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Assignment ID - HomeWork 4

(1a)

Ques (1)

Sol:- Given two random variables X & Y and their joint distribution $P(X, Y)$:

	$X=2$	$X=4$	$X=6$
$Y=$ Red (2)	0.05	0.15	0.07
$Y=$ Blue (4)	0.2	0.06	0.1
$Y=$ Green (6)	0.05	0.15	0.17

(a) $H(X) = ?$

Marginal Entropy $H(X) = -\sum_{x} P(x) \log_2 P(x)$

→ for $x=2$ $\quad P(x=2, y=2) + P(x=2, y=4) + P(x=2, y=6)$

$= 0.05 + 0.2 + 0.05 = 0.3$ Ans

→ for $x=4$

$= 0.15 + 0.06 + 0.15 = 0.36$ Ans

→ for $x=6$

$= 0.07 + 0.1 + 0.17 = 0.34$ Ans

$H(X) = -\left((0.3 \times \log_2(0.3)) + (0.36 \times \log_2(0.36)) + (0.34 \times \log_2(0.34))\right)$

$= -\left((0.3 \times -1.73696) + (0.36 \times -1.4739) + (0.34 \times -1.55639)\right)$

$= 1.5808646$ Ans

(b) $H(Y) = -\sum_y P(y) \log_2 P(y)$

→ for $Y=2$ → $P(x=2, y=2) + P(x=4, y=2) + P(x=6, y=2)$
$= 0.05 + 0.15 + 0.07 = 0.27$ Ans

→ $y=4$ → $0.2 + 0.06 + 0.1 = 0.36$ Ans

→ $Y=6$ → $0.05 + 0.15 + 0.17 = 0.37$ Ans

$H(Y) = -\left((0.27) \log_2 (0.27) + (0.36 \times \log_2 0.36) + (0.37 \times \log_2 0.37)\right)$

$= 1.57136582$ Ans

(c) $D(X||Y) = \sum_x P(x) \log_2 \frac{p(x)}{p(y)}$ i.e. Relative Entropy

$= \left(0.3 \times \log_2 \left(\frac{0.3}{0.27}\right) + \left(0.36 \times \log_2 \left(\frac{0.36}{0.36}\right) + (0.34) \log_2 \left(\frac{0.34}{0.37}\right)\right)\right)$

$= \cancel{(0.3 \times 0.15} = \underline{0.0041242}$ Ans

(d) $D(Y||X) = \sum_y P(y) \log_2 \left(\frac{P(y)}{P(x)}\right)$

$= \left(0.27 \times \log_2 \left(\frac{0.27}{0.3}\right) + 0.36 \times \log_2 (1) + 0.37 \times \log_2 \left(\frac{0.37}{0.34}\right)\right)$

$= \underline{0.0040956}$ Ans

(e) $H(X|Y) = H(X,Y) - H(Y)$

$H(X,Y) = -\sum\sum P(x,y) \log p(x,y)$

$= -\left(2 \times 0.05 \log 0.05 + 2 \times 0.15 \times \log 0.15 + 0.07 \log 0.07\right.$
$+ 0.2 \times \log (0.2) + 0.06 \log (0.06) + 0.1 \log (0.1)$
$\left.+ 0.17 \log (0.17)\right) = \underline{2.996536}$ Ans

$H(X|Y) = 2.996536 - 1.57136582$
$= 1.42517018$ Ans

(f) $H(Y|X) = H(Y,X) - H(X)$

$= \cancel{\sum\sum p(x,y)}$

$H(Y,X) = -\sum\sum P(Y,X) \log(P(Y,X))$

But, since we calculated above

$H(Y|X) = H(X,Y) - H(X)$

$= 2.996536 - 1.5808646$
$= 1.4156714$ Ans.

(g) $H(X,Y)$

already calculated above,

$H(X,Y) = -\sum\sum P(X,Y) \log(P(X,Y))$

$= -(2\times 0.05 \log 0.05 + 2\times 0.15 \log(0.15) + 0.07 \log(0.07)$
$+ 0.2 \log(0.2) + 0.06 \log(0.06) + 0.1 \log(0.1)$
$+ 0.17 \log(0.17))$

$= 2.996536$ Ans

(h) $H(Y) - H(Y|X)$

as calculated above,

Ans $= 1.57136582 - 1.4156714$
$= 0.15569442$ Ans

(i) $I(X,Y) = H(X) - H(X|Y)$

$= H(X) + H(Y) - H(X,Y)$

$= 1.58083646 + 1.57136582 - 2.996536$

$= 0.15569442$ Ans

(j) If X is the number of wheels on a vehicle and Y is the color of the vehicle. what does $I(X;Y)$ tell us?

Y is the color of the vehicle.
X is number of wheels

$I(X;Y)$ measures mutual independence between X & Y.

since X & Y are independent i.e. measure of wheel & color of vehicle are unrelated,

So, $I(X;Y) = 0$ Ans

Ques ② Using a standard die, output is - 5, 4, 2. Compute the amount of information in bits for this event.

Sol:-

when events are independent

$$I_A + I_B = \log\left(\frac{1}{P_A}\right) + \log\left(\frac{1}{P_B}\right)$$

$$P(5) = P(4) = P(2) = \frac{1}{6}$$

$$\therefore \quad I = I_A + I_B + I_C$$

$$= \log\left(\frac{1}{1/6}\right) + \log\left(\frac{1}{1/6}\right) + \log\left(\frac{1}{1/6}\right)$$

$$= 7.75489 \text{ Ans}$$

Ques ③ Consider two die, one is balanced and other is such that $p(1) = 0.25$, $P(5) = 0.35$, rest equal. Compute entropy of event by both dice.

Sol:-

→ For the balanced die, entropy is :-

$$H = -\sum_{i=1}^{n} p(i) \log_2(P_i)$$

$$p(\text{event}) = 1/6$$

$$H = -\sum_{i=1}^{6} \frac{1}{6} \log_2\left(\frac{1}{6}\right) = -\log_2\left(\frac{1}{6}\right)$$

$$= 2.58496 \text{ Ans}$$

→ For unbalanced die, entropy is :-

$$H = -\sum_{i=1}^{n} P(i) \log_2 P(i)$$

$P(1) = 0.25$, $P(5) = 0.35$

$$P(2,3,4,6) = \frac{1-(0.25+0.35)}{4} = \frac{1-0.6}{4} = \frac{0.4}{4} = 0.1$$

$$H = -\sum_{i=1}^{6} P(i) \log_2 P(i)$$

$$H = -\left((0.25 \log_2(0.25)) + 0.35 \log_2(0.35) + 4 \times 0.1 \log_2(0.1)\right)$$

$$= +2.35887$$ Ans

Ques (4) The entropy of a probability distribution is 3.6 bits. ......

Sol:-

Entropy of prob. distribution = 3.6

Given is 1 hartley = 3.322
i.e. $\log_2 10$ bits = 3.322 bits

→ Yes, the given information is enough to calculate the hartleys the entropy of the given distribution.

$$H = \frac{3.6}{3.322} = 1.0837 \text{ hartleys}$$

In Nats,

1 nats = 1.4476

$$H = \frac{3.6}{1.4476} = 2.4869 \text{ nats}$$ Ans