# COMPSCI 250: Introduction to Computation

Lecture #21: Induction for Problem Solving
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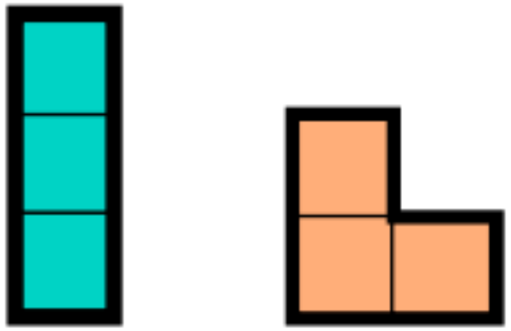
# Induction for Problem Solving

- The L-Shaped Tile Problem
- Recursively Tiling a Chessboard
- Cutting Pizzas
- The Pizza-Cutting Theorem
- Cutting a Block of Cheese With a Katana
- The Speed of the Euclidean Algorithm

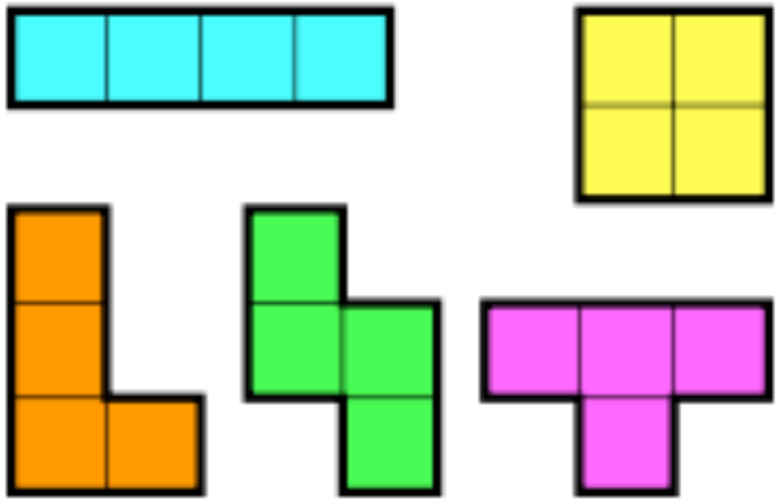
# Polyominoes

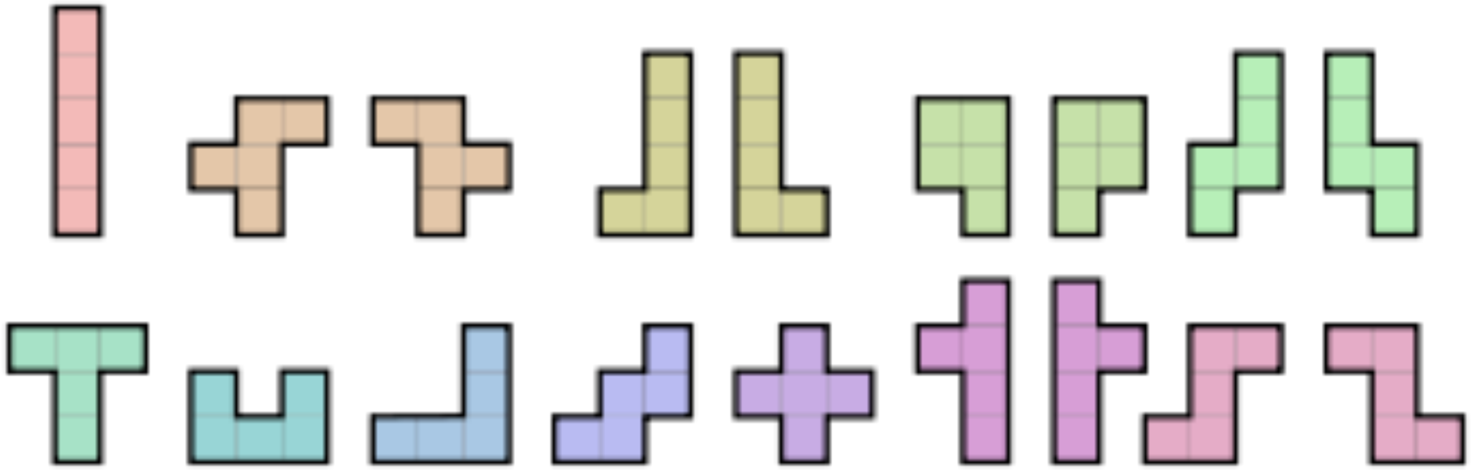
- Samuel Golomb initiated the study of generalized dominos called **polyominos**.
- An ordinary **domino** is made from two connected squares, and there is basically only one way to do it.
- A **tromino** is made from three connected squares, and there are two different ones, I-shaped and L-shaped.



# More Polyominoes

- There are five kinds of tetrominos and twelve kinds of pentominos (six of which are shown twice below, in two reflective forms).





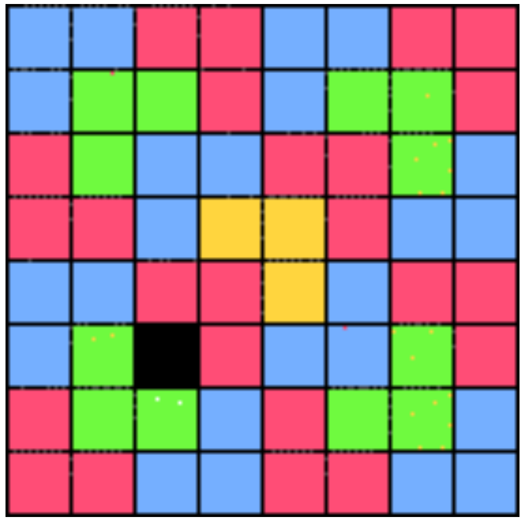
# The L-Shaped Tile Problem

- Golomb posed the question of what kinds of figures can be tiled by various kinds of polyominos.
- In particular, can an $8 \times 8$ chessboard be tiled by L-shaped trominos?
  (No, because $64 \% 3 \neq 0$.)
- What about an $8 \times 8$ board with *one square missing*?

# Recursively Tiling a Chessboard

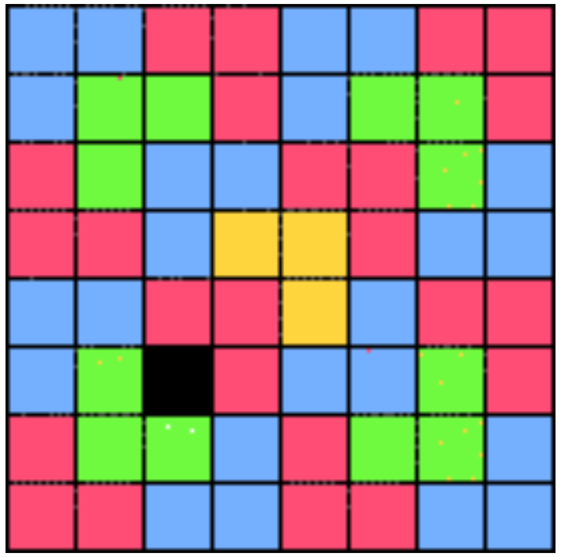
- In the figure below, an 8 × 8 board with one (black) square missing has been tiled by 21 L-shaped trominos. How did we do it, and can we always do it?

- We do it with a recursive algorithm that provides an inductive proof that **any $2^n \times 2^n$ board, with any one square missing, can be tiled**. (The bold-faced statement will be the P(n) of our inductive proof.)



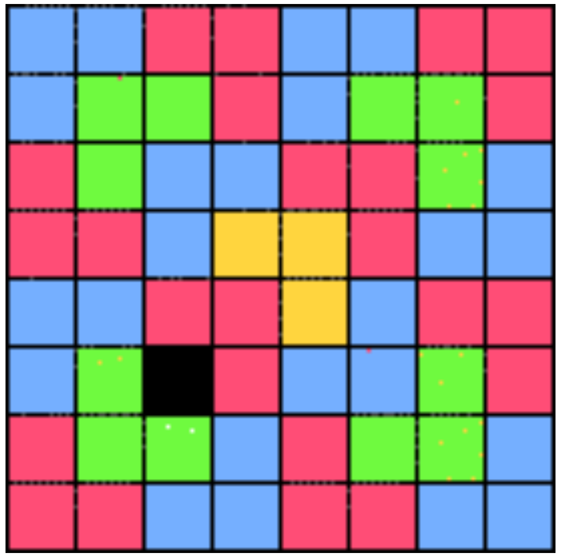
# The Recursive Tiling

- The base case of P(0) says that any 1 × 1 board, with any one square missing, can be tiled. (Use 0 tiles!)
- The key step of the recursive algorithm is to reduce a $2^{n+1} \times 2^{n+1}$ problem to four $2^n \times 2^n$ problems. We do this by placing one tile (the orange one in the figure) in a particular position.



# The Recursive Tiling

- The orange tile takes one square from three of the four quarters, the three that are not already missing a square.
- Then we recursively tile each of the quarters -- in this example by placing the green tile to make 2 × 2 boards with one square missing -- these are covered by the red and blue tiles.



# Clicker Question #1

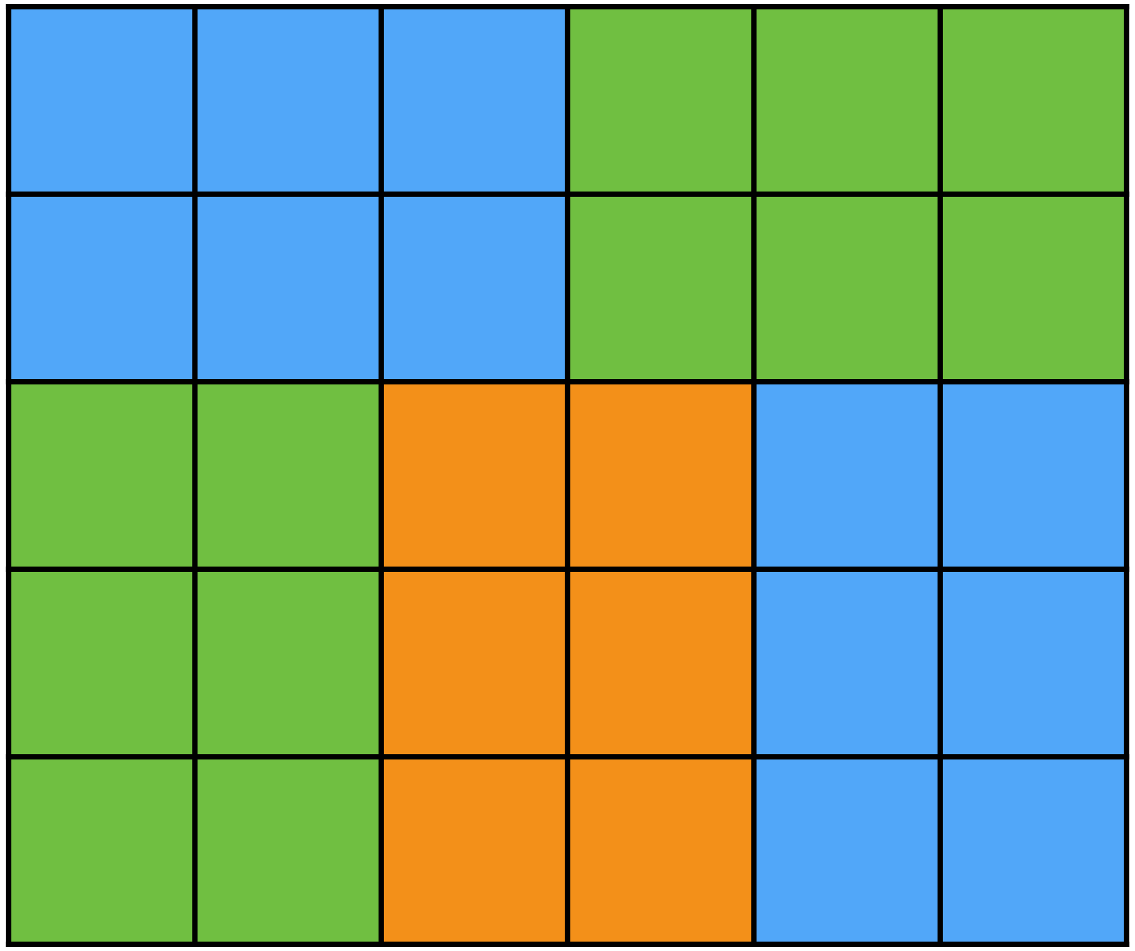
- Consider covering an n by k rectangle with rectangular 2 by 3 hexominos. Which of these four statements *is true*? (Assume n, k > 0.)
- (a) If I can do it, then there exist naturals a and b such that either ($n = 3a$ and $k = 2b$) or ($n = 2a$ and $k = 3b$).
- (b) If I can do it, each one of n and k is either divisible by 2 or divisible by 3.
- (c) If I can do it, $n \times k$ is divisible by 6.
- (d) I can always do it if 6 is a divisor of $n \times k$.

# Not the Answer

# Clicker Answer #1

- Consider covering an n by k rectangle with rectangular 2 by 3 hexominos. Which of these four statements *is true*? (Assume n, k > 0.)
- (a) If I can do it, then there exist naturals a and b such that either (n = 3a and k = 2b) or (n = 2a and k = 3b). (can have 5 by 6, see below)
- (b) If I can do it, each one of n and k is either divisible by 2 or divisible by 3. (5 by 6)
- (c) If I can do it, n × k is divisible by 6.
- (d) I can always do it if 6 is a divisor of n × k. (1 by 6)

# A 5 by 6 made from 2 by 3’s



# Cutting Pizzas

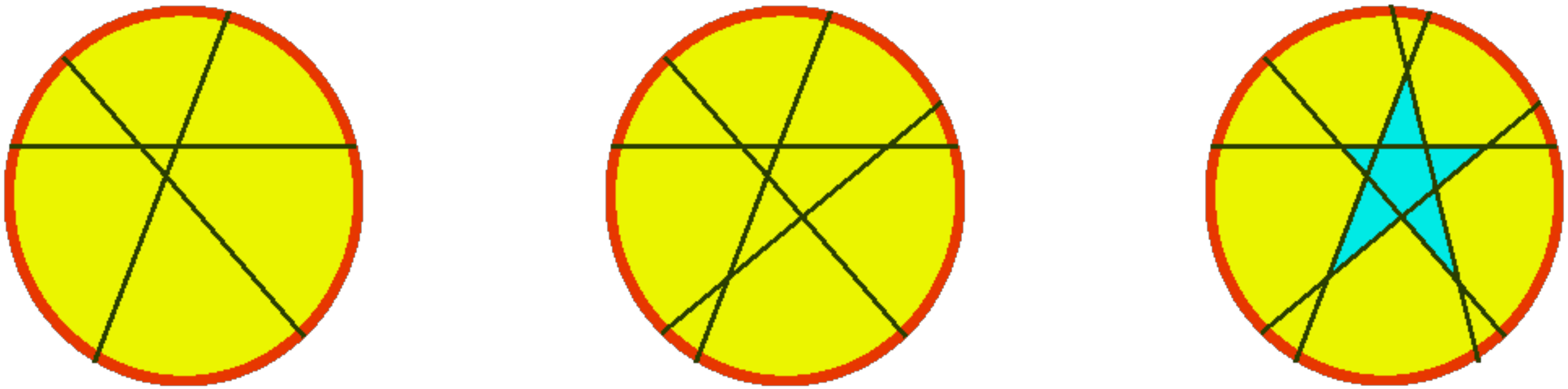
- Our next problem is to determine the maximum number of pieces into which we can divide a circular pizza, using only straight cuts in the plane of the pizza.

- With no cuts we have one piece, with one cut we can make two, and with two cuts we can make four.

- We can’t ever do better than doubling the number of pieces, because the pieces (by induction) are all convex and a single straight cut can divide a convex piece into at most two subpieces.

# Cutting Pizzas

- This leaves the possibility of getting eight pieces with three cuts, but that doesn't seem to be possible.  You can get six by having all three cuts go through a single point, or seven by having then almost do that but leave a triangle.
- Here are also pizzas cut into 11 pieces with four cuts, and 16 with five cuts:



# The Pizza-Cutting Theorem

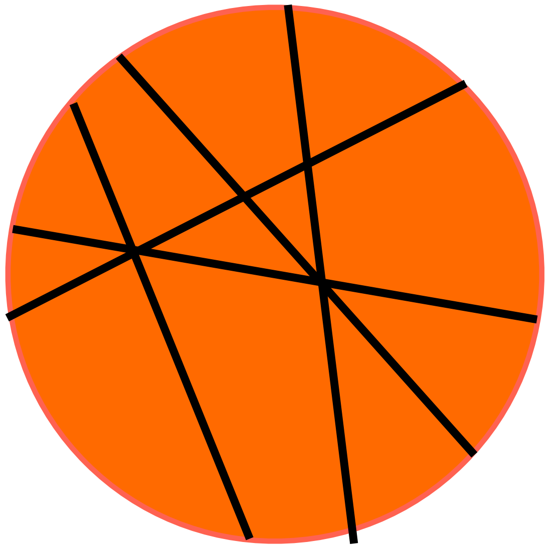
- **Theorem**: The maximum possible number of pizza pieces with n straight cuts is exactly $(n^2 + n + 2)/2$.
- We prove this theorem by ordinary induction on all naturals n.
- The base case says that we have $(0^2 + 0 + 2)/2 = 1$ piece with no cuts, which is true.

# The Pizza-Cutting Theorem

- Suppose that we have $(n^2 + n + 2)/2$ pieces with $n$ cuts, and we want to make cut $n+1$.
- By geometry, two straight lines meet in at most one point, so the new cut can cross each old cut at most once. This means that the new cut passes through at most $n + 1$ old pieces.
- Those pieces are divided into two and the others are not. So after $n + 1$ cuts we have at most $(n^2 + n + 2)/2 + (n + 1) = (n^2 + 3n + 4)/2 = ((n+1)^2 + (n+1) + 2)/2$, *at most* the given number.

# Clicker Question #2

- Suppose we make the first five cuts of our pizza as in this diagram, to make only 12 pieces. What is the *largest* total number of pieces we can get with one more cut?

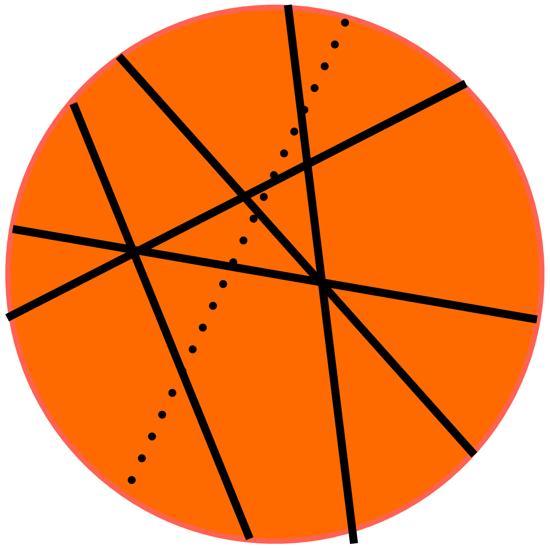


- (a) 17 (b) 19
- (c) 18 (d) 20

# Not the Answer

# Clicker Answer #2

- Suppose we make the first five cuts of our pizza as in this diagram, to make only 12 pieces. What is the *largest* total number of pieces we can get with one more cut?

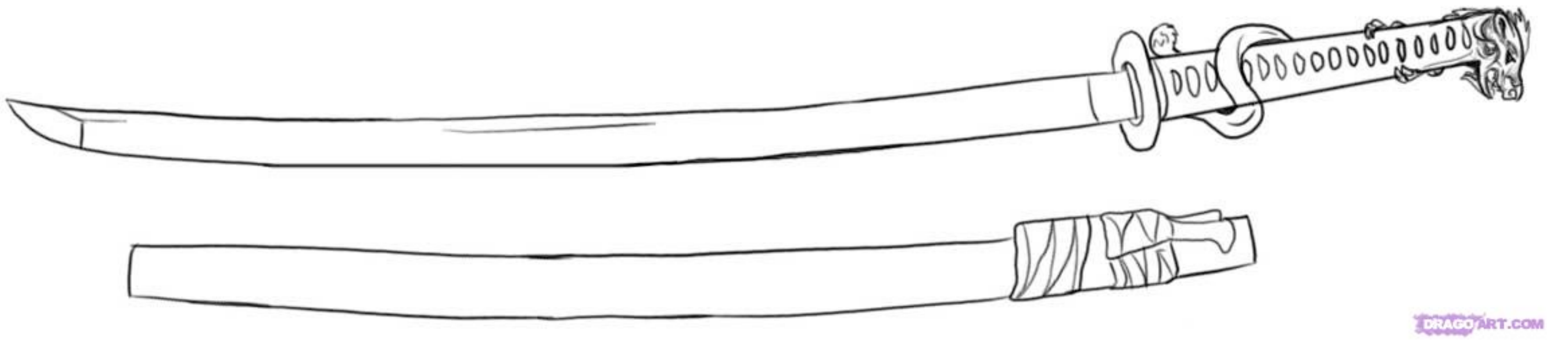


- (a) 17 (b) 19
- (c) 18 (d) 20

If we cross all five existing lines, we make six more pieces.

# Cutting Cheese With a Katana

- A similar analysis works in three dimensions.
- With a **katana** (Japanese samurai's sword), we can make straight cuts through a convex block of cheese in any plane we like. (We are not allowed to move the pieces until all cuts have been made.)
- The first three cuts can be mutually perpendicular, giving us 1 piece with no cuts, 2 pieces with 1, 4 pieces with 2, and 8 with 3.



# Cutting Cheese With a Katana

- The fourth cut is hard to visualize, but if your cut goes near but not through the existing triple intersection point, you cut seven of the eight old pieces and get a new total of 15.
- With a fifth cut, you can get 26 pieces, a number rather hard to figure out by visualizing (but see a way that a UMass alum got the number, at least, at www.gweep.net/~sskoog/block_of_cheese_problem.txt).

# Cutting Cheese With a Katana

- The key to the solution is how many new pieces we add with each cut.
- From the above, the sequence is 1, 2, 4, 7, 11,... which we might recognize as the number of pieces of pizza in the previous problem.
- Why is this?

# The Cheese-Cutting Theorem

- **Theorem**: The maximum number of cheese pieces is exactly $(n^3 + 5n + 6)/6$.
- Where did this polynomial come from?
- It is the only function of the form $f(n) = an^3 + bn^2 + cn + d$ that satisfies $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, and $f(3) = 8$.
- Since the difference function is a quadratic, we expect this to be a cubic.

# The Cheese-Cutting Theorem

- The n+1'st katana cut forms a new plane within the cheese, and the intersection of each of the old cuts is a straight line.
- These n lines divide our new plane into at most $(n^2 + n + 2)/2$ regions by the Pizza Cutting Theorem.
- We can use this in the inductive step of the proof of our new Theorem.

# The Cheese-Cutting Theorem

- If we compute $(n^3 + 5n + 6)/6 + (n^2 + n + 2)/2$, we get $(n^3 + 3n^2 + 8n + 12)/6$, which is exactly $((n + 1)^3 + 5(n + 1) + 6)/6$, the number we want as the conclusion of our inductive step.
- If the cuts are in general position, and the block of cheese is large enough to include all the intersections, we achieve exactly this number.

# Speed of Euclidean Algorithm

- We asserted that the Euclidean Algorithm can quickly test even very large numbers for relative primality.
- For example, starting with 2068 and 1259 gives 809, 450, 359, 91, 86, 5, and 1.
- Starting with 2068 and 1289 gives 779, 510, 269, 241, 28, 17, 11, 6, 5, and 1.

# Speed of Euclidean Algorithm

- Consecutive Fibonacci numbers take a relatively long time, e.g., 233, 144, 89, 55, 34, 21, 13, 8, 5, 3, 2, and 1. But actually F(n) is about $(1.61)^n$, so that if x is a Fibonacci number we take about $\log_{1.61} x$ steps.
- Here we'll show that if the two initial numbers are each at most $2^n$, the EA will terminate in at most $2n + 1$ steps. The base case of our induction says that if both numbers are at most $2^0 = 1$, we need $2(0) + 1 = 1$ step.

# Clicker Question #3

- Let n be a number with exactly 200 decimal digits. Problem 4.11.3 shows that the EA will take at most $\log_{1.61} n$ steps on this input. Which of these numbers is closest to $\log_{1.61} n$?
(Hint: for what x is $1.6^x \simeq 10$?)
- (a) 300
- (b) 500
- (c) 700
- (d) 1000

# Not the Answer

# Clicker Answer #3

- Let n be a number with exactly 200 decimal digits. Problem 4.11.3 shows that the EA will take at most $\log_{1.61} n$ steps on this input. Which of these numbers is closest to $\log_{1.61} n$? (Hint: for what x is $1.6^x \simeq 10$?)
- (a) 300
- (b) 500
- (c) 700
- (d) 1000 $\log_{1.61}(10)$ = about 4.85, so about 970

# Speed of Euclidean Algorithm

- The inductive step uses the contrapositive method.
- We start with a and b, and compute $a = qb + c$ and $b = rc + d$, so $a = (qr + 1)c + qd$. If c or d is *greater than* $2^n$, then a is greater than $2^{n+1}$.
- So if $a \leq 2^{n+1}$, then $c \leq 2^n$. By the IH we need at most $2n + 1$ steps starting with c and d, so we need at most $2n + 3$ total steps.