Raveesh Gupta
Email: rav.gupta11@gmail.com

Homework - Problem Set 4

Problem 1
For given positive integers n and k , determine the number of length-k non decreasing integer sequences such that $1 \leq x_1 \leq x_2 ... \leq x_k \leq n$

Solution:
Simplification: Solving for $1 < x_1 < x_2 ... < x_k < n$ is much easier.

choosing k unique positive integers from n positive integers always has a single non decreasing arrangement. It's one - one mapped function. So the number of length-k non decreasing sequences is $\binom{n}{k}$.

BackTracking: Solving for $1 \leq x_1 \leq x_2 ... \leq x_k \leq n$ is similar to
the old solution as $a \leq b$ can be written as $a < b + 1$

$1 \leq x_1 < x_2 + 1 \ldots \leq x_k + 1 \leq n + 1$
$\implies 1 \leq x_1 < x_2 + 1 < x_3 + 2 < \ldots < x_k \leq n + k - 1$
which again has a solution $\binom{n+k-1}{k}$

Problem 2
The number 8^{2019} is written on the board. At each step it is replaced by the sum of its digits, until a 1-digit number is left. What is the one digit number ?

Solution:
Sum of digits is the property of mod 9 i.e $abcd \equiv a + b + c + d(mod9)$ hence
every time transition operation of summation of digits is done its modulus 9 remains the same , i.e the invariant property through transition.

Final state would be when $0 \leq n < 9$ which is the remainder, so one-digit number is
($8^{2019} \mod 9 \implies (-1)^{2019}(mod9) \implies -1(mod9) \implies 8$

Problem 3
Divisibility by 3

Solution:
$abcd \equiv 0(mod3)$
$\implies a10^3 + b10^2 + c10 + d \equiv 0(mod3)$
$\implies a(1000(mod3)) + b(100(mod3)) + c(10(mod3)) + d \equiv 0(mod3)$
$\implies a(999 + 1(mod3)) + b(99(mod3) + 1) + c(9 + 1(mod3)) + d \equiv 0(mod3)$
$\implies a + b + c + d \equiv 0(mod3)$

Problem 4
Divisibility by 4

Solution:
$abcd \equiv 0(mod4)$
$\implies a(10^3 mod4) + b(10^2 mod4) + c(10mod4) + d \equiv 0(mod4)$
$\implies 10c + d \equiv 0(mod4)$
$\implies cd \equiv 0(mod4)$

Problem 5
Divisibility by 5

Solution:
$abcd \equiv 0(mod5)$
$\implies a10^3 + b10^2 + c10 + d \equiv 0(mod5)$
$\implies a10^3(mod5) + b10^2(mod5)c10(mod5) + d \equiv 0(mod5)$
$\implies 0 + 0 + 0 + d \equiv 0(mod5)$
$d \equiv 0(mod5)$

d can be 0 or 5

Problem 6
Divisibility by 6

Solution: $6 = 2.3$

A number is divisible by 6 if only if it's divisible by 2 and 3.

Problem 7
Divisibility by 7

Solution:
$yabcd \equiv 0(mod7)$
$\implies y10^4 + a10^3 + b10^2 + c10 + d \equiv 0(mod7)$
$\implies y(3 \times 1000 mod7) + a(100(mod7) \times (10(mod7)) + b((10mod7) \times (10mod7)) + c(10mod7) + d \equiv 0(mod7)$
$\implies y(-3) + a(-1)b(2) + c(3) + (1)d \equiv 0(mod7)$

Every 3 consecutive digits must have 3 and 2 and 1. Starting with units place parity of sign changes with three consecutive digits.

Problem 8
Divisibility by 8

Solution:
$abcd \equiv 0(mod 8)$
$\implies a10^3 + b10^2 + c10 + d \equiv 0(mod 8)$
$\implies 0 + bcd \equiv 0(mod 8)$

Problem 9
Divisibility by 9

Solution:
$abcd \equiv 0(mod 9)$
$\implies a10^3 + b10^2 + c10 + d \equiv 0(mod 9)$
$\implies a(1000(mod 9)) + b(100(mod 9)) + c(10(mod 9)) + d \equiv 0(mod 9)$
$\implies a(999 + 1(mod 9)) + b(99(mod 9) + 1) + c(9 + 1(mod 9)) + d \equiv 0(mod 9)$
$\implies a + b + c + d \equiv 0(mod 9)$

Problem 10
Divisibility by 11

Solution:
$abcd \equiv 0(mod 11)$
$\implies a10^3 + b10^2 + c10 + d \equiv 0(mod 11)$
$\implies (-1)a + (1)b + (-1)c + d \equiv 0(mod 11)$

Problem 11
A sequence a_n is given by the recursion
$a_n = \frac{1}{\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} \cdots \frac{1}{a_{n-1}}}$
initial value $a_0 = 1$. Determine closed expression for a_n .

Solution:
Let's define $a_i = \frac{1}{b_i}$,
$\frac{1}{a_n} = \frac{1}{a_0} \cdots \frac{1}{a_{n-1}}$
$b_n = b_0 + b_1 + ... + b_{n-1}\ .\ .\ .\ (1)$
$b_{n-1} = b_0 + b_1 + ... + b_{n-2}\ .\ .\ .\ (2)$

Subtract eq (2) from eq (1) .

$\implies b_n - b_{n-1} = b_{n-1}$
$\implies b_n = 2b_{n-1} \forall n \geq 1$
$\implies b_n = 2^{n-1} \times b_0 \forall n \geq 1$
$\implies a_n = \frac{1}{2^{n-1}} \forall n \geq 1$

Problem 12
Find all positive integers d such that d divides both $n^2 + 1$ and $(n + 1)^2 + 1$ for some integer n.

Solution: $d|(n^2+1)-(n+1)^2-1$
$\implies d|2n+1$
$\implies d|2n^2+n$ and $d|2(n^2+1)$
$\implies d|2(n-2)$ and $d|2n+1$
$\implies d|5$ for any n

Problem 13
The 2-digit integers from 19 to 92 are written consecutively to form the integer $N = 192021...9192$
Suppose 3^k is the highest power of 3 that is a factor of n. What is k ?

Solution:
$3^k|N$
$\implies 3|N$
$N \equiv 9\times 1+18\times 2+17\times(3+4+5+6+7+8)+10\times 9(mod3)$
$\implies N \equiv 9+36+17\times 33+90(mod3)$
$\implies N \equiv 696(mod3)$
$3|N$
$N \equiv 696(mod9) \equiv 0(mod9)$
$k=2$

Problem 14
Let $n = 2^{31}.3^{19}$. How many positive integer divisors of n^2 are less than n but does not divide n.

Solution: Total number of divisors of n is 32×20 . 39×63 are the number of divisors of n^2 . Since perfect square has $2k+1$ divisors where k divisors are less than n and k divisors are more than n and 1 is n. $\frac{39\times 63-1}{2}$ is k implies $k = 1228$ where number of divisors of n is 640. The number of divisors of n^2 which are less than n and do not divide n is 1228 - 640 = 588.

Problem 15
$n \equiv 1mod2 \implies n^2 \equiv 1mod8$

Solution:
$n = 2k+1$
$n^2 = 4k^2+4k+1$
$n^2-1 = 4k^2+4k$

$n^2-1 \equiv 4k(k+1)(mod8)$
$n^2-1 \equiv 4\times 2x(mod8)$ As one of every 2 consecutive integers is even
$n^2-1 \equiv 8x(mod8)$
$n^2-1 \equiv 0(mod(8))$
$n^2 \equiv 1(mod(8))$

Problem 15
Find x_{100} mod 7 closed form where $x_1 = 4$, $x_2 = 10$ and $\forall x \geq 2x_{n+2} = 7x_{n+1} - 12x_n$.

Solution:
Using characteristic equation
$x^2 = 7x - 12$
$x^2 - 7x + 12 = 0$
$(x-4)(x-3) = 0$
$x_n = A.3^n + B.4^n$
$x_1 = 4 \implies 4 = 3A + 4B$
$x_2 = 10 \implies 10 = 9A + 16B$
Solving the system of equation gives $A = 2$ and $B = \frac{-1}{2}$
$x_n = 2.3^n - 2^{2n-1}$
$x_{100} = 2.3^{100} - 2^{199}$
$x_{100} mod7 = 2.3^{100} - 2.4^{99}$
$x_{100} mod7 = 2.3^{100} - 2.(-3)^{99}$
$x_{100} mod7 = 2.3^{99}(3+1)$ $x_{100} mod7 = -2.(27)^{33}.4$
$x_{100} mod7 = 2.(-1).4$
ans $= 6$

Problem 16
Solve for $x_{n+2} = 4x_{n+1} - 4x_n$, $x_1 = 2$ and $x_2 = 5$.

Solution:
Using characteristic equation
$r^2 = 4r - 4$
$r^2 - 4r + 4 = 0$
$(r-2)^2 = 0$
$x_n = A.2^n + B.n2^n$
$x_1 = 2 \implies 2 = 2A + 2B1 = A + B$
$x_2 = 5 \implies 5 = 4A + 8B$
$B = \frac{1}{4}$ and $A = \frac{3}{4}$
$x_n = \frac{3}{4}(2^n) + \frac{1}{2}(n2^n)$
