

# Homework 1

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## Exercise 1

### Part (a)

The matrices

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \quad y = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$$
$$r = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are the generators of the group  $G$  of rotations of  $\theta = \frac{\pi}{4}$  about the conventional  $x$ -,  $y$ -, and  $z$ - axes, respectively. Since  $\theta = \frac{\pi}{4}$ ,  $\cos \theta = \sin \theta = \frac{1}{\sqrt{2}} := t$ , we can rewrite our matrices as:

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & t \\ 0 & -t & t \end{pmatrix} \quad y = \begin{pmatrix} t & 0 & -t \\ 0 & 1 & 0 \\ t & 0 & t \end{pmatrix} \quad r = \begin{pmatrix} t & -t & 0 \\ t & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We seek to find matrices representing  $p^{-1}$  and  $y^{-1}$ ; spatial reasoning indicates that the inverse of a rotation by an angle  $\theta$  would be a rotation by an angle  $-\theta$ . The matrices representing  $p^{-1}$  and  $y^{-1}$  are then:

$$p^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos -\theta & \sin -\theta \\ 0 & -\sin -\theta & \cos -\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & -t \\ 0 & t & t \end{pmatrix} \quad (1)$$
$$y^{-1} = \begin{pmatrix} \cos -\theta & 0 & -\sin -\theta \\ 0 & 1 & 0 \\ \sin -\theta & 0 & \cos -\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} = \begin{pmatrix} t & 0 & t \\ 0 & 1 & 0 \\ -t & 0 & t \end{pmatrix}$$

To verify that these are the inverses of  $p$  and  $y$ , respectively, we multiply  $p$  and  $p^{-1}$ , and  $y$  and  $y^{-1}$ .

$$\begin{aligned} pp^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & t \\ 0 & -t & t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & -t \\ 0 & t & t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2t^2 & 0 \\ 0 & 0 & 2t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ yy^{-1} &= \begin{pmatrix} t & 0 & -t \\ 0 & 1 & 0 \\ t & 0 & t \end{pmatrix} \begin{pmatrix} t & 0 & t \\ 0 & 1 & 0 \\ -t & 0 & t \end{pmatrix} = \begin{pmatrix} 2t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \quad (2)$$

As they multiply to the  $3 \times 3$  identity matrix, they must be inverses.

### Part (b)

We now compute  $p^2y^{-1}p^{-2}$  and  $py^{-1}p^{-1}$ :

$$\begin{aligned} p^2y^{-1}p^{-2} &= \begin{pmatrix} t & -t & 0 \\ t & t & 0 \\ 0 & 0 & 1 \end{pmatrix} = r \\ py^{-1}p &= \begin{pmatrix} t & -t^2 & -t^2 \\ -t^2 & t^2 - t^3 & -t^2 - t^3 \\ t^2 & t^2 + t^3 & -t^2 + t^3 \end{pmatrix} \neq r \end{aligned} \quad (3)$$

$p^2y^{-1}p^{-2} = r$ , but  $py^{-1}p^{-1} \neq r$ .

## Exercise 2

### Part (a)

The set of transpositions  $\{\tau_1, \tau_2, \tau_3\}$  generate  $S_4$ , where  $\tau_i = (i \ i+1)$ . The permutations

$$p = (1 \ 2 \ 3 \ 4) \quad q = (1 \ 3 \ 2 \ 4) \quad r = (1 \ 4 \ 2)$$

can therefore be written as products of (adjacent) transpositions  $\tau_i$  in the following way:

$$\begin{aligned} \tau_1 \circ \tau_2 \circ \tau_3 &= \tau_1 \circ (2 \ 3) \circ (3 \ 4) \\ &= (1 \ 2) \circ (2 \ 3 \ 4) \\ &= (1 \ 2 \ 3 \ 4) = p \\ \tau_2 \circ \tau_1 \circ \tau_3 \circ \tau_2 \circ \tau_3 &= \tau_2 \circ \tau_1 \circ \tau_3 \circ (2 \ 3) \circ (3 \ 4) \\ &= \tau_2 \circ \tau_1 \circ (3 \ 4) \circ (2 \ 3 \ 4) \\ &= \tau_2 \circ (1 \ 2) \circ (2 \ 4) \\ &= (2 \ 3) \circ (1 \ 2 \ 4) \\ &= (1 \ 3 \ 2 \ 4) = q \\ \tau_2 \circ \tau_3 \circ \tau_2 \circ \tau_1 &= \tau_2 \circ \tau_3 \circ (2 \ 3) \circ (1 \ 2) \\ &= \tau_2 \circ (3 \ 4) \circ (1 \ 3 \ 2) \\ &= (2 \ 3) \circ (1 \ 4 \ 3 \ 2) \\ &= (1 \ 4 \ 2) = r \end{aligned} \tag{4}$$

### Part (b)

The symmetric group  $S_4$  is generated by the set of (adjacent) transpositions  $\{\tau_1, \tau_2, \tau_3\}$ . Letting  $p := (1 \ 2 \ 3 \ 4)$ ,  $s := (1 \ 2)$ .  $\{\tau_1, \tau_2, \tau_3\}$  can be generated by  $p, s$  as follows:

$$\begin{aligned} s &= (1 \ 2) = \tau_1 \\ p \circ p \circ s \circ p \circ s &= p \circ p \circ s \circ (1 \ 2 \ 3 \ 4) \circ (1 \ 2) \\ &= p \circ p \circ (1 \ 2) \circ (1 \ 3 \ 4) \\ &= p \circ (1 \ 2 \ 3 \ 4) \circ (1 \ 3 \ 4 \ 2) \\ &= (1 \ 2 \ 3 \ 4) \circ (1 \ 4 \ 3) \\ &= (2 \ 3) = \tau_2 \\ p \circ p \circ s \circ p \circ p &= p \circ p \circ s \circ (1 \ 2 \ 3 \ 4) \circ (1 \ 2 \ 3 \ 4) \\ &= p \circ p \circ (1 \ 2) \circ (1 \ 3) (2 \ 4) \\ &= p \circ (1 \ 2 \ 3 \ 4) \circ (1 \ 3 \ 2 \ 4) \\ &= (1 \ 2 \ 3 \ 4) \circ (1 \ 4 \ 2) \\ &= (3 \ 4) = \tau_3 \end{aligned} \tag{5}$$

Evidently,  $\{\tau_1, \tau_2, \tau_3\}$  is generated by  $\{(1\ 2), (1\ 2\ 3\ 4)\}$ . Since  $S_4$  is generated by  $\{\tau_1, \tau_2, \tau_3\}$ ,  $S_4$  is generated by  $\{(1\ 2), (1\ 2\ 3\ 4)\}$ .

### Exercise 3

#### Part (a)

For some  $n \times n$  permutation matrix  $P$ , the product  $PP^T$  is defined such that:

$$\begin{aligned}(PP^T)_{ij} &= \sum_{k=1}^n P_{ik}P_{kj}^T \\ &= \sum_{k=1}^n P_{ik}P_{jk}\end{aligned}\tag{6}$$

Note that permutation matrices, by definition, have a single 1 in each row and in each column, and 0 for the remaining entries. The above expression then simplifies to:

$$(PP^T)_{ij} = \delta_{ij}\tag{7}$$

As every entry in the product is then equal to the corresponding entry in the identity matrix,  $PP^T = I$ . Therefore  $P^{-1} = P^T$ , and the transpose of a permutation matrix is its inverse.

#### Part (b)

For some  $n \times n$  permutation matrix  $P$ ,  $\det(P^T) = \det(P)$ . Then:

$$\begin{aligned}PP^{-1} &= PP^T = I \\ \det(PP^T) &= \det(I) \\ \det(P) \cdot \det(P^T) &= 1 \\ \det(P) \cdot \det(P) &= \det(P)^2 = 1 \\ 1 - \det(P)^2 &= (1 + \det(P))(1 - \det(P)) = 0 \\ \det(P) &= \pm 1\end{aligned}\tag{8}$$

Therefore the determinant of a permutation matrix is always  $\pm 1$ .

#### Part (c)

The identity matrix, having determinant 1, is an even permutation. We can then express  $p$  as:

$$p = \tau_{i_1} \circ \tau_{i_2} \circ \dots \circ \tau_{i_k} \circ I\tag{9}$$

Since  $\tau_{i_k} \circ I$  is obtained from  $I$  by interchanging two different rows one time,  $\det(\tau_{i_k} \circ I) = -\det(I) = -1$ . This can be extended until  $p$ , which is obtained from  $I$  by interchanging two different rows  $k$  times, which means that  $\det(p) =$

$(-1)^k \det(I) = (-1)^k$ . Evidently,  $\operatorname{sgn}(p) = \det(p) = 1$  when  $k$  is even, and  $\operatorname{sgn}(p) = \det(p) = -1$  when  $k$  is odd. Therefore  $p$  is even if  $k$  is even, and  $p$  is odd if  $k$  is odd.

Let there be some even  $p$  and suppose that  $k$  is odd. Since  $p = \tau_{i_1} \circ \tau_{i_2} \circ \dots \circ \tau_{i_k} \circ I$  is obtained from  $I$  by interchanging two different rows  $k$  times,  $\det(p) = (-1)^k \det(I) = (-1)^k$ . Since  $k$  is odd,  $\operatorname{sgn}(p) = \det(p) = -1$ . However, this is a contradiction; as  $p$  is even,  $\operatorname{sgn}(p) = 1$ . Consequently  $k$  must be even. Now let there be some odd  $p$  and suppose that  $k$  is even. Since  $p = \tau_{i_1} \circ \tau_{i_2} \circ \dots \circ \tau_{i_k} \circ I$  is obtained from  $I$  by interchanging two different rows  $k$  times,  $\det(p) = (-1)^k \det(I) = (-1)^k$ . Since  $k$  is even,  $\operatorname{sgn}(p) = \det(p) = 1$ . However, this is a contradiction; as  $p$  is odd,  $\operatorname{sgn}(p) = -1$ . Consequently  $k$  must be odd.

Therefore  $k$  is even if  $p$  is even, and  $k$  is odd if  $p$  is odd, which means that  $p$  is even iff  $k$  is even, and  $p$  is odd iff  $k$  is odd.