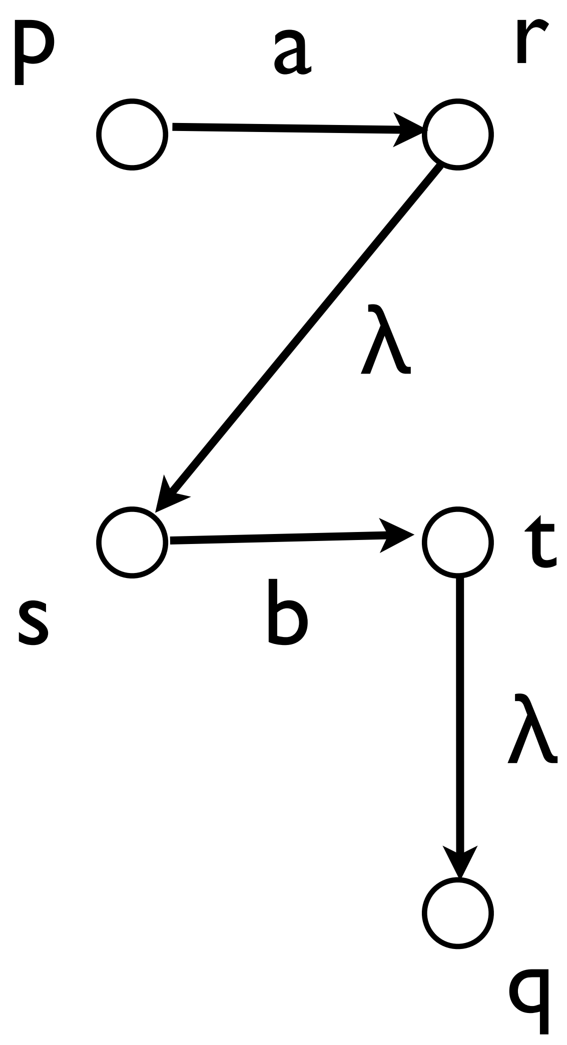
- Formally Δ^* is now more complicated to define. We say that $(p, \lambda, q) \in \Delta^*$ if there is a path of λ-moves from p to q.
- Then we define $\Delta^*(p, wa, q)$ to be true if and only if there exist states r, s, and t such that (p, w, r), (r, λ, s) and (t, λ, q) are all in Δ^*, and (s, a, t) is in Δ.



Paths in a λ-NFA

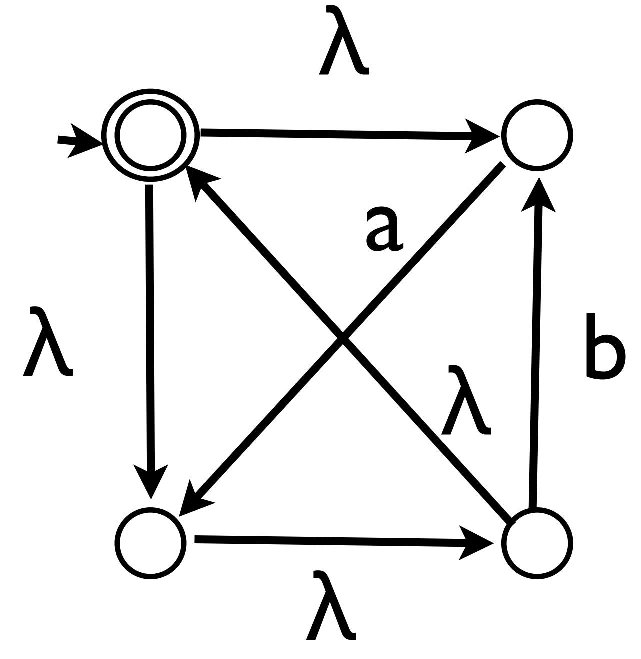
- What this means is that $\Delta^*(p, w, q)$ is true if and only if there exists a path from p to q such that the letters on the path, read in order, spell out w.
- There may be any number of λ-moves in the path as well.
- (Thus we don't even know how long the path from p to q might be.)

$(p, ab, q) \in \Delta^*$



Clicker Question #2

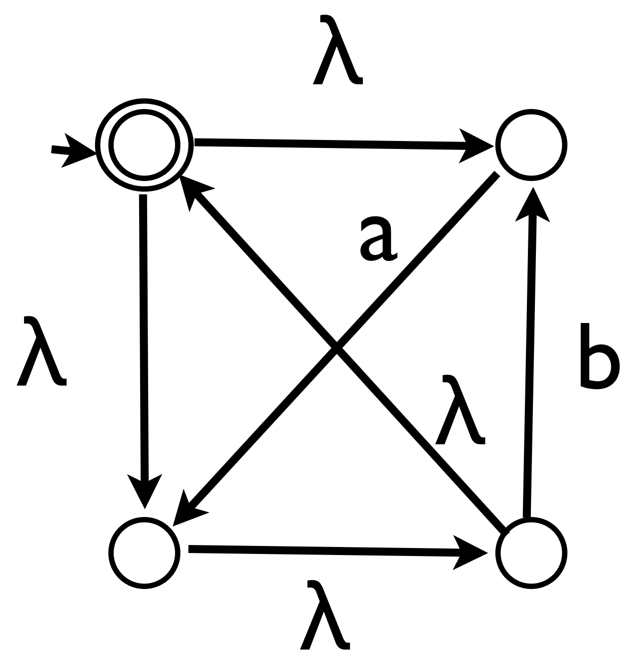
- Which of these strings *is not* in the language of this λ-NFA?
- (a) λ
- (b) aababaa
- (c) babaaba
- (d) Trick question: All three are in the language.



Not the Answer

Clicker Answer #2

- Which of these strings *is not* in the language of this λ-NFA?
- (a) λ
- (b) aababaa
- (c) babaaba
- *(d) Trick question: All three are in the language.* $(a+ba)^*$

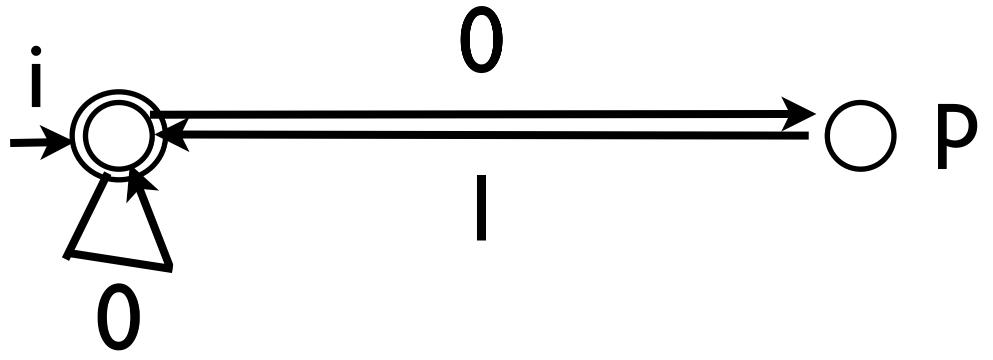


The Subset Construction

- Next lecture we'll see how to convert λ-NFA's to ordinary NFA's.
- Now, though, we will convert ordinary NFA's to DFA's using the **Subset Construction**.
- Given an NFA N with state set Q, we will build a DFA D whose states will be *sets of states* of N -- formally, D's state set is the **power set** of Q.

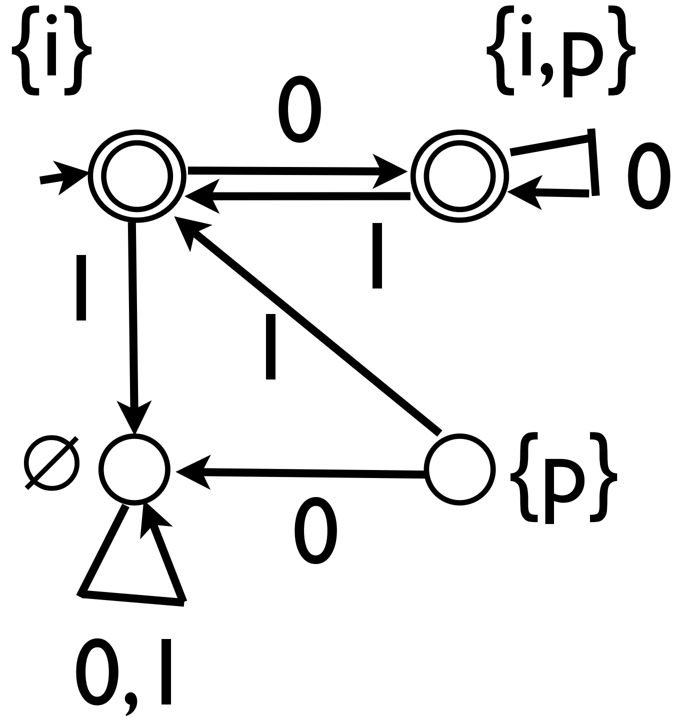
The Subset Construction

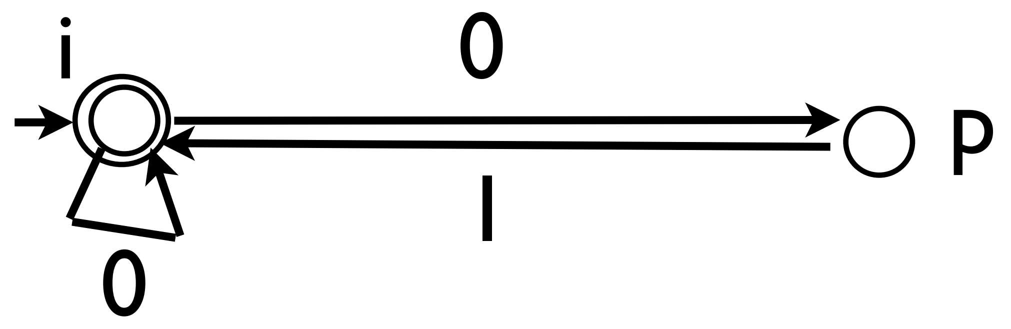
- Here's an example of an NFA N for the language (0 + 01)*, with two states i and p, start state i, final state set {i}, and transitions (i, 0, i), (i, 0, p), and (p, 1, i).
- At the start of its run, N must be in state i.
If the first letter is 0, then it might be in either state i or p after reading the 0.
If the first letter is 1, there is no run of N that reads that letter.



The Subset Construction

- Our DFA D has states ∅, {i}, {p}, and {i, p}.
- Its start state is {i}, final states are {i} and {i, p}, and we have δ({i}, 0) = {i, p}, δ({i}, 1) = ∅,
  δ({i, p}, 0) = {i, p}, δ({i, p}, 1) = {i},
  δ({p}, 0) = ∅, δ({p}, 1) = {i},
  and δ(∅, a) = ∅ for both letters.





Details of the Construction

- The general construction works just like this example.
- The start state of D is {i}, where i is the start state of N.
- The final state set of D is the set of all states of D that contain final states of N, since we want D to accept if and only if N *can* accept.

Details of the Construction

- In general, we need to define $\delta(S, a)$, where S is a state of D, meaning that S is a set of states of N.
- S represents the possible places N might be before reading the a. The set $T = \delta(S, a)$ will be the set of all states q such that the transition (s, a, q) is in Δ for some $s \in S$.
- In the graph, we take the set of destinations of all the a-arrows that start from a state of S.

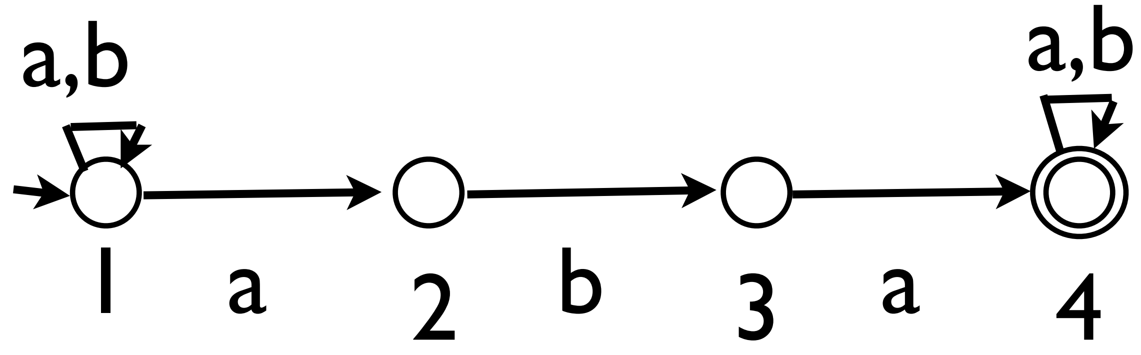
Details of the Construction

- The most common mistake in computing δ comes when one of the states in S has no a-arrows out of it.
- Students often think that ∅ must now be part of δ(S, a). But in fact δ(S, a) is the *union* of the sets {q: Δ(s, a, q)} for each s ∈ S.
- So the empty set is part of the result, but doesn't show up in the description of the result because unioning with ∅ is the identity operation on sets.

Applying This to No-aba

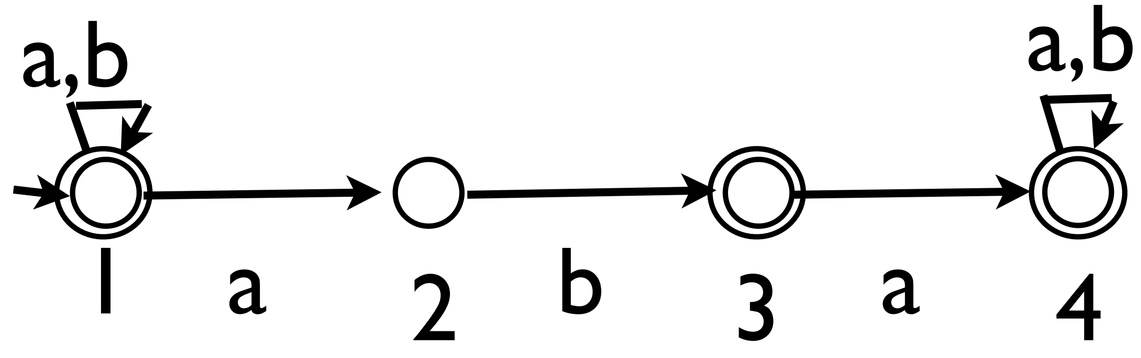
- The language Yes-aba has an easy regular expression $\Sigma^* aba \Sigma^*$. From this expression we can build an NFA N with state set {1, 2, 3, 4}, start state 1, final state set {4}, and $\Delta = \{(1, a, 1), (1, b, 1), (1, a, 2), (2, b, 3), (3, a, 4), (4, a, 4), (4, b, 4)\}$.

- But what if we want a machine for No-aba? Switching the final and non-final states of N will not do -- can you see why?



Clicker Question #3

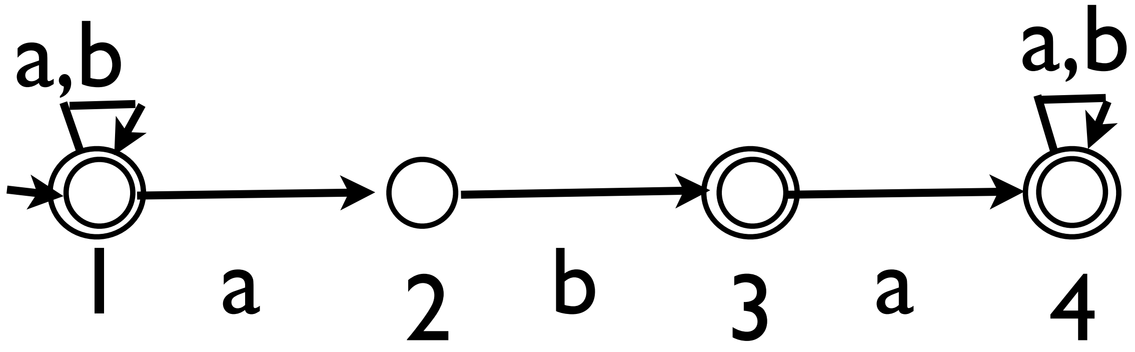
- Which expression *does not* denote the language of this NFA?
- (a) $(a + b)^*$
- (b) $(a + b)^* + (a+b)^*aba(a + b)^*$
- (c) $(a + b)^* + ab(a + b)^*ba + ababaab$
- (d) Trick: All of the expressions are correct.



Not the Answer

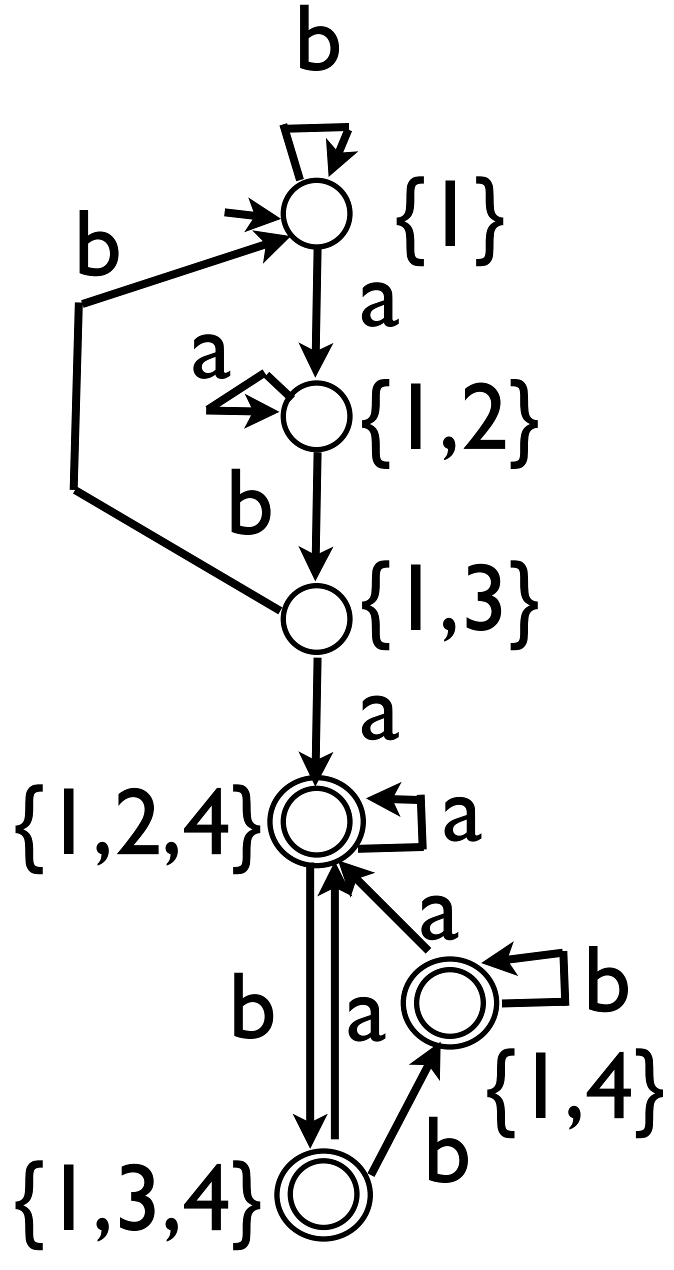
Clicker Answer #3

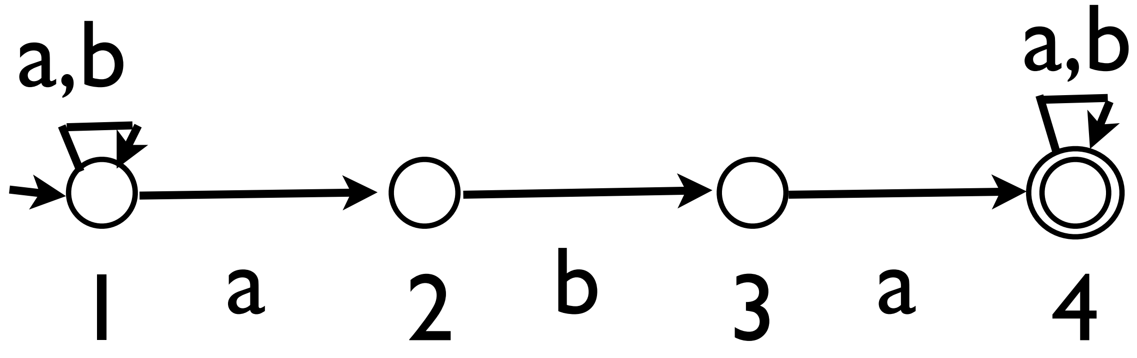
- Which expression *does not* denote the language of this NFA?
- (a) $(a + b)^*$ simplest form
- (b) $(a + b)^* + (a+b)^*aba(a + b)^*$
- (c) $(a + b)^* + ab(a + b)^*ba + ababaab$
- *(d) Trick: All of the expressions are correct.*



Applying This to No-aba

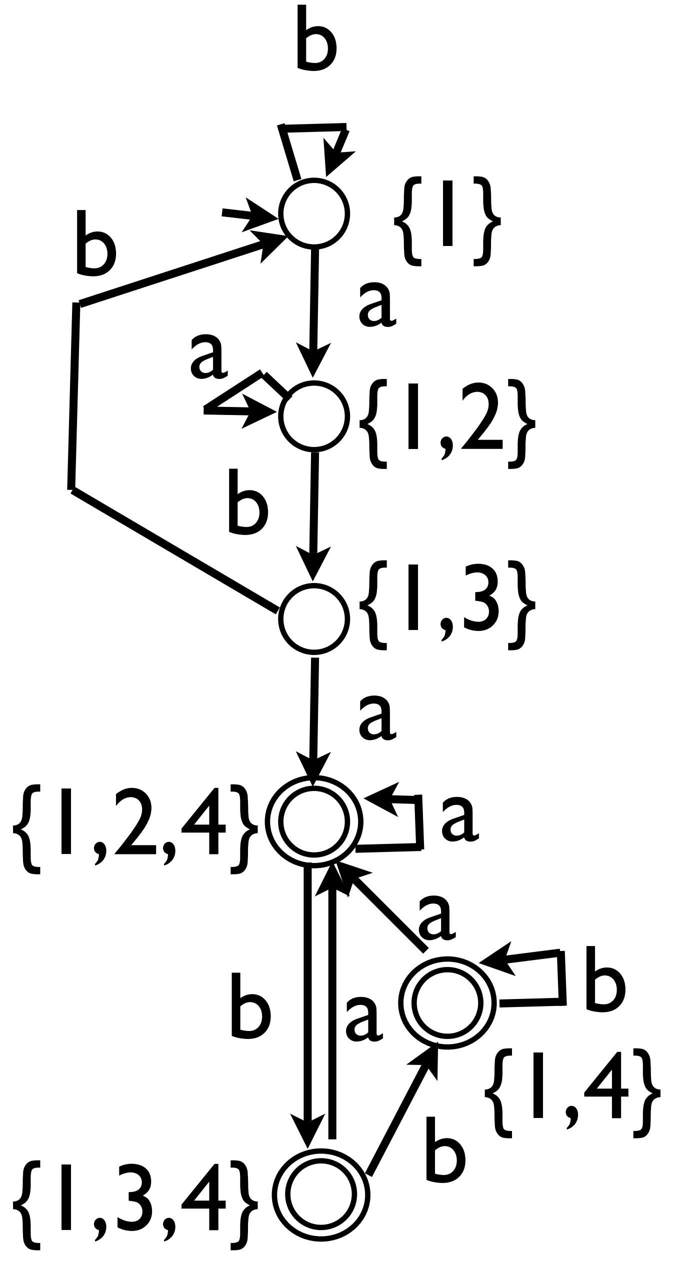
- The best way to get a DFA for No-aba is to first get one for Yes-aba.
- We begin with the start state {1} and compute δ({1}, a) = {1, 2} and δ({1}, b) = {1}.
  Then we compute δ({1, 2}, a) = {1, 2} and δ({1, 2}, b) = {1, 3}.

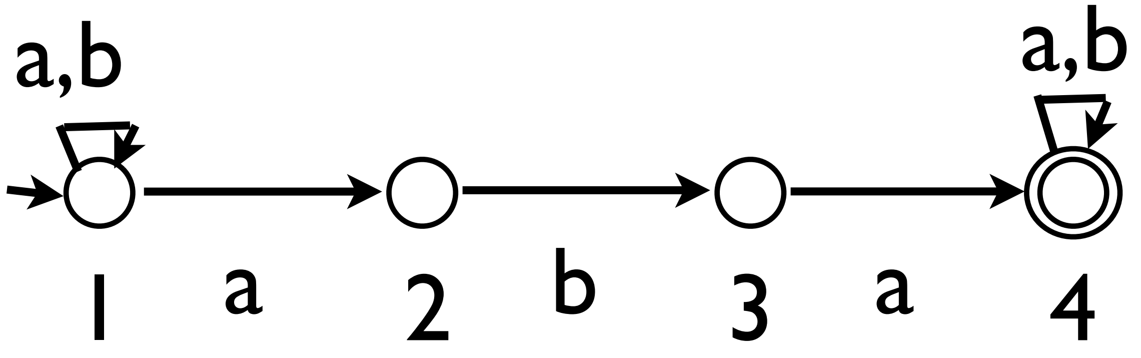




Applying This to No-aba

- Since {1, 3} is new, we must compute δ({1, 3}, a) = {1, 2, 4} and δ({1, 3}, b) = {1}.
- Then we get δ({1, 2, 4}, a) = {1, 2, 4} and δ({1, 2, 4}, b) = {1, 3, 4}.
  Not done yet!
- We have δ({1, 3, 4}, a) = {1, 2, 4}

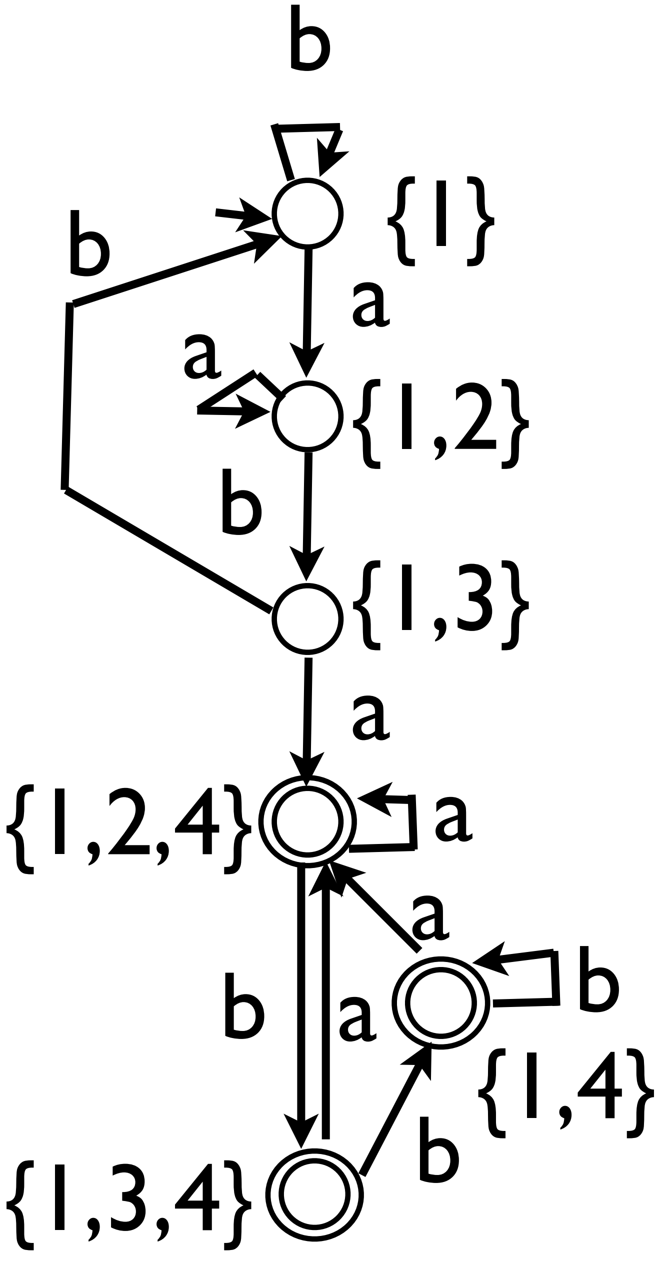


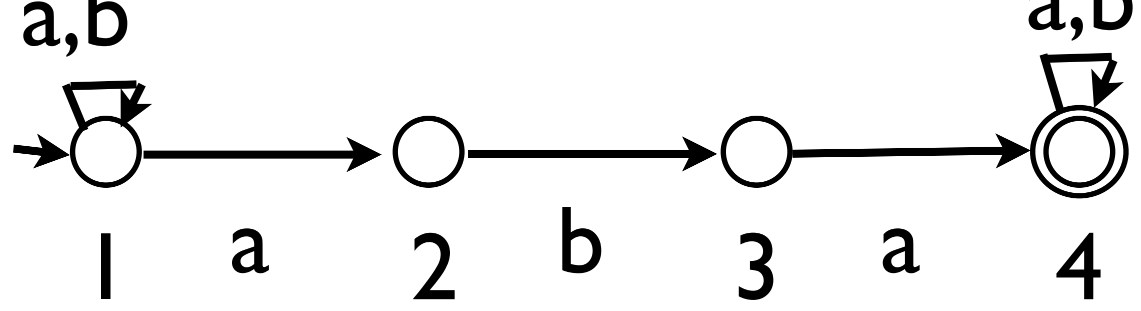


Applying This to No-aba

- Finally, with δ({1, 4}, a) = {1, 2, 4} and δ({1, 4}, b) = {1, 4}, we're done -- we have all reachable states.

- If we minimized this DFA, the three final states would merge into one. This gives us our four-state DFA for Yes-aba, from which we can get one for No-aba.





Validity of the Construction

- How can we prove that for any NFA N, the DFA D that we construct in this way has L(D) = L(N)?
- The key property of D is that for any string w, $\delta^*(\{i\}, w)$ is exactly the set of states $\{q: \Delta^*(i, w, q)\}$ that could be reached from i on a w-path.
- We prove this property by induction -- it is clearly true for λ (though if we had λ-moves it would not be).

Validity of the Construction

- If we assume that $\delta^*(\{i\}, w) = \{q: \Delta^*(i, w, q)\}$, we can then prove $\delta^*(\{i\}, wa) = \{r: \Delta^*(i, wa, r)\}$ for an arbitrary letter a, using the inductive definition of δ^* in terms of δ, of δ in terms of Δ, and of Δ^* in terms of Δ.
- Once this is done, it is clear that $w \in L(D) \leftrightarrow \exists f: f \in \delta^*(\{i\}, w) \leftrightarrow \exists f: \Delta^*(i, w, f) \leftrightarrow w \in L(N)$.
- Note that in general D could have 2^k states when N has k states. But if we leave out unreachable states, D could be much smaller.