§6.4—Integration by *u*-sub & pattern recognition

We can easily integrate something like $\int \sin x dx$, but what about something like $\int 3x\sin\left(x^2+1\right)dx$?
Recall the Chain Rule:

$$\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$$

If we solve this differential equation, take the integral of both sides, we get

$$f\left(g\left(x\right)\right)+C=\int f'\left(g\left(x\right)\right)\cdot g'\left(x\right)dx$$

We state this formally as a theorem for integrating a composite function.

Antidifferentiation of a composite function

Let g be a function whose range is an interval, I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f\left(g\left(x\right)\right)\cdot g'\left(x\right)dx=F\left(g\left(x\right)\right)+C$$

In other words, if we have an integral of either a product or a quotient, and we recognize a factor as being a derivative of the other factor, we can "repackage" the derivative by-product from the chain rule to arrive at the original function. It can be helpful to call the "inside" function, $g\left(x\right)$ above, as u, and the outside function containing u, $f\left(x\right)$ above, to be $f\left(u\right)$. In this case we will use differential form and call u' du.

Rewriting the above rule, we get
Let $u=g\left(x\right)$ and $du=g'\left(x\right)dx$

$$\int f\left(u\right)du=F\left(u\right)+C$$

This formal rewriting of our original integral in terms of a new, temporary, "dummy" variable u can be helpful, but is not always necessary, nor desirable. If you are able to identify both the "inner" and "outer" functions, with a little practice, you will be able to integrate quickly in your head, as the rule of integration will only involve the "outer" function, with a possible "correction" of a scalar multiple (only).

We will work both ways. You decide which way is easier.

Example 1:

Evaluate $\int \left(x^2+1\right)^2 (2x)\,dx$ using u-substitution as well as by pattern recognition.

Example 2:

Evaluate $\int 3x\sin\left(x^2+1\right)dx$ using u-substitution as well as by pattern recognition.

When integrating by pattern recognition, you will collect no more than three different types of scalar/constant multiples out in front of your antiderivative:

- constant multiples that were there in the original integrand (I call these "riders"),
- constant multiples generated from an integration rule like the power rule (I call these "rule" constants), and finally,
- constant multiples that "correct" any unwanted constant multiple generated by the derivative of the "inside function." These values will always be the reciprocal of the unwanted value (I call these "corrections.")

After integrating, you can combine all of these scalar multiples to get your final answer. Oh, and don't forget your $+C$

Example 3:

Evaluate $\int \frac{x\left(x^2+1\right)^2}{\pi}dx$

Example 4:

Each of the following have the same inside function, but a different outside function, and hence, a different rule of integration. Evaluate each.

a) $\int 5\sin 2x e^{\cos 2x}dx$

b) $\int 7\sin 2x\sqrt[3]{\cos^2 2x}dx$

c) $\int 2\sin 2x\cos^2 2x dx$

d) $\int \frac{2\sin 2x}{3\sqrt{1-\cos^2 2x}}dx$

e) $\int \frac{5\sin x}{6\sqrt{\cos^3 x}}dx$

f) $\int (\sin 2x)\sin(\cos 2x)dx$

Example 5:
Evaluate $\int \sec^2 x \tan x dx$ two different ways. Show the antiderivatives are equivalent.

Example 6:
Evaluate $\int 4x^2 5^{x^3+7} \sec^2\left(5^{x^3+7}\right) dx$

Example 7:
Find each of the following by being clever, then memorize the results.

a) $\int \tan x dx$ b) $\int \cot x dx$ c) $\int \sec x dx$ d) $\int \csc x dx$

Example 8:
Evaluate the following:

a) $\int 5x^2 \tan\left(x^3+1\right)dx$

b) $\int 2e^{-x}\sec\left(e^{-x}\right)dx$

Example 9:
Evaluate the following:

a) $\int \frac{x}{x^2-4}dx$

b) $\int \frac{5}{x\ln x}$

c) $\int \frac{\sqrt[3]{\ln^2 x}}{4x}dx$

d) $\int \frac{\arctan x}{1+x^2}dx$

Example 10:

Evaluate the following two ways, using pattern recognition and u-substitution: $\int_0^1 x\left(x^2+1\right)^3\,dx$

Inverse Trig Integral forms: (MEMORIZE)

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\,arc\sec\frac{|u|}{a} + C$$

Example 11:

Evaluate each of the following.

a) $\int \frac{dx}{\sqrt{4-x^2}}$

b) $\int \frac{dx}{2+9x^2}$

c) $\int \frac{dx}{x\sqrt{4x^2-9}}$

d) $\int \frac{dx}{\sqrt{e^{2x}-1}}$

e) $\int \frac{x+2}{\sqrt{4-x^2}}\,dx$

f) $\int \frac{3x^3}{\sqrt{5-x^4}}dx$

g) $\int \frac{3x^2}{\sqrt{5-x^4}}dx$

h) $\int \frac{3x}{\sqrt{5-x^4}}dx$

Example 12:
Find each of the following, if possible.

a) $\int \frac{3}{x^2+6x+9}\,dx$

b) $\int \frac{3}{x^2+6x+10}\,dx$

c) $\int \frac{3}{x^2+6x+8}\,dx$

Example 13:

Find the area of the region bounded by the graph of $f(x)=\dfrac{1}{\sqrt{3x-x^2}}$, the x-axis, and the lines $x=\dfrac{3}{2}$ and $x=\dfrac{9}{4}$. Verify on your calculator.

Example 14:

Find a) $\int \dfrac{x^2+x+1}{x^2+1}dx$ b) $\int \dfrac{x^3-3x^2+4x-9}{x^2+3}dx$

More Trig

Example 15:

Find a) $\int \sin x \cos x dx$ b) $\int \sin^2 x \cos x dx$ c) $\int \sin^2 x dx$ d) $\int \sin^3 x dx$

Example 16:
Find $\int \sqrt{2x-1}\,dx$

Sometimes we MUST use *u*-substitution.

Example 17:
Find the following.

a) $\int x\sqrt{2x-1}\,dx$

b) $\int \frac{x^2}{\sqrt{2x-1}}dx$

c) $\int \frac{2x}{(x+1)^2}\,dx$

d) $\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$

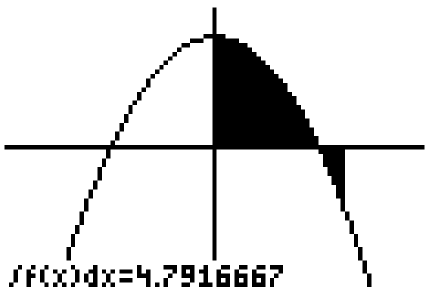
e) $\int \frac{dx}{1+\sqrt{2x}}$

Sometimes symmetry can help us evaluate an integral.

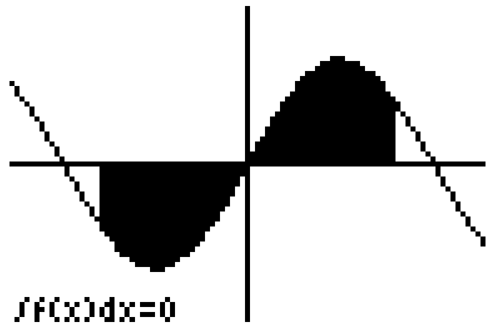
Integration of Even and Odd functions

Let f be an integrable function on a closed interval $[-a,a]$.

1. If f is an EVEN function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
2. If f is an ODD function, then $\int_{-a}^{a} f(x)\,dx = 0$



An Even function on a symmetrical interval



An Odd function on a symmetrical interval

Example 18:

Evaluate each of the following.

a) $\int_{-3}^{3}\left(\sin x + x^3 + \frac{x}{x^2+1}\right)dx$ b) $\int_{-\pi}^{\pi}\left(1+x^2+\cos x\right)dx$ c) $\int_{-1}^{1}\left(5x^5+x^4+11x^3+x^2-15x+1\right)dx$