HW 4

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Exercise 1

Part (a)

Let

$$\begin{aligned} \varphi : \mathbb{Z}/10\mathbb{Z} &\to \mathbb{Z}/7\mathbb{Z} \\ \bar{k} &\mapsto \bar{k} \end{aligned} \tag{1}$$

Note that $5 = 15 \mod 10$, since $5 = 10\,(0) + 5$ and $15 = 10\,(1) + 5$, which means that $\overline{5} = \overline{15}$ in $\mathbb{Z}/10\mathbb{Z}$. However, $5 \neq 15 \mod 7$, since $5 = 7\,(0) + 5$ and $15 = 7\,(2) + 1$, which means that $\overline{5} \neq \overline{15}$ in $\mathbb{Z}/7\mathbb{Z}$. Since $\overline{5} = \overline{15}$ but $\varphi\left(\overline{5}\right) \neq \varphi\left(\overline{15}\right)$, φ is not well-defined, as a single element maps to multiple elements in the codomain.

Part (b)

Let

$$\begin{aligned} \varphi : \mathbb{Z}/10\mathbb{Z} &\to \mathbb{Z}/7\mathbb{Z} \\ \bar{k} &\mapsto \bar{k} \end{aligned} \tag{2}$$

Let $\overline{x}, \overline{y} \in \mathbb{Z}/10\mathbb{Z}$ such that $\overline{x} = \overline{y}$ in $\mathbb{Z}/10\mathbb{Z}$. Then, since $x = y \mod 10$, $x = 10\,(m) + p$ and $y = 10\,(n) + p$ for $m, n, p \in \mathbb{Z}$. Then, since $x = 10\,(m) + p = 5\,(2m) + p$ and $y = 10\,(n) + p = 5\,(2n) + p$ with $2m, 2n \in \mathbb{Z}$, $x = y \mod 5$, which means that $\overline{x} = \overline{y}$ in $\mathbb{Z}/5\mathbb{Z}$. Consequently, since $\overline{x} = \overline{y}$ implies that $\varphi\left(\overline{x}\right) = \varphi\left(\overline{y}\right)$, φ is well-defined.

Exercise 2

Part (a)

Let

$$\begin{aligned} \varphi : \mathbb{Z} &\to \mathbb{Z}/3\mathbb{Z} \\ \bar{k} &\mapsto \bar{k} \end{aligned} \tag{3}$$

Let $\alpha : \mathbb{Z} \to \mathbb{Z}/9\mathbb{Z}$, $k \mapsto \bar{k}$ and $\beta : \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$, $\bar{k} \mapsto \bar{k}$. We first show that α and β are well defined.

Let $x, y \in \mathbb{Z}$ such that $x = y$. Then, since $x = y$, $x = 9(m) + p$ and $y = 9(m) + p$ for $m, p \in \mathbb{Z}$. Then, $x = y \mod 9$, which means that $\overline{x} = \overline{y}$ in $\mathbb{Z}/9\mathbb{Z}$. Consequently, since $x = y$ implies that $\varphi(x) = \varphi(y)$, α is well-defined.

Let $\overline{x}, \overline{y} \in \mathbb{Z}/9\mathbb{Z}$ such that $\overline{x} = \overline{y}$. Then, since $\overline{x} = \overline{y}$, $x = 9(m) + p$ and $y = 9(n) + p$ for $m, n, p \in \mathbb{Z}$. Then, since $x = 9(m) + p = 3(3m) + p$ and $y = 9(n) + p = 3(3n) + p$ with $3m, 3n \in \mathbb{Z}$, $x = y \mod 3$, which means that $\overline{x} = \overline{y}$ in $\mathbb{Z}/3\mathbb{Z}$. Consequently, since $\overline{x} = \overline{y}$ implies that $\varphi(\overline{x}) = \varphi(\overline{y})$, β is well-defined.

Let $a, b \in \mathbb{Z}$ and $\overline{c}, \overline{d} \in \mathbb{Z}/9\mathbb{Z}$. Then, since $\overline{x}\overline{y} = \overline{xy}$:

$$\begin{aligned} \alpha(a)\,\alpha(b) = \overline{a}\overline{b} = \overline{ab} = \alpha(ab) \\ \beta(\overline{c})\,\beta\left(\overline{d}\right) = \overline{c}\overline{d} = \overline{cd} = \beta\left(\overline{cd}\right) \end{aligned} \tag{4}$$

Therefore α and β are homomorphisms. Since $\beta \circ \alpha(x) = \overline{x} = \varphi(x)$, $\beta \circ \alpha = \varphi$, and the homomorphism φ factors through $\mathbb{Z}/9\mathbb{Z}$.

Part (b)

Let $\ell \in \mathbb{N}$. Let $\alpha : \mathbb{Z} \to \mathbb{Z}/3^\ell\mathbb{Z}$, $k \mapsto \bar{k}$ and $\beta : \mathbb{Z}/3^\ell\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$, $\bar{k} \mapsto \bar{k}$. We first show that α and β are well defined.

Let $x, y \in \mathbb{Z}$ such that $x = y$. Then, since $x = y$, $x = 3^\ell(m) + p$ and $y = 3^\ell(m) + p$ for $m, p \in \mathbb{Z}$. Then, $x = y \mod 3^\ell$, which means that $\overline{x} = \overline{y}$ in $\mathbb{Z}/3^\ell\mathbb{Z}$. Consequently, since $x = y$ implies that $\varphi(x) = \varphi(y)$, α is well-defined.

Let $\overline{x}, \overline{y} \in \mathbb{Z}/3^\ell\mathbb{Z}$ such that $\overline{x} = \overline{y}$. Then, since $\overline{x} = \overline{y}$, $x = 3^\ell(m) + p$ and $y = 3^\ell(n) + p$ for $m, n, p \in \mathbb{Z}$. Then, since $x = 3^\ell(m) + p = 3\left(3^{\ell-1}m\right) + p$ and $y = 3^\ell(n) + p = 3\left(3^{\ell-1}n\right) + p$ with $3^{\ell-1}m, 3^{\ell-1}n \in \mathbb{Z}$, $x = y \mod 3$, which means that $\overline{x} = \overline{y}$ in $\mathbb{Z}/3\mathbb{Z}$. Consequently, since $\overline{x} = \overline{y}$ implies that $\varphi(\overline{x}) = \varphi(\overline{y})$, β is well-defined.

Let $a, b \in \mathbb{Z}$ and $\overline{c}, \overline{d} \in \mathbb{Z}/3^\ell\mathbb{Z}$. Then, since $\overline{x}\overline{y} = \overline{xy}$:

$$\begin{aligned} \alpha(a)\,\alpha(b) = \overline{a}\overline{b} = \overline{ab} = \alpha(ab) \\ \beta(\overline{c})\,\beta\left(\overline{d}\right) = \overline{c}\overline{d} = \overline{cd} = \beta\left(\overline{cd}\right) \end{aligned} \tag{5}$$

Therefore α and β are homomorphisms. Since $\beta \circ \alpha(x) = \overline{x} = \varphi(x)$, $\beta \circ \alpha = \varphi$, and the homomorphism φ factors through $\mathbb{Z}/3^\ell\mathbb{Z}$ for all $\ell \in \mathbb{N}$.