Probability Theory
Lecture 3
Random Variables

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Introduction: Random Variables

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$$S = \{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}.$$

Now let X be the number of Heads observed. The possible values of X would be any number from the support set $S_X = \{0, 1, 2, 3\}$.
In this case, the event $\{X = 0\}$ corresponds to the subset $\{TTT\}$, the event $\{X = 1\}$ corresponds to the subset $\{TTH, HTT, THT\}$, the event $\{X = 2\}$ corresponds to the subset $\{HTH, THH, HHT\}$, and $\{X = 3\}$ corresponds to the subset $\{HHH\}$.
We denote the probability of the event $\{X = 0\}$, by $P(\{X = 0\})$ or shortly, $P(X = 0)$. In our example, if the coin is fair then $P(X = 0) = P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{8}$ (Why?)

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Example2: For a statistical experiment of tossing a fair dice twice. We obtain the finite space S consisting of the 36 ordered pairs of numbers between 1 and 6:

$$\{(1,1),(1,2),\ldots,(6,6)\}$$

let X assign to each point $(a,b) \in S$ the maximum of its numbers, i.e. $X(a,b) = \max(a,b)$. Then X is a random variable with support set

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$P(X = 1) = P(\text{The H appears in the first toss itself}) = P(H) = \frac{1}{2}$, and
$P(X = 2) = P(\text{The first H appears in the second toss}) = P(TH) = \frac{1}{4}$.
Similarly, $P(X = 3) = P(TTH) = \frac{1}{8}$,..., that is

$$P(X = x) = (\tfrac{1}{2})^x, \quad x = 1, 2, \ldots$$

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On the other hand, for the discrete probability function of Example 3:

$$p(x) = (\tfrac{1}{2})^x, \quad x = 1, 2, 3, \ldots,$$

we can satisfy condition 2 as follows:

$$\sum_{i=1}^{\infty} (\tfrac{1}{2})^{x_i} = \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^4 + (\tfrac{1}{2})^5 + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

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Example 4: For a statistical experiment of tossing a fair dice twice (as in Example 2). Let Y assign to each point (a, b) in S the sum of its numbers, i.e. $Y(a, b) = a + b$. Then Y is also a random variable on S with support set

$$S_Y = (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$$

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We obtain, for example, $p(4) = P(Y = 4)$ from the fact that $(1, 3), (2, 2)$, and $(3, 1)$ are those points of S for which the sum of the components is 4;

hence $$p(4) = P(Y = 4) = P(\{(1, 3), (2, 2), (3, 1)\}) = \frac{3}{36}$$

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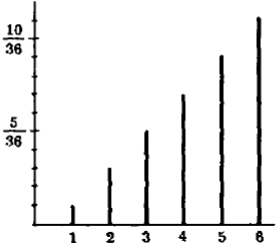
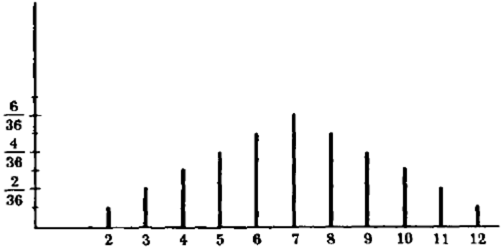
Comparing Graphs of Distributions

The following charts describe the distributions of Example 2 and Example 4 respectively:



Distribution of X

of Example 2



Distribution of Y

of Example 4

Observe that the vertical lines drawn above the numbers on the horizontal axis are proportional to their probabilities.

Example 5:
A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution
Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then x can be any of the numbers 0, 1, and 2. Now,

$$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{5}{2}}{\binom{8}{2}} = \frac{10}{28}$$

Also,

$$f(1) = P(X = 1) = \frac{\begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix}}{\begin{pmatrix} 8 \\ 2 \end{pmatrix}} = \frac{15}{28}$$

$$f(2) = P(X = 2) = \frac{\begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix}}{\begin{pmatrix} 8 \\ 2 \end{pmatrix}} = \frac{3}{28}$$

Thus the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Definition (Cumulative Distribution Function)

The cumulative distribution function (cdf) $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t) \qquad \text{for } -\infty < x < \infty.$$

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For the r.v. X, the # of defective computers purchased in Example 5, we have $F(1.7) = P(X \leq 1.7) = f(0) + f(1) = \frac{10}{28} + \frac{15}{28} = \frac{25}{28}$.

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$$F(x) = \begin{cases} 0, & \text{for } x < 0; \\ \frac{10}{28}, & \text{for } 0 \leq x < 1; \\ \frac{25}{28}, & \text{for } 1 \leq x < 2; \\ 1, & \text{for } x \geq 2. \end{cases}$$

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We have shown that $f(1) = 1/36, f(2) = 3/36, f(3) = 5/36,$ $f(4) = 7/36, f(5) = 9/36,$ and $f(6) = 11/36$. Therefore,

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Remaining Question

Using $F(x)$, verify that $f(2) = \frac{3}{36}$.

Now,

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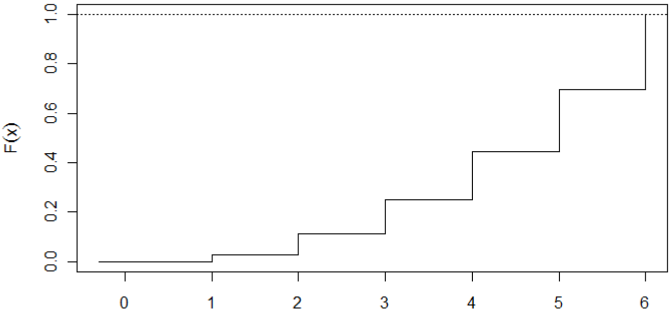

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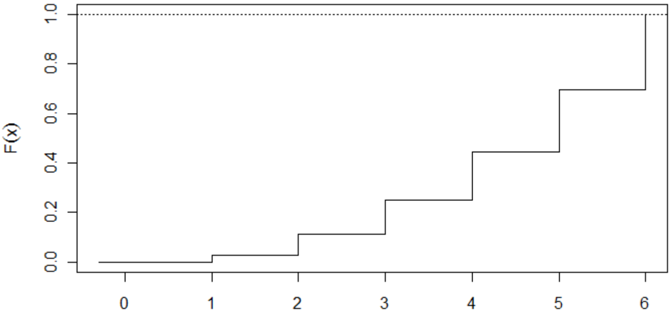

Example 7: Consider the following probability function

x	-2	-1	0	1	2
$f(x)$	1/8	2/8	2/8	2/8	1/8

Find:
(a) $P(X \leq 2)$
(b) $P(X > -2)$
(c) $P(-1 \leq X \leq 1)$
(d) $P(X \leq -1 \text{ or} X = 2)$

Back to Example 2

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Solution:

(a) $P(X \leq 2) = \sum_{x \leq 2} f(x) = f(-2) + f(-1) + f(0) + f(1) + f(2) = 1$

(b) $P(X > -2) = 1 - P(X \leq -2) = 1 - 1/8 = 7/8$

(c) $P(-1 \leq X \leq 1) = P(X \leq 1) - P(X < -1) = 7/8 - 1/8 = 6/8$

(d) $P(X \leq -1 \text{ or} X = 2) = P(X \leq -1) + P(X = 2)$

$= f(-2) + f(-1) + f(2) = 1/8 + 2/8 + 1/8 = 4/8$

Example 8: Consider the following probability function

$$f(x) = \frac{2x+1}{25}, \qquad x = 0, 1, 2, 3, 4$$

Find:

(a) $P(X = 4)$

(b) $P(X \leq 1)$

(c) $P(2 \leq X < 4)$

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Example 8: Consider the following probability function

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Solution:

(a) $P(X = 4) = f(4) = \frac{2*4+1}{25} = 9/25$

(b) $P(X \leq 1) = f(0) + f(1) = 1/25 + 3/25 = 4/25$

(c) $P(2 \leq X < 4) = f(2) + f(3) = 5/25 + 7/25$

(d) $P(X > -10) = 1$

Example 9: Determine the probability mass function of X from the following cumulative distribution function:

$$F(x) = \begin{cases} 0, & \text{for } x < -2; \\ 0.2, & \text{for } -2 \leq x < 0; \\ 0.7, & \text{for } 0 \leq x < 2; \\ 1, & \text{for } x \geq 2. \end{cases}$$

Example 9: Determine the probability mass function of X from the following cumulative distribution function:

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Solution:

The only points that receive nonzero probability are -2, 0, and 2. The probability mass function at each point is the change in the cumulative distribution function at the point. Therefore,

$$f(-2) = F(-2) - F(-2^-) = 0.2 - 0 = 0.2$$
$$f(0) = F(0) - F(-2) = 0.7 - 0.2 = 0.5$$
$$f(2) = F(2) - F(0) = 1 - 0.7 = 0.3$$

The Bernoulli distribution is used to model an experiment with only two possible outcomes, often referred to as "success" and "failure", usually encoded as 1 and 0.

Definition (Bernoulli distribution)

A discrete random variable X has a Bernoulli distribution with parameter p, where $0 \leq p \leq 1$, if its probability mass function is given by

$$f(1) = P(X = 1) = p \text{ and } f(0) = P(X = 0) = 1 - p.$$

We denote this distribution by $Ber(p)$.

Example: Consider the situation that you attend, completely unprepared, a multiple-choice exam. It consists of 10 questions, and each question has four alternatives (of which only one is correct). You will pass the exam if you answer six or more questions correctly. You decide to answer each of the questions in a random way, in such a way that the answer of one question is not affected by the answers of the others. What is the probability that you will pass?
Setting for $i = 1, 2, \ldots, 10$

$$R_i = \begin{cases} 1, & \text{if the ith answer is correct;} \\ 0, & \text{if the ith answer is incorrect.} \end{cases}$$

the number of correct answers X is given by

$$X = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + R_9 + R_{10}.$$

Clearly, X attains only the values $0, 1, \ldots, 10$. Let us first consider the case $X = 0$.

Bernoulli Example (Cont..)

We find

$$
\begin{aligned}
P(X=0) &= P(\text{not a single } R_i \text{ equals} 1) \\
&= P(R_1=0, R_2=0, \ldots, R_{10}=0) \\
&= P(R_1=0)P(R_2=0)\ldots P(R_{10}=0) = \left(\frac{3}{4}\right)^{10}.
\end{aligned}
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In general we find for $k = 0, 1, \ldots, 10$, again using independence, that

$$P(X = k) = \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{10-k} \binom{10}{k},$$

which is the probability that k questions were answered correctly times the probability that the other $10 - k$ answers are wrong, times the number of ways $10Ck = \binom{10}{k}$ this can occur.

Since $P(X \geq 6) = P(X = 6) + \cdots + P(X = 10)$, it is now an easy exercise to determine the probability that you will pass. Check that $P(X \geq 6) = 0.0197$.

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The preceding random variable X is an example of a random variable with a binomial distribution with parameters $n = 10$ and $p = 1/4$.

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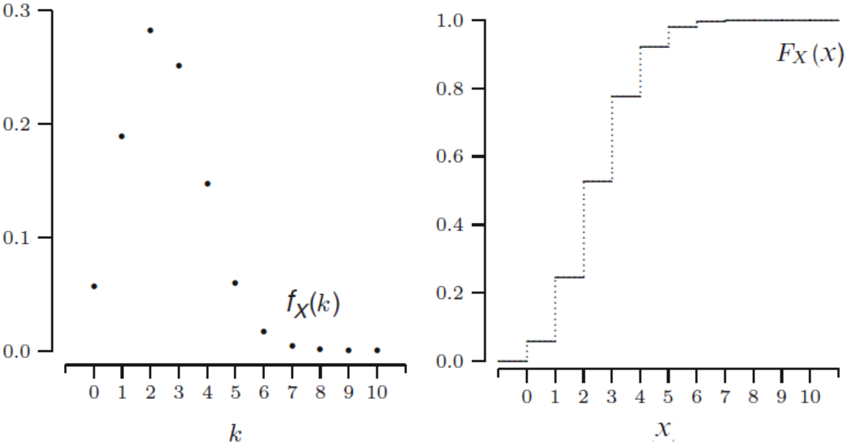

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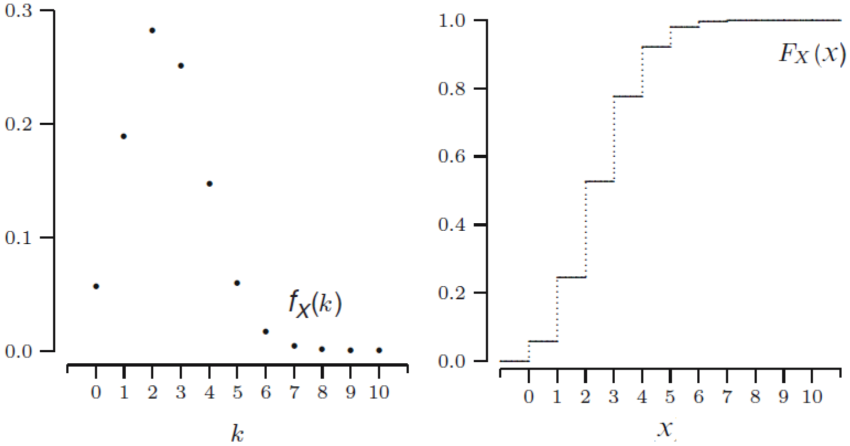

Binomial Examples

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$$P(X=2)=\left(\begin{array}{c}4\\2\end{array}\right)(0.1)^2(0.9)^2=6(0.01)(0.81)=0.0486$$

In general,
$$P(X=x)=\left(\begin{array}{c}4\\x\end{array}\right)(0.1)^x(0.9)^{(4-x)}$$

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Solution:
Let $X =$ the number of samples that contain the pollutant in the next 18 samples analyzed. Then X is a binomial random variable with $p =$ 0.1 and $n =$ 18. Therefore,

$$\begin{aligned} P(X=2) &= \binom{18}{2}(0.1)^2(0.9)^{16} \\ &= 153(0.01)(0.185) = 0.284 \end{aligned}$$

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The requested probability is

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Examples 2 (Cont..)

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Hypergeometric Distribution

A hypergeometric experiment is a statistical experiment that has the following properties:

Hypergeometric: properties

- A sample of size n is randomly selected without replacement from a population of N items.
- In the population, k items can be classified as successes, and N - k items can be classified as failures.

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Consider the following statistical experiment. You have an urn of 10 marbles - 5 red and 5 green. You randomly select 2 marbles without replacement and count the number of red marbles you have selected. This would be a hypergeometric experiment.

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In the beginning, the probability of selecting a red marble is $5/10$. If you select a red marble on the first trial, the probability of selecting a red marble on the second trial is $4/9$. And if you select a green marble on the first trial, the probability of selecting a red marble on the second trial is $5/9$.

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Hypergeometric: Notation

The following notation is helpful, when we talk about hypergeometric distributions and hypergeometric probability.

- N : The number of items in the population.
- k : The number of items in the population that are classified as successes.
- n : The number of items in the sample (sample size).
- x : The number of items in the sample that are classified as successes.

Hypergeometric Distribution (Cont..)

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A hypergeometric random variable is the number of successes that result from a hypergeometric experiment. The probability distribution of a hypergeometric random variable is called a hypergeometric distribution.

Definition (Hypergeometric Formula)

Suppose a population consists of N items, k of which are successes. And a random sample drawn from that population consists of n items, x of which are successes. Then the hypergeometric probability distribution is:

$$f(x) = h(x; N, n, k) = \frac{\begin{pmatrix} k \\ x \end{pmatrix} \begin{pmatrix} N-k \\ n-x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}$$

Example 1: Suppose we randomly select 5 cards without replacement from an ordinary deck of playing cards. What is the probability of getting exactly 2 red cards (i.e., hearts or diamonds)?
Solution: This is a hypergeometric experiment in which we know the following:

- $N = 52$; since there are 52 cards in a deck.
- $k = 26$; since there are 26 red cards in a deck.
- $n = 5$; since we randomly select 5 cards from the deck.
- $x = 2$; since 2 of the cards we select are red.

We substitute these values into the hypergeometric formula as follows:

$$f(2) = h(2; 52, 5, 26) = \frac{\begin{pmatrix} 26 \\ 2 \end{pmatrix} \begin{pmatrix} 26 \\ 3 \end{pmatrix}}{\begin{pmatrix} 52 \\ 5 \end{pmatrix}}$$

$$= \frac{325 * 2600}{2,598,960} = 0.32513$$

Example 2

A committee of size 5 is to be selected at random from 3 chemists and 5 physicists. Find the probability distribution for the number of chemists on the committee.

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$$f(0) = P(X = 0) = h(0; 8, 5, 3) = \frac{\binom{3}{0}\binom{5}{5}}{\binom{8}{5}} = \frac{1}{56}$$

$$f(1) = P(X = 1) = h(1; 8, 5, 3) = \frac{\binom{3}{1}\binom{5}{4}}{\binom{8}{5}} = \frac{15}{56}$$

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Example 2 (Cont..)

$$f(2) = P(X=2) = h(2;8,5,3) = \frac{\left(\begin{array}{c}3\\2\end{array}\right)\left(\begin{array}{c}5\\3\end{array}\right)}{\left(\begin{array}{c}8\\5\end{array}\right)} = \frac{30}{56}$$

$$f(3) = P(X=3) = h(3;8,5,3) = \frac{\left(\begin{array}{c}3\\3\end{array}\right)\left(\begin{array}{c}5\\2\end{array}\right)}{\left(\begin{array}{c}8\\5\end{array}\right)} = \frac{10}{56}$$

Thus the probability distribution function is

x	0	1	2	3
$f(x) = P(X=x)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{30}{56}$	$\frac{10}{56}$

Example 2 (Cont..)

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$$= \frac{40}{56} = 0.7143$$

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$$P(X \geq 1) = 1 - f(0)$$

$$= 1 - \frac{\binom{3}{0}\binom{5}{5}}{\binom{8}{5}} = 1 - \frac{1}{56}$$

$$= 1 - 0.0179 = 0.9821$$

Example 2 (Cont..)

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- What is the probability that at least one chemist on the committee.?

$$\begin{aligned} P(X \geq 1) &= 1 - f(0) \\ &= 1 - \frac{\binom{3}{0}\binom{5}{5}}{\binom{8}{5}} = 1 - \frac{1}{56} \\ &= 1 - 0.0179 = 0.9821 \end{aligned}$$

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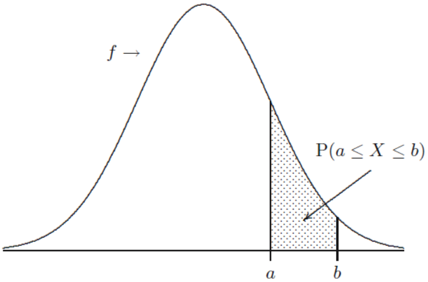

Definition (probability density function)

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1. $f(x) \geq 0$ for all $x \in \mathbb{R}$;
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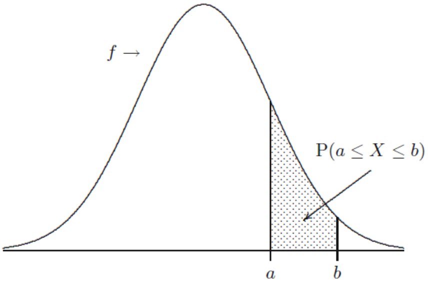

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The cumulative: distribution function $F(x)$ is expressed graphically in the following Figure. Now:

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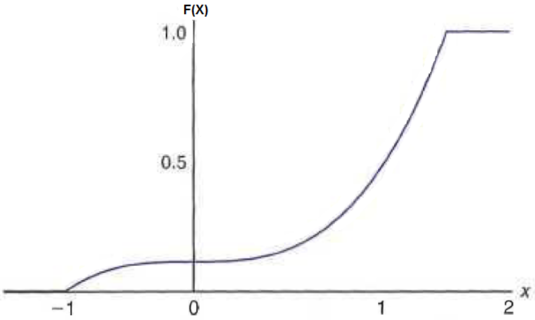

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