HW 1

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October 10, 2023

Exercise 1

Part (a)

The matrices

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \qquad y = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$
$$r = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are the generators of the group G of rotations of $\theta = \frac{\pi}{4}$ about the conventional x-, y-, and $z-$ axes, respectively. Since $\theta = \frac{\pi}{4}$, $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}} := t$, we can rewrite our matrices as:

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & t \\ 0 & -t & t \end{pmatrix} \qquad y = \begin{pmatrix} t & 0 & -t \\ 0 & 1 & 0 \\ t & 0 & t \end{pmatrix} \qquad r = \begin{pmatrix} t & -t & 0 \\ t & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We seek to find matrices representing p^{-1} and y^{-1}; spatial reasoning indicates that the inverse of a rotation by an angle θ would be a rotation by an angle $-\theta$. The matrices representing p^{-1} and y^{-1} are then:

$$\begin{aligned} p^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos -\theta & \sin -\theta \\ 0 & -\sin -\theta & \cos -\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & -t \\ 0 & t & t \end{pmatrix} \\ y^{-1} &= \begin{pmatrix} \cos -\theta & 0 & -\sin -\theta \\ 0 & 1 & 0 \\ \sin -\theta & 0 & \cos -\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} t & 0 & t \\ 0 & 1 & 0 \\ -t & 0 & t \end{pmatrix} \end{aligned} \tag{1}$$

To verify that these are the inverses of p and y, respectively, we multiply p and p^{-1}, and y and y^{-1}.

$$\begin{aligned} pp^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & t \\ 0 & -t & t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & -t \\ 0 & t & t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2t^2 & 0 \\ 0 & 0 & 2t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ yy^{-1} &= \begin{pmatrix} t & 0 & -t \\ 0 & 1 & 0 \\ t & 0 & t \end{pmatrix} \begin{pmatrix} t & 0 & t \\ 0 & 1 & 0 \\ -t & 0 & t \end{pmatrix} = \begin{pmatrix} 2t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{2}$$

As they multiply to the 3×3 identity matrix, they must be inverses.

Part (b)

We now compute $p^2y^{-1}p^{-2}$ and $py^{-1}p^{-1}$:

$$\begin{aligned} p^2y^{-1}p^{-2} &= \begin{pmatrix} t & -t & 0 \\ t & t & 0 \\ 0 & 0 & 1 \end{pmatrix} = r \\ py^{-1}p &= \begin{pmatrix} t & -t^2 & -t^2 \\ -t^2 & t^2 - t^3 & -t^2 - t^3 \\ t^2 & t^2 + t^3 & -t^2 + t^3 \end{pmatrix} \neq r \end{aligned} \tag{3}$$

$p^2y^{-1}p^{-2} = r$, but $py^{-1}p^{-1} \neq r$.

Exercise 2

Part (a)

The set of transpositions $\{\tau_1, \tau_2, \tau_3\}$ generate S_4, where $\tau_i = (i\ i+1)$. The permutations

$$p = (1\ 2\ 3\ 4) \qquad q = (1\ 3\ 2\ 4) \qquad r = (1\ 4\ 2)$$

can therefore be written as products of (adjacent) transpositions τ_i in the following way:

$$\begin{aligned}
\tau_1 \circ \tau_2 \circ \tau_3 &= \tau_1 \circ (2\ 3) \circ (3\ 4) \\
&= (1\ 2) \circ (2\ 3\ 4) \\
&= (1\ 2\ 3\ 4) = p \\
\tau_2 \circ \tau_1 \circ \tau_3 \circ \tau_2 \circ \tau_3 &= \tau_2 \circ \tau_1 \circ \tau_3 \circ (2\ 3) \circ (3\ 4) \\
&= \tau_2 \circ \tau_1 \circ (3\ 4) \circ (2\ 3\ 4) \\
&= \tau_2 \circ (1\ 2) \circ (2\ 4) \\
&= (2\ 3) \circ (1\ 2\ 4) \\
&= (1\ 3\ 2\ 4) = q \\
\tau_2 \circ \tau_3 \circ \tau_2 \circ \tau_1 &= \tau_2 \circ \tau_3 \circ (2\ 3) \circ (1\ 2) \\
&= \tau_2 \circ (3\ 4) \circ (1\ 3\ 2) \\
&= (2\ 3) \circ (1\ 4\ 3\ 2) \\
&= (1\ 4\ 2) = r
\end{aligned} \tag{4}$$

Part (b)

The symmetric group S_4 is generated by the set of (adjacent) transpositions $\{\tau_1, \tau_2, \tau_3\}$. Letting $p := (1\ 2\ 3\ 4)\,, s := (1\ 2)$. $\{\tau_1, \tau_2, \tau_3\}$ can be generated by p, s as follows:

$$\begin{aligned}
s &= (1\ 2) = \tau_1 \\
p \circ p \circ s \circ p \circ s &= p \circ p \circ s \circ (1\ 2\ 3\ 4) \circ (1\ 2) \\
&= p \circ p \circ (1\ 2) \circ (1\ 3\ 4) \\
&= p \circ (1\ 2\ 3\ 4) \circ (1\ 3\ 4\ 2) \\
&= (1\ 2\ 3\ 4) \circ (1\ 4\ 3) \\
&= (2\ 3) = \tau_2 \\
p \circ p \circ s \circ p \circ p &= p \circ p \circ s \circ (1\ 2\ 3\ 4) \circ (1\ 2\ 3\ 4) \\
&= p \circ p \circ (1\ 2) \circ (1\ 3)\,(2\ 4) \\
&= p \circ (1\ 2\ 3\ 4) \circ (1\ 3\ 2\ 4) \\
&= (1\ 2\ 3\ 4) \circ (1\ 4\ 2) \\
&= (3\ 4) = \tau_3
\end{aligned} \tag{5}$$

Evidently, $\{\tau_1, \tau_2, \tau_3\}$ is generated by $\{(1\ 2), (1\ 2\ 3\ 4)\}$. Since S_4 is generated by $\{\tau_1, \tau_2, \tau_3\}$, S_4 is generated by $\{(1\ 2), (1\ 2\ 3\ 4)\}$.

Exercise 3

Part (a)

For some $n \times n$ permutation matrix P, the product PP^T is defined such that:

$$\begin{aligned} \left(PP^T\right)_{ij} &= \sum_{k=1}^{n} P_{ik} P^T_{kj} \\ &= \sum_{k=1}^{n} P_{ik} P_{jk} \end{aligned} \tag{6}$$

Note that permutation matrices, by definition, have a single 1 in each row and in each column, and 0 for the remaining entries. The above expression then simplifies to:

$$\left(PP^T\right)_{ij} = \delta_{ij} \tag{7}$$

As every entry in the product is then equal to the corresponding entry in the identity matrix, $PP^T = I$. Therefore $P^{-1} = P^T$, and the transpose of a permutation matrix is its inverse.

Part (b)

For some $n \times n$ permutation matrix P, $\det\left(P^T\right) = \det(P)$. Then:

$$\begin{aligned} PP^{-1} = PP^T &= I \\ \det\left(PP^T\right) &= \det(I) \\ \det(P) \cdot \det\left(P^T\right) &= 1 \\ \det(P) \cdot \det(P) = \det(P)^2 &= 1 \\ 1 - \det(P)^2 = (1 + \det(P))(1 - \det(P)) &= 0 \\ \det(P) &= \pm 1 \end{aligned} \tag{8}$$

Therefore the determinant of a permutation matrix is always ± 1.

Part (c)

The identity matrix, having determinant 1, is an even permutation. We can then express p as:

$$p = \tau_{i_1} \circ \tau_{i_2} \circ \ldots \circ \tau_{i_k} \circ I \tag{9}$$

Since $\tau_{i_k} \circ I$ is obtained from I by interchanging two different rows one time, $\det(\tau_{i_k} \circ I) = -\det(I) = -1$. This can be extended until p, which is obtained from I by interchanging two different rows k times, which means that $\det(p) =$

$(-1)^k \det(I) = (-1)^k$. Evidently, $\operatorname{sgn}(p) = \det(p) = 1$ when k is even, and $\operatorname{sgn}(p) = \det(p) = -1$ when k is odd. Therefore p is even if k is even, and p is odd if k is odd.

Let there be some even p and suppose that k is odd. Since $p = \tau_{i_1} \circ \tau_{i_2} \circ \ldots \circ \tau_{i_k} \circ I$ is obtained from I by interchanging two different rows k times, $\det(p) = (-1)^k \det(I) = (-1)^k$. Since k is odd, $\operatorname{sgn}(p) = \det(p) = -1$. However, this is a contradiction; as p is even, $\operatorname{sgn}(p) = 1$. Consequently k must be even. Now let there be some odd p and suppose that k is even. Since $p = \tau_{i_1} \circ \tau_{i_2} \circ \ldots \circ \tau_{i_k} \circ I$ is obtained from I by interchanging two different rows k times, $\det(p) = (-1)^k \det(I) = (-1)^k$. Since k is even, $\operatorname{sgn}(p) = \det(p) = 1$. However, this is a contradiction; as p is odd, $\operatorname{sgn}(p) = -1$. Consequently k must be odd.

Therefore k is even if p is even, and k is odd if p is odd, which means that p is even iff k is even, and p is odd iff k is odd.