

Homework 6

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Exercise 1

Algebra (Artin, 2e) Exercise 6.3.4 (extended)

A glide reflection can be written as $g = (t_a \rho_\phi) t_v r (t_a \rho_\phi)^{-1}$ where a, ϕ are determined by the glide line and v is the glide vector. For all $m \in Isom(\mathbb{R}^2)$, $m = t_m \rho_\theta$ or $t_m \rho_\theta r$, for a unique vector a and angle θ . Suppose $m = t_m \rho_\theta$. Then:

$$\begin{aligned} m g m^{-1} &= (t_m \rho_\theta) (t_a \rho_\phi) t_v r (t_a \rho_\phi)^{-1} (t_m \rho_\theta)^{-1} \\ &= (t_m \rho_\theta t_a \rho_\phi) t_v r (t_m \rho_\theta t_a \rho_\phi)^{-1} \\ &= (t_m t_{\rho_\theta(a)} \rho_\theta \rho_\phi) t_v r (t_m t_{\rho_\theta(a)} \rho_\theta \rho_\phi)^{-1} \\ &= (t_{m+\rho_\theta(a)} \rho_{\theta+\phi}) t_v r (t_{m+\rho_\theta(a)} \rho_{\theta+\phi})^{-1} \\ &= (t_{a'} \rho_{\phi'}) t_v r (t_{a'} \rho_{\phi'})^{-1} \end{aligned} \tag{1}$$

Evidently, the conjugate is also a glide vector where $a' = m + \rho_\theta(a)$, $\phi' = \theta + \phi$. As the glide vector is the same glide vector, they naturally have the same length. Now suppose $m = t_m \rho_\theta r$. Then:

$$\begin{aligned} m g m^{-1} &= (t_m \rho_\theta r) (t_a \rho_\phi) t_v r (t_a \rho_\phi)^{-1} (t_m \rho_\theta r)^{-1} \\ &= (t_m \rho_\theta r t_a \rho_\phi) t_v r (t_m \rho_\theta r t_a \rho_\phi)^{-1} \\ &= (t_m \rho_\theta t_{r(a)} r \rho_\phi) t_v r (t_m \rho_\theta t_{r(a)} r \rho_\phi)^{-1} \\ &= (t_m \rho_\theta t_{r(a)} \rho_{-\phi} r) t_v r (t_m \rho_\theta t_{r(a)} \rho_{-\phi} r)^{-1} \\ &= (t_m \rho_\theta t_{r(a)} \rho_{-\phi}) r t_v r r (t_m \rho_\theta t_{r(a)} \rho_{-\phi})^{-1} \\ &= (t_m t_{\rho_\theta(r(a))} \rho_\theta \rho_{-\phi}) r t_v (t_m t_{\rho_\theta(r(a))} \rho_\theta \rho_{-\phi})^{-1} \\ &= (t_{m+\rho_\theta(r(a))} \rho_{\theta-\phi}) t_{r(v)} r (t_{m+\rho_\theta(r(a))} \rho_{\theta-\phi})^{-1} \\ &= (t_{a'} \rho_{\phi'}) t_v r (t_{a'} \rho_{\phi'})^{-1} \end{aligned} \tag{2}$$

Evidently, the conjugate is also a glide vector where $a' = m + \rho_\theta(r(a))$, $\phi' = \theta - \phi$. As the glide vector is the original glide vector under an isometry, length is preserved and $v, r(v)$ have the same length. In both cases, the conjugate of a glide reflection is a glide reflection that has a glide vector with the same length.

Exercise 2

The generators of $Isom(\mathbb{R}^2)$ are known to be t_a, ρ_θ, r where:

$$\begin{aligned} t_a(x) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ \rho_\theta(x) &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ r(x) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \quad (3)$$

Let $z \in \mathbb{C}$. Translation in \mathbb{R}^2 is addition in \mathbb{C} . Consequently:

$$\begin{aligned} t_{z_0}(z) &= z + z_0 \\ |t_{z_0}(z') - t_{z_0}(z)| &= |(z' + z_0) - (z + z_0)| \\ &= |z' - z| \end{aligned} \quad (4)$$

Rotation about the origin in \mathbb{R}^2 is multiplication in \mathbb{C} by a member of S^1 . Consequently:

$$\begin{aligned} \rho_\theta(z) &= ze^{i\theta} \\ |\rho_\theta(z') - \rho_\theta(z)| &= |z'e^{i\theta} - ze^{i\theta}| \\ &= |(z' - z)e^{i\theta}| \\ &= |z' - z||e^{i\theta}| \\ &= |z' - z| \end{aligned} \quad (5)$$

Reflection about the x-axis in \mathbb{R}^2 is taking the complex conjugate in \mathbb{C} . Consequently:

$$\begin{aligned} r(z) &= \bar{z} \\ |r(z') - r(z)| &= |r(x' + iy') - r(x + iy)| = |(x' - iy') - (x - iy)| \\ &= |(x' - x) - i(y' - y)| \\ &= |(x' - x) + i(y' - y)| \quad (6) \\ &= |(x' - x) + i(y' - y)| \\ &= |(x' + iy') - (x + iy)| \\ &= |z' - z| \end{aligned}$$

Evidently, all three are isometries of \mathbb{C} .

Exercise 3

Algebra (Artin, 2e) Exercise 6.6.2

Part (a)

Let G denote the group of symmetries of the following infinite wallpaper pattern constructed from equilateral triangles of side length 1. By inspection, we see that reflection across the x -axis and rotation by a multiple of $\frac{2\pi}{6} = \frac{\pi}{3}$ are symmetries of the wallpaper pattern. Consequently, the point group $\overline{G} = \{\overline{1}, \overline{r}, \overline{\rho_{\frac{\pi}{3}}}, \overline{\rho_{\frac{\pi}{3}}} \overline{r}\} = D_6$. The index of L in G is the the order of $\overline{G} = D_6$, which is $2(6) = 12$.

Part (b)

By inspection, we see that L is the lattice $\mathbb{Z}a + \mathbb{Z}b$ for $a = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, b = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$.