Problem Set - 3

Solutions by Raveesh

0.1 HW

1.1)

Find $1^4 + 2^4 + 3^4 + 4^4 \dots n^4$

Let $P(x)$ be a 5^{th} degree polynomial such that $P(x) - P(x-1) = Ax^4$.

$P(x) = x^5 + ax^4 + bx^3 + cx^2 + d^x + e$

$P(x) - P(x-1) = x^5 - (x-1)^5 + a(x^4 - (x-1)^4) + b(x^3 - (x-1)^3) + c(x^2 - (x-1)^2) + d(x - x + 1) = Ax^4$

$\implies x^4(5) + x^3(-10 + 4a) + x^2(10 - 6a + 3b) + x(-5 + 4a - 3b + 2c) + (d - c + b - a + 1) = Ax^4$

$A = 5$, $a = \frac{5}{2}$, $b = \frac{5}{3}$, $c = 0$ $d = -\frac{1}{6}$

$1^4 = \frac{P(1)-P(0)}{A}$
$2^4 = \frac{P(2)-P(1)}{A}$
.
.
.
$n^4 = \frac{P(n)-P(n-1)}{A}$

Adding all the above equations

$A * S = P(n) - P(0)$
$30 * S = 6n^5 + 15n^4 + 10n^3 - n$
$30 * S = n(6n^4 + 15n^3 + 10n^2 - 1)$
$S = \frac{n(6n^4+15n^3+10n^2-1)}{30}$

1.2

a)

Let S denote a set of all the triplets (i, j, k) of positive integers where $i + j + k = 17$. Find $\Sigma_{(i,j,k)\in S} i.j.k$.

Number of solutions for $\#(S)$

$i \to i'$ where $i > 0$ and $i' geq 0$

$i' + j' + k' = 14$

Let there be 14 stars and 5 bars such that 2 bars are fixed and three are movable
.

$\underbrace{***|**}_{\text{i - 1 stars and 1 bar}}$ $\underbrace{|***|**|}_{\text{j - 1 stars and 2 fixed bar and 1 movable bar}}$ $\underbrace{***|***}_{\text{k - 1 stars and 1 bar}}$

For each $(i, j, k) \in S$ where each $i, j, k > 0$ first bar has i options where the 2nd bar has j options and 3rd bar has k options hence $i \times j \times k$. It's equivalent to $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 14$ where $a_1, a_3, a_5, a_2, a_4, a_6 \geq 0$ question. Hence has a solution $\binom{14+6-1}{6-1}$

b1)

How many $f : A \to A$
functionsare there satisfying $f(f(a)) = a$ for every $a \in A = \{1, 2, 3, 4, 5, 6, 7\}$?

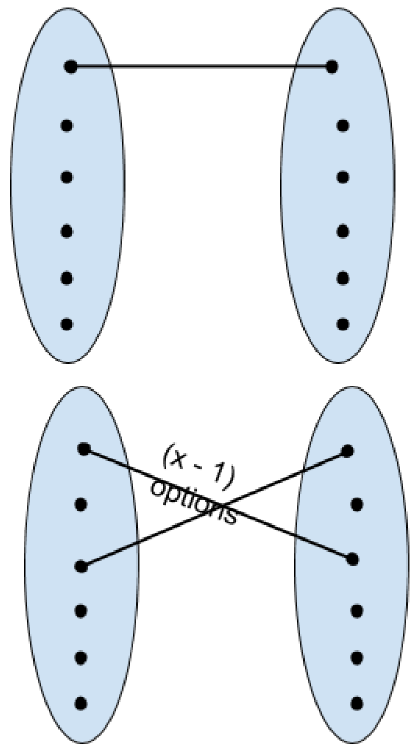

Figure 1: mapping

Figure 1: mapping

Let $T(n)$ be a recursive function that calculate number of functions of elements from A where size of A is n .

$T(n) = T(n-1) + (n-1) \times T(n-2)$
$T(1) = 1, T(2) = 2$
$T(3) = 2 + 2.1 = 4$
$T(4) = 4 + 3.2 = 10$
$T(5) = 10 + 4.4 = 26$
$T(6) = 26 + 5.10 = 76$
$T(7) = 76 + 6.26 = 232$

1.3)

$$
\begin{aligned}
&r = \sqrt{(2019 \times 2020 \times 2021 \times 2022 + 1}\\
&\implies r^2 = 2019 \times 2020 \times 2021 \times 2022 + 1\\
&\implies r^2 = (2020 - 1).(2021 - 1).(2020 + 1).(2021 + 1)\\
&\implies r^2 = (2020 - 1).(2021 + 1).(2021 - 1).(2020 + 1)\\
&\implies r^2 = (2020.2021 - 2).(2020.2021)\\
&\implies r^2 = (2020.2021 - 1 - 1).(2020.2021 - 1 + 1)\\
&\implies r^2 = ((2020.2021 - 1)^2 - 1)\\
&\implies r = 2020.2021 - 1
\end{aligned}
$$

0.2 Counting

2.1)

How many pairs of consecutive integers in $\{1000, 1001, 1002 \ldots 2000\}$ is no carrying around when added ?

$\#\{(A, B = A + 1), A \in [1000, 1999] |$ no carry is done when added $\}$

Let

$A = A_0A_1A_2A_3$

$B = B_0B_1B_2B_3$

where $a - b = 1$

Case 1 $B_0 = A_0 = 1$ $B_1 = A_1$ $B_2 = A_2$ $B_3 = A_3 + 1$ if no carrying is done then $B_3 + A_3 \leq 9$ $\implies 0 \leq A_3 \leq 4$ A_3 has 5 options and for A_1, A_2 $2x \leq 9$ $\implies x \leq 4.5$ $\implies 0 \leq x \leq 4$ that is 5 options. in total $5 \times 5 \times 5$ numbers	Case 2 $B_0 = A_0 = 1$ $B_1 = A_1$ $B_2 = A_2 + 1$ $B_3 = 0, A_3 = 9$ if no carrying is done then $B_2 + A_2 \leq 9$ $\implies 0 \leq A_2 \leq 4$ A_2 has 5 options and for A_1 $2x \leq 9$ $\implies x \leq 4.5$ $\implies 0 \leq x \leq 4$ that is 5 options. in total 5×5 numbers	Case 3 $B_0 = A_0 = 1$ $B_1 = A_1 + 1$ $B_2 = 0, A_2 = 9$ $B_3 = 0, A_3 = 9$ $B_3 + A_3 \leq 9$ if no carrying is done then $\implies 0 \leq A_1 \leq 4$ A_1 has 5 options in total 5 numbers

Special Case:

A = 1999, B = 2000 Hence 1 number

So in conclusion ans is $1 + 5 + 5^2 + 5^3 = 156$

2.2)

For given positive integers n and k , determine the number of length-k non decreasing integer sequences such that $1 \leq x_1 \leq x_2 ... \leq x_k \leq n$

Let $T(m, y)$ be the number of length-y non decreasing integer sequences such that $1 \leq x_1 \leq x_2 ... \leq x_y \leq m$

Transition function:

$T(m, y) = T(m, y-1) + T(m+1, y)$ if $y > 1$ and $m < n$

Base Case:

$T(n, i) = i$ where $\forall i \in [1, k]$

$T(m, 1) = n - m + 1$ where $\forall m \in [1, n-1]$

Ans is $T(1, k)$

Second Approach:

Simplification: Solving for $1 < x_1 < x_2 ... < x_k < n$ is much easier.

choosing k unique positive integers from n positive integers always has a single non decreasing arrangement. It's one - one mapped function. So the number of length-k non decreasing sequences is $\binom{n}{k}$.

BackTracking: Solving for $1 \leq x_1 \leq x_2 ... \leq x_k \leq n$ is similar to the old solution as $a \leq b$ can be written as $a < b + 1$

$1 \leq x_1 < x_2 + 1 \ldots \leq x_k + 1 \leq n + 1$

$\implies 1 \leq x_1 < x_2 + 1 < x_3 + 2 < \ldots < x_k \leq n + k - 1$

which again has a solution $\binom{n+k-1}{k}$

2.3)

A group of n friends wrote a math contest consisting of eight short-answer problems $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$ and four full-solution problems F_1, F_2, F_3, F_4 . Each person in the group correctly solved exactly 11 of the 12 problems. We create an 8×4 table. Inside the square located in the i_{th} row and j_{th} column, we write down the number of people who correctly solved both problem S_ii and problem
F_j
. If the 32 entries in the table sum to 256, what is the value of n?

There are $11n$ solutions submitted.

Its difficult to get number friends if we take table's perspective as $F_iS_j \cap F_iS_k \neq \phi$

So, we look at contribution technique,

Each person is either solving 8 full solutions and 3 short solutions or 7 full-solutions and 4 short solutions.

Implies 24 solutions or 28 solutions by each person.

Let's partition n into m and k such that $n = m + k$ where it represents the number of people solved 24 solutions and 28 solutions respectively.

Total = 256 solutions.

$$24m + 28k = 256$$
$$\implies 6m + 7k = 64$$

which has infinite solutions since $gcd(6, 7) = 1$ but only one solution where m and k are non negative integers.

$$\implies 7k = 4(mod 6)$$
$$\implies k = 4, m = 6$$
$$\implies n = 6 + 4 = 10$$

2.4)

Find the number of ordered quadruples of positive integers (a, b, c, d) such that a , b , c , and d are all (not necessarily distinct) factors of 30 and $abcd > 900$.

Simplification: $\#\{(a, b, c, d), a, b, c, d \in factor(30) | abcd = 900\}$

factors of 30 = $\{1, 2, 3, 5, 6, 10, 15, 30\}$ 8 values

prime factorization of 900 is $2^2 \times 3^2 \times 5^2$

There are 4 containers a, b, c, d that can only accept 2, 3 ,5 each . How many ways to distribute $2, 2, 3, 3, 5, 5$ i.e $\binom{4}{2} \times \binom{4}{2} \times \binom{4}{2} \implies 6^3 = 216$ ways .

Total number quadruples (a, b, c, d) possible $= 8^4$
Number of quadruples such that their product are not equal to 900 $8^4 - 216$

Such products will be strictly less than or strictly greater than 900 and they will be equal in number as in factors of 30 are in pairs a, b where $a < \sqrt{30} \wedge b > \sqrt{30}$. We can map each quadruple greater than 900 to
quadruple lessthan 900.

Replacing all variables $\{x|x < \sqrt{30}\}$ in the quadruple to $\frac{30}{x}$ and variables $\{x|x > \sqrt{30}\}$ to $\frac{30}{x}$ will make a one-one mapping . And all greater than values are mapped as replacing all variables $\{x|x > \sqrt{30}\}$ to $\frac{30}{x}$ and variables $\{x|x < \sqrt{30}\}$ in the quadruple to $\frac{30}{x}$ exists in the domain so it's onto. Hence they are equal number to less than 900 quadruples.

Hence : $\#\{(a, b, c, d), a, b, c, d \in factor(30)|abcd > 900\}$ is $\frac{8^4-216}{2}$

2.5)

Let $T = \{9^k, k \in [0, 4000]\}$ Give that 9^{4000} has 3817 digits and that it's leftmost digit is 9 , How many numbers in T have 9 as it's leftmost digit ?

QUESTION !!!!! HOW DID YOU THINK OF THIS FUNCTION !!!

Let's define a function that denotes number of digits of 9^k

$d(k) = d(k-1) + 1$ if only if 9^k doesn't have 9 as it's leftmost digit.

$d(k) = d(k-1)$ if only if 9^k does have 9 as it's leftmost digit.
$d(1) = d(0) = 1$

since 9^{4000} has 9 as its leftmost digit and has 3817 digits. then $d(3999) = 3817$, number of transitions from n digit number to n digit number is $4000 - 3816$ since 3816 times it was incremented from 1. Ans is 184.

2.6)

Eli, Joy, Paul, and Sam want to form a company; the company will have 16 shares to split among the 4 people. The following constraints are imposed:

Every person must get a positive integer number of shares, and all 16 shares must be given out. • No one person can have more shares than the other three people combined.

Assuming that shares are indistinguishable, but people are distinguishable, in how many ways can the shares be given out?

Let's fix an order,
a is Eli, b is Joy, c is Paul, d is Sam.

$a + b + c + d = 16$
$a + b + c \geq d$

$d = 1, 2, 3, 4, 5, 6, 7, 8$
$a + b + c = 15, 14, 13, 12, 11, 10, 9, 8$
$a' + b' + c' = 12, 11, 10, 9, 8, 7, 6, 5$

$\binom{5+2}{2} + \binom{6+2}{2} + \binom{7+2}{2} + \binom{8+2}{2} + \binom{9+2}{2} + \binom{10+2}{2} + \binom{11+2}{2} + \binom{12+2}{2}$

$(7.6 + 8.7 + 9.8 + 10.9 + 11.10 + 12.11 + 13.12 + 14.13)/2$
$(42 + 56 + 72 + 90 + 110 + 132 + 156 + 182)/2$
$21 + 28 + 36 + 45 + 55 + 66 + 78 + 91$
420

Since there are 4 ways of different selection but 3 are of the same class and 1 is of the other class. $\frac{4!}{3! \times 1!} \times 420 = 1680$

Second Approach

$\implies 2a + 2b + 2c \geq 16$
$\implies 16 > a + b + c \geq 8$ for $a, b, c > 0$
$\implies 13 > a' + b' + c' \geq 5$ for $a', b', c' \geq 0$

Number of ways $a' + b' + c' \leq 12 = \binom{12+4-1}{3}$
Number of ways $a' + b' + c' \leq 4 = \binom{4+4-1}{3}$

Solution is $\binom{15}{3} - \binom{7}{3}$
$455 - 35 = 420$

Since there are 4 ways of different selection but 3 are of the same class and 1 is of the other class. $\frac{4!}{3! \times 1!} \times 420 = 1680$

2.7)

How many squares are there in the xy-plane such that both coordinates of eac
h vertex are integers between 0 and 100 inclusive, and the sides are parallel to the axes?

$$1^2 + 2^2 ... + 100^2 = 338350$$

2.8) Find the number of 20-tuples of integers $x_1, x_2, ..., x_{10}, y_1, y_2, ..., y_{10}$ with the following properties.

- $1 \leq x_i \geq 10$ and $1 \leq y_i \leq 10$
- $x_i \leq x_{i+1}$
- if $x_i = x_{i+1}$, then $y_i \leq y_{i+1}$

Sol)

Let's divide the answer by cases . Simplest of case is when property 3 is not used, that is all x_i are distinct. Hence number of ways to choose 10 different numbers from $\{1, 10\}$ i.e $\binom{10}{10}$ and for each such selection y_i can have any value from $\{1, 10\}$ so answer is $\binom{10}{10} \times 10^{10}$

Case 2: would be if there are 9 distinct x_i and one repeated, hence $x_1 < x_2 ... < x_i = x_{i+1} < x_{i+2} ... < x_{10}$. The number of ways to have such different arrangements is $\binom{10}{9}$ and for each such arrangement there would be a $1 \leq y_i \leq y_{i+1} \leq 10$ which has $\binom{11}{2} \times 10^8$ combinations. so the answer is $\binom{10}{9} \times \binom{11}{2} \times 10^8$.

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Case k: would be if there are k distinct x_i and $10 - k$ repeated, hence $x_1 < x_2 < x_3 ... x_i = x_{i+1} = ... x_{10-k} < ... x_{10}$. The number of ways to have such different arrangements is $\binom{10}{k}$ and for each such arrangement there would be a $1 \leq y_i \leq y_{i+1} ... \leq y_{10-k} \leq 10$ which has $\binom{20-k}{10-k+1} \times 10^{k-1}$ combinations so, answer is $\backslash \binom{10}{k} \times \binom{20-k}{10-k+1} \times 10^{k-1}$.

Final answer is $\Sigma_{k=1}^{1} 0 (\binom{10}{k} \times \binom{20-k}{10-k+1} \times 10^{k-1})$.

1 Invariance Problems

3.1) The number 8^{2019} is written on the board. At each step it is replaced by the sum of its digits, until a 1-digit number is left. What is the one digit number ?

Sol) Sum of digits is the property of mod 9 i.e $abcd \equiv a+b+c+d(mod 9)$ hence every time transition operation of summation of digits is done its modulus 9 remains the same , i.e the invariant property through transition.

Final state would be when $0 \leq n < 9$ which is the remainder, so one-digit number is 8^{2019} mod 9 $\implies$ $(-1)^{2019}(mod 9)$ $\implies$ $-1(mod 9)$ $\implies$ 8

3.2)Starting state is $(20, 19, 2019)$. Every step $(a, b, c) \rightarrow (a+b-c, a+c-b, b+c-a)$ where a, b, c are integers . Can the final set of integers $(20, 20, 2020$ be reached ?
Sol) Invariance property of this problem is that the sum of integers $a+b+c = a+b-c+a+c-b+b+c-a$ remains the same upon transition function. Since initially the sum is 2058 and the wanted sum is 2060 , the wanted final state can't be reached.

3.3) n cards are placed side by side randomly either face up or face down. A move is choosing 2 different cards where the left one is face up and flipping the both (Right can either be face up or down) . Show that after a finite number of steps this process can't continue.

Representing face up as digit 1 and face down as digit 0 would make it easier to solve the problem.

Start state: (101001011)

Transition : Choosing any 10 or 11 $\rightarrow$ 01 or 00 respectively.

Final state : 00000000001 or 00000000000

Since on each transition the binary number is reducing i.e our invariant . it will reach a state where it can't reduce any further since any positive number can't reduce indefinitely hence the process stops.

3.4) There are 24 coins on a table with exactly 3 heads up. In one movement we can reverse exactly 4 coins. Is it possible to make all the coins heads up by less than 100 movements ?

Sol) Simplifying the question is to answer is it possible to make all coins heads in any number of moves. Let's represent the coins by binary digits 1 means tails and 0 means heads.

There are 5 possibilities:

1) 0000 $\rightarrow$ 1111

2) $0001 \rightarrow 1110$

3) $0011 \rightarrow 1100$

4) $0111 \rightarrow 1000$

5) $1111 \rightarrow 0000$

In any case the number of heads decreases or increases by 2, 4, 0 i.e invariant property. Since Initially the number of heads is odd , only an odd number of heads states can be reached upon operating the transition hence, 24 heads can never be reached as it's an even number.

3.5) Consider the number 1 through 6 placed on a circle in order. It is allowed to add one to three consecutive numbers or to subtract one from 3 numbers no two of which are adjacent . Is it possible to make all six numbers equal.

Let's represent the numbers by $(1, 2, 3, 4, 5, 6)$ where it's a cyclic list.

6) Transition: $(a, b, c, d, e, f) \rightarrow (a + 1, b + 1, c + 1, d, e, f)$ or $a - 1, b, c - 1, d, e - 1, f$

Final state: (k, k, k, k, k, k) is it possible to reach it ?

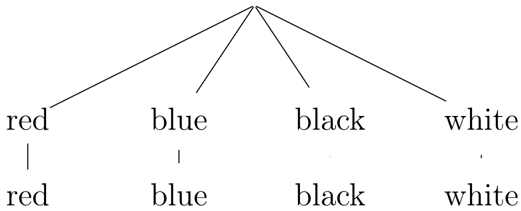
Invariant property : The difference between $\forall a_i$ anda_{i+3} decreases or increases on every transition.

Implies that a state could be reached where $a_i = a_{i+3}$ since $i + 3 + 3(mod6) \equiv i$ it implies that only 2 elements can have the same value a_i and a_{i+3} . Hence all a_i can't have the same value.

4) Miscellaneous

1) There are two red, two black, two white and a positive but unknown number of blue socks in a drawer. It's empirically determined that if two socks are taken from the drawer without the replacement, the probability they are of the same color is $\frac{1}{5}$. How many blue socks are in the drawer ?

   Let x be the unknown number of blue socks.



$$\frac{6}{(6+x)(5+x)} + \frac{x(x-1)}{(5+x)(6+x)} = \frac{1}{5}$$

$$\implies \frac{30+5x(x-1)}{(5+x)(6+x)} = 1$$

$$\implies 30 + 5x^2 - 5x = (6+x)(x+5)$$

$$\implies 30 + 5x^2 - 5x = 6x + 5x + x^2 + 30$$

$$\implies 4x^2 - 16x = 0$$

$$\implies x(x-4) = 0$$

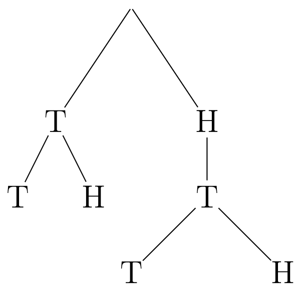
$$\implies x = 4$$

2) A fair coin is to be tossed 10 times. Let $\frac{i}{j}$ in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i + j$.

Let F(x) be the sample size if x coins are tossed where no heads are together.

$F(2) = \{TH, HT, TT\}$

$F(3) = \{TTT, THT, HTH, TTH, HTT\}$



$F(10) = F(9) + F(8) \implies 144$

$\implies F(9) = F(8) + F(7) \implies 89$

$\implies F(8) = F(7) + F(6) \implies 55$

$\implies F(7) = F(6) + F(5) \implies 34$

$\implies F(6) = F(5) + F(4) \implies 21$

$\implies F(5) = F(4) + F(3) \implies 13$

$\implies F(4) = F(3) + F(2) \implies 8$

The probability is $= \frac{144}{2^10} = \frac{(2^4.3^2)}{2^10} = \frac{9}{64}$

ans $= 73$

3) You have infinitely many boxes, and you randomly put 3 balls into them. The boxes are labeled \ (1, 2, . . . \ ) . Each ball has probability \ (\ frac{ 1} { 2ˆ n } \ ) of being put into a box \ (n \ ) . The balls are placed independently of each other. What is the probability that some box will contain at least 2 balls ?

Probability of a ball in any box $(p') = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} ... = 1$

Probability of second ball in the same box $(p') = {\frac{1}{2}}^2 + {\frac{1}{2^2}}^2 + {\frac{1}{2^3}}^2 ... = \frac{1}{3}$

Probability of third ball in any box $(p''') = \frac{1}{2} \times p'' + \frac{1}{2^2} \times p'' + \frac{1}{2^3} \times p'' ... = \frac{1}{3}$

and so on.

4) Define the Fibonacci number by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \forall ngeq 2$. How many $n, 0 \leq n \leq 100$ F_n a multiple of 13 ?

$F_7 = 13$

5) $GCD(F_n, F_7 = 13) = F_{GCD(7,m)}$

m is a multiple of 7 i.e 0 , 7, 14 98

15 fibonacci numbers are multiple of 13.

6) Find a closed expression for a_n . $a_n = \frac{1}{\frac{1}{a_0} + \frac{1}{a_1} + ... + \frac{1}{a_{n-1}}}$ $a_0 = 1$.

Let's define $a_i = \frac{1}{b_i}$,

$\frac{1}{a_n} = \frac{1}{a_0} + \frac{1}{a_1} + ... + \frac{1}{a_{n-1}}$

$b_n = b_0 + b_1 + ... + b_{n-1}$ . . . (1)

$b_{n-1} = b_0 + b_1 + ... + b_{n-2}$ . . . (2)

Subtract eq (2) from eq (1) .

$\implies b_n - b_{n-1} = b_{n-1}$

$\implies b_n = 2b_{n-1}, \forall n \geq 1$

$\implies b_n = 2^{n-1} \times b_0, \forall n \geq 1$

$\implies a_n = \frac{1}{2^{n-1}}, \forall n \geq 1$

7) Let N be number of subsets of $\{1, 2, 3, ..., 15\}$, including an empty set, which does not contain two consecutive numbers. Find the remainder of N when it's divided by 100.

Let's $T(n)$ be the number of subsets for a set of size \ (n \ ).

if $a_0 \in S$ then $T(n) \supset T(n-2)$

if $a_0 \notin S$ then $T(n) \supset T(n-1)$

$T(n) = T(n-1) + T(n-2)$

$T(1) = 2$

$T(2) = 3$

$T(3) = T(1) + T(2)$

$T(4) = T(2) + T(3)$

$T(5) = T(4) + T(5)$

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$T(15) = T(14) + T(13)$

8) What single digit n , does 91 divide $12345n789$?

$10^3 \times 12345n + 789 \equiv 0(mod91)$

Since $91 = 13 \times 7$

$\implies 10^3 \times 12345n + 789 \equiv 0(mod13)$

$\implies 789 - 45n + 123(mod13) \equiv 0$

$\implies 912 - 45n(mod13) \equiv 0$

$\implies 2 - 6n(mod13) \equiv 0$

$\implies 2 \times 1 + 3 \times 2 + 3 - 2 \times 4 - 3 \times 5 - 1 \times n + 2 \times 7 + 3 \times 8 + 9 \equiv 0(mod7)$

$11 - 23 - n + 24 \equiv 0(mod7)$

$-n + 35 \equiv 0(mod7)$

$-n \equiv 0(mod7)$

n is 0 or 7

$60 - 2(mod13) \equiv 0$ or $67 - 2(mod13) \equiv 0$

only $65(mod13) \equiv 0$ hence $n = 7$

9) In decimal representation $34! = 295232799039a0414084761860964 35b0000000$. What are a and b ?

since it's divisible by 3 then the sum of digits must be divisible by 3 and 11.

$136 + a + b \equiv 0(mod3)$

$a + b \equiv 2(mod3)$

and

$18 + a - b \equiv 0(mod11)$

$\implies a - b \equiv 4(mod11)$

10) What is the smallest positive integer that can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers ?

$-3 + -2 + -1 + 0 + 1 + 2 + 3 + 4 + 5 = 9$

$-4 + -3 + -2 + -1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 = 11$

$-5+-4+-3+-2+-1+0+1+2+3+4+5+6+7=13$

11) Let $n = 2^{31}.3^{19}$. How many positive integer divisors of \ (n^ 2\ ) are less than n but do not divide n ?

Total number of divisors of \ (n \ ) is 32×20 . 39×63 are the number of divisors of n^2 . Since a perfect square has $2k+1$ divisors where k divisors are less than n and k divisors are more than n and 1 is n. $\frac{39\times63-1}{2}$ is k implies $k = 1228$ where number of divisors of n is 640. The number of divisors of n^2 which are less than n and do not divide n is 1228 - 640 = 588.