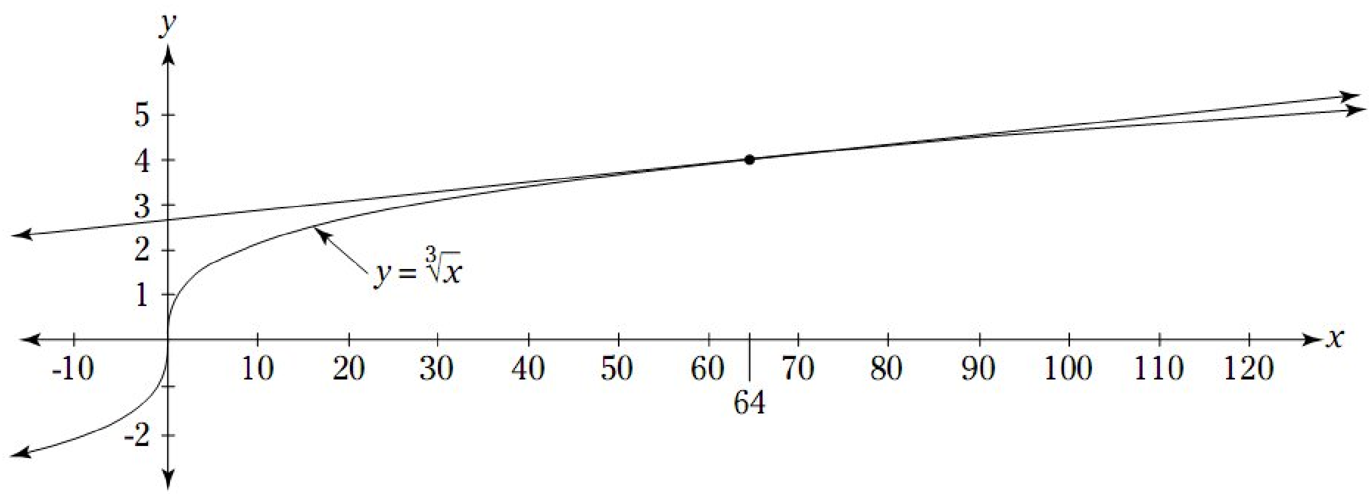
§5.7—Linearization & Differentials

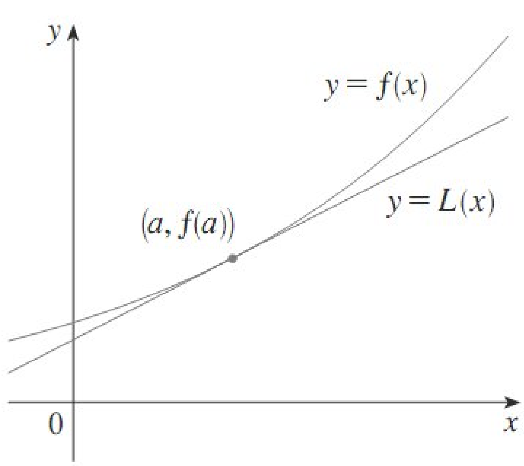
Linear approximation is a very easy thing to do, and once you master it, you can impress all of your friends by calculating things like $\sqrt[3]{70}$ in your head . . . . about 4.125! Impressed? I'll teach you how.

Recall that if a function $f(x)$ is **differentiable** at $x = c$, we say it is **locally linear** at $x = c$. This means that as we zoom in closer and closer and closer and closer around $x = c$, the graph of $f(x)$, regardless of how curvy it is, will begin to look more and more and more and more like the tangent line at $x = c$.

This means that we can use the equation of the tangent line of $f(x)$ at $x = c$ to **approximate** $f(c)$ for values **close to** $x = c$. Let's take a look at $\sqrt[3]{70}$ and the figure below.



The tangent line to $f(x) = \sqrt[3]{x}$ is drawn at $x = 64$. Why there? Well 64 is a perfect cube AND it is the closest perfect cube to 70. Notice that near $x = 64$ on either side, the tangent line and the function itself look indistinguishable from each other, that is, their y-values are very similar (in fact, they're EXACTLY the same at $x = 64$). We call $x = 64$ the **center** of the linear approximation and the actual equation of the tangent line there the **Linearization** of $f(x)$ at $x = 64$. Instead of using y for the equation of the line, since we are going to use it, we'll call it $L(x)$.



We can then say that for values of x **nea**r $x = 64$, $f(x) \approx L(x)$ (IT'S VERY IMPORTANT HERE TO USE THE APPROXIMATION "SQUIGGLEY" LINES!!). 70 is not the closest value to 64, but for this curve, which increases at a decreasing rate it is reasonably close. We can say then that $f(70) \approx L(70)$. Moreover, because the function is concave down, we know the tangent line lies above the curve, so $L(70)$ will be an OVER-approximation of $f(70)$.

This idea can be extended to ANY differentiable function at ANY point!

How to find linear approximations of $f(x)$ at $x=c$, the center to approximate $f(x)$ at $x=a$, a value near the center $x=c$.

1. Find the equation of the tangent line at the center $(c, f(c))$ in point-slope form.
2. Solve for y and rename it $L(x)$.
3. Plug in $x=a$ into $L(x)$ writing the notation VERY CAREFULLY as $f(a) \approx L(a) = \cdots$
4. If asked, determine if $L(a)$ is an over-approximation or an under approximation by examining the concavity of $f(x)$ at the center $x=c$.
    a. If $f''(c)<0$, $f(x)$ is concave down at $x=c$ then $L(a)$ is an over-approximation
    b. If $f''(c)>0$, $f(x)$ is concave up at $x=c$ and $L(a)$ is an under-approximation

Here's the work for this problem:

$$f(x)=\sqrt[3]{x}=x^{1/3}, \quad f(64)=4$$

$$f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3\sqrt[3]{x^2}}, \quad f'(64)=\frac{1}{48}$$

The tangent line of $f(x)$ at $(64,4)$ is

$$y-4=\frac{1}{48}(x-64)$$

So

$$L(x)=4+\frac{1}{48}(x-64)$$

(this form is also called the Taylor-form equation of a line and will show up again later in the year when we study Taylor Polynomials (BC only).)

Plugging in, we get

$$f(70)\approx L(70)=4+\frac{1}{48}(70-64)=4+\frac{1}{8}=4.125$$

Now let's look at the concavity at $x=64$.

$$f''(x)=\frac{-2}{9\left(\sqrt[3]{x}\right)^5} \text{ and } f''(64)=\frac{-2}{9\left(4^5\right)}<0,$$

so $f(x)$ is concave down at $x=64$, and $L(64)$ is and over-approximation of $f(64)$.

For kicks and giggles, the actual value of $\sqrt[3]{70}$ is $4.1212853\ldots$ an error of only 0.0037147002 or a percent error of 0.09%.

Of course, we could have gotten better approximations by evaluating $L(x)$ at values closer to $x=64$.

Let's do some together now.

Example 1:
Estimate the fourth root of 17. Determine if the linearization is and over- or under-approximation.

Example 2:
Approximate 3.01^5. Determine if the linearization is and over- or under-approximation.

Example 3:
Evaluate $\sin\left(\frac{\pi}{180}\right)$ by using the linearization of $f(x) = \sin x$ at an appropriate center. Determine if the linearization is and over- or under-approximation.

Example 4:
Approximate $\ln\left(e^{10}+5\right)$. Determine if the linearization is and over- or under-approximation.

We'll now introduce a notation and a process that will become second nature to you the rest of the year.

Recall Leibniz's notation for the derivative function:

$$\frac{dy}{dx} = f'(x)$$

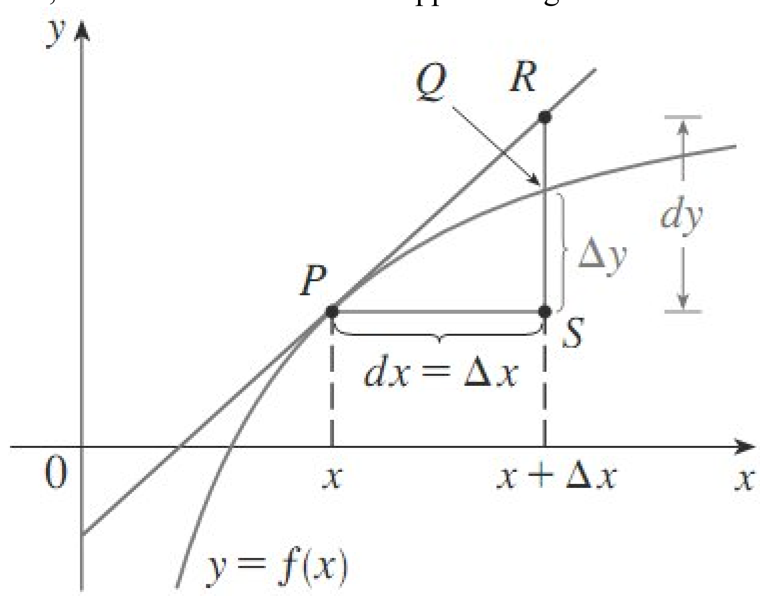
dy is called the differential of y, and dx is called the differential of x.

Similar to Δy and Δx, which are finite, measurable quantities, the differentials denote a change in respective values, however, they are infinitely small, immeasurable differences approaching zero.

We can treat them as individual quantities and we can even solve the equation $\frac{dy}{dx} = f'(x)$ for dy:

$$dy = f'(x)dx$$

This is called the "differential form" of the derivative or y. Let's practice on a few.



Example 5:
Find the derivative of each function in differential form for each of the following.

a) $y = 2t^3 + 5t^2 - 3t + 1$ b) $z = x^3 \sin(3x)$ c) $m = e^{5q^2+1}$

Here's a nice application of differentials I think you'll recognize.

If Δx is small, we can say that $dy \approx \Delta y$.

This is exactly what we did when we did linear approximations! In this new context, though, we can work cooler types of problems like the following.

Example 6:
A machined spherical bearing was measured with a caliper. The bearing's radius was found to be 2.3 inches with a possible error no greater than 0.0001 inches. What is the maximum possible error in the volume of the spherical bearing if we use this measurement for the radius? What is the percent error? (Hint: Assume $\Delta V \approx dV$. Use correct notation.)

Notice this process eliminates the need to find the actual error between $V(45)$ and $V(45.0001)$. This error is sometimes called the **marginal error**. The same principle can be applied to economics when discussing marginal profits, marginal revenue, marginal expenditures, etc.