Homework 6

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Exercise 1

Algebra (Artin, 2e) Exercise 6.3.4 (extended)

A glide reflection can be written as $g = (t_a\rho_\phi)\, t_v r\, (t_a\rho_\phi)^{-1}$ where a, ϕ are determined by the glide line and v is the glide vector. For all $m \in \text{Isom}\left(\mathbb{R}^2\right)$, $m = t_m\rho_\theta$ or $t_m\rho_\theta r$, for a unique vector a and angle θ. Suppose $m = t_m\rho_\theta$. Then:

$$\begin{aligned} mgm^{-1} &= (t_m\rho_\theta)\,(t_a\rho_\phi)\, t_v r\, (t_a\rho_\phi)^{-1}\,(t_m\rho_\theta)^{-1} \\ &= (t_m\rho_\theta t_a\rho_\phi)\, t_v r\, (t_m\rho_\theta t_a\rho_\phi)^{-1} \\ &= \left(t_m t_{\rho_\theta(a)}\rho_\theta\rho_\phi\right) t_v r \left(t_m t_{\rho_\theta(a)}\rho_\theta\rho_\phi\right)^{-1} \\ &= \left(t_{m+\rho_\theta(a)}\rho_{\theta+\phi}\right) t_v r \left(t_{m+\rho_\theta(a)}\rho_{\theta+\phi}\right)^{-1} \\ &= (t_{a'}\rho_{\phi'})\, t_v r\, (t_{a'}\rho_{\phi'})^{-1} \end{aligned} \tag{1}$$

Evidently, the conjugate is also a glide vector where $a' = m + \rho_\theta(a), \phi' = \theta + \phi$. As the glide vector is the same glide vector, they naturally have the same length. Now suppose $m = t_m\rho_\theta r$. Then:

$$\begin{aligned} mgm^{-1} &= (t_m\rho_\theta r)\,(t_a\rho_\phi)\, t_v r\, (t_a\rho_\phi)^{-1}\,(t_m\rho_\theta r)^{-1} \\ &= (t_m\rho_\theta r t_a\rho_\phi)\, t_v r\, (t_m\rho_\theta r t_a\rho_\phi)^{-1} \\ &= \left(t_m\rho_\theta t_{r(a)} r\rho_\phi\right) t_v r \left(t_m\rho_\theta t_{r(a)} r\rho_\phi\right)^{-1} \\ &= \left(t_m\rho_\theta t_{r(a)}\rho_{-\phi} r\right) t_v r \left(t_m\rho_\theta t_{r(a)}\rho_{-\phi} r\right)^{-1} \\ &= \left(t_m\rho_\theta t_{r(a)}\rho_{-\phi}\right) r t_v r r \left(t_m\rho_\theta t_{r(a)}\rho_{-\phi}\right)^{-1} \\ &= \left(t_m t_{\rho_\theta(r(a))}\rho_\theta\rho_{-\phi}\right) r t_v \left(t_m t_{\rho_\theta(r(a))}\rho_\theta\rho_{-\phi}\right)^{-1} \\ &= \left(t_{m+\rho_\theta(r(a))}\rho_{\theta-\phi}\right) t_{r(v)} r \left(t_{m+\rho_\theta(r(a))}\rho_{\theta-\phi}\right)^{-1} \\ &= (t_{a'}\rho_{\phi'})\, t_v r\, (t_{a'}\rho_{\phi'})^{-1} \end{aligned} \tag{2}$$

Evidently, the conjugate is also a glide vector where $a' = m + \rho_\theta(r(a)), \phi' = \theta - \phi$. As the glide vector is the original glide vector under an isometry, length is preserved and $v, r(v)$ have the same length. In both cases, the conjugate of a glide reflection is a glide reflection that has a glide vector with the same length.

Exercise 2

The generators of Isom $\left(\mathbb{R}^2\right)$ are known to be t_a, ρ_θ, r where:

$$\begin{aligned} t_a(x) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ \rho_\theta(x) &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ r(x) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{3}$$

Let $z \in \mathbb{C}$. Translation in $\mathbb{R}^2$ is addition in $\mathbb{C}$. Consequently:

$$\begin{aligned} t_{z_0}(z) &= z + z_0 \\ \left|t_{z_0}(z') - t_{z_0}(z)\right| &= \left|(z' + z_0) - (z + z_0)\right| \\ &= |z' - z| \end{aligned} \tag{4}$$

Rotation about the origin in $\mathbb{R}^2$ is multiplication in $\mathbb{C}$ by a member of S^1. Consequently:

$$\begin{aligned} \rho_\theta(z) &= ze^{i\theta} \\ \left|\rho_\theta(z') - \rho_\theta(z)\right| &= \left|z'e^{i\theta} - ze^{i\theta}\right| \\ &= \left|(z' - z)\, e^{i\theta}\right| \\ &= |z' - z| \left|e^{i\theta}\right| \\ &= |z' - z| \end{aligned} \tag{5}$$

Reflection about the x-axis in $\mathbb{R}^2$ is taking the complex conjugate in $\mathbb{C}$. Consequently:

$$\begin{aligned} r(z) &= \overline{z} \\ |r(z') - r(z)| = |r(x' + iy') - r(x + iy)| &= |(x' - iy') - (x - iy)| \\ &= |(x' - x) - i(y' - y)| \\ &= \left|\overline{(x' - x) + i(y' - y)}\right| \\ &= |(x' - x) + i(y' - y)| \\ &= |(x' + iy') - (x + iy)| \\ &= |z' - z| \end{aligned} \tag{6}$$

Evidently, all three are isometries of $\mathbb{C}$.

Exercise 3

***Algebra* (Artin, 2e) Exercise 6.6.2**

Part (a)

Let G denote the group of symmetries of the following infinite wallpaper pattern constructed from equilateral triangles of side length 1. By inspection, we see that reflection across the x-axis and rotation by a multiple of $\frac{2\pi}{6} = \frac{\pi}{3}$ are symmetries of the wallpaper pattern. Conseqquently, the point group $\overline{G} = \left\{\overline{1}, \overline{r}, \overline{\rho_{\frac{\pi}{3}}}, \overline{\rho_{\frac{\pi}{3}} r}\right\} = D_6$. The index of L in G is the the order of $\overline{G} = D_6$, which is $2\,(6) = 12$.

Part (b)

By inspection, we see that L is the lattice $\mathbb{Z}a + \mathbb{Z}b$ for $a = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, b = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$.