COMPSCI 250: Fall 2023
Homework 2

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Due Date : Friday, October 6

This assignment has 9 problems. There is also 1 Extra Credit problem. The extra credit is 10 points.

Please submit a single PDF file, with the problems in order (as below), and legible. Look at your PDF before submitting it – it is fine to scan or photograph a handwritten document but it the graders can't read it, they won't grade it.

Please assign pages to problems in Gradescope. Graders will click on the problem number. If no page shows up because it's not assigned, the assumption is you have not solved the problem.

Be sure you are doing Problems in the book and not Exercises: the numbers should start with P rather than E.

For full credit, show your work, explaining your reasoning. This also helps assign partial credit.

You are responsible for following the academic honesty guidelines on the Grading and Requirements page. This means that what you present must be your own work in presentation, and you must acknowledge all sources of aid other than course staff and the textbook. You will get 2 extra points if you typeset your Homework.

(8 points) **Problem 2.1.1**

Let A be any set. What are the direct products $\emptyset \times A$ and $A \times \emptyset$? If x is any thing, what are the direct products $A \times \{x\}$ and $\{x\} \times A$? Justify your answers.

(10 points) **Problem 2.1.5**

Let n be a natural and let $I(x)$ be a unary relation on the set $\{0, \ldots, n-1\}$. Let w be the binary string of length n that has 1 in position x whenever $I(x)$ is true and 0 in position x when $I(x)$ is false. (As in Java, we consider the positions of the letters in the string to be numbered starting from 0.) What is the string corresponding to the predicate $I(x)$ meaning "x is an even number" in the case where $n = 5$? The case where $n = 8$? If w is an arbitrary string and $I(x)$ the corresponding unary predicate, describe the set corresponding to the predicate in terms of w.

(12 points) **Problem 2.3.2**

Suppose that for any unary predicate P on a particular type T , you know that the proposition $(\exists x : P(x)) \leftrightarrow (\forall x : P(x))$ is true. What does this tell you about T? Justify your answer – state a property of T and explain why this proposition is always true if T has your property, and not always true if T does not have your property.

(12 points) **Problem 2.5.6**

Suppose that A is a language such that $\lambda \notin A$. Let w be a string of length k. Show that there exists a natural i such that for every natural $j > i$, every string in A^j is longer than k. Explain how this fact can be used to decide whether w is in A^*.

(14 points) **Problem 2.6.3**

Heinlein's second puzzle has the same form as in Problem 2.6.2. Here you get to figure out what the intended conclusion is to be, and prove it as above:

- Everything, not absolutely ugly, may be kept in a drawing room;
- Nothing, that is encrusted with salt, is ever quite dry;
- Nothing should be kept in a drawing room, unless it is free from damp;
- Time-traveling machines are always kept near the sea;
- Nothing, that is what you expect it to be, can be absolutely ugly;
- Whatever is kept near the sea gets encrusted with salt.

(10 points) **Problem 2.8.1**

Let $A = \{1, 2\}$ and $B = \{x, y\}$. There are exactly sixteen different possible relations from A to B. List them. How many are total? How many are well-defined? How many are functions? How many are neither well-defined nor total?

(10 points) **Problem 2.9.3**

Let $f : A \to B$ and $g : B \to C$ be functions such that $g \circ f$ is a bijection. Prove that f must be one-to-one and that g must be onto. Give an example showing that it is possible for neither f nor g to be a bijection.

(12 points) **Problem 2.9.7**

Let A be a set and f a bijection from A to itself. We say that f **fixes** an element x of A if $f(x) = x$.

(a) Write a quantified statement, with variables ranging over A, that says "there is exactly one element of A that f does not fix."

(b) Prove that if A has more than one element, the statement of part (a) leads to a contradiction. That is, if f does not fix x, and there is another element in A besides x, then there is some other element that f does not fix.

(12 points) **Problem 3.1.7**

A **Perfect number** is a natural that is the sum of all its proper divisors. For example, $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$. Prove that if $2^n - 1$ is prime, then $(2^n - 1)2^{n-1}$ is a perfect number. (A prime of the form $2^n - 1$ is called a **Mersenne prime**. Every perfect number known is of the form given here, but it is unknown whether there are any others.)

Extra credit:

(10 points) **Problem 2.10.6**

There is only one partial order possible on the set $\{a\}$, because $R(a, a)$ must be true. On the set $\{a, b\}$, there are three possible partial orders, as $R(a, a)$ and $R(b, b)$ must both be true and either zero or one of $R(a, b)$ and $R(b, a)$ can be true. List all the possible partial orders on the set $\{a, b, c\}$. (**Hint:** There are nineteen of them.) How many are linear orders?