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Homework - # 8

Problem 1
Find all positive integers n such that

$$3^{n-1} + 5^{n-1} \mid 3^n + 5^n \tag{1}$$

Solution:

$3^n + 5^n \equiv 0 \pmod{3^{n-1} + 5^{n-1}}$
$\implies 3 \times 3^{n-1} + 5 \times 5^{n-1} \equiv 0 \pmod{3^{n-1} + 5^{n-1}}$
$\implies 0 + 2 \times 5^{n-1} \equiv 0 \pmod{3^{n-1} + 5^{n-1}}$
$\implies 2 \times 5^{n-1} = 5^{n-1}k + 3^{n-1}k$
$\implies 5^{n-1}(2-k) = 3^{n-1}$
$\implies 5^{n-1}k' = 3^{n-1}$
$\implies k' = \frac{3^{n-1}}{5^{n-1}}$
for k' to be an integer $5^{n-1} \mid 3^{n-1}$ **,** $5^{n-1} > 3^{n-1} \forall n > 1$
$\implies n = 1, (5^0 = 3^0)$
hence only one solution

Problem 2
When 4444^{4444} is written in decimal notation the sum of the digits of A. Find the sum of the digits of B.

Solution:

Problem 3
Prove or Disprove : $2233 \mid a^2 + b^2 \implies 2233 \mid a$

Solution:
Factorization of $2233 = 7 \times 29 \times 11$

Since 7 is a prime of form $4k + 3$ and 11 is a prime 11 of form $4k + 3$
$11 \mid a^2 + b^2 \implies 11 \mid a$
$7 \mid a^2 + b^2 \implies 7 \mid a$
$\implies 77 \mid a^2 + b^2 \implies 77|a$ Since $gcd(7, 11) = 1$

if $a = 5$ and $b = 2$ then $5 \equiv 5 \pmod{29}$ and $2 \equiv 2 \pmod{29}$ d . . . (1) $\implies a^2 + b^2 = 29$
$a = 77k$
$\implies a \equiv 19k \pmod{29}$

From equation (1)
$\implies a \equiv 19k \equiv 5 \pmod{29}$
$k = 26 \times 5$ Since $19^{-1} \pmod{29} \equiv 26$
$a = 26 \times 5 \times 77$ Similarly $b = 26 \times 2 \times 77$

$a^2 + b^2 = 77^2(26^2 \times 5^2 + 26^2 \times 2^2)$
$\implies 77^2(9 \times (-4) + 9 \times 4) \equiv 77^2 \times 9(-4+4) \equiv 0 \pmod{29}$
QED

Problem 4
Find a and b such that $a^2 + b^2 = 26361$

Solution:
Factorization of $26361 = 3^2 \times 29 \times 101$
$\implies 3^2(5^2 + 2^2)(10^2 + 1^2)$
$\implies 3^2((10+10)^2 + (10-5)^2)$
$\implies (60)^2 + (15)^2$
$\implies a = 60 \land b = 15$
or
$\implies a = 15 \land b = 60$

Problem 5
Find a positive integer with property that,l if you move the first digit to the end, the new number is 1.5 times larger than the old one.

Solution:

Problem 6
Find all positive integers solutions (m, n, k) to the following equation

$$m^k = 1! + 2! + ... + n!$$

Solution:
$\forall x^k \pmod{10} =\equiv s \in \{0^k, (\pm 1)^k, (\pm 2)^k, (\pm 3)^k, (\pm 4)^k, 5^k\}$
$m^k \pmod{10} = 1 + 2 + 6 + 24 + 120 + 720 + 720 \times 7 + 720 \times 7 \times 8$
$m^k \pmod{10} = 3$
only power that can give 3 is $\{(\pm 3)^k\}$ in $\pmod{10}$

Problem 7
Find last 2 digits of 2^{2021}

Solution:
Last 2 digits of $2^{22} is 04$
$\implies 2^{19}(2^{22})^{91} \pmod{100}$
$\implies 2^{19}(2^2)^{91} \pmod{100}$
$\implies 2^{201} \pmod{100}$
$\implies 2^{22 \times 9 + 3} \pmod{100}$
$\implies (2^{22})^9 \times 2^3 \pmod{100}$
$\implies (2^2)^9 \times 2^3 \pmod{100}$
$\implies 2^{21} \pmod{100}$
Since last 2 digits of $2^{22} = 04 \implies 2^{21} = 52$

Hence 52 is 2 last digits of 2^{2021}

Problem 8
What is the smallest positive integer that can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers ?

Solution:
Sum of 9 consecutive integers can be written as $a = 9n + 36 \implies gcd(9, 45) \mid a = 9 \mid a$
Sum of 10 consecutive integers can be written as $a = 10m + 45 \implies gcd(10, 45) \mid a = 5 \mid a$
Sum of 11 consecutive integers can be written as $a = 11k + 55 \implies gcd(11, 55) \mid a = 11 \mid a$
$\implies gcd(11, 5, 9) = 1 \implies 495 \mid a$
Minimal a is when $495 = a$ with $n = 51$, $m = 45$, $k = -10$

Problem 9
Find the number of positive integer solutions to the equation $x^2 - y^2 = 2^{10}.37^5.41$

Solution:
Let's solve for $x^2 - y^2 = k$
$\implies (x - y)(x + y) = ab$ $\{k = ab$ **where a, b are integers** $\}$

$x - y = a$ and $x + y = b$
Solving the system of equation
$x = \frac{a+b}{2}$ and $y = \frac{b-a}{2}$

For x and y to have integer values a and b both needs to odd or both needs to be even.

k is an even number so calculate number of ordered pairs (a, b) where a and b are even

In this case both $a = 2r$ and $b = 2s$ needs to be even .
Hence $x = r + s$ and $y = s - r$ where $rs = 2^8.37^5.41$
Number of different divisors of rs is $9 * 6 * 2 = 108$.
Total divisors $= \#\{a < b\} + \#\{a > b\}$
Total divisors $= 2 \times \#\{a < b\}$
Hence ans is 54.

Problem 10
Let a and b be integer solutions to $17a + 6b = 13$. What is the smallest possible value for $a - b$?

Solution:
$17x + 6y = 1$ Since $gcd(17, 6) = 1$
$\implies x = 1 , y = -3$
$\implies 17(13x) + 6(13y) = 13$
$\implies 17(13) + 6(-39) = 13$
$\implies a = 13 \pm 6k$ and $b = -39 - \pm 17k$
$Min(a - b) \mid a - b > 0$
$\implies (min(13 + 39 \pm 6k \pm 17k))$

$\implies 52 \pm 23k$
$\implies k = 2 \mid min(a-b) = 6$
$a = 13 - 12 = 1,\ b = -5$
QED

Problem 11
Prove that there is no prime triplet of form p, $p+2$, $p+4$ except for 3, 5, 7

Solution:
Assume that all primes > 3 hence $p = 6k \pm 1$
$\implies p+2 = 6k+1 or 6k+3$
$\implies p+4 = 6k+3, 6k-1$
So triplet is $(6k-1, 6k+1, 6k+3)$ or $(6k+1, 6k+3, 6k-1)$
In either way 6k + 3 is not a prime except when its 3 Contradiction!!.
What about p = 2 ? But then p+2k is even and no prime except 2 is even . Hence triplet is rejected.
QED

Problem 12
Find $2^{50^{50}} \pmod{13}$

Solution:
$2^{-2} \times 2^{50^{50}+2} \pmod{13}$
$\implies 4^{-1} \times (2^6)^{\frac{50^{50}+2}{6}} \pmod{13}$
$\implies 4^{-1} \times (64)^{\frac{50^{50}+2}{6}} \pmod{13}$

$50^{50} + 2$ is divisible by 2
$\implies (50 \pmod 3)^{50} + 2 \equiv (-1)^{50} + 2 \equiv 0 \pmod 3$

$50^{50} + 2$ is divisible by 6

$(50 \pmod 4)^{50} - 2 \pmod 4$
$\implies (2^{50}) + 2 \pmod 4$
$\implies 2 \pmod 4$
$\implies k = \frac{50^{50}+2}{6} is odd$

$4^{-1} \times (-1)^k \pmod{13}$ k is odd
$\implies 7 \times (-1) \equiv 5 \pmod{13}$ Since $4 \times 7 \equiv 1 \pmod 1)3$
5 is the ans

Problem 13
For how many positive integers $n \mid \frac{n^4-n+2}{n+2}$ is also an integer ?

Solution:
$n \equiv -2 \pmod{n+2}$

$(-2)^4 - (-2) + 2 \pmod{n+2}$
$16 + 2 + 2 \pmod{n+2}$
$20 \pmod{n+2}$
$2^2 \times 5 \equiv 0 \pmod{n+2}$
$n = 2$ or $n = 3$ or $n = 8$ or $n = 18$
4 positive values of n

Problem 14
$p > 7$ is a prime number and $m = \frac{8^p-1}{7}$ chosen . Prove

$$2^{m-1} \equiv 1 \pmod{m} \tag{2}$$

Solution:

Problem 15
Solve $x^y + 1 = z$ in prime numbers.

Solution:
Every prime number except 2 is an odd number.
$x^y = z - 1$
Assume$z > 2$ hence always of form $2k + 1$
$\implies x^y = 2k$
x must be even . Only even prime is 2 implies $x = 2$
Solve $2^y + 1 = z$.
Theorem: If there exist an odd divisor of n then $2^n + 1$ is not a prime.
Counter-positive Theorem
If $2^n + 1$ is a prime them n is form 2^k.
$\implies 2^k = y$ which is a prime
$\implies 2 = y$, $k = 1$
Hence the prime triplet solution is (2, 2, 5)

Problem 15
How many triples $a, b, c \in \{1, 2, 3, ..., 10\}$ exist so that $a < b < c$?

Solution:
$\binom{10}{3}$

Problem 16
How many triples $a, b, c \in \{1, 2, 3, ..., 10\}$ exist so that $a \leq b \leq c$?

Solution:
$\binom{10+2}{3}$

Problem 17
In how many ways 10 couples can be selected from 10 woman and 15 men?

Solution:
$\binom{10}{10} \times \binom{15}{10} \times (10)^2$

Problem 18
In how many ways we can place 8 black and 12 white marbles on 8 by 8 chessboard?

Solution:

Problem 19
How three element subset of $\{1, 2, 3, ..., 30\}$ exist so that the sum of element are divisible by 3?

Solution:
$\{1, 2, 3, ..., 30\}$
$\{1, -1, 0, ..., 0\} \pmod 3$
There are 10 labeled one's , 10 labeled negative one and 10 zeroes.
{#ways to choose 3 labeled one } + {# ways to choose 1 labeled one and 1 labeled negative one and 1 labeled zero } + { all 3 labeled zeros }
$\binom{10}{3} + (\binom{10}{1})^3 + \binom{10}{3}$

Problem 20
How many 6-digit number can be constructed using 1, 2, 2, 3, 3, 3 ?

Solution:
$\frac{6!}{2!3!}$

Problem 21
How many 7-digit number can be constructed using 0, 1, 2, 2, 3, 3, 3 ?

Solution:
$\frac{6 \times 6!}{2! \times 3!}$

Problem 22
How many 5-digit number can be constructed using 1, 2, 2, 3, 3, 3?

Solution:
$\binom{6}{5} \times \frac{5!}{2!3!}$

Problem 23
How many 6-digit number can be constructed using 0, 1, 2, 2, 3, 3, 3?

Solution:
There are 6 ways to choose the digits with 0 and 1 way to choose without it.
Case 1 (2 NOT INCLUDED):$5 \times \frac{5!}{3!}$

Case 2 (3 NOT INCLUDED): $5 \times \frac{5!}{2!2!}$
Case 3 (0 NOT INCLUDED): $\frac{6!}{2!3!}$
Case 4 (1 NOT INCLUDED): $5 \times \frac{5!}{2!3!}$
Total of $\binom{2}{1}$ Case 1 and $\binom{3}{1}$ Case 2 and $\binom{1}{1}$ Case 3 and $\binom{1}{1}$ Case 4
$2 \times 5 \times \frac{5!}{3!} + 3 \times 5 \times \frac{5!}{2!2!} + \frac{6!}{2!3!} + 5 \times \frac{5!}{2!3!}$

Problem 24
How many 5 digit palindrome numbers exist?

Solution:
$9 \times 10 \times 10$

Problem 25
Two dices are thrown, and sum of numbers facing up is a. Again, two dices are thrown, and sum of numbers facing up is now b. Find probability of $a > b$.

Solution:
Favourable cases $\binom{11}{2} = 55$
Total cases 11^2
$p = \frac{5}{11}$

Problem 26
Two dices are thrown, and sum of numbers facing up is a. Then, a dice is thrown, and the numbers facing up is now b. Find probability of $a \leq b$.

Solution:
Favourable cases $\binom{11}{2} + 11 = 66$
Total cases 11^2
$p = \frac{6}{11}$

Problem 27
The English alphabet contains 26 letters. Find the number of 3-length words that contains either ”a” or ”e”.

Solution:
Total number of 3 letter words that can be made with 26 letters $-$ Total number of 3 letter words that can be made4 without $\{'a','e'\}$
$(26)^3 - (24)^3 = 3752$

Problem 28
Using numbers 1,2,3,4,5 without repetition, how many 5 digit number that is divisible by 4 can be constructed?

Solution:
{ 12, 24, 32, 52 } are the last 2 digits that can ensure that the number formed is divisible by 4 .
Rest can be filled up with 3! each.
$\implies 4 \times 3! = 24$

Problem 29
Using numbers 1,2,3,4,5 with repetition allowed, how many 5 digit number that is divisible by 4 can be constructed?

Solution:
{ 12, 24, 32, 52, 44 } are the last 2 digits that can ensure that the number formed is divisible by 4 .
Rest can be filled up with 5^3 each.
$\implies 5 \times 5^3 = 5^4 = 625$

Problem 30
Let $A = \{1, 2, 3\}$. Find the number of functions $f : A \to A$ such that $f(f(f(a))) = a$ for all $a \in \{A\}$

Solution:
f has two options
either be identity function
or be assigned to value $f(a) = b \mid f(f(b)) = a \wedge b \neq a$
if $f(b) = b \implies f(f(b)) = b \implies f(f(f(b))) = b \mid a \neq b$ so that case is rejected
$f(b) = c \mid f(c) = a \implies f(f(f(a))) = a \mid b \neq c \neq a$
so b has $(n-1)$ options and c has $(n-2)$ options

$f(n) = f(n-1) + (n-1)(n-2) * f(n-3) \mid \forall n > 3$
$f(1) = 1 f(2) = 1 f(3) = 3$
