NUMERICAL METHODS LABORATORY(MA29202) & NUMERICAL TECHNIQUES LABORATORY(MA39110)

Assignment-6 based on Time Dependent PDEs [1]

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One-Dimensional System of Shallow Water Equations

Let us consider one-dimensional inviscid homogeneous shallow water system which is given by

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} = 0. \tag{1}$$

This is a system of two nonlinear coupled equations, where $\boldsymbol{U}$ is vector of conserved variables and $\boldsymbol{F}(\boldsymbol{U})$ is the flux vector, and are given as

$$\boldsymbol{U} = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}$$

respectively. Here, h represents depth of water level, u is fluid velocity and $g = 9.81$ m/s^2 represents gravity force.

We consider two initial cases defined on $[0, 5]$ of the form:

$$h(x,0) = e^{\frac{-(x-2)^2}{0.1}} \quad \text{and} \quad u(x,0) = 0, \tag{2}$$

and

$$h(x,0) = \begin{cases} 2, & x < 2.5, \\ 1, & x \geq 2.5 \end{cases} \quad \text{and} \quad u(x,0) = 0, \tag{3}$$

which corresponds to the **dam-break problem**.

Exercise

1. Compute the eigenvalues of the Jacobian of system (1).

2. Implement first-order finite-difference Lax-Friedrichs scheme to the shallow-water system (1) with initial data (2) and (3) on the domain $[0, 5]$ with $\Delta x = 1/100$ and T = 1. The time step will be adapted at every step according to the stability condition. We will use Neumann boundary conditions. Also, plot the reference solution with $\Delta x = 1/1000$.

Finite-Difference Lax-Friedrichs Method: To derive the Lax-Friedrichs method, first consider Forward in Time and Centre in Space (FTCS) discretization:

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\Delta t}{2\Delta x}\left(\boldsymbol{F}\left(\boldsymbol{U}_{i+1}^n\right) - F\left(\boldsymbol{U}_{i-1}^n\right)\right).$$

Now, replace $\boldsymbol{U}_i^n$ with $\dfrac{\left(\boldsymbol{U}_{i+1}^n + \boldsymbol{U}_{i-1}^n\right)}{2}$ in the first-term on the right hand side. With this modification, FTCS becomes the Lax-Friedrichs scheme:

$$\boldsymbol{U}_i^{n+1} = \frac{1}{2}\left(\boldsymbol{U}_{i+1}^n + \boldsymbol{U}_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(\boldsymbol{F}\left(\boldsymbol{U}_{i+1}^n\right) - \boldsymbol{F}\left(\boldsymbol{U}_{i-1}^n\right)\right).$$

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The stability condition is equal to

$$\Delta t \leq \frac{\Delta x}{\max \rho\left(\boldsymbol{F'(U)}\right)}.$$

Here $\rho\left(M\right)$ denotes the spectral radius of the matrix M .