HW 7

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Exercise 1

***Algebra* (Artin, 2e) Exercise 6.7.1 (partial)**

Part (a)

Let $G = D_4$ be the dihedral group of symmetries of the square $[-1,1]^2 \subset \mathbb{R}^2$, generated by $\rho = \rho_{\pi/2}$ and τ reflection across the e_1-axis. By inspection, the elements of D_4 that fix $v = (1,1)$ are $\{e, \rho\tau\}$. These correspond, respectively, with performing no actions and performing a reflection across the $e - 1$ axis followed by a rotation of $\theta = \frac{\pi}{2}$.

$$\begin{aligned} v &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \tau v &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \rho\tau v &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v \end{aligned} \tag{1}$$

Evidently, the vertex is mapped to itself.

Part (b)

By inspection, the elements of D_4 that fix the top edge e connecting $(-1,1)$ and $(1,1)$ are $\left\{e, \rho^2\tau\right\}$. These correspond, respectively, with performing no actions and performing a reflection across the $e - 1$ axis followed by two rotations of

$\theta = \frac{\pi}{2}$.

$$\begin{aligned}
e &= \begin{bmatrix} x \\ 1 \end{bmatrix} \\
\tau e &= \begin{bmatrix} x \\ -1 \end{bmatrix} \\
\rho\tau e &= \begin{bmatrix} 1 \\ x \end{bmatrix} \\
\rho^2\tau e &= \begin{bmatrix} -x \\ 1 \end{bmatrix} = e
\end{aligned} \tag{2}$$

Evidently, every point on the edge is bijectively mapped to a point on the edge.

Exercise 2

***Algebra* (Artin, 2e) Exercise 6.3.4 (extended)**

Part (a)

Let $G = GL_n(\mathbb{R})$ act on the set $V = \mathbb{R}^n$ by left multiplication. Since $g0 = 0$ for all $g \in G$ and the null space of an invertible matrix is only the zero vector, $O_0 = \{0\}$. Further, for some nonzero vectors v, there must exist some $g \in G$ such that $ge_1 = v$. Since ge_1 is the first column of g, we see that g is a matrix where the first column is v with the other columns chosen such that the matrix is invertible. Since e_1 can therefore be mapped to every nonzero vector in $\mathbb{R}^n$, $O_{e_1} = V - \{0\}$. The two orbits are then the set containing only the zero vector, and the set containing the rest of $V = \mathbb{R}^n$.

Part (b)

The stabilizer of e_1 is the set of all $g \in G$ such that $ge_1 = e_1$. Since ge_1 is the first column of g, the stabilizer of e_1 is therefore the set of all invertible matrices where the first column is e_1.

Part (c)

The action of G on $V - \{0\}$ is then observed to have a single orbit. Consequently, for $v, v' \in V - \{0\}$, there exists $g \in G$ such that $gv = v'$, which means that the action is transitive. We see that for all $v \in V$, $Iv = v$. Suppose, for arbitrary v, there exists some $g' \in G$ such that $g'v = v = Iv$. Since a matrix is uniquely determined by the linear transformation it performs, $g' = I$. Consequently, $gv = v$ iff $g = I$, so the action is free.