Assignment- 9

Use the Crank-Nicolson method to approximate the solution to the following parabolic PDEs:

(i) $u_t = \frac{1}{\pi^2} u_{xx}$; $0 < x < 1$, $0 < t < 0.1$

$u(0,t) = u(1,t) = 0$; $0 \le t \le 0.1$

$u(x,0) = \sin \pi x$, $0 \le x \le 1$.

Use $\Delta x = 0.1$, $\Delta t = 0.01$. Compare your results with the explicit scheme (done in assignment 6) using $0 < \lambda = \frac{\alpha \Delta t}{(\Delta x)^2} \le 1/2$. Also plot both solutions together. For explicit scheme, ~~trake~~ partion the domain $[0, 1]$ into 50 equally spaced cells, i.e, $\Delta x = 1/50$.

~~[illegible]~~.

(ii) $u_t = 3u_{xx}$; $0 < x < \pi$, $0 < t < 0.1$;

$u(0,t) = u(\pi,t) = 0$; $0 \le t \le 0.1$;

$u(x,0) = \sin x$; $0 \le x \le \pi$.

Use $\Delta x = 0.52$ & $\Delta t = 0.01$. Again, Compare your results with the explicit scheme with $\Delta x = \frac{\pi}{50}$.

use $0 < \lambda = \frac{\alpha \Delta t}{(\Delta x)^2} \leq 1/2$ to fix Δt.

(xiii) $u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t < 0.05;$

$u(0,t) = u(\pi, t) = 0, \quad 0 \leq t \leq 0.05;$

$u(x,0) = \sin \pi x + \sin 2\pi x, \quad 0 \leq x \leq 1.$

Use $\Delta x = 0.2$ & $\Delta t = 0.01$.

Again, compare the results with the explicit scheme with $\Delta x = 1/50$ & use $0 < \lambda \leq 1/2$ to fix Δt.

[A Brief Review of Crank-Nicolson method is provided on pages 3 & 4.]

Parabolic PDEs & the Crank–Nicolson method.

Consider; $u_t = \alpha u_{xx}$

Then explicit scheme ~~is~~ can be written as:

$$\circledast \longleftarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right).$$

$$\text{or} \quad u_i^{n+1} = u_i^n - \frac{\alpha \Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$\lambda = \frac{\alpha \Delta t}{\Delta x^2}$$

$$0 < \lambda \leq 1/2 .$$

Let us replace the terms of RHS of $\circledast$ as follows:

$$u_{i+1}^n = \frac{u_{i+1}^n + u_{i+1}^{n+1}}{2}, \quad u_i^n = \frac{u_i^n + u_i^{n+1}}{2}$$

$$\& \quad u_{i-1}^n = \frac{u_{i-1}^n + u_{i-1}^{n+1}}{2}$$

$\therefore$ $\circledast$ becomes

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2(\Delta x)^2}\left[\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) + \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)\right]$$

set $\lambda = \dfrac{\alpha \Delta t}{\cancel{2}(\Delta x)^2}$

Then;

$$-\lambda u_{i-1}^{n+1} + 2(1+\lambda) u_i^{n+1} - \lambda u_{i+1}^{n+1}$$
$$= \lambda u_{i-1}^{n} + 2(1-\lambda) u_i^{n} + \lambda u_{i+1}^{n}$$

for $\quad i = 1, 2, \ldots, N-1$.

This method is called the Crank-Nicolson method.

In the matrix form, one can write

$$A u^{(n+1)} = B u^{(n)} \quad ; \quad n = 0, 1, 2, \ldots$$

where $\quad u^{(n+1)} = \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \end{bmatrix}$