Ravi Rajpurohit
UTA ID- 1002079916
Homework 1

Ques 1. Parameter $= q$
Geometric Distribution $\Rightarrow$ $P_q(K) = (1-q)^{K-1} q$
datapoints are independent
(a) Dataset $= \{K_1, \ldots\ldots, K_n\}$
Soln Max. likelihood for the distributions
$$L(q/K_1 \ldots K_n) = P(K_1 \ldots K_n/q)$$
since data is independent
$$L(q/K_1 \ldots K_n) = P(K_1/q) * P(K_2/q) \ldots\ldots \times P(K_n/q)$$
$$= \prod_{i=1}^{n} P(K_i/q)$$
$$= \prod_{i=1}^{n} (1-q)^{K_i - 1} q$$

adding log for log likelihood
$$\log(L(q/K_1 \ldots K_n)) = \log\left(\prod_{i=1}^{n} (1-q)^{K_i-1} q\right)$$
$$= \sum_{i=1}^{n} \log\left((1-q)^{K_i-1} q\right)$$
$$= \sum_{i=1}^{n} \log(1-q)(K_i - n) + n \log q$$

Finding MLE
$$\frac{d}{dq} \log(L(q/K_1 \ldots K_n)) = \frac{d}{dq}\left(n \log q + \sum_{i=1}^{n} K_i - n \log(1-q)\right) = 0$$
$$= \frac{n}{q} - \sum_{i=1}^{n} \frac{K_i - n}{(1-q)} = 0$$

$$\frac{n}{q} - \frac{\sum_{i=1}^{n} (K_i - n)}{1-q} = 0$$

$$n(1-q) = \left(\sum_{i=1}^{n} K_i - n\right) = 0 \quad \Big| \quad q = \frac{n}{\sum_{i=1}^{n} K_i}$$

(b) Dataset $D = [2,2,2,4,1,1,2,2,2,3,2,2,3]$

$n = 13$

$\sum_{i=1}^{n} k_i = 28$

$\Rightarrow q = \frac{n}{\sum_{1}^{n} k_i} = \frac{13}{28} = 0.464$

Hence, $q = 0.464$ for given data.

(c) Derivation for optimization to find MAP

$P_{\alpha,\beta}(q) = \frac{q^{\alpha-1}(1-q)^{\beta-1}}{B(\alpha,\beta)}$

$q = \text{argmax}\left(\frac{P(D/q) \times P(q)}{P(D)}\right)$

$= \text{argmax}\left(P(D/q) \times p(q)\right)$

$\Rightarrow$ we know already, $P(D/q) = \prod_{i=1}^{n} (1-q)^{k_i-1} \times q$

$P(D/q) \times p(q) = \left(\prod_{i=1}^{n}(1-q)^{k_i-1} \times q\right) \times \left(\frac{q^{\alpha-1}(1-q)^{\beta-1}}{B(\alpha,\beta)}\right)$

$\Rightarrow$ log on both sides

$\log(P(D/q) \times p(q)) = n \log q + \left(\sum_{i=1}^{n} k_i - n\right) \times \log(1-q)$
$+ (\alpha-1)\log q$
$+ \beta - 1 \log(1-q) - \log(B(\alpha,\beta))$

$= \text{argmax}\left(n \log q + \sum_{i=1}^{n} k_i - n\right) \times \log(1-q) + (\alpha-1)\log q$
$+ (\beta-1)\log(1-q) - \log(B(\alpha,\beta))\Big)$

→ derivation :-

$$\frac{d}{dq}\left(n\log q + \left(\sum_{i=1}^{n} k_i - n\right) \times \log(1-q) + (\alpha-1)\log q + (\beta-1)\log(1-q) - \log(B(\alpha,\beta))\right) = 0$$

$$\Rightarrow \frac{n}{q} - \sum_{i=1}^{n} \frac{k_i - n}{1-q} + \frac{\alpha-1}{q} - \frac{\beta-1}{1-q} = 0$$

$$\Rightarrow \frac{\alpha-1+n}{q} - \frac{\sum_{i=1}^{n} k_i - n + \beta - 1}{1-q} = 0$$

$$(1-q)(\alpha-1+n) - q\left(\sum_{i=1}^{n} k_i - n\right) - q\beta - q = 0$$

$$\alpha - 1 + n - q\alpha + q + nq - q\left(\sum_{j=1}^{n} k_j - n\right) - q\beta - q = 0$$

$$\alpha - 1 + n = q\alpha + q\beta + q\sum_{i=1}^{n} k_i$$

$$\frac{\alpha-1+n}{\alpha+\beta+\sum_{i=1}^{n} k_i} = q$$

→ optimize for q, $\frac{\alpha-1+n}{\alpha+\beta+\sum_{i=1}^{n} k_i}$

given $\alpha = 4.66$, $\beta = 4.66$, $n = 13$, $\sum_{i=1}^{n} = 28$

$$\Rightarrow q = \frac{4.46 - 1 + 13}{4.46 + 4.46 + 28} = \frac{16.46}{36.92} = 0.446 \text{ Ans}$$

using MAP, $q = 0.446$ Soln