Foundations of Fractal Dimension

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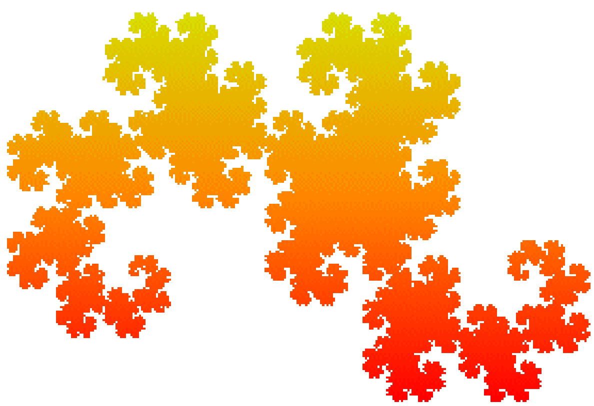
Bachelor Mathematics
Project on Fractal Geometry



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November 2021

Abstract

In this short paper we discuss some introductory topics related to the computation of fractal dimension. We aim for the public to be third year mathematics bachelor students. Thus, we give a brief introduction Hausdorff measure and Outer measure. This allows us to write about our topics of interest. These are Hausdorff dimension, Self-similar dimension, their computation and properties and most importantly Moran's theorem which relates both.

We finish by giving an interesting example of how to compute the Hausdorff dimension of a more complex set called the Heighway dragon boundary using the self-similar dimension and Moran's open set condition. This final example will hopefully give a visualisation of the concepts previously introduced and a further understanding of fractal geometry.

Contents

1 Introduction

In the past mathematicians disregarded the study of irregular sets, they focused on functions that could be observed through classical calculus. This has changed as we now know that more complicated non smooth objects such as fractals and self similar sets have a relevant place in geometry. In fact, these non-smooth objects have a better way to describe nature than any forms. Fractal geometry is the field where such irregular sets are studied.

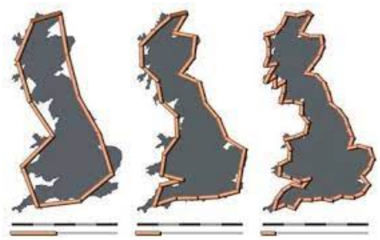


Figure 1: Fractal seen in coast of Britain

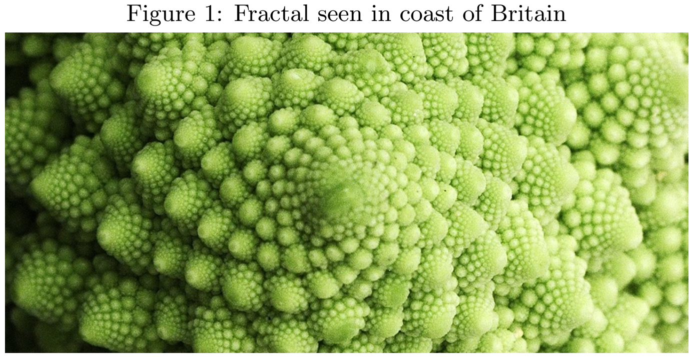


Figure 2: Fractal seen in a cauliflower

One can say and study many things about fractal geometry. For instance, the notion of dimension is completely redefined as some fractals do not have an integer dimension. At first, this might seem counter intuitive and unrealistic. Nevertheless, it helps us understand more the behaviour of these sets. For example, the Sierpiński gasket, as we will see later, has a fractal dimension of $\frac{\log 3}{\log 2} \approx 1.585$. This implies that it is not exactly a line but also not exactly a plane, thus it stays somewhat 'in between'.

As you can imagine, the computation of these dimensions can be something very tricky as we can no longer use classical geometry. That is why, we write this paper in the hopes that one might understand better how one can think of this new way to see dimension of irregular sets even the very complicated ones. We will use notions of measure, and different dimensions to give an intuition on how to compute the fractal dimension of certain sets such as the Highway dragon, the coast of England, Barnsely leaf outline....etc.

This paper is also concerned with the definition of Hausdorff measure and its dimension, that is originally a fractal but bigger than its topological dimension.

For instance, it's known that the Hausdorff dimension of a single point is zero, of a line segment is 1, of a square is 2, and of a cube is 3 [5]. But what we do not know is the Hausdorff dimension of difficult fractals and that is what we are going to explain in details.

In addition to all that, Morans theorem which will describe the relation between the Hausdorff dimension and self similarity using some extra theorems and properties.

2 Outer measure

In this chapter we start by defining the outer measure. We then look at a general method to construct the outer measures, the 'method I'. This method will be used in the rest of the article.

Definition 2.1. An outer measure on a set X is a map $m^* : \mathcal{P}(X) \to [0, \infty]$ satisfying the following properties:

(i) $m^*(\emptyset) = 0$;

(ii) If $A \subset B$ then $m^*(A) \leq m^*(B)$;

(iii) If $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ then $m^*\left(\bigcup_{n\in\mathbb{N}}\right) \leq \sum_{n\in\mathbb{N}} m^*(A_n)$.

We now look at a general method to construct outer measures, we will call this method I. This is theorem 4.7.1 from Sternberg [4].

Let X be a set and $\mathcal{C}$ a family of subsets of X that covers X. Let $l : \mathcal{C} \to [0, \infty]$ be any function. The method I given by the following theorem.

Theorem 2.2. *There exists a unique outer measure m^* on X such that*

(i) $m^*(A) \leq l(A)$ for all $A \in \mathcal{C}$

(ii) If n^* is any outer measure satisfying the preceding condition then $n^*(A) \leq m^*(A)$ for all subsets A of X.

Proof. Suppose we have two outer measures m^* and n^* that satisfy (i) and (ii). Then we see that $n^*(A) \leq m^*(A)$ and $m^*(A) \leq n^*(A)$ for all subsets A of X. So the measures are equal, hence we find the uniqueness.

For a subset A of X we define the outer measure by

$$m^*(A) = \inf \sum_{D\in\mathcal{D}} l(D),$$

where the infinum is taken over all countable covers $\mathcal{D}$ of A.

We check now if m^* is an outer measure. First we see that $m^*(\emptyset) = 0$, since an empty set is covered by an empty cover, and the sum over this is zero. Secondly we take $A \subseteq B$. A cover of B is also a cover of A, so we have

$m^*(A) \leq m^*(B)$. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. When $m^*(A_n) = \infty$ for some $n \in \mathbb{N}$ we have

$$m^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

So suppose we have $m^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ and let $\mathcal{D}_n$ a countable cover of A_n where

$$\sum_{D\in\mathcal{D}_n} l(D) \leq m^*(A_n) + \frac{\epsilon}{2^n}.$$

We will use this and $\mathcal{D} = \cup_{n\in\mathbb{N}}\mathcal{D}_n$ and see that

$$m^*\left(\bigcup_{n\in\mathbb{N}}\right) \leq \sum_{D\in\mathcal{D}} l(D) \leq \sum_{n\in\mathbb{N}} \sum_{D\in\mathcal{D}_n} l(D) \leq \sum_{n\in\mathbb{N}} m^*(A_n) + \sum_{n\in\mathbb{N}} \frac{\epsilon}{2^n} = \sum_{n\in\mathbb{N}} m^*(A_n) + \epsilon.$$

We conclude that m^* is an outer measure.

To check (i) we see that $A \in \mathcal{C}$ can by covered by $\{A\}$. So $m^*(A) \leq \sum_{D\in\{A\}} l(D) = l(A)$.

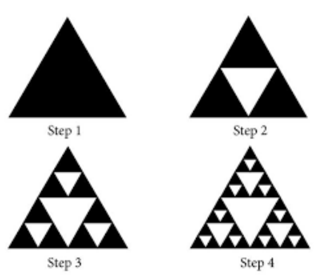
To check (ii) let n^* an outer measure satisfying $n^*(D) \leq l(D)$ for all $D \in \mathcal{C}$. For a countable cover $\mathcal{D}$ of a set A we have that

$$\sum_{D\in\mathcal{D}} l(D) \geq \sum_{D\in\mathcal{D}} n^*(D) \geq n^*\left(\bigcup_{D\in\mathcal{D}} D\right) \geq n^*(A).$$

We used the properties of outer measures. Now we conclude by minimizing over all the covers $\mathcal{D}$ we get that $m^*(A) \geq n^*(A)$. □

3 Hausdorff Measure

When we try to measure sets, until now we work in a very limited manner. We can compute the area of a circle or a sphere, but what about more complex structures such as the Sierpiński gasket.



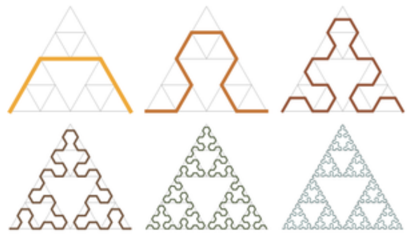


Figure 3: Sierpiński gasket.

We have that the Sierpiński gasket, has zero area but infinite perimeter. This lack of precision in measurement is frustrating. We need a way to assign a finite number to the size of a Sierpiński gasket and that is why we introduce a new measure that will enable us to compute this number: the Hausdorff measure.

Definition 3.1. Let (X, d) be a metric space and $A \subset X$. We define the $s-$dimensional outer Hausdorff measure of A where $s \in \mathbb{R}_+$ as the following function:

$$\mathcal{H}^s_\delta = \inf \sum_{i=1}^{\infty} (diam E_i)^s$$

where the inf is over all countable $\delta-$covers of A, namely $A \subset \cup_{i=1}^{\infty} E_i$ with $diam E_i \leq \delta$. (Note $0^s = 0$).(Note s can be any real number, not just an integer.)

It could happen that there is no countable covering with a given δ for A so if you cannot cover it, we agree that it is infinite.

Note that smaller delta (> 0) gives a bigger $\mathcal{H}^s_\delta(A)$. So, $\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$. Restricting this to measurable sets of this measure we get the **s-dimensional Hausdorff measure of set** A. .

We can visualise this with the following figure where the Hausdorff measure (area) of a piece of surface A is approximated by the cross-sections of little balls which cover it.

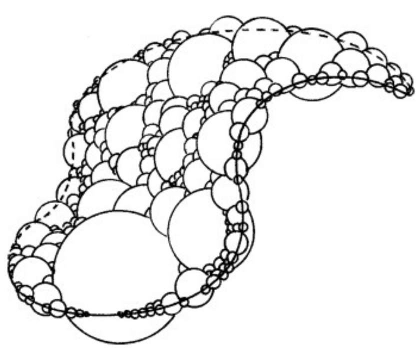


Figure 4: Visualisation of Hausdorff measure.

We see that there is a relation between the Lebesgue measure and the Hausdorff measure. Every Borel set is $\mathcal{H}^s$-measurable as this measure is a metric outer measure. Most importantly for all sets in X, there exists a Borel set containing it and equal in $\mathcal{H}^s$-measure.

Theorem 3.2. *If s is some integer then $\mathcal{H}^s = C_n \mathbb{L}^n$ where C_n is a constant and L is the Lebesgue measure.*

some people put the constant inside of the definition of the Hausdorff measure such that they are equal.

For example, in the metric space $\mathbb{R}$, the one dimensional Hausdorff measure $\mathcal{H}^1$ coincides with the Lebesgue measure $\mathcal{L}$. When we take $A \subset \mathbb{R}$ with diameter r, then A is contained in a closed interval of length r. So we have $\mathcal{L}(A) \leq r$. Using the method I theorem 2.2 we see that the measure $\mathcal{H}^1_\delta$ is the largest measure that satisfies $\mathcal{H}^1_\delta(A) \leq diam(A) = r$. So we have $\mathcal{L}(A) \leq r$. Take the half-open interval $[a, b)$. This interval is a finite union of half open intervals of length $\epsilon > 0$. Where the sum of the diameters of these intervals is $b - a$. So $\mathcal{H}^1_\delta([a, b)) \leq b - a$. Again using the method I theorem 2.2 we see that the Lebesgue measure $\mathcal{L}$ has the largest measure that satisfies $\mathcal{L}([a, b)) \leq b - a$. So $\mathcal{H}^1_\delta([a, b)) \leq \mathcal{L}([a, b))$. We conclude that $\mathcal{H}^1_\delta = \mathcal{L}$.

Suppose you are given a set that is nonzero dimensional, what happens if you apply the wrong Hausdorff measure to it? Finding what is the right Hausdorff measure is the beginning of finding the Hausdorff dimension of the set we are working with. We can see this with definition of the Hausdorff dimension.

4 Hausdorff dimension

We introduce the definition of Hausdorff dimension and state some useful theorems. Later on we give some examples of the Hausdorff dimension of some fractals.

In the book of Goodson [3] we find that the Hausdorff metric D which is defined on $C(X)$ by

$$D(A, B) = inf\{\delta > 0 : A \subseteq U_\delta(B) \quad \text{and} \quad B \subseteq U_\delta(A)\}, \quad A, B \in C(X) \quad (1)$$

Where $C(X) = \{A : A \quad \text{is a non-empty compact subset of X}\}$, so that $F : C(X) \to C(X)$. Hausdorff measure is already explained above, so we will move on to the dimension of it which is the main part of this section.

Theorem 4.1. *Define a Borel set F , if $\mathcal{H}^s(F) < \infty$ then $\mathcal{H}^t(F) = 0$. Where $0 < s < t$, and if $\mathcal{H}^t(F) > 0$ then $\mathcal{H}^s(F) = \infty$.*

We notice that if diam A$\leq \epsilon$, then

$$\bar{\mathcal{H}}^t_\epsilon(A) \leq (diamA)^t \leq \epsilon^{t-s}(diamA)^s$$

Thus, $\bar{\mathcal{H}}^t_\epsilon(F) \leq \epsilon^{t-s}\bar{\mathcal{H}}^s_\epsilon(F)$ for every F using the Method I theorem (2.2).

Therefore, for a set F, there is a unique "critical value" which is the **Hausdorff dimension** of this set.

Definition 4.2. The Hausdorff dimension will be a critical value $s_0 \in [0, \infty]$ such that

$$\mathcal{H}^s(F) = \infty \qquad \text{for all} \quad s < s_0;$$

$$\mathcal{H}^s(F) = 0 \qquad \text{for all} \quad s > s_0.$$

Notation: $s_0 = \text{dim}F$.

If $\mathcal{H}^s(F) = 0$ for any $s > 0$, then $\text{dim}F = 0$. Also, if $\mathcal{H}^s(F) = \infty$ for all s, then $\text{dim}F = \infty$.

Let A be a smooth rectifiable curve, to know what size does it have we can use its length; whilst its area and volume will be zero. Consider the surface of a sphere B with positive and finite area, its length could be infinite. But its volume is 0, because it is in a solid spherical shell which has small thickness that can be adjusted according to what we want. The sets A and B have different dimensions; for A, the dimensions 2 and 3 are huge, whilst for B, the dimension 1 is tiny, dimension 3 is very big, and dimension 2 is exactly right.

Theorem 4.3. *Consider the two Borel sets A, B;*

1. *if $A \subseteq B$, then dim $A \leq$ dim B.*
2. *dim $(A \cup B)$= max $\{dimA, dimB\}$*

Proof. (1) We will assume that $A \subseteq B$. If $s > \text{dim}B$, then $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$. Thus dim $A \leq s$ for all $s > \text{dim}B$ which leads to $\text{dim}A \leq \text{dim}B$.

(2) To prove this part we will suppose that $s > \max\{\text{dim}, \text{dim}B\}$. Then $s >\text{dim}A$, so $\mathcal{H}^s(A) = 0$ and $\mathcal{H}^s(B) = 0$. Also $\mathcal{H}^s(A \cup B) \leq \mathcal{H}^s(A) + \mathcal{H}^s(B) = 0$, thus dim $(A \cup B) \leq s$ for every $s > \max\{\text{dim}, \text{dim}B\}$. Therefore dim $(A \cup B)$ $\leq \max\{\text{dim}, \text{dim}B\}$ and by the first part the proof will complete. □

The s-dimensional Hausdorff measure gave us a way of measuring the size of a set for dimensions s other than the integers 1,2,3,... .

Theorem 4.4. *Consider $f : S \to T$ to be a similarity with ratio $r > 0$, with a positive real number s, and a set $F \subseteq S$. Then $\bar{\mathcal{H}}^s(f[F]) = r^s\bar{\mathcal{H}}^s(F)$. Thus dim $f[F]$ = dimF.*

Proof. By assuming $T = f[S]$, f will have an inverse f^{-1}. Let $A \subseteq S$ be a set that satisfies $\text{diam}f[A] = r \text{ diam}A$. Thus $(\text{diam}f[A])^s = r^s(\text{diam}A)^s$. We notice that we can use the Method I theorem and we will apply it twice to get, $\bar{\mathcal{H}}^s_{r\epsilon}(f[F]) = r^s\bar{\mathcal{H}}^s_\epsilon(F)$.

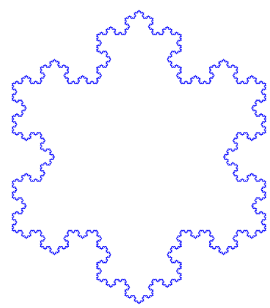
Which leads us to $\bar{\mathcal{H}}^s(f[F]) = r^s\bar{\mathcal{H}}^s(F)$ and dim $f[F] = \text{dim}F$.

□

5 Self-similarity dimension

There are different ways to interpret fractal dimension. Among the most widely used of these are the Hausdorff dimension, the packing dimension, and the box dimension. Let us define the similarity dimension, a fractal dimension that is easier to compute, but not that useful.

A self-similar object is exactly or approximately similar to a part of itself. For example, Koch Snowflake has an infinitely repeating self-similarity when it is magnified that is why its determined to be a fractal.



Self similarity is one of the most common ways to build up fractals. Luckily for us, the similarity dimension is simpler to compute than the Hausdorff dimension. Further in this project we will relate the two. This is very interesting, when the two coincide the similarity dimension is easy to compute, and the Hausdorff dimension is more generally applicable and has many useful properties, so we will have a win - win situation.

5.1 Ratio Lists and iterated function systems

Typically, a fractal looks irregular; but more importantly, after it is magnified it still looks irregular which is very different from what happens normally in classical geometry. Nevertheless, self-symmetric fractals have the same irregular shape when magnified at a certain scale. This behavior of magnification can be described with the use of iterated function system. The ratio will be 'how much we magnify' to obtain the same structure.

This is a finite list of positive numbers $(r_1, \cdots, r_n)$. Fractals are similar with certain parts of themselves by a ratio number. This can be thought as an iterated function system in a metric space S, which is a list $(f_1, \cdots, f_n)$, where $f_i : S \to S$ is a similarity with ratio r_i. A set $K \subset S$ is called invariant, when the union of all elements of the iterated function system at K is K itself: $K = f_1[K] \cup \cdots \cup f_n[K]$.

Theorem 5.1. *Let $(r_1, \cdots, r_n)$ be a ratio list (so $r_i < 1$. Then there is a unique positive number s such that $\sum_{i=1}^{n} r_i^s = 1$.*

We denote s as the sim-value or similarity value of a set K when $K \subset S$ is an invariant set.

It is, of course, conceivable that a given set admits two different decompositions, and therefore two different similarity dimensions.

5.2 Computing the Similarity Dimension of the Sierpiński gasket

Let us introduce an example where we compute the similarity dimension of a fractal such as the Sierpiński gasket. We construct the Sierpiński gasket by starting with a filled-in equilateral triangle with side length 1, this is called S_0. This triangle can be divided into three smaller triangles. Where the triangle in the middle is bigger and rotated by 180 degrees. The other triangles have side length $\frac{1}{2}$. When removing the interior of the middle triangle we get the set S_1. To get S_2 we divide the remaining 3 triangles into smaller triangles. Where we again remove that bigger inner triangle. Now we have 9 triangles left with side length $\frac{1}{4}$. Continuing this way we get the sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$. The Sierpiński gasket is given by $S = \bigcap_{k \in \mathbb{N}} S_k$.

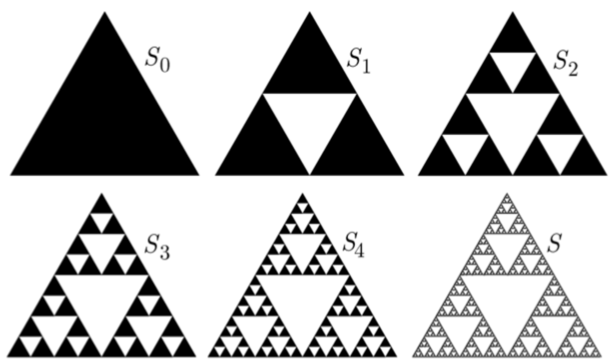


Figure 5: The Sierpiński gasket.

The Sierpiński gasket S_k is constructed above using approximations of itself. Let f_1, f_2, f_3 be the three dilations with ratio $\frac{1}{2}$ and centers at the three vertices of size length 1 of the triangle S_0. Now it follows by induction that

$$S_{k+1} = f_1[S_k] \cup f_2[S_k] \cup f_3[S_k].$$

And we can see that $S = f_1[S] \cup f_2[S] \cup f_3[S]$.

So S is an invariant set of the iterated function system (f_1, f_2, f_3) with ratio list $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. By theorem 5.1 we can say that the similarity value is s such that

$$\sum_{i=1}^{3} \left(\frac{1}{2}\right)^s = 1.$$

Thus the Sierpiński's similarity dimension is $s = \frac{\log 3}{\log 2}$.

Again, this is only the similarity dimension of the Sierpiński gasket which is in fact equal to the Hausdorff dimension. This is indeed not very complicated to compute and we will see further in this project how it is related to the Fractal dimension.

6 Computing the Hausdorff Dimension of the Cantor Dust

To compute the Hausdorff dimension of a fractal, we want to look for upper and lower bounds of such. Let us first work with the simple fractal Cantor Dust. It is constructed following this image:

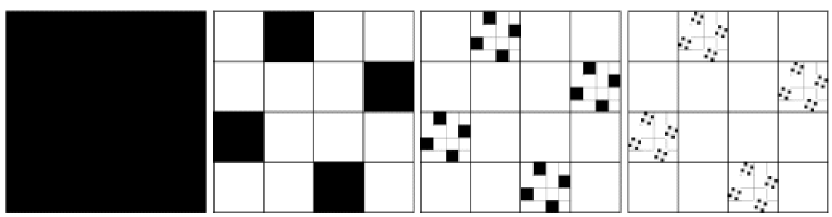


Figure 6: Cantor Dust.

Denote the Cantor Dust by F, we construct this fractal from the unit square. At each stage of the construction the squares are divided into 16 squares with a quarter of the side length, of which the same pattern of four squares is retained.

Observe that E_k, the k^{th} stage of the construction, consists of 4^k squares of side 4^{-k} and thus of diameter $4^{-k}\sqrt{2}$. Taking the squares of E_k as a δ-cover of F where $\delta = 4^{-k}\sqrt{2}$, we get an estimate $\mathcal{H}^1(F) \leq 4^k 4^{-k}\sqrt{2}$ where $k \to \infty$ and $\delta \to 0$.

For the lower bound we are lucky as we can use an orthogonal projection onto the x axis, denote it by proj. These projections do not increase in distances so we have that $|\text{proj}(x) - \text{proj}(y)| \leq |x - y|$ for $x, y \in \mathbb{R}$. We have that the projection of F onto the x axis is the interval $[0, 1]$.

$$1 = \text{length}([0,1]) = \mathcal{H}^1([0, 1]) = \mathcal{H}^1(\text{proj}(F)) \leq \mathcal{H}^1(F)$$

The Hausdorff measure of the Cantor Dust is 1.

7 Sierpiński gasket

We already calculated the similarity dimension of the Sierpiński gasket. Now we want to compute the Hausdorff dimension of the Sierpiński gasket. This is unfortunately not as simple as computing the Hausdorff dimension of the Cantor Dust. We first compute the upper bound of the Hausdorff dimension with help of the similarity value we already found. Then we have a more involved proof of the lower bound.

Take the set $E = \{L, U, R\}$, with these three letters we can make a string. Now is $E^{(n)}$ the set of all string of length n from the set E. (A string of length 0 is an empty string.) The set

$$E^{(*)} = E^{(0)} \cup E^{(1)} \cup E^{(2)} \cup \cdots$$

is the set of all finite strings. The set $E^{(\omega)}$ is the set of all infinite strings.

Let ϱ be the metric on $E^{(\omega)}$ for the ratio list $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. ϱ is defined such that $\text{diam}[\alpha] = 2^{-|\alpha|}$ for all finite strings $\alpha \in E^{(*)}$. Now when we apply the right shift we put a letter from E in front of the string α. So by the right shifts we get the ratio list:

$$\varrho(L\sigma, L\tau) = \frac{1}{2}\varrho(\sigma, \tau), \ \varrho(U\sigma, U\tau) = \frac{1}{2}\varrho(\sigma, \tau) \text{ and } \varrho(R\sigma, R\tau) = \frac{1}{2}\varrho(\sigma, \tau).$$

The measure $\mathcal{M}$ is given by $\mathcal{M}([\alpha]) = 3^{-|\alpha|}$. Notice that

$$\mathcal{M}([\alpha]) = (\text{diam}[\alpha])^s \text{ for all } \alpha \in E^{(*)},$$

with $s = \log 3/\log 2$. According to theorem 6.4.3 in Edgar [1] we have that the Hausdorff dimension of the string space $E^{(\omega)}$ equal to $s = \log 3/\log 2$.

Let $h : E^{(\omega)} \to \mathbb{R}^2$ be the function that sends $E^{(\omega)}$ onto the Sierpiński gasket S. If the iterated function system in $\mathbb{R}^2$ is (f_L, f_U, f_R), we then have

$$h(L\sigma) = f_L(h(\sigma)), \ h(U\sigma) = f_U(h(\sigma)) \text{ and } h(R\sigma) = f_R(h(\sigma)).$$

The function h satisfies the Lipschitz condition

$$|h(\sigma) - h(\tau)| \leq \varrho(\sigma, \tau).$$

Now we find that

$$\text{dim} S \leq \text{Dim} S \leq \text{dim} E^{(\omega)} = s = \frac{\log 3}{\log 2}.$$

We have found the upper bound of the Hausdorff dimension. Now we will need to prove the lower bound in order to prove that the Hausdorff dimension is equal to the similarity dimension.

Proposition 7.1. *The Sierpiński gasket S has Hausdorff dimension equal to the similarity dimension* $\log 3/\log 2$.

Proof. Let V be the interior of the triangle S_0. The triangle has height $\sqrt{3}/2$ and width 1 so we have $\mathcal{L}^2(V) = \sqrt{3}/4$. If we have two strings α and β of the same length that are not equal to each other. Then we have $\alpha[V] \cap \beta[V] = \emptyset$ and we have $h([\alpha]) = \overline{[V]} \cap S$. Now we see that

$$S_k = \bigcup_{\alpha \in E^{(k)}} \overline{\alpha[V]}.$$

Let a set $A \subseteq S$, and k a positive integer such that

$$2^{-k} < \text{diam}A \leq 2^{-k+1}.$$

Let

$$T = \{\alpha \in E^{(k)} \ : \ \overline{\alpha[V]} \cap A \neq \emptyset\}.$$

Let m the number of elements in T. We claim that $m \leq 100$. The set $\alpha[V]$ is the image of V under a similarity with ratio 2^{-k}, so that means that

$$\mathcal{L}(\alpha[V]) = 4^{-k}\frac{\sqrt{3}}{4}.$$

For all the strings $\alpha \in T$ we have that all the sets $\alpha[V]$ are disjoint. Let x a point in A, now are all of the elements of all the $\alpha[V]$ with $\alpha \in T$ within distance $\text{diam}A + 2^{-k} \leq 3 \cdot 2^{-k}$ of x. So m disjoint sets of area $4^{-k}\sqrt{3}/4$ are contained in the ball with center x and radius $3 \cdot 2^{-k}$. So we find

$$m \cdot 4^{-k}\frac{\sqrt{3}}{4} \leq \pi(3 \cdot 2^{-k})^2.$$

If we solve this we find that $m \leq 36\pi/\sqrt{3} < 100$. So we have proven our claim.

We also claim that $\mathcal{M}(h^{-1}[A]) \leq 100(\text{diam}A)^s$ for all Borel sets $A \subseteq S$. Let $A \subseteq \bigcup_{\alpha \in T} \overline{\alpha[V]}$, so $h^{-1}[A] \subseteq \bigcup_{\alpha \in T}[\alpha]$. So we have

$$\mathcal{M}(h^{-1}[A]) \leq \sum_{\alpha \in T} \mathcal{M}([\alpha]) \leq 100 \cdot 3^{-k} = 100 \cdot (2^{-k})^s \leq 100 \cdot (\text{diam}A)^s.$$

Now we can conclude by Method I theorem (2.2) that $1 = \mathcal{M}(h^{-1}[A]) \leq 100 \cdot \mathcal{H}^s(A)$ for all Borel sets A. So we find that $\dim S \geq s$. $\square$

8 Morans Open Set Condition

In the example above, we saw it is somewhat simple to come across an upper and lower bound that can help us determine the Hausdorff dimension. Nevertheless, in most cases this is not so straight forward, hence it is necessary to introduce other tools such as the Moran's open set condition. This will help us with the computation of the Hausdorff dimension.

Define $(r_1, ..., r_n)$ to be a contracting ratio list with dimension s. Its corresponding iterated function system of similarities in $\mathbb{R}^d$ is $(f_1, ..., f_n)$, and let

K be its invariant set. Generally, it is not true that dim $K = s$, where s denotes the sim-value. For instance, take the iterated function system (f_L, f_U, f_R) for the Sierpiński gasket, with ratio list (1/2,1/2,1/2). The iterated function system (f_L, f_L, f_U, f_R) will have the same invariant set with longer ratio list (1/2,1/2,1/2,1/2).

The Hausdorff dimension of the invariant set K is log3/log2, where the iterated function system has sim-value of 2. Now, the problem that we have is that the first two images $f_L[K]$ and $f_L[K]$ overlap a lot. In the Cantor dust (6), the images do not overlap at all. But we can not do the same here because it will exclude a lot of the interesting examples such as the Sierpiński gasket itself, which has nonempty overlap sets $f_L[K] \cap f_U[K]$. There are inequality between the Hausdorff dimension, packing dimension, and similarity dimension. If s is the similarity dimension, then the string model will have packing dimension s and the addressing function is lipschitz, Thus

$$\dim K \leq \mathrm{Dim} K \leq s.$$

Theorem 8.1. *Define a contracting ratio list $(r_e)_{e\in E}$, with sim-value s and a realization $(f_e)_{e\in E}$ in $\mathbb{R}^d$. Let K be the invariant set. If Moran's open set condition is satisfied, then dim $K = s$.*

Proof. Let $c > 0$ be a constant such that if $A \subseteq K$, then the set T has at most c elements. Where

$$T = \left\{\alpha \in E^{(*)} : \overline{\alpha[U]} \cap A \neq \varnothing, \mathrm{diam}\alpha[U] < \mathrm{diam} A \leq \mathrm{diam}\bar{\alpha}[U]\right\}$$

Suppose that there is a positive constant b so that for any Borel set $A \subseteq K$, we will have

$$\mathcal{M}(h^{-1}[A]) \leq b(\mathrm{diam} A)^s.$$

Assume U is a Moran open set, and $w = \mathrm{diam} U$. Where $A \subseteq \sum_{\alpha\in T} \overline{\alpha[U]}$ and $h^{-1}[A] \subseteq \sum_{\alpha\in T}[\alpha]$. If $\alpha \in T$, then $\mathcal{M}([\alpha]) = (\mathrm{diam}[\alpha])^s = ((1/w)\mathrm{diam}\alpha[U])^s \leq (1/w^s)(\mathrm{diam} A)^s$. Thus,

$$\mathcal{M}(h^{-1}[A]) \quad \leq \Sigma_{\alpha\in T}\mathcal{M}([\alpha]) \quad \leq c(1/w^s)(\mathrm{diam} A)^s.$$

So b should be equal c/w^s. Using Method I theorem (2.2), we get $1 = \mathcal{M}(h^{-1}[K]) \leq b\mathcal{H}^s(K)$. Therefore dim $K \geq s$. □

We have used in the proof the properties of Lebesgue measure in $\mathbb{R}^d$

9 Heighway Dragon Boundary

One might say that the using the Open Set condition is unnecessary for finding the Hausdorff dimension of a fractal. However it turns out to be quite useful, especially for more complicated fractals. One such fractal is the Heighway Dragon Boundary. Let us first consider the Heighway Dragon itself. Thereafter, the Hausdorff dimensional is computed, making use of the Open Set condition.

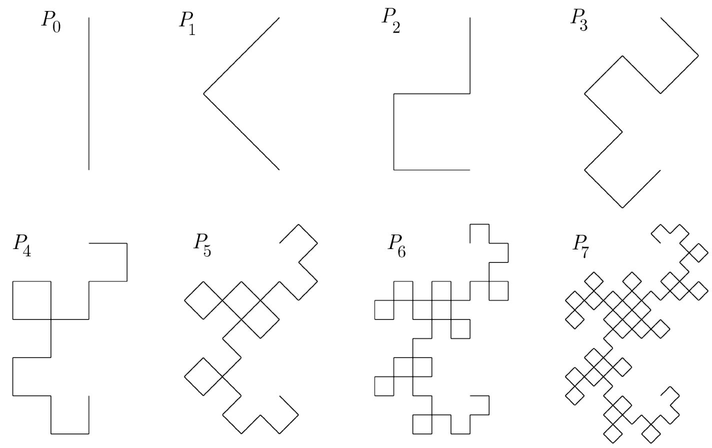


Figure 7: The first seven iteration of the creation of Heighway's Dragon. Heighway's Dragon is the limit of the series P_n.

9.1 Heighway Dragon

The Heighway Dragon is a set in the plane, which is constructed in the following way. Let P_0 be a line segment of unit length. The next step is the line segment P_1, which is obtained from P_0 by replacing it with a polygon of two line segments each with length $\frac{1}{2}\sqrt{2}$ joined at a right angle. The endpoints of P_1 are the same endpoints as of P_0. The next iteration is obtained by applying the latter to each segment of P_1, and is denoted by P_2. The new length is $1/\sqrt{2}$ times the length of the last segment. Note the there are two choice of replacing a segment with a polygon. From the bottom up, we have the choice 'left' and 'right'. By convention, we always choose the first segment (the bottom one) to get left, and then alternate to the top. The inductive process is shown in figure 7. The limit of the sequence P_n is Heighway's Dragon, hereafter denote by P.

Heighway's Dragon has some interesting properties. In order to compute its boundary, let us make sure that it is a bounded set.

Proposition 9.1. *All P_n remain within some bounded region of the plane.*

Proof. Let the top of P_0, with P_0 taken as in the figure, be the endpoint of the line segment. Note that every point in P_0 is at most a distance 1 away from the endpoint. By construction of P_1, every point in P_1 is at most a distance $1/2$ from the segment P_0. By induction, it is clear the every point in P_k is at most $(1/\sqrt{2})^{k+1}$ away from a point in P_{k-1}. Hence the maximum distance to the endpoint for points in P_k is

$$1 + \sum_{i=1}^{k} \left(\frac{1}{\sqrt{2}}\right)^{i+1} \leq 1 + \sum_{i=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^{i+1}.$$

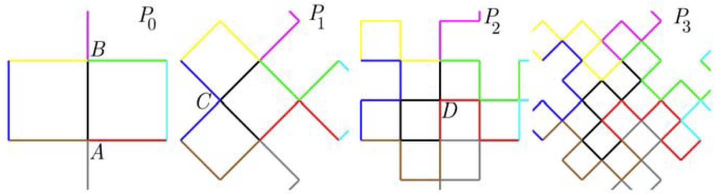


Figure 8: Heighway Tilling generators.

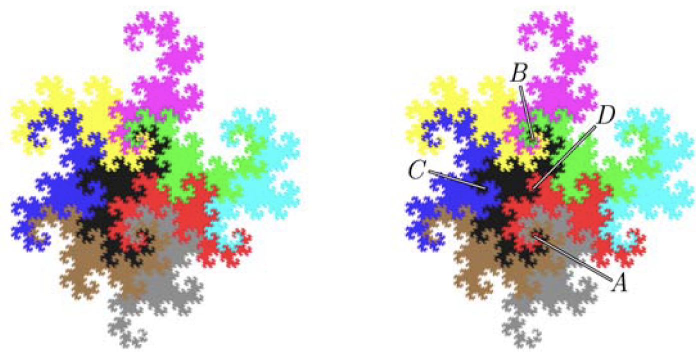


Figure 9: Heighway Tilling result.

The latter converges to a finite value, since it is a geometric series with ratio less than 1. □

It can also be shown that the iteration P_n never crosses itself, although we omit the proof. The interested reader can find a proof in Edgar [1]. It can even be shown that Heighway's Dragon tiles the plane, however the proof is beyond the scope of the report. Interestingly, this means that P itself is not a fractal in the Mandelbrot sense. It turns out that the boundary, ∂P, is a fractal. The main question is, how do we compute the Hausdorff dimension of this boundary.

9.2 The Hausdorff Dimension of the Heighway Dragon Boundary

In figure 8 the black segment from A to B generates a sequence P_n of polygons that converge to P, depicted as the black tile in figure 9. Note that the point A in P_0 also belongs to all P_n and that it is a boundary point of P. That is because A lies not only in the black tile P, but also in the brown, gray, and red tiles. Similarly B is in ∂P. We will write $\partial P = U \cup V$ in the following way. The set U is the portion of the boundary to the left of curve AB, i.e. the points that belong not only to the black tile, but also to at least one of the brown, blue, or yellow tiles. The set V is the same but right of curve AB,

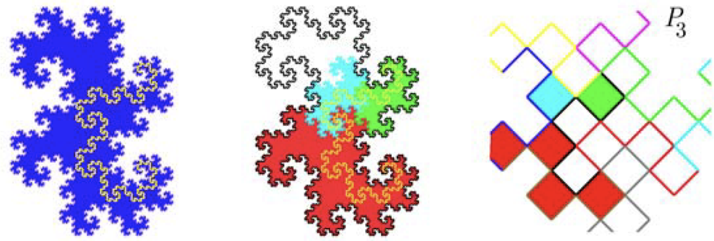


Figure 10: Caption

i.e. the points that belong not only to the black tile, but also to at least one of the red, green, or cyan tiles. No other tiles can touch the black tile, because the plane has topological dimension 2, so they would have to cross one of the curves shown to reach the black tile. First observe the set U. According to P_1 in figure 8, the "midpoint", denoted by C, of P is a boundary point, since it lies in both the black and blue tiles. The portion of U that is between A and C is a copy of U shrunk by factor $1/\sqrt{2}$. Similarly the part of U between B and C is a copy of V shrunk by factor $1/\sqrt{2}$. Now observe the set V. The "three-quarter" point, denoted by D, of P is a boundary point, because it lies in both the black and red tiles. Again, the portion of V between B and D is a copy of U shrunk by factor $1/\sqrt{2} \cdot 1/\sqrt{2} = 1/2$. And the portion of V between D and A is the boundary between black and red; looking at it from the red point of view, we see it is a copy of U shrunk by the same factor, $1/2$. Thus the set V is made up of two copies of U, so $\dim U = \dim V$. The set U is made up of one copy of U and one copy of V, both shrunk by a factor $1/\sqrt{2}$. The latter is made up of two copies of U each shrunk by a further factor of $1/2$. So the complete decomposition gives that U is made up of three copies of itself, and has a ratio list $(2^{-1/2}, 2^{-3/2}, 2^{-3/2})$. This ratio list has a sim-value, s, equal to ~ 1.52, which is given by $s = 2\frac{\log \lambda}{\log 2}$, where λ is a solution of $\lambda^3 = 2 + \lambda^2$. In order to claim that this value is also the Hausdorff of U, we need an open set condition. Define an open set G as follows: in figure 8 start with the four segments: black, yellow, blue, brown. The curve U lies in the union of the four tiles they produce: see the black, yellow, blue, brown in figure 9. The open set G will be the interior of this union. Set G is shown in blue in figure 10, with U shown in yellow. The images of G under the three maps that make up the iterated function system for U are shown in the second picture. The large red set is an image of G shrunk by factor $1/\sqrt{2}$. The cyan and green sets are images of G shrunk by factor $2^{-3/2}$. Since the tiles are generated from different segments in figure 8, these images are disjoint and have disjoint interiors. The three images descend from the edges bordering the like-colored squares shown in P_3 of figure 10.

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