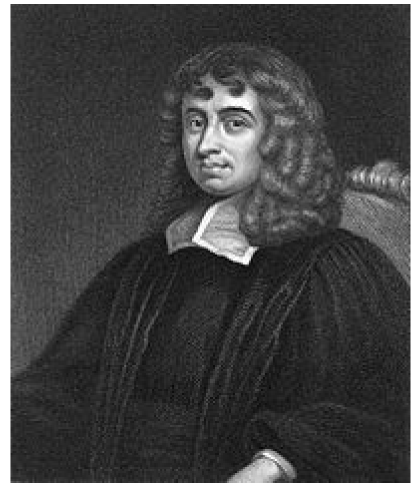
§6.3—The Fundamental Theorem of Calculus

We've learned two different branches of calculus so far: differentiation and integration. Finding slopes of tangent lines and finding areas under curves seem unrelated, but in fact, they are very closely related. It was Isaac Newton's teacher at Cambridge University, a man name **Isaac Barrow** (1630 – 1677), who discovered that these two processes are actually **inverse operations** of each other in much the same way division and multiplication are. It was Newton and Leibniz who exploited this idea and developed the calculus into it current form.

The Theorem Barrow discovered that states this inverse relation between differentiation and integration is called **The Fundamental Theorem of Calculus**.



The lunar crater "Barrow" is named after Isaac Barrow. The Wheel Barrow is not.

We're now ready for the "shortcut" rule for integration. This is what we've been waiting for: an easier way to calculate definite integrals.

The Fundamental Theorem of Calculus, Part 1 (FTOC1)

If *f* is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Where F is any antiderivative of *f*, such that $F' = f$.

This integral gives us the NET change!!!

Example 1:

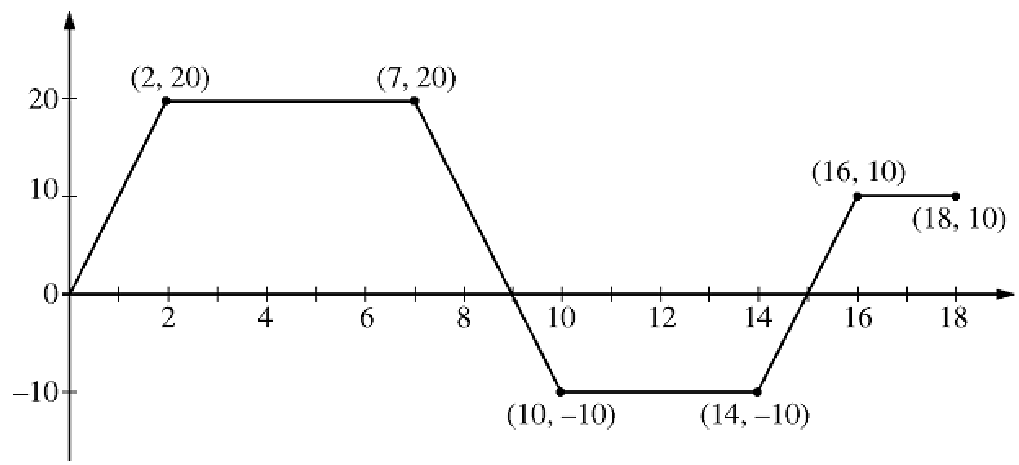
Evaluate the integral a) $\int_1^3 e^x dx$ b) $\int_0^{1/2} \frac{2}{\sqrt{1-x^2}} dx$

Example 2:
Find the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$.

The second part of the theorem deals with integral equations of the form

$$F(x) = \int_a^x f(t)\,dt$$

where *f* is a continuous function on $[a,b]$, and *x* varies between *a* and *b*. Notice that this integral equation is a function of *x*, which appears as the upper limit of integration. If $f(t)$ happens to be positive, and we let $x \in (a,b]$, then we can define $F(x)$ as the area under the curve from *a* to *x*.



Example 3:
The graph of $f(t)$ is given at right.

Let $F(x) = \int_2^x f(t)\,dt$.

Find the following:

a) $F(2)$

b) $F(9)$

c) $F(15)$

d) $F(18)$

e) $F(0)$

The Fundamental Theorem of Calculus Part 2 (FTOC2)

If f is a continuous function on $[a,b]$, then the function F defined by

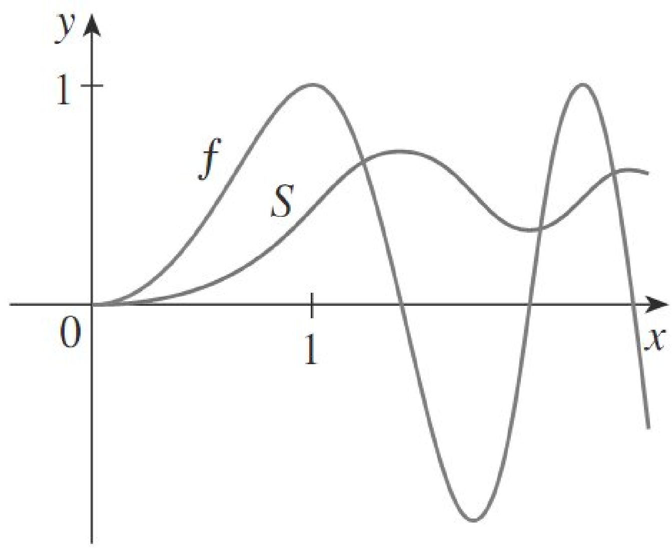
$$F(x)=\int_a^x f(t)\,dt\,,\ a\le x\le b$$

is continuous on $[a,b]$ and differentiable on (a,b). Additionally, $F'(x)=f(x)$. We can also say that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$$

Example 4:

The function $S(x)=\int_0^x \sin\left(\pi t^2/2\right)dt$ is called the **Fresnel function**, after French physicist Augustin-Jean Fresnel (1788 – 1827). It is used in optics to describe the diffraction of light waves, but has recently been applied to the design of highways. Find $f(x)=S'(x)$.



Example 5:

Evaluate a) $\frac{d}{dx}\left[\int_1^x \sec t\ dt\right]$ b) $\frac{d}{dx}\left[\int_5^x \sqrt{t^2}\ \sin t\ dt\right]$ c) $\frac{d}{dx}\left[\int_\pi^x \frac{e^t\sqrt{t+t^2}}{\ln t}\ dt\right]$

Chain Rule anyone?

The FTOC2, most general form:

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dt\right] = f\left(g(x)\right)\cdot g'(x) - f\left(h(x)\right)\cdot h'(x)$$

Example 6:
Evaluate the following using the FTOC2, then verify by doing in the Looooooong way.

a) $\dfrac{d}{dx}\left[\displaystyle\int_{1}^{2x^3} \sec^2 t\ dt\right]$

b) $\dfrac{d}{dx}\left[\displaystyle\int_{e^x}^{7} \left(t^2+5t\right)dt\right]$

Example 7:

If $F(x) = \displaystyle\int_{2+\sin 2x}^{3^x} \ln t\ dt$, find $F'(x)$

Before we proceed, we need to restate some basic integration patterns (before it gets more complicated).

Table of Indefinite Integrals

$\int cf(x)dx = c\int f(x)dx$	$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
$\int k\ dx = kx + C$	
$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$	$\int \dfrac{1}{x}dx = \int x^{-1}dx = \ln\lvert x\rvert + C$
$\int e^x dx = e^x + C$	$\int b^x dx = \dfrac{b^x}{\ln b} + C$
$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int \sec^2 x = \tan x + C$	$\int \csc^2 x = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \csc x \cot x dx = -\csc x + C$
$\int \dfrac{1}{x^2+1}dx = \tan^{-1} x + C$	$\int \dfrac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$

*We must always preserve the domain of the integrand upon antidifferentiating.

Example 8:

Find the exact value of $\int_2^e \left(10x^4 - 2\left(1-x^2\right)^{-1/2} - \dfrac{5}{x}\right)$

Example 9:

Evaluate each definite integral. Verify on the calculator.

a) $\int_1^2 \left(x^2 - 3\right)dx$ b) $\int_1^4 3\sqrt{x}dx$ c) $\int_0^{\pi/4} \sec^2 xdx$ d) $\int_0^2 \lvert 2x-1\rvert dx$

Example 10:

Find the area under the region bounded by the curves $y = 0$, $y = \frac{2}{x}$, $x = 1$ and $x = e$. Verify on the calculator.

Example 11:

Find the area under the region bounded by the x-axis and the function $y = \frac{20}{x^2 + 1}$ on the interval $-1 \le x \le 1$. Use symmetry if you can, then verify on your calculator.

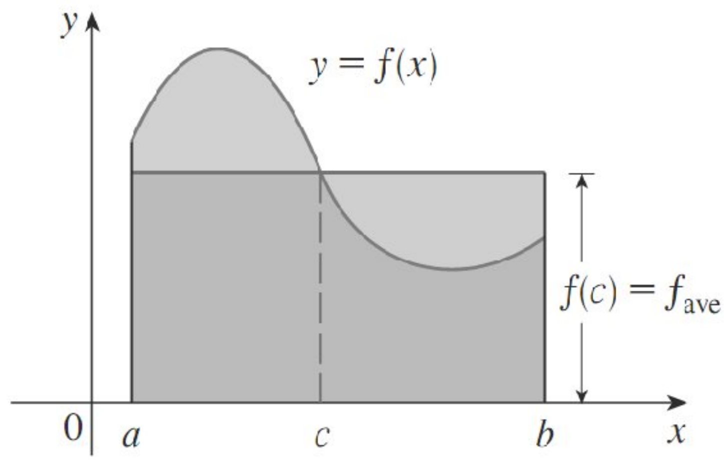
The Mean Value Theorem (for Integrals)

If *f* is continuous on the closed interval $[a,b]$, then there exists a number $x=c$ in the CLOSED interval $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)\cdot(b-a)$$

Where $f(c)$ is called the **average value** of the function *f* on the interval $[a,b]$. The above equation above can be explicitly solved for $f(c)$.

$$f(c)=\frac{\int_a^b f(x)\,dx}{b-a} \quad \text{or} \quad f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$$



Example 12:

Find the value of *c* guaranteed by the MVT for integrals for $f(x)=1+x^2$ on $[-1,2]$.

Example 13:

(Calculator) In New Braunfels, the temperature (in $^\circ F$) *t* hours after 9 a.m. was modeled by the function $T(t)=50+14\sin\frac{\pi t}{12}$. Find the average temperature during the 12-hour period from 9 a.m. to 9 p.m.

Example 14:
Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the entire trip.

Example 15:
The table below gives values for a continuous function. Using the values given, find the arithmetic mean of $f(x)$. Using a left endpoint Riemann approximation using 6 subintervals, estimate the average value of f on $[20, 50]$. For this continuous function, which is more accurate? Why? Oh, and also, find an upper and a lower bound for approximate $f'(35)$, then approximate $f'(37)$.

x	20	25	30	35	40	45	50
$f(x)$	42	38	31	29	35	48	60

Example 15:
Find the number(s) b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

Example 16:

If $F(x)=\int_1^x 2t\,dt$ find $F(7)$ by evaluating the given function, then find the simplified function for $F(x)$ by evaluating the definite integral. Find $F(7)$ again using this new, simplified function. What do you notice? Which to you prefer. What if the antiderivative of the integrand was unknown. How could you evaluate $F(7)$ on your calculator? Could you also evaluate this integral graphically?

Example 17:
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(* means VERY IMPORTANT)
Using your calculator, if $\frac{dy}{dx}=4\sin^2(2x)$ and $y(2)=-2$, find a) and integral equation for *y*, then find b) $y(3)$ c) $y(5)$ c) $y(-2)$

Example 18:
Find all the values of *k* such that

a) $\int_{-5}^{k}\left(x^3-7x\right)dx=0$

b) $\int_{-3}^{k}\left(x^2-4\right)dx=0$