Rate of combined heat loss per unit length

$$\frac{Q}{L} = h \pi D (T_s - T_\infty) + \varepsilon \pi D \sigma (T_s^4 - T_{surr}^4)$$

$$= (577 + 421)\ W/m = 998\ W/m$$

In this example, the rates of heat transfer by convection and radiation are comparable, because of (a) large temperature difference between surface and surroundings and (b) free convection heat transfer coefficient is small. But in case of forced convection, radiation heat transfer is often relatively small.

Thermal Resistance

In all three modes of heat transfer, the heat transfer rate can be written as

$$Q = \frac{\Delta T}{R_t}$$

where ΔT is the relevant temperature difference
and A is the surface area normal to the direction of heat transfer.

Mechanism:	conduction	convection	radiation
R_t :	$\frac{L}{kA}$	$\frac{1}{hA}$	$\frac{1}{h_r A}$

This equation is analogous to electric current flow rate

$$I = \frac{\Delta V}{R_e}$$

With this analogy, one can borrow concepts of series and parallel resistances.

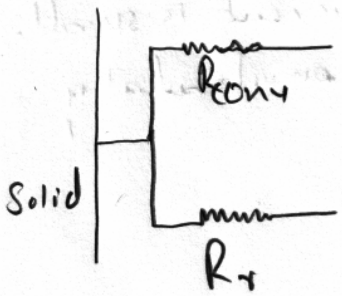
For example, a solid wall surrounded by a gas involves radiation effects in addition to convection.

We had used

$$h_{combined} = h_{conv.} + h_r$$

This is equivalently described by two thermal resistances in parallel (convection and radiation). The net thermal resistance

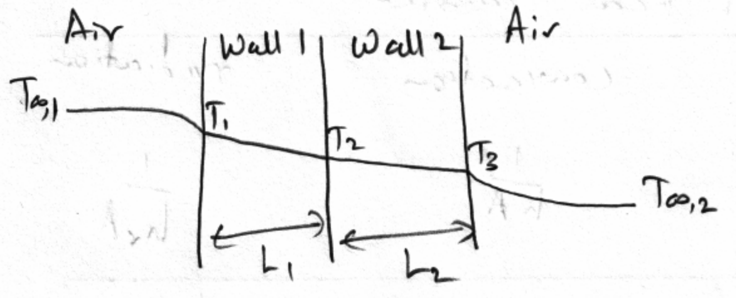
$$\frac{1}{R_{combined}} = \frac{1}{R_{conv.}} + \frac{1}{R_r} \quad \text{(parallel)}$$



$$\Rightarrow \frac{1}{\left(\frac{1}{h_{combined} A}\right)} = \frac{1}{\left(\frac{1}{h_{conv} A}\right)} + \frac{1}{\left(\frac{1}{h_r A}\right)}$$

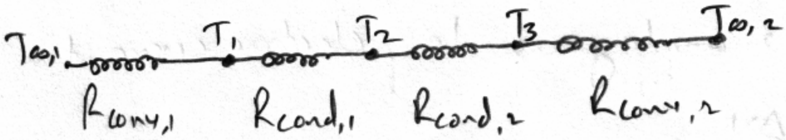
$$\Rightarrow h_{combined} = h_{conv} + h_r$$

A composite (multi-layer) plane wall surrounded by a gas on both sides. Even if the radiation heat transfer is ignored, there are multiple ressistances (thermal) in series (convection, conduction), resulting into a net thermal resistance



$R_{combined}$ (series)

$= R_{conv,1} + R_{cond,1} + R_{cond,2} + R_{conv,2}$



$\frac{1}{h_1 A}$ $\quad \frac{L_1}{k_1 A}$ $\quad \frac{L_2}{k_2 A}$ $\quad \frac{1}{h_2 A}$

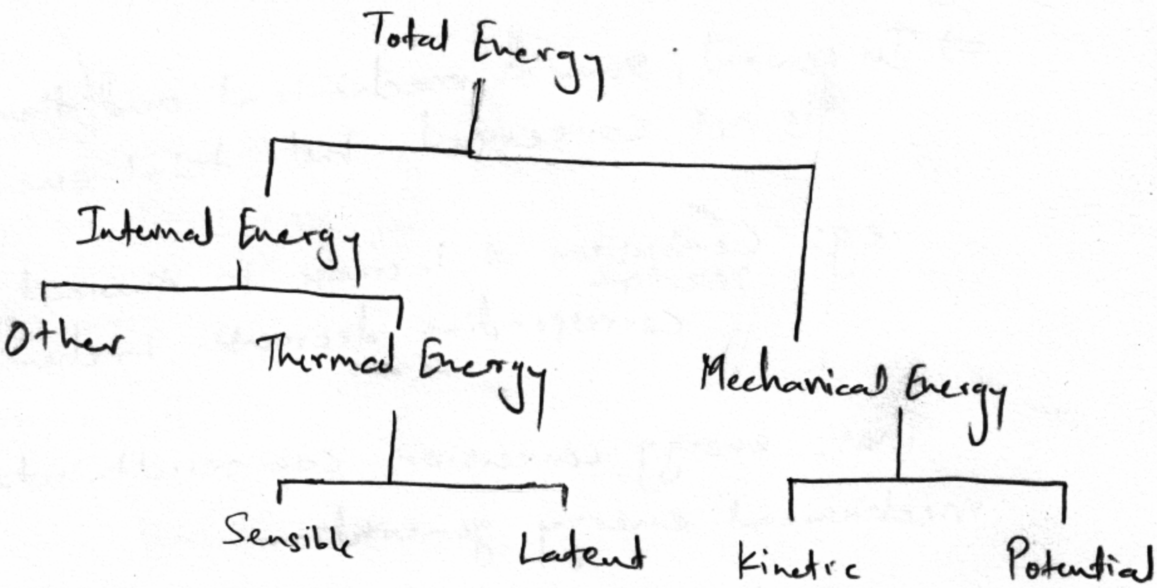
Forms of Energy

First law of thermodynamics

"total energy of a system is conserved"

"energy can neither be created nor destroyed during a process; it can only change forms"

Total Energy is a sum of numerous forms of energy such as thermal, mechanical, kinetic, potential, chemical, nuclear, magnetic, sensible, latent, Coulomb, etc.



Thermal Energy ≡ heat

Sensible heat ≡ Kinetic energy of the molecules
(Changes in temperature) (translation, vibration, rotation)

Latent heat ≡ intermolecular forces
(Changes in phase)

Energy Balance

First law of thermodynamics

Closed system, over a time interval

Change in total energy = Net heat transferred to the system − Work done by the system

Control volume, at an instant (or over a time interval Δt)

Change in total energy = Energy entering the control volume − Energy leaving the control volume

⇒ In general, sum of mechanical and thermal energy is not conserved, but total energy is.

eg: Combustion reaction ⇒ increase in thermal energy with corresponding decrease in chemical energy.

Thus, energy conversion can result into thermal or mechanical energy generation

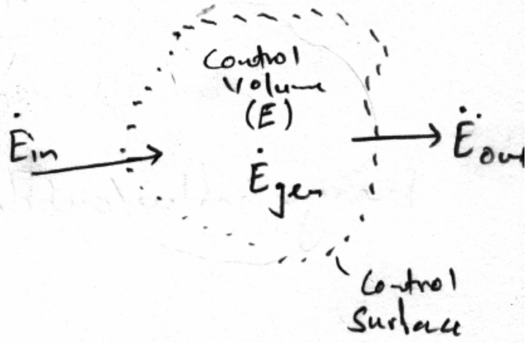
Change in thermal + mechanical energy = Thermal & mechanical energy (entering − leaving) the control volume + Thermal & mechanical energy generated

* generation can be positive or negative

⇒ at steady state, entering − leaving + generation = 0

Denoting thermal + mechanical energy by the symbol E, its rate of change is given by

$$\dot{E} = \frac{dE}{dt} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen}$$



Note:
Including $\dot{E}_{gen}$ does not violate the first law of thermodynamics.

Reminder:

$E \equiv$ Thermal plus mechanical energy

Mechanical $\equiv$ kinetic and potential energy. (often unchanged)

Thermal $\equiv$ sensible and latent heat

No phase change $\Rightarrow$ No latent heat change

For incompressible substances

change in sensible heat $\equiv m \, C_v \, \Delta T$

(m: mass; ΔT: change in temperature)

$C_v \equiv$ specific heat at constant volume

Specific heat $\equiv$ energy required to change the temperature of a unit mass by one degree

$C_v \equiv$ Specific heat at constant volume

$C_p \equiv$ specific heat at constant pressure

For a control volume, C_v is the relevant specific heat

for incompressible liquids $\quad C_p = C_v$

for an ideal gas $\quad C_p - C_v = R$

Energy generation: conversion of some other form of energy to thermal and/or mechanical energy
Volumetric phenomenon

Energy inflow/outflow: Surface phenomena
thermal + mechanical energy transferred at the boundaries (control surface enclosing the control volume)

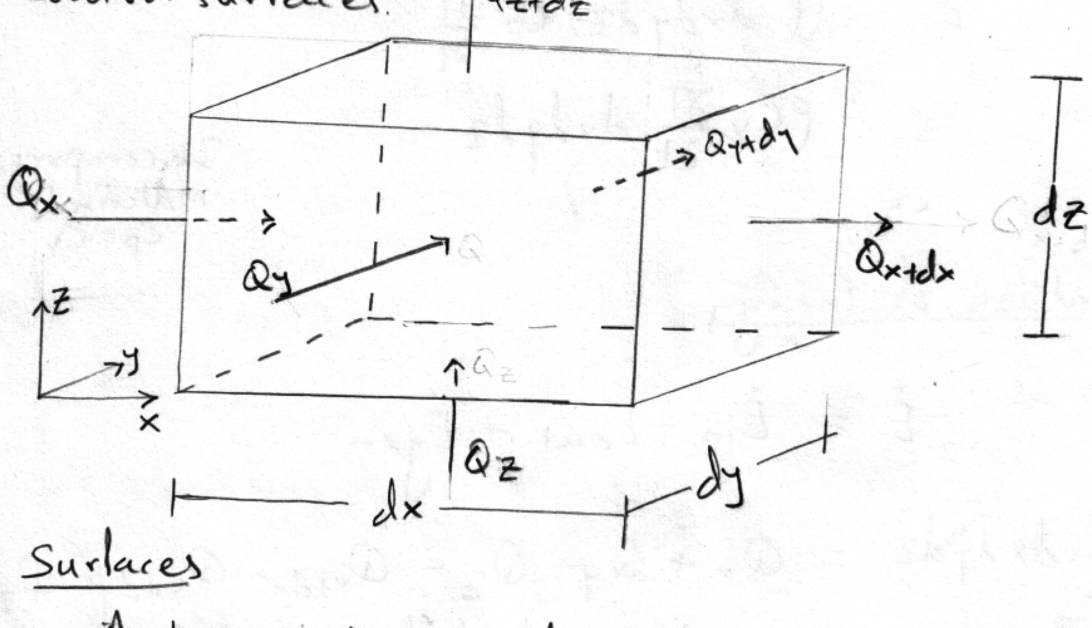
Heat Conduction Equation

Major objective in many problems → knowing the temperature field.

Cartesian Coordinates:

- Consider a homogeneous medium eg. a metal object
- Assume that the medium is incompressible and there is no bulk motion
- Incompressible ⇒ "Constant density"
- No bulk motion ⇒ no changes in mechanical energy

Defining an infinitesimally small volume as our control volume ($dx \cdot dy \cdot dz$) enclosed by a control surface as shown in the figure, we only consider thermal energy in the control volume and across the control surfaces.



Surfaces

A temperature gradient in the x-direction will give rise to conduction in the x-direction. Considering the rates of heat transfer at surfaces located at x and $x+dx$, as Q_x and Q_{x+dx}, Taylor series expansion (truncated at first order) gives

$$Q_{x+dx} = Q_x + \frac{\partial Q_x}{\partial x} dx$$

Similarly, in the y- and z- directions

$$Q_{y+dy} = Q_y + \frac{\partial Q_y}{\partial y} dy$$

$$Q_{z+dz} = Q_z + \frac{\partial Q_z}{\partial z} dz$$

Control volume

If there is an energy source associated with the rate of thermal energy generation $\dot{E}_{gen}$

$$\dot{E}_{gen} = \dot{e}_{gen} \, dx dy dz \qquad \dot{e}_{gen} \equiv \dot{E}_{gen}/\text{volume}$$

In absence of phase change, the rate of thermal energy change is equal to the rate of change of sensible heat

$$\dot{E} = (\rho \, dx dy dz) \, C_p \frac{\partial T}{\partial t}$$
$$= \rho C_v \frac{\partial T}{\partial t} dx dy dz$$

Incompressible material
$C_p = C_v$

Conservation of Energy

$$\dot{E} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen}$$

$$\rho c_p \frac{\partial T}{\partial t} dx dy dz = Q_x + Q_y + Q_z - Q_{x+dx} - Q_{y+dy} - Q_{z+dz} + \dot{e}_{gen} dx dy dz$$

$$= -\frac{\partial Q_x}{\partial x} dx - \frac{\partial Q_y}{\partial y} dy - \frac{\partial Q_z}{\partial z} dz + \dot{e}_{gen} dx dy dz$$

Thus,

$$\rho c_p \frac{\partial T}{\partial t} = -\frac{1}{A_x}\frac{\partial Q_x}{\partial x} - \frac{1}{A_y}\frac{\partial Q_y}{\partial y} - \frac{1}{A_z}\frac{\partial Q_z}{\partial z} + \dot{e}_{gen}$$

when $A_x = dy\,dz$ ≡ area of each of the two surfaces along x-direction

$A_y = dx\,dz$

$A_z = dx\,dy$

<u>Fourier's law of heat conduction</u>

$$Q_x = -k A_x \frac{\partial T}{\partial x}$$

$$Q_y = -k A_y \frac{\partial T}{\partial y}$$

$$Q_z = -k A_z \frac{\partial T}{\partial z}$$

Substituting in the energy balance,

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_{gen}$$

— Heat conduction equation in Cartesian coordinates

<u>Notes</u>:

- Energy balance + Fourier's law of conduction
- Solution: $T(x, y, z, t)$
- $k\frac{\partial T}{\partial x} = q_x$ = flux. Thus, at steady state, with no energy source/generation term, heat conduction equation ⇒ q_x ≡ constant in the x-direction

• In many common problems, k = constant. The conduction equation reduces to

$$\nabla^2 T + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

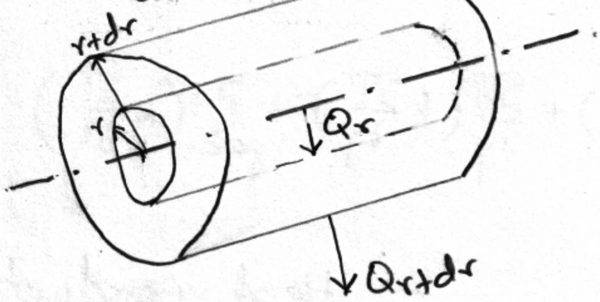
where

$$\alpha = \frac{k}{\rho C_p} \equiv \text{thermal diffusivity}$$

(a material property)
(how fast heat propagates through the material)

Cylindrical Coordinates

For the case of 1-dimensional heat conduction (in the radial direction), a control volume corresponding to a thin cylindrical shell can be defined as shown in the figure



Surface

$$Q_{r+dr} = Q_r + \frac{\partial Q_r}{\partial r}dr$$

Control volume

$$\dot{E}_{gen} = \dot{e}_{gen}\, 2\pi r L\, dr$$

Area of both surfaces can be approximated as $2\pi r L$, since $dr \to 0$

$$A_r = 2\pi r L$$

$$\dot{E} = \rho C_p \frac{\partial T}{\partial t} A_r dr$$

Conservation of Energy

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{1}{A_r}\frac{\partial Q_r}{\partial r} + \dot{e}_{gen}$$

Fourier's law of heat conduction

$$Q_r = -k A_r \frac{\partial T}{\partial r}$$

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r k \frac{\partial T}{\partial r}\right) + \dot{e}_{gen}$$

— Heat conduction equation in cylindrical coordinates, (1-D; radial)

Three-Dimensional Heat Conduction equation in Cylindrical coordinates

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r k \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k \frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{e}_{gen}$$

<u>Spherical Coordinates</u>

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(k r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k \frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k \sin\theta \frac{\partial T}{\partial \theta}\right) + \dot{e}_{gen}$$

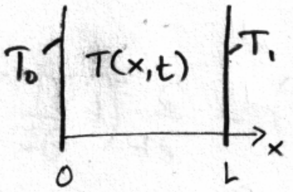
After choosing the appropriate coordinate system and simplifying the heat conduction equation, depending on the problem, a number of "boundary conditions" and an "initial condition" are required to solve the differential equation.

<u>Initial Condition</u>

$T(x,y,z,0)$ specified

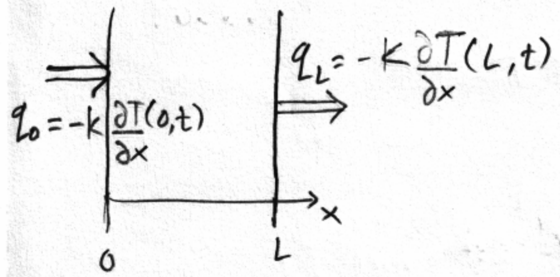
Boundary Conditions

- Thermal conditions at the boundary
- Required to integrate/solve the governing heat conduction equation
- Commonly encountered boundary conditions

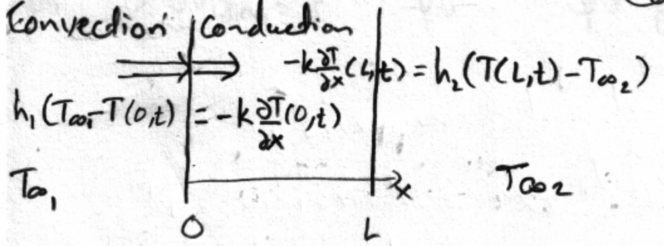


$T(0,t)=T_0$
$T(L,t)=T_1$

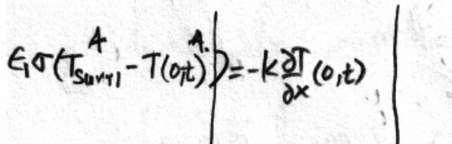
- Specified temperature (Dirichlet)
  - direct measurement



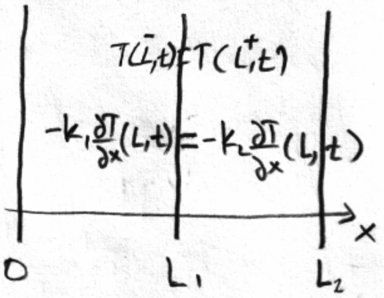
- Specified heat flux (Newmann)
  - known energy interactions
  - eg: insulated boundary
    q (insulated boundary) $= 0$
  - eg: thermal symmetry



- Convection boundary condition
  - most common
  - Conduction at surface = convection at surface in the same direction



- Radiation boundary condition
  - less common
  - conducting material surrounded by vacuum ⇒ no convection
  - typically negligible compared to convection, especially forced convection.



- Interface boundary Condition
  - two layers in contact
  - Same temperature at contact surface
  - matching flux at contact surface

- Combined/Generalized boundary conditions
  - heat transfer to the surface in all modes = heat transfer from the surface in all modes

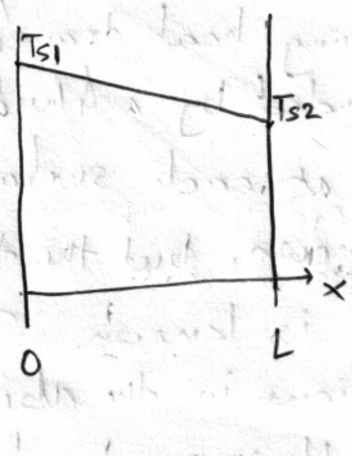
Steady State Conduction: One Dimensional

Consider steady heat transfer through a wall. If temperature at different locations of the same surface of the wall is a constant, heat transfer can be assumed to be 1-D (one dimensional) in the direction perpendicular to the surface, because heat transfer is governed by a temperature gradient.

Choice: Cartesian coordinate system

Assumptions:

- Steady State
- No heat source or sink
- One dimensional heat conduction



Simplified Heat conduction equation:

$$0 = \frac{d}{dx}\left(k \frac{dT}{dx}\right)$$

Farther assuming $k \equiv$ constant

$$T = C_1 x + C_2$$

Boundary conditions:

$$T = T_{s1} \quad \text{at } x = 0$$
$$T = T_{s2} \quad \text{at } x = L$$

The condition at $x=0$ gives $C_2 = T_{s1}$

The condition at $x=L$ gives $C_1 = \dfrac{T_{s2} - T_{s1}}{L}$

The temperature profile in the wall is given as

$$T(x) = \frac{T_{s2} - T_{s1}}{L}\, x + T_{s1}$$ ---- Linearly varying temperature

Once the temperature profile is known, quantities such as rate of heat transfer, heat flux can be calculated.

Flux: $$q = -k \frac{dT}{dx}$$ Fourier's Law

$$= -k \frac{T_{s2} - T_{s1}}{L}$$

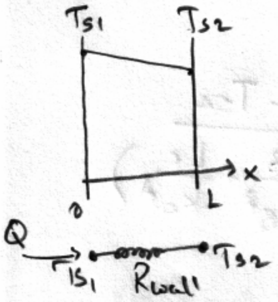
Rate of heat transfer across an area A

$$Q = -kA \frac{dT}{dx} = -kA \frac{T_{s2} - T_{s1}}{L}$$

Note: In many problems involving heat transfer through a wall, the wall is surrounded by a fluid (liquid or gas). Further, temperature at each surface of the wall is often not known, but the temperature of the surrounding fluid is known. Thus, dirichlet boundary conditions in the above example are replaced by Neumann boundary condition

Revisiting Thermal Resistance

Rearranging the heat transfer rate equation



$$Q = \frac{T_{s1} - T_{s2}}{R_{wall}}$$

where

$$R_{wall} = \frac{L}{kA}$$

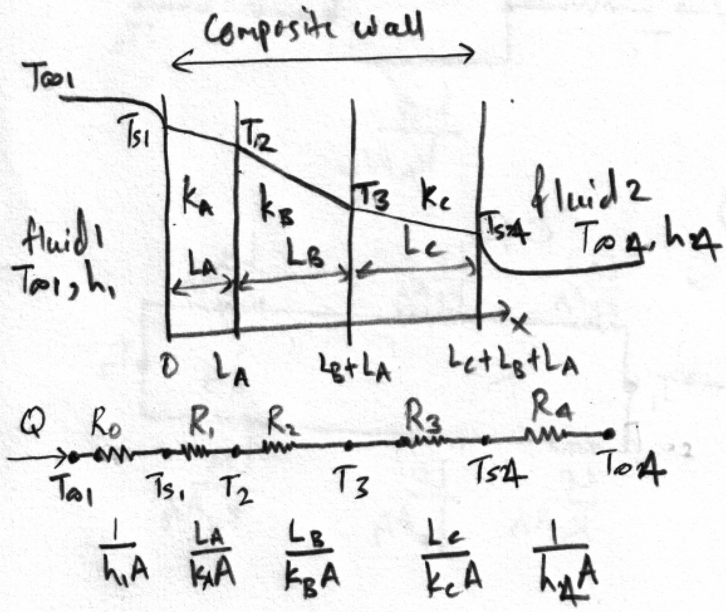
Analogy:

$$I = \frac{V_1 - V_2}{R_e}$$

Flow rate of charge = Driving potential difference / resistance to flow of charge

Trivial result for this problem, but a powerful concept in solving complicated problems. Resistances (thermal) can be in series or parallel arrangements, and an equivalent resistance can be computed.

Example: Composite Wall with convection boundary conditions.



• 1D heat transfer rate

$$Q = \frac{T_{\infty 1} - T_{\infty 4}}{R_{tot}}$$

where

$$R_{tot} = \frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_4 A}$$

$$= R_0 + R_1 + R_2 + R_3 + R_4$$

Overall heat transfer coefficient

With composite systems, a "black box" approach would be to use an expression analogous to Newton's Law of cooling

$$Q = U A \Delta T$$

where

$U \equiv$ overall heat transfer coefficient

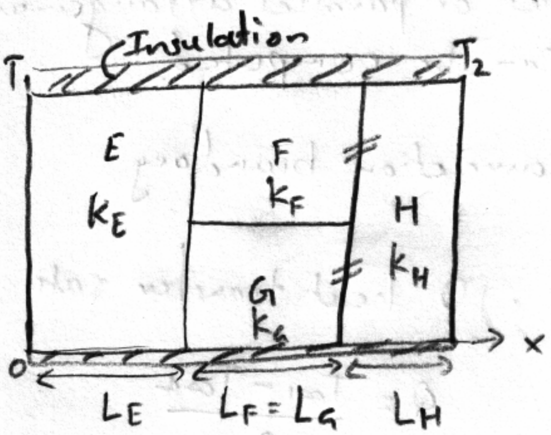
$$U = \frac{1}{R_{tot} A}$$

Note: At steady state, heat is not accumulated in the composite. Thus, all of the expressions below are equivalent.

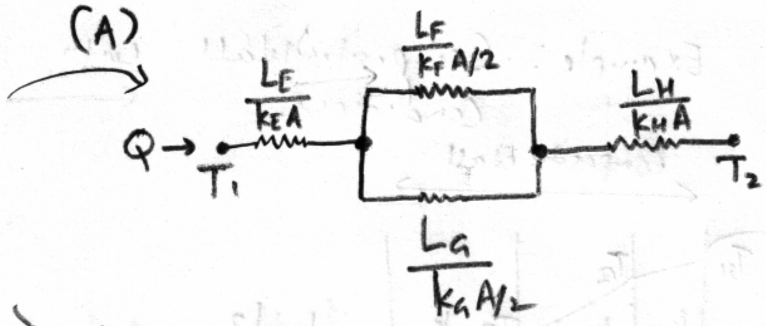
$$Q = \frac{T_{\infty 1} - T_{s1}}{(1/h_1 A)} = \frac{T_{s1} - T_2}{(L_A/k_A A)} = \frac{T_2 - T_{s4}}{\left(\frac{L_B}{k_B A} + \frac{L_C}{k_C A}\right)}$$

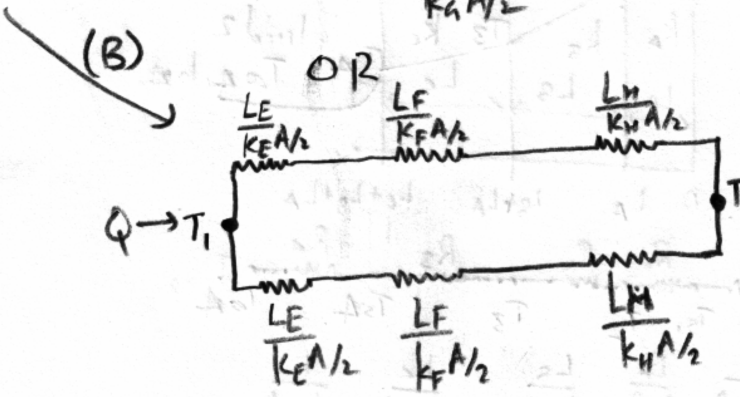
– Generally true as long as Q is a constant

Example: A series-parallel composite wall



Equivalent thermal





Case A) Temperature gradient is only in the x-direction
i.e.
any plane wall normal to x-axis is isothermal

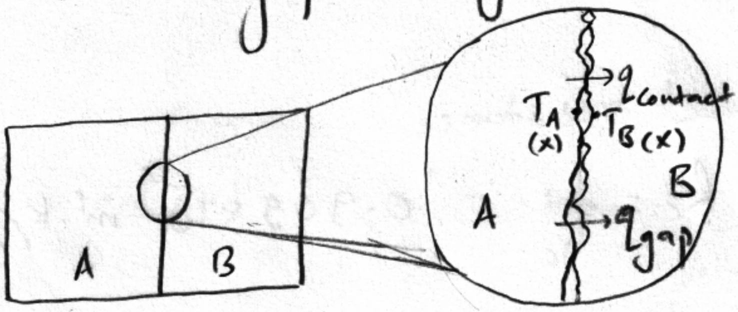
Case B) Heat transfer is only in the x-direction
i.e.
any plane wall parallel to x-axis is adiabatic

Note: Although heat flow is multidimensional, it is reasonable to assume 1D heat conduction in many cases.

Contact Resistance

In composite systems, lack of perfect contact at the interface leads to "contact resistance".

A practical interface that is "smooth" is microscopically rough. Thus, there are contact spots along with gaps that are typically filled with air. Conduction heat transfer takes place across actual contacts, and Conduction and/or radiation heat transfer takes place across the gaps through the filled fluid (air) or vacuum.



If the heat transfer rate is

$$Q = Q_{contact} + Q_{gap}$$

and the effective temperature difference across the interface is ΔT, with A being the apparent contact area;

$$h_{contact} = \frac{Q/A}{\Delta T} \quad \text{and} \quad R_c = \frac{1}{h_c} = \frac{\Delta T}{Q/A}$$

Note:

- Contact resistance can be decreased
  - → by selecting an interfacial fluid of large thermal conductivity
    - eg: thermal grease used at interfaces involving electronic components
    - "thermal paste (Cpu heat sink)
  - → by increasing the contact area

Example : Thermal contact conductance at the interface of two 1cm thick aluminium plates is measured to be 11,000 $W/m^2 \cdot K$. Determine the thickness of an aluminium plate whose thermal resistance is equal to the thermal resistance of the interface between the plates. Note that thermal conductivity of aluminium at the temperature of the problem is $k = 237\ W/m \cdot K$.

Thermal contact resistance

$$R_c = \frac{1}{h_c} = 0.909 \times 10^{-4}\ m^2 \cdot K/W$$

For a unit surface area, an equivalent resistance due to conduction across an aluminium plate of length L is

$$R = \frac{L}{k}$$

Equating the two,

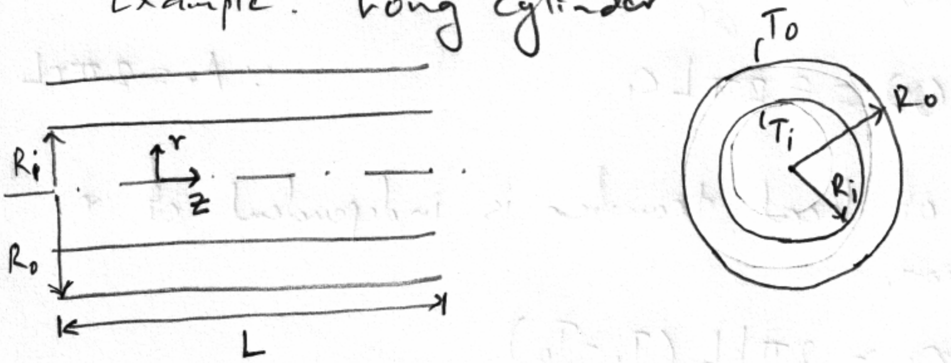
<u>Equivalent Thickness for Contact Resistance</u>

$$L = kR_c = 0.0215\ m = 2.15\ cm$$

Note : In this example, the thermal contact resistance is greater than the sum of thermal resistances of the two plates.
Thus, ignoring contact resistance would have resulted in a significant error.

Pipe

Example: long cylinder



Assumptions
- steady state
- No gradient in temperature along z-direction or ϕ-direction
- No heat generation

1D heat conduction in cylindrical coordinates with $T = T(r)$ only

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) = \frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right)$$

$$0 = \frac{d}{dr}\left(rk\frac{dT}{dr}\right)$$

$$C_1 = rk\frac{dT}{dr} \Rightarrow \frac{C_1}{rk} = \frac{dT}{dr}$$

$$T(r) = \frac{C_1}{k}\ln(r) + C_2$$

Boundary conditions

$T = T_i$ at $r = R_i$

$T = T_o$ at $r = R_o$

$$T_i = \frac{C_1}{k}\ln(R_i) + C_2$$

$$T_o = \frac{C_1}{k}\ln(R_o) + C_2$$

$$C_1 = \frac{k(T_i - T_o)}{\ln(R_i/R_o)} \qquad C_2 = T_i - (T_i - T_o)\frac{\ln(R_i)}{\ln(R_i/R_o)}$$

$$Q_r(r) = -kA_r\frac{dT}{dr} = -A_r\frac{C_1}{r}$$

$$Q_r(r) = -2\pi L C_1 \qquad \because A_r = 2\pi r L$$

Thus, rate of heat transfer is independent of r.

For completion,

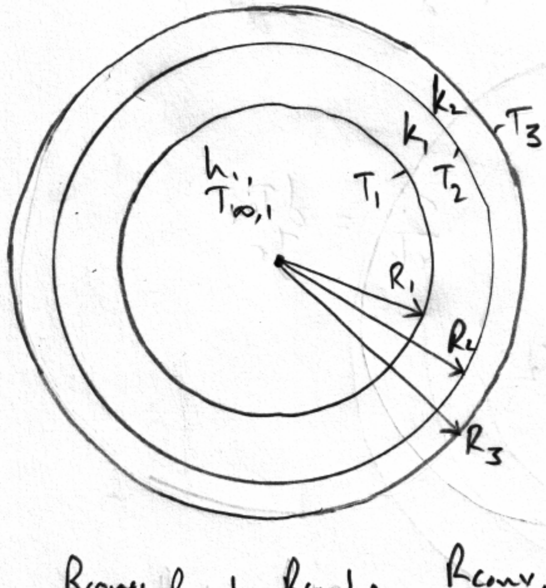
$$Q_r = 2\pi k L \frac{(T_i - T_0)}{\ln(R_o/R_i)}$$

Thus,

$$R_{th} = \frac{\ln(R_o/R_i)}{2\pi k L}$$

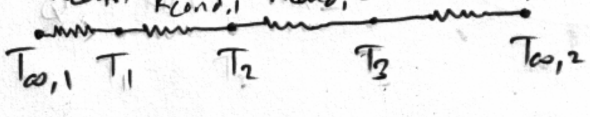
Again, a trivial result for this problem, but a powerful concept in solving complicated problems.

Example: Composite long cylinder (insulated pipe)



$$R_{tot} = R_{conv,1} + R_{cond,1} + R_{cond,2} + R_{conv,2}$$

$$= \frac{1}{h_1(2\pi R_1 L)} + \frac{\ln(R_2/R_1)}{2\pi k_1 L} + \frac{\ln(R_3/R_2)}{2\pi k_2 L} + \frac{1}{h_2(2\pi R_3 L)}$$



$$\frac{1}{h_1 2\pi R_1 L} \quad \frac{\ln(R_2/R_1)}{2\pi k_1 L} \quad \frac{\ln(R_3/R_2)}{2\pi k_2 L} \quad \frac{1}{h_2 2\pi R_3 L}$$