HW 8

Ravi Kini

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Exercise 1

***Algebra* (Artin, 2e) Exercise 6.8.1**

Let $P * A := PAP^T$ for $P \in GL_n, A \in M_{n\times n}$. The identity of GL_n is I; since $IAI^T = IAI = AI = A$ for arbitrary $A \in M_{n\times n}$, the identity property is satisfied. Let $P_1, P_2 \in GL_n$ and $A \in M_{n\times n}$. Then:

$$\begin{aligned}(P_1P_2) * A &= (P_1P_2)\, A\, (P_1P_2)^T \\ &= (P_1P_2)\, A \left(P_2^T P_1^T\right) \\ &= P_1 \left(P_2 A P_2^T\right) P_1^T \\ &= P_1 \left(P_2 * A\right) P_1^T \\ &= P_1 * (P_2 * A)\end{aligned} \tag{1}$$

Consequently, the associativity property is fulfilled; $*$ is therefore an operation of GL_n on $M_{n\times n}$.

Exercise 2

***Algebra* (Artin, 2e) Exercise 6.8.2**

Let group $G, H \leq G$, and consider the operation of G on G/H. Let some arbitary $k \in aHa^{-1}$. Then $k = aha^{-1}$ for some $h \in H$. Then:

$$\begin{aligned} kaH &= aha^{-1}aH \\ &= ahH \\ &= aH \end{aligned} \tag{2}$$

Evidently k is a stabilizer of aH, so $aHa^{-1} \subseteq \mathrm{stab}_G(aH)$. Now let some arbitrary $l \in \mathrm{stab}_G(aH)$. Then:

$$\begin{aligned} laH &= aH \\ a^{-1}laH &= H \end{aligned} \tag{3}$$

Consequently, $a^{-1}la = h$ for some $h \in H$, which means that $l = aha^{-1}$ and $\mathrm{stab}_G(aH) \subseteq aHa^{-1}$. Evidently, $\mathrm{stab}_G(aH) = aHa^{-1}$.

Exercise 3

***Algebra* (Artin, 2e) Exercise 6.8.3**

Let G be the dihedral group D_4 and S be the set of vertices of a square. A given vertex s has two stabilizers: e and either yx or yx^3 (the identity transformation and reflection about the diagonal through the vertex). Without loss of generality, let the second stabilizer be yx. A similar proof proceeds if the second stabilizer is yx^3. The cosets of H are then:

$$\begin{aligned} eH &= H & xH &= \{x, y\} \\ x^2H &= \left\{x^2, yx^3\right\} & x^3H &= \left\{x^3, yx^2\right\} \\ yH &= \{x, y\} & yxH &= H \\ yx^2H &= \left\{x^3, yx^2\right\} & yx^3H &= \left\{x^2, yx^3\right\} \end{aligned} \tag{4}$$

Consequently, we define $D_4/4 = \left\{\overline{e}, \overline{x}, \overline{x^3}, \overline{x^4}\right\}$, noting that each coset has exactly one element that can be reduced to a form that does not contain y. The orbit of the vertex O_s is S, as it can be mapped to any other vertex through an arbitrary number of rotations, such that $S = \left\{s, xs, x^2s, x^3s\right\}$. We then construct the bijective mapping $\varepsilon : D_4/H \to S$ that maps $\overline{d} \to ds$.

Exercise 4

***Algebra* (Artin, 2e) Exercise 6.9.2 (extended)**

Part (a)

Let G be the group of **rotational symmetries** of the cube. Let V, E, and F denote the sets of vertices, edges, and faces of the cube, respectively. Check for yourself that the size of these sets are

$$|V| = 8 \quad |E| = 12 \quad |F| = 6. \tag{5}$$

The stabilizer of a particular vertex G_v is the group of rotations by multiples of $\frac{2\pi}{3}$ about the vertex, which has order 3. The orbit of a particular vertex is the set of all vertices, which has order 8. The stabilizer of a particular edge G_e is the group of rotations by multiples of $\frac{2\pi}{2} = \pi$ about the center of the edge, which has order 2. The orbit of a particular edge is the set of all edges, which has order 12. The stabilizer of a particular face G_f is the group of rotations by multiples of $\frac{2\pi}{4} = \frac{\pi}{2}$ about the center of the face, which has order 4. The orbit of a particular face is the set of all faces, which has order 6. By the counting formula, $|G| = 3 \cdot 8 = 2 \cdot 12 = 4 \cdot 6 = 24$.

Part (b)

Let G_v, G_e, G_f be the stabilizers of a vertex v, and edge e, and a face f of the cube. Let the columns G_v, G_e, G_f be the groups and the rows V, E, F be the sets in the group action. Each cell represents the partition of the set in the corresponding row into orbits under the group action of the group in the corresponding column on the set in the corresponding row.

	G_v	G_e	G_f
V	$8 = 1+1+3+3$	$8 = 2+2+2+2$	$8 = 4+4$
E	$12 = 3+3+3+3$	$12 = 1+1+2+2+2+2+2$	$12 = 4+4+4$
F	$6 = 3+3$	$6 = 2+2+2$	$6 = 1+1+4$