CMPSCI 250: Introduction to Computation

Lecture #10: Equivalence Relations
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Equivalence Relations

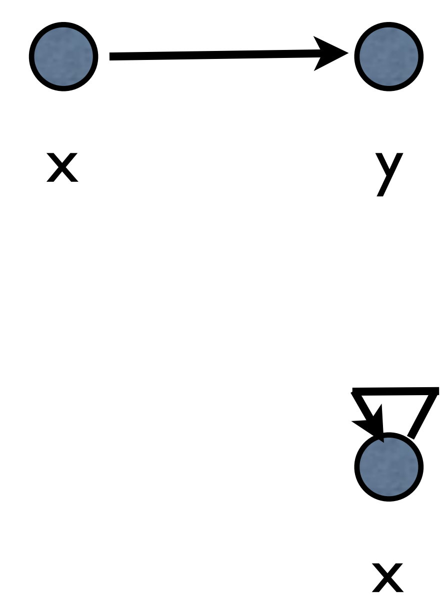
- Diagrams of Binary Relations
- Definition of Equivalence Relations
- Examples and Their Graphs
- Partitions and the Partition Theorem
- “Same Set” is an Equivalence Relation
- Equivalence Classes Form a Partition

Properties of Relations

- A binary relation on a set A is a subset of the set A $\times$ A, which contains all ordered pairs from A.
- Last time we defined several properties of binary relations on a set: reflexive, symmetric, antisymmetric, and transitive.
- These properties will allow us to define two special kinds of such relations: equivalence relations today and partial orders next time.

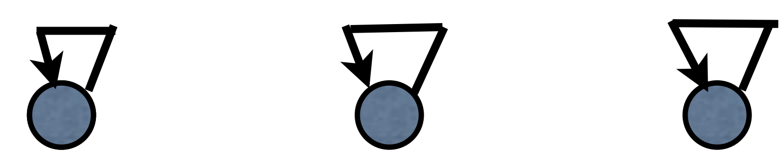
Diagrams of Binary Relations

- If A is a finite set and R is a binary relation on A, we can draw R in a diagram called a graph. We make a dot for each element of A, and draw an arrow from the dot for x to the dot for y whenever R(x, y) is true. If R(x, x), we draw a loop from the dot for x to itself.



Seeing the Properties

- Our relation properties are perhaps easier to see in one of these diagrams.

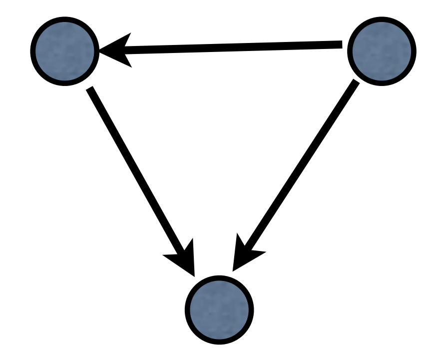


- A relation is reflexive if its diagram has a loop at every dot.

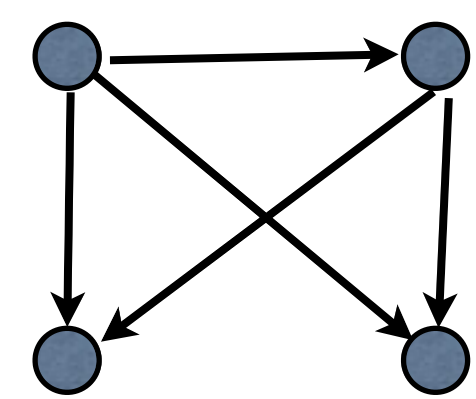
- It is symmetric if every arrow (except loops) has a matching opposite arrow.

Seeing the Properties

- It is antisymmetric if there are never two arrows in opposite directions between two different nodes.

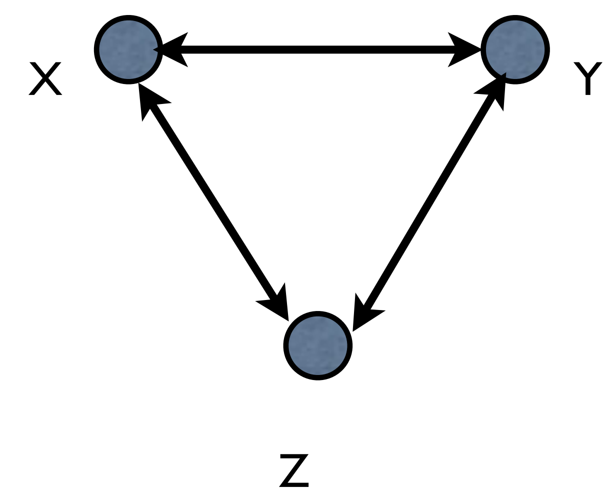


- It is transitive if for every path of arrows (a chain where the start of each arrow is the end of the previous one) there is a single arrow from the start of the chain to the end.



Clicker Question #1

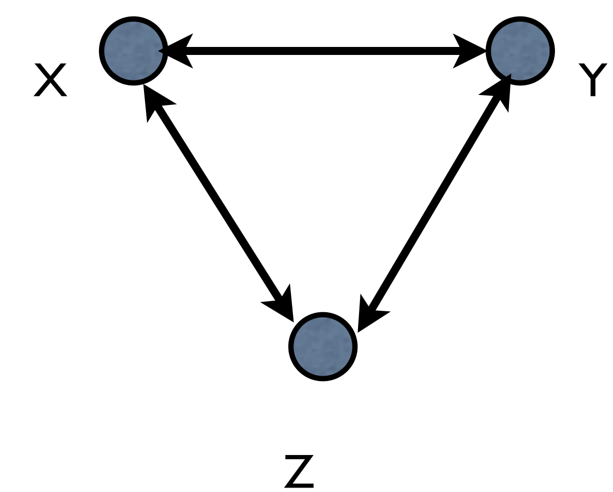
- Which property does the diagrammed relation have?
- (a) reflexive
- (b) antisymmetric
- (c) transitive
- (d) none of these



Not the Answer

Clicker Answer #1

- Which property does the diagrammed relation have?
- (a) reflexive (no, missing loops)
- (b) antisymmetric (it's symmetric)
- (c) transitive (xy and yx, not xx)
- *(d) none of these*



Defining an Equivalence Relation

- We'll soon look at partial orders, which are reflexive, antisymmetric, and transitive. Now we look at **equivalence relations**: binary relations on a set that are reflexive, symmetric, and transitive.

- Recall the definitions:
  R is **reflexive** if $\forall x: R(x, x)$,
  R is **symmetric** if $\forall x: \forall y: R(x, y) \rightarrow R(y, x)$,
  and R is **transitive** if
  $\forall x: \forall y: \forall z: (R(x, y) \wedge R(y, z)) \rightarrow R(x, z)$.

Defining an Equivalence Relation

- You should be familiar with these properties of the equality relation: "x = x" is always true, from "x = y" we can get "y = x", and we know that if x = y and y = z, then x = z.
  The idea of equivalence relations is to formalize the property of acting like equality in this way.
- To prove that a relation is an equivalence relation, we formally need to use the Rule of Generalization, though we often skip steps if they are obvious.

Some Equivalence Relations

- If A is any set, we can define the **universal relation** U on A to *always be true*. Formally, U is the entire set $A \times A$ consisting of all possible ordered pairs.

- Of course $U(x, x)$ is always true, and the implications in the definitions of symmetry and transitivity are always true because their conclusions are true.

- The **always false** relation $\neg U$ (or $\emptyset$) is symmetric and transitive but not reflexive (unless the set A is empty).

More Equivalence Relations

- The **parity relation** on naturals is perhaps more interesting. We define P(i, j) to be true if i and j are either both even or both odd. Later we'll call this "being congruent modulo 2" and we'll define "being congruent modulo n" in general.
- Any relation of the form "x and y are the same in this respect" will normally be reflexive, symmetric, and transitive, and thus be an equivalence relation.

Clicker Question #2

- S is the set of the fifty states of the Union. Which of the following is *not* an equivalence relation?
- (a) A = {(x, y): state x and y both have a capital}
- (b) B = {(x, y): states x and y border one another}
- (c) C = {(x, y): states x and y border the same number of states}
- (d) D = {(x, y): states x and y both have borders with other states or both don't }

Not the Answer

Clicker Answer #2

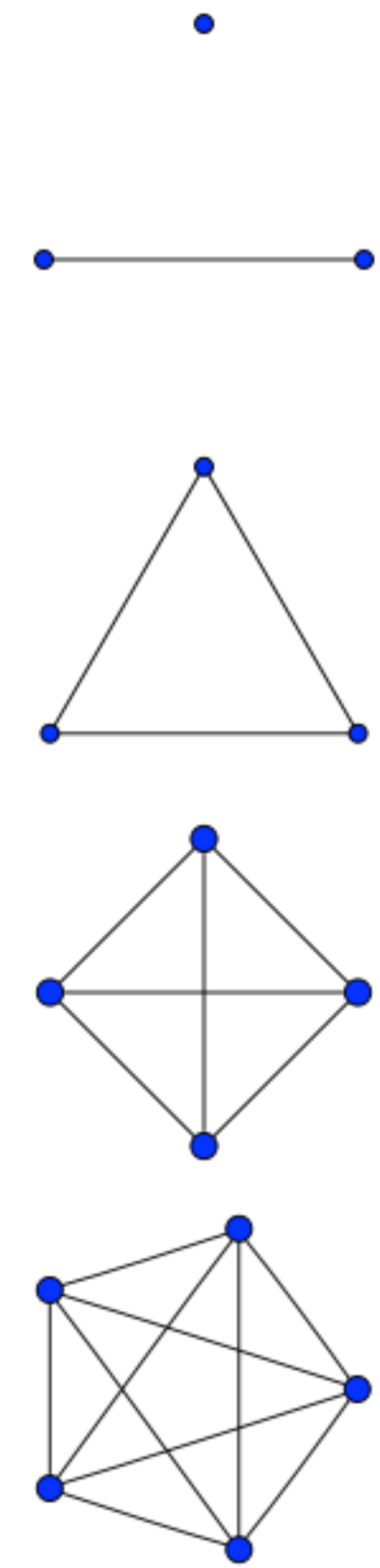
- S is the set of the fifty states of the Union. Which of the following is *not* an equivalence relation?
- (a) A = {(x, y): state x and y both have a capital}
- (b) B = {(x, y): states x and y border one another} not transitive, e.g., MA, NY, PA
- (c) C = {(x, y): states x and y border the same number of states}
- (d) D = {(x, y): states x and y both have borders with other states or both don't }

Graphs of Equivalence Relations

- What happens when we draw the diagram of an equivalence relation?
- Because it is reflexive, we have a loop on every vertex, but we can leave those out for clarity. The arrows are bidirectional because the relation is symmetric.
- The effect of transitivity on the diagram is a bit harder to see.

Complete Graphs

- If we have a set of points that have some connection from each point to each other point, transitivity forces us to have all possible direct connections among those points.
- A graph with all possible undirected edges is called a **complete graph** on its points. The graph of an equivalence relation has a complete graph for each **connected component**.



Partitions

- We've claimed a characterization of the graph of any equivalence relation, using complete graphs. Let's prove that this characterization is correct -- we will need a new definition.

- If A is any set, a **partition** of A is a set of subsets of A -- a set $P = \{S_1, S_2, \ldots, S_k\}$ where
(1) each S_i is a subset of A,
(2) the union of all the S_i's is A, and
(3) the sets are **pairwise disjoint** --
$\forall i: \forall j: (i \neq j) \rightarrow (S_i \cap S_j = \emptyset)$.

Clicker Question #3

- Let D be the set {Cardie, Duncan, Mia, Scout}. Which of these sets of sets *is not* a partition of D?
- (a) {{Mia}, {Cardie}, {Duncan}, {Scout}}
- (b) {{Mia,Duncan}, {Scout,Cardie}}
- (c) {{Duncan, Cardie}, Mia, Scout}
- (d) {{Cardie, Duncan, Mia, Scout}}

Not the Answer

Clicker Answer #3

- Let D be the set {Cardie, Duncan, Mia, Scout}. Which of these sets of sets *is not* a partition of D?
- (a) {{Mia}, {Cardie}, {Duncan}, {Scout}}
- (b) {{Mia,Duncan}, {Scout,Cardie}}
- *(c) {{Duncan, Cardie}, Mia, Scout}* *not set of sets*
- (d) {{Cardie, Duncan, Mia, Scout}}

The Partition Theorem

- The **Partition Theorem** relates equivalence relations to partitions. It says that a relation is an equivalence relation if and only if it is the "same-set" relation of some partition. In symbols, the same-set relation of P is given by the predicate SS(x, y) defined to be true if $\exists i: (x \in S_i) \wedge (y \in S_i)$.
- So we need to get a partition from any equivalence relation, and an equivalence relation from any partition.

“Same-Set” is an E.R.

- Let $P = \{S_1, S_2, \ldots, S_k\}$ be a partition of A and let SS be its same set relation. We need to show that SS is an equivalence relation.
- It is easy to check that SS is reflexive, symmetric, and transitive by working with the definition.
- Of course any element x is in the same set as itself. So SS is reflexive.

“Same-Set” is an E.R.

- Recall that $P = \{S_1, S_2, \ldots, S_k\}$ is a partition of A and that SS is its same-set relation. We are showing that SS is an equivalence relation.
- If x is in the same set as y, then y is also in the same set as x. So $SS(x, y) \rightarrow SS(y, x)$ and SS is symmetric.
- If x and y are in the same set, and so are y and z, then x and z are also in the same set. (The element y is in some set, and x and z are both in that same set.) So SS is transitive.

Equivalence Classes

- If R is an equivalence relation on A, and x is any element of A, we define the **equivalence class** of x, written [x], as the set {y: R(x, y)}, that is, the set of elements of A that are related to x by R.
- The universal relation U has a single equivalence class consisting of all the elements. The equality relation has a separate equivalence class for each element.

Equivalence Classes

- In the parity relation, the set of even numbers forms one equivalence class and the set of odd numbers forms another.
- If we let A be the set of people in the USA, and define R(x, y) to mean "x and y are legal residents of the same state", we get fifty equivalence classes, one for each state. One of them is {x: x is a legal resident of Massachusetts}.

The Classes Form a Partition

- To finish the proof of the Partition Theorem, we must prove that if R is any equivalence relation on A, the set of equivalence classes forms a partition.
- Clearly the classes are a set of sets of elements of A, and every element is in at least one class because it is in its own class (by reflexivity). So the union of the classes must be exactly A.

The Classes Form a Partition

- It remains to show that the classes are pairwise disjoint.
- This is done in the text, where we show that if two equivalence classes [a] and [b] share a member, they have exactly the same elements and are thus equal.
- If x is in both [a] and [b], we then know that R(a, x) and R(b, x) are both true. We can get R(x, b) by symmetry and then R(a, b) by transitivity.

The Classes Form a Partition

- Now that we know $R(a, b)$, we must prove that $[a] = [b]$.
- Let y be any element. Then $R(a, y)$ implies that $R(y, a)$ by symmetry. $R(y, a)$ and $R(a, b)$ together imply $R(y, b)$ by transitivity, and this gives $R(b, y)$ by symmetry.
- So $R(a, y) \to R(b, y)$ and we can prove $R(b, y) \to R(a, y)$ the same way, and thus we have that $R(a, y) \leftrightarrow R(b, y)$ and so $[a] = [b]$.