Wigner Monte Carlo Approach to Neutrino Oscillation

PHY455

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Abstract

Neutrino flavour is considered as a mixed state of mass eigenstates ($\nu_a, a = 1, 2, 3$) which are governed by time-dependent Schrodinger Equation. Thus, the evolution of an initial neutrino flavour can be determined through solving TDSE. In this approach the Sellier's generalization of Wigner Monte Carlo (based on signed particle formulation) is used to evaluate the real time evolution of the mixed flavour state to show the flavour change phenomenon and thus neutrino oscillation.

1 Background

The subject of neutrino mass can uncover many fundamental questions in particle physics such as the asymmetry in amount of matter and anti-matter in the universe. That's why it is like the holy grail of particle physics.The concept of neutrino was first introduced by *Wolfgang Pauli* in 1930 to explain the continuous energy spectrum of beta decays .Under the condition of energy-momentum conservation the only possibility of explaining continuous beta spectra was to think of the beta decay as a three body process where a new neutral,spin 1/2 particle was emitted alongside electron in the beta decay process which was undetected in experiment[1].Later *E.Fermi* in 1934 established the first theory of beta decay of nuclei incorporating the idea of light, neutral, spin $\frac{1}{2}$ particle proposed by *Pauli* coining the term **Neutrino**[2].In 1946 *Shoichi Sakata* and *Takesi Inoue* discussed the decay of *Yukawa* particle as a three body decay of muon introducing a second type of neutrino(muon neutrino) as a distinct neutrino form the neutrino in beta decay(electron neutrino)[3]. In 1956 neutrino(electron-antineutrino) was first observed experimentally which was led by *Fred Reines* and *Clyde Cowan Jr.* [4]accompanied by discovering muon neutrino in 1962 by *Leon Lederman, Melvin Schwartz* and*Jack Steinberger*.The idea of another neutrino(tau-neutrino) was first implied by *Martin Perl* and colleagues after the discovery of tau particle in a series of experiment done in 1974-1977.The existence of tau neutrino was first observed from **DONUT** experiment from fermilab in 2000[5]. This completes the three generation of fermions in standard model where the 1st, 2nd and 3rd generation consists up and down, charm and strange, top and bottom quarks and electron and electron-neutrino, muon and muon-neutrino, tau and tau-neutrino respectively.

The standard model of particle physics works pretty well in describing most of the phenomena regarding particle physics. One such exception is neutrino-mass. According to the standard model neutrino is neutral, spin $\frac{1}{2}$ particle with zero mass. But some experiments tell otherwise. One such experiment was done in Kolar Gold Field (KGF) mine in india in 1965[6] and successfully detected cosmic ray produced neutrinos. But the paradigm shift in the nature of neutrino came from experiment conducted by *Ray Davis* and *John N.Bahcall* resulting in solar neutrino problem[7].Their experiment was to get solar neutrinos(electron neutrino) predicted by standard solar model (SSM). But *Davis* found only 1/3 of the value of the predicted theoretical value by SSM. The confirmation of this deficit was further confirmed by Kamiokande in 1989 and in 1990 IMB confirmed muon neutrino deficit resulting in atmospheric neutrino anomaly[8]. The only explanation for this discrepancy is for neutrino to oscillate meaning one flavor of neutrino can transform into another flavor. *Bruno Pontecorvo* in 1957 first showed that neutrino-antineutrino transition may occur in analogy with neutral kaon mixing[9] which has not been observed but paved the way for quantitative description of neutrino flavor oscillation which was first developed by *Maki, Nakagawa*, and *Sakata* in 1962[10] and further elaborated by *Pontecorvo* in 1967[11] one year before solar neutrino problem being observed followed by the famous work of *Gribov* and *Pontecorvo* which shows that neutrino oscillation can solve solar neutrino problem[12].Flavor conversion arises directly from phase difference caused by energy difference caused by mass difference. Hence this experiments suggest that neutrinos have non-zero mass. Then in 1998 Super-Kamiokande collaboration announced evidence of neutrino oscillation consistent with the theory of non-zero neutrino mass[13]. Sudbury Neutrino Observatory (SN0) also gave convincing evidence of neutrino oscillation cause of the solar neutrino problem in 2001-02 [14]. So, neutrino points out that we need physics beyond standard model to describe the origin of neutrino mass.

2 Introduction

Three known neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ are massless in the standard model (SM) as a straightforward consequence of its simple structure and renormalizability. On the one hand, the SM does not contain any right-handed neutrinos, and thus there is no way to write out the Dirac neutrino mass term. On the other hand, the SM conserves the $SU(2)_L$ gauge symmetry and only contains the Higgs doublet, and thus the Majorana mass term is forbidden. Although the SM possesses the (B-L) symmetry and "naturally" allows neutrinos to be mass less, the vanishing of neutrino masses in the SM is not guaranteed by any fundamental symmetry or conservation law. There exists a lot of evidence for neutrino oscillations from solar, atmospheric, reactor and accelerator neutrino experiments. The phenomenon of neutrino oscillations implies that at least two neutrinos must be massive and three neutrino flavors must be mixed. This is the first convincing evidence for new physics beyond the SM.
The fact that neutrinos have masses means that there is a spectrum of neutrino mass eigenstates ν_a, a = 1, 2, . . ., each with a mass m_a. Leptonic mixing implies that in a decay process (i.e. $W^+ \rightarrow \nu_i + \bar{l}_a$) the flavour neutrino eigenstate is accompanied by the neutrino mass eigenstates which is not always the same ν_a, but can be any of the normalized combination of ν_a. The amplitude for W^+ decay to produce the specific combination $\bar{l}_\alpha + \nu_i$ is denoted by $U^*_{\alpha a}$ The neutrino state emitted in W^+ decay together with the particular charged lepton $\bar{l}_\alpha$ is then

$$|\nu_\alpha\rangle = \sum_a \mathbf{U_{\alpha a}} |\nu_a\rangle$$

The mass eigenstates ν_a can be regarded as a quantum particle with definite mass m_a governed by Schrodinger equation or rather the Dirac equation in case of relativistic consideration.

3 Theory

Neutrino oscillation are transition between different flavor neutrino $(\nu_e, \nu_\mu, \nu_\tau)$.The time dependent mass eigen states are governed by Schrödinger equation.

$$i\frac{\partial}{\partial t} |\nu_a(t)\rangle = H |\nu_a(t)\rangle \tag{1}$$

Taking $\hbar = 1$ and H is the total Hamiltonian of forms

$$H = (p + \frac{\sum m_i}{4p})\mathbf{I} + \frac{1}{4p}\begin{pmatrix} m_1^2 - m_2^2 - m_3^2 & 0 & 0 \\ 0 & m_2^2 - m_3^2 - m_1^2 & 0 \\ 0 & 0 & m_3^2 - m_1^2 - m_2^2 \end{pmatrix} \tag{2}$$

where m's are the masses of ν_a. $\mathbf{I}$ is the identity matrix and considering momentum p is the same for all neutrino mass eigenstate. The general solution of (1) is,

$$|\nu_a(t)\rangle = e^{-iHt}|\nu_a(0)\rangle$$

Now, flavor neutrino states $|\nu(\alpha)\rangle$ and mass states $|\nu_a\rangle$ are related by leptonic mixing matrix or PMNS (Pontecorvo-Maki-Nakagawa-Sakata matrix) $\mathbf{U}$.

$$|\nu_\alpha\rangle = \sum_a \mathbf{U_{\alpha a}}|\nu_a\rangle \tag{3}$$

where a=1,2,3 and $\alpha = \nu_e, \nu_\mu, \nu_\tau$
thus,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \mathbf{U_{e1}} & \mathbf{U_{e2}} & \mathbf{U_{e3}} \\ \mathbf{U_{\mu 1}} & \mathbf{U_{\mu 2}} & \mathbf{U_{\mu 3}} \\ \mathbf{U_{\tau 1}} & \mathbf{U_{\tau 2}} & \mathbf{U_{\tau 3}} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$\mathbf{U}$ is a unitary matrix and expressed in terms of 3 rotation angle θ_{12},θ_{23},θ_{31} and a complex phase δ (cp violation phase).Using the notation $S_{ij} = sin\theta_{ij}$ and $C_{ij} = cos\theta_{ij}$

$$\begin{pmatrix} \mathbf{U_{e1}} & \mathbf{U_{e2}} & \mathbf{U_{e3}} \\ \mathbf{U_{\mu 1}} & \mathbf{U_{\mu 2}} & \mathbf{U_{\mu 3}} \\ \mathbf{U_{\tau 1}} & \mathbf{U_{\tau 2}} & \mathbf{U_{\tau 3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix}\begin{pmatrix} C_{13} & 0 & S_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -S_{13}e^{-i\delta} & 0 & C_{13} \end{pmatrix}\begin{pmatrix} C_{12} & S_{12} & 0 \\ -S12 & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

writing this out in full form:

$$\mathbf{U} = \begin{pmatrix} C_{12}C_{13} & S_{12}C_{13} & S_{13}e^{-i\delta} \\ -S_{12}C_{23} - C_{12}S_{23}S_{13}e^{i\delta} & C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\delta} & S_{23}C_{13} \\ S_{12}S_{23} - C_{12}C_{23}S_{13}e^{i\delta} & -C_{12}S_{23} - S_{12}C23S_{13}e^{i\delta} & C_{23}C_{13} \end{pmatrix}$$

To satisfy uncertainty relation it is mere logical to consider wave packet treatment rather than plain wave treatment for neutrino propagation.
The normalized mass-eigenstate wave packet is co-ordinate space in and momentum space is,

$$\nu_a(x,t) = (\sqrt{2\pi}\sigma_x)^{-1/2}e^{i(p_a x - E_a t) - \frac{(x - v_a t)^2}{4\sigma_x^2}}$$

$$\nu_a(p) = (\sqrt{2\pi}\sigma_p)^{1/2}e^{\frac{(p-p_a)^2}{4\sigma_p^2}}$$

We assume that the gaussian functions are sharply peaked around the corresponding average momentum with $E_a \equiv E_a(p_a)$. Under this condition the energy $E_a(p)$ can be approximated by $E_a(p) \simeq E_a + v_a(p - p_a)$ where $v_a = p_a/E_a$ is the groupn velocity of each wave packet. In this case, the neutrino wave function in coordinate space $|v_\alpha(x,t)\rangle = \langle x|v_\alpha(t)\rangle$ is easily obtained

$$|\nu_\alpha(x,t)\rangle = (\sqrt{2\pi}\sigma)^{1/2}\sum_a \mathbf{U}_{\alpha a}e^{i(p_a x - E_a t) - \frac{(x - v_a t)^2}{4\sigma n^2}}|\nu_a\rangle \tag{4}$$

Let us consider a neutrino at coordinates $x = 0$, $t = 0$ by a weak process as a flavour neutrino ν_α. The quantum mechanical probability to find the flavour state ν_β detected at a distance $x = L$ and time $t = T$ is given by [15] [16]

$$P_{\alpha \longrightarrow \beta}(L,T) = \frac{1}{\sqrt{2\pi\sigma_x}}\sum_{a,b}\mathbf{U}^*_{\beta a}\mathbf{U}_{\alpha a}\mathbf{U}_{\beta b}\mathbf{U}^*_{\alpha b}\exp\left\{ i(p_a - p_b)L - i(E_a - E_b)T - \frac{(L - v_a T)^2}{4\sigma_x^2} - \frac{(L - v_b T)^2}{4\sigma_x^2} \right\} \tag{5}$$

The mass eigenstate energies can be approximated by

$$E_a \simeq E + \zeta\frac{m_a^2}{2E}$$

Where, E is the average energy of the neutrino flavour state and ζ is a dimensionless correction factor for the relativistic energy approximation which depends upon neutrino production process. The oscillation probability can be written as following in terms of relativistic approximation

$$P_{\alpha\longrightarrow\beta}(L,T) \propto \sum_{a,b} \mathbf{U}^*_{\beta a}\mathbf{U}_{\alpha a}\mathbf{U}_{\beta b}\mathbf{U}^*_{\alpha b} \exp\left\{ i\frac{\delta m^2_{ab}}{2E}\Big[\zeta T + (1-\zeta)L)\Big] - \frac{(L-v_aT)^2}{4\sigma_x^2} - \frac{(L-v_bT)^2}{4\sigma_x^2}\right\} \tag{6}$$

where, $\delta m^2_{ab} = m^2_a - m^2_b$ In principle, $P_{\alpha\longrightarrow\beta}(L,T)$ is a measurable quantity, but in all realistic experiments the distance L is a fixed and known quantity whereas the time T is not measured. Therefore, the quantity that is measured in all experiments is the oscillation probability $P_{\alpha\longrightarrow\beta}(L)$ at a fixed distance L given by the time average of $P_{\alpha\longrightarrow\beta}(L,T)$. After integrating over T and imposing the normalization condition $\sum_\beta P_{\alpha\longrightarrow\beta}(L) = 1$, we obtain

$$P_{\alpha\longrightarrow\beta}(L) = \sum_{a,b} \mathbf{U}^*_{\beta a}\mathbf{U}_{\alpha a}\mathbf{U}_{\beta b}\mathbf{U}^*_{\alpha b} \exp\left\{-2\pi i\frac{L}{L^{osc}_{ab}} - \left(\frac{L}{L^{coh}_{ab}}\right)^2\right\} \exp\left\{-2\pi^2(1-\zeta^2)\left(\frac{\sigma_x}{L^{osc}_{ab}}\right)^2\right\}$$

where, $L^{osc}_{ab} = \frac{4\pi E}{\delta m^2_{ab}}$ is the oscillation length and $L^{coh}_{ab} = \frac{4\sqrt{2}E^2}{\delta m^2_{ab}}\sigma_x$ is the coherent length.

The Phase Factor

$$\exp\left\{-2\pi i\frac{L}{L^{osc}_{ab}}\right\}$$

This factor describes the actual oscillation.

The Localization Term

$$\exp\left\{-2\pi^2(1-\zeta^2)\left(\frac{\sigma_x}{L^{osc}_{ab}}\right)^2\right\}$$

This term suppresses if $\sigma_x >> L^{osc}_{ab}$. This means in order to get oscillation, the production and detection process must be localized in a region much smaller than oscillation length.

Coherence term

$$\exp\left\{-\frac{L}{L^{coh}_{ab}}\right\}$$

This term suppresses the oscillations if the distance L becomes lager than the coherence length, that is $L >> Lab^{coh}$. Therefore, it describes the suppression of the oscillations due to the separation of the wave packets for different mass neutrino. If the packets are too much separated they can not have an overlap with the detection process and thus they ca not be detected coherently.

3.1 Wigner Formulation

Given that the mass eigenstate wave function is $\nu_a(x,t)$, it is possible to construct the following expression:

$$f_w(x,p,t) \propto \int_{-\infty}^{+\infty} dx'\nu^*_a(x+x',t)\nu_a(x-x',t)\exp\left\{-\frac{2i}{\hbar}x'.p\right\} \tag{7}$$

Which is the Wigner function [17] Taking the time derivation of $f_w(x,p,t)$, the Wigner equation can be obtained

$$\frac{\partial f_w}{\partial t} + \frac{p}{m}.\nabla_x f_w = \int_{-\infty}^{+\infty} dp' V_w(x,p',t) f_w(x,p+p',t) \tag{8}$$

where

$$V_w(x,p',t) = \frac{i}{\pi^d\hbar^{d+1}}\int dx'\Big[U(x+\frac{x'}{2},t) - U(x-\frac{x'}{2},t)\Big]\exp\left\{-\frac{i}{\hbar}x'.p\right\} \tag{9}$$

V_w is the wigner kernel (defined in d-dimensional space) and $U(x,t)$ is the external potential. Eq. (7) describes the dynamics of a system consisting of a single particle in a presence of a external potential. In the case of neutrino propagation in vacuum, there is no external potential acting on the system. Neutrino flavour state ν_α can be interpreted as a mixed state of mass eigenstates ν_a, for which the density matrix can be written as,

$$\rho(x, x', t) = \sum_a \mathbf{U}_{\alpha a} \mathbf{U}^*_{\alpha a} \nu_a(x, t) \nu^*_a(x', t) \tag{10}$$

The mixed state Wigner function is of the following form which also satisfies the Wigner equation [18]

$$f_w(x, p, t) \propto \int dx' \rho(x - x', x + x', t) \exp\left\{\frac{2i}{\hbar}\right\} x'p \tag{11}$$

4 Numerical Scheme

4.1 Semi-discrete Phase Space

In Wigner formulation of quantum mechanics, a quantum system consisting of one particle is completely described in terms of a phase space quasi distribution function $f_w(x, p, t)$ evolving according to (11). Thus, our aim is to reconstruct the function f_w at a given time. We start by reformulating the Wigner equation in a semi discrete phase space with a continuous spatial variable and discredited momentum p described in terms of a step $\delta p = \frac{\hbar\pi}{L}$, where L is a free parameter defining the discrimination. Now, the semi discrete Wigner equation reads, [18]

$$\frac{\partial f_w}{\partial t} + \frac{\hbar}{m} \frac{M \delta p}{\hbar} \nabla_x f_w = 0 \tag{12}$$

where, $M = (M_1, ..., M_d$ a set of integers of d elements and $M\delta p = (M_1 \delta p_1, ..., M_d \delta p_d$

4.2 Wigner Monte Carlo

There are two basic Monte Carlo approaches to Wigner equation. In the time-dependent case, the so-called particle affinity and, in the stationary case, integer particle signs. We considered Jean Michel Sellier's generalization of signed particle approach to time-dependent scenario. [19]. In the signed particle formulation of quantum mechanics can be constructed from three postulates. [20].

1. Physical systems can be described by means of virtual Newtonian particles with a definite position x, a momentum p with a sign which can be either positive or negative.
2. A signed particle evolving in a potential $V = V(x)$ behaves as a field less classical point particle which during the time interval dt, creates a new pair of signed particles with a probability $\gamma(x(t))dt$ where

   $$\gamma(x) = \int_{-\infty}^{+\infty} Dp' V_w^+(x, p') \equiv \lim_{\delta p' \to 0^+} \sum_{M=-\infty}^{+\infty} V_w^+(x, M\delta p')$$

   If, at a moment of creation, the parent particle has sign s, position x and momentum p, the new particles are both located at x, has signs $+s$, $-s$ and momentum $p - p'$, $p + p$ respectively with p' chosen randomly according to the (normalized) probability $\frac{V_w^+(x,p)}{\gamma(x)}$
3. The two particles with opposite sign and same phase space coordinates $(x, p$ annihilate

In free space there is no external potential U which nullify the Wigner kernel V_w and thus, in free propagation of neutrino wave packet there will be no creation or annihilation events in Monte Carlo steps. The Wigner Monte Carlo algorithm requires an initial Wigner function which can be easily determined from the neutrino flavour density matrix Since the flavour state of neutrino is a mixed state of mass states (pure state).

$$f_w^0(x, p) = \sum_{a=1,2,3} p_{ea} \exp\left\{-\left(\frac{x - x_0}{\sigma_x}\right)^2\right\} \exp\left\{-\left((p - p_0^a)\sigma_x\right)^2\right\}$$

This initial Wigner function describes an electron neutrino produced at a coherent source. The density matrix components come from the PMNS matrix; $p_{ea} = \mathbf{U}_{ea}$. The p_0^a denotes the momentum for a^{th} mass eigenstates which is implied by the energy-momentum conservation in neutrino creation process and can be approximated by the mass-squared differences (δm^2_{ab}) of the mass eigenstates. The experimental values of (δm^2_{ab}), σ_x are incompatible with our numerical method to produce any visible result in limited time. Thus, we estimated new values of these terms which are analogous to the experimental values.

5 Analysis

5.1 Oscillation

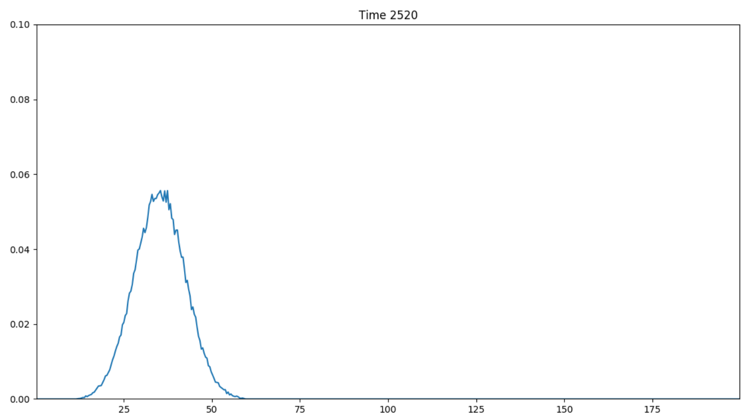

Figure 1: Wave propagation in space. The vertical axes indicate amplitude of the wave and the horizontal one indicates length.

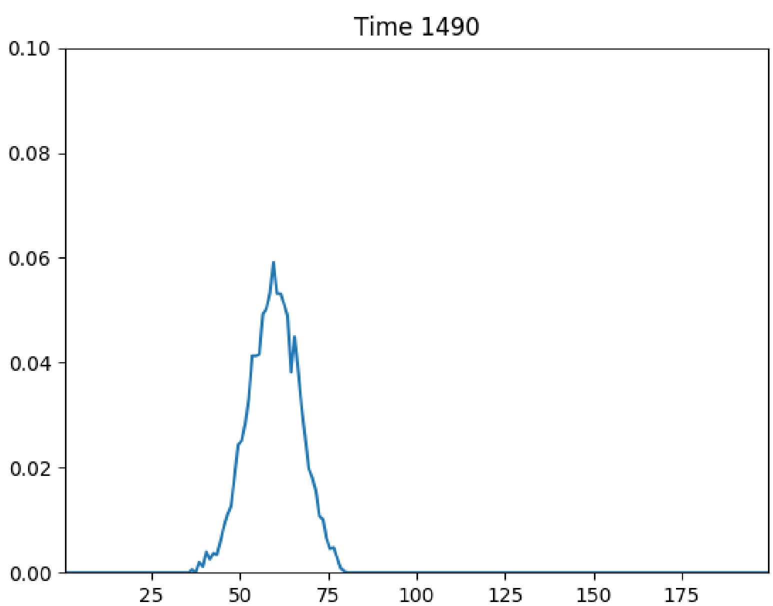

Figure 2: Wave propagation in space. The vertical axes indicate amplitude of the wave and the horizontal one indicates length.

If at distance L, a different flavour of neutrino is detected, that implies that the group velocity of the super imposed wave will be different than the initial one. Thus the velocity over time plot can actually can show us the information of flavour changing phenomenon.

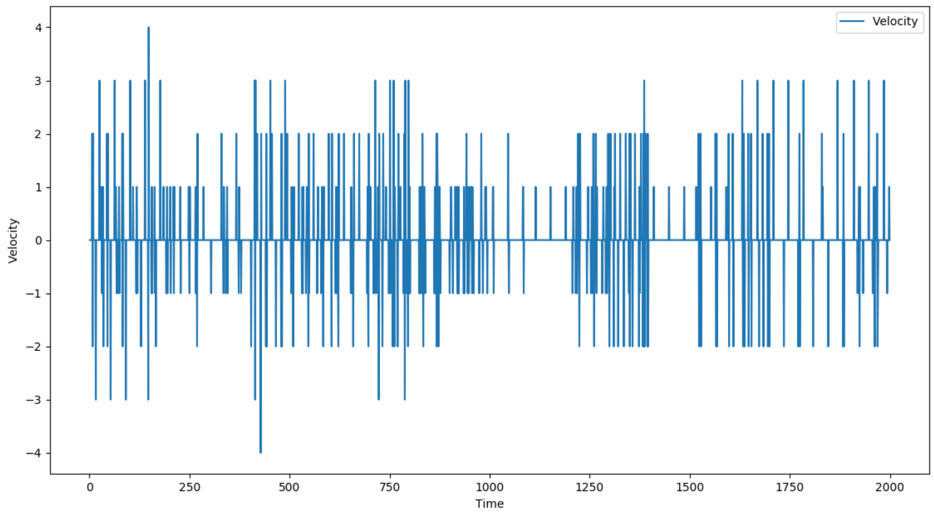

Figure 3: Velocity vs. time graph

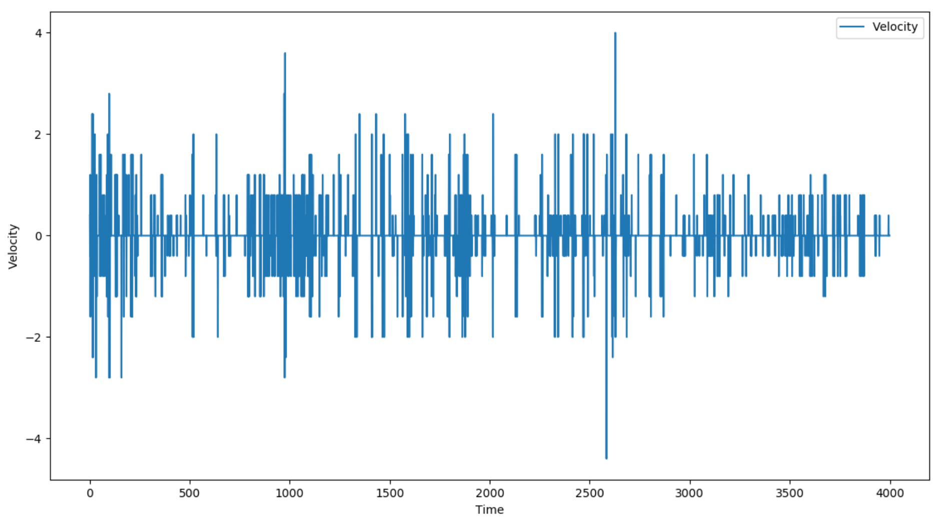

Figure 4: Velocity vs. time graph

These plots clearly shows an oscillating pattern in velocity term. Thus, flavour changing is taking place while propagating over tiny distances.

5.2 Decoherence

The coherence of neutrino mass eigenstates depends upon the mass-squared differences and their relation can be shown to be inversely proportional.

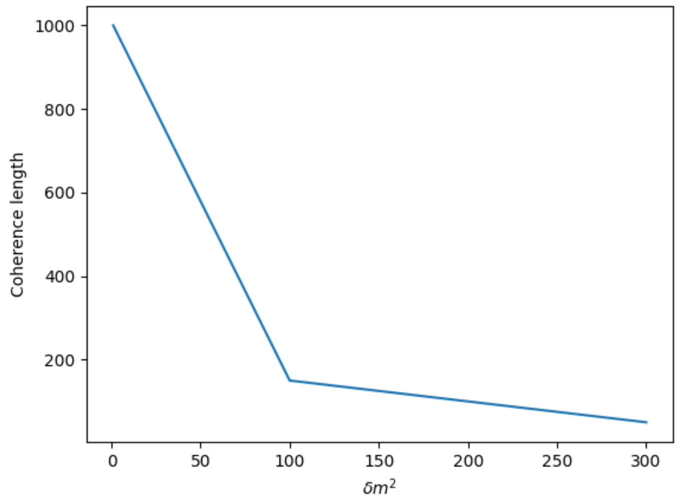

Figure 5: Coherence length vs. δm^2

6 Discussion

The conventional procedure to simulate neutrino oscillation by Monte Carlo is evaluating multidimensional integral of scattering matrix which can be obtained from Dyson series. The other Monte Carlo methods like path integral Monte Carlo fails for determining real time evolution of quantum system as they are based on imaginary time formulation. Wigner Monte Carlo in terms of singed particle formulation can be exploited for evolving quantum system in real time. Hence, the choice for Wigner Monte Carlo is justified.

7 Conclusion

Neutrino oscillation refers to physics beyond standard model. The origin of neutrino mass generation is still unknown. In this work we did not go into the complexities of neutrino mass generation but rather tried to analyze the phenomena of neutrino oscillation from an alternative formulation of quantum mechanics-Wigner formulation, which is analogous to Feynman's path integral formulation of quantum mechanics. In future work we will work on the details behind neutrino oscillation from quantum field theory approach.

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