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Homework - Problem Set 5

Problem 1
Find all positive integers d such that d divides both n^2+1 and $(n+1)^2+1$ for some integer n.

Solution: $d|(n^2+1)-(n+1)^2-1$
$\implies d|2n+1$
$\implies d|2n^2+n$ and $d|2(n^2+1)$
$\implies d|2(n-2)$ and $d|2n+1$
$\implies d|5$ for any n

Problem 2
How many subsets of the set $\{1,2,3,4,5...,10\}$ are there , that does not contain 4 consecutive integers ?

Solution: Let's $T(n)$ be the number of subsets for a set of size n.
if $a_0 \in S$ then $T(n) => T(n-1)$
if $a_0 \notin S$, $a_1 \in S$ then $T(n) => T(n-2)$
if $a_0 \notin S$, $a_1 \notin S$, $a_2 \in S$ then $T(n) => T(n-3)$
if $a_0 \notin S$, $a_i \notin S$, $a_2 \notin S$, $a_3 \in S$ then $T(n) => T(n-4)$

$T(n) = T(n-1) + T(n-2) + T(n-3) + T(n-4)$

Problem 3
Let $n = 2^{31}.3^{19}$. How many positive integer divisors of n^2 are less than n but does not divide n.

Solution: Total number of divisors of n is 32×20 . 39×63 are the number of divisors of n^2 . Since perfect square has $2k+1$ divisors where k divisors are less than n and k divisors are more than n and 1 is n. $\frac{39\times63-1}{2}$ is k implies $k = 1228$ where number of divisors of n is 640. The number of divisors of n^2 which are less than n and do not divide n is 1228 - 640 = 588.

Problem 4
In decimal representation
$34! = 295232799039a041408476186096435b0000000$.
What are a and b ?

Solution: Since it's divisible by 3 then the sum of digits must be divisible by 3 and 11.
$136 + a + b \equiv 0(mod3)$
$a + b \equiv 2(mod3)$
and

$18 + a - b \equiv 0(mod 11)$
$\implies a - b \equiv 4(mod 11)$

Problem 5
Find all positive integers n for which $3n - 4$, $4n - 5$, $5n - 3$ are all prime numbers.

Solution: Every prime number except 2 is an odd number. so
$3n - 4(mod(2)) \equiv n(mod(2)) \equiv 1$
$\implies n$ must be odd.

$4n - 5 \equiv -5 \equiv 1(mod 2)$
$1 \equiv 1(mod(2))$
$5n - 3 \equiv n + 1 \equiv 1(mod(2))$
$\implies n \equiv 0(mod 2)$

So there must be one prime number 2 and the other two are odd.
The smallest in the given is $3n - 4$ equal to 2.
$n = 2$ i.e the only n for which one prime is even and the other two are odd.

2, 3, 7 are the given prime

Problem 6
Find the last digit of 7^{7^7}

Solution: $7^{7^7} \equiv x(mod(10))$
$7^{7^7+1} \equiv 7x(mod(10)$
$49^{\frac{7^7+1}{2}} \equiv 7x(mod 10)$
$(-1)^{\frac{7^7+1}{2}} \equiv 7x(mod(10))$
$(7)^{-1} \times (-1)^{\frac{7^7+1}{2}} \equiv x(mod(10))$

$7^7 + 1 \equiv 0(mod 4)$
$(-1)^7 + 1 \equiv 0(mod 4)$
$-1 + 1 \equiv 0(mod 4)$
$0 \equiv 0(mod 4)$

Since $7^{-1} = 3(mod 10)$

$3(-1)^{2k} \equiv x(mod(10))$
$3 \equiv x(mod(10))$

ans $= 3$

Problem 7
$a - c|ab + cd \implies a - c|ad + bc$

Solution: $a - c \equiv 0(mod(a - c))$
$\implies a \equiv c(mod(a - c))$

$\implies ad \equiv cd(mod(a-c))$ .. eq (1)
$\implies ab \equiv cb(mod(a-c))$ .. eq(2)
Adding (1) and (2)
$\implies ab + cd \equiv cb + ad(mod(a-c))$
$\implies bc + ad \equiv 0(mod(a-c))$

Problem 8
$a \equiv b \equiv 1mod2 \implies a^2 + b^2 \neq c^2$

Solution:
$a = (2k+1)$, $b = (2k'+1)$
$a^2 = 4k^2 + 4k + 1$, $b^2 = 4k'^2 + 4k' + 1$
$a^2 + b^2 = 4(k^2 + k'^2) + 4(k+k') + 2$
$\implies 2|c^2 \implies 2|c \implies 4|c^2$
$a^2 + b^2 \equiv 2(mod4) \equiv c^2$
So, if $a^2 + b^2 = c^2$ exists
then $c^2 \equiv 0mod(4)$
Contradiction

Problem 9
$6|n^3 + 5n$

Solution: $n^3 + 5n \equiv 0(mod6) \implies (n^3 - n) \equiv 0(mod6)$
$\implies n(n-1)(n+1) \equiv 0(mod6)$
It must be divisible 2 and 3
$\implies n(n-1)(n+1) \equiv 0mod(2)$ Because every 2 consecutive integers are represented as n , (n -1) , one of which is even
$\implies n(n-1)(n+1) \equiv 0(mod3)$ Because every 3 consecutive integers are represented as k, k-1, k+1 where one of them is divisible by 3.

Problem 10
$30|n^5 - n$

Solution: $30|n^5 - n \implies 30|n(n-1)(n^2+1)(n+1)$
$30 = 2 \times 3 \times 5$ if 30 — k then 2 —k and 3 —k and 5—k
$6|(n+1)n(n-1)$ as shown earlier
$n(n-1)(n+1)(n^2+1) \equiv n(n-1)(n+1)(n^2-4)(mod(5))$
$\implies n(n-1)(n+1)(n-2)(n+2) \equiv 0(mod(5))$ Because every 5 consecutive integers are represented as k, k-1, k+1, k- 2, k+2 where one of them is divisible by 5.

Problem 11
Find n for which $120|n^5 - n$

Solution: $120 = 6 \times 5 \times 4$
$n^5 - n$ is divisible by 3 and 5 as shown in 3b.

$n^5 - n \equiv 0(mod(2))$
$n(n-1)(n+1)(n^2+1) \equiv n(n-1)(n+1)(n^2-1)(mod 2)$
$\implies n(n-1)^2(n+1)^2 \equiv 0(mod 2)$
$\implies n(n-1)^4 \equiv 0(mod 2)$

if n is odd then
$(2k+1)(2k)^4 \equiv 16 \times k^4(2k+1)(mod 2) \implies 0(mod 8)$
if n is even then
$2k(2k-1)^4 \equiv 0(mod 2) \implies$ 8 will not divide it

So for every $n = 2k+1$ the given expression is divisible by 120

Problem 11
$3|a$, $3|b \iff 3|a^2+b^2$

Solution:
if $3|a^2+b^2$ then $3|a$ and $3|b$
$a^2+b^2 \equiv 0(mod(3))$
$a^2 \equiv -b^2(mod(3))$
$a^2 \equiv 2b^2(mod(3))$
then
$a^2+b^2 \equiv 3b^2(mod 3)$
$3b^2 \equiv 0(mod(3))$
$\implies 3|b$

WLOG , Similarly $3|a$

if $3|a$ and $3|b$ then $3|a^2+b^2$

$a = 3k$, $b = 3k'$
$a^2+b^2 \equiv 9(k^2+k'^2)(mod(3))$
$\implies 0 \equiv a^2+b^2(mod(3))$
QED

Problem 12
$7|a$, $7|b \iff 7|a^2+b^2$

Solution:
if $7|a^2+b^2$ then $7|a)and7|b$
$a^2+b^2 \equiv 0(mod(7))$
$a^2 \equiv -b^2(mod(7))$
$a^2 \equiv 6b^2(mod(7))$
then
$a^2+b^2 \equiv 7b^2(mod 7)$
$7b^2 \equiv 0(mod(7))$
$\implies 7|b$

WLOG , Similarly $7|a$

if $7|a$ and $7|b$ then $7|a^2+b^2$
$a = 7k$, $b = 7k'$
$a^2+b^2 \equiv 49(k^2+k'^2)(mod(7))$
$\implies 0 \equiv a^2+b^2(mod(7))$
QED

Problem 13
$21|a^2+b^2 \implies 441|a^2+b^2$

Solution:
$a^2 \equiv -b^2(mod(21))$
$implies a^2 \equiv 20b^2(mod(21)$
$\implies a^2+b^2 \equiv 21b^2(mod(21))$
$\implies 21|b^2 \implies 3\times 7|b^2 \implies 3\times 7|b$

WLOG , $3\times 7|a$

$21|a \implies 441|a^2$
$21|b \implies 441|b^2$
Adding above two equations
$441|a^2+b^2$

Problem 14
$n \equiv 1mod2 \implies n^2 \equiv 1mod8$

Solution:
$n = 2k+1$
$n^2 = 4k^2+4k+1$
$n^2-1 = 4k^2+4k$

$n^2-1 \equiv 4k(k+1)(mod8)$
$n^2-1 \equiv 4\times 2x(mod8)$ As one of every 2 consecutive integers is even
$n^2-1 \equiv 8x(mod8)$
$n^2-1 \equiv 0(mod(8))$
$n^2 \equiv 1(mod(8))$

Problem 15
$6|a+b+c \iff 6|a^3+b^3+c^3$

Solution:
if $6|a^3+b^3+c^3$ then $6|a+b+c$
Using Property 1
$6|a^3+b^3+c^3-(a+b+c)$
$\implies 6|a^3-a+b^3-b+c^3-c$
$\implies 6|a(a-1)(a+1)+b(b+1)(b-1)c(c+1)(c-1)$
Earlier shown in 3a that $6|n(n+1)(n-1)$
$\implies 6|a+b+c$

if $6|a+b+c$ then $6|a^3+b^3+c^3$

Similarly WLOG , Replacing $a^3 + b^3 + c^3$ by $a + b + c$ in the above proof gives
$6|a^3 + b^3 + c^3$

QED

Problem 16
Alexei Ivanovich tosses a fair coin infinitely many times. He gains 1 ruble for each head that turns up and gains 2 rubles for each tail that turns up. Prove that the probability of scoring n rubles at some point is $\frac{1}{3}.(2 + (\frac{-1}{2})^n)$

Solution:
Let r_i be the probability of total cost i
$r_i \to \{probabilityofheads\} \times r_{i-1}$
$r_i \to \{probabilityoftails\} \times r_{i-2}$
$r_i = \frac{r_{i-1}+r_{i-2}}{2}$
$r_1 = \frac{1}{2}$
$r_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
Using the characteristic equation.
$2x^2 = x + 1$
$(x - 1)(2x + 1) = 0$
$r_i = A.(\frac{-1}{2})^n + B$
$r_1 = \frac{1}{2} = \frac{-A}{2} + B$
$r_2 = \frac{3}{4} = \frac{A}{4} + B$
Solving both equations get A $= \frac{1}{3}$ and B $= \frac{2}{3}$
$r_i = \frac{1}{3} \times (2 + (\frac{-1}{2})^i$

QED
