NUMERICAL METHODS LABORATORY(MA29202) & NUMERICAL TECHNIQUES LABORATORY(MA39110)

Assignment-5 based on Time Dependent PDEs [1]

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1. Find $u(x,t)$ using Lax-Friedrichs Method for the following linear convection equation with $f(u) = u$, that is,
$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$
and the initial conditions are given as,
$$u(x,0) = \begin{cases} 1, & |x| < 1/3, \\ 0, & 1/3 < |x| \leq 1. \end{cases}$$
Use 201 evenly spaced grid points to divide the given domain $[-1,1]$ into 200 equally spaced intervals with each of length Δx. The final solutions are required to be obtained at $t = 4$ units. Also, take $\lambda = \frac{\Delta t}{\Delta x} = 0.8$ and periodic boundary conditions are imposed at the boundary, *i.e*, $u\left(x(0),t\right) = u\left(x(200),t\right)$. Similarly, use $u\left(x(201),t\right) = u\left(x(1),t\right)$. Also, plot a reference solution using 5001 grid points and plot both the solutions together.

2. Find $u(x,t)$ using Lax-Friedrichs Method for the following inviscid Burgers' equation with $f(u) = \dfrac{u^2}{2}$, that is,
$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$
and initial conditions are given as,
$$u(x,0) = \begin{cases} +1, & |x| < 1/3, \\ -1, & 1/3 < |x| \leq 1. \end{cases}$$
Use 201 evenly spaced grid points to obtained final solutions at $t = 0.3$ units. Also, take $\lambda = \frac{\Delta t}{\Delta x} = 0.8$ and periodic boundary conditions are imposed at each boundary. Also, plot a reference solution using 5001 grid points and plot both the solutions together.

Lax-Friedrichs Method: To derive the Lax-Friedrichs method, first consider Forward in Time and Centre in Space (FTCS) discretization:
$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left(f\left(u_{i+1}^n\right) - f\left(u_{i-1}^n\right)\right).$$
Now, replace u_i^n with $\frac{\left(u_{i+1}^n + u_{i-1}^n\right)}{2}$ in the first-term on the right hand side. With this modification, FTCS becomes the Lax-Friedrichs Method:
$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f\left(u_{i+1}^n\right) - f\left(u_{i-1}^n\right)\right).$$

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