RAVI KANDA
01/16/2020

1. $V_P = \sqrt{\dfrac{\lambda+2\mu}{\rho}} = \sqrt{\dfrac{K+\frac{4}{3}\mu}{\rho}}$ ... (I) | $V_S = \sqrt{\dfrac{\mu}{\rho}}$ ... (II) | $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ ... (III)

(a) From (I) : $\lambda + 2\mu = K + \frac{4}{3}\mu \Rightarrow \boxed{K = \lambda + \frac{2}{3}\mu}$ ... (IV)

(b) From (III) : $\lambda = \mu\left(\dfrac{2\nu}{1-2\nu}\right)$ ... (V)

- plugging (V) into (IV) : $K = \mu\left[\dfrac{2\nu}{1-2\nu} + \dfrac{2}{3}\right] = \dfrac{\mu}{3(1-2\nu)}[6\nu+2-4\nu]$
- So, $K = \frac{2}{3}\mu\left(\dfrac{1+\nu}{1-2\nu}\right) \Rightarrow \boxed{\mu = \frac{3}{2}K\left(\dfrac{1-2\nu}{1+\nu}\right)}$ ... (VI)

(c) From (VI) : INCOMPRESSIBLE FLUID : $\mu = 0 \Rightarrow 1-2\nu = 0 \Rightarrow \boxed{\nu = \frac{1}{2}}$ ... (VII)

(d) From (II) : $\mu = \rho V_S^2$
From (I) : $\lambda = \rho V_P^2 - 2\mu = \rho\left(V_P^2 - 2V_S^2\right)$ ... (VIII)

(e) from (I) & (II) : $\dfrac{V_P}{V_S} = \sqrt{\dfrac{\lambda+2\mu}{\mu}} = \sqrt{\dfrac{\lambda}{\mu}+2}$

From (V) : $\dfrac{\lambda}{\mu} = \dfrac{2\nu}{1-2\nu} \Rightarrow \dfrac{V_P}{V_S} = \sqrt{\dfrac{2\nu}{1-2\nu}+2}$

$\therefore \boxed{\dfrac{V_P}{V_S} = \sqrt{\dfrac{2(1-\nu)}{1-2\nu}}}$ ... (IX)

From (VII) : $\nu = \frac{1}{2}$ & $\lim\limits_{\nu\to\frac{1}{2}}\left(\dfrac{V_P}{V_S}\right) \to \infty$
or $\lim\limits_{\nu\to\frac{1}{2}}\left(\dfrac{V_S}{V_P}\right) \to 0$
$\Rightarrow V_S \to 0!$ in an incompressible fluid

(f) Plot : See Solution Spread-sheet "HW1.xls".

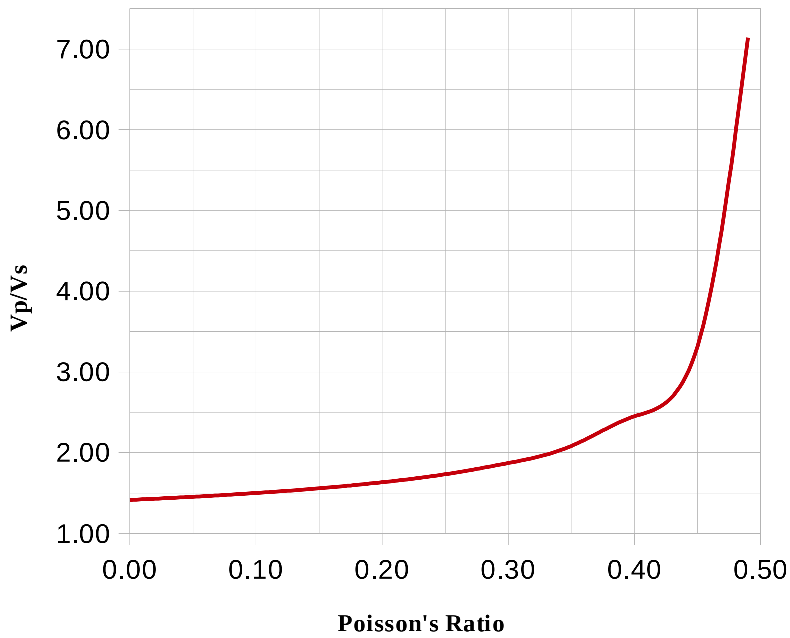
If $\nu = 0.25$, from (IX), $\dfrac{V_P}{V_S} = \sqrt{\dfrac{2(0.75)}{(0.5)}} = \sqrt{3} \simeq \underline{\underline{1.73}}$ ... (X)

As $\nu \to 0$, $\dfrac{V_P}{V_S} = \sqrt{\dfrac{2(1)}{1}} = \sqrt{2} \simeq \underline{\underline{1.41}}$

For most Earth materials $\boxed{\nu < 0.35,\ \& \ 1.4 \le \frac{V_P}{V_S} \le 2.1}$

Vp/Vs ratio as a function of Poisson's ratio, ν:

ν	Vp/Vs
0.00	1.41
0.05	1.45
0.10	1.50
0.15	1.56
0.20	1.63
0.25	1.73
0.30	1.87
0.35	2.08
0.40	2.45
0.45	3.32
0.49	7.14



For most rocks, $0.01 < \nu < 0.35$, and $1.4 < (Vp/Vs) < 2.1$

2. (a) P-wave: $u_x = A\sin(\omega t - kx)$

$$\therefore\ e = \begin{bmatrix} \overbrace{\frac{\partial u_x}{\partial x}}^{e_{xx}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -Ak\cos(\omega t - kx) & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \theta \neq 0 \Rightarrow \boxed{\text{Dilational/ Volumetric Change!}}$$

$$\sigma \equiv \begin{bmatrix} (\lambda+2\mu)e_{xx} & 0 \\ 0 & \lambda e_{xx} \end{bmatrix} = \begin{bmatrix} -(\lambda+2\mu)Ak\cos(\omega t - kx) & 0 \\ 0 & -\lambda Ak\cos(\omega t - kx) \end{bmatrix}$$

Thus: $e_{xx} = -Ak\cos(\omega t - kx)$; $e_{yy} = 0 = e_{xy}$

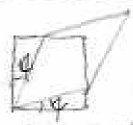
$\sigma_{xx} = -Ak(\lambda+2\mu)\cos(\omega t - kx)$ | & $\sigma_{xy} = 0$ → No Shear!

$\sigma_{yy} = -Ak\lambda\cos(\omega t - kx)$

(b) S-wave: $u_y = A\sin(\omega t - kx)$

$$e = \begin{bmatrix} 0 & -\frac{k}{2}A\cos(\omega t - kx) \\ -\frac{k}{2}A\cos(\omega t - kx) & 0 \end{bmatrix} \Rightarrow \theta = 0 \Rightarrow \text{No Volume change!}$$

$\boxed{\text{SIMPLE SHEAR}}$



$\gamma = 2\tan\psi$

$$\sigma = \begin{bmatrix} 0 & -Ak\mu\cos(\omega t - kx) \\ -Ak\mu\cos(\omega t - kx) & 0 \end{bmatrix}$$

Thus: $e_{xx} = 0 = e_{yy}$; $e_{xy} = -\frac{1}{2}Ak\cos(\omega t - kx)$ → Only Shear!

$\sigma_{xx} = 0 = \sigma_{yy}$; $\sigma_{xy} = -Ak\mu\cos(\omega t - kx)$

→ P.T.O

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2(c) * Recall: $\frac{\omega}{k} = V_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$; $\omega = \frac{2\pi}{T}$; $k = \frac{2\pi}{\lambda}$

For P-Waves [from (a)]: $e_{xx} = -Ak\cos(\omega t - kx)$

$$\|e_{xx}\|_{max} = Ak = \frac{A\omega}{V_p} = \frac{2\pi A}{V_p T}$$

Given $\|e_{xx}\|_{max} = e_0 = 10^{-8}$

$$\therefore \boxed{A = \frac{e_0 V_p T}{2\pi} = \frac{10^{-8} \times 10^{4}\ m/s}{2\pi} T\ s} \quad \cdots (\mathrm{I})$$

or $A = 1.60 \times 10^{-5}\, T$ m (T in seconds)

(i) $T = 1s$: $A = 1.60 \times 10^{-5}$ m or $\sim 16\ \mu m$

(ii) $T = 10s$: $A = 1.60 \times 10^{-4}$ m or ~ 0.16 mm

(iii) $T = 100s$: $A = 1.60 \times 10^{-3}$ m or ~ 1.6 mm

* Recall: $\hat{k} \equiv \hat{p} \equiv \frac{1}{4}\rho A^2 \omega^2 = \frac{1}{4}\rho A^2\left(\frac{4\pi^2}{T^2}\right) = \rho\left(\frac{\pi A}{T}\right)^2$

$\hat{e} = \hat{k} + \hat{p} = \frac{1}{2}\rho A^2 \omega^2 = \longrightarrow 2\rho\left(\frac{\pi A}{T}\right)^2$

Thus, $\boxed{\hat{k} \equiv \hat{p} \equiv \frac{1}{2}\hat{e} = 2\rho\left(\frac{\pi^2 A^2}{T^2}\right)} \quad \cdots\cdots (\mathrm{II})$

But from (I) above: $\frac{\pi^2 A^2}{T^2} = \frac{e_0^2 V_p^2}{4}$

$\therefore \hat{k} \equiv \hat{p} \equiv \frac{1}{2}\hat{e} = \frac{1}{2}\rho e_0^2 V_p^2 \rightarrow \left(\text{INDEPENDENT OF } T\right)$!!!

$= \frac{1}{2}\left(2800 \frac{kg}{m^3}\right)\left(10^{-8}\right)^2\left(10^4\right)^2 \frac{m^2}{s^2}$

$\therefore \hat{k} \equiv \hat{p} \equiv \frac{1}{2}\hat{e} = 1400 \times 10^{-16} \times 10^{8}\ J/m^3 = \boxed{14\ \mu J/m^3}$

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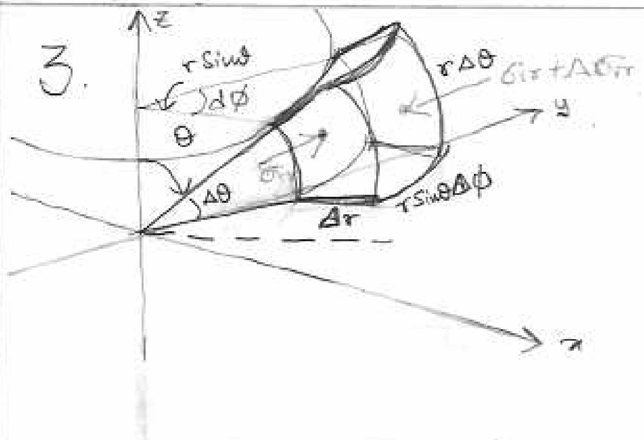
3.



$$\Delta V = (r\sin\theta)\Delta\phi(r\Delta\theta)(\Delta r)$$

$$\boxed{\Delta V = r^2 \sin\theta \Delta r \Delta\theta \Delta\phi} \quad \cdots (I)$$

$$\Delta m = \rho \Delta V$$

$$\boxed{\Delta m = \rho r^2 \sin\theta \Delta r \Delta\theta \Delta\phi} \quad \cdots (II)$$

(a) We ignore body forces in 1D:

$$\sum F_{rr} = \Delta m\, a_r$$

$$F_{rr} = \sigma_{rr}(r\Delta\theta)(r\sin\theta\Delta\phi)$$
$$= \sigma_{rr} r^2 \sin\theta\Delta\theta\Delta\phi$$

$$\sum F_{rr} = \begin{bmatrix} (\sigma_{rr} + \Delta\sigma_{rr})(r+\Delta r)^2 \cancel{(\sin\theta\Delta\theta\Delta\phi)} \\ - \sigma_{rr} r^2 \cancel{(\sin\theta\Delta\theta\Delta\phi)} \end{bmatrix} = \rho r^2 \cancel{(\sin\theta\Delta\theta\Delta\phi)} \Delta r \frac{\partial^2 u_r}{\partial t^2}$$

$$\therefore \quad (\sigma_{rr} + \Delta\sigma_{rr})(r^2 + 2r\Delta r + (\Delta r)^2) - \sigma_{rr} r^2 = \rho r^2 \Delta r \frac{\partial^2 u_r}{\partial t^2}$$

Dividing throughout by $r^2 \Delta r$, we get

$$\frac{\begin{bmatrix} \cancel{\sigma_{rr} r^2} + 2\sigma_{rr} r\Delta r + \boxed{\sigma_{rr}(\Delta r)^2} + \Delta\sigma_{rr} r^2 + \boxed{2\Delta\sigma_{rr} r\Delta r + \Delta\sigma_{rr}(\Delta r)^2} \\ - \cancel{\sigma_{rr} r^2} \end{bmatrix}}{r^2 \Delta r} = \rho \frac{\partial^2 u_r}{\partial t^2}$$

In the limit as $\Delta r \to 0$ we can ignore these 2nd & 3rd order terms by setting them ≈ 0

$$\therefore \quad \frac{2r}{r^2}\sigma_{rr} + \frac{\Delta\sigma_{rr}}{\Delta r} = \rho \frac{\partial^2 u_r}{\partial t^2}$$

Multiplying throughout by r^2 & setting Lim $\Delta r \to 0$ $\Big\}$ $\boxed{2r\sigma_{rr} + r^2 \frac{\partial \sigma_{rr}}{\partial r} = r^2 \rho \frac{\partial^2 u_r}{\partial t^2}}$ $\rightarrow$

3(a) (continued)

From the previous page: $\underbrace{2r\sigma_{rr} + r^2 \frac{\partial \sigma_{rr}}{\partial r}}_{LHS} = r^2 \rho \frac{\partial^2 u_r}{\partial t^2}$

But LHS $= \frac{\partial}{\partial r}\left[r^2 \sigma_{rr}\right]$!

$$\therefore \quad \frac{\partial}{\partial r}\left[r^2 \sigma_{rr}\right] = r^2 \rho \frac{\partial^2 u_r}{\partial t^2}$$

Finally, $\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \sigma_{rr}\right] = \rho \frac{\partial^2 u_r}{\partial t^2}}$ ....(III)

3(b) In 1D: $\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r}$

Using this in (III), we obtain after dividing throughout by $(\lambda+2\mu)$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u_r}{\partial r}\right] = \frac{\rho}{(\lambda+2\mu)}\frac{\partial^2 u_r}{\partial t^2} = \frac{1}{C^2}\frac{\partial^2 u_r}{\partial t^2}} \quad \text{..(IV)}$$

where, $C = \sqrt{\frac{\lambda+2\mu}{\rho}}$

3(c) $\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u_r}{\partial r}\right] = \frac{1}{r^2}\left[2r\frac{\partial u_r}{\partial r} + r^2 \frac{\partial^2 u_r}{\partial r^2}\right] = \underbrace{\frac{2}{r}\frac{\partial u_r}{\partial r} + \frac{\partial^2 u_r}{\partial r^2}}_{(Va)}$

$$\frac{1}{r}\frac{\partial^2 (r u_r)}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}(r u_r)\right] = \frac{1}{r}\frac{\partial}{\partial r}\left[u_r + r\frac{\partial u_r}{\partial r}\right]$$

$$= \frac{1}{r}\left[\frac{\partial u_r}{\partial r} + \frac{\partial u_r}{\partial r} + r\frac{\partial^2 u_r}{\partial r^2}\right]$$

$$= \frac{2}{r}\frac{\partial u_r}{\partial r} + \frac{\partial^2 u_r}{\partial r^2} \equiv (Va)\,!$$

$$\therefore \underbrace{\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u_r}{\partial r}\right] = \frac{1}{r}\frac{\partial^2 (r u_r)}{\partial r^2}}}_{\text{...(Vb)}}$$

3(d) Using (Vb) in (IV), & multiplying throughout by r, we get

$\boxed{\frac{\partial^2 (r u_r)}{\partial r^2} = \frac{r}{C^2}\frac{\partial^2 u_r}{\partial t^2} = \frac{1}{C^2}\frac{\partial^2 (r u_r)}{\partial t^2}}$ ∴ THE WAVE EQN for $(r u_r)$!! ..(VI)

3(e) We thus have, repeating (VI) from previous page:

$$\frac{\partial^2 (r u_r)}{\partial r^2} = \frac{1}{C^2}\frac{\partial^2 (r u_r)}{\partial t^2}$$, the wave Eqn for $(r u_r)$

* General solutions are (as for Cartesian case discussed in class):

$$r u_r = f(kr \pm \omega t)$$

$$\Rightarrow \quad u_r = \frac{f(kr \pm \omega t)}{r}$$

* If we consider only the OUTWARD propagating solutions, we have:

$$\boxed{u_r = \frac{f(kr - \omega t)}{r}}$$

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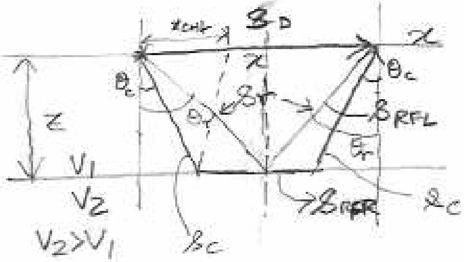
4. (a) Direct Wave:



$s_D \equiv x$

$$\boxed{t_D = \frac{x}{V_1}}$$ (zero intercept!)

4 (b) Reflected Wave:

$$t_{RFL} = \frac{s_{RFL}}{V_1} = \frac{2s_r}{V_1} = \frac{2}{V_1}\sqrt{z^2 + (x/2)^2} = \frac{\sqrt{x^2 + 2z^2}}{V_1}$$

$$\therefore \boxed{t_{RFL} = \frac{\sqrt{x^2+2z^2}}{V_1}}$$

4 (c) Refracted Wave: $\boxed{\sin\theta_c = V_1/V_2}$ $(V_2 > V_1)$

$$t_{RFR} = \frac{s_{RFR}}{V_2} + \frac{2s_c}{V_1} = \frac{x - 2z\tan\theta_c}{V_2} + \frac{2z}{\cos\theta_c V_1}$$

$$= \frac{x}{V_2} + 2z\left[\frac{1}{V_1\cos\theta_c} - \frac{\tan\theta_c}{V_2}\right]$$

$$= \frac{x}{V_2} + 2z\left[\frac{V_2 - V_1\sin\theta_c}{V_1V_2\cos\theta_c}\right]$$

$$= \frac{x}{V_2} + 2z\left[\frac{V_2 - V_2\sin^2\theta_c}{V_1V_2\cos\theta_c}\right]$$

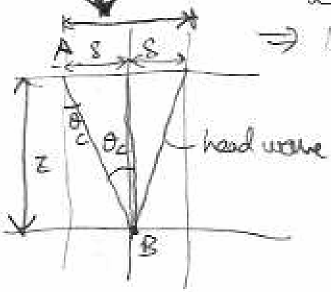
$$= \frac{x}{V_2} + \frac{2z}{V_1\cos\theta_c}(\cos^2\theta_c)$$

$$\therefore \boxed{t_{RFR} = \frac{x}{V_2} + \left\{\frac{2z\cos\theta_c}{V_1}\right\}}$$ → (t-intercept: all "constants")

OR $$t_{RFR} = \frac{x}{V_2} + \left\{\frac{2z\sqrt{V_2^2 - V_1^2}}{V_1V_2}\right\}$$ ← using Snell's law again.

4(d) See Solution Spread-sheet (tab 2)

4(e) $x_{critical} \equiv$ Minimum distance, x, after which refracted waves show up.



$\Rightarrow \delta_{RFR} = 0$, headwave arrives from the point of incidence, B

$\therefore x_{critical} = 2\delta = 2z\tan\theta_c$

$$\boxed{x_{critical} = 2z\tan\theta_c = \frac{2zV_1}{\sqrt{V_2^2 - V_1^2}}}$$

4(f) $x_{cross\text{-}over} \equiv$ distance at which headwaves arrive sooner than the direct wave

So, we set $t_{RFR} \leq t_D$ using expressions from 4(a) & 4(c)

$$\therefore \frac{x}{V_2} + \frac{2z\sqrt{V_2^2 - V_1^2}}{V_1V_2} \leq \frac{x}{V_1}$$

$$\Rightarrow x\left(\frac{1}{V_1} - \frac{1}{V_2}\right) \geq \frac{2z\sqrt{V_2^2 - V_1^2}}{V_1V_2}$$

$$\Rightarrow x \geq \frac{2z\sqrt{V_2^2 - V_1^2}}{\cancel{V_2V_1}} \cdot \frac{\cancel{V_2V_1}}{(V_2 - V_1)}$$

use $V_2^2 - V_1^2 = (V_2 - V_1)(V_2 + V_1)$

$$\Rightarrow x \geq 2z\sqrt{\frac{V_2 + V_1}{V_2 - V_1}} \quad \therefore \boxed{x_{cross\text{-}over} = 2z\sqrt{\frac{V_2 + V_1}{V_2 - V_1}}}$$

4(g) From 4(c): $\boxed{t_{intercept} \equiv t_{RFR}(x=0) = \frac{2z\sqrt{V_2^2 - V_1^2}}{V_1V_2}}$

Travel Time Curves

V1 =	5000	m/s
V2 =	8000	m/s
z =	30000	m

x	t_D	t_RFL	t_RFR
0	0.00	8.49	9.37
5000	1.00	8.54	9.99
10000	2.00	8.72	10.62
15000	3.00	9.00	11.24
20000	4.00	9.38	11.87
25000	5.00	9.85	12.49
30000	6.00	10.39	13.12
35000	7.00	11.00	13.74
40000	8.00	11.66	14.37
45000	9.00	12.37	14.99
50000	10.00	13.11	15.62
55000	11.00	13.89	16.24
60000	12.00	14.70	16.87
65000	13.00	15.52	17.49
70000	14.00	16.37	18.12
75000	15.00	17.23	18.74
80000	16.00	18.11	19.37
85000	17.00	19.00	19.99
90000	18.00	19.90	20.62
95000	19.00	20.81	21.24
100000	20.00	21.73	21.87
105000	21.00	22.65	22.49
110000	22.00	23.58	23.12
115000	23.00	24.52	23.74
120000	24.00	25.46	24.37
125000	25.00	26.40	24.99
130000	26.00	27.35	25.62
135000	27.00	28.30	26.24
140000	28.00	29.26	26.87
145000	29.00	30.22	27.49
150000	30.00	31.18	28.12
155000	31.00	32.14	28.74
160000	32.00	33.11	29.37
165000	33.00	34.07	29.99
170000	34.00	35.04	30.62
175000	35.00	36.01	31.24
180000	36.00	36.99	31.87
185000	37.00	37.96	32.49
190000	38.00	38.94	33.12
195000	39.00	39.91	33.74
200000	40.00	40.89	34.37
205000	41.00	41.87	34.99
210000	42.00	42.85	35.62
215000	43.00	43.83	36.24
220000	44.00	44.81	36.87
225000	45.00	45.79	37.49
230000	46.00	46.78	38.12
235000	47.00	47.76	38.74
240000	48.00	48.74	39.37
245000	49.00	49.73	39.99
250000	50.00	50.71	40.62
255000	51.00	51.70	41.24
260000	52.00	52.69	41.87
265000	53.00	53.67	42.49
270000	54.00	54.66	43.12
275000	55.00	55.65	43.74
280000	56.00	56.64	44.37
285000	57.00	57.63	44.99
290000	58.00	58.62	45.62
295000	59.00	59.61	46.24
300000	60.00	60.60	46.87
305000	61.00	61.59	47.49
310000	62.00	62.58	48.12
315000	63.00	63.57	48.74
320000	64.00	64.56	49.37
325000	65.00	65.55	49.99