



DEPARTMENT OF INFORMATICS

TECHNISCHE UNIVERSITÄT MÜNCHEN

Master's Thesis in Informatics: Computational Science and  
Engineering

**Thesis title**

**Ravil Dorozhinskii**





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**Titel der Abschlussarbeit**

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Submission Date:	Submission date



I confirm that this master's thesis in informatics: computational science and engineering  
is my own work and I have documented all sources and material used.

Munich, Submission date

Ravil Dorozhinskii

## Acknowledgments

# Abstract

Parallel linear equation solvers are one of the most important components determining the scalability and efficiency of many supercomputing applications.

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In many HPC applications, the optimized single-node performance is one of the most crucial building blocks for the optimization of probabilistic analysis or hybrid MPIOpenMP implementation.

# Nomenclature

- ATHLET Analysis of THermal-hydraulics of LEaks and Transients
- NUT Numerical Toolkit
- PETSc Portable, Extensible Toolkit for Scientific Computation
- MPI Message Passing Interface library
- OpenMP Open Multi-Processing library
- BLAS Basic Linear Algebra Subprograms
- LAPACK Linear Algebra Package
- ScaLAPACK Scalable Linear Algebra Package
- GEMM General Matrix-matrix Multiplication (BLAS subroutine)
- TRSM Triangular Solver with Multiple right-hand sides (LAPACK subroutine)
- GETRF General Triangular Factorization (LAPACK subroutine)
- NUMA non-uniform memory access
- MUMPS MULTifrontal Massively Parallel sparse direct Solver
- WSMP Watson Sparse Matrix Package
- PDE Partial Differential Equation
- SPD Symmetric Positive Definite
- PE Processing Element
- GRS Die Gesellschaft für Anlagen- und Reaktorsicherheit (GRS) gGmbH
- LRZ Leibniz-Rechenzentrum (Leibniz Supercomputing Centre of the Bavarian Academy of Sciences and Humanities)
- HW1 Hardware installed on the GRS cluster
- HW2 Hardware installed on a LRZ CoolMUC-2 Linux cluster

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# 1. Introduction

Nowadays, nuclear energy is one of the main sources of electricity. It comes from splitting atoms in a reactor which, as the results, heats water up to the point when it converts into pressurized steam. In its turn, steam turns turbines which, finally, produce electricity. According to the recent estimations, thermal efficiency of modern nuclear power plants lies in the range of 35-45% which is comparable to conventional fossil fueled power plants [Jor19]. In spite of considerable initial investment, nuclear power plants have low running costs and longevity which makes them particularly cost effective.

In recent years, nuclear power plants have become even more attractive because of relatively low emission of carbon dioxide. As a result, the emissions of green house gases to the atmosphere and thus the contribution of nuclear power plants to global warming is relatively little [Tim07].

Today, nuclear power plants generate almost 30% of the electricity produced in the European Union (EU). There are almost 130 nuclear reactors in operation in 14 the EU, namely: Belgium, Bulgaria, Czech Republic, Finland, France, Germany, Hungary, Netherlands, Romania, Slovakia, Slovenia, Spain, Sweden, and the United Kingdom [Eur18].

The main problem of nuclear power is radioactive waste which is extremely dangerous for people and environment and has to be carefully looked after for several thousand years after utilization. Any accident in a plant can cause of grave consequences at a level similar to Chernobyl disaster. For this reason, nuclear power safety is one of the most important topics in this field. It requires to perform a huge amount of testing and analysis before and during operation of a nuclear power plant in order to predict any possible outcome and device protective means from any type of accidents. The topic has become even more prominent after 2011 Fukushima accident. In response to the disaster, a numerous of stress tests were conducted to measure the ability of the EU nuclear industry to withstand any kind of natural disaster [Eur18].

Since 1977, Gesellschaft für Anlagen- und Reaktorsicherheit (GRS) has been the main

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## 1. Introduction

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German scientific research institute in the field of nuclear safety and radioactive waste management [Ges]. Today, the organization carries out advanced research and analysis in its fields of reactor safety, radioactive waste management as well as radiation and environmental protection [Ges]. Due to a huge amount of testing, different types of possible accidents and disability to conduct natural experiments, GRS provides and develops numerous simulation software products to cope with this problem. Table 1.1 represents the main software packages that have been developed by GRS and their short description.

Name	Description
ASTEC	Integral code for determination of the source term during core meltdown for the primary circuit and containment of Light-Water Reactors (LWRs)
ATHLET	Thermohydraulic safety analyses for the primary circuit of LWRs
ATHLET-CD	Analyses of accidents with core meltdown and fission product release for LWRs
ATLAS	Analysis simulator for interactive handling and visualisation of several computer codes
COCOSYS	Analyses of severe incidents in the containment of LWRs
DORT/TORT	Solution of time-dependant neutron transport equations for 2D/3D transients analyses
QUABOX/CUBBOX	3-D neutron kinetics core model
SUSA	Uncertainty and sensitivity analyses
TESPA-ROD	Core rod code for design basis accidents

Table 1.1.: A general overview of software developed by GRS [Ges]

The main focus of this study is dedicated to ATHLET software package as well as its Numerical Toolkit. During the study, we will try to identify the most computer-intensive part of the ATHLET-NuT code and accelerate its execution time.

## 2. Overview of ATHLET and NuT software

### 2.1. ATHLET

The thermal-hydraulic system code ATHLET (Analysis of THermal-hydraulics of LEaks and Transients) is developed by GRS for the analysis of the whole spectrum of operational conditions, incidental transients, design-basis accidents and beyond design-basis accidents without core damage anticipated for nuclear energy facilities [Ges16]. The code provides specific models and methods for the simulation of many types of nuclear power plants comprising current light water reactors (PWR, BWR, VVER, RBMK), advanced Generation III+ and IV reactors as well as Small Modular Reactors [Ges16].

Physical processes inside of hydraulic circuits of light-water reactors can be naturally described by a two-phase thermo-fluiddynamic model based on conservation equations of mass, momentum and energy for liquid and vapor.

1. Liquid mass

$$\frac{\partial((1-\alpha)\rho_l)}{\partial t} + \nabla((1-\alpha)\rho_l\vec{w}_l) = -\psi \quad (2.1)$$

2. Vapor mass

$$\frac{\partial(\alpha\rho_v)}{\partial t} + \nabla(\alpha\rho_v\vec{w}_v) = \psi \quad (2.2)$$

3. Liquid momentum

$$\frac{\partial((1-\alpha)\rho_l\vec{w}_l)}{\partial t} + \nabla((1-\alpha)\rho_l\vec{w}_l\vec{w}_l) + \nabla((1-\alpha)p) = \vec{F}_l \quad (2.3)$$

4. Vapor momentum

$$\frac{\partial(\alpha\rho_v\vec{w}_v)}{\partial t} + \nabla(\alpha\rho_v\vec{w}_v\vec{w}_v) + \nabla(\alpha p) = \vec{F}_v \quad (2.4)$$

5. Liquid energy

$$\frac{\partial\left[(1-\alpha)\rho_l(h_l + \frac{1}{2}\vec{w}_l\vec{w}_l - \frac{p}{\rho_l})\right]}{\partial t} + \nabla\left[(1-\alpha)\rho_l\vec{w}_l(h_l + \frac{1}{2}\vec{w}_l\vec{w}_l)\right] = -p\frac{\partial(1-\alpha)}{\partial t} + E_l \quad (2.5)$$

### 6. Vapor energy

$$\frac{\partial \left[ \alpha \rho_v (h_v + \frac{1}{2} \vec{w}_v \vec{w}_v - \frac{p}{\rho_v}) \right]}{\partial t} + \nabla \left[ \alpha \rho_v \vec{w}_v (h_v + \frac{1}{2} \vec{w}_v \vec{w}_v) \right] = -p \frac{\partial \alpha}{\partial t} + E_v \quad (2.6)$$

### 7. Volume vapor fraction

$$\alpha = \frac{V_v}{V} \quad (2.7)$$

where  $p$  - pressure of mixture,  $\psi$  - mass source term,  $\vec{F}$  - external composite force acted on a control volume,  $E$  - external composite energy source term within a control volume, subscripts  $l$  and  $v$  denote liquid and vapor phases, respectively.

Spacial integration of the conservation equations, the system 2.1 - 2.7, is performed on the basis of finite-volume method with using one dimensional formulation, figure 2.1.

place for  
figure 2.1

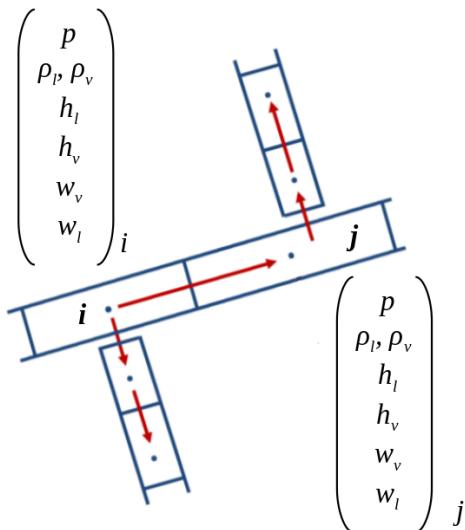


Figure 2.1.: ATHLET: one dimensional finite volume formulation of the problem

Finally, the system is transformed to a non-autonomous system of ordinary differential equations and expressed as an initial value problem, equation 2.8, after spacial finite-volume integration and many mathematical transformations with the aim of decoupling the initial system REF.

$$\frac{dy}{dt} = f(t, y), \quad t_0 \leq t \leq t_F \quad y(t_0) = y_0 \quad (2.8)$$

where  $y \in \mathbb{R}^N$  is a composite vector of variables,  $f$  is a non-linear function such that  $f : \mathbb{R} \times \mathbb{R}^N \supset \Omega \rightarrow \mathbb{R}^N$ .

Analysis of system 2.8 shows the problem is rather stiff and thus must to be solved with an implicit solver. Rosenbrock methods are a class of linear implicit methods which is capable of solving such stiff systems of ODEs efficiently. The methods replace non-linear systems with a sequence of linear ones, however, some stability and accuracy properties are usually lost [Blo+13]. An additional drawback of the methods is evaluation of the exact Jacobian at every time step which affects computational cost.

To decrease the cost and preserve sufficient accuracy of numerical integration, ATHLET, instead, uses a W-method of the third order. W-methods belong to the family of Rosenbrock methods, however, calculate the Jacobian matrix occasionally. The ATHLET developers spent much of their time and efforts to develop heuristics to identify instances of time when evaluation of the Jacobian must be performed. In other words, the algorithm can re-use the same Jacobian matrix approximation between steps with some partial matrix updates. However, when a hydraulic circuit state drastically changes due to transitivity, the evaluation of the full Jacobian is required.

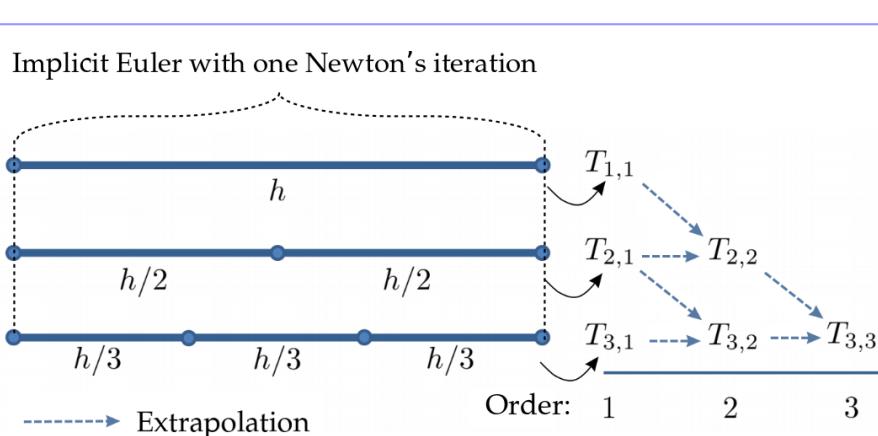


Figure 2.2.: A general view on the W-method solver implemented in ATHLET

In the general case, a step of the W-method method, implemented in ATHLET, can be viewed as a sequence of three stages in the following way. Each stage uses implicit Euler method with different sub-steps and exactly one Newton's iteration to evaluate the value of vector  $y$  at the next integration step with different accuracy. Then, the obtained values are extrapolated, in order explained in figure 2.2, to achieve desired

order of integration. By and large, the algorithm can be expressed in a compact form of equation 2.9.

$$((h\gamma)^{-1}I - J)\Delta z_i^l = -h^{-1}z_i^l + f(t_0 + \tau_i h, y_0 + z_i^l) \quad (2.9)$$

where  $\Delta z_i^l = z_i^{l+1} - z_i^l$ ,  $z_i^l = y_i^l - y_{i-1}^l$ ,  $J \approx \frac{\partial f}{\partial y}$  - approximation of Jacobian matrix,  $l = 1, 2$  - Newton's iteration index,  $i = 1, 2, 3$  - integration step index.

## 2.2. NuT

Numerical Toolkit, or just NuT, can be viewed as a container of various dense and sparse linear algebra subroutines which can run in parallel on distributed-memory machines. NuT design follows ideas of the *Adapter/Wrapper* pattern which provides a uniform common interface for its services to various GRS simulation tools, table 1.1, and thus helps to achieve re-usability, flexibility and extensibility properties of the code.

Currently, NuT is based heavily on Portable, Extensible Toolkit for Scientific Computation, known as PETSc library. It is known as the world's most widely used parallel numerical software library [Wik18c]. It includes a large suite of parallel linear and nonlinear equation solvers as well as its software-infrastructure to handle computations on distributed-memory machines by means of Message Passing Interface (MPI) and specific data structures. Fortunately, because of an appropriate choice of the design pattern, NuT can be easily extended to provide an extra service or an external library access which has not been available in PETSc yet.

## 2.3. ATHLET-NuT coupling

Coupling of NuT with other GRS tools can be viewed as a client-server architecture where NuT acts as a server and the tools can be treated as clients. Communication between two parts is performed via Message Passing Interface.

place for figure 2.3

To provide a clear and concise external interface, NuT contains a client module called "NuT Plug-in" which is viewed as a socket from the client side using the analogy of Transmission Control Protocol (TCP). Plug-in hides all MPI calls to the sever which considerably improves readability of the code.

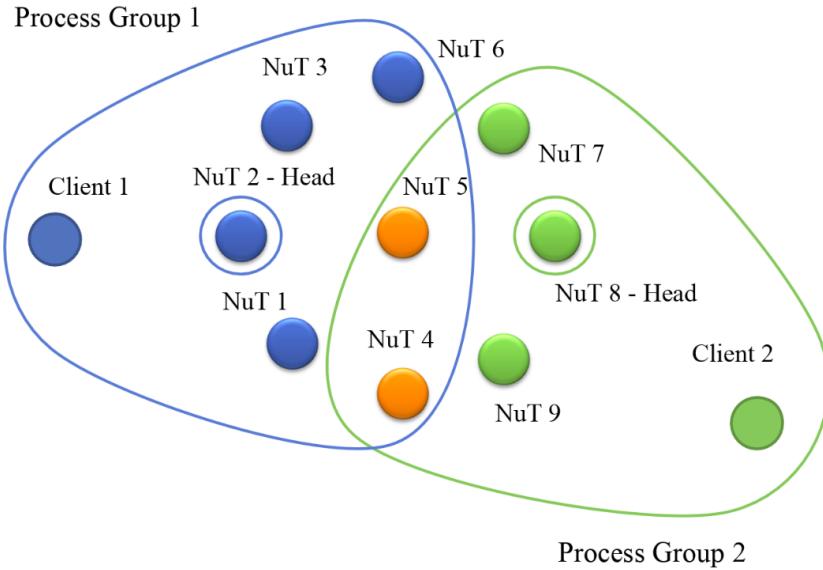


Figure 2.3.: NuT process groups

In the general case, NuT allows multiple clients to work concurrently with the server. To handle the traffic, the library splits the default MPI communicator into appropriate process groups, as it is shown in figure 2.3, at start-up time of the application.

The design of NuT allows to share some NuT-MPI processes between different process groups due to performance reasons i.e. finite number of processing units on hardware. To resolve possible deadlocks, each process group has its representative, called the head. Each client has two views on its group which is done by means of distinct MPI communicators. The first communicator is responsible for client-head communication whereas the second one allows the client to talk to any NuT process within the group.

A general view of client-server communication looks like a 3-way handshake in the following way. A client sends a request to the head which is a signal to reserve all compute-units of the group for an upcoming task. Having possessed the resources and prepared them for a specific service, the head notifies the client about resource acquisition and the entire process group waits for data. Afterwards, the client sends data either to a specific NuT-process or to the entire group using the second communicator and waits for a result of the service. In the current implementation of NuT, the

## 2. Overview of ATHLET and NuT software

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communication between client and server is synchronous i.e. the client gets blocked while waiting for a result from the server.

As an example, figure 2.4 represent a general view of ATHLET-NuT coupling where ATHLET is responsible for marching of the numerical integration solver whereas NuT computes solutions of systems 2.9.

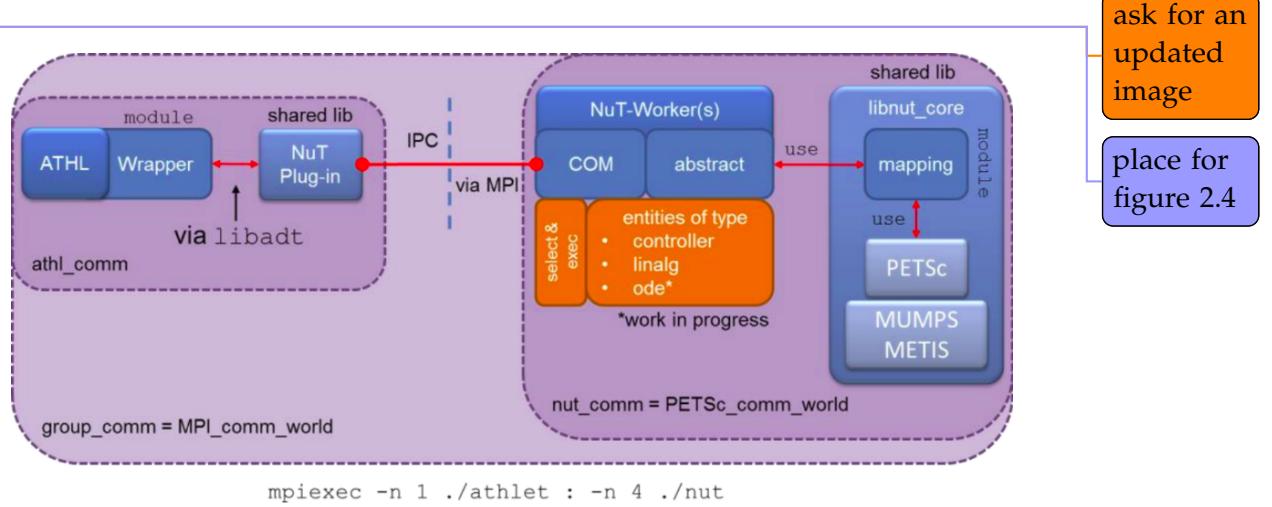


Figure 2.4.: ATHLET-NuT software coupling

Partial and full Jacobian matrix updates, by means of finite differences, are computed on the client side since only the client has the access to function  $f(t, y)$ , equation 2.8. Due to decoupling of the underlying system of PDEs and specifics of finite volume discretization, Jacobian matrix is rather sparse and, therefore, ATHLET uses Jacobian matrix compression algorithm, described in chapter BRA, to reduce the amount of Jacobian columns evaluations. Having computed a matrix column, ATHLET immediately broadcasts it to its entire NuT process group, by means of 3-way handshake mechanism described above. It is worth mentioning this approach allows to circumvent potential memory limits on the client side and thus store the entire sparse Jacobian matrix in distributed manner on the server. In other words, ATHLET never holds the entire Jacobian matrix in its memory; conversely, the matrix is distributed across multiple NuT processes according to block-row distribution induced by PETSc.

In its turn, NuT is waiting for the entire Jacobian matrix information from ATHLET and starts solution of systems 2.9 right after that.

### 3. Problem Statement

Integration of a system of ODEs by means of W-methods can be considered as a solution of a sequence of linear systems from another point of view. Equations 2.9 can be rewritten in a form 3.1, after grouping both the right- and left-hand sides in a single matrix and vector, respectively.

$$A_i \Delta z_i^l = b_i^l \quad (3.1)$$

where  $A = ((h\gamma)^{-1}I - J)$  is a  $\mathbb{R}^N \times \mathbb{R}^N$  non-singular sparse matrix,  $\Delta z_i^l$  and  $b_i^l$  are  $\mathbb{R}^N$  vectors.

According to the integration scheme, figure 2.2, and definition of the method, each step of numerical integration requires to solve 6 linear systems with 3 distinct matrices, arisen from the Jacobian matrix by means of the corresponding shift of the main diagonal. Therefore, the computational burden of the W-method mainly lies on solution of sparse linear systems.

It is well known there exist two families of linear sparse solver, namely: iterative and direct sparse methods. In general case, execution time of any algorithm, regardless of the solver family, is bounded by  $O(N^2)$  complexity due to matrix sparsity, where  $N$  is number of equations in the system. However, the constant in front of the factor  $N^2$  can vary significantly between the methods which has its direct effect on solver execution time. Additionally, it is important to mention the families use absolutely different approaches for solution of sparse linear systems and thus posses different numerical properties. Among all properties there are some which are particular important for efficient execution of W-methods, namely:

- robustness (or numerical stability) with respect to ill-conditioner problems
- numerical accuracy
- parallel efficiency, with emphasis on strong scaling

### *3. Problem Statement*

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This, above mentioned, properties can be treated as non-functional requirements to sparse linear solver for efficient numerical time integration.

Finding solutions of sparse linear systems is a well-known and commonly occurring problem in the field of scientific computing. For that reason, numerous implementations of different kind of linear solvers exist. However, the NuT project imposes some extra constrains due to its software design:

- open-source license
- direct interface to PETSc

In this study, we are primarily concerned with selection and configuration of a linear sparse solver that could cover all requirements listed above; this, as we have described above, is the main source of runtime acceleration.

This report is organized as follows. Chapter 4 provides full information about the experimental setup and matrix sets used in the study. Chapters BRA and BRA explain theory and some parallelization aspects of iterative and sparse direct methods, respectively, using a well-known representative-algorithm of each type as an example. In chapter BRA, we give a short summary and conclude which type of sparse linear solvers is the best suited for time integration governed by the W-method. In chapter BRA, we represent a list of solvers, according to the chosen type, available on the market, at the moment of writing, and perform initial tests with aim of finding the fastest. From chapter BRA onwards, we focus on configuration of a specific solver to reduce its execution time. Chapter BRA sums up overall results of performed configuration and, in chapter BRA, we give some recommendations to the ATHLET users about which solver options are better to use for a specific matrix size: small, medium, large.

## 4. Experimental setup and matrix sets

In this study, we used two matrix sets: GRS and SuiteSparse. In our case, the SiteSparse matrix set was, in fact, few matrices downloaded from SuiteSparse Matrix Collection [DH11], [DGL89]. We tried to choose different matrices from the collection with respect to both the number of equations  $n$  in a system and ratio  $R$  between the number of non-zero elements  $nnz$  and the number of equations.

To generate GRS matrix set, we ran the most common GRS simulations in ATHLET and stopped the simulations somewhere in the middle saving corresponding shifted Jacobian matrices in the PETSc binary format.

Tables 4.1 and 4.2 shows matrix properties of both matrix sets.

Name	n	nnz	nnz / n
pwr-3d	6009	32537	5.4147
cube-5	9325	117897	12.6431
cube-64	100657	1388993	13.7993
cube-645	1000045	13906057	13.9054
k3-2	130101	787997	6.0568
k3-18	1155955	7204723	6.2327

add sparsity plots to appendix

Table 4.1.: GRS matrix set

From now, we are going to introduce and use a definition of **skinny sparse matrices**. It is matrices with relatively low ration  $R$  i.e. less than 15.

add sparsity plots to appendix

The objective of this study was to find and configure a solver which could fulfill all requirements listed above for the GRS matrix set. From time to time, we used the SuiteSparse set for comparison reasons. It seems to us that GRS matrix set was different because of some specifics of 1D pipeline discretization. For example, GRS matrices are

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#### 4. Experimental setup and matrix sets

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Name	n	nnz	nnz / n	Field
cant	62451	4007383	64.1684	2D/3D Problem
consph	83334	6010480	72.1251	2D/3D Problem
CurlCurl_3	1219574	13544618	11.1060	Model Reduction Problem
Geo_1438	1437960	63156690	43.9210	2D/3D Problem
memchip	2707524	13343948	4.9285	Circuit Simulation Problem
PFlow_742	742793	37138461	49.9984	2D/3D Problem
pkustk10	80676	4308984	53.4110	Structural Problem
torso3	259156	4429042	7.0903	2D/3D Problem
x104	108384	8713602	80.3956	Structural Problem

Table 4.2.: SuiteSparse matrix set

skinny and blocked, where each block is a small, approximately 3-by-3, dense matrix. Additionally, some rows can contain only one element i.e. on the diagonal. It is a result of dynamic pipeline switching of a reactor cooling system. As we will see in section 5.2, sparse direct solvers can be quite sensitive to the sparse structure of a matrix.

We used different hardware to measure performance of different solvers. The first machine was the GRS cluster (HW1) which was our main target. We used a LRZ CoolMUC-2 Linux cluster (HW2) every time when we got some ambiguous results in order to check whether a problem was hardware or software specific. Table 4.3 shows a single node specification of both machines.

We decided to stick to the OpenMPI library which is an implementation of the MPI standard. OpenMPI is an open-source project and it is very well documented. The library has many options for processes pinning which was a crucial part in our study because we had to deal with multi-socket machines which, in turn, had multiple NUMA domains.

To make process pinning deterministic, we developed a python script which automatically generated rank-files based on the number of MPI processes, OpenMP threads per MPI process, the maximum number of processing elements and the number of NUMA domains. The script always leaves appropriate gaps between MPI processes to allow each process to fork the corresponding number of threads within a parallel region.

A rank-file specifies explicit mapping between MPI processes (ranks) and actual processing elements (cores) within a machine. The script has two modes, namely:

#### 4. Experimental setup and matrix sets

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	HW1 (GRS)	HW2 (LRZ Linux)
Architecture	x86_64	x86_64
CPU(s)	20	28
On-line CPU(s) list	0-19	0-27
Thread(s) per core	1	1
Core(s) per socket	10	14
Socket(s)	2	2
NUMA node(s)	2	4
Model	62	63
Model name	E5-2680 v2	E5-2697 v3
Stepping	4	2
CPU MHz	1200.0	2036.707
Virtualization	VT-x	VT-x
L1d cache	32K	32K
L1i cache	32K	32K
L2 cache	256K	256K
L3 cache	25600K	17920K
NUMA node0 CPU(s)	0-9	0-6
NUMA node1 CPU(s)	10-19	7-13
NUMA node2 CPU(s)	-	14-20
NUMA node3 CPU(s)	-	21-27

Table 4.3.: Hardware specification

*spread* and *close*. Given a certain number of ranks, the spread mode tries to distribute them as spread as possible across available NUMA domains in round-robin fashion. In contrast to the spread strategy, the close mode groups ranks as close as possible to keep the maximum number of ranks within a single NUMA domain. Figure 4.1 shows an example of how these two modes work in case of 5 MPI ranks, 2 OpenMP threads per rank, on a compute node equipped with 20 cores and 2 NUMA domains (HW1).

We chose **Intel 18.2** compiler for this study as the newest and the most efficient Intel compiler at the time of writing.

place for  
figure 4.1

#### 4. Experimental setup and matrix sets

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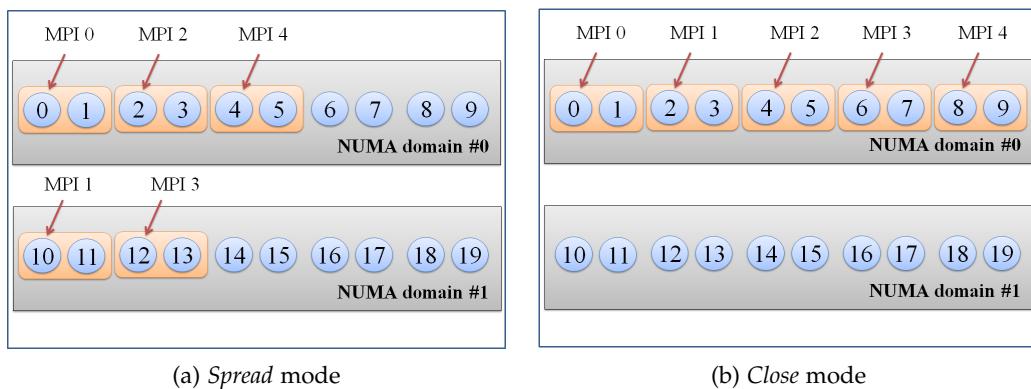


Figure 4.1.: An example of process pinning of 5 MPI processes with 2 OpenMP threads per rank in case of HW1

## 5. Theoretical and parallelization aspects of iterative and direct sparse methods

### 5.1. Iterative methods

Iterative methods, especially Krylov subspace methods that we are going to discuss in this section, are well known for their relatively low storage requirements  $O(nnz)$  and computation cost  $O(N^2)$  in case of sparse linear systems of equations and good condition number. It turns out that sometimes it might be only one way to solve huge systems with millions unknowns.

The most well known methods are Conjugate Gradient (CG) for symmetric positive definite matrices, Minimal Residual Method (MINRES) for symmetric indefinite systems, Generalized Minimal Residual Method (GMRES) for non-symmetric systems of linear equations as well as different variants of GMRES such Biconjugate Gradient Method (BiCG), Biconjugate Gradient Stabilized Method (BiCGSTAB) and so on.

All Krylov methods solve a system of equation as a minimization problem. For example, the goal of CG algorithm is to minimize the energy functional  $f(x) = 0.5x^T Ax - b^T x + c$ , whereas, MINRES and GMRES tries to minimize residual norm  $r_j$  for  $x_j$  in a subspace.

The methods construct an approximate solution of a system as a linear combination of vectors  $b$ ,  $Ab$ ,  $A^2b$ ,  $A^3b$  and so on which defines the Krylov subspace. At each iteration we expand the subspace adding and evaluating a next vector in the combination.

Let's consider GMRES, as the most popular and general iterative solver, without preconditioning to just analyze its strong scaling behavior and potential problems.

As we mentioned above GMRES minimizes the residual norm in a subspace  $U_m$ .

$$\min_{x \in U_m} ||Ax - b||^2 \quad (5.1)$$

We can consider a solution vector  $x$  in the subspace  $U_m$  in a form  $x = U_m y$ . Thus, equation 5.1 can be written as following:

$$\min_{x \in U_m} \|AU_m y - b\|^2 \quad (5.2)$$

The most natural way to choose a proper subspace  $U_m$  is the corresponding Krylov subspace  $\mathcal{K}_m$  because it can be easily generated on the fly. However, decomposition of vector  $x$  in that subspace can be a problem. Since the subspace  $\mathcal{K}_m$  is spanned by the sequence of  $b, Ab, A^2b, \dots, A^{m-1}b$  and due to round-off error the sequence can become linear dependent. Therefore, we have to compute and use the orthonormal base of the given Krylov subspace. Saad and Schultz in their work [SS86] used Arnoldi process for constructing an  $l_2$ -orthogonal basis. As the results equation 5.2 can be written in the following form:

$$\min_{x \in U_m} \|U_{m+1}H_{m+1,m}y - \|b\|u_1\|^2 = \min_{x \in U_m} \|H_{m+1,m}y - \|b\|e_1\|^2 \quad (5.3)$$

where  $H_m$  is an upper Hessenberg matrix. We can apply Givens rotation algorithm to compute QR decomposition to convert  $H_m$  to a strictly upper triangular matrix. Thus,

$$\min_{x \in K_m} \|Ax - b\|^2 = \min_{x \in U_m} \|Q^T R y - \|b\|e_1\|^2 = \min_{x \in U_m} \left\| \begin{pmatrix} R_m \\ 0 \end{pmatrix} y - \begin{pmatrix} \tilde{b}_m \\ \tilde{b}_{n-m} \end{pmatrix} \right\|^2 \quad (5.4)$$

Given 5.4, we can compute the solution as following:

$$R_m y = \tilde{b}_m \quad (5.5)$$

$$x_m = U_m y \quad (5.6)$$

Because of large computational and storage costs, in case of evaluation of the full Krylov subspace, only small a subspace is computed, typically first 20 - 50 column vectors. Then the algorithm is restarted using the computed approximate solution as a initial guess for the next iteration.

We can see that some operations, for example 5.6, can be efficiently done in parallel. However, operations like sparse triangular solve 5.5 can introduce some effect on strong scaling behavior. Figure 5.1 shows strong scaling performance results of a sparse parallel triangular solver with a two dimensional matrix distribution. Performance considerations of the solver can be found in [Jos+98].

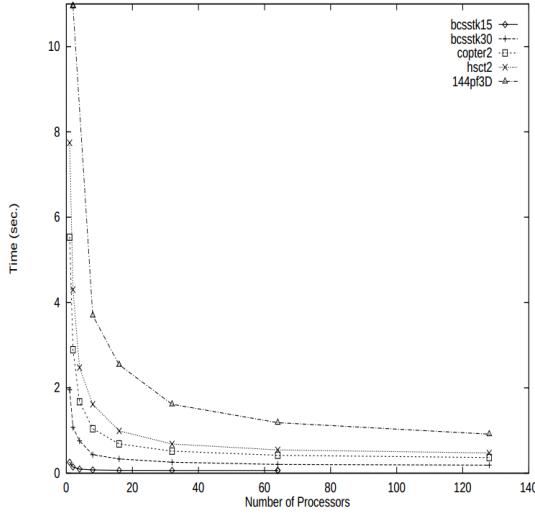


Figure 5.1.: Performance of sparse triangular solve [Jos+98]

It is interesting to notice that performance of the triangular solver depends on a matrix sparsity structure as well as the matrix size.

place for figure 5.1

Triangular solve 5.5 can be computed in a single processor because matrix  $R_m$  is usually small and depends on the number iterations before the restart. In this case the triangular solve can become a bottleneck again.

Figure 5.2 shows strong scaling performance results of the default GMRES solver from the PETSc library. The solver was set up without any preconditioner and 50 iterations as the restart. Additionally no stop criteria was specified except the maximum number iterations which was equal to 100. The *spread* process pinning strategy, described in section 4, was used. It is well-known that all iterative methods are memory bound due to indirect memory addressing caused by sparse matrix storage schemes. Hence equal process distribution can help reduce the load on memory channels since it is an obvious bottleneck for this type of applications.

Four matrices were chosen from the GRS matrix set for the tests, namely: cube-64, cube-645, k3-2 and k3-18. The information about the matrices is summarized in table 4.1. As we expected, we can observe strong deviation of our results from the ideal speed-up line when the number of processes exceeds 10.

It should be mentioned that parallelization overheads, introduced by such MPI operations as MPI\_Send, MPI\_Recv, MPI\_Allreduce, etc., also have their impact on performance of the algorithm.

place for figure 5.2

Other Krylov methods such as CG, for example, scales much better than GMRES. Because of the nature of the CG algorithm the next search direction can be found using a recurrent expression and the algorithms boils down to simple operations such as dot products and matrix vector multiplications. These operations can be easily parallelized and drop of performance comes only from MPI overheads. A quite comprehensive study about parallel CG algorithm performance can be found in [CCV18]. The authors also introduced a deeply pipelined version of CG algorithm that scales even better due to overlapping the time-consuming global communication phase, induced by parallel dot product computations, with useful independent computations [CCV18].

place for figure 5.2

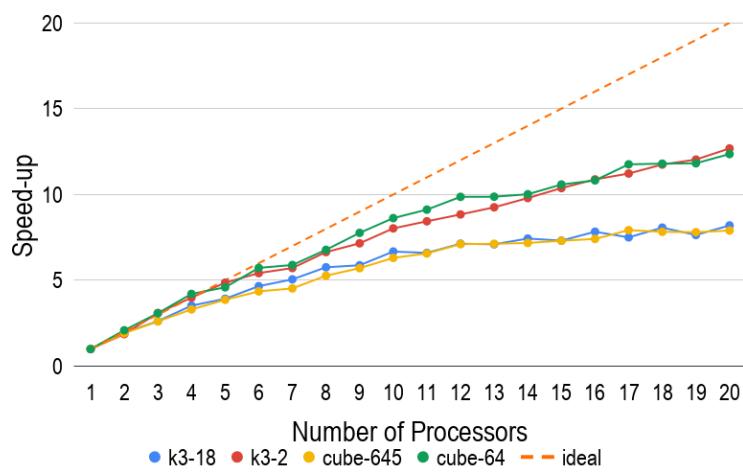


Figure 5.2.: GMRES strong scaling speed-up

The most important criteria of Krylov methods is convergence rate. The convergence rate of iterative methods strongly depends on a matrix and, in particular, on its condition number. For instance, equation 5.7 shows dependence of the convergence rate from the matrix condition number. It can be clearly seen that a big condition number leads to very slow error reduction and, as the results, to huge number of iterations.

$$\|e^i\|_A \leq 2 \left( \frac{\sqrt{k} - 1}{\sqrt{k} + 1} \right)^i \|e^0\|_A \quad (5.7)$$

where  $k = \frac{\lambda_{\max}}{\lambda_{\min}}$  - condition number of the corresponding matrix.

An obvious solution of such a problem is to reduce the condition number of the original system ?? . A general method is to transform the original system in such a way that the conditional number of the transformed system gets significantly smaller. The transformation of ?? can be done from the left side 5.8 or from the right one 5.9.

$$PAx = Pb \quad (5.8)$$

$$AP(P^{-1}x) = b \quad (5.9)$$

where matrix  $P$  is called preconditioner.

As an extreme example we can consider the inverse matrix  $A^{-1}$  as the best preconditioner since it directly leads to the solution of the problem ?? and, thus, it requires only one iteration. However, it is obvious that computation of inverse  $A^{-1}$  is extremely expensive operation and it is not an objective of any iterative methods. That example helps understand and set requirements for preconditioners, namely:

1. cheap to compute e.g. a 5-10 iterations of the corresponding Krylov solver
2. should lead to a small conditioner number of the transposed system
3. should be sparse, otherwise storage requirements will considerably increase

There exist numerous techniques to compute preconditioners given a matrix  $A$  e.g. (point) Jacobi, Block-Jacobi, incomplete LU decomposition (ILU), multilevel ILU (ILU(p)), threshold ILU (ILUT), incomplete Cholesky factorization (IC), sparse approximate inverse (SPAI), multigrid as a preconditioner, etc. Almost all methods listed above have some tuning parameters which allow to get a better preconditioner i.e. a smaller condition number of the transformed system. However, it usually leads to increase of computational and storage costs.

Some methods can work particularly well for matrices derived from certain PDEs e.g. Poisson, NavierStokes, etc. problems discretized using the cartesian grid. However sometimes it can take a considerable amount of time to choice right parameters for a certain preconditioning algorithm. It can become a challenge to fulfill all requirements

1, 2, 3 mentioned above.

Table

Table [] shows results of different preconditioning algorithms application to our test case. It can be seen that some algorithms failed even after tuning.

It is interesting to notice that Gupta, Koric, and George came to approximately the same results working on their set of matrices in their work [GKG09]. They observed that preconditioned iterative solvers worked efficiently only for 2 out 5 cases in contrast to direct sparse solvers.

Table with comparisons of different preconditioning for our test cases

We can summarize that it is vital to perform careful parameter tuning of any preconditioning algorithms combining results from [table] and [GKG09]. In general the search can take a considerable amount of time. Moreover, it becomes impractical for time integration problems where topology of an underlying problem and, as the results, the computational mesh, discretization, Jacobian matrix can be changed over time of a simulation. It is obvious that parameters chosen for a particular time step can become not optimal for consecutive steps and, at the end, it can lead to divergence. If divergence happens at any time step the entire time integration algorithm fails and the simulation has to be restarted with different preconditioning parameters or with a different preconditioning algorithm.

By and large we come to a conclusion that preconditioned iterative solvers are not robust and thus cannot fully fulfill requirements listed in section 6.

## 5.2. Direct sparse methods

Direct sparse methods combine main advantages of direct and iterative methods i.e. numerical robustness and usage of sparsity structures. As a result, there is no need for preconditioning and the computation complexity is  $O(n^2)$  [Wu12]. The problem is that storage cost can significantly increase during factorization i.e. the inverse of a sparse matrix can be sufficiently dense. To reduce storage space of *LU* decomposition this group of methods performs fill-in reduction reordering as a pre-processing step before actual factorization. If storage space is still huge even after fill-in reduction reordering out-of core factorization can be used where partial results are stored in the secondary memory.

The most widely known sparse direct method is multifrontal method introduced by Duff and Reid in their work [DR83]. Multifrontal method is an improved extension of a frontal method [IB70] that can compute independent fronts in parallel. A front, or also called frontal matrix, can be considered as small dense matrix which is a result of Gaussian Elimination for a particular column. The algorithm, in fact, is as a variant of Gaussian Elimination process. There also exist left-looking and right-looking sparse direct methods. The difference between all of them is explained and can be found in [PT01].

In order to understand and analyze strong scaling behavior of the algorithm we have to briefly discuss the theory of the method. For simplicity we will assume that matrix  $A$  is real symmetric and  $LU$  decomposition boils down to the Cholesky factorization 5.10. It allows us to focus on the Cholesky factor  $L$  and its sparsity pattern only.

$$A = LDL^T \quad (5.10)$$

The algorithm usually starts with symbolic factorization to predict sparsity pattern of  $L$ . Once it is done the corresponding elimination tree has to be constructed.

place for figure 5.3

$$A = \begin{pmatrix} a & & & & \bullet & \bullet & \bullet \\ b & \bullet & & \bullet & & & \\ c & & \bullet & & \bullet & & \\ \bullet & d & & & \bullet & \bullet & \\ \bullet & e & \bullet & & \bullet & & \\ \bullet & & f & & & \bullet & \\ \bullet & & & g & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & h & & \\ \bullet & & \bullet & \bullet & & i & \end{pmatrix}$$

$$L = \begin{pmatrix} a & & & & & & \\ b & & & & & & \\ c & & & & & & \\ \bullet & d & & & & & \\ \bullet & & e & & & & \\ \bullet & & & f & & & \\ \bullet & & & & g & & \\ \bullet & & & & & h & \\ \bullet & & & & & & i \end{pmatrix}$$

Figure 5.3.: An example of a sparse matrix and its Cholesky factor [Liu92]

Figure 5.3 shows an illustrative example of a sparse matrix and its Cholesky factor from [Liu92]. The solid circles represent original non-zero elements whereas hollow ones define fill-in factors of  $L$ .

The elimination tree is a crucial part of the method. It can be considered as a structure of  $n$  nodes that node  $p$  is the parent of  $j$  if and only if it satisfies equation 5.11. It is

worth pointing out the definition 5.11 is not only one possible and one can define a strucutre of an elimination tree in a different way as well. As an example one can find a definition of a general assembly tree in [Liu92] proposed by Liu.

$$p = \min(i > j | l_{ij} \neq 0) \quad (5.11)$$

It is important to notice that node  $p$  represents elimination process of the corresponding column  $p$  of matrix  $A$  as well as all dependencies of column  $p$  factorization on the results of its descendants.

Given definition 5.11 we can build the corresponding elimination tree as it is shown in figure 5.4.

place for  
figure 5.4

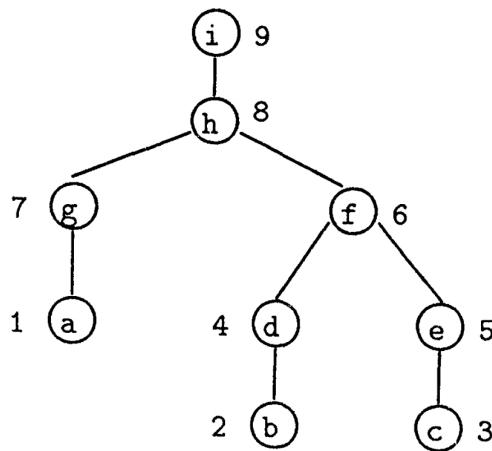


Figure 5.4.: The elimination tree for the matrix example in Figure 5.3 [Liu92]

The fundamental idea of multifrontal method spins around frontal and update matrices. A frontal matrix is used to perform Gaussian Elimination for a specific column  $j$ . It is a sum of a frame and update matrices as it can be seen from equation 5.12

$$F_j = Fr_j + \hat{U}_j = \begin{bmatrix} a_{j,j} & a_{j,i_1} & a_{j,i_2} & \dots & a_{j,i_r} \\ a_{i_1,j} & & & & \\ a_{i_2,j} & & & & \\ \vdots & & & & 0 \\ a_{i_r,j} & & & & \end{bmatrix} + \hat{U}_j \quad (5.12)$$

where  $i_0, i_1, \dots, i_r$  are the row subscripts of non-zeros in  $L_{*j}$  with  $i_0 = j$  and  $r$  is number of off-diagonal non-zero elements.

The frame matrix  $Fr_j$  is filled with zeros except the first row and column. The first row and column contain non-zeros elements of the  $j$ th row and column of the original matrix  $A$ . Because we consider matrix  $A$  to be symmetric the frame matrix is square and symmetric as well.

In order to describe parts of the elimination tree we will use the notation  $T[j]$  to represent all descendants of the node  $j$  in the tree and node  $j$  itself. In this way we can define the update matrix  $\hat{U}_j$  as following:

$$\hat{U}_j = - \sum_{k \in T[j]-j} \begin{bmatrix} l_{j,k} \\ l_{i_1,k} \\ \vdots \\ l_{i_r,k} \end{bmatrix} \begin{bmatrix} l_{j,k} & l_{i_1,k} & \dots & l_{i_r,k} \end{bmatrix} \quad (5.13)$$

The update matrix  $\hat{U}_j$  is, in fact, can be considered as the second term of the Schur complement i.e. update contributions from already factorized columns of  $A$ .

The subscript  $k$  represents descendant columns of node  $j$ . Thus we include and consider only those elements of descendant columns which correspond to the non-zero pattern of the  $j$ th column that we are currently factorizing.

Let's consider the partial factorization of 2-by-2 block dense matrix to better understand essence of update matrix  $\hat{U}_j$ .

$$A = \begin{bmatrix} B & V^T \\ V & C \end{bmatrix} = \begin{bmatrix} L_B & 0 \\ VL_B^{-T} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C - VB^{-1}V^T \end{bmatrix} \begin{bmatrix} L_B^T & L_B^{-1}V^T \\ 0 & I \end{bmatrix} \quad (5.14)$$

Again we assume that  $B$  has already been factorized and can be expressed as:

$$B = L_B L_B^T \quad (5.15)$$

The Schur complement from equation 5.14 can be viewed as the original sub-matrix  $C$  and update  $-VB^{-1}V^T$ . It can be written in a vector form as well:

$$-VB^{-1}V^T = -(VL_B^{-T})(L_B^{-1}V^T) = -\sum_{k=1}^{j-1} \begin{bmatrix} l_{j,k} \\ \vdots \\ l_{n,k} \end{bmatrix} [l_{j,k} \ \dots \ l_{n,k}] \quad (5.16)$$

As it can be easily seen that equations 5.16 and 5.13 are identical. The difference is that equation 5.13 exploits sparsity of the corresponding row and column of  $L$  and thus masks unnecessary information.

We can also notice from equation 5.14 that the frame matrix  $Fr_j$  corresponds to the block matrix  $C$  and brings information from the original matrix  $A$  whereas matrix  $\hat{U}_j$  adds information about the columns that have already been factorized.

As soon as the frontal matrix  $F_j$  is assembled i.e. we have the complete update of column  $j$ , we can perform elimination of the first column and get non-zero entries of factor column  $L_{*j}$ .

Let's denote  $\hat{F}_j$  as a result of the first column factorization of the frontal matrix  $F_j$ . Then we can express the results as following:

$$\hat{F}_j = \begin{bmatrix} l_{j,j} & \dots & 0 \\ \vdots & I & \\ l_{i_r,j} & & \end{bmatrix} \begin{bmatrix} 1 & \dots & 0 \\ \vdots & U_j & \\ 0 & & \end{bmatrix} \begin{bmatrix} l_{j,j} & \dots & l_{i_r,j} \\ \vdots & I & \\ 0 & & \end{bmatrix} \quad (5.17)$$

where sub-matrix  $U_j$  represents the full update from all descendants of node  $j$  and node  $j$  itself. Equation 5.18 express the sub-matrix  $U_j$  in a vector form.

$$\hat{U}_j = -\sum_{k \in T[j]} \begin{bmatrix} l_{i_1,k} \\ \vdots \\ l_{i_r,k} \end{bmatrix} [l_{i_1,k} \ \dots \ l_{i_r,k}] \quad (5.18)$$

Together with the frontal  $F_j$  and update  $\hat{U}_j$  matrices, the update column matrix  $U_j$  (also called contribution matrices) forms the key concepts of the multifrontal method. To consider the importance of sub-matrix  $U_j$  let's consider and example illustrated in

Figure 5.5.

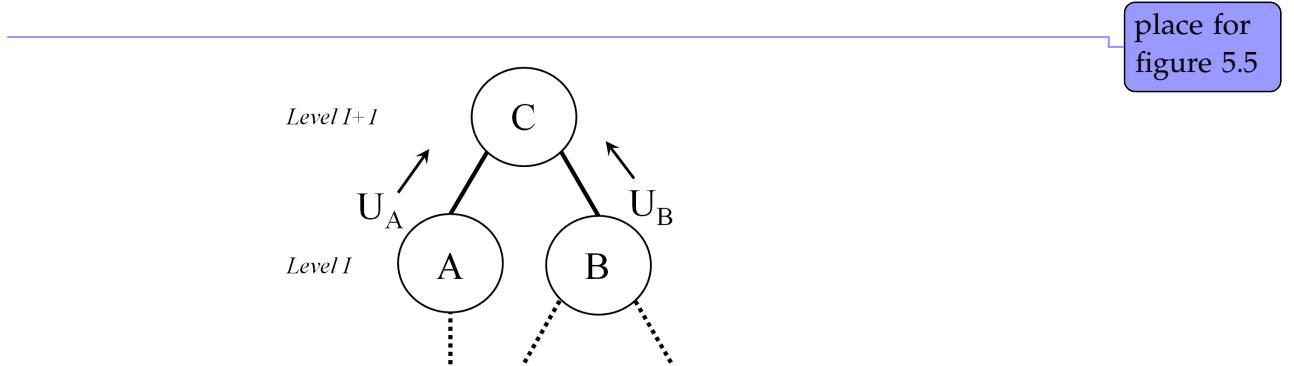


Figure 5.5.: Information flow of the multifrontal method

We assume that factorization of columns A and B have already been done and corresponding contribution matrices  $U_A$  and  $U_B$  have been computed. From equation 5.18 we have already known that both  $U_A$  and  $U_B$  contain the full updates of all their descendants including updates from factorization of columns A and B as well. Therefore update column matrices  $U_A$  and  $U_B$  have already got all necessary information to construct update matrix  $\hat{U}_C$ . The detailed proof and careful explanation can be found in [Liu92].

It might happen that we do not need all rows and columns of  $U_A$  and  $U_B$  i.e. we need only some subset of them, because of sparsity of column C. It is also important to place all necessary rows and columns of matrices  $U_A$  and  $U_B$  in a right place within matrix  $\hat{U}_C$ . For that reason, an additional matrix operation, called *extend-add*, must be introduced.

Let's consider an example from [Liu92] of an extend-add operation for 2-by-2 matrices R and S which correspond to the indices 5, 8 and 5, 9 of some matrix B, respectively.

$$R = \begin{bmatrix} p & q \\ u & v \end{bmatrix}, S = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \quad (5.19)$$

The result of the operation is going to be a 3-by-3 K matrix which looks as following:

$$K = R \ddot{+} S = \begin{bmatrix} p & q & 0 \\ u & v & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} w & 0 & x \\ 0 & 0 & 0 \\ y & 0 & z \end{bmatrix} = \begin{bmatrix} p+w & q & x \\ u & v & 0 \\ y & 0 & z \end{bmatrix} \quad (5.20)$$

Hence we can express formation of the frontal matrix  $F_j$  using the extend-add operation and all direct children of node  $j$  in the following way:

$$F_j = \begin{bmatrix} a_{j,j} & a_{j,i_1} & a_{j,i_2} & \dots & a_{j,i_r} \\ a_{i_1,j} & & & & \\ a_{i_2,j} & & & & \\ \vdots & & & & 0 \\ a_{i_r,j} & & & & \end{bmatrix} \ddot{+} U_{c_1} \ddot{+} \dots \ddot{+} U_{c_s} \quad (5.21)$$

where  $c_1, c_2, \dots, c_n$  are indices of direct children of the node  $j$ .

Now it can be clearly seen that the resultant frontal matrix  $F_j$  is a small dense one and it can be efficiently computed using BLAS level 3 subroutines.

After factorization we have to build the contribution matrix  $U_j$  i.e. add columns and rows of  $U_{c_1}, U_{c_2}, \dots, U_{c_s}$  to  $U_j$  that have not been used in factorization of  $F_j$  due to sparsity of column  $j$ . After that we can continue to move up along the tree. The complete update matrices grow in size as we move to the top of the tree. Therefore they have to be stored in a sparse matrix format to stay within memory constraints of the computer.

Another important aspect is storage and manipulation of frontal and contribution matrices. Sometimes we have to store contribution matrices produced in previous steps into some temporary buffer and efficiently retrieve them later during factorization. This can require some matrix re-ordering. In case of symmetric matrices, one can apply postordering on a tree to be able to use the stack data structure to alleviate the process of contribution matrix manipulations during factorization. A tree postordering is based on topological ordering and it has been proven that it is equivalent to the original matrix ordering and thus leads to the same filled graph [Liu92]. We refer to the original matrix ordering as the ordering received from fill-in reduction operation.

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ing

A tree postordering means that a node is ordered before its parent and, additionally, nodes in each subtree are numbered consecutively. Figure 5.6 shows an example of posrordering applied to the elimination tree of the matrix from figure 5.3. The results of this can be see in figure 5.7 where consecutive *push* and *pop* operations are efficiently

used during factorization and thus simplify the program logic.

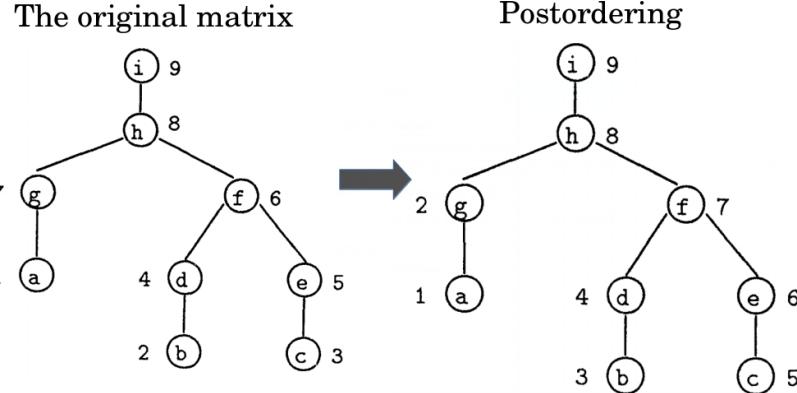


Figure 5.6.: An example of matrix postordering from [Liu92]

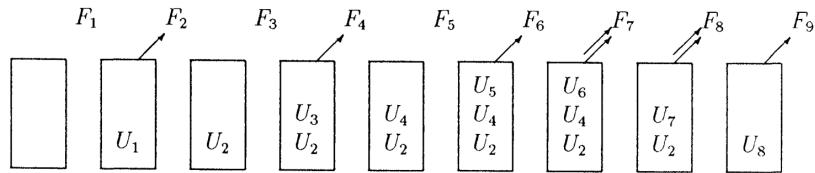


Figure 5.7.: The stack contents for the postordering [Liu92]

We can see the algorithm requires to perform some preprocessing steps in order to estimate the size of working space for matrix manipulations. If the working space has not been predicted correctly the algorithm will terminate during factorization. Additionally it can happen that even with the correct estimation we can be run out of space in the main memory, in case of huge sparse matrices. This fact can require to use the secondary memory and, as a result, the execution time will increase significantly. Therefore, different optimal postordering schemes have been proposed which allow to shrink the amount of space needed during factorization [Liu86] [Liu88]. Some schemes, for example elimination tree rotations [Liu88], can lead to deep and unbalanced trees which might have their negative effect on task parallelism as we will see later.

In general, the estimation of working space can be tricky due to pivoting. Because pivoting happens only during the numerical factorization it is not always possible to

estimate enough space correctly beforehand. There exist some heuristics which allow to use some numerical matrix information during symbolic factorization to better predict the amount of required space [Gup00].

It can be clearly observed the method consists of three distinct phases, namely: analysis, numerical factorization and solution. The analysis phase includes all pre-processing steps that have been discussed above i.e. fill-in reduction, postordering, symbolic factorization, building elimination tree and so on. During the numerical factorization phase the  $L$  and  $D$  (or  $U$ ) parts of a matrix  $A$  are computed based on sequence of dense factorization on frontal matrices. At the solution step, the solution vector  $x$  is computed by means of backward and forward substitutions (equations ?? and ??).

In practice, an improved version of multifrontal method, called supernodal method, is used. The idea of the supernodal method is to shrink the final elimination tree by grouping some particular nodes/columns in one computational unit. As a result, more useful floating point operations per memory access can be performed by eliminating few columns at once within the same frontal matrix.

A supernode is formed by a set of contiguous columns with identical off-diagonal sparsity structure forms. Thus, a supernode has few important properties. Firstly, it can be expressed as a set of indices, namely:  $\{j, j+1, \dots, j+t\}$ , where node  $j+k$  is parent of  $j+k-1$  in the elimination tree. Secondly, the size of the supernodal frontal matrix is equal to the frontal matrix of the  $j$ th column within a supernode. As an example, Figure 5.8 shows a postordered matrix  $A$  and its Cholesky factor  $L$  as well as the corresponding supernodal elimination tree.

place for figure 5.8

$$\bar{A} = \begin{pmatrix} 1 & a & \bullet & & & \bullet & \bullet \\ 2 & \bullet & g & & & \bullet & \bullet \\ 3 & & b & \bullet & & \bullet & \bullet \\ 4 & & \bullet & d & & \bullet & \bullet \\ 5 & & c & \bullet & \bullet & & \bullet \\ 6 & & \bullet & e & \bullet & \bullet & \bullet \\ 7 & & \bullet & & \bullet & f & \bullet \\ 8 & \bullet & \bullet & \bullet & \bullet & h & \bullet \\ 9 & \bullet & \bullet & \bullet & \bullet & & i \end{pmatrix}$$

$$\bar{L} = \begin{pmatrix} 1 & a & & & & & \\ 2 & \bullet & g & & & & \\ 3 & & b & & & & \\ 4 & & \bullet & d & & & \\ 5 & & c & & & & \\ 6 & & \bullet & e & & & \\ 7 & & \bullet & \circ & \bullet & f & \bullet \\ 8 & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & h \\ 9 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & i \end{pmatrix}$$

Figure 5.8.: An example of a supernodal elimination tree [Liu92]

Equation 5.22 expresses the building process of a frontal matrix of a supernode. In

contrast to 5.21, the frame matrix  $\mathcal{F}_j$  contains more dense rows and columns. As before, we use *extend-add* operation to get the full block update from children contribution matrices.

It should be mentioned there exist more sophisticated variants of supernodes. Most of the time, it intends to improve efficiency of the algorithm. Liu pointed out that supernodes could be defined without using the contiguous constrains [Liu92]. On another hand, Wu defines supernodes corresponded to separators from the nested dissection step [Wu12] which was used for fill-in reduction.

check grammar

$$\mathcal{F}_j = \begin{bmatrix} a_{j,j} & a_{j,j+1} & \dots & a_{j,j+t} & a_{j,i_1} & \dots & a_{j,i_r} \\ a_{j+1,j} & a_{j+1,j+1} & \dots & a_{j+1,j+t} & a_{j+1,i_1} & \dots & a_{j+1,i_r} \\ \vdots & \vdots & \dots & \vdots & & & \\ a_{j+t,j} & a_{j+t,j+1} & \dots & a_{j+t,j+t} & a_{j+t,i_1} & \dots & a_{j+t,i_r} \\ a_{i_1,j} & a_{i_1,j+1} & \dots & a_{i_1,j+t} & & & \\ \vdots & \vdots & \dots & \vdots & & 0 & \\ a_{i_r,j} & a_{i_r,j+1} & \dots & a_{i_r,j+t} & & & \end{bmatrix} \oplus U_{c_1} \oplus \dots \oplus U_{c_s} \quad (5.22)$$

Up to this point we have already seen all key concepts of the multifrontal method and discussed how the algorithm works. We will move to the discussion of parallelization of the method.

The elimination tree, in fact, represents dependencies among columns. Conversely, the tree also shows independent steps of elimination process. Hence the tree forms independent problems that can be executed in parallel. Task parallelism is the main and primary source the algorithm parallelisation. Figure 5.9 shows task parallelism, for the example given in Figure 5.8, where each color represents a set concurrent tasks.

place for figure 5.9

For example, nodes on separate branches of the tree are totally independent and can processed in parallel. However, as soon as at least two branches run into the same node it forms a dependency and we have to wait all contribution matrices of its children and cannot proceed further.

We can observe the amount of task parallelism is rapidly decreasing while moving towards the root along the tree. Once we reach the root of the tree the algorithm becomes totally sequential. This fact can play the significant role in strong scaling

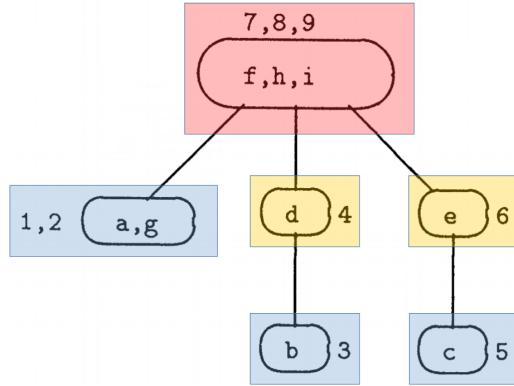


Figure 5.9.: Parallel steps of the multifrontal method based on the example in Figures 5.8

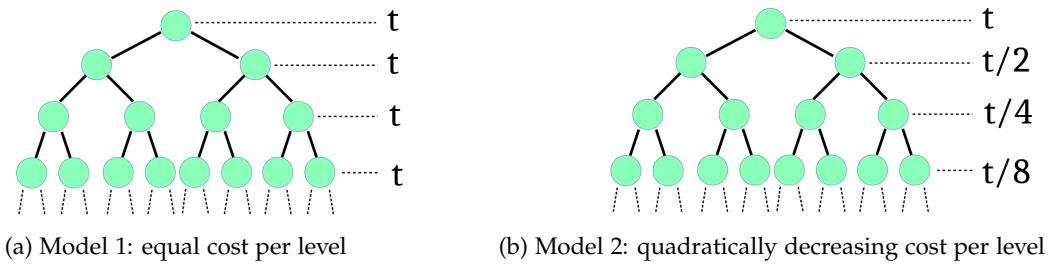


Figure 5.10.: Simple parallel models of the multifrontal method

behavior of the method.

We developed two simple models based on perfectly balanced binary trees to better understand strong scaling of the algorithm. The main concept of the models is so-called cost per level or cost per node. This idea is similar to the recursion trees in [Cor+09] which explains and computes complexity of recurrent algorithms.

Figure 5.10a represents the first model where we keep the same cost per level whereas the second model (Figure 5.10b) simulates quadratic cost decay from level to level. Additionally we assume that computational cost distributed uniformly between nodes at the same level for both models.

place for  
figure  
5.10

We have to say that our models mimic only numerical factorization and do not include time spent on any per-processing steps, for example, fill-in reduction reordering. A cost per level can be interpreted in different ways e.g. increase of partial factorization time due to growth of frontal matrices in size, time increase spent on numerical pivoting, increase of MPI communication overheads due to growth of contribution matrices, etc. It should be mentioned that real computer implementations of the multifrontal algorithm (MUMPS, SuperLU, etc.) are quite sophisticated in many aspects and our models do not have any intention to analyze performance of a particular package. Instead the objective of these models is to show possible strong scaling behavior and possible bottlenecks.

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We will consider only task parallelism at the beginning to a first approximation and later we will discuss how additional data parallelism can affect algorithm performance.

Instead of coloring given in Figure 5.9, we assume that each level has the same color and thus can be executed fully in parallel if we have enough processing elements. We cannot go to the next level till the current one has not been completed yet i.e. free processing elements, that do not have nodes to execute at the current level, have to wait.

As we mentioned above the root of the tree can be processed purely sequentially if we only consider task parallelism. As a first approximation, time spent on the root factorization determines the minimal execution time according to the Amdahl's law [Wik18a]. More precisely, the minimal execution time is equal to a sum of time spent on single node partial factorization at each level. This time determines the asymptote on the corresponding speed-up graph.

We considered a perfectly balanced tree with 16 levels, 65535 nodes and the maximum of 20 processing elements as an example. The numerical results of linear and quadratic models can be viewed in Figures 5.10a and 5.10b, respectively. The figures show a rapid drop of performance, especially in case of the quadratic model. Table 5.1 demonstrates the maximum potential speed-up, having 32768 processing elements which is equal to the number of leaves of the tree, against the speed-up we have got using only 20 processing elements.

We can see that model 1 still has some potential to grow whereas the second model has already reached its asymptote and further increase of processing elements does not make sense. In spite of a potential growth of the first model, both models have very

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5.11

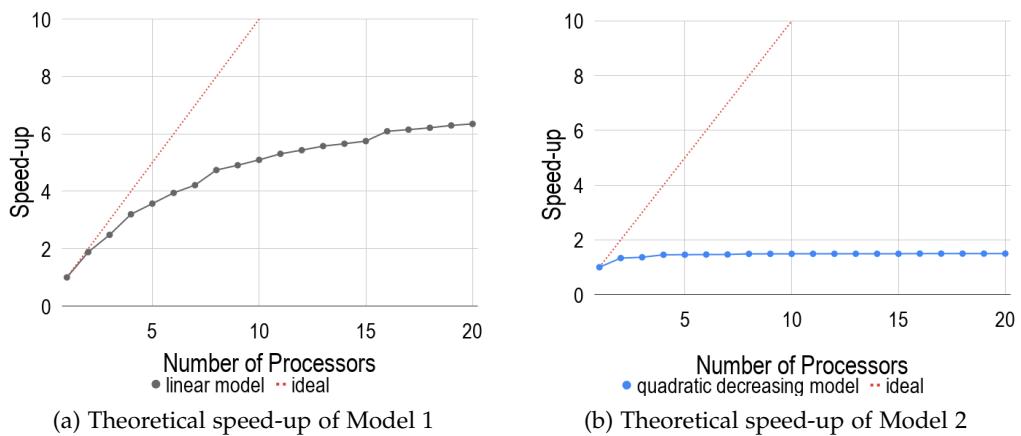


Figure 5.11.: Theoretical speed-up

	20 PEs	32768 PEs
Model 1	6.3492	8.0000
Model 2	1.4972	1.5000

Table 5.1.: Potential speed-up of linear and quadratic models

low parallel efficiency even with 20 processing elements which can be observed from table 5.2.

	20 PEs
Model 1	0.3175
Model 2	0.0749

Table 5.2.: Efficiency of linear and quadratic models using 20 PEs

Both models shows that computational intensity per node grows from bottom to top. It is easy to conclude from Figure 5.10 that intensity per node is equal  $t/2^i$  and  $t/2^{2i}$  for the first and second models, respectively (where  $i$  is a level of the tree). It reflects that the most intensive part of the method is centered on the top part of the tree i.e. first few level. Liu discussed application of the multifrontal method to a  $k - by - k$  regular model problem with nine-point difference operator in his paper [Liu92]. He observed that factorization of the last 6 nodes took slightly more than 25% of the total amount of arithmetical operations. As a comparison, table 5.3 shows fractions of time spent on processing first few top levels of our models: 1 and 2.

	Model 1	Model 2
Level 0	6.25%	50.00%
Level 1	12.50%	75.00%
Level 2	18.75%	87.50%

Table 5.3.: Distribution workload per level in case model 1 and 2

As we can see, the result of our first model is relatively close to 25% and, therefore, it looks quite optimistic. However, the second model shows that 87% of workload is focused on the top part of the tree and, as a result, we can consider that model as extremely pessimistic.

By and large, reduction of time spent on the top nodes is a way to improve strong scaling behavior. To do so, data parallelism can be additionally exploited for these nodes. It is worth noting that data parallelism at bottom levels does not make sense because it leads to increase of granularity there and thus increase communication overheads which can lead to significant performance drop.

Figure 5.12 shows an example of two types of parallelism applied to the algorithm.

First of all, we can see the leaves are grouped in subtrees and a single PE is assigned to each subtree. Other nodes are distributed among three different types. Nodes of the first type uses task parallelism only, which is induced by the tree, and each node is executed in a single processor. The second type exploits data parallelism with 1D block row distribution among the processors. The root belongs to the third type where data parallelism is used with 2D block cyclic distribution. The details of MUMPS parallelism management is carefully explained and can be found in [OA07].

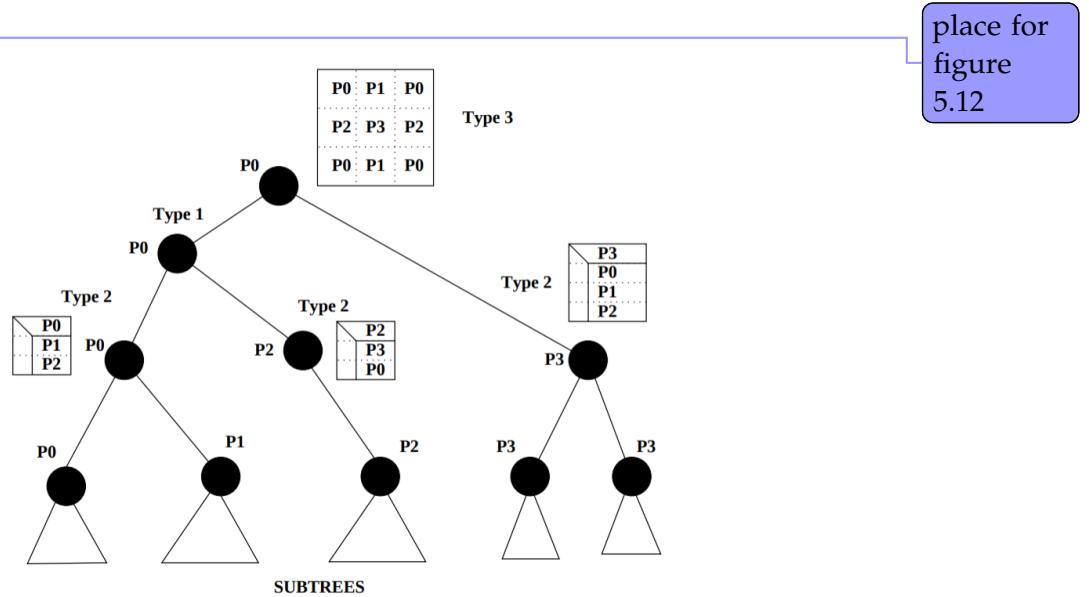


Figure 5.12.: MUMPS parallelism management in case of 4 PEs [OA07]

All the techniques mentioned above were designed to improve strong scaling behavior by splitting the most intensive parts among all available processors. Going back to our models, we can also think about that in a slightly different way, namely: *data parallelism helps to re-distribute cost per node/level on the corresponding elimination tree*. However, we have to notice that efficiency of data parallelism totally depends on sizes of frontal matrices at the top part of the tree. In case of skinny sparse matrices, oversubscription of processing elements can lead to strong performance penalties as we could see from section ???. A machine-dependent minimal frontal matrix size was introduced in MUMPS in order to control whether to use ScaLAPACK at the root node or not [17]. It can happen that the algorithm uses only task parallelism, due to the threshold, and, as a results, scaling will only depend on the tree structure that can be deep and unbalanced.

Figure 5.13 shows comparison of strong scaling between model 1 and parallel numerical factorization of the matrix *memchip* (Table 4.2) done with using MUMPS library. The sparsity pattern before and after fill-in reduction is shown in figure 5.14.

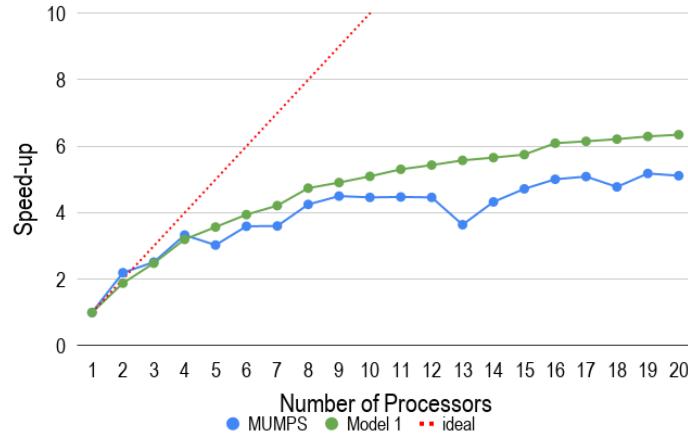


Figure 5.13.: Comparison between model 1 and numerical factorization of the matrix *memchip* using MUMPS library

add some examples to the appendix

place for figure 5.13. Show standard deviation

place for figure 5.14

As we can see, our model and the experiment show the same trend and the results are pretty much close to each other. However, our model takes into consideration only task parallelism whereas MUMPS exploits both data and task parallelism. Additionally, we have to mention that our model 1 works well only for relatively big sparse matrices. It can be quite inaccurate in case of small/skinny sparse systems.

In general, it is possible to refine our models and make them more accurate, using Bulk Synchronous Parallel (BSP) approach, for example. However, it will require to possess a real postordered elimination tree, extracted from a specific implementation of multifrontal method, together with information about supernodes sizes. We strongly believe the new enhanced model can explain the jagged strong scaling behavior of the MUMPS solver that we can observe in figure 5.13. But, this approach seems to be quite cumbersome and requires to delve into the source code of a particular library. It is needless to say that data can be retrieved only during run time and only after the analysis phase. This makes it less valuable and we can see that it is definitely a wrong way to go.

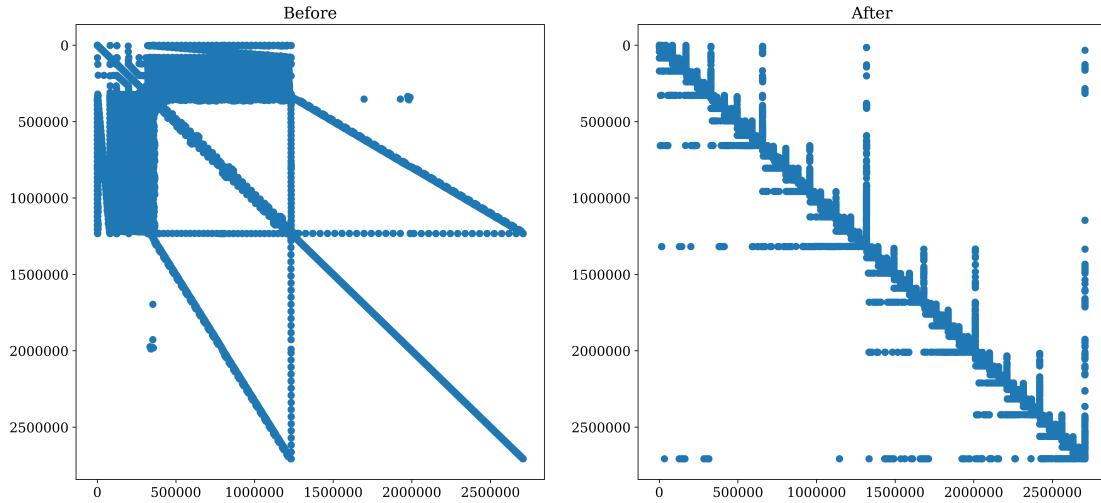


Figure 5.14.: Sparsity structure of the matrix *memchip* before and after fill-in reduction

There are few important aspects to discuss at the end of the section. Numerical robustness is the main advantage of the multifrontal method. It does not require any preconditioner to solve a system of equations. As we discussed in the previous section, tuning a specific preconditioning algorithm can take a considerable amount of time, especially in case of our systems. As another advantage, the method (heavily) exploits matrix sparsity which lowers computational complexity up to  $O(n^2)$ . In case of massively huge matrices, the algorithm can utilize the secondary memory which sometimes is only one way to solve a system.

We can conclude, from the analysis above, the method has inherently bad scaling behavior and it is quite sensitive to a matrix structure. We will see later that it is almost impossible to predict the saturation point i.e. a point after which performance either drops or stays at the same level. We assume that scaling becomes better with growth of a matrix size. However, we cannot expect such behavior for small and medium systems.

Secondly, we can see the algorithm requires many pre-processing steps to be done before numerical factorization phase. All these steps must run in parallel and be highly scalable. Apart from performance constraints of the steps, they must lead to wide and well balanced elimination trees which becomes crucial during the numerical phase.

Lastly, the algorithm can fail due to incorrect working space prediction. As a result, factorization has to be restarted with some modification of input solver parameters.

### 5.3. Hybrid method

We have observed almost all available methods and we could see that none of them can fully cover all our requirements at once, namely:

- robustness
- numerical stability
- parallel efficiency
- open source licenses

The analysis from sections 5.1 and 5.1 shows that iterative methods scale much better in contrast to sparse direct ones. However, they are only efficient in case of very well preconditioned systems. We showed in section 5.1 that search of preconditioning parameters usually takes lots of time and efforts. Additionally, we cannot guarantee that the settings found for our GRS matrix set will always work well in subsequent steps of time integration or for other different simulations.

Sparse direct methods do not have such a problem. They always produce the right solution. The methods can only fail in case of underestimation of the working space due to numerical pivoting during the numerical factorization phase. In order to cope with that problems, some implementations of direct sparse methods provide two options to the user, namely: to increase predicted working space by some factor e.g. 2, 3, 4, etc. or to lower constraints of numerical pivoting which allows small numerical values to stay on the diagonal.

The drawback of the second option is that it can lead to out-of-core execution with using the secondary memory which makes numerical factorization significantly slow. While the second option has lower chance of out-of-core factorization it can lead to a numerically inaccurate solution.

After many considerations we decided to stick to the sparse direct solvers because **robustness** criteria had the highest priority in our case. To circumvent problems mentioned above, we proposed a so-called hybrid solver, in spite of the fact that the definition of *hybrid linear solvers* had already been used in scientific computing literature in a slightly different way [RBH12]. *The idea is to switch off numerical pivoting (or*

*significantly lower the constraints) of sparse direct solvers and use the resultant LU decomposition as a preconditioner for an iterative method, for example GMRES.*

```

1 # compute LU decomposition with a sparse direct solver
2 LU = SparseDirectSolver(matrix=A, pivoting="switch_off")
3
4 # compute inverse of A using backward-forward substitutions
5 # sovle: LU * A_inv = I
6 A_inv = ComputeInverse(decomposition=LU)
7
8 # apply a Krylov method to a preconditioned system
9 # i.e A_inv * A * x = A_inv * b
10 GMRES(matrix=A, rhs=b, preconditioner=A_inv)

```

Listing 5.1: A pseudo-code of the Hybrid approach

According to our primary tests, the hybrid approach showed us that it required from 1 to 5 iterations of the GMRES solver on average to converge to a desired residual.

write  
which  
solver we  
used

The main problem of our approach is parallel efficiency because sparse *LU* decomposition takes the most of computational time. We discussed reasons of possible bad strong scaling behavior of sparse direct solvers in section 5.2. We could see, in case of the multifrontal method, these methods consist of multiple steps and implementation of each step has its strong effect on parallel performance. We also mentioned that the main source of performance improvement is data parallelism and it can be achieved in many different ways. Hence, performance of the same method can vary from library to library.

In the next section we are going to investigate all available open-source implementations of sparse direct solvers, compare their efficiency and choose one of them. At the beginning, we will only consider libraries that have their direct interface to PETSc [Bal+18]. PETSc is a scientific numerical library that contains various algorithms and methods, especially the Krylov methods. It is highly efficient in parallel and provides numerous interfaces to other libraries such as MUMPS, SuperLU, Hypre, PaStiX, ViennaCL and etc.

The subsequent sections will be dedicated to tuning and optimization of a specific library with the aim to reduce execution time.

proofreading

# 6. Solver selection and configuration

## 6.1. Choice of a Sparse Linear Direct Solver Library

Fair to say, there is no single algorithm or software that is best for all types of linear systems [Li18]. Nowadays there exist many different sparse direct solvers on the market. Some of them are tuned for specific linear systems i.e. symmetric positive definite, systems with symmetric sparsity pattern, system with complex numbers, etc., some are targeted for the most general cases. Some packages can handle data parallelism in different ways even within the same library depending on the system size and other criteria. Hence, parallel performance highly depends on a specific implementation of a method. Table 6.1 displays a short summary of almost all available and well-known packages, at the time of writing, in this field based on works [Li18] and [Bal+18].

We only listed libraries that can run on distributed memory parallel machines. Almost all of them also support shared memory environment in some degree. Nonetheless there also exist libraries that run either only sequentially (UMFPACK, SPARSE, TAUCS, SuperLU) or only on shared memory machines (PanelLLT, SuperLU\_MT).

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notes of  
Open\*

We can see, from table 6.1, that only MUMPS, PaStiX and SuperLU\_DIST cover all our initial requirements: open-source license and direct interface to the PETSc library. However, these libraries implement different sparse direct methods, namely: multifrontal, left-looking and right-looking, respectively. Moreover, they handle partial pivoting in different ways.

It is known that partial pivoting is necessary to achieve a good numerical accuracy during Gaussian Elimination. It interchanges rows and columns of a matrix in such a way to avoid small numerical values along the diagonal. In case of sparse direct solvers, the numerical pivoting, in run-time, usually distorts all predictions that have been made during the analysis phase and can lead to significant fill-in and load unbalanced, with respect to floating-point operations, during factorization. Hence, implementation of numerical pivoting, especially during parallel execution, plays the most important role in performance for this group of methods.

## 6. Solver selection and configuration

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Package	Method	Matrix Types	PETSc Interface	License
Clique	Multifrontal	Symmetric	Not Officially	Open
MF2	Multifrontal	Symmetric pattern	No	-
DSCPACK	Multifrontal	SPD	No	Open*
MUMPS	Multifrontal	General	Yes	Open
PaStiX	Left looking	General	Yes	Open
PSPASES	Multifrontal	SPD	No	Open*
SPOOLES	Left-looking	Symmetric pattern	No	Open*
SuperLU_DIST	Right-looking	General	Yes	Open
symPACK	Left-Right looking	SPD	No	Open
S+	Right-lookin	General	No	-
PARDISO	Multifrontal	General	No	Commercial
WSMP	Multifrontal	General	No	Commercial

Table 6.1.: List of packages to solve sparse linear systems using direct methods on distributed memory parallel machines [Li18], [Bal+18]

Both PaStiX and SuperLU\_DIST libraries use so-called static pivoting where the pivot order is chosen before numerical factorization and kept fixed during factorization.

The main advantage of static pivoting is that it allows to better optimize the data layout, load balance, and communication scheduling [11]. However, it leads to a higher risk of numeric instability. Therefore, both PaStiX and SuperLU\_DIST provide a few ways to perform the solution refinement.

For instance, SuperLU\_DIST uses diagonal scaling, setting very tiny pivots to larger values, and iterative refinement (listing 6.2). While PaStiX allows the user to choose a refinement strategy between GMRES, CG (for SPD systems) and iterative refinement as well. At this point it is interesting to notice that we came to the same conclusion as the PasTiX developers with respect to the solution refinement using Krylov iterative methods.

Iterative refinement, shown in listing 6.2, is claimed to converge to a decent precision

can we use decent here?

within 2 or 3 steps in work [ADD89]. However, in practice, we noticed that the iterative refinement can work not as expected, especially in case of lowered partial pivoting constrains.

For completeness, we have to mention that the variable *too\_large* is, in fact, an estimation of the backward error [ADD89] which can be expressed as following:

$$\frac{|b - A\hat{x}|_i}{(|b| + |A||\hat{x}|)_i} \quad (6.1)$$

where  $\hat{x}$  is the computed solution and  $|\bullet|$  is the element-wise module operation.

```

1 # perform analysis and numerical factorization
2 # phases
3 LU = SparseDirectSolver(matrix=A)
4
5 # compute initial solution
6 x = Solve(factorization=LU, rhs=b)
7
8 # compute initial residual
9 r = A * x - b
10
11 while r > too_large:
12     # find correction
13     d = Solve(factorization=LU, rhs=r)
14
15     # update solution
16     x = x - d
17
18     # update residual
19     r = A * x - b

```

Listing 6.1: A simple iterative refinement

In contrast to PaStiX and SuperLU\_DIST, MUMPS performs partial pivoting in run-time during the numerical factorization phase. To limit the amount of numerical pivoting, and stick better to the sparsity predictions done during the symbolic factorization, partial pivoting can be relaxed, leading to the partial threshold pivoting strategy [17].

listing of  
iterative  
refine-  
ment

A pivot  $|a_{i,i}|$  is accepted if it satisfies:

$$|a_{i,i}| \geq u \times \max_{k=i \dots n} |a_{k,j}| \quad (6.2)$$

where  $u$  is value between 0 and 1.

To improve solution accuracy, MUMPS, as PasTiX and SuperLU\_DIST, provides the iterative refinement as a post-processing step as well.

The most important feature which MUMPS introduces is so-called delayed pivots. It can happen that equation 6.2 cannot be satisfied within a fully-summed block of a frontal matrix (equation 5.22) and we also cannot consider elements outside the block since the corresponding rows are not fully-summed. In this case, some rows and columns will remain unfactored, or delayed, in the front. They are going to be sent the frontal matrix of the parent, as part of the contribution block and the process will repeat. The delayed pivot approach helps to improve numerical accuracy, however, it causes additional fill-in in the parent node.

In spite of obvious complexity of dynamic partial pivoting, MUMPS allows the user to explicitly control run-time behavior of the algorithm due to partial threshold pivoting strategy. This provides an opportunity for optimization and tuning in some degree.

PETSc (version 3.10) provides the full interface to both SuperLU\_DIST and MUMPS, whereas the interface to the PasTiX library is quite limited. In fact, in case of PasTiX, the user can only control the number of threads per MPI process and a level of verbosity, which makes this library to be less interesting for our subsequent (following) research.

data parallelism  
difference  
between  
MUMPS  
and SuperLU

In order to evaluate the overall parallel performance of the libraries, we performed a few flat-MPI tests with the GRS matrix using the HW1 machine. Before testing, we downloaded and configured the libraries, MUMPS version 5.1.2, PasTiX version 6.0.0, SuperLU\_DIST version 5.4, within the PETSc environment with their **default settings**. As a profiling tool, we used the internal PETSc profiler. A time limit of 15 minutes was set up for each test case to prevent blocking of a compute node in case of out-of-core execution. Results of the tests are summarized in tables 6.2, 6.3, 6.4 and in appendix A. Numerical values in tables are given in **seconds**.

We ran into a few problems with the SuperLU\_DIST library during the tests. Firstly, factorization exceeded the set time limit in case of **cube-64** and **k3-2** matrices. Secondly, we noticed the library crashed during processing of **k3-18**, **cube-645** and (partially) **pwr-3d** matrices. A debugging process showed that a segmentation fault occurred in **pdgstrf** function during the numerical factorization phase. We still keep working on this problem together with the PETSc team in order to find a solution.

## 6. Solver selection and configuration

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MPI	MUMPS	PaStiX	SuperLU
1	7.02E-02	8.72E-02	3.17E+00
2	6.73E-02	7.10E-02	1.43E+00
3	6.36E-02	7.01E-02	1.07E+00
4	6.28E-02	7.11E-02	8.17E-01
5	6.50E-02	7.15E-02	7.51E-01
6	6.72E-02	7.62E-02	6.15E-01
7	6.91E-02	7.69E-02	6.48E-01
8	6.89E-02	8.17E-02	5.41E-01
9	7.50E-02	8.28E-02	5.02E-01
10	7.22E-02	8.52E-02	4.64E-01
11	7.55E-02	8.89E-02	5.82E-01
12	7.61E-02	1.06E-01	4.37E-01
13	7.84E-02	9.72E-02	5.43E-01
14	8.06E-02	1.02E-01	4.22E-01
15	8.20E-02	1.19E-01	3.91E-01
16	8.07E-02	1.19E-01	4.44E-01
17	8.38E-02	1.22E-01	5.19E-01
18	8.40E-02	1.26E-01	3.77E-01
19	8.58E-02	1.33E-01	5.47E-01
20	8.64E-02	1.49E-01	3.39E-01

Table 6.2.: Results of a flat-MPI test of MUMPS, PasTiX and SuperLU\_DIST libraries with their default settings and the matrix **cube-5** (9352 equations)

To complete and perform a fair comparison, an additional flat-MPI test was conducted with a 5-point stencil Poisson matrix with 100000 equations. We think this test can partially allow us to estimate parallel performance of systems like **k3-18**, **cube-645** where SuperLU\_DIST crashed. The results of the test are given in figure 6.1.

According to the test results, it is clear that MUMPS significantly outperforms both SuperLU\_DIST and PaStiX. A literature review showed that Gupta, Koric, and George, in paper [GKG09], came to nearly the same results comparing parallel performance of WSMP, MUMPS and SuperLU\_DIST libraries with respect to execution time for their matrix set. However, paper [KBK16] showed an almost opposite outcome. According to Kwack, Bauer, and Koric, SuperLU\_DIST spent less time on factorization and solution phases, which almost always determines the total execution time, and, even more interesting, it scaled much better overall. We must note that both research groups used different machines and matrix sets. This fact actually seconds our idea that a choice of a suitable library can depend heavily on data and hardware.

place for  
figure 6.1

Taking into consideration the results of our primary flat-MPI tests, the MUMPS library was chosen as a sparse direct solver for our hybrid approach and the following study. Furthermore, an overview of the MUMPS documentation also showed some room for performance tuning that we were going to discuss in detail in sections [bla], [bla] and [bla].

## 6. Solver selection and configuration

---

MPI	MUMPS	PaStiX	SuperLU
1	1.36E+00	1.39E+00	time-out
2	1.00E+00	9.82E-01	time-out
3	8.83E-01	1.06E+00	time-out
4	8.17E-01	8.74E-01	time-out
5	7.85E-01	8.50E-01	time-out
6	8.06E-01	8.52E-01	time-out
7	7.71E-01	8.33E-01	time-out
8	7.66E-01	8.33E-01	time-out
9	7.93E-01	8.35E-01	time-out
10	8.07E-01	8.15E-01	time-out
MPI	MUMPS	PaStiX	SuperLU
11	7.75E-01	8.15E-01	time-out
12	7.81E-01	8.10E-01	time-out
13	7.85E-01	8.35E-01	time-out
14	7.85E-01	8.18E-01	time-out
15	7.88E-01	8.46E-01	time-out
16	7.81E-01	8.23E-01	time-out
17	6.83E-01	8.49E-01	time-out
18	7.96E-01	8.44E-01	time-out
19	8.04E-01	8.65E-01	time-out
20	6.85E-01	8.87E-01	time-out

Table 6.3.: Results of a flat-MPI test of MUMPS, PaStiX and SuperLU\_DIST libraries with their default settings and the matrix **cube-64** (100657 equations)

MPI	MUMPS	PaStiX	SuperLU
1	1.55E+02	6.44E+01	crashed
2	6.28E+01	4.84E+01	crashed
3	5.06E+01	5.02E+01	crashed
4	4.17E+01	4.50E+01	crashed
5	2.52E+01	3.98E+01	crashed
6	2.58E+01	4.29E+01	crashed
7	2.65E+01	4.30E+01	crashed
8	2.59E+01	3.73E+01	crashed
9	1.95E+01	4.08E+01	crashed
10	1.91E+01	3.81E+01	crashed
MPI	MUMPS	PaStiX	SuperLU
11	1.77E+01	3.81E+01	crashed
12	1.60E+01	3.75E+01	crashed
13	1.42E+01	3.58E+01	crashed
14	1.45E+01	3.59E+01	crashed
15	1.47E+01	3.57E+01	crashed
16	1.41E+01	3.52E+01	crashed
17	1.54E+01	3.45E+01	crashed
18	1.52E+01	3.31E+01	crashed
19	1.52E+01	3.31E+01	crashed
20	1.38E+01	3.16E+01	crashed

Table 6.4.: Results of a flat-MPI test of MUMPS, PaStiX and SuperLU\_DIST libraries with their default settings and the matrix **k3-18** (1155955 equations)

However, it should be mentioned that we cannot exclude that SuperLU\_DIST and PaStiX can perform similar, or even better, as the MUMPS library with appropriate parameters tuning or for another matrix set.

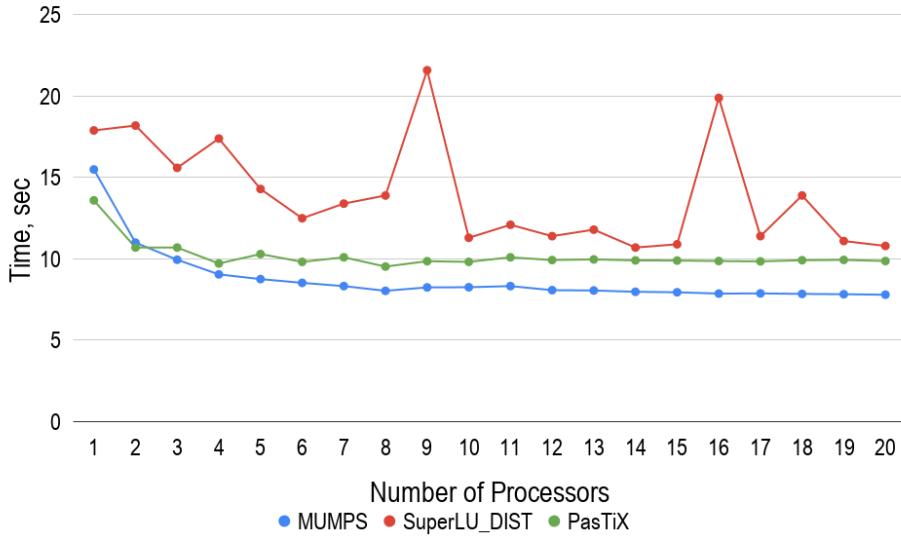


Figure 6.1.: Results of a flat-MPI test of MUMPS, PasTiX and SuperLU\_DIST libraries with their default settings and a 5 point-stencil Poisson matrix (1000000 equations)

## 6.2. Review of MUMPS Library

Originally, MUMPS library was a part of the PARASOL Project. The project was an ESPRIT IV Long Term Research whose main goal was to build and test a portable library for solving large sparse systems of equations on distributed memory systems [Ame+98]. An important aspect of the research was the strong link between the developers of the sparse solvers and the industrial end users, who provided a range of test problems and evaluated the solvers [Ame+02]. Since 2000 MUMPS had continued as an ongoing project and, by the moment of writing, the library have contained almost 5 main releases.

It was mentioned in section 6.1 that MUMPS is an implementation of the multifrontal method. Hence, MUMPS sequentially performs all three phases: analysis, numerical factorization and solution. The numerical factorization and solution phases were fully described in detail in section 5.2. It is important to examine the analysis phase of MUMPS library because this phase varies from library to library and plays a significant role on parallel performance.

According to the documentation, the MUMPS analysis phases consists of several pre-processing steps:

1. Fill-reducing pivot order
2. Symbolic factorization
3. Scaling
4. Amalgamation
5. Mapping

1) To handle both symmetric and unsymmetric cases, MUMPS performs fill-in reordering based on  $A + A^T$  sparsity pattern. The library provides numerous sequential algorithms for reordering such as Approximate Minimum Degree (AMD) [ADD96], Approximate Minimum Fill (AMF), Approximate Minimum Degree with automatic quasi-dense row detection (QAMD) [Ame97], Bottom-up and Top-down Sparse Reordering (PORD) [Sch01], Nested Dissection coupled with AMD (Scotch) [Pel08], Multilevel Nested Dissection coupled with Multiple Minimum Degree (METIS) [KK09]. Additionally, MUMPS can work together with ParMETIS and PT-Scotch which are extensions of METIS and Scotch libraries for parallel execution. MUMPS also provides the user with an automatic choice option where an appropriate reordering algorithm is selected in run-time based on matrix type and size and the number of processors [17].

2) Sparsity structures of factors  $L$  and  $U$  are computed during the step, based on permuted matrix  $A$  after fill reducing reordering, in order to build the corresponding elimination tree. All computations are performed on a directed graph  $G(A)$  associated with the matrix  $A$ .

3) At this step, matrix  $A$  is scaled in such a way to get absolute values of *one* along the main diagonal and *less than one* for all off-diagonal entries. Scaling algorithms are based on studies described in detail in works [DK99], [DK01] (for the unsymmetric case) and [DP05] (for the symmetric case). This preprocessing step is supposed to improve numerical accuracy and makes all estimations performed during analysis more reliable [17]. MUMPS also provides an option to switch off scaling or perform it during the factorization phase.

4) During amalgamation, sets of columns with the same off-diagonal sparsity pattern are grouped together to create bigger nodes, also known as supernodes. The process leads to restructuring of the initial elimination tree to an amalgamated tree of supernodes which is also known as the *assembly tree*. The main purpose of that step is to improve efficiency of dense matrix operations. An example of the amalgamation

process is shown in section 5.2.

5) A host process, chosen by MUMPS, creates a pool of tasks where each task can be either a subtree or type 2 or type 3 node (figure 5.12). Then each task is mapped by the host among all available processes in such a way to achieve good memory and compute balance.

Type 1 nodes are grouped in subtrees, according to the Geist-Ng algorithm [GN89], and each subtree is processed only by one single process to avoid the finest granularity, which can cause high communication overheads.

In case of type 2 nodes, the host process assigns each node to one process, which is called the master, which holds fully summed rows and columns of the node and perform pivoting and factorization of these rows. During the numerical factorization phase, in run-time, the master process first receives symbolic information which describes the structure of the contribution blocks sent by its children. At the next step, the master collects information concerning the load of all other processes and decides which of them (*slaves*) are going to participate. Then the master informs the chosen slaves that a new task has been allocated for them and sends them the frontal matrix distribution. After that, the slaves communicates the the children of the master process and collects the corresponding numerical elements. The slaves are in charge of all the assembly and computation of the partly summed rows.

The root node belongs to the type 3. The host statically assigns the master for the root, as it is in case of type 2 nodes, to hold all the indices describing the structure of the frontal matrix. Before factorization, the structure of the frontal matrix of the root is statically mapped onto a 2D grid of processes using block cyclic distribution. This allows to determine, during the analysis phase, which process an entry of the root is assigned. Hence, the original matrix entries and the part of the contribution blocks can be assembled as soon as they are available. Due to partial pivoting, the master process collects the index information for all delayed variables of its sons, builds the final structure of the root frontal matrix and broadcast the corresponding symbolic information to all slave processes. The slave, in turn, adjust their local data structure. After that numerical factorization can be perform in parallel.

It is important to mention that if the size of the root node is less than a certain computer depended parameter, defined by MUMPS, the root node will be treated as the type 2.

An illustrative example of process a mapping together with a combination of static and dynamic scheduling is given in figure 6.2.

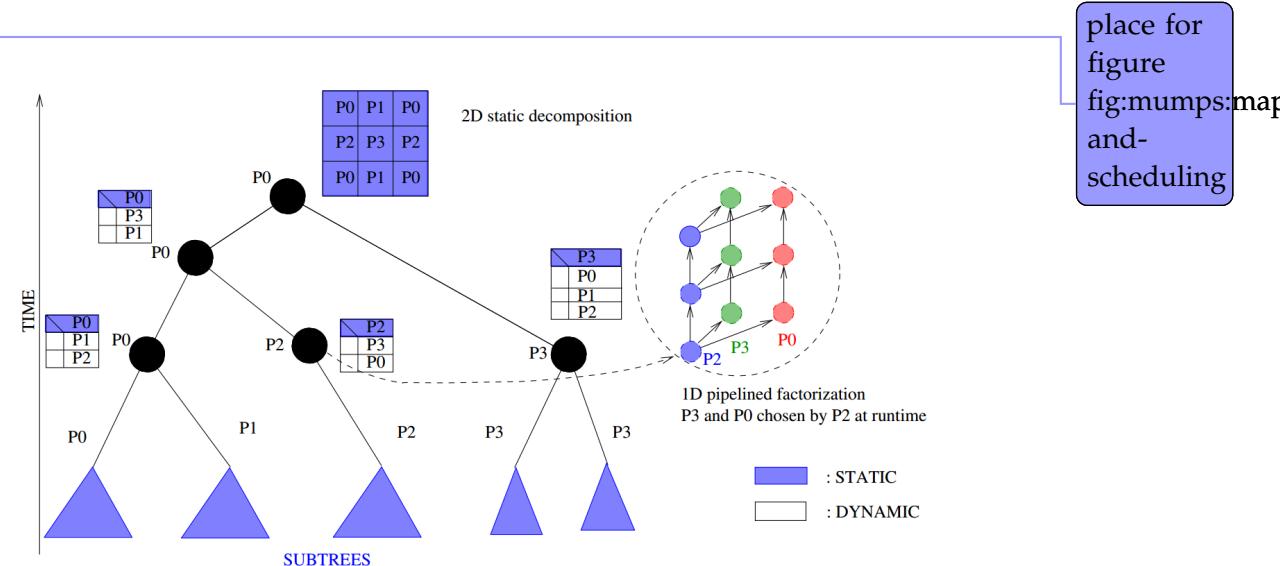


Figure 6.2.: MUMPS: static and dynamic scheduling [LEx12]

Another outstanding feature of MUMPS is treatment of partial pivoting during the numerical factorization phase. To handle this, MUMPS uses threshold pivoting and delayed pivots approaches which are fully described in section 6.1 where different implementations of direct sparse solvers are compared.

### 6.3. Choice of Fill Reducing Reordering

Fill reducing reordering is the first and the most important step of sparse matrix decomposition since it has its direct impact on the assembly tree structure. As we mentioned above, the tree structure defines the task parallelism as well as sizes of frontal matrices and thus performance of the method.

MUMPS provides various algorithms for reordering described in section 6.2. A detailed study and comparison between different methods were done by Guermouche, L'Excellent, and Utard in work [GLU03] for sequential execution of the analysis phase. Guermouche, L'Excellent, and Utard noticed that the trees generated by METIS and SCOTCH were rather wide (because of the global partitioning performed at the top),

while the trees generated by AMD, AMF and PORD tend to be deeper. In addition, they came to two important conclusions. Firstly, they noticed both SCOTCH and METIS generated much better balanced trees in contrast to other methods. Secondly, according to their results, SCOTCH and METIS produced trees with bigger frontal matrices than those generated by the other reorderings [GLU03].

In this section we are going to investigate influence of two different parallel fill reducing packages, namely: PT-Scotch and ParMETIS, on parallel performance of MUMPS. The algorithmic difference between PT-Scotch and ParMETIS was explained in section 6.2.

To perform a test, the default PETSc, MUMPS, PT-Scotch and ParMETIS libraries were downloaded, compiled, configured and link together. The test was carried out using only flat-MPI mode without any explicit process pinning. The results are shown in figure 6.3 as well as in appendix B.

According to the results, parallel performance of MUMPS can vary significantly and very sensitive to a used fill-in reducing reordering algorithm. In average, the difference between application of different algorithms achieves almost 15%. However, in some particular cases, *cube-5* and *pwr-3d*, the difference varies around 40-55%.

place for  
figure 6.3

It is important to mention that both packages, PT-Scotch and ParMetis, use heuristic approaches with the main aim to reduce fill-in of the factors and thus do not directly consider quality of the resultant elimination tree. It is relevant to assume that efficiency of a particular heuristic can be very sensitive to a matrix structure and size. This fact makes it difficult to predict which algorithm is better to use for a specific case in advance. Taking GRS matrix set as an example, we can observe that PT-Scotch is the best choice for small and medium matrices, namely: *cube-5*, *cube-64*, *k3-2* and *pwr-3d* cases. However, at the same time PerMetis tends to work better for relatively big systems such as *cube-645* and *k3-18*.

During the test, we noticed that application of ParMetis for small systems of equations showed a strong negative effect on parallel performance of MUMPS. The results showed the factorization time of *pwr-3d* and *cube-5* matrices grew with the increase of processing units (figure 6.4).

A simple profiling showed two important things. Firstly, numerical factorization

place for  
figure 6.4

## 6. Solver selection and configuration

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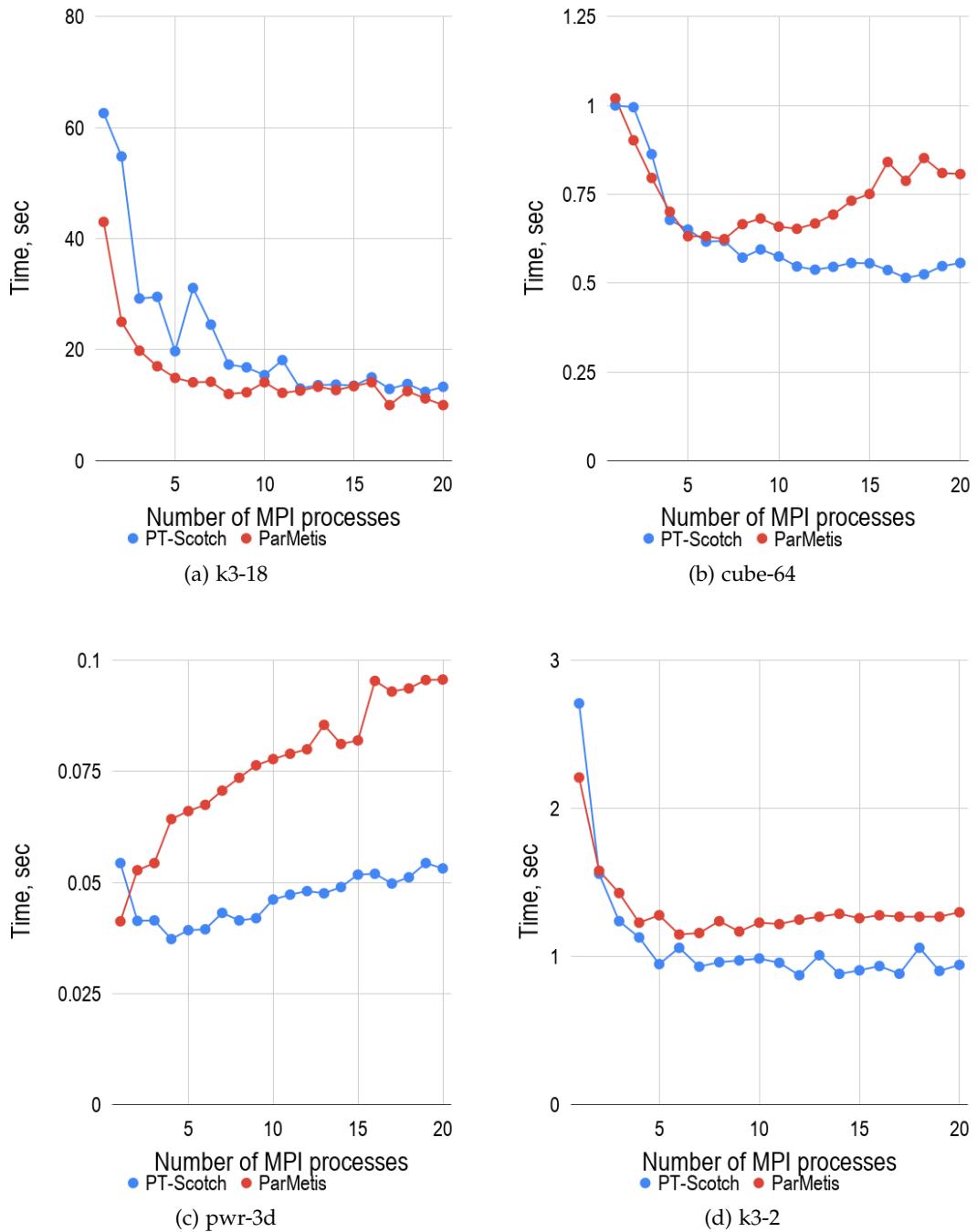


Figure 6.3.: Comparison of different fill-reducing algorithms

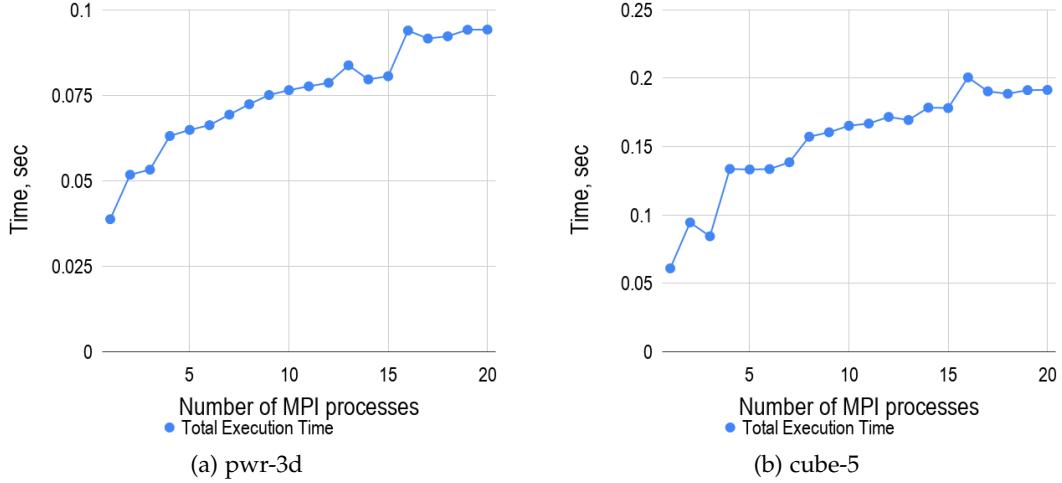


Figure 6.4.: MUMPS-ParMetis parallel performance in case of relatively small matrices

time and time spent on the analysis phase had approximately the same order in case of sequential execution i.e. 1 MPI process. Secondly, while numerical factorization time barely decreased with increase of number of processing elements, time spent on analysis phase significantly grew. By and large, we can observe the slow-down of MUMPS mainly comes from only the analysis phase.

A careful investigation revealed the analysis phase contained several peaks at points where the processor count was equal to a power of two. We assumed that cause could be due to either fill reducing or process mapping pre-processing steps. However, a detailed profiling and tracing of the analysis phase, which are out of the scope of this study, are required in order to give the exact answer. The results of profiling are shown in figure 6.5.

In this section, we have presented the influence of different fill-in reducing algorithms on parallel performance of MUMPS. We have observed the right choice of the algorithm can lead to significant improvements in terms of the overall execution time. We have showed there is no a single algorithm that performs the best for all test cases. At the moment of writing, we came to the conclusion there was no an indirect metric to predict the best algorithm in advance for a specific system of equations and only a flat-MPI test could be used for that purposed. Sometimes PT-Scotch and ParMetis can

place for  
figure 6.5

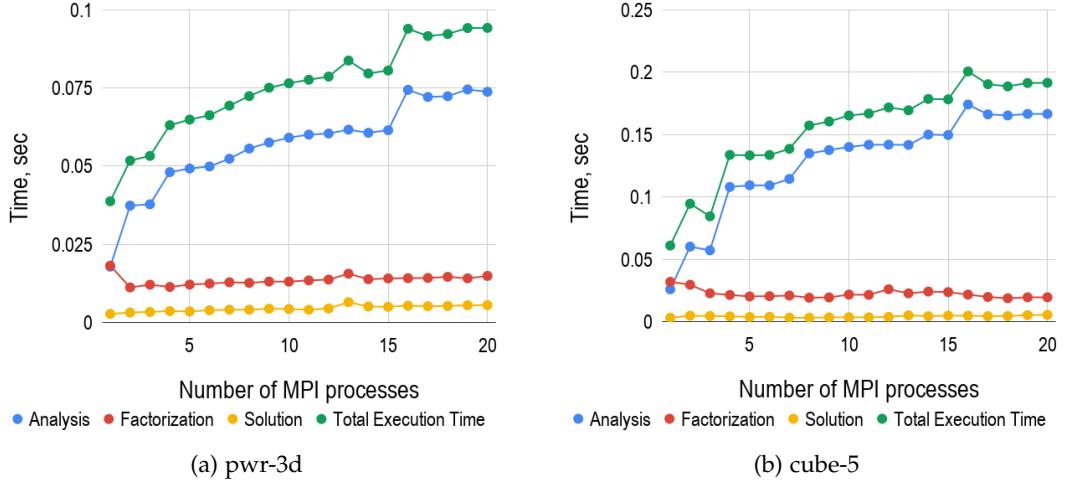


Figure 6.5.: Profiling of MUMPS library with using ParMetis as a fill-in reducing algorithm in case of factorization of relatively small matrices

perform nearly the same as it is in case of factorization of *CurlCurl\_3* and *cant* matrices, for example (see appendix B). Therefore, from time to time, it can be quite difficult to make a decision which package to use. At the end, we have assigned each test case to a specific fill reducing reordering method based on results of the conducted experiments and our subjective opinion. The results are summarized in tables 6.5 and 6.6.

Matrix Name	Ordering	n	nnz	nnz / n
cube-5	PT-Scotch	9325	117897	12.6431
cube-64	PT-Scotch	100657	1388993	13.7993
cube-645	ParMetis	1000045	13906057	13.9054
k3-2	PT-Scotch	130101	787997	6.0568
k3-18	ParMetis	1155955	7204723	6.2327
pwr-3d	PT-Scotch	6009	32537	5.4147

Table 6.5.: GRS matrix set: assignment of matrices to a specific fill-in reducing algorithm based on parallel performance of flat-MPI tests

From now onwards, assignments mentioned in tables 6.5, 6.6 will be used without explicitly referring to it.

Matrix Name	Ordering	n	nnz	nnz / n
cant	ParMetis	62451	4007383	64.1684
consph	PT-Scotch	83334	6010480	72.1252
memchip	PT-Scotch	2707524	13343948	4.9285
PFlow_742	PT-Scotch	742793	37138461	49.9984
pkustk10	PT-Scotch	80676	4308984	53.4110
torso3	ParMetis	259156	4429042	17.0903
x104	PT-Scotch	108384	8713602	80.3956
CurlCurl_3	PT-Scotch	1219574	13544618	11.1060
Geo_1438	ParMetis	1437960	63156690	43.9210

Table 6.6.: SuiteSparse matrix set: assignment of matrices to a specific fill-in reducing algorithm based on parallel performance of flat-MPI tests

## 6.4. MUMPS: Process Pinning

Due to intensive and complex manipulations with frontal and and contribution matrices, we can assume that MUMPS belongs to memory bound applications. In this case memory access can be a bottleneck for the library. A common way to improve performance of memory bound applications running on distributed memory machines is to distribute processes equally among NUMA domains within a compute node. Given the fact that each NUMA domain has its own system bus, this strategy allows to reduce conjunction of memory traffic by balancing data requests equally among the memory channels.

However, because MUMPS uses both task and data parallelism as well as a complex hybrid, both static and dynamic, task scheduling, it becomes difficult to decide which pinning strategy is better i.e. *close* or *spread*, described in section 4.

Therefore, a couple of tests were conducted with both GRS and SuiteSparse matrix sets in order to investigate influence of different strategies on MUMPS performance. For this group of tests, MUMPS was used with default settings and a specific fill-in reducing algorithm for each test case, mentioned in section 6.3. The tests were performed on both HW1 and HW2 machines using only flat-MPI mode. This comparison allows to investigate influence of number independent system buses of a compute node on MUMPS overall performance since HW1 and HW2 machines have different number of NUMA domains, 2 and 4, respectively. Results are shown in figures 6.6, 6.7, 6.8 and in appendix C. The graphs depict the total time of MUMPS, i.e. time spent on analysis,

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factorization and solution phases.

The tests revealed that, in general, *spread*-pinning performed better on both machines. In average, the strategy allows to reduce run-time by approximately 5.5% and 13.8% on HW1 and HW2 machines, respectively. The main performance gain can be observed in the middle range of process count i.e. the range from 2 to 12, where the view on the process distribution, between *close* and *spread* strategies, varies considerably. On another hand, the performance gain becomes less prominent while moving towards the tail of process count since difference of process distribution becomes negligible. As expected for HW1, the points where process count is equal to 1 and 20 show the same performance because they basically represent the same process distribution.

It is also important to investigate the gain of performance around the saturation point. It is worth pointing out that from time to time it becomes very difficult to decide where the saturation point locates. For that reason, a careful analysis was performed for each graph based on values of speed-up, efficiency and our subjective opinion. The results are summarized in tables 6.7 and 6.8.

HW1					HW2				
Matrix Name	MPI	Gain w.r.t "close", %	Speed up	Efficiency	MPI	Gain w.r.t "close", %	Speed up	Efficiency	
pwr-3d	4	11.594	1.386	0.347	4	6.616	1.626	0.406	
cube-5	4	8.261	1.139	0.285	4	10.640	1.156	0.289	
cube-64	8	5.645	1.812	0.226	8	7.521	1.729	0.216	
cube-645	6	9.985	2.152	0.359	8	9.078	2.521	0.315	
k3-2	7	7.788	2.899	0.414	8	9.947	3.298	0.412	
k3-18	8	6.716	3.472	0.434	8	9.567	3.896	0.487	

Table 6.7.: Analysis and comparison of MUMPS performance at the saturation point between HW1 and HW2 for GRS matrix set

A study of tables 6.7 and 6.7 reveals HW2 machine performs slightly better in contrast to HW1 with respect to performance gain around the saturation points. This result is considerably different from the overall performance gain that was mentioned above where the difference was as twice as much. However, the results also show that increase

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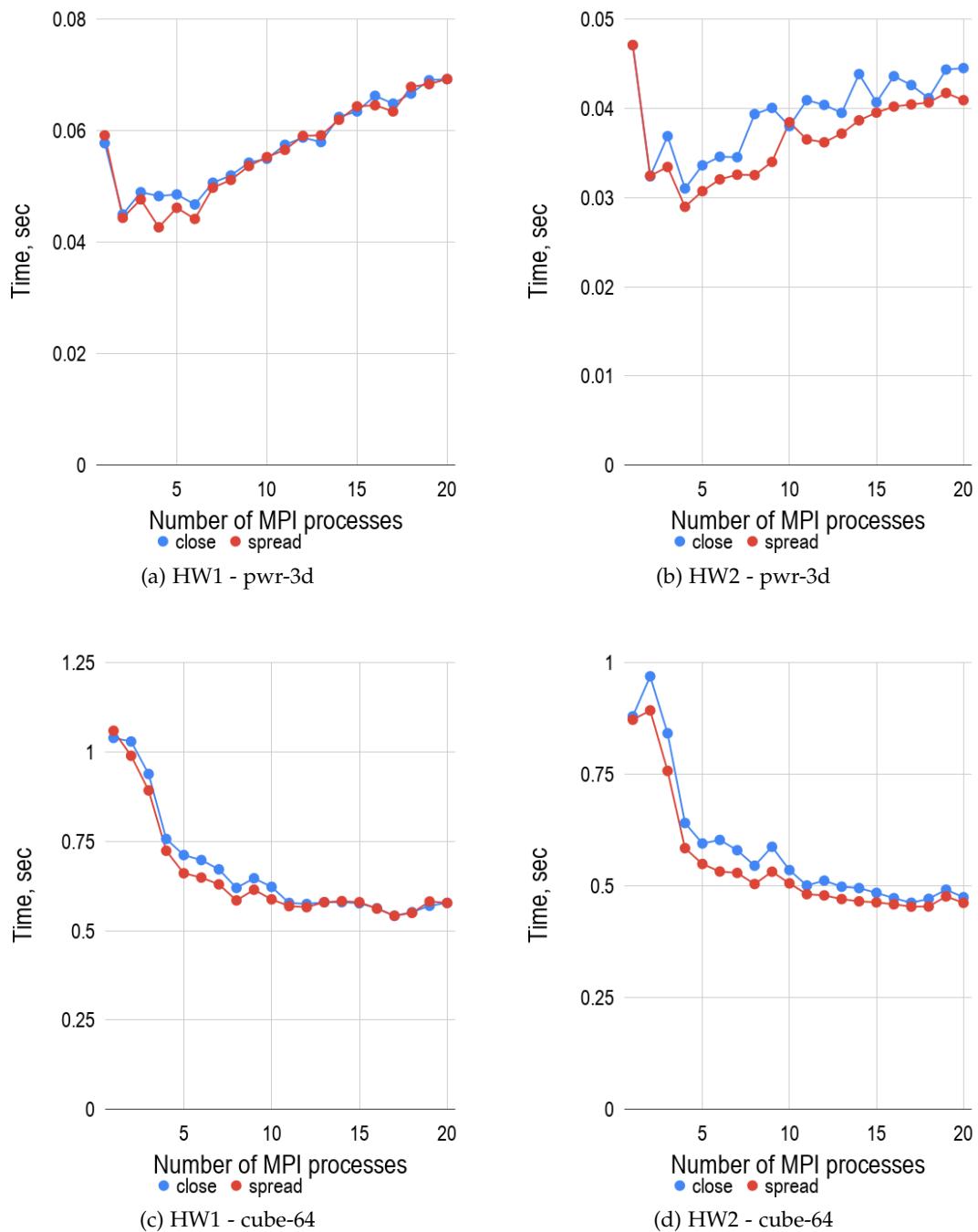


Figure 6.6.: Comparison of *close* and *spread* pinning strategies

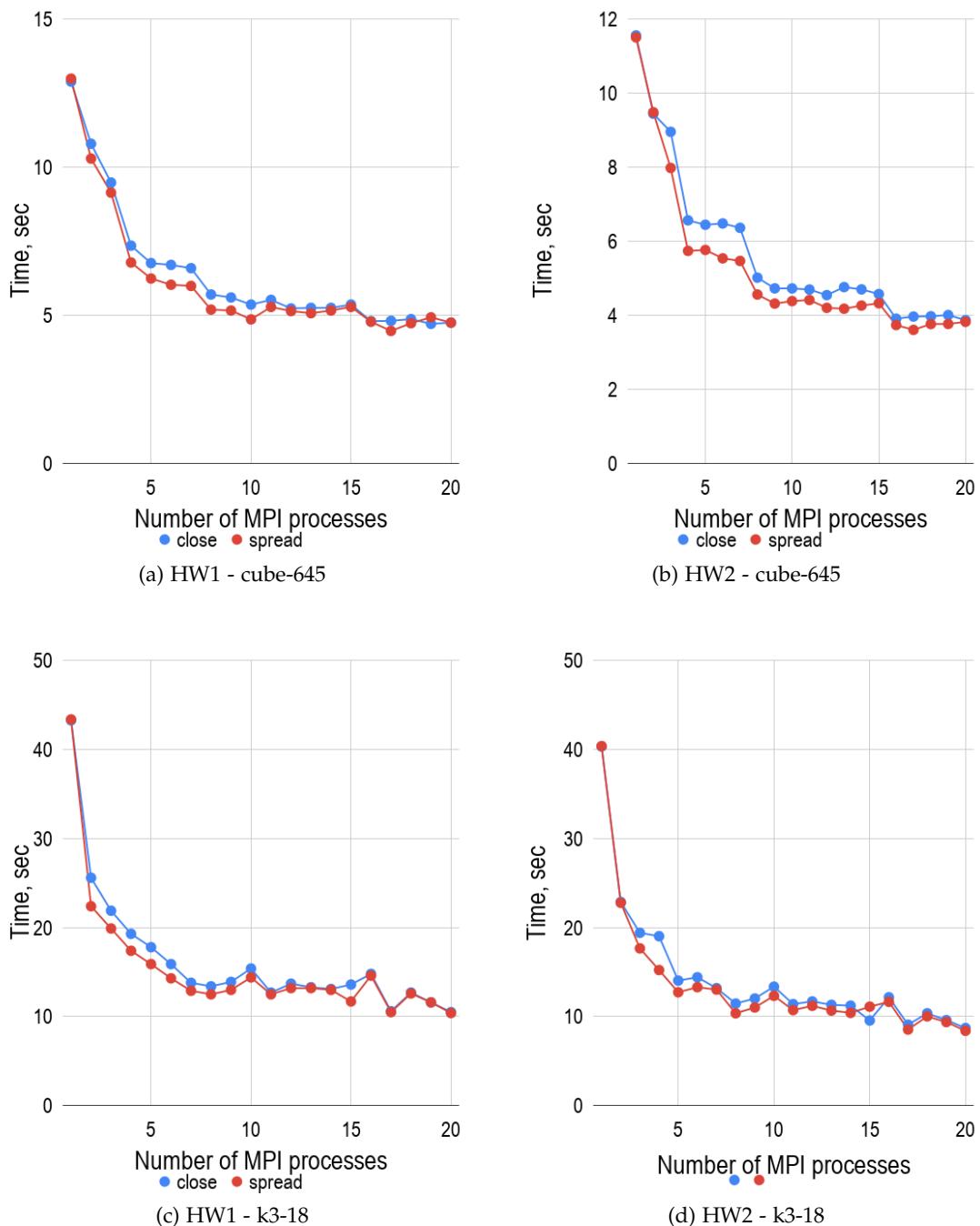


Figure 6.7.: Comparison of *close* and *spread* pinning strategies

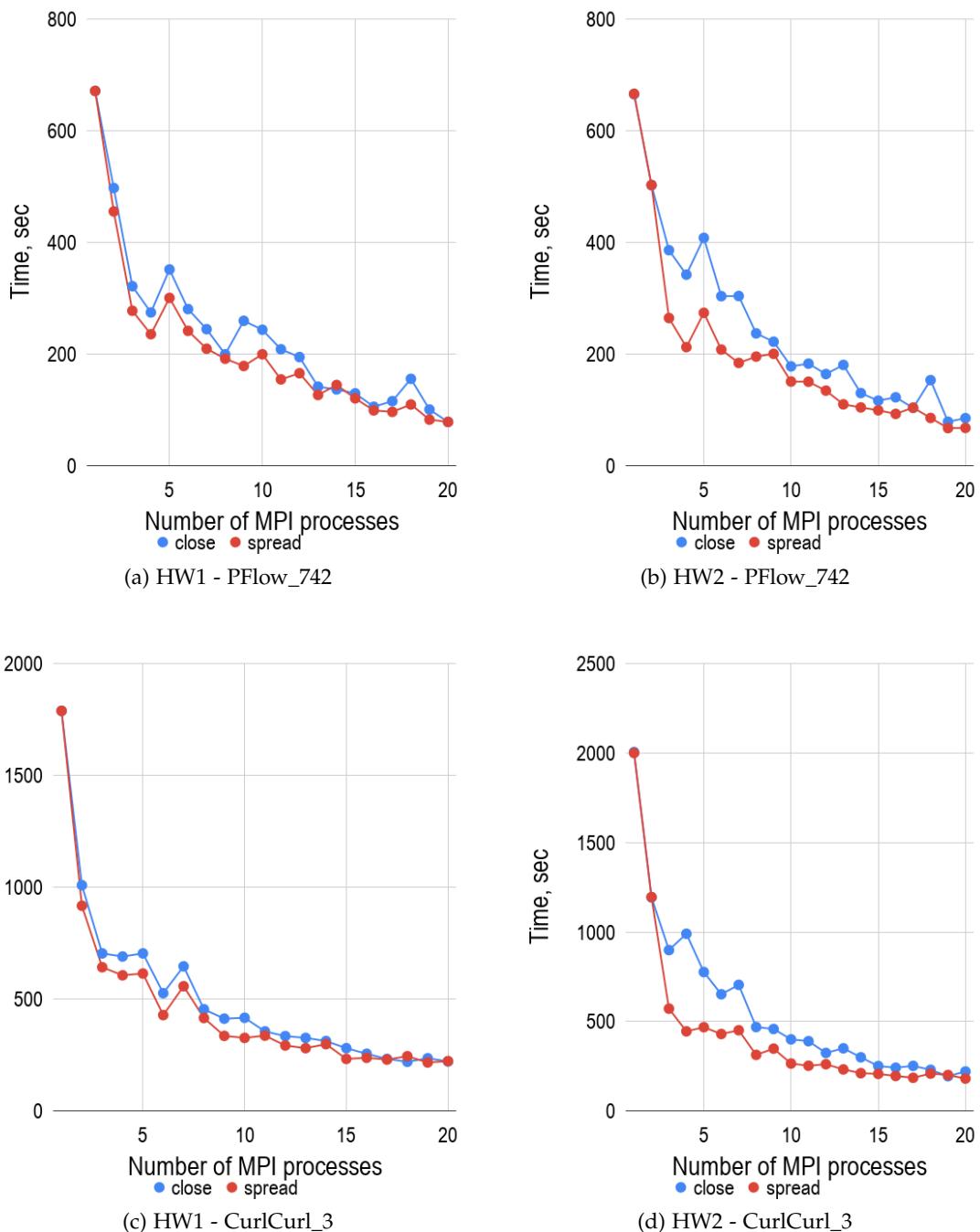


Figure 6.8.: Comparison of *close* and *spread* pinning strategies

## 6. Solver selection and configuration

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Matrix Name	MPI	HW1			HW2			
		Gain w.r.t "close", %	Speed up	Efficiency	MPI	Gain w.r.t "close", %	Speed up	Efficiency
cant	8	7.914	3.297	0.412	8	12.437	3.407	0.426
consph	15	0.110	6.147	0.410	15	2.409	6.667	0.444
CurlCurl_3	19	8.051	8.249	0.434	20	17.908	11.039	0.552
Geo_1438	13	21.609	4.548	0.350	ROM	ROM	ROM	ROM
memchip	9	11.290	4.299	0.477	9	11.102	4.213	0.468
PFlow_742	19	17.921	8.106	0.427	20	20.469	9.798	0.490
pkustk10	17	-0.664	3.872	0.228	17	-1.108	4.036	0.237
torso3	18	5.607	8.149	0.453	19	6.028	9.493	0.499
x104	6	9.537	1.789	0.298	6	7.829	1.763	0.294

Table 6.8.: Analysis and comparison of MUMPS performance at the saturation point between HW1 and HW2 for SuiteSparse matrix set.

\*ROM - run out of memory

of NUMA domains always help to improve the values of efficiency and speed-up and thus strong scaling behavior of MUMPS.

In this section, we showed influence of process distribution as well as the NUMA domain count on overall MUMPS parallel performance. We show the *spread* process distribution always has some beneficial effect. The increase of NUMA domains also brings some extra performance but not too much as it was expected at the beginning.

This result of this study can be relevant for energy-efficient parallel computing where a strong requirements to program efficiency are applied. This fact usually forces the user to decrease process count and go a slightly away from saturation point in order to keep the values of efficiency around 0.7-0.8. In this case performance of MUMPS can be improved in 15-20% in contrast to the straight forward process pinning.

Taken into account results of the tests, *spread*-pinning has been chosen for the rest of the study. This process distribution can be easily achieved by means of some advanced OpenMPI command line options, for example *-rank-by* and *-bind-to*, as following::

```
1 mpiexec --rank-by numa --bind-to core -n $num_proc $executable_name
$parameters
```

---

Listing 6.2: An example of *spread-pinning* with using advanced OpenMPI command line options

## 6.5. Choice of BLAS Library

To perform columns elimination of fully summed block of frontal matrices, MUMPS intensively uses GEMM, TRSM and GETRF subroutines which are parts of BLAS and LAPACK libraries. As an example, figure 6.10 demonstrates factorization of a type 2 node.

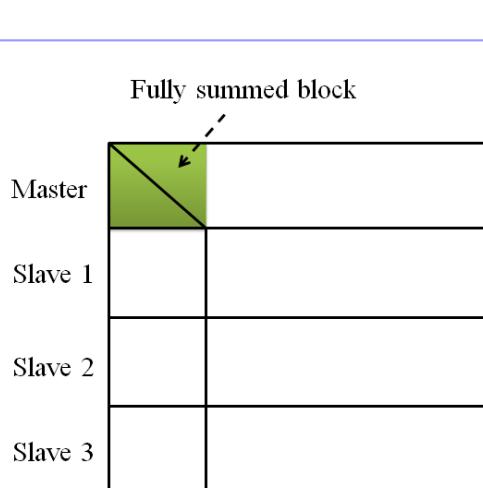


Figure 6.9.: MUMPS: static and dynamic scheduling

Both BLAS and LAPACK originate from the Netlib project which is a repository of numerous scientific computing software maintained by AT&T Bell Laboratories, the University of Tennessee, Oak Ridge National Laboratory and other scientific communities [06].

The goal of BLAS library is provision of a high efficient implementation of common dense linear algebra kernels by means of high rate of floating point operations per memory access, low cache and Translation Lookaside Buffer (TLB) miss rates.

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figure 6.9

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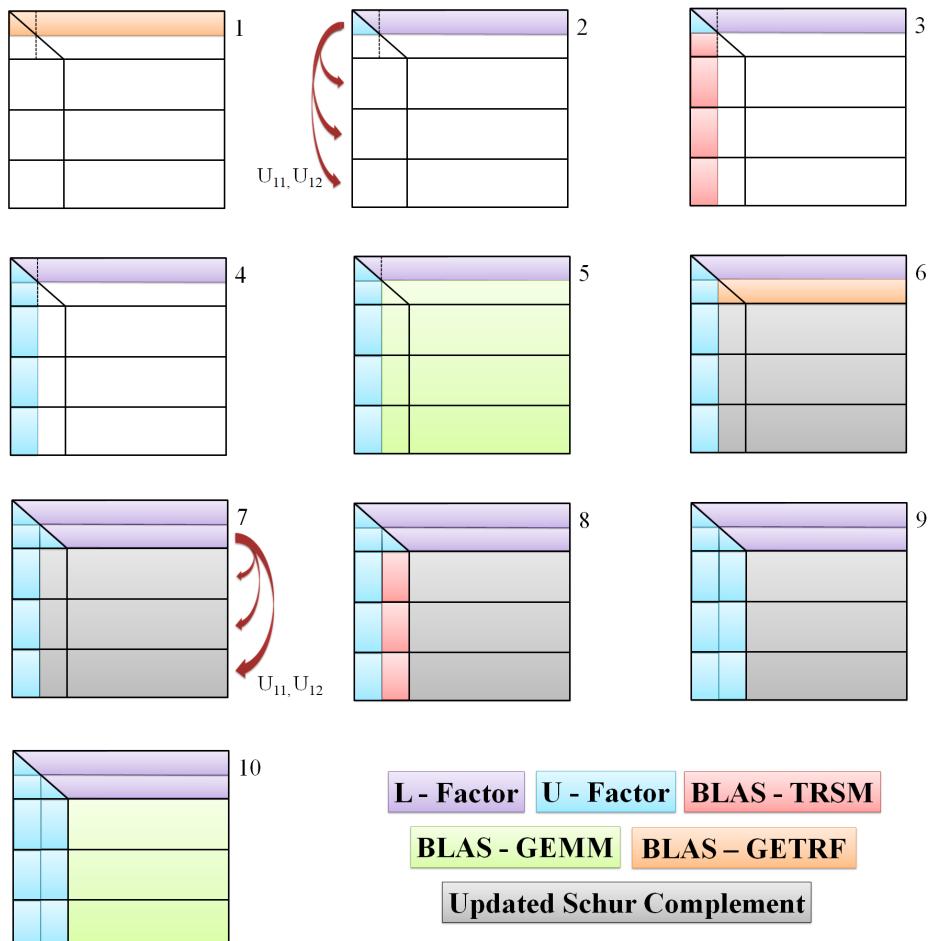


Figure 6.10.: MUMPS: An example of a type 2 node factorization

In its turn, LAPACK is designed in such a way so that as much as possible computations is performed by calls to BLAS library. This allows to achieve high efficiency for operations such as *LU*, *QR*, *SVD* decompositions, triangular solve, etc. on modern computers. However, the Netlib BLAS implementation is written for an abstract general-purpose central processing unit, in mind, where hardware parameters are based on market statistics. Hence, it is not possible to achieve the maximum possible performance on a specific machine.

Hence, there exist special-purpose, hardware-specific implementations of the library developed by hardware vendors i.e. IBM, Cray, Intel, AMD, etc., as well as open-source tuned implementations such as ATLAS, OpenBLAS, etc. To achieve full compatibility, the developers consider the Netlib implementation of BLAS library as the standard (or reference) and thus overwrite all subroutines with additional tuning and optimization. This approach makes it possible to easily replace different BLAS implementations during object files linking without any modifications of the source code.

Table 6.9 shows commercial and open-source tunned BLAS implementations available on the market today.

Among all libraries listed in table 6.9 there were only four available on HW1 machine, namely: Netlib BLAS, Intel MKL, OpenBLAS and ATLAS. However, installation of ATLAS requires to switch off dynamic frequency scaling, also called CPU throttling, to allow an ATLAS configuration routines to find the best loop transformation parameters for a specific hardware. In order to turn off CPU throttling, one has to reboot the entire machine and make appropriate changes in Basic Input/Output System (BIOS). This fact made ATLAS library not suitable for the rest of the study and we excluded it from the primary list of candidates. Moreover, during installation, one has to explicitly provide the number of OpenMP threads that are going to be used once a BLAS subroutine is called. This means there is no way to change the number of threads per MPI process in run-time without re-installation of ATALS library. Thus, only 3 versions of MUMPS-PETSc (Netlib BLAS, Intel MKL and OpenBLAS) library were compiled, installed and tested with using both GRS and SuiteSparse matrix sets and 1 thread per MPI process. The test results are obtained on HW1 machine only are represented in figures 6.11, 6.12 and appendix D.

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The tests show that OpenBLAS outperforms both Netlib and Intel MKL libraries in case of GRS matrix set. In average, OpenBLAS is about **13%** faster than the default

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6.11

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6.12

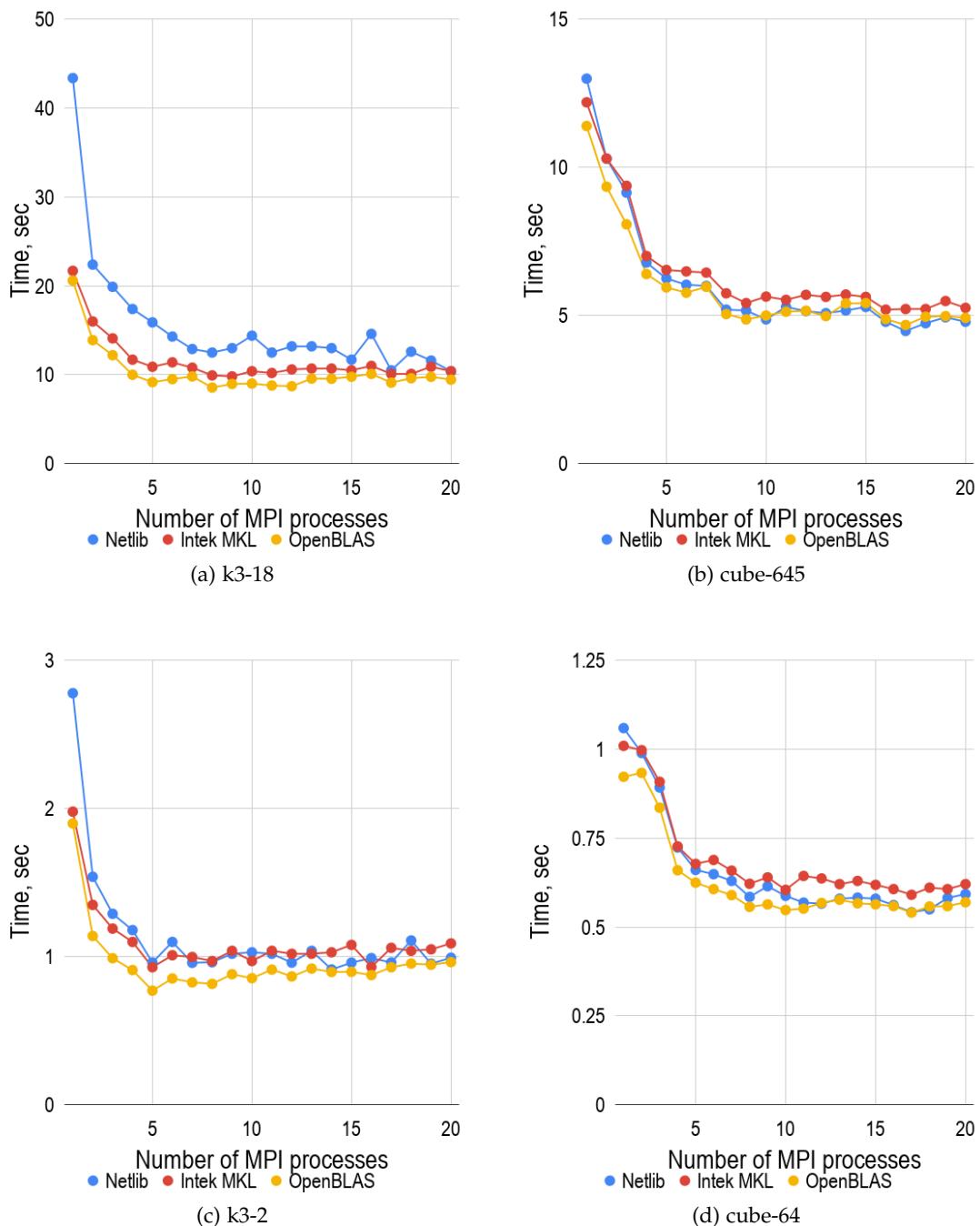


Figure 6.11.: MUMPS: comparison of different BLAS libraries with using GRS matrix set on HW1 machine

## 6. Solver selection and configuration

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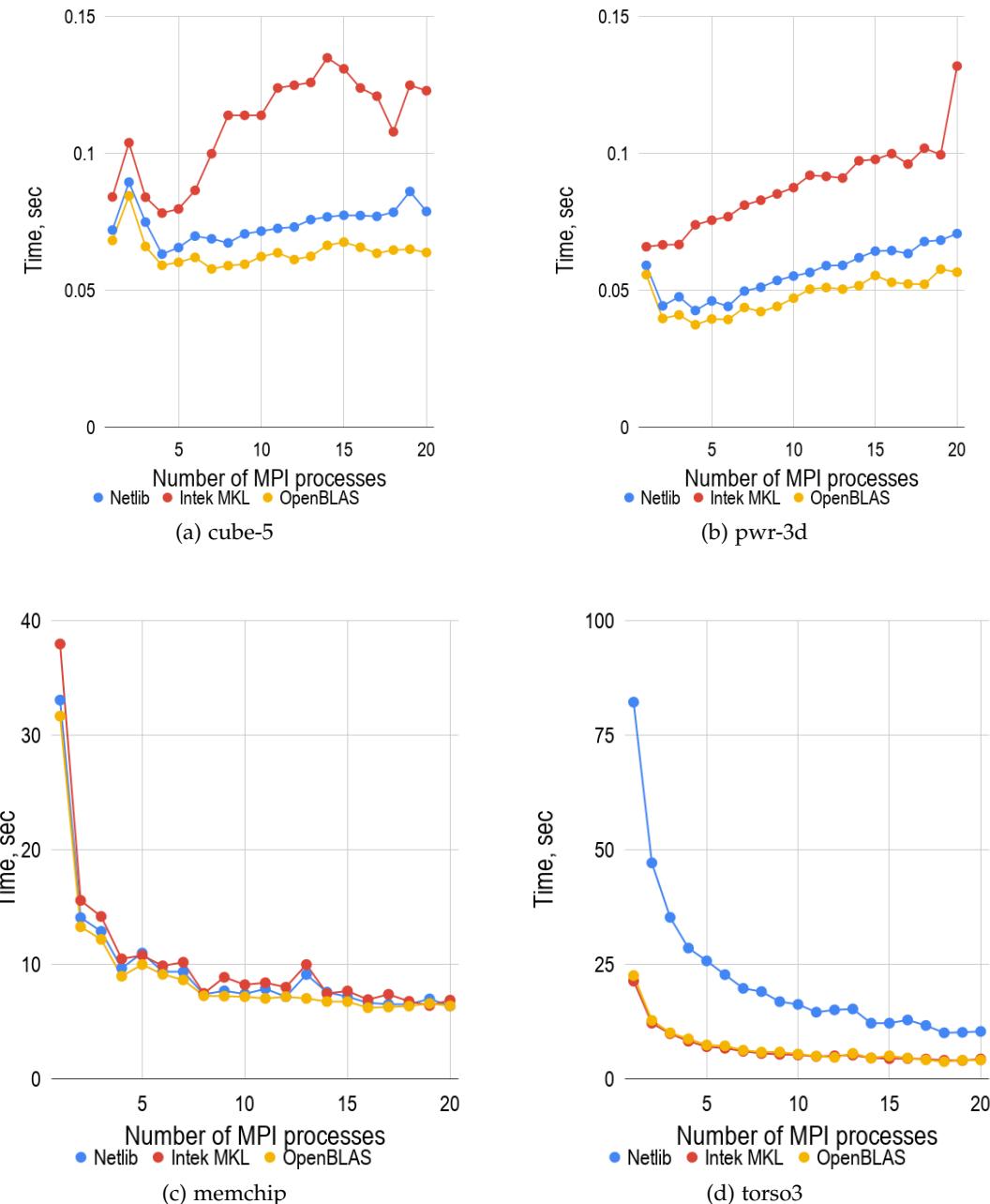


Figure 6.12.: MUMPS: comparison of different BLAS libraries with using both GRS and SuiteSparse matrix sets on HW1 machine

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Name	Description	License
Accelerate	Apple's implementation for macOS and iOS	proprietary license
ACML	BLAS implementation for AMD processors	proprietary license
C++ AMP	Microsoft's AMP language extension for Visual C++	open source
ATLAS	Automatically tuned BLAS implementation	open source
Eigen BLAS	BLAS implemented on top of the MPL-licensed Eigen library	open source
ESSL	optimized BLAS implementation for IBM's machines	proprietary license
GotoBLAS	Kazushige Goto's implementation of BLAS	proprietary license
HP MLIB	BLAS implementation supporting IA-64, PA-RISC, x86 and Opteron architecture	proprietary license
Intel MKL	Intel's implementation of BLAS optimized for Intel Pentium, Core, Xeon and Xeon Phi	proprietary license
Netlib BLAS	The official reference implementation on Netlib	open source
OpenBLAS	Optimized BLAS library based on GotoBLAS	open source
PDLIB/SX	BLAS library targeted to the NEC SX-4 system	proprietary license
SCSL	BLAS implementations for SGI's Irix workstations	proprietary license
Sun Performance Library	Optimized BLAS and LAPACK for SPARC, Core and AMD64 architectures under Solaris 8, 9, and 10 as well as Linux	proprietary license

Table 6.9.: Commercial and open source BLAS implementations [Wik18b]

Netlib implementation and approximately **21%** faster than Intel MKL library. It is interesting to notice Intel MKL turned out to be slower than the default Netlib BLAS implementation for small and medium size GRS matrices in almost **52%** and **2%**, respectively. At the same time, both tuned libraries, OpenBLAS and Intel MKL, showed significant performance gain relatively to the standard Netlib BLAS implementation in case of SuiteSparse matrix set. The libraries reduced the execution time in almost **50%** in average. In opposite to GRS matrix set, it turned out that Intel MKL was often faster than OpenBLAS for almost all test cases from SuiteSparse matrix set. However, the difference between them is negligibly small. The result of comparison are summarized

in tables 6.10 and 6.11.

Matrix Name	Performance gain of OpenBLAS relatively to Netlib %	Performance gain of IntelMKL relatively to Netlib %	Performance gain of OpenBLAS relatively to Intel MKL %
pwr-3d	14.607	-56.249	44.695
cube-5	13.569	-47.797	39.931
cube-64	4.385	-5.483	9.323
cube-645	1.897	-7.474	8.702
k3-2	13.906	0.833	13.057
k3-18	29.914	21.03	11.29

Table 6.10.: Comparison of different MUMPS-BLAS configurations applied to GRS matrix set

Matrix Name	Performance gain of OpenBLAS relatively to Netlib %	Performance gain of IntelMKL relatively to Netlib %	Performance gain of OpenBLAS relatively to Intel MKL %
cant	26.981	25.964	1.233
consph	67.617	68.252	-2.327
CurlCurl_3	78.804	79.37	-3.371
Geo_1438	83.106	83.565	-2.857
memchip	6.066	-6.909	11.883
PFlow_742	75.574	74.943	1.416
pkustk10	35.089	34.536	0.502
torso3	66.185	66.988	-2.837
x104	41.82	41.936	-0.445

Table 6.11.: Comparison of different MUMPS-BLAS configurations applied to SuiteSparse matrix set

In this section, we have shown how and where MUMPS utilizes common third-party libraries. The design of MUMPS can be considered as a good example of a scientific software architecture which reuses well-known, common blocks which appear quite

often in scientific programs i.e. dense linear algebra kernels. Such design allows to easily configure and adopt software for a specific hardware which we demonstrated in this section.

We have also shown that MUMPS-OpenBLAS configuration is the best choice for GRS matrix set. It has been interesting to observe that the tuned Intel implementation of BLAS compiled with the latest Intel compiler demonstrated slow-down in contrast to the standard not optimized implementation. However, we can assume that amount of GRS test cases is not enough to make such a conclusion and thus the matrix set has to be extended considerably.

## 6.6. MPI-OpenMP Tuning of MUMPS Library

As it was mentioned in section 6.2, the development of MUMPS began in 1996 when message-passing programming paradigm dominated in parallel computing. Therefore, the library originally was designed only for distributed-memory machines.

In 2010, Chowdhury and L'Excellent published their first experiments and some issues, in [CL10], of exploiting shared memory parallelism in MUMPS. The authors showed that it was possible to achieve some improvements in multicore systems with multithreading, given a purely MPI application. However, later L'Excellent and Sid-Lakhdar mentioned, in [LS13], that adaptation of existing code for NUMA architecture was still a challenge because of memory allocation, memory affinity, thread pinning and other related issues.

In spite of natural data locality of message-passing applications which is always beneficial, a general motivation of switching to a hybrid mode, mixed MPI/OpenMP, is to reduce communication overheads between processes. According to the profiling results done by Chowdhury and L'Excellent, MUMPS contained four main initial sources of shared-memory parallelization, namely:

1. BLAS Level 1, 2, 3 operations during both factorization and solution phases
2. Assembly operations, where contribution blocks of children nodes of the assembly tree are assembled at the parent level
3. Copying contribution blocks during stacking operations
4. Pivot search operations

Almost all customized BLAS libraries, for example Intel MKL and OpenBLAS, are multi-threaded and can efficiently work in shared-memory environment. Thus, parallelization of region 1 can be achieved by linking a suitable BLAS library whereas regions 2, 3 and 4 are multithreaded by inserting appropriate OpenMP directives above the corresponding loop statements.

A detailed review of works [LS13] and [CL10] reveals that, in general, a pure OpenMP or mixed MPI/OpenMP strategy can reduce run-time of MUMPS. In average, factorization time is reduced by **14.3%** and in some special cases improvement reaches about **50.4%**, according to the data provided in the papers. However, at the same time, the results demonstrate that pure-MPI mode sometimes can significantly outperform any hybrid mixed strategy.

By and large, the results show two important aspects. Firstly, performance of a specific strategy depends heavily on the resultant assembly tree and thus on the matrix sparsity pattern and applied fill reducing algorithm. Secondly, it is not possible to guess in advance which strategy gives the best parallel performance without detailed information about the tree structure and computational cost per node. L'Excellent and Sid-Lakhdar showed that performance of a particular mode depended on the ratio of large and small fronts. For example, they noticed more threads per MPI process leaded to better parallel performance when the ratio was high. On the other hand, they observed the absolutely opposite result with relatively small ratios. Unfortunately, L'Excellent and Sid-Lakhdar did not provide any quantitative measure for that in their work [LS13].

It is also interesting to notice that parallelization of region 1 with using a multi-threaded BLAS library brings most of parallel performance for mixed or pure OpenMP strategies, according to the results from[LS13]. But, at the same time, performance of regions 2, 3, 4 multithreaded by OpenMP directives is marginal. In average, it allows to reduce numerical factorization run-time by only **0.66%**.

This outcome is expected because BLAS subroutines, especially level 3, have a high ratio of floating point operations per memory access and thus PEs re-use data stored in caches as much as possible. Meanwhile, regions 2, 3, 4 mainly perform initialization, data movement and execution of *if-statements* which lead to a low compute ratio.

Additionally, it is worth noticing that both works, [CL10] and [LS13], were mainly focused on the numerical factorization phase assuming that both analysis and solution phases do not take lots of time. In spite of credibility of this assumption, it still should

be pointed out the solution phase runs faster in case of flat-MPI mode. This fact becomes even more interesting because, in our case, a system with multiple right-hand sides has to be solved in order to generate a preconditioner.

We have to admit that both works, [CL10] and [LS13], are relatively old and the analysis above may be not complete and full. Because MUMPS is a dynamic developing project, we can expect that adaptation of shared-memory parallelization in MUMPS has been significantly advanced since that time. Since the release of MUMPS version 4, the developers have persistently recommended to use only a hybrid mode like *one MPI process per socket and as many threads as the number of cores* [17].

As an initial test, we compared influence of both Intel MKL and OpenBLAS libraries on parallel performance of MUMPS using GRS matrix set only. In order to pin OpenMP threads in a right way, without any conflict between them, the following OpenMP environment variables were set as:

- OMP\_PLACES=cores
- OMP\_PROC\_BIND=spread

During the test, we found that run-time of MUMPS-OpenBLAS configuration abnormality increased for some test cases. For instance, in case of matrix *cube-645*, the increase reached almost 450% in contrast to the pure sequential execution.

Multiple conflicts between application and system threads were observed using *htop* as an interactive process viewer. Figure 6.14 shows a snapshot taken during factorization of matrix *k3-18* running with 1 MPI process and 20 threads.

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figure  
6.13

It is difficult to say what exactly caused such behavior. However, Chowdhury and L'Excellent also reported about the same problem with using GotoBLAS (OpenBLAS). They assumed that GotoBLAS created and kept some threads active even after the main threads returned to the calling application which could lead to interference with threads created in other OpenMP regions [CL10]. For that reason, we decided to stick only to Intel MKL library for the rest of the study of this section because there were no conflicts detected during initial runs.

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6.14

Only common mixed MPI/OpenMP modes were tested in order to check influence of shared-memory parallelism on parallel performance of MUMPS as well as to limit

## 6. Solver selection and configuration

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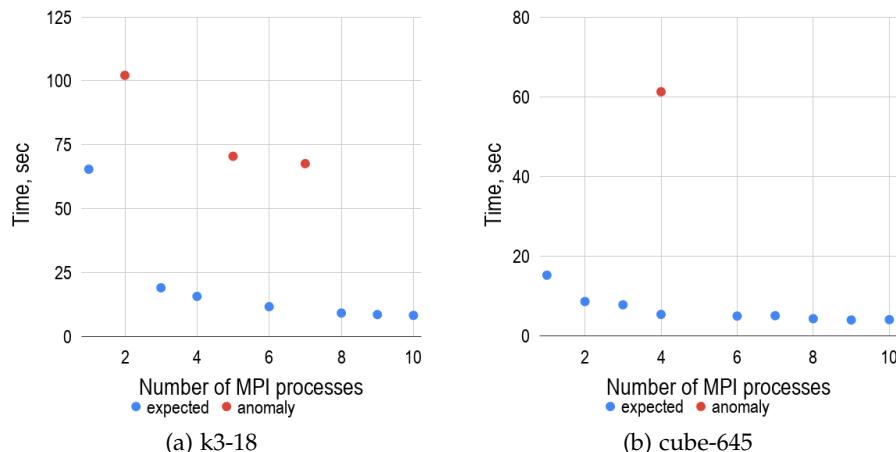


Figure 6.13.: Anomalies of MUMPS-OpenBLAS configuration running with 2 OpenMP threads per MPI process

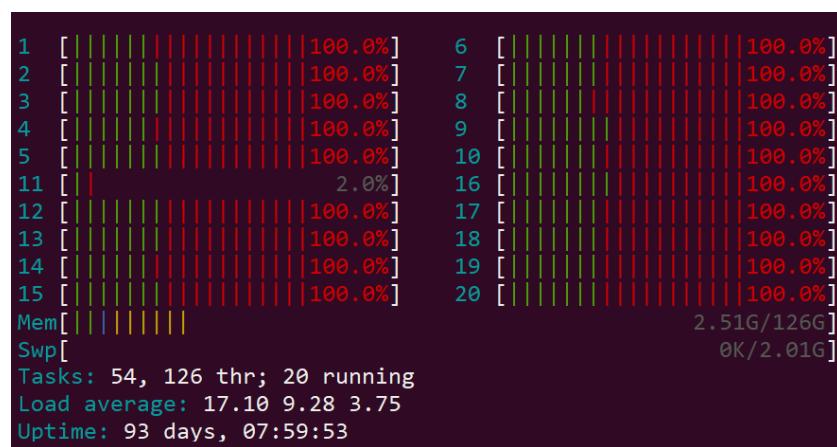


Figure 6.14.: A MUMPS-OpenBLAS thread conflict in case of  $k3-18$  matrix factorization  
(green - application threads, red - system threads)

## 6. Solver selection and configuration

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the amount of testing. The following strategies were chosen, namely: 20 MPI - 1 thread (flat-MPI), 10 MPI - 2 threads, 4 MPI - 5 threads, 2 MPI - 10 threads, 1 MPI - 20 threads (flat-OpenMP). The tests were conducted on both HW1 and HW2 machines in order to check whether the results are consistent across different machines with different operating and environment settings. Additionally, MUMPS was set as a preconditioning algorithm for the GMRES solver with only *one iteration*. This allowed to force MUMPS library to solve systems multiple right-hand sides. According to our assumption, time spent on one GMRES iteration is negligible in contrast time spent on sparse direct matrix factorization. The test results are represented in tables 6.12 6.13, 6.14 and 6.15. Numerical values in tables are given in seconds.

Matrix Name	20 MPI 1 thread	10 MPI 2 threads	4 MPI 5 threads	2 MPI 10 threads	1 MPI 20 threads	Gain w.r.t. flat-MPI
k3-18	<b>12.520</b>	12.630	14.010	18.020	19.170	-
k3-2	1.341	<b>1.250</b>	1.470	1.671	2.052	1.073
cube-645	<b>6.585</b>	6.859	8.552	12.010	14.080	-
cube-64	0.756	<b>0.749</b>	0.874	1.178	1.354	1.010
cube-5	0.181	0.132	<b>0.104</b>	0.126	0.117	1.744
pwr-3d	0.130	0.114	0.0972	<b>0.077</b>	0.109	1.691

Table 6.12.: Compassion of different hybrid MPI/OpenMP modes used for GRS matrix set on HW1

Matrix Name	20 MPI 1 thread	10 MPI 2 threads	4 MPI 5 threads	2 MPI 10 threads	1 MPI 20 threads	Gain w.r.t. flat-MPI
k3-18	8.558	<b>7.819</b>	8.165	11.330	14.320	1.095
k3-2	1.168	<b>0.788</b>	0.956	1.131	1.651	1.482
cube-645	5.735	<b>4.859</b>	6.069	9.360	11.040	1.180
cube-64	0.805	<b>0.541</b>	0.664	0.947	0.918	1.490
cube-5	0.241	0.121	<b>0.093</b>	0.129	0.126	2.582
pwr-3d	0.234	0.095	0.098	<b>0.070</b>	0.094	3.341

Table 6.13.: Compassion of different hybrid MPI/OpenMP modes used for GRS matrix set on HW2

According to the results, we have noticed the optimal hybrid MPI/OpenMP mode

## 6. Solver selection and configuration

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Matrix Name	20 MPI 1 thread	10 MPI 2 threads	4 MPI 5 threads	2 MPI 10 threads	1 MPI 20 threads	Gain w.r.t. flat-MPI
cant	1.400	<b>0.990</b>	1.050	1.605	2.019	1.414
consph	3.495	<b>2.652</b>	3.015	3.706	3.714	1.318
memchip	<b>7.470</b>	9.080	13.301	20.198	45.800	-
PFflow_742	26.802	24.204	<b>21.897</b>	30.389	54.501	1.224
pkustk10	<b>0.748</b>	0.879	0.972	1.459	1.280	-
torso3	<b>3.922</b>	4.285	4.642	5.603	8.144	-
x104	<b>1.597</b>	1.644	2.024	3.208	2.167	-
CurlCurl_3	49.250	44.120	<b>39.909</b>	43.311	63.001	1.234
Geo_1438	478.101	234.697	<b>151.603</b>	157.697	158.102	3.154

Table 6.14.: Comparison of different hybrid MPI/OpenMP modes used for SuiteSparse matrix set on HW1

Matrix Name	20 MPI 1 thread	10 MPI 2 threads	4 MPI 5 threads	2 MPI 10 threads	1 MPI 20 threads	Gain w.r.t flat-MPI
cant	2.128	<b>0.955</b>	1.011	1.577	2.058	2.229
consph	3.840	<b>2.852</b>	3.111	3.695	3.897	1.346
memchip	<b>7.811</b>	7.816	9.811	15.160	31.969	-
PFflow_742	24.190	29.241	<b>19.686</b>	27.530	55.431	1.230
pkustk10	1.373	<b>0.904</b>	1.022	1.421	1.403	1.520
torso3	4.733	<b>4.080</b>	4.483	5.648	8.217	1.160
x104	2.676	<b>1.597</b>	2.025	3.204	2.133	1.676
CurlCurl_3	39.890	<b>34.579</b>	38.620	41.171	67.760	1.154
Geo_1438	ROM	ROM	ROM	ROM	ROM	ROM

Table 6.15.: Comparison of different hybrid MPI/OpenMP modes used for SuiteSparse matrix set on HW2

\*ROM - run out of memory

locates near the saturation point of the corresponding flat-MPI test. Therefore, search of the optimal mode can take considerable amount of time. It is needless to say that the mode varies from matrix to matrix and there is no way to predict the mode in advance. Moreover, the results show that performance gain is around **2.1%** in case of GRS matrix set on HW1 hardware, excluding small test cases such as *cube-5* and *pwr-3d* where runs

with 20 MPI processes were slower in contrast to sequential execution. Much optimistic results were obtained for experiments conducted on HW2 machine where performance gain reached almost 31% for the same test cases.

SuiteSparse matrix set demonstrates much better performance gain out of hybrid parallel computing on both hardware. In average, run-time is improved in more than 15% for HW1 and approximately 41% in case of HW2, excluding *Geo\_1438* from statistics. The best result was obtained for *Geo\_1438* matrix where execution time dropped in about **3 times** for all hybrid modes relatively to flat-MPI. We suppose it could happen because *Geo\_1438* matrix had a very huge ratio of large and small fronts. *Geo\_1438* is the best example which shows the benefit of hybrid parallel programming usage.

By far, we have noticed some negligible improvement of execution time in case of GRS matrix set on HW1 machine with using the optimal hybrid MPI/OpenMP configuration. However, in spite of the improvement, the parallel efficiency drops significantly. As the result, we have come to the conclusion that flat-MPI mode is the best for GRS matrix set on HW1 machine for three reasons. Firstly, this mode is deterministic and requires less amount of testing to find the optimal process count. Secondly, efficiency of hardware usage is better in case of flat-MPI which can be relevant for energy-efficient computing. Thirdly, improvement obtained with using Intel MKL library running with the optimal hybrid MPI/OpenMP mode can deteriorate performance gain of MUMPS-OpenBLAS flat-MPI configuration.

## 6.7. Conclusion

In this chapter, we have examined different types of sparse linear solvers for integration of ODEs systems arisen from thermo-hydraulic simulations. We have come to the conclusion that, in spite of better scalability and parallel efficiency of iterative methods, direct sparse linear solvers are most suitable for this purpose due to its robustness.

In section 6.1, we have tested different direct sparse solvers, namely: SuperLU\_DIST, PasTiX and MUMPS. MUMPS library showed better parallel efficiency among the others according to the test results and thus was chosen for the rest of the study where we focused on performance tuning of the library.

We have shown in subsequent sections there have been four main sources of MUMPS library tuning, namely:

1. careful choice of a fill reducing reordering method
2. distributed MPI process pinning across a compute node
3. configuration of MUMPS with an optimal, tuned BLAS implementaion
4. execution of MUMPS with an optimal hybrid MPI/OpenMP mode

To perform accurate testing and careful analysis, two different matrix sets have been used, namely: GRS and SuiteSparse. The fist one is problem specific for thermo-hydraulic simulations whereas the second one was built from SuiteSparse Matrix Collection [DH11], [DGL89] by downloading a dozen different matrices with respect to the number of equations and the number of non-zeros. Additionally, some tests were performed on two different compute clusters, called HW1 and HW2 (see section 4), to check whether a problem was hardware specific or not.

1. In section 6.3 it has been shown that parallel performance of MUMPS is quite sensitive to a used fill-in reducing reordering algorithm. A right choice of the algorithm can lead to significant improvements in terms of the overall execution time. The average performance gain was almost **15%** and in some particular cases it was possible to reduce execution time in approximately **40-55%** in case of GRS matrix set. During experiments, we came to the conclusion there was no an indirect metric to predict the best algorithm in advance for a specific system of equations.
2. In section 6.4, influence of different process pinning strategies on MUMPS parallel performance has been demonstrated. The tests have shown that equal distribution of MPI process among all available NUMA domains always bring extra performance. In average, *spread*-pinning allows to reduce execution time of MUMPS by almost **5.5%** and **13.8%** on HW1 and HW2 machines, respectively, in case of GRS matrix set.
3. We have discussed and examined in section 6.5 the way how MUMPS library has been implemented to preform partial factorization of type 2 nodes in section. It has turned out the library intensively calls *GEMM*, *TRSM* and *GETRF* BLAS subroutines. Therefore, replacement of standard Netlib BLAS library by tuned BLAS implemen-tations makes it possible to improve overall performance of the solver in case of a sufficient amount of type 2 nodes in a matrix assembly tree.

The results have shown that OpenBLAS outperforms both Netlib and Intel MKL libraries, which were available for the tests, in case of GRS matrix set. The average performance gain turned out to be about **13%** relatively to the default Netlib implemen-tation and approximately **21%** in contrast to Intel MKL library. We have also noticed

## 6. Solver selection and configuration

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that Intel MKL was even slower than Netlib BLAS for small and medium size GRS matrices in almost **52%** and **2%**, respectively. At the same time, both libraries, OpenBLAS and Intel MKL, showed significant performance gain relatively to the standard Netlib BLAS in case of SuiteSparse matrix set and allowed to reduce the execution time by almost **50%** in average.

4. In section 6.6, we have discussed where and how developers of MUMPS library applied shared-memory parallelism based on review of works [CL10] and [LS13]. We have also found severe slow-down of some hybrid MPI/OpenMP modes due to system and application thread conflicts while MUMPS-OpenMP configuration was running on HW1 cluster. For that reason, the following study was continued with only using Intel MKL library which, in its turn, turned out to be thread-safe.

problems  
with open  
blas  
intel mkl

We have shown that in some particular cases, factorization of *Geo\_1438* matrix for example, hybrid MPI-OpenMP approach can bring significant performance improvement. However, application of hybrid computing to GRS matrix set gives negligible improvement on HW1 machine i.e. around **2.1%**, and significantly deteriorates parallel efficiency. Much optimistic results were obtained for experiments conducted on HW2 machine where performance gain reached almost **31%** for the same matrix set.

During the study, we have also discovered the optimal hybrid MPI/OpenMP mode locates near the saturation point of the corresponding flat-MPI test. This fact can allow to reduce the amount of testing in general. However, we do not recommend to proceed the study in this direction due to above mention parallel efficiency issues detected on HW1 machine and lack of ability to run OpenBLAS library without thread conflicts.

Figures 6.15 and 6.16 show a comparison of MUMPS parallel performance before and after application of above mentioned tuning to GRS matrix set. In case of experiments labeled as *default*, we used ParMetis as a fill reducing reordering algorithm because it had been used by GRS engineers before the current study.

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figure  
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6.16  
include  
statistics

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By and large, we have observed significant improvement of applied tuning techniques, mentioned above, for the entire GRS matrix set.

In average, the factorization time has been reduced in **51.4%** for small sized systems. The significant performance gain mainly came from the optimal choice of fill-in reduction algorithm. Moreover, application of PT-Scotch for small sized systems allowed to

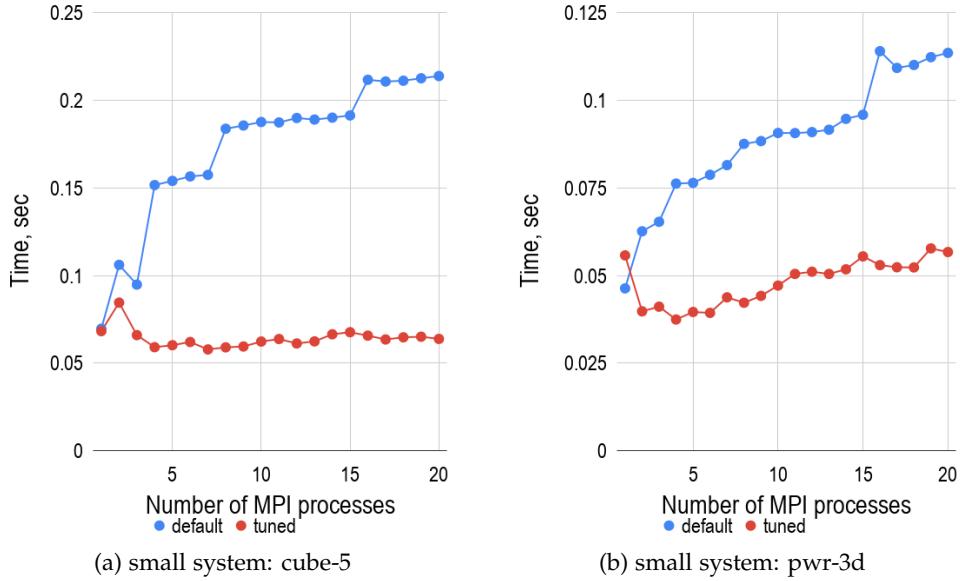


Figure 6.15.: MUMPS: comparison of different BLAS libraries with using both GRS and SuiteSparse matrix sets on HW1 machine

drastically change strong scaling behavior and reduce the execution time by approximately **17%** in contrast to sequential execution of the default MUMPS configuration that was a challenge before the study.

The execution time spent on factorization of medium sized systems dropped in **1.4** times. Additionally, we have noticed that scaling of *cube-64* test case has been considerably changed as it happened in case of small sized systems. Application of PT-Scotch, to *cube-64*, allowed to shift the saturation point from 5 to 10 process count without significant drop of parallel efficiency. All in all, tuning of MUMPS made it possible to reduce the execution time around the saturation points in almost **31%** in average for medium sized GRS systems of equations.

The factorization performance gain of large sized systems mainly came from configuration of MUMPS with optimized BLAS library, OpenBLAS, and MPI *spread* process distribution because it turned out that ParMetis was the best fill-in reduction algorithm for large systems. A noticeable performance gain difference was observed between *cube-645* and *k3-18* test cases. In average, run-time of MUMPS-OpenBLAS configuration reduced by almost **20%** in contrast to the default installation in case of *k3-18* whereas

factorization time of *cube-645* was improved only by **1.3%**. However, the saturation points of both cases were shifted towards lower values of the process count which allows to considerably improve hardware utilization together with improvement of library performance. For example, a detailed study of *k3-18* performance graph, figure 6.16d, shows that the saturation point has been moved from 17 to 8 process count. At the same time, the execution time drops by almost **19%** and parallel efficiency jumped in about **13%**. The same but less significant results can be observed for *cube-645* test case, figure 6.16c.

## 6.8. Recommendations

According to results and conclusion obtained in this study, we are able to make a general guidance for ATHLET-NuT users and developers.

First of all, we insist to configure PETSc-MUMPS module of NuT with OpenBLAS library and use only flat-MPI mode together with *spread* process pinning strategy because, as it was shown in sections 6.4 and 6.5, these settings always have their positive effect on parallel performance regardless of the matrix size and structure.

Secondly, we recommend the user to check table 6.16 before submitting a job on a production compute node in order to define a matrix type of a problem i.e. *small*, *medium* or *large*. Based on the matrix type, the user can select both a fill reducing reordering algorithm and the number of MPI processes which, we believe, can be pretty close to the optimal MUMPS settings.

Matrix type	Number of equations
Small	Less than 25000
Medium	form 50000 to 250000
Large	More than 500000

Table 6.16.: Matrix type reference table

We recommend to use only PT-Scotch and no more than 4 MPI processes for **small** systems. The same reordering algorithm, PT-Scotch, is better to apply to **medium** sized systems of equations as well, however, the optimal number of MPI processes lays in a range from 5 to 10. In case of **large** systems, we insistently recommend to switch to ParMetis package to perform fill-in reduction and start parallel execution from 8 MPI processes.

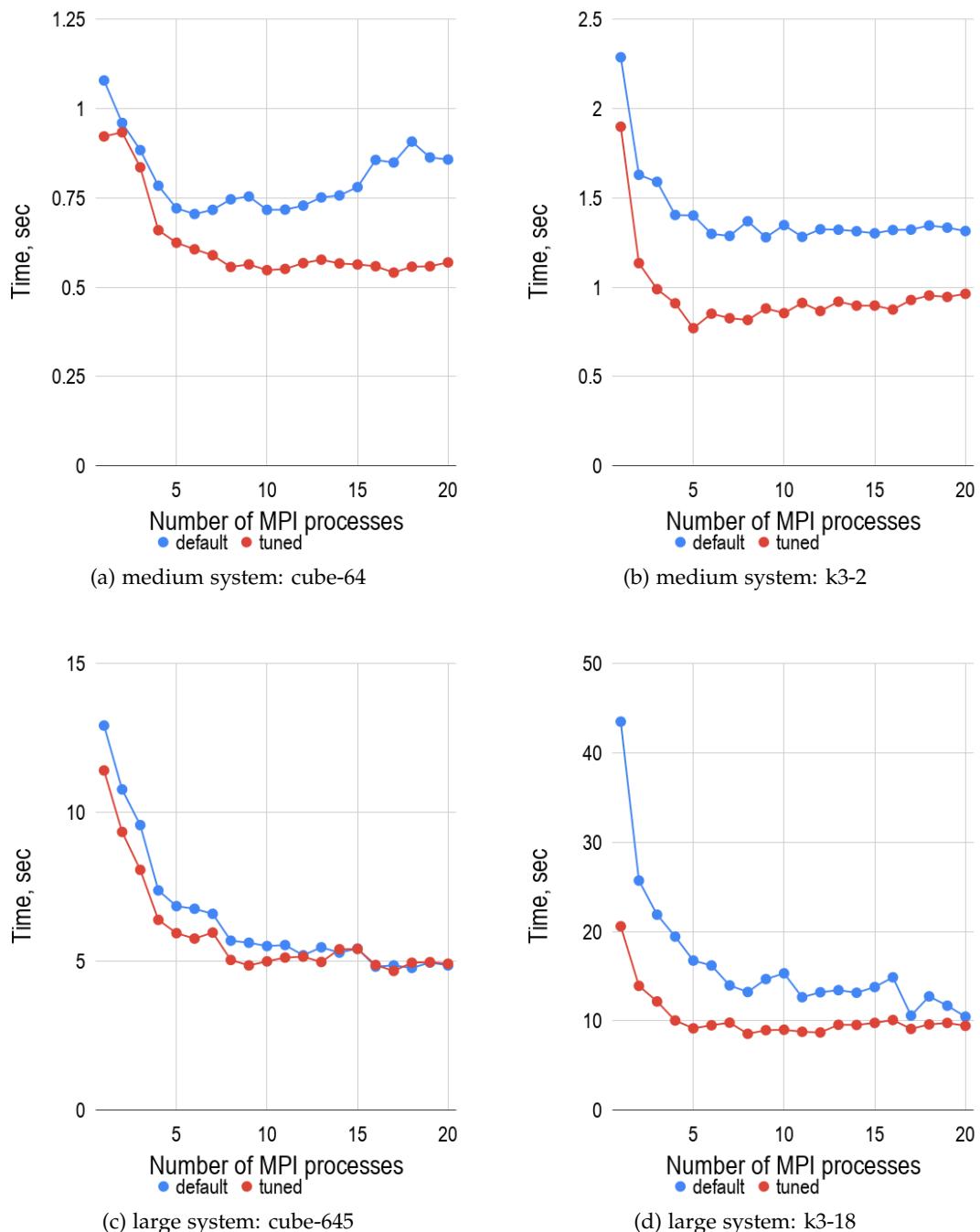


Figure 6.16.: MUMPS: comparison of different BLAS libraries with using both GRS and SuiteSparse matrix sets on HW1 machine

# Appendices

## A. Choice of a Sparse Direct Solver Library

MPI	MUMPS	PaStiX	SuperLU
1	4.58E-02	5.60E-02	4.64E+00
2	4.31E-02	5.14E-02	1.89E+00
3	4.51E-02	5.28E-02	1.22E+00
4	4.61E-02	5.64E-02	9.13E-01
5	4.92E-02	5.97E-02	7.70E-01
6	5.37E-02	6.14E-02	6.04E-01
7	5.42E-02	6.51E-02	crashed
8	5.41E-02	6.60E-02	4.81E-01
9	5.69E-02	6.84E-02	4.35E-01
10	5.86E-02	7.22E-02	4.08E-01

MPI	MUMPS	PaStiX	SuperLU
11	5.93E-02	8.97E-02	crashed
12	6.07E-02	9.20E-02	3.61E-01
13	6.26E-02	8.25E-02	crashed
14	6.28E-02	9.75E-02	crashed
15	6.43E-02	1.03E-01	3.05E-01
16	6.55E-02	1.05E-01	2.99E-01
17	6.61E-02	9.46E-02	crashed
18	6.73E-02	1.24E-01	2.65E-01
19	6.84E-02	1.14E-01	crashed
20	7.02E-02	1.32E-01	2.60E-01

Table A.1.: Results of a flat-MPI test of MUMPS, PaStiX and SuperLU\_DIST libraries with their default settings and the matrix **pwr-3d** (6009 equations)

MPI	MUMPS	PaStiX	SuperLU
1	1.55E+02	6.44E+01	time-out
2	6.28E+01	4.84E+01	time-out
3	5.06E+01	5.02E+01	time-out
4	4.17E+01	4.50E+01	time-out
5	2.52E+01	3.98E+01	time-out
6	2.58E+01	4.29E+01	time-out
7	2.65E+01	4.30E+01	time-out
8	2.59E+01	3.73E+01	time-out
9	1.95E+01	4.08E+01	time-out
10	1.91E+01	3.81E+01	time-out

MPI	MUMPS	PaStiX	SuperLU
11	1.77E+01	3.75E+01	time-out
12	1.60E+01	3.58E+01	time-out
13	1.42E+01	3.59E+01	time-out
14	1.45E+01	3.57E+01	time-out
15	1.47E+01	3.52E+01	time-out
16	1.41E+01	3.45E+01	time-out
17	1.54E+01	3.31E+01	time-out
18	1.52E+01	3.31E+01	time-out
19	1.52E+01	3.16E+01	time-out
20	1.38E+01	3.15E+01	time-out

Table A.2.: Results of a flat-MPI test of MUMPS, PaStiX and SuperLU\_DIST libraries with their default settings and the matrix **k3-2** (130101 equations)

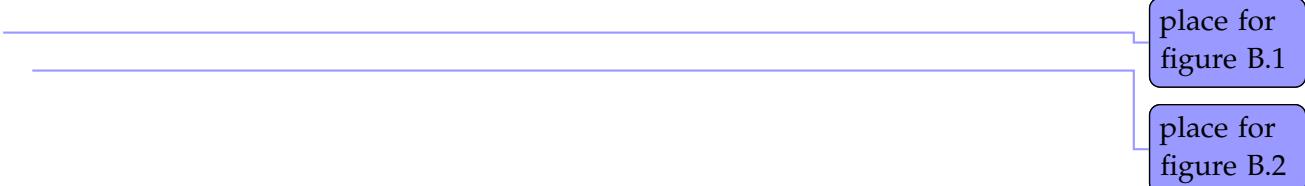
MPI	MUMPS	PaStiX	SuperLU
1	1.52E+01	1.61E+01	crashed
2	1.13E+01	1.13E+01	crashed
3	1.00E+01	1.03E+01	crashed
4	9.29E+00	1.05E+01	crashed
5	8.85E+00	9.84E+00	crashed
6	8.43E+00	8.99E+00	crashed
7	8.64E+00	9.69E+00	crashed
8	8.70E+00	9.12E+00	crashed
9	8.91E+00	8.94E+00	crashed
10	8.76E+00	9.26E+00	crashed

MPI	MUMPS	PaStiX	SuperLU
11	8.62E+00	9.09E+00	crashed
12	8.53E+00	8.92E+00	crashed
13	8.44E+00	9.13E+00	crashed
14	8.52E+00	9.00E+00	crashed
15	8.54E+00	9.19E+00	crashed
16	8.56E+00	9.05E+00	crashed
17	8.65E+00	9.12E+00	crashed
18	8.62E+00	8.96E+00	crashed
19	8.66E+00	9.30E+00	crashed
20	8.66E+00	9.16E+00	crashed

Table A.3.: Results of a flat-MPI test of MUMPS, PaStiX and SuperLU\_DIST libraries with their default settings and the matrix **cube-645** (1000045 equations)

## B. Choice of Fill Reducing Reordering



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figure B.1

place for  
figure B.2

### B. Choice of Fill Reducing Reordering

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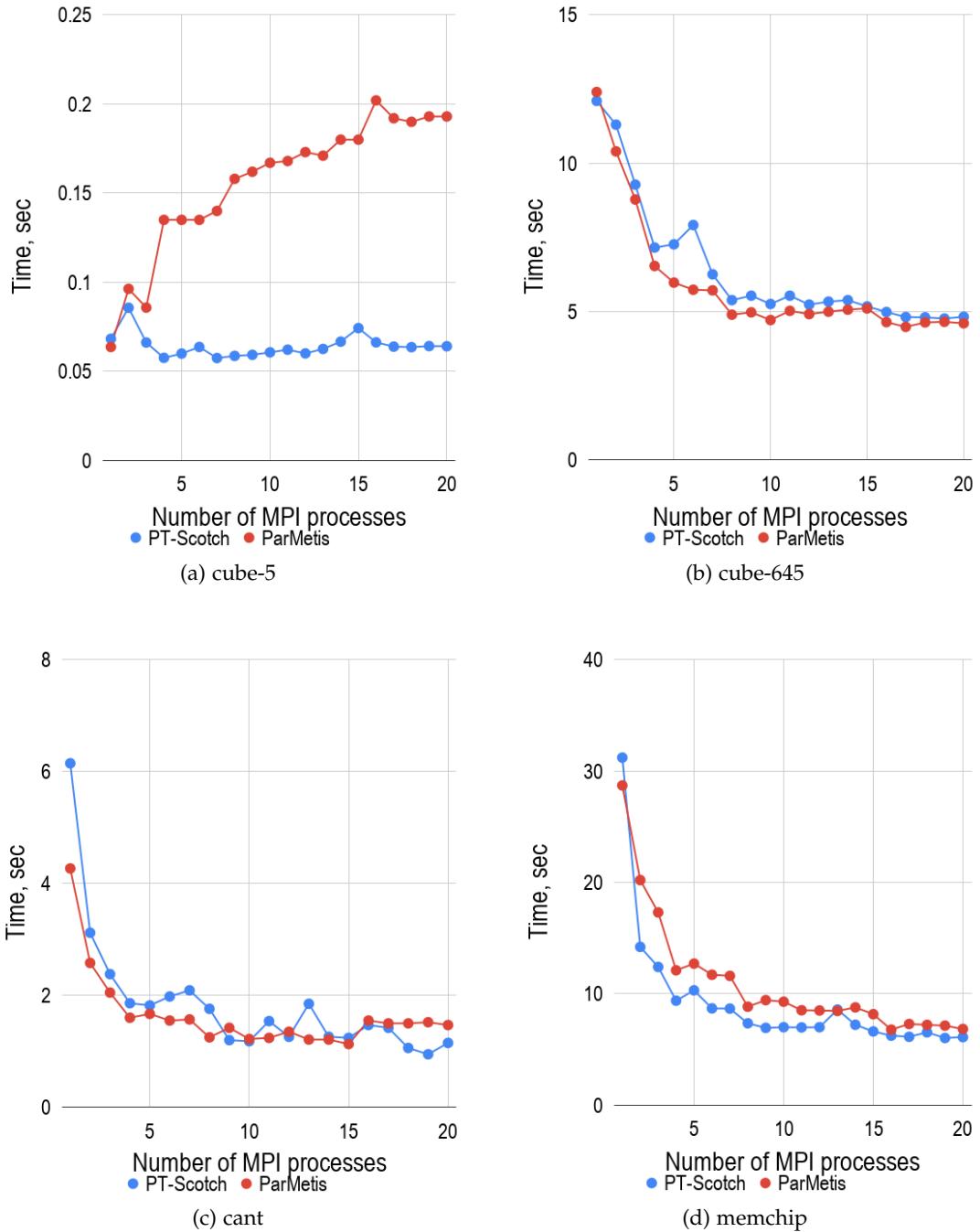


Figure B.1.: Comparison of different fill-reducing algorithms

### B. Choice of Fill Reducing Reordering

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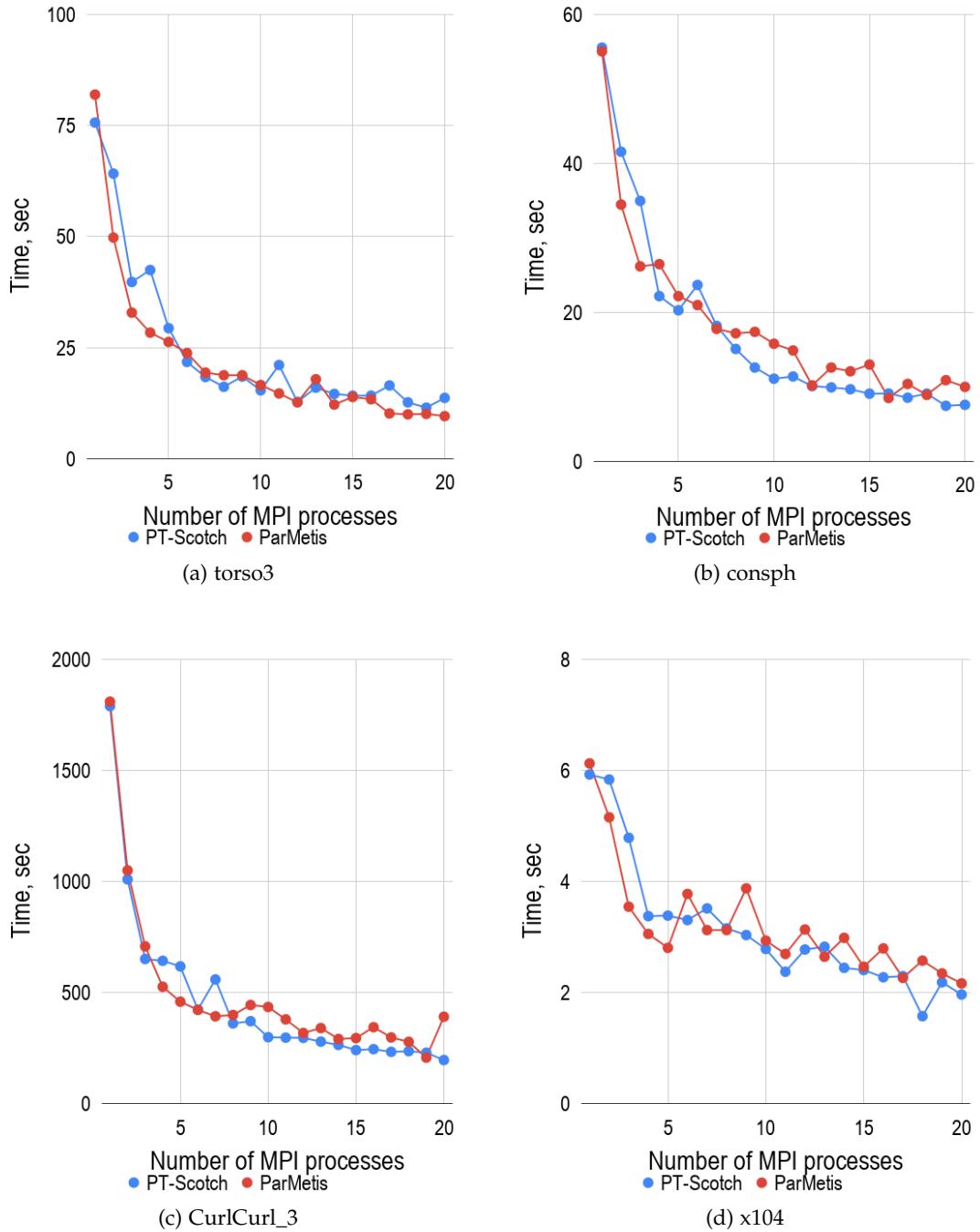


Figure B.2.: Comparison of different fill-reducing algorithms

## C. MUMPS: Process Pinning

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cube-64  
by Geo  
matrix

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figure C.1

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### C. MUMPS: Process Pinning

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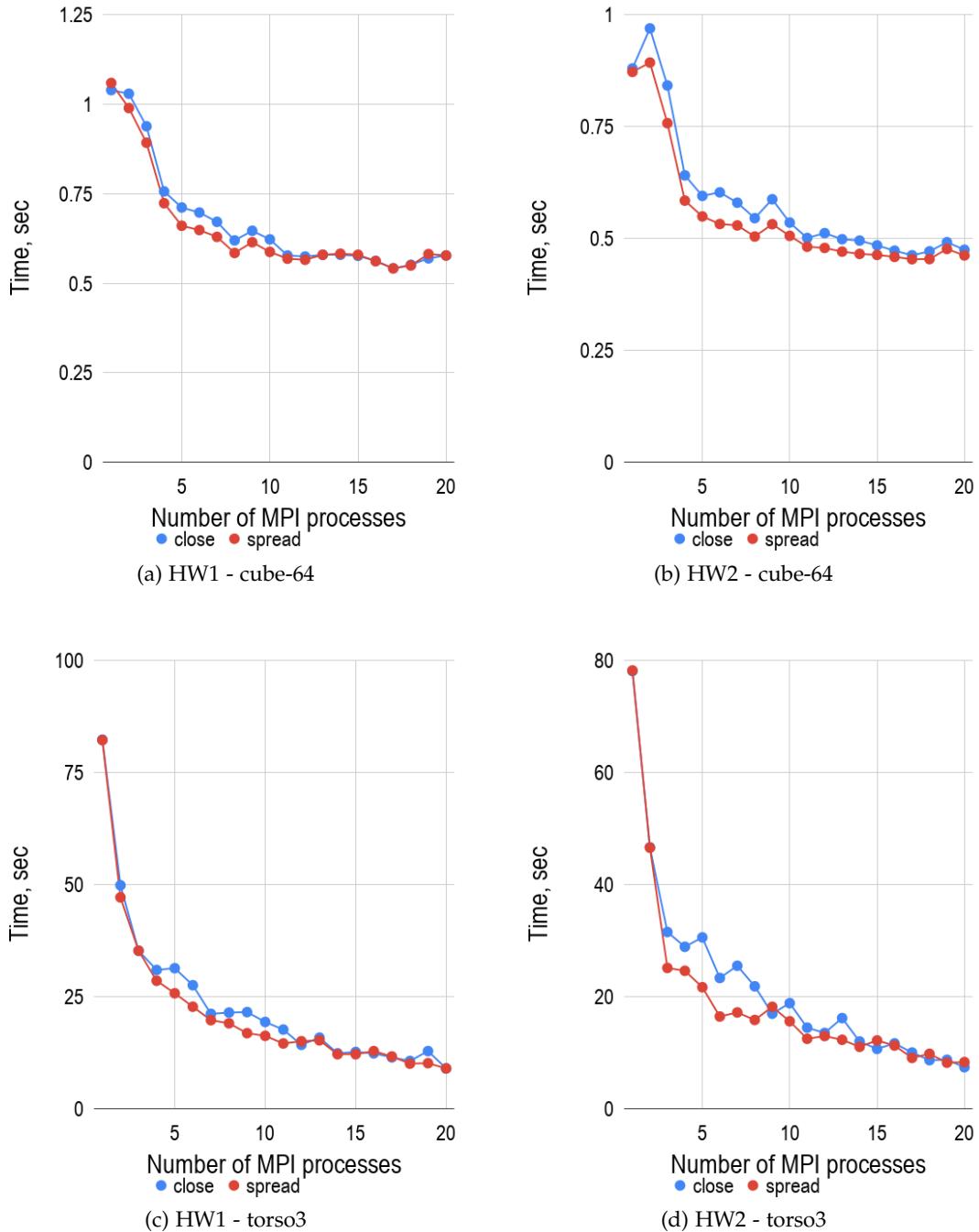


Figure C.1.: Comparison of *close* and *spread* pinning strategies

### C. MUMPS: Process Pinning

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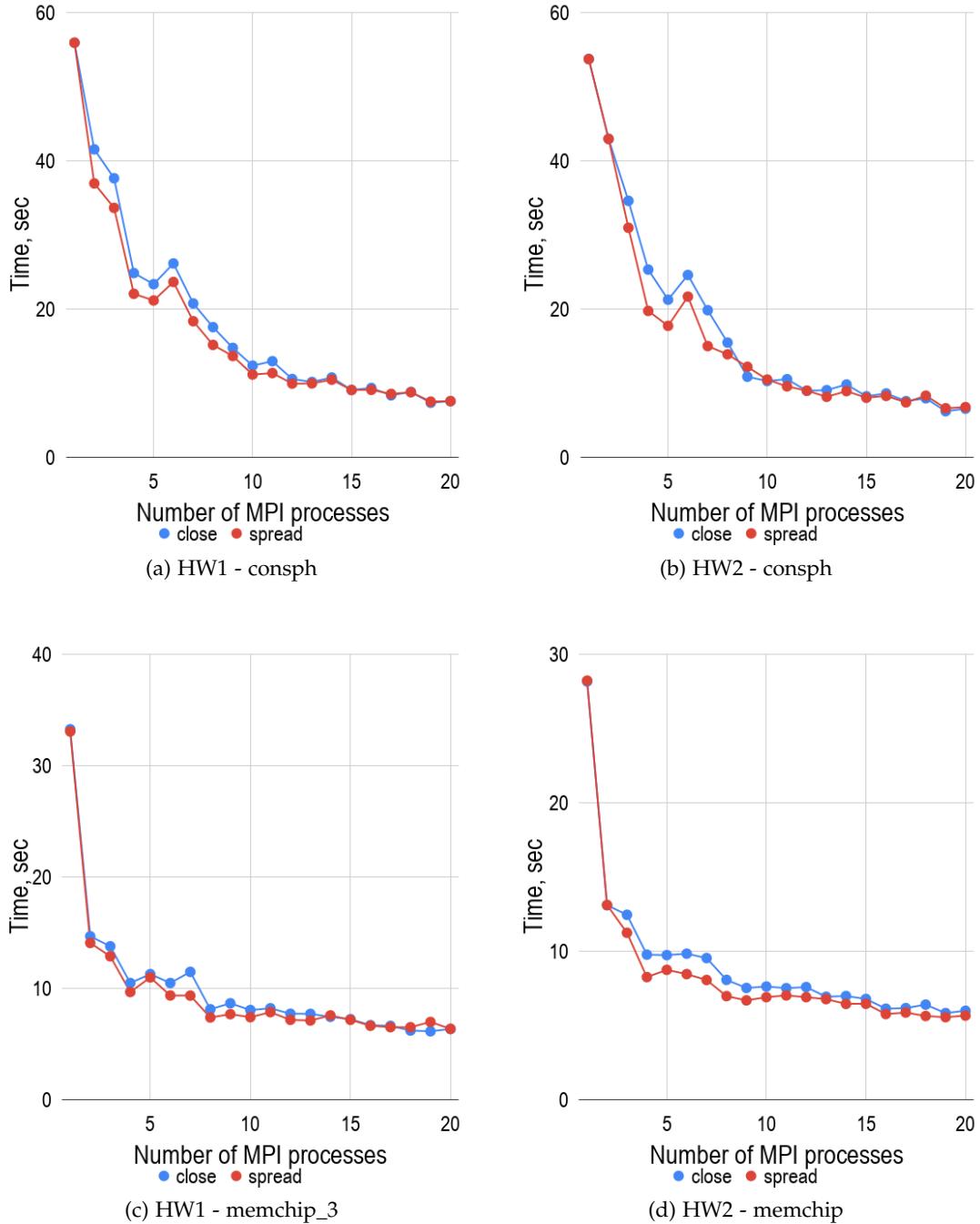


Figure C.2.: Comparison of *close* and *spread* pinning strategies

## **D. Choice of BLAS Library**

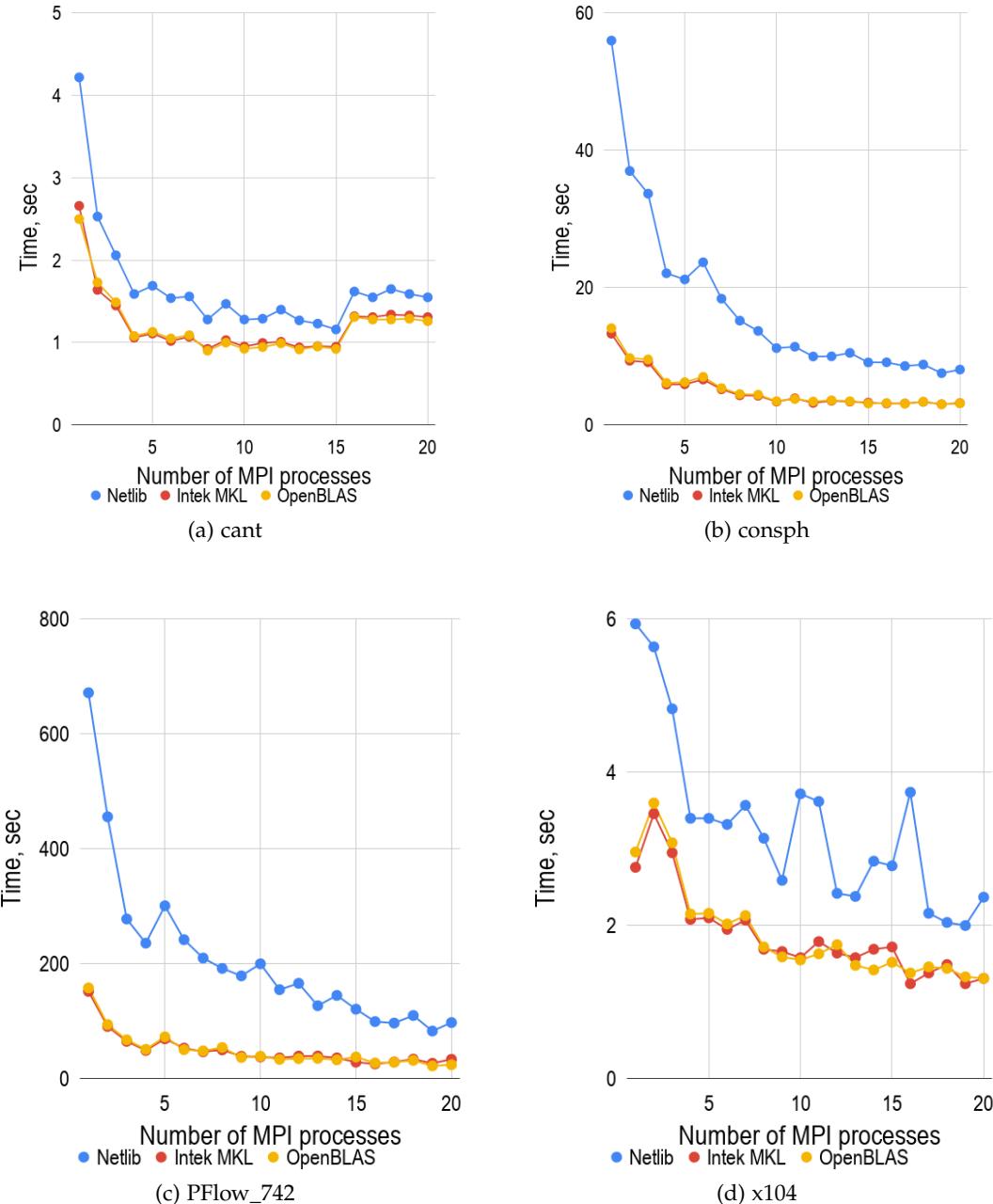


Figure D.1.: MUMPS: comparison of different BLAS libraries with using both GRS and SuiteSparse matrix sets on HW1 machine

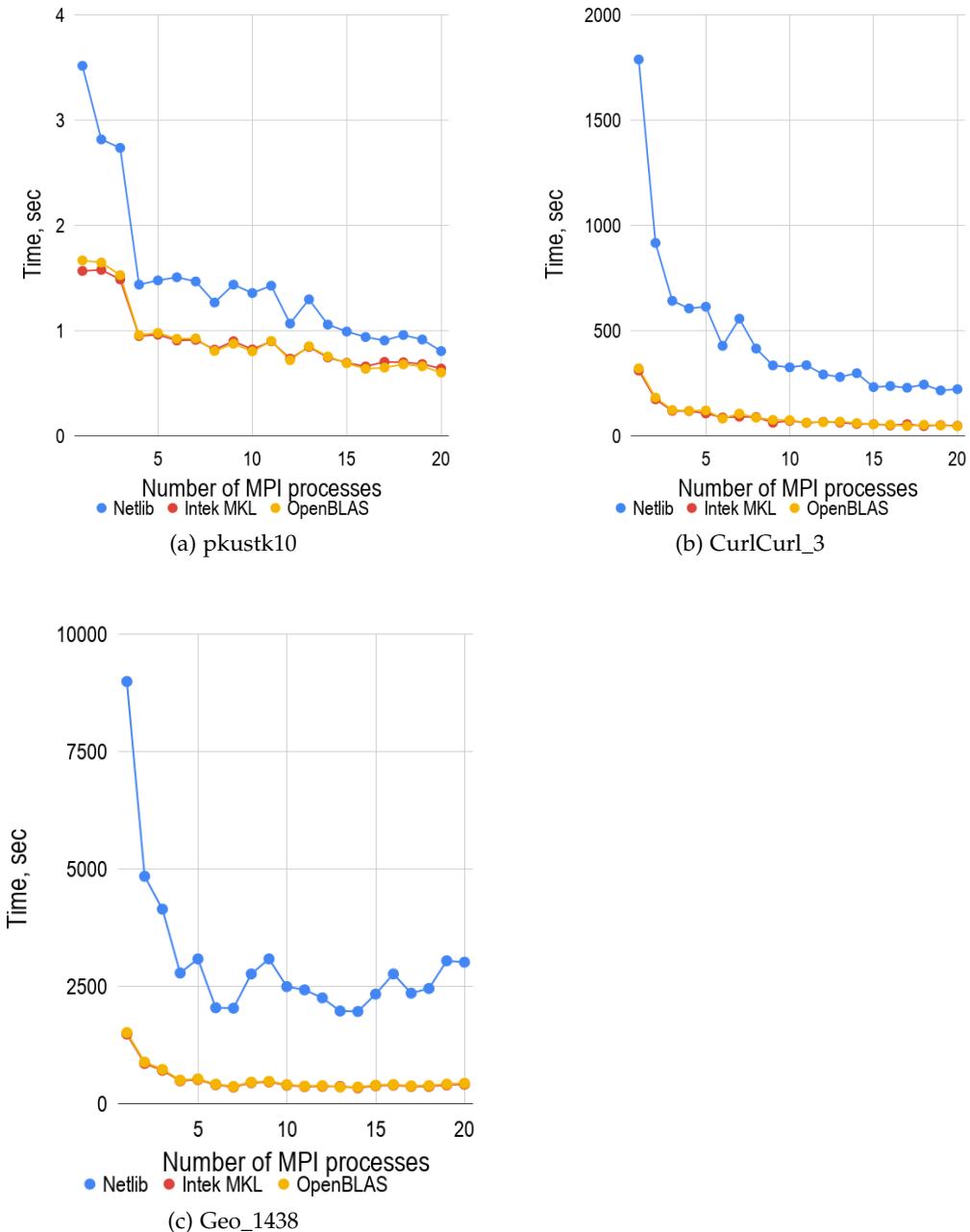


Figure D.2.: MUMPS: comparison of different BLAS libraries with using both GRS and SuiteSparse matrix sets on HW1 machine

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