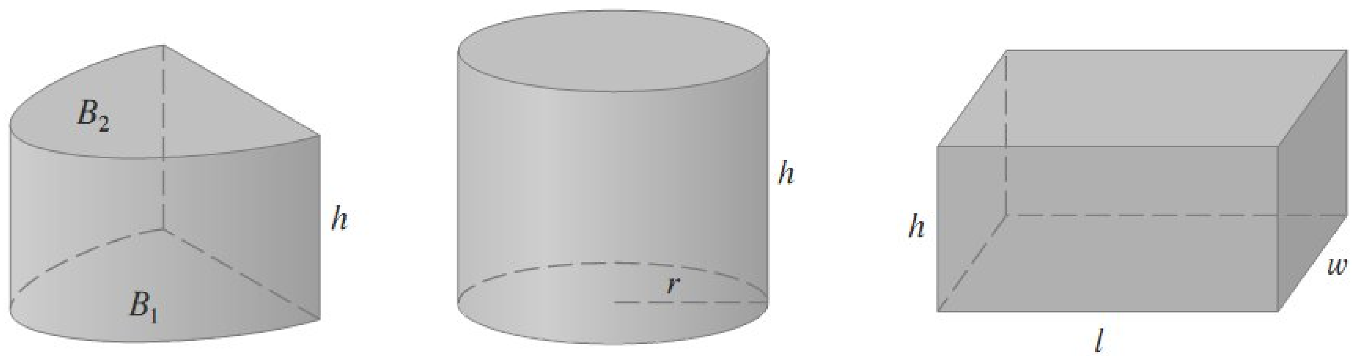
§8.3—Volumes

Just as area is always positive, so is volume.

Let's review how to find the volume of a regular geometric prism, that is, a 3-dimensional object with two regular faces separated by some distance, h. Whether it be a circular prism (cylinder), triangular prism, rectangular prism, etc., if we can find the area of the face, we need only multiply by the distance between the two faces, h. In general

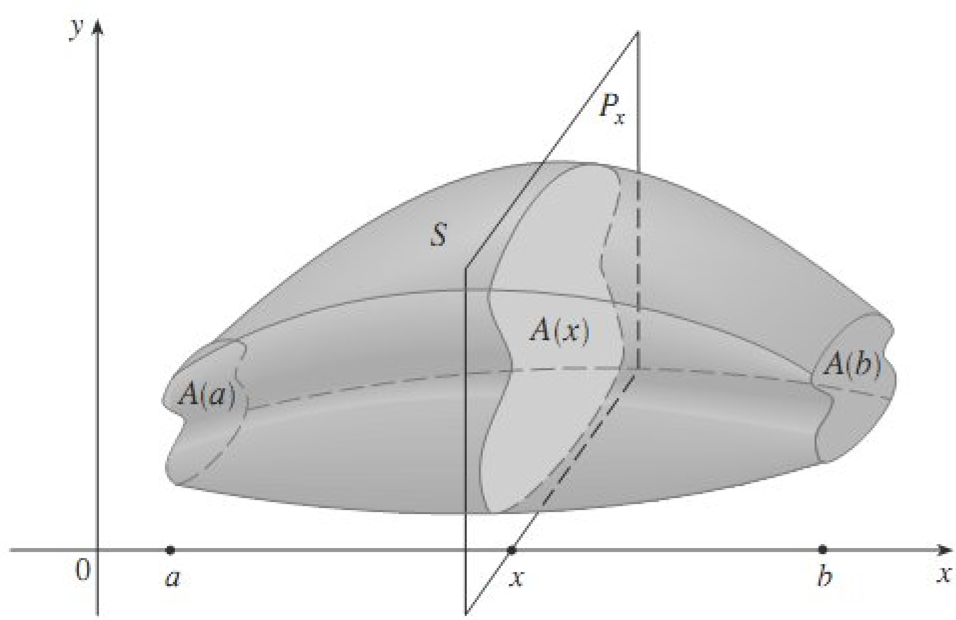
$$V = A \cdot h$$



$V = B_2 \cdot h$ or $V = B_1 \cdot h$ $\quad$ $V = \pi r^2 \cdot h$ $\quad$ $V = lw \cdot h$

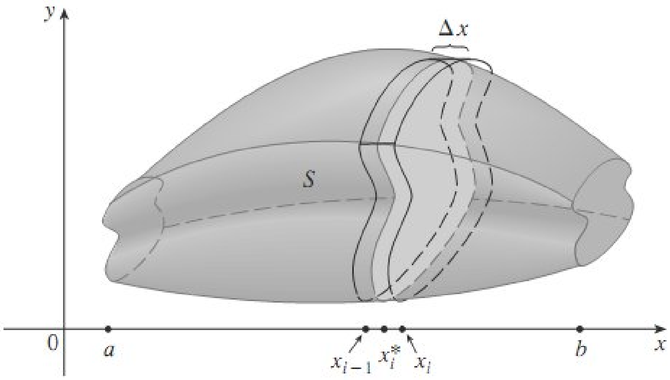
What would each of the sections above look like if we thinly sliced them parallel to their faces (perpendicular to h)? Could we then find the volume of each of these representative slices?

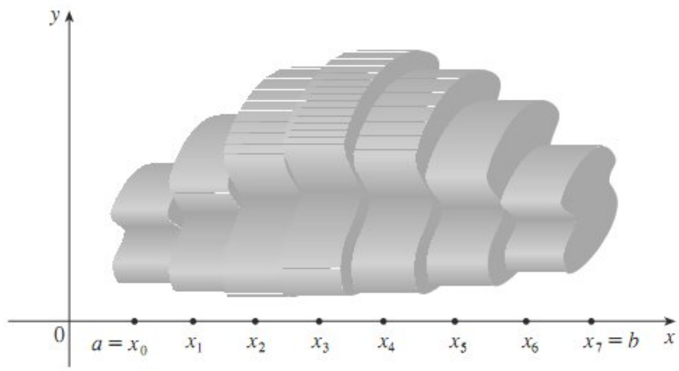
Imagine slicing a loaf of bread (mathematically). It might look like this



At each slice, we have a face with a different area, but one that is a function of where along the loaf the slice was taken.

The measurable thickness of the slice, whether it be for a sandwich or Texas Toast, is Δx. This thickness forms the distance *h*, between the two faces. We can slice up the loaf from left to right using a uniform Δx.





We can approximate the volume of the entire loaf by finding the volumes of each slice $V(x_i) = A(x_i)\cdot\Delta x$ and adding them up (This works much better than finding volume by the Archimedean method of fluid displacement, which leaves the bread rather soggy. Incidentally, Archimedes is called the "Father of Integral Calculus" since he was the first person to envision finding volumes by this thin, slicing method).

As we slice the regions thinner and thinner and thinner, approaching infinitely thin, we get increasingly better approximations of the volume. Here's the summary:

Definition of Volume

Let *S* be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of *S* in the plane through *x* and perpendicular to the *x*-axis is $A(x)$, where *A* is a continuous function, then the **volume** of *S* is

$$V = \lim_{\Delta x \to 0} \sum A(x_i)\Delta x = \int_a^b A(x)\,dx$$

Example 1:

Show that the volume of a sphere of radius *r* is $V = \frac{4}{3}\pi r^3$.

Anytime our cross-sections, perpendicular to an axis of rotation, are circles (or thin discs), we can us a similar approach. Very often we will have to create/envision our solids by rotating or revolving a given region around or about an axis. ***When we create solids by revolving around an axis that is perpendicular to our slices, our cross-sections will always be circular.***

Disc Method for Volumes of Solids of Rotation

When the volume of solid is obtained by rotating a region **perpenDISCular** to the axis of rotation and the cross-sections are discs or circles, the volume of the solid is given by

$$V = \pi \int_a^b R(x)^2 \, dx$$

Where $R(x)$ is the radius of rotation as a function of *x*.

Example 2:
Find the volume of the solid formed by rotating the region bounded by the *x*-axis, $y = \sqrt{x}$, and $x = 1$ around the *x*-axis.

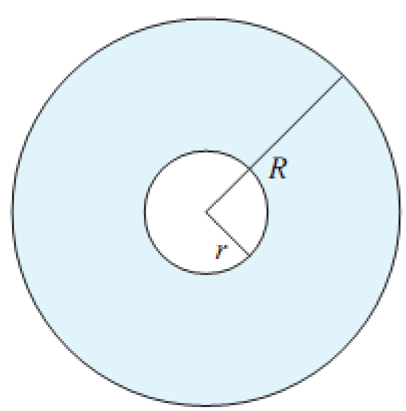
perpenDISCular

Example 3:
Find the volume of the solid formed by rotating the region bounded by the $y=1$, $y=\sqrt{x}$, and $x=0$ around the line $y=1$.

Example 4:
Find the volume of the solid obtained by rotating the region bounded by $y=x^3$, $y=8$, and $x=0$ about the y-axis.

Sometimes, our cross sections are circles but have a void or hole in them. In this case, our circular cross-section, perpendicular to the axis of rotation, will resemble a **washer**, with an inner, smaller radius r, and a larger, outer radius R.

In this case, the area of the face of the cross section will be
$A(x)=\pi R(x)^2-\pi r(x)^2=\pi\left(R(x)^2-r(x)^2\right)$



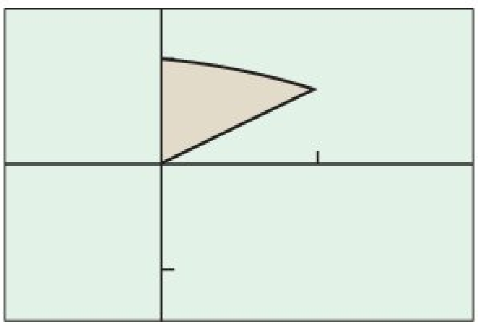
Washer Method for Volumes of Solids of Rotation

When the volume of solid is obtained by rotating a region **perpenWASHular** to the axis of rotation and the cross-sections are washers, the volume of the solid is given by

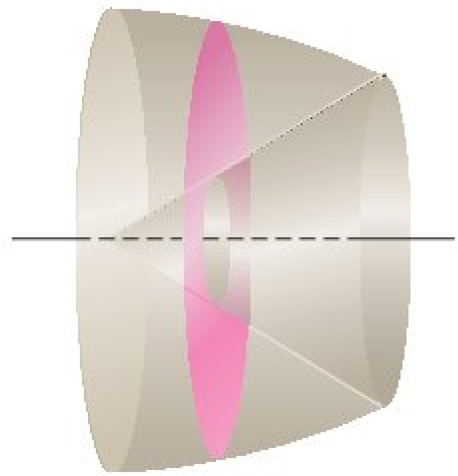
$$V = \pi \int_a^b \left[R(x)^2 - r(x)^2 \right] dx$$

Where $R(x)$ is the larger, outer radius of rotation and $r(x)$ is the smaller, inner radius rotation.

Example 5:
The region in the first quadrant enclosed by the y-axis and the graphs of $y = \cos x$ and $y = \sin x$ is revolved about the x-axis to form a solid. Find its volume.



$[-\pi/4, \pi/2]$ by $[-1.5, 1.5]$



Important things to consider when using the Washer method:

- Draw a picture, draw a picture, draw a picture, . . .
- Like the Disc method, the cross-sections (slices/representative rectangles) must be PERPENDICULAR to the axis of rotation.
- Before writing an equation for *R* and *r*, draw them on your diagram. If you can draw them, you can write them.
- When writing an equation for *R* and *r*, it will still involve $TOP - BOTTOM$ (vertical slice) or $RIGHT - LEFT$ (horizontal slice). **One of these in each case will be the axis of rotation itself.**
- **DON'T FORGET TO SQUARE EACH RADIUS BEFORE SUBTRACTING THEM.** The most common error is to integrate as $V = \pi\int_a^b \left[\left(R(x) - r(x)\right)^2\right]dx$. This is **WRONG**. Keep telling yourself that you're subtracting two separate volumes: $\pi R^2 dx - \pi r^r dx$. The π and dx are simply factored out.

perpenWASHular

Example 6:

Find the volume of the solid formed when the *R* enclosed by the curves $y = x$ and $y = x^2$ is rotated about the following axes:

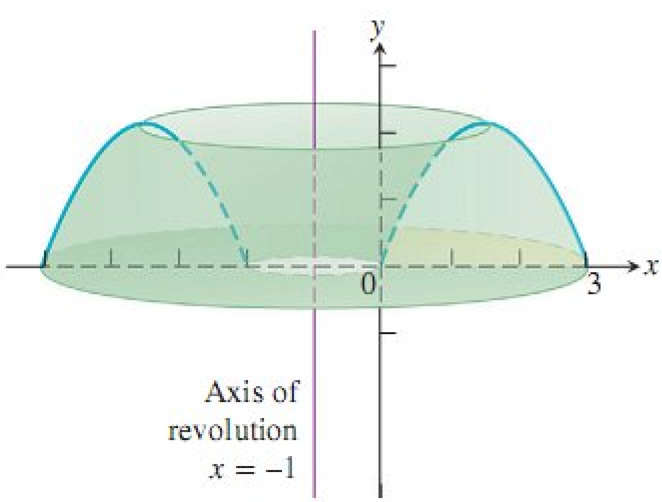
a) the *x*-axis. b) the line $y = 1$ c) the line $y = 2$ d) the line $y = -5$ e) the line $x = -1$

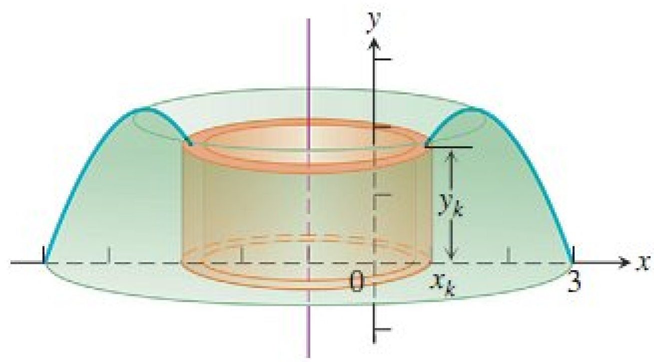
Example 7:

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

Is there a way to find the volume of the solid from example 7 using a single integral? Not if we slice the solid perpendicular to the axis of rotation . . . . . but what if we sliced it more cleverly! Aha! 'tis fun to be clever.

What if we sliced the region **PARALLEL** to the axis of rotation. What would happen to a representative rectangle when taken for a spin around an axis parallel to it? Think "Bundt Cake."





We get a thing called a **Cylindrical Shell**. Can we find it's volume?

The shell, with finite thickness has two radii, but as we slice thinner and thinner and thinner, the two radii approach each other. For an infinitely thin slice, we can use a single radius (think school-bought toilet paper).

Shell Method for Volumes of Solids of Rotation

When the volume of solid is obtained by rotating a region **paraSHELL** to the axis of rotation and the cross-sections are cylindrical shells with radius $r(x)$ and height $h(x)$, the volume of the solid is given by

$$V = 2\pi\int_a^b r(x)\cdot h(x)\,dx$$

We don't need to worry about holes, since we are only integrating, slicing over the interval $[a,b]$, parallel to the axis of rotation.

Let's return to the region from example 7:

Example 8:

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the *y*-axis.

paraSHELL

Let's now find the volume of that Bundt cake.

Example 8:

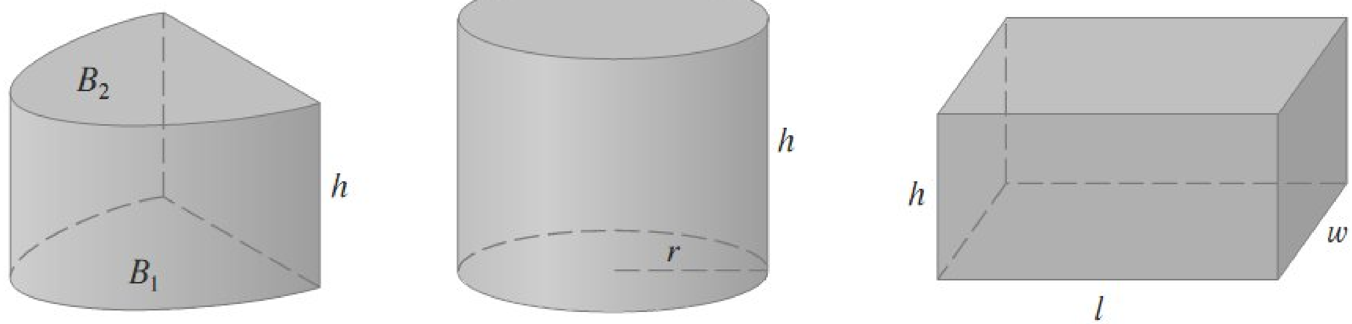
The region enclosed by the x-axis and the parabola $f(x)=3x-x^2$ is revolved about the line $x=-1$ to generate a solid of revolution resembling a Bundt cake. What is the volume of the cake? What would happen if the graph was rotated about the line $x=4$ instead?

Example 9:

Find the volume of solid of revolution formed by revolving the region formed by $y=x^2$ and $y=\sqrt{x}$ about the y-axis. Use both vertical and horizontal slices. Compare your results.

So what if we don't generate our solid by revolving it around an axis? Remember these guys from the beginning of the lesson?

$$V = A \cdot h$$



$V = B_2 \cdot h$ or $V = B_1 \cdot h$ $\qquad$ $V = \pi r^2 \cdot h$ $\qquad$ $V = lw \cdot h$

Only the middle one was a solid of revolution. The first and third are not. Remember, though, that if we are still able to find the formula for the area of the cross-sectional face, we're just fine. To remind you:

Definition of Volume

Let *S* be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of *S* in the plane through *x* and perpendicular to the *x*-axis is $A(x)$, where *A* is a continuous function, then the **volume** of *S* is

$$V = \int_a^b A(x)\,dx$$

Imagine that a concrete slab has been poured. Upon that slab, walls are built perpendicular to the slab. If we can find the area of the face of one of these walls, we kind find the volume of that panel, and thus, the entire house. The slab represents the area of the region enclosed by the curves.

If we can find $A(x)$ (for slices perpendicular to the *x*-axis) or $A(y)$ (for slices perpendicular to the *y*-axis), we just need to integrate over the interval $[a,b]$ with respect to *x* (dx being the infinitely thin width of each slice).

Here's some geometric formulas for the areas of some shapes with flat bases that show up quite a bit as cross-sections:

Squares: $A = s^2$ $\qquad$ Rectangles: $A = lw$

Triangles: $A = \frac{1}{2}bh$ $\qquad$ Equilateral Triangles: $A = \frac{\sqrt{3}}{4}s^2$

Semicircles: $A = \frac{\pi}{8}d^2$

Example 10:

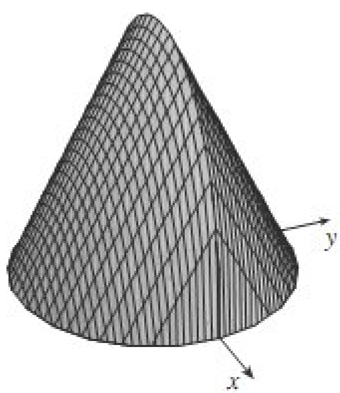
Find the volume of the solid whose base is bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with the following cross sections taken perpendicular to the x-axis:

a) Squares b) Rectangles with height 2 c) Rectangles whose height is twice the base

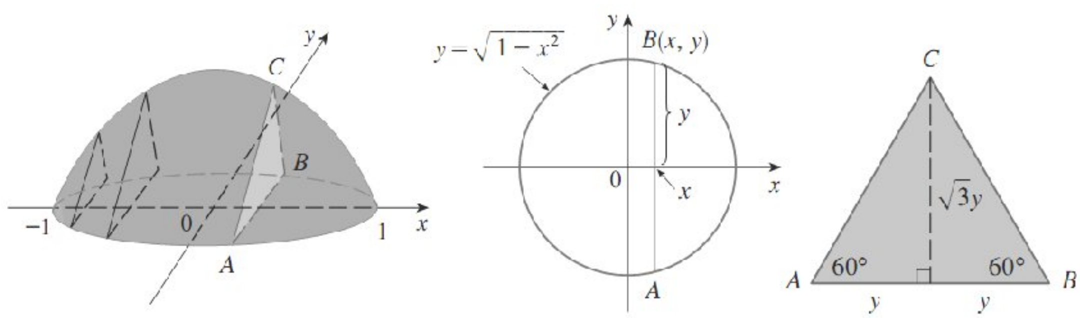
e) Semicircles f) Equilateral Triangles g) Isosceles Right Triangles whose base is a short leg

h) Isosceles Right Triangles whose hypotenuse is the base

Example 11:



The figure shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



(a) The solid (b) Its base (c) A cross-section

Example 12:
(Calculator) An oil spill on the surface of the water has a surface shape defined by the **first positive region** of the intersections of the equations $f(x) = \sin x$ and $g(x) = -\sin\left(x - \frac{\pi}{6}\right)$. The depth of the oil spill has a depth given by $d(x) = 2\cos(x/2)$. Find the volume of the oil spill.

Example 13:
A region *R*, defined by the intersections of the graphs of $y = 5x$, $y = -\frac{x}{5} + 3$, and $y = 0$, is the the base of the solid. For this solid, at each *x*, the cross section perpendicular to the *y*-axis has area $A(y) = y^2 + 1$. Find the volume of the solid.