§5.2—Rolle's Theorem & the MVT

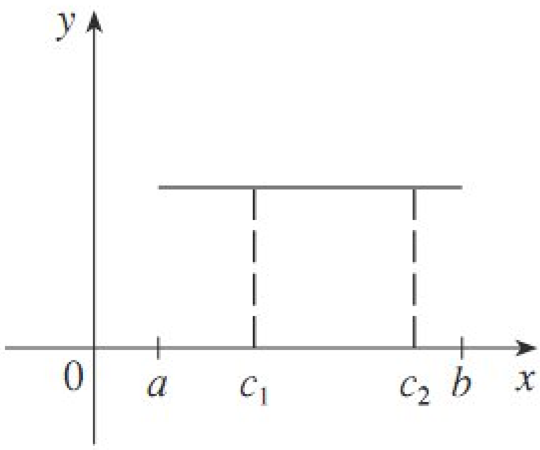
Rolle's Theorem

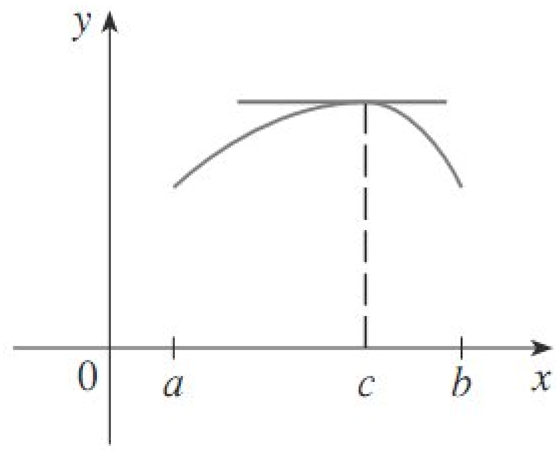
Let *f* be a function that satisfies the following three hypothesis:

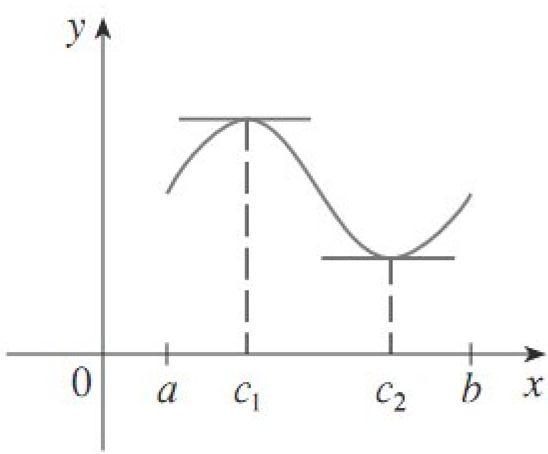
1. *f* is continuous on the closed interval $[a,b]$.
2. *f* is differentiable on the open interval (a,b).
3. $f(a)=f(b)$

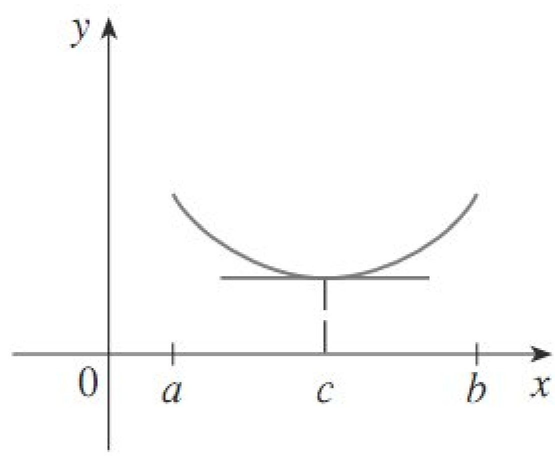
Then there is a number $x=c$ in (a,b) such that $f'(c)=0$

Here are some functions that satisfy all three hypotheses.







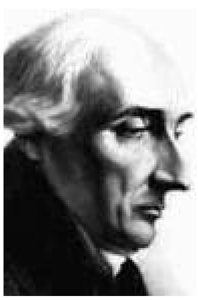


Example 1:

If we apply Rolle's Theorem to the position function $s=f(t)$ of a moving object. If the object is in the same place at two different instances $t=a$ and $t=b$, then $f(a)=f(b)$. Rolle's Theorem says that there is some instant of time $t=c$ between *a* and *b* when $f'(c)=0$; that is, the velocity is 0. This can be visualized when a ball is thrown directly upward.

Example 2:
Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

French mathematician Joseph-Louis Lagrange (1736 – 1813) first stated the next very important theorem.



The Mean Value Theorem:

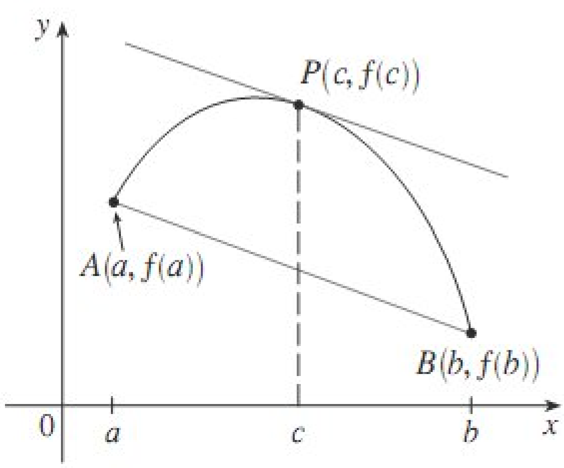
Let *f* be a function that satisfies the following two hypothesis:

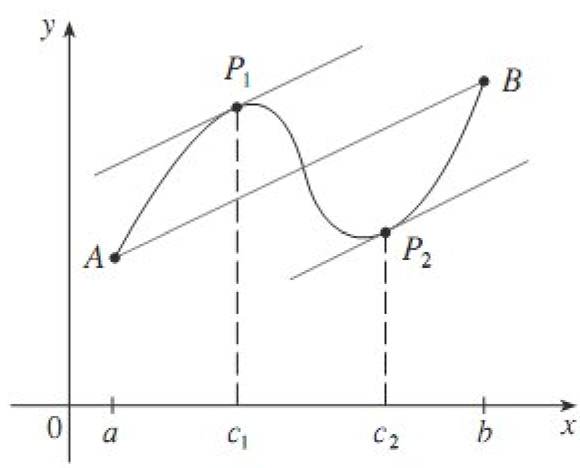
1. *f* is continuous on the closed interval $[a,b]$.
2. *f* is differentiable on the open interval (a,b).

Then there is a number $x = c$ in (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Or $f(b) - f(a) = f'(c)(b - a)$

Here's the graphical implication.





Example 3:
Determine if the MVT applies to $f(x) = x^3 - x$ on $[0,2]$, if so, find the value(s) guaranteed by the theorem.

Example 4:
Calculator permitted: Determine all the numbers c which satisfy the conclusion of the Mean Value Theorem for the function $f(x) = x^3 + 2x^2 - x$ on $[-1,2]$.

Example 5:
For the following functions, determine of the MVT applies. If so, find the value of c guaranteed by the theorem. If not, specifically state why the theorem does not apply.

a) $f(x) = \dfrac{x+5}{x-1}$ on $[-3,5]$

b) $g(x) = x^{2/3}$ on $[-3,3]$

Example 6:

The calculus cops have set up their elaborate speed trap on a busy, stretch of road. A suspected pumpkin farmer who uses the road daily to haul his pumpkins to market is suspected of chronic speeding well beyond the posted limit of 55 mph. The calculus cops aim to finally ticket this unlawful transporter of seasonal gourds. The calculus cops set up 5 miles apart from each other, each parked furtively behind a piece of scenery. Calculus cop A spots the farmer with his jalopy loaded down with would-be jack-o-lanterns. As the farmer passes cop A, he is clocked at a paltry 50 mph. Cop A could have sworn the farmer waves at him as he drives by. Cop A immediately radios calculus cop B 5 miles down the road, whereby cop A starts his timer. Four minutes later, cop B clocks the farmer cruising by at only 55 mph. He clearly sees the farmer wave at him with a giant grin that would make a jack-o-lantern jealous. After a quick calculation on his field-issued TI-88 ½ calculator, he pulls out with his lights on to issue a speeding ticket to the ornery pumpkin farmer. For what speed can calculus cop B ticket the farmer? Would this ticket hold up in a court of law? Why or why not?

Example 8:

Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x. How large can $f(2)$ possibly be?

The MVT can help establish some basic facts of differential calculus, such as the following:

Theorem:

If $f'(x)=0$ for all x in an interval (a,b), then f is constant on (a,b).

Example 8:
Prove the theorem above.

Example 9:
If $f(x)=\frac{x}{|x|}$, find $f'(x)$

Theorem:

If $f'(x)=g'(x)$ for all x in an interval (a,b), then $f-g$ is constant on (a,b). This means that $f(x)=g(x)+c$, where c is a constant.

Example 10:
Find the function f whose derivative is $\sin x$ and whose graph passes through the point $(0,2)$.