Homework 8

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Exercise 1

An Introduction to Thermal Physics **(Schroeder, 1e) Exercise 7.33 (partial)**

Part (a)

The density of states is seen to be symmetric about ϵ_F. Further, as:

$$\begin{aligned}\overline{n}_{FD}(\epsilon) = \frac{1}{e^{\frac{\epsilon-\mu}{k_B T}}+1} &= 1 - \frac{e^{\frac{\epsilon-\mu}{k_B T}}}{e^{\frac{\epsilon-\mu}{k_B T}}+1} \\ &= 1 - \frac{1}{1+e^{-\frac{\epsilon-\mu}{k_B T}}} \\ &= 1 - \frac{1}{e^{\frac{(2\mu-\epsilon)-\mu}{k_B T}}+1} \\ &= 1 - \overline{n}_{FD}(2\mu-\epsilon)\end{aligned} \tag{1}$$

Evidently, the probability of a state at ϵ being occupied is equal to the probability of a state at $2\mu - \epsilon$ being unoccupied. These states are symmetric about μ. For a semiconductor at nonzero temperatures, there will be some electrons in the conduction band and an equal number of holes in the valence band. Consequently, letting $\epsilon_c = \epsilon_F + \epsilon'$:

$$\begin{aligned}\int_{\epsilon_F+\epsilon'}^{\infty} g_0\sqrt{\epsilon-(\epsilon_F+\epsilon')}\,\overline{n}_{FD}(\epsilon)\ \mathrm{d}\epsilon &= -\int_{\epsilon_F-\epsilon'}^{-\infty} g_0\sqrt{(\epsilon_F-\epsilon')-\epsilon}\,(1-\overline{n}_{FD}(\epsilon))\ \mathrm{d}\epsilon \\ &= \int_{2\mu-\epsilon_F+\epsilon'}^{\infty} g_0\sqrt{\epsilon-(2\mu-\epsilon_F+\epsilon')}\,(1-\overline{n}_{FD}(2\mu-\epsilon))\ \mathrm{d}\epsilon\end{aligned} \tag{2}$$

This is only true when $\epsilon_F + \epsilon' = 2\mu - \epsilon_F + \epsilon'$, or $\mu = \epsilon_F$, which means that the chemical potential is exactly in the middle of the gap.

Part (b)

For gap width $\delta\epsilon = 2\epsilon'$:

$$
\begin{aligned}
N &= \int_{\epsilon_c}^{\infty} g\left(\epsilon\right) \overline{n}_{FD}\left(\epsilon\right)\, \mathrm{d}\epsilon \\
&= g_0 \int_{\epsilon_c}^{\infty} \sqrt{\epsilon - \epsilon_c} \frac{1}{e^{\frac{\epsilon-\epsilon_F}{k_B T}} + 1}\, \mathrm{d}\epsilon \\
&\approx g_0 \int_{\epsilon_c}^{\infty} \sqrt{\epsilon - \epsilon_c} e^{-\frac{\epsilon-\epsilon_F}{k_B T}}\, \mathrm{d}\epsilon \\
&\approx g_0 e^{-\frac{\epsilon'}{k_B T}} \int_{\epsilon_c}^{\infty} \sqrt{\epsilon - \epsilon_c} e^{-\frac{\epsilon-\epsilon_c}{k_B T}}\, \mathrm{d}\epsilon \\
&\approx g_0 e^{-\frac{\epsilon'}{k_B T}} \left(k_B T\right)^{\frac{3}{2}} \int_{\epsilon_c}^{\infty} \sqrt{x} e^{-x}\, \mathrm{d}\epsilon \\
&\approx \frac{\pi (8m)^{\frac{3}{2}}}{2h^3} V e^{-\frac{\epsilon'}{k_B T}} \left(k_B T\right)^{\frac{3}{2}} \sqrt{\pi} 2 \\
\frac{N}{V} &\approx 2\left(\frac{2\pi k_B T m}{h^2}\right)^{\frac{3}{2}} e^{-\frac{\epsilon'}{k_B T}} \\
&\approx \frac{2}{v_Q} e^{-\frac{\delta\epsilon}{2k_B T}}
\end{aligned}
\tag{3}
$$

Part (c)

$$
\begin{aligned}
\frac{N}{V}_{\text{Si}} &\approx 2\left(\frac{2\pi k_B T m}{h^2}\right)^{\frac{3}{2}} e^{-\frac{\delta\epsilon_{\text{Si}}}{2k_B T}} \\
&\approx 2\left(\frac{2\pi \cdot 1.381 \cdot 10^{-23}\ \text{J/K} \cdot 298\ \text{K} \cdot 9.11 \cdot 10^{-31}\ \text{kg}}{\left(6.63 \cdot 10^{-34}\ \text{J s}\right)^2}\right)^{\frac{3}{2}} e^{-\frac{1.11\ \text{eV}}{2 \cdot 8.617 \cdot 10^{-5}\ \text{eV/K} \cdot 298\ \text{K}}} \\
&\approx 1.018 \cdot 10^{16}\ \frac{\text{e}}{\text{m}^3} \approx 1.018 \cdot 10^{10}\ \frac{\text{e}}{\text{cm}^3} \\
\frac{N}{V}_{Cu} &\approx \frac{N_A}{V_{\text{mol}}} \approx \frac{6.022 \cdot 10^{23}}{\frac{63.5}{8.93}\ \text{cm}^3} \approx 8.469 \cdot 10^{22}\ \frac{\text{e}}{\text{cm}^3}
\end{aligned}
\tag{4}
$$

Conduction electrons are far more dense in copper in comparison to silicon, so copper conducts electricity several times better than silicon.

Exercise 2

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 7.41**

Part (a)

The number of atoms N_1 in state s_1 obeys the differential equation:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = AN_2 - Bu\left(f\right)N_1 + B'u\left(f\right)N_2 \tag{5}$$

Part (b)

At equilibrium, $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$, with $\frac{N_2}{N_1} = e^{-\frac{E(s_2)-E(s_1)}{k_B T}} = e^{-\frac{\epsilon}{k_B T}} = e^{-\frac{hf}{k_B T}}$.

$$\begin{aligned}
\frac{dN_1}{dt} &= 0 = AN_2 - Bu\left(f\right)N_1 + B'u\left(f\right)N_2 \\
\frac{N_1}{N_2} = e^{\frac{hf}{k_B T}} &= \frac{A + B'u\left(f\right)}{Bu\left(f\right)} \\
&= \frac{\frac{A}{u(f)} + B'}{B} \\
\frac{A}{Be^{\frac{hf}{k_B T}} - B'} &= \frac{8\pi h}{c^3}\frac{f^3}{e^{\frac{hf}{k_B T}} - 1} \\
\frac{A}{B}\frac{1}{e^{\frac{hf}{k_B T}} - \frac{B'}{B}} &= \frac{8\pi h f^3}{c^3}\frac{1}{e^{\frac{hf}{k_B T}} - 1}
\end{aligned} \tag{6}$$

Evidently, $\frac{A}{B} = \frac{8\pi hf^3}{c^3}$ and $B = B'$.

Exercise 3

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 7.44**

Part (a)

The number of photons in equilibrium N in a box of volume V and temperature T is:

$$\begin{aligned} N &= 2 \sum_{n_x, n_y, n_z} \frac{1}{e^{\frac{hcn}{2Lk_B T}} - 1} \\ &= 2(\frac{4\pi}{8}) \int_0^\infty \frac{n^2}{e^{\frac{hcn}{2Lk_B T}} - 1} \, \mathrm{d}n \\ &= \pi \left(\frac{2Lk_B T}{hc} \right)^3 \int_0^\infty \frac{x^2}{e^x - 1} \, \mathrm{d}x \\ &= 8\pi V \left(\frac{k_B T}{hc} \right)^3 \int_0^\infty \frac{x^2}{e^x - 1} \, \mathrm{d}x \end{aligned} \tag{7}$$

Part (b)

The entropy per photon $\frac{S}{N}$ is:

$$\begin{aligned} \frac{S}{N} &= \frac{\frac{32\pi^5}{45} V k_B \left(\frac{k_B T}{hc} \right)^3}{8\pi V \left(\frac{k_B T}{hc} \right)^3 \int_0^\infty \frac{x^2}{e^x - 1} \, \mathrm{d}x} \\ &= k_B \frac{\frac{4\pi^4}{45}}{\int_0^\infty \frac{x^2}{e^x - 1} \, \mathrm{d}x} \\ &\approx 3.602 k_B \end{aligned} \tag{8}$$

There is about $3.602k_B$ entropy per photon.

Part (c)

The photons per cubic meter at 300 K is:

$$\begin{aligned} \frac{N}{V} (T = 300 \text{ K}) &= 8\pi \left(\frac{k_B T}{hc} \right)^3 \int_0^\infty \frac{x^2}{e^x - 1} \, \mathrm{d}x \\ &\approx 8\pi \left(\frac{1.381 \cdot 10^{-23} \text{ J/K} \cdot 300 \text{ K}}{6.63 \cdot 10^{-34} \text{ J s} \cdot 3 \cdot 10^8 \text{ m/s}} \right)^3 (2.404) \\ &\approx 5.460 \cdot 10^{14} \frac{\gamma}{\text{m}^3} \end{aligned} \tag{9}$$

The photons per cubic meter at 1500 K is:

$$\begin{aligned}
\frac{N}{V}\,(T = 1500\ \mathrm{K}) &= 8\pi\left(\frac{k_B T}{hc}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1}\ \mathrm{d}x \\
&\approx 8\pi\left(\frac{1.381 \cdot 10^{-23}\ \mathrm{J/K} \cdot 1500\ \mathrm{K}}{6.63 \cdot 10^{-34}\ \mathrm{J\,s} \cdot 3 \cdot 10^{8}\ \mathrm{m/s}}\right)^3 (2.404) \\
&\approx 6.825 \cdot 10^{16}\ \frac{\gamma}{\mathrm{m}^3}
\end{aligned} \tag{10}$$

The photons per cubic meter at 2.73 K is:

$$\begin{aligned}
\frac{N}{V}\,(T = 2.73\ \mathrm{K}) &= 8\pi\left(\frac{k_B T}{hc}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1}\ \mathrm{d}x \\
&\approx 8\pi\left(\frac{1.381 \cdot 10^{-23}\ \mathrm{J/K} \cdot 2.73\ \mathrm{K}}{6.63 \cdot 10^{-34}\ \mathrm{J\,s} \cdot 3 \cdot 10^{8}\ \mathrm{m/s}}\right)^3 (2.404) \\
&\approx 4.115 \cdot 10^{8}\ \frac{\gamma}{\mathrm{m}^3}
\end{aligned} \tag{11}$$

Exercise 4

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 7.51 (partial)**

Part (a)

The surface area of the filament A is:

$$\begin{aligned} P &= \sigma e A T^4 \\ A &= \frac{P}{\sigma e T^4} \\ &= \frac{100\ \mathrm{W}}{5.67 \cdot 10^{-8}\ \mathrm{W/(m^2\,K^4)} \cdot \frac{1}{3} \cdot (3000\ \mathrm{K})^4} \\ &\approx 6.532 \cdot 10^{-5}\ \mathrm{m}^2 \end{aligned} \tag{12}$$

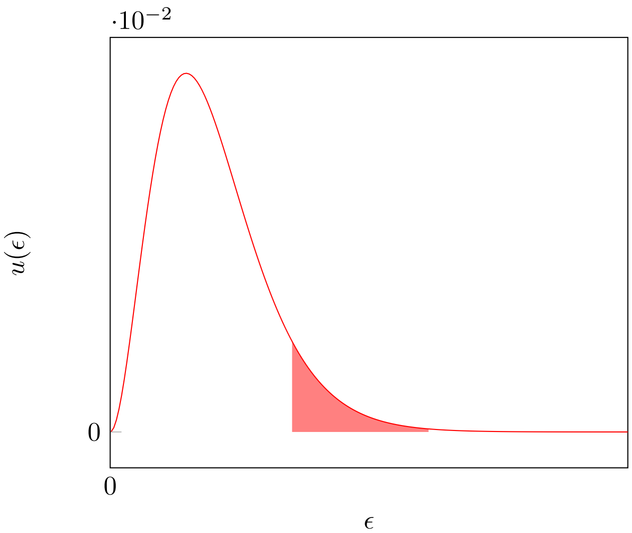
Part (b)

The peak in the bulb's spectrum occurs at an energy ϵ of:

$$\begin{aligned} \epsilon &= 2.82 k_B T \\ &= 2.82 \cdot 1.381 \cdot 10^{-23}\ \mathrm{J/K} \cdot 3000\ \mathrm{K} \\ &\approx 1.168 \cdot 10^{-19}\ \mathrm{J} \approx 0.729\ \mathrm{eV} \\ \lambda &= \frac{hc}{\epsilon} \\ &= \frac{1230\ \mathrm{eVnm}}{0.729\ \mathrm{eV}} \\ &\approx 1687.243\ \mathrm{nm} \approx 1.687\ \mathrm{\mu m} \end{aligned} \tag{13}$$

Part (c)

The shaded area in the plot is the visible light spectrum.



Part (d)

The fraction of energy that comes out as visible light p_{vis} is:

$$
\begin{aligned}
x_r &= \frac{\epsilon_r}{k_B T} = \frac{\frac{1230\ \text{eVnm}}{700\ \text{nm}}}{8.617 \cdot 10^{-5}\ \text{eV/K} \cdot 3000\ \text{K}} \approx 6.797 \\
x_v &= \frac{\epsilon_v}{k_B T} = \frac{\frac{1230\ \text{eVnm}}{400\ \text{nm}}}{8.617 \cdot 10^{-5}\ \text{eV/K} \cdot 3000\ \text{K}} \approx 11.895 \\
p_{\text{vis}} &= \frac{\int_{x_r}^{x_v} \frac{x^3}{e^x - 1}\ \mathrm{d}x}{\int_0^\infty \frac{x^3}{e^x - 1}\ \mathrm{d}x} \\
&\approx \frac{\int_{6.797}^{11.895} \frac{x^3}{e^x - 1}\ \mathrm{d}x}{\int_0^\infty \frac{x^3}{e^x - 1}\ \mathrm{d}x} \\
&\approx 0.084
\end{aligned}
\tag{14}
$$

About 8% of the light is visible, which matches with the plot.

Part (e)

Increasing the temperature would move the peak to the right and closer to the range of visible light.

Exercise 5

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 7.66**

The theoretical Debye temperature $T_{D,\text{theoretical}}$ is:

$$\begin{aligned} T_{D,\text{theoretical}} &= \frac{hc_s}{2k_B}\left(\frac{6}{\pi}\frac{N}{V}\right)^{\frac{1}{3}} \\ &\approx \frac{6.626 \cdot 10^{-34}\ \text{J s} \cdot 3560\ \text{m/s}}{2 \cdot 1.381 \cdot 10^{-23}\ \text{J/s}}\left(\frac{6}{\pi} \cdot 8.469 \cdot 10^{28}\ 1/\text{m}^3\right)^{\frac{1}{3}} \\ &\approx 465.324\ \text{K} \end{aligned} \tag{15}$$

The experimental Debye temperature $T_{D,\text{experimental}}$ is:

$$\begin{aligned} C_V &= \frac{12\pi^4}{5}\left(\frac{T}{T_{D,\text{experimental}}}\right)^3 Nk_B \\ \frac{C_V}{T} &= \frac{12\pi^4 Nk_B}{T^3_{D,\text{experimental}}} \cdot T^2 \\ \frac{12\pi^4 Nk_B}{T^3_{D,\text{experimental}}} &= m \approx \frac{1}{18} \cdot 10^{-3}\ \text{kg} \\ T_{D,\text{experimental}} &\approx \left(\frac{12\pi^4 Nk_B}{5m}\right)^{\frac{1}{3}} \\ &\approx \left(\frac{12\pi^4 12\pi^4 \cdot 6.022 \cdot 10^{23} \cdot 1.381 \cdot 10^{-23}\ \text{J/K}}{5 \cdot \frac{1}{18} \cdot 10^{-3}\ \text{kg}}\right)^{\frac{1}{3}} \\ &\approx 327.094\ \text{K} \end{aligned} \tag{16}$$

The theoretical Debye temperature is about 50% larger than the experimental Debye temperature.

Exercise 6

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 7.73 (partial)**

Part (a)

The energy of the ground state ϵ_0 is:

$$\begin{aligned}\epsilon_0 &= \frac{3h^2}{8mL^2}\\ &= \frac{3\big(6.63\cdot 10^{-34}\ \mathrm{J\,s}\big)^2}{8\cdot 87\cdot 1.66\cdot 10^{-27}\ \mathrm{kg}\cdot(10^{-5}\ \mathrm{m})^2}\\ &\approx 1.141\cdot 10^{-32}\ \mathrm{J}\approx 7.122\cdot 10^{-14}\ \mathrm{eV}\end{aligned}\tag{17}$$

Part (b)

The condensation temperature T_c is:

$$\begin{aligned}T_c &= \frac{0.527}{k_B}\frac{h^2}{2\pi m}\left(\frac{N}{V}\right)^{\frac{2}{3}}\\ &= \frac{0.527}{1.381\cdot 10^{-23}\ \mathrm{J/K}}\frac{\big(6.63\cdot 10^{-34}\ \mathrm{J\,s}\big)^2}{2\pi\cdot 87\cdot 1.66\cdot 10^{-27}\ \mathrm{kg}}\left(\frac{10000}{(10^{-5}\ \mathrm{m})^3}\right)^{\frac{2}{3}}\\ &\approx 8.580\cdot 10^{-8}\ \mathrm{K}\end{aligned}\tag{18}$$

Consequently, $k_BT_c \approx 1.185\cdot 10^{-30}\ \mathrm{J}\approx 100\epsilon_0$.

Part (c)

When $T = 0.9T_c$:

$$\begin{aligned}
N_0 &= \left(1 - \frac{T}{T_c}^{\frac{3}{2}}\right) N \\
&= \left(1 - 0.9^{\frac{3}{2}}\right) \cdot 10000 \approx 1462 \text{ atoms} \\
\epsilon_0 - \mu &= \frac{k_B T}{N_0} \\
&= \frac{8.617 \cdot 10^{-5} \cdot 0.9 \cdot 8.580 \cdot 10^{-8}}{1462} \approx 4.551 \cdot 10^{-15} \text{ eV} \\
\epsilon_1 &= \frac{2^2 + 1^2 + 1^2}{3} \epsilon_0 = 2\epsilon_0 \\
N_1 &= \frac{1}{e^{\frac{\epsilon_1 - \mu}{k_B T}} - 1} \\
&= \frac{1}{e^{\frac{4.551 \cdot 10^{-15} \text{ eV} + 7.122 \cdot 10^{-14} \text{ eV}}{8.617 \cdot 10^{-5} \text{ eV/K} \cdot 0.9 \cdot 8.580 \cdot 10^{-8} \text{ K}}} - 1} \approx 87
\end{aligned} \tag{19}$$

There are 87 atoms in each of the three first excited states, for a total of $3 \cdot 87 = 261$ atoms in the first excited state.

Part (d)

For $N = 10^6$ atoms:

$$\begin{aligned}
T_c &= \frac{0.527}{k_B} \frac{h^2}{2\pi m} \left(\frac{N}{V}\right)^{\frac{2}{3}} \\
&= \frac{0.527}{1.381 \cdot 10^{-23} \text{ J/K}} \frac{\left(6.63 \cdot 10^{-34} \text{ J s}\right)^2}{2\pi \cdot 87 \cdot 1.66 \cdot 10^{-27} \text{ kg}} \left(\frac{10^6}{(10^{-5} \text{ m})^3}\right)^{\frac{2}{3}} \\
&\approx 1.848 \cdot 10^{-6} \text{ K}
\end{aligned} \tag{20}$$

Consequently, $k_B T_c \approx 2.552 \cdot 10^{-29}$ J $\approx 100\epsilon_0$. When $T = 0.9T_c$:

$$\begin{aligned}
N_0 &= \left(1 - \frac{T}{T_c}^{\frac{3}{2}}\right) N \\
&= \left(1 - 0.9^{\frac{3}{2}}\right) \cdot 10^6 \approx 1.462 \cdot 10^5 \text{ atoms} \\
\epsilon_0 - \mu &= \frac{k_B T}{N_0} \\
&= \frac{8.617 \cdot 10^{-5} \cdot 0.9 \cdot 8.580 \cdot 10^{-8}}{1462} \approx 9.802 \cdot 10^{-16} \text{ eV} \\
\epsilon_1 &= \frac{2^2 + 1^2 + 1^2}{3} \epsilon_0 = 2\epsilon_0 \\
N_1 &= \frac{1}{e^{\frac{\epsilon_1 - \mu}{k_B T}} - 1} \\
&= \frac{1}{e^{\frac{9.802 \cdot 10^{-16} \text{ eV} + 7.122 \cdot 10^{-14} \text{ eV}}{8.617 \cdot 10^{-5} \text{ eV/K} \cdot 0.9 \cdot 8.580 \cdot 10^{-8} \text{ K}}} - 1} \approx 1985
\end{aligned} \tag{21}$$

There are 1985 atoms in each of the three first excited states, for a total of $3 \cdot 1985 = 5955$ atoms in the first excited state. The number of atoms in the ground state is much greater than the number of atoms in the first excited state for large N within a range of temperatures that gets wider as N increases, for fixed $\frac{T}{T_c}$.

Exercise 7

Part (a)

For $n \gg 1$, the degeneracy is approximately $\frac{n^2}{2}$. The density of states $g\left(\epsilon\right)$ is then:

$$\begin{aligned} g\left(\epsilon\right) &= \frac{\frac{n^2}{2}}{hf} \\ &= \frac{n^2}{2hf} = \frac{\epsilon^2}{2(hf)^3} \end{aligned} \tag{22}$$

Part (b)

The condensation temperature T_c is then:

$$\begin{aligned} N &= \int_0^\infty g\left(\epsilon\right) \frac{1}{e^{\frac{\epsilon-\mu}{k_B T_c}} - 1} \, \mathrm{d}\epsilon \\ &= \frac{1}{2(hf)^3} \int_0^\infty \frac{\epsilon^2}{e^{\frac{\epsilon-\mu}{k_B T_c}} - 1} \, \mathrm{d}\epsilon \\ &= \frac{1}{2}\left(\frac{k_B T_c}{hf}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1} \, \mathrm{d}x \\ T_c &= \left(\frac{2N}{\int_0^\infty \frac{x^2}{e^x-1} \, \mathrm{d}x}\right)^{\frac{1}{3}} \left(\frac{hf}{k_B}\right) \\ &\approx \frac{hf}{k_B}\left(\frac{N}{1.202}\right)^{\frac{1}{3}} \end{aligned} \tag{23}$$