Question 2

$$J_\pi(x_0) = \mathbb{E}\left[\exp\left(g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(xk), x_{k+1})\right)\right]$$

Define:

$$J_k^*(x_k) = \min_{\pi_k} \mathbb{E}\left[\exp\left(g_N(x_N) + \sum_{i=k}^{N-1} g_i(x_i, \mu_k(xi), x_{i+1})\right)\right]$$

Part A

Now we will prove the DP-algorithm by induction on $N-k$

Inductive Hypothesis: $J_k^* = J_k$ obtained by the given DP algorithm.

Base Case:

$$J_N^*(x_N) = \min_{\pi_N} \mathbb{E}\left[\exp\left(g_N(x_N) + 0\right)\right] = exp\left(g_N(x_N)\right)$$

Inductive Step:

$$J_k^*(x_k) = \min_{\pi_k} \mathbb{E}\left[\exp\left(g_N(x_N) + \sum_{i=k}^{N-1} g_i(x_i, \mu_k(xi), x_{i+1})\right)\right]$$

$$J_k^*(x_k) = \min_{\pi_k} \mathbb{E}\left[\exp\left(g_k(x_k, \mu_k(x_k), x_{k+1}) + g_N(x_N) + \sum_{i=k+1}^{N-1} g_i(x_i, \mu_k(xi), x_{i+1})\right)\right]$$

$$J_k^*(x_k) = \min_{\pi_k} \mathbb{E}\left[\exp(g_k(x_k, \mu_k(x_k), x_{k+1})) * \exp\left(g_N(x_N) + \sum_{i=k+1}^{N-1} g_i(x_i, \mu_k(xi), x_{i+1})\right)\right]$$

$$J_k^*(x_k) =$$

$$\min_{a_k \in A(k)} \mathbb{E}_{x_{k+1}}\left[\exp(g_k(x_k, a_k, x_{k+1})) * min_{\mu_{k+1}}\left(\mathbb{E}\left[\exp\left(g_N(x_N) + \sum_{i=k+1}^{N-1} g_i(x_i, \mu_k(xi), x_{i+1}\right]\right)\right)\right]$$

$$J_k^*(x_k) = \min_{a_k \in A(k)} \mathbb{E}_{x_{k+1}}\left[\exp(g_k(x_k, a_k, x_{k+1})) * J_{k+1}^*(x_{k+1})\right]$$

$$J_k^*(x_k) = \min_{a_k \in A(k)} \mathbb{E}_{x_{k+1}}\left[\exp(g_k(x_k, a_k, x_{k+1})) * J_{k+1}(x_{k+1})\right] = J_k(x_k) \quad (1)$$

Thus by induction DP-Algorithm works. **Q.E.D**

PART B

After taking log since g_k doesn't depend on x_{k+1}, we can take the expectation inside.

$$V_k(x_k) = log(J_k(x_k))$$

$$V_N(x_N) = g_N(x_N)$$

$$V_k(x_k) = log(J_k(x_k))$$

$$V_k(x_k) = log\left(\min_{a_k \in A(k)} \mathbb{E}_{x_{k+1}}\left[\exp(g_k(x_k, a_k)) * J_{k+1}(x_{k+1})\right]\right)$$

Taking log inside and $\mathbb{E}$ to J_{k+1} we get

$$V_k(x_k) = (\min_{a_k \in A(k)}\left(g_k(x_k, a_k) + log\left(\mathbb{E}_{x_{k+1}} \exp([V_{k+1}(x_{k+1}))]\right)\right)$$

Q.E.D