COMPSCI 250 Discussion #6: More Induction Problems
Individual Handout

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Today's exercise is to solve some more varied problems using mathematical induction, and write careful proofs of each of the four statements you are given. Remember that each proof needs a base case, a clear statement of the inductive hypothesis, and a clear argument for the inductive step.

- (a) Recall that the **Fibonacci numbers** are defined recursively by the rules $F(0) = 0$, $F(1) = 1$, and for any $n \geq 1$, $F(n+1) = F(n) + F(n-1)$. Prove that for any positive integer n, the numbers $F(n)$ and $F(n+1)$ are relatively prime.

  We asserted in lecture that the Euclidean Algorithm reaches 1 *when started on two consecutive Fibonacci numbers, and in Problem 4.11.3 it is shown that it takes the maximum possible number of steps in this case, relative to the starting numbers. The induction is very straightforward.*

- (b) In this problem we consider tiling a large equilateral triangle made up of smaller triangles in the same way a chessboard is made of smaller squares. Given any equilateral triangle, we can divide it into four smaller triangles by connecting the midpoints of the three sides. If we similarly subdivide each of these triangles, we get 16 total pieces, and if we carry out this process k times we get 4^k small triangles. Our basic tile will consist of three of the smallest triangles, arranged in a trapezoid – we can get this figure by taking a triangle made up of four small triangles, and removing any one of the three triangles at a point. Your task is to prove that for any k, if we divide a big triangle into 4^k small ones and remove *one* of the small ones at one of the points, we can tile the remainder of the figure with these trapezoids.

  Clearly this is pretty similar to the L-shaped tile problem. You can use ordinary induction.

- (c) Begin with a rectangle in the Euclidean plane and divide it into regions by drawing any number of straight lines or circles. Show that the resulting figure may be **two-colored**, meaning that we can paint each region either red or blue such that no two red regions touch at more than a point, and that no blue regions touch at more than a point.

  Use ordinary induction on the combined number of circles and straight lines. Show how you can adapt a correct two-coloring to deal with one more line or circle.

- (d) We are told that a particular county has n cities, with $n > 1$, and that from each city x to each other city y there is a road that may traversed in only one direction – we are not told which. We want to prove that for any such county, there is a path starting at one city, visiting each city exactly once, and ending (if $n > 1$) at a different city. This path uses $n - 1$ of the roads and traverses each road in the correct direction.

  Here we naturally use induction on the number of cities. Given a path through n *of the cities, we want to adapt it into a path that includes a new city, possibly by changing the start or end city. We must show that for any assignment of directions to the roads involving the new city, we may extend the path in this way.*