MODULI OF HIGGS BUNDLES

1. INTRODUCTORY MOTIVATION

Definition 1.1. In mathematics, a reductive group is an algebraic group G over an algebraically closed field such that the unipotent radical of G is trivial (i.e., the group of unipotent elements of the radical of G).

Suppose given a compact Riemann surface C of genus $g \geq 2$ and a compact reductive (1.1) Lie group G, e.g. $G = U(1)$, $G = SU(2)$. Built from these data there is a moduli space

$$\mathcal{M} = \mathcal{M}^H(C, G)$$

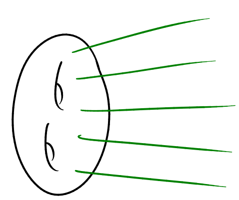
It is *almost* a manifold — has some singularities, but also some components without singularities, and at first we can focus on the parts without singularities. It can be seen in various ways:

- $\mathcal{M}$ is the (twisted) *character variety*, i.e. moduli space of (twisted) reductive representations[1] $\pi_1(C)$1.2 $\to G_{\mathbb{C}}$. e.g. for $g = 2$ and $G = SU(2)$, this means

$$\mathcal{M} = \{A_1, A_2, B_1, B_2 \in SL(2, \mathbb{C}) : A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} = \pm 1\} / \sim \qquad (1.1)$$

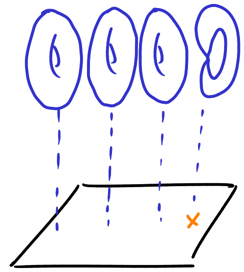
  Definition 1.2. The fundamental group is a mathematical group associated to any given pointed topological space that provides a way to determine when two paths, starting and ending at a fixed base point, can be continuously deformed into each other.

- $\mathcal{M}$ is the moduli space parameterizing (stable) *flat $G_{\mathbb{C}}$-connections* over C. (Certain sheaves on this moduli space are basic objects on "B side" of the geometric Langlands correspondence.)
- $\mathcal{M}$ is a partial compactification of $T^* \operatorname{Bun}(C, G)$, where $\operatorname{Bun}(C, G)$ is the moduli space of semistable G-bundles on C. (Lagrangian submanifolds are related to *D-modules* on $\operatorname{Bun}(C, G)$, basic objects on "A side" of the geometric Langlands correspondence.)



- $\mathcal{M}$ is a *complex integrable system* [**MR88i:58068**], i.e. a holomorphic symplectic space fibered over a complex base with Lagrangian fibers, generic fiber a compact complex torus.

[1] "Reductive" means the closure of the image is a reductive subgroup of $G_{\mathbb{C}}$.



- $\mathcal{M}$ is a noncompact *Calabi-Yau space*, i.e. a Kähler space admitting a Ricci-flat metric, in some sense a close cousin of the K3 surface; from this point of view it is a paradigmatic example of the Strominger-Yau-Zaslow philosophy [**Strominger:1996it**], which says that every Calabi-Yau space arises naturally as a special Lagrangian *torus fibration* over a complex base, and that its *mirror* can be obtained by a natural fiberwise duality operation; moreover in this case the mirror is a space of the same kind, namely $\mathcal{M}^\vee = \mathcal{M}^H(C, {}^L G)$ where ${}^L G$ is the *Langlands dual* group [**mlh, MR2957305**]. (The mirror symmetry exchanges the two sides of the geometric Langlands correspondence.)
- $\mathcal{M}$ is a *cluster variety*, built by gluing together very simple pieces $(\mathbb{C}^\times)^n$ in an essentially *combinatorial* way. (Almost: to make this precisely true, we have to include *punctures* on C; but even without the punctures, some cluster-like structure seems to persist.)
- $\mathcal{M}$ is the space of solutions of an interesting PDE, *Hitchin's equations* [**MR89a:32021**], containing as special cases various sorts of harmonic maps (including *uniformization* in the case $G = PSU(2)$).

How can one space $\mathcal{M}$ be so many different things at once?

A partial answer comes from another structure $\mathcal{M}$ carries, namely the *hyperkähler* structure. This says in short that $\mathcal{M}$ has a metric compatible with many different complex structures, fitting together in a specific way; thus $\mathcal{M}$ gives rise to many complex manifolds which look quite different from one another, but are nevertheless canonically diffeomorphic. Loosely speaking, one complex structure comes from the Riemann surface C, another comes from $G_{\mathbb{C}}$. An hyperkähler structure is rather rigid and gives a lot of constraints, e.g. it implies that the metric on $\mathcal{M}$ is Ricci-flat, and even lets us say some things about what the metric looks like (much more than we can say for "generic" Ricci-flat metrics); also allows us to study the *topology* of $\mathcal{M}$, e.g. its Betti numbers.

Our first major aim is to understand this structure — first we will study some simpler "baby" examples of hyperkähler geometry, then we will study $\mathcal{M}(C, G)$ for $G = U(1)$, finally we will come to $\mathcal{M}(C, G)$ for general G.

(A fuller answer should come from the way $\mathcal{M}$ fits into supersymmetric quantum field theory; but this is mostly beyond the scope of this course.)

2. Local complex and Kähler geometry: a quick review

This is only intended as a review and to fix notation. There are many references for this material: one good one is [**MR2093043**].

2.1. Complex manifolds.

In this section X is a smooth manifold.

Definition 2.1 (Almost complex structure). An *almost complex structure* on X is a smooth section I of $\mathrm{End}(TX)$[2] with $I^2 = -1$. An *almost complex manifold* is a pair (X, I) where I is an almost complex structure. If X has real dimension $2n$, an almost complex structure I equips TX with the structure of *complex* vector bundle[3] over X, of rank n, and we say the *complex dimension* $\dim_{\mathbb{C}} X$ is n.

Example 2.2 (Flat complex space). $\mathbb{C}^n$ has a canonical almost complex structure I, as follows. Each tangent space $T_p\mathbb{C}^n \simeq \mathbb{C}^n$ canonically; I is multiplication by i, thought of as an endomorphism of the underlying $2n$-dimensional real vector space. Writing $z_i = x_i + \mathrm{i}y_i$, and taking the coordinate basis $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, \partial_{y_1}, \partial_{y_2}, \ldots, \partial_{y_n}\}$ for $T_p\mathbb{C}^n$, I is represented by the matrix

$$I = \begin{pmatrix} \mathbf{0}_{n\times n} & -\mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}. \tag{2.1}$$

Definition 2.3 (Holomorphic maps). If (X, I_X) and (Y, I_Y) are almost complex manifolds, a *holomorphic map* $\phi : X \to Y$ is one obeying

$$I_Y \circ \mathrm{d}\phi = \mathrm{d}\phi \circ I_X. \tag{2.2}$$

Exercise 2.1. Show that, if both (X, I_X) and (Y, I_Y) are $\mathbb{C}$ with its canonical almost complex structure, Definition 2.3 becomes the standard definition of holomorphic function (Cauchy-Riemann equations).

Definition 2.4 (Complex structures). An almost complex structure I on X is *integrable*, or a *complex structure*, if there is a covering of X by open sets U_α with holomorphic diffeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$ (where on $\mathbb{C}^n$ we take the canonical almost complex structure.) A *complex manifold* is an almost complex manifold (X, I) with I integrable.

Note: The almost complex structure doenot necessarilly mean the locally $\mathbb{C}^n$ manifold. The complex structure (with integrable structure) gaurantees the locally $\mathbb{C}^n$ space.

Exercise 2.2. Show that Definition 2.4 is equivalent to the usual definition of a complex manifold as a space X with a covering by charts $\phi_\alpha : U_\alpha \to \mathbb{C}^n$, where the transition maps are holomorphic (obey Cauchy-Riemann equations).

Example 2.5 (Complex structure on $\mathbb{C}^n$). A tautological example is $X = \mathbb{C}^n$ itself with its canonical almost complex structure: just take a single open set $U = \mathbb{C}^n$, and $\phi : U \to \mathbb{C}^n$ to be the identity map. So the canonical almost complex structure on $\mathbb{C}^n$ is, tautologically, a complex structure.

There are various equivalent ways of formulating the integrability condition. One which will be useful for us is:

Proposition 2.6 (Infinitesimal characterization of integrability). I is integrable if and only if the *Nijenhuis tensor* $N_I \in \Omega^0(\wedge^2 T^*X \otimes TX)$,

$$N_I(v, w) = [v, w] + I[v, Iw] + I[Iv, w] - [Iv, Iw], \tag{2.3}$$

[2] TX is the tangent bundle.

[3] I think that the fibre of the bundle is the complex space now of dimension n instead of $2n$ of the real space.

vanishes:

$$N_I = 0. \tag{2.4}$$

2.2. **Type decompositions.** Suppose (X, I) is a complex manifold. We have a decomposition of $T_{\mathbb{C}}X = TX \otimes_{\mathbb{R}} \mathbb{C}$,

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X \tag{2.5}$$

where $T^{1,0}X$ and $T^{0,1}X$ are respectively the $+\mathrm{i}$ and $-\mathrm{i}$ eigenspaces of I. Both TX and $T^{1,0}X$ are complex vector bundles of rank n; it is sometimes convenient to identify them, by projection on the $(1,0)$ part.

Exercise 2.3. Show that this is indeed an isomorphism of complex vector bundles. (This reduces essentially to a question of linear algebra, concerning a vector space V with complex structure I, and its complexification $V_{\mathbb{C}}$.)

There is also a dual decomposition

$$T^*_{\mathbb{C}}X = (T^*)^{1,0}X \oplus (T^*)^{0,1}X, \tag{2.6}$$

where $(T^*)^{1,0}X$ is the annihilator of $T^{0,1}X$, and $(T^*)^{0,1}X$ is the annihilator of $T^{1,0}X$. This decomposition induces

$$\wedge^* T^*_{\mathbb{C}}X = \bigoplus_{p,q=0}^{n} \wedge^{p,q} T^*X, \qquad \Omega^*_{\mathbb{C}}X = \bigoplus_{p,q=0}^{n} \Omega^{p,q}(X) \tag{2.7}$$

and a corresponding decomposition

$$\mathrm{d} = \partial + \bar{\partial}, \quad \partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \quad \bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X). \tag{2.8}$$

Complex conjugation is an $\mathbb{R}$-linear map $\Omega^{p,q}(X) \to \Omega^{q,p}(X)$; thus it maps $\Omega^{p,p}(X)$ to itself; we let $\Omega^{p,p}_{\mathbb{R}}(X)$ denote the fixed subspace.

2.3. **Holomorphic vector bundles.** In this section (X, I) is always a complex manifold.

Definition 2.7 (Holomorphic vector bundle). A *holomorphic vector bundle* over X is a complex vector bundle E over X, equipped with an operator

$$\bar{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E) \tag{2.9}$$

obeying

$$\bar{\partial}_E(\alpha\psi) = (\bar{\partial}\alpha)\psi + (-1)^{|\alpha|}\alpha \wedge \bar{\partial}_E\psi \qquad \alpha \in \Omega^*(X), \quad \psi \in \Omega^0(E) \tag{2.10}$$

and the integrability condition

$$\bar{\partial}_E^2 = 0. \tag{2.11}$$

The structure of holomorphic vector bundle is much more rigid than that of a merely complex vector bundle. We emphasize that this structure makes sense only when X is a complex manifold, while complex vector bundles make sense over any X.

Exercise 2.4. Show that a structure of holomorphic vector bundle on E is equivalent to a maximal atlas of preferred trivializations of E, such that the transition maps $U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$ are holomorphic.

Example 2.8 (Tangent bundle as a holomorphic bundle). The tangent bundle TX carries a canonical structure of holomorphic vector bundle. (Indeed, the coordinate charts give rise to preferred trivializations corresponding to the bases $\{\partial_{z_1}, \ldots, \partial_{z_n}\}$ for $TX \simeq T^{1,0}X$, and the transition maps are given by the Jacobians, which are holomorphic.)

Definition 2.9 (Connection compatible with holomorphic structure). If E is a holomorphic vector bundle over X, a connection D in E is *compatible with the holomorphic structure* if, for all $\psi \in \Omega^0(E)$, the $(0,1)$ part of $D\psi$ is $\bar{\partial}_E \psi$.

Definition 2.10 (Chern connection). If E is a holomorphic vector bundle over X with a Hermitian metric h, the *Chern connection* in E is the unique connection which is h-unitary and compatible with the holomorphic structure.

2.4. **Hermitian and Kähler metrics.** In this section (X, I) is always a complex manifold.

Definition 2.11 (Hermitian metric on complex manifold). A Hermitian metric on X is a Riemannian metric g obeying

$$g(v, w) = g(Iv, Iw).$$

Equivalently, with respect to the decomposition

$$\mathrm{Sym}^2(T_{\mathbb{C}}X) = \mathrm{Sym}^{2,0}\, TX \oplus \mathrm{Sym}^{1,1}\, TX \oplus \mathrm{Sym}^{0,2}\, TX, \tag{2.12}$$

we have $g \in \mathrm{Sym}^{1,1}\, TX$, i.e. g is of "type $(1,1)$."

Definition 2.12 (Fundamental form). If g is a Hermitian metric on X, the *fundamental form* $\omega \in \Omega^{1,1}_{\mathbb{R}}(X)$ is

$$\omega(v, w) = g(Iv, w). \tag{2.13}$$

Exercise 2.5. If g is a Hermitian metric on X, verify that

$$h = g - \mathrm{i}\omega \tag{2.14}$$

defines a Hermitian metric on the complex vector bundle TX.

Definition 2.13 (Kähler metric). If g is a Hermitian metric on X, g is Kähler if the corresponding ω obeys

$$\mathrm{d}\omega = 0. \tag{2.15}$$

In this case we say (X, g, I) is a Kähler manifold, and ω is the Kähler form.

The Kähler property has various useful consequences, some local and some global. Here we recall some of the local ones: the global ones will come later. Let ∇ denote the Levi-Civita connection on TX induced by the metric g.

Proposition 2.14 (Kähler means covariant constancy of I). If g is a Hermitian metric on X, g is Kähler if and only if $\nabla I = 0$.

Corollary 2.15 (Kähler means covariant constancy of other things). If g is a Hermitian metric on X, with fundamental form ω, then the following are equivalent:

- g is Kähler,
- $\nabla I = 0$,
- $\nabla \omega = 0$,

- ∇ agrees with the Chern connection on TX, when we view TX as a complex vector bundle with the induced Hermitian metric h of Exercise 2.5.

Finally we quickly recall the notion of special holonomy. Recall that for any Riemannian metric g the parallel transport of Levi-Civita preserves g, so that for any $p \in X$ the holonomy group $Hol_g(p) \subset GL(T_pX)$ is contained in the subgroup $O(h_p) \simeq O(2n)$. For a Kähler metric, since the Chern connection agrees with the Levi-Civita connection, the parallel transport of Levi-Civita preserves the Hermitian metric h on the complex vector bundle TX. Thus, for any $p \in X$, the holonomy group $Hol_g(p) \subset GL(T_pX)$ is contained in the smaller group $U(h_p) \simeq U(n)$. This proves one-half of the following:

Proposition 2.16 (Special holonomy of Kähler manifolds). Given any Riemannian metric g on a manifold M of dimension $2n$, g is a Kähler metric (for some complex structure I on M) if and only if the holonomy group at a point is contained in a subgroup isomorphic to $U(n)$.

3. Hyperkähler manifolds

Useful (and inspiring) references are [**MR88f:53087**, **Hitchin-hk**, **MR1798605**, **boalch-notes**].

3.1. Basic definitions and examples.

Definition 3.1 (Hyperkähler manifold). A *hyperkähler manifold* is a tuple (X, g, I_1, I_2, I_3), where (X, g) is a Riemannian manifold equipped with three complex structures I_i obeying the algebra of the quaternions ($I_1 I_2 = -I_2 I_1 = I_3$), such that g is Hermitian and Kähler with respect to any of the I_i.

It is crucial that we require the *single* metric g to be Kähler for *all* of the I_i: this is a very strong condition! We denote the three corresponding Kähler forms ω_i. Sometimes it is convenient to use instead the notation $(I_1, I_2, I_3) = (I, J, K)$ and $(\omega_1, \omega_2, \omega_3) = (\omega_I, \omega_J, \omega_K)$.

Definition 3.2 (Holomorphic symplectic form). If (X, I) is a complex manifold, $\Omega \in \Omega^{2,0}(X)$ is a *holomorphic symplectic form* if $\mathrm{d}\Omega = 0$ and Ω is nondegenerate in the holomorphic sense, i.e. it induces an isomorphism $T^{1,0}X \to (T^{1,0}X)^*$.

(Note that this definitely does *not* mean that Ω is nondegenerate on the whole $T_{\mathbb{C}}X$. Indeed, since Ω is of type $(2,0)$ its contraction with any $v \in T^{0,1}X$ vanishes.)

Proposition 3.3 (Hyperkähler manifolds are holomorphic symplectic). If X is hyperkähler then $\Omega_1 = \omega_2 + \mathrm{i}\omega_3$ is a *holomorphic symplectic form* with respect to structure I_1.

Proof.

$$\Omega_1(v, w) = \omega_2(v, w) + \mathrm{i}\omega_3(v, w) \tag{3.1}$$
$$= g(I_2 v, w) + \mathrm{i}g(I_3 v, w) \tag{3.2}$$

Thus

$$\Omega_1(I_1 v, w) = g(I_2 I_1 v, w) + \mathrm{i}g(I_3 I_1 v, w) \tag{3.3}$$
$$= -g(I_3 v, w) + \mathrm{i}g(I_2 v, w) \tag{3.4}$$
$$= \mathrm{i}\Omega_1(v, w) \tag{3.5}$$

and similarly

$$\Omega_1(v, I_1 w) = \mathrm{i}\Omega_1(v, w). \tag{3.6}$$

It follows that Ω_1 is of type $(2,0)$ for I_1, $\Omega_1 \in \Omega^{2,0}_{I_1}(X)$. The nondegeneracy follows from the nondegeneracy for the ω_i: namely, for any $v \in T^{1,0}_{I_1}X$,

$$\Omega_1(v, \cdot) = 0 \implies \Omega_1(v + \bar{v}, \cdot) = 0 \implies \omega_2(v + \bar{v}, \cdot) = 0 \implies v + \bar{v} = 0 \implies v = 0. \tag{3.7}$$

Corollary 3.4 (Hyperkähler manifolds have dimension $4n$). If X is hyperkähler then $\dim_{\mathbb{R}} X$ is a multiple of 4.

Proof. There is a standard bit of "symplectic linear algebra" saying that a vector space with a nondegenerate antisymmetric pairing is always even-dimensional (to prove it, one inductively constructs a standard "Darboux" basis). Thus the existence of the holomorphic symplectic form Ω_1 implies that $T^{1,0}_{I_1}X$ has even complex dimension.

Example 3.5 (Flat quaternionic space). The quaternions $\mathbb{H}$ can be identified with $\mathbb{R}^4$ via the map

$$a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \mapsto (a, b, c, d). \tag{3.8}$$

This makes $\mathbb{H}$ into a manifold of real dimension 4. The standard metric on $\mathbb{R}^4$ induces a metric g on $\mathbb{H}$. If we identify $T_p\mathbb{H} \simeq \mathbb{H}$ in the obvious way, the operations of left-multiplication by i, j and k give complex structures I, J, K on $\mathbb{H}$, obeying the quaternion algebra. The metric g is Kähler for all three of these complex structures. Thus $\mathbb{H}$ is hyperkähler, in a canonical way.

Exercise 3.1. Write explicit formulas for the symplectic forms ω_i on $\mathbb{H}$, and for complex coordinates with respect to the complex structures I_i. Thus verify directly that ω_i is of type $(1,1)$ for structure I_i, that $\Omega_1 = \omega_2 + \mathrm{i}\omega_3$ is of type $(2,0)$ with respect to complex structure I_1, and that when considered as a complex manifold in structure I_i, $\mathbb{H}$ is biholomorphic to $\mathbb{C}^2$.

The group $O(4)$ acts on $\mathbb{H}$ by isometries, but these do *not* generally preserve the hyperkähler structure. However, we do have the following. Recall that the unit sphere in $\mathbb{H}$ is isomorphic as a group to $SU(2)$. Thus we have an action of $SU(2) \times SU(2)$ on $\mathbb{H}$ by

$$(q, q') \cdot x = qxq'^{-1}. \tag{3.9}$$

This gives a map $SU(2) \times SU(2) \to SO(4)$, which has kernel $\{(1,1), (-1,-1)\} \simeq \mathbb{Z}_2$, thus an isomorphism $SO(4) \simeq (SU(2) \times SU(2))/\mathbb{Z}_2$. Said otherwise, $O(4)$ has two canonical $SU(2)$ subgroups, which we could call $SU(2)_L$ and $SU(2)_R$ (for "left" and "right.")

Exercise 3.2. Show that $SU(2)_R$ acts on $\mathbb{H}$ by *triholomorphic* isometries, i.e. isometries which are holomorphic for all of I, J, and K.

Example 3.6 (Incomplete Gibbons-Hawking spaces). Fix some open set $U \subset \mathbb{R}^3$ and let $V : U \to \mathbb{R}_{>0}$ be a positive harmonic function. Then $\Delta V = \mathrm{d} \star \mathrm{d}V = 0$, so if we write

$$F = \star \mathrm{d}V \tag{3.10}$$

then we have $\mathrm{d}F = 0$. Suppose F is moreover integrally quantized, i.e. $[F/2\pi] \in H^2(U, \mathbb{Z})$. Then there exists a principal $U(1)$ bundle X over U, carrying a connection Θ whose curvature is F. Concretely we represent Θ as a 1-form on X, with integral 1 over each fiber, locally of the form $\Theta = A + \mathrm{d}\chi$ with $A \in \Omega^1(U)$. Then the metric on X given by

$$g = V(\vec{x})\mathrm{d}\vec{x}^2 + V(\vec{x})^{-1}\Theta^2 \tag{3.11}$$

is hyperkähler. The three symplectic forms are [check]

$$\omega_i = \mathrm{d}x_i \wedge \Theta + \frac{1}{2}\epsilon^{ijk}V(\vec{x})\mathrm{d}x_j \wedge \mathrm{d}x_k. \tag{3.12}$$

The principal $U(1)$ action on X is by isometries preserving the hyperkähler structure (this is clear since nothing in g or ω_i depends on the fiber coordinates).

Example 3.7 (Gibbons-Hawking spaces). Extending Example 3.6 we can allow V to have quantized (distributional) sources on U:

$$\Delta V = 2\pi \sum_{i=1}^{n} \delta(\vec{x} - \vec{x}_i) \tag{3.13}$$

In this case, consider a small S^2 around $\vec{x}_i$:

$$\int_{S^2} F = \int_{S^2} \star \mathrm{d}V = \int_{B^3} \mathrm{d} \star \mathrm{d}V = \int_{B^3} 2\pi\delta(\vec{x} - \vec{x}_i) = 2\pi \tag{3.14}$$

so the $U(1)$ bundle X restricted to this S^2 has degree 1. X thus doesn't extend as a $U(1)$ bundle over the point $\vec{x}_i$. Nevertheless it turns out that it *does* extend as a hyperkähler manifold. Indeed, near $\vec{x}_i$ we have [fix wrong factor of 2π somewhere!] $V = 1/4\pi\|\vec{x} - \vec{x}_i\| +$ regular, so the circle fibers of X are shrinking to zero length, and it turns out to be possible to add a single point over $\vec{x}_i$, in such a way that the total space is a hyperkähler manifold with non-free $U(1)$ action, and the quotient is U.

Example 3.8 (Taub-NUT space). This is the case of Example 3.7 with one singularity. Take $U = \mathbb{R}^3$ and

$$V(\vec{x}) = 1 + \frac{1}{4\pi\|\vec{x}\|}. \tag{3.15}$$

[...]

Exercise 3.3. Show that, in any of its complex structures I_i, Taub-NUT space is biholomorphic to $\mathbb{C}^2$.

Example 3.9 (Eguchi-Hanson space). This is the case of Example 3.7 with two singularities. Fix two points, say $\vec{x}_1 = (1, 0, 0)$ and $\vec{x}_2 = (-1, 0, 0)$. Take $U = \mathbb{R}^3$ and

$$V(\vec{x}) = 1 + \frac{1}{4\pi\|\vec{x} - \vec{x}_1\|} + \frac{1}{4\pi\|\vec{x} - \vec{x}_2\|}. \tag{3.16}$$

[...]

Example 3.10 (ALE spaces). It follows from Exercise 3.2 that, if we choose a subgroup $\Gamma \subset SU(2)$, the quotient $\mathbb{H}/\Gamma$ is a hyperkähler orbifold: in particular, it carries a natural hyperkähler structure where it is a manifold. For example, if Γ is a discrete subgroup, it acts freely away from the origin, so

$$X_\Gamma^\circ = (\mathbb{H} \setminus \{0\})/\Gamma \tag{3.17}$$

is a hyperkähler manifold. However, this hyperkähler manifold is *incomplete*, since the origin is at finite distance. One can consider the so-called *minimal resolution* X_Γ of the singularity at the origin; then X_Γ is an honest smooth manifold, carrying a natural family of *complete* hyperkähler metrics [**MR90d:53055**]. These metrics asymptotically approach the metric we started with on X_Γ° (induced from the flat metric on $\mathbb{H}$); thus the X_Γ are called "ALE spaces", for "asymptotically locally Euclidean."

3.2. Twistor space.

Exercise 3.4. Suppose (X, g, I_1, I_2, I_3) is a hyperkähler manifold. Fix any $\vec{s} = (s_1, s_2, s_3) \in S^2 \subset \mathbb{R}^3$, and set

$$I_{\vec{s}} = \sum_{i=1}^{3} s_i I_i, \qquad \omega_{\vec{s}} = \sum_{i=1}^{3} s_i \omega_i. \tag{3.18}$$

Show that $(X, I_{\vec{s}}, g)$ is a Kähler manifold, with Kähler form $\omega_{\vec{s}}$.

In other words, a hyperkähler metric is Kähler for a whole S^2 of complex structures, not only three of them; moreover specifying I_1, I_2, I_3 is equivalent to specifying the whole collection of $I_{\vec{s}}$.

Exercise 3.5. Given a Riemannian manifold (X, g) and a hyperkähler structure thereon, specified by complex structures $I_{\vec{s}}$, show that we get another hyperkähler structure by choosing an element $T \in SO(3)$ and defining

$$I'_{\vec{s}} = I_{T\vec{s}}. \tag{3.19}$$

Thus $SO(3)$ naturally acts on the set of hyperkähler manifolds.

[reduced holonomy]
[Ricci-flatness]
[twistor family]