

Homework 4

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October 31, 2023

Exercise 1

Part (a)

Let

$$\begin{aligned}\varphi : \mathbb{Z}/10\mathbb{Z} &\rightarrow \mathbb{Z}/7\mathbb{Z} \\ \bar{k} &\mapsto \bar{k}\end{aligned}\tag{1}$$

Note that $5 = 15 \pmod{10}$, since $5 = 10(0) + 5$ and $15 = 10(1) + 5$, which means that $\bar{5} = \bar{15}$ in $\mathbb{Z}/10\mathbb{Z}$. However, $5 \not\equiv 15 \pmod{7}$, since $5 = 7(0) + 5$ and $15 = 7(2) + 1$, which means that $\bar{5} \neq \bar{15}$ in $\mathbb{Z}/7\mathbb{Z}$. Since $\bar{5} = \bar{15}$ but $\varphi(\bar{5}) \neq \varphi(\bar{15})$, φ is not well-defined, as a single element maps to multiple elements in the codomain.

Part (b)

Let

$$\begin{aligned}\varphi : \mathbb{Z}/10\mathbb{Z} &\rightarrow \mathbb{Z}/7\mathbb{Z} \\ \bar{k} &\mapsto \bar{k}\end{aligned}\tag{2}$$

Let $\bar{x}, \bar{y} \in \mathbb{Z}/10\mathbb{Z}$ such that $\bar{x} = \bar{y}$ in $\mathbb{Z}/10\mathbb{Z}$. Then, since $x = y \pmod{10}$, $x = 10(m) + p$ and $y = 10(n) + p$ for $m, n, p \in \mathbb{Z}$. Then, since $x = 10(m) + p = 5(2m) + p$ and $y = 10(n) + p = 5(2n) + p$ with $2m, 2n \in \mathbb{Z}$, $x = y \pmod{5}$, which means that $\bar{x} = \bar{y}$ in $\mathbb{Z}/5\mathbb{Z}$. Consequently, since $\bar{x} = \bar{y}$ implies that $\varphi(\bar{x}) = \varphi(\bar{y})$, φ is well-defined.

Exercise 2

Part (a)

Let

$$\begin{aligned}\varphi : \mathbb{Z} &\rightarrow \mathbb{Z}/3\mathbb{Z} \\ \bar{k} &\mapsto \bar{k}\end{aligned}\tag{3}$$

Let $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}/9\mathbb{Z}$, $k \mapsto \bar{k}$ and $\beta : \mathbb{Z}/9\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, $\bar{k} \mapsto \bar{k}$. We first show that α and β are well defined.

Let $x, y \in \mathbb{Z}$ such that $x = y$. Then, since $x = y$, $x = 9(m) + p$ and $y = 9(m) + p$ for $m, p \in \mathbb{Z}$. Then, $x = y \pmod{9}$, which means that $\bar{x} = \bar{y}$ in $\mathbb{Z}/9\mathbb{Z}$. Consequently, since $x = y$ implies that $\varphi(x) = \varphi(y)$, α is well-defined.

Let $\bar{x}, \bar{y} \in \mathbb{Z}/9\mathbb{Z}$ such that $\bar{x} = \bar{y}$. Then, since $\bar{x} = \bar{y}$, $x = 9(m) + p$ and $y = 9(n) + p$ for $m, n, p \in \mathbb{Z}$. Then, since $x = 9(m) + p = 3(3m) + p$ and $y = 9(n) + p = 3(3n) + p$ with $3m, 3n \in \mathbb{Z}$, $x = y \pmod{3}$, which means that $\bar{x} = \bar{y}$ in $\mathbb{Z}/3\mathbb{Z}$. Consequently, since $\bar{x} = \bar{y}$ implies that $\varphi(\bar{x}) = \varphi(\bar{y})$, β is well-defined.

Let $a, b \in \mathbb{Z}$ and $\bar{c}, \bar{d} \in \mathbb{Z}/9\mathbb{Z}$. Then, since $\overline{xy} = \bar{x}\bar{y}$:

$$\begin{aligned}\alpha(a)\alpha(b) &= \overline{ab} = \overline{ab} = \alpha(ab) \\ \beta(\bar{c})\beta(\bar{d}) &= \overline{cd} = \overline{cd} = \beta(\overline{cd})\end{aligned}\tag{4}$$

Therefore α and β are homomorphisms. Since $\beta \circ \alpha(x) = \bar{x} = \varphi(x)$, $\beta \circ \alpha = \varphi$, and the homomorphism φ factors through $\mathbb{Z}/9\mathbb{Z}$.

Part (b)

Let $\ell \in \mathbb{N}$. Let $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}/3^\ell\mathbb{Z}$, $k \mapsto \bar{k}$ and $\beta : \mathbb{Z}/3^\ell\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, $\bar{k} \mapsto \bar{k}$. We first show that α and β are well defined.

Let $x, y \in \mathbb{Z}$ such that $x = y$. Then, since $x = y$, $x = 3^\ell(m) + p$ and $y = 3^\ell(m) + p$ for $m, p \in \mathbb{Z}$. Then, $x = y \pmod{3^\ell}$, which means that $\bar{x} = \bar{y}$ in $\mathbb{Z}/3^\ell\mathbb{Z}$. Consequently, since $x = y$ implies that $\varphi(x) = \varphi(y)$, α is well-defined.

Let $\bar{x}, \bar{y} \in \mathbb{Z}/3^\ell\mathbb{Z}$ such that $\bar{x} = \bar{y}$. Then, since $\bar{x} = \bar{y}$, $x = 3^\ell(m) + p$ and $y = 3^\ell(n) + p$ for $m, n, p \in \mathbb{Z}$. Then, since $x = 3^\ell(m) + p = 3(3^{\ell-1}m) + p$ and $y = 3^\ell(n) + p = 3(3^{\ell-1}n) + p$ with $3^{\ell-1}m, 3^{\ell-1}n \in \mathbb{Z}$, $x = y \pmod{3}$, which means that $\bar{x} = \bar{y}$ in $\mathbb{Z}/3\mathbb{Z}$. Consequently, since $\bar{x} = \bar{y}$ implies that $\varphi(\bar{x}) = \varphi(\bar{y})$, β is well-defined.

Let $a, b \in \mathbb{Z}$ and $\bar{c}, \bar{d} \in \mathbb{Z}/3^\ell\mathbb{Z}$. Then, since $\overline{xy} = \bar{x}\bar{y}$:

$$\begin{aligned}\alpha(a)\alpha(b) &= \overline{ab} = \overline{ab} = \alpha(ab) \\ \beta(\bar{c})\beta(\bar{d}) &= \overline{cd} = \overline{cd} = \beta(\overline{cd})\end{aligned}\tag{5}$$

Therefore α and β are homomorphisms. Since $\beta \circ \alpha(x) = \bar{x} = \varphi(x)$, $\beta \circ \alpha = \varphi$, and the homomorphism φ factors through $\mathbb{Z}/3^\ell\mathbb{Z}$ for all $\ell \in \mathbb{N}$.