HW 5

Ravi Kini

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Exercise 1

Let G be a group generated by some $x, y \in G$, let $N \trianglelefteq G$, and let $\bar{G} = G/N$. Let $g \in G$. Since G is generated by $x, y \in G$, for some $g_1, g_2, \ldots g_n \in \left\{x, y, x^{-1}, y^{-1}\right\}$, $g = g_1 g_2 \ldots g_n$. Consequently, for $gN \in \overline{G}$:

$$\begin{aligned} gN &= (g_1 g_2 \ldots g_n) N \\ &= (g_1 N)(g_2 N) \ldots (g_n N) \\ &= \overline{g}_1 \overline{g}_2 \ldots \overline{g}_n \end{aligned} \tag{1}$$

Evidently, every element in $\overline{G}$ can be expressed as $\overline{g}_1 \overline{g}_2 \ldots \overline{g}_n$ for some $\overline{g}_1, \overline{g}_2 \ldots \overline{g}_n \in \left\{\overline{x}, \overline{y}, \overline{x^{-1}}, \overline{y^{-1}}\right\}$. Therefore $\overline{G}$ is generated by $\overline{x}, \overline{y} \in \overline{G}$.

Exercise 2

***Algebra* (Artin, 2e) Exercise 2.12.2 (partial)**

Part (a)

Evidently, the identity element is in H, as it is the element of H where $a, b, c = 0$. We then compute the product of two arbitrary elements of H:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+a & y+az+b \\ 0 & 1 & z+c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} \in H \tag{2}$$

From the above computation, the group exhibits closure, as the product is the element of H where $a' = x + a, b' = y + az + b, c' = z + c$. From the above computation, the group has inverses, as when $x = -a, y = ac - b, z = -c$, the product is the identity. Since H satisfies the identity, closure, and inverse properties, $H \leq GL_3(\mathbb{F})$.

Part (b)

We compute the following product:

$$\begin{aligned} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} &= \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a & y+b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in K \end{aligned} \tag{3}$$

Evidently, for arbitrary $h \in H$ and $k \in K$, $hkh^{-1} \in K$, so $K \trianglelefteq H$.

Part (c)

Let $A = \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{bmatrix} \in H$. From the computation done in (b), we see that A and B are in the same coset of K when $a_1 = b_1$ and $a_3 = b_3$. We can then construct the surjective homomorphism mapping $h = \begin{bmatrix} 1 & h_1 & h_2 \\ 0 & 1 & h_3 \\ 0 & 0 & 1 \end{bmatrix} \in H$ to $\overline{h} = \overline{\begin{bmatrix} 1 & h_1 & 0 \\ 0 & 1 & h_3 \\ 0 & 0 & 1 \end{bmatrix}} \in \overline{H}$. Evidently, K is the kernel of this homomorphism, as it maps to the identity matrix.

Exercise 3

Part (a)

Let the Klein four group $V = \{1, a, b, ab\} = \langle a, b \ : \ a^2 = b^2 = [a, b] = 1\rangle \cong C_2 \times C_2$. Let the subgroup $N = \{e, (1\ 2)\,(3\ 4), (1\ 3)\,(2\ 4), (1\ 4)\,(2\ 3)\}$ in S_4. We construct the bijective mapping $\varphi : V \to N$ where $1 \to e$, $a \to (1\ 2)\,(3\ 4)$, $b \to (1\ 3)\,(2\ 4)$, $ab \to (1\ 4)\,(2\ 3)$. To show that this is a homomorphism, we construct Cayley tables:

	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

	e	$(1\ 2)\,(3\ 4)$	$(1\ 3)\,(2\ 4)$	$(1\ 4)\,(2\ 3)$
e	e	$(1\ 2)\,(3\ 4)$	$(1\ 3)\,(2\ 4)$	$(1\ 4)\,(2\ 3)$
$(1\ 2)\,(3\ 4)$	$(1\ 2)\,(3\ 4)$	e	$(1\ 4)\,(2\ 3)$	$(1\ 3)\,(2\ 4)$
$(1\ 3)\,(2\ 4)$	$(1\ 3)\,(2\ 4)$	$(1\ 4)\,(2\ 3)$	e	$(1\ 2)\,(3\ 4)$
$(1\ 4)\,(2\ 3)$	$(1\ 4)\,(2\ 3)$	$(1\ 3)\,(2\ 4)$	$(1\ 2)\,(3\ 4)$	e

By inspecting the Cayley tables, we see that $\varphi(x)\,\varphi(y) = \varphi(xy)$ for all $x, y \in V$. As φ is a bijective homomorphism, it is an isomorphism and as an isomorphism exists between V and N, N is isomorphic to the Klein four group.

Part (b)

The subgroup N is composed of 1 permutation of cycle type $1, 1, 1, 1$ and permutations of cycle type $2, 2$. We note that e is the only permutation of cycle type $1, 1, 1, 1$, as $\frac{\frac{P_1^4}{1}\frac{P_1^3}{1}\frac{P_1^1}{1}\frac{P_1^1}{1}}{4!} = 1$, and that $(1\ 2)\,(3\ 4), (1\ 3)\,(2\ 4), (1\ 4)\,(2\ 3)$ are the only permutations of cycle type $2, 2$, as $\frac{\frac{P_2^4}{2}\frac{P_2^2}{2}}{2!} = 3$. As conjugation preserves cycle type, e must be mapped back to itself, the only permutation of its cycle type, and $(1\ 2)\,(3\ 4), (1\ 3)\,(2\ 4), (1\ 4)\,(2\ 3)$ must be mapped to cycles of type $2, 2, 2$, all of which are elements of V. Since for all $p \in V$ and all $q \in S_4$, $qpq^{-1} \in V$, $V \trianglelefteq S_4$.

Part (c)

Let $H = \langle (1\ 2), (3\ 4)\rangle \leq S_4$. We note that $H = \langle (1\ 2), (3\ 4)\rangle = \{e, (1\ 2), (3\ 4), (1\ 2)\,(3\ 4)\}$. We construct the bijective mapping $\varphi : V \to H$ where $1 \to e$, $a \to (1\ 2)$, $b \to (3\ 4)$, $ab \to (1\ 2)\,(3\ 4)$. To show that this is a homomorphism, we construct Cayley tables:

	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

	e	(1 2)	(3 4)	(1 2) (3 4)
e	e	(1 2)	(3 4)	(1 2) (3 4)
(1 2)	(1 2)	e	(1 2) (3 4)	(1 2)
(3 4)	(3 4)	(1 2) (3 4)	e	(3 4)
(1 2) (3 4)	(1 2) (3 4)	(3 4)	(1 2)	e

By inspecting the Cayley tables, we see that $\varphi(x)\varphi(y) = \varphi(xy)$ for all $x, y \in V$. As φ is a bijective homomorphism, it is an isomorphism and as an isomorphism exists between V and H, H is isomorphic to the Klein four group.

However, H is not normal in S_4. To see this, conjugate (1 2) by (1 2 3 4). Then:

$$\begin{aligned} (1\ 2\ 3\ 4)(1\ 2)(1\ 2\ 3\ 4)^{-1} &= (1\ 2\ 3\ 4)(1\ 2)(4\ 3\ 2\ 1) \\ &= (1\ 2\ 3\ 4)(1\ 4\ 3) \\ &= (2\ 3) \end{aligned} \tag{4}$$

Since $(2\ 3) \notin H$, evidently H is not normal in S_4.

Part (d)

We compute the cosets of N in S_4.

$$\begin{aligned} eN &= N & (1\ 2)(3\ 4)\,N &= N \\ (1\ 3)(2\ 4)\,N &= N & (1\ 4)(2\ 3)\,N &= N \end{aligned}$$

$$\begin{aligned}
(1\ 2)\,N &= \{(1\ 2),(3\ 4),(1\ 4\ 2\ 3),(1\ 3\ 2\ 4)\} & (3\ 4)\,N &= \{(3\ 4),(1\ 2),(1\ 4\ 2\ 3),(1\ 3\ 2\ 4)\} \\
(1\ 3\ 2\ 4)\,N &= \{(1\ 3\ 2\ 4),(1\ 4\ 2\ 3),(3\ 4),(1\ 2)\} & (1\ 4\ 2\ 3)\,N &= \{(1\ 4\ 2\ 3),(1\ 3\ 2\ 4),(1\ 2),(3\ 4)\} \\[1ex]
(2\ 3)\,N &= \{(2\ 3),(1\ 3\ 4\ 2),(1\ 2\ 4\ 3),(1\ 4)\} & (1\ 3\ 4\ 2)\,N &= \{(1\ 3\ 4\ 2),(2\ 3),(1\ 4),(1\ 2\ 4\ 3)\} \\
(1\ 2\ 4\ 3)\,N &= \{(1\ 2\ 4\ 3),(1\ 4),(2\ 3),(1\ 3\ 4\ 2)\} & (1\ 4)\,N &= \{(1\ 4),(1\ 2\ 4\ 3),(1\ 3\ 4\ 2),(2\ 3)\} \\[1ex]
(1\ 3\ 2)\,N &= \{(1\ 3\ 2),(2\ 3\ 4),(1\ 2\ 4),(1\ 4\ 3)\} & (2\ 3\ 4)\,N &= \{(2\ 3\ 4),(1\ 3\ 2),(1\ 4\ 3),(1\ 2\ 4)\} \\
(1\ 4\ 3)\,N &= \{(1\ 4\ 3),(1\ 2\ 4),(2\ 3\ 4),(1\ 3\ 2)\} & (1\ 2\ 4)\,N &= \{(1\ 2\ 4),(1\ 4\ 3),(1\ 3\ 2),(2\ 3\ 4)\} \\[1ex]
(2\ 4\ 3)\,N &= \{(2\ 4\ 3),(1\ 4\ 2),(1\ 2\ 3),(1\ 3\ 4)\} & (1\ 4\ 2)\,N &= \{(1\ 4\ 2),(2\ 4\ 3),(1\ 3\ 4),(1\ 2\ 3)\} \\
(1\ 2\ 3)\,N &= \{(1\ 2\ 3),(1\ 3\ 4),(2\ 4\ 3),(1\ 4\ 2)\} & (1\ 3\ 4)\,N &= \{(1\ 3\ 4),(1\ 2\ 3),(1\ 4\ 2),(2\ 4\ 3)\} \\[1ex]
(1\ 3)\,N &= \{(1\ 3),(1\ 2\ 3\ 4),(2\ 4),(1\ 4\ 3\ 2)\} & (1\ 4\ 3\ 2)\,N &= \{(1\ 4\ 3\ 2),(2\ 4),(1\ 2\ 3\ 4),(1\ 3)\} \\
(2\ 4)\,N &= \{(2\ 4),(1\ 4\ 3\ 2),(1\ 3),(1\ 2\ 3\ 4)\} & (1\ 2\ 3\ 4)\,N &= \{(1\ 2\ 3\ 4),(1\ 3),(1\ 4\ 3\ 2),(2\ 4)\}
\end{aligned}$$

Consequently, we define $S_4/N = \left\{\overline{e}, \overline{(1\ 2)}, \overline{(2\ 3)}, \overline{(1\ 3\ 2)}, \overline{(1\ 2\ 3)}, \overline{(1\ 3)}\right\}$, noting that each coset has exactly one element where 4 permutes to itself. We then define the biijection $\varphi : S_4/N \to S_3$ that maps $\overline{\sigma} \to \sigma$. It is easily verified that φ is a homomorphism and that N is the kernel of the homomorphism. As φ is a bijective homomorphism, φ is an isomorphism and as an isomorphism exists between S_4/N and S_3, $S_4/N \cong S_3$.

Part (e)

By the Correspondence Theorem, since φ is a subjective homomorphism of which N is the kernel, there exists a bijection between the subgroups of S_4 containing N and the subgroups of S_3. We note that the inverse of an element of S_3 (and, in general, S_n) can be found by reversing the order of the individual cycles that compose the permutation (e.g. $(1\ 2\ 3)^{-1} = (3\ 2\ 1) = (1\ 3\ 2)$). Consequently, the three permutations of cycle type $2,1$ are their own inverses, and the two permutations of cycle type 3 are inverses of each other (the identity permutation of cycle type $1,1,1$ is, of course, its own inverse). Further, the composition of two different permutations of cycle type $2,1$ is a permutation of cycle type 3. We then enumerate the subgroups of S_3 as such: the trivial subgroup, three subgroups containing the identity and one of the permutations of cycle type $2, 1$, the subgroup containing the identity and both permutations of cycle type 3, and S_3 itself. As there exist 6 subgroups of S_3, there are therefore 6 subgroups of S_4 that contain N.