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**Homework - # 7**

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**Problem 1**
Prove $\forall p = 4n+3, p|a^2+b^2 \implies p|a, p|b$

*Solution:*
Claim : $\forall p = 4n+3, p|a^2+b^2 \implies p|a, p|b$
Assume $p|(a^2+b^2)$ but $p \not| a, p \not| b$
$a^2+b^2 \equiv 0 \pmod{p}$
$\implies a^2 \equiv -b^2 \pmod{p}$
$\implies a^{4k+2} \equiv -b^{4k+2} \pmod{p}$ **Power both sides by** $2k+1$
Fermat's theorem says $a^{p-1} \equiv 1 \pmod{p}$ hence, $a^{4k+2} \equiv 1 \pmod{p}$
$\implies 1 \equiv -1 \pmod{p}$
$\implies 2 \equiv 0 \pmod{p}$
$\implies p = 2$ Contradiction
QED

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**Problem 2**
Prove $21|(a^2+b^2) \implies 441|(a^2+b^2)$

*Solution:*
$21 \mid a^2+b^2 \implies 3 \mid a^2+b^2$
**Using property in Problem 1** $3 \mid a^2+b^2 \implies 3|a, 3|b \implies 9|a^2, 9|b^2$
$\implies 9 \mid a^2+b^2$

$21 \mid a^2+b^2 \implies 7 \mid a^2+b^2$
**Using property in Problem 1** $7 \mid a^2+b^2 \implies 7|a, 7|b \implies 49|a^2, 49|b^2$
$\implies 49 \mid a^2+b^2$

GCD of 49 and 9 is 1 $\implies 441 \mid a^2+b^2$
QED

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**Problem 3**
Prove if prime $p \equiv 3 \pmod{4}$, then $x^2 \equiv -1 \pmod{p}$ has no integer solution.

*Solution:*
Assume if $p \equiv 3 \pmod{4}$ then $x^2 \equiv -1 \pmod{p}$ has integer solution.

Fermat's theorem says $x^{4k+2} \equiv 1 \pmod{p}$
$\implies (x^2)^{2k+1} \equiv 1^{2k+1} \pmod{p}$
$\implies x^2 \equiv 1 \pmod{p}$
$\implies -1 \equiv 1 \pmod{p} \quad \{Given x^2 \equiv -1 \pmod{p}\}$
$\implies p = 2$

Contradiction
QED

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**Problem 4**
$\#n \in Z^+$ does $\frac{n^3+10}{n+3}$ have integer solution.

*Solution:*
$\frac{n^3+10}{n+3}$
$\implies \frac{n^3+27-17}{n+3}$
$\implies \frac{n^3+3^3}{n+3} - \frac{17}{n+3}$
$\implies -17 \equiv 0 \pmod{n+3}$
**17 is a prime number so $n+3$ is a factor of 17**
$\implies n+3 = 17$
$\implies n = 14$
or
$\implies n+3 = 1$
$\implies n = -2$
$n < 0$ its rejected
Only 1 solution

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**Problem 5**
Prove or Disprove $1463 \mid a^2+b^2 \implies 1463 \mid a,\ 1463 \mid b$

*Solution:*
$1463 = 11 \times 7 \times 19$
$\implies 11 \mid a^2+b^2 \implies 11 \mid a,\ 11 \mid b \ \ \{11 \equiv 3 \pmod 4\}$

$\implies 7 \mid a^2+b^2 \implies 7 \mid a,\ 7 \mid b\ \{7 \equiv 3 \pmod 4\}$

$19 \mid a^2+b^2 \implies 19 \mid a, 19 \mid b\ \{19 \equiv 3 \pmod 4\}$

QED

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**Problem 6**
Prove or Disprove $1001 \mid a^2+b^2 \implies 1001 \mid a,\ 1001 \mid b$

*Solution:*

$1001 = 7 \times 13 \times 11$

$\implies 11 \mid a^2+b^2 \implies 11 \mid a,\ 11 \mid b \ \ \{11 \equiv 3 \pmod 4\}$

$\implies 7 \mid a^2+b^2 \implies 7 \mid a,\ 7 \mid b\ \{7 \equiv 3 \pmod 4\}$

$13 \mid a^2+b^2$
For $a = 12$ and $b = 5$, $13 \mid a^2+b^2$ but $12 \equiv -1 \pmod{13}$ and $5 \equiv 5 \pmod{13}$

Since $13 \not| a$, $13 \not| b \implies 1001 \not| a$, $1001 \not| b$

$\implies 77|a \implies a = 77k$ , $77|b \implies b = 77l$
$\implies a \equiv 77k \equiv -k \equiv -1 \pmod{13}, \implies b \equiv 77l \equiv -l \equiv 5 \pmod{13}$

$\implies k = 1$ , $l = 8$
$\implies a = 77$ and $b = 8 \times 77$
But $13 \mid a^2 + b^2 \implies 13 \mid (77)^2(8^2 + 1^2) \implies 13 \mid (1001).(77).5$
QED

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**Problem 7**
Find the number of positive integer solutions to the equation $x^2 - y^2 = 37^5 \times 41$

*Solution:*
Let's solve for $x^2 - y^2 = k$
$\implies (x - y)(x + y) = ab$ $\{k = ab$ **where a, b are integers** $\}$

$x - y = a$ and $x + y = b$
Solving the system of equation
$x = \frac{a+b}{2}$ and $y = \frac{b-a}{2}$

For $x$ and $y$ to have integer values $a$ and $b$ both needs to odd or both needs to be even.

Since there 12 divisors of $37^5 \times 41$ and all divisors are odd since 2 isn't a factor.
There are 12 pairs $(a, b)$ having unique relation to ordered pair integer solution $(x, y)$ to the equation $x^2 - y^2 = 37^5 \times 41$ .

$y > 0 \implies b - a > 0 \implies b > a$ Hence 6 pairs of (a, b) gives (x, y).

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**Problem 8**
Can the number A consisting of 600 sixes and some zeros be a square?

*Solution:*
Any $x^2 \pmod 4 \equiv s \in \{0, 1\}$
To Prove: A consisting of 600 sixes and some zeros is not a square
Since it's a square it must have even number of zeros at the end.
Removing it completely should also give us a square since $k^2 = 10^{2k} \times m^2$
$m^2$ can have 66 or 06 in the end.
Since $66 \pmod 4 \equiv 2$ and $06 \pmod 4 \equiv 2$
Both cases $k^2 \pmod 4 \equiv 2$ Which is a contradiction.
QED

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**Problem 9**
How often does the factor 2 occur in the product $(n + 1)(n + 2)\Delta\Delta\Delta(2n)$ ?

*Solution:*
Property 1: $\left\lfloor \frac{a}{b} \right\rfloor$ is number of multiples of $b \in [1, a]$

In other words find the sum of multiples of $2, 2^2, ..., 2^k$ in $(n+1)(n+2)\Delta\Delta\Delta(2n)$ such that $2^{k+1} > 2n$

Number of multiples of 2 in the given expression is $\implies \left\lfloor \frac{2n}{2} \right\rfloor - \left\lfloor \frac{2n}{2^2} \right\rfloor$

Number of multiples of $2^2$ in the given expression is $\implies \left\lfloor \frac{2n}{2^2} \right\rfloor - \left\lfloor \frac{2n}{2^3} \right\rfloor$
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Number of multiples of $2^k$ in the given expression is $\implies \left\lfloor \frac{2n}{2^k} \right\rfloor - \left\lfloor \frac{2n}{2^{k+1}} \right\rfloor \implies \left\lfloor \frac{2n}{2^k} \right\rfloor$ Since $2^{k+1} > 2n \implies \left\lfloor \frac{2n}{2^{k+1}} \right\rfloor = 0$

Summing up all the multiples
$\implies \left\lfloor \frac{2n}{2} \right\rfloor = n$

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**Problem 10**
Prove $9 \mid a^2 + b^2 + ab \implies 3 \mid a,\ 3 \mid b$

*Solution:*
Any $k^2 \pmod 9 \equiv s \in \{0, 1, 4, 7, -8, -5, -2\}$
Also any $3k^2 \pmod 9 \equiv s \in \{0, 3, -6\}$

$9 \mid a^2 + b^2 + ab \implies 4(a^2 + b^2 + ab) \equiv 0 \pmod 9$
$\implies (2a + b)^2 + 3b^2 \equiv 0 \pmod 9$
$\implies (2a + b)^2 \equiv -3b^2 \pmod 9$

There isn't any $\{-0, -3, 6\}$ in $\{0, 1, 4, 7, -8, -5, -2\}$ except 0
$\implies 3b^2 \equiv 0 \pmod 9$
$\implies 9k = 3b^2$
$\implies 3k = b^2$
$\implies 3 \mid b^2 \implies 3 \mid b \ . \ . \ . \ (1)$

Also
$(2a + b)^2 \equiv 0 \pmod 9$
$\implies 2a + b \equiv 0 \pmod 3$ Using if $p^2 \mid n^2 \implies p \mid n$
Using (1)
$\implies 2a \equiv 0 \pmod 3$
$\implies a \equiv 0 \pmod 3$
$\implies 3 \mid a$
$\square$

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**Problem 11**
Find all positive integers solutions $(m, n)$ to the following equations

$$m^2 = 1! + 2! + 3! + ... + n!$$

*Solution:*
Claim: To prove there doesn't exist a solution $(m, n)$ for $n > 3$
Assume there exist a solution $(m, n)$ for $n > 3$
$m^2 \equiv 1 + 2 + 6 + 24 (mod(5))$
$\implies m^2 \equiv 33 (mod(5))$
Any $x^2$ mod 5 is in $\{0, 1, 4\}$
$\implies m^2 \equiv 3 (mod(5))$
$m^2$ mod 5 can't be 3 hence, contradiction

QED.

for $n = \{1, 3\}$
$1^2 = 1!$
$3^2 = 1! + 2! + 3!$
Hence (3, 3) and (1,1 ) are the only solutions

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**Problem 12**
Is it possible to arrange the numbers $1^1, 2^2, ..., 2008^{2008}$ one after the other in such a way the obtained number is a perfect square ?

*Solution:*

Sum of the digits is not dependent upon the order of the numbers.
Assume given numbers make a square let it be $k^2$ then $k^2$ mod 3 must be in $\{1, 0\}$

$1 + -1 + 0 + 1 + 1 + 0 + 1 + -1 + 0 + .... + 1 + 1 + 0 + 1 + -1 + 0 + 1 \mod 3$
$\equiv 2 \times 334 + 1 + 1 + 0 + 1 \mod 3$
$\equiv 668 + 3 \mod 3$
$\equiv 671 \mod 3$
$\equiv 2 \mod 3$
$2 \notin \{0, 1\}$
CONTRADICTION !

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**Problem 13**
Find the number of positive integer solutions to the equation $x^2 - y^2 = 2^{10}.37^5.41$

*Solution:*
Let's solve for $x^2 - y^2 = k$
$\implies (x - y)(x + y) = ab$ $\{k = ab$ **where a, b are integers** $\}$

$x - y = a$ and $x + y = b$
Solving the system of equation
$x = \frac{a+b}{2}$ and $y = \frac{b-a}{2}$

For $x$ and $y$ to have integer values $a$ and $b$ both needs to odd or both needs to be even.

$k$ is an even number so calculate number of ordered pairs $(a, b)$ where a and b are even

In this case both $a = 2r$ and $b = 2s$ needs to be even .
Hence $x = r + s$ and $y = s - r$ where $rs = 2^8.37^5.41$
Number of different divisors of $rs$ is $9 * 6 * 2 = 108$.
Total divisors = $\#\{a < b\}$ + $\#\{a > b\}$
Total divisors = $2 \times \#\{a < b\}$
Hence ans is 54.

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