Homework 1

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Exercise 1

Part (a)

Let M be an $n \times n$ matrix, with eigenvectors $u = \{u_1, \ldots, u_n\}$ and corresponding eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_n\}$. Then, since $Mu_i = \lambda_i u_i$:

$$M \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \ldots & u_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 u_1 & \lambda_2 u_2 & \ldots & \lambda_n u_n \\ | & | & & | \end{bmatrix} \tag{1}$$

Part (b)

For some small n:

$$\begin{gathered} M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \\ \lambda = \{-1, 3\} \\ u = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} & -1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix} \end{gathered} \tag{2}$$

$$\begin{gathered} M = \begin{bmatrix} 2 & 2 \\ 8 & 2 \end{bmatrix} \\ \lambda = \{-2, 6\} \\ u = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \\ \begin{bmatrix} 2 & 2 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -4 & 12 \end{bmatrix} = \begin{bmatrix} -2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} & 6 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} \end{gathered} \tag{3}$$

Exercise 2

***Algebra* (Artin, 2e) Exercise 8.1.1 (extended)**

Part (a)

Let $\langle v, w\rangle$ be a bilinear form on a real vector space V. Let $\varphi(v, w) = \langle v, w\rangle + \langle w, v\rangle$. For arbitrary $r \in \mathbb{R}$, $v, w, v', w' \in V$:

$$\begin{aligned}
\varphi(v, w) &= \langle v, w\rangle + \langle w, v\rangle \\
\varphi(rv, w) &= \langle rv, w\rangle + \langle w, rv\rangle \\
&= r\langle v, w\rangle + r\langle w, v\rangle \\
&= r(\langle v, w\rangle + \langle w, v\rangle) = r\varphi(v, w) \\
\varphi(v, rw) &= \langle v, rw\rangle + \langle rw, v\rangle \\
&= r\langle v, w\rangle + r\langle w, v\rangle \\
&= r(\langle v, w\rangle + \langle w, v\rangle) = r\varphi(v, w) \\
\varphi(v + v', w) &= \langle v + v', w\rangle + \langle w, v + v'\rangle \\
&= \langle v, w\rangle + \langle v', w\rangle + \langle w, v\rangle + \langle w, v'\rangle \\
&= (\langle v, w\rangle + \langle w, v\rangle) + (\langle v', w\rangle + \langle w, v'\rangle) = \varphi(v, w) + \varphi(v', w) \\
\varphi(v, w + w') &= \langle v, w + w'\rangle + \langle w + w', v\rangle \\
&= \langle v, w\rangle + \langle v, w'\rangle + \langle w, v\rangle + \langle w', v\rangle \\
&= (\langle v, w\rangle + \langle w, v\rangle) + (\langle v, w'\rangle + \langle w', v\rangle) = \varphi(v, w) + \varphi(v, w')
\end{aligned} \tag{4}$$

Since the map φ is bilinear in both variables, it is a bilinear form.

Part (b)

For an arbitrary bilinear form $\langle v, w\rangle$, consider the following bilinear forms (which can be proven to be so similarly to the proof in part (a)):

$$\begin{aligned}
\varphi(v, w) &= \frac{1}{2}(\langle v, w\rangle + \langle w, v\rangle) \\
\varphi(w, v) &= \frac{1}{2}(\langle w, v\rangle + \langle v, w\rangle) = \varphi(v, w) \\
\phi(v, w) &= \frac{1}{2}(\langle v, w\rangle - \langle w, v\rangle) \\
\phi(w, v) &= \frac{1}{2}(\langle w, v\rangle - \langle v, w\rangle) = -\phi(v, w)
\end{aligned} \tag{5}$$

Since φ is evidently symmetric and ϕ skew-symmetric, and $\langle v, w\rangle = \varphi(v, w) + \phi(v, w)$, every bilinear form on a real vector space is the sum of a symmetric form and a skew symmetric form.

Exercise 3

***Algebra* (Artin, 2e) Exercise 8.3.4**

Let invertible matrix A. Then:

$$\begin{aligned} \left(A^* A\right)^* &= A^* \left(A^*\right)^* \\ &= A^* A \end{aligned} \tag{6}$$

Since $\left(A^* A\right)^* = A^* A$, $A^* A$ is Hermitian. Let nonzero column vector v:

$$v^* A^* A v = \left(A v\right)^* \left(A v\right) \tag{7}$$

Since the nullspace of an invertible matrix is the zero vector, $w^* w > 0$ for all nonzero $w = Av$ and $v^* A^* A v = \left(Av\right)^* \left(Av\right) > 0$, which means that $A^* A$ is positive definite.

Exercise 4

Part (a)

Let W be the subspace of $\mathbb{R}^3$ (with the standard dot product) that is generated by $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. To find an orthogonal basis from $\{v_1, v_2\}$, we find $\{w_1, w_2\}$ where $w_1 = v_1, w_2 = v_2 - \text{proj}_{v_1} v_2$.

$$\begin{aligned} v_1 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ w_1 = v_1 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ w_2 = v_2 - \text{proj}_{v_1} v_2 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{aligned} \tag{8}$$

The orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$.

Part (b)

The projection of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ onto W is:

$$\begin{aligned} \text{proj}_W \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= \text{proj}_{w_1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \text{proj}_{w_2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{\frac{1}{2}}{\frac{3}{2}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \end{aligned} \tag{9}$$

Part (c)

An alternative orthogonal basis can be found by changing the order of the vectors in the Gram-Schmidt procedure.

$$\begin{aligned} v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_2 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ w_1 = v_1 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ w_2 = v_2 - \text{proj}_{v_1} v_2 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \end{aligned} \tag{10}$$

The orthogonal basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\}$. The projection of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ onto W is:

$$\begin{aligned} \text{proj}_W \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= \text{proj}_{w_1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \text{proj}_{w_2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{\frac{1}{2}}{\frac{3}{2}} \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \end{aligned} \tag{11}$$

The result is the same, regardless of the orthogonal basis used.

Exercise 5

Take the bilinear form with associated matrix of the form A with respect to the standard $\mathbb{R}^3$ basis and subspace W where:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ W &= \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \ : \ y = -z\} \end{aligned} \tag{12}$$

However, for all $w \in W$:

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}^T Aw = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} w = 0 \tag{13}$$

The orthogonal complement is then $W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \ : \ x = 0, y = -z \right\}$.

Consequently, $W \cap W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \ : \ x = 0, y = -z \right\}$, which is one-dimensional.