§6.2—Definite Integrals & Numeric Integration

Calculus answers two very important questions. The first, how to find the instantaneous rate of change, we answered with our study of the derivative. We are now ready to answer the second question: how to find the area of irregular regions.

We start by introducing sigma notation.

The sum S of n terms $a_1,\ a_2,\ a_3, \ldots, a_n$ is written as

$$S = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

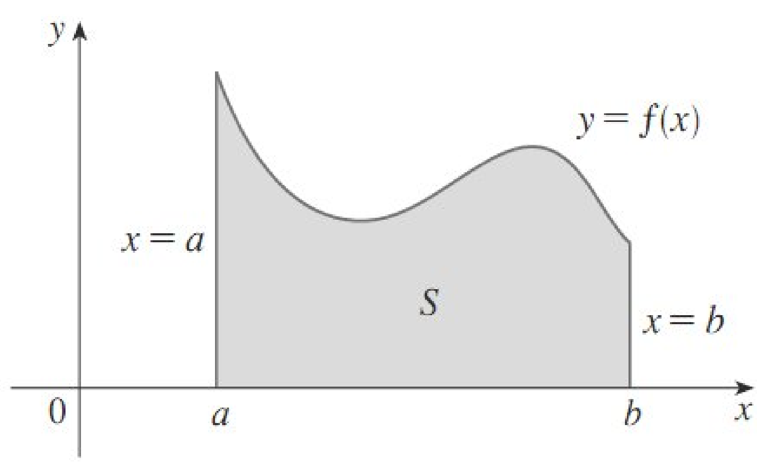
where i is called the **index of summation**, a_i is the ***i*th term** of the sum, and the **lower and upper bounds** of the summation are 1 and n.

Example 1:

Evaluate a) $\displaystyle\sum_{i=1}^{3} i^2$

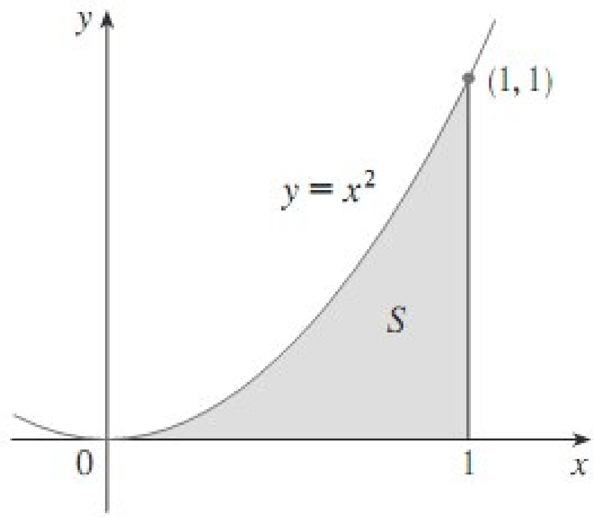
b) $\displaystyle\sum_{n=0}^{5} \frac{x^{n+1}}{2^n}$

We will now approximate an irregular area bounded by a function, the x-axis between the vertical lines $x = a$ and $x = b$, like the one below, by finding the areas of many rectangles and summing them up.



Example 2:

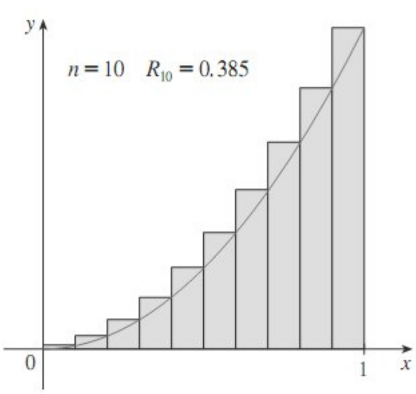
Use 4 subintervals of equal width to approximate the area under the parabola $f(x) = x^2$ from $x = 0$ to $x = 1$, notated as region *S*. Use L_4, R_4, M_4, and T_4. Compare to the actual area using your calculator's numeric integration capabilities.

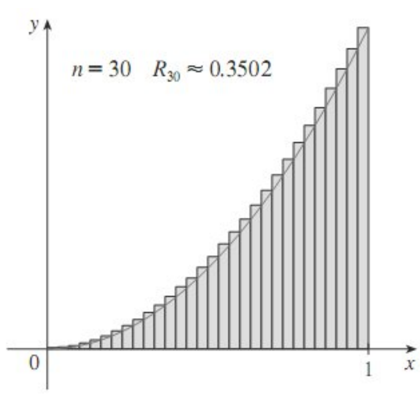


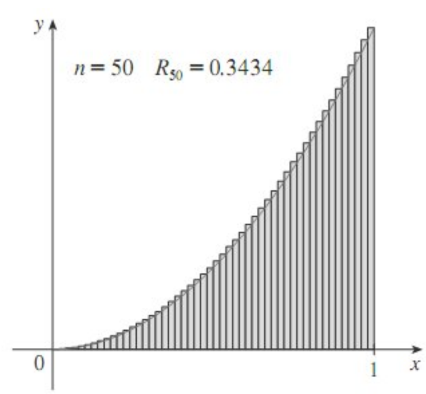
In this case, finding the area approximation using the left-endpoints of the intervals, L_4, gave us an under-approximation for the actual area. Using the right-endpoints, R_4, gives us an over-approximation. Together, these give us an upper and lower bound for the actual area (Note: depending on whether the function is increasing or decreasing, L_n or R_n could either be an upper or lower bound.)

If we desire better approximations of the area, we could partition our area into smaller subintervals using more rectangles. The following chart shows the areas of the same region *S*, using *n* rectangles of equal width using both the left-endpoint and right-endpoint methods.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3282500	0.3383500
1000	0.3328335	0.3338335







One can see the limiting process in action from the chart above. As n approaches infinity, the area approximations approach the actual area, each converging on the true value of the area.

The process of finding the sum of the areas of rectangles to approximate area of a region is called **Riemann Sums**, after Bernhard Riemann, who pioneered the process.

Riemann proved that the finite process of adding up rectangular areas could be found by a routine analytic process know as **definite integration**. Here's the essence of his great, time-saving work.

$$A = \lim_{n\to\infty} \sum_{i=a}^{b} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

Example 3:

Evaluate $\int_0^4 2x\,dx$ using the limit of infinite Riemann sums. Also, find the area geometrically as well as on your calculator. Hint: use Gauss's result that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Area Under the Curve:

If $y = f(x)$ is **nonnegative** and integrable over a closed interval $[a,b]$, then the **area under the curve** $y = f(x)$ **from *a* to *b*** is the definite integral of f from a to b.

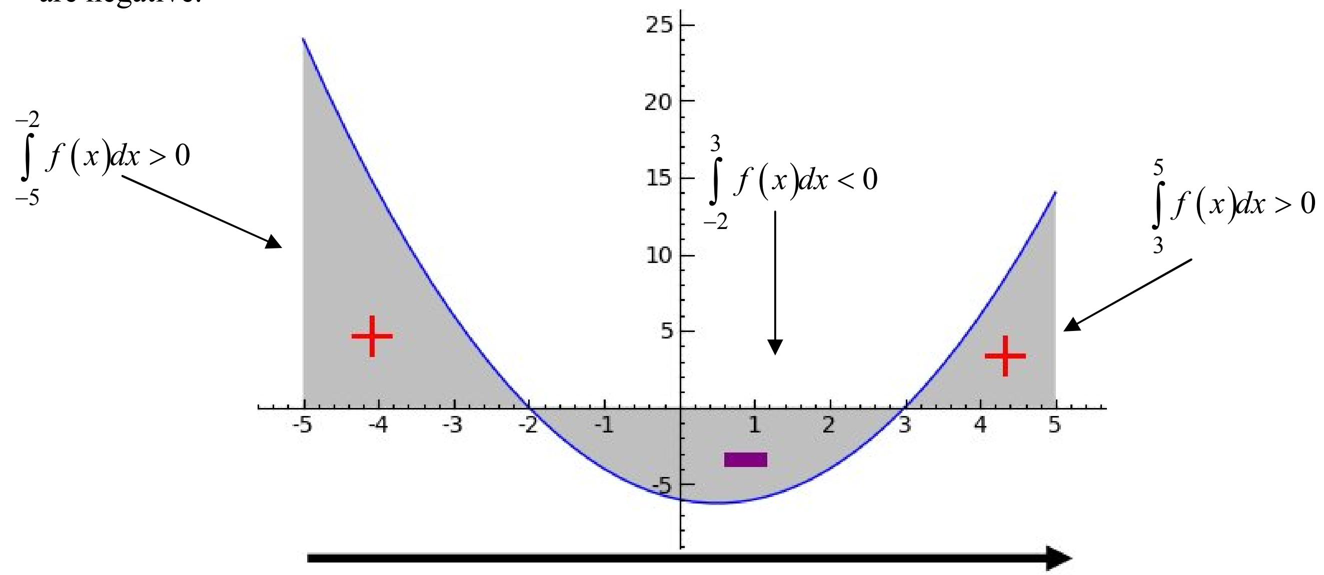
$$A = \int_a^b f(x)\,dx$$

If If $y = f(x)$ is **negative** and integrable over a closed interval $[a,b]$, then the **area under the curve** $y = f(x)$ **from *a* to *b*** is the definite integral of f from a to b.

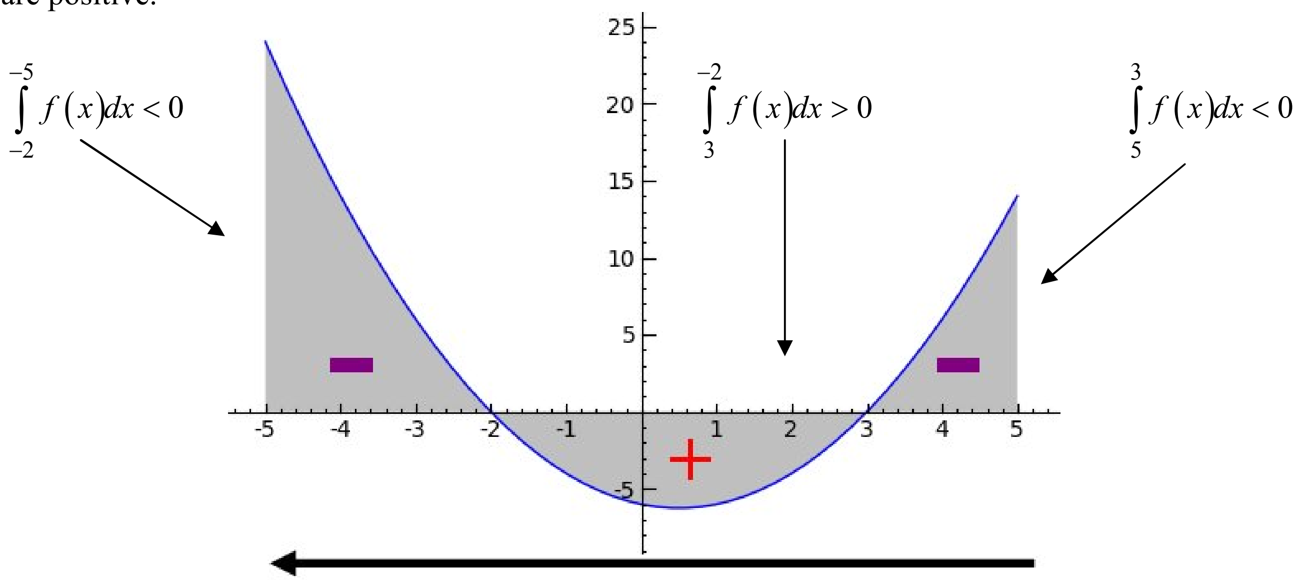
$$A = -\int_a^b f(x)\,dx$$

****In General, $\int_a^b f(x)\,dx$ does **NOT** give us the area but rather the **NET** accumulation over the interval from $x = a$ to $x = b$. If $y = f(x)$ is both positive and negative on closed interval $[a,b]$, then $\int_a^b f(x)\,dx$ will NOT give us the area.

- When integrating from **left to right**, regions above the *x*-axis are positive and regions below the *x*-axis are negative.



- When integrating from **right to left**, regions above the *x*-axis are negative and regions below the *x*-axis are positive.



This second result can be summarized, in general, this way:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

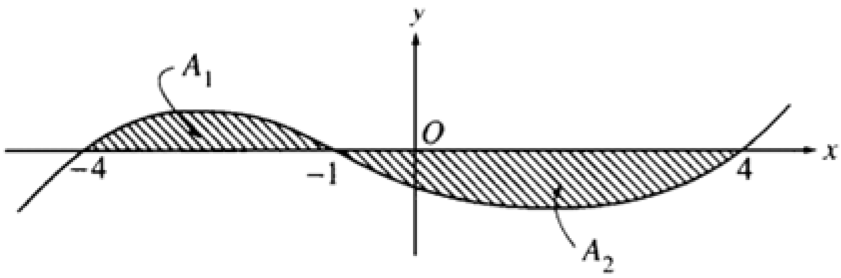
When using the definite integral to help you find area, you cannot always simply evaluate $\int_a^b f(x)\,dx$ if you are integrating over a interval containing negative y-values. It is important to remember the following:

AREA IS ALWAYS POSITIVE! AREA IS ALWAYS POSITIVE! AREA IS ALWAYS POSITIVE!

When using area to find definite integrals, we are responsible for assigning the regions the correct sign (positive or negative).

So, we can use area to help us find definite integrals, and we can use definite integrals to help us find areas!!

Example 4:
The graph of $y = f(x)$ is shown below. If A_1 and A_2 are positive numbers that represent the areas of the shaded regions, then find, in terms of A_1 and A_2, the following:



a) $\int_{-4}^{-1} f(x)\,dx$ b) $\int_{-1}^{4} f(x)\,dx$ c) $\int_{4}^{-1} f(x)\,dx$ d) $\int_{-4}^{4} f(x)\,dx$ e) $\int_{-4}^{4} f(x)\,dx - 2\int_{-1}^{4} f(x)\,dx$

If we are integrating by hand, we must decide if and where the graph crosses the x-axis, then split up our interval, manually making negative regions positive. The following property will help accomplish this:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 5:

Approximate $\int_1^5 x^2 - 4dx$ using four subintervals of equal length using the Midpoint Riemann and the Trapezoidal methods. Do these approximations represent the area of the region? What is the area of the region?

The other way to find area of $f(x)$ on an interval $[a,b]$, if the calculator is permitted, is to graph the transformation $y = |f(x)|$, then evaluate $\int_a^b |f(x)|\,dx$.

Theorem: Area of a region on a calculator

If $f(x)$ is a function defined on an interval $[a,b]$, the area of the region, A, bounded by $f(x)$ and the x-axis is given by

$$A = \int_a^b |f(x)|\,dx$$

Example 6:
Using your calculator's **fnint(** command, find the area of the region from Example 5.

Theorem: Continuity Implies Integrability

All continuous functions are integrable. That is, if a function *f* is continuous on a the closed interval $[a,b]$, then *f* is integrable on $[a,b]$.

$$C \to I$$
$$\neg I \to \neg C$$

Example 7:
Approximate the area of the region bounded by the *x*-axis and the function $f(x)=\sqrt{x}-3$ on the interval $[0,12]$ using 4 subintervals of equal length using all 4 numeric methods. Compare you answer the actual answer you find on your calculator. Is $\int_0^{12} f(x)\,dx$ positive or negative? Explain.

We can find areas when our function is given to us in either data form or graph form as well.

Example 8:
If $f(x)$ is a continuous function such that $f(x)\geq 0$ for all *x*, given selected values of *f* below, approximate $\int_0^{3} f(x)\,dx$ using numeric methods. Do these represent the area. **Also**, approximate $f'(1)$.

x		0	0.5	1	1.5	2	2.5	3
$f(x)$		2	4	6	7	4	1	5

Example 9:

If $f(x)$ is a continuous function for all *x*, given selected values of *f* below, approximate $\int_1^8 f(x)\,dx$ using numeric methods (reread the definite integral). Do these represent the area?. **Also**, approximate $f'(7)$.

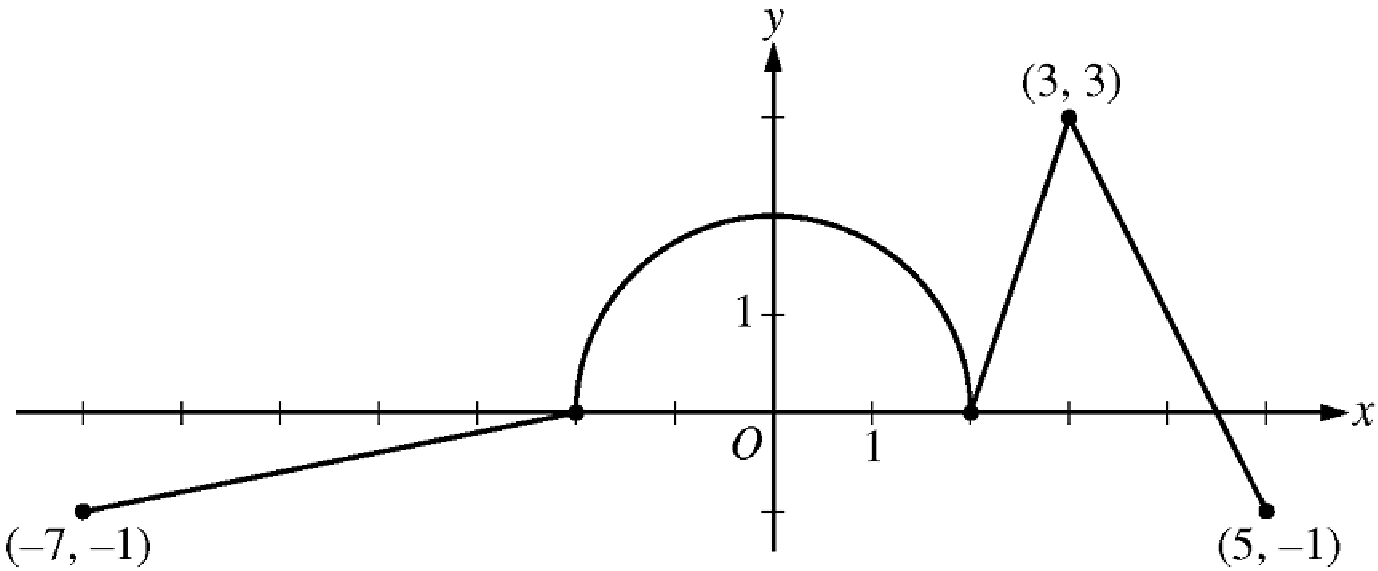
x		0	1	3	6	6.6	8	10
$f(x)$		4	3	3	1	5	8	10

Example 10:
Sketch the region corresponding to each definite integral, then evaluate each integral using a geometric formula. Decide if the integral represents the area of the region

a) $\int_1^3 4dx$ b) $\int_0^3 (x-2)dx$ c) $\int_{-6}^6 \sqrt{36-x^2}\,dx$ d) $\int_{-1}^4 \frac{|x|}{x}dx$

Example 11:

The graph is of a function $y = f(x)$. It is composed of three line segments and a semicircle. Evaluate a) $\int_{-7}^{0} f(x)\,dx$ and b) $\int_{2}^{5} f(x)\,dx$. Do these integrals represent the area of the region? Why or why not? If not, what is the area of each region?



Properties of integrals:

1. If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$
2. If f is integrable on $[a,b]$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b c\,dx = c(b-a)$
4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Example 12:

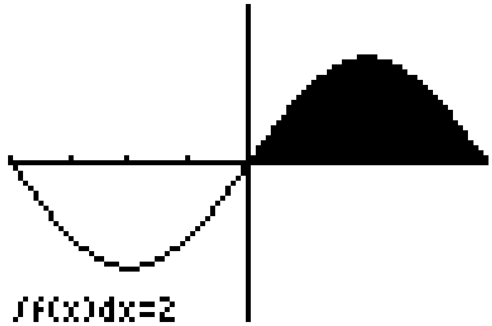
If $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} (3f(x)+2)\,dx$

Example 13:

$\int_{-5}^{6} \frac{9-x^2}{x-3}\,dx$ using area of the region to help evaluate the integral.

Example 14:

If $\int_0^{\pi} \sin x dx = 2$, use this fact and the symmetry of the graph of $f(x) = \sin x$ to find the following:



a) $\int_{\pi}^{2\pi} \sin x dx$ b) $\int_{0}^{2\pi} \sin x dx$ c) $\int_{\pi}^{\pi/2} \sin x dx$ d) $\int_0^{\pi} (2+\sin x)dx$ e) $\int_0^{\pi} 2\sin x dx$

f) $\int_2^{\pi+2} \sin(x-2)dx$ g) $\int_0^{2\pi} \sin\left(\frac{x}{2}\right)dx$ h) $\int_{-\pi}^{\pi} \sin x dx$ i) $\int_0^{\pi} \cos x dx$ j) $\int_0^{2\pi} \cos x dx$