SPECIAL FIBERS OF SHIMURA VARIETIES IN THE TORELLI LOCUS

BY

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DISSERTATION

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Abstract

In this dissertation, we study the special fibers of thirteen Shimura varieties with an irreducible component containing a dense subset of Jacobians. The relevant Jacobians arise from branched non-abelian Galois covers of the projective line. Using the action of the Galois group of the cover, we are able to determine the position of the special fibers of these Shimura varieties inside the Siegel modular variety with respect to the Newton and Ekedahl–Oort stratifications in positive characteristic.

For Akka

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Chapter 1

Introduction

1.1 General introduction

This thesis is an attempt at a mathematical analog of taxonomy. Algebraic geometry is the study of geometric objects called *algebraic varieties*. Like any kind of object, algebraic varieties have various attributes or invariants which can be used to classify them into various collections. Often times, the collections themselves turn out to exhibit unexpected properties making them worthy of study in their own right. The main motif of this thesis is the interaction of two such collections in arithmetic algebraic geometry.

The most basic attribute of an algebraic variety is a natural number called its *dimension*. So the collection of algebraic varieties can be sorted into various bins labeled $0, 1, 2, \ldots$ based on their dimension. Algebraic varieties of dimension 0 are points, those of dimension 1 curves, those of dimension three surfaces and so on. We now consider the task of further organizing each bin based on the objects inside it.

As individual points are not particularly interesting, the next interesting bin to further organize is that of curves. Each curve, an object of dimension 1, has another interesting attribute called it's *genus*, also a natural number. This attribute is the dimension of space of differentials on the curve that are regular everywhere. More geometrically, it just so happens that curves over the complex numbers look like many-holed donuts, and the genus of a curve simply counts the number of holes the corresponding donut has.

The invariants, dimension and genus, that we have used so far are *discrete*. But to further organize the bin of curves with a fixed genus g, we must use some continuous invariants. As an analogy, consider how the taxonomist first neatly sorts the varied living creatures on earth into successively finer but discrete categories such as Kingdom, Phylum, Class, Order, Family, Genus and Species. But to distinguish the creatures in a fixed species, say *homo sapiens*, we must use continuous attributes of individuals such as height, weight, etc.

In algebraic geometry, once one has used enough discrete invarants, we can often build a geometric space called a *moduli space* to neatly arrange and parameterize objects with a fixed set of discrete invariants. The points of this space correspond to the objects we want to describe and the closer two points are in this space, the more similar the corresponding objects are. It is quite common in the history of mathematics that moduli spaces have often become interesting objects meriting study for their own sake.

One of the two main types of objects of study of this thesis are the moduli spaces of smooth projective curves of genus g, denoted $\mathfrak{M}_g$. The other is the moduli space of principally polarized abelian varieties (typically shortened to *ppav*) of dimension g, denoted $\mathcal{A}_g$. Both of these moduli spaces are very heavily studied in algebraic and arithmetic geometry. As the names suggest, these spaces parametrize objects called curves and ppavs respectively. There is a connection between these two spaces that goes by the name of the *Jacobian construction*: Namely, given a curve C in $\mathfrak{M}_g$, we can construct an associated ppav $J(C)$ called the Jacobian of C in such a way that if $J(C_1)$ and $J(C_2)$ are the same ppav in $\mathcal{A}_g$, then C_1 and C_2 are the same curve in $\mathfrak{M}_g$. From the perspective of $\mathcal{A}_g$, we may think of the space $\mathfrak{M}_g$ as a very familiar source of abelian varieties.

Now, as the integer g grows larger, the spaces $\mathfrak{M}_g$ and $\mathcal{A}_g$ grow in dimension as well. However, the dimension of the space $\mathcal{A}_g$ grows far faster than the spaces $\mathfrak{M}_g$. Thus there is no chance for the Jacobian construction to yield every point of $\mathcal{A}_g$ for

$g \geq 4$. Thus, a natural problem is to try and characterize those $A \in \mathcal{A}_g$ for which there is a curve $C \in \mathfrak{M}_g$ such that $J(C) = A$ as a ppav.

This thesis focuses on this problem not over the classical fields of characteristic 0 but over finite fields $\mathbb{F}_q$ and their algebraic closures $\overline{\mathbb{F}_q}$ of characteristic p, an odd prime. In this setting, the spaces $\mathcal{A}_g$ can be naturally further sub-divided into smaller spaces that fit together to form $\mathcal{A}_g$. So, even though the Jacobian construction fails to capture all of $\mathcal{A}_g$, we can still ask if each of the smaller spaces that compose $\mathcal{A}_g$ at least contain one Jacobian of a curve in $\mathfrak{M}_g$. We address this issue in genera $4, 5$ and 7 in this thesis.

1.2 Technical introduction

On a more technical note, the purpose of this thesis is to continue the search for new Newton polygons in the Torelli locus for new genera $g \geq 4$ and various positive characteristics. This most closely builds on the recent work of Li, Mantovan, Pries and Tang in [LMPT19]. In this section, we review some background, state our main results and describe the structure of this thesis.

Given a geometrically integral, smooth projective curve C over $\mathbb{F}_q$ (a *nice* curve), denote its Jacobian by $J(C)$. If C has genus g, then $J(C)$ is an abelian variety of dimension g. The curve C induces a canonical principal polarization on the abelian variety $J(C)$.

The moduli space of principally polarized abelian varieties of a fixed dimension g, denoted $\mathcal{A}_g$, is a central object of study in arithmetic geometry. For $g \geq 2$, it has dimension $\frac{g(g+1)}{2}$. The moduli space of smooth curves of genus g, denoted $\mathfrak{M}_g$, is $3g-3$ dimensional for $g \geq 2$. The Jacobian construction yields a map $J : \mathfrak{M}_g \to \mathcal{A}_g$. The image of this map is called the open Torelli locus. It is known that the Torelli morphism is injective and is, in fact, an immersion outside the hyperelliptic locus.

For $g \geq 3$, the map J is not surjective purely for dimension reasons. It is a classical problem to characterize the image of $\mathfrak{M}_g$ inside $\mathcal{A}_g$. If k were, unlike in our situation, a field of characteristic 0, then the Torelli locus could be studied using differential geometry and the Hodge theory of period mappings. However, in positive characteristic, this is not an option and we must employ different tools, most notably the Frobenius endomorphism.

A common approach to this problem in characterstic p involves stratifications of $\mathcal{A}_g$ that are defined using the Frobenius endomorphism. A *stratification* is simply a decomposition of the space into a finite number of disjoint sets (called *strata*) that are locally closed and such that the closure of a given stratum is a union of strata. Even though the map J is typically not surjective, we can still ask if its image intersects every stratum of a given stratification.

In characteristic $p > 0$, the space $\mathcal{A}_g$ admits many stratifications through which it is traditionally studied and we describe them now.

These stratifications are not available in the characteristic 0 setting due to some fundamental differences in the geometry of $\mathcal{A}_g$ over an algebraically closed field of characteristic 0 and a field k of characteristic $p > 0$. In the former setting, the n-torsion points of any abelian variety A of dimension g are always n^{2g} in number. In fact, they form a subgroup $A[n] \subset A$ that is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$. Thus, when we examine the behavior of abelian varieties with a focus on its n-torsion over an algebraically closed field in characteristic 0, we find that the moduli space $\mathcal{A}_g$ appears to be identical everywhere.

However, in the characteristic $p > 0$ setting, the number of n-torsion points of an abelian variety A of dimension g is only equal to n^{2g} when n is coprime to p. If $n = p$, then the number of p-torsion points of A is p^i where i is an integer in the range $0 \leq i \leq g$. The integer i is called the p-rank of the abelian variety. This immediately suggests a partition of the space $\mathcal{A}_g(\overline{k})$ into $g + 1$ disjoint subsets based on their p-rank, and this is essentially the p-rank stratification.

Two more stratifications that are finer than (and refine the) p-rank stratification are the Ekedahl–Oort stratification and the Newton stratification. On the geometric points of $\mathcal{A}_g$, a stratification can be expressed as a map to a finite set such that the fibers of this map yield the different strata. The p-rank stratification assigns to each $A \in \mathcal{A}_g$ an integer i in the range $0 \leq i \leq g$ such that the number of p-torsion points in $A(\overline{k})$ is p^i. The integer i is called the *p-rank* of A. The possible isomorphism classes of $A[p]$, the p-torsion group scheme of A, is a finite set E_g for abelian varieties of a fixed dimension g. The assignment $A \mapsto A[p]$ thus yields a discrete invariant called the *Ekedahl–Oort type*. Finally, there are only finitely many possible isogeny classes of the Barsotti–Tate group $A[p^\infty]$. The set of these possibilities is denoted $\mathcal{N}_g$ and its elements are called *Newton polygons*. For all these stratifications and the codimension of each stratum in $\mathcal{A}_g(\overline{k})$, the closure relations among the strata are explicitly known.

Both the E_g and $\mathcal{N}_g$ have natural combinatorial interpretations as directed posets with maximum and minimum elements. The set of abelian varieties in the maximum Ekedahl–Oort stratum is the same as those in the maximum Newton stratum. These are called *ordinary* abelian varieties. The abelian varieties corresponding to the minimum Newton polygon stratum are called *supersingular*. These abelian varieties are isogenous to a product of elliptic curves. The abelian varieties corresponding to the minimum Ekedahl–Oort stratum are called *superspecial*. These abelian varieties are isomorphic to a product of elliptic curves. Roughly speaking, the higher the stratum in these directed posets, the higher its dimension is inside $\mathcal{A}_g$. For example, the ordinary stratum is always open and dense in $\mathcal{A}_g$ while the superspecial stratum always has dimension 0.

For abelian varieties of dimension $g = 1$, namely elliptic curves, there are only two possible Newton polygons (equivalently Ekedahl–Oort types) corresponding to ordinary and supersingular elliptic curves respectively. In this setting of $g = 1$, the p-rank, Newton polygon and Ekedahl–Oort strata all coincide. Even for slightly larger genera, i.e. $g = 2, 3$, for each possible Newton polygon or Ekedahl–Oort type, smooth curves have been found in every characteristic $p > 0$ with these invariants.

For $p = 2$, van der Geer and van der Vlugt have shown that for every genus $g \geq 1$, there exists a supersingular curve of genus g over $\mathbb{F}_2$. For higher genera, the situation becomes difficult very quickly. Despite a great deal of interest in this question, even basic questions regarding this interaction remain unanswered. For instance, it is not known whether the supersingular stratum of $\mathcal{A}_g$ always intersects the open Torelli locus for every genus g and characteristic p.

In recent work, Li–Mantovan–Pries–Tang [LMPT19] have combined work of Moonen [Moo10] with work of Viehmann–Wedhorn [VW13] to produce many hitherto-unknown Newton polygons in the open Torelli locus. More precisely, they consider twenty Shimura varieties described by Moonen and compute the exact Newton polygons that appear in each family using the main results of Viehmann–Wedhorn [VW13]. Since these Shimura varieties arise from applying the Torelli morphism to families of cyclic covers of the projective line, they are contained in the open Torelli locus. Consequently, all the Newton polygons computed above are realizable in the Torelli locus. Moonen's original work implies that these twenty families are the only positive dimensional Shimura varieties that both arise from cyclic covers of $\mathbb{P}^1$ and are entirely contained in the open Torelli locus. So this method is exhausted for families of cyclic covers of $\mathbb{P}^1$.

We apply the same strategy to some PEL-type Shimura varieties arising from non-abelian Galois families of curves lying in the Torelli locus. In [FGP15], the authors find twenty families of non-cyclic Galois covers of $\mathbb{P}^1$ whose images under the Torelli morphism are Shimura varieties. There are two significant points of departure from their method. First, since our curves are non-abelian, the structure of the set of possible Newton polygons realized using these families is more complicated. Second, we do not always have explicit equations defining the universal family of curves in the non-abelian setting. Generalizing the method of [LMPT19] using the Chevalley–Weil formula, we compute the set of Newton polygons occuring in thirteen PEL-type Shimura varieties with an irreducible component containing a dense subset of Jacobians.

1.3 Results

We now describe the main results of this thesis more precisely. We use the definition of PEL-type Shimura variety from Section 2.1 of [VW13]. There is some ambiguity in the definition of a Shimura datum and a Shimura variety which we now clarify for this thesis. Some authors require that the algebraic group G appearing in the Shimura datum (G, X) be connected to call the resulting moduli space a Shimura variety. We require no such condition here. The only difference is that what we call a Shimura variety here could be a disjoint union of connected Shimura varieties in the other sense.

Furthermore, it is known that in every connected component of a PEL-type Shimura variety, there is a unique (open) stratum of maximum dimension called the μ-ordinary stratum and a unique (closed) stratum of minimum dimension called the basic stratum. For instance, the μ-ordinary stratum of the moduli space of principally polarized abelian varieties $\mathcal{A}_g(\overline{k})$ consists exactly of the ordinary abelian varieties, i.e. those $A \in \mathcal{A}_g$ for which $|A[p](\overline{k})| = p^g$. The basic locus of $\mathcal{A}_g(\overline{k})$ consists of abelian varieties that are isogenous to a product of supersingular elliptic curves. If the connected component of a PEL-type Shimura variety is one dimensional then the μ-ordinary and basic strata are the only ones that appear.

1.3.1 Newton Polygons

In this thesis, we calculate the possible Newton Polygons appearing in the thirteen families together with any congruence restriction that may need to be imposed. Here the notation ord^n refers to a Newton polygon with n slopes of 0 and n slopes of 1. Similarly the notation ss^k signifies a Newton polygon with $2k$ slopes of $\frac{1}{2}$. The symbol $\oplus$ is used between two Newton polygons to signify a new, larger Newton polygon that has all the slopes of the original two counted with multiplicity.

Theorem 1 (Theorem 20)**.** *Let Sh be an integral model of one of the* 13 *PEL-type Shimura varieties arising from families of non-abelian covers of* $\mathbb{P}^1$ *defined in Table 2.1 in Chapter* 3*. At each sufficiently large prime p of good reduction of Sh, the possible* μ*-ordinary and basic polygons that can be realized using the points of Sh are tabulated below:*

However, unlike in Moonen's result on the completeness of the list of cyclic families, the list of families found in [FGP15] is not proved to be complete. This is becuse their computation only relies on a sufficient condition for when the image of a family of Galois covers is a Shimura subvariety of $\mathcal{A}_g$. Since the work of [FGP15], there have been partial results precluding other families of non-cyclic abelian covers from yielding Shimura subvarieties, for example in [MZ18]. Nevertheless, the computations of [FGP15] naturally lead one to guess that there are no positive dimensional Shimura varieties in the Torelli locus for genus $g \geq 8$.

Of all the families in the table, the most interesting ones are those of genus $4, 5$ and 7 as many families of curves of lower genera have been studied previously. Even among the families of curves of high genus, family (36) is particularly amenable to further examination and we study it in greater detail.

Id	g	Conditions	μ-ordinary	Basic
(28)	2	-	ord^2	ss^2
(29)	2	-	ord^2	ss^2
(30)	2	-	ord^2	ss^2
(31)	3	-	ord^3	ss^3
(32)	3	-	ord^3	ss^3
(33)	3	-	ord^3	ss^3
(34)	3	$p \equiv 1 \pmod 4$, $p \equiv 3 \pmod 4$	ord^3 $\mathrm{ord} \oplus \mathrm{ss}^2$	$\mathrm{ord}^2 \oplus \mathrm{ss}$ ss^3
(35)	3	-	ord^3	ss^3
(36)	4	$p \equiv 1, 7 \pmod 8$ $p \equiv 3, 5 \pmod 8$	ord^4 $ord^2 \oplus \mathrm{ss}^2$	ss^4 ss^4
(37)	4	$p \equiv 1 \pmod 3$ $p \equiv 2 \pmod 3$	ord^4 $\mathrm{ord}^3 \oplus \mathrm{ss}$	$\mathrm{ord} \oplus \mathrm{ss}^3$ ss^4
(38)	4	$p \equiv 1 \pmod 3$, $p \equiv 2 \pmod 3$	ord^4 $(1 \pmod 3)$, $\mathrm{ord}^2 \oplus \mathrm{ss}^2$ $(2 \pmod 3)$	$\mathrm{ord}^2 \oplus \mathrm{ss}^2$ $(1 \mod 3)$, $\mathrm{ss}^4(2 \mod 3)$
(39)	5	$1 \pmod 8$ $3 \pmod 8$ $5 \pmod 8$ $7 \pmod 8$	ord^5 $\mathrm{ord}^2 \oplus \mathrm{ss}^3$ $\mathrm{ord}^3 \oplus \mathrm{ss}^2$ $\mathrm{ord}^4 \oplus \mathrm{ss}$	$\mathrm{ord} \oplus \mathrm{ss}^4$ ss^5 $\mathrm{ord} \oplus \mathrm{ss}^4$ ss^5
(40)	7	$1 \pmod{24}$ $5 \pmod{24}$ $7 \pmod{24}$ $11 \pmod{24}$ $13 \pmod{24}$ $17 \pmod{24}$ $19 \pmod{24}$ $23 \pmod{24}$	ord^7 $\mathrm{ord}^2 \oplus \mathrm{ss}^3$ ord^7 $\mathrm{ord}^2 \oplus \mathrm{ss}^5$ $\mathrm{ord}^5 \oplus \mathrm{ss}^2$ $\mathrm{ord}^4 \oplus \mathrm{ss}^3$ $\mathrm{ord}^5 \oplus \mathrm{ss}^2$ $\mathrm{ord}^4 \oplus \mathrm{ss}^3$	$\mathrm{ord}^3 \oplus \mathrm{ss}^4$ ss^7 $\mathrm{ord}^3 \oplus \mathrm{ss}^4$ ss^7 $\mathrm{ord}^3 \oplus \mathrm{ss}^4$ ss^7 $\mathrm{ord}^3 \oplus \mathrm{ss}^4$ ss^7

Table 1.1: Newton polygons realized in the special fibers of Shimura varieties attached to the 13 non-abelian families of genus g for all sufficiently large p.

1.3.2 The Q_8 family of curves

Of the families in table 2.1, family (36) is the most interesting for multiple reasons:

1. It is a family of curves of relatively high genus (4).
2. Each curve in this family is a hyperelliptic curve.
3. The above table reveals that the special fiber of the associated Shimura variety at a prime p contains points with supersingular Newton polygon without a restriction on the congruence class of p.
4. We are able to find an explicit equation for the universal curve of the family.

Thus we undertake a detailed study of the curves in this family and discover several regular patterns.

Let p be an odd prime number and $q = p^n$. Consider the family of genus 4 curves defined over the finite field $\mathbb{F}_p$ given by the equation

$$C_t : y^2 = g_t(x) \quad \text{where} \quad g_t(x) = x(x^4 - 1)(x^4 + tx^2 + 1).$$

This family yields a singular curve exactly when $t = 2, -2$.

We prove in Proposition 21 that this is the equation for the universal family of curves in family (36).

Define the polynomials $c_\alpha(t)$ by the equation

$$\sum_\alpha c_\alpha(t) x^\alpha := g_t(x)^{\frac{p-1}{2}}.$$

Now construct the following 4×4 matrix whose entries are the above described polynomials c_α:

$$M(t) := \begin{pmatrix} c_{p-1} & c_{p-2} & c_{p-3} & c_{p-4} \\ c_{2p-1} & c_{2p-2} & c_{2p-3} & c_{2p-4} \\ c_{3p-1} & c_{3p-2} & c_{3p-3} & c_{3p-4} \\ c_{4p-1} & c_{4p-2} & c_{4p-3} & c_{4p-4} \end{pmatrix}.$$

For any value $t_0 \in \overline{\mathbb{F}}_p$, the above matrix is called a Cartier–Manin matrix of the curve C_{t_0}. It models the action of Verschiebung on the k-vector space $H^0(C_{t_0}, \Omega^1)$ of regular differentials on k. Its importance is derived from (among other things) the fact that the curve C_{t_0} is ordinary if and only if $\det(M(t_0)) \neq 0$. We have proven the following results about the Cartier–Manin matrix:

Theorem 2. *1. The degree of the determinant $M(t)$ as a polynomial in t is*

$$\deg(\det M(t)) = \begin{cases} 3(\frac{p-1}{2}) & p \equiv 1 \pmod 8, \\ 3(\frac{p-1}{2}) - 1 & p \equiv 7 \pmod 8. \end{cases}$$

2. *For $p \equiv 3, 5 \pmod 8$, det$(M(t_0)) = 0$ for every choice of $t \in \overline{\mathbb{F}}_p$. In other words, det$(M(t))$ is the zero polynomial.*

3. *For $p \equiv 1, 7 \pmod 8$, if det$(M(t_0)) = 0$ for any $a \neq 2, -2$, then every entry of the matrix $M(t_0)$ is zero for that value of a.*

4. *Every entry $f(t)$ in the first and fourth column of the matrix $M(t)$ satisfies the following differential equation:*

$$(21 + (804t)\frac{d}{dt} + (1548t^2 - 2160)\frac{d^2}{dt^2} + 640t(t^2-4)\frac{d^3}{dt^3} + 64(t^2-4)^2\frac{d^4}{dt^4})f(t) = 0.$$

5. *Every entry in the second and third column of the matrix $M(t)$ satisfies the*

following differential equation:

$$(45+(900t)\frac{d}{dt}+(1580t^2-2288)\frac{d^2}{dt^2}+640t(t^2-4)\frac{d^3}{dt^3}+64(t^2-4)^2\frac{d^4}{dt^4})f(t)=0.$$

Given the bounded multiplicity of roots to solutions of the differential equations, we have the following results that bear striking parallels to Igusa's results in Theorem 22. In particular, they provide a uniform construction of supersingular curves over any characteristic $p > 7$. These curves are actually superspecial when $p \equiv 1, 7 \pmod 8$ This demonstrates an extremely unlikely intersection of the 9-dimensional Torelli locus with the 0-dimensional superspecial locus inside the 10 dimensional space of principally polarized abelian varieties. 31.

Corollary 3. 1. *The determinant of $M(t)$ is always divisible by $(t-2)$ and $(t+2)$ to the same power k and this power is equal to*

$$\begin{cases} \frac{p-1}{2} & p \equiv 1 \pmod 8 \\ \frac{p+1}{2} & p \equiv 7 \pmod 8 \end{cases}$$

2. *For $p > 7$, if $p \equiv 1$ or $7 \pmod 8$, there is always an element $t_0 \in \overline{\mathbb{F}}_p$ such that $M(t_0)$ is identically the 0 matrix, i.e. the corresponding curve $y^2 = x(x^4 - 1)(x^4 + t_0x^2 + 1)$ is superspecial.*

3. *For $p \equiv 1 \pmod 8$, $\gcd(c_{p-1}, c_{p-3})$ has only linear and quadratic irreducible factors in $\mathbb{F}_p$, i.e. all the roots are in $\mathbb{F}_{p^2}$. Here the c_α are polynomials in the ring $\mathbb{F}_p[t]$ and are defined by $(x(x^4-1)(x^4+tx^2+1))^{\frac{p-1}{2}} = \sum_\alpha c_\alpha x^\alpha$.*

4. *For $p \equiv 7 \pmod 8$, $\gcd(c_{p-2}, c_{p-4})$ has only linear and quadratic irreducible factors in $\mathbb{F}_p$, i.e. all the roots are in $\mathbb{F}_{p^2}$. Again, the c_α are polynomials in the ring $\mathbb{F}_p[t]$ and are defined by $(x(x^4-1)(x^4+tx^2+1))^{\frac{p-1}{2}} = \sum_\alpha c_\alpha x^\alpha$.*

5. *For $p \equiv 1 \pmod{16}$, there exists $t_0 \in \mathbb{F}_p$ such that C_{4,t_0} is superspecial if and*

only if p is represented by one of the quadratic forms $x^2 + 32y^2$ or $x^2 + 64y^2$. [1]

1.4 Outline

In the first chapter of this thesis, we include a general introduction to the taxonomic side of algebraic geometry in studying shapes and highlight the dichotomy of discrete vs. continuous invariants in building useful collections of objects to study. We then begin a more technical introduction to the methods and results of this thesis. The main idea is that the Torelli locus, consisting of the Jacobians of curves, is a more accessible part of $\mathcal{A}_g$. Even within the Torelli locus, curves with non-trivial automorphisms are easier to study than those without any because of the induced group actions on linear algebraic data such as (co)homology. These group actions make it much easier to calculate the discrete invariants attached to the curves with automorphisms than it is for arbitrary Jacobians, or worse, arbitrary abelian varieties.

In Chapter 2, we focus on the characteristic 0 side of the story. We study the Riemann existence theorem describing combinatorial data $(\mathbf{m}, G, \theta)$ of a surjection from the fundamental group of a punctured sphere to a finite group determines a family $\mathcal{C}(\mathbf{m}, G, \theta)$ of Riemann surfaces arising as Galois covers of $\mathbb{P}^1$ with signature $(\mathbf{m}, G, \theta)$. We use the Chevalley–Weil formula to determine how the $\mathbb{C}$-vector space of regular differentials on a curve in this family decomposes into irreducible representations under the induced action of G. We then discuss the work of Frediani, Ghigi and Penegini [FGP15] in this context and their findings of families $\mathcal{C}(\mathbf{m}, G, \theta)$ whose image under the Jacobian map is (Zariski) dense in an irreducible component of a PEL-type Shimura variety. We present tables of the relevant families, discuss their details, and mention which of th are already considered by the work of previous authors in [LMPT19]. We then describe 13 families of covers arising from non-abelian Galois groups G and describe the restrictions on their Jacobians imposed by the group

[1] By a theorem of Kaplansky [Kap03], a prime $p \equiv 1 \pmod{16}$ is either represented by both or none of the forms $x^2 + 32y^2$ and $x^2 + 64y^2$.

action.

In Chapter 3, we discuss the general theory of abelian varieties in positive characteristic, group schemes and Dieudonné modules. We begin our study of the various stratifications by using Dieudonné modules to analyze p-torsion group schemes. We then discuss characterizations of Ekedahl–Oort types using the more linear algebraic description of Weyl group cosets in the algebraic group Sp_{2n}.

We discuss Barsotti–Tate groups (or p-divisible groups), and in particular the Barsotti–Tate group of p-th power torsion of the abelian variety. We use the Dieudonné-Manin classification to describe the isogeny classes of Barsotti–Tate groups using the combinatorial data of Newton polygons. While our focus is on Newton polygons, it is only by closely scrutinizing the interaction between between the Newton polygon and Ekedahl–Oort type in terms of their induced stratifications on the Shimura varieties that we are able to compute the Newton polygons appearing in the special fiber of the attached Shimura variety. For all thirteen of the non-abelian families of Galois covers $\mathcal{C}(\mathbf{m}, G, \theta)$ whose Torelli image we calculate the possible Ekedahl–Oort types of the corresponding PEL-type Shimura varieties, except for family (36), whose resolution happens at the end of Chapter 4.

In Chapter 4, we focus on family (36) in the list of [FGP15]. We explore the geometric aspects of this family first, such as its automorphisms, hyperellipticity, genus of quotient curves, etc. We then use the work of [Sha06] to show that we have an explicit affine formula for the universal curve of family (36). We then describe the possible automorphisms of individual curves in this family over $\mathbb{C}$ and $\overline{\mathbb{F}}_p$ for $p > 7$. We discuss the classical example of the Legendre pencil of elliptic curves studied by Igusa and discuss how the Picard-Fuchs differential equation helps us to deduce the number of supersingular curves in this pencil. We then introduce the Cartier–Manin matrix of the curves in family (36) and analyze combinatorial properties. This allows us to showcase some striking parallels between this family and the Legendre pencil analyzed by Igusa. We combine the combinatorial calculations regarding the determinants of the Cartier–Manin matrix with the results of the BT_1 computations of the previous chapter to complete the computation of the μ-ordinary and basic Newton polygons in each case with congruence constraints on the prime at which we are considering the curves finishing the computation of the Newton polygons for the family (36). Finally we describe some partial progress made towards explicitly finding smooth hyperelliptic supersingular curves of every characteristic $p > 7$ in the family in every characteristic and end with some conjectures.

Chapter 2

Branched Galois covers of $\mathbb{P}^1$

In this section, we first follow the set up and terminology of [FGP15] to describe families of G-covers of the projective line combinatorially in characteristic 0. Then, for a curve C_t in such a family, we decompose the $\mathbb{C}$-vector space $H^0(C_t, \Omega^1)$ of global holomorphic differentials on the curves into irreducible G-representations. Matching the irreducible representations of G appearing in $H^0(C, \Omega^1)$ with the simple components of the group ring $\mathbb{Q}[G]$ lets us decompose the Jacobians $J(C)$ into smaller dimensional abelian varieties to facilitate easier study. These results will be useful in the subsequent chapters as the reduction of the curves in the families of G at various primes will yield curves over finite fields whose Jacobians have interesting Newton polygons and Ekedahl–Oort types.

2.1 Families of branched covers of $\mathbb{P}^1$

For an integer $r \geq 3$, let Γ_r denote the group defined by generators and relations as:

$$\Gamma_r := \langle \gamma_1, \dots, \gamma_r | \gamma_1 \cdots \gamma_r = 1 \rangle.$$

Let U_r be the complex projective line $\mathbb{P}^1$ punctured at r points, and t_0 an arbitrary element of X_r. We can then interpret the γ_i as homotopy classes of loops around the r punctures since

$$\pi_1(U_r, t_0) \simeq \Gamma_r.$$

Definition 4. *A* datum *is a triple* $(\mathbf{m}, G, \theta)$ *where* $\mathbf{m} = (m_1, \dots, m_r)$ *is an* r*-tuple* *of integers such that each $m_i \geq 2$, G is a finite group and $\theta : \Gamma_r \to G$ is a surjection such that the images $\theta(\gamma_i)$ have order m_i for each i.*

The following classical result is indispensable in our study. See [Mir95] for a proof.

Proposition 5 (The Riemann-Hurwitz formula)**.** *Let X be a compact Riemann surface and G a group faithfully acting on X by holomorphic automorphisms. Then the quotient X/G has a unique complex structure making it a Riemann surface such that the quotient map $\pi : X \to X/G$ is a holomorphic branched covering of Riemann surfaces. Furthermore, let g and g_0 denote the genera of X and X/G respectively and $m_1, m_2, \dots m_k$ denote the orders of the stabilizers above the k branch points in X/G. Then the integers $g, g_0, |G|, m_1, m_2 \dots, m_k$ satisfy the following relation:*

$$2g - 2 \;=\; |G| \cdot \left(2g_0 - 2 + \sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right)\right).$$

Given a datum $(\mathbf{m}, G, \theta)$, let Y_r be the complement of the weak diagonal in $\mathbb{C}^r$, namely,

$$Y_r = \{(z_1, z_2, \dots z_r) \in \mathbb{C}^r | z_i \neq z_j \text{ for all } i \neq j\}.$$

For any $t = (t_1, \dots t_n) \in Y_r$, once we set

$$U_r = \mathbb{P}^1 \backslash \{t_1, t_2, \dots t_n\},$$

choose a point $t_0 \in U_r$ and choose an isomorphism $\Gamma_r \to \pi_1(U_r, t_0)$, we get a Galois covering of compact Riemann surfaces $C_t \to \mathbb{P}^1$ with automorphism group G, with branch points exactly at $t_1, t_2, \dots t_r$ with (cyclic) monodromy groups generated by the elements $\theta(\gamma_r)$ of order $m_1, m_2, \dots, m_r$ respectively. In [FGP15] the authors prove that in fact, there is a universal family of curves $C \to Y_r \times \mathbb{P}^1$ such that the fiber over each $t = (t_1, \dots, t_r) \in Y_r$ is the branched G-cover $C_t \to \mathbb{P}^1$ branched precisely at $t_1, \dots t_r$. Denote this family by $\mathcal{C}(\mathbf{m}, G, \theta)$.

In addition, the authors of [FGP15] find a sufficient condition for the image of a family $\mathcal{C}(\mathbf{m}, G, \theta)$ under the Torelli map to be dense in a irreducible component attached to a PEL-type Shimura variety. They implement a procedure in the software program Magma and find forty such families $\mathcal{C}(\mathbf{m}, G, \theta)$ in genus up to 7. Of these families, the group G in twenty of them is cyclic and these are exactly the families studied in [LMPT19]. Seven of them consist of an abelian, but non-cyclic group G, while thirteen of them consist of a non-abelian group G.

In the next chapter, we will describe general procedures to facilitate the calculation of possible discrete invariants in positive characteristic when the abelian varieties we are studying are points of a PEL moduli space. Examples of such invariants include the p-rank, Newton polygon and Ekedahl–Oort type. This is our main justification and interest in studying the families found above. Curves with action by cyclic (and more generally abelian) groups tend to have Jacobians whose special fibers are ordinary at primes that split in certain cyclotomic fields. As a consequence these types of curve families typically do not yield curves with less common Newton polygons when p is congruent to 1 modulo the discriminant of the appropriate cyclotomic field.

We attempt to remedy this problem in this thesis by studying the thirteen families of curves with action by non-abelian groups. We tabulate below the genus of the curves in the family, the size of the group G, the isomorphism class of the group G, the order of the monodromy group at each branch point, the dimension of the base space of the family and the original id number of each family used in [FGP15]. We will continue using these id numbers to refer to the families. The symbol $G(16, 13)$ below corresponds in the row corresponding to family (34) refers to a non-abelian group of 16 elements in the GAP small groups library [Mic00].

g	$\|G\|$	G	$\mathbf{m}$	dim	Id
2	6	S_3	$(2,2,3,3)$	1	(28)
2	8	D_4	$(2,2,2,4)$	1	(29)
2	12	D_6	$(2,2,2,3)$	1	(30)
3	6	S_3	$(2,2,2,2,3)$	2	(31)
3	8	D_4	$(2,2,2,2,2)$	2	(32)
3	12	A_4	$(2,2,3,3,3)$	1	(33)
3	16	$G(16,13)$	$(2,2,2,4)$	1	(34)
3	24	S_4	$(2,2,2,3)$	1	(35)
4	8	Q_8	$(2,4,4,4)$	1	(36)
4	12	A_4	$(2,3,3,3)$	1	(37)
4	18	$(\mathbb{Z}/3\mathbb{Z}) \times S_3$	$(2,2,3,3)$	1	(38)
5	12	$(\mathbb{Z}/3\mathbb{Z}) \rtimes (\mathbb{Z}/4\mathbb{Z})$	$(2,3,4,4)$	1	(39)
7	24	$\mathrm{SL}_2(\mathbb{F}_3)$	$(2,3,3,3)$	1	(40)

Table 2.1: List of non-abelian data yielding a Shimura subvariety

Our next task is to ensure that the families $C(\mathbf{m}, G, \theta)$ of curves are defined over the integers so that they may be reduced modulo p for various primes. The strategy is to first consider the much larger moduli problem of covers that only specify the finite group G and the number of branch points of the cover. Then the various families $C(\mathbf{m}, G, \theta)$ for varying $\mathbf{m}$ and θ can be recovered as the connected components of the complexified scheme representing this space.

Over the complex numbers, consider the more general moduli space of isomorphism classes of G-covers of the projective line $\mathbb{P}^1$ that are branched at r points with the Galois group of the branched cover isomorphic to G. We have the following result of Wewers that shows that this analytic construction, a priori over the complex numbers, actually descends to over $\mathbb{Z}$:

Theorem 6 (Wewers, [Wew98])**.** *Let G be a finite group and $r > 0$. There is a scheme $H_r^{in}(G)$ that is smooth and of finite type over $\mathbb{Z}$ such that for every algebraically closed field k, the k-rational points of $H_r^{in}(G)$ correspond to G-Galois covers of $\mathbb{P}^1_k$ that are tamely ramified over k.*

See Section 1.3 of [RW06] for a more detailed discussion of this topic. In particular, $H_r^{in}(G) \otimes \mathbb{C}$ parametrizes the G-covers of $\mathbb{P}^1$ with r branch points. The connected components of this smooth moduli scheme are exactly the various moduli spaces $\mathcal{C}(\mathbf{m}, G, \theta)$. Even though the spaces $H_r^{in}(G)$ are defined over $\mathbb{Q}$, their geometric connected components may require an extension to a number field $K/\mathbb{Q}$ to be defined. In the next chapter, as we study G-covers of the projective line over fields of positive characteristic, the freedom to choose k to be an arbitrary field is especially important to us.

2.2 Decomposition of $H^0(C, \Omega^1)$ as a G-representation

Since the group G acts on each fiber C_t by holomorphic automorphisms, there is a naturally induced linear action of G on the $\mathbb{C}$-vector space of holomorphic differentials $H^0(C_t, \Omega^1_{C_t})$. This action is determined, up to isomorphism, only by the combinatorial datum $(\mathbf{m}, G, \theta)$ and not on the particular choice of fiber. Thus we suppress the C_t in the subscript, and write the decomposition of $H^0(C, \Omega^1)$ into irreducible G-representations:

$$H^0(C, \Omega^1) \simeq \bigoplus_{\chi \in \mathrm{Irr}(G)} (V_\chi)^{\mu_\chi}.$$

Here $\mathrm{Irr}(G)$ is the set of irreducible characters of the group G_d and V_χ is the corresponding irreducible (complex) representation of χ. The Chevalley–Weil formula lets us explicitly compute the multiplicities μ_χ. We first set up some notation and state the formula.

For an irreducible character $\chi \in \mathrm{Irr}(G)$, let σ_χ be the corresponding irreducible representation and d_χ the degree of σ_χ. Denote by μ_χ the multiplicity of σ_χ in $H^0(C, \Omega^1)$. For the datum $(\mathbf{m}, G, \theta)$, let $x_i = \theta(\gamma_i)$, an element of order m_i. The x_i represent the monodromy of the covering at the various corresponding branch points P_i. Let ζ_m denote the primitive m-th root of unity $e^{\frac{2\pi i}{m}}$. Finally, let $E_{i,\alpha}$ denote the number of eigenvalues of $\sigma_\chi(x_i)$ that are equal to $\zeta_{m_i}^\alpha$. We can now decompose

$H^0(C, \Omega^1)$ as a direct sum of irreducible G-representations.

Theorem 7 (Chevalley–Weil formula, [FGP15], Theorem 2.10)**.** *For a finite group G and a G-cover of compact Riemann surfaces $X \to \mathbb{P}^1$, branched at r points. With m_i and $E_{i,\alpha}$ as above, the multiplicity μ_χ of a given irreducible character χ in $H^0(C, \Omega^1)$ is given by the following formula:*

$$\mu_\chi = -d_\chi + \sum_{i=1}^{r} \sum_{\alpha=0}^{m_i - 1} E_{i,\alpha} \langle -\frac{\alpha}{m_i} \rangle + \epsilon$$

where $\epsilon = 1$ if χ is the principal character and $\epsilon = 0$ otherwise.

A direct consequence of this theorem is an algorithm to compute the multiplicites μ_χ from the data of $(\mathbf{m}, G, \theta)$.

Corollary 8. *With notation as in Theorem 7, there is an algorithm (recorded in Appendix A.1), that takes as input the group G, its conjugacy classses, the genus $g(C)$, the number of branch points r of the cover, and the monodromy group at each branch point as a subgroup of G and computes the multiplicity with which each irreducible representation of G appears in $H^0(C, \Omega^1)$.*

A weak version of the Chevalley–Weil formula is used in the work of [LMPT19] in the following sense: the families of curves considered in the authors are branched cyclic covers of $\mathbb{P}^1$. Once we fix a primitive n-th root of unity $\zeta_n \in \mathbb{C}$, all the other n-th roots of unity are given by ζ_n^i for $1 \leq i \leq n$. Now the irreducible representations of the cyclic group μ_n are determined by a choice of which n-th root of unity to map a generator of μ_n to. Thus, any representation ρ of μ_n can be encoded in an n-tuple of non-negative integers called the *signature*. Thus, the Chevalley–Weil formula in the context of cyclic groups simply outputs an n-tuple as the signature. However, in our examples of families of covers with non-abelian Galois groups, the groups do not enjoy a uniform representation theory like that of cyclic groups. As such, the signature of $H^0(C_t, \Omega^1)$ as a representation of a non-abelian group G is simply expressed as a direct sum of irreducible G-representations.

To each of the 13 families of non-abelian covers of $\mathbb{P}^1$, we apply the Chevalley–Weil formula to decompose the vector space $H^0(C, \Omega^1)$ as a direct sum of irreducible G-representations using the software program Magma. The code we use is directly taken from the main procedures of the authors in [FGP15]. We include it in Appendix A.1 with no claim to orignality. In what follows, we we continue to use the family ids introduced by [FGP15] while discussing the representation theory of the relevant non-abelian groups.

(28) Consider $G = S_3$, the symmetric group on 3 letters. It has three irreducible representations: the trivial and sign representations, both 1-dimensional irreducible representations, as well as a unique irreducible 2-dimensional representation ρ_1.

For the genus 2 family (28), the Chevalley–Weil formula reveals that the 2-dimensional vector space $H^0(C, \Omega^1)$ is irreducible and isomorphic to ρ_1 as a G-representation.

(29) Consider $G = D_4$, the dihedral group with eight elements. It has five irreducible representations: four 1-dimensional irreducible representations, as well as a unique irreducible 2 dimensional representation ρ_2.

For the genus 2 family (29), the Chevalley–Weil formula reveals that the 2-dimensional vector space $H^0(C, \Omega^1)$ is irreducible and isomorphic to ρ_2.

(30) Consider $G = D_6$, the dihedral group with twelve elements. It has six irreducible representations: Four 1-dimensional irreducible representations as well as two irreducible 2-dimensional representations ρ_3 and ρ_3'. The representations ρ_3 and ρ_3' can be distinguished by evaluating their trace character at the unique conjugacy class C containing the group elements of order 6. At this conjugacy class, $\mathrm{Tr}(\rho_3(C)) = 1$ while $\mathrm{Tr}(\rho_3'(C)) = -1$

For the genus 2 family (30), the Chevalley–Weil formula reveals that the 2-dimensional vector space $H^0(C, \Omega^1)$ is irreducible and isomorphic to ρ_3.

(31) Consider $G = S_3$, the symmetric group on 3 letters. It has three irreducible representations: the trivial and sign representations. Both of these are 1-dimensional irreducible representations. Denote the sign representation by ρ_{sgn}. It also has a unique irreducible 2-dimensional representation ρ_1.

For the genus 3 family (31), the 3-dimensional vector space $H^0(C, \Omega^1)$ is reducible as an S_3-representation for dimension reasons alone. The Chevalley–Weil formula induces the following decomposition of G-representations:

$$H^0(C, \Omega^1) \simeq \rho_1 \oplus \rho_{sgn}.$$

(32) Consider $G = D_4$, the dihedral group with eight elements. It has five irreducible representations: four 1-dimensional irreducible representations as well as a unique irreducible 2-dimensional representation ρ_2. We focus on one of the four 1-dimensional representations that is not the trivial representation and has trace 1 on the unique conjugacy class of G containing the elements of order 4. Denote this representation by ρ_4.

For the genus 3 family (32), the Chevalley–Weil formula reveals that the 3-dimensional vector space $H^0(C, \Omega^1)$ admits the following decomposition of G-representations:

$$H^0(C, \Omega^1) \simeq \rho_2 \oplus \rho_4.$$

(33) Consider $G = A_4$, the alternating group on 4 letters. It has four irreducible representations: three 1-dimensional irreducible representations as well as a unique irreducible 3-dimensional representation ρ_5.

For the genus 3 family (33), the Chevalley–Weil formula reveals that the 3-dimensional vector space $H^0(C, \Omega^1)$ is irreducible and isomorphic to ρ_5.

(34) Consider $G = (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$. It has eight 1-dimensional irreducible representations and two 2-dimensional irreducible representations. Among the

group's 8 conjugacy classes, three of them contain all the elements of G that have order 4. Denote these conjugacy classes C_1, C_2, C_3. We use these conjugacy classes to identify one of the 1-dimensional irreducible representation ρ_6 by specifying the following conditions:

1. It is not the trivial representation.
2. Its trace character evaluated at each of C_1, C_2, C_3 always yields 1.

The two 2-dimensional representations are complex conjugates so we denote them by ρ_7 and $\overline{\rho}_7$.

For the genus 3 family (34), the Chevalley–Weil formula reveals that the 3-dimensional vector space $H^0(C, \Omega^1)$ admits the following decomposition of G-representations:

$$H^0(C, \Omega^1) \simeq \rho_6 \oplus \rho_7.$$

(35) Consider $G = S_4$, the symmetric group on 4 letters. It has five irreducible representations: two 1-dimensional irreducible representations, one irreducible 2-dimensional representation and two irreducible 3-dimensional representations ρ_8 and ρ_9. The representations ρ_8 and ρ_9 may be distinguished by evaluating their trace at the conjugacy class C of 4-cycles. In particular, $\rho_8(C) = 1$ and $\rho_9(C) = -1$. By Theorem 2.1 of [Ung19], both ρ_8 and ρ_9 are both defined over $\mathbb{Q}$.

For the genus 3 family (35), the Chevalley–Weil formula reveals that the 3-dimensional vector space $H^0(C, \Omega^1)$ is irreducible and isomorphic to ρ_8 as a G-representation.

(36) Consider $G = Q_8$, the quaternion group, i.e. the group of pure quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$. This is a group of order 8 and has four irreducible 1-dimensional representations and a single irreducible 2-dimensional representations $\rho_{10} : Q_8 \to \mathrm{GL}(V)$

For the genus 4 family (36), the Chevalley–Weil formula reveals that the 4-dimensional vector space $H^0(C,\Omega^1)$ is reducible and admits the following decomposition:

$$H^0(C,\Omega^1) \simeq \rho_{10} \oplus \rho_{10}.$$

(37) Consider $G = A_4$, the alternating group on 4 letters. It has four irreducible representations: three 1-dimensional irreducible representations as well as a unique irreducible 3-dimensional representation ρ_5. The two non-trivial 1-dimensional irreducible representations only differ by complex conjugation. Thus we denote them ρ_{12} and $\overline{\rho_{12}}$.

For the genus 4 family (37), the Chevalley–Weil formula reveals that the 4-dimensional vector space $H^0(C,\Omega^1)$ is reducible and admits the following decomposition of G-representations:

$$H^0(C,\Omega^1) \simeq \rho_5 \oplus \rho_{12}.$$

(38) Consider $G = S_3 \times (\mathbb{Z}/3\mathbb{Z})$. It has nine irreducible representations: six 1-dimensional irreducible representations and three 2-dimensional irreducible representations. The trace characters to associated to four of the 1-dimensional irreducible representations do not have all their values in $\mathbb{Q}$. The field $\mathbb{Q}(\zeta_3)$ contains all their values, where ζ_3 is a primitive third root of unity. These four representations come in complex conjugate pairs: $\rho_{13}, \overline{\rho_{13}}$ and $\rho_{14}, \overline{\rho_{14}}$. They can be distinguished by evaluating their trace characters at either of the conjugacy classes C_1 or C_2 consisting of the order 6 elements of G. More precisely, for $i = 1, 2$, the elements $\mathrm{Tr}(\rho_{13})(C_i)$ are primitive sixth roots of unity while the elements $\mathrm{Tr}(\rho_{14})(C_i)$ are primitive third roots of unity. Similarly, the trace characters of two of the 2-dimensional irreducible representations have values outside $\mathbb{Q}$. Again these trace character values are all contained in the field $\mathbb{Q}(\zeta_3)$. Furthermore, these representations only differ by complex conjugation

and thus can be denoted ρ_{14} and $\overline{\rho_{14}}$.

For the genus 4 family (38), the Chevalley–Weil formula reveals that the 4-dimensional vector space $H^0(C, \Omega^1)$ is reducible and admits the following decomposition of G-representations:

$$H^0(C, \Omega^1) \simeq \rho_{13} \oplus \overline{\rho_{13}} \oplus \rho_{15}.$$

(39) Consider $G = \mathbb{Z}/3\mathbb{Z} \rtimes (\mathbb{Z}/4\mathbb{Z})$. It has six irreducible representations: four 1-dimensional irreducible representations and two 2-dimensional irreducible representations. The trace characters to associated to two of the 1-dimensional irreducible representations do not have all their values in $\mathbb{Q}$. The field $\mathbb{Q}(i)$ contains all their values, where i is a primitive fourth root of unity. These two representations, denoted ρ_{15} and $\overline{\rho_{15}}$, only differ by complex conjugation. The two 2-dimensional irreducible representations ρ_{16} and ρ_{17} with rational trace values can be distinguished by evaluating their trace characters on the unique conjugacy class C of order 6 elements: $\mathrm{Tr}(\rho_{16})(C) = 1$, while $\mathrm{Tr}(\rho_{17})(C) = -1$.

For the genus 5 family (39), the Chevalley–Weil formula reveals that the 5-dimensional vector space $H^0(C, \Omega^1)$ is reducible and admits the following decomposition of G-representations:

$$H^0(C, \Omega^1) \simeq \rho_{15} \oplus \rho_{16} \oplus \rho_{16}.$$

(40) Consider $G = \mathrm{SL}(2, \mathbb{F}_3)$, the special linear group over the finite field $\mathbb{F}_3$. It has seven irreducible representations: three 1-dimensional irreducible representations, three 2-dimensional irreducible representations and one 3-dimensional irreducible representation. The two non-trivial 1-dimensional representations ρ_{18} and $\overline{\rho_{18}}$ only differ only by complex conjugation. The trace characters of these two representations do not have all their values in $\mathbb{Q}$. The field $\mathbb{Q}(\zeta_3)$

contains all their values, where ζ_3 is a primitive third root of unity. Denote the unique 2-dimensional irreducible representation whose trace characters has all its values in $\mathbb{Q}$ by ρ_{19}. Finally, some of the trace character values of the two remaining 2-dimensional irreducible representations are also not rational but rather in $\mathbb{Q}(\zeta_3)$. These two representations are also complex conjugates and therefore denoted by ρ_{20} and $\overline{\rho_{20}}$.

$$\rho'_{19} \otimes_{\mathbb{Q}} K \simeq \rho_{19} \oplus \rho_{19}.$$

For the genus 7 family (40), the Chevalley–Weil formula reveals that the 7-dimensional vector space $H^0(C, \Omega^1)$ is reducible and admits the following decomposition of G-representations:

$$H^0(C, \Omega^1) \simeq \rho_{18} \oplus \rho_{19} \oplus \rho_{19} \oplus \rho_{20}.$$

Recall that the Jacobian of a curve can be constructed as a quotient of degree zero divisors on the curve modulo principal divisors. If the curve is considered over a field, divisors are in a one-to-one correspondance with formal linear combinations of points on the curve. For a commutative ring R and a finite group G, denote the group ring of G over R by $R[G]$. If $R = k$, a field, then $k[G]$ is also called a group algebra.

For any curve C_t in the family $\mathcal{C}(\mathbf{m}, G, \theta)$, there is, by definition, an action of the group G. We thus have an induced action of the integral group ring $\mathbb{Z}[G]$ on the free $\mathbb{Z}$-module generated by the points on the curve. Because principal divisors are zero loci of rational functions on the curve and the group G acts by isomorphisms, it follows that the induced action of $\mathbb{Z}[G]$ descends to the Jacobian $J(C_t)$. Working in the isogeny category of abelian varieties, we can extend this to an action of $\mathbb{Q}[G]$. By the Artin-Wedderburn theorem, the group algebra $\mathbb{Q}[G]$ can be decomposed into a direct sum of matrix rings over division algebras:

$$\mathbb{Q}[G] \simeq \bigoplus M_{r_i}(D_i).$$

Then, by a result of Kani and Rosen ([KR89], Theorem A), we have a corresponding decomposition (up to isogeny) of the Jacobian $J(C_t)$ using the idempotents of the above direct sum analogous to the Chevalley–Weil theorem:

$$J(C_t) \sim \bigoplus A_i^{r_i}.$$

Here A_i is an abelian variety (not necessarily simple!) with action by the corresponding division algebra D_i over $\mathbb{Q}$ in the above decomposition of $\mathbb{Q}[G]$. We first use the "Wedderga" package [OdR09] of the mathematical software GAP to compute the Artin-Wedderburn decomposition of the group ring $\mathbb{Q}[G]$. Our code is documented in Appendix A.2. Then we use the information of the Chevalley–Weil formula to isolate those simple components $M_{r_i}(D_i)$ of $\mathbb{Q}[G]$ for which the corresponding A_i are non-trivial.

The Artin-Wedderburn decompositions of the thirteen group algebras associated to the non-abelian Galois families $Z(\mathbf{m}, G, \theta)$ yielding special subvarieties of $\mathcal{A}_g$, together with the components of the decomposition that embed into the endomorphism ring of the curve are tabulated below:

The signatures refer to the representations of the non-abelian groups discussed previously. In Table 2.2, the symbols D_2, D_3 and $G(16, 13)$ are used without explanation. We now explain them:

- D_2 is the quaternion division algebra over $\mathbb{Q}$ ramified at 2 and ∞ and split at every other place.

- D_3 is the quaternion division algebra over $\mathbb{Q}$ ramified at 3 and ∞ and split at every other place.

- The group $G(16, 13)$ refers to a non-abelian group of 16 elements in the GAP

Id	G	$\mathbb{Q}[G]$	Signature
28	S_3	$\mathbb{Q}^2 \oplus M_2(\mathbb{Q})$	ρ_1
29	D_4	$\mathbb{Q}^4 \oplus M_2(\mathbb{Q})$	ρ_2
30	D_6	$\mathbb{Q}^4 \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q})$	ρ_3
31	S_3	$\mathbb{Q}^2 \oplus M_2(\mathbb{Q})$	$\rho_1 \oplus \rho_{sgn}$
32	D_4	$\mathbb{Q}^4 \oplus M_2(\mathbb{Q})$	$\rho_2 \oplus \rho_4$
33	A_4	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_3) \oplus M_3(\mathbb{Q})$	ρ_5
34	$G(16, 13)$	$\mathbb{Q}^8 \oplus M_2(\mathbb{Q}(i))$	$\rho_6 \oplus \rho_7$
35	S_4	$\mathbb{Q}^2 \oplus M_2(\mathbb{Q}) \oplus M_3(\mathbb{Q})^2$	ρ_8
36	Q_8	$\mathbb{Q}^4 \oplus D_2$	$\rho_{10} \oplus \rho_{10}$
37	A_4	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_3) \oplus M_3(\mathbb{Q})$	$\rho_5 \oplus \rho_{12}$
38	$\mathbb{Z}/3\mathbb{Z} \times S_3$	$\mathbb{Q}^2 \oplus \mathbb{Q}(\zeta_3)^2 \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q}(\zeta_3))$	$\rho_{13} \oplus \overline{\rho_{13}} \oplus \rho_{15}$
39	$(\mathbb{Z}/3\mathbb{Z}) \rtimes (\mathbb{Z}/4\mathbb{Z})$	$\mathbb{Q}^2 \oplus \mathbb{Q}(i) \oplus M_2(\mathbb{Q}) \oplus D_3$	$\rho_{15} \oplus \rho_{16} \oplus \rho_{16}$
40	$\mathrm{SL}_2(\mathbb{F}_3)$	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_3) \oplus D_3 \oplus M_2(\mathbb{Q}(\zeta_3))$	$\rho_{18} \oplus \rho_{19} \oplus \rho_{19} \oplus \rho_{20}$

Table 2.2: Wedderburn decompositions of group algebras $\mathbb{Q}[G]$ in the non-abelian families and the relevant pieces of the decomposition acting non trivially on $\mathrm{Jac}(C_t)$

small groups library [Mic00].

The families (28), (29) and (30) are actually all the same family of curves (Theorem 1.9 and Table 1 of [FGP15]). By the structure of $H^0(C_t, \Omega^1)$ as a G-representation specified by the Chevalley–Weil formula combined with the decomposition of $Q[G]$ for these families, we see that $M_2(\mathbb{Q})$ is the part of the group ring $\mathbb{Q}[G]$ that acts non-trivially on the Jacobians of these curves. Again, we conclude that

Proposition 9. *The abelian varieties arising as Jacobians of curves C_t in the family 40 (in characteristic* 0*) admit the following decomposition into a product of smaller dimensional abelian varieties, up to isogeny:*

$$J(C_t) \sim E_1 \times E_2^2 \times A_1,$$

where E_1 and E_2 are elliptic curves with complex multiplication by (possibly different) orders in $\mathbb{Q}(\zeta_3)$ and A_1 is an abelian fourfold with action by an order in the $\mathbb{Q}$-division algebra D_2.

Proof. As seen from Table 2.2, the abelian varieties appearing as the Jacobians of curves in family (40) have action by the group ring

$$\mathbb{Q}[\mathrm{SL}(2,3)] \simeq \mathbb{Q} \oplus \mathbb{Q}(\zeta_3) \oplus D_2 \oplus M_2(\mathbb{Q}(\zeta_3)).$$

However, by the Chevalley–Weil formula, the only simple parts of this group ring that act non-trivially on the Jacobians $J(C_t)$ are those corresponding to the irreducible representations $\rho_{18} \oplus \rho_{19} \oplus \overline{\rho_{19}} \oplus \rho_{20}$. Thus by matching the irreducible representations with the simple pieces of the group ring, the effective part of the group ring that acts non-trivially on $J(C_t)$ is $\mathbb{Q}(\zeta_3) \oplus D_2 \oplus M_2(\mathbb{Q}(\zeta_3))$. Now, we apply the Kani-Rosen theorem ([KR89], Theorem 1) and deduce the isogeny relation

$$J(C_t) \sim E_1 \times E_2^2 \times A_1 \tag{2.1}$$

where E_1 and E_2 are elliptic curves with complex multiplication by an order in $\mathbb{Q}(\zeta_3)$ and A_1 is an abelian fourfold with action by an order in the $\mathbb{Q}$-division algebra D_2. We were able to deduce the dimension of the abelian varieties appearing in this decomposition of $J(C_t)$ because $H^0(C, \Omega^1)$ is the (complex) cotangent space of $J(C_t)$ at the identity and it's decomposition into irreducible $\mathbb{C}$-representations according to the Chevalley-Weil formula yield the dimensions of the factors of $J(C_t)$ in Equation 2.1. An alternative method involving Hurwitz characters is elaborated upon in [Pau08]. □

For many detailed computations using the Kani-Rosen theorem to decompose Jacobians using group actions, see [Pau07].

However, there is a direct connection of this Shimura variety to another appearing earlier in the list! The quaternion group Q_8 is the unique 2-Sylow subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$ and is thus a normal subgroup. The irreducible representation ρ_{10} of Q_8 is in fact the restriction of the irreducible representation ρ_{19} of the larger group $\mathrm{SL}_2(\mathbb{F}_3)$. As a consequence we deduce that the Shimura variety attached to family (40) is simply a product of the 1-dimensional Shimura variety attached to family (36) with the two 0-dimensional Shimura varieties described above.

Using similar arguments to the ones in Lemma 9, we can obtain structural results on the decomposition of $J(C_t)$ for curves C_t in any of the Galois families. We tabulate the results of these propositions in the following two tables. Table 2.3 explains the symbols for the various types of abelian varieties appearing in the last column of Table 2.4

The following table describes the various types of abelian varieties appearing in Table 2.4.

Symbol	Meaning	dim of moduli space
E	An elliptic curve	1
$E_{CM,i}$	An elliptic curve with CM by an order in $\mathbb{Q}(i)$	0
E_{CM,ζ_3}	An elliptic curve with CM by an order in $\mathbb{Q}(\zeta_3)$	0
A_{2,ζ_3}	An abelian surface with action by an order in $\mathbb{Q}(\zeta_3)$	1
A_4	An abelian fourfold with action by an order in D_2	1
A'_4	An abelian fourfold with action by an order in D_2	1

Table 2.3: Description of the abelian varieties appearing in Table 2.4

Id	Relevant piece of $\mathbb{Q}[G]$	Decomposition of $J(C_t)$
28	$M_2(\mathbb{Q})$	E^2
29	$M_2(\mathbb{Q})$	E^2
30	$M_2(\mathbb{Q})$	E^2
31	$\mathbb{Q} \oplus M_2(\mathbb{Q})$	$E_0 \times E_1^2$
32	$\mathbb{Q} \oplus M_2(\mathbb{Q})$	$E_0 \times E_1^2$
33	$M_3(\mathbb{Q})$	E^3
34	$\mathbb{Q} \oplus M_2(\mathbb{Q}(i))$	$E \times E_{CM,i}^2$
35	$M_3(\mathbb{Q})$	E^3
36	D_2	A_4
37	$\mathbb{Q}(\zeta_3) \oplus M_3(\mathbb{Q})$	$E_{CM,\zeta_3} \times E^3$
38	$\mathbb{Q}(\zeta_3) \oplus M_2(\mathbb{Q}(\zeta_3))$	$A_{2,\zeta_3} \times E_{CM,\zeta_3}$
39	$\mathbb{Q}(i) \oplus D_2$	$E_{CM,i} \times A_4'$
40	$\mathbb{Q}(\zeta_3) \oplus M_2(\mathbb{Q}(\zeta_3)) \oplus D_2$	$E_{CM,\zeta_3} \times E_{CM,\zeta_3}^2 \times A_4$

Table 2.4: The components of the Wedderburn decomposition of $\mathbb{Q}[G]$ acting non-trivially on the Jacobians of curves in the thirteen non-abelian families and the induced decompositon of the Jacobians.

The Chevalley–Weil theorem actually furnishes us with a complete decomposition of the action of G on the deRham cohomology $H^1_{\mathrm{dR}}(C)$ for curves in families $\mathcal{C}(\mathbf{m}, G, \theta)$. In preparation for the following chapter, we record this in a lemma.

Lemma 10. *Consider the Chevalley–Weil decomposition of a curve in a family $\mathcal{C}(\mathbf{m}, G, \theta)$:*

$$H^0(C, \Omega^1) \simeq \bigoplus_{\chi \in Irr(G)} \rho_\chi^{\mu_\chi}$$

as G-representations. Then the deRham cohomology $H^1_{\mathrm{dR}}(C)$ is also naturally a G-representation and its decomposition into irreducibles is given by

$$H^1_{\mathrm{dR}}(C) \simeq \bigoplus_{\chi \in Irr(G)} (\rho_\chi^{\mu_\chi} \oplus \rho_{\overline{\chi}}^{\mu_\chi}),$$

where $\rho_{\overline{\chi}}$ is the irreducible representation associated with the character $\overline{\chi}$.

Proof. We notice that since G acts on the curve C (in characteristic 0) by holomorphic automorphisms, it also acts on the deRham cohomology $H^1_{\mathrm{dR}}(C_t)$ and $H^1(C, O_C)$ by functoriality. The Hodge decomposition

$$H^1_{\mathrm{dR}}(C) \simeq H^0(C, \Omega^1) \oplus H^1(C, O_C)$$

is equivariant for this group action. By Hodge symmetry combined with the Dolbeault theorem, we have the isomorphism

$$H^1(C, O_C) \simeq \overline{H^0(C, \Omega^1)}$$

which is also equivariant for the G-action. The G-representations occurring in $H^1(C, O_C)$ are thus exactly the complex conjugates of the G-representations occurring in $H^0(C, \Omega^1)$.

$\square$

Chapter 3

Dieudonné modules and stratifications in positive characteristic

In this chapter, we continue our study of the thirteen families of curves highlighted in the previous chapter but in the positive characteristic setting. As described in Theorem 6, each of the above families, originally defined over $\mathbb{C}$, admit a model over a number field K. Over this number field, we may reduce the curves in these families at all but a finite number of primes of K. We can then commence our study of the discrete invariants attached to these curves and and their Jacobians in this positive characteristic setting. More precisely, we can study the Ekedahl–Oort types and Newton polygons of the Jacobians of the curves in these thirteen families. However, because these Jacobians are dense in the irreducible component of a PEL-type Shimura variety, we can obtain a description of the Ekedahl–Oort and Newton stratification in purely group theoretic terms involving Weyl groups of appropriate algebraic groups.

3.1 Preliminaries

Let k be a perfect field of characteristic $p > 0$. We denote by $\mathrm{Fr}_k : k \to k$ the absolute Frobenius, i.e. the map $x \to x^p$. Typically we will take k to be a finite field $\mathbb{F}_q$ but if we need k to be algebraically closed we can let k be an algebraic closure $\overline{\mathbb{F}}_q$ of a finite field. We usually denote abelian varieties over k by A. For such an A, we denote its p^n-torsion group subscheme (i.e. the kernel of the multiplication by p^n map) by $A[p^n]$. We now review some group schemes and related objects in characteristic p that arise in our study of abelian varieties.

A *Barsotti–Tate group* (also called a *p-divisible group*) over k of height h is an inductive system $H = (G_n, \iota_n)$ for $n \geq 0$ where G_n is a finite group scheme over k of order p^{nh} such that the sequences

$$0 \longrightarrow G_n \xrightarrow{\iota_n} G_{n+1} \xrightarrow{[p^n]} G_{n+1}$$

are exact for all $n \geq 0$.

A group scheme that appears as the p-torsion subscheme of a Barsotti–Tate group is called a *truncated Barsotti–Tate group of level* 1 or a BT_1 for short.

Our running example of a Barsotti–Tate group is given by the inductive system of p^n-torsion of an abelian variety A defined over k. We denote it $A[p^\infty] = (A[p^n], \iota_n)$. Here the maps $\iota_n : A[p^n] \to A[p^{n+1}]$ are given by the obvious closed immersion giving $A[p^n]$ the reduced induced subscheme structure inside $A[p^{n+1}]$. The BT_1 associated to $A[p^\infty]$ is simply the p-torsion subgroup scheme of the abelian variety, namely $A[p]$.

3.2 Dieudonné theory

We have the following theorem that lets us study finite commutative group schemes of order p by studying modules over a certain non-commutative ring called the *Dieudonné ring*. The contents of the theorem often go by the name of Dieudonné theory.

Now let k be a perfect, algebraically closed field of characteristic $p > 0$. Let $W(k)$ denote the ring of infinite Witt vectors over k with Frobenius automorphism $\sigma : W(k) \to W(k)$. Denote by D_k the Dieudonné ring, the non-commutative ring over $W(k)$ generated by the symbols F and V with the relations

$$\begin{aligned}
&FV = VF = p, \\
&Fw = w^\sigma F \text{ for all } w \in W(k), \\
&wV = Vw^\sigma \text{ for all } w \in W(k).
\end{aligned}$$

Proposition 11 (Dieudonné Theory [CCO14])**.** *There is an additive anti-equivalence of categories $\mathscr{G} \rightsquigarrow M(\mathscr{G})$ from the category of finite, p-th power order, commutative group schemes over k to the category of left modules over D_k of finite length as $W(k)$-modules. The module $M(\mathscr{G})$ is called the Dieudonné module of $\mathscr{G}$. This anti-equivalence has the following properties:*

1. *A group scheme $\mathscr{G}$ has order p^ℓ where ℓ is the length of $M(\mathscr{G})$ as a $W(k)$-module.*

2. *Let $k \to k'$ be an extension of perfect fields of characteristic p and the induced map on Witt rings $W(k) \to W(k')$. Then the functor $W(k') \otimes_{W(k)} (\cdot)$ on Dieudonné modules is naturally identified with the base-change functor on finite commutative group schemes over k. In particular $M(\mathscr{G}^{(p)}) \simeq \sigma^*(M(\mathscr{G}))$ as $W(k)$-modules.*

3. *If $\mathrm{Fr}_{\mathscr{G}/k} : \mathscr{G} \to \mathscr{G}^p$ is relative Frobenius morphism, then the σ-semilinear action on $M(\mathscr{G})$ induced by $M(\mathrm{Fr}_{\mathscr{G}/k})$ through the isomorphism $M(\mathscr{G}^{(p)} \simeq \sigma^*(M(\mathscr{G}))$ is the same as the action of F, and $\mathscr{G}$ is connected if and only if F is nilpotent on $M(\mathscr{G})$.*

4. *There is a natural k-linear isomorphism $M(\mathscr{G})/FM(\mathscr{G}) \simeq \mathrm{Lie}(\mathscr{G})^\vee$.*

5. *For the Cartier dual $\mathscr{G}^D$, the following isomorphism of Dieudonné modules holds:*

$$M(\mathscr{G}^D) \simeq \mathrm{Hom}_{W(k)}(M(\mathscr{G}), W(k)[\frac{1}{p}]/W(k)).$$

On elements (i.e. linear forms) $f \in M(\mathscr{G}^D)$, the action of F and V are given by the formulae $F(\ell) : m \mapsto \sigma(\ell(V(m)))$ and $V(\ell) : m \mapsto \sigma^{-1}(\ell(F(m)))$.

The next result describes some restrictions on the group scheme structure of the p-torsion subscheme of a Barsott–Tate group.

Theorem 12 ([Moo01])**.** *A group scheme $\mathscr{G}$ appears as the p-torsion subscheme of a Barsotti–Tate group if and only if the following sequence is exact:*

$$\mathscr{G} \xrightarrow{\mathrm{Fr}} \mathscr{G}^{(p)} \xrightarrow{\mathrm{Ver}} \mathscr{G}$$

Let $C(1)_k$ be the subcategory of finite commutative group schemes over k that are killed by p. The corresponding Dieudonné module, $M(\mathscr{G})$ is a priori a $W(k)$-module but because $\mathscr{G}$ is annihilated by p, so is $M(\mathscr{G})$. Thus $pW(k) \subset \mathrm{Ann}(M)$ and we may consider M as a module (vector space, in fact) over $W(k)/pW(k)$, which is just the field k.

In particular, any BT_1 over k is an object of $C(1)_k$. The equivalence of categories in Proposition 11 combined with the above discussion yields the following characterization of the Dieudonné modules of BT_1s.

Proposition 13. *The category of BT_1s over k is equivalent to the category of tuples (M, V_M, F_M) where*

- *M is a finite dimensional vector space over k*
- *$F_M : M \to M$ is a Fr_k-linear map*
- *$V_M : M \to M$ is a Fr_k^{-1}-linear map.*
- *We have $\mathrm{Ker}(F_M) = \mathrm{Im}(V_M)$ and $\mathrm{Ker}(V_M) = \mathrm{Im}(F_M)$*

Let A an abelian variety defined over k. Then the map $[n] : A \to A$ given by multiplication by an integer n is an isogeny of degree n^{2g}. If n is coprime to p, the map $[n]$ is separable and its kernel $A[n]$ is a group scheme over k. $A[n]$ is isomoprhic to the constant group scheme $\underline{(\mathbb{Z}/n\mathbb{Z})}^{2g}$, an étale group scheme of order n^{2g}. As a consequence, we also have that the group of k-valued points of $A[n]$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$. More generally let S be a scheme over k. If $A \to S$ is an abelian scheme of relative dimension g, then each of the k-valued points of A are g-dimensional abelian varieties over k. Then there are only finitely many possibilities for the isomorphism class of $A[p]$. This is the invariant of A called its *Ekedahl–Oort type*. We can also study the isogeny class of the Barsotti–Tate group $A[p^\infty]$ which is an invariant of A called the *Newton polygon* of A for reasons that we now explain.

3.3 The Newton Stratification

By the Dieudonné–Manin classfication of Barsotti–Tate groups over an algebraically closed field k, the isogeny category of Barsotti–Tate groups over k is semi-simple and the simple objects are parametrized by ordered pairs (r, s) of non-negative integers such that $\gcd(r, s) = 1$ and $s \neq 0$.

To a Barsotti–Tate group over k that is simple, we may thus assign a non-negative rational number $\frac{r}{s}$ called its *slope*. More generally, to any Barsotti–Tate group over k, we can assign a non-decreasing sequence $\frac{r_1}{s_1} \leq \frac{r_2}{s_2} \leq, \dots, \leq \frac{r_t}{s_t}$, called its *slopes*. We can use this information to construct a piece-wise linear curve in $\mathbb{R}^2$ called the *Newton polygon* of the Barsotti–Tate group by the following procedure:

Construction 14 (Newton Polygon)**.** *Given a non-decreasing sequence*

$$\frac{r_1}{s_1} \leq \frac{r_2}{s_2} \leq, \dots, \leq \frac{r_t}{s_t}$$

of rational numbers, we construct a piece-wise linear curve in $\mathbb{R}^2$ *starting at* $(0, 0)$*. First, construct a line segment of horizontal length* 1 *and a slope of* $\frac{r_1}{s_1}$*. Next, starting at the end point of the previously constructed segment, construct a line segment of* *horizontal length* 1 *and slope of* $\frac{r_2}{s_2}$. *Continue this process until all the slopes are exhausted.*

However, not every sequence of slopes occurs when we consider the possible isogeny classes of the Barsotti–Tate group attached to an abelian variety. For instance, we know that every abelian variety A is isogenous to its dual abelian variety $A^\vee$ (via the dual isogeny). This in turn implies that the Barsotti–Tate groups $A[p^\infty]$ and $A^\vee[p^\infty]$ have the same slopes. Since the dual of a Barsotti–Tate group with slopes $\lambda_1, \lambda_2, \ldots, \lambda_t$ has slopes $1-\lambda_1, 1-\lambda_2, \ldots, 1-\lambda_t$, it follows that the Newton polygon of the Barsotti–Tate group $A[p^\infty]$ attached to an abelian variety has slopes between 0 and 1. We can also deduce that any rational number $\lambda \geq 0$ appears as a slope of $A[p^\infty]$ with the same multiplicity as $1-\lambda$.

In fact, with an additional integrality requirement, we have a complete classification of exactly which Newton polygons appear as invariants of $A[p^\infty]$ where A is an abelian variety over k. For a fixed g, consider the set $\mathcal{N}_g$ of lower convex, piece-wise linear curves in the plane with the following four properties:

1. They have endpoints $(0,0)$ and $(2g, g)$.
2. The points of non-differentiability are at integer lattice points.
3. The slope of each line segment that occurs is in $[0,1]$.
4. They are symmetric, namely if a line segment of horizontal length m and slope λ occurs, then so does a line segment of the same horizontal length and slope $1-\lambda$.

Proposition 15 ([Oor00], 5.3)**.** *For* $g \geq 1$*, the set* $\mathcal{N}_g$ *consists exactly of all the Newton polygons realizable by* $A[p^\infty]$ *for an abelian variety* A *defined over* k.

Given the Barsotti–Tate group $A[p^\infty]$ attached to an abelian variety A of dimension g defined over a field that is not algebraically closed, once we make a base-change

A to an algebraic closure of the field of definition, we may attach an element of $\mathcal{N}_g$ to $A[p^\infty]$ using the Dieudonné–Manin classfication. We refer to this element as the associated Newton polygon of A. As the definition of Newton polygon implies, the base-change of an abelian variety to a bigger field does not affect the Newton polygon so we may work over a field $k = \overline{k}$ without loss of generality. An abelian variety all of whose slopes are $\frac{1}{2}$ is called supersingular while an abelian variety all of whose slopes are either 0 or 1 is called ordinary.

The set $\mathcal{N}_g$ of Newton polygons has the natural structure of a ranked poset $(\mathcal{N}_g, \preccurlyeq)$ with maximum and minimum elements. For two elements $\nu_1, \nu_2 \in \mathcal{N}_g$, we say that $\nu_1 \leq \nu_2$ precisely if the piece-wise linear convex curve attached to ν_1 lies on or above the curve attached to ν_2. The ordinary Newton polygon is the maximum element in this order and the minimum element is given by the supersingular Newton polygon.

Now consider the map $\mu : \mathcal{A}_g(k) \to \mathcal{N}_g$ from the k-points of the space of principally polarized abelian varieties of dimension g to the poset $\mathcal{N}_g$ which attaches to an abelian variety A its Newton polygon. The fibers of this map yield locally closed subsets of $\mathcal{A}_g(k)$ called the *Newton strata*. Moreover, for any $\nu \in \mathcal{N}_g$, the closure of a Newton stratum $\mu^{-1}(\nu)$ is exactly the union all the strata $\mu^{-1}(\nu_1)$ for which $\nu_1 \preccurlyeq \nu$:

$$\overline{\mu^{-1}(\nu)} = \bigcup_{\nu_1 \preccurlyeq \nu} \mu^{-1}(\nu_1).$$

This particular decomposition of $A_g(k)$ into locally closed subsets is called the *Newton stratification*.

3.4 The Ekedahl–Oort Stratification

Now let A be a *principally polarized* abelian variety over a perfect, algebraically closed field k of characteristic $p > 0$. Then the isomorphism class of the group scheme $A[p]$ is an invariant of the abelian variety A called its *Ekedahl–Oort type*. The isomorphism class of $A[p]$ (as a group scheme) can be described combinatorially in many ways: Final types, Kraft diagrams, Young diagrams, Linear types and Circular types.

Using Proposition 13, we know that the isomorphism class of $A[p]$ as a group scheme is encoded entirely in a k-vector space M with two maps F_M and V_M. This fact is true for all A, polarized or otherwise. Since we are additionally assuming that A is principally polarized, the k-vector space M is equipped with a canonical symplectic form $\psi : M \times M \to k$ such that

$$\psi(F_M(m_1), m_2) = \psi(m_1, V_M(m_2))^\sigma.$$

This is detailed in Section 2.6 of [Moo01].

Given the data of (M, F_M, V_M, ψ), the *canonical flag* is the coarsest flag of ψ-isotropic k-linear subspaces of M that is stable under taking images of F_M and pre-images of V_M:

$$0 \subsetneq M_1 \subset \cdots \subsetneq F_M(M) = \mathrm{Ker}(V_M) \subsetneq \cdots \subsetneq M$$

In addition to this flag, we have a one-step flag given by the kernel of F_M, which is a maximal isotropic subspace for the polarization form ψ_M:

$$0 \subsetneq \mathrm{Ker}(F_M) \subsetneq M.$$

The relative position of these two flags is measured by an appropriate Weyl group coset of the symplectic group $\mathrm{Sp}(M, \psi_M)$ (see Section 3.2 of [Moo01]). This Weyl group coset captures the isomorphism class of the original group scheme $A[p]$ and will be our primary tool in calculating the Ekedahl–Oort type (and consequently the Newton polygon) of the abelian varieties in the Shimura varieties attached to the 13 families of the previous chapter.

3.5 Special fibers of PEL-type Shimura varieties

In the previous chapter, for each fiber C_t in the family $\mathcal{C}(\mathbf{m}, G, \theta)$ of branched Galois covers of $\mathbb{P}^1$, we have determined the structure of $H^0(X_t, \Omega^1)$ as a G-module. As the result of Theorem 6 and the following discussion implies, the families $C(\mathbf{m}, G, \theta)$ are actually defined over a number field $K(\mathbf{m}, G, \theta)$ which will often be abbreviated K as we only deal with one family of curves at a time. Let $S = S(\mathbf{m}, G, \theta)$ be the finite set of primes of K that lie over the primes of $\mathbb{Q}$ that divide $|G|!$. Then the family $\mathcal{C}(\mathbf{m}, G, \theta)$ is actually defined over $O_{K,S}$. Here $O_{K,S}$ is the set of S-integers of K:

$$O_{K,S} = \{x \in K | v_{\mathfrak{p}}(x) \geq 0 \text{ for all } \mathfrak{p} \notin S\}.$$

For any prime $\mathfrak{p}$ not in S, we can reduce any curve C_t in the family $\mathcal{C}(\mathbf{m}, G, \theta)$ to obtain a curve $\overline{C_t}$ over the finite field $O_{K,S}/\mathfrak{p}$.

The PEL-type Shimura varieties attached to semi-simple algebras B with involution such that the algebras B are not simple can be decomposed as a product of PEL-type Shimura varieties attached to simple algebras with involution. See Section 2.2 of [Wed99] for more details. We apply this principle to the families of Table 2.4.

If a PEL-moduli space is parameterizing powers of elliptic curves, then the only Newton polygons possible in this family are the ordinary and supersingular ones. Similarly, if the PEL-moduli space is 0-dimensional, then it parameterizes CM abelian varieties of dimension g with endomorphisms by an order in a CM-field L of $\mathbb{Q}$-dimension $2g$. In this case, the CM method or the Shimura-Taniyama formula may be used to determine the Newton polygon at all primes of good reduction and this only depends on the decomposition behavior of a rational prime in the field L.

Thus when the PEL-moduli space is either a modular curve or consists only of CM abelian varieties, since we already know the contribution of their Newton polygons, we can already determine the Newton polygons in the families of Table 2.4. Thus the remainder of our work boils down to whether we can calculate the position of the special fibers of three specific PEL moduli spaces with respect to the Ekedahl–Oort and Newton stratifications. Using the notation and definition for PEL-type Shimura varieties as in [VW13], Section 2.1, we describe these three moduli spaces now.

1. The first moduli space corresponds to the simple quaternion division algebra D_2 acting with signature $\rho_{10} \oplus \rho_{10}$. This space parametrizes abelian fourfolds with endomorphisms by an order in D_2. This information is relevant for families (36) and (40).

2. The second moduli space corresponds to the simple quaternion division algebra D_3 acting with signature $\rho_{16} \oplus \rho_{16}$. This space parametrizes abelian fourfolds with endomorphisms by an order in D_3. This information is relevant for family (39).

3. The third moduli space corresponds to the simple algebra $\mathbb{Q}(\zeta_3))$ acting with signature $\rho_{10} \oplus \overline{\rho_{10}}$. This space parametrizes abelian surfaces with endomorphisms by an order in $\mathbb{Q}(\zeta_3)$. This information is relevant for family (38).

3.6 BT_1 with extra structures

In this section, we follow the notation of [Moo01] to describe the possible BT_1s that appear in the special fiber of a PEL-type Shimura variety.

As in Section (6.11) of [MW04], consider a PEL-datum $\mathcal{D} = (B, *, V, \psi, O_B, \Lambda, h)$ that is integral and unramified at an odd prime p. We now explain this notation.

- B is a semi-simple $\mathbb{Q}$-algebra such that $B_{\mathbb{Q}_p} = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is isomorphic to a product of matrix algebras over finite unramified extensions of $\mathbb{Q}_p$.

- $*$ is a positive involution of B.

- V is a finitely generated left module over B.

- ψ is a non-degenerate, alternating, $\mathbb{Q}$-valued form on V such that

$$\psi(bv_1, v_2) = \psi(v_1, b^* v_2) \quad \text{for all } \; b \in B \text{ and all } v_1, v_1 \in V$$

- $O_B \subset B$ is a B-order that is stable under $*$ such that $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order in $B_{\mathbb{Q}_p}$

- Λ is O_B-invariant $\mathbb{Z}_p$-lattice in $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that the restriction $\psi|_{\Lambda \times \Lambda}$ is a perfect pairing of $\mathbb{Z}_p$-modules.

- $\mathbf{G}$ is the algebraic group (defined over $\mathbb{Q}$, not necessarily connected) of B-linear symplectic similitudes of V, i.e. for any $\mathbb{Q}$-algebra R, the R-points of $\mathbf{G}$ are given by

$$\mathbf{G}(R) = \{g \in GL_B(V \otimes_{\mathbb{Q}} R) | \psi(gv, gw) = c(g) \cdot \psi(v, w) \text{ for some } c(g) \in R^*\}$$

- $h : \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}) \to \mathbf{G}_{\mathbb{R}}$ is a homomorphism defining a complex structure on V that is compatible with ψ.

- $[\mu]$ is the $\mathbf{G}(\mathbb{C})$-conjugacy class of the cocharacter μ_h associated with h.

Let E be the reflex field of $\mathcal{D}$, i.e. the field of definition of μ. Further fix an open compact subgroup $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$. Denote by $\mathbf{A}_{\mathcal{D},K^p}$ to be the corresponding moduli space as in [Kot92]. If K^p is small enough, the moduli problem is represented by a smooth, equidimensional, quasi-projective scheme over the localization of O_E at p. As detailed in loc. cit, the T points of the corresponding moduli problem parametrize tuples $(A, \overline{\lambda}, \iota, \eta)$, where

- A is an abelian scheme up to prime-to-p isogeny,

- $\overline{\lambda}$ is $\mathbb{Q}$-homogeneous polarization,

- $\iota : O_B \to \mathrm{End}_T(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a map of $\mathbb{Z}_{(p)}$-algebra homomorphism with $\iota(b^*) = \iota(b)^\dagger$ where $\dagger$ is the Rosati involution associated to $\overline{\lambda}$ and
- η is a K^p-level structure,
- Further more $(A, \overline{\lambda}, \iota, \eta)$ should satisfy a determinant condition as in [Kot92].

In our case, the determinant condition is most prominently visible in the structure of $H^0(C, \Omega^1)$ as a G-representation as dictated by the Chevalley-Weil theorem. Our PEL-moduli problem arises from studying curves C in a family with a group G acting on them by automorphisms such that the induced action of G on the cotangent space of the Jacobian $J(C)$ is always the same G-representation up to isomorphism. Said differently, for any two curves C and C' in the same family, the $\mathbb{C}[G]$-modules $\mathrm{Lie}(C)$ and $\mathrm{Lie}(C')$ are isomorphic.

Now let k be an algebraically closed field of characteristic not dividing $|G|$. Then the group ring $k[G]$ is semi-simple and there is a canonical bijection between irreducible $k[G]$-modules and $\mathbb{C}[G]$-modules. Thus we may use this bijection to compare $k[G]$-modules and $\mathbb{C}[G]$-modules based on the multiplicity of the irreducible submodules appearing inside them. For any k-point $(A, \overline{\lambda}, \iota, \eta)$ of the moduli space $\mathbf{A}_{\mathcal{D},K^p}$, the Lie algebra of A, $\mathrm{Lie}(A)$ has the structure of a $k[G]$-module. The determinant condition simply means that the $k[G]$-submodules appearing in $\mathrm{Lie}(A)$ match the irreducible $\mathbb{C}[G]$-submodules in $\mathrm{Lie}(J(C))$ with the same multiplicities.

For a prime $\mathfrak{p}$ of E lying above p, the algebra $D := O_B/pO_B$ acts on the abelian varieties in the special fiber of the moduli space $\mathbf{A}_{\mathcal{D},K^p}$. Then the BT_1 group schemes $\mathscr{G}$ that appear as p-torsion of these abelian varieties also have an action of D given as $\iota : D \hookrightarrow \mathrm{End}_k(\mathscr{G})$ and an induced quasi-polarization λ on $\mathscr{G}$. The data of such a tuple $(\mathscr{G}, \lambda, \iota)$ is referred to as a BT_1 *with extra structure*.

The main discovery of [Moo01] is that the possible triples $(\mathscr{G}, \lambda, \iota)$ can be explicitly described combinatorially using Weyl group cosets of appropriate algebraic groups. But first, we can use Morita theory to simplify our calculations. As discussed in Section 5.1 of [Moo01], we may decompose the triples $(\mathscr{G}, \lambda, \iota)$ into a product of BT_1s with extra structure where the algebra D is either a field or a product of fields. There are three cases labeled (C), (D) and (A). Case (D) is relevant for the first two moduli spaces described in the previous section (relevant to families (36), (39) and (40)) while case (A) is relevant to the third moduli space described in the previous section (relevant to family (38)).

More precisely, Moonen considers the following problem in [Moo01]:

Problem 16. *Classify triples $(\mathscr{G}, \lambda, \iota)$ where $\mathscr{G}$ is a BT_1 over k with principal quasi-polarization λ and a homomorphism $D \hookrightarrow \mathrm{End}_k(\mathscr{G})$ such that $\iota(1_D) = 1_{\mathscr{G}}$ and $i(d^*) = \iota(d)^\dagger$, where $\dagger$ is the Rosati involution on $\mathrm{End}_k(\mathscr{G})$.*

Moonen further introduces discrete invariants of the triples $(\mathscr{G}, \lambda, \iota)$ called $q, \mathfrak{f}, \delta$ to facilitate the classification. Using Dieudonné theory, this classification problem is equivalent to that of classifying triples $(\mathscr{H}, \mu, \theta)$, where $\mathscr{H}$ is a BT_1 with an action of a finite field $\kappa/\mathbb{F}_p$ and $\mu : \mathscr{H} \to \mathscr{H}^D$ is a κ-ϵ-duality where $\epsilon \in \{0, 1\}$. See [Moo01], Section 5.2. All notation in the following theorem, which solves the above stated problem 16, is from [Moo01].

Let S_n denote the Symmetric group on n letters. The Weyl group of the Symplectic group Sp_{2q} has 2^q elements and can be explicitly identified with a subgroup of S_{2q} as follows:

$$\mathbb{H}_q = \{\omega \in S_{2q} | \omega(i) + \omega(2q+1-i) = 2q+1 \text{ for all } 1 \leq i \leq q\}. \tag{3.1}$$

We define a subgroup $\mathbb{H}_{q,\mathrm{even}}$ of index 2 and it's non-identity coset $\mathbb{H}_{q,\mathrm{odd}}$ inside $\mathbb{H}_q$ as follows:

$$\mathbb{H}_{q,\mathrm{even}} := \left\{\sigma \in S_{2q} \;\middle|\; \substack{\sigma(j)+\sigma(2q+1-j)=2q+1 \text{ for all } j, \text{ and} \\ \text{the number of } j \leq q \text{ with } \sigma(j)>q \text{ is even}}\right\}, \tag{3.2}$$

$$\mathbb{H}_{q,\mathrm{odd}} := \left\{\sigma \in S_{2q} \;\middle|\; \substack{\sigma(j)+\sigma(2q+1-j)=2q+1 \text{ for all } j, \text{ and} \\ \text{the number of } j \leq q \text{ with } \sigma(j)>q \text{ is odd}}\right\} \tag{3.3}$$

So as not to duplicate too much notation from the original source, we simply state the theorem and deduce the consequences for our families of curves.

Theorem 17 ([Moo01], section 5.6, ii')**.**

1. *(Case (A)) There is a bijection:*

$$\{tuples\ (\mathscr{H}, \mu, \theta)\ of\ type\ (2q, q)\} \xrightarrow{\sim} \prod_{i \in \mathscr{J}} S_q \backslash \mathbb{H}_q$$

2. *(Case (D)) In this case, there is an extra discrete invariant δ we need to keep track of to complete the classification. There is a bijection:*

$$\{tuples\ (\mathscr{H}, \mu, \theta)\ of\ type\ (2q, q, \delta)\} \xrightarrow{\sim} W_{X^0} \backslash W_{G^0} \simeq \prod_{i \in \mathscr{J}} S_q \backslash \mathbb{H}_{q,even}$$

Now we calculate the possible Ekedahl–Oort types of the curves in the families $\mathcal{C}(\mathbf{m}, G, \theta)$ in each positive characteristic p that do not divide $|G|!$. We first have a theorem of Oda that lets us deduce more information about the BT_1s arising in the families $\mathcal{C}(\mathbf{m}, G, \theta)$.

Theorem 18 ([Oda69], Corollary 5.11)**.** *Let k be a perfect, algebraically closed field of characteristic p. A/k be a principally polarized abelian variety. The structures $H^1_{\mathrm{dR}}(A)$ and $M(A[p])$ are isomorphic as Dieudonné modules.*

Let p be a prime such that $p \nmid |G|$ and assume that $k = \overline{\mathbb{F}}_p$ and consider the k points of the moduli scheme $\mathcal{H}_{r,G}$. By Theorem 2.1 of [RW06]), these are in bijection with the tamely ramified G-covers of $\mathbb{P}^1_k$ branched at r points.

Consider a monodromy datum $(\mathbf{m}, G, \theta)$ as in Definition 4. This specifies the order of the cyclic monodromy group at each of the r branch points as well as a generator $g_i \in G$ of the monodromy group. Now consider a G-cover $\overline{C} \in \mathcal{H}_{r,G,k}$ such that the monodromy group at each branch point of the cover is $\langle g_i \rangle \subset G$. There is an induced action of G on the k-vector space $H^1_{\mathrm{dR}}(J(\overline{C}))$. By Lemma 4.5 of [RW06], we can lift $\overline{C}$ to a G-cover C over the ring of Witt vectors $W(k)$. Denote by K the algebraic closure of $W(k)[\frac{1}{p}]$ and choose an isomorphism of fields $\iota : K \to \mathbb{C}$. Denote the base-change of C to K by $\tilde{C}$. Due to the isomorphism ι, we may consider $\tilde{C}$ as a complex curve and the deRham cohomology $H^1_{\mathrm{dR}}(J(\tilde{C}))$ as a $\mathbb{C}$-vector space. There is, once again, an induced G-action on $H^1_{\mathrm{dR}}(J(\tilde{C}))$.

By Proposition 2.5.2 of [BBM82], $H^1_{\mathrm{dR}}(J(C))$ is a torsion free $W(k)$-module. The same proposition reveals that the formation of the sheaves $\mathcal{H}^q_{\mathrm{dR}}(J(C))$ commutes with base-change. By Proposition 43 of [Ser77], the irreducible representations of G on finite dimensional $W(k)[\frac{1}{p}]$- vector spaces are in bijection with irreducible representations of G on finite dimensional k-vector spaces. As a consequence, the decomposition of the k-vector space $H^1_{\mathrm{dR}}(J(\overline{C}))$ into irreducible G-representations can be deduced from the decomposition of the $\mathbb{C}$-vector space $H^1_{\mathrm{dR}}(J(\tilde{C}))$. The latter decomposition is explicitly described in Lemma 10. This implies that the possible BT_1s with extra structure arising from the families $\mathcal{C}(\mathbf{m}, G, \theta)$ can be deduced from Table 2.2 and Table 2.4.

We now describe our strategy to use the information obtained about the thirteen families $C(\mathbf{m}, G, \theta)$ and compute the Newton polygons appearing in their their special fiber. We closely follow the work of Moonen in this computation.

1. For each of the thirteen families of non-abelian cyclic covers of $\mathbb{P}^1$ and a large enough prime number p, compute the possible $\mathscr{G}$, we calculate the attached polarized BT_1 with action by a semi-simple $D \hookrightarrow \mathrm{End}(\mathscr{G})$ using Moonen's description in example D of Section 3.9 in [Moo01]. The result is a list of Weyl group cosets.

2. Construct the "standard objects" associated to the elements of the Weyl group coset and identify them with Ekedahl–Oort types in standard form using the method of [Moo01], Section 4.8.

3. Check that the possible Ekedahl–Oort types appearing as invariants of curves

in our families determine their Newton polygon uniquely.

The last step in the strategy only works for us because Morita theory ensures that our examples do not deal with group schemes that could arise as p-torsion of abelian varieties of dimension higher than 2. As described in Section 4 of [Pri08], since the Ekedahl–Oort stratification *refines* the Newton stratification at this dimension, we can easily determine the possible Newton strata from the Ekedahl–Oort strata without any ambiguity.

Here we tabulate the results of the above procedure and record the possible Newton polygons appearing in the thirteen families together with any congruence restriction on the prime p that may need to be imposed. Here the notation ord^n refers to a Newton polygon with n slopes of 0 and n slopes of 1. Similarly the notation ss^k signifies a Newton polygon with $2k$ slopes of $\frac{1}{2}$. The symbol $\oplus$ is used between two Newton polygons to signify the Newton polygon that is formed by pooling together the slopes of the original two Newton polygons.

Before we proceed to the tabulation, there is a subtlety that needs to be dealt with. Moonen's procedure reveals the following:

Proposition 19. *For a sufficiently large prime p, the possible sets of Newton polygons appearing in the reduction to characteristic p of the curve family (36) are either:*

1. $\{\mathrm{ord}^4, \mathrm{ss}^4\}$ *or*
2. $\{\mathrm{ord}^2 \oplus \mathrm{ss}^2, \mathrm{ss}^4\}$.

Moreover, the supersingular curves in case 1 are superspecial while those in case 2 are supersingular but not superspecial.

Proof. Recall that D_2 denotes the unique quaternion division algebra over $\mathbb{Q}$ that is ramified at 2 and ∞. This division algebra can also be identified with Then $D_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$ for any odd prime p. The group Q_8 can be identified with the subgroup of pure quaternions in the division algebra D_2:

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset D_2.$$

Since the group Q_8 acts on the curves in family (36), the order $\mathbb{Z}[i,j,k] \subset D_2$ acts on the Jacobians of these curves and the associated involution is quaternionic conjugation. For every odd prime p, $\mathbb{Z}[i,j,k] \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq M_2(\mathbb{Z}_p)$ is a maximal order inside $M_2(\mathbb{Q}_p)$ and the induced involution is the standard involution. Therefore the p-torsion of these Jacobians $A[p]$ admit an action by $M_2(\mathbb{F}_p)$ with a symplectic involution of the first kind.

The Dieudonné module of $A[p]$ as above is an 8-dimensional k-vector space M with a symplectic form on M . To further simplify computations, as in section 5.2 of [Moo01], we can use Morita theory to study a 4-dimensional k-vector space N with a symmetric bilinear form ψ on it.

Let G be the orthogonal group $\mathrm{O}(N,\psi)$ where N and ψ are as above. Since the group G is not connected, we consider the neutral component $G^0 = \mathrm{SO}(N,\psi)$ instead. Consider the kernel of F acting on the vector space N. This is an isotropic subspace for the form ψ and it's stabilizer X^0 is a subgroup of G^0. Let W_{X^0} and W_{G^0} denote the Weyl groups of X^0 and G^0 respectively. The inclusion $X^0 \hookrightarrow G^0$ induces an inclusion of Weyl groups: $W_{X^0} \hookrightarrow W_{G^0}$.

As discussed in Section 3.4, the Ekedahl–Oort type can be interpreted as a Weyl group coset of an algebraic group. Following the notation of [Moo01], Section 5.3, since $q = 2$, for a fixed $\delta \in \{0,1\}$ there is only one $i \in \mathscr{I}$ and $|S_2| = 2$ and $|\mathbb{H}_{q,\mathrm{even}}| = 4$, so there are exactly two Ekedahl–Oort types for the moduli space at any given prime by Theorem 17, part 2.

The two cosets of W_{X^0} in W_{G^0} are $\{\mathrm{id}, s_1\}$ and $\{s_2 s_1 s_2, s_2 s_1 s_2 s_1\}$. Here the s_i are the simple reflections generating the Weyl group of the algebraic group G. The reduced $(X^0, \emptyset)$-representatives (in the notation of [Moo01], Section 3.1) in these cosets are id and $s_2 s_1 s_2$ respectively. By Theorem 17, these are the isomorphism classes of BT_1s with extra structure in the case when $\delta = 0$. If $\delta = 1$, relevant isomorphism classes are obtained by multiplying id and $s_2 s_1 s_2$ on the right by s_2 as in [Moo01], Secction 3.8. We obtain the elements s_2 and $s_2 s_1$.

Let $\mathscr{G}$ be a BT_1 over k. The p-rank of $\mathscr{G}$ is defined as $\dim_k \mathrm{Hom}(\mu_p, \mathscr{G})$, where μ_p is the kernel of Fr_k on the multiplicative group scheme $\mathbb{G}_{m,k}$. In what follows, we express the Weyl group elements above as permutations using the embedding of Weyl groups induced by the inclusion $\mathrm{O}(N, \psi) \hookrightarrow \mathrm{GL}(N)$ since the Weyl group of GL_n is the symmetric group S_n. We use the recipe in [Moo01], Section 4.9 to construct the Dieudonné modules encoded by the four Weyl group cosets above. In each of the cases, the vector space underlying the Dieudonné module has k-basis e_1, e_2, e_3, e_4. We now specify the action of the semilinear maps F and V on it.

1. The identity element corresponds to the Dieudonné module M_1 with F and V action
$$F(e_1) = 0, \quad F(e_2) = 0 \quad F(e_3) = e_1 \quad F(e_4) = e_2,$$
$$V(e_1) = 0, \quad V(e_2) = 0 \quad V(e_3) = e_1 \quad V(e_4) = e_2.$$
Since $V|_{\mathrm{Ker}\, F}$ is zero, M_1 has the superspecial Ekedahl–Oort type. The corresponding group schemes have p-rank 0.

2. The element $s_2 s_1 s_2 = (1,3)(2,4)$ corresponds to the Dieudonné module M_2 with F and V action given by
$$F(e_1) = e_1, \quad F(e_2) = e_2 \quad F(e_3) = 0 \quad F(e_4) = 0,$$
$$V(e_1) = 0, \quad V(e_2) = 0 \quad V(e_3) = e_3 \quad V(e_4) = e_4.$$
Since $V|_{\mathrm{Ker}\, F}$ is the identity (hence is invertible), M_2 has the ordinary Ekedahl–Oort type. The corresponding group schemes have p-rank 2.

3. The element $s_2 = (2,3)$ corresponds to the Dieudonné module M_3 with F and

V action given by

$$F(e_1) = 0, \quad F(e_2) = e_1 \quad F(e_3) = 0 \quad F(e_4) = e_2,$$

$$V(e_1) = 0, \quad V(e_2) = 0 \quad V(e_3) = e_1 \quad V(e_4) = e_3.$$

Since the operator $V^2|_{\operatorname{Ker} F}$ is zero but $V|_{\operatorname{Ker} F}$ is not zero, the group scheme associated to M_3 has p-rank 0 but is not superspecial.

4. The element $s_2 s_1 = (2, 1, 3, 4)$ corresponds to the Dieudonné module M_4 with F and V action given by

$$F(e_1) = e_1, \quad F(e_2) = 0 \quad F(e_3) = e_2 \quad F(e_4) = 0,$$

$$V(e_1) = 0, \quad V(e_2) = 0 \quad V(e_3) = e_2 \quad V(e_4) = e_4.$$

Since the operator V^n has rank 1 for every positive integer n, we conclude that the group scheme associated to M_4 has p-rank 1.

At every odd prime p, the Barsotti–Tate groups $A[p^\infty]$ arising from Jacobians of curves in family (36) admit an action by $M_2(\mathbb{Q}_p) = D_2 \otimes \mathbb{Q}_p$. By Morita theory, each of these $A[p^\infty]$ is the direct product of a Barsotti–Tate group with itself, up to isogeny. It follows that the only three possible Newton polygons that can occur as invariants of a point in the special fiber of the Shimura variety attached to family (36) are $\mathrm{ord}^4, \mathrm{ss}^4$ and $\mathrm{ord}^2 \oplus \mathrm{ss}^2$. These three Newton polygons can be distinguished using p-rank alone. Combining this with the descriptions of the Ekedahl–Oort types attached to the four Weyl group cosets, our result follows. $\square$

Let A be an abelian variety arising as a Jacobian of a smooth curve in family (36). Alternatively, we can think of A as a point in the special fiber of the Shimura variety attached to family (36). Since D_2 acts on A, we have that $D_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$ acts on the Barsotti–Tate group $A[p^\infty]$.

However, we do not know at which primes the BT_1s have signature $\delta = 0$ and at which primes they have $\delta = 1$. These two values of δ correspond, by Wedhorn's calculations [Wed99] of μ-ordinary Newton polygons, to the two cases when the attached algebraic group $\mathbf{G}(\mathbb{Q}_p)$ is either split or only quasi-split. We will need to use the fact that we have an explicit affine equation for the universal curve over the one-dimensional curve parametrizing family (36). More precisely, Theorem 27 in the next chapter (which is entirely independant of the following theorem) reveals exactly which congruence classes of primes correspond to which set of Newton polygons in the above Proposition 19. In summary, we have:

Theorem 20. *At each sufficiently large prime p of good reduction of Sh_i, the possible μ-ordinary and basic Newton polygons that can be realized in the special fibers of the* 13 *PEL-type Shimura varieties arising from families of non-abelian covers of $\mathbb{P}^1$ defined in table 2.1 in Chapter* 3 *are tabulated below.*

In addition, exactly two families (namely (31) and (32)) have additional Newton polygons other than the μ-ordinary and basic Newton polygons. In both cases, the additional Newton polygons arising are $\mathrm{ord} \oplus \mathrm{ss}^2$ *as well as* $\mathrm{ord}^2 \oplus \mathrm{ss}$ *and there is no congruence restriction on the prime p.*

Id	g	Conditions	μ-ordinary	Basic
(28)	2	-	ord^2	ss^2
(29)	2	-	ord^2	ss^2
(30)	2	-	ord^2	ss^2
(31)	3	-	ord^3	ss^3
(32)	3	-	ord^3	ss^3
(33)	3	-	ord^3	ss^3
(34)	3	$p \equiv 1 \pmod 4$, $p \equiv 3 \pmod 4$	ord^3 $\mathrm{ord} \oplus \mathrm{ss}^2$	$\mathrm{ord}^2 \oplus \mathrm{ss}$ ss^3
(35)	3	-	ord^3	ss^3
(36)	4	$p \equiv 1, 7 \pmod 8$ $p \equiv 3, 5 \pmod 8$	ord^4 $\mathrm{ord}^2 \oplus \mathrm{ss}^2$	ss^4 ss^4
(37)	4	$p \equiv 1 \pmod 3$ $p \equiv 2 \pmod 3$	ord^4 $\mathrm{ord}^3 \oplus \mathrm{ss}$	$\mathrm{ord} \oplus \mathrm{ss}^3$ ss^4
(38)	4	$p \equiv 1 \pmod 3$, $p \equiv 2 \pmod 3$	ord^4 , $\mathrm{ord}^2 \oplus \mathrm{ss}^2$	$\mathrm{ord}^2 \oplus \mathrm{ss}^2$, ss^4
(39)	5	$1 \pmod 8$ $3 \pmod 8$ $5 \pmod 8$ $7 \pmod 8$	ord^5 $\mathrm{ord}^2 \oplus \mathrm{ss}^3$ $\mathrm{ord}^3 \oplus \mathrm{ss}^2$ $\mathrm{ord}^4 \oplus \mathrm{ss}$	$\mathrm{ord} \oplus \mathrm{ss}^4$ ss^5 $\mathrm{ord} \oplus \mathrm{ss}^4$ ss^5
(40)	7	$1 \pmod{24}$ $5 \pmod{24}$ $7 \pmod{24}$ $11 \pmod{24}$ $13 \pmod{24}$ $17 \pmod{24}$ $19 \pmod{24}$ $23 \pmod{24}$	ord^7 $\mathrm{ord}^2 \oplus \mathrm{ss}^3$ ord^7 $\mathrm{ord}^2 \oplus \mathrm{ss}^5$ $\mathrm{ord}^5 \oplus \mathrm{ss}^2$ $\mathrm{ord}^4 \oplus \mathrm{ss}^3$ $\mathrm{ord}^5 \oplus \mathrm{ss}^2$ $\mathrm{ss}^3 \oplus \mathrm{ord}^4$	$\mathrm{ord}^3 \oplus \mathrm{ss}^4$ ss^7 $\mathrm{ord}^3 \oplus \mathrm{ss}^4$ ss^7 $\mathrm{ord}^3 \oplus \mathrm{ss}^4$ ss^7 $\mathrm{ord}^3 \oplus \mathrm{ss}^4$ ss^7

Table 3.1: Newton polygons realized in the special fibers of Shimura varieties attached to the 13 non-abelian families of genus g for all sufficiently large p.

Chapter 4

A family of hyperelliptic curves with Q_8- action

In this chapter, we undertake a detailed study of the family (36) described first in Chapter 2, corresponding to the choice of datum $(\mathbf{m}, G, \theta) = ((2,4,4,4), \mathbb{Q}_8, \theta)$ where $G = Q_8$ is the quaternion group of order eight.

The group G is explicitly given by the presentation

$$Q_8 = \langle y_1, y_2, y_3 | y_1^2 = y_3, y_2^2 = y_3, y_3^2 = 1, y_1 y_2 y_1^{-1} = y_2 y_3 \rangle.$$

Here we may think of y_1, y_2, y_3 as being the familiar elements $i, j, -1$ respectively in D_2, the algebra of Hamilton quaternions. Under this identification,

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset D_2.$$

Recall that the fundamental group Γ_r of $\mathbb{P}^1_{\mathbb{C}}$ punctured at r points has presentation $\Gamma_r = \langle \gamma_1, \ldots, \gamma_r | \gamma_1 \cdots \gamma_r = 1 \rangle$.

For family (36), the map $\theta : \Gamma_4 \to G$ is given by $\theta(\gamma_1) = y_3 = -1, \theta(\gamma_2) = y_2 y_3 = -j, \theta(\gamma_3) = y_1 y_2 = k, \theta(\gamma_4) = y_1 y_3 = -j$.

By the Riemann-Hurwitz formula,

$$2g(Y) - 2 = 8(0 - 2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4})$$

and this so monodromy datum corresponds to a family of genus 4 curves covering the projective line. Further more, constructing the quotient of any curve in this family by the subgroup $\langle -1 \rangle \subset Q_8$ yields $\mathbb{P}^1$ again by the Riemann-Hurwitz formula. This shows that each curve in this family is hyperelliptic with the automorphism y_3 playing the role of the hyperelliptic involution.

Since the curves are branched at $N = 4$ points, this specifies a 1 dimensional family. Explicitly, we can find a (singular) affine model for this family of curves using Equation (14) of section 4.2 of [Sha03].

Proposition 21. *An affine model for the family, defined over $\mathbb{Z}[\frac{1}{2}]$, is given by*

$$\mathcal{C}_t : y^2 = x(x^4 - 1)(x^4 + tx^2 + 1)$$

where a is a parameter that is not equal to ± 2.

However, to realize the entire group Q_8 as automorphisms of this curve, we need to base-change to $\mathbb{Z}[\frac{1}{2}, i]$ where i is a fourth root of unity and the automorphisms of Q_8 on this model are explicitly given by

$$(x, y) \mapsto (x, y), \qquad (x, y) \mapsto (x, -y), \qquad (x, y) \mapsto (-x, iy), \qquad (x, y) \mapsto (-x, -iy)$$

$$(x, y) \mapsto (\frac{1}{x}, \frac{y}{x^5}), \quad (x, y) \mapsto (\frac{1}{x}, \frac{-y}{x^5}), \quad (x, y) \mapsto (\frac{-1}{x}, \frac{iy}{x^5}), \quad (x, y) \mapsto (\frac{-1}{x}, \frac{-iy}{x^5})$$

Denote the right side of the equation of the curve $\mathcal{C}_t$ as $f_t(x)$. Then, for $a \neq \pm 2$, the roots of this polynomial in any algebraically closed field of characteristic not equal to 2 are $\{0, 1, -1, i, -i\} \cup \{\lambda, -\lambda, \lambda^{-1}, -\lambda^{-1}\}$, where λ is any root of the polynomial $x^4 + ax^2 + 1$. When t approaches 2, two of the roots depending on a approach i and the other two roots approach $-i$ leading to $f_2(x)$ having a two roots of order 3 at i and $-i$. Similarly, when t approaches -2, the polynomial $f_{-2}(x)$ has two roots of order 3 at 1 and -1 respectively. In particular, this is not a semistable family of curves.

We next recall a few facts about algebraic curves over perfect fields of characteristic $p > 0$. The reader should consult the works [AH19] to carefully avoid the significant number of typos in the mathematical literature regarding some of the following results.

1. Over fields k of odd characteristic p, any hyperelliptic curve C has an affine (usually singular) model of the form $y^2 = f(x)$ for a square-free polynomial $f(x) \in k[x]$. If the curve C has genus g, then the polynomial $f(x)$ has degree $2g+1$ or $2g+2$.

2. Let Ω^1_C be the sheaf of regular differentials on C . Then the global regular differentials, $H^0(C, \Omega^1_C)$ forms a g-dimensional k-vector space by the definition of genus. Using the affine model above, a k-basis of regular differentials is given by the set
$$\{x^i \frac{dx}{y} \mid 0 \leq i \leq g-1\}.$$
See [AH19], formula (3.1).

3. There is a short exact sequence of left-modules over the Dieudonné ring D_k:
$$0 \to H^0(C, \Omega^1) \to H^1_{\mathrm{dR}}(C) \to H^1(C, O_C) \to 0.$$
The Frobenius F and Verschiebung V act on $H^1_{\mathrm{dR}}(C)$ and $V(H^1_{\mathrm{dR}}(C)) \subset H^0(C, \Omega^1)$ ([AH19] Sections 2.2 and 2.3). Moreover, $\mathrm{Ker}(F) = \mathrm{Im}(V) = H^0(C, \Omega^1)$. See [AH19], Section 2.1.

4. There is an operator $\mathcal{C}$, called the Cartier operator on the sheaf Ω^1, that is Fr_k^{-1}-linear, preserves the global differentials $H^0(C, \Omega^1)$, kills exact differentials and preserves logarithmic differentials. See [Yui78], Proposition 2.1.

5. Choosing any k-basis for $H^0(C, \Omega^1)$ yields a matrix representation for the Cartier operator called the Cartier-Manin matrix.

6. The Cartier-Manin matrix of a smooth curve over k is invertible if and only if the curve is ordinary. See [Yui78], Corollary 2.3.

7. Let M denote a Cartier-Manin matrix of C. For a matrix A and positive integer n, let $A^{(n)}$ denote the matrix obtained by raising the entries of A to the n-th power. Then the p-rank of the Jacobian of C is equal to the rank of the matrix $M^{(p^{g-1})} \cdots M^{(p)} M$. See [Yui78], Theorem 2.2 as well as [AH19] regarding errors in the literature regarding semi-linear algebra.

8. The Cartier-Manin matrix of a smooth curve C over k is identically zero if and only if the curve C is superspecial, ie. i.e. $J(C) \simeq \prod E_i$ where the E_i are supersingular elliptic curves over k. (Theorem 4.1 of [Nyg81])

4.1 The classical case of the Legendre pencil

Let p be an odd prime number and $q = p^n$. Consider the Legendre family of elliptic (genus 1) curves defined over the finite field $\mathbb{F}_p$ given by the equation

$$E_t : y^2 = f_t(x) \quad \text{where} \quad f_t(x) = x(x-1)(x-t).$$

This family yields a singular curve exactly when $t = 0, 1$. Define

$$\sum_\alpha H_\alpha(t) x^\alpha := f_t(x)^{\frac{p-1}{2}}.$$

The polynomial $H_{p-1}(t)$ is very important because the curve E_t is ordinary if and only if $H_{p-1}(t) \neq 0$.

A classical result of Igusa [Igu58] shows the following.

Theorem 22. *The polynomial $H_{p-1}(t)$ has the following properties*

1. *It has degree $\frac{p-1}{2}$.*

2. *It satisfies the differential equation*

$$\left(1 + 4(2t-1)\frac{d}{dt} + 4t(t-1)\frac{d^2}{dt^2}\right) H(t) = 0.$$

3. *It has* $\frac{p-1}{2}$ *distinct roots in* $\overline{\mathbb{F}}_p$ *and they are all different from* 0 *and* 1*.*

4. *There are exactly* $\frac{p-1}{2}$ *values of* $t \in \overline{\mathbb{F}}_p$ *for which* $y^2 = x(x-1)(x-t)$ *is the affine equation of a supersingular curve.*

5. $H(t)$ *has only linear and quadratic irreducible factors in* $\mathbb{F}_p$*, i.e. all it roots are in* $\mathbb{F}_{p^2}$*.*

We briefly sketch the proof of parts 1 and 3 of Theorem 22 highlighting their relationship to the differential equation in part 2.

Proof. We directly calculate that

$$H(t) = (-1)^{\frac{p-1}{2}} \sum_{i=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{i}^2 t^i$$

which makes it clear that the degree is $\frac{p-1}{2}$.

Also $H(0)$ and $H(1)$ are both non-zero. Suppose that $H(t)$ has a multiple root $t = a$. Then $H(a) = \frac{dH}{dt}(a)$=0. Then we can differentiate the above differential equation with respect to the variable t to show that unless $a = 0$ or 1 (which we have ruled out), $\frac{d^2H}{dt^2}(a) = 0$ as well. By induction, we can show that $\frac{d^nH}{dt^n}(a) = 0$ for every positive integer, which is impossible since H is a non-constant polynomial. Thus $H(t)$ must have distinct roots all different from $0, 1$. $\square$

In the next section, we attempt to generalize these results to a higher genus family of curves.

4.2 Generalization to a genus 4 family

Let p be an odd prime number and $q = p^n$. Consider the family (36) of genus 4 curves defined over the finite field $\mathbb{F}_p$ given by the equation

$$C_t : y^2 = g_t(x) \text{ where } g_t(x) = x(x^4 - 1)(x^4 + tx^2 + 1).$$

This family yields a singular curve exactly when $t = 2, -2$. As above, define

$$\sum_\alpha c_\alpha(t)x^\alpha := g_t(x)^{\frac{p-1}{2}}.$$

Now make the 4×4 Cartier-Manin matrix of C_t with respect to the usual basis $\{\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}, \frac{x^3dx}{y}\}$ of $H^0(C_t, \Omega^1)$ and denote it $M(t)$. Then, explicitly,

$$M(t) := \begin{pmatrix} c_{p-1} & c_{p-2} & c_{p-3} & c_{p-4} \\ c_{2p-1} & c_{2p-2} & c_{2p-3} & c_{2p-4} \\ c_{3p-1} & c_{3p-2} & c_{3p-3} & c_{3p-4} \\ c_{4p-1} & c_{4p-2} & c_{4p-3} & c_{4p-4} \end{pmatrix}.$$

The entries of this matrix are the polynomials in the variable t appearing as coefficients of x^i in the expansion of $f_t(x)^{\frac{p-1}{2}}$ above.

We now prove some results on the structure of this matrix. For the remainder of this section, for a positive integer n and integers $0 \leq k_1, k_2, \ldots k_l \leq n$, we define the $m$ultinomial coefficients

$$\binom{n}{k_1, k_2, k_3, \ldots, k_l} = \frac{n!}{k_1!k_2! \ldots k_l!(n - k_1 - k_2 \cdots - k_l)!}.$$

These numbers will feature heavily in our analysis of the combinatorics governing the behavior of the entries c_α of the matrix $M(t)$.

Proposition 23. *For every prime $p > 2$, the matrix M is centrosymmetric. In other* *words, if* $(i,j),(i',j')$ *are indices such that* $pi-j+pi'-j'=5p-5$*, then the* (i,j)*th entry* c_{pi-j} *and the* (i',j')*th entry of* $c_{pi'-j'}$ *are equal if* $p\equiv 1 \pmod 4$ *and are additive inverses if* $p\equiv 3 \pmod 4$

Proof. Set $\alpha = pi-j$ and $\beta = pi'-j'$ so that $\alpha+\beta = 5p-5$. By expanding $g^{\frac{p-1}{2}}$ and gathering terms of the x-degree together, we find that

$$c_\alpha = \sum_{(i,j,k)\in I_\alpha} (-1)^{\frac{p-1}{2}-i}\binom{\frac{p-1}{2}}{i}\binom{\frac{p-1}{2}}{j,k,\frac{p-1}{2}-j-k}t^k$$

where I_α is the set of tuples $(i,j,k)\in\mathbb{Z}^3$ satisfying

1. $0\le i,j,k\le \frac{p-1}{2}$,
2. $0\le j+k\le \frac{p-1}{2}$, and
3. $\frac{p-1}{2}+4i+4j+2k=\alpha$.

Consider the map $\phi : I_\alpha \to I_\beta$ given by

$$\phi(i,j,k) = (\frac{p-1}{2}-i, \frac{p-1}{2}-j-k, k).$$

Under ϕ, the term corresponding to (i,j,k) in the sum defining c_α, namely

$$(-1)^{\frac{p-1}{2}-i}\binom{\frac{p-1}{2}}{i}\binom{\frac{p-1}{2}}{j,k,\frac{p-1}{2}-j-k}t^k$$

transforms into

$$(-1)^{i}\binom{\frac{p-1}{2}}{\frac{p-1}{2}-i}\binom{\frac{p-1}{2}}{\frac{p-1}{2}-j-k,k,j}t^k.$$

The symmetry of binomial coefficients implies that these two expressions are equal, except possibly for the signs which are the same when $p\equiv 1 \pmod 4$ and opposite when $p\equiv 3 \pmod 4$. This proves our claim. $\square$

Lemma 24. *Let p be a prime number congruent to* 1 *or* $7 \pmod 8$. *Then the determinant of the Cartier-Matrix $M_p(t)$ is a non-constant polynomial in t of degree*

$$\deg(\det M_p(t)) = \begin{cases} \frac{3(p-1)}{2} & p \equiv 1 \pmod 8 \\ \frac{3(p-1)}{2} - 1 & p \equiv 7 \pmod 8. \end{cases}$$

Proof. Suppose $p \equiv 1 \pmod 8$. We compute the entries of the Cartier-Manin matrix as in the above lemma. The third condition in the description of I_α implies that if α is odd, then $c_\alpha = 0$. This combined with the above lemma implies that the Cartier-Manin matrix has the following form:

$$M_p(t) := \begin{pmatrix} c_{p-1} & 0 & c_{p-3} & 0 \\ 0 & c_{2p-2} & 0 & c_{2p-4} \\ c_{2p-4} & 0 & c_{2p-2} & 0 \\ 0 & c_{p-3} & 0 & c_{p-1} \end{pmatrix}.$$

We will show that each row has a unique element of highest degree as a polynomial in t. Furthermore, this degree will be positive. By expanding along minors, this will imply that the highest degree term of the determinant is non-constant proving our claim.

First notice that an element $(i, j, k) \in I_\alpha$ contributes a coefficient to the monomial t^k in the polynomial c_α. For $\alpha = p - 1$, the largest degree monomial in c_α has degree $\frac{p-1}{4}$ coming from the element $(0, 0, \frac{p-1}{4})$ in I_{p-1}. The other non-zero element in the first row, c_{p-3} has degree smaller than $\frac{p-1}{4}$ because if $(i, j, k) \in I_{p-3}$, then $k \leq \frac{p-3}{2} - \frac{p-1}{4} = \frac{p-1}{4} - 1$.

The degree of c_{2p-2} is $\frac{p-1}{2}$, with the coefficient of the leading monomial coming from the unique element $(\frac{p-1}{8}, 0, \frac{p-1}{2}) \in I_{2p-2}$. Now, the degree of c_{2p-4} clearly cannot be greater than $\frac{p-1}{2}$ by condition 2 in the definition of I_α. If $k = \frac{p-1}{2}$, then condition 3 implies that $4i + 4j = \frac{p-1}{2} - 2$, a contradiction since the left side is $0 \pmod 4$ but the right side is $2 \pmod 4$.

This shows that each row of the Cartier-Manin matrix has a unique element of the highest degree and furthermore that the determinant is a polynomial in t of degree $\frac{3(p-1)}{2}$.

Suppose $p \equiv 7 \pmod 8$. Now, if α is even, then $c_\alpha = 0$. Thus the Cartier-Manin matrix has the following form:

$$M_p(t) := \begin{pmatrix} 0 & c_{p-2} & 0 & c_{p-4} \\ c_{2p-1} & 0 & c_{2p-3} & 0 \\ 0 & -c_{2p-3} & 0 & -c_{2p-1} \\ -c_{p-4} & 0 & -c_{p-2} & 0 \end{pmatrix}.$$

Again, we show that each row has a unique element of highest degree as a polynomial in t.

The largest degree contribution to c_{p-2} has degree $\frac{p-3}{4}$ coming from the element $(0, 0, \frac{p-3}{4})$ in I_{p-1}. The polynomial c_{p-4} has degree smaller than $\frac{p-3}{4}$ because if $(i, j, k) \in I_{p-3}$, then $k \leq \frac{p-4}{2} - \frac{p-1}{4} = \frac{p-3}{4} - 1$.

The degree of c_{2p-1} is $p-4$, with the coefficient of the leading monomial coming from the unique element $(\frac{p+1}{8}, 0, \frac{p-1}{2}) \in I_{2p-2}$. Again, the degree of c_{2p-3} cannot be greater than $\frac{p-1}{2}$ by condition 2 in the definition of I_α. If $k = \frac{p-1}{2}$, then condition 3 implies that $4i + 4j = \frac{p-1}{2} - 1$, a contradiction since the left side is $0 \pmod 4$ but the right side is $2 \pmod 4$.

Thus the highest degree polynomials in each row have degree $\frac{p-3}{4}, \frac{p-1}{2}, \frac{p-1}{2}, \frac{p-3}{4}$, showing that the determinant is a polynomial in t of degree $\frac{3(p-1)}{2} - 1$. □

Proposition 25. *For an odd prime p, the curve $C_0 : y^2 = x(x^8 - 1)$ is ordinary at p when $p \equiv 1$ or 7 and supersingular (but not superspecial) when $p \equiv 3$ or 5 mod p.*

Proof. This result is classically known but we can directly calculate that the Cartier-Manin matrix for the curve C_0 is diagonal for $p \equiv 1 \pmod 8$ and anti-diagonal for

$p \equiv 7 \pmod 8$. In the $p \equiv 1 \pmod 8$ (resp. $p \equiv 7 \pmod 8$) case, each entry on the diagonal (resp. antidiagonal) is a multinomial coefficient $\binom{\frac{p-1}{2}}{i,j,k}$ with $0 \le i,j,k \le \frac{p-1}{2}$ and this is non-zero in $\mathbb{F}_p$. Therefore, their product is also non-zero, demonstrating that the Cartier-Manin matrix is invertible and thus that the curve C_0 is ordinary.

Similarly, when $p \equiv 3 \pmod 8$, we construct the Cartier-Manin matrix of $y^2 = x(x^8-1)$:

Let

$$(x(x^8-1))^{\frac{p-1}{2}} = x^{\frac{p-1}{2}} \left(\sum_{i=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{i} (-1)^{i+1} x^{8i} \right).$$

For a coefficient of a monomial x^α to be non-zero, it is necessary and sufficient for α to satisfy $\alpha = \frac{p-1}{2} + 8k$ for some integer $0 \le i \le \frac{p-1}{2}$.

There are two cases: $p \equiv 3 \pmod{16}$ and $p \equiv 11 \pmod{16}$. The former case requires that any α for which the coefficient of x^α is non-zero satisfies $\alpha \equiv 1 \pmod 8$.

Then the Cartier-Manin matrix of $y^2 = x(x^8-1)$ is given by

$$\begin{pmatrix} c_{p-1} & c_{p-2} & c_{p-3} & c_{p-4} \\ c_{2p-1} & c_{2p-2} & c_{2p-3} & c_{2p-4} \\ c_{3p-1} & c_{3p-2} & c_{3p-3} & c_{3p-4} \\ c_{4p-1} & c_{4p-2} & c_{4p-3} & c_{4p-4} \end{pmatrix} = \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix}$$

where the entries marked with a $*$ represent non-zero elements. While this matrix M is non-zero, the corresponding matrix $M^{(p)}M$ is zero, proving that the corresponding curve has p-rank 0 but is not superspecial. By Theorem 19, the only possible Newton polygon that can occur in this family with p-rank 0 is the supersingular Newton polygon.

Similarly, if $p \equiv 11 \pmod{16}$, then that any α such that the coefficient of x^α is non-zero is required to satisfy $\alpha \equiv 5 \pmod 8$.

Then the Cartier-Manin matrix of $y^2 = x(x^8 - 1)$ is given by

$$\begin{pmatrix} c_{p-1} & c_{p-2} & c_{p-3} & c_{p-4} \\ c_{2p-1} & c_{2p-2} & c_{2p-3} & c_{2p-4} \\ c_{3p-1} & c_{3p-2} & c_{3p-3} & c_{3p-4} \\ c_{4p-1} & c_{4p-2} & c_{4p-3} & c_{4p-4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the entries marked with a $*$ represent non-zero elements. Again this matrix M is non-zero, but $M^{(p)}M$ is zero, proving that the corresponding curve has p-rank 0 but is not superspecial. By Theorem 19, the only possible Newton polygon that can occur in this family with p-rank 0 is the supersingular Newton polygon. The argument for $p \equiv 5 \pmod 8$ is nearly identical so we omit it. □

Corollary 26. *If $p \equiv 1, 7 \pmod 8$, then the μ-ordinary locus of the PEL moduli space Sh_p attached to family (36) consists of ordinary abelian varieties.*

Proof. Lemma 24 shows that for all but finitely many values of $a \in \overline{\mathbb{F}}_p$, the Cartier-Manin matrix $M_p(t)$ is invertible. Since the Cartier-Manin matrix models exactly the action of the Verschiebung on $H^0(C, \Omega^1)$, this is equivalent to the Verschiebung being separable and thus the corresponding Jacobian being ordinary. Since the image of the Jacobians of this curve family is dense in Sh_p, a non-empty open subset of Sh_p corresponds to ordinary abelian varieties and exactly form the μ-ordinary locus. □

Now we are set to clarify the ambiguity at the end of Chapter 2 in the calculation of Newton polygons of family (36).

Theorem 27. *Let p be a sufficiently large prime number. The algebraic group G attached to the Shimura datum (G, X) corresponding to family (36) is split exactly when $p = 1, 7 \pmod 8$ (and thus has invariant signature $\delta = 0$ and is quasisplit (has invariant signature $\delta = 1$ when $p \equiv 3, 8 \pmod 8$).*

Proof. By the main computations of Wedhorn ([Wed99] pp. 590-591), at every prime p of good reduction for the algebraic group G, the μ-ordinary Newton polygon of the representing moduli scheme associated to the algebraic group G is ord^4 when G is split at p and is $ord^2 \oplus ss^2$, when G is only quasi-split (but not split) at p. Since Corollary 26 shows that the μ-ordinary Newton polygon is ord^4 when the $p \equiv 1, 7 \pmod 8$, it must be the case that the algebraic group G is split at these primes. The Jacobians of the smooth curve with affine model $y^2 = x(x^8 - 1)$ in family (36) are supersingular but not superspecial when $p \equiv 3, 5 \pmod 8$ by Proposition 25. Combining this result with the computation of BT_1s with extra structure in Proposition 19 reveals decisively that primes $p \equiv 3, 5 \pmod 8$ correspond to $\delta = 1$ and therefore the quasi-split (but not split) case. $\square$

Corollary 28. *Let p be a prime number congruent to 3 or 5 $\pmod 8$. Then the determinant of the Cartier-Manin matrix $M_p(t)$, as a polynomial in the variable t, is identically 0.*

Proof. Since Proposition 25 shows that the family of curves (over $\overline{\mathbb{F}}_p$) contains a curve C_0 that is supersingular but not superspecial, our computations of the Ekedahl–Oort types in Proposition 19 show that this family can never contain an ordinary curve. As a consequence, the determinant of the Cartier-Manin matrix is always 0 for any parameter value $t \in \overline{\mathbb{F}}_q$. Since $\overline{\mathbb{F}}_p$ is infinite, this implies that the polynomial $M_p(t)$ has to vanish identically. $\square$

Motivated by Igusa's attempt to use the Picard-Fuchs differential equation of the Legendre pencil, we compute some differential equations satisfied by the entries of the Cartier-Manin matrix.

Proposition 29. 1. *Every entry $f(t)$ in the first and fourth column of the Cartier-Manin matrix $M(t)$ of the pencil $y^2 = x(x^4 - 1)(x^4 + tx^2 + 1)$ is annihilated by the differential operator*

$$\left(21 + (804t)\frac{d}{dt} + (1548t^2 - 2160)\frac{d^2}{dt^2} + 640t(t^2 - 4)\frac{d^3}{dt^3} + 64(t^2 - 4)^2\frac{d^4}{dt^4}\right) \tag{4.1}$$

2. *Every entry in the second and third column of the Cartier-Manin matrix $M(t)$ of the pencil $y^2 = x(x^4-1)(x^4+tx^2+1)$ is annihilated by the differential operator*

$$\left(45 + (900t)\frac{d}{dt} + (1580t^2 - 2288)\frac{d^2}{dt^2} + 640t(t^2-4)\frac{d^3}{dt^3} + 64(t^2-4)^2\frac{d^4}{dt^4}\right). \tag{4.2}$$

Proof. We use the mathematical computation package Singular ([DGPS20]) based on the **foliation** library accompanying [Mov17]. Our procedures can be found in Appendix A.3. While this is technically a computation describing the periods of the same curve family considered over the complex numbers, the classical result of Manin [Man61] assures us that the rows of the Cartier-Manin matrix of this curve family in positive characteristic still satisfy the same differential equation. Our result follows. □

Just as Igusa proves that the supersingular polynomial $H_{p-1}(t)$ has no multiple roots, we are able to prove a similar result away from the roots 2 and -2.

Remark 30. *Let $M(t)$ be the Cartier-Manin matrix of the family*

$$C_t : y^2 = x(x^4-1)(x^4+tx^2+1)$$

. Let $c_\alpha(t)$ be one of the non-zero entries of $M(t)$ (as a polynomial in the variable t) and t_0 a root of $c_\alpha(t)$. If we further assume that $t \neq \pm 2$, then its multiplicity as a root of $c_\alpha(t)$ is at most 2.

This is a straightforward but tedious calculation that begins with the supposition that $t_0 \neq \pm 2$ is a root of multiplicity $k \geq 4$ of a non and repeatedly differentiating the appropriate differential equation from proposition 29 to prove that t_0 is in fact a root of multiplicty $k+1$, yielding a contradiction. Then the same argument may be repeated with $k = 3$ to prove our claim.

$$21(t-t_0)^k g(t) + 804t(k(t-t_0)^{k-1} + g(t-t_0)^k$$

Given the bounded multiplicity of roots to solutions of the differential equations, we have the following results that bear striking parallels to Igusa's results in Theorem 22. In particular, they provide a uniform construction of supersingular curves over any characteristic $p > 7$. These curves are actually superspecial when $p \equiv 1, 7 \pmod 8$ This demonstrates an extremely unlikely intersection of the 9-dimensional Torelli locus with the 0-dimensional superspecial locus inside the 10 dimensional space of principally polarized abelian varieties.

Corollary 31. *1. The determinant of $M(t)$ is always divisible by $(t-2)$ and $(t+2)$ to the same power k and this power is equal to*

$$\begin{cases} \frac{p-1}{2} & p \equiv 1 \pmod 8 \\ \frac{p+1}{2} & p \equiv 7 \pmod 8 \end{cases}$$

2. *For $p > 7$, if $p \equiv 1$ or $7 \pmod 8$, there is always an element $t_0 \in \overline{\mathbb{F}}_p$ such that $M(t_0)$ is identically the 0 matrix, i.e. the corresponding curve $y^2 = x(x^4 - 1)(x^4 + t_0x^2 + 1)$ is superspecial.*

3. *For $p \equiv 1 \pmod 8$, $\gcd(c_{p-1}, c_{p-3})$ has only linear and quadratic irreducible factors in $\mathbb{F}_p$, i.e. all the roots are in $\mathbb{F}_{p^2}$. Here the c_α are polynomials in the ring $\mathbb{F}_p[t]$ and are defined by $(x(x^4-1)(x^4+tx^2+1))^{\frac{p-1}{2}} = \sum_\alpha c_\alpha x^\alpha$.*

4. *For $p \equiv 7 \pmod 8$, $\gcd(c_{p-2}, c_{p-4})$ has only linear and quadratic irreducible factors in $\mathbb{F}_p$, i.e. all the roots are in $\mathbb{F}_{p^2}$. Again, the c_α are polynomials in the ring $\mathbb{F}_p[t]$ and are defined by $(x(x^4-1)(x^4+tx^2+1))^{\frac{p-1}{2}} = \sum_\alpha c_\alpha x^\alpha$.*

5. *For $p \equiv 1 \pmod{16}$, there exists $t_0 \in \mathbb{F}_p$ such that C_{4,t_0} is superspecial if and only if p is represented by one of the quadratic forms $x^2 + 32y^2$ or $x^2 + 64y^2$.* [1]

[1] By a theorem of Kaplansky [Kap03], a prime $p \equiv 1 \pmod{16}$ is either represented by both or

none of the forms $x^2 + 32y^2$ and $x^2 + 64y^2$.

Appendix A

Code

This appendix records the procedures and algorithms used as part of our main results. Our code is written in three different programming languages: Magma, GAP and Singular.

A.1 Chevalley-Weil decomposition of $H^0(C, \Omega^1)$ in *Magma*

In the paper [FGP15], the authors describe the classical Chevalley-Weil decomposition of the complex vectorspace $H^0(C, \Omega^1)$ for a curve in the family $\mathcal{C}(\mathbf{m}, G, \theta)$. We describe it's implementation below. In the second subsection, we apply the procedure "CW" to the signatures $(\mathbf{m}, G, \theta)$ relevant to our work in Chapter 2, Table 2.2.

A.1.1 Implementation of the Chevalley-Weil formula

```
CharPolyFromTraces:=function(pp,S,x)

dd:=#pp;
ans:=x^dd;
s:=[pp[1]];
ans:=ans-s[1]*x^(dd-1);
 if dd gt 1 then
  for k:=2 to dd do
```

```
    s[k]:=0;
     for l:=1 to k-1 do
      s[k]:=s[k]+ (-1)^(l-1) * s[k-l]*pp[l];
     end for;
    s[k]:=s[k]+(-1)^(k-1) *pp[k];
    s[k]:=(1/k)*s[k];
    ans:=ans+(-1)^(k)*s[k]*x^(dd-k);
   end for;
  end if;

return ans;
end function;

////////////////////////////////////////

TraceSequence:=function(G,A,chi,K)

f:=ClassMap(G);
eQ:=Order(A);

return [K!chi[f(A^k)] : k in [1..chi[1]]];
end function;

////////////////////////////////////////
n:=function(G,alpha,A,chi)

eQ:=Order(A);
K<z>:=CyclotomicField(#G);
```

```
t := z^Round(#G/eQ);
p:=TraceSequence(G,A,chi,K);
S<x>:=PolynomialRing(K);
rootlist:=Roots(CharPolyFromTraces(p,S,x));
ans:=0;
 for k :=1 to #rootlist do
  if t^alpha eq rootlist[k][1] then
   ans:= rootlist[k][2];
  end if;
 end for;

return ans;
end function;

////////////////////////////////////////
FractionalPart:=function(x)

return x-Floor(x);
end function;
```

```
CW:= function(G,gprime,T,CCL,M,m)

H0mK:=[];
e:=[Order(M[j]) : j in [1..#M]];
 for i:=1 to #CCL do
  H0mK[i]:=(2*m-1)*(gprime-1)*T[i][1];
   for j:=1 to #M do
    for alpha:=0 to e[j]-1 do
     H0mK[i]:=H0mK[i]+( (m-1)*(1-1/e[j]) +
      FractionalPart( (m-1-alpha)/e[j]) )*n(G,alpha,M[j],T[i]);
    end for;
   end for;
   if m eq 1 then
    if i eq 1 then
     H0mK[i]:=H0mK[i]+1;
    end if;
   end if;
 end for;

return H0mK;
end function;
```

A.1.2 Application of the Chevalley-Weil formula

We apply the function CW in the above Magma code to the thirteen families of Table 2.1 in Chapter 2 to produce the last column of Table 2.2.

```
// Family (28)
>G<x,y>:=SymmetricGroup(3);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[y,y,x,x^2];
>CW(G,0,T,CCL,M,1);
[ 0, 0, 1 ]

// Family (29)
>G<x,y>:=DihedralGroup(4);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[x^3*y,x^2,y,x^3];
>CW(G,0,T,CCL,M,1);
[0,0,0,0,1]

// Family (30)
>G<x,y>:=DihedralGroup(6);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[x^3*y,x^4*y,x^3,x^4];
>CW(G,0,T,CCL,M,1);
[0,0,0,0,1]

// Family (31)
>G<x,y>:=SymmetricGroup(3);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[x*y,x^2*y,y,x*y,x^2];
>CW(G,0,T,CCL,M,1);
[0,1,1]

// Family (32)
```

```
>G<x,y>:=DihedralGroup(4);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[x*y,x^2*y,x^2,x^2*y,x^3*y];
>CW(G,0,T,CCL,M,1);
[0,0,0,1,1]

// Family (33)
>G<x,y>:=AlternatingGroup(4);y1:=G!(1,2,3);
>y2:= G!(1,2)(3,4);y3:=G!(1,3)(2,4);
>T:=CharacterTable(G);CCL:=ConjugacyClasses(G);
>M:=[y3,y2,y1*y3,y1^2*y3];CW(G,0,T,CCL,M,1);
[0,0,0,1]
// Family (34)
>G<y1,y2,y3>:=SmallGroup(16,13);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[y1,y1*y2*y3^3,y2*y3^2,y3^3];
>CW(G,0,T,CCL,M,1);
[ 0, 0, 0, 0, 0, 1, 0, 0, 0, 1 ]

// Family (35)
>G:=SymmetricGroup(4);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);
>M:=[G!(1,3),G!(1,2)(3,4),G!(1,2),G!(1,4,3)];
>CW(G,0,T,CCL,M,1);
[ 0, 0, 0, 1, 0 ]

// Family (36)
>G<y1,y2,y3>:=SmallGroup(8,4);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[y3,y2*y3,y1*y2,y1*y3];
>CW(G,0,T,CCL,M,1);
```

```
[ 0, 0, 0, 0, 2 ]

// Family (37)
>G<y1,y2,y3>:=SmallGroup(12,3);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[y3,y1,y1,y1*y3];
>CW(G,0,T,CCL,M,1);
[ 0, 1, 0, 1 ]

// Family (38)
>G<y1,y2,y3>:=SmallGroup(18,3);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[y1*y3^2,y1*y3,y2*y3,y2^2];
>CW(G,0,T,CCL,M,1);
[ 0, 0, 1, 1, 0, 0, 0, 1, 0 ]

// Family (39)
>G<y1, y2, y3>:=SmallGroup(12,1);T:=CharacterTable(G);
>CCL:=ConjugacyClasses(G);M:=[y1^2,y3,(y1^3)*(y3^2),(y1^3)*y3];
>CW(G,0,T,CCL,M,1);
[ 0, 0, 1, 0, 2, 0 ]

// Family (40)
>G:=SL(2,3);y1:=G![ [2 ,1],[2 ,0]];y2:=G![[0,2],[1,0]];
>y3:=G![[1,2],[2,2]];y4:=G![[2,0],[0,2]];
>T:=CharacterTable(G);CCL:=ConjugacyClasses(G);
>M:=[y4,y1^2*y2*y3*y4,y1^2*y2*y4,y1^2*y3*y4];
>CW(G,0,T,CCL,M,1);
[ 0, 0, 1, 2, 0, 1, 0 ]
```

A.2 Artin-Wedderburn decomposition of group-rings in *GAP*

Let G be a finite group. We use the **wedderga** library of the GAP programming language ([Mic00]) developed in ([OdR09]) to compute the Artin-Wedderburn decomposition of the group-ring $\mathbb{Q}[G]$ into a direct sum of matrix algebras over division rings. This is used in Table 2.2 of Chapter 2.

Lines beginning with gap> correspond to user input to the program. The lines beginning with # correspond to user comments. We use comments to indicate which family of curves in Table 2.2 the procedure is relevant to. All other lines correspond to the output of the program.

```
gap> LoadPackage("wedderga");
true

#family (28)
gap> G:=SymmetricGroup(3);
Sym( [ 1 .. 3 ] )
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, Rationals ], [ 2, Rationals ] ]

#family (29)
gap> G:=DihedralGroup(8);
<pc group of size 8 with 3 generators>
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ],
[ 1, Rationals ], [ 2, Rationals ] ]

#family (30)
gap> G:=DihedralGroup(12);
<pc group of size 12 with 3 generators>
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ],
[ 1, Rationals ], [ 2, Rationals ], [ 2, Rationals ] ]
```

```
#family (31)
gap> G:=SymmetricGroup(3);
Sym( [ 1 .. 3 ] )
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, Rationals ], [ 2, Rationals ] ]

#family (32)
gap> G:=DihedralGroup(8);
<pc group of size 8 with 3 generators>
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ],
[ 1, Rationals ], [ 2, Rationals ] ]

#family (33)
gap> G:=AlternatingGroup(4);
Alt( [ 1 .. 4 ] )
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, CF(3) ], [ 3, Rationals ] ]
```

```
#family (34)
gap> G:=SmallGroup(16,13);
<pc group of size 16 with 4 generators>
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ],
[ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ],
[ 1, Rationals ], [ 1, Rationals ], [ 2, GaussianRationals ] ]

#family (35)
gap> G:=SymmetricGroup(4);
Sym( [ 1 .. 4 ] )
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, Rationals ], [ 2, Rationals ],
[ 3, Rationals ], [ 3, Rationals ] ]

#family (36)
gap> G:=SmallGroup(8,4);
<pc group of size 8 with 3 generators>
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ],
[ 1, Rationals ],
[ 1, rec( Center := Rationals, DivAlg := true,
```

```
  LocalIndices := [ [ 2, 2 ], [ infinity , 2 ] ],
  SchurIndex := 2 ) ] ]

#family (37)
gap> G:=SmallGroup(12,3);
<pc group of size 12 with 3 generators>
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, CF(3) ], [ 3, Rationals ] ]

#family (38)
gap> G:=SmallGroup(18,3);
<pc group of size 18 with 3 generators>
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, Rationals ], [ 1, CF(3) ],
[ 1, CF(3) ], [ 2, Rationals ], [ 2, CF(3) ] ]

#family (39)
gap> G:=SmallGroup(12,3);
<pc group of size 12 with 3 generators>
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, CF(3) ], [ 3, Rationals ] ]
```

```
#family (40)
gap> G:=SL(2,3);
SL(2,3)
gap> A:=GroupRing(Rationals,G);
<algebra-with-one over Rationals, with 2 generators>
gap> WedderburnDecompositionWithDivAlgParts(A);
[ [ 1, Rationals ], [ 1, CF(3) ], [ 3, Rationals ],
  [ 1, rec( Center := Rationals, DivAlg := true,
  LocalIndices := [ [ 2, 2 ], [ infinity, 2 ] ],
  SchurIndex := 2)], [ 2, CF(3) ] ]
```

A.3 Picard-Fuchs differential equations in *Singular*

This section records the code used to obtain the differential equations satisfied by the entries of the Cartier-Manin matrix in Lemma 29. The code is written in the programming language *Singular* ([DGPS20]) and uses the libraries **linalg** and **foliation**. The former is a standard library of the Singular programming language but the latter is code accompanying [Mov17].

The following procedure first defines the pencil of curves given by equation

$$y^2 = x(x^4 - 1)(x^4 + tx^2 + 1)$$

in weighted projective space. Then, it computes the Picard-Fuchs differential equations satisfied by each element in the basis $\{\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}, \frac{x^3dx}{y}\}$ of the vector space of global holomorphic differentials. The lines starting with $>$ correspond to user input while the lines starting with - correspond to the output of the program.

```
>LIB "foliation.lib"; LIB "linalg.lib"
> ring r=(0,t), (x,y), wp(2,9);
>poly L=-y^2+x*(x^4-1)*(x^4+t*(x^2)+1);
> PFeq(L,1,t);
_[1,1]=21
_[1,2]=(804t)
_[1,3]=(1548t2-2160)
_[1,4]=(640t3-2560t)
_[1,5]=(64t4-512t2+1024)
_[1,6]=0
_[1,7]=0
_[1,8]=0
_[1,9]=0
> PFeq(L,x,t);
_[1,1]=45
_[1,2]=(900t)
_[1,3]=(1580t2-2288)
_[1,4]=(640t3-2560t)
_[1,5]=(64t4-512t2+1024)
_[1,6]=0
_[1,7]=0
_[1,8]=0
_[1,9]=0
> PFeq(L,x2,t);
_[1,1]=45
_[1,2]=(900t)
_[1,3]=(1580t2-2288)
_[1,4]=(640t3-2560t)
```

```
_[1,5]=(64t4-512t2+1024)
_[1,6]=0
_[1,7]=0
_[1,8]=0
_[1,9]=0
> PFeq(L,x3,t);
_[1,1]=21
_[1,2]=(804t)
_[1,3]=(1548t2-2160)
_[1,4]=(640t3-2560t)
_[1,5]=(64t4-512t2+1024)
_[1,6]=0
_[1,7]=0
_[1,8]=0
_[1,9]=0
```

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