§3.4—Algebraic Limits

At this point, you should be very comfortable finding limits both graphically and numerically with the help of your graphing calculator. Now it's time to practice finding limits without a graph, without a calculator, and without your brilliant friend sitting next to you, relying entirely upon your algebraic prowess.

Here's a list of some of the more common algebraic methods for evaluating limits. The names of the methods are not important, but recognizing the situations in which each method arises **IS** important.

1. Direct Substitution

This is the method you should **ALWAYS** try first, never **NOT** first. If direct substitution does NOT yield one of the two indeterminate forms, $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then 99 times out of 99 ½ , this method works and the value obtained from direct substitution will be the limit value. This is because the function is continuous at that point. Remember to try it **first**! <u>When direct substitution does not work, the goal will be to remove the "bad guy" factor from the denominator that's causing the division by zero. Once the "bad guy" factor is gone, you will use direct substitution again, as you shall see.</u>

Ex 1) $\lim\limits_{x\to 4}\frac{\sqrt{x}}{x^2}=\frac{\sqrt{4}}{4^2}=\frac{1}{8}$

Ex 2) $\lim\limits_{x\to \frac{5\pi}{3}}\csc x=\csc\frac{5\pi}{3}=-\frac{2\sqrt{3}}{3}$ (Know the Unit Circle!!)

2. Factor and Cancel

We've already seen this method. This is an ideal method to use when the function is a rational function containing a ratio of two polynomial functions. A $\frac{0}{0}$ in this case guarantees that the "bad guy" factor, the one causing the zero in the denominator, can be divided out with its common factor in the numerator, leaving a function that "paves" over the hole in the original graph at that point. This allows us to use direct substitution on the remaining function.

Ex 1) $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}=\lim\limits_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}=\lim\limits_{x\to 2}(x+2)=4$

Ex 2) $\lim\limits_{x\to 1}\frac{x^3-1}{x-1}=\lim\limits_{x\to 1}\frac{\cancel{(x-1)}\left(x^2+x+1\right)}{\cancel{x-1}}=\lim\limits_{x\to 1}\left(x^2+x+1\right)=3$

HINT: Synthetic Division can help you find the remaining factors when you know a root

Ex 3) $\lim\limits_{x\to -1}\frac{x^4-x^3-x^2-3x-4}{2x^2-x-3}=?$

3. **Knowing a VA exists**

Remember any time direct substitution yields $\frac{\neq 0}{0}$, there is a VA at that x-value

Ex 1) $\lim\limits_{x\to 1}\frac{x^3+1}{x-1}=\frac{2}{0}=DNE$

Ex 2) $\lim\limits_{x\to\frac{\pi}{2}}\tan x = DNE$

4. **RATCON (short for RATionalization CONjugation)**

This method works well in the $\frac{0}{0}$ case when there is a sum or difference of one or more radical terms.

It involves multiplying by a clever form of one that is the $\frac{conjugate}{conjugate}$ of the radical binomial.

Ex 1)

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)=\lim_{x\to 0}\frac{x+1-1}{x\left(\sqrt{x+1}+1\right)}=\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}=\lim_{x\to 0}\frac{1}{1\left(\sqrt{x+1}+1\right)}=\frac{1}{2}$$

**NOTICE that we leave the denominator in factored form so that we can divide out the "bad guy" factor.

Ex 2)

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}=\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right)=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x+1}+2\right)}{x+1-4}=\lim_{x\to 3}\frac{\cancel{(x-3)}\left(\sqrt{x+1}+2\right)}{\cancel{x-3}}=\sqrt{3+1}+2=4$$

**NOTICE that we leave the numerator in factored form so that we can divide out the "bad guy" factor.

5. **LCM/LCM**

This method works well in the $\frac{0}{0}$ case when there is a complex or compound fraction (a fraction within a fraction.) It involves multiplying by a clever form on one that involves the Least Constant Multiple of all the denominators in the fraction.

Ex 1)

$$\lim_{x\to 0}\frac{\left[1/(2+x)\right]-(1/2)}{x}=\lim_{x\to 0}\frac{\left[1/(2+x)\right]-(1/2)}{x}\left(\frac{2(2+x)}{2(2+x)}\right)=\lim_{x\to 0}\frac{2-(2+x)}{2x(2+x)}=\lim_{x\to 0}\frac{-\cancel{x}}{2\cancel{x}(2+x)}=\lim_{x\to 0}\frac{-1}{2(2+x)}=-\frac{1}{4}$$

**NOTICE that we again leave the denominator in factored form so that we can divide out the "bad guy" factor.

Ex 2) $\lim\limits_{x\to 0}\frac{\left[1/(x+4)\right]-(1/4)}{x}=?$

Ex 3) $\lim\limits_{x\to 0}\frac{x}{\frac{1}{6}+\frac{1}{x-6}}=?$

6. Sledgehammer or Expand and Cancel and Factor and Cancel . . . .

This method works well in the $\frac{0}{0}$ case when there is a some obvious math to do, like expanding a binomial or distributing factors. Usually expanding everything out and collecting like terms will lead to something in which the "bad guy" factor can then be isolated by factoring, then removed.

Ex 1) $\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}=\lim\limits_{h\to 0}\dfrac{x^2+2xh+h^2-x^2}{h}=\lim\limits_{h\to 0}\dfrac{\cancel{h}(2x+h)}{\cancel{h}}=2x$

Ex 2) $\lim\limits_{h\to 0}\dfrac{(x+h)^2-2(x+h)+1-\left(x^2-2x+1\right)}{h}=?$

7. Getting' Triggy Wit It

We've already seen this method. It involves cleverly rewriting trig expressions as equivalent ones that are easier to work with. It's basically the reason we spent time in precal doing trig proofs. This method requires memorizing two important limits that will repeatedly appear in many problems. If you need them again, one last time, they're below. You'll also have fun doing a little numeric and algebraic gymnastics with these.

MEMORIZE the following:

If $f(0)=0$, then

$$\lim_{x\to 0}\frac{\sin f(x)}{f(x)}=1=\lim_{x\to 0}\frac{f(x)}{\sin f(x)} \qquad \text{AND} \qquad \lim_{x\to 0}\frac{1-\cos f(x)}{f(x)}=0=\lim_{x\to 0}\frac{\cos f(x)-1}{f(x)}$$

Ex. 1) $\lim\limits_{x\to 0}\dfrac{\tan(4x)}{x}=\lim\limits_{x\to 0}\dfrac{\sin 4x}{x\cos 4x}\left(\dfrac{4}{4}\right)=\lim\limits_{x\to 0}\dfrac{\sin 4x}{4x}\left(\dfrac{4}{\cos 4x}\right)=\lim\limits_{x\to 0}\dfrac{4}{\cos 4x}=4$

Ex 2) $\lim\limits_{\theta\to 0}\dfrac{\sec 2\theta-1}{\theta\sec 2\theta}=?$

Ex 3) $\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta\sec\theta}=$

Ex 4) $\lim\limits_{x\to 0}\dfrac{\tan 5x}{\sin 7x}=$

Ex 5) $\lim\limits_{t\to 0}\dfrac{\sin^2 2t}{3t^2}=$

Ex 6) $\lim\limits_{x\to \pi/4}\dfrac{1-\tan x}{\sin x-\cos x}=$

Ex 7) $\lim\limits_{x\to \pi/3}\dfrac{12\cos^2 x+2\cos x-4}{4\cos x-2}=$

8. <u>General Cleverness</u>:
Try this when all else fails. What can you do to an expression without changing its values? Multiply by a clever form of one, add a clever form of zero, make equivalent substitutions (like trig identities, $|x|$ with its piecewise definition or $\sqrt{x^2}$), factor, expand, pray, and anything else you can think of.

Ex1) $\lim\limits_{x\to 0}\dfrac{5x^2}{1-\cos x}=$

Ex2) $\lim\limits_{u\to 5^+}\dfrac{-u^2+4u+5}{|5-u|}=$

Ex3) $\lim\limits_{x\to 5}\dfrac{3x-15}{\sqrt{x^2-10x+25}}=$

9. <u>A Limit Sandwich</u>:
It's worth mentioning the Squeeze Theorem or Sandwich Theorem again.

Before we close out limits altogether, let's summarize a bit:

1. The limit is a y-value
2. The limit can exist independently of a function value
3. The limit only exists if the two one-sided limits BOTH exist and are the SAME value.
4. When the left-sided limit equals the right-sided limit and both equal the function value, the function is said to be continuous at that point.
5. If we have our calculator and don't know how to evaluate the limit, we can always graph it and plug values in on either side of our target value.
6. When evaluating limits algebraically, indeterminate forms such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, and $\infty-\infty$ require us to roll up our sleeves and get to work. The limit may or may not exist.
7. Piecewise functions are the exception to every rule.
8. Our friend $f(x)=\frac{|x|}{x}=\frac{x}{|x|}$ and his family members will always have a jump discontinuity at the x-value that yields $\frac{0}{0}$. Evaluating if from either side of this involves plugging in ANY value from that side and evaluating, since the graph is composed two horizontal lines.
9. A Limit can fail to exist at a point in any of the following circumstances:
    a. A jump discontinuity $\lim\limits_{x\to 0}\frac{|x|}{x}$
    b. A Vertical Asymptote $\lim\limits_{x\to 0}\frac{1}{x}$
    c. Graph not defined from both sides Ex) $\lim\limits_{x\to 0}\ln x$
    d. Oscillation between two fixed values Ex) $\lim\limits_{x\to 0}\sin\left(\frac{1}{x}\right)$
    e. Oscillation between two increasing large values Ex) $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)\sin\left(\frac{1}{x}\right)$
10. We can squeeze functions through a point even if the function doesn't exist there. **Ex)** $\lim\limits_{x\to 0}x\sin\left(\frac{1}{x}\right)$
11. Limits form the basis of both differential and integral Calculus
12. Limits are fun