COMPSCI 250: Fall 2023
Homework 3

David A. Mix Barrington , Ghazaleh Parvini

Due Date : Friday, October 20

This assignment has 7 problems. There is also 1 Extra Credit problem. The extra credit is 10 points.

Please submit a single PDF file, with the problems in order (as below), and legible. Look at your PDF before submitting it – it is fine to scan or photograph a handwritten document but it the graders can't read it, they won't grade it.

Please assign pages to problems in Gradescope. Graders will click on the problem number. If no page shows up because it's not assigned, the assumption is you have not solved the problem.

Be sure you are doing Problems in the book and not Exercises: the numbers should start with P rather than E.

For full credit, show your work, explaining your reasoning. This also helps assign partial credit.

You are responsible for following the academic honesty guidelines on the Grading and Requirements page. This means that what you present must be your own work in presentation, and you must acknowledge all sources of aid other than course staff and the textbook. You will get 2 extra points if you typeset your Homework.

(15 points) **Problem 3.1.4**

The least common multiple of two naturals x and y is the smallest natural that both x and y divide. For example, $lcm(8, 12) = 24$ because 8 and 12 each divide 24, and there is no smaller natural that both 8 and 12 divide.

(a) Find the least common multiple of 60 and 339.

(b) Find the least common multiple of $2^3 3^2 5^4$ and $2^2 3^4 5^3$.

(c) Describe a general method to find the least common multiple of two naturals, given their factorization into primes (and assuming that the factorization exists and is unique).

(15 points) **Problem 3.3.4**

We have defined the factorial $n!$ of a natural n to be the product of all the naturals from 1 through n, with $0!$ being defined as 1. Let p be an odd prime number. Prove that $(p-1)!$ is congruent to -1 modulo p. (**Hint:** Pair as many numbers as you can with their multiplicative inverses.)

(17 points) **Problem 3.5.4**

Suppose that the naturals $m_1, \ldots, m_k$ are pairwise relatively prime and that for each i from 1 through k, the natural x satisfies $x \equiv x_i \pmod{m_i}$ and the natural y satisfies $y \equiv y_i \pmod{m_i}$. Explain why for each i, xy satisfies $xy \equiv x_i y_i \pmod{m_i}$ and $x + y$ satisfies $(x + y) \equiv (x_i + y_i) \pmod{m_i}$. Now suppose that $z_1, \cdots, z_j$ are some naturals and that we have an arithmetic expression in the z_i's (a combination of them using sums and products) whose result is guaranteed to be less than M, the product of the m_i's. Explain how we can compute the exact result of this arithmetic expression using the Chinese Remainder Theorem only once, no matter how large j is.

(8 points) **Problem 4.1.6**

(uses Java) Give a recursive definition of and a recursive static method for the **natural subtraction** function, with pseudo-Java header

natural minus (natural x, natural y).

On input x and y this function returns $x - y$ if this is a natural (i.e., if $x \geq y$) and 0 otherwise.

(15 points) **Problem 4.3.2**

Let the finite sequence $a_0, a_1, \cdots, a_n$ be defined by the rule $a_i = b + i \cdot c$. Prove by induction on n that the sum of the terms in the sequence is $(n + 1)(a_0 + a_n)/2$. (**Hint:** In the base case, $n = 0$ and so a_0 is equal to a_n. For the induction case, note that the sum for $n + 1$ is equal to the sum for n plus the one new term a_{n+1}.)

(15 points) **Problem 4.3.6**

Define $S(n)$ to be the sum, for all i from 1 through n, of $\frac{1}{i(i+1)}$. Prove by induction on all naturals n (including 0) that $S(n) = 1 - \frac{1}{n+1}$.

(15 points) **Problem 4.4.1**

Consider a variant of Exercise 4.4.3, for \$4 and \$11 bills (made, we might suppose, by a particularly inept counterfeiter). What is the minimum number k such that you can make up \$$n$ for all $n \geq k$? Prove that you can do so.

Extra credit:

(10 points) **Problem 3.4.6**

A **Fermat number** is a natural of the form $F_i = 2^{2^i} + 1$, where i is any natural. In 1730 Goldbach used Fermat numbers to give an alternate proof that there are infinitely many primes.

(a) List the Fermat numbers F_0, F_1, F_2, F_3, and F_4,

(b) Prove that for any n the product $F_0 \cdot F_1 \cdot ... \cdot F_n$ is equal to $F_{n+1} - 2$.

(c) Argue that no two different Fermat numbers can share a prime factor. Since there are infinitely many Fermat numbers, there must thus be infinitely many primes.