Surface Area &

Surface integrals

Consider a surface 'S' $f(x,y,z) = c$

(or) $z = f(x,y)$.

parametric rep. can be

$$r(u,v) = u\hat{i} + v\hat{j} + f(u,v)\hat{k}$$

$$= x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

Let 'C' be a curve on 'S' then,

Parametric rep. of C

$$u = f(t) \ ; \quad v = g(t)$$

$$r = r(t) = r(u(t), v(t))$$

$$= r(f(t), g(t))$$

Then tangent vector to 'C' for any value of parameter t is given by $\frac{dr(t)}{dt}$

$$\frac{dr(t)}{dt} = \frac{\partial r}{\partial u} \cdot \frac{du}{dt} + \frac{\partial r}{\partial v} \cdot \frac{dv}{dt}$$

$$= r_u \frac{du}{dt} + r_v \frac{dv}{dt}$$

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* Consider a point 'P' for any point $P(r)$ on surface 'S', the eq of tangent plane at P is

$$\boxed{(r^* - r) \cdot r_u \times r_v = [r^* - r \quad r_u \quad r_v] = 0}$$

where $[\cdot \cdot \cdot]$ is scalar triple product and r^* is position vector of any point in tangent plane.

Now, $(r_u \times r_v)$ is ⊥ler to both r_u and r_v.

Hence, it is normal vector to tangent plane. i.e., normal vector to surface 'S' at point P.

$\therefore$ $\boxed{\text{normal vector} = (r_u \times r_v)}$

unit normal vector,

$$\hat{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$$

Now, in 2D, ds of curve 'C' is given

by $ds^2 = dx^2 + dy^2$

$= dr \cdot dr$

where $[r = x\hat{i} + y\hat{j}]$

In 3D, $d\vec{r} = \frac{\partial r}{\partial u} du + \frac{\partial r}{\partial v} dv$

$$\boxed{\begin{aligned} \Rightarrow ds^2 &= (r_u du + r_v dv) \cdot (r_u du + r_v dv) \\ &= \boxed{r_u^2 du^2 + 2 r_u r_v du dv + r_v^2 dv^2} \end{aligned}}$$

$r_u^2 = r_u \cdot r_u$

$r_v^2 = r_v \cdot r_v$

This diff. form of du is 1st fundamental form of surface 'S'.

Surface Area:

Surface area A of surface 'S' is given by

$$A = \iint_R |r_u \times r_v| \, du \, dv$$

where R is region in uv plane.

* $|\vec{a} \times \vec{b}|^2 = a^2 b^2 - (a \cdot b)^2$

$$A = \iint_R \sqrt{r_u^2 r_v^2 - (r_u \cdot r_v)^2}\, du\, dv$$

If $r(u,v) = u\hat{i} + v\hat{j} + f(u,v)\hat{k}$

$$x = u, \quad y = v$$

$$r_u = 1\hat{i} + f_u \hat{k}$$

$$r_v = \hat{j} + f_v \hat{k}$$

$$r_u \times r_v = -f_u \hat{i} - f_v \hat{j} + \hat{k}$$

$$|r_u \times r_v| = \sqrt{f_u^2 + f_v^2 + 1}$$

$$A = \iint_R \sqrt{1 + f_u^2 + f_v^2}\, du\, dv$$

$$= \iint_{R^*} \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

R^* is orthogonal projection of $S(z = f(x,y))$ on xy plane.

$$dA = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

Remark:

* Let $z = f(x,y)$, Eqn of Surface. Let element area be projected on XY plane.

Then $$A = \iint_{R^*} \sqrt{1+f_x^2+f_y^2}\, dx\, dy$$

* Let $x = g(y,z)$, be project 'S' on YZ plane then,

$$A = \iint_{R^*} \sqrt{1+g_y^2+g_z^2}\, dy\, dz$$

$R^* \longrightarrow$ projection of 'S' on YZ plane.

* Let $y = h(x,z)$, then

$$A = \iint_{R^*} \sqrt{1+h_x^2+h_z^2}\, dx\, dz$$

$R^* \longrightarrow$ projection of 'S' on XZ plane.

$\Rightarrow$ write parametric rep. of surface $z^2 = x^2+y^2$ & hence find eq. of tangent plane at point $(0,1,1)$ on the surface.

Sol:

$r(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u\,\hat{k}$

$(x-0)(2x) + (y-1)(2y) + (z-1)(2z) = 0$

$\Rightarrow x(0) + \cancel{(y-1)(2)} + \cancel{(z-1)(-2)} = 0$

$\Rightarrow y = z$ //

$\longrightarrow$ obtain, The first fundamental form or diff. form of a sphere

$x^2+y^2+z^2 = a^2$.

Sol:

$ds^2 = r_u^2 du^2 + 2 r_u r_v\, du\,dv + r_v^2 dv^2$

$r(u,v) = a\cos u\cos v\,\hat{i} + a\sin u\cos v\,\hat{j} + a\sin v\,\hat{k}$

$r_u = -a\sin u\cos v\,\hat{i} + a\cos u\cos v\,\hat{j}$

$r_v = -a\cos u\sin v\,\hat{i} - a\sin u\sin v\,\hat{j} + a\cos v\,\hat{k}$

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$ds^2 = (a^2\cos^2 v \sin^2 u + a^2\cos^2 u \cos^2 v) du^2$

$2(\cdot a^2 \sin u \cos v \cancel{\sin v \cos u} - a^2 \sin u \cancel{\cos u} \sin v \cos u)$

$du\,dv + (a^2 \sin^2 v \cos^2 u + a^2 \sin^2 u \sin^2 v$
$+ a^2\cos^2 v) dv^2$

$= a^2 \cos^2 v\, du^2 + a^2 dv^2$

$\Rightarrow$ The cylinder $y^2+z^2=9$ intersects sphere $x^2+y^2+z^2=25$. find the surface area of the portion of the sphere cut by the cylinder above the yz plane and within cylinder.

Sol: we project the sphere on yz plane.

$$\therefore A = \iint_R \sqrt{1+f_y^2+f_z^2}\, dy\, dz$$

$\Rightarrow$ $x = \sqrt{25-y^2-z^2} = f(y,z)$

$$f_y = \frac{1\times(-2y)}{2\sqrt{25-y^2-z^2}} = \frac{-y}{\sqrt{25-y^2-z^2}}$$

$$f_z = \frac{-z}{\sqrt{25-y^2-z^2}}$$

$$1 + f_y^2 + f_z^2 = 1 + \frac{y^2}{25 - y^2 - z^2} + \frac{z^2}{25 - y^2 - z^2}$$

$$= \frac{25}{25 - y^2 - z^2}$$

$$\rightarrow A = \iint_R \frac{5}{\sqrt{25 - y^2 - z^2}} \, dy \, dz$$

R is projection of surface of sphere on yz plane within the cylinder given by

$$R: y^2 + z^2 \leq 9$$

$$x = 0$$

$$y = r \cos\theta, \quad z = r \sin\theta$$

$$0 \leq r \leq 3$$

$$0 \leq \theta \leq 2\pi$$

$$= \int_0^3 \int_0^{2\pi} \frac{5}{\sqrt{25 - r^2}} \, r \, d\theta \, dr$$

$$= \int_0^3 \frac{10\pi r}{\sqrt{25-r^2}}\,dr$$

$25 - r^2 = t$

$-2r\,dr = dt$

$r = 0,\ t = 25$

$r = 3,\ t = 16$

$$= \frac{-10\pi}{2}\int_{25}^{16} \frac{1}{\sqrt{t}}\,dt$$

$$= 5\pi \int_{16}^{25} t^{-1/2}\,dt$$

$$= 5\pi \times 2\, t^{1/2} \Big|_{16}^{25}$$

$$= 10\pi\,[5 - 4]$$

$$= 10\pi //$$

Surface Integral:

It can be evaluated in any one of the following ways:

Let S be rep. in parametric form $r = r(u,v)$ then the surface integral

1) $$\iint_S g(x,y,z)\,dA = \iint_{R^*} g[x(u,v), y(u,v), z(u,v)]\,|r_u \times r_v|\,du\,dv$$

$$= \iint_{R^*} g[x(u,v), y(u,v), z(u,v)]\sqrt{r_u^2 r_v^2 - (r_u \cdot r_v)^2}\,du\,dv$$

where R^* is the region corresponding to region in uv form.

2) S be a surface,

$z = f(x,y)$

$$\iint_S g(x,y,z)\,dA = \iint_{R^*} g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dx\,dy$$

$R^* \to$ orthogonal proj. of S on XY plane.

3) $y = h(x, z)$

$$\iint_S g(x,y,z)\,dA = \iint_R g(x, h(x,z), z)\sqrt{1 + h_x^2 + h_z^2}\,dx\,dz$$

4) $x = h(y, z)$

$$\iint_S g(x,y,z)\,dA = \iint_R g(h(y,z), y, z)\sqrt{1 + h_y^2 + h_z^2}\,dy\,dz$$

If S be piecewise smooth and consist of surfaces $S_1, S_2 \ldots S_K$ then

$$\iint_S g(x,y,z)\,dA = \iint_{S_1} g(x,y,z)\,dA + \iint_{S_2} g(x,y,z)\,dA + \ldots \iint_{S_K} g(x,y,z)\,dA$$

Orientable Surfaces:

S be an orientable surface, if it has 2 sides. A Smooth Surface S is said to be orientable if $\exists$ cont. unit normal vector field ($\hat{n}$) defined at each point (x, y, z) on the surface.

Here, $\vec{n}(x, y, z)$ is orientation of S.

* An orientable Surface has 2 orientations.

Since a unit normal to the surface S at (x, y, z) can be $\hat{n}(x, y, z)$ or $-\hat{n}(x, y, z)$

Let S be an orientable Surface, we choose a unit normal ($\hat{n}$) and orient the surface S.

$\hat{n} = \cos\alpha\, \hat{i} + \cos\beta\, \hat{j} + \cos\gamma\, \hat{k}$

where α, β, γ are the angles which the unit normal makes with the +ve direction of x, y, z resp.

$\hat{n} \cdot \hat{i} = \cos\alpha$

$\hat{n} \cdot \hat{j} = \cos\beta$

$\hat{n} \cdot \hat{k} = \cos\gamma$

Let $v(x,y,z) = v_1(x,y,z)\hat{i} + v_2(x,y,z)\hat{j} + v_3(x,y,z)\hat{k}$

then

$$\iint_S \vec{v} \cdot \hat{n}\, dA = \iint_S (v_1 \cos\alpha + v_2 \cos\beta + v_3 \cos\gamma)\, dA$$

$$= \iint_S v_1(\cos\alpha\, dA) + v_2(\cos\beta\, dA) + v_3(\cos\gamma\, dA)$$

$$\cos\alpha\, dA = \hat{n} \cdot \hat{i}\, dA = dy\, dz \quad \text{——} \quad (1)$$

$$= \iint_S v_1\, dy\, dz + v_2\, dx\, dz + v_3\, dx\, dy$$

These (1) expressions are for elementary projections of area on co-ordinate planes.

Flux of a vector field:

flux of a vector field of surface S.

Let $\vec{v}(x,y,z)$

~~Let~~ be a vector field rep. the velocity of a fluid. The flux of a velocity vector field through the area ΔA is

app. by $(\vec{v} \cdot \hat{n})\, dA$ where $\hat{n}$ is unit normal vector to the surface.

* The total volume of the fluid flowing through S per unit time is called flux of V through S.

$$\text{flux} = \iint_S (\vec{V} \cdot \hat{n})\, dA$$

$$dA = \frac{dx\,dy}{\hat{n} \cdot \hat{k}} \quad (or) \quad \frac{dy\,dz}{\hat{n} \cdot \hat{i}} \quad (or) \quad \frac{dx\,dz}{\hat{n} \cdot \hat{j}}$$

the Find the Eq of tangent plane of the surface $x^2+z^2=25$ at $P(3,-2,4)$.

Sol: $(x-3)6+(z-4)8=0$

$\Rightarrow 6x-18+8z-32=0$

$\Rightarrow 6x+8z-50=0$

$\Rightarrow 3x+4z-25=0$

$\Rightarrow 36x^2+16y^2+9z^2=144$ at $\left(0, \frac{3\sqrt{2}}{2}, 2\sqrt{2}\right)$

Sol: $\frac{\partial f}{\partial x} = 72x\Big|_P = 0$

$\frac{\partial f}{\partial y} = 32y = 48\sqrt{2}$

$\frac{\partial f}{\partial z} = 36\sqrt{2}$

$\left(y-\frac{3\sqrt{2}}{2}\right)48\sqrt{2}+(z-2\sqrt{2})(36\sqrt{2})=0$

$48\sqrt{2}y+36\sqrt{2}z-72-144=0$

$\Rightarrow \boxed{4y+3z=12\sqrt{2}}$

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→ Obtain 1st fund. form of Surface $x^2+y^2=a^2$.

Sol:

$$r(u,v) = a\cos u\,\hat{i} + a\sin u\,\hat{j} + v\hat{k}$$

$$r_u = \frac{\partial r}{\partial u} = -a\sin u\,\hat{i} + a\cos u\,\hat{j}$$

$$r_v = \frac{\partial r}{\partial v} = \hat{k}$$

$$r_u^2 = a^2\sin^2 u + a^2\cos^2 u = a^2$$

$$r_v^2 = 1$$

$$r_u r_v = 0$$

$$ds^2 = r_u^2 du^2 + 2r_u r_v\, du\, dv + r_v^2 dv^2$$

$$\boxed{ds^2 = a^2 du^2 + dv^2}$$

→ $z = x^2 + y^2$

$$r(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u^2\hat{k}$$

$$r_u = \cos v\,\hat{i} + \sin v\,\hat{j} + 2u\,\hat{k}$$

$$r_v = -u\sin v\,\hat{i} + u\cos v\,\hat{j}$$

$$r_u^2 = 1+4u^2 \qquad r_u \cdot r_v = 0$$

$$r_v^2 = u^2$$

$$\boxed{\therefore ds^2 = (1+4u^2)\,du^2 + u^2 dv^2}$$

→ Evaluate the surface integral

$$\iint_S \vec{F} \cdot \hat{n} \, dA$$

where $\vec{F} = 6z\hat{i} + 6\hat{j} + 3y\hat{k}$ as S is portion of plane $2x + 3y + 4z = 12$ which is in 1st octact.

Sol: $\text{grad } f = 2\hat{i} + 3\hat{j} + 4\hat{k} \rightarrow \vec{n}$

$$\hat{n} = \frac{2\hat{i} + 3\hat{j} + 4\hat{k}}{\sqrt{29}}$$

$$\iint_S \frac{12z + 18 + 12y}{\sqrt{29}} \, dA$$

consider projection of S on XY plane;

So, proj. in 1st octant is bounded by

$2x + 3y = 12$, $x = 0$, $y = 0$

$$dA = \frac{dx\,dy}{\hat{n} \cdot \hat{k}} = \frac{\sqrt{29}}{4} dx\,dy$$

$$\rightarrow \iint_R \frac{1}{\cancel{\sqrt{29}}} (36 - 6x - 9y + 18 + 12y) \frac{\cancel{\sqrt{29}}}{4} dx\,dy$$

$$= \frac{1}{4}\iint_R 54 - 6x + 3y)\,dx\,dy$$

$$= \frac{1}{4}\int_0^6 \int_0^{\frac{12-2x}{3}} (54 - 6x + 3y)\,dy\,dx$$

$$= \frac{1}{4}\int_0^6 \cancel{54}^{18}\left(\frac{12-2x}{\cancel{3}}\right) - 6x^{2}\left(\frac{12-2x}{\cancel{3}}\right) + \frac{\cancel{3}}{2}\left(\frac{12-2x}{3}\right)^{\cancel{2}} \Bigg|$$

$$= \frac{1}{4}\int_0^6 (18-2x)(12-2x) + \frac{1}{6}(12-2x)^2\,dx$$

$$= \frac{1}{4}\int_0^6 (18\times12) - 60x + 4x^2 + \frac{1}{6}\left[144 - 48x + \overset{36}{\cancel{4}}x^2\right]dx$$

Same question

$$= \frac{1}{4}\int_0^6 216 - 60x + 4x^2 + \frac{1}{6}\left[144 - 48x + 4x^2\right]dx$$

$$= 138$$

$$\begin{array}{r} 18 \\ \times 12 \\ \hline 216 \end{array}$$

$\Rightarrow$ ~~Find values of~~ Show that the line integral

$$\int_{(1,1)}^{(3,4)} \frac{-x\,dy + y\,dx}{x^2} = d\left(-y/x\right)$$

$$= \frac{-y}{x}\Bigg|_{(1,1)}^{(3,4)} = -\frac{4}{3} + 1 = \frac{-1}{3}$$

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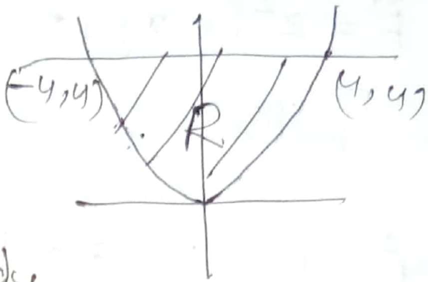
$$\Rightarrow \oint_C (x^2+y^2)\,dx + (5x^2 + -3y)\,dy$$

$C: x^2 = 4y$

$y = 4$



$$= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$0 \le y \le 4$

$$= \iint_R (10x - 2y)\,dx\,dy$$

$-2\sqrt{y} \le x \le 2\sqrt{y}$

$$= \int_0^4 \int_{-2\sqrt{y}}^{2\sqrt{y}} (10 - 2y)\,dx\,dy \rightarrow$$ to be cont

$\Rightarrow$ Evaluate the Surface Integral

$\iint_S \vec{F}\cdot\hat{n}\,dA$; $\vec{F} = 6z\hat{i} + 6\hat{j} + 3y\hat{k}$ and S is portion of the plane $\underbrace{2x + 3y + 4z = 12}_{f}$, which is in 1st octant.

$$\nabla f = \vec{n} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\hat{n} = \frac{2\hat{i} + 3\hat{j} + 4\hat{k}}{\sqrt{29}}$$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dA = \iint_S \frac{12z + 18 + 12y}{\sqrt{29}}\,dA$$

→ Evaluate the Surface Integral.

$$\iint_S \vec{F}\cdot\hat{n}\, dA$$

$$\vec{F} = z^2\hat{i} + xy\hat{j} - y^2\hat{k}$$

S is portion of surface of cylinder

$x^2+y^2=36$, $0\le z\le 4$ included in 1st octant

Sol:

$$\nabla f = \vec{n} = 2x\hat{i} + 2y\hat{j}$$

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{4} = \frac{x\hat{i}+y\hat{j}}{2}$$

$$\Rightarrow \iint_S \frac{xz^2+xy^2}{2}\, dA$$

(or)

$\operatorname{div} F = x$

D: $-6\le x\le 6$

$-\sqrt{36-x^2}\le y\le\sqrt{36-x^2}$

$0\le z\le 4$

$$\int_0^6\int_{-y}^{y}\int_0^4 (x)\, dz\,dy\,dx = \int_0^6\int_{-y}^{y} 4x\, dy\,dx$$

$$= 8\int_0^6 x\left(\sqrt{36-x^2}\right)dx = 8\left[6(0)-0\right] = 0$$