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Homework - # 6

Problem 1
$a - c|ab + cd \implies a - c|ad + bc$

Solution:
$a - c \equiv 0(mod(a - c))$
$\implies a \equiv c(mod(a - c))$
$\implies ad \equiv cd(mod(a - c))$ .. eq (1)
$\implies ab \equiv cb(mod(a - c))$ .. eq(2)
Adding (1) and (2)
$\implies ab + cd \equiv cb + ad(mod(a - c))$
$\implies bc + ad \equiv 0(mod(a - c))$

Problem 2
$a \equiv b \equiv 1(mod(2)) \implies a^2 + b^2 \neq c^2$

Solution:
$a = (2k + 1)$, $b = (2k' + 1)$
$a^2 = 4k^2 + 4k + 1$, $b^2 = 4k'^2 + 4k' + 1$
$a^2 + b^2 = 4(k^2 + k'^2) + 4(k + k') + 2$
$\implies 2|c^2 \implies 2|c \implies 4|c^2$
$a^2 + b^2 \equiv 2(mod(4)) \equiv c^2$

So, if $a^2 + b^2 = c^2$ exists then $c^2 \equiv 0 mod(4)$
Contradiction

Problem 3
Prove $6|n^3 + 5n$

Solution:
$n^3 + 5n \equiv 0(mod(6))$
$\implies (n^3 - n) \equiv 0(mod(6))$
$\implies n(n - 1)(n + 1) \equiv 0(mod(6))$
$\implies n(n - 1)(n + 1) \equiv 0(mod(2))$ Because every 2 consecutive integers are represented as n , (n-1) , one of which is even
$\implies n(n - 1)(n + 1) \equiv 0(mod(3))$ Because every 3 consecutive integers are represented as k, k-1, k+1 where one of them is divisible by 3.

Problem 4
Prove $30|n^5 - n$

Solution: $n^5 - n = 0(mod(30))$

$\implies n(n-1)(n^2+1)(n+1) \equiv 0(mod(30))$
$\implies n(n-1)(n+1)(n-2)(n+2) \equiv 0(mod(30)$
Because every 5 consecutive integers are represented as k, k-1, k+1, k- 2, k+2 where one of them is divisible by 5.
Because every 2 consecutive integers are represented as n , (n -1) , one of which is even
Because every 3 consecutive integers are represented as k, k-1, k+1 where one of them is divisible by 3.

Problem 5
Find n , $120|n^5 - n$

Solution:

$n^5 - n$
$\implies n^5 - n = n(n^4 - 1)$
$\implies n(n^2 - 1)(n^2 + 1)$
$\implies n(n-1)(n+1)(n^2+1)$
$120 = 8.5.3$
Find n, $8|n(n-1)(n+1)(n^2+1)$
$n(n-1)(n+1)(n^2+1) = 0(mod(8))$
if $n = 2k+1$
$(2k+1)(2k)(2k+2)(4k^2+4k+2) \equiv 0(mod(8))$
$2^3(2k+1)(k)(k+1)(2k^2+2k+1) \equiv 0(mod(8))$
So for every $n = 2k+1, k \geq 1$ 120 divides $n^5 - n$

Problem 6
Prove $3|a$, $3|b \iff 3|a^2 + b^2$

Solution:

if $3|a^2 + b^2$ then $3|a$ and $3|b$
any x (mod(3)) is $\{0, 1, -1\}$ and $x^2(mod(3))$ is $\{0, 1\}$
$\implies a^2 + b^2 \equiv 0(mod(3))$
that is only possible when $a^2(mod(3)) \equiv 0$ and $b^2(mod(3)) \equiv 0$
$\implies 3|a$ and $3|b$

if $3|a$ and $3|b$ then $3|a^2 + b^2$
$a = 3k$, $b = 3k'$
$a^2 + b^2 \equiv 9(k^2 + k'^2)(mod(3))$
$\implies 0 \equiv a^2 + b^2(mod(3))$
QED

Problem 7
Prove $7|a$, $7|b \iff 7|a^2 + b^2$

Solution:

if $7|a^2+b^2$ then $7|a$ and $7|b$
any x (mod(7)) is $\{0,1,2,3,-3,-2,-1\}$ and $x^2(mod(7))$ is $\{0,1,-3,2\}$
$\implies a^2+b^2 \equiv 0(mod(7))$
that is only possible when $a^2(mod(7)) \equiv 0$ and $b^2(mod(7)) \equiv 0$
$\implies 7|a$ and $7|b$

if $7|a$ and $7|b$ then $7|a^2+b^2$
$a = 7k$, $b = 7k'$
$a^2+b^2 \equiv 49(k^2+k'^2)(mod(7))$
$\implies 0 \equiv a^2+b^2(mod(7))$
QED

Problem 8
Prove $21|a^2+b^2 \implies 441|a^2+b^2$

Solution:
$21|a^2+b^2$
$a^2+b^2 = 0(mod(21))$
$a^2+b^2 = 0(mod(3)) \implies 3|a$ and $3|b$
$a^2+b^2 = 0(mod(7)) \implies 7|a$ and $7|b$
$3|a \implies 9|a^2$ and $3|b \implies 9|b^2$
$7|a \implies 49|a^2$ and $7|b \implies 49|b^2$
$\implies 3^2|a^2+b^2$ and $\implies 7^2|a^2+b^2$
$\implies 441|a^2+b^2$ Since gcd (9, 49) = 1
QED

Problem 9
Prove $n \equiv 1(mod(2)) \implies n^2 \equiv 1(mod(8))$

Solution:
$n = 2k+1$
$\implies n^2 = 4k^2+4k+1$
$\implies n^2-1 = 4k^2+4k$
$\implies n^2-1 \equiv 4k(k+1)(mod8)$
$\implies n^2-1 \equiv 4 \times 2x(mod8)$ As one of every 2 consecutive integers is even
$\implies n^2-1 \equiv 8x(mod8)$
$\implies n^2-1 \equiv 0(mod(8))$
$\implies n^2 \equiv 1(mod(8))$
QED

Problem 10
$6|a+b+c \iff 6|a^3+b^3+c^3$

Solution:
if $6|a^3+b^3+c^3$ then $6|a+b+c$
Using Property if $m|n$ and $m|n-k$ then $m|k$

$6|a^3+b^3+c^3-(a+b+c)$
$\implies 6|a^3-a+b^3-b+c^3-c$
$\implies 6|a(a-1)(a+1)+b(b+1)(b-1)c(c+1)(c-1)$ Earlier shown in 3a that $6|n(n+1)(n-1)$
$\implies 6|a+b+c$

if $6|a+b+c$ then $6|a^3+b^3+c^3$
Similarly WLOG, Replacing $a^3+b^3+c^3$ by $a+b+c$ in the above proof gives
$6|a^3+b^3+c^3$

QED

Problem 11
Let $A = 3^{105} + 4^{105}$. Show that $7|A$. Find $A(mod(11))$ and $A(mod(13))$

Solution:

$A \equiv 3^{105} + (-3)^{105}(mod(7))$
$\implies A \equiv 0(mod(7))$

$A \equiv 3^{105} + 4^{105}(mod(11))$
$\implies A \equiv (3^5)^{21} + (4^5)^{21}(mod(11))$
$\implies A \equiv (243)^{21} + (1024)^{21}(mod(11))$
$\implies A \equiv 1^{21} + 1^{21}(mod(11))$
$\implies 2(mod(11))$

$A \equiv 3^{105} + 4^{105}(mod(13)$
$\implies A \equiv (3^3)^{35} + (4^3)^{35}(mod(13))$
$\implies A \equiv (27)^{35} + (64)^{35}(mod(13))$
$\implies A \equiv 1 - 1(mod(13))$
$\implies A \equiv 0(mod(13))$

Problem 12
Show that $3n-1$, $5n-2$, $5n+2$, $7n-1$, $7n-2$, $7n+3$

Solution:

any $x^2(mod(3))$ is $\{0,1\}$
any $x^2(mod(5))$ is $\{0,1,4\}$
any $x^2(mod(7))$ is $\{0,1,2,4\}$
Let $k^2 \equiv 3n-1(mod(3)) \equiv -1$ which is not possible hence $3n-1$ is not a square.

Let $k^2 \equiv 5n-2(mod(5)) \equiv -2$ which is not possible hence $5n-2$ is not a square.

Let $k^2 \equiv 5n+2(mod(5)) \equiv 2$ which is not possible hence $5n+2$ is not a square.

Let $k^2 \equiv 7n-1(mod(7)) \equiv -1$ which is not possible hence $7n-1$ is not a square.

Let $k^2 \equiv 7n-2(mod(7)) \equiv -2$ which is not possible hence $7n-2$ is not a square.

Let $k^2 \equiv 7n + 3(mod(7)) \equiv 3$ which is not possible hence $7n + 3$ is not a square.

Problem 13
If n is not a prime then $2^n - 1$ is not a prime

Solution:

$n = d_1 d_2$, $d_1, d_2 > 1$ Since n is not a prime
$2^n - 1 \implies (2^{d_1})^{d_2} - 1^{d_2}$
$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + ... + b^{n-1}$
$\implies (2^{d_1} - 1)((2^{d_1})^{d_2-1} + (2^{d_1})^{d_2-2} + ... + (2^{d_1})^0)$
$\implies 2^{d_1} - 1 > 1$ is a factor of $2^n - 1$
$\implies 2^n - 1$ is not a prime
QED

Problem 14
If n has an odd divisor $\implies 2^n + 1$ is not a prime.

Solution:

$n = d_1 d_2$ and $d_1, d_2 > 1$ and d_1 is odd
$2^n + 1 \implies (2^{d_2})^{d_2} + 1^{d_2}$
$a^n + 1^n \implies (a + 1)(a^{n-1} - a^{n-2} + ... - a + 1^{n-1})$ where $a = 2^{d_1}$
$2^{d_1} + 1 > 1$ is a factor of $2^n + 1$
$\implies 2^n + 1$ is not a prime
QED

Problem 15
if $n > 2 \implies 2^n - 1$ is not a power of 3

Solution:
$\equiv 2^n - 1 = 3^k \implies n \leq 2$
$\implies 1 + 2^1 + 2^2 + 2^3 ... 2^{n-1} = (2 + 1)^k$
$\implies 1 + 2 + 2^2 ... + 2^{n-1} = (1 + \binom{k}{1}2^1 + \binom{k}{2}2^2 + ... + \binom{k}{k}2^k)$
Polynomials on both sides are equal if only if each coefficient is equal.
Case 1

$\implies \binom{k}{1} = 1$ and $\implies 2^k = 2^{n-1} = 2^k$
$\implies k = 1$ and $\implies k = n - 1$
$\implies k = 1$ and $\implies n = 2$
Case 2

$\implies 1 \times 2^0 = \binom{0}{0}2^0$
$n = 1$ and $k = 0$
Hence $n \leq 2$ QED

Problem 16
If $n > 3 \implies 2^n + 1$ is not a power of 3

Solution:
$\equiv 2^n + 1 = 3^k \implies n \leq 3$
$\implies 2^n - 1 = 3^k - 2$
$\implies 1 + 2 + 2^2 + 2^3 + ... + 2^{n-1} = (2+1)^k - 2$
$\implies 1 + 2 + 2^2 + ... + 2^{n-1} = (\binom{k}{0}2^0 + (\binom{k}{1}2^1 - 2) + \binom{k}{2}2^2 ... 2^k$
Polynomials on both sides are equal if only if each coefficient is equal
Case 1.

$\implies n - 1 = k$ and $2 = 2\binom{k}{1} - 2$
$\implies n = k + 1$ and $\binom{k}{1} = 2$
$\implies n = k + 1$ and $k = 2$
$\implies n = 3$
Case 2.

$\implies 1 \times 2^0 = \binom{1}{0}2^0 + \binom{1}{1}2 - 2$
$\implies 2^0 = 2^0 + 0$
$n = 1$ and $k = 1$
Hence $n \leq 3$ QED

Problem 17
A number with 3^n digits is divisible by 3^n

Solution:

$3^n|(111111...11)$ i.e number with $3^n digits.$
$\implies 3^n|(10^(3^n - 1) + ... + 10^(0)$
$\implies 3^n|\frac{(10^{3^n} - 1)}{9}$
$\implies 3^{n+2}|(10^{3^n} - 1)$ ... eq(1)

By the Principle Of Mathematical Induction
Assume eq(1) true for $n = k$

$\implies 3^{k+2}|10^{3^k} - 1$ ... eq(2)

To prove that $3^{k+3}|(10^{3^{k+1}} - 1)$

$\implies 3^{k+3}|(10^{3^k \times 3} - 1^3$
$\implies 3^{k+3}|((10^{3^k 3}) - 1^3$
$\implies 3^{k+3}|((10^{3^k} - 1)((10^{3^k})^2 + 1 + 10^{3^k}))$
$\implies 3^{k+2} \times 3|((10^{3^k} - 1)((10^{3^k})^2 + 1 + 10^{3^k}))$

Using eq(2)

$\implies 3|((10^{3^k})^2 + 10^{3^k} + 1)$
$((10^{3^k})^2 + 10^{3^k} + 1) \equiv 0(mod(3))$
$((1^{3^k})^2 + 1^{3^k} + 1) \equiv 0(mod(3))$

$3 \equiv 0(mod(3))$
For base case $n = 0$
$\implies 3^2|(10^1 - 1$

QED

Problem 18
Find all primes p and q so that $p^2 - 2q^2 = 1$

Solution:

$p^2 - 1 = 2q^2$
$\implies (p-1)(p+1) = 2q^2$
Factors of $2q^2 = \{(1, 2q^2), (q, 2q), (2, q^2)\}$

$a = (p-1)$
$b = (p+1)$
$\implies b - a = 2$

for $(a, b) = (1, 2q^2)$
$\implies 2^{q2} = 2 + 1$
$\implies q^2 = \frac{3}{2}$
no such prime exists

for $(a, b) = (q, 2q)$
$\implies 2q - q = 2$
$\implies q = 2$

$p^2 = 8 + 1 \implies p = 3$

for $(a, b) = (2, q^2)$
$\implies q^2 - 2 = 2$
$\implies q^2 = 4$
$\implies q = 2, p = 3$
QED

Another approach
$p^2 - 2q^2 \equiv 1 \pmod 3$
$\implies p^2 + q^2 \equiv 13$
$\forall x^2 \equiv s \in \{0, 1\}$
Case 1: Let $p^2 \pmod 3 \equiv 0 \implies p \pmod 3 \equiv 0$
p is a prime hence $p = 3$
$q^2 = \frac{9-1}{2}$
$\implies q^2 = 4$
$\implies q = 2$

Case 2: if $q^2 \equiv 0 \pmod 3 \implies q = 3$ WLOG
$\implies p = \sqrt{19}$ Hence not a prime or an integer.

Hence $p = 3$, $q = 2$

QED

Problem 19
If $2n+1$ and $3n+1$ are squares, then $5n+3$ is not a prime.

Solution:

Let $a^2 = 2n+1$ and $b^2 = 3n+1$
$\implies 4a^2 - b^2 = 8n - 3n + 3$
$\implies (2a)^2 - b^2 = 5n + 3$
$\implies (2a-b)(2a+b) = 5n+3$

$5n+3$ is not a prime $\iff 2a - b > 1$
Let's say $b = 2a - 1$
$3a^2 - 2b^2 = 1$
$\implies 3a^2 - 2(4a^2 - 4a + 1) - 1 = 0$
$\implies 5a^2 - 8a + 3 = 0$
$\implies a = 1$
$\implies b = 1$
$2n+1 = 1$ and $3n+1 = 1$
$\implies n = 0$
Hence if $n > 0$ $2a - b \neq 1 \implies 5n+3$ is not a prime
QED

Problem 20
If p is a prime , then $p^2 \equiv 1(mod(24))$

Solution:
For $p = 3 \implies 9 \not\equiv 1 \pmod 24$
For $p \neq 3$, $p = 6k \pm 1$
To prove $\implies (6k \pm 1)^2 \equiv 1 \pmod 24$

$\implies 36k^2 + 1 \pm 12k - 1 \equiv 0 \pmod 24$
$\implies 12k^2 \pm 12k \equiv 0 \pmod 24$
$\implies 12k(k \pm 1) \equiv 0 \pmod 24$
Since $k(k \pm 1)$ is always even it can be written s as $2l$

$\implies 24l \equiv 0 \pmod 24$
$\implies 0 \equiv 0 \pmod 24$ for $n > 3$
QED

Problem 21
If $9|(a^2+b^2+c^2)$ then $9|(a^2-b^2)$ or $9|(b^2-c^2)$ or $9|a^2-c^2$

Solution:

Any x^2 mod 9 is in $\{0, 1, 4, 7, -8, -5, -2\}$

different combinations of $a^2 + b^2 + c^2 (mod(9)) \equiv 0$
$(4, 4, 8)$
$(1, 4, -5)$
$(1, 7, -8)$
$(1, 1, -2)$
$(7, -5, -2)$
$(4, -2, -2)$
Converting them to all positive.
$(1, 4, 4)$
$(1, 1, 7)$
$(7, 4, 7)$
$(0, 0, 0)$
In every combination 2 elements are always equal
Since $(a^2 + b^2 + c^2)$ is symmetrical implies $a^2 - b^2$ or $b^2 - c^2$ or $c^2 - a^2$ always will be 0 mod 9

Problem 22
Find the last digit of 7^{7^7}

Solution:
$7^{7^7} \equiv x(mod(10))$
$\implies 7^{7^7+1} \equiv 7x(mod(10)$
$\implies 49^{\frac{7^7+1}{2}} \equiv 7x(mod10)$
$\implies (-1)^{\frac{7^7+1}{2}} \equiv 7x(mod(10))$
$\implies (7)^{-1} \times (-1)^{\frac{7^7+1}{2}} \equiv x(mod(10))$

$7^7 + 1 \equiv 0(mod4)$
$\implies (-1)^7 + 1 \equiv 0(mod4)$
$\implies -1 + 1 \equiv 0(mod4)$
$\implies 0 \equiv 0(mod4)$

Since $7^{-1} = 3(mod10)$

$\implies 3(-1)^{2k} \equiv x(mod(10))$
$\implies 3 \equiv x(mod(10))$

Problem 23
Find all positive integers solutions (m, n) to the following equations

$$m^2 = 1! + 2! + 3! + ... + n!$$

Solution:
Claim: To prove there doesn't exist a solution (m, n) for $n > 3$
Assume there exist a solution (m, n) for $n > 3$
$m^2 \equiv 1 + 2 + 6 + 24(mod(5))$
$\implies m^2 \equiv 33(mod(5))$
Any x^2 mod 5 is in $\{0, 1, 4\}$
$\implies m^2 \equiv 3(mod(5))$

m^2 mod 5 can't be 3 hence, contradiction
QED.

for $n = \{1, 3\}$
$1^2 = 1!$
$3^2 = 1! + 2! + 3!$
Hence (3, 3) and (1,1) are the only solutions

Problem 24
Find all positive integers n for which $3n - 4$, $4n - 5$, $5n - 3$ are all prime numbers.

Solution:
Every prime number except 2 is an odd number.

$3n - 4(mod(2)) \equiv n \equiv 1$
$\implies$ n must be odd.

$4n - 5 \equiv -5 \equiv 1(mod(2))$
$\implies 1 \equiv 1(mod(2))$

$5n - 3 \equiv n + 1 \equiv 1(mod(2))$
$\implies n \equiv 0(mod2)$ n must be even

So there must be one prime number 2 and the other two are odd.

The smallest in the given is $3n - 4$ equal to 2.
$n = 2$ i.e the only n for which one prime is even and the other two are odd.

2, 3, 7 are the given primes

Problem 25
Is it possible to arrange the numbers $1^1, 2^2, ..., 2008^{2008}$ one after the other in such a way the obtained number is a perfect square ?

Solution:

Sum of the digits is not dependent upon the order of the numbers.
Assume given numbers make a square let it be k^2 then k^2 mod 3 must be in $\{1, 0\}$

$1 + -1 + 0 + 1 + 1 + 0 + 1 + -1 + 0 + ... + 1 + 1 + 0 + 1 + -1 + 0 + 1$ mod 3
$\equiv 2 \times 334 + 1 + 1 + 0 + 1$ mod 3
$\equiv 668 + 3$ mod 3
$\equiv 671$ mod 3
$\equiv 2$ mod 3
$2 \notin \{0, 1\}$
CONTRADICTION !

Problem 26
Find all possible n for which $3^{n-1}+5^{n-1}|3^n+5^n$

Solution:
$3^n+5^n \equiv 0 \pmod{3^{n-1}+5^{n-1}}$
$z=3^{n-1}+5^{n-1}$
$\implies 3^n-3^{n-1}+5^n-5^{n-1} \equiv 0 \pmod{z}$
$\implies 3^{n-1}(3-1)+5^{n-1}(5-1) \equiv 0 \pmod{z}$
$\implies 2\times 3^{n-1}+4\times 5^{n-1} \equiv 0 \pmod{z}$
$\implies 2\times 3^{n-1}+2\times 5^{n-1} \equiv -2\times 5^{n-1} \pmod{z}$
$\implies 0 \equiv -2\times 5^{n-1} \pmod{z}$
$\implies (3^{n-1}+5^{n-1})k=-2(5^{n-1})$
$\implies 3^{n-1}=5^{n-1}(-2-k)$
$\implies 5^{n-1} \mid 3^{n-1}$
For $n>1 \implies 5^{\geq 1}>3^{\geq 1}$
Since if $m>n \implies m \not| n$
for $n=1$
$\implies 5^0 \mid 3^0$
QED

Problem 27
What single digit n , does 91 divide $12345n789$?

Solution:
$10^3\times 12345n+789 \equiv 0 (mod 91)$
$91=13\times 7$
$\implies 10^3\times 12345n+789 \equiv 0 (mod 13)$
$\implies 789-45n+123 (mod 13) \equiv 0$
$\implies 912-45n (mod 13) \equiv 0$
$\implies 2-6n (mod 13) \equiv 0$
$\implies 2\times 1+3\times 2+3-2\times 4-3\times 5-1\times n+2\times 7+3\times 8+9 \equiv 0 (mod 7)$
$\implies 11-23-n+24 \equiv 0 (mod 7)$
$\implies -n+35 \equiv 0 (mod 7)$
$\implies -n \equiv 0 (mod 7)$
$\implies n$ is 0 or 7

$\implies 60-2 (mod 13) \equiv 0$ or $67-2 (mod 13) \equiv 0$ only $65 (mod 13) \equiv 0$ hence $n=7$

Problem 28
What is the smallest positive integer that can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers ?

Solution:
$-3+-2+-1+0+1+2+3+4+5=9$
$-4+-3+-2+-1+0+1+2+3+4+5+6=11$
$-5+-4+-3+-2+-1+0+1+2+3+4+5+6+7=13$
