Linear Independence & Dependence

* Let $f_1(x), f_2(x), \ldots f_n(x)$ be n functions, then these functions are said to L.I on some interval I if the eq.

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{—(1)}$$

has unique Sol. if $c_1 = c_2 = c_3 = \cdots c_n = 0$

* These functions are said to be L.D on I if Eq.(1) holds for $c_1, c_2, \ldots c_n$ not all zero. (infinite Sol.)

In this case, one or more functions can be expressed as a Linear Combination of the remaining functions.

Note:

Check $f_1(x) = x^2$

$f_2(x) = x^3$

$f_3(x) = 6x^2 - x^3$ is L.D / L.I

$\Rightarrow f_3 = 6f_1 - f_2$ it is L.D

Since, there is a relation b/w the 3 function.

$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

$c_1 x^2 + c_2 (x^3) + c_3 (6x^2 - x^3) = 0$

$(c_1 + 6c_3) x^2 + (c_2 - c_3) x^3 = 0x^2 + 0 \cdot x^3$

$\rightarrow \quad c_1 = -6c_3 = -6c_2$

$\Rightarrow \quad x^2 - 1, \quad 3x^2, \quad 2 - 5x^2$
$\qquad f_1 \qquad f_2 \qquad f_3$

$-2f_1 - f_2 = f_3$

$\Rightarrow 2f_1 + f_2 + f_3 = 0$ It is L.D

(or)

$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

$c_1 (x^2 - 1) + c_2 (3x^2) + c_3 (2 - 5x^2) = 0$

$(c_1 + 3c_2 - 5c_3) x^2 - c_1 + 2c_3 = 0$

$\Rightarrow c_1 + 3c_2 - 5c_3 = 0 \qquad \Rightarrow c_1 = 2c_3$

$\Rightarrow \quad 2c_3 + 3c_2 - 5c_3 = 0$

$c_2 = c_3$

$\Rightarrow \quad c_1 = 2c_3 = 2c_2$

If $c_2 = 1, \ c_3 = 1, \ c_1 = 2$

$\Rightarrow \quad 2f_1 + f_2 + f_3 = 0$

* A Procedure to test L.I or L.D for a given set of functions is the application of WRONSKIANS.

Let $f_1(x), f_2(x) \ldots f_n(x)$ be n functions, then the Wronskians of this function is denoted by

$$w(f_1(x), f_2(x), \ldots f_n(x)) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

$$= W(x)$$

Theorem:

If the coefficients $a_0(x), a_1(x), \ldots a_n(x)$ in the linear Homogenous Eq, $a_0 y^{(n)} + \ldots + a_n y = 0$, $a_0 \neq 0$ are Continuous on I, then the LDE is normal and $y_1(x), \ldots, y_n(x)$ are n Sol. of this eq; Then

i) $W(x) = W(y_1, y_2 \ldots y_n) \neq 0 \; \forall \; x \in I$

$\Leftrightarrow$ $y_1(x), y_2(x), \ldots y_n(x)$ are L.I on I

ii) $W(x_0) = 0$ for $x_0 \in I$, a fixed point

$\Rightarrow W(x) = 0 \ \forall \ x \in I$ and

$y_1(x), y_2(x) \dots y_n(x)$ are L.D on I.

$\Rightarrow x, x^2, x^3$ are

a) L.I on any interval I.

b) L.D

c) L.I on $\mathbb{R}$

d) L.I on any interval I not containing zero

$$W(x, x^2, x^3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x(12x^2 - 6x^2) - 1(6x^3 - 6x^3) + (0)$$

$$= 6x^3 \rightarrow x^3 \neq 0 \quad \because x \in I - \{0\}$$

$\Rightarrow 1, \sin x, \cos x$

$$= \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = 1(-\cos^2 x - \sin^2 x) = -1$$

$\therefore$ L.I on $\mathbb{R}$

$\Longrightarrow$ e^x, $\sinh x$, $\cosh x$. Find interval:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$W(x) = \begin{vmatrix} e^x & \sinh x & \cosh x \\ e^x & \cosh x & \sinh x \\ e^x & \sinh x & \cosh x \end{vmatrix} = 0 \quad \text{L.D on } \mathbb{R}$$

$\Rightarrow$ Find an interval on which the functions

$1, \cos x, \sec x, \quad x > 0 \rightarrow$ are L.I

$$W(x) = \begin{vmatrix} 1 & \cos x & \sec x \\ 0 & -\sin x & \sec x \tan x \\ 0 & -\cos x & \sec x \tan^2 x + \sec^3 x \end{vmatrix}$$

$$= 1\,(-\sin x)(\sec x \tan^2 x + \sec^3 x)$$

$$- (-\cancel{\cos x})(\cancel{\sec x} \tan x)$$

$$= -\sin x \sec x \tan^2 x - \sin x \sec^3 x + \tan x$$

$$= -\tan^3 x - \tan x \sec^2 x + \tan x$$

$$= -\tan x(\sec^2 x - 1) - \tan^3 x$$

$$= -2\tan^3 x \neq 0 \Rightarrow \begin{array}{l} \tan x \neq 0 \\ x \neq n\pi \end{array}$$

$x \neq 0, \pi, 2\pi, 3\pi \ldots \rightarrow x \in (0, \pi)$

$x \neq \pi/2, 3\pi/2, 5\pi/2 \ldots \rightarrow x \in (0, \pi/2)$

$\therefore$ functions are L.I on $(0, \pi/2)$

Theorem:

If Linear Homo. Eq $a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots a_n y = 0$ —①

is normal on I, Then this Eq. has

n L.I Solution.

* If $y_1(x), y_2(x), \ldots y_n(x)$ are n L.I Sol.

then the general sol. is given by

the linear combination:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x).$$

* The n L.I Sol. $y_1(x), y_2(x), \ldots y_n(x)$

are also called fundamental sol. of

Eq. ① on I.

This set of fundamental sol. forms a

basis of n^{th} order LHDE.

Eg: Check whether e^{2x} and e^{3x} are fundamen

sol. of $y'' - 5y' + 6y = 0$.

Sol:

Step 1: $y_1 = e^{2x}$ is sol. of

$y_1' = 2e^{2x}$; $y_1'' = 4e^{2x}$

$\Rightarrow 4e^{2x} - 10e^{2x} + 6e^{2x} = 0$

Step 2: $y_2 = e^{3x}$; $y_2' = 3e^{3x}$; $y_2'' = 9e^{3x}$

$\Rightarrow 9e^{3x} - 15e^{3x} + 6e^{3x} = 0 = R.H.S$

$y_2 = e^{3x}$ is a sol.

Step 3: y_1 & y_2 are L.I

$$W(x) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{2x}e^{3x}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$= e^{5x} \neq 0$

(L.I)

Hence $y_1 = e^{2x}$, $y_2 = e^{3x}$ are fundamental

Sol. of given Eq.

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$\Rightarrow$ S.T e^x, e^{2x}, e^{3x} are f. Sol. of

$y''' - 6y'' + 11y' - 6y = 0$ on I.

Q: $\Rightarrow$ S.T the set of functions $\{x, 1/x\}$ forms a basis of $x^2 y'' + xy' - y = 0$.

Obtain a particular Sol. when $y(1) = 1$, $y'(1) = 2$

Sol: $y_1 = x \Rightarrow y_1' = 1, \quad y_1'' = 0$

$\Rightarrow 0 + x - \cancel{x} = 0$

$y_2 = 1/x \Rightarrow y_2' = -1/x^2 = -x^{-2}$

$y_2'' = 2x^{-3}$

$\Rightarrow x^2(2x^{-3}) - x(x^{-2}) - 1/x = 0$

$\Rightarrow 2/x - 2/x = 0$

$$W(x) = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{2}{x} \Rightarrow x \neq 0$$

$\therefore$ x, $1/x$ are L.I Sol.

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$$G.S = c_1 x + c_2 \, 1/x$$

$$\Rightarrow y(x) = c_1 x + c_2 \cdot 1/x$$

$$y'(x) = c_1 - \frac{c_2}{x^2}$$

$$\left.\begin{aligned} y(1) &= c_1 + c_2 = 1 \\ & c_1 - c_2 = 2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 3/2 \\ c_2 &= -1/2 \end{aligned}$$

$\therefore$ particular sol. is $\frac{3}{2}x - 1/2x$.

Q. $\Rightarrow y = xp^2 + p$

$$p = 2px\frac{dp}{dx} + p^2 + \frac{dp}{dx}$$

$$p - p^2 = (2px + 1)\frac{dp}{dx}$$

$$\Rightarrow \frac{dx}{dp} = \frac{2px + 1}{p - p^2}$$ $\left(\because y = x f(p) + \phi(p)\right)$

lagrange's eq.

$$\Rightarrow \frac{dx}{dp} - \frac{2p}{p(1-p)}x = \frac{1}{p(1-p)}$$

$\Rightarrow$ Check whether $\sin x$ is a sol. to a diff. Eq $y'' + y = 0$.

Sol: $-\sin x + \sin x = 0$

$\therefore$ $\sin x$ is sol.

$\Rightarrow$ $4\sin x$ is also a sol.

$\Rightarrow$ Check whether $y_1 = \sin x$, $y_2 = 4\sin x - 2\cos x$ for the D.E $y'' + y = 0$

Sol: $W(x) = \begin{vmatrix} \sin x & 4\sin x - 2\cos x \\ \cos x & 4\cos x + 2\sin x \end{vmatrix}$

$y_2' = 4\cos x + 2\sin x$

$y_2'' = -4\sin x + 2\cos x$

$\Rightarrow -4\sin x + 2\cos x + 4\sin x - 2\cos x = 0$

$W(x) = \sin x(4\cos x + 2\sin x) - (4\sin x - 2\cos x)\cos x$

$= 4\sin x\cos x + 2\sin^2 x - 4\sin x\cos x + 2\cos^2 x$

$= 2 \neq 0$

$\therefore$ They are L.I and ...

$\Rightarrow$ $y'' + 5y' + 6y = 0, \; x \in \mathbb{R}$

$y(1) = 0$

$y'(1) = 0$

Sol: $y \equiv 0$ is a sol.

$\Rightarrow$ Find Sol. of $y'' - 5y' + 4y = 0$

$y(0) = 2$

$y'(0) = 1$ $\quad (\because D = \frac{dy}{dx})$

Sol: $(D^2 - 5D + 4)y = 0$

$\Rightarrow$ $D^2 - 5D + 4 = 0 \longrightarrow$ Auxiliary eq.

$\Rightarrow$ $D^2 - 4D - D + 4 = 0$

$\Rightarrow$ $(D-4)(D-1) = 0$

$\Rightarrow$ $D = 4, 1$

$$\boxed{y(x) = \frac{7}{3}e^{x} - \frac{1}{3}e^{4x}}$$

L.I $= e^{4x}, e^{x}$

G.S $= y(x) = c_1 e^{x} + c_2 e^{4x}$

$\rightarrow$ $c_1 e^{x} + 4c_2 e^{4x} = y'(x)$

$c_1 + c_2 = 2$

$\Rightarrow$ $-c_1 + -4c_2 = -1$

$-3c_2 = 1 \rightarrow c_2 = -1/3, \; c_1 = 7/3$

$\Rightarrow 4y'' - 8y' + 3y = 0$

$y(0) = 1$

$y'(0) = 3$

Sol: $4m^2 - 8m + 3 = 0$

$$m = \frac{8 \pm \sqrt{64 - 48}}{8}$$

$= 1 \pm 1/2$

$= 3/2$ (or) $1/2$

L.I Sol. $= e^{3/2 x}, e^{x/2}$

G.S $\rightarrow y(x) = Ae^{3/2 x} + Be^{x/2}$

$y'(x) = \frac{3}{2} Ae^{3/2 x} + \frac{B}{2} e^{x/2}$

$y(0) = A + B = 1$

$y'(0) = 3A + B = 6$

$\Rightarrow A = 5/2, B = -3/2$

$\Rightarrow y(x) = \frac{5}{2} e^{3/2 x} - 3/2 e^{x/2}$

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$\Rightarrow$ Find a HLDE with Real const. coeff. of lowest order which has the following particular Sol.

1) $5 + e^{x} + 2e^{3x} \longrightarrow G.S = A + Be^{x} + Ce^{3x}$

$m = 0, 1, 3$

$(m-0)(m-1)(m-3) = 0$

$\Rightarrow m^3 - m^2(a+b+c) + m(ab+bc+ca) - (abc) = 0$

$\Rightarrow m^3 - 4m^2 + 3m = 0$

$\Rightarrow y''' - 4y'' + 3y' = 0$

2) $e^{-x} + \cos 5x + 3\sin 5x$

Sol: $m = -1, \pm 5i$

$m^3 + m^2 + (-5i + 5i + 25)m + 25 = 0$

$\Rightarrow m^3 + m^2 + 25m + 25 = 0$

$\Rightarrow y''' + y'' + 25y' + 25y = 0$

3) $xe^{-x} + e^{2x} \Rightarrow Ae^{-x} + Bxe^{-x} + Ce^{2x}$

$m = -1, -1, 2$

$m^3 - 3m - 2 = 0$

$y''' - 3y' - 2y = 0$ //

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4) $1 + x + e^{x} - 3e^{3x} \longrightarrow G.S = A + Bx + Ce^{x} + D$

$\Rightarrow m = 0, 0, 1, 3,$

$\Rightarrow (m-0)^2(m-1)(m-3) = 0$

$\Rightarrow m^2(m^2 - 4m + 3) = 0$

$\Rightarrow m^4 - 4m^3 + 3m^2 = 0$

$\Rightarrow y'''' - 4y''' + 3y'' = 0$

5) $x^2 e^{2x} + 2e^{-2x} \Rightarrow Ae^{2x} + Bxe^{2x} + Cx^2e^{2x} + D$

$\Rightarrow m = 2, 2, 2, -2$

$\Rightarrow (m-2)^3(m+2) = 0$

$\Rightarrow m^3 - 3$

6) $xe^{-x} + x^2e^{-x} - \cos x$

$\Rightarrow m = -1, -1, -1, \pm i$

$\Rightarrow (m+1)^3(m+i)(m-i)$

⟹ Find all nontrivial sol. of boundary value problem.

$$y'' + \omega^2 y = 0$$

$$y(0) = 0$$

$$y(\pi) = 0$$

$\Rightarrow$ $m^2 + \omega^2 = 0$

$$m^2 = -\omega^2$$

$$m = \underline{\pm \omega i}$$

L.I Sol. $\cos \omega x$, $\sin \omega x$

G.S $y(x) = A \cos \omega x + B \sin \omega x$

$$y(0) = A = 0$$

$$y(\pi) = A \cos \omega\pi + B \sin \omega\pi = 0$$

$\rightarrow$ $B \sin \omega\pi = 0$

$B = 0$ $\sin \omega\pi = 0 = \sin n\pi$

Trivial sol. X

$$\omega\pi = n\pi$$

$$\omega = n = 0, 1, 2, \ldots$$

$y(x) = B \sin nx$, $n = 1, 2, 3, \ldots$

$n \neq 0$; Since it gives trivial sol.

$\therefore$ There are infinite L.I sol.

$$\boxed{y(x) = \sum_{n=1}^{\infty} B_n \sin nx}$$