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Linear D.E

* A linear ordinary diff. Eq of order n is written as

$$a_0(x)\,y^{(n)}(x) + a_1(x)\,y^{(n-1)}(x) + \cdots + a_{n-1}(x)\,y'(x) + a_n(x)\,y(x) = r(x) \quad \text{--- (1)}$$

$\Rightarrow$ where y is dependent and x is independent variable; $a_0 x \neq 0$

* If $r(x) = 0$ then it is called Homogeneous L.D.E otherwise non. Homogeneous L DE

(1) $y'' + 4y' = x^2 e^x \longrightarrow$ linear, non. Homo,

(2) $y'' + 4yy' = 0 \longrightarrow$ non linear, Homo.

(3) $x^2 y'' + (x^2 - 4)y = 0 \longrightarrow$ linear, Homog.

(4) $(1-x^2)y'' - 2xy' + 20y = 0 \longrightarrow$ linear, Homog.

<u>Note</u>: Trignometric functions with as L.D.E donot be linear.

<u>Eg</u>: $y'' + \sin y' = 0 \longrightarrow$ non linear.

$y'' + (\sin x)y = 0 \longrightarrow$ linear

Note: All the a_i should be constant then it is called const. coff. & else variable coefficient.

Eg: $y'' + 4y' + 2y = 0 \longrightarrow$ c. coff

$(1-x^2)y'' + xy' = 0 \rightarrow$ v. coff.

Solution of linear d.E:

Let eq. (1) be a LDE, then

$y_1(x)$ is a solution of this eq if $y_1(x)$ satisfies the eq. identically. Hence $y_1(x)$ must be continuously diff. $(n-1)$ times and n^{th} derivative of y_1 is continuous on I.

Uniqueness of solutions:

1. If the functions $a_0(x), a_1(x), \ldots a_n(x)$ & $r(x)$ are continuous on I and
2. $a_0(x) \neq 0$ on I then $\exists$ a unique sol. to the initial value problem.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = r(x)$$

$$y(x_0) = c_1, \quad y'(x_0) = c_2, \ldots \quad y^{(n-1)}(x_0) = c_n$$

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where $x_0 \in I$ & $c_1, c_2, \cdots c_n$ are

known constants (or) given constants.

Eg: $x^2y'' + xy' - 4y = 0$, $x \in (0, \infty)$ $\Rightarrow$

$y(0) = 0$

$y'(0) = 0$

It can't be done as 3 condition are

not satisfied.

$\Rightarrow$ If the conditions stated in the theorem

are satisfied then Eq ①, is said to be

normal on I.

* A point $x_0 \in I$ for which $a_0(x) \neq 0$ is

called an ordinary point (or) Regular point

of D.E.

Eg: Find the intervals on which following D.E

are Normal.

a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, n is integer

Sol: $a_0(x) = 1 - x^2$ (cont. on R)

$a_1(x) = -2x$ "

$a_2(x) = n(n+1)$ "

$h(x) = 0$ "

$a_0(x) = 1 - x^2 = 0$ I

$\Rightarrow x \neq 0$ in $\mathbb{R} - \{\pm 1\}$

$\Rightarrow x \in (-\infty, -1), (-1, 1), (1, \infty)$

Eg. $x^2 y'' + xy' + (n^2 - x^2) y = 0$, n - Real

$$\left.\begin{aligned} a_0(x) &= x^2 \\ a_1(x) &= x \\ a_2(x) &= n^2 - x^2 \end{aligned}\right\} \text{Cont. on } R$$

$r(x) = 0$

$a_0(x) = x^2 = 0$

$\Rightarrow x \neq 0$ in $R - \{0\}$

$\Rightarrow x \in (-\infty, 0), (0, \infty)$

Q: $\sqrt{x}\, y'' + 6xy' + 15y = \ln(x^4 - 256)$ is normal on

a) R

b) $(0, \infty)$

c) $(3, \infty)$

d) $(5, \infty)$ ✓

$a_0(x) = \sqrt{x} \Rightarrow x \geq 0$

$\Rightarrow x > 0$

$\ln(x^4 - 256)$

for domain

$\rightarrow x^4 - 256 > 0$

$\Rightarrow x > 4$

$x \in (4, \infty)$

Remark :

If the function $a_0(x);\ a_1(x),\ a_2(x) \dots$ are cont. on I and $a_0(x) \neq 0$ then the only sol. of Homogeneous initial value problem

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots a_n y = 0$$

$$y(x_0) = 0$$

$$y'(x_0) = 0$$

$$\vdots$$

$$y^{(n-1)}(x_0) = 0, \quad x_0 \in I$$

is the trivial Solution $y \equiv 0$ on I.

Eg: $x^2 y'' + xy' + y = 0 \ ; \ x \in (0, \infty)$

$$y(1) = 0$$

$$y'(1) = 0$$

Then the Solution is $y = 0$.

$\Longrightarrow$ Find the interval in which D.E becomes normal.

1) $x(1-x)y'' - 3xy' - y = 0$

$a_0 = x(1-x)$ cts on $\mathbb{R}$ $\qquad a_0 \neq 0$

$\qquad x(1-x) \neq 0$

$a_1 = -3x$ $\qquad \mathbb{R}$ $\qquad x \neq 0, 1$

$a_2 = -1$ $\qquad \mathbb{R}$ $\qquad x \in (-\infty, 0), (0, 1), (1, \infty)$ //

2) $y'' + 3y' + \sqrt{x}\, y = \sin x$

$a_0 = 1$ cts $\mathbb{R}$

$a_1 = 3$ $\mathbb{R}$

$a_2 = \sqrt{x}$ cts on $x \geq 0 \Rightarrow [0, \infty)$

$r = \sin x$ $\mathbb{R}$

$\therefore x \in [0, \infty)$

Linear Combination of functions:

Let $f_1(x), f_2(x) \ldots\ldots f_n(x)$ be n functions, then

$c_1 f_1(x) + c_2 f_2(x) + \ldots\ c_n f_n(x)$ is called a linear combination of given functions.

where $c_1, c_2, \ldots\ c_n$ are constants.

Super position / Linearity principle:

If $y_1(x), y_2(x), \ldots\ y_m(x)$ are m solutions of linear homogenous Eq.

$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots\ a_n y = 0$ on I, then the linear combination of the solutions

$c_1 y_1 + c_2 y_2 + \cdots + c_m y_m$ is also a

Solution of the Equation on I.

Remark: This principle doesnot hold for non-homogenous, non linear dE.

For non. Homogenous linear D.E, this principle holds if $c_1 + c_2 + \cdots + c_m = 1$

Eg: $a_0 y'' + a_1 y' + a_2 y = 0$ where y_1 and y_2 are Sol. which of these will not be Sol?

a) $c_1 y_1 + y_2$, c_1 is constant

b) $y_1 + y_2$

c) $y_1 - \frac{y_2}{2}$

d) $y_1 y_2 + y_2^2$

$\rightarrow$ $a_0 y'' + a_1 y' + a_2 y = g(x) \neq 0$. which of the following is a Sol if y_1, y_2 be Sol's.

a) $y_1 - y_2$

b) $c_1 y_1 + c_2 y_2$

c) $y_1 - 2y_2$

d) $2y_1 - y_2$

→ verify $y_1 = e^x$, $y_2 = e^{-2x}$, $y_3 =$ l.c of y_1 & y_2

are sol. of $y'' + y' - 2y = 0$

Sol: for $y_1 = e^x$

L.H.S = R.H.S

$\Rightarrow$ $y_1 = e^x$ is a Sol.

for $y_2 = e^{-2x}$ $\Rightarrow$ $y_2' = -2e^{-2x}$ $\Rightarrow$ $y_2'' = 4e^{-2x}$

L.H.S $= 4e^{-2x} - 2e^{-2x} - 2e^{-2x}$

$= 0$

$=$ R.H.S

$\Rightarrow$ $y_2 = e^{-2x}$ is also sol.

for $y_3 = c_1 e^x + c_2 e^{-2x}$ is also be

the solution.

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