Gauss Divergence Theorem:

Let D be a closed and bounded in 3D Space, whose boundary is piece wise smooth Surface S, that is oriented outward.

Let $V(x,y,z) = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}$ be a vector field for which V_1, V_2, V_3 are cont. and have cont. 1st order partial derivatives in some domain containing D.

Then,

$$\iint_S (\vec{V}\cdot\hat{n})\, dA = \iiint_D \operatorname{div}\vec{V}\, dv$$

$\hat{n}$ is outer unit normal

Interms of component of V, it can be written as,

$$\Rightarrow \iint_S V_1\, dy\, d_z + V_2\, d_x\, d_z + V_3\, dx\, d_y$$

$$= \iiint_D \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) dx\, dy\, dz$$

(or) $\iint_S (V_1 \cos\alpha + V_2 \cos\beta + V_3 \cos\gamma)\, dA$

→ Evaluate $\iint_S (\vec{v} \cdot \hat{n})\, dA$ using divergence theorem, if $v = 3x^2\hat{i} + 6y^2\hat{j} + z\hat{k}$, D be the region bounded by cylinder $x^2+y^2=16$

$z=0$ & $z=4$.

Sol: $\operatorname{div} \vec{v} = \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right)$

$= 6x + 12y + 1$

D:

$-4 \le x \le 4$

$-\sqrt{16-x^2} \le y \le \sqrt{16-x^2}$

$0 \le z \le 4$

$$\Rightarrow \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{0}^{4} (6x+12y+1)\, dz\, dy\, dx$$

$$= 4 \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (6x+12y+1)\, dy\, dx$$

$$= 4 \int_{-4}^{4} \left[(6x+1)y + 6y^2 \right]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx$$

$$= 8 \int_{-4}^{4} (6x+1)\sqrt{16-x^2}\, dx$$

$$= 8\left[\int_{-4}^{4} 6x\left(\sqrt{16-x^2}\right) dx + \int_{-4}^{4} \sqrt{16-x^2}\, dx\right]$$

$$= 16 \int_{0}^{4} \sqrt{16-x^2}\, dx$$

$$= 16 \left[\frac{x}{2}\sqrt{16-x^2} + 8 \sin^{-1}\left(x/4\right) \Big|_0^4 \right]$$

$$= 16 \times \cancel{8}^{4} \left(\pi/2\right) = 64\pi$$

$\Rightarrow$ Use divergence theorem to evaluate

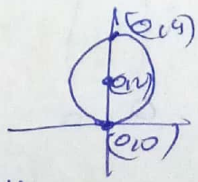
$\iint_S \vec{V}\cdot\hat{n}\, dA$, $V = xz^2\hat{i} + y\hat{j} - xz^2\hat{k}$ and

S is boundary of region bounded by

paraboloid, $z = x^2+y^2$ and plane $z = 4y$.

Sol: $$= \iiint_D 2xz + 1 - 2xz \; dx\, dy\, dz$$

$$= \iiint_D dx\, dy\, dz$$



D:

$x^2+y^2 = 4y$ $\qquad 0 \le y \le 4$

$\rightarrow x^2 + (y-2)^2 = 4$ $\qquad -\sqrt{4y-y^2} \le x \le \sqrt{4y-y^2}$

$x^2+y^2 \le z \le 4y$

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$$= \int_0^4 \int_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} \int_{x^2+y^2}^{4y} dz\, dx\, dy$$

$$= \int_0^4 \int_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} (4y-(x^2+y^2))\, dx\, dy$$

$$= \int_0^4 \left(4xy - \frac{x^3}{3} - xy^2\right)\Big|_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} dy$$

→ Evaluate $\iint_S (\vec{F} \cdot \hat{n})\, dA$

$\vec{F} = x\hat{i} + zx\hat{j} - xy\hat{k}$

$0 \le x \le a$

$0 \le y \le b$

$0 \le z \le c$

Sol: $= \iiint_D (\text{div}\, \vec{F})\, dv$

$$= \int_0^a \int_0^b \int_0^c 1\, dz\, dy\, dx$$

$= abc$ //

Green's Identities:

Let f, g be Scalar functions which are cont. and have 1^{st} and 2^{nd} order partial derivatives in some region of the 3D Space.

Let S be a piece wise smooth Surface bounding a Domain D in this region.

Let $\vec{v} = f \text{ grad } g$ Then

$$\nabla \cdot \vec{v} = \nabla \cdot (f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$$

By divergence theorem, The Surface integral

$$\iint_S (\vec{v} \cdot \hat{n})\, dA = \iint_S (f \nabla g \cdot \hat{n})\, dA$$

$$= \iint_S f (\nabla g \cdot \hat{n})\, dA$$

$$= \iiint_D \text{div}(f \nabla g)\, dv$$

$$= \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g)\, dv$$

$\frac{\partial g}{\partial n} = (\nabla g \cdot \hat{n})$ is directional derivative of g in direction of unit normal vector $\hat{n}$.

Therefore, Green's 1^{st} identity is given by

$$\boxed{\iint_S (f\,\nabla g\cdot\hat{n})\,dA = \iint_S f\,\frac{\partial g}{\partial n}\,dA = \iiint_D (f\nabla^2 g + \nabla f\cdot\nabla g)\,dv} \quad \text{——} (1)$$

$$\iint_S (g\,\nabla f\cdot\hat{n})\,dA = \iint_S g\,\frac{\partial f}{\partial n}\,dA = \iiint_D (g\nabla^2 f + \nabla g\cdot\nabla f)\,dv \quad \text{——} (2)$$

(1) − (2) $\Rightarrow$

$$\boxed{\iint_S (f\nabla g - g\nabla f)\cdot\hat{n}\,dA = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dA = \iiint_D (f\nabla^2 g - g\nabla^2 f)\,dv}$$

ie, the green's 2^{nd} identity.

* If $f=1$

$$\iint_S \nabla g\cdot\hat{n}\,dA = \iint_S \frac{\partial g}{\partial n}\,dA = \iiint_D \nabla^2 g\,dv$$

* If g is harmonic function;

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0$$

$$\Rightarrow \nabla^2 g = 0$$

So the integral of the normal derivative of

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g over any piece wise smooth closed orientable surface is Zero.

$$\therefore \boxed{\iint_S \nabla g \cdot \hat{n}\, dA = 0}$$

Stoke's Theorem:

Let S be a piece wise smooth orientable surface bounded by a piece wise smooth simple curve (closed) C

Let $\vec{v}(x,y,z) = v_1(x,y,z)\hat{i} + v_2(x,y,z)\hat{j} + v_3(x,y,z)\hat{k}$ be a vector function which is continuous and has cont. 1^{st} order partial derivatives in a domain, which contains 'S'. If C is ~~diversed~~ traversed in +ve direction then

$$\oint_C \vec{v} \cdot d\vec{r} = \oint_C (\vec{v} \cdot T)\, ds = \iint_S (\nabla \times \vec{v}) \cdot \hat{n}\, dA$$

where $\hat{n}$ is unit normal vector to S in direction of orientation of C. T is tangent vector to C.

In terms of components of $\vec{v}$, Stoke's theorem is

$$\oint_C v_1 dx + v_2 dy + v_3 dz = \iint_S (\nabla \times \vec{v}) \cdot \hat{n} \, dA$$

→ Verify Stoke's Theorem for the vector field

$$\vec{v}(x,y,z) = (3x - y)\hat{i} - 2yz^2\hat{j} - 2y^2z\hat{k}$$

where S is the surface of sphere

$S: x^2 + y^2 + z^2 = 16, \quad z > 0$

Sol: Consider, the projection of 'S' on XY plane.

Projection is $x^2 + y^2 \leq 16$ with bounding curve $C: x^2 + y^2 = 16, \ z = 0$

$$\oint_C v_1 dx + v_2 dy + v_3 dz = \oint_C (3x - y) dx - 2yz^2 dy - 2y^2z \, dz$$

$\Rightarrow x = 4\cos\theta$

$y = 4\sin\theta$

$z = 0$

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$$= \int_0^{2\pi} (12\cos\theta - 4\sin\theta)(-4\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} \left(-12\cdot 4 \sin\theta\cos\theta + 16\sin^2\theta\right) d\theta$$

$$= \int_0^{2\pi} -24\sin 2\theta + 8(1-\cos 2\theta)\, d\theta$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\frac{1-\cos 2\theta}{2}$

$$= 12\cos 2\theta + 8\theta - 4\sin 2\theta \Big|_0^{2\pi}$$

$$= 12(\cancel{1}-\cancel{1}) + 16\pi - 4(0-0)$$

$$= 16\pi$$

$$\iint_S (\nabla\times\vec{v})\cdot\hat{n}\, dA$$

$$\text{curl } \vec{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x-2y & -2yz^2 & -2y^2z \end{vmatrix} = \hat{k}$$

$$\nabla f = \vec{n} = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\hat{n} = \frac{x\,\hat{i} + y\,\hat{j} + z\,\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{4}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$(\nabla\times\vec{v})\cdot\hat{n} = z/4$$

$$dA = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{4}{z}\, dx\, dy$$

$$= \iint_R \frac{\cancel{z}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{z}}\, dx\, dy$$

$$= \pi (4)^2 \quad (\because \text{ area of circle} = \pi r^2)$$

$$= 16\pi$$

* If $f = \tan^{-1}(y/x)$ then

$$\boxed{\text{div}(\text{grad} f) = 0}$$

$$\nabla^2 f = \frac{\cancel{x^2}}{y^2+x^2} \left(-\frac{1}{\cancel{x^2}}\right) + \frac{x^2}{y^2+x^2}.$$

$$= \frac{x^2-1}{x^2+y^2}$$

$\nabla^2 f =$

$$f^{I}(y/x) = \frac{x^2-1}{x^2+y^2}$$

$$f^{II}(y/x) = \frac{(x^2-1)(2x) - (x^2+y^2)(2x)}{(x^2+y^2)^2}.$$

* $$\boxed{\text{curl}(\text{grad} f) = 0}$$

* $\oint_C \vec{F} \cdot d\vec{r}$ necessary cond. for, so that it vanishes is

$$= \iint_S (\text{curl} \vec{F}) \hat{n} \, dA$$

$$\text{curl} \vec{F} = 0$$

$$\boxed{\text{curl} \vec{F} = 0}$$

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* $\int_C y^2\,dx + x^2\,dy$ C: $-1 \le x \le 1$, $-1 \le y \le 1$ square

$$= \iint_R (2x - 2y)\,dx\,dy$$

$$= 2\int_{-1}^{1}\int_{-1}^{1} (x - y)\,dx\,dy$$

$$= 2\int_{-1}^{1} \left[\left(\frac{x^2}{2} - xy\right)\right]_{-1}^{1} dy$$

$$= 2\int_{-1}^{1} \left(\frac{1}{2} - y\right) - \left(\frac{1}{2} + y\right)\,dy$$

$$= -2\int_{-1}^{1} y\,dy = -2\left[y^2/2\right]_{-1}^{1}$$

$$= -2\left[1/2 - 1/2\right] = 0 \,//$$

* $S: x^2 + 2y^2 + z^2 = 7$ at $(1, -1, 2)$

$$\nabla f = 2x + 4y + 2z$$

$$\vec{n} = 2\hat{i} - 4\hat{j} + 4\hat{k}$$

$$\cancel{= (x-1)\hat{i} + (y+4)\hat{j} + (z-2)\hat{k}}$$

$$|\hat{n}| = \sqrt{4 + 16 + 16} = 6 \,//$$

* value of λ such that

$f = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+\lambda z)\hat{k}$ is solenoidal.

$\operatorname{div} f = 1 + 1 + \lambda = 0$

$\Rightarrow \underline{\lambda = -2}$

* $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$ moving from

$(1,1,1)$ to $(3,3,2)$. find work done.

$x = 1+2t$ $\qquad 1 \le x \le 3$

$y = 1+2t$

$z = 1+t$

$\Rightarrow$ we have to check whether it is independent of path.

$\Rightarrow xyz + f(y,z)$ $\qquad xyz + g(x,z)$

$\qquad xyz + h(x,y)$

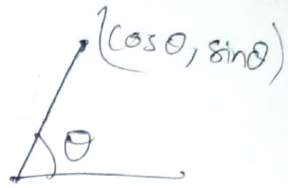
$\Rightarrow \left. xyz + C \right|_{(1,1,1)}^{(3,3,2)}$

$= 18 - 1$

$= 17$

* $f(x,y) = \frac{x^2-y^2}{xy} = \frac{x}{y} - \frac{y}{x}$

if $D_b = 0$ at $(1,1)$ along a ray making angle with +ve X axis.



$\nabla f = \left(-\frac{x}{y^2} - \frac{1}{x}\right)\hat{j} + \left(\frac{1}{y} + \frac{y}{x^2}\right)\hat{i}$

$\vec{b} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\nabla f_{(1,1)} = \left(-\frac{1}{1} - 1\right)\hat{j} + (1+1)\hat{i}$

$= 2\hat{i} - 2\hat{j}$

$\nabla f \cdot b = 0$

$\Rightarrow 2\cos\theta = 2\sin\theta$

$\boxed{\theta = 45^\circ}$

* $\vec{v} = e^x \sin y + e^x \cos y\,\hat{j}$ is solenoidal

* Tangent at $t=1$ to curve

$R = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$ is

$r'(t) = 2t\,\hat{i} + 2\hat{j} - 3t^2\hat{k}$

$r'(t)_1 = 2\hat{i} + 2\hat{j} - 3\hat{k}$ //

* PDE for $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$$\frac{\partial z}{\partial x} = \frac{2x}{a^2} ; \quad \frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

$$\Rightarrow 2z = x . \frac{2x}{a^2} + y . \frac{2y}{b^2}$$

$$\boxed{2z = px + qy}$$

* $\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y} + 2u$

Trail Sol. by Sep. of variables is
$u(x,y) = f(x)\, g(y)$

* PDE for $f(x+y, z-xy) = 0$

$\cancel{\frac{\partial z}{\partial x} = f'(x+y, z-xy)}$ Suppose,

$$x+y = z - xy$$

$$\Rightarrow z = x + y + xy$$

$$\frac{\partial z}{\partial x} = p = 1 + y \quad \Rightarrow \quad 1 = p - y$$

$$\frac{\partial z}{\partial y} = q = 1 + x \quad \quad 1 = q - x$$

$$\Rightarrow \quad \Rightarrow p + x = y + q$$

* Trivial Sol. for any Eq is

$y = 0$

* Solving laplace eqn, best Sol. is

Case (ii) if

$$\left.\begin{aligned} u(0, x) &= 0 \\ u(1, x) &= 0 \end{aligned}\right\} \checkmark$$

$u(x, t) =$ ——

$u(x, 0) = 0.$