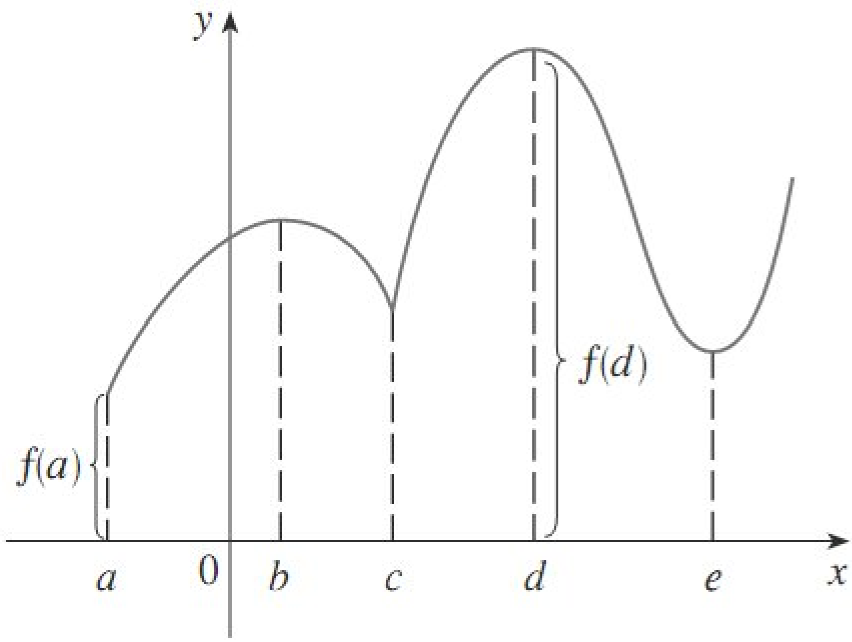
# §5.1—Extrema on an Interval

**(Absolute/Global) Extrema**

If $f$ is a function on an interval $I$, then $f(c)$ is the

I. (Absolute/Global) **Maximum** on $I$, IFF $f(c) \geq f(x)$ for all $x$ in $I$.

II. (Absolute/Global) **Minimum** on $I$, IFF $f(c) \leq f(x)$ for all $x$ in $I$.




$f(a)$ is the minimum. It is the lowest $y$-value on the graph

$f(d)$ is the maximum. It is the highest $y$-value on the graph

**Example 1:**

Sketch the following functions on the given interval, then, determine the extrema, if they exist.

a) $f(x) = x^2 + 1$ on $[-1, 2]$ b) $f(x) = x^2 + 1$ on $(-1, 2)$ c) $f(x) = \begin{cases} x^2 + 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$ on $[-1, 2]$

What other type of extrema are there?

**Relative/Local Extrema**

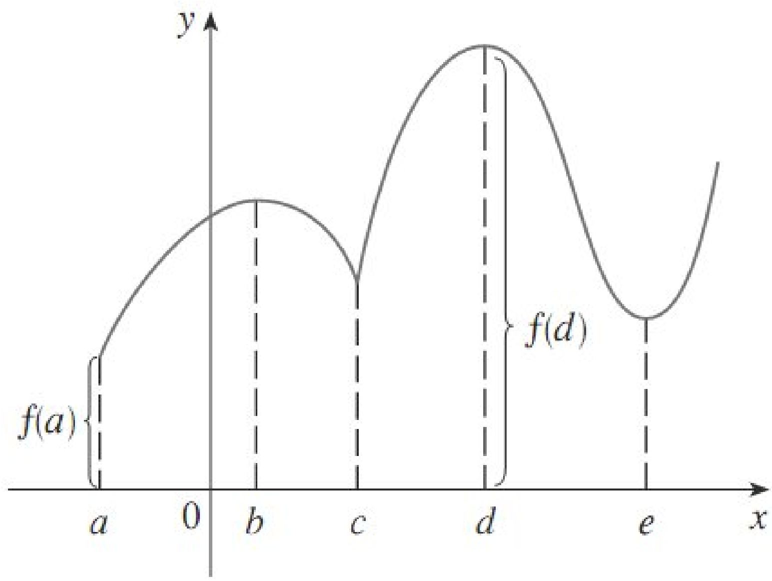
If there exists an **open** interval containing $x = c$ on which $f(c)$ is a . . .

I. . . . maximum, then $f(c)$ is called a **relative/local maximum** of *f*. Equivalently, we can say that *f* has a relative/local max at $(c, f(c))$.

II. . . . minimum, then $f(c)$ is called a **relative/local minimum** of *f*. Equivalently, we can say that *f* has a relative/local min at $(c, f(c))$.

In the graph of function $f$ below, the max was also a relative max, but not the only one. There is another relative max at $(b, f(b))$.

In the graph of function *f* below, the min was not at a relative min, but rather at an endpoint ***(can a relative extrema be at an endpoint???!!)***. The graph does have a relative min at $(c, f(c))$ and $(e, f(e))$.

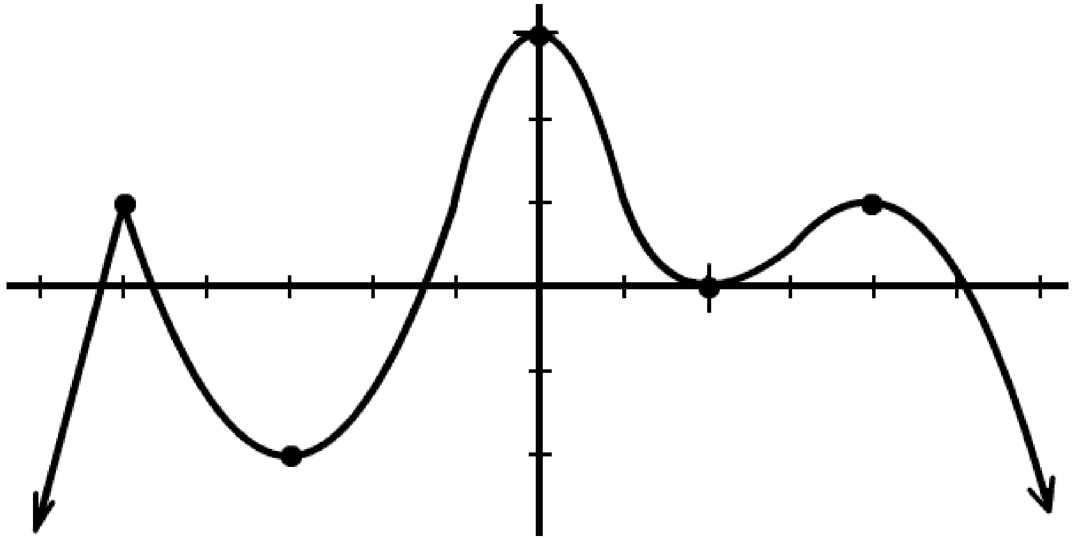



Notes:

- The plural term for maximums and minimums is **maxima** and **minima**, respectively. Also, we can collectively refer to both types as **extrema**. The singular for extrema is **extremum**.
- Extrema are *y*-values. We say they occur at either $x = c$ or at the point $(c, f(c))$.
- Absolute extrema and global extrema are interchangeable terms. In fact, when referring to simply "maximum" and/or "minimums," it is implied to be an absolute/global extrema.
- Relative extrema and local extrema are also interchangeable terms. These types MUST be qualified.
- When finding absolute extrema, an interval, whether it's open, closed, or half-open, is usually specified. If no interval is specified, we assume we are talking about the entire domain of the function.
- When we say relative extrema exist on an open interval, we mean "if an open interval exists" such that on either side of it, there are points lower or equal to (rel max) or higher or equal to it (rel min). This means that relative extrema must be on the interior of the interval and can NOT occur at endpoints of an interval (while absolute extrema can)**.

**There is actually some debate on this. Another school of thought accepts that relative extrema can occur on the end points of a domain. However, in this class we will be using the definition that says that they **can't** occur at the end points of a domain.

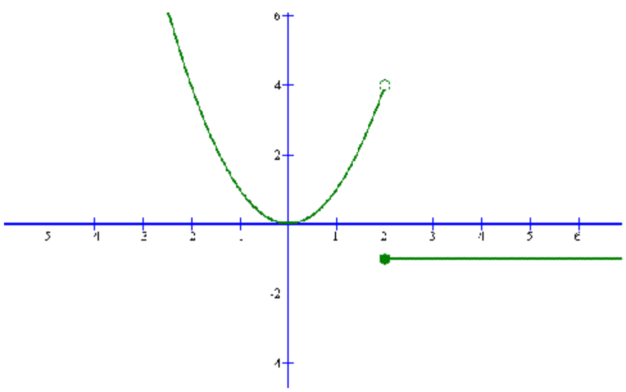
**Example 2:**
The graph of *f* is given below. Identify the extrema, both relative and absolute, on the interval $\left[-6,5\right]$.
Sketch $f'(x)$ if you have time.



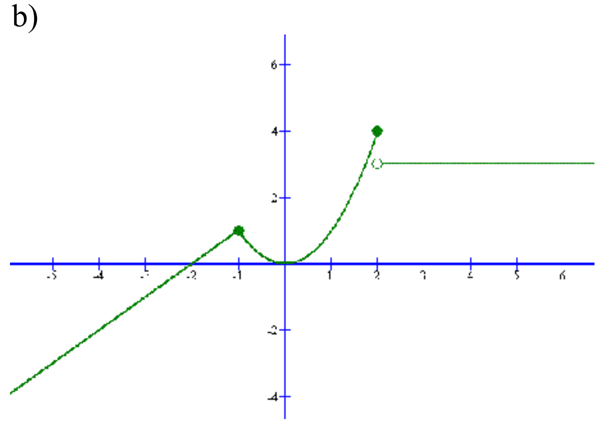
Relative Extrema may not always be at nice, tidy hills and valleys.

**Example 3:**
For each of the following, find any relative extrema on the shown intervals.

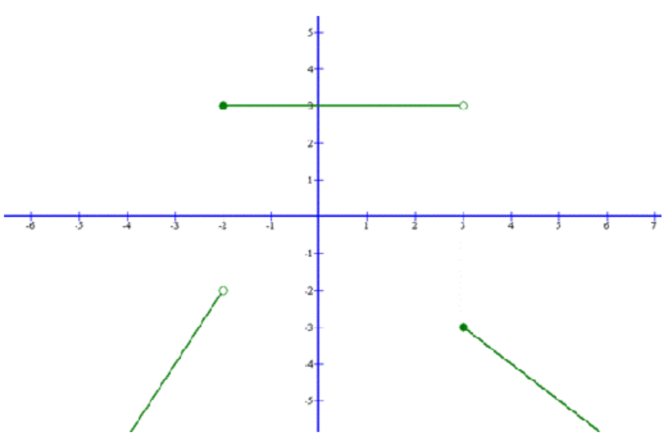
a)



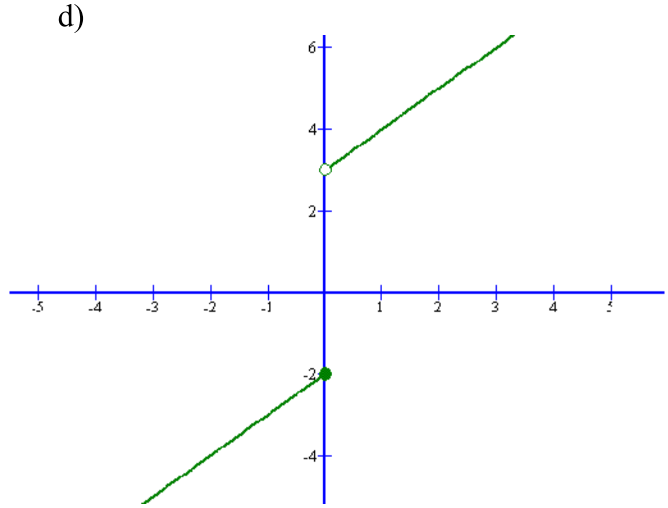
b)



c)



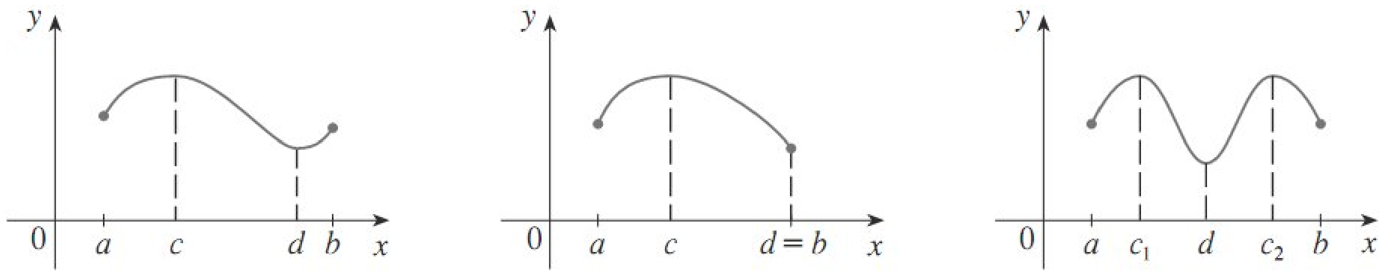
d)



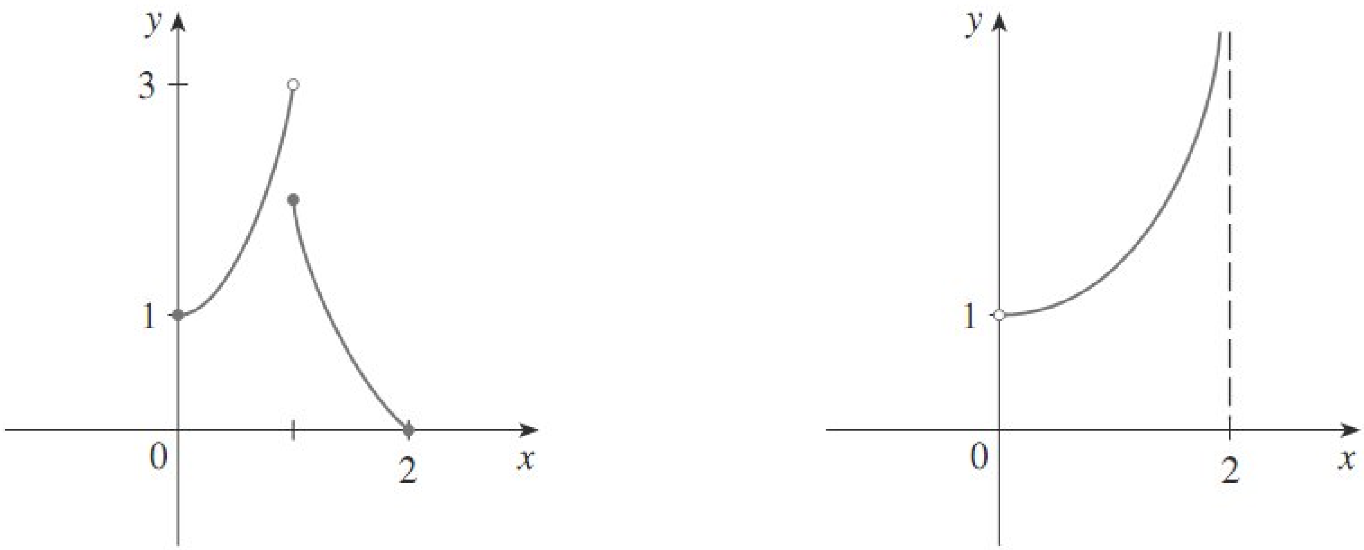
Under what conditions will a max and a min both occur?

**The Extreme Value Theorem (EVT)**

If $f$ is continuous on a closed interval $[a,b]$, then $f$ has both a maximum value and a minimum value on the interval.




If the hypothesis is not met, either the continuity or the closed interval part, there is not guarantee of the conclusion.

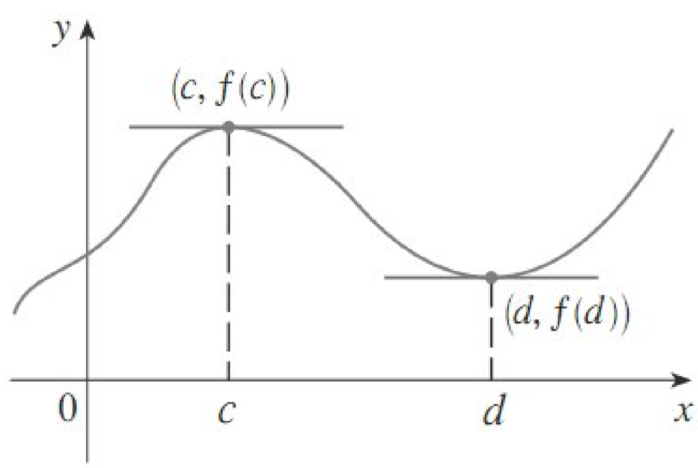



So how do we find extreme values? The theorem doesn't tell us ***that***! At what different values of $x$ can extreme values occur?

The graph of the function at right has a local max and a local min shown. It appears the tangent lines at these points are horizontal, and so we know that

$f'(c) = 0$ and $f'(d) = 0$.

If the derivative is defined at a local extremum, then this will always be the case. This is called **Fermat's Theorem**.



Is every value where $f'(x)=0$ a local Extremum?

**Example 4:**
Find the coordinates of any horizontal tangents of $f(x)=2x^3-3$ and determine if a relative extremum occurs there.

Can relative extrema occur at any other place other than where $f'(x)=0$?

**Example 5:**
Sketch the graph of $f(x)=2-|4x-8|$ and graphically find any relative extrema.

**Critical Values and Critical Points**

A **critical value** of a function *f* is a value $x=c$ <u>in the domain</u> of *f* such that either

$$f'(c)=0 \text{ or } f'(c)=DNE$$

If $x=c$ is a critical value, then $(c, f(c))$ is a critical point. A critical value is sometimes called a **critical number.**

**Fermat's Corollary:**

Relative Extrema can only occur on an open interval at a critical value.

**Example 6:**

Find the critical values of $f(x) = x^{3/5}(4-x)$.

So where can Absolute Extrema occur?

**Theorem:**

(Aboslute/Global) Extrema can only occur on at a **critical value** OR at an **endpoint** of an interval.

We now have one way to find the absolute extrema guaranteed by the EVT. It's called the **Closed Interval Method**.

To find absolute extrema of a continuous function $f(x)$ on a closed interval $[a,b]$.

1. Identify the endpoints.
2. Identify any critical values in $(a,b)$.
3. Find the function values at both endpoints and at all critical values in $(a,b)$.
4. The largest value will be the max. The smallest value will be the min.


*Who will be the biggest y-value? Who will be crowed the Global Maximum??? Will it be contestant A, the left endpoint? Will it be contestant B, the right endpoint? Or will it be contestant C, the critical value?*

*The good news is that the loser . . . is also a winner!!!!! Winner of the smallest y-value contest, and officially the Global Minimum. The only real loser is the one in the middle. So Sorry. Thanks for playing.*

**Example 7:**
Determine if the EVT applies. If so, find the extrema of $f\left(x\right)=3x^4-4x^3$ on the interval $\left[-1,2\right]$.

**Example 8:**
Determine if the EVT applies. If so, find the extrema of $f\left(x\right)=2x-3x^{2/3}$ on the interval $\left[-1,3\right]$

**Example 9:**
Determine if the EVT applies. If so, find the extrema of $f\left(x\right)=2\sin x-\cos 2x$ on $\left[0,2\pi\right]$.

What do we do if we don't have endpoints???

**Example 10:**

Find the extrema of $f\left(x\right)=\dfrac{1}{\sqrt{4-x^2}}$. Use your calculator to verify your result.

**Example 11:**

Find the extreme values of $f\left(x\right)=\begin{cases}5-2x^2, & x\le 1\\ x+2, & x>1\end{cases}$

It's worth pointing out that even though the sufficient conditions of the EVT are not met (the "if" part), a function may still have both an absolute max and min. We don't want to misinterpret the theorem. Here'an example.

**Example 12:**

Find the absolute maximum and minimum of the following function.

$$h(t)=\begin{cases}t^{2/3}, & -8<t\le 1\\ (t-1)^2+2, & 1<t\le 4\end{cases}$$

**Example 13:**

a) Sketch the graph of a function that has a local max at 2 and is differentiable at 2

b) Sketch the graph of a function that has a local max at 2 and is continuous but not differentiable at 2.

c) Sketch the graph of a function that has a local max at 2 and is not continuous at 2.