In Chapter 3 we developed the foundations of fluid flow through porous media and introduced Darcy's Law; the fundamental flow equation to describe this behavior. In this chapter we will expand Darcy's Law and investigate various forms and applications useful for engineers. We will begin with various geometries for single phase, incompressible flow, migrate to compressible flow equations, and end with multiphase flow equations, by applying multiphase flow concepts from Chapter 5. The second section of this chapter presents the mathematical development of governing equations and associated conditions for describing fluid flow. This will lay the framework to solutions in preceding chapters.

6.1 Applications of Darcy's Law

6.1.1 Radial Flow

Up to now the equations developed for fluid flow assume linear flow. However, flow into a wellbore from a cylindrical drainage region is radial as shown in Figure 6.1.

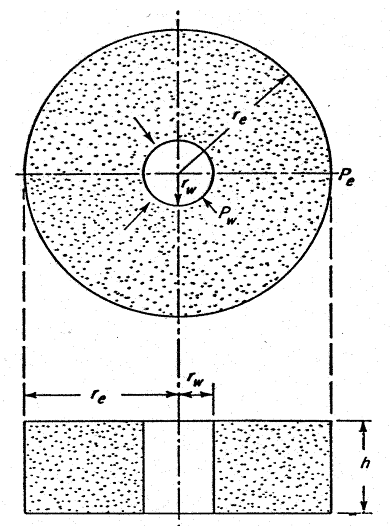


Figure 6.1 Model for radial flow of fluids to the wellbore

The two conditions are given by an external boundary pressure, p_e, located at boundary radius, r_e and the internal bottomhole flowing pressure, p_{wf} at the wellbore radius, r_w. In terms of Darcy velocity,

$$v = \frac{q}{A} = \frac{q}{2\pi rh} = \frac{k}{\mu}\frac{dp}{dr} \qquad (6.1)$$

Note since the radius direction is measured positive in the direction opposite to flow, the potential gradient is positive. Eq. (6.2) can be separated and integrated with respect to the two boundary conditions, resulting in,

$$q = \frac{2\pi kh(p_e - p_{wf})}{\mu B_o \ln\left(r_e / r_w\right)} \tag{6.2}$$

Equation (6.2) is expressed in Darcy units. Conversion to field units yields the following widely used expression for radial flow of a single-phase fluid.

$$q = \frac{kh(p_e - p_{wf})}{141.2\mu B_o \ln\left(r_e / r_w\right)} \tag{6.3}$$

Example 6.1

A well is producing in a radial reservoir at a bottom hole pressure of 5,500 psi. The reservoir pressure is 6,000 psi. Oil viscosity is 0.25 cp and the formation volume factor is 1.5 bbl/STB. If the permeability of the reservoir is 20 md, the thickness is 30 ft and the drainage radius is 1,000 ft., at what rate will the well produce? The well bore radius is 6".

If, by applying artificial lift method, the bottom hole pressure is reduced to 3,000 psi, at what rate will the well produce?

Solution

Using Eq. 6.3,

$$q = \frac{(20)(30)(6{,}000 - 5{,}500)}{141.2(0.25)(1.5)\ln\left(1{,}000 / 0.5\right)}$$
$$= 745\, stbd$$

If the bottomhole pressure is reduced to 3,000 psia, we will get,

$$q = \frac{(20)(30)(6{,}000 - 3{,}000)}{141.2(0.25)(1.5)\ln\left(1{,}000 / 0.5\right)}$$
$$= 4{,}471\, stbd$$

a substantial increase in production by lowering the bottom hole flowing pressure through artificial lift.

Example 6.2

A well is producing under reservoir pressure maintenance project. The reservoir pressure is 8,000 psi and the minimum possible wellbore pressure without using artificial lift is 7,500 psi. Well test analysis shows a skin factor of +4. Determine:

1. Well flow rate under these conditions (steady-state radial flow)
2. Well flow rate after an acid job that improves the skin factor to S = 0.

Also, given:

Bo = 1.5 Res. bbl/STB

μ_o = 0.5 cp

k = 100 md

h = 30 ft

r_w = 6 inches

r_e = 1,000 ft.

Solution

The radial flow equation can readily be modified to include steady state skin,

$$q = \frac{kh(p_e - p_{wf})}{141.2\mu B_o \ln\left(r_e / r_w + S \right)} \tag{6.4}$$

Subsequently, the flow rate for the damaged case is,

$$q = \frac{(100)(30)(8{,}000 - 7{,}500)}{141.2(0.5)(1.5)\left[\ln\left(1000/.5\right) + 4\right]}$$
$$= 1{,}220\, stbd$$

After the acid treatment the rate increases to,

$$q = \frac{(100)(30)(8{,}000 - 7{,}500)}{141.2(0.5)(1.5)\left[\ln\left(1000/.5\right) + 0\right]} = 1{,}863\, stbd$$

6.1.2 Permeability of Combination Layers

Most porous rocks have spatial variations of permeability and may be comprised of distinct layers, blocks, or concentric rings of constant permeability. To determine the average permeability of such a system, consider the following cases.

Case I: Layered reservoirs without crossflow

Reservoir rocks are interbedded with impermeable shales or silts such that no crossflow exists between sand beds (Figure 6.2).

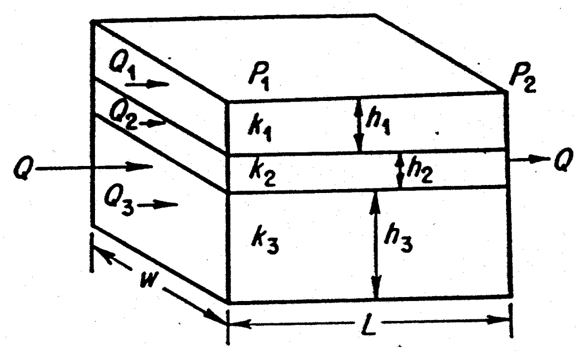


Figure 6.2 Linear flow, parallel combination of beds

In this case, $q_t=q_l+q_2+q_3$ and $\Delta p_t = \Delta p_1 = \ldots$

Substituting Darcy's equation in for flow rates, results in,

$$\frac{\bar{k}wh_t(p_1-p_2)}{\mu L}=\frac{k_1wh_1(p_1-p_2)}{\mu L}+\frac{k_2wh_2(p_1-p_2)}{\mu L}+\frac{k_3wh_3(p_1-p_2)}{\mu L}$$

which reduces to,

$$\bar{k}h_t\frac{w(p_1-p_2)}{\mu L}=\frac{w(p_1-p_2)}{\mu L}(k_1h_1+k_2h_2+k_3h_3)$$

or

$$\bar{k}=\frac{\sum_{i=1}^{n}k_ih_i}{\sum_{i=1}^{n}h_i} \tag{6.5}$$

Example 6.3

What is the equivalent linear permeability of four parallel beds having equal widths and lengths under the following conditions?

Bed	thickness, h, ft.	permeability, k, md.
1	5	250
2	8	200
3	15	130
4	20	80

Solution

Applying Eq. (6.5) where n = 4,

$$\bar{k} = \frac{(250)(5) + (200)(8) + (130)(15) + (80)(20)}{(5 + 8 + 15 + 20)} = 133md$$

Similar to linear flow, we can develop average permeability equations for radial flow systems. If the beds are arranged in parallel as shown in Figure 6.3, it can be shown that Eq. (6.5) is still valid.

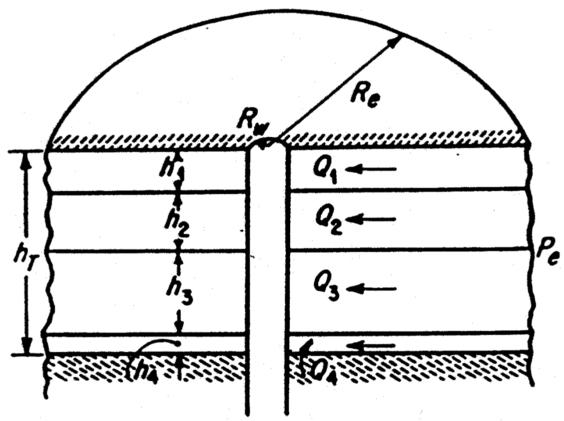


Figure 6.3 Radial flow, parallel combination of beds

Case II: Composite Reservoirs

A composite reservoir is described by variations in properties occurring away from the wellbore. These variations could be induced by drilling and completion practices (invasion of fluids into the reservoir), by injection of water during waterflooding operations or could be natural to the reservoir. The system is simplified to a set of different blocks arranged in series as shown in Figure 6.4.

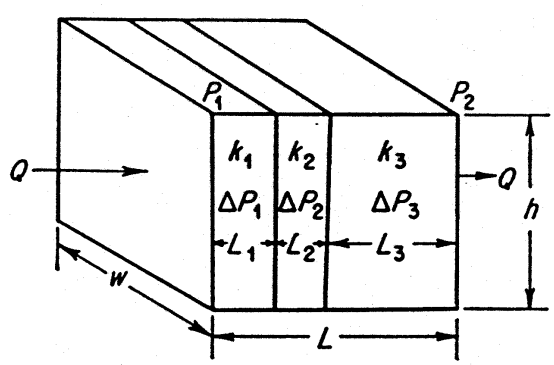


Figure 6.4 Linear flow, combination of beds in series

In this case, $q_t=q_l=q_2=q_3$ and $\Delta p_t = \Delta p_1 + \Delta p_2 + \Delta p_3$,

Substituting Darcy's equation in for the pressure drop, results in,

$$\frac{q\mu}{wh}\frac{L}{\bar{k}} = \frac{q\mu}{wh}\left(\frac{L_1}{k_1}+\frac{L_2}{k_2}+\frac{L_3}{k_3}\right)$$

which can be written in general form as:

$$\bar{k} = \frac{L}{\sum_{i=1}^{n}\frac{L_i}{k_i}} \tag{6.6}$$

A similar expression can be developed for radial flow of multiple beds (Figure 6.5)

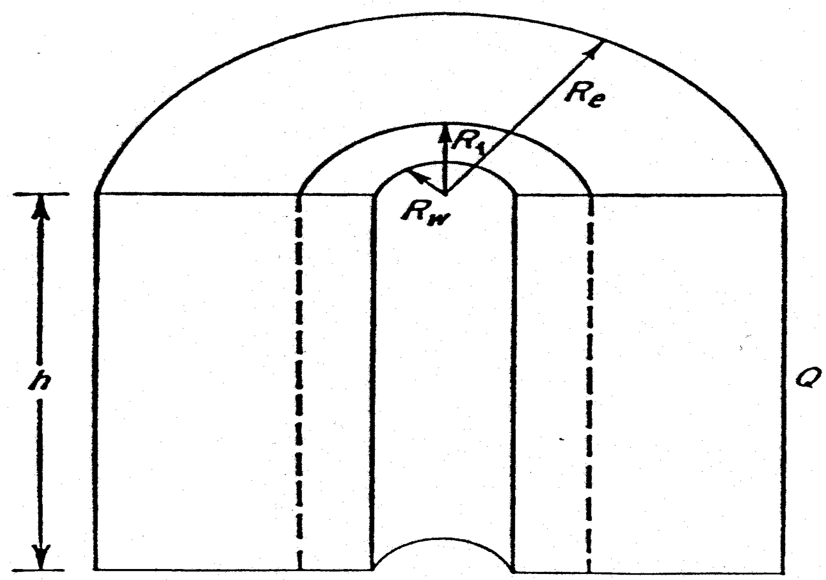


Figure 6.5 Radial flow, combination of beds in series

$$\frac{q\mu\ln\left({r_e}/{r_w}\right)}{2\pi\bar{k}h} = \frac{q\mu\ln\left({r_1}/{r_w}\right)}{2\pi k_1 h} + \frac{q\mu\ln\left({r_e}/{r_1}\right)}{2\pi k_2 h}$$

Simplifying to general terms,

$$\bar{k} = \frac{\ln\left({r_e}/{r_w}\right)}{\sum_{i=1}^{n}\frac{\ln\left({r_i}/{r_{i-1}}\right)}{k_i}} \tag{6.7}$$

Example 6.4

Consider a radial system comprised of three zones with the following properties.

Layer	Inner Radius	Outer Radius	Permeability
1	0.25	5.0	10
2	5.0	150.0	80
3	150.0	750.0	150

Calculate the average permeability.

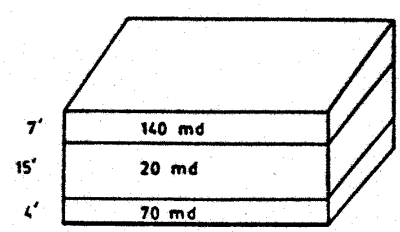
Solution

Using Eq. (6.7),

$$\bar{k} = \frac{\ln\left(750/.25\right)}{\frac{1}{10}\ln\left(5/.25\right)+\frac{1}{80}\ln\left(150/5\right)+\frac{1}{150}\ln\left(750/150\right)}$$
$$= 23.4\,md$$

Example 6.5

For a configuration shown below, calculate the horizontal and the vertical average permeabilities.



Solution

a. If flow is in the horizontal direction, then the beds are arranged in a parallel sequence. Therefore Eq. (6.5) results in the average horizontal permeability.

$$\bar{k}_h = \frac{(7)(140)+(15)(20)+(4)(70)}{7+15+4} = 60\,md$$

b. If the flow is vertical, the beds are in series and applying Eq. (6.6) provides the average vertical permeability.

$$\bar{k}_v = \frac{7+15+4}{\frac{7}{140}+\frac{15}{20}+\frac{4}{70}} = 30.3\,md$$

Darcy’s Law and Filtration

The API static filtration test is indicative of the rate at which permeable formations are sealed by the deposition of a mudcake after being penetrated by the bit. The flow of mud filtrate through a mudcake is described by Darcy's law. Thus, the rate of filtration is given by,

$$\frac{dV_f}{dt} = \frac{k_{mc} A}{\mu_{mf}} \frac{\Delta p}{h_{mc(t)}} \tag{6.8}$$

where the variables are defined below and on Figure 6.6.

dV_f/dt = the filtration rate, cc/s,

k = the permeability of the mudcake, darcies,

A = the area of the filter paper, cm,

Δp = the pressure drop across the mudcake, atm,

μ_{mf} = the viscosity of the mud filtrate, cp, and

h_{mc} = the thickness of the filter (mud) cake, cm.

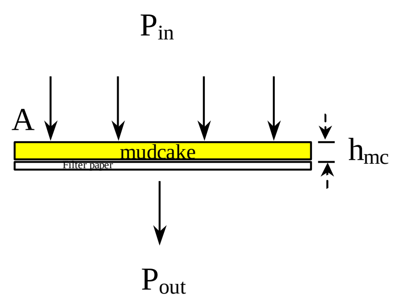


Figure 6.6 Schematic of filtration process

At any time, t, during the filtration process, the volume of solids in the mud that has been filtered is equal to the volume of solids deposited in the filter cake. Defining f_{sm}, as the volume fraction of solids in the mud and f_{sc}, is the volume fraction of solids in the cake, a volume balance equation can be written.

$$f_{sm} V_m = f_{sc} h_{mc} A \tag{6.9}$$

therefore,

$$h_{mc} = \frac{V_f}{A\left(\frac{f_{sc}}{f_{sm}} - 1\right)} \tag{6.10}$$

Inserting this expression for h_{mc} and integrating, results in [Bourgoyne, 1991],

$$V_f = A\sqrt{\frac{2k\Delta p}{\mu}\left(\frac{f_{sc}}{f_{sm}}-1\right)} * \sqrt{t} \qquad (6.11)$$

Notice the filtrate volume is proportional to the square root of time; therefore typical filtrate loss data is plotted as shown below in Figure 6.7. V_{sp} is the spurt loss volume which

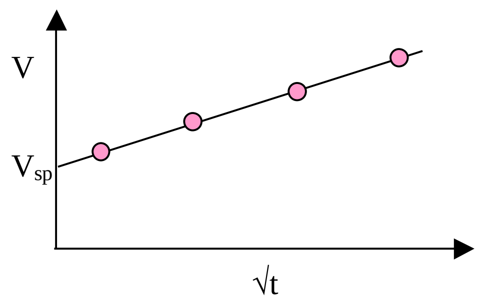


Figure 6.7 Example of filtrate volume loss during filtration test

occurs prior to the buildup of mudcake.

Example 6.6

A standard filtration test is operated at 100 psig (6.8 atm) through a cross-sectional area of 45 cm^2. Consider 5 ml of filtrate with a viscosity of 1.0 cp is collected in 30 minutes. The cell is disassembled and the mudcake thickness is measured to be 3/32". Determine the permeability of the mudcake.

Solution

Rearranging Eq. (6.10) to solve for the volume fraction ratio, we obtain,

$$\left(\frac{f_{sc}}{f_{sm}}-1\right) = \frac{V_f}{Ah_{mc}} = \frac{5cc}{45cm^2 * 3/32in * 2.54\frac{cm}{in}} = 0.4666$$

Substituting this value into Eq. (6.11) and solving for permeability,

$$\frac{5^2\{cc\}^2}{2} = \frac{k\{d\}}{1cp} * 45^2\{cm^2\}^2 * 0.4666 * 6.8atm * 30\,\text{min}$$

$$k = 0.065md$$

6.1.3 Compressible Flow

Equations developed earlier for liquid flow considered the fluids to be incompressible, with a constant density in the flow system, which is not a bad assumption for many liquid flow problems. However, gas is so highly compressible that equations for liquid flow cannot be used. Derivation of gas flow rate for a linear horizontal system follows. Darcy's equation in differential form can be written as,

$$\begin{aligned} q &= 1.127x10^{-3}\frac{kA}{\mu}\frac{dP}{dL}*5.615\frac{ft^3}{bbl} \\ &= 6.328x10^{-3}\frac{kA}{\mu}\frac{dP}{dL} \end{aligned} \tag{6.12}$$

where q is the flow rate at reservoir conditions, rcf/d.

Naturally, it is most desirable to determine gas flow rate at surface, or standard conditions in units of standard cubic feet per day (scf/d). For steady-state conditions, the flow rate in scfd (q_{sc}) is constant everywhere in the system at any pressure. Beginning with Boyle's law for a real gas under isothermal conditions,

$$\left(\frac{PV}{zT}\right)_{res} = \left(\frac{PV}{zT}\right)_{sc} \tag{6.13}$$

where:

P = absolute pressure, psia;

V = volume, cu ft,

Z = compressibility factor;

T_{res} = reservoir temperature, °R;

T_{sc} = temperature at standard conditions, °R.

For volumetric flow rates Equation 6.13 can be written as,

$$\left(\frac{Pq}{zT}\right)_{res} = \left(\frac{Pq}{zT}\right)_{sc} \tag{6.14}$$

which can be rearranged to,

$$q_{res} = q_{sc}\frac{P_{sc}}{P_{res}}\frac{T_{res}}{T_{sc}}\frac{z_{res}}{z_{sc}} \tag{6.15}$$

Substituting Eq (6.15) for reservoir flow rate in Eq. (6.12), and separating variables results in,

$$q_{sc}\int_0^L dx = 0.006328\frac{kAT_{sc}}{P_{sc}T}\int_{p_1}^{p_2}\frac{p}{\mu(p)z(p)}dp \qquad (6.16)$$

Note that viscosity and z-factor are strong functions of pressure; therefore to solve Eq. (6.16) two approximations were developed. Figure 6.8 illustrates the variation in viscosity and z-factor with pressure. In the low-pressure region (p < 2000 psi), μz = constant and can be evaluated at average pressure $(p_1+p_2)/2$.

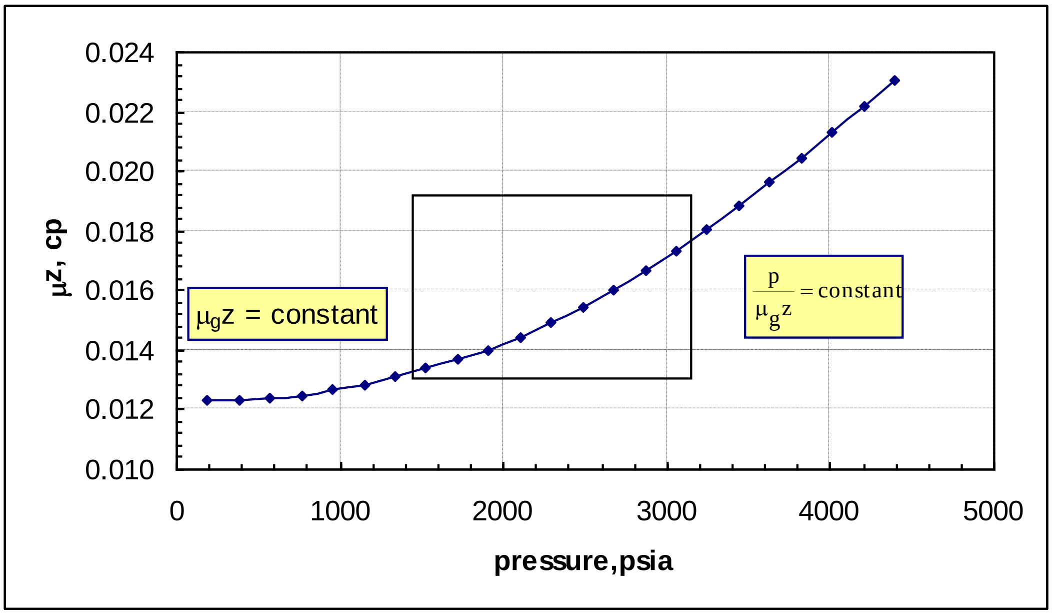


Figure 6.8 viscosity-z-factor variation with pressure

Subsequently, Eq. (6.16) can be integrated, resulting in

$$q_{sc} = 0.003164\frac{kAT_{sc}(p_1^2 - p_2^2)}{P_{sc}T(\mu z)_i L} \qquad (6.17)$$

where,

qsc	{scfd}	k	{md}
A	{ft^2}	T	{°R}
P	{psia}	L	{ft}
μ	{cp}		

In terms of lab use we change to darcy units. The linear flow equation for a compressible gas becomes,

$$q_{sc} = \frac{kAT_{sc}(p_{in}^2 - p_{out}^2)}{2000 P_{sc} T (\mu z)_i L} \tag{6.18}$$

where,

qsc	{scc/sec}	k	{md}
A	{cm^2}	T	{°R}
P	{atm}	L	{cm}
μ	{cp}		

A second approximation occurs at high pressures (p > 4000 psia). Inspection of Figure 6.8, shows that at high pressures the slope is constant; therefore

$$\mu z = \frac{\overline{\mu z}}{\overline{p}} p \tag{6.19}$$

Substituting Eq. (6.19) in for the viscosity/z-factor product of Eq. (6.16) and integrating, results in

$$q_{sc} = 0.006328 \frac{kAT_{sc}\overline{p}(p_1 - p_2)}{P_{sc} T (\overline{\mu z}) L} \tag{6.20}$$

In laboratory work, the linear flow of compressible gas is in Darcy units and is given by,

$$q_{sc} = \frac{kAT_{sc}\overline{p}(p_{in} - p_{out})}{1000 P_{sc} T (\overline{\mu z}) L} \tag{6.21}$$

Shortcomings exist for both approximations when put to practical use. The derivations of both were based on the assumption that μ and z were constants and were removed from the integral. In fact, μ and z both vary substantially with pressure. Also, both equations require μ and z to be evaluated at some "average pressure," which is difficult to estimate.

These difficulties involving true average pressure and the assumption that μ and z are constants can be avoided by using the real-gas pseudopressure, or real-gas potential m(p), presented by Al-Hussainy et al in 1966.

$$m(p) = 2\int_{p_b}^{p} \frac{p}{\mu z}\,dp \qquad (6.22)$$

Replacing the integral term in Eq. 6.16 with m(p) and solving results in,

$$q_{sc} = 0.003164\frac{kAT_{sc}[(m(p_1) - m(p_2))]}{P_{sc}TL} \qquad (6.23)$$

The advantages of Eq. (6.23) are it's applicability for the entire pressure range and the avoidance of averaging pressure and assuming μz is constant. The disadvantage is the calculations for m(p).

Example 6.7

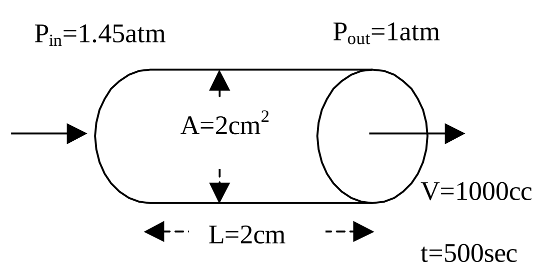
Assume steady state, horizontal flow of a compressible gas through a core as shown in the diagram below. Find the permeability of the sample.

μ_g = 0.02 cp

P_{sc} = 1 atm

T_{sc} = 60°F

Tout = 70°F



The outlet flow rate q_o = 1000/500 = 2 cc/sec. The flow rate in standard conditions from Eq. (6.15) is,

$$q_{sc} = \left(\frac{1}{1}\right)\left(\frac{520}{530}\right)\left(\frac{1}{1}\right)2 = 1.96 cc/\sec$$

Solving Eq. (6.18) for permeability,

$$k = \frac{2000(1)(.02)(2)(1.96)}{2(1.45^2 - 1^2)}\left(\frac{530}{520}\right) = 72.5md$$

Similar equations can be derived for radial flow. For the sake of completeness the three forms are given below.

Low-pressure approximation:

$$q_{sc} = 0.01988\frac{khT_{sc}(p_e^2 - p_w^2)}{P_{sc}T(\mu z)_i \ln\left(r_e/r_w\right)} \tag{6.24}$$

High-pressure approximation:

$$q_{sc} = 0.03976\frac{khT_{sc}\,\bar{p}(p_e - p_w)}{P_{sc}T(\overline{\mu z})\ln\left(r_e/r_w\right)} \tag{6.25}$$

Real Gas potential:

$$q_{sc} = 0.01988\frac{khT_{sc}[m(p_e) - m(p_w)]}{P_{sc}T\ln\left(r_e/r_w\right)} \tag{6.26}$$

6.1.4 High-velocity flow

Darcy's experiments and the resulting equations used by petroleum engineers were based on strictly laminar flow; that is, $q \propto \Delta p$. At increased flow rates, this relationship is no longer valid (see Figure 6.9). At high velocity flow, the use of Darcy's Law in the form presented so far would lead to calculated permeabilities that are less than the true permeability of the rock.

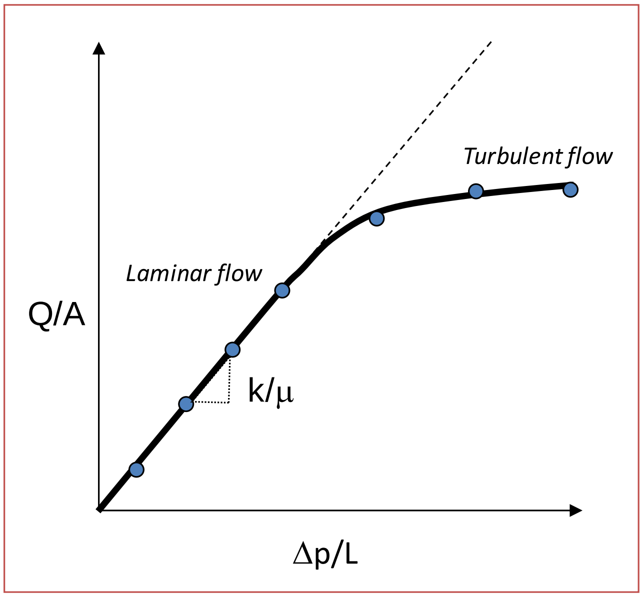


Figure 6.9 Effect of high-velocity flow on Darcys Equation

It should be evident that for a given pressure differential, the velocity of gas is greater than the velocity of liquid due to the viscosity difference. Therefore, this high velocity flow effect has a significant impact on gas well testing such as isochronal and deliverability tests. In the laboratory, accurate results require the correction for Klinkenberg Effect at low pressures and the non-Darcy flow a high pressures.

To resolve this issue of non-Darcy flow, Forchheimer in 1901 proposed a quadratic flow equation to describe both laminar and turbulent conditions.

$$\frac{dP}{dL} = \frac{\mu v}{k} + \beta \rho v^2 \qquad (6.27)$$

Units of the parameters are:

p	{atm}	k	{darcy}
L	{cm}	ρ	{gm/cc}

μ {cp} v {cm/sec}

β {atm-sec²/gm}

where β is the non-Darcy flow coefficient. Dimensional analysis reveals that β has dimensions of 1/length; therefore the conversion factor is given by,

$$\beta\left\{\frac{1}{ft}\right\} = \beta\left\{\frac{atm - s^2}{gm}\right\} * 3.0889x10^7$$

A key to applying the Forchheimer equation is to estimate a value for β. Methods developed to calculate β are based on experimental work, correlations, and from the Forchheimer equation.

To determine β experimentally, the procedure is to first measure the absolute permeability of each of the core samples and then to apply a series of increasing pressure differentials across each sample by flowing air through the core plugs at ever increasing rates. Knowing the flow rates and pressure differentials across the plugs, the coefficient of inertial resistance can be directly calculated using a linear version of the Forchheimer equation (6.27). The results are shown in Figure 6.10 in which β is plotted as a function of the absolute permeability over the range of core samples tested.

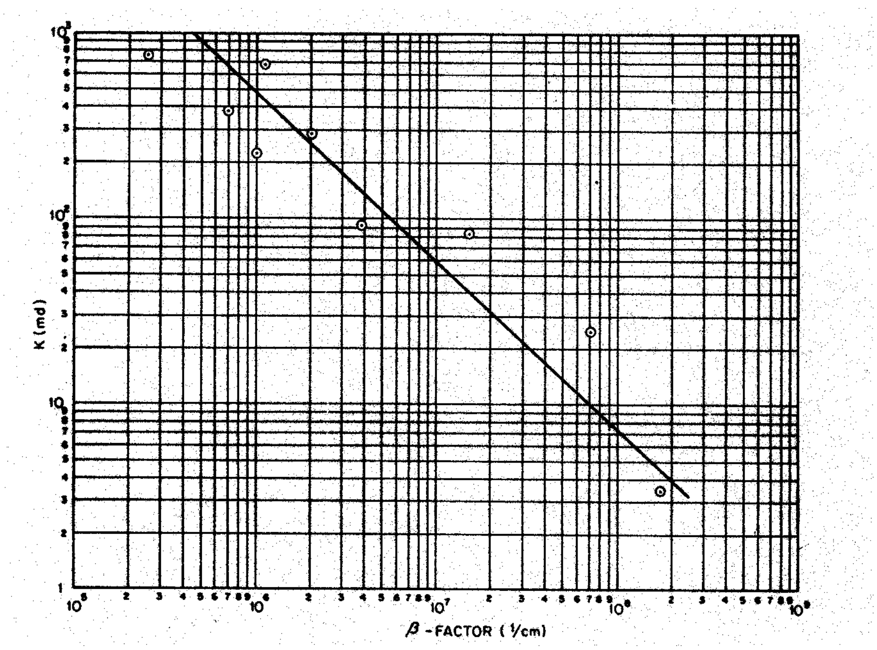


Figure 6.10 Laboratory determined β for non-Darcy flow [Dake, 1978]

A relationship is usually derived of the form,

$$\beta = \frac{constant}{k^{a}} \tag{6.28}$$

in which the exponent, a is a constant. For the experimental results shown in Figure 6.10, the specific relationship is,

$$\beta\{ft^{-1}\} = \frac{2.73x10^{10}}{k^{1.1045}\{md\}} \tag{6.29}$$

This example is for single-phase gas flow with a limited range of porosity variation (neglecting the effect of porosity on non-Darcy flow).

Many correlations exist for β in the literature, with most using absolute permeability as the correlating parameter. See Li, et al., 2001 for a comprehensive literature review of β correlations. Figure 6.11 is widely popular correlation developed by Firoozabadi and Katz in 1979.

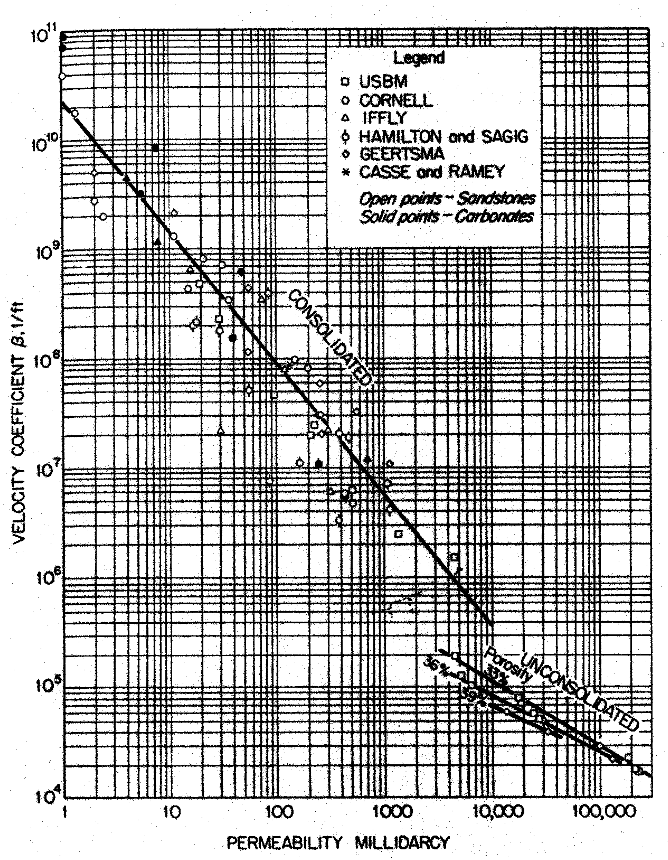


Figure 6.11 Correlation of β with absolute permeability [Firoozabadi & Katz, 1979]

The empirical relationship developed was,

$$\beta\{ft^{-1}\} = \frac{2.33x10^{10}}{k^{1.201}\{md\}} \tag{6.30}$$

Li (2001) attempted to unify the various correlations into a single general equation. The results of his work lead to,

$$\beta = \frac{c_1(\tau,\delta)}{k^{.5+.5c_2}\phi^{1+.5c_3}} \tag{6.31}$$

where c_1, c_2 and c_3 are constants for a given porous media, with $c_2 + c_3 = 1$. It was determined for parallel capillary tubes that $c_2 = 0$ and $c_3 = 1$, while for capillary tubes in series, $c_2 = 1$ and $c_3 = 0$. These are upper and lower limits, with actual porous media residing between the two endpoints.

The last alternative to estimate β is to rearrange the Forchheimer equation in the form

$$\frac{\Delta p^2}{q_{sc}} = a + bq_{sc} \tag{6.32}$$

A cartesian plot of $\Delta p^2/q_{sc}$ vs. q_{sc} will result in a slope of *b* and a y-intercept of *a* (Figure 6.12).

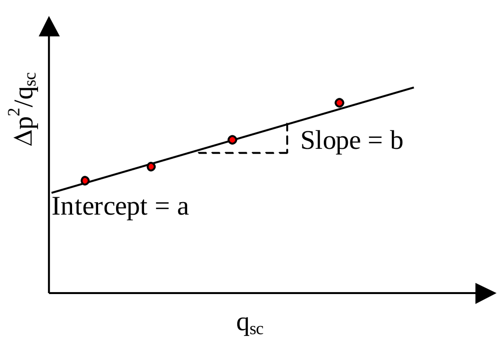


Figure 6.12 Cartesian plot of Forchheimer equation

In laboratory applications, *a* and *b* are derived for linear, steady state flow in Darcy units.

$$a = \frac{2\mu P_{sc}L}{k_g A}\left(\frac{T}{T_{sc}}\right)z \tag{6.33}$$

$$b = 8.276x10^{-5}\frac{ML\beta}{A^2}\left(\frac{P_{sc}Tz}{T_{sc}}\right) \tag{6.34}$$

Temperature is expressed in °R, molecular weight (M) in gm/gm-mole and β is in cm^{-1}. The gas permeability is solved from the y-intercept (a), and the non-Darcy coefficient is solved from the slope (b); respectively.

In reservoir conditions the flow is radial and the units are frequently in field units. Therefore,

$$a = \frac{1422\mu zT}{k_g h}\ln\left(\frac{r_e}{r_w}\right) \tag{6.35}$$

$$b = \frac{3.161x10^{-12}\beta zT\gamma_g}{h^2 r_w} \tag{6.36}$$

where β is in ft^{-1} and q_{sc} is mscfd.

Example 6.8

The following isochronal test data are available from a gas well. Additional information available are:

S_g	= 48%	ϕ	= 13%
k	= 5.5 md	r_e	= 2980 ft.
h	= 54 ft.	μ_g	= 0.0144 cp
r_w	= 0.276 ft	z	= 0.8743
p_r	= 1798 psia	T	= 164 °F
γ_g	= 0.65		

Find the non-Darcy coefficient from the a) experimental method, b) empirical correlation and c) from the Forchheimer equation.

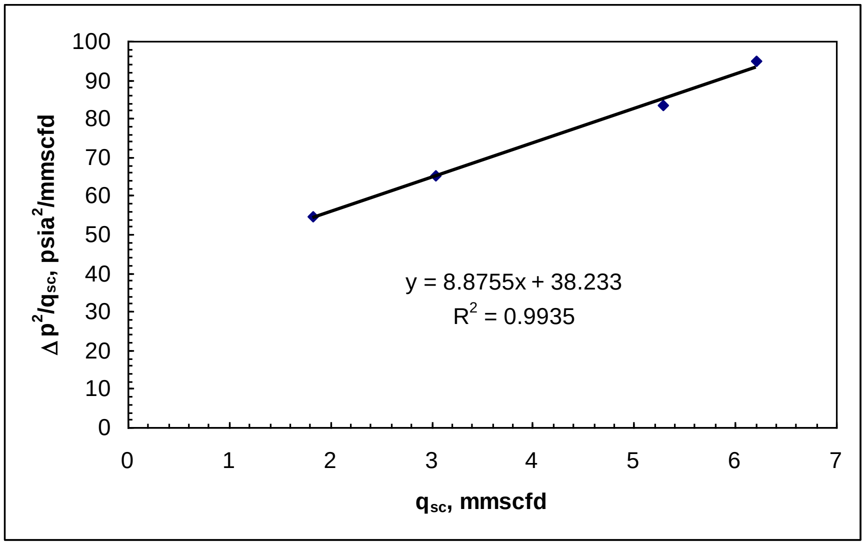
q_{sc},mmscfd	p_{wf}, psia
6.208	1626
5.291	1671
3.041	1742
1.822	1770

Solution

a. Applying the experimental relationship of Eq. (6.29), results in β = 4.15 x 10^9 ft^{-1}. Recall, this relationship was derived for a specific set of samples and therefore is not a general equation to use.

b. Applying the empirical equation (6.30) gives β = 3.01 x 10^9 ft^{-1}.

c. Plotting the isochronal test data,



Results in a slope (b) = 8.8755 $psia^2/mmscfd^2$ and an intercept (a) = 38.233 $psia^2$/mmscfd. Substituting this slope into Eq. (6.36) results in,

$$\beta = \frac{8.8755x10^{-3}(54)^2(.276)}{3.161x10^{-12}(.8743)(624)(0.65)} = 6.37x10^9\ ft^{-1}$$

and the intercept into (Eq. 6.35) gives.

$$k_g = \frac{1422(.0155)(.8743)(624)}{(38.233)(54)}\ln\left(\frac{2980}{.276}\right) = 5.41md$$

The last value of β is considered the most accurate because actual data for the reservoir was used to generate the answer.

6.1.5 Fracture Flow

In Chapter 2 we discussed a simple dual porosity model composed of matrix and fracture porosity. The majority of the storage of fluids is in the matrix. However, in terms of fluid flow and flow capacity the fractures dominate and the matrix typically contributes a minor amount.

Consider a matrix block of negligible porosity intersected by a fracture of width, w_f and height, h_f as shown in Figure 6.12. The flow rate is parallel to the fracture length, which coincides to the principal direction. The volumetric flow through a slot can be

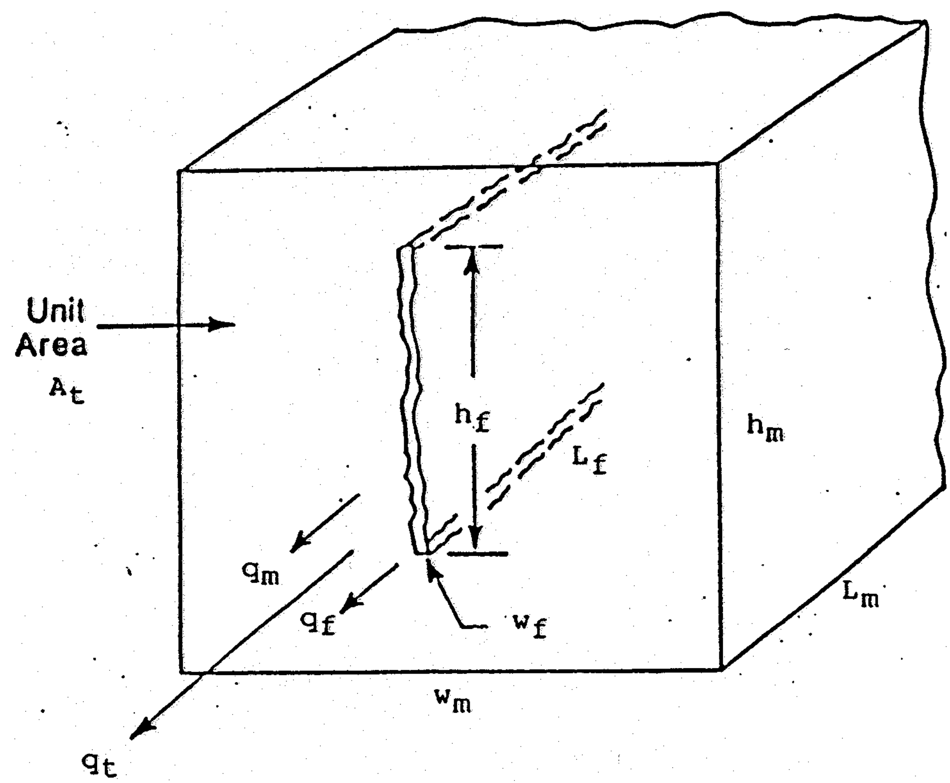


Figure 6.12. Unit model for fracture permeability

described by Buckingham's equation for flow through slots of fine clearance,

$$q_f = \frac{(n_f A) h_f w_f^3}{12} \left(\frac{\Delta p}{\mu L_f} \right) \tag{6.37}$$

where n_f is the number of fractures per unit area.

Tortuosity can be included in the flow equation be multiplying and dividing the right-hand side by the unit length, L. Equation (6.37) becomes,

$$q_f = \frac{(n_f A) h_f w_f^3}{12\sqrt{\tau}} \left(\frac{\Delta p}{\mu L} \right) \tag{6.38}$$

Furthermore, we can substitute (Eq. 2.12) into Eq. (6.38) to include porosity in the flow equation.

$$q_f = \frac{A \phi_f w_f^2}{12\tau} \left(\frac{\Delta p}{\mu L} \right) \tag{6.39}$$

Equation (6.39) is a modified Buckingham equation for flow through multiple fractures of tortuous path. If we compare Eq. (6.39) with Darcy's Law for flow through porous media,

$$q = -\frac{kA}{\mu} \left(\frac{\Delta p}{L} \right) \tag{6.40}$$

we can obtain an expression for permeability.

$$k = \frac{\phi_f w_f^2}{12\tau} \tag{6.41}$$

This expression is in similar form to the capillary tube models where the capillary radius, r and constant 8 are replaced by the fracture width, w_f and constant 12. In terms of Darcy units the resulting relationships are:

$$\begin{aligned} k_f\{darcy\} &= 84.4x10^5 w_f^2\{cm^2\} * \phi_f / \tau_f \\ k_f\{darcy\} &= 54.5x10^6 w_f^2\{in^2\} * \phi_f / \tau_f \end{aligned} \tag{6.42}$$

Note that the effect of multiple fractures on permeability is accounted for in the porosity term.

Example 6.9

A cubic block (1 ft^3) of a carbonate rock has a single horizontal fracture of width 0.00635 cm. What is the fracture permeability? If the number of fractures was doubled; i.e., 2 fractures/ft^2 what is the increase in fracture permeability? If flow direction is parallel to the fracture and matrix system, what is the average permeability of the block if the matrix permeability is 1 md.? Water was injected with a viscosity of 1 cp and a pressure drop of 10 psia was recorded. What is the matrix and fracture flow rates for these conditions?

Solution

a. From Chapter 2, the fracture porosity, ϕ_f = .00635/(12*2.54) = 0.02%. The fracture permeability is calculated from Eq. (6.42),

$$k_f = 84.4x10^5 (0.00635)^2 * (0.0002) = 71md$$

Recall that the fracture porosity is defined as the fracture pore volume with respect to the bulk volume of the rock. Therefore this permeability is the fracture permeability with respect to that same volume. This is different from isolating the fracture only. The permeability would be infinite in the fracture in this case. A further case would be if we define the fracture porosity as unity with flow respect to the block area, then permeability would be 340 darcys. This is the average permeability for the fracture layer.

b. The number of fracture for the block is n_f = 2 fractures/ft^2 * 1 ft^2 = 2. Subsequently, the permeability would double to 142 md.

c. If flow is parallel to the alternating layers of matrix and fractures, we can use Eq. (6.5) to determine the average permeability.

$$\bar{k} = \frac{k_m h_m + k_f h_f}{h_t}$$

$$\bar{k} = \frac{(1md)(1-.000208)\{ft\} + (340000md)(.000208\,ft)}{1\,ft} = 72md$$

or

$$\bar{k} = \bar{k}_m + \bar{k}_f = 1 + 71 = 72md$$

d. The fracture flow rate is,

$$q_f = 1.127x10^{-3} \frac{(71md)(1\,ft^2)(10\,psia)}{(1cp)(1\,ft)} = 0.80bpd$$

and a similar equation for the matrix results in q_m = 0.011 bpd. Notice the production rate is 70 times greater in the fracture than the matrix, illustrating the importance of fractures on production rate.

Using the definition of S_{pv} and applying to fractures, we obtain,

$$S_{pv} = \frac{A_s}{V_p} = \frac{(n_f A) * 2(w_f + h_f) L_f}{(n_f A) * w_f h_f L_f}$$
$$S_{pv} = 2\left(\frac{1}{h_f} + \frac{1}{w_f}\right) \tag{6.43}$$

Since the fracture width is much smaller than the fracture height, then $1/w_f >> 1/h_f$, and Eq. (6.43) simplifies to,

$$S_{pv} = \frac{2}{w_f} \tag{6.44}$$

Substituting this relationship for specific surface areas into the expression for permeability (Eq. 6.41), results in a Carmen – Kozeny type equation for fractures.

$$k = \frac{\phi_f}{3\tau S_{pv}^2} \tag{6.45}$$

We can generalize this relationship further by replacing the constant with a fracture shape factor, k_{sf}. Subsequently, the effective zoning factor, k_{Tf} can be defined as,

$$k_{Tf} = k_{sf} * \tau \tag{6.46}$$

Substituting into Eq. (6.45) results in a generalized Carmen-Kozeny equation for fractures.

$$k = \frac{\phi_f}{k_{Tf} S_{pv}^2} \tag{6.47}$$

This expression can be applied to identify and characterize hydraulic flow units; subsequently, we can define H_c as the hydraulic unit characterization factor for naturally fractured systems as,

$$H_c = k_{Tf} S_{pv}^2 \tag{6.48}$$

This factor can be quantified with the aid of core analysis and petrographic image analysis.

Effect of Fracture Shape

The constant in Eq. (6.45) is only valid for a fracture shape of a rectangular parallelopiped. To investigate this behavior, consider a fracture with an ellipsoidal shape as shown in Figure 6.13.

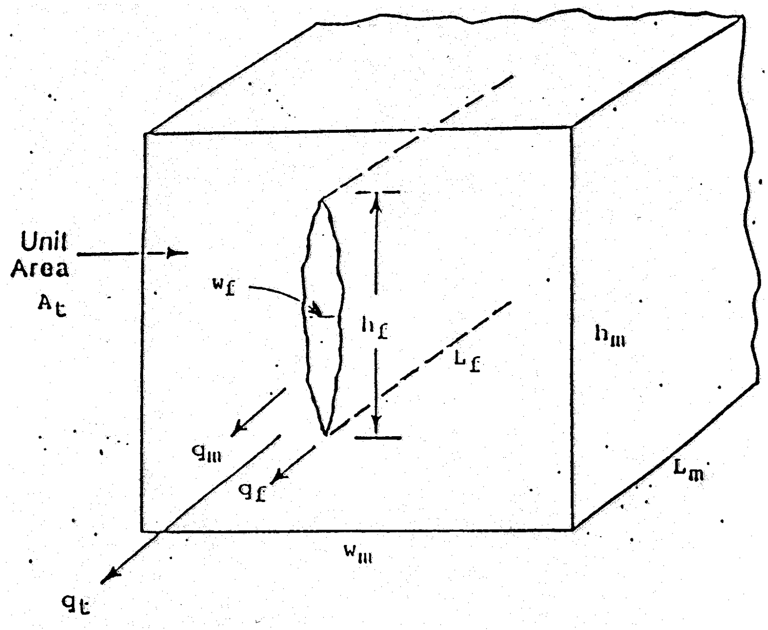


Figure 6.13. Effect of fracture shape on the permeability – porosity relationship

Assuming the flow only occurs in the fracture network and that no matrix porosity exists, then the specific surface area per unit pore volume is defined only by the ellipsoidal cylinder.

$$V_p = \frac{\pi}{4} w_f h_f L_f \tag{6.49}$$

$$A_s = \pi L_f * \left[\frac{3}{4}\left(w_f + h_f \right) - \frac{1}{2}\sqrt{w_f h_f} \right] \tag{6.50}$$

Combining Eqs. (6.49) and (6.50) and simplifying,

$$S_{pv} = 3\left(\frac{1}{h_f} + \frac{1}{w_f} \right) - \frac{2}{\sqrt{w_f h_f}} \tag{6.51}$$

Based on the premise that fracture width is much smaller than the fracture height, $1/w_f >> 1/h_f$, and also assuming the square root of $w_f h_f >> w_f$ (less than 5% error), then the expression for specific surface area is reduced to,

$$S_{pv} = \frac{3}{w_f} \tag{6.52}$$

Thus clearly demonstrating the effect of fracture shape and the reliance of fracture geometry in the hydraulic unit characterization factor.

Hydraulic radius of a fracture

An effective hydraulic radius for the fracture can be derived by equating the Hagen – Poiseuille equations for a capillary tube and fracture. The resulting relationship is,

$$\frac{\pi r_{hf}^4}{8} = \frac{h_f w_f^3}{12} \tag{6.53}$$

solving for hydraulic radius, r_{hf},

$$r_{hf} = \left(\frac{2}{3\pi} h_f w_f^3 \right)^{1/4} \tag{6.54}$$

We can relate the hydraulic radius to permeability through the Kozeny equation for capillary tubes,

$$k = \frac{\phi r^2}{8} = \frac{\phi}{8} \sqrt{\frac{2}{3\pi} h_f w_f^3} \tag{6.55}$$

If we consider the fracture as a slit, then $S_{pv} = 2/w_f$ can be substituted into Eq. (6.55),

$$k = \frac{\phi}{2} \sqrt{\frac{1}{3\pi} \frac{h_f}{S_{pv}^3}} \tag{6.56}$$

The importance of Equations (6.55) and (6.56) is related to the effect of fracture height on the permeability – porosity relationship. For example, fracture height can be determined if the other parameters are estimated by core, log or other type of analysis.

6.1.5 Multiphase Flow

Darcy's Law can be extended to multiphase flow using concepts developed in Chapter 5 on relative permeability. In terms of superficial velocity for a 1D problem,

$$u_i = -\frac{kk_{ri}}{\mu_i}\frac{\partial \psi_i}{\partial x} \tag{6.57}$$

where i = o, g and w depending on the phase and permeability is the effective permeability of the phase. The potential for each phase is given by

$$\psi_i = g(z - z_d) + \int_{p_{id}}^{p_i} \frac{dp_i}{\rho_i} \tag{6.58}$$

where z is the elevation above a horizontal datum, z_d is the elevation of the reference datum and p_{id} is the phase pressure at the reference datum. The phase pressures are related through capillary pressure curves. The effective permeability of a phase are functions of the saturation, and vary according to the relative permeability curves. An example of applying relative permeability curves to flow rates is given in Section 5.3.4 for review.

6.2 Differential Equations for Fluid Flow

The objective of this section is to understand and develop fundamental equations for fluid flow through porous media. Direct applications occur in well testing, simulation, and production decline analysis, and will form the basis for later development of secondary recovery processes. Detailed derivations can be found in Collins (1961), Lee (1982), Matthews and Russell (1967), and Muskat (1982).

A mathematical description of fluid flow in a porous medium can be obtained by combining the following physical principles: (1) the law of conservation of mass, (2) a momentum equation, e.g., Darcy's law and (3) an equation of state. In flow of any type the conservation principle is simply a statement that some physical quantity is conserved, i.e., neither created nor destroyed. In fluid flow in a porous medium, the most significant quantity conserved is mass.

1. Law of conservation of mass

$$\left\{\begin{matrix}\text{mass into}\\ \text{the element}\end{matrix}\right\}-\left\{\begin{matrix}\text{mass out of}\\ \text{the element}\end{matrix}\right\}+\left\{\begin{matrix}\text{net amount of}\\ \text{mass introduced}\end{matrix}\right\}=\left\{\begin{matrix}\text{rate of}\\ \text{accumulation}\end{matrix}\right\}$$

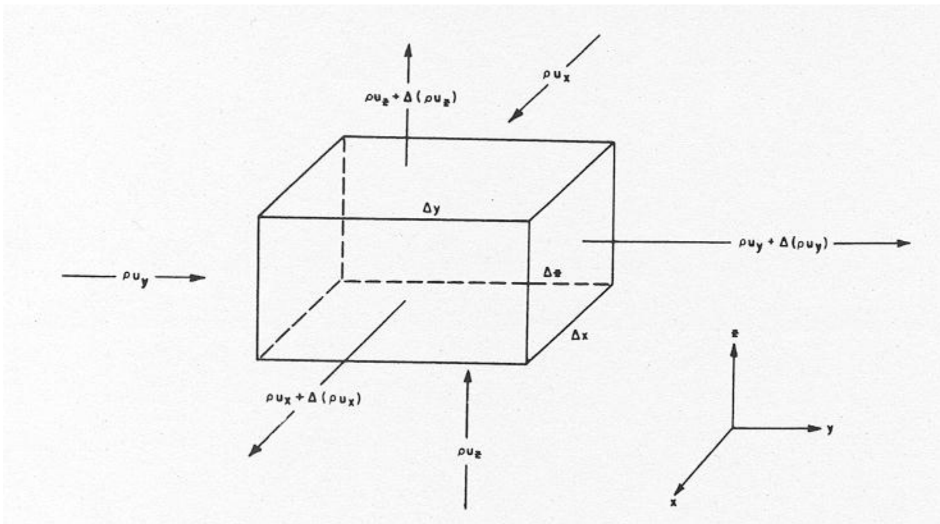


Figure 6.14. Volume element in Cartesian coordinates

Begin by applying the principle of continuity to an arbitrary volume element within the flow region. As an example, consider the three-dimensional case shown in Figure 6.14. The volumetric components of flow into the element in the x, y and z directions are denoted by v_x, v_y, and v_z, respectively. These are volumetric flow rates per unit of cross-sectional area. Thus, the mass flow rate into the element in the x-direction is

$$\rho u_x \Delta y \Delta z$$

The mass flow rate in the x-direction out of the element is

$$\left[\rho u_x + \Delta(\rho u_x)\right]\Delta y \Delta z$$

where $\Delta(\rho u_x)$ is the change in mass flux that occurs within the element. The net flow rate in the x-direction (amount-in less the amount-out) is

$$-\Delta(\rho u_x)\Delta y \Delta z$$

Similar expressions can be written for the y and z directions. Assuming no mass is generated or lost in the element, the amount of net mass change in the element in a time element Δt is,

$$-\Delta t\left[\Delta(\rho u_x)\Delta_y\Delta_z + \Delta(\rho u_y)\Delta_x\Delta_z + \Delta(\rho u_z)\Delta_x\Delta_y\right] = \left[\phi\rho\big|_{t+\Delta t} - \phi\rho\big|_t\right]\cdot V$$

Dividing by the volume(V)-Δt product and taking the limit as Δx, Δy, Δz, and Δt approach zero, results in the continuity equation for flow of fluid in a porous media, and can be written as,

$$-\nabla(\rho\vec{v}) + G = \frac{\partial}{\partial t}(\rho\phi) \tag{6.59}$$

where G represents mass introduced (source) or mass removed (sink) within the system.

2. Momentum Equation

Darcy's law expresses the fact that the volumetric rate of flow per unit cross-sectional area (v) at any point in a uniform porous medium is proportional to the gradient in potential (∇ψ)in the direction of flow at that point. The law is valid for laminar flow and can be written as:

$$\vec{v} = -\overline{\overline{k}}\frac{\rho}{\mu}\nabla\psi \tag{6.60}$$

In the above equation, note that flow occurs in the direction of decreasing potential. Hubbert has studied Darcy's law and its implications quite extensively and showed that

$$\psi = \int_{p_o}^{p}\frac{dp}{\rho} + gz \tag{6.61}$$

where z is the height above an arbitrary datum plane. Subsequently, Darcy’s Law becomes

$$\vec{v} = -\frac{\overline{\overline{k}}}{\mu}\left[\nabla p + \rho g\hat{i}_z\right] \tag{6.62}$$

Also, note in Eqs. (6.60 and 6.62) that permeability is a tensor property. Including all cross terms, permeability can be expanded to:

$$\begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \tag{6.63}$$

If the permeability is aligned with the principal directions, then only the three diagonal terms are present. If the media is considered isotropic, then the three diagonal terms reduce to a single valued permeability. Combining Darcy’s Law with the continuity equation results in,

$$\nabla(\rho \frac{k_i}{\mu} \nabla p) \pm G = \frac{\partial}{\partial t}(\rho\phi) \qquad (6.64)$$

3. Equation of state

Various equations of state are used in deriving the flow equations. An equation of state specifies the dependence of fluid density ρ on the fluid pressure p and temperature T. Thus, depending on the actual fluid(s) present, an appropriate equation of state must be chosen. The simpliest example is to consider the fluid incompressible,

$$\frac{d\rho}{dP} = 0 \qquad (6.65)$$

Therefore the density is constant and can easily be eliminated from Eq. (6.64).

An important class of flow equations is the isothermal flow of fluids of small and constant compressibility; i.e., an ideal liquid.

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial P} \qquad (6.66)$$

The resulting governing differential equation becomes,

$$\nabla^2 P = \frac{\phi\mu c}{k}\frac{\partial P}{\partial t} \qquad (6.67)$$

The assumptions to obtain Eq. (6.67) are:

a. compressibility is small and constant
b. permeability is isotropic and constant
c. porosity is constant
d. gradient squared terms are small
e. …?

Alternative solutions can be obtained given constraints or assumptions are relaxed. Several of the possibilities are:

1. Pressure dependent porosity and permeability
2. incompressible fluid
3. anisotropy (aligned with principal directions)
4. compressible fluid
5. multiphase flow

A final important class of flow equations describes the flow of a real gas through a porous media. The equation of state for a compressible fluid; i.e., real gas is given by,

$$\rho = \frac{PM}{z_{(p,T)}RT} \tag{6.68}$$

The resulting governing equation becomes,

$$\nabla\left(\frac{p}{\mu z}\nabla p\right) = \frac{\phi\mu c}{k}\frac{\partial}{\partial t}\left(\frac{p}{\mu z}\right) \tag{6.69}$$

To account for the non-linearity in the gas governing equation, three approaches have been taken to handle the pressure dependent fluid properties.

1. Low pressure approximation
2. High pressure approximation
3. Real gas potential or pseudopressure function (Al-Hussainy, et al.,1966)

$$m(p) = 2\int_{p_b}^{p} \frac{p}{\mu z}\,dp \tag{6.70}$$

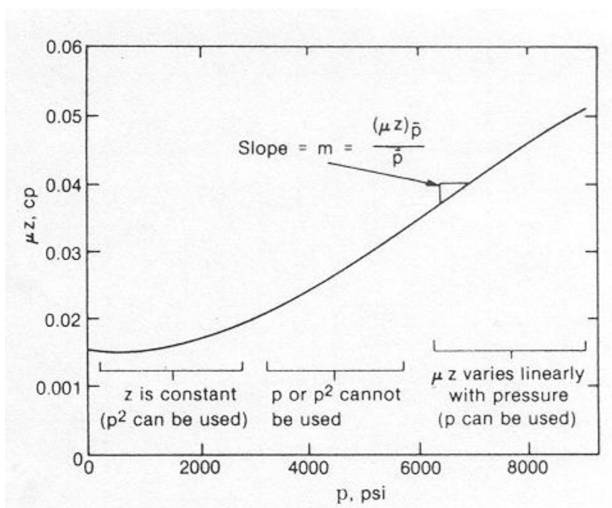


Figure 6.15 Diagram illustrating variations of μz with pressure.

Before developing solutions to the differential equations for flow through porous media, we should point out that a differential equation describes only the physical law or laws which apply to a situation. To obtain a solution to a specific flow problem, one must have not only the differential equation, but also the boundary and initial conditions that characterize the particular situation of interest. Common conditions used in fluid flow are:

Initial and Boundary Conditions

Initial condition $P(r,0) = P_i$

The cases of interest are (a) an infinite acting reservoir – the case where a well is assumed to be situated in a porous medium of infinite extent, (b) bounded reservoir – the case in which the well is assumed to be located in the center of reservoir with no flow across the external boundary, and (c) constant potential outer boundary - the case in which the well is assumed to be located in the center of an area with constant pressure along the external boundary. Listed below and shown in Figure 6.16 is a description of these boundary conditions.

a. infinite reservoir

$$\psi(\infty, t) = \psi_i$$

b. Constant potential boundary

$$\psi(r_e, t) = \psi_e$$

c. Bounded cylindrical reservoir

$$\frac{\partial \psi}{\partial l_n} = 0$$

where l_n is the distance measured parallel to the unit vector n.

d. Constant flux boundary

$$v_n = -\frac{k\rho}{\mu}\frac{\partial \psi}{\partial l_n}$$

e. Discontinuity in porous media

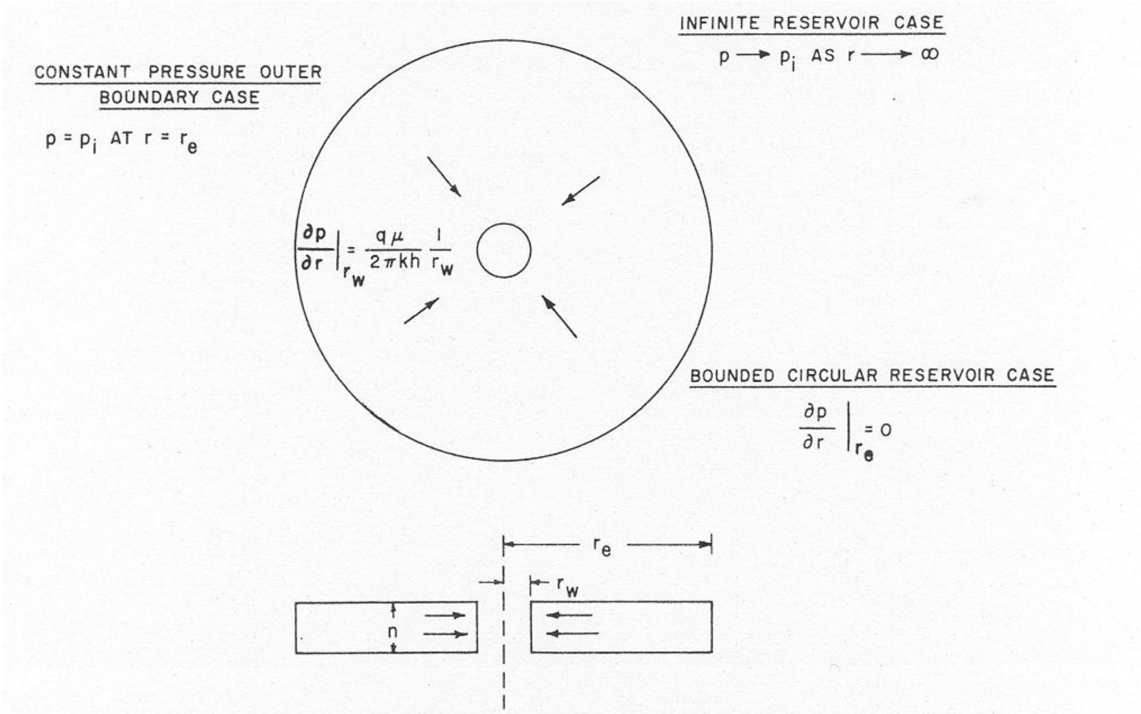


Figure 6.16 Schematic of the boundary conditions for radial flow, constant flow rate (after Matthews and Russell)

Frequently flow problems must be considered in which a discontinuity in the permeability of the medium exists. The proper boundary conditions at such a boundary are arrived at from physical considerations. Suppose, for example, that the domain of flow is composed of two regions, 1 and 2, having different permeabilities, k_1 and k_2. Since pressure at any given point must be single-valued, the first requirement is $p_1 = p_2$. Also since what enters the boundary from one side must come out on the other side, the velocities normal to the boundary must be equal on the two sides.

$$\rho \frac{k_1}{\mu}\frac{\partial \psi_1}{\partial l_n} = \rho \frac{k_2}{\mu}\frac{\partial \psi_2}{\partial l_n}$$

where l_n is distance measured normal to the boundary.

The last set of boundary conditions describes the physical conditions that occur at the inner condition (at the wellbore). If production rate is constant then from Darcy's Law,

$$q = -\frac{kA}{\mu}\nabla \psi$$

If potential is constant then $\psi(r_w, t) = \psi_w$.