# COMPSCI 250: Introduction to Computation

Lecture #36: State Elimination

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# State Elimination

- Kleene’s Theorem Overview
- Another New Model: The r.e.-NFA
- Overview of the Construction
- Eliminating a State
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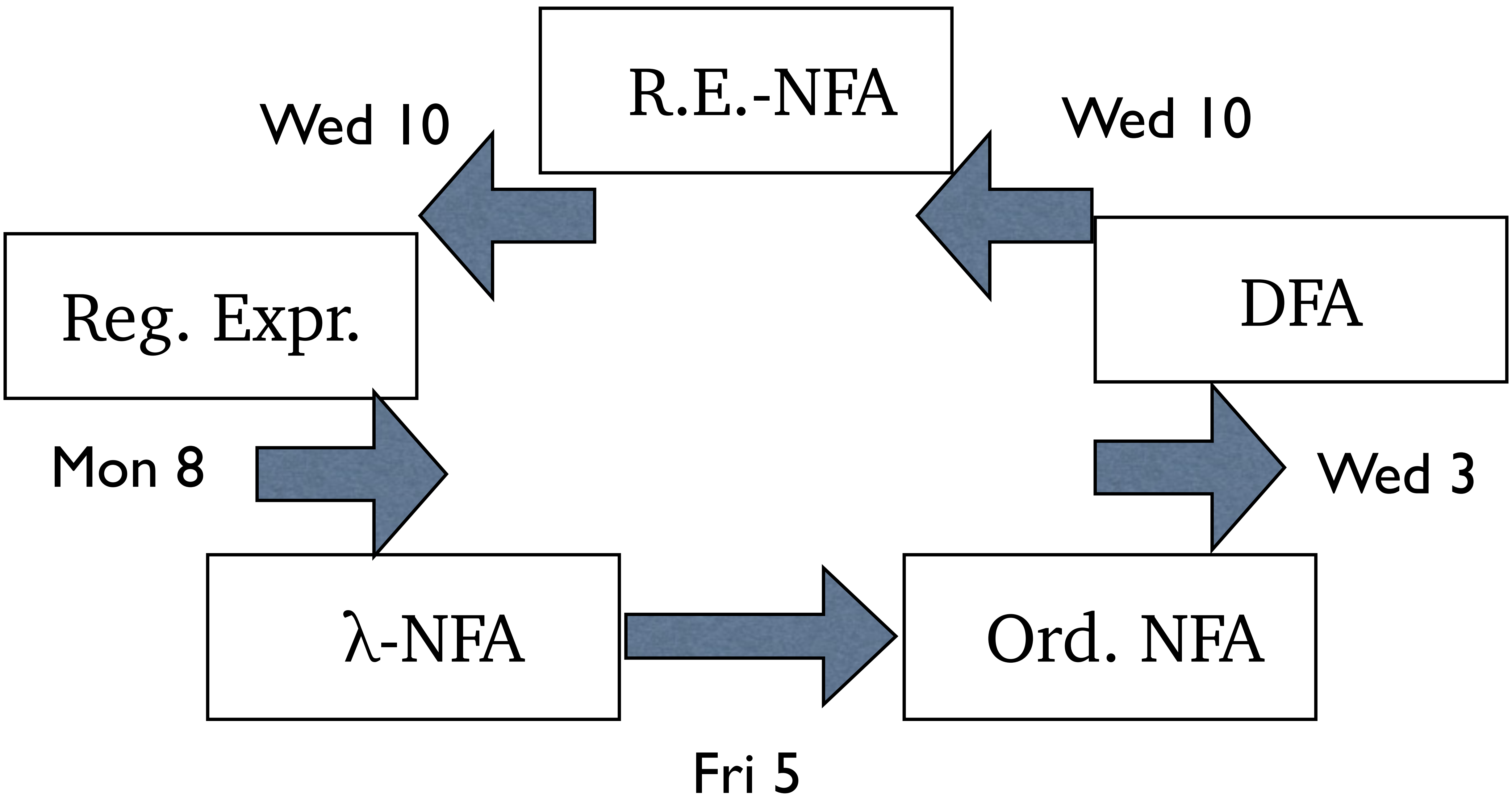
- Example: The Language No-aba
- Example: Number of a’s Divisible by 3

# Kleene's Theorem Overview

- We are finally ready to finish Kleene's Theorem, proving that a language has a regular expression if and only if it has a DFA.
- We have shown how to:
  - take a regular expression, produce a λ-NFA from it by the recursive construction,
  - kill the λ-moves to get an ordinary NFA,
  - use the Subset Construction to get a DFA,
  - and then (if we want to and know how) minimize that DFA.

# Kleene's Theorem Chart



# Final Step of Kleene's Theorem

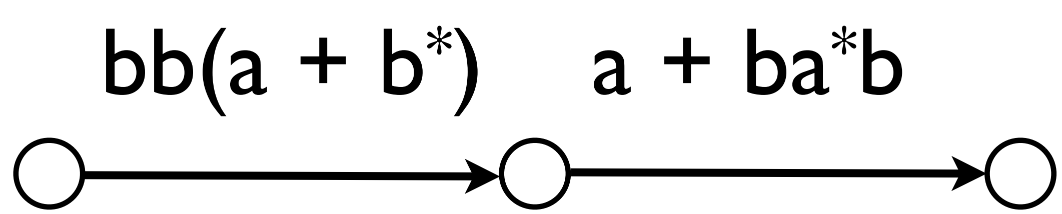
- The remaining step is to take a DFA and produce a regular expression for its language.
- As it turns out, the State Elimination Construction works equally well to get a regular expression for the language of any ordinary NFA or $\lambda$-NFA as well.

# Final Step of Kleene's Theorem

- The first two steps of converting a regular expression to a DFA roughly preserve the size.
- The Subset Construction in general takes an NFA with k states to a DFA with $2^k$ states.
- Though we won't prove this, State Elimination can also cause a large blowup, creating a long regular expression from a small DFA.
- (Excursion 14.11 in the text takes a closer look at this.)

# Another Model: The R.E.-NFA

- The State Elimination Construction operates on yet another kind of NFA, which we will call an r.e.-NFA because the labels on its moves can be arbitrary regular expressions instead of just letters (as in an ordinary NFA) or either letters or λ (as in a λ-NFA).



# Normal Form for R.E.-NFA's

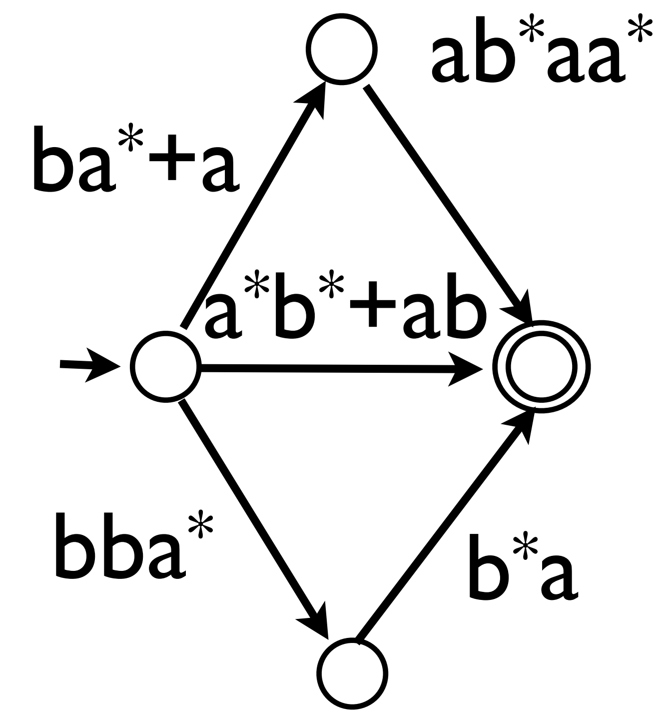
- Not every diagram with regular expressions on its edges is an r.e.-NFA -- we need to satisfy some rules.
- The first three are the same as the rules in our construction of $\lambda$-NFA's from regular expressions:
- (1) Exactly one final state, not equal to the start state,
- (2) No moves into the start state, and
- (3) No moves out of the final state.

# Normal Form for R.E.-NFA's

- The last rule is new: (4) no **parallel edges**, that is, no two edges with the same start node and end node.
- We have to redefine the $\Delta^*$ relation. We still have $\forall s$: $\Delta^*(s, \lambda, s)$, but now we have the rule $[\Delta^*(s, v, u) \wedge \Delta(u, R, t) \wedge (w \in L(R))] \rightarrow \Delta^*(s, vw, t)$.
- This rule isn't very useful for computing, as we have no equivalent top-down form for it.

# Clicker Question #1

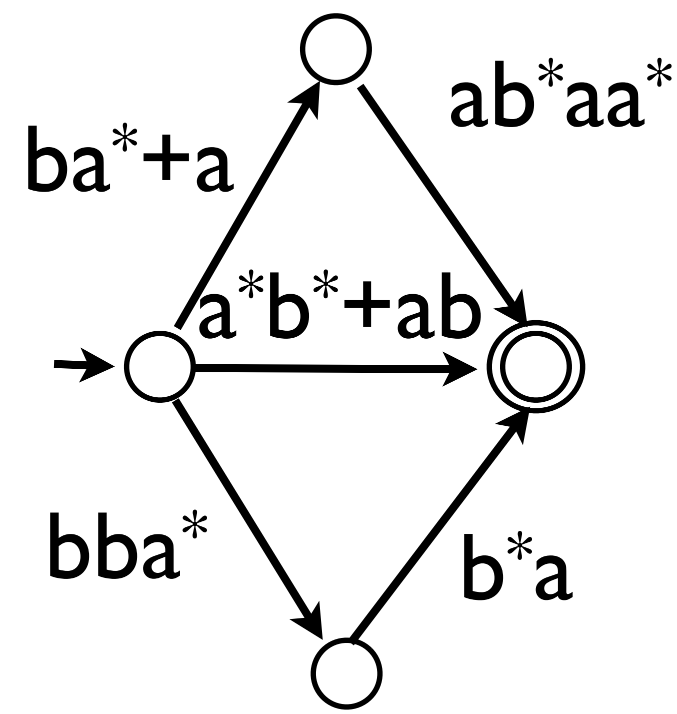
- Which of these strings *is not* in the language of the r.e.-NFA pictured at right?
- (a) aaabb
- (b) baaaa
- (c) babaa
- (d) baabb



# Not the Answer

# Clicker Answer #1

- Which of these strings *is not* in the language of the r.e.-NFA pictured at right?
- (a) aaabb  center
- (b) baaaa  top
- (c) babaa  top
- *(d) baabb*  none of them can start with b and end in b



# Overview of the Construction

- The basic idea is to take our original DFA (or NFA, or λ-NFA), modify it so that it obeys the r.e.-NFA rules but still has the same language (how?) and then **eliminate states** one by one until there are only two left.
- Each elimination will preserve the language of the automaton and ensure that the r.e.-NFA rules still hold.

# Overview of the Construction

- An r.e.-NFA with two states must have one of them as the start state and the other as the only final state, by rule (1).
- By rules (2), (3), and (4), there can be only one edge, going from the start state to the final state, and the only possible path from the start state to a final state has exactly one edge, this one.
- This edge is labeled by a regular expression R, and the language of the r.e.-NFA is exactly L(R).

# Overview of the Construction

- Thus L(R) is also the language of the original DFA.
- The states we eliminate are every state except the start state and final state.
- We can eliminate them in any order and get a *correct* final regular expression, but if we choose the order wisely we may get a simpler regular expression.

# Eliminating a State

- Suppose we have a state q that is neither initial nor final, and we want to eliminate it.
- We don't care about paths that *start* or *end* at q, because the language is defined only in terms of paths that start at the initial state and end at the final state.
- To safely delete q, we have to *replace* any two-step path, that had q as its middle node, by a single edge.

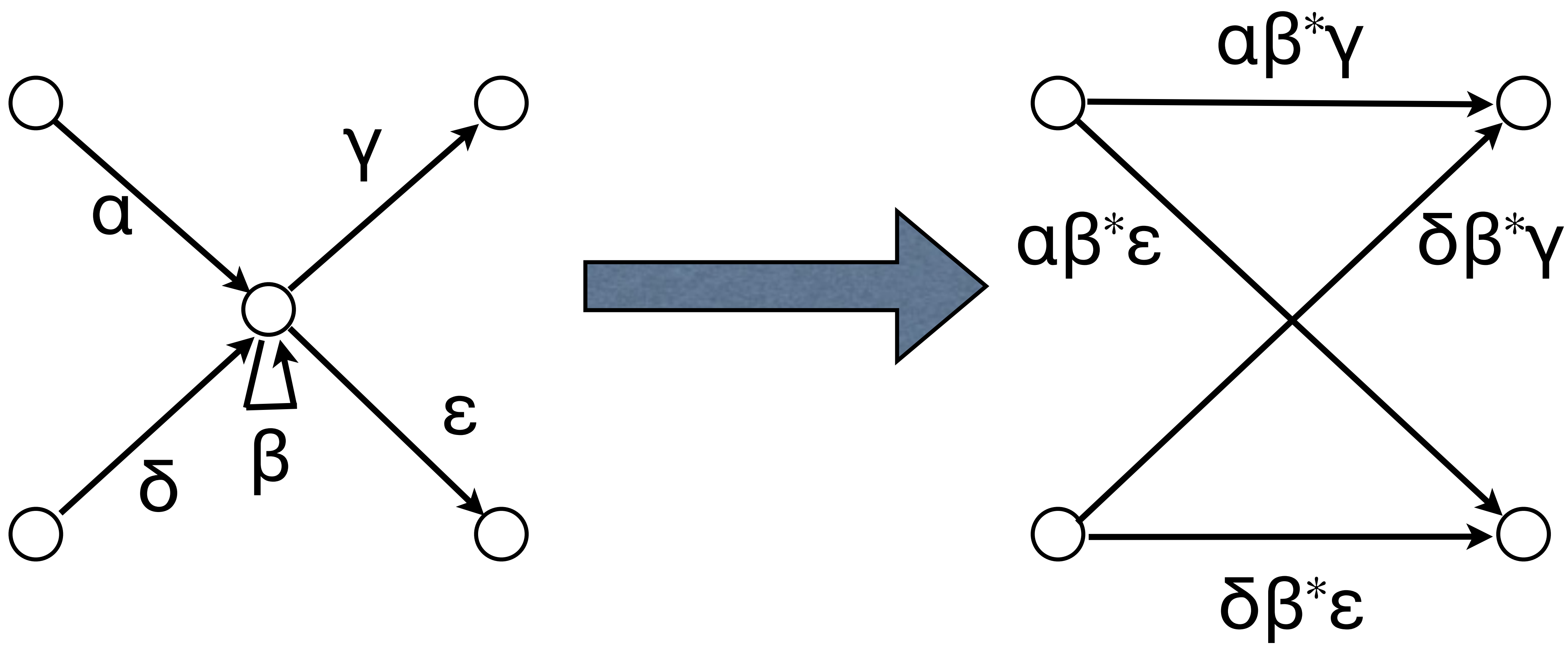
# Eliminating a State

- If $(p, \alpha, q)$ and $(q, \beta, r)$ are any two edges, and $(q, \gamma, q)$ is the loop on q, then when we delete q we must add a new edge $(p, \alpha\gamma^*\beta, r)$.
- (Here $\alpha$, $\beta$, and $\gamma$ are regular expressions. Note also that $p = r$ is possible.)
- If there is already an edge from p to r, though, we add the new edge by changing the existing $(p, \delta, r)$ to $(p, \delta + \alpha\gamma^*\beta, r)$.

# Eliminating a State

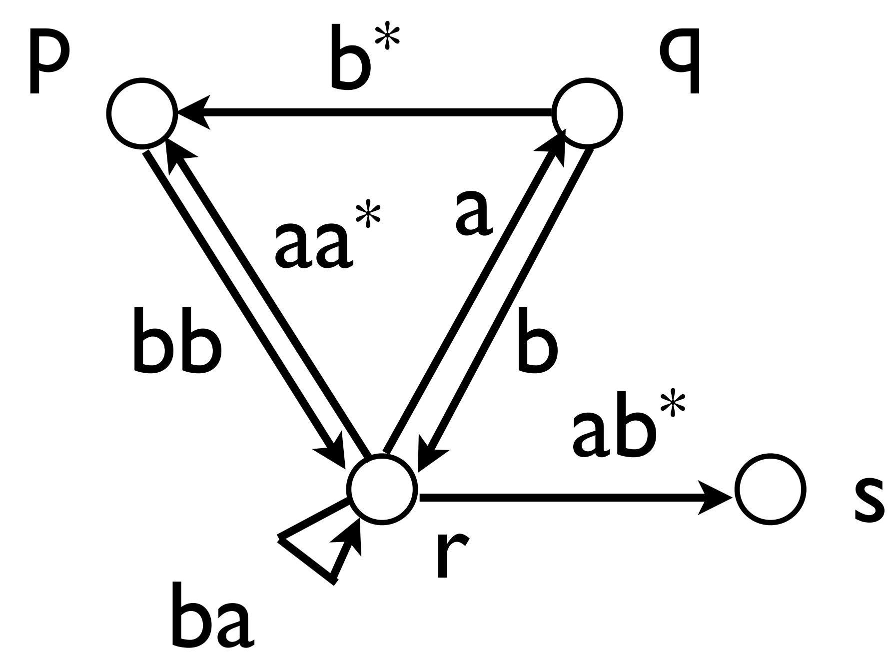
- Note that if there is no loop on q we can take $\gamma$ to be $\varnothing$ and then $\gamma^* = \varnothing^*$ which is the identity for concatenation, so that $\alpha\gamma^*\beta = \alpha\beta$.
- When we delete q, we should count all the m edges into q and all the n edges out of q, and make sure that we have added mn new edges. The loop on q, if it exists, does not count toward either m or n.

# A General Example



# Clicker Question #2

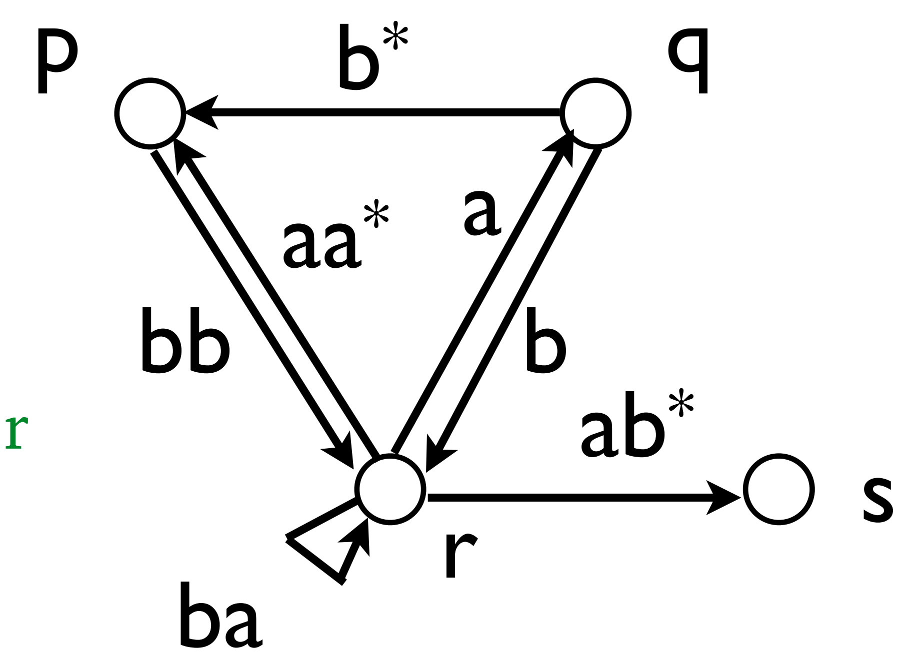
- Which transition *will* be present in the new r.e.-NFA if we eliminate state r?
- (a) (q, ba, q)
- (b) (q, b(ba)$^*$aa$^*$, p)
- (c) (q, b(ba)$^*$aa$^*$ + b$^*$ , p)
- (d) (r, (ba)$^*$, r)



Not the Answer

# Clicker Answer #2

- Which transition *will* be present in the new r.e.-NFA if we eliminate state r?



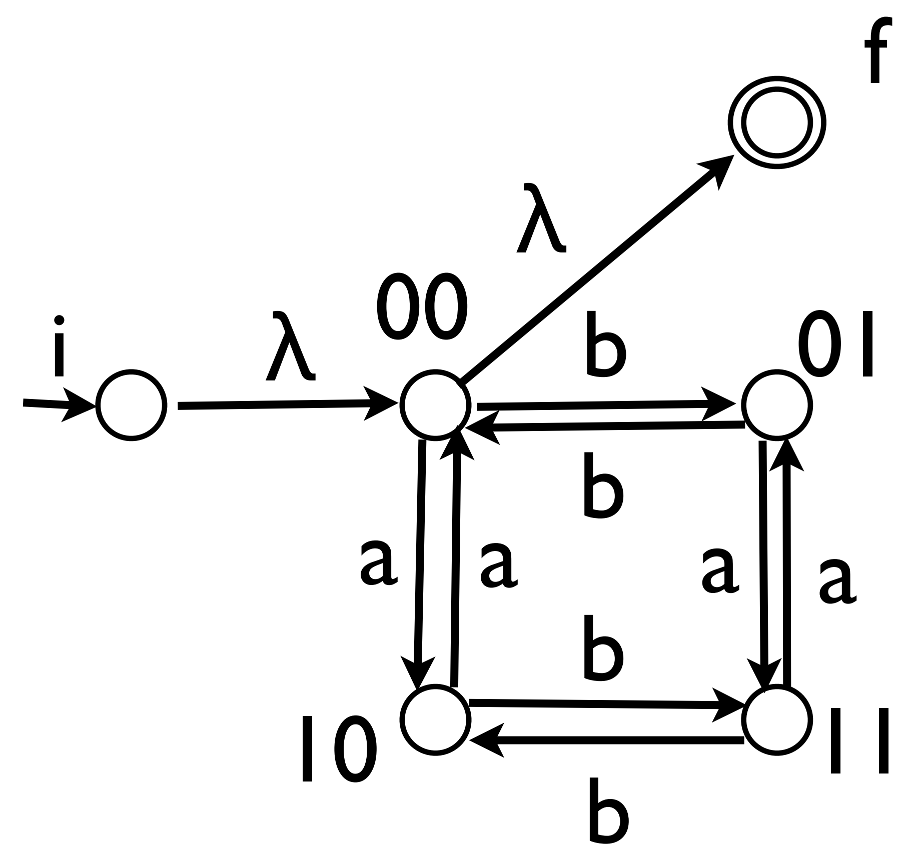

- (a) (q, ba, q) Missing former loop on r
- (b) (q, $b(ba)^*aa^*$, p) Missing old transition on $b^*$
- *(c) (q, $b(ba)^*aa^* + b^*$, p)*
- (d) (r, $(ba)^*$, r) State r is being eliminated

# Example: The Language EE

- In Discussion #8, we designed a regular expression for the language EE, of strings over {a, b} that have both an even number of a's and an even number of b's.
- We'll now use State Elimination to get such an expression from a DFA.
- The natural DFA has state set {00, 01, 10, 11}. Here 00 is the start state and the only final state, a's change the first bit of the state, and b's change the second bit.

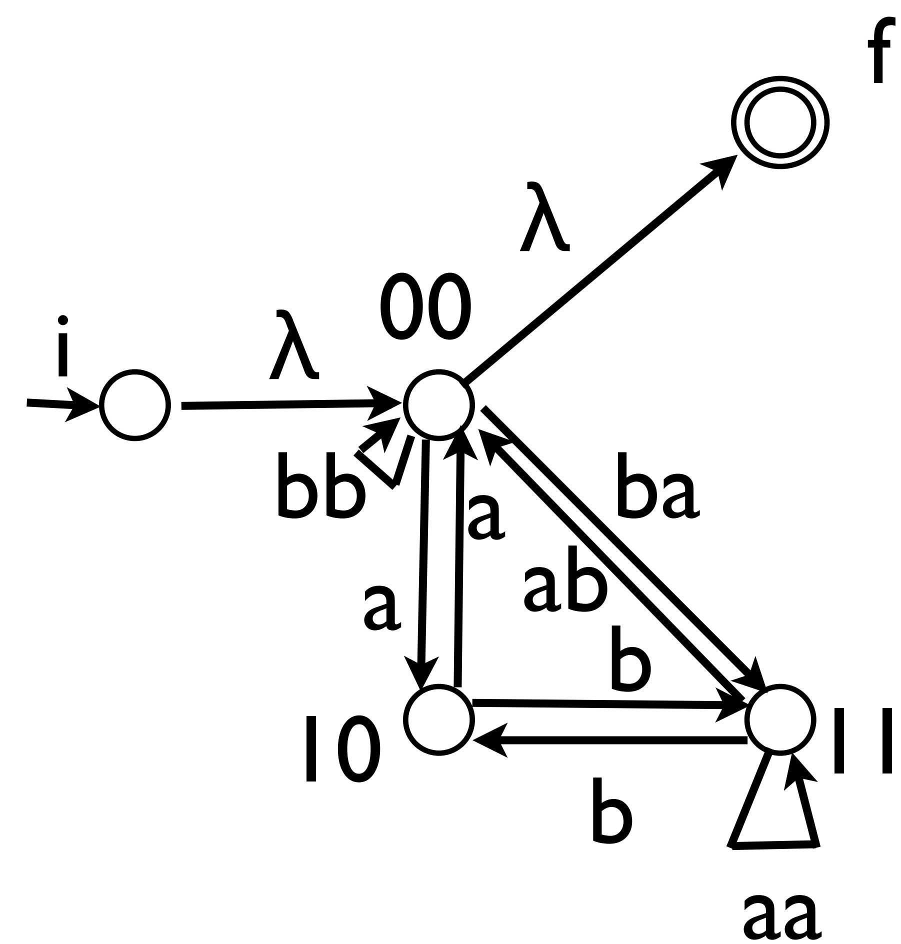
# Example: The Language EE

- But this DFA violates the rules for an r.e.-NFA -- we have to add a new start state i and a new final state f, and add transitions (i, λ, 00) and (00, λ, f).
- Now all we have to do is eliminate four states to get our regular expression.



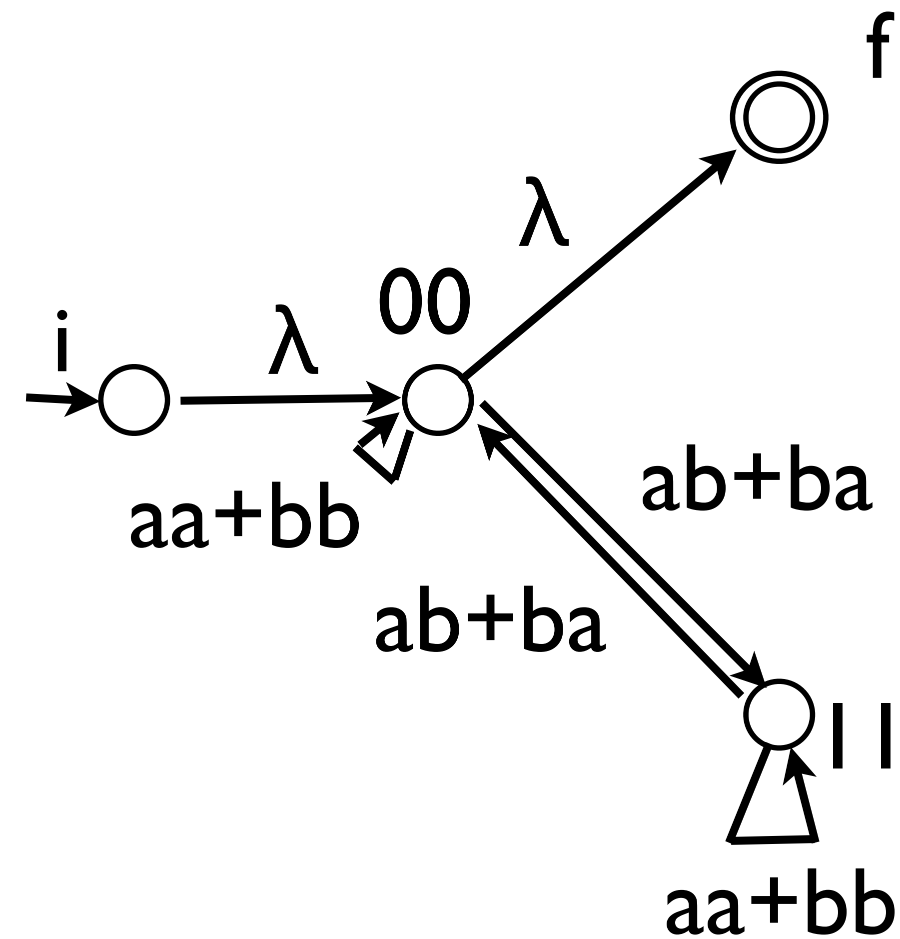
# Example: The Language EE

- We begin by killing 01, which has two edges in and two out.
- We need four new edges: (00, bb, 00), (00, ba, 11), (11, ab, 00), and (11, aa, 11).



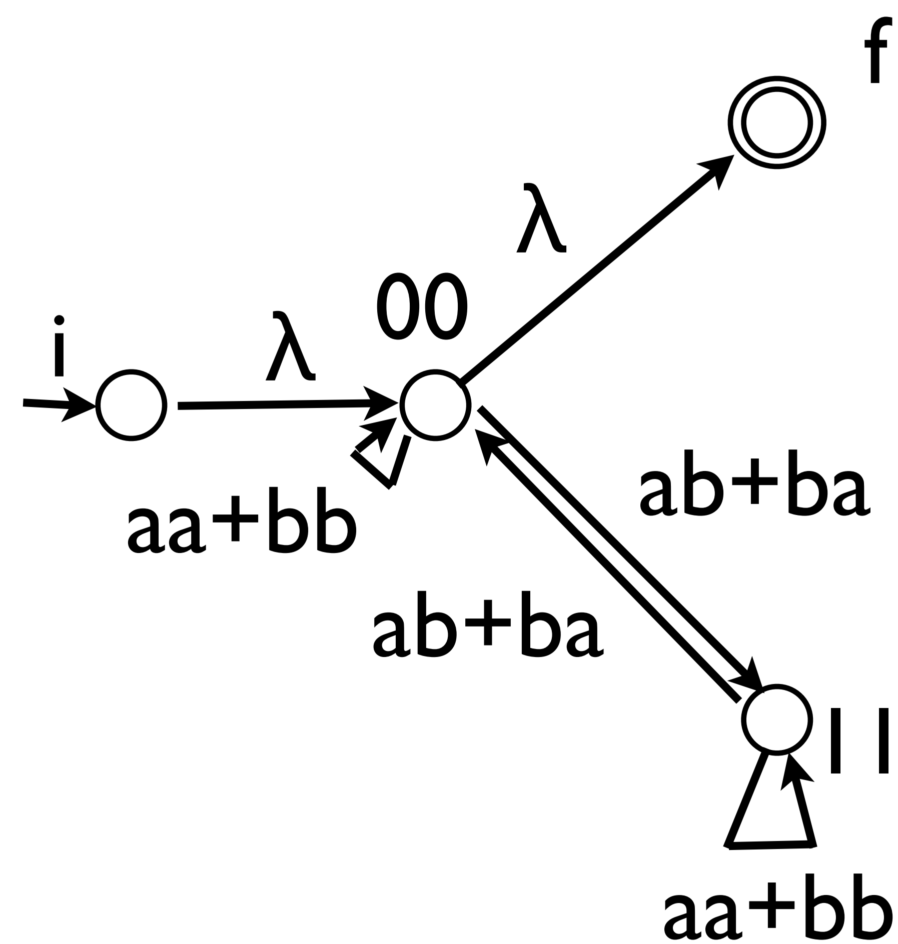
# Example: The Language EE

- Next we eliminate 10 (which looks like a good idea as it has no loop and fewer edges).
- Again we get four new edges, each of which is parallel to an existing edge: (00, aa+bb, 00), (00, ab+ba, 11), (11, ab+ba, 00), and (11, aa+bb, 00).
- This gives us four states.



# Clicker Question #3

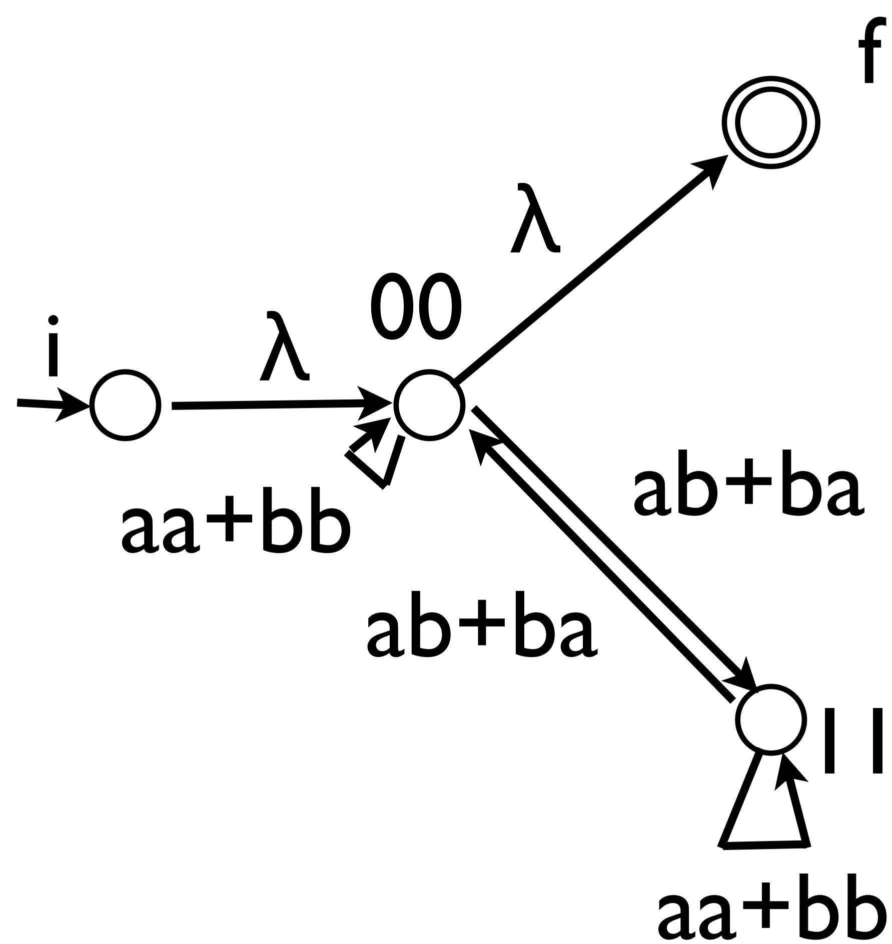
- If we now eliminate 00, which of these transitions *would not* appear in the new r.e.-NFA?
- (a) (i, aa+bb+(ab+ba)(aa+bb)*(ab+ba) , f)
- (b) (i, (aa+bb)*(ab+ba), 11)
- (c) (11, (ab+ba)(aa+bb)*, f)
- (d) (i, (aa+bb)*, f)



# Not the Answer

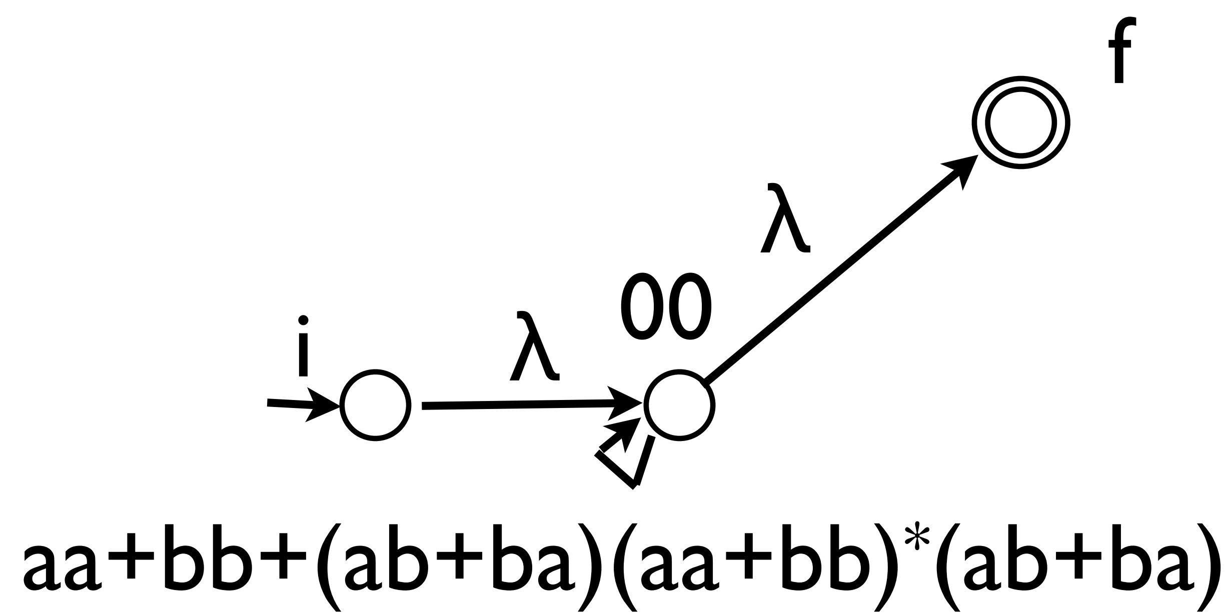
# Clicker Answer #3

- If we now eliminate 00, which of these transitions *would not* appear in the new r.e.-NFA?
- *(a) (i, aa+bb+(ab+ba)(aa+bb)*(ab+ba) , f)* also eliminates 11 at the same time
- (b) (i, (aa+bb)*(ab+ba), 11)
- (c) (11, (ab+ba)(aa+bb)*, f)
- (d) (i, (aa+bb)*, f)



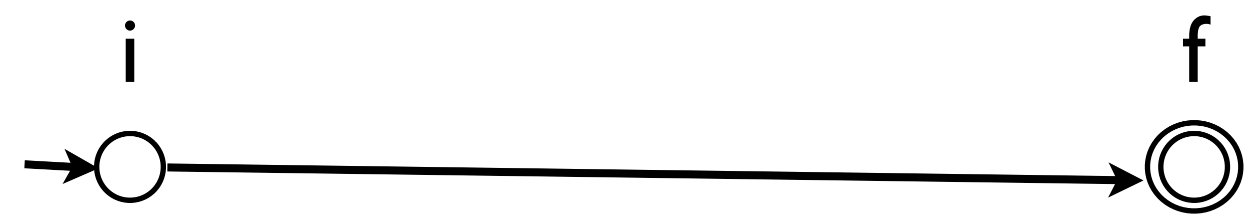
# Finishing the EE Example

- The four remaining states are i, 00, 11, and f.
- State 11 now has one edge in and one edge out, along with a loop.
- When we eliminate 11 we create only one edge, (00, (ab+ba)(aa+bb)*(ab+ba), 00).



# Finishing the EE Example

- The last state to eliminate is now 00, which also has one edge in, one edge out, and one loop.
- (Note that *any* three-state r.e.-NFA must have a form similar to this, maybe with another edge from initial to final state.)




$[aa + bb + (ab+ba)(aa+bb)^*(ab+ba)]^*$

# Finishing the EE Example

- The one edge that we create is (i, $[aa+bb+(ab+ba)(aa+bb)^*(ab+ba)]^*$, f), and our final regular expression is the label of this edge.
- We would get a grubbier, equivalent regular expression by eliminating the states in a different order.
- The expression $aa+bb+(ab+ba)(aa+bb)^*(ab+ba)$ represents the language EEP of “primitive” (non-factorable) strings in EE.

# Example: The Language No-aba

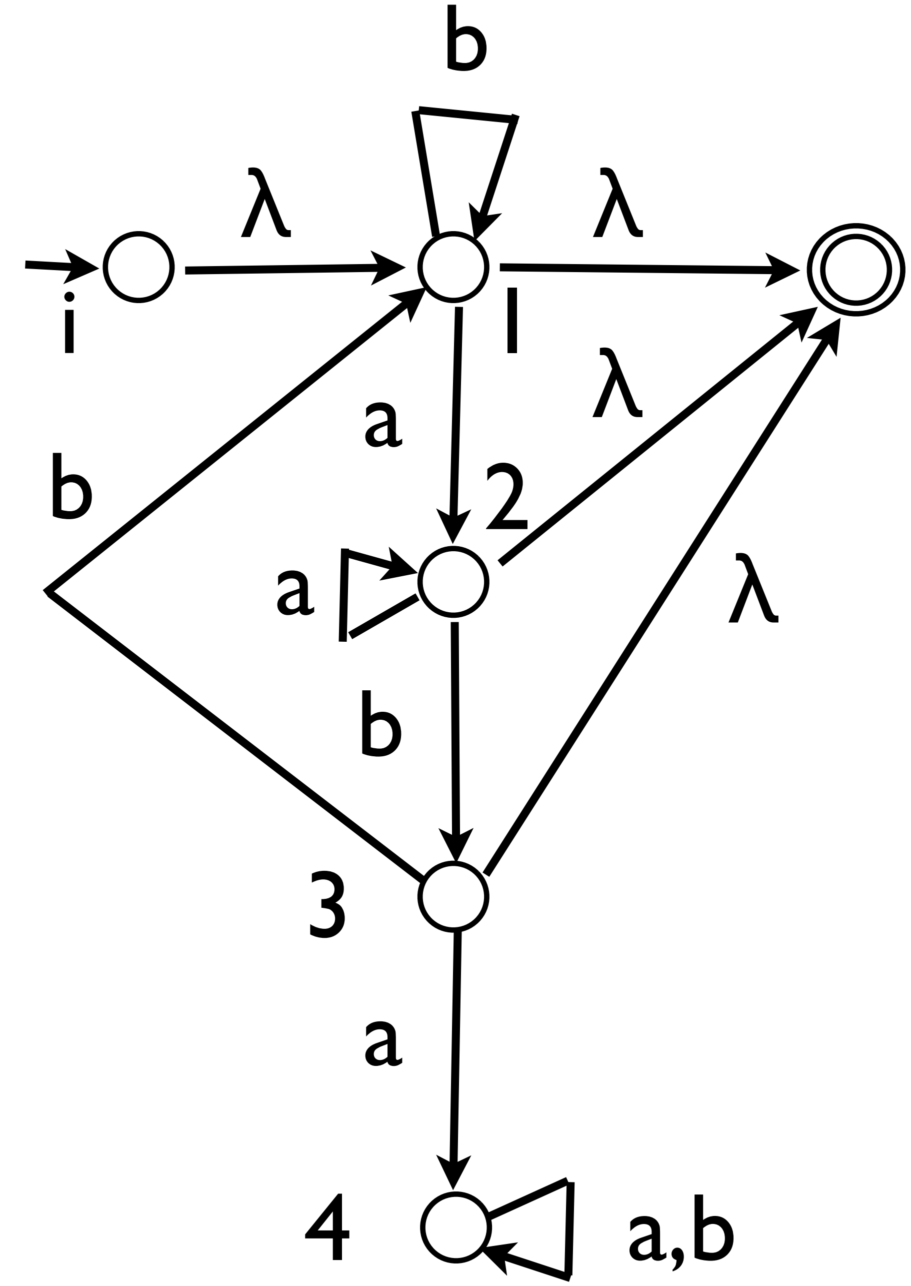
- We've seen the language Yes-aba = $\Sigma^*aba\Sigma^*$ and its complement No-aba several times now. We have a four-state DFA for No-aba -- let's turn this into a regular expression.

- The state set is {1,2,3,4}, the start state 1, final state set {1,2,3}, and edges (1,a,2), (1,b,1), (2,a,2), (2,b,3), (3,a,4), (3,b,1), (4,a,4), and (4,b,4).

- Again we need new start states i and f, with new edges (i,λ,1), (1,λ,f), (2,λ,f), and (3,λ,f).

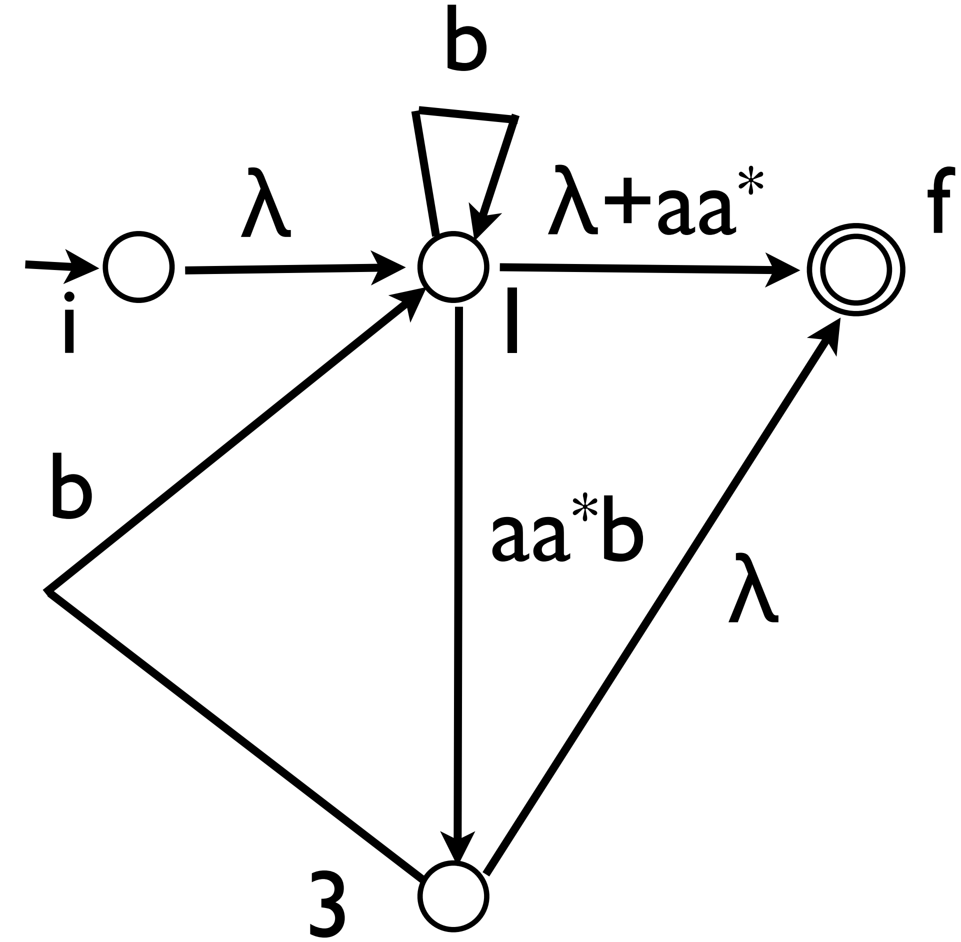
# Example: The Language No-aba

- The first thing to do is to kill state 4, which requires adding no new edges, because it has no paths through it from i to f.
- Next, state 2 looks like a good target. It has one edge in and two out, for two new edges.



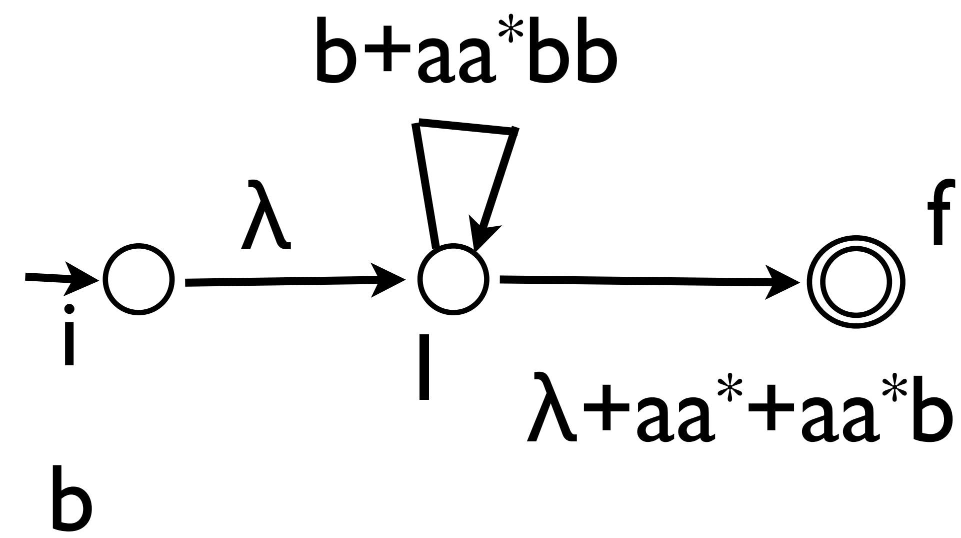
# Example: The Language No-aba

- Killing state 2 produces two new edges, (1, $aa^*b$, 2) and (1, $aa^*$, f).
- The latter edge is merged with the existing edge (1, λ, f).

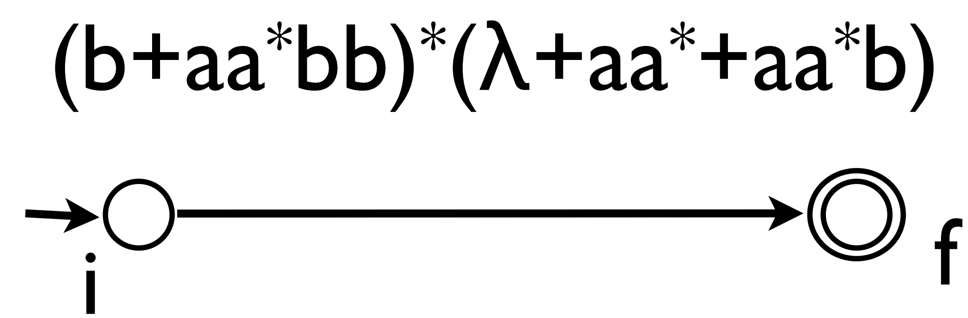


# Example: The Language No-aba

- Now if we kill state 3 we create (1, $aa^*bb$, 1) which becomes (1, $b + aa^*bb$, 1) and (1, $aa^*b$, f) which becomes (1, $\lambda + aa^* + aa^*b$, f).




- Killing state 1 gives the final expression $(b + aa^*bb)^*(\lambda + aa^* + aa^*b)$.

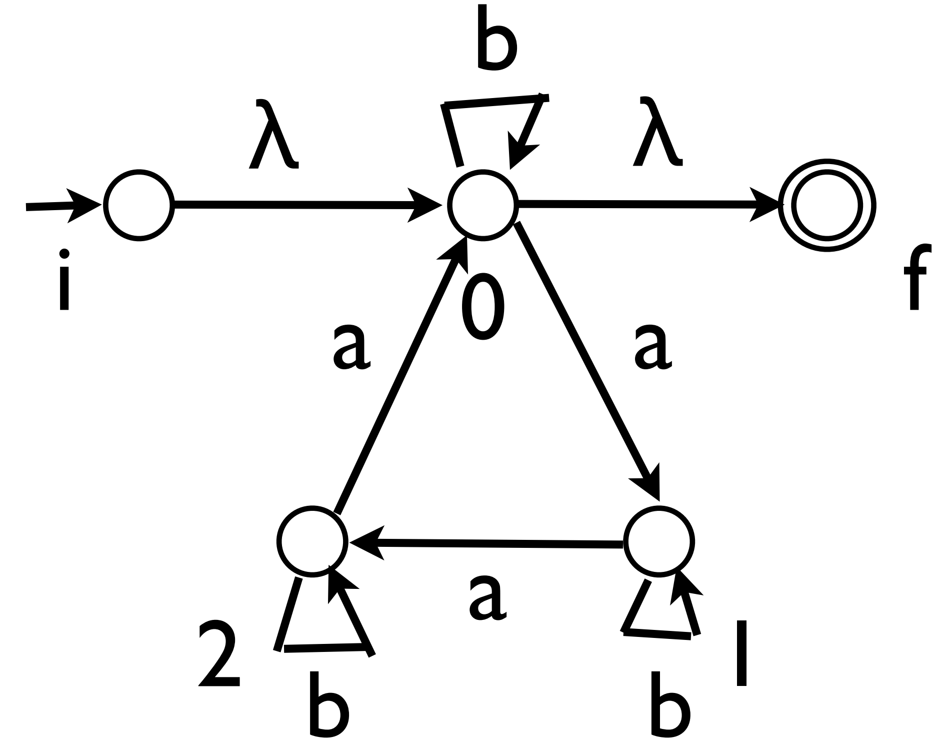


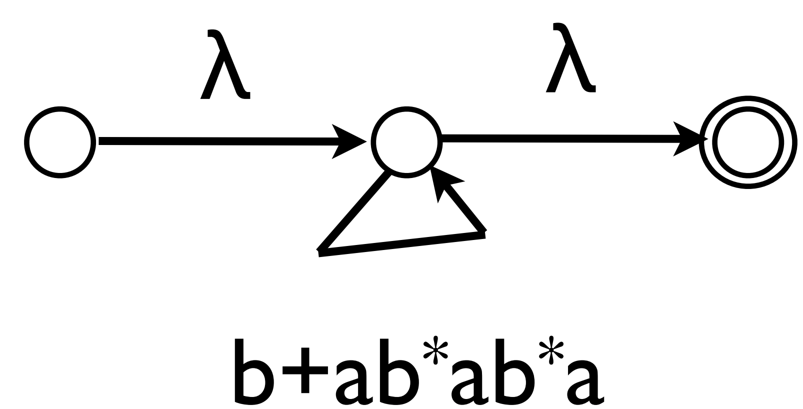
# Example: # of a's Divisible by 3

- Here's another example (Exercise 14.10.3 in the text). Let D be the language of strings over {a, b} where the number of a's is divisible by 3.
- It's clear how to make a DFA for this: states {0, 1, 2}, start state and only final state 0, edges (p, b, p) for each state p, and edges (0, a, 1), (1, a, 2), and (2, a, 0).
- To make an r.e.-NFA, we once again add a new start state i and new final state f, with edges (i, λ, 0) and (0, λ, f). We have five states now and must kill three.

# Example: # of a's Divisible by 3

- We first kill 2, creating one new edge (1, $ab^*a$, 0).
- Then killing 1 creates a new edge (0, $ab^*ab^*a$, 0), which adds to the existing (0, b, 0) to get (0, b + $ab^*ab^*a$, 0).



# Example: # of a's Divisible by 3

- Finally, killing 0 gives the expression $[b + ab^*ab^*a]^*$, which makes sense because we can break any string in D into pieces that are either b's or have exactly three a's.
- A more challenging problem is the language of strings where *both* the number of a's and the number of b's are divisible by three.
- How about the strings where the number of a's and the number of b's are congruent to *one another* modulo 3?

# Counting Mod 3

- Let i be the number of a's in the input and let j be the number of b's.
- To build a DFA for the set of strings with $i \equiv 0$ and $j \equiv 0 \pmod 3$, we need nine states.
- By making (0, 0), (1, 1), and (2, 2) all final, we can get a nine-state DFA for "$i \equiv j \pmod 3$".
- There are also five-state and three-state DFA's for this language. The three-state one keeps track of the number (i - j) % 3.