NUMERICAL METHODS LABORATORY(MA29202) &
NUMERICAL TECHNIQUES LABORATORY(MA39110)
Assignment-7 based on Time Dependent PDEs [1]

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One-Dimensional System of Pressureless Gas Dynamics Equations

Let us consider one-dimensional inviscid homogeneous system of Pressureless Gas Dynamics Equations which is given by

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} = 0. \tag{1}$$

This is a system of two nonlinear coupled equations, where $\boldsymbol{U}$ is vector of conserved variables and $\boldsymbol{F}(\boldsymbol{U})$ is the flux vector, and are given as

$$\boldsymbol{U} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \rho u \\ \rho u^2 \end{bmatrix}$$

respectively. Here, ρ represents density, u is fluid velocity.

We consider a following initial cases defined on $[-1, 1]$ of the form:

$$u(x,0) = \begin{cases} +2, & x < 0, \\ -2, & x > 0, \end{cases} \quad \text{and} \quad \rho(x,0) = 1. \tag{2}$$

Exercise

1. Compute the eigenvalues and eigenvectors of the Jacobian of system (1).
2. Implement first-order finite-difference Lax-Friedrichs scheme to the given system (1) with initial data (2) with $\Delta x = 1/100$ and T = 0.5. The time step will be adapted at every step according to the stability condition. We will use Neumann boundary conditions.

Finite-Difference Lax-Friedrichs Method: To derive the Lax-Friedrichs method, first consider Forward in Time and Centre in Space (FTCS) discretization:

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\Delta t}{2\Delta x}\left(\boldsymbol{F}\left(\boldsymbol{U}_{i+1}^n\right) - F\left(\boldsymbol{U}_{i-1}^n\right)\right).$$

Now, replace $\boldsymbol{U}_i^n$ with $\frac{\left(\boldsymbol{U}_{i+1}^n + \boldsymbol{U}_{i-1}^n\right)}{2}$ in the first-term on the right hand side. With this modification, FTCS becomes the Lax-Friedrichs scheme:

$$\boldsymbol{U}_i^{n+1} = \frac{1}{2}\left(\boldsymbol{U}_{i+1}^n + \boldsymbol{U}_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(\boldsymbol{F}\left(\boldsymbol{U}_{i+1}^n\right) - \boldsymbol{F}\left(\boldsymbol{U}_{i-1}^n\right)\right).$$

The stability condition is equal to

$$\Delta t \leq \frac{\Delta x}{\max \rho\left(\boldsymbol{F}'(\boldsymbol{U})\right)}.$$

Here $\rho(M)$ denotes the spectral radius of the matrix M .

[1] Sent on: `March 20, 2024`.