§5.6—Optimization

Here's our first, REAL application of the derivative: Optimization. As students, you are well experienced at optimizing, trying to get the absolute BEST grade by exerting a FIXED amount of effort. Any time you use the superlative case—biggest, smallest, cheapest, strongest, ugliest, etc—you're trying to optimize. Why are these problems at all? What's keeping us from making something infinitely big, infinitely small, infinitely ugly? Well, there is always some limiting factor, a constraint that prevents that from happening.

In this unit, we will be looking for Absolute Extrema, and we won't always have endpoints. We'll learn 3 different methods for showing the value we find is indeed the Global Extrema. Each problem is unique, and we'll have to decide which justification method to use so that we expend the LEAST amount of effort.

These types of problems are among the most challenging for Calculus students because they combine Math with English, they're the much dreaded WORD PROBLEMS. Careful reading can go a long way in translating the written language into the language of mathematics.

Steps to a successful Optimization Problem:

1. Read the problem carefully
2. Draw a picture. Label unknowns as variables. Label constants as numbers. IT IS EASIER TO LABEL SMALL PARTS AS SINGLE VARIABLES SO THAT YOU CAN AVOID FRACTIONS AND/OR SUBTRACTING.
3. Write a primary equation for the quantity to be optimized.
4. Identify the limiting factor/constraing and write a secondary equation (sometimes) involving this constraint.
5. Solve the secondary equation for any convenient variable and plug it into the primary equation to establish an equation for the optimal quantity in terms of a single variable.
6. Simplify the primary equation and think about a relevant domain.
7. Differentiate and find critical values within the relevant domain.
8. Determine the Absolute Extrema using one of 3 methods.
9. Answer the question in a complete sentence using appropriate units.
10. Smile ☺

Here's a relatively simple problem with which to get our feet wet and appetite whetted.

Example 1:
The sum of two positive numbers is 8. What are the two numbers that will have the largest product?

How do we know that our values were indeed the ones that produced the largest product? We have to be able to justify our response beyond than saying it was "obvious," or that our intuition told us so. There are actually three different ways.

WAYS TO JUSTIFY AN ABSOLUTE EXTREMA

Method 1: Closed interval argument
We can play the "biggest *y*-value" game with Team Endpoint and Team Critical Value. This sure-fire method requires finite endpoints and a continuous function over the relevant interval. This method will not work every time, because some examples may have at least one infinite (unbounded) endpoint. Also, once we reduce the primary equation down to a single variable, many equations will no longer be continuous. Alas, the method fails.

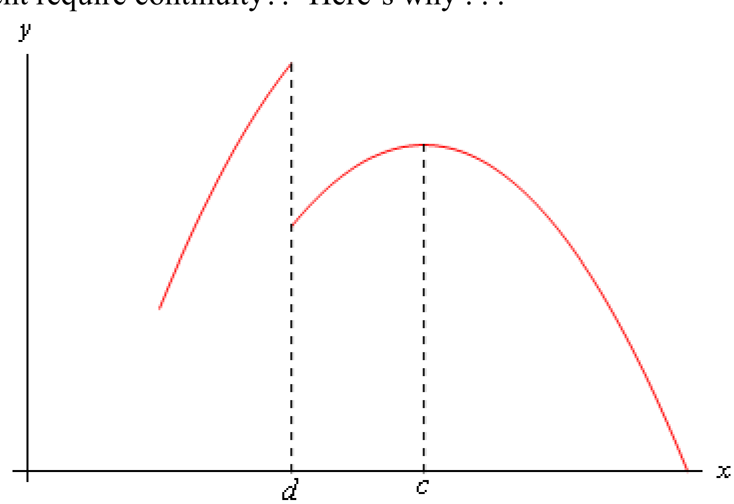
Method 2: 1^{st} Derivative Test for Relative Extrema modified for Absolute Extrema
We will use this method when we don't have both endpoints at a closed interval, but instead have either a half-open interval or open interval. It does require a continuous function, though. Here's how it will work.

Suppose we have a critical value $x = c$ of a function $f(x)$ we're trying to optimize. The First Derivative Test tells us that if $f'(x) > 0$ immediately to the left of $x = c$ and if $f'(x) < 0$ immediately to the right of $x = c$, then $f(x)$ will have a relative maximum at $x = c$. This does **NOT** mean that the absolute max will occur at $x = c$.

What if, though we also knew that $f'(x) > 0$ for ALL $x < c$ in an interval and also that $f'(x) < 0$ for ALL $x > c$ in an interval??!!?? This means that EVERYWHERE to the left of $x = c$ (as long as we're still in the interval *I*), the graph of $f(x)$ is increasing, and EVERYWHERE to the right of $x = c$ (again, as long as we're still in the interval *I*), the graph of $f(x)$ is decreasing. We could then conclude that $f(x)$ has an ABSOLUTE maximum at $x = c$!!

Similarly, if we knew that $f'(x) < 0$ for ALL $x < c$ in an interval and also that $f'(x) > 0$ for ALL $x > c$ in an interval, then $f(x)$ has an ABSOLUTE minimum at $x = c$!!

So why does this argument require continuity?? Here's why . . .



First Derivative Test for Absolute Extrema

Let *I* be the interval of all possible optimal values of $f(x)$ and further suppose that $f(x)$ is continuous on *I*, except possibly at the endpoints. Finally suppose that $x = c$ is a critical value of $f(x)$ and that *c* is in the interval *I*. If we restrict *x* to values from *I* (*i.e.* we only consider possible optimal values of the function) then,

1. If $f'(x) > 0$ for all $x < c$ and if $f'(x) < 0$ for all $x > c$ then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval *I*.
2. If $f'(x) < 0$ for all $x < c$ and if $f'(x) > 0$ for all $x > c$ then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval *I*.

Method 3 : Use the second derivative.
This method is similar to the method above, but uses the Second Derivative Test:

Second Derivative Test for Absolute Extrema

Let *I* be the range of all possible optimal values of $f(x)$ and further suppose that $f(x)$ is continuous on *I*, except possibly at the endpoints. Finally suppose that $x = c$ is a critical value of $f(x)$ and that *c* is in the interval *I*. Then,

1. If $f''(x) > 0$ **for all** *x* in *I* then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval *I*.
2. If $f''(x) < 0$ **for all** *x* in *I* then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval *I*.

Example 2:
We need to enclose a rectangular field with a fence. We have 500 feet of fencing material and a building is on one of the longer sides of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

Example 3:
We need to enclose a rectangular field with a fence. We want to enclose 10,000 square feet. A building is on one of the longer sides of the field and so won't need any fencing. Determine the dimensions of the field that will require the least amount of fencing.

Example 4:
We need to enclose a rectangular field with a fence. A river is on one side of the fence, so we won't need to fence along the river. We would like to enclose 10,000 square feet. Because of the added construction costs of building perpendicular to the river, it costs \$8 a liner foot to buy and build the fence. The portion of the fence parallel to the river only cost \$5 per liner foot to buy and build. What are the dimensions that will minimize the cost?

Example 5:
A cylindrical aluminum soda can is to hold 12 ounces of tasty beverage. What are the dimensions of the can that will minimize the amount of aluminum used? Hint: 12 US fluid ounces equals 21.6562 cubic inches. If the dimensions of the actual can are 2.6 inch diameter and 4.75 inches tall, does the actual can have the optimal dimensions?

Example 6:
Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$. HINT: If a function can be expressed as a radical, then optimizing the radicand will optimize the radical.

Example 7:
Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Example 8:
Find the volume of the largest cylinder that can be inscribed inside a sphere of radius R.

Example 9:

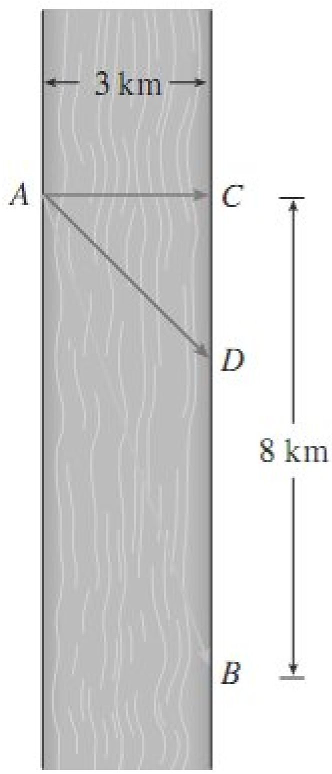
An arch top window is being built whose bottom is a rectangle and the top is a semicircle. If there is 12 meters of framing materials what must the dimensions of the window be to let in the most light?

Example 10:

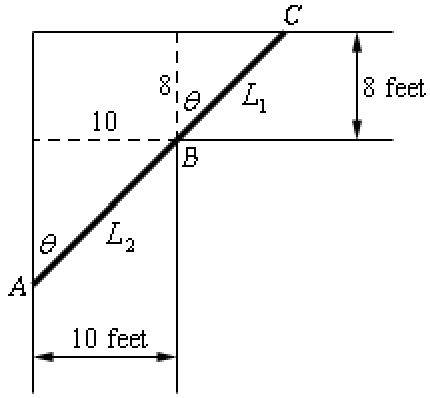
A 2 feet piece of wire is cut into two pieces and once piece is bent into a square and the other is bent into an equilateral triangle. Where should the wire cut so that the total area enclosed by both is minimum and maximum?

OPTIONAL EXAMPLES

Example 11:
A calculus student launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank. His impatient calculus buddies are there waiting there for him, so he needs to get there as quickly as he can. His options are to row straight across to point C, then run to point B, or he could row directly to B, or he could row to some point D on shore between C and B, and then run the rest of the way to B. If he can row at 6 km/hr and run at 8 km/hr, where should he land his boat to reach B as soon as possible. (We'll assume that he wastes no time in making these calculations in the boat before he starts.)



Example 12:
A piece of pipe is being carried down a hallway that is 10 feet wide. At the end of the hallway the there is a right-angled turn and the hallway narrows down to 8 feet wide. What is the longest pipe that can be carried (always keeping it horizontal) around the turn in the hallway?



Example 13:
A trough for holding water is be formed by taking a piece of sheet metal 60 cm wide and folding the 20 cm on either end up as shown below. Determine the angle θ that will maximize the amount of water that the trough can hold. (Hint: Maximizing the cross-sectional area will maximize the volume.)