Homework 2

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Exercise 1

***Introduction to Elementary Particles* (Griffiths, 2e) Exercise 3.7**

$$
\begin{aligned}
x^\mu &= M^\mu_\nu x^{\nu\prime} \\
M &= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\Lambda M &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & \gamma^2(1-\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I
\end{aligned} \tag{1}
$$

M is then the matrix inverse of the Lorentz transformation matrix Λ.

Exercise 2

Introduction to Elementary Particles **(Griffiths, 2e) Exercise 3.12**

Applying the Lorentz transformation:

$$\begin{gathered} p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu \\ \Lambda_\nu^\mu \left(p_A^\mu + p_B^\mu\right) = \Lambda_\nu^\mu \left(p_C^\mu + p_D^\mu\right) \\ p_A^\nu{}' + p_B^\nu{}' = p_C^\nu{}' + p_D^\mu{}' \end{gathered} \tag{2}$$

Energy and momentum are conserved in S'.

Exercise 3

***Introduction to Elementary Particles* (Griffiths, 2e) Exercise 3.15**

By conservation of the energy-momentum 4-vector:

$$\begin{aligned}
p_\pi &= p_\mu + p_\nu \\
p_\mu &= p_\pi - p_\nu \\
p_\mu^2 &= p_\pi^2 + p_\nu^2 - 2p_\pi \cdot p_\nu \\
m_\mu^2 c^2 &= m_\pi^2 c^2 - 2\left(\frac{E_\pi}{c}\frac{E_\nu}{c} - \mathbf{p}_\pi \cdot \mathbf{p}_\nu\right) \\
&= m_\pi^2 c^2 - 2\frac{E_\pi E_\nu}{c^2} \\
&= m_\pi^2 c^2 - 2\gamma m_\pi \left|\mathbf{p}_\nu\right| c \\
\left|\mathbf{p}_\nu\right| &= \frac{(m_\pi^2 - m_\mu^2)c}{2\gamma m_\pi}
\end{aligned} \tag{3}$$

Calculating the scattering angle:

$$\begin{aligned}
\tan\theta &= \frac{\left|\mathbf{p}_\nu\right|}{\left|\mathbf{p}_\pi\right|} = \frac{\frac{\left(m_\pi^2 - m_\mu^2\right)c}{2\gamma m_\pi}}{\gamma m_\pi \beta c} \\
&= \frac{m_\pi^2 - m_\mu^2}{2\beta\left(\gamma m_\pi\right)^2} \\
&= \frac{1 - \left(\frac{m_\mu}{m_\pi}\right)^2}{2\beta\gamma^2} \\
\theta &= \tan^{-1}\frac{1 - \left(\frac{m_\mu}{m_\pi}\right)^2}{2\beta\gamma^2}
\end{aligned} \tag{4}$$

Exercise 4

Introduction to Elementary Particles **(Griffiths, 2e) Exercise 3.16**

Before the collision:

$$
\begin{aligned}
p^\mu &= \left(\frac{E_A}{c} + m_B c, \mathbf{p}_A\right) \\
p^2 &= \left(\frac{E_A}{c} + m_B c\right)^2 - \mathbf{p}_A^2 \\
&= \frac{E_A^2}{c^2} + 2E_a m_B + m_B^2 c^2 - \frac{1}{c^2}\left(E_A^2 - m_A^2 c^4\right) \\
&= 2E_a m_B + \left(m_B^2 + m_A^2\right) c^2
\end{aligned}
\tag{5}
$$

After the collision:

$$
p^{\mu\prime} = \left(\sum_{i=1}^{n} m_n c, \mathbf{0}\right) = (Mc, \mathbf{0}) \tag{6}
$$

Using the invariance of the dot product:

$$
\begin{aligned}
p^2 = p^{2\prime} &= M^2 c^2 \\
E_a &= \frac{M^2 - m_B^2 - m_A^2}{2m_B} c^2
\end{aligned}
\tag{7}
$$

Exercise 5

***Introduction to Elementary Particles* (Griffiths, 2e) Exercise 3.17**

Part (a)

Applying the previous results to $p + p \to p + p + \pi^0$:

$$\begin{aligned} E_p &= \frac{\left(2m_p + m_{\pi^0}\right)^2 - m_p^2 - m_p^2}{2m_p} c^2 \\ &\approx 1217.94 \text{ MeV} \end{aligned} \tag{8}$$

Part (b)

Applying the previous results to $p + p \to p + p + \pi^+ + \pi^-$:

$$\begin{aligned} E_p &= \frac{\left(2m_p + m_{\pi^+} + m_{\pi^-}\right)^2 - m_p^2 - m_p^2}{2m_p} c^2 \\ &\approx 1538.07 \text{ MeV} \end{aligned} \tag{9}$$

Part (c)

Applying the previous results to $\pi^- + p \to p + \bar{p} + n$:

$$\begin{aligned} E_{\pi^-} &= \frac{\left(m_p + m_{\bar{p}} + m_n\right)^2 - m_p^2 - m_{\pi^-}^2}{2m_p} c^2 \\ &\approx 3746.60 \text{ MeV} \end{aligned} \tag{10}$$

Part (d)

Applying the previous results to $\pi^- + p \to K^0 + \Sigma^0$:

$$\begin{aligned} E_{\pi^-} &= \frac{\left(m_{K^0} + m_{\Sigma^0}\right)^2 - m_p^2 - m_{\pi^-}^2}{2m_p} c^2 \\ &\approx 1042.94 \text{ MeV} \end{aligned} \tag{11}$$

Part (e)

Applying the previous results to $p + p \to p + \Sigma^+ + K^0$:

$$\begin{aligned} E_p &= \frac{\left(m_p + m_{\Sigma^+} + m_{K^0}\right)^2 - m_p^2 - m_p^2}{2m_p} c^2 \\ &\approx 2734.61 \text{ MeV} \end{aligned} \tag{12}$$

Exercise 6

***Introduction to Elementary Particles* (Griffiths, 2e) Exercise 3.18**

For the second collision:

$$\begin{aligned} E_{K^-} &= \frac{(m_{\Omega^-} + m_{K^0} + m_{K^+})^2 - m_p^2 - m_{K^-}^2}{2m_p} c^2 \\ &\approx 3182.32 \text{ MeV} \end{aligned} \tag{13}$$

For the first collision, treating the non-K^- products as a particle P:

$$\begin{aligned} p_{p_1} + p_{p_2} &= p_P + p_{K^-} \\ (p_{p_1} - p_{K^-})^2 &= (p_P - p_{p_2})^2 \\ & m_{p_1}^2 c^2 + m_{K^-}^2 c^2 - 2\left(\frac{E_p}{c}\frac{E_{K^-}}{c} - \mathbf{p}_{p_1} \cdot \mathbf{p}_{K^-}\right) \\ & = m_P^2 c^2 + m_{p_2}^2 c^2 - 2\left(\frac{E_P}{c} m_p c\right) \\ & m_{K^-}^2 c^2 - 2\left(\frac{E_p E_{K^-} - \sqrt{\left(E_p^2 - m_p^2 c^4\right)\left(E_{K^-}^2 - m_{K^-}^2 c^4\right)}}{c^2}\right) \\ & = m_P^2 c^2 - 2m_p(E_p + m_p c^2 - E_{K^-}) \\ & E_p\left(E_{K^-} - m_p c^2\right) + \frac{m_P^2 - m_{K^-}^2 - 2m_p^2 + 2m_p \frac{E_{K^-}}{c^2}}{2} c^4 \\ & = \sqrt{\left(E_p^2 - m_p^2 c^4\right)\left(E_{K^-}^2 - (m_{K^-} c)^2\right)} \end{aligned} \tag{14}$$

Let $a = E_{K^-} - m_p c^2, b = \frac{m_P^2 - m_{K^-}^2 - 2m_p^2 + 2m_p \frac{E_{K^-}}{c^2}}{2} c^4, d = E_{K^-}^2 - (m_{K^-} c)^2, e =$

$m_p^2 c^4$.

$$
\begin{aligned}
a &= E_{K^-} - m_p c^2 \\
&\approx 2244.05 \text{ MeV} \\
b &= \frac{m_P^2 - m_{K^-}^2 - 2m_p^2 + 2m_p \frac{E_{K^-}}{c^2}}{2} c^4 \\
&\approx 4792636.24 \text{ MeV}^2 \\
d &= E_{K^-}^2 - (m_{K^-} c)^2 \\
&\approx 9883440.64 \text{ MeV}^2 \\
e &= (m_p c)^2 \\
&\approx 880350.59 \text{ MeV}^2 \\
aE_p + b &= \sqrt{(E_p^2 - (m_p c)^2) d} \\
a^2 E_p^2 + 2abE_p + b^2 &= (E_p^2 - e) d \\
0 &= (a^2 - d) E_p^2 + (2ab) E_p + (b^2 + de) \\
E_p &= \frac{-2ab \pm \sqrt{(2ab)^2 - 4(a^2 - d)(b^2 + de)}}{2(a^2 - d)} \\
&= -1165.97, 5603.11 \text{ MeV}
\end{aligned}
\tag{15}
$$

The incident proton has minimum kinetic energy $E_p - m_p c^2 = 5603.11 - 938.27 = 4664.84$ MeV.

Exercise 7

***Introduction to Elementary Particles* (Griffiths, 2e) Exercise 3.21**

Applying conservation of the energy-momentum 4-vector:

$$
\begin{aligned}
\mathbf{p}_{\pi^-} &= 0 = \mathbf{p}_{\mu^-} + \mathbf{p}_\nu \\
\mathbf{p}_{\mu^-} &= \mathbf{p} = -\mathbf{p}_\nu \\
E_{\pi^-} &= E_{\mu^-} + E_{\bar{\nu}_\mu} \\
m_{\pi^-}c^2 &= c\sqrt{(m_{\mu^-}c)^2 + \mathbf{p}^2} + |\mathbf{p}|c \\
|\mathbf{p}_\mu| = |\mathbf{p}| &= \frac{m_{\pi^-}^2 - m_{\mu^-}^2}{2m_{\pi^-}}c \\
E_\mu &= \sqrt{m_\mu^2 c^4 + \mathbf{p}^2 c^2} = \frac{m_{\pi^-}^2 + m_{\mu^-}^2}{2m_{\pi^-}}c^2 \\
v_\mu &= \frac{\mathbf{p}_\mu c^2}{E} = \frac{m_{\pi^-}^2 - m_{\mu^-}^2}{m_{\pi^-}^2 + m_{\mu^-}^2}c \\
\gamma &= \frac{1}{\sqrt{1-(\frac{v}{c})^2}} = \frac{1}{\sqrt{1-(\frac{m_{\pi^-}^2 - m_{\mu^-}^2}{m_{\pi^-}^2 + m_{\mu^-}^2})^2}} = \frac{m_{\pi^-}^2 + m_{\mu^-}^2}{2m_{\pi^-}m_{\mu^-}} \\
d = v\gamma\tau &= \frac{m_{\pi^-}^2 - m_{\mu^-}^2}{m_{\pi^-}^2 + m_{\mu^-}^2}c\frac{m_{\pi^-}^2 + m_{\mu^-}^2}{2m_{\pi^-}m_{\mu^-}}\tau \\
&= \frac{m_{\pi^-}^2 - m_{\mu^-}^2}{2m_{\pi^-}m_{\mu^-}}c\tau \approx 185.87 \text{ m}
\end{aligned}
\tag{16}
$$

Exercise 8

***Introduction to Elementary Particles* (Griffiths, 2e) Exercise 3.27**

Applying conservation of the energy-momentum 4-vector:

$$
\begin{aligned}
p\sin\phi &= p'_\gamma \sin\theta \\
\sin\phi &= \frac{E'}{pc}\sin\theta \\
p_\gamma &= p\cos\phi + p'_\gamma\cos\theta \\
\frac{E}{c} &= p\sqrt{1-(\frac{E'}{pc}\sin\theta)^2} + \frac{E'}{c}\cos\theta \\
&= \sqrt{p^2-(\frac{E'}{c}\sin\theta)^2} + \frac{E'}{c}\cos\theta \\
p &= (\frac{E}{c}-\frac{E'}{c}\cos\theta)^2 + (\frac{E'}{c}\sin\theta)^2 \\
p^2c^2 &= E^2 - 2EE'\cos\theta + E'^2 \\
E + mc^2 &= \sqrt{m^2c^4+p^2c^2} + E' \\
E + mc^2 - E' &= \sqrt{m^2c^4+E^2-2EE'\cos\theta+E'^2} \\
-EE' + (E-E')mc^2 &= -EE'\cos\theta \\
E' &= \frac{Emc^2}{E(1-\cos\theta)+mc^2} \\
\frac{hc}{\lambda'} &= \frac{\frac{hc}{\lambda}mc^2}{\frac{hc}{\lambda}(1-\cos\theta)+mc^2} \\
\lambda' &= \frac{hc(1-\cos\theta)+mc^2\lambda}{mc^2} \\
&= \lambda + \frac{h}{mc}(1-\cos\theta)
\end{aligned}
\tag{17}
$$