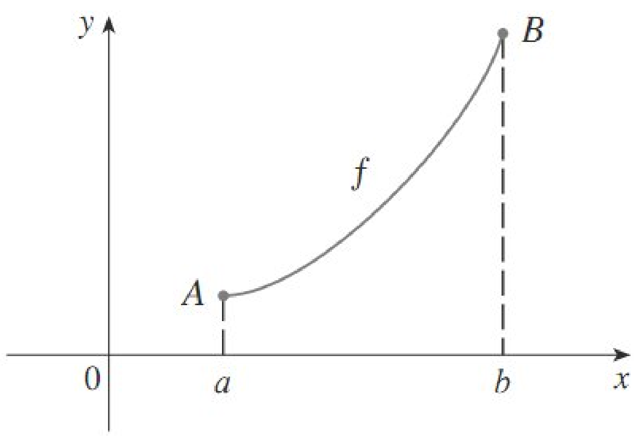
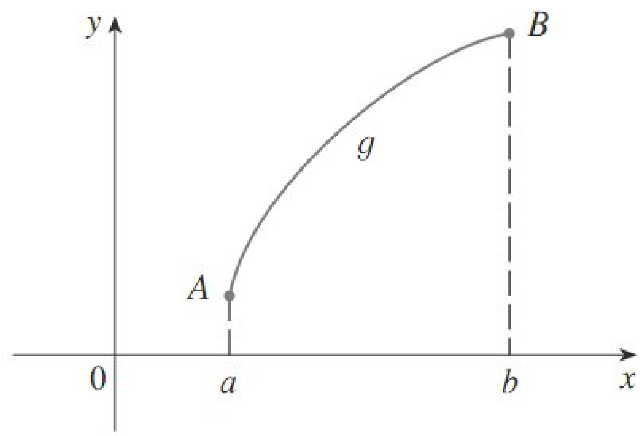
§5.4—Concavity and the Second Derivative Test

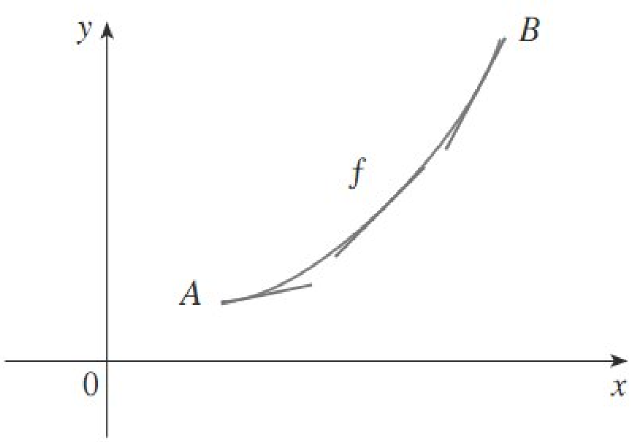
If we know that a function has a positive derivative over an interval, we know the graph of the function is increasing on that interval, but HOW is it increasing? At a constant rate? An increasing rate? A decreasing rate?

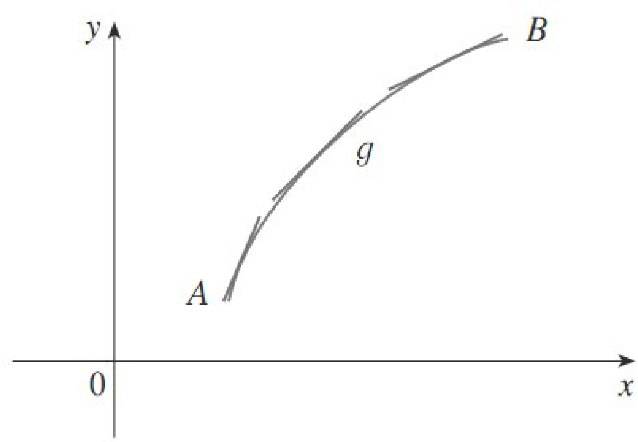
The two functions below both increase, but they bend differently, and, therefore, have different curvature. The function on the left increases at an increasing rate and the second increases at a decreasing rate (functions that increase at a constant rate are linear, boring, and don't require calculus.)





If we analyze the tangent lines in each of these cases at several points, we can begin to talk about how the slopes, and not just the y-values are changing.

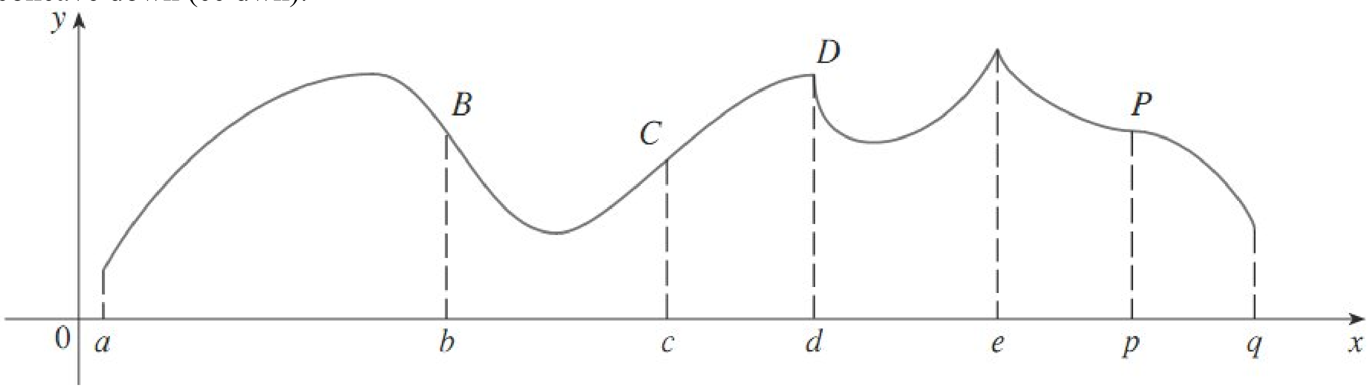




In the graph on the left, the **tangent lines are below the curve** and are increasing from left to right. In this case, we say the graph is **<u>concave up</u>** (like a cup).

In the graph on the right, the **tangent lines are above the curve** and are decreasing from left to right. In this case, we say that graph is **<u>concave down</u>** (like a frown).

Example 1:
In the graph below, list the open intervals on which the graph of the function is concave up (cc up) and concave down (cc dwn).



Anytime we talk about something changing, we're talking about the derivative. When we talk about the slopes of the tangent lines of a function changing, we're talking about how the derivative function is changing. This means we're talking about the second derivative!!

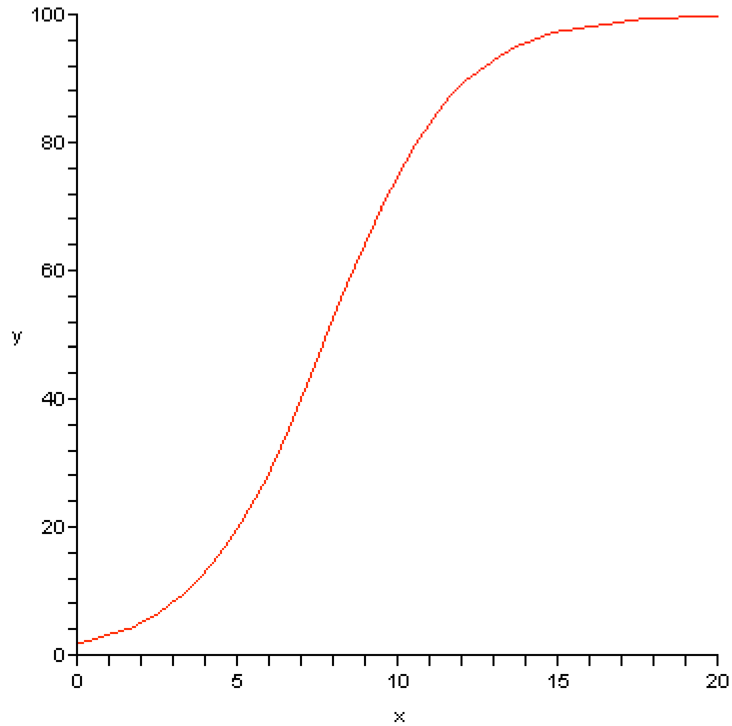
If the slopes of *f* are increasing, $f''(x) > 0$. If the slopes of *f* are decreasing, $f''(x) < 0$.

Concavity Test

1. If $f''(x) > 0$ for all x in an interval, then $f(x)$ is **concave up** (like a cup) on that interval.
2. If $f''(x) < 0$ for all x in an interval, then $f(x)$ is **concave down** (like a frown) on that interval.

Example 2:

The graph below shows the population of virulent bacteria in Mr. Korpi's knee (in millions) over a 20-hour period. How does the rate of population increase over time? When is this rate highest? Over what intervals is the graph concave up or concave down?



Example 3:
Sketch a possible graph of a function *f* that satisfies the following conditions:

i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

ii) $f''(x) > 0$ on $(-\infty, -2) \cup (2, \infty)$ and $f''(x) < 0$ on $(-2, 2)$.

iii) $\lim_{x \to -\infty} f(x) = -2$ and $\lim_{x \to \infty} f(x) = 0$

Definition

A point $\left(c, f\left(c\right)\right)$ on a curve $y = f\left(x\right)$ is called an **inflection point** if the graph of f changes from concave up to concave down OR from concave down to concave up at $\left(c, f\left(c\right)\right)$.

*Concavity can also change at a discontinuity, such as a VA, but it won't be an inflection point.
*To find possible inflection values (p.i.v.'s), find any $c \in D_f$ such that $f''\left(c\right) = 0$ or $f''\left(c\right)$ is undefined at $x = c$ (as long as $f\left(c\right)$ is defined).
*Not every p.i.v. is an inflection value.
*A chart can help you efficiently test for concavity in between p.i.v.'s and discontinuities.

Example 4:
Determine the open intervals on which the graphs of the following functions are concave up or concave down, then find any inflection points.

a) $y = 3 + \sin x \quad x \in \left[0, 2\pi\right]$

b) $f\left(x\right) = 6\left(x^2 + 3\right)^{-1}$

c) $g\left(x\right) = \dfrac{x^2 + 1}{x^2 - 4}$

d) $f\left(x\right) = x^4 - 4x^3$

e) $y = e^{-x^2}$

f) $g\left(x\right) = x^4$

g) $f(x) = \sqrt[3]{x}$

h) $y = \begin{cases} \sqrt{-x}, & x \le 0 \\ x^2, & x > 0 \end{cases}$

Here's a convenient application of the second derivative.

The Second Derivative Test (for Relative Extrema)

Let *f* be a function such that $f'(c) = 0$ and $f''(x)$ exists on an open interval containing $x = c$.

1. If $f''(c) > 0$, then $(c, f(c))$ is a relative minimum.
2. If $f''(c) < 0$, then $(c, f(c))$ is a relative maximum.
3. If $f''(c) = 0$, then the test fails and the First Derivative Test must be used.

Example 5:

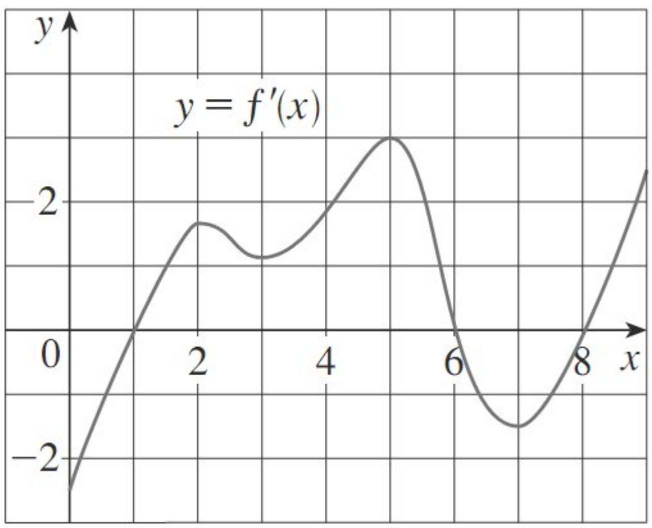
Find the relative extrema for $f(x) = 2x^5 - 20x^2$

Example 6:

The graph of the derivative f' of a two continuous functions on $[0,9]$ *f* are shown below. Answer the following questions for each.

a) On what intervals is *f* increasing or decreasing? Justify.
b) On what intervals is $f'(x) < 0$, $f'(x) > 0$.
c) At what values of *x* does *f* have a local maximum or minimum? Justify.
d) On what intervals is *f* concave upward or downward? Justify.
e) State the *x*-coordinate(s) of the point(s) of inflection.
f) Assuming that $f(0) = 0$, sketch a graph of *f*.

I.



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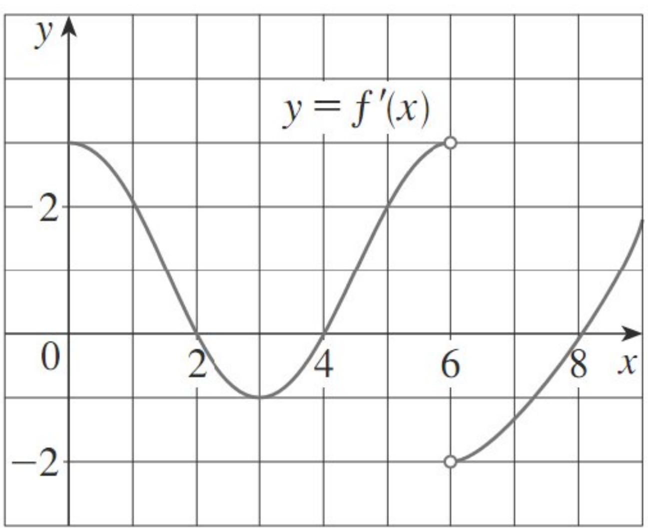

Putting it all together . . .

Example 7:
Sketch the graph of the following functions using whatever math you need. Verify (only) on the calculator.

a) $f\left(x\right)=x^{2/3}\left(6-x\right)^{1/3}$

b) $f\left(x\right)=e^{1/x}$