§8.5—L'Hôpital's Rule & Indeterminate forms

We're back to evaluating limits again. Recall the two **indeterminate forms** $\frac{0}{0}$ and $\frac{\infty}{\infty}$. When direct substitution yielded either one of these results, we had to roll up our sleeves and try various algebraic methods to compute the limit, because, even though the function value failed to exist there, the limit still could, and it could be ANYTHING from 5 to $-\pi$ to 1000000 to "peaches," . . . . well, maybe not peaches.

In this section, we will be evaluating limits with similarly interesting, perhaps surprising, results. We will also be encountering several more indeterminate forms and ways to remedy them.

Example 1:
Make a guess as to what you think the limit will be, then check by evaluating the expression at the indicated values using your calculator.

a) $\lim\limits_{x\to 0}\left(\dfrac{e^{3x}-1}{x}\right)=?$

$\left.\dfrac{e^{3x}-1}{x}\right|_{x=0.1}=$ $\left.\dfrac{e^{3x}-1}{x}\right|_{x=0.01}=$ $\left.\dfrac{e^{3x}-1}{x}\right|_{x=0.001}=$

b) $\lim\limits_{x\to 1^+}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)=?$

$\left.\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)\right|_{x=1.1}=$ $\left.\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)\right|_{x=1.01}=$ $\left.\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)\right|_{x=1.001}=$

c) $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x=?$

$\left.\left(1+\dfrac{2}{x}\right)^x\right|_{x=100}=$ $\left.\left(1+\dfrac{2}{x}\right)^x\right|_{x=1000}=$ $\left.\left(1+\dfrac{2}{x}\right)^x\right|_{x=10000}=$

Let's review some earlier algebraic techniques for dealing with indeterminate forms.

Example 2:

a) $\lim\limits_{x\to -1}\dfrac{2x^2-2}{x+1}=$ b) $\lim\limits_{x\to\infty}\dfrac{3x^2-1}{2x^2+1}=$ c) $\lim\limits_{x\to 7}\dfrac{\sqrt{x+2}-3}{x-7}=$

Not all indeterminate forms can be reconciled via algebraic techniques. Another method, discovered by Swiss mathematician Johann Bernoulli, is called **L'Hôpital's Rule**, named after the 17th century French mathematician Guillaume de L'Hôpital who did not discover it, but who first published it in the first-ever textbook on differential calculus. From the looks on their faces, can you guess which one is L'Hôpital and which one is Bernoulli?

L'Hôpital's Rule (or Bernoulli's Rule)

If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ yields either of the indeterminate forms $\dfrac{0}{0}$ or $\pm\dfrac{\infty}{\infty}$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

Example 3:
Evaluate the following.

a) $\lim\limits_{x\to 0}\dfrac{e^{3x}-1}{x}=$ b) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}=$ c) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sec x}{1+\tan x}=$ d) $\lim\limits_{x\to\infty}\dfrac{\ln x}{2\sqrt{x}}=$

Sometimes we may need to repeat ourselves. Sometimes we may need to repeat ourselves.

Example 4:

a) $\lim\limits_{x\to-\infty} \dfrac{x^2}{e^{-x}} =$

b) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1-x/2}{x^2} =$

Be careful when evaluating one-sided limits:

Example 5:

a) $\lim\limits_{x\to 0^+} \dfrac{\sin x}{x^2} =$

b) $\lim\limits_{x\to 0^-} \dfrac{\sin x}{x^2} =$

The rule works great, but it only works with the two forms $\dfrac{0}{0}$ or $\pm\dfrac{\infty}{\infty}$. There are other indeterminate forms including 0^0, 1^∞, $\infty-\infty$, $0\cdot\infty$, and ∞^0. We can still use the rule, but we have to first convert them to $\dfrac{0}{0}$ or $\pm\dfrac{\infty}{\infty}$.

Example 6:

a) $\lim\limits_{x\to\infty} e^{-x}\sqrt{x} =$

b) $\lim\limits_{x\to 1^+} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right) =$

Sometimes we need logs to come to our rescue.

Example 7:

a) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x =$

b) $\lim_{x\to 0^+} x^x =$

c) $\lim_{x\to\infty} x^{1/x} =$

Here's a fun one.

Example 8:

$$\lim_{x\to 1}\frac{\int_1^x \cos t\,dt}{x^2-1} =$$