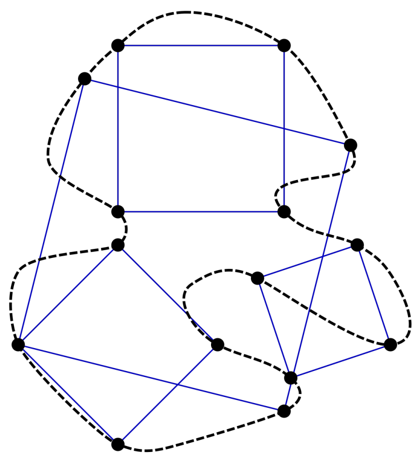
Topological and Symplectic Resolution of the Inscribed Square Problem



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Abstract

The aim of this project is to give an understanding of the Square Peg Problem. This involves studying in detail Vaughan's proof [1] of the Rectangle Peg Problem which uses a new way to think about rectangles, topology and a theorem about embedding non-orientable manifolds in $\mathbb{R}^n$. We spend some time on this theorem (Theorem 2.2.1) and give a proof for it. This proof involves concepts such as non-orientable manifolds and intersection theory. The last part of the project involves an introduction to symplectic geometry. This enables us to study of Joshua Evan Greene and Andrew Lobb's proof [2] regarding the Square Peg Problem. This conjecture is proved at the end. A student of the Pure Mathematics Bachelor is expected follow with the help of some nice coffee.

Contents

Chapter 1

Introduction

In the first years of the 20th Century, Otto Toeplitz discussed a topology problem for which he never published a proof. The conjecture claims that in any Jordan curve (curve that is closed and non-intersecting) one can find an inscribed square. Inscribed square means that the corners of the square intersect the curve.

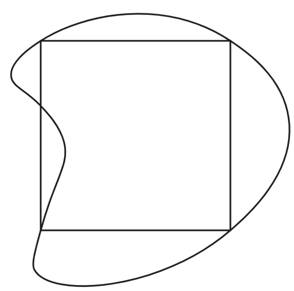


Figure 1.1: Inscribed square on a Jordan curve [3].

Theorem 1.0.1. *(Square Peg Problem) For every Jordan curve on the plane, there exists an inscribed square on this curve.*

There have been many attempts to prove Theorem 1.0.1. It was not until a worldwide pandemic hit our world in 2020, that Joshua Evan Greene, Andrew Lobb managed to publish a complete proof. It involves symplectic spaces, Lagrangian Klein bottles and Shevchishin's Theorem. Nevertheless, this becomes easier to understand once we get an intuition on how to tackle this problem. That is why, in this project we first focus on getting a clear understanding of a simplified version of this conjecture: the Rectangle Peg Problem.

Chapter 2

Rectangle Peg Problem

In this chapter we discuss the Rectangle Peg Problem. This is a simplified version of the Square Peg Problem. This conjecture was proved to be true by Vaughan [1] in the late 1970s.

Theorem 2.0.1. *(Rectangle Peg Problem) For every Jordan curve on the plane, there exists an inscribed rectangle on this curve.*

Definition 2.0.1. (Jordan curve) A Jordan curve is a curve which is homeomorphic to the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$, i.e. it is simple and closed.

2.1 Rectangles and Topological Surfaces

First, lets think about rectangles as two pairs of points that form equal-length line segments with the same midpoint.

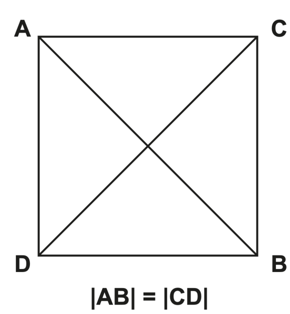


Figure 2.1: Rectangle as two unordered pairs of points [3].

Thus, a rectangle inscribed on a Jordan curve γ can be thought of as two unordered pairs of points on γ, with those properties. Hence we study the unordered pairs in γ to find an inscribed rectangle.

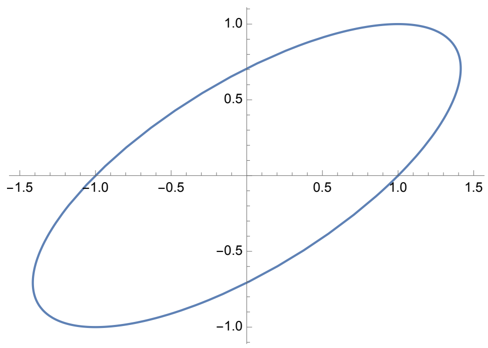


Figure 2.2: 2D plot of an example of γ [4].

Take an arbitrary Jordan curve γ. Recall, by definition, the Jordan curve γ is homeomorphic to the circle S^1. We also know that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to $[0,1]/\sim_1$ where $\sim_1$ identifies 0 with 1. Thus, to prove that γ is homeomorphic to the interval $[0,1]/\sim_1$, it is sufficient to prove that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.

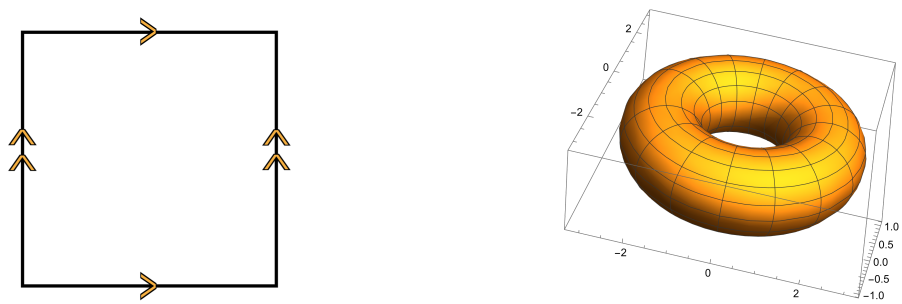
Proposition 2.1.1. *The map $f : \mathbb{R}/\mathbb{Z} \to S^1$ defined by $f([t]) = (\cos 2\pi t, \sin 2\pi t)$ is a homeomorphism.*

Proof. Let $g : \mathbb{R} \to S^1$ be defined by $g(t) = (\cos 2\pi t, \sin 2\pi t)$. We have that g is a surjective continuous open map, hence a quotient map. We have that $\mathbb{R}/\mathbb{Z}$ is equal to the following set $\mathbb{R}/\mathbb{Z} = \{g^{-1}(\{x\})|x \in S^1\}$. The map $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ defined by $p(x) = [x]$ (where $[x] = \{x + k|k \in \mathbb{Z}\}$), is a surjective map. So there exists a unique quotient topology $\mathcal{T}$ on $\mathbb{R}/\mathbb{Z}$ induced by p. Then by Corollary [5, Cor 22.3] the map g induces a continuous bijective map $f : \mathbb{R}/\mathbb{Z} \to S^1$ which is a homeomorphism because g is a quotient map.

$$\begin{tikzcd} \mathbb{R} \arrow[d, "p"] \arrow[dr, "g"] & \\ \mathbb{R}/\mathbb{Z} \arrow[r, "f"] & S^1 \end{tikzcd}$$

□

Hence, we can map the points in γ homeomorphically to $[0,1]/\sim_1$, where $x = 0$ and $x = 1$ correspond to the same point in γ. Now we represent the ordered pairs on γ as $[0,1] \times [0,1]/\sim_2$, where $\sim_2$ identifies $(0,y)$ with $(1,y)$ and $(x,0)$ with $(x,1)$. These identifications can be made as these pairs represent the same ordered pair. This can be visualized in Figure 2.3a. If we glue the indicated part together, we arrive to a torus such as Figure 2.3b. This surface represents all ordered pairs of points on the loop.



(a) Ordered pairs of points in γ: 2D plot of the torus [3].

(b) 3D plot of the torus [4].

Figure 2.3: Torus

Recall, that we are interested in finding the surface that represents all unordered pairs of points on the loop. This essentially means that the points (x, y) will be identified with (y, x) for all $x, y \in [0, 1]$. We go back to the square $[0, 1] \times [0, 1]$, and we draw the diagonal $y = x$. Fold it thorough this line and remembering $(0, y)$,$(1, y)$ and $(x, 0)$,$(x, 1)$ are identified respectively. This results in the surface that represents all unordered pairs of points on γ. The following two propositions helps us understand this surface.

Proposition 2.1.2. *The torus $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.*

Proof. Consider two spaces A, B with equivalence relations R, S. Then $R \times S$ is an equivalence relation. We know that the natural map

$$(A \times B)/(R \times S) \to A/R \times B/S$$

is then a continuous bijection. In our case

$$f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$

is then a continuous bijection. Since both of our spaces are Hausdorff compact, it follows that f is a homeomorphism. □

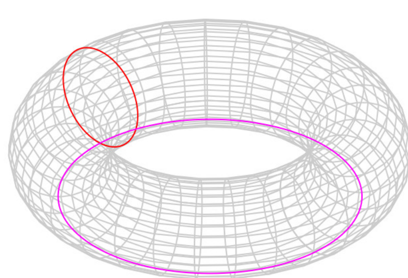


Figure 2.4: Torus as a circle of circles [6, p.6].

Proposition 2.1.3. *The Mobiüs strip is homeomorphic to* $(\mathbb{R}^2/\mathbb{Z}^2)/\mathbb{Z}_2$.

Proof. Define the Möbius strip by the quotient space $[0,1]^2/[(0,x) \sim (1,1-x)]$, for $x \in [0,1]$. Also $\mathbb{Z}_2$ identifies points (x,y) with (y,x) $\forall x, y \in \mathbb{R}^2/\mathbb{Z}^2$.

Define $g : T \to M$, where T is the torus and M is the Möbius strip. Define L, to be the lower half triangle of the square $[0,1] \times [0,1]$. The map g is the composition of two smooth maps $g = g_2 \circ g_1$. Where $g_1 : T \to L$ defined by $g_1(x,y) = \begin{cases} (x,y) & x \geq y \\ (y,x) & x < y \end{cases}$ is open continuous and surjective. Also, $g_2(x,y) : U \to M$ defined by $g_1(x,y) = (x+y-1, -x+y+1)$ is also open, continuous and surjective. So the composition of both is open, continuous and surjective. Hence, the map g is a quotient map.

Then, define T^* to be the set $\{g^{-1}(\{x\})|x \in M\}$. Hence, T^* is homeomorphic to the quotient $T/\mathbb{Z}_2$. The map $p : T \to T^*$ is therefore a surjective map. Thus, there is a unique quotient topology $\mathcal{T}$ on T^* induced by p. Then by Corrolary [5, Cor 22.3], since g is a quotient map, g induces a bijective continuous map $f : T^* \to M$ that is a homeomorphism. But T^* is homeomorphic to $T/\mathbb{Z}_2$ which is homeomorphic to $(\mathbb{R}^2/\mathbb{Z}^2)/\mathbb{Z}_2$. So f is a homeomorphism between $(\mathbb{R}^2/\mathbb{Z}^2)/\mathbb{Z}_2$ and the Möbius strip.

$$\begin{tikzcd} T \arrow[d, "p"] \arrow[dr, "g"] & \\ T^* \arrow[r, "f"] & M \end{tikzcd}$$

□

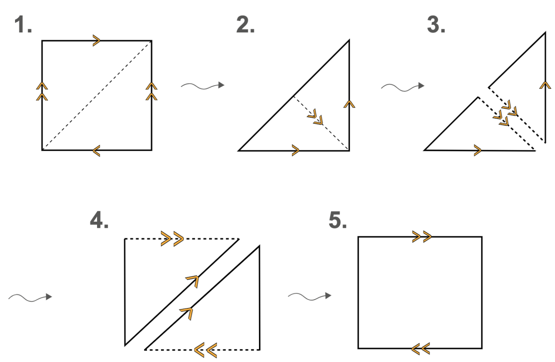


Figure 2.5: The Möbius strip is homeomorphic to $(\mathbb{R}^2/\mathbb{Z}^2)/\mathbb{Z}_2$ [3].

Since the unordered pairs of points on γ are the ordered pairs of points with the quotient $\mathbb{Z}_2$, ie. $(\mathbb{R}^2/\mathbb{Z}^2)/\mathbb{Z}_2$. Hence, the unordered pairs of points on γ can be represented by the Möbius strip.

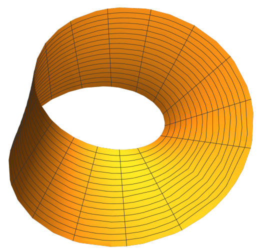


Figure 2.6: Möbius Strip [4].

2.2 Embedding of Topological Surface in $\mathbb{R}^3$

We have established a relation between rectangles on γ and the Möbius strip. Our aim now, is to prove that two different pairs of unordered points on the curve γ have the same midpoint and distance. If this happens, then we have found four points that lie on γ and make an inscribed rectangle.

Define the map f as follows:

$$f : (\mathbb{R}^2/\mathbb{Z}^2)/\mathbb{Z}_2 \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$$
$$f(x, y) = ((x, y)/2, |x - y|)$$

If f is not one to one, then we must have that two different pairs make two segments that have equal midpoint and length. Thus, they form a rectangle whose vertices lie on γ.

From Section 2.1, we know that the Möbius strip represents the set of unordered pairs of points on γ. Thus, we map the Möbius strip by f to $\mathbb{R}^3$. Recall, that the edge of the Möbius strip represents the points (x, x) for $x \in \gamma$. Thus, the edge is mapped by f to γ on the plane $\mathbb{R}^2 \times \{0\}$. The image of the Möbius under f can be visualised by Figure 2.7.

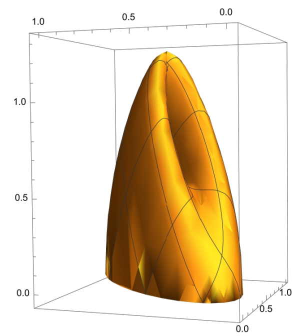


Figure 2.7: 3D plot of $f(M)$ [4].

Essentially we are embedding the Möbius strip into $\mathbb{R}^n$ with the map f. Proving this embedding is not smooth is sufficient to prove f is not one to one. Hence, we use the following theorem that will be discussed and proved in Chapter 2:

Theorem 2.2.1. *A closed non-orientable differentiable manifold (without boundary) of dimension $n-1$ cannot be smoothly embedded as a subset of $\mathbb{R}^n$.*

2.3 Rectangle Peg Problem Proof

Rectangle Peg Problem [Theorem 2.0.1]: *For every Jordan curve on the plane, there exists an inscribed rectangle on this curve.*

Proof. Let $M = \{\{x, y\} | x, y \in \gamma\}$ and $f : M \to \mathbb{R}^3$, where $f(x, y) = ((x, y)/2, |x - y|)$.

From Section 2.1, M is a Möbius strip. If γ doesn't contain the vertices of a rectangle f is one-to-one. If it is, then there cannot be two pairs of different unordered pairs of points that have the same midpoint and length segment which make a rectangle on γ. For the sake of contradiction assume f is one-to-one.

Then $f(M)$ is the image of a Möbius strip onto $\mathbb{R}^3$. This image (as we see in figure 2.7) lies in the closed half-plane $z \geq 0$ and meets $z = 0$ in its boundary which is γ. Adding the interior of γ we get a closed two dimensional nonorientable manifold, ie. the projective plane $\mathbb{RP}^2$, embedded as a closed subset of $\mathbb{R}^3$. Thus, by theorem 2.2.1 this embedding cannot be smooth. So f is not one-to-one. Hence, there exists at least one rectangle in our Jordan curve γ. $\square$

Chapter 3

Embedding Non-orientable Hypersurfaces $\in \mathbf{R}^n$

In this chapter we discuss Theorem 2.2.1 which is a very powerful statement regarding embeddings of non-orientable hypersurfaces in $\mathbb{R}^n$. Recall the theorem states the following: a closed non-orientable differentiable manifold (without boundary) of dimension $n-1$ cannot be smoothly embedded as a subset of $\mathbb{R}^n$.

For example, the Möbius strip M can be embedded smoothly in $\mathbb{R}^n$. This does not contradict Theorem 2.2.1 as it is a manifold with boundary. If we glue the boundary of M to a disc, this is homeomorphic to the real projective plane. Therefore, the real projective plane $\mathbb{RP}^2$ is a closed, non-orientable manifold with no boundary of dimension 2. Therefore, by Theorem 2.2.1, it cannot be smoothly embedded in $\mathbb{R}^3$, and we are limited to represent it by the following figure.

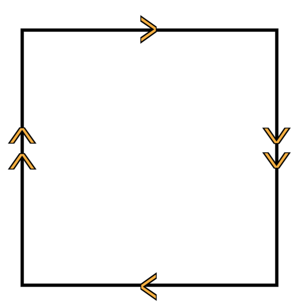


Figure 3.1: Real Projective Plane

3.1 Orientability of manifolds

Orientation will be crucial for the understanding of Theorem 2.2.1. It is a concept that tells us where we are in respect to our surroundings. So, in the case of a line, orientation is just the direction we are going along it.

Now, we can think of orientation of manifolds of higher dimensions in a similar way. If we can consistently give a manifold an orientation (we explain this further later on) we call it **orientable**, otherwise it is **non-orientable**. For example in the case of $\mathbb{R}^n$, the orientation is standard, and similarly for a sphere. Thus, we call this surfaces orientable, as we can consistently choose an orientation. To the contrary, the Möbius strip has the property that when we move around it, we can come back to the starting point completely upside down. So there is no consistent choice of orientation, therefore we call the Möbius strip non-orientable.

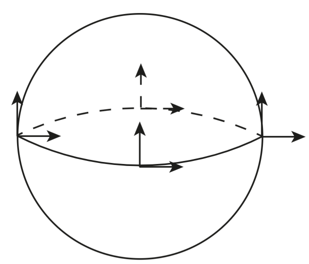


Figure 3.2: Sphere with consistent orientation [3].

For a vector space V we have the following definition for consistent orientation of its basis. We define an orientation for V as an equivalence class of ordered bases.

Definition 3.1.1. (Consistently Oriented Basis) Let V be a real vector space of dimension $n \geq 1$. We say that two ordered bases $(E_1, \cdots E_n)$ and $(\tilde{E}_1, \cdots, \tilde{E}_n)$ for V are consistently oriented if the transition matrix (B_i^j), defined by

$$E_i = B_i^j \tilde{E}_j,$$

has positive determinant [7, p.378].

For manifolds, we first have the **pointwise orientation**. Let M be a smooth n-dimensional manifold with or without boundary. We define a pointwise orientation on M to be a choice of orientation of each tangent space.

Definition 3.1.2. (Orientation of Local Frames) Let M be endowed with a pointwise orientation. If (E_i) is a local frame for its tangent space TM; we say that (E_i) is (positively) oriented if $(E_1|_p, \cdots E_n|_p)$ is a positively oriented basis for T_pM at each point $p \in U$. A negatively oriented frame is defined analogously.[7, p.380]

A pointwise orientation is said to be continuous if every point of M is in the domain of an oriented local frame.

Definition 3.1.3. (Oriented Manifold) An orientation of M is a continuous point-wise orientation. We say that M is orientable if there exists an orientation for it, and non-orientable if not. An oriented manifold is an ordered pair $(M, \mathcal{O})$, where M is an orientable smooth manifold and $\mathcal{O}$ is a choice of orientation for M ; an oriented manifold with boundary is defined similarly [7, p.380].

In Theorem 2.2.1, we treat a non-orientable smooth manifold. Thus, M will be non-orientable if there exists a loop M such that the normal to M at a point on that loop, when transported continuously around the loop, comes back in opposite direction.

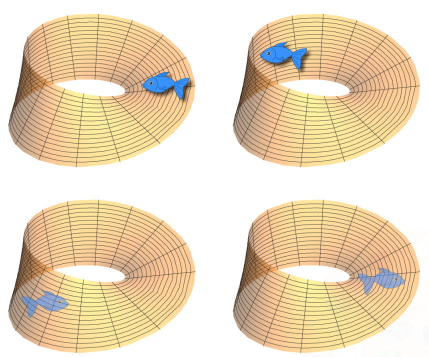


Figure 3.3: Non-orientable Möbius - Fish arrives to the opposite side when completing one loop [3].

3.2 Intersection Theory mod2

To be able to prove Theorem 2.2.1, we must introduce some crucial concepts that will give some insight to the reader regarding intersection theory.

First, lets introduce the concept of transversality between submanifolds that will depend on their ambient manifold.

Definition 3.2.1. (Transversality) Transversality between two smooth submanifolds implies that at every point of intersection their respective tangent spaces generate together at that point the tangent space of the ambient space. This is denoted by the symbol $\pitchfork$.

Two submanifolds without boundaries $X, Y \subset Z$ have **complementary dimension** if $\dim X + \dim Y = \dim Z$. If, additionally $X \pitchfork Y$, then $X \cap Y$ is a zero dimensional manifold [8, p.77].

Theorem 3.2.1. *Let f be a smooth map from a manifold X with boundary onto a boundaryless manifold Y, and suppose that both $f : X \to Y$ and $\partial f : \partial X \to Y$ are transversal with repect to a boundaryless submanifold Z in Y. Then the preimage $f^{-1}(Z)$ is a manifold with boundary*

$$\partial\{f^{-1}(Z)\} = f^{-1}(Z) \cap \partial X$$

and the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y [8].

For example, if we take $X = D^2$, $Y = \mathbb{R}^3$ and the smooth map f, for which $f : X \to Y$ and $\partial f : \partial X \to Y$ are transversal with respect to a boundaryless Möbius M which is a submanifold of Y. Then we have that $f^{-1}(M)$ is a manifold of dimension 1, and its boundary has dimension 0 and lies on the boundary of X.

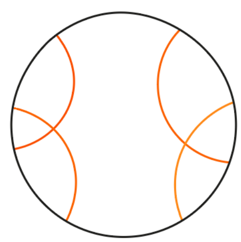


Figure 3.4: Pre-image $f^{-1}(M)$ in orange on D^2 [3].

Let X, Y be two smooth manifolds inside Z which have complementary dimension. Also X is compact and Y is closed. It follows that $X \cap Y$ must be a finite set of points, namely the **intersection number** $\#(X \cap Y)$. Note that transversality between these two manifolds might not always be a property. Nevertheless, we can alter X ever so slightly and the intersection number will agree $\mod 2$.

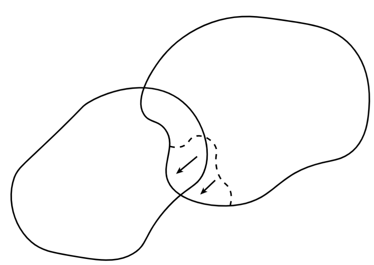


Figure 3.5: Deforming until transversality [3].

Definition 3.2.2. (Intersection Number $\mod 2$) Let X, Y be smooth manifolds, and $Z \subset Y$ be a submanifold. Then the intersection number $\mod 2$ of a smooth map $f : X \to Y$ with Z, is the number of points in $f^{-1}(Z) \mod 2$, and its denoted by notation $I_2(f, Z)$.

Theorem 3.2.2. *(Boundary Theorem) Let Y be a smooth manifold. Suppose that X is the boundary of some compact manifold W and $g : X \to Y$ is a smooth map. If g may be extended to all of W then $I_2(g, Z) = 0$ for any closed submanifold Z in Y of complementary dimension. [8, p.80]*

3.3 Proof of Theorem 2.2.1

With this background we can prove Theorem 2.2.1. The proof is inspired by Hans Samelson's paper [9]. Recall the statement of Theorem 2.2.1:

A closed non-orientable differentiable manifold (without boundary) of dimension $n-1$ cannot be smoothly embedded as a subset of $\mathbb{R}^n$.

Proof. Assume by contradiction that such a nonorientable hypersurface M in $\mathbb{R}^n$ exists. As we saw in Section 3.1, nonorientable implies that there exists a closed loop in M such that the normal to M, when transported around the loop continuously, gets to the initial point with opposite direction with respect to the original direction of the normal 3.7. We will call this loop around the manifold β.

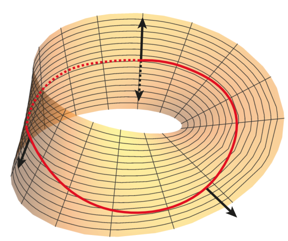


Figure 3.6: Möbius non-orientable manifold with loop β [3].

Let $\epsilon > 0$ be sufficiently small. Consider the loop β. Now take a point on this loop, and take the vector normal to M at this point. Next, take a point on this normal that is ϵ distance from M, call it p. Now transport this normal around β. We will arrive near the starting point, and the only way to reach it is by intersecting M because of non-orientability of M, see figure 3.7. This forms a smooth closed curve δ in $\mathbb{R}^n$. Moreover, δ meets M at exactly one point , and its transversal to M at this point (as the tangent of δ is the not tangent of M) .

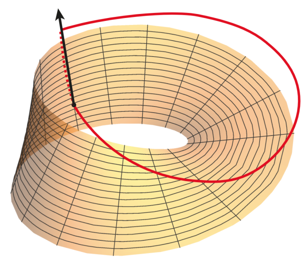


Figure 3.7: Möbius with δ represented [3].

Define the disc to be $D^2 = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$, whose boundary is S^1. Now, construct a smooth inclusion map $f : D^2 \to \mathbb{R}^n$ that maps the boundary S_1 of D^2 to δ. Hence, δ is in this way contracted by f to a point of $\mathbb{R}^n$.

Now, we can 'wiggle' the image of f, deforming it ever so slightly, making it transversal to M. We have that $f(D^2)$ is not tangent to M whenever they intersect, this is already the case in the boundary S^1 of D^2 as $f(S^1) = \delta$ is transversal to M at their intersection by how we defined f.

By the Theorem 3.2.1 it follows that the preimage $f^{-1}(M)$ is a manifold with boundary. The boundary $\delta f^{-1}(M) = f^{-1}(M) \cap \delta(D^2)$, and the codimension of $f^{-1}(M)$ in D^2 is equal to the codimension of M on $\mathbb{R}^n$, which is 1. And the boundary of the preimage of M has dimension 0. Therefore the boundary are points on S^1 in D^2, as we saw in Figure 3.4. We know these boundary points must be even by the boundary Theorem 3.2.2. Thus, we have reached a contradiction as we stated that γ_1 only intersects once with M. Hence, a closed non-orientable manifold without boundary of dimension $n - 1$ cannot be smoothly embedded in R^n.

$\square$

Chapter 4

Discussion on Square Peg Problem's Proof

The square peg problem's proof was written by Andrew Lobb and Joshua Even Greene in 2020. The proof itself is one page long and very dense. The paper they published is six pages, and is slightly out of reach for a bachelor student with no background in symplectic geometry.

In this chapter, I give a proof in a more extended manner. First, I give a brief introduction to symplectic geometry. Namely giving definitions for symplectic vector spaces and such. Second, I go through the proof while giving some intuition (comparing it with the rectangle peg problem's proof and adding some explanations).

4.1 Brief Introduction to Symplectic Geometry

Symplectic geometry has useful properties that are needed to complete the proof of the inscribed square problem. In a symplectic space we have the notion of areas and it offers greater rigidity for controlling intersections. For example, on this space we can use Shechishin's Theorem.

Theorem 4.1.1. *(Shechishin's Theorem) A Klein bottle does not admit a smooth Lagrangian embedding in* $\mathbb{C}^2$.

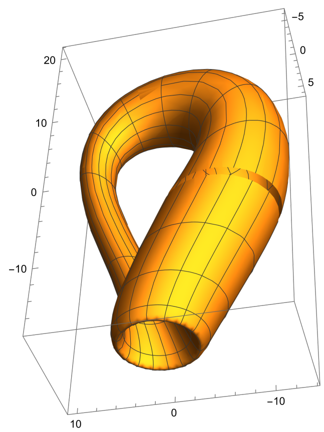


Figure 4.1: Klein Bottle embedded in $\mathbb{R}^3$ [4].

In Andrew and Joshua's proof we identify the Klein bottle with the set of unordered pairs of points on a curve with certain distances between them, midpoints and aspect ratios. It will become clear in Section 4.2 why we can do this.

We will work with a vector space V, over a field $\mathbb{F}$. Let the field $\mathbb{F}$ be $\mathbb{R}$, as this is what we are interested in for the inscribed square conjecture.

Definition 4.1.1. (Bilinear map) A bilinear map defined by $\Omega : V \times V \to \mathbb{R}$ is linear in each of its components.

The bilinear map Ω is a **skew-symmetric bilinear map** if $\Omega(u, v) = -\Omega(v, u)$ for all $u, v \in V$.

Now define a linear map $\tilde{\Omega} : V \to V^*$, where V^* is the dual space of V, by $\tilde{\Omega}(v)(u) = \Omega(u, v)$. Then the subspace $U = \{u \in V |\ \Omega(u, v) = 0\ \forall v \in V\}$ is the kernel of $\tilde{\Omega}$.

Definition 4.1.2. (Linear symplectic structure) A skew-symmetric bilinear map Ω is symplectic if $\tilde{\Omega}$ is bijective ie. $U = \{0\}$. The map Ω is then called a linear symplectic structure on V, and (V, Ω) is called a symplectic vector space.

Otherwise said, we can recognise a linear symplectic structure to be a skew-symmetric bilinear form that is **non-degenerate** (ie. $\Omega(u, v) = 0\ \forall v \in V$ then $u = 0$).

The subspaces of (V, Ω) can behave very differently. We have two important scenarios for which we give names: A subspace W of (V, Ω) is called a **symplectic subspace** if Ω restricted to W is nondegenerate. If the subspace W has that Ω restricted to W vanishes, we call it an **isotropic subspace** [8, p.4]. These subspaces can help us define symplectic manifolds and Lagrangian submanifolds.

Definition 4.1.3. (Differential 2-form) A differential 2-form, call it ω, on a manifold M is defined by

$$\omega_p : T_pM \times T_pM \to \mathbb{R}$$

this gives at each point $p \in M$ a **skew-symmetric bilinear form** on the tangent space T_pM, where ω varies smoothly on p.

This ω is closed if it satisfies the differential equation $d\omega = 0$, where d is the Rham differential (i.e., exterior derivative) [8, p.6]. We say that ω is a **symplectic 2-form** if ω is closed and ω_p is symplectic for all $p \in M$.

Now we can define what a symplectic manifold is.

Definition 4.1.4. (Symplectic manifold) A Symplectic manifold is a pair (M, ω) where M is a manifold and ω a symplectic 2-form.

For example, symplectic manifolds can be symplectic vector spaces such as $\mathbb{C}^2$ endowed with a symplectic structure ω. Symplectic manifolds are necessarily even-dimensional, because if ω is symplectic $\dim T_pM = \dim M$ must be even.

Now we can classify symplectic manifolds up to symplectomorphisms, as we have done in previous vector spaces where we clasified manifolds up to diffeomorphisms (up to bijective homeomorphisms of smooth manifolds).

Definition 4.1.5. (Symplectomorphism) A symplectomorphism is a diffeomorphism Φ from the symplectic manifold (X_1, ω_1) to the symplectic manifold (X_2, ω_2)

$$\Phi : X_1 \to X_2$$

that preserves the symplectic form. [8, p.7]

If both symplectic manifolds are of dimension 2, then a symplectictomorphism between them is a diffeomorphism that preserves the area.

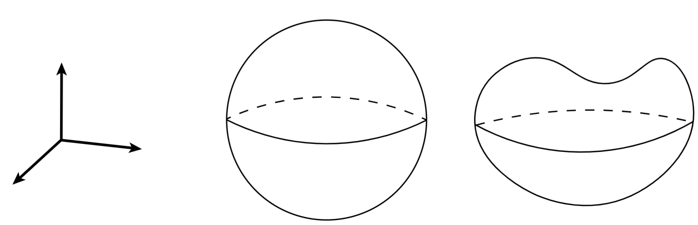


Figure 4.2: Two symplectic manifolds of dimension 2 with the same area. There is a symplectomorphism between them [3].

Lastly, we want to introduce the concept of a specific type of submanifold: a Lagrangian submanifold. For this we work with the isotropic subspaces defined above. Essentially, an isotropic subspace is a subspace of a sympletic vector space for which the symplectic form vanishes on the tangent space. Note: we think of **sumbanifolds** of M as a manifold L for which we can define a closed embedding $i : L \to M$.

Definition 4.1.6. (Lagrangian subspace) Lagragian subspace is defined in a sympletic vector space as a maximal isotropic subspace. We say its maximal when the subspace has half the dimension of the space.

Definition 4.1.7. (Maximal isotropic submanifold) A maximal isotropic submanifold L of a symplectic manifold (M^{2n}, ω), where ω is a symplectic 2-form, is a submanifold L of M for which ω restricts to zero on the tangent bundle of L. It is maximal when $\dim(TL) = \frac{1}{2}\dim(TM)$.

Definition 4.1.8. (Lagrangian submanifold) Lagrangian submanifolds of a sympletic manifold, is a submanifold which is a maximal isotropic submanifold.

Otherwise said, we define the inclusion map $i : L \to M$. Then L is Lagrangian if and only if $i^*\omega = 0$ and $\dim(TL) = \frac{1}{2}\dim(TM)$. An example of a Lagrangian submanifold is the Lagrangian torus in $\mathbb{C}^2$. This is a submanifold which is diffeomorphic to the torus $\mathbb{T} \cong \mathbb{S}^1 \times \mathbb{S}^1$, and it's also a Lagrangian submanifold of $\mathbb{C}^2$.

Lastly, a Lagrangian embeddings ι is an embedding from an n-dimensional sumbanifold L of a 2n-dimensional a symplectic manifold (M, ω)

$$\iota : L \to M$$

such that $\iota^*\omega = 0$. Thus, this embedding makes the submanifold L Lagrangian.

4.2 Square Peg Problem Demonstration

In this section we discuss the square peg problem's proof. We will essentially prove that in any Jordan curve we can find an inscribed rectangle of any aspect angle θ, therefore we can find a square. Moreover, the proof concludes a much stronger statement than the square peg problem.

The proof [2] was written by Joshua Evan Greene and Andrew Lobb during the Covid pandemic. The inspiration behind their solution was to recast the problem within the framework of symplectic geometry, which offers greater rigidity for controlling intersections. Then associating the concept of a rectangle being inscribed on a Jordan curve with the self intersection of a geometric-topological object. The self-intersection is then proved using the fact that a Klein bottle does not admit a smooth Lagrangian embedding in $\mathbb{C}^2$, ie. Schechishin's Theorem.

Square Peg Problem [Theorem 1.0.1]: *For every Jordan curve on the plane, there exists an inscribed rectangle on this curve.*

Proof. We form the symplectic vector space $\mathbb{C}^2$. We do this is by taking the Euclidean space $\mathbb{C}$ with coordinates $z = x + iy$ for $x, y \in \mathbb{R}$ and a copy of it with coordinates $\omega = re^{i\theta}$. This vector space is equipped with the standard symplectic bilinear form $\omega = dx \wedge dy + rdr \wedge d\theta$.

Now we define the map l by

$$l : \mathbb{C}^2 \to \mathbb{C}^2 : (z, \omega) \to (\frac{z+\omega}{2}, \frac{z-\omega}{2}),$$

this is a diffeomorphism and satisfies $l^*(\omega) = \frac{\omega}{2}$.

Let γ be the Jordan curve for which we are trying to find an inscribed square. Then, $\gamma \times \gamma$ is the set of ordered pairs of points on this curve. From the discussion in Section 2.1, we know that this set is homeomorphic to a smooth manifold, namely the torus $\mathbb{T}^2 = S^1 \times S^1$.

Theorem 4.2.1. *(Cartesian Product of Lagrangian submanifolds) Take two Lagrangian submanifolds L_1 and L_2 of a symplectic space. Now $L_1 \times L_2$ is a Lagrangian submanifold too, where $\times$ is the Cartesian product.*

Proof. Let (M_A, ω_A) and (M_B, ω_B) be symplectic manifolds, and let L_A and L_B be two respective Lagrangian submanifolds. Define $M = M_A \times M_B$ and $L = L_A \times L_B$. Thus, L is canonically a submanifold of M. Since the dimension of L_A is half the dimension of M_A, and similarly for L_B, then L is half the dimension of M.

Define $\pi_A : M \to M_A$ and $\pi_B : M \to M_B$ to be canonical projections. We can equip M with the following symplectic form $\omega = \pi_A^*\omega_A + \pi_B^*\omega_B$. Take $u \in TL$, $v \in TL$. Then $u = u_A + u_B$ and $v = v_A + v_B$ for $u_A, v_A \in TL_A$ and $u_B, v_B \in TL_B$. For all $u, v \in TL$ it follows that

$$\omega(u, v) = \omega(u_A + u_B, v_A + v_B)$$

$$= \omega_A(u_A, v_A) + \omega_B(u_B, v_B) = 0$$

because L_A and L_B are Lagrangian. So ω vanishes in the tangent space of L. Lastly,

$$d\omega = d(\pi_A^*\omega_A + \pi_B^*\omega_B)$$

$$= \pi_A^* d\omega_A + \pi_B^* d\omega_B = 0$$

by properties of the exterior derivative and because both ω_A and ω_B are closed. So ω is closed. Hence, L is a Lagrangian submanifold.

□

Notice that γ is Lagrangian in $\mathbb{C}^2$. It follows by Theorem 4.2.1 that $\gamma \times \gamma$ is Lagrangian in $\mathbb{C}^2$. Hence, $\gamma \times \gamma$ is a Lagrangian torus in $\mathbb{C}^2$.

Since l is a diffeomorphism, then $L = l(\gamma \times \gamma)$ is a smooth manifold, ei. a torus. For any vectors $u, v \in T_p(\gamma \times \gamma)$ we have that $u = dlu'$ and $v = dlv'$ for $u', v' \in T_{l^{-1}(p)}L$. It follows that for all $p \in \gamma \times \gamma$,

$$\omega(u, v) = \omega(dlu', dl'v)$$
$$= l^*\omega(u', v') = \frac{1}{2}\omega(u', v')$$

Since $\gamma \times \gamma$ is Lagrangian, then $\omega(u, v) = 0$. It follows that $\frac{1}{2}\omega(u', v') = 0$. Hence L is a submanifold equipped with a closed symplectic 2-form which vanishes on its tangent bundle, therefore Lagrangian. So L is a smooth Lagrangian torus in $\mathbb{C}^2$.

For any $\phi \in \mathbb{R}$ (the fact that this is arbitrary will make possible the existence of a square), the map

$$R_\phi : \mathbb{C}^2 \to \mathbb{C}^2 : (z, r, \theta) \to (z, r, \theta + \phi)$$

is a symplectomorphism. For a fixed θ such that $0 < \theta \geq \pi/2$, $L_\theta = R_\phi(L)$ is another smooth Lagrangian torus. This is because L_ϕ is the torus L rotated by an angle ϕ in $\mathbb{C}^2$, so its properties are preserved.

Now define the map g by

$$g : \mathbb{C}^2 \to \mathbb{C}^2 : (z, r, \theta) \to (z, r/\sqrt{2}, 2\theta).$$

Away from the boundary $\mathbb{C} \times \{0\}$, the map g is smooth and satisfies $g*(\omega) = \omega$. Now

$$g \circ l(z, \omega) = g(\frac{z + \omega}{2}, \frac{z - \omega}{2})$$

and we can write the complex number $\frac{z-\omega}{2} = re^{i\theta}$, so

$$g \circ l(z, \omega) = (\frac{z + \omega}{2}, \frac{r}{\sqrt{2}}e^{i2\theta}).$$

So we have that $g \circ l(z, \omega) = g \circ l(z', \omega')$ iff $(\frac{z+\omega}{2}, \frac{r}{\sqrt{2}}e^{i2\theta}) = (\frac{z'+\omega'}{2}, \frac{r'}{\sqrt{2}}e^{i2\theta'})$ where now $\frac{z'-\omega'}{2} = r'e^{i\theta'}$. But this is only possible when $\{z, \omega\} = \{z', \omega'\}$. That is why g makes the set of ordered pairs unordered. Thus, $M = g(l(\gamma \times \gamma))$ will be the set of unordered pairs of points of γ. As discussed in Section 2.1, $M = g(l(\gamma \times \gamma))$ is homeomorphic to the Möbius strip.

For all $u, v \in T_pM$ where $u = dgu'$ and $v = dgv'$ with $u', v' \in T_{g^{-1}(p)}L$, we have that

$$\omega(u, v) = \omega(dgu', dgv')$$
$$= g^*\omega(u', v') = \omega(u', v') = 0.$$

It vanishes because L is Lagrangian. Thus M is a smooth Lagrangian Möbius strip away from $\mathbb{C} \times \{0\}$. Similarly, $M_\phi = R_\phi(M)$ is also a smooth Lagrangian

Möbius strip away from $\mathbb{C} \times \{0\}$, as R_ϕ is a symplectomorphism.

The map R_π preserves each of L and L_ϕ. That is, for every point in $p \in L$ we have that $R_\pi(p) \in L$ (similarly for L_ϕ). Also, R_π fixes $\gamma \times \{0\}$.

Definition 4.2.1. (Clean intersection) Two manifolds M_1 and M_2 intersect cleanly if $M_1 \cap M_2$ is a smooth manifold such that

$$T_p(M_1 \cap M_2) = T_pM_1 \cap T_pM_2$$

$\forall p \in M_1 \cap M_2$.

In our case, it follows that $L \cap L_\phi = \gamma \times \{0\}$. This intersection is a closed loop that lies on a plane. So T_pL will be the same as T_pL_ϕ at $\gamma \times \{0\}$. Everywhere else, their tangent spaces are different as its the only 1-dimensional manifold where the manifolds intersect. Thus, $T_p(\gamma \times \{0\}) = T_pL \cap T_pL_\phi$ $\forall p \in \gamma \times \{0\}$. Hence, L and L_ϕ intersect cleanly at $\gamma \times \{0\}$.

Remark. A Lagrangian smoothing is understood in this paper to be a procedure that transforms a geometric topological object into a smooth Lagrangian manifold.

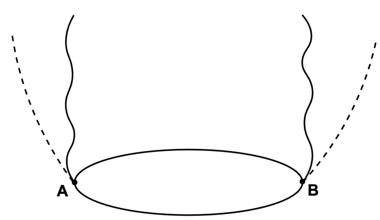


Figure 4.3: Intuitive drawing of $L \cup L_\phi$ in $\mathbb{C}^2$ that coincide at $\gamma \times \{0\}$ [3].

In Figure 4.3 we can see that at points A and B $L \cup L_\phi$ is not smooth or Lagrangian, therefore a Lagrangian smoothing will make this object a Lagrangian submanifold. Unfortunately, we do not go into detail about how we do this, although it can be further understood in Joshua and Andrew's proof [2, prop 1.1].

We perform a Lagrangian smoothing of $L \cup L_\phi$ along $\gamma \times \{0\}$ following [2, Prop.1.1]. We end up with a smoothly immersed Lagrangian torus in $\mathbb{C}^2$ that coincides with $L \cup L_\phi$ away from the neighbourhood of $\gamma \times \{0\}$. This manifold is disjoint from $\mathbb{C} \times \{0\}$.

Definition 4.2.2. (Fixed point free involution) A fixed-point free involution on a finite set S is defined as a bijection $I : S \to S$ such that $\forall p \in S$, $I(I(p)) = p$ and $I(p) = p$.

Here R_π acts as a fixed point free involution. The image of this Lagrangian smoothing under g is a smoothly immersed Klein bottle in $\mathbb{C}^2$. This coincides with $M \cup M_\phi$ outside the neighbourhood of $\gamma \times \{0\}$.

By Shechishin's Theorem (Theorem 4.1.1), we cannot have a smooth Lagrangian embedding of the Klein bottle in $\mathbb{C}^2$. Hence, M and M_ϕ intersect at a point away from $\gamma \times \{0\}$. This is a point in $\mathbb{C}^2$ which represents two pairs of points on the curve gamma that form a rectangle of aspect ratio ϕ. Since $\phi \in (0, \pi/2]$ was arbitrary we have that we can find a square whose vertices lie on γ. $\square$

We can not only find a square inscribed on γ, but also a rectangle of any aspect ratio. More specifically, since M and M_ϕ intersect at a point away from $\gamma \times \{0\}$, so do L and L_ϕ. Denote the point they intersect $(z, re^{i(\theta+\phi)})$. It follows that the four points $z \pm re^{i\theta}$ and $z \pm re^{i(\theta+\phi)}$ all lie on the Jordan curve γ, and they form a rectangle of any aspect ratio ϕ.

Chapter 5

Conclusion

In this project we covered both the Rectangle Peg Problem and the square peg problem. First, we discussed the simplified version: the Rectangle Peg Problem. The discussion of its proof aimed to give some intuition to understand the more complicated proof of the inscribed square problem. It involves associating the inscribed rectangle on a Jordan curve, to the self intersection of a topological structure. Thus, we define a map from the set of unordered pairs of points on any Jordan curve to $\mathbb{R}^3$. For the sake of contradiction, we assume this embedding is smooth. The contradiction arises because of Theorem 2.2.1 on embedings of non-orientable manifolds. This one is discussed extensively in chapter 3 where one can find its proof.

Lastly, we discus the Square Peg Problem where the proof has many similarities with the one just explained. Here, we define a the symplectic space $\mathbb{C}^2$ and define three maps that will essentially behave like the one's above. We are interested in the images of the compositions of these maps. This images gives rise to two Lagrangian Möbius that intersect at their boundary. If we glue these two together by their boundary, we get a Lagrangian Klein bottle embedded smoothly in $\mathbb{C}^2$. Here we use Shechishin's Theorem and arrive at the fact that we cannot make a smooth embedding of this object to the space $\mathbb{C}^2$. Therefore it must intersects at a point. This intersection point generates a rectangle of an arbitrary aspect ratio on an arbitrary Jordan curve. This results is stronger than the conjecture of the Rectangle Peg Problem.

In the future it would be interesting to study the proof of Shechishin's Theorem. Additionally, this project remains without a discussion on the Lagrangian smoothing that permitted the creation of a smooth Lagrangian submanifold. Another of the many ideas one can proceed with this project is by studying intersections of transversal manifolds of complementary dimension which always make a zero dimensional manifold. Moreover, this project gives rise to many interesting questions that open doors for further understanding on mathematics.

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