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Unit 6

Line Integrals & Green's Theorem

Let 'C' be a simple curve, whose parametric representation is written as

$$\left.\begin{aligned} x &= x(t) \\ y &= y(t) \\ z &= z(t)\ ; \quad a \le t \le b \end{aligned}\right\} \quad \text{— (1)}$$

Therefore, the position vector of a point on the curve 'C' can be written as

$$r(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \quad \text{— (2)}$$

Initial point of curve: $(x(a), y(a), z(a))$

final point: $(x(b), y(b), z(b))$

When we have closed curve,

$x(a) = x(b)$ similarly ...

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Line Integral w.r.t arc length:

Let C be a simple curve, whose parametric rep. is given in (1) & (2).

Let $f(x,y,z)$ be continuous on C, then we define, the line integral of f over C w.r.t arc length (s) by

$$\int_C f(x,y,z)\, ds$$

$$s(t) = \int_a^t \sqrt{(x'(\xi))^2 + (y'(\xi))^2 + (z'(\xi))^2}\; d\xi$$

$$= \int_a^t |r'(\xi)|\, d\xi$$

$$\frac{d\,s(t)}{dt} = |r'(t)|$$

$$\Rightarrow ds = |r'(t)|dt$$

$$\therefore \int_C f(x,y,z)\, ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\; dt$$

$$= \int_a^b f\big(x(t),\; y(t),\; z(t)\big)$$

→ Evaluate $\int_C x^2 y \, ds$

$C: x = 3\cos t$

$y = 3\sin t, \quad 0 \le t \le \pi/2$

$$= \int_0^{\pi/2} (3\cos t)^2 (3\sin t) \sqrt{(-3\sin t)^2 + (3\cos t)^2} \, dt.$$

$$= 81 \int_0^{\pi/2} \sin t \cos^2 t \, dt$$

Let $\cos t = u$

$-\sin t \, dt = du$

$t = 0 \Rightarrow u = 1$

$t = \pi/2, u = 0$

$$= -81 \int_1^0 u^2 \, du$$

$$= \cancel{81}^{27} \int_0^1 \left(\frac{u^3}{\cancel{3}}\right) = 27 \, u^3 \Big|_0^1$$

$= 27$ //

⟹ Evaluate $\int_C (x^2 + yz) \, ds$

$x = 4y$

$z = 3$

from $(2, 1/2, 3)$ to $(4, 1, 3)$

Let $x = t$ $\quad (\because 2 \le x \le 4)$

$\Rightarrow y = t/4 \quad \Rightarrow 2 \le t \le 4$ //

$z = 3$

$$= \int_2^4 \left(t^2 + \frac{3t}{4}\right) \sqrt{1^2 + \frac{1}{16} + 0}\; dt$$

$$= \frac{\sqrt{17}}{4} \int_2^4 t^2 + \frac{3t}{4} \; dt$$

$$= \frac{\sqrt{17}}{4} \left(\frac{t^3}{3} + \frac{3t^2}{8} \right) \Big|_2^4$$

$$= \frac{\sqrt{17}}{4} \left[\frac{64}{3} - \frac{8}{3} + 6 - \frac{3}{2} \right]$$

$$= \frac{\sqrt{17}}{4} \left[\frac{128 - 16 + 36 - 9}{6} \right]$$

$$= \frac{139\sqrt{17}}{24}$$

128
36
164
25
139
6

Line integrals of vector field:

Let C be a smooth curve whose parametric rep. are

$x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$

Let

$$\vec{v}(x,y,z) = v_1(x,y,z)\hat{i} + v_2(x,y,z)\hat{j} + v_3(x,y,z)\hat{k}$$

Let $\vec{v}$ be a vector field, cont. on C.

Then the line integral of $\vec{v}$ over C

is $\boxed{\int_C \vec{v} \cdot d\vec{r} = v_1\,dx + v_2\,dy + v_3\,dz}$

$$= \int_a^b \vec{v}(x(t), y(t), z(t)) \cdot \frac{dr}{dt}\,dt$$

If $\vec{v} = v_1(x, y, z)\,\hat{i}$

$$\rightarrow \int_C \vec{v} \cdot d\vec{r} = \int_C v_1\,dx = \int_a^b v_1(x(t), y(t), z(t))\,\frac{dx}{dt}\,dt$$

$\Rightarrow$ If C is piece wise smooth containing the arcs $C_1, C_2, C_3 \dots C_n$ then this integral is

$$\int_C \vec{v} \cdot d\vec{r} = \int_{C_1} \vec{v} \cdot d\vec{r} + \int_{C_2} \vec{v} \cdot d\vec{r} + \dots \int_{C_n} \vec{v} \cdot d\vec{r}$$

$\longrightarrow$ Evaluate line integral of $v = xy\hat{i} + y^2\hat{j} + e^2\hat{k}$ over C, whose parametric rep. is given by

$x = t^2$

$y = 2t$

$z = t$, $0 \le t \le 1$

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Sol: $= \int_0^1 (2t^3\hat{i} + 4t^2\hat{j} + e^t\hat{k}) \cdot (2t\hat{i} + 2\hat{j} + \hat{k})\, dt$

$\therefore \frac{dr}{dt} = 2t\hat{i} + 2\hat{j} + \hat{k}$

$\frac{dr}{dt} \to \boxed{\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}}$

$= \int_0^1 (4t^4 + 8t^2 + e^t)\, dt$

$= \left[\frac{4}{5}(t^5) + \frac{8}{3}t^3 + e^t\right]_0^1$

$= \frac{4}{5} + \frac{8}{3} + e - 1$

$= \frac{37}{15} + e$ //

$\Rightarrow$ Evaluate $\int_C (x^2 + yz)\, dz$

$C : x = t$

$y = t^2$

$z = 3t \quad , \quad 1 \le t \le 2 \qquad dz = 3dt$ //

$= \int_1^2 (t^2 + 3t^3)(3dt)$

$= \int_1^2 (3t^2 + 9t^3)\, dt = \left(t^3 + \frac{9}{4}t^4\right)\Big|_1^2$

$= \left(8 + \frac{9}{\cancel{4}}(\overset{4}{\cancel{16}})\right) - \left(1 + \frac{9}{4}\right) = 36 + 8 - \left(\frac{13}{4}\right)$

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Parametric form if two point given:

(x_1, y_1, z_1) to (x_2, y_2, z_2)

$$\Rightarrow \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

If a and b are the two points then p. rep. of line joining them is

$$\boxed{r = a + t(b - a)}$$

$\int \vec{v} \cdot d\vec{r}$

$v(t) \cdot \frac{dr}{dt} dt$

$\Rightarrow$ Evaluate the line integral of $\vec{v} = x^2\hat{i} - 2y\hat{j} + z^2\hat{k}$ over a path b/w $(-1, 2, 3)$ to $(2, 3, 5)$

$2 \le 2 + t \le 3$

$0 \le t \le 1$

Sol:
$x = -1 + 3t$

$y = 2 + t$

$z = 3 + 2t \quad , \quad 0 \le t \le 1$

$$= \int_0^1 \left[(3t-1)^2\hat{i} + 2(2+t)\hat{j} + (3+2t)^2\hat{k}\right] \cdot \left[3\hat{i} + \hat{j} + 2\hat{k}\right] dt$$

$$= \int_0^1 \left[3(3t-1)^2 + 2(2+t) + 2(3+2t)^2\right] dt$$

$$= \frac{(3t-1)^3}{3} - 4t - t^2 + \frac{(3+2t)^3}{3} \Big|_0^1$$

$$= \frac{(3-1)^3}{3} - 4 - 1 + \frac{(3+2)^3}{3}$$

$$= \frac{8}{3} - 5 + \frac{125}{3} = \frac{8 - 15 + 125}{3} = \frac{118}{3}$$

Evaluate $\int_C (x^2 - y^2)\, ds$, where C is closed curve given by

$x = 3\cos t$

$y = 3\sin t$, $0 \le t \le 2\pi$

$$= \int_0^{2\pi} (9\cos^2 t - 9\sin^2 t)\sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= 27\int_0^{2\pi} (\cos^2 t - \sin^2 t)\, dt$$

$$= 27\int_0^{2\pi} \cos 2t\, dt$$

$$= \frac{27}{2} \sin 2t \Big|_0^{2\pi}$$

$$= 0$$

$\Rightarrow \int_C f(x,y)\,dx$ & $\int_C f(x,y)\,dy$

$f(x,y) = x^2 + 2x^2y + 3y^2$

$x = t$

$y = 2t^2$

$0 \le t \le 2$

$\Rightarrow \int_0^2 (t^2 + 4t^4 + 12t^4)\,dt$

$= \frac{t^3}{3} + 16\frac{t^5}{5} \Big|_0^2 = \frac{8}{3} + \frac{16 \times 32}{5}$

Line integral of Scalar fields:

Let 'C' be a Smooth curve, $f(x,y,z)$ $g(x,y,z)$, $h(x,y,z)$ be scalar fields which are continuous at points over C.

Then we define as

$$\int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$$

$$= \int_a^b \left[f(x(t),y(t),z(t))\frac{dx}{dt} + g(x(t),y(t),z(t))\frac{dy}{dt} + h(x(t),y(t),z(t))\frac{dz}{dt} \right] dt \quad \text{--- (1)}$$

The line integral doesnot contain any vector field but involves 3 scalar fields.

if $\vec{v} = f\hat{i} + g\hat{j} + h\hat{k}$ Then (1) is Same as $\int_C \vec{v}\cdot d\vec{r}$

If C is a closed curve, then the line integral is defined as

$$\oint_C \vec{v}\cdot\vec{dr}.$$

Evaluate $\int_C (x+y)\,dx - x^2 dy + (y+z)\,dz$

$C: \; x^2 = 4y \qquad 0 \le x \le 2$

$z = x$

Let $x \Rightarrow t$

$\Rightarrow y = \frac{t^2}{4}$

$z = t \; ; \; 0 \le t \le 2$

$$\int_0^2 \left[\left(t + \frac{t^2}{4}\right) - t^2\left(\frac{t}{2}\right) + \left(\frac{t^2}{4} + t\right)\right] dt$$

$$= \int_0^2 \left(2t + \frac{\cancel{2}t^2}{\cancel{4}_{\,2}} - \frac{t^3}{2}\right) dt$$

$$= \left[t^2 + \frac{t^3}{6} - \frac{t^4}{8}\right]_0^2$$

$$= 4 + \frac{4}{3} - 2 = 2 + \frac{4}{3} = 10/3$$

Application of line integrals:

work done by a force:

Let $F = v = v_1(x,y,z)\hat{i} + v_2(x,y,z)\hat{j} + v_3(x,y,z)\hat{k}$

A variable force acting on a particle which moves along a curve C.

Then the work done by force F in displacing the particle from point P to q along the curve 'C' is given by

$$W = \int_P^q \vec{F} \cdot d\vec{r} = \int_{C^*} \vec{F} \cdot d\vec{r}$$

where C^* is part of C whose initial and terminal points are P and q.

Suppose, F is conservative vector field.

Then $\vec{F} = \text{grad} f$, F is called gradient field and f is called scalar potential field. $\rightarrow W = \int_{C^*} (\text{grad} f) \, d\vec{r}$

$$= \int_{C^*} (\text{grad } f) \cdot d\vec{x}$$

$$= \int_{C^*} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \int_{C^*} df = \int_P^q df = f(x,y,z)\Big|_P^q$$

<u>Note</u>: work done depends only on initial and terminal points of curve C^*

ie., work done is independent of path of integration.

* The <u>units of work</u> depend on the <u>units of magnitude</u> <u>of force and on units of</u> distance.

→ Find the work done by the force $F = -xy\hat{i} + y^2\hat{j} + z\hat{k}$ in moving a particle over a circular path $x^2+y^2=4$, $z=0$, from $(2,0,0)$ to $(0,2,0)$

Sol: $x = 2\cos t$ $\quad 2 \le x \le 0$

$y = 2\sin t$ $\quad 2 \le 2\cos t \le 0$

$z = 0$ $\quad 1 \le \cos t \le 0$

$0 \le t \le \pi/2$

$$W = \int_C \vec{F} \cdot ds = \int_0^{\pi/2} \left(-4\sin t\cos t\,\hat{i} + 4\sin^2 t\,\hat{j}\right) \cdot \left(-2\sin t\,\hat{i} + 2\cos t\,\hat{j}\right) dt$$

$$= \int_0^{\pi/2} \left(8\sin^2 t\cos t + 8\sin^2 t\cos t\right) dt$$

$$= 16\int_0^{\pi/2} \sin^2 t\cos t\, dt$$

$\sin t = u$

$\cos t\, dt = du$

$$= 16\int_0^1 u^2\, du$$

$t = 0 \Rightarrow u = 0$

$t = \pi/2 \Rightarrow u = 1$

$$= 16 \frac{u^3}{3} \Big|_0^1$$

$$= \frac{16}{3}$$

⇒ Find the work done by the force

$F = (2x+y)\hat{i} + (4y-x)\hat{j}$ along the curve

C taken once around the triangle, with

vertices at $(2,2), (4,2), (4,4)$.

Sol: $W = \int_C \vec{F} \cdot \vec{dr}$

$$= \int_C \left[(2x+y)\hat{i} + (4y-x)\hat{j}\right] \cdot \left[dx\,\hat{i} + dy\,\hat{j}\right]$$

$$= \int_C \underbrace{(2x+y)}_{f}\,dx + \underbrace{(4y-x)}_{g}\,dy$$

$$= \iint_R (-1-1)\,dx\,dy \quad \left(\because \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

$$= -2 \iint_R dx\,dy$$

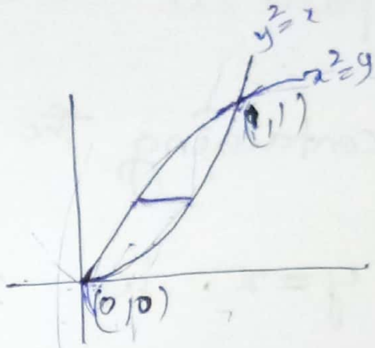
$$= -2 \times \frac{1}{\cancel{2}} \times \cancel{2} \times 2$$

$$= -4$$

→ Evaluate

$\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$, If the session is in 1st quad, bounded by curves

$y^2 = x$, $x^2 = y$.



Sol: $= \iint_R (2-2y)\,dx\,dy$

$0 \le x \le 1$

$x^2 \le y \le \sqrt{x}$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-2y)\,dy\,dx$$

$$= \int_0^1 (2y - y^2)\Big|_{x^2}^{\sqrt{x}}\,dx$$

$$= \int_0^1 (2\sqrt{x} - x) - (2x^2 - x^4)\,dx$$

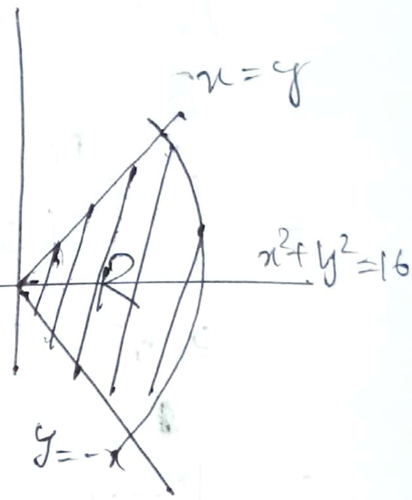
$$= \int_0^1 (2\sqrt{x} - 2x^2 - x + x^4)\,dx$$

$$= \left[2 \times \frac{2}{3} x^{3/2} - \frac{2}{3}x^3 - \frac{x^2}{2} + \frac{x^5}{5}\right]_0^1$$

$=$

→ Find the work done by force $\vec{F} = (x^2 - y^3)\hat{i} + (x+y)\hat{j}$ in moving a particle along a closed path C containing the curves $x+y=0$, $x^2+y^2=16$, $y=x$, in 1^{st} & 4^{th} quad.

$$W = \oint_C \vec{F} \cdot d\vec{r}$$



$$= \oint_C (x^3 - y^3)\,dx + (x+y)\,dy$$

$$= \iint_R (1 + 3y^2)\,dx\,dy \quad \text{(Green's theorm)}$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= \int_{-\pi/4}^{\pi/4} \int_0^4 (r + 3r^3 \sin^2\theta)\,dr\,d\theta$$

$0 \le r \le 4$

$\frac{-\pi}{4} \le \theta \le \pi/4$

$$= \int_{-\pi/4}^{\pi/4} \frac{r^2}{2} + \frac{3}{4} r^4 \sin^2\theta \Big|_0^4$$

$dx\,dy = |J|\,dr\,d\theta$

$= r\,dr\,d\theta$

$$= \int_{-\pi/4}^{\pi/4} 8 + 192 \sin^2\theta \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} 8 + 96(1 - \cos 2\theta) \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} (104 - 96 \cos 2\theta) \, d\theta$$

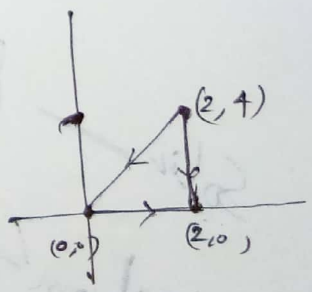
$$= (104\theta - 48 \sin 2\theta) \Big|_{-\pi/4}^{\pi/4}$$

$$= 104(\pi/4 + \pi/4) - 48(1+1)$$

$$= 52\pi - 96$$

⇒ Evaluate

$$\oint_C (x+y)dx + x^2 dy \qquad (0,0), (2,0) \,\&\, (2,4)$$



Sol:

$$= \iint_R (2x - 1) \, dx \, dy$$

$0 \le x \le 2$

$0 \le y \le 2x$

$$= \int_0^2 \int_0^{2x} (2x-1) \, dy \, dx$$

$$= \int_0^2 (2x-1)(2x) \, dx = \int_0^2 (4x^2 - 2x) \, dx$$

$$= \frac{4x^3}{3} - x^2 = \frac{32}{3} - 4$$

$$= \frac{20}{3}$$

$\Longrightarrow$ Find the const's a, b, c such that

$$\vec{v} = (3x + ay + z)\hat{i} + (2x - y + bz)\hat{j} + (x + cy + z)\hat{k}$$

Such that $\vec{v}$ is irrotational.

Sol:
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = 0$$

$$= \cancel{\hat{i}(3x+ay+z)()} \; i(c-b) - \hat{j}(1-1)$$

$$+ \hat{k}(2-a)$$

$\rightarrow$ $c = b, \; a = 2$

Sol — div = 0

irr / conv → curl = 0

Parametric Rep. of Surfaces

Let $f(x,y,z) = c$ (or) $g(x,y,z)=0$

(or) $z = h(x,y)$, this be the Eq. of surface.

If $x = u$, $y = v$

$\Rightarrow z = h(u,v)$ then parametric form

of surface can be written as

$$r(u,v) = u\hat{i} + v\hat{j} + h(u,v)\hat{k}$$

alternatively choose

$x = x(u,v)$

$y = y(u,v)$

$z = z(u,v)$

$$r(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

Cylinder:

$$x^2+y^2=a^2$$

$$\boxed{r(u,v) = a\cos u\,\hat{i} + a\sin u\,\hat{j} + v\,\hat{k}}$$

Parametric rep.

Sphere:

$$x^2+y^2+z^2=a^2$$

$$r(u,v) = a\cos u\cos v\,\hat{i} + a\sin u\cos v\,\hat{j} + a\sin v\,\hat{k}$$

Paraboloid of Revolution:

$$z = x^2+y^2$$

$$r(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u^2\,\hat{k}$$

Cone of Revolution:

$$z^2 = x^2+y^2$$

$$r(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u\,\hat{k}$$

Ellipsoid:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$r(u,v) = a\cos u\cos v\,\hat{i} + b\sin u\cos v\,\hat{j} + c\sin v\,\hat{k}$$