

# Homework 5

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## Exercise 1

Let  $G$  be a group generated by some  $x, y \in G$ , let  $N \trianglelefteq G$ , and let  $\bar{G} = G/N$ . Let  $g \in G$ . Since  $G$  is generated by  $x, y \in G$ , for some  $g_1, g_2, \dots, g_n \in \{x, y, x^{-1}, y^{-1}\}$ ,  $g = g_1 g_2 \dots g_n$ . Consequently, for  $gN \in \bar{G}$ :

$$\begin{aligned} gN &= (g_1 g_2 \dots g_n) N \\ &= (g_1 N) (g_2 N) \dots (g_n N) \\ &= \bar{g}_1 \bar{g}_2 \dots \bar{g}_n \end{aligned} \tag{1}$$

Evidently, every element in  $\bar{G}$  can be expressed as  $\bar{g}_1 \bar{g}_2 \dots \bar{g}_n$  for some  $\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n \in \{\bar{x}, \bar{y}, \bar{x}^{-1}, \bar{y}^{-1}\}$ . Therefore  $\bar{G}$  is generated by  $\bar{x}, \bar{y} \in \bar{G}$ .

## Exercise 2

*Algebra (Artin, 2e) Exercise 2.12.2 (partial)*

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### Part (a)

Evidently, the identity element is in  $H$ , as it is the element of  $H$  where  $a, b, c = 0$ . We then compute the product of two arbitrary elements of  $H$ :

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+a & y+az+b \\ 0 & 1 & z+c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} \in H \quad (2)$$

From the above computation, the group exhibits closure, as the product is the element of  $H$  where  $a' = x + a, b' = y + az + b, c' = z + c$ . From the above computation, the group has inverses, as when  $x = -a, y = ac - b, z = -c$ , the product is the identity. Since  $H$  satisfies the identity, closure, and inverse properties,  $H \leq GL_3(\mathbb{F})$ .

### Part (b)

We compute the following product:

$$\begin{aligned} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} &= \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a & y+b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in K \end{aligned} \quad (3)$$

Evidently, for arbitrary  $h \in H$  and  $k \in K$ ,  $hkh^{-1} \in K$ , so  $K \trianglelefteq H$ .

### Part (c)

Let  $A = \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{bmatrix} \in H$ . From the computation done

in (b), we see that  $A$  and  $B$  are in the same coset of  $K$  when  $a_1 = b_1$  and  $a_3 = b_3$ . We can then construct the surjective homomorphism mapping  $h =$

$\begin{bmatrix} 1 & h_1 & h_2 \\ 0 & 1 & h_3 \\ 0 & 0 & 1 \end{bmatrix} \in H$  to  $\bar{h} = \begin{bmatrix} 1 & h_1 & 0 \\ 0 & 1 & h_3 \\ 0 & 0 & 1 \end{bmatrix} \in \bar{H}$ . Evidently,  $K$  is the kernel of this

homomorphism, as it maps to the identity matrix.

### Exercise 3

#### Part (a)

Let the Klein four group  $V = \{1, a, b, ab\} = \langle a, b : a^2 = b^2 = [a, b] = 1 \rangle \cong C_2 \times C_2$ . Let the subgroup  $N = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$  in  $S_4$ . We construct the bijective mapping  $\varphi : V \rightarrow N$  where  $1 \rightarrow e$ ,  $a \rightarrow (1\ 2)(3\ 4)$ ,  $b \rightarrow (1\ 3)(2\ 4)$ ,  $ab \rightarrow (1\ 4)(2\ 3)$ . To show that this is a homomorphism, we construct Cayley tables:

	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

	e	(1 2)(3 4)	(1 3)(2 4)	(1 4)(2 3)
e	e	(1 2)(3 4)	(1 3)(2 4)	(1 4)(2 3)
(1 2)(3 4)	(1 2)(3 4)	e	(1 4)(2 3)	(1 3)(2 4)
(1 3)(2 4)	(1 3)(2 4)	(1 4)(2 3)	e	(1 2)(3 4)
(1 4)(2 3)	(1 4)(2 3)	(1 3)(2 4)	(1 2)(3 4)	e

By inspecting the Cayley tables, we see that  $\varphi(x)\varphi(y) = \varphi(xy)$  for all  $x, y \in V$ . As  $\varphi$  is a bijective homomorphism, it is an isomorphism and as an isomorphism exists between  $V$  and  $N$ ,  $N$  is isomorphic to the Klein four group.

#### Part (b)

The subgroup  $N$  is composed of 1 permutation of cycle type 1, 1, 1, 1 and permutations of cycle type 2, 2. We note that  $e$  is the only permutation of cycle type 1, 1, 1, 1, as  $\frac{P_1^4 P_1^3 P_1^1 P_1^1}{4!} = 1$ , and that  $(1\ 2)(3\ 4)$ ,  $(1\ 3)(2\ 4)$ ,  $(1\ 4)(2\ 3)$  are the only permutations of cycle type 2, 2, as  $\frac{P_2^2 P_2^2}{2!} = 3$ . As conjugation preserves cycle type,  $e$  must be mapped back to itself, the only permutation of its cycle type, and  $(1\ 2)(3\ 4)$ ,  $(1\ 3)(2\ 4)$ ,  $(1\ 4)(2\ 3)$  must be mapped to cycles of type 2, 2, 2, all of which are elements of  $V$ . Since for all  $p \in V$  and all  $q \in S_4$ ,  $qpq^{-1} \in V$ ,  $V \trianglelefteq S_4$ .

#### Part (c)

Let  $H = \langle (1\ 2), (3\ 4) \rangle \leq S_4$ . We note that  $H = \langle (1\ 2), (3\ 4) \rangle = \{e, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$ . We construct the bijective mapping  $\varphi : V \rightarrow H$  where  $1 \rightarrow e$ ,  $a \rightarrow (1\ 2)$ ,  $b \rightarrow (3\ 4)$ ,  $ab \rightarrow (1\ 2)(3\ 4)$ . To show that this is a homomorphism, we construct Cayley tables:

	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

	$e$	$(1\ 2)$	$(3\ 4)$	$(1\ 2)(3\ 4)$
$e$	$e$	$(1\ 2)$	$(3\ 4)$	$(1\ 2)(3\ 4)$
$(1\ 2)$	$(1\ 2)$	$e$	$(1\ 2)(3\ 4)$	$(1\ 2)$
$(3\ 4)$	$(3\ 4)$	$(1\ 2)(3\ 4)$	$e$	$(3\ 4)$
$(1\ 2)(3\ 4)$	$(1\ 2)(3\ 4)$	$(3\ 4)$	$(1\ 2)$	$e$

By inspecting the Cayley tables, we see that  $\varphi(x)\varphi(y) = \varphi(xy)$  for all  $x, y \in V$ . As  $\varphi$  is a bijective homomorphism, it is an isomorphism and as an isomorphism exists between  $V$  and  $H$ ,  $H$  is isomorphic to the Klein four group.

However,  $H$  is not normal in  $S_4$ . To see this, conjugate  $(1\ 2)$  by  $(1\ 2\ 3\ 4)$ . Then:

$$\begin{aligned}
(1\ 2\ 3\ 4)(1\ 2)(1\ 2\ 3\ 4)^{-1} &= (1\ 2\ 3\ 4)(1\ 2)(4\ 3\ 2\ 1) \\
&= (1\ 2\ 3\ 4)(1\ 4\ 3) \\
&= (2\ 3)
\end{aligned} \tag{4}$$

Since  $(2\ 3) \notin H$ , evidently  $H$  is not normal in  $S_4$ .

### Part (d)

We compute the cosets of  $N$  in  $S_4$ .

$$\begin{array}{ll}
eN = N & (1\ 2)(3\ 4)N = N \\
(1\ 3)(2\ 4)N = N & (1\ 4)(2\ 3)N = N \\
\\
(1\ 2)N = \{(1\ 2), (3\ 4), (1\ 4\ 2\ 3), (1\ 3\ 2\ 4)\} & (3\ 4)N = \{(3\ 4), (1\ 2), (1\ 4\ 2\ 3), (1\ 3\ 2\ 4)\} \\
(1\ 3\ 2\ 4)N = \{(1\ 3\ 2\ 4), (1\ 4\ 2\ 3), (3\ 4), (1\ 2)\} & (1\ 4\ 2\ 3)N = \{(1\ 4\ 2\ 3), (1\ 3\ 2\ 4), (1\ 2), (3\ 4)\} \\
\\
(2\ 3)N = \{(2\ 3), (1\ 3\ 4\ 2), (1\ 2\ 4\ 3), (1\ 4)\} & (1\ 3\ 4\ 2)N = \{(1\ 3\ 4\ 2), (2\ 3), (1\ 4), (1\ 2\ 4\ 3)\} \\
(1\ 2\ 4\ 3)N = \{(1\ 2\ 4\ 3), (1\ 4), (2\ 3), (1\ 3\ 4\ 2)\} & (1\ 4)N = \{(1\ 4), (1\ 2\ 4\ 3), (1\ 3\ 4\ 2), (2\ 3)\} \\
\\
(1\ 3\ 2)N = \{(1\ 3\ 2), (2\ 3\ 4), (1\ 2\ 4), (1\ 4\ 3)\} & (2\ 3\ 4)N = \{(2\ 3\ 4), (1\ 3\ 2), (1\ 4\ 3), (1\ 2\ 4)\} \\
(1\ 4\ 3)N = \{(1\ 4\ 3), (1\ 2\ 4), (2\ 3\ 4), (1\ 3\ 2)\} & (1\ 2\ 4)N = \{(1\ 2\ 4), (1\ 4\ 3), (1\ 3\ 2), (2\ 3\ 4)\} \\
\\
(2\ 4\ 3)N = \{(2\ 4\ 3), (1\ 4\ 2), (1\ 2\ 3), (1\ 3\ 4)\} & (1\ 4\ 2)N = \{(1\ 4\ 2), (2\ 4\ 3), (1\ 3\ 4), (1\ 2\ 3)\} \\
(1\ 2\ 3)N = \{(1\ 2\ 3), (1\ 3\ 4), (2\ 4\ 3), (1\ 4\ 2)\} & (1\ 3\ 4)N = \{(1\ 3\ 4), (1\ 2\ 3), (1\ 4\ 2), (2\ 4\ 3)\} \\
\\
(1\ 3)N = \{(1\ 3), (1\ 2\ 3\ 4), (2\ 4), (1\ 4\ 3\ 2)\} & (1\ 4\ 3\ 2)N = \{(1\ 4\ 3\ 2), (2\ 4), (1\ 2\ 3\ 4), (1\ 3)\} \\
(2\ 4)N = \{(2\ 4), (1\ 4\ 3\ 2), (1\ 3), (1\ 2\ 3\ 4)\} & (1\ 2\ 3\ 4)N = \{(1\ 2\ 3\ 4), (1\ 3), (1\ 4\ 3\ 2), (2\ 4)\}
\end{array}$$

Consequently, we define  $S_4/N = \{\bar{e}, \overline{(1\ 2)}, \overline{(2\ 3)}, \overline{(1\ 3\ 2)}, \overline{(1\ 2\ 3)}, \overline{(1\ 3)}\}$ , noting that each coset has exactly one element where 4 permutes to itself. We then define the bijection  $\varphi: S_4/N \rightarrow S_3$  that maps  $\bar{\sigma} \rightarrow \sigma$ . It is easily verified that  $\varphi$  is a homomorphism and that  $N$  is the kernel of the homomorphism. As  $\varphi$  is a bijective homomorphism,  $\varphi$  is an isomorphism and as an isomorphism exists between  $S_4/N$  and  $S_3$ ,  $S_4/N \cong S_3$ .

### Part (e)

By the Correspondence Theorem, since  $\varphi$  is a subjective homomorphism of which  $N$  is the kernel, there exists a bijection between the subgroups of  $S_4$  containing  $N$  and the subgroups of  $S_3$ . We note that the inverse of an element of  $S_3$  (and, in general,  $S_n$ ) can be found by reversing the order of the individual cycles that compose the permutation (e.g.  $(1\ 2\ 3)^{-1} = (3\ 2\ 1) = (1\ 3\ 2)$ ). Consequently, the three permutations of cycle type 2, 1 are their own inverses, and the two permutations of cycle type 3 are inverses of each other (the identity permutation of cycle type 1, 1, 1 is, of course, its own inverse). Further, the composition of two different permutations of cycle type 2, 1 is a permutation of cycle type 3. We then enumerate the subgroups of  $S_3$  as such: the trivial subgroup, three subgroups containing the identity and one of the permutations

of cycle type 2, 1, the subgroup containing the identity and both permutations of cycle type 3, and  $S_3$  itself. As there exist 6 subgroups of  $S_3$ , there are therefore 6 subgroups of  $S_4$  that contain  $N$ .