(A) Deformed K-map ⇒ Our original K-map is as:-

$$K(x) = \frac{\mu x(1-x)}{1+x} \quad \text{—(A)} \quad \text{where } \mu \geq 0 \text{ and } x \in [0,1]$$

$$[x]_\epsilon = \frac{x}{1+\epsilon(1-x)} \quad \text{—(B)}$$

$$K([x]_\epsilon) = \frac{\mu [x]_\epsilon (1-[x]_\epsilon)}{1+[x]_\epsilon}$$

$$K([x]_\epsilon) = \frac{\mu \frac{x}{1+\epsilon(1-x)}\left(1-\frac{x}{1+\epsilon(1-x)}\right)}{1+\frac{x}{1+\epsilon(1-x)}}$$

$$\boxed{K([x]_\epsilon) = \frac{\mu x\,(1-x+\epsilon(1-x))}{(1+\epsilon(1-x))\,(1+x+\epsilon(1-x))}} \quad \text{—(C)}$$

A.1 Fixed point analysis of deformed K-map ⇒ The fixed point of the deformed map can be obtained by the solving following equation:-

$$K([x]_\epsilon) = x$$

$$\Rightarrow \frac{\mu x\,(1-x+\epsilon(1-x))}{(1+\epsilon(1-x))\,(1+x+\epsilon(1-x))} = x$$

On simplifying the above equation, we get

$$x\left[(1+\epsilon(1-x))(1+x+\epsilon(1-x)) - \mu(1+\epsilon)(1-x)\right] = 0 \quad \text{—(D)}$$

One fixed point is

$$x^* = 0 \quad \text{——} (E)$$

Another fixed points are

$$x^* = \frac{-(1-\epsilon-2\epsilon^2+\mu(\epsilon+1)) \pm \sqrt{(1-\epsilon-2\epsilon^2+\mu(\epsilon+1))^2 - 4\epsilon(\epsilon+1)(\epsilon-1)(1+\epsilon-\mu)}}{2\epsilon(\epsilon-1)} \quad \text{——} (F)$$

Remark 1. The existance of the fixed point $x^*=0$ is independent of the parameter μ and ϵ.

Remark 2: But the existence of the other fixed points depends on parameter μ and ϵ.

Now, from here we will start the discussion of fixed points given by the equation (F).

The fixed point is ~~also~~ only possible if

$$(1-\epsilon-2\epsilon^2+\mu(1+\epsilon))^2 - 4\epsilon(\epsilon+1)(\epsilon-1)(1+\epsilon-\mu) \geq 0 \quad \text{——} (G)$$

$$\Rightarrow (1+\epsilon)\left[(1+\epsilon)(1-2\epsilon+\mu)^2 - 4\epsilon(\epsilon-1)(1+\epsilon-\mu)\right] \geq 0 \quad \text{——} (H)$$

Now, there are two cases ~~of~~ possible for solving the above equation.

Case I :-

$$1+\epsilon \geq 0 \Rightarrow \epsilon \geq -1$$

and

$$(1+\epsilon)(1-2\epsilon+\mu)^2 - 4\epsilon(\epsilon-1)(1+\epsilon-\mu) \geq 0$$

$$\Rightarrow \mu^2(1+\epsilon) + 2\mu(1-3\epsilon) + (1+\epsilon) \geq 0$$

$$\Rightarrow \left(\mu + \frac{(1-3\epsilon)+\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}\right) \times \left(\mu + \frac{(1-3\epsilon)-\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}\right) \geq 0$$

Sub case 1:

$$u + \frac{(1-3\epsilon)+\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \geq 0 \quad \text{and} \quad u + \frac{(1-3\epsilon)-\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \geq 0$$

$$\Rightarrow \quad u \geq \frac{-(1-3\epsilon)-\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \quad \text{and} \quad u \geq \frac{-(1-3\epsilon)+\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}$$

So, the common values of parameter u will lie an (feasible)

$$u \geq \frac{-(1-3\epsilon)+\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}$$

This u will be applicable when

$$8\epsilon^2-8\epsilon \geq 0$$

$$\Rightarrow 8\epsilon(\epsilon-1) \geq 0$$

$$\Rightarrow \epsilon(\epsilon-1) \geq 0 \quad \text{——} \quad (\mathrm{I})$$

Here are two scenario for eq^n (I) as

Ist scenario $\Rightarrow$ $\quad \epsilon \geq 0$ and $\epsilon - 1 \geq 0$

$$\Rightarrow \epsilon \geq 0 \quad \text{and} \quad \epsilon \geq 1$$

Common Values of parameter ϵ will be-

$$\boxed{\epsilon \geq 1}$$

2nd Scenario $\Rightarrow$ $\quad \epsilon \leq 0$ and $\epsilon - 1 \leq 0$

$$\Rightarrow \epsilon \leq 0 \quad \text{and} \quad \epsilon \leq 1$$

Common values of parameter ϵ will be-

$$\boxed{\epsilon \leq 0}$$

So, we can say that in this subcase u will be applicable if $\epsilon \notin (0,1)$.

Subcase 2:

$$\mu + \frac{(1-3\epsilon) + \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \leq 0 \quad \text{and}$$

$$\mu + \frac{(1-3\epsilon) - \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \leq 0$$

$$\Rightarrow \mu \leq \frac{-(1-3\epsilon) - \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \quad \text{and} \quad \mu \leq \frac{-(1-3\epsilon) + \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}$$

Common values of parameter μ will be

$$\mu \leq \frac{-(1-3\epsilon) - \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} = \mu_a \qquad \mu \in [0, \mu_a]$$

Same as previous subcase, μ will be applicable at given ϵ when $\epsilon \notin (0,1)$. i.e. $\epsilon \geq 1$ or $\epsilon \in (-1, 0)$ *

Case 2:

$$1+\epsilon \leq 0 \quad \text{and}$$

$$(1+\epsilon)(1-2\epsilon+\mu)^2 - 4\epsilon(\epsilon+1)(1+\epsilon-\mu) \leq 0$$

$$\Rightarrow \mu^2(1+\epsilon) + \mu(2-6\epsilon) + (1+\epsilon) \leq 0$$

$$\Rightarrow \left(\mu + \frac{(1-3\epsilon) + \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}\right) \times \left(\mu + \frac{(1-3\epsilon) - \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}\right) \leq 0$$

Subcase 1:→

$$\mu + \frac{(1-3\epsilon) + \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \leq 0 \quad \text{and}$$

$$\mu + \frac{(1-3\epsilon) - \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \geq 0$$

$$\Rightarrow \mu \leq \frac{-(1-3\epsilon) - \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \quad \text{and} \quad \mu \geq \frac{-(1-3\epsilon) + \sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}$$

There is no common values of parameter μ, so we will not go further for the discussion.

Subcase 2:

$$\mu + \frac{(1-3\epsilon)+\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \geq 0 \text{ and}$$

$$\mu + \frac{(1-3\epsilon)-\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \leq 0$$

$$\Rightarrow -\frac{(1-3\epsilon)-\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \leq \mu \leq \frac{-(1-3\epsilon)+\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}$$

where $\epsilon \notin (0,1)$

*. Now, focus on subcase 2 of case 1.

~~Remark 1: On decreasing the value of parameter μ, the deformed map going outside the $[0,1]\times[0,1]$, which is not the part of this report.~~

~~Remark 2:~~

Remark 1: We have deformed map for given value of parameter ϵ, and the allowed value of μ will be in range $[0, \mu_b]$, ~~here~~ but here bifurcation doesnot happens at $\mu = \mu_b$.

Remark 2: We have not fixed points other than $x^* = 0$, on variation of μ and ϵ.

Now, we will not procede further for this case.

Now, focus on sub case 2 of case 2:-

where $\epsilon \leq -1$

$$\frac{-(1-3\epsilon)-\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \leq \mu \leq \frac{-(1-3\epsilon)+\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}$$

<u>Remark</u> :⇒ We don't get the fixed points for this subcase for given parameters values μ and ϵ.

Now, we will focus on subcase 1 of case 1.

$$\mu \geq \frac{-(1-3\epsilon)+\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon} \quad \text{where} \quad \epsilon \geq 1$$

Let $$\mu_a = \frac{-(1-3\epsilon)+\sqrt{8\epsilon^2-8\epsilon}}{1+\epsilon}$$

At $\mu = \mu_a$ bifurcation happens.

i.e. <u>Lemma 1</u> :⇒ For $\epsilon \geq 1$ $\mu < \mu_a$ the deformed K-map doesnot have fixed points (except $x^* = 0$)

<u>Lemma 2</u> :⇒ For $\epsilon \in [1, 2]$ and $\mu \geq \mu_a$ the deformed map has one fixed point except $x^* = 0$.

<u>Lemma 3</u> :⇒ For $\epsilon > 2$ and $\mu_a \leq \mu \leq \epsilon + 1$ the deformed map has two fixed points except $x^* = 0$.

Lemma 4: $\Rightarrow$ For $\epsilon > 2$ and $\mu > \epsilon + 1$ the deformed map has one fixed point except $x^* = 0$.

Stability Analysis $\Rightarrow$ A fixed point x^* is stable fixed point if

$|K'([x]_\epsilon)| < 1$ and

Unstable if $|K'([x]_\epsilon)| > 1$.

$$K'([x]_\epsilon) = \frac{-\mu(1+\epsilon)}{(1+\epsilon(1-x))^2}\left[\frac{x^2 + 2x(1+\epsilon(1-x)) - (1+\epsilon(1-x))^2}{(1+x+\epsilon(1-x))^2}\right]$$

At $x^* = 0$,

$$K'([x^*]_\epsilon) = \frac{-\mu(1+\epsilon)}{(1+\epsilon)^2}\left[\frac{-(1+\epsilon)^2}{(1+\epsilon)^2}\right]$$

$$= \frac{\mu}{1+\epsilon}$$

$x^* = 0$ is stable if $\left|\frac{\mu}{1+\epsilon}\right| < 1$

$\Rightarrow |\mu| < |1+\epsilon|$

Since, $\mu > 0$ and $\epsilon \geq 1$

$\Rightarrow \boxed{\mu < 1+\epsilon}$

and unstable if $\mu \geq 1+\epsilon$.

$$\text{At } \quad x^* = \frac{-(1-\epsilon-2\epsilon^2+\mu(\epsilon+1)) \pm \sqrt{(1-\epsilon-2\epsilon^2+\mu(\epsilon+1))^2 - 4\epsilon(\epsilon+1)(\epsilon-1)\times(1+\epsilon-\mu)}}{2\epsilon(\epsilon-1)}$$