Homework 3

Ravi Kini

October 19, 2023

Exercise 1

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 2.26, 2.32**

Part (a)

For a single gas atom, the multiplicity Ω should be proportional to the area of position space and area of momentum space.

$$\begin{aligned}
\Omega_1 &\propto A \cdot A_p \\
U &= \frac{1}{2} m \left(v_x^2 + v_y^2\right) = \frac{1}{2m} \left(p_x^2 + p_y^2\right) \\
2mU &= p_x^2 + p_y^2 \\
\Omega_1 &= \frac{A A_p}{h^2}
\end{aligned} \tag{1}$$

For N gas atoms, the multiplicity Ω_N is:

$$\begin{aligned}
\Omega_N &= \frac{1}{N!} \frac{A^N}{h^{2N}} \times (\text{area of momentum hypersphere}) \\
&= \frac{1}{N!} \frac{A^N}{h^{2N}} \frac{2\pi^N}{(N-1)!} \left(\sqrt{2mU}\right)^{2N-1} \\
&\approx \frac{1}{N!} \frac{A^N}{h^{2N}} \frac{\pi^N}{N!} (2mU)^N
\end{aligned} \tag{2}$$

Part (b)

The entropy S is:

$$\begin{aligned}
S = k_B \ln \Omega_N &= k_B \ln \left(\frac{1}{N!} \frac{A^N}{h^{2N}} \frac{\pi^N}{N!} (2mU)^N \right) \\
&= k_B \left(N \ln \left(\frac{A}{h^2} \pi \left(2mU\right) \right) - 2 \left(N \ln N - N\right) \right) \\
&= N k_B \left(\ln \left(\frac{A}{N} \frac{2\pi m U}{N h^2} \right) + 2 \right)
\end{aligned} \tag{3}$$

Exercise 2

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 2.27 (modified)**

Part (a)

The probability of finding all N molecules in the left 99.9% of the container is 0.999^N. For $N = 100$ molecules:

$$P(N = 100) = 0.999^{100} \approx 0.905 \tag{4}$$

Part (b)

For $N = 10000$ molecules:

$$P(N = 10000) = 0.999^{10000} \approx 4.517 \cdot 10^{-5} \tag{5}$$

Part (c)

For $N = 10^{23}$ molecules:

$$\begin{aligned} P\left(N = 10^{23}\right) &= 0.999^{10^{23}} \\ &= e^{10^{23} \ln 0.999} = e^{10^{23} \ln(1-0.001)} \approx e^{10^{23} \cdot -0.001} \\ &\approx e^{-10^{20}} \approx 0 \end{aligned} \tag{6}$$

Part (d)

If there is a $4.517 \cdot 10^{-5}$ chance of the configuration occurring, as all configurations are equally likely, it will take $\frac{1}{4.517 \cdot 10^{-5}} \approx 22139$ trials for the configuration to be observed. As 10^{12} trials are occurring per second, observing the configuration will take $2.2139 \cdot 10^{-8}$ s, or 22.139 ns, or $7.02 \cdot 10^{-16}$ y.

Exercise 3

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 2.37**

There are $N(1-x)$ molecules of gas A, and it expands to a volume $\frac{1}{1-x}$ as big as its original volume. There are Nx molecules of gas B, and it expands to a volume $\frac{1}{x}$ as big as its original volume. The entropy of mixing ΔS_{mixing} is:

$$\begin{aligned}
\Delta S_A &= N_A k_b \ln \frac{V_{f,A}}{V_{i,A}} = N(1-x)\, k_B \ln \frac{1}{1-x} \\
\Delta S_B &= N_B k_b \ln \frac{V_{f,B}}{V_{i,B}} = N x k_B \ln \frac{1}{x} \\
\Delta S_{\text{mixing}} = \Delta S_A + \Delta S_B &= N(1-x)\, k_B \ln \frac{1}{1-x} + N x k_B \ln \frac{1}{x} \\
&= -N k_B \left((1-x) \ln (1-x) + x \ln x\right)
\end{aligned} \tag{7}$$

For $x = \frac{1}{2}$:

$$\begin{aligned}
\Delta S_{\text{mixing}} &= -N k_B \left(\left(1 - \frac{1}{2}\right) \ln \left(1 - \frac{1}{2}\right) + \frac{1}{2} \ln \frac{1}{2}\right) \\
&= -N k_B \left(\frac{1}{2} \ln \left(\frac{1}{2}\right) + \frac{1}{2} \ln \frac{1}{2}\right) \\
&= -N k_B \ln \frac{1}{2} = N k_B \ln 2
\end{aligned} \tag{8}$$

Exercise 4

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 3.8**

Part (a)

In the low-temperature limit where $q \ll N$, the multiplicity of an Einstein solid Ω is:

$$\begin{aligned}
\Omega &= \binom{q+N-1}{q} \approx \frac{(q+N)!}{q!N!} \\
\ln\Omega &\approx ((q+N)\ln(q+N) - (q+N)) - (q\ln q - q) - (N\ln N - N) \\
&\approx \left((q+N)\left(\ln N + \ln\left(1+\frac{q}{N}\right)\right) - (q+N)\right) - (q\ln q - q) - (N\ln N - N) \\
&\approx \left((q+N)\left(\ln N + \frac{q}{N}\right) - (q+N)\right) - (q\ln q - q) - (N\ln N - N) \\
&\approx q\ln N + \frac{q^2}{N} + q - q\ln q \approx q\ln\frac{N}{q} + q \\
\Omega &\approx e^{q\ln\frac{N}{q}+q} = \left(\frac{eN}{q}\right)^q
\end{aligned} \tag{9}$$

Part (b)

In the low-temperature limit, the energy of an Einstein solid U is:

$$\begin{aligned}
S &= k_B\ln\Omega \approx qk_B\ln\frac{eN}{q} \\
&\approx qk_B\left(\ln\frac{N}{q}+1\right) \\
&\approx \frac{Uk_B}{\epsilon}\left(\ln\frac{N\epsilon}{U}+1\right) \\
\frac{1}{T} = \frac{\partial S}{\partial U} &\approx \frac{k_B}{\epsilon}\left(\ln\frac{N\epsilon}{U}+1\right) + \frac{k_BU}{\epsilon}\left(-\frac{1}{U}\right) = \frac{k_B}{\epsilon}\ln\frac{N\epsilon}{U} \\
T &\approx \frac{\epsilon}{k_B\ln\frac{N\epsilon}{U}} \\
U &\approx N\epsilon e^{-\frac{\epsilon}{k_BT}}
\end{aligned} \tag{10}$$

Exercise 5

***An Introduction to Thermal Physics* (Schroeder, 1e) Exercise 3.10**

Part (a)

The change in the entropy of the ice cube as it melts $\Delta S_{\text{water},1}$ is:

$$\begin{aligned}\Delta S_{\text{water},1} &= \frac{Q}{T} = \frac{ml}{T} \\ &= \frac{30\ \text{g} \cdot 333\ \text{J/g}}{273\ \text{K}} \approx 36.593\ \text{J/K}\end{aligned} \tag{11}$$

Part (b)

The change in the entropy of the water as it heats up $\Delta S_{\text{water},2}$ is:

$$\begin{aligned}\Delta S_{\text{water},2} &= \int_{T_i}^{T_f} \frac{C_V}{T}\ \mathrm{d}T = mc_V \ln\frac{T_f}{T_i} \\ &= (30\ \text{g} \cdot 4.186\ \text{J/(g\,K)}) \ln\frac{298\ \text{K}}{273\ \text{K}} \approx 11.004\ \text{J/K}\end{aligned} \tag{12}$$

Part (c)

The change in the entropy of the kitchen $\Delta S_{\text{kitchen}}$ is:

$$\begin{aligned}\Delta S_{\text{kitchen}} &= \frac{Q}{T} = \frac{-ml - mc_v\Delta T}{T} \\ &= \frac{-30\ \text{g} \cdot 333\ \text{J/g} - 30\ \text{g} \cdot 4.186\ \text{J/(g\,K)} \cdot 25\ \text{K}}{298\ \text{K}} \approx -44.059\ \text{J/K}\end{aligned} \tag{13}$$

Part (d)

The net change in the entropy of the universe $\Delta S_{\text{universe}}$ is:

$$\Delta S_{\text{universe}} = \Delta S_{\text{water},1} + \Delta S_{\text{water},2} + \Delta S_{\text{kitchen}} \approx 3.538\ \text{J/K} \tag{14}$$

The net change is positive, as expected, as the entropy of the universe can only ever increase.