Chapter: Special Relativity

Objectives: proper frame of reference, space-time unification, relativistic/proper vectors

1 Relativistic kinematics

In the begning of the course, we studied the kinematical equations. Let us consider the equation

$$x(t) = ut + \frac{1}{2}at^2 \tag{1}$$

for a particle moving with constant acceleration a with initial speed u. For simplicity, we set $a = 0$. Thus the equation converts into

$$x(t) = ut \tag{2}$$

We can interpret equation 2 as "the displacement or *space* of the particle as function of *time*". According to this (Galilean) interpretation, time runs in the background (measured by the clocks) and spatial configurations *depend* on the time.

Einstein's relativity aligns the *space* and *time* on the same footing. According to this interpretation, "space-time" is a single physical entity on which the physical events take place. Furthermore, each physical quantity that we encountered earlier (distance, speed and momentum), has both space and time components!

Consider the spacetime graph of a shuttle 'O' seen by Alice standing at the origin as shown. We consider

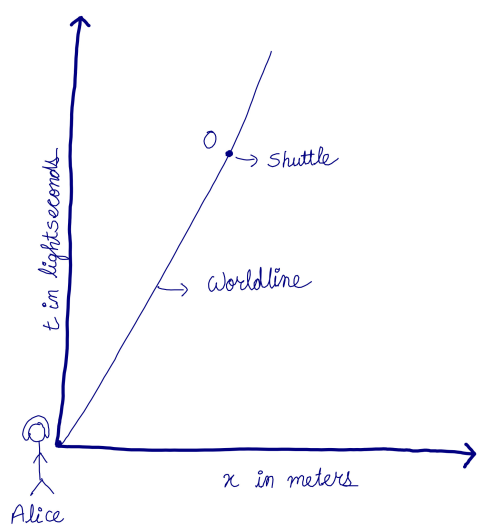

Figure 1: Spacetime graph with respect to Alice

the functional dependance of spatial and temporal coordinates as observed by Alice as follows

$$\begin{aligned} t &= t(\tau) \\ x &= x(\tau) \end{aligned} \tag{3}$$

where τ is called the proper time. It is the time measured by the oberver Bob standing inside the shuttle 'O' (in the exercises, you will draw the spacetime graph of the shuttle with respect to Bob). According to Einstein's relativity, Bob's clock (τ) ticks differently when compared with Alice's clock (t) *in the spacetime graph drawn above.*

Now we will write the space and time components of the physical quantities we studied in the class. We will start with the distance.

1.1 Proper displacement

Earlier we just required the displacement to specify the state of the point particle. Now we need *both* displacement and time to specify the state of particle in terms of displacement vector

$$\vec{d_p} = t\hat{a}_t + x\hat{a}_x \tag{4}$$

Here $\hat{a}_t$ and $\hat{a}_x$ are the unit vectors in time and space directions respectively.
Now recall when (with only spatial dimensions) $\vec{A}$ and $\vec{B}$ were orthogonal to each other and were oriented along x and y axis as shown

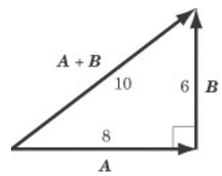

Figure 2: Perpendicular vectors

Then the resultant vector $\vec{C} = A\hat{a}_x + B\hat{a}_y$ had the magnitude

$$|\vec{C}| = \sqrt{A^2 + B^2} \tag{5}$$

where A and B were the magnitudes of $\vec{A}$ and $\vec{B}$ respectively.
For the proper (relativistic) vectors there is a minor difference. The magnitude is given by

$$|\vec{d_p}| = \sqrt{t^2 - x^2} \tag{6}$$

Note the '$-$' sign instead of '$+$' sign.

1.2 Proper speed

In one spatial dimention, the speed is defined as $v = \Delta x/\Delta t$. Now we define the proper *space* speed as

$$v_p^1 = \frac{\Delta x}{\Delta \tau} = \frac{\text{displacement observed by Alice}}{\text{time elapsed in Bob's clock}} \tag{7}$$

In exercises below you will show that

$$v_p^1 = \gamma \times v \tag{8}$$

where $\gamma = \frac{\Delta t}{\Delta \tau}$. Now we define proper *time* speed as

$$v_p^0 = \frac{\Delta t}{\Delta \tau} = \gamma. \tag{9}$$

Thus in our usual vector notation, we have proper speed vector given by

$$\vec{v}_p = v_p^0\hat{a}_t + v_p^1\hat{a}_x \tag{10}$$

In other words

$$\vec{v}_p = \frac{\Delta\vec{d_p}}{\Delta\tau}. \tag{11}$$

where $\Delta\vec{d_p}$ is given by equation 4. The magnitude of the resultant is given by

$$||\vec{v}_p|| = \sqrt{(v_p^0)^2 - (v_p^1)^2} \tag{12}$$

In the exercises you will show that for $||\vec{v}_p|| = 1$,

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \tag{13}$$

For γ to be a real number $v < 1$ (why?).

1.3 Proper momentum

Recall that momentum $\vec{p}$ was defined as $m \times \vec{v}$. Now we have proper momentum or energy-momentum as

$$\vec{p}_p = m\vec{v}_p \tag{14}$$

where $\vec{v}_p$ is given by equation 10. In the exercises you will compute the p_p^0 and p_p^1. According to Einstein's relativity, p_p^0 is energy and p_p^1 is the momentum associated with the shuttle as observed by Alice in her spacetime.

2 Sample questions

1. Draw the spacetime graph of the shuttle O with respect to Bob.

2. Show that the equation 8 can be obtained form the definition 7.

3. From the statement $||\vec{v}_p|| = 1$ prove equation 13.

4. Time dilation: Assume that the shuttle O is moving with speed 0.5 m/s. Find the time elapsed in Alice's clock, Δt, when Bob's clock ticks $\Delta\tau = 1$ s.
   Hint: Use the definition of proper time speed and the relation 13 you proved in the previous question.

   In the spacetime graph drawn by Alice, we can say that Bob ages __________________ (faster/slower) than Alice. This relativistic phenomenon is called time dialation!

5. From the definition 14 and equation 10, find the expressions for p_p^0 and p_p^1 in terms of γ, m and v.

6. Dimensional analysis: In this chapter we have been neglecting the dimensions of physical quantities. Let us rectify that in this exercise. In the section 1.3, we said that p_p^0 proportional to the Energy. Let

$$\frac{E}{\text{constant}} = \text{constant} \times p_p^0 \tag{15}$$

Find the SI units for the constant.

Again, we consider the spacetime of Alice in which the shuttle is moving with the speed .5 m/s. Write down the $\vec{p}_p$ (14) of the shuttle.

Finally write down the $\vec{p}_p$ of the shuttle with respect to Bob (who is moving with the shuttle).

Let the constant $= c$ in equation 15. Find the energy E of the shuttle with respect to Bob.

Voila! You just wrote Einstein's famous relativistic equation!