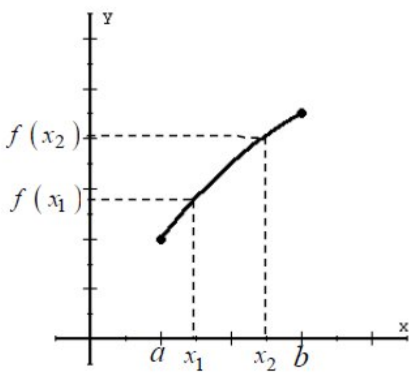
§5.3—Increasing, Decreasing, and the 1st Derivative Test

If a graph exists on an interval, it is doing one of three things:
1. Increasing (y-values rise as x-values increase)
2. Decreasing (y-values fall as x-values increase)
3. Staying Constant (y-values stay the same as x-values increase)

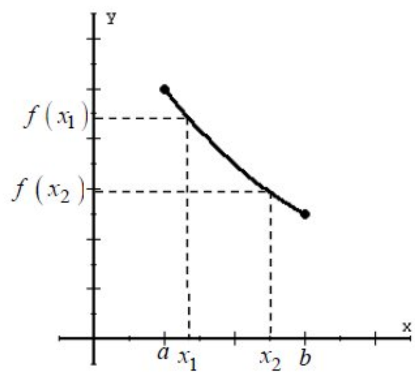
We would like to be able to interpret information about a function f by analyzing information about its derivative $f'(x)$. Since $f'(x)$ tells us the slope of the curve $y=f(x)$ at any point $(x, f(x))$, it tells us whether the curve is going up, going down, or staying the same **at each point**. If we, therefore know the values of $f'(x)$ on a given interval, we know whether the graph is increasing, decreasing, or constant on that interval.



$f(x_1) < f(x_2)$

f is Increasing

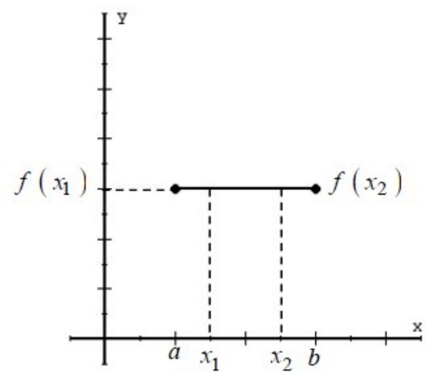
$f'(x) > 0$



$f(x_1) > f(x_2)$

f is Decreasing

$f'(x) < 0$



$f(x_1) = f(x_2)$

f is Constant

$f'(x) = 0$

Theorem: Test for Increasing and Decreasing Functions

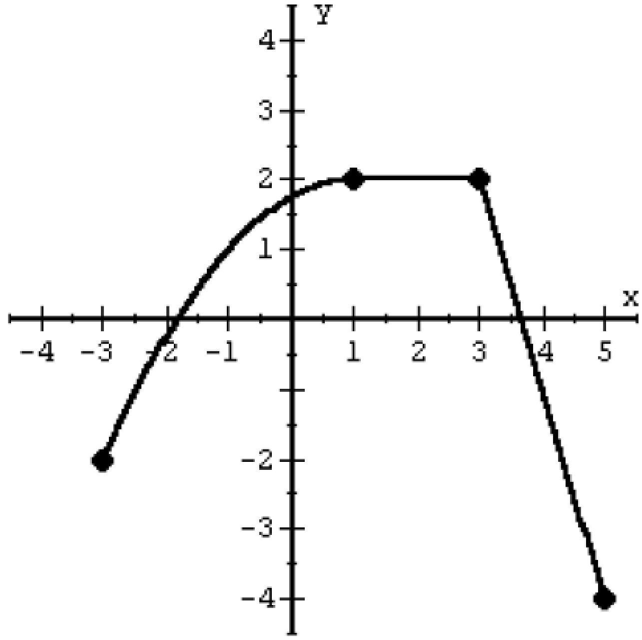
Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).
1. If $f'(x) > 0 \ \ \forall x \in (a,b)$, then f is increasing on $[a,b]$.
2. If $f'(x) < 0 \ \ \forall x \in (a,b)$, then f is decreasing on $[a,b]$.
3. If $f'(x) = 0 \ \ \forall x \in (a,b)$, then f is constant on $[a,b]$.

If $f'(x) > 0$ on an interval, then f is **monotonic increasing** on that interval.

If $f'(x) < 0$ on an interval, then f is **monotonic decreasing** on that interval.

If $f'(x) = 0$ on an interval, then f is **constant** (and boring) on that interval.

Example 1:
The graph of a function $f(x)$ defined on $[-3,5]$ is shown. List the **open** intervals over which the function is increasing, decreasing, and/or constant.

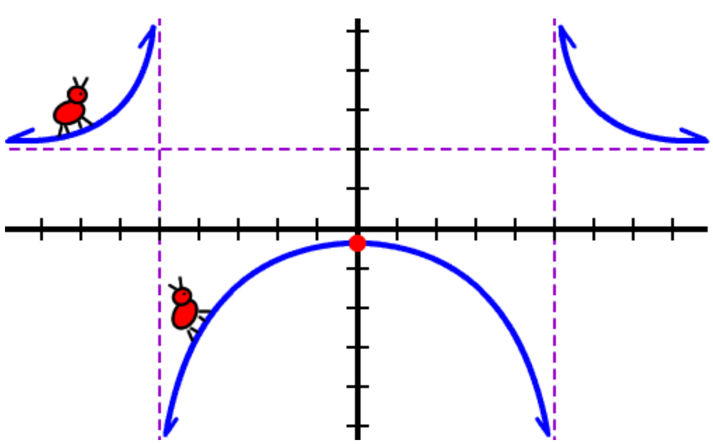


At what values of x can the graph of a function change its increasing/decreasing/constant status?

The graph of a **continuous function** can only change its increasing/decreasing/constant status at a **critical value**.

The graph of a **discontinuous function** can change its increasing/decreasing/constant status at a **critical point** OR a **discontinuity**.

Example 2:
The graph of a function $f(x)$ defined $\forall x \neq -5, 5$ is shown. List the **open** intervals over which ants walk uphill (increasing) or downhill (decreasing).



Without the graph, we must find any critical values and discontinuities, then test the intervals in between them.

Example 3:

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Example 4:

Find the open intervals on which $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing or decreasing.

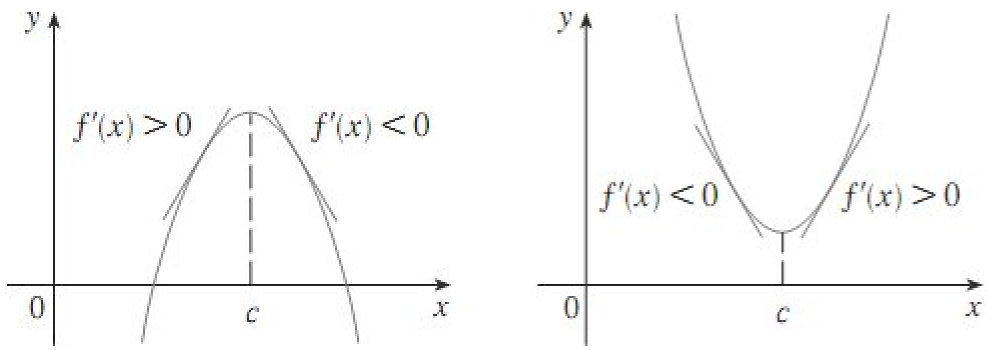
For a continuous function, knowing when and where the sign of the derivative changes lends great insight into existence of any Relative Maximums or Relative Minimums.

Theorem: The First Derivative Test (for Relative Extrema)

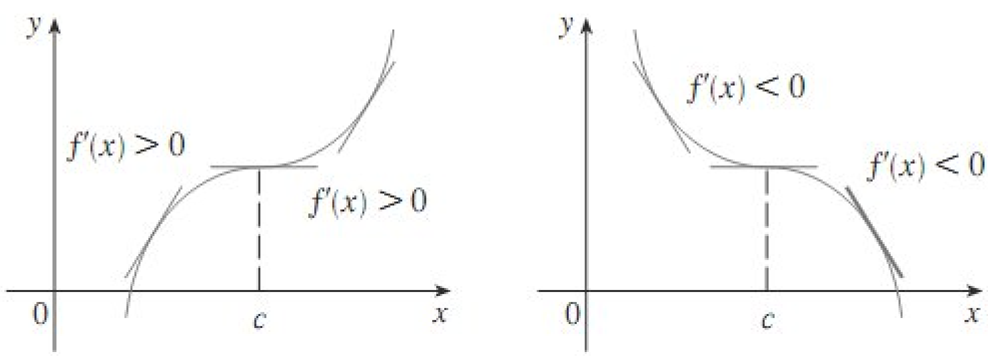
Let $x = c$ be a critical value of a continuous function *f*.

1. If $f'(x)$ changes from negative to positive at $x = c$, then *f* has a **relative minimum** at $x = c$ (or at $(c, f(c))$).
2. If $f'(x)$ changes from positive to negative at $x = c$, then *f* has a **relative maximum** at $x = c$ (or at $(c, f(c))$).
3. If $f'(x)$ is positive on both sides of $x = c$ or negative on both sides of $x = c$, then $f(c)$ is neither a relative maximum nor a relative maximum.

Here's the visualization:



(a) Local maximum (b) Local minimum



(c) No maximum or minimum (d) No maximum or minimum

**When using the First Derivative Test to justify Relative Extrema, you MUST write a concluding statement clearly communicating the type of sign change of *f* at each $x = c$. Thou mustn't use pronouns either!!

Example 5:

Find the local extrema of the function $f(x)=\frac{1}{2}x-\sin x$ on the interval $[0,2\pi]$. Justify.

Example 6:

Find the relative extrema of $f(x)=\left(x^2-4\right)^{2/3}$. Justify.

Example 7:

Find the relative extrema of $f(x)=-\frac{x^4+1}{x^2}$. Justify.