Chapter 4

Matrices

Chapter Overview and Pacing

LESSON OBJECTIVES	PACING (days)			
	Regular		Block	
	Basic/ Average	Advanced	Basic/ Average	Advanced
4-1 **Introduction to Matrices** *(pp. 154–159)* • Organize data in matrices. • Solve equations involving matrices. ***Follow-Up:*** Organizing Data	1	1	0.5	0.5
4-2 **Operations with Matrices** *(pp. 160–166)* • Add and subtract matrices. • Multiply by a matrix scalar.	2 (with 4-1 Follow-Up)	2 (with 4-1 Follow-Up)	1 (with 4-1 Follow-Up)	1.5 (with 4-1 Follow-Up)
4-3 **Multiplying Matrices** *(pp. 167–174)* • Multiply matrices. • Use the properties of matrix multiplication.	2	2	1	1
4-4 **Transformations with Matrices** *(pp. 175–181)* • Use matrices to determine the coordinates of a translated or dilated figure. • Use matrix multiplication to find the coordinates of a reflected or rotated figure.	2	2	1	1
4-5 **Determinants** *(pp. 182–188)* • Evaluate the determinant of a 2×2 matrix. • Evaluate the determinant of a 3×3 matrix.	2	2	1	1
4-6 **Cramer's Rule** *(pp. 189–194)* • Solve systems of two linear equations by using Cramer's Rule. • Solve systems of three linear equations by using Cramer's Rule.	2	2	1	1
4-7 **Identity and Inverse Matrices** *(pp. 195–201)* • Determine whether two matrices are inverses. • Find the inverse of a 2×2 matrix.	2	2	1	1
4-8 **Using Matrices to Solve Systems of Equations** *(pp. 202–208)* • Write matrix equations for systems of equations. • Solve systems of equations using matrix equations. ***Follow-Up:*** Augmented Matrices	2	2 (with 4-8 Follow-Up)	0.5	1
Study Guide and **Practice Test** *(pp. 209–215)* **Standardized Test Practice** *(pp. 216–217)*	1	1	0.5	0.5
Chapter Assessment	1	1	0.5	0.5
TOTAL	**17**	**17**	**8**	**9**

Pacing suggestions for the entire year can be found on pages T20–T21.

Timesaving Tools

TeacherWorks™

All-In-One Planner and Resource Center

See pages T12–T13.

Chapter Resource Manager

Chapter 4 Resource Masters: Study Guide and Intervention	Practice (Skills and Average)	Reading to Learn Mathematics	Enrichment	Assessment	Applications*	5-Minute Check Transparencies	Interactive Chalkboard	Alge2PASS: Tutorial Plus (lessons)	Materials
169–170	171–172	173	174			4-1	4-1		(*Follow-Up:* spreadsheet software)
175–176	177–178	179	180	231		4-2	4-2	6	graphing calculator
181–182	183–184	185	186			4-3	4-3		
187–188	189–190	191	192	231, 233	GCS 33	4-4	4-4		colored pencils
193–194	195–196	197	198		GCS 34, SC 7	4-5	4-5		posterboard, colored markers
199–200	201–202	203	204	232		4-6	4-6	7	
205–206	207–208	209	210			4-7	4-7		
211–212	213–214	215	216	232	SC 8	4-8	4-8		graphing calculator (*Follow-Up:* graphing calculator)
				217–230, 234–236					

**Key to Abbreviations:* GCS = Graphing Calculator and Speadsheet Masters,
SC = School-to-Career Masters,
SM = Science and Mathematics Lab Manual

Chapter 4

Mathematical Connections and Background

Continuity of Instruction

Prior Knowledge

The main idea of this chapter, matrices, introduces a new notation in which the position of a number within that notation is important. Students have seen this idea, to focus on the position of a number, for exponents and subscripts. An important topic of the chapter, solving systems of equations, has been studied in Chapter 3.

This Chapter

Students explore matrices as a notation in which they have to attend to the position of a number as well as to its magnitude. They learn when matrices can be added, subtracted, or multiplied, and learn how to find the sums, differences, products, and scalar products of matrices. They use matrices to represent transformations and to represent and solve systems of equations.

Future Connections

Attending to the position of a number within a notation occurs frequently in mathematics; students will see such notation when they read and write symbols for permutations and combinations. Also, in later courses they will derive and justify the matrix properties used in this chapter and will explore other algebraic properties of matrices.

4-1 Introduction to Matrices

Matrices are introduced as a way to organize data. In a matrix students have to attend to both the *magnitude* and the *position* of a number within a notation. Students explore equal matrices. They identify corresponding elements in the two matrices and write statements equating the corresponding elements. Then they apply previously-developed skills to solve equations or a system of equations.

4-2 Operations with Matrices

Students continue to explore the positions of elements in matrices and also look at some general properties of matrices. Students find the sum or difference of two matrices by checking that the matrices have the same dimensions and then calculating the sums or differences of corresponding elements. Students find the product of a scalar and a matrix by multiplying every element in the matrix by that scalar. For addition, subtraction, and scalar multiplication, the resulting matrix and the given matrices (or matrix) have identical dimensions.

For general properties of matrices, students look at examples that illustrate the commutative and associative properties of matrix addition.

4-3 Multiplying Matrices

Matrix multiplication extends two ideas of working with matrices. One idea deals with identifying the dimensions of a matrix. Students explore this idea by determining when matrix products are defined and, if so, stating the dimensions of the product matrix. The second idea deals with combining elements in two matrices to find an element in the resulting matrix. Unlike matrix addition or subtraction, which combined single elements of two matrices, matrix multiplication combines all the elements of a row in one matrix with all the elements of a column in the other matrix. Like matrix addition and subtraction, the result of a combination is a single element.

Students also explore matrix multiplication to show that there is an Associative Property of Matrix Multiplication and a Distributive Property of Matrix Multiplication over Matrix Addition. However, there cannot be a Commutative Property of Matrix Multiplication.

4-4 Transformations with Matrices

Students practice matrix addition, scalar multiplication, and matrix multiplication by using matrices to describe transformations. To describe a translation, the coordinates of the n vertices of the object are written as the n columns of a 2-by-n matrix. The translation is written as a 2-by-n matrix, each column the same. The sum of the two matrices gives the coordinates of the result of the translation. To describe a dilation (change in size) of a figure by a factor of n, the matrix of the coordinates for the figure is multiplied by the scalar n.

Reflections and rotations of figures are described using matrix multiplication. Each reflection and each rotation has a unique 2-by-2 matrix. When the matrix of the coordinates for a figure is multiplied by such a matrix, the resulting matrix gives the coordinates of the reflected or rotated figure.

4-5 Determinants

Students learn how to find the determinant of a 2-by-2 and a 3-by-3 matrix. This lesson introduces just a single application of a determinant, finding the area of a triangle, but other applications are seen in other lessons in this chapter. The determinant of a matrix is a single number or expression; for a 2-by-2 matrix it is the difference of two 2-factor terms, the products of the two diagonals. For a 3-by-3 matrix, the determinant is the sum or difference of six 3-factor terms.

To find the area of a triangle, students write a matrix where each row has the x-coordinate and y-coordinate of a vertex, and the number 1. The area of the triangle is the absolute value of one-half of the determinant for the matrix.

4-6 Cramer's Rule

An important algebraic application of determinants, called Cramer's Rule, is to solve systems of equations. When using Cramer's Rule to solve a system of equations, each equation is first written in standard form. Then the value of each variable in the solution is given by a fraction whose numerator and denominator are both determinants. The denominator is always the determinant of the coefficients of the variables. The numerators begin with the same determinant, but the column of coefficients for that particular variable is replaced with the column of constants. Since the determinant of the coefficients appears in each denominator, so the system has no solution if the value of that determinant is zero.

4-7 Identity and Inverse Matrices

This lesson returns to a focus on the positions of elements within a matrix and uses matrices to explore two algebraic ideas. One idea is that an element I of a set is a multiplicative identity for that set if, for any element A in the set, $A \cdot I = I \cdot A = A$; that is, multiplying by the multiplicative identity element results in no change. The other idea is that two elements are multiplicative inverses if their product is the multiplicative identity element.

For matrices, the multiplicative identity element is a square matrix with 1's down the left-to-right diagonal and 0's in every other position. Students verify that this matrix serves as a multiplicative identity. The multiplicative inverse for a 2-by-2 matrix is given as a product of two factors. One factor is a unit fraction whose denominator is the determinant of the given matrix. The other factor is the given matrix, but down the left-to-right diagonal the elements are switched and along the other diagonal the elements are multiplied by –1. The lesson points out that since the determinant appears in a denominator, a matrix must have a non-zero determinant in order to have an inverse.

4-8 Using Matrices to Solve Systems of Equations

In this culminating lesson of the chapter, students see how matrices can provide an efficient way to represent a system of equations and how to use matrix inverses and matrix multiplication as an efficient way to solve the system. To represent a system of equations, they write a single matrix equation. The left-hand side shows the matrix of coefficients multiplied by a column matrix of the variables. The right-hand side is a column matrix of the constants. To solve the system, students first find the inverse of the matrix of coefficients. When they multiply both sides of the original matrix equation by that inverse, the result gives the column matrix of variables equal to another column matrix. The value of each variable can be read directly from that equation.

Chapter 4

DAILY INTERVENTION and Assessment

INTERVENTION

ASSESSMENT

Type	Student Edition	Teacher Resources	Technology/Internet
Ongoing	Prerequisite Skills, pp. 153, 158, 166, 174, 181, 188, 194, 201 Practice Quiz 1, p. 174 Practice Quiz 2, p. 194	5-Minute Check Transparencies Quizzes, *CRM* pp. 231–232 Mid-Chapter Test, *CRM* p. 233 Study Guide and Intervention, *CRM* pp. 169–170, 175–176, 181–182, 187–188, 193–194, 199–200, 205–206, 211–212	Alge2PASS: Tutorial Plus www.algebra2.com/self_check_quiz www.algebra2.com/extra_examples
Mixed Review	pp. 158, 166, 174, 181, 188, 194, 201, 207	Cumulative Review, *CRM* p. 234	
Error Analysis	Find the Error, pp. 185, 205	Find the Error, *TWE* pp. 185, 205 Unlocking Misconceptions, *TWE* pp. 161, 183 Tips for New Teachers, *TWE* pp. 159, 166, 167	
Standardized Test Practice	pp. 158, 166, 173, 176, 179, 181, 187, 194, 201, 207, 215, 216–217	*TWE* p. 176 Standardized Test Practice, *CRM* pp. 235–236	Standardized Test Practice CD-ROM www.algebra2.com/standardized_test
Open-Ended Assessment	Writing in Math, pp. 158, 166, 173, 181, 187, 193, 200, 207 Open Ended, pp. 156, 163, 171, 178, 185, 192, 198, 205	Modeling: *TWE* pp. 174, 188 Speaking: *TWE* pp. 166, 194, 207 Writing: *TWE* pp. 158, 181, 201 Open-Ended Assessment, *CRM* p. 229	
Chapter Assessment	Study Guide, pp. 209–214 Practice Test, p. 215	Multiple-Choice Tests (Forms 1, 2A, 2B), *CRM* pp. 217–222 Free-Response Tests (Forms 2C, 2D, 3), *CRM* pp. 223–228 Vocabulary Test/Review, *CRM* p. 230	TestCheck and Worksheet Builder (see below) MindJogger Videoquizzes www.algebra2.com/vocabulary_review www.algebra2.com/chapter_test

Key to Abbreviations: TWE = Teacher Wraparound Edition; CRM = Chapter Resource Masters

Additional Intervention Resources

The Princeton Review's *Cracking the SAT & PSAT*

The Princeton Review's *Cracking the ACT*

ALEKS

TestCheck and Worksheet Builder

This **networkable** software has three modules for intervention and assessment flexibility:

- **Worksheet Builder** to make worksheet and tests
- **Student Module** to take tests on screen (optional)
- **Management System** to keep student records (optional)

Special banks are included for SAT, ACT, TIMSS, NAEP, and End-of-Course tests.

Intervention Technology

Alge2PASS: Tutorial Plus CD-ROM offers a complete, self-paced algebra curriculum.

Algebra 2 Lesson	Alge2PASS Lesson	
4-2	**6**	*Adding and Subtracting Matrices and Scalar Multiplication*
4-6	**7**	*Multiplying Matrices*

ALEKS is an online mathematics learning system that adapts assessment and tutoring to the student's needs. Subscribe at www.k12aleks.com.

Intervention at Home

Log on for student study help.

- For each lesson in the Student Edition, there are Extra Examples and Self-Check Quizzes.
  www.algebra2.com/extra_examples
  www.algebra2.com/self_check_quiz
- For chapter review, there is vocabulary review, test practice, and standardized test practice.
  www.algebra2.com/vocabulary_review
  www.algebra2.com/chapter_test
  www.algebra2.com/standardized_test

For more information on Intervention and Assessment, see pp. T8–T11.

Reading and Writing in Mathematics

Glencoe Algebra 2 provides numerous opportunities to incorporate reading and writing into the mathematics classroom.

Student Edition

- Foldables Study Organizer, p. 153
- Concept Check questions require students to verbalize and write about what they have learned in the lesson. (pp. 156, 163, 171, 178, 185, 192, 198, 205, 209)
- Writing in Math questions in every lesson, pp. 158, 166, 173, 181, 187, 193, 200, 207
- Reading Study Tip, pp. 154, 175, 182
- WebQuest, pp. 192, 207

Teacher Wraparound Edition

- Foldables Study Organizer, pp. 153, 209
- Study Notebook suggestions, pp. 156, 163, 171, 178, 185, 192, 198, 205
- Modeling activities, pp. 174, 188
- Speaking activities, pp. 166, 194, 207
- Writing activities, pp. 158, 181, 201
- Differentiated Instruction, (Verbal/Linguistic), p. 162
- ELL Resources, pp. 152, 157, 162, 165, 173, 180, 187, 193, 200, 206, 209

Additional Resources

- Vocabulary Builder worksheets require students to define and give examples for key vocabulary terms as they progress through the chapter. (*Chapter 4 Resource Masters*, pp. vii-viii)
- Reading to Learn Mathematics master for each lesson (*Chapter 4 Resource Masters*, pp. 173, 179, 185, 191, 197, 203, 209, 215)
- *Vocabulary PuzzleMaker* software creates crossword, jumble, and word search puzzles using vocabulary lists that you can customize.
- *Teaching Mathematics with Foldables* provides suggestions for promoting cognition and language.
- *Reading and Writing in the Mathematics Classroom*
- *WebQuest and Project Resources*

For more information on Reading and Writing in Mathematics, see pp. T6–T7.

Chapter 4 Notes

What You'll Learn

Have students read over the list of objectives and make a list of any words with which they are not familiar.

Why It's Important

Point out to students that this is only one of many reasons why each objective is important. Others are provided in the introduction to each lesson.

Lesson	NCTM Standards	Local Objectives
4-1	1, 2, 5, 8, 9, 10	
4-1 Follow-Up	1, 5, 10	
4-2	1, 2, 6, 8, 9, 10	
4-3	1, 2, 3, 6, 8, 9, 10	
4-4	1, 2, 6, 8, 9, 10	
4-5	1, 2, 4, 6, 7, 8, 9	
4-6	1, 2, 6, 8, 9, 10	
4-7	1, 2, 6, 8, 9, 10	
4-8	1, 2, 6, 7, 8, 9, 10	
4-8 Follow-Up	1, 2, 10	

Key to NCTM Standards:

1=Number & Operations, 2=Algebra, 3=Geometry, 4=Measurement, 5=Data Analysis & Probability, 6=Problem Solving, 7=Reasoning & Proof, 8=Communication, 9=Connections, 10=Representation

Chapter 4 Matrices

What You'll Learn

- **Lesson 4-1** Organize data in matrices.
- **Lessons 4-2, 4-3, and 4-5** Perform operations with matrices and determinants.
- **Lesson 4-4** Transform figures on a coordinate plane.
- **Lessons 4-6 and 4-8** Use matrices to solve systems of equations.
- **Lesson 4-7** Find the inverse of a matrix.

Key Vocabulary

- matrix (p. 154)
- determinant (p.182)
- expansion by minors (p. 183)
- Cramer's Rule (p. 189)
- matrix equation (p. 202)

Why It's Important

Data are often organized into matrices. For example, the National Federation of State High School Associations uses matrices to record student participation in sports by category for males and females. To find the total participation of both groups in each sport, you can add the two matrices. *You will learn how to add matrices in Lesson 4-2.*

Vocabulary Builder ELL

The Key Vocabulary list introduces students to some of the main vocabulary terms included in this chapter. For a more thorough vocabulary list with pronunciations of new words, give students the Vocabulary Builder worksheets found on pages vii and viii of the *Chapter 4 Resource Masters*. Encourage them to complete the definition of each term as they progress through the chapter. You may suggest that they add these sheets to their study notebooks for future reference when studying for the Chapter 4 test.

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 4.

For Lesson 4-1 — Solve Equations

Solve each equation. *(For review, see Lesson 1-3.)*

1. $3x = 18$ **6**
2. $2a - 3 = -11$ **−4**
3. $4t - 5 = 14$ $\mathbf{4\frac{3}{4}}$
4. $\frac{1}{3}y + 5 = 9$ **12**
5. $3k + 5 = 2k - 8$ **−13**
6. $5m - 6 = 7m - 8$ **1**

For Lessons 4-2 and 4-7 — Additive and Multiplicative Inverses

Name the additive inverse and the multiplicative inverse for each number. *(For review, see Lesson 1-2.)*

7. 3 $\mathbf{-3; \frac{1}{3}}$
8. -11 $\mathbf{11; -\frac{1}{11}}$
9. 8 $\mathbf{-8; \frac{1}{8}}$
10. -0.5 **0.5; −2**
11. 1.25 **−1.25; 0.8**
12. $\frac{5}{9}$ $\mathbf{-\frac{5}{9}; \frac{9}{5}}$
13. $-\frac{8}{3}$ $\mathbf{\frac{8}{3}; -\frac{3}{8}}$
14. $-1\frac{1}{5}$ $\mathbf{\frac{6}{5}; -\frac{5}{6}}$

For Lesson 4-4 — Graph Ordered Pairs

Graph each set of ordered pairs on a coordinate plane. *(For review, see Lesson 2-1.)*

15. $\{(0, 0), (1, 3), (-2, 4)\}$
16. $\{(-1, 5), (2, -3), (4, 0)\}$
17. $\{(-3, -3), (-1, 2), (1, -3), (3, -6)\}$
18. $\{(-2, 5), (1, 3), (4, -2), (4, 7)\}$

15–18. See pp. 217A–217B.

For Lessons 4-6 and 4-8 — Solve Systems of Equations

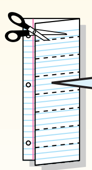
Solve each system of equations by using either substitution or elimination. *(For review, see Lesson 3-2.)*

19. $x = y + 5$, $3x + y = 19$ **(6, 1)**
20. $3x - 2y = 1$, $4x + 2y = 20$ **(3, 4)**
21. $5x + 3y = 25$, $4x + 7y = -3$ **(8, −5)**
22. $y = x - 7$, $2x - 8y = 2$ **(9, 2)**
23. $5x - 3y = 16$, $x - 3y = 8$ **(2, −2)**
24. $9x + 4y = 17$, $3x - 2y = 29$ **(5, −7)**

FOLDABLES™ Study Organizer

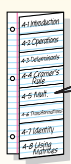
Make this Foldable to record information about matrices. Begin with one sheet of notebook paper.

Step 1 Fold and Cut



Fold lengthwise to the holes. Cut eight tabs in the top sheet.

Step 2 Label



Label each tab with a lesson number and title.

Reading and Writing As you read and study the chapter, write notes and examples for each topic under the tabs.

Getting Started

This section provides a review of the basic concepts needed before beginning Chapter 4. Page references are included for additional student help.

Prerequisite Skills in the Getting Ready for the Next Lesson section at the end of each exercise set review a skill needed in the next lesson.

For Lesson	Prerequisite Skill
4-2	Evaluating Expressions (p. 158)
4-3	Properties of Equality (p. 166)
4-4	Graphing Ordered Pairs (p. 174)
4-6	Solving Systems of Equations (p. 188)
4-7	Multiplying Matrices (p. 194)
4-8	Solving Multi-Step Equations (p. 201)

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Organizing Data and Descriptive Writing After students make their Foldable, have them use their Foldable to take notes, define terms, record concepts, and write examples about matrices. At the end of each lesson, ask students to use their notes to write a descriptive paragraph sharing their learning experiences with matrices. For example, a student might descriptively write about how they use Cramer's Rule to solve systems of two or three linear equations.

4-1 Lesson Notes

1 Focus

5-Minute Check Transparency 4-1 Use as a quiz or review of Chapter 3.

Mathematical Background notes are available for this lesson on p. 152C.

Building on Prior Knowledge

Students have used tables and arrays of data in a number of contexts. In previous courses, they may have been introduced to the concept of a matrix.

How are matrices used to make decisions?

Ask students:

- How many rows of numbers appear in the SUV matrix? **4**
- How many columns of numbers are there in the SUV matrix? **5**

4-1 Introduction to Matrices

What You'll Learn

- Organize data in matrices.
- Solve equations involving matrices.

Vocabulary

- matrix
- element
- dimension
- row matrix
- column matrix
- square matrix
- zero matrix
- equal matrices

How are matrices used to make decisions?

Sabrina wants to buy a sports-utility vehicle (SUV). There are many types of SUVs in many prices and styles. So, Sabrina makes a list of the qualities for different models and organizes the information in a matrix.

$$\begin{array}{rc} & \begin{array}{ccccc} \text{Base Price} & \text{Horse-power} & \text{Towing Capacity (lb)} & \text{Cargo Space (ft}^3\text{)} & \text{Fuel Economy (mpg)} \end{array} \\ \begin{array}{r} \text{Large SUV} \\ \text{Standard SUV} \\ \text{Mid-Size SUV} \\ \text{Compact SUV} \end{array} & \begin{bmatrix} \$32{,}450 & 285 & 12{,}000 & 46 & 17 \\ \$29{,}115 & 275 & 8700 & 16 & 17.5 \\ \$27{,}975 & 190 & 5700 & 34 & 20 \\ \$18{,}180 & 127 & 3000 & 15 & 26.5 \end{bmatrix} \end{array}$$

Source: *Car and Driver Buyer's Guide*

When the information is organized in a matrix, it is easy to compare the features of each vehicle.

Study Tip

Reading Math
The plural of *matrix* is *matrices*.

ORGANIZE DATA A **matrix** is a rectangular array of variables or constants in horizontal rows and vertical columns, usually enclosed in brackets.

Example 1 Organize Data in a Matrix

Sharon wants to install cable television in her new apartment. There are two cable companies in the area whose prices are listed below. Use a matrix to organize the information. When is each company's service less expensive?

Metro Cable	
Basic Service (26 channels)	$11.95
Standard Service (53 channels)	$30.75
Premium Channels (in addition to Standard Service)	
• One Premium	$10.00
• Two Premiums	$19.00
• Three Premiums	$25.00

Cable City	
Basic Service (26 channels)	$9.95
Standard Service (53 channels)	$31.95
Premium Channels (in addition to Standard Service)	
• One Premium	$8.95
• Two Premiums	$16.95
• Three Premiums	$22.95

Organize the costs into labeled columns and rows.

$$\begin{array}{rc} & \begin{array}{ccccc} \text{Basic} & \text{Standard} & \text{Standard Plus One Premium} & \text{Standard Plus Two Premiums} & \text{Standard Plus Three Premiums} \end{array} \\ \begin{array}{r} \text{Metro Cable} \\ \text{Cable City} \end{array} & \begin{bmatrix} 11.95 & 30.75 & 40.75 & 49.75 & 55.75 \\ 9.95 & 31.95 & 40.90 & 48.90 & 54.90 \end{bmatrix} \end{array}$$

Metro Cable has the best price for standard service and standard plus one premium channel. Cable City has the best price for the other categories.

Resource Manager

Workbook and Reproducible Masters

Chapter 4 Resource Masters
- Study Guide and Intervention, pp. 169–170
- Skills Practice, p. 171
- Practice, p. 172
- Reading to Learn Mathematics, p. 173
- Enrichment, p. 174

Teaching Algebra With Manipulatives Masters, pp. 229–230

Transparencies

5-Minute Check Transparency 4-1
Real-World Transparency 4
Answer Key Transparencies

Technology

Interactive Chalkboard

Study Tip

Element

The elements of a matrix can be represented using double subscript notation. The element a_{ij} is the element in row i column j.

In a matrix, numbers or data are organized so that each position in the matrix has a purpose. Each value in the matrix is called an **element**. A matrix is usually named using an uppercase letter.

$$A = \underbrace{\begin{bmatrix} 2 & 6 & 1 \\ 7 & 1 & 5 \\ 9 & 3 & 0 \\ 12 & 15 & 26 \end{bmatrix}}_{\text{3 columns}} \Big\} \text{4 rows}$$

The element 15 is in row 4, column 2.

A matrix can be described by its **dimensions**. A matrix with m rows and n columns is an $m \times n$ matrix (read "m by n"). Matrix A above is a 4×3 matrix since it has 4 rows and 3 columns.

Example 2 Dimensions of a Matrix

State the dimensions of matrix B if $B = \begin{bmatrix} 1 & -3 \\ -5 & 18 \\ 0 & -2 \end{bmatrix}$.

$$B = \underbrace{\begin{bmatrix} 1 & -3 \\ -5 & 18 \\ 0 & -2 \end{bmatrix}}_{\text{2 columns}} \Big\} \text{3 rows}$$

Since matrix B has 3 rows and 2 columns, the dimensions of matrix B are 3×2.

Certain matrices have special names. A matrix that has only one row is called a **row matrix**, while a matrix that has only one column is called a **column matrix**. A matrix that has the same number of rows and columns is called a **square matrix**. Another special type of matrix is the **zero matrix**, in which every element is 0. The zero matrix can have any dimension.

EQUATIONS INVOLVING MATRICES Two matrices are considered **equal matrices** if they have the same dimensions and if each element of one matrix is equal to the corresponding element of the other matrix.

$\begin{bmatrix} 6 & 3 \\ 0 & 9 \\ 1 & 3 \end{bmatrix} \neq \begin{bmatrix} 6 & 0 & 1 \\ 3 & 9 & 3 \end{bmatrix}$ The matrices have different dimensions. They are not equal.

$\begin{bmatrix} 1 & 2 \\ 8 & 5 \end{bmatrix} \neq \begin{bmatrix} 1 & 8 \\ 2 & 5 \end{bmatrix}$ Corresponding elements are not equal. The matrices are not equal.

$\begin{bmatrix} 5 & 6 & 0 \\ 0 & 7 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 0 \\ 0 & 7 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ The matrices have the same dimensions and the corresponding elements are equal. The matrices are equal.

The definition of equal matrices can be used to find values when elements of equal matrices are algebraic expressions.

Example 3 Solve an Equation Involving Matrices

Solve $\begin{bmatrix} y \\ 3x \end{bmatrix} = \begin{bmatrix} 6 - 2x \\ 31 + 4y \end{bmatrix}$ for x and y.

Since the matrices are equal, the corresponding elements are equal. When you write the sentences to show this equality, two linear equations are formed.

$y = 6 - 2x$

$3x = 31 + 4y$

(continued on the next page)

www.algebra2.com/extra_examples

2 Teach

ORGANIZE DATA

In-Class Examples Power Point®

1 **COLLEGE** Kaitlin wants to attend one of three Iowa universities next year. She has gathered information about tuition (T), room and board (R/B), and enrollment (E) for the universities. Use a matrix to organize the information. Which university's total cost is lowest?

Iowa State University: T–\$3132, R/B–\$4432, E–26,845

University of Iowa: T–\$3204, R/B–\$4597, E–28,311

University of Northern Iowa: T–\$3130, R/B–\$4149, E–14,106

Source: The World Almanac 2002

$$\begin{array}{c} \\ \text{ISU} \\ \text{UI} \\ \text{UNI} \end{array} \begin{array}{c} \begin{array}{ccc} \text{T} & \text{R/B} & \text{E} \end{array} \\ \begin{bmatrix} 3132 & 4432 & 26845 \\ 3204 & 4597 & 28311 \\ 3130 & 4149 & 14106 \end{bmatrix} \end{array}$$

The University of Northern Iowa has the lowest total cost.

2 State the dimensions of matrix G if $G = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 5 & -3 & -1 \end{bmatrix}$. 2×4

Teaching Tip Stress that matrix dimensions are always given as "rows by columns."

EQUATIONS INVOLVING MATRICES

In-Class Example Power Point®

3 Solve $\begin{bmatrix} y \\ 3 \end{bmatrix} = \begin{bmatrix} 3x - 2 \\ 2y + x \end{bmatrix}$ for x and y. (1, 1)

Teaching Tip Caution students to set equal those elements of the matrices that are in corresponding positions.

DAILY INTERVENTION **Differentiated Instruction**

Kinesthetic To help students associate the terms *row* and *column* with the correct parts of a matrix, use this activity. Instruct students to say either "row" or "column" when you point to them. When a student says "row," tell students to stretch out their arms parallel to the floor. When a student calls out "column," everyone is to stretch both of their arms overhead to form a column. Relate these movements to the side-to-side aspect of matrix rows and the up-and-down aspect of matrix columns.

Study Tip

Look Back

To review **solving systems of equations by using substitution**, see Lesson 3-2.

This system can be solved using substitution.

$3x = 31 + 4y$	Second equation
$3x = 31 + 4(6 - 2x)$	Substitute $6 - 2x$ for y.
$3x = 31 + 24 - 8x$	Distributive Property
$11x = 55$	Add $8x$ to each side.
$x = 5$	Divide each side by 11.

To find the value for y, substitute 5 for x in either equation.

$y = 6 - 2x$	First equation
$y = 6 - 2(5)$	Substitute 5 for x.
$y = -4$	Simplify.

The solution is $(5, -4)$.

Check for Understanding

Concept Check

1–2. See margin.

1. **Describe** the conditions that must be met in order for two matrices to be considered equal.
2. **OPEN ENDED** Give examples of a row matrix, a column matrix, a square matrix, and a zero matrix. State the dimensions of each matrix.
3. **Explain** what is meant by corresponding elements. Corresponding elements are elements in the same row and column positions.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4, 5	2
6, 7	3
8, 9	1

State the dimensions of each matrix.

4. $[3 \quad 4 \quad 5 \quad 6 \quad 7]$ 1×5

5. $\begin{bmatrix} 10 & -6 & 18 & 0 \\ -7 & 5 & 2 & 4 \\ 3 & 11 & 9 & 7 \end{bmatrix}$ 3×4

Solve each equation.

6. $\begin{bmatrix} x + 4 \\ 2y \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$ (5, 6)

7. $[9 \quad 13] = [x + 2y \quad 4x + 1]$ (3, 3)

Application

WEATHER **For Exercises 8 and 9, use the table that shows a five-day forecast indicating high (H) and low (L) temperatures.**

Fri	Sat	Sun	Mon	Tue
H 88	H 88	H 90	H 86	H 85
L 54	L 54	L 56	L 53	L 52

8. Organize the temperatures in a matrix. See margin.
9. What are the dimensions of the matrix? 2×5

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
10–15	2
16–25	3
26–31	1

Extra Practice

See page 834.

State the dimensions of each matrix.

10. $\begin{bmatrix} 6 & -1 & 5 \\ -2 & 3 & -4 \end{bmatrix}$ 2×3

11. $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ 3×1

12. $\begin{bmatrix} 0 & 0 & 8 \\ 6 & 2 & 4 \\ 1 & 3 & 6 \\ 5 & 9 & 2 \end{bmatrix}$ 4×3

13. $\begin{bmatrix} -3 & 17 & -22 \\ 9 & 31 & 16 \\ 20 & -15 & 4 \end{bmatrix}$ 3×3

14. $\begin{bmatrix} 17 & -2 & 8 & -9 & 6 \\ 5 & 11 & 20 & -1 & 4 \end{bmatrix}$ 2×5

15. $\begin{bmatrix} 16 & 8 \\ 10 & 5 \\ 0 & 0 \end{bmatrix}$ 3×2

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms in this lesson to their Vocabulary Builder worksheets for Chapter 4.
- add labeled examples of various kinds of matrices to their notebook.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective

Organize Data: 10–15, 26–33

Equations Involving Matrices: 16–25

Odd/Even Assignments

Exercises 10–25 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 11–23 odd, 26, 27, 32–57

Average: 11–25 odd, 26–29, 31–57

Advanced: 10–24 even, 26–49 (optional: 50–57)

Answers

1. The matrices must have the same dimensions and each element of one matrix must be equal to the corresponding element of the other matrix.
2. Sample answers: row matrix, $[1 \quad 2 \quad 3]$, 1×3; column matrix, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, 2×1; square matrix, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, 2×2; zero matrix, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, 2×2

8. $\begin{array}{c} \\ \text{High} \\ \text{Low} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Fri} & \text{Sat} & \text{Sun} & \text{Mon} & \text{Tue} \end{array} \\ \begin{bmatrix} 88 & 88 & 90 & 86 & 85 \\ 54 & 54 & 56 & 53 & 52 \end{bmatrix} \end{array}$

26. $\begin{array}{c} \\ \text{Adult} \\ \text{Child} \\ \text{Senior} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Evening} & \text{Matinee} & \text{Twilight} \end{array} \\ \begin{bmatrix} 7.50 & 5.50 & 3.75 \\ 4.50 & 4.50 & 3.75 \\ 5.50 & 5.50 & 3.75 \end{bmatrix} \end{array}$

Solve each equation.

16. $[2x \quad 3 \quad 3z] = [5 \quad 3y \quad 9]$ **(2.5, 1, 3)**

17. $[4x \quad 3y] = [12 \quad -1]$ **$\left(3, -\frac{1}{3}\right)$**

18. $\begin{bmatrix} 4x \\ 5 \end{bmatrix} = \begin{bmatrix} 15 + x \\ 2y - 1 \end{bmatrix}$ **(5, 3)**

19. $\begin{bmatrix} 4x - 3 & 3y \\ 7 & 13 \end{bmatrix} = \begin{bmatrix} 9 & -15 \\ 7 & 2z + 1 \end{bmatrix}$ **(3, −5, 6)**

20. $\begin{bmatrix} x + 3y \\ 3x + y \end{bmatrix} = \begin{bmatrix} -13 \\ 1 \end{bmatrix}$ **(2, −5)**

21. $\begin{bmatrix} 2x + y \\ x - 3y \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$ **(4, −3)**

22. $\begin{bmatrix} 2x \\ 2x + 3y \end{bmatrix} = \begin{bmatrix} y \\ 12 \end{bmatrix}$ **(1.5, 3)**

23. $\begin{bmatrix} 4x \\ y - 1 \end{bmatrix} = \begin{bmatrix} 11 + 3y \\ x \end{bmatrix}$ **(14, 15)**

★ 24. $\begin{bmatrix} x^2 + 1 & 5 - y \\ x + y & y - 4 \end{bmatrix} = \begin{bmatrix} 5 & x \\ 5 & 3 \end{bmatrix}$ **(−2, 7)**

★ 25. $\begin{bmatrix} 3x - 5 & x + y \\ 12 & 9z \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 12 & 3x + y \end{bmatrix}$ **(5, 3, 2)**

More About. . .

Movies

Adjusting for inflation, *Cleopatra* (1963) is the most expensive movie ever made. Its $44 million budget is equivalent to $306,867,120 today.

Source: *The Guinness Book of Records*

MOVIES For Exercises 26 and 27, use the advertisement shown at the right.

26. Write a matrix for the prices of movie tickets for adults, children, and seniors. **See margin.**

27. What are the dimensions of the matrix? **3 × 3**

DINING OUT For Exercises 28 and 29, use the following information.
A newspaper rated several restaurants by cost, level of service, atmosphere, and location using a scale of ★ being low and ★★★★ being high.

Catalina Grill: cost ★★, service ★, atmosphere ★, location ★

Oyster Club: cost ★★★, service ★★, atmosphere ★, location ★★

Casa di Pasta: cost ★★★★, service ★★★, atmosphere ★★★, location ★★★

Mason's Steakhouse: cost ★★, service ★★★★, atmosphere ★★★★, location ★★★

28. Write a 4 × 4 matrix to organize this information. **28–29. See pp. 217A–217B.**

29. Which restaurant would you select based on this information and why?

HOTELS For Exercises 30 and 31, use the costs for an overnight stay at a hotel that are given below.

Single Room: $60 weekday; $79 weekend

Double Room: $70 weekday; $89 weekend

Suite: $75 weekday; $95 weekend

30. Write a 3 × 2 matrix that represents the cost of each room. **See margin.**

★ 31. Write a 2 × 3 matrix that represents the cost of each room. **See margin.**

CRITICAL THINKING For Exercises 32 and 33, use the matrix at the right.

$$\begin{bmatrix} 1 & 3 & 6 & 10 & 15 & \dots \\ 2 & 5 & 9 & 14 & 20 & \dots \\ 4 & 8 & 13 & 19 & 26 & \dots \\ 7 & 12 & 18 & 25 & 33 & \dots \\ 11 & 17 & 24 & 32 & 41 & \dots \\ 16 & 23 & 31 & 40 & 50 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

32. Study the pattern of numbers. Complete the matrix for column 6 and row 7.

33. In which row and column will 100 occur? **row 6, column 9**

32.
$$\begin{bmatrix} 1 & 3 & 6 & 10 & 15 & 21 \\ 2 & 5 & 9 & 14 & 20 & 27 \\ 4 & 8 & 13 & 19 & 26 & 34 \\ 7 & 12 & 18 & 25 & 33 & 42 \\ 11 & 17 & 24 & 32 & 41 & 51 \\ 16 & 23 & 31 & 40 & 50 & 61 \\ 22 & 30 & 39 & 49 & 60 & 72 \end{bmatrix}$$

www.algebra2.com/self_check_quiz

Answers

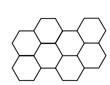
30.
$$\begin{array}{r} \\ \text{Single} \\ \text{Double} \\ \text{Suite} \end{array} \begin{array}{c} \begin{array}{cc} \text{Weekday} & \text{Weekend} \end{array} \\ \begin{bmatrix} 60 & 79 \\ 70 & 89 \\ 75 & 95 \end{bmatrix} \end{array}$$

31.
$$\begin{array}{r} \\ \text{Weekday} \\ \text{Weekend} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Single} & \text{Double} & \text{Suite} \end{array} \\ \begin{bmatrix} 60 & 70 & 75 \\ 79 & 89 & 95 \end{bmatrix} \end{array}$$

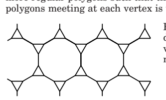
Enrichment, p. 174

Tessellations

A **tessellation** is an arangement of polygons covering a plane without any gaps or overlapping. One example of a tessellation is a honeycomb. Three congruent regular hexagons meet at each vertex, and there is no wasted space between cells. This tessellation is called a regular tessellation since it is formed by congruent regular polygons.



A **semi-regular tessellation** is a tessellation formed by two or more regular polygons such that the number of sides of the polygons meeting at each vertex is the same.



For example, the tessellation at the left has two regular dodecagons and one equilateral triangle meeting at each vertex. We can name this tessellation a 3-12-12 for the number of sides of each polygon that meet at one vertex.

Study Guide and Intervention, p. 169 (shown) and p. 170

Organize Data

Matrix	a rectangular array of variables or constants in horizontal rows and vertical columns, usually enclosed in brackets.

A matrix can be described by its **dimensions**. A matrix with m rows and n columns is an $m \times n$ matrix.

Example 1 **Owls' eggs incubate for 30 days and their fledgling period is also 30 days. Swifts' eggs incubate for 20 days and their fledgling period is 44 days. Pigeon eggs incubate for 15 days, and their fledgling period is 17 days. Eggs of the king penguin incubate for 53 days, and the fledgling time for a king penguin is 360 days. Write a 2 × 4 matrix to organize this information.** **Source:** *The Cambridge Factfinder*

$$\begin{array}{r} \\ \text{Incubation} \\ \text{Fledgling} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Owl} & \text{Swift} & \text{Pigeon} & \text{King Penguin} \end{array} \\ \begin{bmatrix} 30 & 20 & 15 & 53 \\ 30 & 44 & 17 & 360 \end{bmatrix} \end{array}$$

Example 2 **What are the dimensions of matrix A if $A = \begin{bmatrix} 13 & 10 & -3 & 45 \\ 2 & 8 & 15 & 80 \end{bmatrix}$?**

Since matrix A has 2 rows and 4 columns, the dimensions of A are 2×4.

Exercises

State the dimensions of each matrix.

1. $\begin{bmatrix} 15 & 5 & 27 & -4 \\ 23 & 6 & 0 & 5 \\ 14 & 70 & 24 & -3 \\ 63 & 3 & 42 & 90 \end{bmatrix}$ **4 × 4**

2. $[16 \quad 12 \quad 0]$ **1 × 3**

3. $\begin{bmatrix} 71 & 44 \\ 39 & 27 \\ 45 & 16 \\ 92 & 53 \\ 78 & 65 \end{bmatrix}$ **5 × 2**

4. A travel agent provides for potential travelers the normal high temperatures for the months of January, April, July, and October for various cities. In Boston these figures are 36°, 56°, 82°, and 63°. In Dallas they are 54°, 76°, 97°, and 79°. In Los Angeles they are 68°, 72°, 84°, and 79°. In Seattle they are 46°, 58°, 74°, and 60°, and in St. Louis they are 38°, 67°, 89°, and 69°. Organize this information in a 4 × 5 matrix. **Source:** *The New York Times Almanac*

$$\begin{array}{r} \\ \text{January} \\ \text{April} \\ \text{July} \\ \text{October} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Boston} & \text{Dallas} & \text{Los Angeles} & \text{Seattle} & \text{St. Louis} \end{array} \\ \begin{bmatrix} 36 & 54 & 68 & 46 & 38 \\ 56 & 76 & 72 & 58 & 67 \\ 82 & 97 & 84 & 74 & 89 \\ 63 & 79 & 79 & 60 & 69 \end{bmatrix} \end{array}$$

Skills Practice, p. 171 and Practice, p. 172 (shown)

State the dimensions of each matrix.

1. $[-3 \quad -3 \quad 7]$ **1 × 3**

2. $\begin{bmatrix} 5 & 8 & -1 \\ -2 & 1 & 8 \end{bmatrix}$ **2 × 3**

3. $\begin{bmatrix} -2 & 2 & -2 & 3 \\ 5 & 16 & 0 & 0 \\ 4 & 7 & -1 & 4 \end{bmatrix}$ **3 × 4**

Solve each equation.

4. $[4x \quad 42] = [24 \quad 6y]$ **(6, 7)**

5. $[-2x \quad 22 \quad -3z] = [6x \quad -2y \quad 45]$ **(0, −11, −15)**

6. $\begin{bmatrix} 6x \\ 2y + 3 \end{bmatrix} = \begin{bmatrix} -36 \\ 17 \end{bmatrix}$ **(−6, 7)**

7. $\begin{bmatrix} 7x - 8 \\ 8y - 3 \end{bmatrix} = \begin{bmatrix} 20 \\ 2y + 3 \end{bmatrix}$ **(4, 1)**

8. $\begin{bmatrix} -4x - 3 \\ 6y \end{bmatrix} = \begin{bmatrix} -3x \\ -2y + 16 \end{bmatrix}$ **(−3, 2)**

9. $\begin{bmatrix} 6x - 12 \\ -3y + 6 \end{bmatrix} = \begin{bmatrix} -3x - 21 \\ 8y - 5 \end{bmatrix}$ **(−1, 1)**

10. $\begin{bmatrix} -5 & 3x + 1 \\ 2y - 1 & 3z - 2 \end{bmatrix} = \begin{bmatrix} -5 & x - 1 \\ 3y & 5z - 4 \end{bmatrix}$ **(−1, −1, 1)**

11. $\begin{bmatrix} 3x \\ y + 4 \end{bmatrix} = \begin{bmatrix} y + 8 \\ 17 \end{bmatrix}$ **(7, 13)**

12. $\begin{bmatrix} 5x + 8y \\ 3x - 11 \end{bmatrix} = \begin{bmatrix} -1 \\ y \end{bmatrix}$ **(3, −2)**

13. $\begin{bmatrix} 2x + y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ **(2, −4)**

14. **TICKET PRICES** The table at the right gives ticket prices for a concert. Write a 2 × 3 matrix that represents the cost of a ticket.

	Child	Student	Adult
Cost Purchased in Advance	$6	$12	$18
Cost Purchased at the Door	$8	$15	$22

$\begin{bmatrix} 6 & 12 & 18 \\ 8 & 15 & 22 \end{bmatrix}$

CONSTRUCTION **For Exercises 15 and 16, use the following information.**
During each of the last three weeks, a road-building crew has used three truck-loads of gravel. The table at the right shows the amount of gravel in each load.

Week 1	Week 2	Week 3
Load 1 40 tons	Load 1 40 tons	Load 1 32 tons
Load 2 32 tons	Load 2 40 tons	Load 2 24 tons
Load 3 24 tons	Load 3 32 tons	Load 3 24 tons

15. Write a matrix for the amount of gravel in each load.

$\begin{bmatrix} 40 & 32 & 24 \\ 40 & 40 & 32 \\ 32 & 24 & 24 \end{bmatrix}$ **or** $\begin{bmatrix} 40 & 40 & 32 \\ 32 & 40 & 24 \\ 24 & 32 & 24 \end{bmatrix}$

16. What are the dimensions of the matrix? **3 × 3**

Reading to Learn Mathematics, p. 173

ELL

Pre-Activity **How are matrices used to make decisions?**

Read the introduction to Lesson 4-1 at the top of page 154 in your textbook.

What is the base price of a Mid-Size SUV?
$27,975

Reading the Lesson

1. Give the dimensions of each matrix.

   a. $\begin{bmatrix} 3 & 2 & 5 \\ -1 & 0 & 6 \end{bmatrix}$ **2 × 3**

   b. $[1 \quad 4 \quad 0 \quad -8 \quad 2]$ **1 × 5**

2. Identify each matrix with as many of the following descriptions that apply: *row matrix*, *column matrix*, *square matrix*, *zero matrix*.

   a. $[6 \quad 5 \quad 4 \quad 3]$ **row matrix**

   b. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **column matrix; zero matrix**

   c. $[0]$ **row matrix; column matrix; square matrix; zero matrix**

3. Write a system of equations that you could use to solve the following matrix equation for x, y, and z. (Do not actually solve the system.)

   $\begin{bmatrix} 3x \\ x + y \\ y - z \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \\ 6 \end{bmatrix}$ **$3x = -9$, $x + y = 5$, $y - z = 6$**

Helping You Remember

4. Some students have trouble remembering which number comes first in writing the dimensions of a matrix. Think of an easy way to remember this.
   Sample answer: Read the matrix from top to bottom, then from left to right. Reading down gives the number of rows, which is written first in the dimensions of the matrix. Reading across gives the number of columns, which is written second.

4 Assess

Open-Ended Assessment

Writing Have students write a matrix equation that is equivalent to the following system of equations.

$2x - y = -4$

$3x + y = -1$

$\begin{bmatrix} 2x - y \\ 3x + y \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$

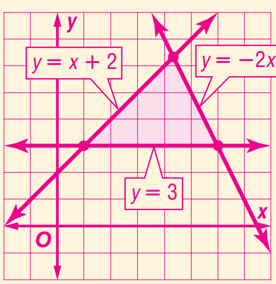
Getting Ready for Lesson 4-2

PREREQUISITE SKILL Lesson 4-2 presents the addition and subtraction of matrices, and the multiplication of a matrix by a scalar. These operations involve evaluating various expressions with real numbers. Exercises 50–57 should be used to determine your students' familiarity with evaluating expressions.

Answers

34. Matrices are used to organize information so it can be read and compared more easily. Answers should include the following.
    - If you want the least expensive vehicle, the compact SUV has the best price; the large SUV has the most horsepower, towing capacity, and cargo space, and the standard SUV has the best fuel economy.
    - Sample answer: Matrices are used to report stock prices in the newspaper.

40.



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34. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

    How are matrices used to make decisions?

    Include the following in your answer:
    - the circumstances under which each vehicle best fits a person's needs, and
    - an example of how matrices are used in newspapers or magazines.

Standardized Test Practice

35. In matrix $A = \begin{bmatrix} 1 & 5 & -2 \\ -4 & 0 & 6 \\ 3 & 7 & 8 \end{bmatrix}$, element 3 is in which row and column? **B**

    (A) row 1, column 3 (B) row 3, column 1
    (C) row 1, column 1 (D) row 3, column 3

36. What is the value of y if $\begin{bmatrix} 3x \\ y + 5 \end{bmatrix} = \begin{bmatrix} 9 + y \\ x \end{bmatrix}$? **C**

    (A) 2 (B) 4 (C) −3 (D) −1

Maintain Your Skills

Mixed Review

Solve each system of equations. *(Lesson 3-5)*

37. $3x - 3y = 6$, $-6y = -30$, $5z - 2x = 6$ **(7, 5, 4)**

38. $3a + 2b = 27$, $5a - 7b + c = 5$, $-2a + 10b + 5c = -29$ **(7, 3, −9)**

39. $3r - 15s + 4t = -57$, $9r + 45s - t = 26$, $-6r + 10s + 3t = -19$

39. $\left(-\frac{4}{3}, \frac{3}{5}, -11\right)$

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region. *(Lesson 3-4)* **40–42. See margin for graphs.**

40. $y \ge 3$, $y \le x + 2$, $y \le -2x + 15$, $f(x, y) = 2x + 3y$

41. $y \ge \frac{1}{3}x$, $y \ge -5x + 16$, $y \le -x + 10$, $f(x, y) = 5x - y$

42. $y \ge \frac{1}{2}x$, $y \ge -x + 3$, $y \le -\frac{3}{2}x + 12$, $f(x, y) = 3y - x$

40. vertices: (1, 3), (6, 3), $\left(\frac{13}{3}, \frac{19}{3}\right)$; max: $f\left(\frac{13}{3}, \frac{19}{3}\right) = \frac{83}{8}$, min: $f(1, 3) = 11$

41. vertices: (3, 1), $\left(\frac{15}{2}, \frac{5}{2}\right)$, $\left(\frac{3}{2}, \frac{17}{2}\right)$; max: $f\left(\frac{15}{2}, \frac{5}{2}\right) = 35$, min: $f\left(\frac{3}{2}, \frac{17}{2}\right) = -1$

42. vertices: (2, 1), (6, 3); no maximum, min: $f(2, 2) = 1$

BUSINESS **For Exercises 43–45, use the following information.**
The parking garage at Burrough's Department Store charges \$1.50 for each hour or fraction of an hour for parking. *(Lesson 2-6)*

43. Graph the function. **See pp. 217A–217B.**

44. What type of function represents this situation? **step function**

45. Jada went shopping at Burrough's Department Store yesterday. She left her car in the parking garage for two hours and twenty-seven minutes. How much did Jada pay for parking? **\$4.50**

Find each value if $f(x) = x^2 - 3x + 2$. *(Lesson 2-1)*

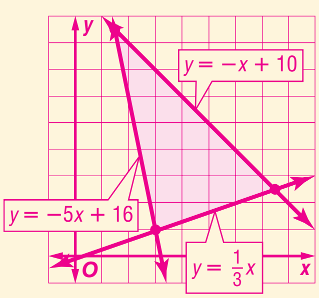
46. $f(3)$ **2**
47. $f(0)$ **2**
48. $f(2)$ **0**
49. $f(-3)$ **20**

Getting Ready for the Next Lesson

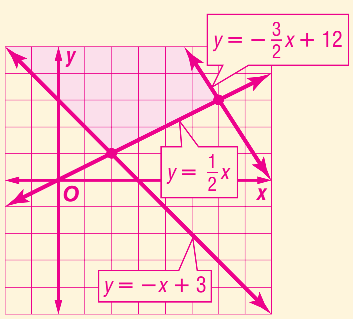
PREREQUISITE SKILL **Find the value of each expression.**
*(To review **evaluating expressions**, see Lesson 1-2.)*

50. $8 + (-5)$ **3**
51. $-2 - 8$ **−10**
52. $3.5 + 2.7$ **6.2**
53. $6(-3)$ **−18**
54. $\frac{1}{2}(34)$ **17**
55. $6(4) + 3(-9)$ **−3**
56. $-5(3 - 18)$ **75**
57. $14\left(\frac{1}{4}\right) - 12\left(\frac{1}{6}\right)$ **$\frac{3}{2}$**

41.



42.



Spreadsheet Investigation

A Follow-Up of Lesson 4-1

Organizing Data

You can use a computer spreadsheet to organize and display data. Then you can use the data to create graphs or perform calculations.

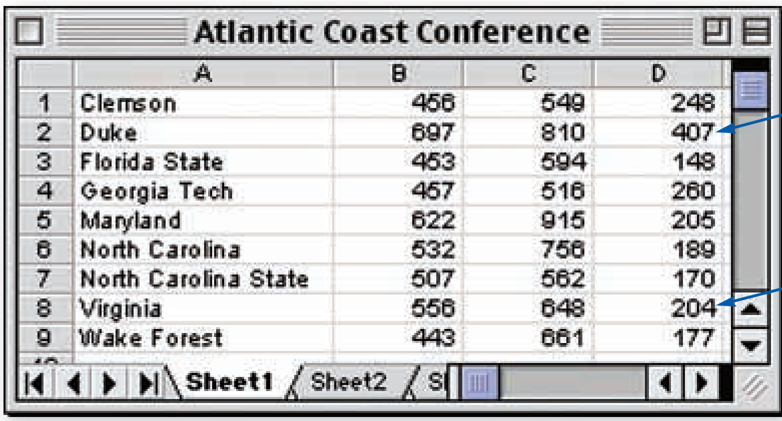
Example

Enter the data on the Atlantic Coast Conference Men's Basketball scoring into a spreadsheet.

Atlantic Coast Conference 2000–2001 Men's Basketball			
Team	**Free Throws**	**2-Point Field Goals**	**3-Point Field Goals**
Clemson	456	549	248
Duke	697	810	407
Florida State	453	594	148
Georgia Tech	457	516	260
Maryland	622	915	205
North Carolina	532	756	189
North Carolina State	507	562	170
Virginia	556	648	204
Wake Forest	443	661	177

Source: Atlantic Coast Conference

Use Column A for the team names, Column B for the numbers of free throws, Column C for the numbers of 2-point field goals, and Column D for the numbers of 3-point field goals.


Atlantic Coast Conference

	A	B	C	D
1	Clemson	456	549	248
2	Duke	697	810	407
3	Florida State	453	594	148
4	Georgia Tech	457	516	260
5	Maryland	622	915	205
6	North Carolina	532	756	189
7	North Carolina State	507	562	170
8	Virginia	556	648	204
9	Wake Forest	443	661	177

Each **row** contains data for a different team. Row 2 represents Duke.

Each **cell** of the spreadsheet contains one piece of data. Cell 8D contains the value 204, representing the number of 3-point field goals made by Virginia.

Model and Analyze **1. See pp. 217A–217B.**

1. Enter the data about sports-utility vehicles on page 154 into a spreadsheet.
2. Compare and contrast how data are organized in a spreadsheet and how they are organized in a matrix. **See margin.**

Answer

2. Both use rows and columns. In a spreadsheet, the rows are designated by numbers and the columns are designated by letters. In a matrix, both rows and columns are designated by numbers.

Spreadsheet Investigation

A Follow-Up of Lesson 4-1

Getting Started

You may want to have students format the cells before they start to enter the data. Use the Format Cells command. You can use the General or Text format for all the data unless number operations are going to be performed with the contents of the cells.

Teach

- Have students practice their skills in using a spreadsheet by entering the data in the example.
- Clear up any confusion that students may have about how to move from cell to cell in a spreadsheet.
- Have students complete Exercises 1 and 2.

Tips for New Teachers

Intervention
Many sources of confusion can be avoided when students work with a partner on technology investigations. Assign partners so that a student with more knowledge of spreadsheets is paired with one who has less experience with them.

Assess

Ask students about the structure of a given spreadsheet. For example, ask them which cells contain data values and which contain labels to identify the data.

4-2 Lesson Notes

1 Focus

5-Minute Check Transparency 4-2 Use as a quiz or review of Lesson 4-1.

Mathematical Background notes are available for this lesson on p. 152C.

How can matrices be used to calculate daily dietary needs?

Ask students:

- Suppose a 3 × 3 matrix is created for the breakfast data with the days as the rows and the Calories, protein, and fat contents as the columns. What is the row and column location for the entry for 17? **row 2, column 3**
- Suppose a matrix for the dinner data is created with the days as the rows. What is the meaning of the entry that will be located in row 3, column 2? **The entry 29 means that there are 29 grams of protein in the dinner meal on day 3.**

4-2 Operations with Matrices

What You'll Learn

- Add and subtract matrices.
- Multiply by a matrix scalar.

Vocabulary

- scalar
- scalar multiplication

How can matrices be used to calculate daily dietary needs?

In her job as a hospital dietician, Celeste designs weekly menus for her patients and tracks various nutrients for each daily diet. The table shows the Calories, protein, and fat in a patient's meals over a three-day period.

Day	Breakfast			Lunch			Dinner		
	Calories	Protein (g)	Fat (g)	Calories	Protein (g)	Fat (g)	Calories	Protein (g)	Fat (g)
1	566	18	7	785	22	19	1257	40	26
2	482	12	17	622	23	20	987	32	45
3	530	10	11	710	26	12	1380	29	38

These data can be organized in three matrices representing breakfast, lunch, and dinner. The daily totals can then be found by adding the three matrices.

ADD AND SUBTRACT MATRICES Matrices can be added if and only if they have the same dimensions.

Key Concept *Addition of Matrices*

- **Words** If A and B are two $m \times n$ matrices, then $A + B$ is an $m \times n$ matrix in which each element is the sum of the corresponding elements of A and B.
- **Symbols** $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix}$

Example 1 *Add Matrices*

a. Find $A + B$ if $A = \begin{bmatrix} 4 & -6 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 7 \\ 5 & -9 \end{bmatrix}$.

$A + B = \begin{bmatrix} 4 & -6 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ 5 & -9 \end{bmatrix}$ Definition of matrix addition

$= \begin{bmatrix} 4+(-3) & -6+7 \\ 2+5 & 3+(-9) \end{bmatrix}$ Add corresponding elements.

$= \begin{bmatrix} 1 & 1 \\ 7 & -6 \end{bmatrix}$ Simplify.

b. Find $A + B$ if $A = \begin{bmatrix} 3 & -7 & 4 \\ 12 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 9 \\ 4 & -6 \end{bmatrix}$.

Since the dimensions of A are 2×3 and the dimensions of B are 2×2, you cannot add these matrices.

Resource Manager

Workbook and Reproducible Masters

Chapter 4 Resource Masters

- Study Guide and Intervention, pp. 175–176
- Skills Practice, p. 177
- Practice, p. 178
- Reading to Learn Mathematics, p. 179
- Enrichment, p. 180
- Assessment, p. 231

Transparencies

5-Minute Check Transparency 4-2
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 6
Interactive Chalkboard

You can subtract matrices in a similar manner.

Key Concept — *Subtraction of Matrices*

- **Words** If A and B are two $m \times n$ matrices, then $A - B$ is an $m \times n$ matrix in which each element is the difference of the corresponding elements of A and B.
- **Symbols** $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a-j & b-k & c-l \\ d-m & e-n & f-o \\ g-p & h-q & i-r \end{bmatrix}$

Example 2 Subtract Matrices

Find $A - B$ if $A = \begin{bmatrix} 9 & 2 \\ -4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 8 & -2 \end{bmatrix}$.

$A - B = \begin{bmatrix} 9 & 2 \\ -4 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 8 & -2 \end{bmatrix}$ Definition of matrix subtraction

$= \begin{bmatrix} 9-3 & 2-6 \\ -4-8 & 7-(-2) \end{bmatrix}$ Subtract corresponding elements.

$= \begin{bmatrix} 6 & -4 \\ -12 & 9 \end{bmatrix}$ Simplify.

Example 3 Use Matrices to Model Real-World Data

ANIMALS **The table below shows the number of endangered and threatened species in the United States and in the world. How many more endangered and threatened species are there on the world list than on the U.S. list?**

Endangered and Threatened Species

Type of Animal	United States		World	
	Endangered	Threatened	Endangered	Threatened
Mammals	61	8	309	24
Birds	74	15	252	21
Reptiles	14	22	79	36
Amphibians	9	8	17	9
Fish	69	42	80	42

Source: Fish and Wildlife Service, U.S. Department of Interior

The data in the table can be organized in two matrices. Find the difference of the matrix that represents species in the world and the matrix that represents species in the U.S.

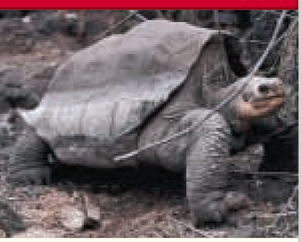
$$\begin{array}{c} \text{World} \\ \begin{bmatrix} 309 & 24 \\ 252 & 21 \\ 79 & 36 \\ 17 & 9 \\ 80 & 42 \end{bmatrix} \end{array} - \begin{array}{c} \text{U.S.} \\ \begin{bmatrix} 61 & 8 \\ 74 & 15 \\ 14 & 22 \\ 9 & 8 \\ 69 & 42 \end{bmatrix} \end{array} = \begin{bmatrix} 309-61 & 24-8 \\ 252-74 & 21-15 \\ 79-14 & 36-22 \\ 17-9 & 9-8 \\ 80-69 & 42-42 \end{bmatrix}$$

Subtract corresponding elements.

(continued on the next page)

www.algebra2.com/extra_examples

More About. . .



Animals

The rarest animal in the world today is a giant tortoise that lives in the Galapagos Islands. "Lonesome George" is the only remaining representative of his species *(Geochelone elephantopus abingdoni)*. With virtually no hope of discovering another specimen, this species is now effectively extinct.

Source: www.ecoworld.com

2 Teach

ADD AND SUBTRACT MATRICES

In-Class Examples Power Point®

1

a. Find $A + B$ if $A = \begin{bmatrix} 6 & 4 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ 0 & 3 \end{bmatrix}$. $\begin{bmatrix} 3 & 5 \\ -1 & 3 \end{bmatrix}$

b. Find $A + B$ if $A = \begin{bmatrix} 4 & -2 & 0 \\ 1 & 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 7 \\ -9 & 3 \end{bmatrix}$. **Since their dimensions are different, these matrices cannot be added.**

2 Find $A - B$ if $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$. $\begin{bmatrix} 5 & 1 \\ -1 & 1 \end{bmatrix}$

Teaching Tip Remind students that subtraction can be rewritten as "adding the opposite."

3 The table below shows the total number of households in the United States and the number of U.S. households that had a computer (PC) in given years. Use matrices to find the number of U.S. households without a PC.

Year	U.S. Households: Total Households (millions)	U.S. Households: Households with PCs (millions)
1995	97.7	33.2
1996	98.9	38.7
1997	100.0	44.0
1998	101.0	47.8
1999	101.7	51.9

Source: The New York Times Almanac, 2000

64.5 million households in 1995, 60.2 million households in 1996, 56.0 million households in 1997, 53.2 million households in 1998, 49.8 million households in 1999

DAILY INTERVENTION

Unlocking Misconceptions

In order to help students see why matrices must have the same dimensions if they are to be added or subtracted, suggest that they try to add a 2×3 matrix to a 3×2 matrix.

Lesson 4-2 Operations with Matrices 161

SCALAR MULTIPLICATION

Building on Prior Knowledge

Point out that scalar multiplication of matrices is similar to using the Distributive Property to multiply the expression $3(x + y)$.

In-Class Example Power Point®

4 If $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 5 \end{bmatrix}$, find $2A$.

$\begin{bmatrix} 4 & 2 \\ -2 & 6 \\ 0 & 10 \end{bmatrix}$

Interactive Chalkboard
PowerPoint® Presentations

This CD-ROM is a customizable Microsoft® PowerPoint® presentation that includes:

- Step-by-step, dynamic solutions of each In-Class Example from the Teacher Wraparound Edition
- Additional, Your Turn exercises for each example
- The 5-Minute Check Transparencies
- Hot links to Glencoe Online Study Tools

$$\begin{array}{ccl} & \begin{array}{cc}\text{Endangered} & \text{Threatened}\end{array} & \\ = & \begin{bmatrix} 248 & 16 \\ 178 & 6 \\ 65 & 14 \\ 8 & 1 \\ 11 & 0 \end{bmatrix} & \begin{array}{l}\text{mammals}\\ \text{birds}\\ \text{reptiles}\\ \text{amphibians}\\ \text{fish}\end{array} \end{array}$$

The first column represents the difference in the number of endangered species on the world and U.S. lists. There are 248 mammals, 178 birds, 65 reptiles, 8 amphibians, and 11 fish species in this category.

The second column represents the difference in the number of threatened species on the world and U.S. lists. There are 16 mammals, 6 birds, 14 reptiles, 1 amphibian, and no fish in this category.

SCALAR MULTIPLICATION You can multiply any matrix by a constant called a **scalar**. This operation is called **scalar multiplication**.

Key Concept — Scalar Multiplication

- **Words** The product of a scalar k and an $m \times n$ matrix is an $m \times n$ matrix in which each element equals k times the corresponding elements of the original matrix.
- **Symbols** $k\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$

Example 4 Multiply a Matrix by a Scalar

If $A = \begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix}$, find $3A$.

$3A = 3\begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix}$ Substitution.

$= \begin{bmatrix} 3(2) & 3(8) & 3(-3) \\ 3(5) & 3(-9) & 3(2) \end{bmatrix}$ Multiply each element by 3.

$= \begin{bmatrix} 6 & 24 & -9 \\ 15 & -27 & 6 \end{bmatrix}$ Simplify.

Many properties of real numbers also hold true for matrices.

Study Tip

Additive Identity

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called a *zero matrix*. It is the *additive identity matrix* for any 2×2 matrix. How is this similar to the additive identity for real numbers?

Concept Summary — Properties of Matrix Operations

For any matrices A, B, and C with the same dimensions and any scalar c, the following properties are true.

Commutative Property of Addition	$A + B = B + A$
Associative Property of Addition	$(A + B) + C = A + (B + C)$
Distributive Property	$c(A + B) = cA + cB$

DAILY INTERVENTION — Differentiated Instruction — ELL

Verbal/Linguistic Students may find it helpful to talk softly, or even silently, to themselves as they work with matrices. For example, they might recite the words “row by column” to remind themselves how to write the dimensions of a matrix. When multiplying by a scalar, students may find that it helps to say, for example, “5 times 1 is 5 and 5 times negative 3 is negative 15.” In this way, students use more than one of their senses to check their calculations.

Example 5 Combination of Matrix Operations

If $A = \begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$, find $5A - 2B$.

Perform the scalar multiplication first. Then subtract the matrices.

$$5A - 2B = 5\begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix} - 2\begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$$ Substitution

$$= \begin{bmatrix} 5(7) & 5(3) \\ 5(-4) & 5(-1) \end{bmatrix} - \begin{bmatrix} 2(9) & 2(6) \\ 2(3) & 2(10) \end{bmatrix}$$ Multiply each element in the first matrix by 5 and multiply each element in the second matrix by 2.

$$= \begin{bmatrix} 35 & 15 \\ -20 & -5 \end{bmatrix} - \begin{bmatrix} 18 & 12 \\ 6 & 20 \end{bmatrix}$$ Simplify.

$$= \begin{bmatrix} 35 - 18 & 15 - 12 \\ -20 - 6 & -5 - 20 \end{bmatrix}$$ Subtract corresponding elements.

$$= \begin{bmatrix} 17 & 3 \\ -26 & -25 \end{bmatrix}$$ Simplify.

Study Tip

Matrix Operations
The order of operations for matrices is similar to that of real numbers. Perform scalar multiplication before matrix addition and subtraction.

Graphing Calculator Investigation

Matrix Operations

Most graphing calculators can perform operations with matrices. On the TI-83 Plus, [2nd] [MATRX] accesses the matrix menu. Choose EDIT to define a matrix. Enter the dimensions of the matrix A using the [▶] key. Then enter each element by pressing [ENTER] after each entry. To display and use the matrix in calculations, choose the matrix under NAMES from the [MATRX] menu.

Think and Discuss

1. Enter $A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$ with a graphing calculator. Does the calculator enter elements row by row or column by column? **row by row**
2. Notice that there are two numbers in the bottom left corner of the screen. What do these numbers represent? **2. the row and column of the element being entered**
3. Clear the screen. Find the matrix $18A$. $\begin{bmatrix} 54 & -36 \\ 90 & 72 \end{bmatrix}$
4. Enter $B = \begin{bmatrix} 1 & 9 & -3 \\ 8 & 6 & -5 \end{bmatrix}$. Find $A + B$. What is the result and why? **There is an error on the screen because the dimensions are not equal.**

Check for Understanding

Concept Check

1. **Describe** the conditions under which matrices can be added or subtracted. **1. They must have the same dimensions.**
2. **OPEN ENDED** Give an example of two matrices whose sum is a zero matrix. **2. Sample answer: [−3 1], [3 −1]**
3. **Write** a matrix that, when added to a 3×2 matrix, increases each element in the matrix by 4. **See margin.**

Guided Practice

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. **5–7. See margin.**

4. $[5 \quad 8 \quad -4] + [12 \quad 5]$ **impossible**
5. $\begin{bmatrix} 3 & 7 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix}$
6. $3\begin{bmatrix} 6 & -1 & 5 & 2 \\ 7 & 3 & -2 & 8 \end{bmatrix}$
7. $4\begin{bmatrix} 2 & 7 \\ -3 & 6 \end{bmatrix} + 5\begin{bmatrix} -6 & -4 \\ 3 & 0 \end{bmatrix}$

GUIDED PRACTICE KEY

Exercises	Examples
4–10	1, 2, 4, 5
11–13	3

In-Class Example

Power Point®

5 If $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$, find $4A - 3B$. $\begin{bmatrix} 14 & 9 \\ -4 & 3 \end{bmatrix}$

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms in this lesson to their Vocabulary Builder worksheets for Chapter 4.
- write the general formulas for matrix addition, matrix subtraction, and scalar multiplication, along with an example of the use of each formula.
- copy the properties of matrix operations, including the study tips shown on pp. 162–163, into their notebooks.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

Answers

3. $\begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 10 \\ -7 & 5 \end{bmatrix}$

6. $\begin{bmatrix} 18 & -3 & 15 & 6 \\ 21 & 9 & -6 & 24 \end{bmatrix}$

7. $\begin{bmatrix} -22 & 8 \\ 3 & 24 \end{bmatrix}$

Graphing Calculator Investigation

If students are using a TI-83 and not the TI-83 Plus, point out that the [MATRX] key is not a **2nd** function. Also, point out that the row and column designation of the element being entered appears at the bottom of the screen, and that the matrix elements are entered row by row.

About the Exercises...

Organization by Objective

- **Add and Subtract Matrices:** 14–16, 20, 24, 25, 36–39
- **Scalar Multiplication:** 17–19, 21–23, 26–29

Odd/Even Assignments

Exercises 14–29 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Alert! Exercise 35 involves research on the Internet or other reference materials.

Assignment Guide

Basic: 15–19 odd, 25–29 odd, 30–35, 40–62

Average: 15–29 odd, 30–35, 40–62

Advanced: 14–28 even, 33–58 (optional: 59–62)

Use matrices A, B, and C to find the following.

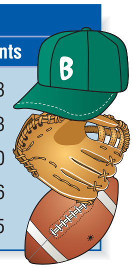
$A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 7 \\ 0 & -4 \end{bmatrix}$ $C = \begin{bmatrix} 9 & -4 \\ -6 & 5 \end{bmatrix}$

8. $A + B + C$ $\begin{bmatrix} 10 & 6 \\ -1 & 7 \end{bmatrix}$ **9.** $3B - 2C$ $\begin{bmatrix} -21 & 29 \\ 12 & -22 \end{bmatrix}$ **10.** $4A + 2B - C$ $\begin{bmatrix} -3 & 30 \\ 26 & 11 \end{bmatrix}$

Application

SPORTS For Exercises 11–13, use the table below that shows high school participation in various sports.

Sport	Males		Females	
	Schools	Participants	Schools	Participants
Basketball	16,763	549,499	16,439	456,873
Track and Field	14,620	477,960	14,545	405,163
Baseball/Softball	14,486	455,305	12,679	340,480
Soccer	9041	321,416	7931	257,586
Swimming and Diving	5234	83,411	5450	133,235



Source: National Federation of State High School Associations

11. Write two matrices that represent these data for males and females. **See margin.**

12. Find the total number of students that participate in each individual sport expressed as a matrix.

12. $\begin{bmatrix} 1{,}006{,}372 \\ 883{,}123 \\ 795{,}785 \\ 579{,}002 \\ 216{,}646 \end{bmatrix}$

13. Could you add the two matrices to find the total number of schools that offer a particular sport? Why or why not? **No; many schools offer the same sport for males and females, so those schools would be counted twice.**

★ **indicates increased difficulty**

Practice and Apply

Homework Help

For Exercises	See Examples
14–29	1, 2, 4, 5
30–39	3

Extra Practice

See page 834.

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

14. $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 6 \\ -5 \\ 8 \end{bmatrix}$ $\begin{bmatrix} 10 \\ -4 \\ 5 \end{bmatrix}$

15. $\begin{bmatrix} -5 & 7 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -2 \\ 9 & 0 & 1 \end{bmatrix}$ **impossible**

16. $\begin{bmatrix} 12 & 0 & 8 \\ 9 & 15 & -11 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 4 \\ 9 & 2 & -6 \end{bmatrix}$

17. $-2\begin{bmatrix} 2 & -4 & 1 \\ -3 & 5 & 8 \\ 7 & 6 & -2 \end{bmatrix}$

16. $\begin{bmatrix} 15 & 0 & 4 \\ 0 & 13 & -5 \end{bmatrix}$

17. $\begin{bmatrix} -4 & 8 & -2 \\ 6 & -10 & -16 \\ -14 & -12 & 4 \end{bmatrix}$

18. $5[0 \quad -1 \quad 7 \quad 2] + 3[5 \quad -8 \quad 10 \quad -4]$ $[15 \quad -29 \quad 65 \quad -2]$

19. $5\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + 6\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 8 \\ -4 \end{bmatrix}$ $\begin{bmatrix} -13 \\ -3 \\ 23 \end{bmatrix}$

★ **20.** $\begin{bmatrix} 1.35 & 5.80 \\ 1.24 & 14.32 \\ 6.10 & 35.26 \end{bmatrix} + \begin{bmatrix} 0.45 & 3.28 \\ 1.94 & 16.72 \\ 4.31 & 21.30 \end{bmatrix}$

★ **21.** $8\begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 1.5 \end{bmatrix} - 2\begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 1.5 \end{bmatrix}$ $\begin{bmatrix} 1.5 & 3 \\ 4.5 & 9 \end{bmatrix}$

20. $\begin{bmatrix} 1.8 & 9.08 \\ 3.18 & 31.04 \\ 10.41 & 56.56 \end{bmatrix}$

★ **22.** $\frac{1}{2}\begin{bmatrix} 4 & 6 \\ 3 & 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 9 & 27 \\ 0 & 3 \end{bmatrix}$ $\begin{bmatrix} -4 & -15 \\ \frac{3}{2} & -2 \end{bmatrix}$

★ **23.** $5\begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 2 & \frac{1}{3} & -1 \end{bmatrix} + 4\begin{bmatrix} -2 & \frac{3}{4} & 1 \\ \frac{1}{6} & 0 & \frac{5}{8} \end{bmatrix}$

23. $\begin{bmatrix} -5\frac{1}{2} & 3 & 9 \\ 10\frac{2}{3} & 1\frac{2}{3} & -2\frac{1}{2} \end{bmatrix}$

Use matrices A, B, C, and D to find the following. **24–29. See margin.**

$A = \begin{bmatrix} 5 & 7 \\ -1 & 6 \\ 3 & -9 \end{bmatrix}$ $B = \begin{bmatrix} 8 & 3 \\ 5 & 1 \\ 4 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 4 \\ -2 & 5 \\ 7 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 6 & 2 \\ 9 & 0 \\ -3 & 0 \end{bmatrix}$

24. $A + B$ **25.** $D - B$ **26.** $4C$

27. $6B - 2A$ **28.** $3C - 4A + B$ **29.** $C + \frac{1}{3}D$

Answers

11. Males = $\begin{bmatrix} 16{,}763 & 549{,}499 \\ 14{,}620 & 477{,}960 \\ 14{,}486 & 455{,}305 \\ 9041 & 321{,}416 \\ 5234 & 83{,}411 \end{bmatrix}$,

Females = $\begin{bmatrix} 16{,}439 & 456{,}873 \\ 14{,}545 & 405{,}163 \\ 12{,}679 & 340{,}480 \\ 7931 & 257{,}586 \\ 5450 & 133{,}235 \end{bmatrix}$

24. $\begin{bmatrix} 13 & 10 \\ 4 & 7 \\ 7 & -5 \end{bmatrix}$ **25.** $\begin{bmatrix} -2 & -1 \\ 4 & -1 \\ -7 & -4 \end{bmatrix}$

26. $\begin{bmatrix} 0 & 16 \\ -8 & 20 \\ 28 & -4 \end{bmatrix}$ **27.** $\begin{bmatrix} 38 & 4 \\ 32 & -6 \\ 18 & 42 \end{bmatrix}$

28. $\begin{bmatrix} -12 & -13 \\ 3 & -8 \\ 13 & 37 \end{bmatrix}$ **29.** $\begin{bmatrix} 2 & 4\frac{2}{3} \\ 1 & 5 \\ 6 & -1 \end{bmatrix}$

36. Residents: $\begin{array}{r c} & \begin{array}{cc} \text{Child} & \text{Adult} \end{array} \\ \begin{array}{r} \text{Before 6} \\ \text{After 6} \end{array} & \begin{bmatrix} 3.00 & 4.50 \\ 2.00 & 3.50 \end{bmatrix} \end{array}$,

Nonresidents: $\begin{array}{r c} & \begin{array}{cc} \text{Child} & \text{Adult} \end{array} \\ \begin{array}{r} \text{Before 6} \\ \text{After 6} \end{array} & \begin{bmatrix} 4.50 & 6.75 \\ 3.00 & 5.25 \end{bmatrix} \end{array}$

37. $\begin{bmatrix} 1.50 & 2.25 \\ 1.00 & 1.75 \end{bmatrix}$

38. Before 6:00: $\begin{array}{r c} & \begin{array}{cc} \text{Child} & \text{Adult} \end{array} \\ \begin{array}{r} \text{Residents} \\ \text{Nonresidents} \end{array} & \begin{bmatrix} 3.00 & 4.50 \\ 4.50 & 6.75 \end{bmatrix} \end{array}$,

After 6:00: $\begin{array}{r c} & \begin{array}{cc} \text{Child} & \text{Adult} \end{array} \\ \begin{array}{r} \text{Residents} \\ \text{Nonresidents} \end{array} & \begin{bmatrix} 2.00 & 3.50 \\ 3.00 & 5.25 \end{bmatrix} \end{array}$

39. $\begin{bmatrix} 1.00 & 1.00 \\ 1.50 & 1.50 \end{bmatrix}$

30. Friday: $\begin{bmatrix} 120 & 97 & 64 & 75 \\ 80 & 59 & 36 & 60 \\ 72 & 84 & 29 & 48 \end{bmatrix}$

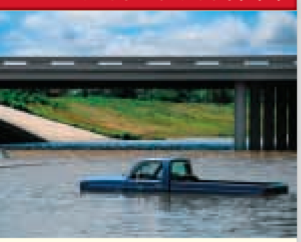
Saturday: $\begin{bmatrix} 112 & 87 & 56 & 74 \\ 84 & 65 & 39 & 70 \\ 88 & 98 & 43 & 60 \end{bmatrix}$

31. $\begin{bmatrix} 232 & 184 & 120 & 149 \\ 164 & 124 & 75 & 130 \\ 160 & 182 & 72 & 108 \end{bmatrix}$

32. $\begin{bmatrix} -8 & -10 & -8 & -1 \\ 4 & 6 & 3 & 10 \\ 16 & 14 & 14 & 12 \end{bmatrix}$

33. $\begin{bmatrix} 245 \\ 228 \\ 319 \\ 227 \\ 117 \end{bmatrix}$ 34. $\begin{bmatrix} 15 \\ 41 \\ 35 \\ 27 \\ 51 \end{bmatrix}$

More About. . .



Weather

Flash floods and floods are the number 1 weather-related killer recorded in the U.S. each year. The large majority of deaths due to flash flooding are a result of people driving through flooded areas.

Source: National Oceanic & Atmospheric Administration

BUSINESS **For Exercises 30–32, use the following information.**
The Cookie Cutter Bakery records each type of cookie sold at three of their branch stores. Two days of sales are shown in the spreadsheets below.

30. Write a matrix for each day's sales.
31. Find the sum of the two days' sales expressed as a matrix.
32. Find the difference in cookie sales from Friday to Saturday expressed as a matrix.

	A	B	C	D	E
1	Friday	chocolate chip	peanut butter	sugar	cut-out
2	Store 1	120	97	64	75
3	Store 2	80	59	36	60
4	Store 3	72	84	29	48

	A	B	C	D	E
1	Saturday	chocolate chip	peanut butter	sugar	cut-out
2	Store 1	112	87	56	74
3	Store 2	84	65	39	70
4	Store 3	88	98	43	60

WEATHER **For Exercises 33–35, use the table that shows the total number of deaths due to severe weather.**

Year	Lightning	Tornadoes	Floods	Hurricanes
1996	52	25	131	37
1997	42	67	118	1
1998	44	130	136	9
1999	46	94	68	19
2000	51	29	37	0

Source: National Oceanic & Atmospheric Administration

33. Find the total number of deaths due to severe weather for each year expressed as a column matrix.
34. Write a matrix that represents how many more people died as a result of lightning than hurricanes for each year.
35. What type of severe weather accounted for the most deaths each year? **1996, floods; 1997, floods; 1998, floods; 1999, tornadoes; 2000, lightning**

Online Research **Data Update** What are the current weather statistics? Visit www.algebra2.com/data_update to learn more.

RECREATION **For Exercises 36–39, use the following price list for one-day admissions to the community pool.** **36–39. See margin.**

36. Write a matrix that represents the cost of admission for residents and a matrix that represents the cost of admission for nonresidents.
37. Find the matrix that represents the additional cost for nonresidents.
38. Write a matrix that represents the cost of admission before 6:00 P.M. and a matrix that represents the cost of admission after 6:00 P.M.
39. Find a matrix that represents the difference in cost if a child or adult goes to the pool after 6:00 P.M.

Daily Admission Fees

Residents		
Time of day	Child	Adult
Before 6:00 P.M.	\$3.00	\$4.50
After 6:00 P.M.	\$2.00	\$3.50
Nonresidents		
Time of day	Child	Adult
Before 6:00 P.M.	\$4.50	\$6.75
After 6:00 P.M.	\$3.00	\$5.25

www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 175 (shown) and p. 176

Add and Subtract Matrices

Addition of Matrices	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix}$
Subtraction of Matrices	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a-j & b-k & c-l \\ d-m & e-n & f-o \\ g-p & h-q & i-r \end{bmatrix}$

Example 1 **Find $A + B$ if $A = \begin{bmatrix} 6 & -7 \\ 2 & -12 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ -5 & -6 \end{bmatrix}$.**

$$A + B = \begin{bmatrix} 6 & -7 \\ 2 & -12 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -5 & -6 \end{bmatrix} = \begin{bmatrix} 6+4 & -7+2 \\ 2+(-5) & -12+(-6) \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ -3 & -18 \end{bmatrix}$$

Example 2 **Find $A - B$ if $A = \begin{bmatrix} -2 & 8 \\ 3 & -4 \\ 10 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ -2 & 1 \\ -6 & 8 \end{bmatrix}$.**

$$A - B = \begin{bmatrix} -2 & 8 \\ 3 & -4 \\ 10 & 7 \end{bmatrix} - \begin{bmatrix} 4 & -3 \\ -2 & 1 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} -2-4 & 8-(-3) \\ 3-(-2) & -4-1 \\ 10-(-6) & 7-8 \end{bmatrix} = \begin{bmatrix} -6 & 11 \\ 5 & -5 \\ 16 & -1 \end{bmatrix}$$

Exercises

Perform the indicated operations. If the matrix does not exist, write *impossible*.

1. $\begin{bmatrix} 8 & 7 \\ -10 & -6 \end{bmatrix} - \begin{bmatrix} -4 & 3 \\ 2 & -12 \end{bmatrix}$ $\begin{bmatrix} 12 & 4 \\ -12 & 6 \end{bmatrix}$
2. $\begin{bmatrix} 6 & -5 & 9 \\ -3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} -4 & 3 & 2 \\ 6 & 9 & -4 \end{bmatrix}$ $\begin{bmatrix} 2 & -2 & 11 \\ 3 & 13 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix} + [-6 \quad 3 \quad -2]$ **impossible**
4. $\begin{bmatrix} 5 & -2 \\ -4 & 6 \\ 7 & 9 \end{bmatrix} + \begin{bmatrix} -11 & 6 \\ 2 & -5 \\ 4 & -7 \end{bmatrix}$ $\begin{bmatrix} -6 & 4 \\ -2 & 1 \\ 11 & 2 \end{bmatrix}$
5. $\begin{bmatrix} 8 & 0 & -6 \\ 4 & 5 & -11 \\ -7 & 3 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 7 \\ 3 & -4 & 3 \\ -8 & 5 & 6 \end{bmatrix}$ $\begin{bmatrix} 10 & -1 & -13 \\ 1 & 9 & -14 \\ 1 & -2 & -2 \end{bmatrix}$
6. $\begin{bmatrix} \frac{3}{4} & \frac{2}{5} \\ -\frac{1}{2} & \frac{4}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$ $\begin{bmatrix} \frac{1}{4} & -\frac{4}{15} \\ -\frac{7}{6} & \frac{11}{6} \end{bmatrix}$

Skills Practice, p. 177 and Practice, p. 178 (shown)

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

1. $\begin{bmatrix} 2 & -1 \\ 3 & 7 \\ 14 & -9 \end{bmatrix} + \begin{bmatrix} -6 & 9 \\ 7 & -11 \\ -8 & 17 \end{bmatrix}$ $\begin{bmatrix} -4 & 8 \\ 10 & -4 \\ 6 & 8 \end{bmatrix}$
2. $\begin{bmatrix} 4 \\ -71 \\ 18 \end{bmatrix} - \begin{bmatrix} -67 \\ 45 \\ -24 \end{bmatrix}$ $\begin{bmatrix} 71 \\ -116 \\ 42 \end{bmatrix}$
3. $-3\begin{bmatrix} -1 & 0 \\ 17 & -11 \end{bmatrix} + 4\begin{bmatrix} -3 & 16 \\ -21 & 12 \end{bmatrix}$ $\begin{bmatrix} -9 & 64 \\ -135 & 81 \end{bmatrix}$
4. $7\begin{bmatrix} 2 & -1 & 8 \\ 4 & 7 & 9 \end{bmatrix} - 2\begin{bmatrix} -1 & 4 & -3 \\ 7 & 2 & -6 \end{bmatrix}$ $\begin{bmatrix} 16 & -15 & 62 \\ 14 & 45 & 75 \end{bmatrix}$
5. $-2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 10 \\ 18 \end{bmatrix}$ $\begin{bmatrix} -12 \\ -2 \end{bmatrix}$
6. $\frac{3}{4}\begin{bmatrix} 8 & 12 \\ -16 & 20 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 27 & -9 \\ 54 & -18 \end{bmatrix}$ $\begin{bmatrix} 24 & 3 \\ 24 & 3 \end{bmatrix}$

Use $A = \begin{bmatrix} 4 & -1 & 0 \\ -3 & 6 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 & 5 \\ 1 & 0 & -9 \end{bmatrix}$, and $C = \begin{bmatrix} 10 & -8 & 6 \\ -6 & -4 & 20 \end{bmatrix}$ to find the following.

7. $A - B$ $\begin{bmatrix} 6 & -5 & -5 \\ -4 & 6 & 11 \end{bmatrix}$
8. $A - C$ $\begin{bmatrix} -6 & 7 & -6 \\ 3 & 10 & -18 \end{bmatrix}$
9. $-3B$ $\begin{bmatrix} 6 & -12 & -15 \\ -3 & 0 & 27 \end{bmatrix}$
10. $4B - A$ $\begin{bmatrix} -12 & 17 & 20 \\ 7 & -6 & -38 \end{bmatrix}$
11. $-2B - 3C$ $\begin{bmatrix} -26 & 16 & -28 \\ 16 & 12 & -42 \end{bmatrix}$
12. $A + 0.5C$ $\begin{bmatrix} 9 & -5 & 3 \\ -6 & 4 & 12 \end{bmatrix}$

ECONOMICS **For Exercises 13 and 14, use the table that shows loans by an economic development board to women and men starting new businesses.**

	Women		Men	
	Businesses	Loan Amount (\$)	Businesses	Loan Amount (\$)
1999	27	\$567,000	36	\$864,000
2000	41	\$902,000	32	\$672,000
2001	35	\$777,000	28	\$562,000

13. Write two matrices that represent the number of new businesses and loan amounts, one for women and one for men. $\begin{bmatrix} 27 & 567{,}000 \\ 41 & 902{,}000 \\ 35 & 777{,}000 \end{bmatrix}, \begin{bmatrix} 36 & 864{,}000 \\ 32 & 672{,}000 \\ 28 & 562{,}000 \end{bmatrix}$
14. Find the sum of the numbers of new businesses and loan amounts for both men and women over the three-year period expressed as a matrix. $\begin{bmatrix} 63 & 1{,}431{,}000 \\ 73 & 1{,}574{,}000 \\ 63 & 1{,}339{,}000 \end{bmatrix}$
15. **PET NUTRITION** Use the table that gives nutritional information for two types of dog food. Find the difference in the percent of protein, fat, and fiber between Mix B and Mix A expressed as a matrix. $[2 \quad -4 \quad 3]$

	% Protein	% Fat	% Fiber
Mix A	22	12	5
Mix B	24	8	8

Reading to Learn Mathematics, p. 179 ELL

Pre-Activity **How can matrices be used to calculate daily dietary needs?**

Read the introduction to Lesson 4-2 at the top of page 160 in your textbook.

- Write a sum that represents the total number of Calories in the patient's diet for Day 2. (Do not actually calculate the sum.) **482 + 622 + 987**
- Write the sum that represents the total fat content in the patient's diet for Day 3. (Do not actually calculate the sum.) **11 + 12 + 38**

Reading the Lesson

1. For each pair of matrices, give the dimensions of the indicated sum, difference, or scalar product. If the indicated sum, difference, or scalar product does not exist, write *impossible*.

$A = \begin{bmatrix} 3 & 5 & 6 \\ -2 & 8 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}$

$C = \begin{bmatrix} 5 & 10 \\ -3 & 6 \\ 4 & 12 \end{bmatrix}$ $D = \begin{bmatrix} -3 & 6 & 0 \\ -8 & 4 & 0 \end{bmatrix}$

$A + D$: **2 × 3** $C + D$: **impossible** $5B$: **2 × 2**

$-4C$: **3 × 2** $2D - 3A$: **2 × 3**

2. Suppose that M, N, and P are nonzero 2×4 matrices and k is a negative real number. Indicate whether each of the following statements is *true* or *false*.
   a. $M + (N + P) = M + (P + N)$ **true**
   b. $M - N = N - M$ **false**
   c. $M - (N - P) = (M - N) - P$ **false**
   d. $k(M - N) = kM - kN$ **true**

Helping You Remember

3. The mathematical term *scalar* may be unfamiliar, but its meaning is related to the word *scale* as used in a *scale of miles* on a map. How can this usage of the word *scale* help you remember the meaning of *scalar*?
Sample answer: A scale of miles tells you how to multiply the distances you measure on a map by a certain number to get the actual distance between two locations. This multiplier is often called a *scale* factor. A *scalar* represents the same idea: It is a real number by which a matrix can be multiplied to change all the elements of the matrix by a uniform *scale* factor.

Enrichment, p. 180

Sundaram's Sieve

The properties and patterns of prime numbers have fascinated many mathematicians. In 1934, a young East Indian student named Sundaram constructed the following matrix.

4	7	10	13	16	19	22	25	. . .
7	12	17	22	27	32	37	42	. . .
10	17	24	31	38	45	52	59	. . .
13	22	31	40	49	58	67	76	. . .
16	27	38	49	60	71	82	93	. . .
.	.	.	.	.	.	.	.	. . .

A surprising property of this matrix is that it can be used to determine whether or not some numbers are prime.

Complete these problems to discover this property.

1. The first row and the first column are created by using an arithmetic pattern. What is the common difference used in the pattern? **3**

Lesson 4-2 Operations with Matrices 165

4 Assess

Open-Ended Assessment

Speaking Have students discuss and explain how adding and subtracting matrices, and multiplying a matrix by a scalar, are similar to these operations with numbers.

Tips for New Teachers

Intervention Adding and subtracting matrices offers numerous chances for students to make errors. They might make a mistake in choosing which elements to combine when adding or subtracting the entries. They may not write the result in the correct position in the answer matrix. Ask students to develop tips for keeping track of where they are in the matrix. Students may find it helpful to circle the entries they are combining, to use color coding, or to point with their fingers to the two entries to be added or subtracted as they do the calculation.

Assessment Options

Quiz (Lessons 4-1 and 4-2) is available on p. 231 of the *Chapter 4 Resource Masters*.

Getting Ready for Lesson 4-3

PREREQUISITE SKILL Lesson 4-3 presents matrix multiplication and the properties pertaining to it. Students will use their familiarity with the properties of real numbers to compare them to matrix properties. Exercises 59–62 should be used to determine your students' familiarity with the properties of equality.

40. $2\begin{bmatrix} 0.5 & 0.75 & 3 \\ 1 & 4 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 1.5 & 6 \\ 2 & 8 & 0.2 \end{bmatrix}$

40. CRITICAL THINKING Determine values for each variable if $d = 1$, $e = 4d$, $z + d = e$, $f = \frac{x}{5}$, $ay = 1.5$, $x = \frac{d}{2}$, and $y = x + \frac{x}{2}$.

$$a\begin{bmatrix} x & y & z \\ d & e & f \end{bmatrix} = \begin{bmatrix} ax & ay & az \\ ad & ae & af \end{bmatrix}$$

41. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. **See pp. 217A–217B.**

How can matrices be used to calculate daily dietary needs?

Include the following in your answer:
- three matrices that represent breakfast, lunch, and dinner over the three-day period, and
- a matrix that represents the total Calories, protein, and fat consumed each day.

Standardized Test Practice

42. Which matrix equals $\begin{bmatrix} 5 & -2 \\ -3 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix}$? **D**

(A) $\begin{bmatrix} 2 & 2 \\ -8 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 8 & -6 \\ -8 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix}$

43. Solve for x and y in the matrix equation $\begin{bmatrix} x \\ 7 \end{bmatrix} + \begin{bmatrix} 3y \\ -x \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$. **A**

(A) $(-5, 7)$ (B) $(7, 5)$ (C) $(7, 3)$ (D) $(5, 7)$

Maintain Your Skills

Mixed Review

State the dimensions of each matrix. *(Lesson 4-1)*

44. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **2 × 2**

45. $[2 \quad 0 \quad 3 \quad 0]$ **1 × 4**

46. $\begin{bmatrix} 5 & 1 & -6 & 2 \\ -38 & 5 & 7 & 3 \end{bmatrix}$ **2 × 4**

47. $\begin{bmatrix} 7 & -3 & 5 \\ 0 & 2 & -9 \\ 6 & 5 & 1 \end{bmatrix}$ **3 × 3**

48. $\begin{bmatrix} 8 & 6 \\ 5 & 2 \\ -4 & -1 \end{bmatrix}$ **3 × 2**

49. $\begin{bmatrix} 7 & 5 & 0 \\ -8 & 3 & 8 \\ 9 & -1 & 15 \\ 4 & 2 & 11 \end{bmatrix}$ **4 × 3**

Solve each system of equations. *(Lesson 3-5)*

52. $\left(\frac{1}{4}, 6, -\frac{1}{6}\right)$

50. $2a + b = 2$
$5a = 15$
$a + b + c = -1$ **(3, −4, 0)**

51. $r + s + t = 15$
$r + t = 12$
$s + t = 10$ **(5, 3, 7)**

52. $6x - 2y - 3z = -10$
$-6x + y + 9z = 3$
$8x - 3y = -16$

Solve each system by using substitution or elimination. *(Lesson 3-2)*

53. $2s + 7t = 39$
$5s - t = 5$ **(2, 5)**

54. $3p + 6q = -3$
$2p - 3q = -9$ **(−3, 1)**

55. $a + 5b = 1$
$7a - 2b = 44$ **(6, −1)**

SCRAPBOOKS **For Exercises 56–58, use the following information.**
Ian has \$6.00, and he wants to buy paper for his scrapbook. A sheet of printed paper costs 30¢, and a sheet of solid color paper costs 15¢. *(Lesson 2-7)*

56. Write an inequality that describes this situation. **$0.30p + 0.15s \le 6$**

57. Graph the inequality. **See margin.**

58. Does Ian have enough money to buy 14 pieces of each type of paper? **No, it would cost \$6.30.**

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Name the property illustrated by each equation.**
(To review the ***properties of equality****, see Lesson 1-2.)*

59. $\frac{7}{9} \cdot \frac{9}{7} = 1$ **Mult. Inverse**

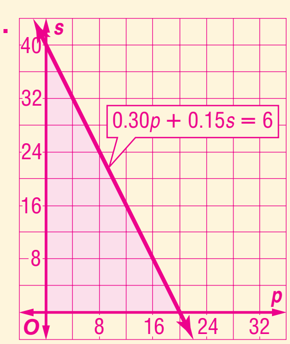
60. $7 + (w + 5) = (7 + w) + 5$ **Assoc. (+)**

61. $3(x + 12) = 3x + 3(12)$ **Dist.**

62. $6(9a) = 9a(6)$ **Comm. (×)**

Answer

57.



4-3 Multiplying Matrices

What You'll Learn

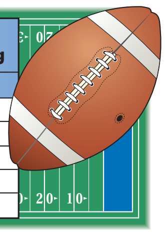
- Multiply matrices.
- Use the properties of matrix multiplication.

How can matrices be used in sports statistics?

Professional football teams track many statistics throughout the season to help evaluate their performance. The table shows the scoring summary of the Oakland Raiders for the 2000 season. The team's record can be summarized in the record matrix R. The values for each type of score can be organized in the point values matrix P.

Oakland Raiders Regular Season Scoring

Type	Number
Touchdown	58
Extra Point	56
Field Goal	23
2-Point Conversion	1
Safety	2



Source: National Football League

Record

$$R = \begin{array}{c}\begin{bmatrix} 58 \\ 56 \\ 23 \\ 1 \\ 2 \end{bmatrix}\end{array}\begin{array}{l} \text{touchdown} \\ \text{extra point} \\ \text{field goal} \\ \text{2-point conversion} \\ \text{safety} \end{array}$$

Point Values

$$P = \begin{array}{ccccc} \text{touchdown} & \text{extra point} & \text{field goal} & \text{2-point conversion} & \text{safety} \\ [6 & 1 & 3 & 2 & 2] \end{array}$$

You can use matrix multiplication to find the total points scored.

MULTIPLY MATRICES You can multiply two matrices if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix. When you multiply two matrices $A_{m \times n}$ and $B_{n \times r}$, the resulting matrix AB is an $m \times r$ matrix.

outer dimensions = dimensions of AB

$$\begin{array}{ccccc} A & \cdot & B & = & AB \\ 2 \times 3 & & 3 \times 4 & = & 2 \times 4 \end{array}$$

inner dimensions are equal

Example 1 Dimensions of Matrix Products

Determine whether each matrix product is defined. If so, state the dimensions of the product.

a. $A_{2 \times 5}$ **and** $B_{5 \times 4}$

$$\begin{array}{ccccc} A & \cdot & B & = & AB \\ 2 \times 5 & & 5 \times 4 & & 2 \times 4 \end{array}$$

The inner dimensions are equal so the matrix product is defined. The dimensions of the product are 2×4.

b. $A_{1 \times 3}$ **and** $B_{4 \times 3}$

$$\begin{array}{ccc} A & \cdot & B \\ 1 \times 3 & & 4 \times 3 \end{array}$$

The inner dimensions are not equal, so the matrix product is not defined.

4-3 Lesson Notes

1 Focus

5-Minute Check Transparency 4-3 Use as a quiz or review of Lesson 4-2.

Mathematical Background notes are available for this lesson on p. 152C.

How can matrices be used in sports statistics?

Ask students:

- The table shows that the Raiders scored 58 touchdowns. A touchdown is worth 6 points. How many points were scored by touchdowns? **348 points**
- A field goal is worth 3 points. How many points did the Raiders score by field goals? **69 points**

Tips for New Teachers

Ask a volunteer to explain the various football plays for the benefit of those who may not be familiar with them.

Resource Manager

Workbook and Reproducible Masters

Chapter 4 Resource Masters
- Study Guide and Intervention, pp. 181–182
- Skills Practice, p. 183
- Practice, p. 184
- Reading to Learn Mathematics, p. 185
- Enrichment, p. 186

Teaching Algebra With Manipulatives Masters, p. 231

Transparencies

5-Minute Check Transparency 4-3
Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

MULTIPLY MATRICES

In-Class Examples Power Point®

1 Determine whether each matrix product is defined. If so, state the dimensions of the product.

a. $A_{3 \times 4}$ and $B_{4 \times 2}$ **The matrix product is defined. The dimensions are 3 × 2.**

b. $A_{3 \times 2}$ and $B_{4 \times 3}$ **The matrix product is not defined.**

2 Find RS if $R = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$. $\begin{bmatrix} -6 & 1 \\ 2 & -1 \end{bmatrix}$

The product of two matrices is found by multiplying columns and rows. The entry in the first row and first column of AB is found by multiplying corresponding elements in the first row of A and the first column of B and then adding.

Key Concept — *Multiply Matrices*

- **Words** The element a_{ij} of AB is the sum of the products of the corresponding elements in row i of A and column j of B.
- **Symbols** $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}$

Example 2 *Multiply Square Matrices*

Find RS if $R = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $S = \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix}$.

$RS = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix}$

Step 1 Multiply the numbers in the first row of R by the numbers in the first column of S, add the products, and put the result in the first row, first column of RS.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & \\ & \end{bmatrix}$$

Step 2 Multiply the numbers in the first row of R by the numbers in the second column of S, add the products, and put the result in the first row, second column of RS.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ & \end{bmatrix}$$

Step 3 Multiply the numbers in the second row of R by the numbers in the first column of S, add the products, and put the result in the second row, first column of RS.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & \end{bmatrix}$$

Step 4 Multiply the numbers in the second row of R by the numbers in the second column of S, add the products, and put the result in the second row, second column of RS.

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & 3(-9) + 4(7) \end{bmatrix}$$

Step 5 Simplify the product matrix.

$$\begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & 3(-9) + 4(7) \end{bmatrix} = \begin{bmatrix} 1 & -25 \\ 29 & 1 \end{bmatrix}$$

So, $RS = \begin{bmatrix} 1 & -25 \\ 29 & 1 \end{bmatrix}$.

Study Tip

Multiplying Matrices

To avoid any miscalculations, find the product of the matrices in order as shown in Example 2. It may also help to cover rows or columns not being multiplied as you find elements of the product matrix.

When solving real-world problems, make sure to multiply the matrices in the order for which the product is defined.

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Example 3 Multiply Matrices with Different Dimensions

TRACK AND FIELD **In a four-team track meet, 5 points were awarded for each first-place finish, 3 points for each second, and 1 point for each third. Find the total number of points for each school. Which school won the meet?**

School	First Place	Second Place	Third Place
Jefferson	8	4	5
London	6	3	7
Springfield	5	7	3
Madison	7	5	4

Explore The final scores can be found by multiplying the track results for each school by the points awarded for each first-, second-, and third-place finish.

Plan Write the results of the races and the points awarded in matrix form. Set up the matrices so that the number of rows in the points matrix equals the number of columns in the results matrix.

Results **Points**

$$R = \begin{bmatrix} 8 & 4 & 5 \\ 6 & 3 & 7 \\ 5 & 7 & 3 \\ 7 & 5 & 4 \end{bmatrix} \qquad P = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

Solve Multiply the matrices.

$$RP = \begin{bmatrix} 8 & 4 & 5 \\ 6 & 3 & 7 \\ 5 & 7 & 3 \\ 7 & 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 8(5) + 4(3) + 5(1) \\ 6(5) + 3(3) + 7(1) \\ 5(5) + 7(3) + 3(1) \\ 7(5) + 5(3) + 4(1) \end{bmatrix}$$ Multiply columns by rows.

$$= \begin{bmatrix} 57 \\ 46 \\ 49 \\ 54 \end{bmatrix}$$ Simplify.

The labels for the product matrix are shown below.

Total Points

$$\begin{array}{l} \text{Jefferson} \\ \text{London} \\ \text{Springfield} \\ \text{Madison} \end{array} \begin{bmatrix} 57 \\ 46 \\ 49 \\ 54 \end{bmatrix}$$

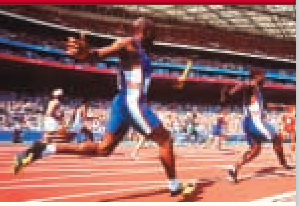
Jefferson won the track meet with a total of 57 points.

Examine R is a 4×3 matrix and P is a 3×1 matrix; so their product should be a 4×1 matrix. *Why?*

MULTIPLICATIVE PROPERTIES Recall that the same properties for real numbers also held true for matrix addition. However, some of these properties do *not* always hold true for matrix multiplication.

www.algebra2.com/extra_examples

More About. . .



Track and Field

Running and hurdling contests make up the track events. Jumping and throwing contests make up the field events. More than 950,000 high school students participate in track and field competitions each year.

Source: www.encarta.msn.com

TEACHING TIP

Ask the students if they could multiply the number of points times the results. Ask them to explain their answer.

In-Class Example

Power Point®

3 **CHESS** Three teams competed in the final round of the Chess Club's championships. For each win, a team was awarded 3 points and for each draw a team received 1 point.

Team	Wins	Draws
Blue	5	4
Red	6	3
Green	4	5

Find the total number of points for each team. Which team won the tournament? **The Blue Team had 19 points, the Red Team had 21 points, and the Green Team had 17 points. So, the Red Team won the tournament.**

Lesson 4-3 Multiplying Matrices 169

MULTIPLICATIVE PROPERTIES

In-Class Examples

Power Point®

4 Find each product if $K = \begin{bmatrix} -3 & 2 & 2 \\ -1 & -2 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & -2 \\ 4 & 3 \\ 0 & -1 \end{bmatrix}$.

a. KL $\begin{bmatrix} 5 & 10 \\ -9 & -4 \end{bmatrix}$

b. LK $\begin{bmatrix} -1 & 6 & 2 \\ -15 & 2 & 8 \\ 1 & 2 & 0 \end{bmatrix}$

5 Find each product if $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix}$.

a. $A(B + C)$ $\begin{bmatrix} 6 & -5 \\ 2 & -2 \end{bmatrix}$

b. $AB + AC$ $\begin{bmatrix} 6 & -5 \\ 2 & -2 \end{bmatrix}$

Building on Prior Knowledge

Ask students to compare the process in Example 5 to the procedures they have used previously when applying the Distributive Property to real numbers and algebraic expressions.

Example 4 Commutative Property

Find each product if $P = \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix}$.

a. PQ

$$PQ = \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 72 - 42 & -24 + 7 & 16 + 35 \\ -18 + 24 & 6 - 4 & -4 - 20 \\ 0 + 18 & 0 - 3 & 0 - 15 \end{bmatrix} \quad \text{Multiply columns by rows.}$$

$$= \begin{bmatrix} 30 & -17 & 51 \\ 6 & 2 & -24 \\ 18 & -3 & -15 \end{bmatrix} \quad \text{Simplify.}$$

b. QP

$$QP = \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 72 + 6 + 0 & -63 - 12 + 6 \\ 48 + 2 + 0 & -42 - 4 - 15 \end{bmatrix} \quad \text{Multiply columns by rows.}$$

$$= \begin{bmatrix} 78 & -69 \\ 50 & -61 \end{bmatrix} \quad \text{Simplify.}$$

In Example 4, notice that $PQ \neq QP$ because $\begin{bmatrix} 30 & -17 & 51 \\ 6 & 2 & -24 \\ 18 & -3 & -15 \end{bmatrix} \neq \begin{bmatrix} 78 & -69 \\ 50 & -61 \end{bmatrix}$. This demonstrates that the Commutative Property of Multiplication does not hold for matrix multiplication. The order in which you multiply matrices is very important.

Example 5 Distributive Property

Find each product if $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix}$, **and** $C = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$.

a. $A(B + C)$

$$A(B + C) = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \right) \quad \text{Substitution}$$

$$= \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 6 \\ 1 & 10 \end{bmatrix} \quad \text{Add corresponding elements.}$$

$$= \begin{bmatrix} 3(-1) + 2(1) & 3(6) + 2(10) \\ -1(-1) + 4(1) & -1(6) + 4(10) \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix} \quad \text{Multiply columns by rows.}$$

b. $AB + AC$

$$AB + AC = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 3(-2) + 2(6) & 3(5) + 2(7) \\ -1(-2) + 4(6) & -1(5) + 4(7) \end{bmatrix} + \begin{bmatrix} 3(1) + 2(-5) & 3(1) + 2(3) \\ -1(1) + 4(-5) & -1(1) + 4(3) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 29 \\ 26 & 23 \end{bmatrix} + \begin{bmatrix} -7 & 9 \\ -21 & 11 \end{bmatrix} \quad \text{Simplify.}$$

$$= \begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix} \quad \text{Add corresponding elements.}$$

DAILY INTERVENTION — Differentiated Instruction

Auditory/Musical Ask students to write their own verse or rap to explain the steps in multiplying a 2 × 3 matrix and a 3 × 2 matrix. Invite some students to share their efforts with the class.

Notice that in Example 5, $A(B + C) = AB + AC$. This and other examples suggest that the Distributive Property is true for matrix multiplication. Some properties of matrix multiplication are shown below.

Concept Summary — *Properties of Matrix Multiplication*

For any matrices A, B, and C for which the matrix product is defined, and any scalar c, the following properties are true.

Associative Property of Matrix Multiplication	$(AB)C = A(BC)$
Associative Property of Scalar Multiplication	$c(AB) = (cA)B = A(cB)$
Left Distributive Property	$C(A + B) = CA + CB$
Right Distributive Property	$(A + B)C = AC + BC$

To show that a property is true for all cases, you must show it is true for the general case. To show that a property is *not* true for all cases, you only need to find a counterexample.

Check for Understanding

Concept Check

1. Sample answer: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$

1. **OPEN ENDED** Give an example of two matrices whose product is a 3×2 matrix.
2. **Determine** whether the following statement is *always*, *sometimes*, or *never* true. Explain your reasoning. **Never; the inner dimensions will never be equal.**
   For any matrix $A_{m \times n}$ for $m \neq n$, A^2 is defined.
3. **Explain** why, in most cases, $(A + B)C \neq CA + CB$. **See margin.**

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4, 5	1
6–9	2
10	4, 5
11, 12	3

Determine whether each matrix product is defined. If so, state the dimensions of the product.

4. $A_{3 \times 5} \cdot B_{5 \times 2}$ **3 × 2**
5. $X_{2 \times 3} \cdot Y_{2 \times 3}$ **undefined**

Find each product, if possible.

6. $[3 \ \ -5] \cdot \begin{bmatrix} 3 & 5 \\ -2 & 0 \end{bmatrix}$ $[19 \ \ 15]$
7. $\begin{bmatrix} 5 \\ 8 \end{bmatrix} \cdot [3 \ \ -1 \ \ 4]$ $\begin{bmatrix} 15 & -5 & 20 \\ 24 & -8 & 32 \end{bmatrix}$
8. $\begin{bmatrix} 5 & -2 & -1 \\ 8 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$ **not possible**
9. $\begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 24 \\ 41 \end{bmatrix}$

10. Use $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 1 \\ 8 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$ to determine whether $A(BC) = (AB)C$ is true for the given matrices. **See margin.**

Application

SPORTS For Exercises 11 and 12, use the table below that shows the number of kids registered for baseball and softball.

The Westfall Youth Baseball and Softball League charges the following registration fees: ages 7–8, \$45; ages 9–10, \$55; and ages 11–14, \$65.

Team Members		
Age	Baseball	Softball
7–8	350	280
9–10	320	165
11–14	180	120

11. $[45 \ \ 55 \ \ 65]$, $\begin{bmatrix} 350 & 280 \\ 320 & 165 \\ 180 & 120 \end{bmatrix}$

11. Write a matrix for the registration fees and a matrix for the number of players.
12. Find the total amount of money the League received from baseball and softball registrations. **\$74,525**

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms in this lesson to their Vocabulary Builder worksheets for Chapter 4.
- copy the general formula for multiplying matrices into their notebooks.
- write their own examples for testing whether two matrices can be multiplied and for multiplying two matrices.
- copy the properties of matrix multiplication.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

Answers

3. The Right Distributive Property says that $(A + B)C = AC + BC$, but $AC + BC \neq CA + CB$ since the Commutative Property does not hold for matrix multiplication in most cases.

10. yes;

$A(BC)$

$= \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \cdot \left(\begin{bmatrix} -4 & 1 \\ 8 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \right)$

$= \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -13 & -6 \\ 24 & 16 \end{bmatrix}$

$= \begin{bmatrix} -50 & -28 \\ 81 & 62 \end{bmatrix}$

$(AB)C$

$= \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 \\ 8 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -16 & 2 \\ 28 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -50 & -28 \\ 81 & 62 \end{bmatrix}$

About the Exercises...

Organization by Objective

- **Multiply Matrices:** 13–26, 31–34, 36–42
- **Multiplicative Properties:** 27–30, 35

Odd/Even Assignments

Exercises 13–30 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 13–29 odd, 31–39, 43–60

Average: 13–29 odd, 31–39, 43–60

Advanced: 14–30 even, 35–56 (optional: 57–60)

All: Practice Quiz 1 (1–10)

Practice and Apply

Homework Help

For Exercises	See Examples
13–18	1
19–26	2
27–30	4, 5
31–41	3

Extra Practice

See page 834.

Determine whether each matrix product is defined. If so, state the dimensions of the product. **15. undefined**

13. $A_{4 \times 3} \cdot B_{3 \times 2}$ **4×2**
14. $X_{2 \times 2} \cdot Y_{2 \times 2}$ **2×2**
15. $P_{1 \times 3} \cdot Q_{4 \times 1}$
16. $R_{1 \times 4} \cdot S_{4 \times 5}$ **1×5**
17. $M_{4 \times 3} \cdot N_{4 \times 3}$ **undefined**
18. $A_{3 \times 1} \cdot B_{1 \times 5}$ **3×5**

Find each product, if possible.

19. $[2 \quad -1] \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ **$[6]$**

20. $\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix}$ **$\begin{bmatrix} 8 & -11 \\ 22 & 12 \end{bmatrix}$**

21. $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$ **not possible**

22. $\begin{bmatrix} 4 & -2 & -7 \\ 6 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$ **$\begin{bmatrix} -39 \\ 18 \end{bmatrix}$**

23. $\begin{bmatrix} 1 & -25 & 2 \\ 29 & 1 & -30 \end{bmatrix}$

23. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 & -2 \\ 5 & 7 & -6 \end{bmatrix}$

24. $\begin{bmatrix} 7 & 3 \\ 0 & 2 \\ 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 4 \\ 3 & -5 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ **not possible**

26. $\begin{bmatrix} 0 & 64 & -40 \\ 9 & 11 & -11 \\ -3 & 39 & -23 \end{bmatrix}$

25. $\begin{bmatrix} 4 & 0 \\ -3 & 7 \\ -5 & 9 \end{bmatrix} \cdot \begin{bmatrix} 6 & 4 \\ -2 & 1 \end{bmatrix}$ **$\begin{bmatrix} 24 & 16 \\ -32 & -5 \\ -48 & -11 \end{bmatrix}$**

26. $\begin{bmatrix} 0 & 8 \\ 3 & 1 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & -2 \\ 0 & 8 & -5 \end{bmatrix}$

Use $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$, and scalar $c = 3$ to determine whether the following equations are true for the given matrices.

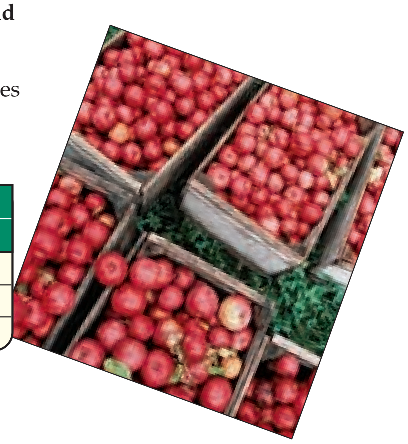
27–30. See pp. 217A–217B for work.

27. $AC + BC = (A + B)C$ **yes**
28. $c(AB) = A(cB)$ **yes**
29. $C(A + B) = AC + BC$ **no**
30. $ABC = CBA$ **no**

PRODUCE **For Exercises 31–34, use the table and the following information.**
Carmen Fox owns three fruit farms on which he grows apples, peaches, and apricots. He sells apples for $22 a case, peaches for $25 a case, and apricots for $18 a case.

Number of Cases in Stock of Each Type of Fruit

Farm	Apples	Peaches	Apricots
1	290	165	210
2	175	240	190
3	110	75	0



31. Write an inventory matrix for the number of cases for each type of fruit for each farm.
32. Write a cost matrix for the price per case for each type of fruit.
33. Find the total income of the three fruit farms expressed as a matrix.
34. What is the total income from all three fruit farms combined? **$31,850**

31. $\begin{bmatrix} 290 & 165 & 210 \\ 175 & 240 & 190 \\ 110 & 75 & 0 \end{bmatrix}$

32. $\begin{bmatrix} 22 \\ 25 \\ 18 \end{bmatrix}$

33. $\begin{bmatrix} 14{,}285 \\ 13{,}270 \\ 4295 \end{bmatrix}$

35. any two matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ where $bg = cf$, $a = d$, and $e = h$

35. **CRITICAL THINKING** Give an example of two matrices A and B whose product is commutative so that $AB = BA$.

Answer

44. Sports statistics are often listed in columns and matrices. In this case, you can find the total number of points scored by multiplying the point matrix, which doesn't change, by the record matrix, which changes for each season. Answers should include the following.

- **$P \cdot R = [479]$**
- **Basketball and wrestling use different point values in scoring.**

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FUND-RAISING **For Exercises 36–39, use the table and the information below.** Lawrence High School sold wrapping paper and boxed cards for their fund-raising event. The school receives $1.00 for each roll of wrapping paper sold and $0.50 for each box of cards sold.

Total Amounts for Each Class		
Class	**Wrapping Paper**	**Cards**
Freshmen	72	49
Sophomores	68	63
Juniors	90	56
Seniors	86	62

36. $\begin{bmatrix} 72 & 49 \\ 68 & 63 \\ 90 & 56 \\ 86 & 62 \end{bmatrix}, \begin{bmatrix} 1.00 \\ 0.50 \end{bmatrix}$

37. $\begin{bmatrix} 96.50 \\ 99.50 \\ 118 \\ 117 \end{bmatrix}$

36. Write a matrix that represents the amounts sold for each class and a matrix that represents the amount of money the school earns for each item sold.

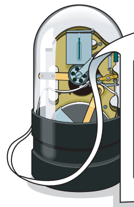
37. Write a matrix that shows how much each class earned.

38. Which class earned the most money? **Juniors**

39. What is the total amount of money the school made from the fund-raiser? **$431**

FINANCE **For Exercises 40–42, use the table below that shows the purchase price and selling price of stock for three companies.**

For a class project, Taini "bought" shares of stock in three companies. She bought 150 shares of a utility company, 100 shares of a computer company, and 200 shares of a food company. At the end of the project she "sold" all of her stock.



Company	Purchase Price (per share)	Selling Price (per share)
Utility	$54.00	$55.20
Computer	$48.00	$58.60
Food	$60.00	$61.10

40. Organize the data in two matrices and use matrix multiplication to find the total amount she spent for the stock. **$24,900**

41. Write two matrices and use matrix multiplication to find the total amount she received for selling the stock. **$26,360**

42. Use matrix operations to find how much money Taini "made" or "lost." **$1460**

43. $a = 1$, $b = 0$, $c = 0$, $d = 1$; the original matrix

43. **CRITICAL THINKING** Find the values of a, b, c, and d to make the statement $\begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix}$ true. If the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is multiplied by any other matrix containing two columns, what do you think the result would be?

44. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How can matrices be used in sports statistics?

Include the following in your answer:
- a matrix that represents the total points scored in the 2000 season, and
- an example of another sport where different point values are used in scoring.

Standardized Test Practice

45. If C is a 5×1 matrix and D is a 3×5 matrix, what are the dimensions of DC? **B**

(A) 5×5 (B) 3×1 (C) 1×3 (D) DC is not defined.

46. What is the product of $[5 \quad -2 \quad 3]$ and $\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$? **A**

(A) $[11 \quad -1]$ (B) $\begin{bmatrix} 11 \\ -1 \end{bmatrix}$ (C) $\begin{bmatrix} 5 & -10 \\ 0 & -6 \\ 6 & -15 \end{bmatrix}$ (D) undefined

www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 181 (shown) and p. 182

Multiply Matrices You can multiply two matrices if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Multiplication of Matrices	$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}$

Example Find AB if $A = \begin{bmatrix} -4 & 3 \\ 2 & -2 \\ 1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}$.

$AB = \begin{bmatrix} -4 & 3 \\ 2 & -2 \\ 1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}$ Substitution

$= \begin{bmatrix} -4(5) + 3(-1) & -4(-2) + 3(3) \\ 2(5) + (-2)(-1) & 2(-2) + (-2)(3) \\ 1(5) + 7(-1) & 1(-2) + 7(3) \end{bmatrix}$ Multiply columns by rows.

$= \begin{bmatrix} -23 & 17 \\ 12 & -10 \\ -2 & 19 \end{bmatrix}$ Simplify.

Exercises

Find each product, if possible.

1. $\begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ $\begin{bmatrix} 12 & 3 \\ -6 & 9 \end{bmatrix}$
2. $\begin{bmatrix} -1 & 0 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$ $\begin{bmatrix} -3 & -2 \\ 2 & 34 \end{bmatrix}$
3. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ $\begin{bmatrix} 7 & -7 \\ 14 & 14 \end{bmatrix}$
4. $\begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix}$ $\begin{bmatrix} -15 & 1 & 7 \\ 26 & -2 & -12 \end{bmatrix}$
5. $\begin{bmatrix} 3 & -2 \\ 0 & 4 \\ -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 4 \\ 8 & 4 \\ -3 & -9 \end{bmatrix}$
6. $\begin{bmatrix} 5 & -2 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$ $\begin{bmatrix} 24 & -15 \\ 14 & -17 \end{bmatrix}$
7. $\begin{bmatrix} 6 & 10 \\ -4 & 3 \\ -2 & 7 \end{bmatrix} \cdot [0 \quad 4 \quad -3]$ not possible
8. $\begin{bmatrix} 7 & -2 \\ 5 & -4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ -2 & 0 \end{bmatrix}$ $\begin{bmatrix} 11 & -21 \\ 13 & -15 \end{bmatrix}$
9. $\begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & -2 \\ -1 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$ $\begin{bmatrix} 10 & -16 \\ 18 & -6 \\ 5 & 9 \end{bmatrix}$

Skills Practice, p. 183 and Practice, p. 184 (shown)

Determine whether each matrix product is defined. If so, state the dimensions of the product.

1. $A_{7 \times 4} \cdot B_{4 \times 3}$ 7×3
2. $A_{3 \times 5} \cdot M_{5 \times 8}$ 3×8
3. $M_{2 \times 1} \cdot A_{1 \times 6}$ 2×6
4. $M_{3 \times 2} \cdot A_{3 \times 2}$ undefined
5. $P_{1 \times 9} \cdot Q_{9 \times 1}$ 1×1
6. $P_{9 \times 1} \cdot Q_{1 \times 9}$ 9×9

Find each product, if possible.

7. $\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 7 \\ 6 & 0 & -5 \end{bmatrix}$ $\begin{bmatrix} 30 & -4 & -6 \\ 3 & -6 & 26 \end{bmatrix}$
8. $\begin{bmatrix} 2 & 4 \\ 7 & -1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 0 \\ 2 & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & 20 \\ -23 & -5 \end{bmatrix}$
9. $\begin{bmatrix} -3 & 0 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 7 & -1 \end{bmatrix}$ $\begin{bmatrix} -6 & -12 \\ 39 & 3 \end{bmatrix}$
10. $\begin{bmatrix} 3 & -2 & 7 \\ 6 & 0 & -5 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 7 \\ 6 & 0 & -5 \end{bmatrix}$ not possible
11. $[4 \quad 0 \quad 2] \cdot \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ $[2]$
12. $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \cdot [4 \quad 0 \quad 2]$ $\begin{bmatrix} 4 & 0 & 2 \\ 12 & 0 & 6 \\ -4 & 0 & -2 \end{bmatrix}$
13. $\begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $\begin{bmatrix} -30 & 10 \\ 15 & -5 \end{bmatrix}$
14. $[-15 \quad -9] \cdot \begin{bmatrix} 6 & 11 \\ 23 & -10 \end{bmatrix}$ $[-297 \quad -75]$

Use $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ -2 & -1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and scalar $c = 3$ to determine whether the following equations are true for the given matrices.

15. $AC = CA$ yes
16. $A(B + C) = BA + CA$ no
17. $(AB)c = c(AB)$ yes
18. $(A + C)B = B(A + C)$ no

RENTALS **For Exercises 19–21, use the following information.**

For their one-week vacation, the Montoyas can rent a 2-bedroom condominium for $1796, a 3-bedroom condominium for $2165, or a 4-bedroom condominium for $2538. The table shows the number of units in each of three complexes.

	2-Bedroom	3-Bedroom	4-Bedroom
Sun Haven	36	24	22
Surfside	29	32	42
Seabreeze	18	22	18

19. Write a matrix that represents the number of each type of unit available at each complex and a matrix that represents the weekly charge for each type of unit. $\begin{bmatrix} 36 & 24 & 22 \\ 29 & 32 & 42 \\ 18 & 22 & 18 \end{bmatrix}, \begin{bmatrix} \$1796 \\ \$2165 \\ \$2538 \end{bmatrix}$
20. If all of the units in the three complexes are rented for the week at the rates given the Montoyas, express the income of each of the three complexes as a matrix. $\begin{bmatrix} 172{,}452 \\ 227{,}960 \\ 125{,}642 \end{bmatrix}$
21. What is the total income of all three complexes for the week? $526,054

Reading to Learn Mathematics, p. 185 ELL

Pre-Activity **How can matrices be used in sports statistics?**

Read the introduction to Lesson 4-3 at the top of page 167 in your textbook.

Write a sum that shows the total points scored by the Oakland Raiders during the 2000 season. (The sum will include multiplications. Do not actually calculate this sum.)

$6 \cdot 58 + 1 \cdot 56 + 3 \cdot 23 + 2 \cdot 1 + 2 \cdot 2$

Reading the Lesson

1. Determine whether each indicated matrix product is defined. If so, state the dimensions of the product. If not, write *undefined*.
   a. $M_{3 \times 2}$ and $N_{2 \times 3}$ MN: 3×3 NM: 2×2
   b. $M_{1 \times 2}$ and $N_{1 \times 2}$ MN: undefined NM: undefined
   c. $M_{4 \times 1}$ and $N_{1 \times 4}$ MN: 4×4 NM: 1×1
   d. $M_{3 \times 4}$ and $N_{4 \times 4}$ MN: 3×4 NM: undefined

2. The regional sales manager for a chain of computer stores wants to compare the revenue from sales of one model of notebook computer and one model of printer for three stores in his area. The notebook computer sells for $1850 and the printer for $175. The number of computers and printers sold at the three stores during September are shown in the following table.

Store	Computers	Printers
A	128	101
B	205	166
C	97	73

Write a matrix product that the manager could use to find the total revenue for computers and printers for each of the three stores. (Do not calculate the product.)

$\begin{bmatrix} 128 & 101 \\ 205 & 166 \\ 97 & 73 \end{bmatrix} \cdot \begin{bmatrix} 1850 \\ 175 \end{bmatrix}$

Helping You Remember

3. Many students find the procedure of matrix multiplication confusing at first because it is unfamiliar. Think of an easy way to use the letters R and C to remember how to multiply matrices and what the dimensions of the product will be. **Sample answer: Just remember RC for "row, column." Multiply each *row* of the first matrix by each *column* of the second matrix. The dimensions of the product are the number of rows of the first matrix and the number of *columns* of the second matrix.**

Enrichment, p. 186

Fourth-Order Determinants

To find the value of a 4×4 determinant, use a method called **expansion by minors**.

First write the expansion. Use the first row of the determinant. Remember that the signs of the terms alternate.

$$\begin{vmatrix} 6 & -3 & 2 & 7 \\ 0 & 4 & 3 & 5 \\ 0 & 2 & 1 & -4 \\ 6 & 0 & -2 & 0 \end{vmatrix} = 6\begin{vmatrix} 4 & 3 & 5 \\ 2 & 1 & -4 \\ 0 & -2 & 0 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 3 & 5 \\ 0 & 1 & -4 \\ 6 & -2 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & 4 & 5 \\ 0 & 2 & -4 \\ 6 & 0 & 0 \end{vmatrix} - 7\begin{vmatrix} 0 & 4 & 3 \\ 0 & 2 & 1 \\ 6 & 0 & -2 \end{vmatrix}$$

Then evaluate each 3×3 determinant. Use any row.

$$\begin{vmatrix} 4 & 3 & 5 \\ 2 & 1 & -4 \\ 0 & -2 & 0 \end{vmatrix} = -(-2)\begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix} \qquad \begin{vmatrix} 0 & 3 & 5 \\ 0 & 1 & -4 \\ 6 & -2 & 0 \end{vmatrix} = -3\begin{vmatrix} 0 & -4 \\ 6 & 0 \end{vmatrix} + 5\begin{vmatrix} 0 & 1 \\ 6 & -2 \end{vmatrix}$$

$= 2(-16 - 10)$ $\qquad = -3(24) + 5(-6)$

4 Assess

Open-Ended Assessment

Modeling Have students use the class seating arrangement to form matrices, with each student's desk as an element. Have students make 24 large cards showing the values for the elements of two matrices. Form two matrices $A_{4 \times 3}$ and $B_{3 \times 4}$ with student seats and give each student sitting in these "matrices" a card to show the element in that position. Have students model the matrix multiplication $AB = C$. Begin by drawing a blank 4×4 matrix on the board. For each element of matrix C, have students walk out the products, compute the sums, and then write the results in the correct locations of the matrix on the board.

Assessment Options

Practice Quiz 1 The quiz provides students with a brief review of the concepts and skills in Lessons 4-1 through 4-3. Lesson numbers are given to the right of exercises or instruction lines so students can review concepts not yet mastered.

Getting Ready for Lesson 4-4

PREREQUISITE SKILL Lesson 4-4 shows how to use a matrix to find the coordinates of a figure that has been transformed by translation, dilation, reflection, or rotation. Students will use their familiarity with graphing ordered pairs to graph transformed figures. Exercises 57–60 should be used to determine your students' familiarity with graphing ordered pairs on a coordinate plane.

Answer (Practice Quiz 1)

4. $\begin{bmatrix}120 & 80 & 64 & 75\\ 65 & 105 & 77 & 53\end{bmatrix}$, $\begin{bmatrix}112 & 79 & 56 & 74\\ 69 & 95 & 82 & 50\end{bmatrix}$

Maintain Your Skills

Mixed Review

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. *(Lesson 4-2)*

47. $3\begin{bmatrix}4 & -2\\ -1 & 7\end{bmatrix}$ $\begin{bmatrix}12 & -6\\ -3 & 21\end{bmatrix}$

48. $\begin{bmatrix}3 & 5 & 9\end{bmatrix} + \begin{bmatrix}5\\ 2\\ 6\end{bmatrix}$ impossible

49. $2\begin{bmatrix}6 & 3\\ -8 & -2\end{bmatrix} - 4\begin{bmatrix}8 & 1\\ 3 & -4\end{bmatrix}$ $\begin{bmatrix}-20 & 2\\ -28 & 12\end{bmatrix}$

Solve each equation. *(Lesson 4-1)*

50. $\begin{bmatrix}3x + 2\\ 15\end{bmatrix} = \begin{bmatrix}23\\ -4y - 1\end{bmatrix}$ (7, −4)

51. $\begin{bmatrix}x + 3y\\ 2x - y\end{bmatrix} = \begin{bmatrix}-22\\ 19\end{bmatrix}$ (5, −9)

52. $\begin{bmatrix}x + 3z\\ -2x + y - z\\ 5y - 7z\end{bmatrix} = \begin{bmatrix}-19\\ -2\\ 24\end{bmatrix}$ (2, −5, −7)

53. CAMERA SUPPLIES Mrs. Franklin is planning a family vacation. She bought 8 rolls of film and 2 camera batteries for \$23. The next day, her daughter went back and bought 6 more rolls of film and 2 batteries for her camera. This bill was \$18. What is the price of a roll of film and a camera battery? *(Lesson 3-2)* **\$2.50; \$1.50**

Find the x-intercept and the y-intercept of the graph of each equation. Then graph the equation. *(Lesson 2-2)* **54–56. See margin for graphs.**

54. $y = 3 - 2x$ $\frac{3}{2}$; 3 **55.** $x - \frac{1}{2}y = 8$ 8; −16 **56.** $5x - 2y = 10$ 2; −5

Getting Ready for the Next Lesson

PREREQUISITE SKILL Graph each set of ordered pairs on a coordinate plane.
*(To review **graphing ordered pairs**, see Lesson 2-1.)* **57–60. See pp. 217A–217B.**

57. {(2, 4), (−1, 3), (0, −2)} **58.** {(−3, 5), (−2, −4), (3, −2)}

59. {(−1, 2), (2, 4), (3, −3), (4, −1)} **60.** {(−3, 3), (1, 3), (4, 2), (−1, −5)}

Practice Quiz 1 — *Lessons 4-1 through 4-3*

Solve each equation. *(Lesson 4-1)*

1. $\begin{bmatrix}3x + 1\\ 7y\end{bmatrix} = \begin{bmatrix}19\\ 21\end{bmatrix}$ (6, 3) **2.** $\begin{bmatrix}2x + y\\ 4x - 3y\end{bmatrix} = \begin{bmatrix}9\\ 23\end{bmatrix}$ (5, −1) **3.** $\begin{bmatrix}2 & x\\ y & 5\end{bmatrix} = \begin{bmatrix}2 & 1\\ 3 & z\end{bmatrix}$ (1, 3, 5)

BUSINESS **For Exercises 4 and 5, use the table and the following information.**
The manager of The Best Bagel Shop keeps records of each type of bagel sold each day at their two stores. Two days of sales are shown below.

Day	Store	Type of Bagel			
		Sesame	Poppy	Blueberry	Plain
Monday	East	120	80	64	75
	West	65	105	77	53
Tuesday	East	112	79	56	74
	West	69	95	82	50

4. Write a matrix for each day's sales. *(Lesson 4-1)* **See margin.**

5. Find the sum of the two days' sales using matrix addition. *(Lesson 4-2)* $\begin{bmatrix}232 & 159 & 120 & 149\\ 134 & 200 & 159 & 103\end{bmatrix}$

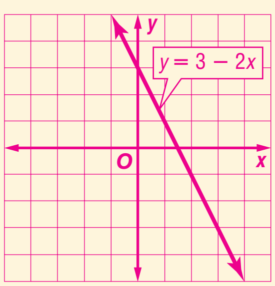
Perform the indicated matrix operations. *(Lesson 4-2)*

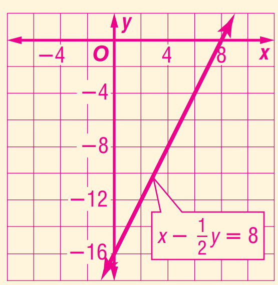
6. $\begin{bmatrix}3 & 0\\ 7 & 12\end{bmatrix} - \begin{bmatrix}6 & -5\\ 4 & -1\end{bmatrix}$ $\begin{bmatrix}-3 & 5\\ 3 & 13\end{bmatrix}$ **7.** $\frac{2}{3}\begin{bmatrix}9 & 0\\ 12 & 15\end{bmatrix} + \begin{bmatrix}-2 & 3\\ -7 & -7\end{bmatrix}$ $\begin{bmatrix}4 & 3\\ 1 & 3\end{bmatrix}$ **8.** $5\begin{bmatrix}-2 & 4 & 5\\ 0 & -4 & 7\end{bmatrix}$ $\begin{bmatrix}-10 & 20 & 25\\ 0 & -20 & 35\end{bmatrix}$

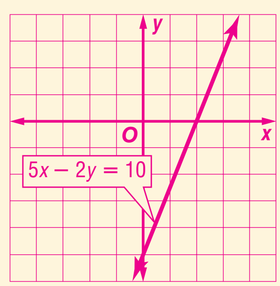
Find each product, if possible. *(Lesson 4-3)*

9. $\begin{bmatrix}4 & 0 & -8\\ 7 & -2 & 10\end{bmatrix} \cdot \begin{bmatrix}-1 & 3\\ 6 & 0\end{bmatrix}$ not possible **10.** $\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix} \cdot \begin{bmatrix}4 & -1 & -2\\ -3 & 5 & 4\end{bmatrix}$ $\begin{bmatrix}15 & -8 & -10\\ -7 & 23 & 16\end{bmatrix}$

Answers

54.


55.


56.


4-4 Transformations with Matrices

What You'll Learn

- Use matrices to determine the coordinates of a translated or dilated figure.
- Use matrix multiplication to find the coordinates of a reflected or rotated figure.

Vocabulary

- vertex matrix
- transformation
- preimage
- image
- isometry
- translation
- dilation
- reflection
- rotation

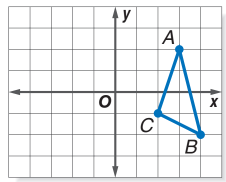
How are transformations used in computer animation?

Computer animation creates the illusion of motion by using a succession of computer-generated still images. Computer animation is used to create movie special effects and to simulate images that would be impossible to show otherwise. An object's size and orientation are stored in a computer program. Complex geometric figures can be broken into simple triangles and then moved to other parts of the screen.

TRANSLATIONS AND DILATIONS Points on a coordinate plane can be represented by matrices. The ordered pair (x, y) can be represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. Likewise, polygons can be represented by placing all of the column matrices of the coordinates of the vertices into one matrix, called a **vertex matrix**.

Triangle ABC with vertices $A(3, 2)$, $B(4, -2)$, and $C(2, -1)$ can be represented by the following vertex matrix.

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \triangle ABC = & \begin{bmatrix} 3 & 4 & 2 \\ 2 & -2 & -1 \end{bmatrix} \begin{array}{l} \leftarrow x\text{-coordinates} \\ \leftarrow y\text{-coordinates} \end{array} \end{array}$$



One of the ways that matrices are used is to perform transformations. **Transformations** are functions that map points of a **preimage** onto its **image**. If the image and preimage are congruent figures, the transformation is an **isometry**.

One type of isometry is a translation. A **translation** occurs when a figure is moved from one location to another without changing its size, shape, or orientation. You can use matrix addition and a *translation matrix* to find the coordinates of a translated figure.

Study Tip

Reading Math
A matrix containing coordinates of a geometric figure is also called a *coordinate matrix.*

Example 1 Translate a Figure

Find the coordinates of the vertices of the image of quadrilateral $QUAD$ with $Q(2, 3)$, $U(5, 2)$, $A(4, -2)$, and $D(1, -1)$, if it is moved 4 units to the left and 2 units up. Then graph $QUAD$ and its image $Q'U'A'D'$.

Write the vertex matrix for quadrilateral $QUAD$. $\begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix}$

To translate the quadrilateral 4 units to the left, add -4 to each x-coordinate. To translate the figure 2 units up, add 2 to each y-coordinate. This can be done by adding the translation matrix $\begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix}$ to the vertex matrix of $QUAD$.

(continued on the next page)

4-4 Lesson Notes

1 Focus

5-Minute Check Transparency 4-4 Use as a quiz or review of Lesson 4-3.

Mathematical Background notes are available for this lesson on p. 152D.

How are transformations used in computer animation?

Ask students:

- What does it mean to "simulate" an image? **Sample answer: to use a series of drawings that together create the illusion of real life**
- How would you break a geometric figure (a polygon) into simple triangles? **Draw all the diagonals from one vertex.**

Resource Manager

Workbook and Reproducible Masters

Chapter 4 Resource Masters
- Study Guide and Intervention, pp. 187–188
- Skills Practice, p. 189
- Practice, p. 190
- Reading to Learn Mathematics, p. 191
- Enrichment, p. 192
- Assessment, pp. 231, 233

Graphing Calculator and Spreadsheet Masters, p. 33

Transparencies

5-Minute Check Transparency 4-4
Answer Key Transparencies

Technology

Interactive Chalkboard

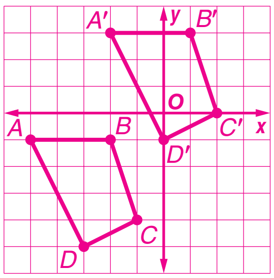
2 Teach

TRANSLATIONS AND DILATIONS

In-Class Examples

Power Point®

1 Find the coordinates of the vertices of the image of quadrilateral $ABCD$ with $A(-5, -1)$, $B(-2, -1)$, $C(-1, -4)$, $D(-3, -5)$, if it is moved 3 units to the right and 4 units up. Then graph $ABCD$ and its image $A'B'C'D'$. **$A'(-2, 3)$, $B'(1, 3)$, $C'(2, 0)$, $D'(0, -1)$**



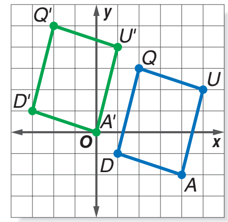
2 Rectangle $E'F'G'H'$ is the result of a translation of rectangle $EFGH$. A table of the vertices of each rectangle is shown. Find the coordinates of F and G'.

Rectangle EFGH	Rectangle E′F′G′H′
E(−2, 2)	E′(−5, 0)
F	F′(1, 0)
G(4, −2)	G′
H(−2, −2)	H′(−5, −4)

The coordinates of F are (4, 2) and the coordinates of G' are (1, −4).

$$\begin{array}{ccccc} \text{Vertex Matrix of } QUAD & & \text{Translation Matrix} & & \text{Vertex Matrix of } Q'U'A'D' \\ \begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix} & + & \begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix} & = & \begin{bmatrix} -2 & 1 & 0 & -3 \\ 5 & 4 & 0 & 1 \end{bmatrix} \end{array}$$

The coordinates of $Q'U'A'D'$ are $Q'(-2, 5)$, $U'(1, 4)$, $A'(0, 0)$, and $D'(-3, 1)$. Graph the preimage and the image. The two quadrilaterals have the same size and shape.



Standardized Test Practice

Example 2 Find a Translation Matrix

Short-Response Test Item

Rectangle $A'B'C'D'$ is the result of a translation of rectangle $ABCD$. A table of the vertices of each rectangle is shown. Find the coordinates of A and D'.

Rectangle ABCD	Rectangle A′B′C′D′
A	A′(−1, 1)
B(1, 5)	B′(4, 1)
C(1, −2)	C′(4, −6)
D(−4, −2)	D′

Read the Test Item

- You are given the coordinates of the preimage and image of points B and C. Use this information to find the translation matrix. Then you can use the translation matrix to find the coordinates of A and D'.

Solve the Test Item

- Write a matrix equation. Let (a, b) represent the coordinates of A and let (c, d) represent the coordinates of D'.

$$\begin{bmatrix} a & 1 & 1 & -4 \\ b & 5 & -2 & -2 \end{bmatrix} + \begin{bmatrix} x & x & x & x \\ y & y & y & y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$

$$\begin{bmatrix} a+x & 1+x & 1+x & -4+x \\ b+y & 5+y & -2+y & -2+y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$

- Since these two matrices are equal, corresponding elements are equal.

Solve an equation for x.

$$1 + x = 4$$
$$x = 3$$

Solve an equation for y.

$$5 + y = 1$$
$$y = -4$$

- Use the values for x and y to find the values for $A(a, b)$ and $D'(c, d)$.

$$\begin{aligned} a + x &= -1 \\ a + 3 &= -1 \\ a &= -4 \end{aligned} \qquad \begin{aligned} b + y &= 1 \\ b + (-4) &= 1 \\ b &= 5 \end{aligned} \qquad \begin{aligned} -4 + x &= c \\ -4 + 3 &= c \\ -1 &= c \end{aligned} \qquad \begin{aligned} -2 + y &= d \\ -2 + (-4) &= d \\ -6 &= d \end{aligned}$$

So the coordinates of A are $(-4, 5)$, and the coordinates for D' are $(-1, -6)$.

The Princeton Review

Test-Taking Tip

Sometimes you need to solve for unknown value(s) before you can solve for the value(s) requested in the question.

When a geometric figure is enlarged or reduced, the transformation is called a **dilation**. In a dilation, all linear measures of the image change in the same ratio. For example, if the length of each side of a figure doubles, then the perimeter doubles, and vice versa. You can use scalar multiplication to perform dilations.

Standardized Test Practice

Example 2 Test items involving figures in coordinate planes can often be solved quickly by sketching the figure. After plotting points B, C, and D, students can use their knowledge of rectangles to quickly determine the location of point A. Students can then plot point B' or point C', determine the direction of the translation, and then apply it to point D to find the coordinates of point D'.

Example 3 Dilation

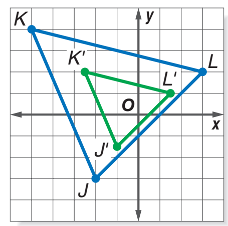
$\triangle JKL$ has vertices $J(-2, -3)$, $K(-5, 4)$, and $L(3, 2)$. Dilate $\triangle JKL$ so that its perimeter is one-half the original perimeter. What are the coordinates of the vertices of $\triangle J'K'L'$?

If the perimeter of a figure is one-half the original perimeter, then the lengths of the sides of the figure will be one-half the measure of the original lengths. Multiply the vertex matrix by the scale factor of $\frac{1}{2}$.

$$\frac{1}{2}\begin{bmatrix} -2 & -5 & 3 \\ -3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & 2 & 1 \end{bmatrix}$$

The coordinates of the vertices of $\triangle J'K'L'$ are $J'\left(-1, -\frac{3}{2}\right)$, $K'\left(-\frac{5}{2}, 2\right)$, and $L'\left(\frac{3}{2}, 1\right)$.

Graph $\triangle JKL$ and $\triangle J'K'L'$. The triangles are not congruent. The image has sides that are half the length of those of the original figure.



REFLECTIONS AND ROTATIONS In addition to translations, reflections and rotations are also isometries. A **reflection** occurs when every point of a figure is mapped to a corresponding image across a line of symmetry using a *reflection matrix*. The matrices used for three common reflections are shown below.

Concept Summary — *Reflection Matrices*

For a reflection over the:	x-axis	y-axis	line $y = x$
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

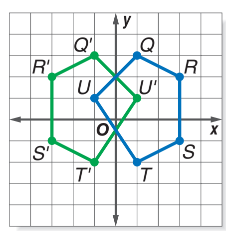
Example 4 Reflection

Find the coordinates of the vertices of the image of pentagon $QRSTU$ with $Q(1, 3)$, $R(3, 2)$, $S(3, -1)$, $T(1, -2)$, and $U(-1, 1)$ after a reflection across the y-axis.

Write the ordered pairs as a vertex matrix. Then multiply the vertex matrix by the reflection matrix for the y-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 3 & 1 & -1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 & -1 & 1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix}$$

The coordinates of the vertices of $Q'R'S'T'U'$ are $Q'(-1, 3)$, $R'(-3, 2)$, $S'(-3, -1)$, $T'(-1, -2)$, and $U'(1, 1)$. Notice that the preimage and image are congruent. Both figures have the same size and shape.



www.algebra2.com/extra_examples

In-Class Example — Power Point®

3 $\triangle XYZ$ has vertices $X(1, 2)$, $Y(3, -1)$, and $Z(-1, -2)$. Dilate $\triangle XYZ$ so that its perimeter is twice the original perimeter. What are the coordinates of the vertices of $\triangle X'Y'Z'$? **$X'(2, 4)$, $Y'(6, -2)$, $Z'(-2, -4)$**

REFLECTIONS AND ROTATIONS

In-Class Example — Power Point®

4 Find the coordinates of the vertices of the image of pentagon $PENTA$ with $P(-3, 1)$, $E(0, -1)$, $N(-1, -3)$, $T(-3, -4)$, and $A(-4, -1)$ after a reflection across the x-axis. **$P'(-3, -1)$, $E'(0, 1)$, $N'(-1, 3)$, $T'(-3, 4)$, $A'(-4, 1)$**

Teaching Tip Ask students to describe the pattern of coordinates for corresponding points after a reflection over either axis. **When a figure is reflected over the x-axis, the sign of the y-coordinate for each point on the figure changes. When a figure is reflected over the y-axis, the sign of the x-coordinate for each point on the figure changes.** Then ask them to describe the pattern after a reflection over the line $y = x$. **When a figure is reflected over the line $y = x$, the x- and y-coordinates of each point are transposed.**

DAILY INTERVENTION — Differentiated Instruction

Visual/Spatial When students are drawing the figures and their images in a coordinate plane in this lesson, have them use different colored pencils for the original figure and its image.

In-Class Example

Power Point®

5 Find the coordinates of the vertices of the image of $\triangle DEF$ with $D(4, 3)$, $E(1, 1)$, and $F(2, 5)$ after it is rotated 90° counterclockwise about the origin. **$D'(-3, 4)$, $E'(-1, 1)$, and $F'(-5, 2)$**

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms in this lesson to their Vocabulary Builder worksheets for Chapter 4.
- write their own examples for translating, dilating, reflecting, and rotating polygons.
- add the reflection matrices for reflections over the x-axis, the y-axis, and the line $y = x$, and the rotation matrices for counterclockwise rotations of 90°, 180°, and 270° about the origin.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

A **rotation** occurs when a figure is moved around a center point, usually the origin. To determine the vertices of a figure's image by rotation, multiply its vertex matrix by a *rotation matrix*. Commonly used rotation matrices are summarized below.

Concept Summary — Rotation Matrices

For a counterclockwise rotation about the origin of:	90°	180°	270°
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

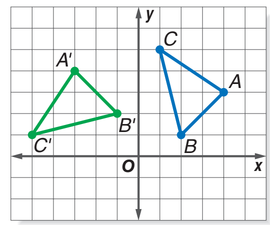
Example 5 Rotation

Find the coordinates of the vertices of the image of $\triangle ABC$ with $A(4, 3)$, $B(2, 1)$, and $C(1, 5)$ after it is rotated 90° counterclockwise about the origin.

Write the ordered pairs in a vertex matrix. Then multiply the vertex matrix by the rotation matrix.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & -1 & -5 \\ 4 & 2 & 1 \end{bmatrix}$$

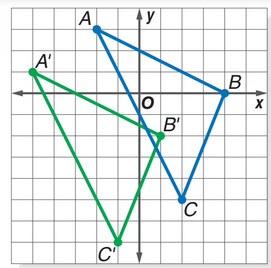
The coordinates of the vertices of $\triangle A'B'C'$ are $A'(-3, 4)$, $B'(-1, 2)$, and $C'(-5, 1)$. The image is congruent to the preimage.



Check for Understanding

Concept Check

1. **Compare and contrast** the size and shape of the preimage and image for each type of transformation. Tell which transformations are isometries. **See margin.**
2. **Write** the translation matrix for $\triangle ABC$ and its image $\triangle A'B'C'$ shown at the right.
3. **OPEN ENDED** Write a translation matrix that moves $\triangle DEF$ up and left on the coordinate plane. **Sample answer: $\begin{bmatrix} -4 & -4 & -4 \\ 1 & 1 & 1 \end{bmatrix}$**



2. $\begin{bmatrix} -3 & -3 & -3 \\ -2 & -2 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 5 & 5 & 0 \\ 4 & 4 & 0 & 0 \end{bmatrix}$

Guided Practice

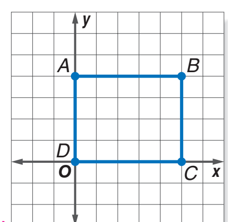
GUIDED PRACTICE KEY	
Exercises	**Examples**
4–6	1
7, 8	3
9	4
10	5
11	2

Triangle ABC with vertices $A(1, 4)$, $B(2, -5)$, and $C(-6, -6)$ is translated 3 units right and 1 unit down.

4. Write the translation matrix. $\begin{bmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix}$
5. Find the coordinates of $\triangle A'B'C'$. **$A'(4, 3)$, $B'(5, -6)$, $C'(-3, -7)$**
6. Graph the preimage and the image. **See margin.**

For Exercises 7–10, use the rectangle at the right.

7. Write the coordinates in a vertex matrix.
8. Find the coordinates of the image after a dilation by a scale factor of 3.
9. Find the coordinates of the image after a reflection over the x-axis.
10. Find the coordinates of the image after a rotation of 180°. **$A'(0, -4)$, $B'(-5, -4)$, $C'(-5, 0)$, $D'(0, 0)$**



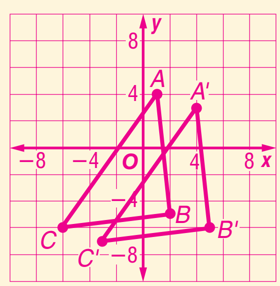
8. **$A'(0, 12)$, $B'(15, 12)$, $C'(15, 0)$, $D'(0, 0)$**
9. **$A'(0, -4)$, $B'(-5, -4)$, $C'(5, 0)$, $D'(0, 0)$**

Answers

1.

Transformation	Size	Shape	Isometry
reflection	same	same	yes
rotation	same	same	yes
translation	same	same	yes
dilation	changes	same	no

6.

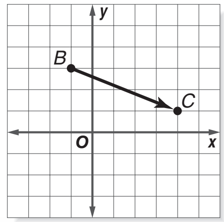


Standardized Test Practice

11. A point is translated from B to C as shown at the right. If a point at $(-4, 3)$ is translated in the same way, what will be its new coordinates? **B**

(A) (3, 4) (B) (1, 1)

(C) (−7, 8) (D) (1, 6)



★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
12–14, 35, 36	1
15–17, 33, 34	3
18–20	4
21–23, 25, 37	5
24	2
26–32, 38–41	1–5

Extra Practice

See page 835.

For Exercises 12–14, use the following information.
Triangle DEF with vertices $D(1, 4)$, $E(2, -5)$, and $F(-6, -6)$ is translated 4 units left and 2 units up.

12. Write the translation matrix. $\begin{bmatrix} -4 & -4 & -4 \\ 2 & 2 & 2 \end{bmatrix}$

13. Find the coordinates of $\triangle D'E'F'$. **$D'(-3, 6)$, $E'(-2, -3)$, $F'(-10, -4)$**

14. Graph the preimage and the image. **See margin.**

For Exercises 15–17, use the following information.
The vertices of $\triangle ABC$ are $A(0, 2)$, $B(1.5, -1.5)$, and $C(-2.5, 0)$. The triangle is dilated so that its perimeter is three times the original perimeter.

15. Write the coordinates for $\triangle ABC$ in a vertex matrix.

16. Find the coordinates of the image $\triangle A'B'C'$. **$A'(0, 6)$, $B'(4.5, -4.5)$, $C'(-7.5, 0)$**

17. Graph $\triangle ABC$ and $\triangle A'B'C'$. **See margin.**

15. $\begin{bmatrix} 0 & 1.5 & -2.5 \\ 2 & -1.5 & 0 \end{bmatrix}$

For Exercises 18–20, use the following information.
The vertices of $\triangle XYZ$ are $X(1, -1)$, $Y(2, -4)$, and $Z(7, -1)$. The triangle is reflected over the line $y = x$.

18. Write the coordinates of $\triangle XYZ$ in a vertex matrix. $\begin{bmatrix} 1 & 2 & 7 \\ -1 & -4 & -1 \end{bmatrix}$

19. Find the coordinates of $\triangle X'Y'Z'$. **$X'(-1, 1)$, $Y'(-4, 2)$, $Z'(-1, 7)$**

20. Graph $\triangle XYZ$ and $\triangle X'Y'Z'$. **See margin.**

For Exercises 21–23, use the following information.
Parallelogram $DEFG$ with $D(2, 4)$, $E(5, 4)$, $F(4, 1)$, and $G(1, 1)$ is rotated 270° counterclockwise about the origin.

21. Write the coordinates of the parallelogram in a vertex matrix. $\begin{bmatrix} 2 & 5 & 4 & 1 \\ 4 & 4 & 1 & 1 \end{bmatrix}$

22. Find the coordinates of parallelogram $D'E'F'G'$.

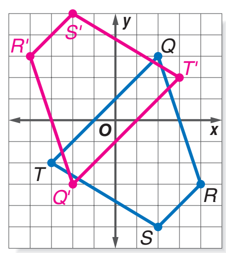
23. Graph the preimage and the image. **See margin.**

22. **$D'(4, -2)$, $E'(4, -5)$, $F'(1, -4)$, $G'(1, -1)$**

★ 24. Triangle DEF with vertices $D(-2, 2)$, $E(3, 5)$, and $F(5, -2)$ is translated so that D' is at $(1, -5)$. Find the coordinates of E' and F'. **$E'(6, -2)$, $F'(8, -9)$**

★ 25. A triangle is rotated 90° counterclockwise about the origin. The coordinates of the vertices are $J'(-3, -5)$, $K'(-2, 7)$, and $L'(1, 4)$. What were the coordinates of the triangle in its original position? **$J(-5, 3)$, $K(7, 2)$, $L(4, -1)$**

For Exercises 26–28, use quadrilateral $QRST$ shown at the right.



26. Write the vertex matrix. Multiply the vertex matrix by −1.

27. Graph the preimage and image. **See right.**

28. What type of transformation does the graph represent? **180° rotation**

26. $\begin{bmatrix} 2 & 4 & 2 & -3 \\ 3 & -3 & -5 & -2 \end{bmatrix} \cdot (-1) = \begin{bmatrix} -2 & -4 & -2 & 3 \\ -3 & 3 & 5 & 2 \end{bmatrix}$

www.algebra2.com/self_check_quiz

About the Exercises...

Organization by Objective
- **Translations and Dilations:** 12–17, 24, 33–36
- **Reflections and Rotations:** 18–23, 25–32, 37

Assignment Guide

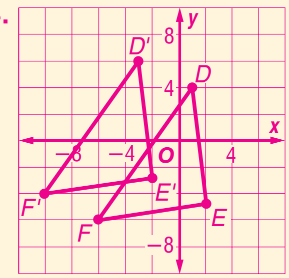
Basic: 12–23, 26–28, 33–36, 42–64

Average: 12–23, 25–32, 35–37, 42–64

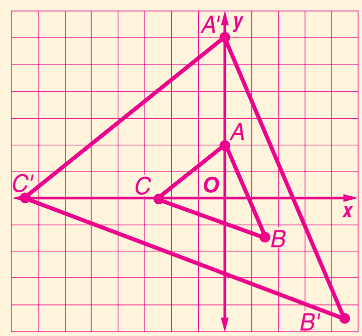
Advanced: 12–24, 26–32, 38–58 (optional: 59–64)

Answers

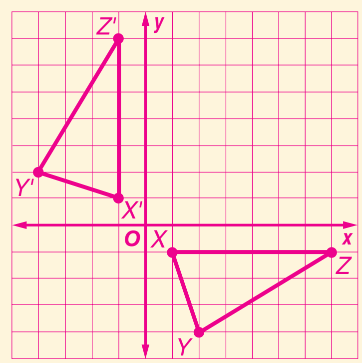
14.



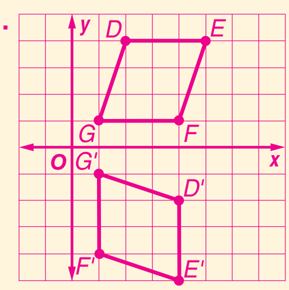
17.



20.



23.



Lesson 4-4 Transformations with Matrices 179

Study Guide and Intervention, p. 187 (shown) and p. 188

Translations and Dilations Matrices that represent coordinates of points on a plane are useful in describing transformations.

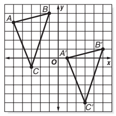
Translation	a transformation that moves a figure from one location to another on the coordinate plane

You can use matrix addition and a translation matrix to find the coordinates of the translated figure.

Dilation	a transformation in which a figure is enlarged or reduced

You can use scalar multiplication to perform dilations.

Example **Find the coordinates of the vertices of the image of $\triangle ABC$ with vertices $A(-5, 4)$, $B(-1, 5)$, and $C(-3, -1)$ if it is moved 6 units to the right and 4 units down. Then graph $\triangle ABC$ and its image $\triangle A'B'C'$.**



Write the vertex matrix for $\triangle ABC$. $\begin{bmatrix} -5 & -1 & -3 \\ 4 & 5 & -1 \end{bmatrix}$

Add the translation matrix $\begin{bmatrix} 6 & 6 & 6 \\ -4 & -4 & -4 \end{bmatrix}$ to the vertex matrix of $\triangle ABC$.

$\begin{bmatrix} -5 & -1 & -3 \\ 4 & 5 & -1 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ -4 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & -5 \end{bmatrix}$

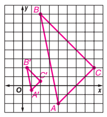
The coordinates of the vertices of $\triangle A'B'C'$ are $A'(1, 0)$, $B'(5, 1)$, and $C'(3, -5)$.

Exercises

For Exercises 1 and 2 use the following information. Quadrilateral *QUAD* with vertices $Q(-1, -3)$, $U(0, 0)$, $A(5, -1)$, and $D(2, -5)$ is translated 3 units to the left and 2 units up.

1. Write the translation matrix. $\begin{bmatrix} -3 & -3 & -3 & -3 \\ 2 & 2 & 2 & 2 \end{bmatrix}$
2. Find the coordinates of the vertices of $Q'U'A'D'$.
   $Q'(-4, -1)$, U', $(-3, 2)$, $A'(2, 1)$, $D'(-1, -3)$

For Exercises 3–5, use the following information. The vertices of $\triangle ABC$ are $A(4, -2)$, $B(2, 8)$, and $C(8, 2)$. The triangle is dilated so that its perimeter is one-fourth the original perimeter.

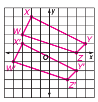


3. Write the coordinates of the vertices of $\triangle ABC$ in a vertex matrix. $\begin{bmatrix} 4 & 2 & 8 \\ -2 & 8 & 2 \end{bmatrix}$
4. Find the coordinates of the vertices of image $\triangle A'B'C'$.
   $A'\left(1, -\frac{1}{2}\right)$, $B'\left(\frac{1}{2}, 2\right)$, $C'\left(2, \frac{1}{2}\right)$
5. Graph the preimage and the image.

Skills Practice, p. 189 and Practice, p. 190 (shown)

For Exercises 1–3, use the following information.

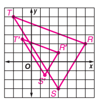
Quadrilateral $WXYZ$ with vertices $W(-3, 2)$, $X(-2, 4)$, $Y(4, 1)$, and $Z(3, 0)$ is translated 1 unit left and 3 units down.



1. Write the translation matrix. $\begin{bmatrix} -1 & -1 & -1 & -1 \\ -3 & -3 & -3 & -3 \end{bmatrix}$
2. Find the coordinates of quadrilateral $W'X'Y'Z'$.
   $W'(-4, -1)$, $X'(-3, 1)$, $Y'(3, -2)$, $Z'(2, -3)$
3. Graph the preimage and the image.

For Exercises 4–6, use the following information.

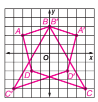
The vertices of $\triangle RST$ are $R(6, 2)$, $S(3, -3)$, and $T(-2, 5)$. The triangle is dilated so that its perimeter is one half the original perimeter.



4. Write the coordinates of $\triangle RST$ in a vertex matrix. $\begin{bmatrix} 6 & 3 & -2 \\ 2 & -3 & 5 \end{bmatrix}$
5. Find the coordinates of the image $\triangle R'S'T'$.
   $R'(3, 1)$, $S'(1.5, -1.5)$, $T'(-1, 2.5)$
6. Graph $\triangle RST$ and $\triangle R'S'T'$.

For Exercises 7–10, use the following information.

The vertices of quadrilateral $ABCD$ are $A(-3, 2)$, $B(0, 3)$, $C(4, -4)$, and $D(-2, -2)$. The quadrilateral is reflected over the y-axis.



7. Write the coordinates of $ABCD$ in a vertex matrix. $\begin{bmatrix} -3 & 0 & 4 & -2 \\ 2 & 3 & -4 & -2 \end{bmatrix}$
8. Write the reflection matrix for this situation. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
9. Find the coordinates of $A'B'C'D'$.
   $A'(3, 2)$, $B'(0, 3)$, $C'(-4, -4)$, $D'(2, -2)$
10. Graph $ABCD$ and $A'B'C'D'$.

11. **ARCHITECTURE** Using architectural design software, the Bradleys plot their kitchen plans on a grid with each unit representing 1 foot. They place the corners of an island at (2, 8), (8, 11), (3, 5), and (9, 8). If the Bradleys wish to move the island 1.5 feet to the right and 2 feet down, what will the new coordinates of its corners be?
    (3.5, 6), (9.5, 9), (4.5, 3), and (10.5, 6)

12. **BUSINESS** The design of a business logo calls for locating the vertices of a triangle at (1.5, 5), (4, 1), and (1, 0) on a grid. If design changes require rotating the triangle 90º counterclockwise, what will the new coordinates of the vertices be?
    (−5, 1.5), (−1, 4), and (0, 1)

Reading to Learn Mathematics, p. 191 ELL

Pre-Activity **How are transformations used in computer animation?**

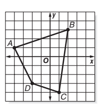
Read the introduction to Lesson 4-4 at the top of page 175 in your textbook.

Describe how you can change the orientation of a figure without changing its size or shape.

Flip (or reflect) the figure over a line.

Reading the Lesson

1. a. Write the vertex matrix for the quadrilateral $ABCD$ shown in the graph at the right.



   $\begin{bmatrix} -4 & 2 & 1 & -2 \\ 1 & 3 & -4 & -3 \end{bmatrix}$

   b. Write the vertex matrix that represents the position of the quadrilateral $A'B'C'D'$ that results when quadrilateral $ABCD$ is translated 3 units to the right and 2 units down.

   $\begin{bmatrix} -1 & 5 & 4 & 1 \\ -1 & 1 & -6 & -5 \end{bmatrix}$

2. Describe the transformation that corresponds to each of the following matrices.

   a. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **counterclockwise rotation about the origin of 180°**

   b. $\begin{bmatrix} 3 & 3 & 3 \\ -4 & -4 & -4 \end{bmatrix}$ **translation 4 units down and 3 units to the right**

   c. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **reflection over the y-axis**

   d. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **reflection over the line $y = x$**

Helping You Remember

3. Describe a way to remember which of the reflection matrices corresponds to reflection over the x-axis.
   Sample answer: The only elements used in the reflection matrices are 0, 1, and −1. For such a 2 × 2 matrix M to have the property that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$, the elements in the top row must be 1 and 0 (in that order), and elements in the bottom row must be 0 and −1 (in that order).

29. $\begin{bmatrix} 4 & -4 & -4 & 4 \\ -4 & -4 & 4 & 4 \end{bmatrix}$

30. $\begin{bmatrix} 4 & -4 & -4 & 4 \\ -4 & -4 & 4 & 4 \end{bmatrix}$

31. $\begin{bmatrix} 4 & 4 & -4 & -4 \\ -4 & 4 & 4 & -4 \end{bmatrix}$

For Exercises 29–32, use rectangle *ABCD* with vertices $A(-4, 4)$, $B(4, 4)$, $C(4, -4)$, and $D(-4, -4)$.

★ **29.** Find the coordinates of the image in matrix form after a reflection over the x-axis followed by a reflection over the y-axis.

30. Find the coordinates of the image after a 180° rotation about the origin.

31. Find the coordinates of the image after a reflection over the line $y = x$.

32. What do you observe about these three matrices? Explain.

LANDSCAPING **For Exercises 33 and 34, use the following information.**
A garden design is plotted on a coordinate grid. The original plan shows a fountain with vertices at (−2, −2), (−6, −2), (−8, −5), and (−4, −5). Changes to the plan now require that the fountain's perimeter be three-fourths that of the original.

33. Determine the new coordinates for the fountain.

34. The center of the fountain was at (−5, −3.5). What will be the coordinate of the center after the changes in the plan have been made? **(−3.75, −2.625)**

TECHNOLOGY **For Exercises 35 and 36, use the following information.**
As you move the mouse for your computer, a corresponding arrow is translated on the screen. Suppose the position of the cursor on the screen is given in inches with the origin at the bottom left-hand corner of the screen. **35. See margin.**

35. You want to move your cursor 3 inches to the right and 4 inches up. Write a translation matrix that can be used to move the cursor to the new position.

36. If the cursor is currently at (3.5, 2.25), what are the coordinates of the position after the translation? **(6.5, 6.25)**

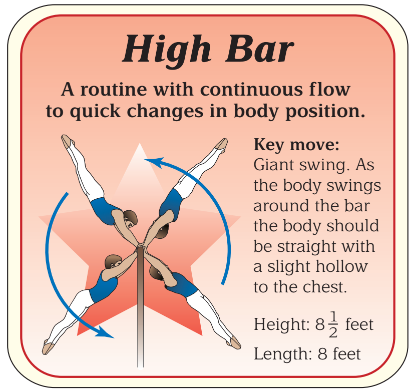
More About. . .

Technology

Douglas Engelbart invented the "X-Y position indicator for a display system" in 1964. He nicknamed this invention "the mouse" because a tail came out the end.

Source: www.about.com

37. **GYMNASTICS** The drawing at the right shows four positions of a man performing the giant swing in the high bar event. Suppose this drawing is placed on a coordinate grid with the hand grips at $H(0, 0)$ and the toe of the figure in the upper right corner at $T(7, 8)$. Find the coordinates of the toes of the other three figures, if each successive figure has been rotated 90° counterclockwise about the origin.
(−8, 7), (−7, −8), and (8, −7)



32. The figures in Exercise 29 and Exercise 30 have the same coordinates, but the figure in Exercise 31 has different coordinates.

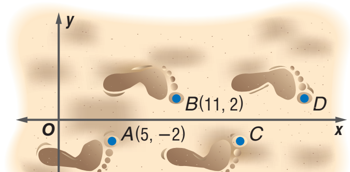
33. (−1.5, −1.5), (−4.5, −1.5), (−6, −3.75), (−3, −3.75)

38. The object is reflected over the x-axis, then translated 6 units to the right.

39. Multiply the coordinates by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then add the result to $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$.

FOOTPRINTS **For Exercises 38–41, use the following information.**
The combination of a reflection and a translation is called a *glide reflection*. An example is a set of footprints.

★ **38.** Describe the reflection and transformation combination shown at the right.



★ **39.** Write two matrix operations that can be used to find the coordinates of point C.

★ **40.** Does it matter which operation you do first? Explain. **See margin.**

★ **41.** What are the coordinates of the next two footprints? **(17, −2), (23, 2)**

Enrichment, p. 192

Properties of Determinants

The following properties often help when evaluating determinants.

- If all the elements of a row (or column) are zero, the value of the determinant is zero.
  $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$ **$(a \cdot 0) - (0 \cdot b) = 0$**
- Multiplying all the elements of a row (or column) by a constant is equivalent to multiplying the value of the determinant by the constant.
  $3\begin{vmatrix} 4 & -1 \\ 5 & 3 \end{vmatrix} = \begin{vmatrix} 12 & -3 \\ 5 & 3 \end{vmatrix}$ **$3[4(3) - 5(-1)] = 3[12 + 5] = 51$**
  $12(3) - 5(-3) = 51$
- If two rows (or columns) have equal corresponding elements, the value of the determinant is zero.

Answer

35. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

42. There is no single matrix to achieve this. However, you could reflect the object over the y-axis and then translate it 2(3) or 6 units to the right.

42. **CRITICAL THINKING** Do you think a matrix exists that would represent a reflection over the line $x = 3$? If so, make a conjecture and verify it.

43. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. See pp. 217A–217B.

How are transformations used in computer animation?

Include the following in your answer:
- an example of how a figure with 5 points (coordinates) could be written in a matrix and multiplied by a rotation matrix, and
- a description of the motion that is a result of repeated dilations with a scale factor of one-fourth.

Standardized Test Practice

44. Which matrix represents a reflection over the y-axis followed by a reflection over the x-axis? **B**

(A) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (C) $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ (D) none of these

45. Triangle ABC has vertices with coordinates $A(-4, 2)$, $B(-4, -3)$, and $C(3, -2)$. After a dilation, triangle $A'B'C'$ has coordinates $A'(-12, 6)$, $B'(-12, -9)$, and $C'(9, -6)$. How many times as great is the perimeter of $A'B'C'$ as ABC? **A**

(A) 3 (B) 6 (C) 12 (D) $\frac{1}{3}$

Maintain Your Skills

Mixed Review

Determine whether each matrix product is defined. If so, state the dimensions of the product. *(Lesson 4-3)*

46. $A_{2 \times 3} \cdot B_{3 \times 2}$ **2 × 2**
47. $A_{4 \times 1} \cdot B_{2 \times 1}$ **undefined**
48. $A_{2 \times 5} \cdot B_{5 \times 5}$ **2 × 5**

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. *(Lesson 4-2)*

49. $2\begin{bmatrix} 4 & 9 & -8 \\ 6 & -11 & -2 \\ 12 & -10 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

50. $4\begin{bmatrix} 3 & 4 & -7 \\ 6 & -9 & -2 \\ -3 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -8 & 6 & -4 \\ -7 & 10 & 1 \\ -2 & 1 & 5 \end{bmatrix}$

49. $\begin{bmatrix} 11 & 24 & -7 \\ 18 & -13 & 8 \\ 33 & -8 & 21 \end{bmatrix}$

50. $\begin{bmatrix} 20 & 10 & -24 \\ 31 & -46 & -9 \\ -10 & 3 & 7 \end{bmatrix}$

Graph each relation or equation and find the domain and range. Then determine whether the relation or equation is a function. *(Lesson 2-1)*

51. $(3, 5), (4, 6), (5, -4)$
52. $x = -5y + 2$
53. $x = y^2$

51–53. See margin for graphs.

51. D = {3, 4, 5}, R = {−4, 5, 6}; yes
52. D = {all real numbers}, R = {all real numbers}; yes
53. D = $\{x \mid x \geq 0\}$, R = {all real numbers}; no

Write an absolute value inequality for each graph. *(Lesson 1-6)*

54. 

55. 

56. 

$|x + 1| > 2$

57. 

$|x - 1| < 1$

54. $|x| \geq 4$
55. $|x| < 2.8$

58. **BUSINESS** Reliable Rentals rents cars for \$12.95 per day plus 15¢ per mile. Luis Romero works for a company that limits expenses for car rentals to \$90 per day. What is the maximum number of miles that Mr. Romero can drive each day? *(Lesson 1-5)* $513\frac{2}{3}$ **mi**

Getting Ready for the Next Lesson

BASIC SKILL Use cross products to solve each proportion.

59. $\frac{x}{8} = \frac{3}{4}$ **6**
60. $\frac{4}{20} = \frac{1}{m}$ **5**
61. $\frac{2}{3} = \frac{a}{42}$ **28**
62. $\frac{5}{6} = \frac{k}{4}$ $\frac{10}{3}$
63. $\frac{2}{y} = \frac{8}{9}$ $\frac{9}{4}$
64. $\frac{x}{5} = \frac{x+1}{8}$ $\frac{5}{3}$

4 Assess

Open-Ended Assessment

Writing Have students write steps or procedures that they can use to translate, dilate, reflect, and rotate a polygon using matrices as described in this lesson.

Assessment Options

Quiz (Lessons 4-3 and 4-4) is available on p. 231 of the *Chapter 4 Resource Masters*.

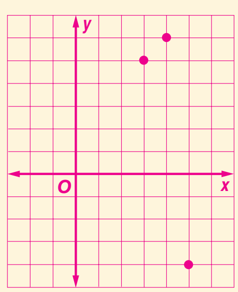
Mid-Chapter Test (Lessons 4-1 through 4-4) is available on p. 233 of the *Chapter 4 Resource Masters*.

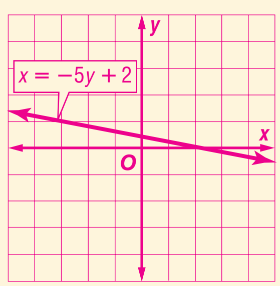
Getting Ready for Lesson 4-5

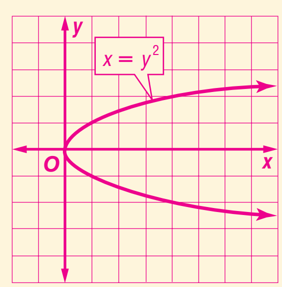
BASIC SKILL Lesson 4-5 shows how to evaluate the determinant of a 2 × 2 matrix and the determinant of a 3 × 3 matrix. Students will use their familiarity with finding cross products as they compute the determinant. Exercises 59–64 should be used to determine your students' familiarity with using cross products.

Answers

40. No; since the translation does not change the y-coordinate, it does not matter whether you do the translation or the reflection over the x-axis first. However, if the translation did change the y-coordinate, then order would be important.

51. 

52. 

53. 

4-5 Lesson Notes

1 Focus

5-Minute Check Transparency 4-5 Use as a quiz or review of Lesson 4-4.

Mathematical Background notes are available for this lesson on p. 152D.

How are determinants used to find areas of polygons?

Ask students:

- What method do you already know for finding the area of a triangle? **the formula $A = \frac{1}{2}bh$**
- Why would it be difficult to use this method in this situation? **The height of the triangle would be difficult to find.**

4-5 Determinants

What You'll Learn

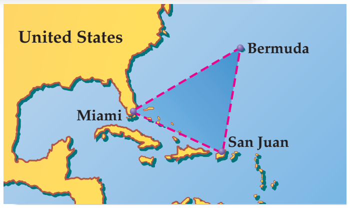
- Evaluate the determinant of a 2×2 matrix.
- Evaluate the determinant of a 3×3 matrix.

Vocabulary

- determinant
- second-order determinant
- third-order determinant
- expansion by minors
- minor

How are determinants used to find areas of polygons?

The "Bermuda Triangle" is an area located off the southeastern Atlantic coast of the United States that is noted for a high incidence of unexplained losses of ships, small boats, and aircraft. You can estimate the area of this triangular region by finding the determinant of the matrix that contains the coordinates of the vertices of the triangle.



DETERMINANTS OF 2 × 2 MATRICES Every square matrix has a number associated with it called its determinant. A **determinant** is a square array of numbers or variables enclosed between two parallel lines. For example, the determinant of $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ can be represented by $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$ or det $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$.

The determinant of a 2×2 matrix is called a **second-order determinant**.

Key Concept — *Second-Order Determinant*

- **Words** The value of a second-order determinant is found by calculating the difference of the products of the two diagonals.
- **Symbols** $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example 1 *Second-Order Determinant*

Find the value of each determinant.

a. $\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix}$

$\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix} = (-2)(8) - 5(6)$ Definition of determinant

$= -16 - 30$ Multiply.

$= -46$ Simplify.

b. $\begin{vmatrix} 7 & 4 \\ -3 & 2 \end{vmatrix}$

$\begin{vmatrix} 7 & 4 \\ -3 & 2 \end{vmatrix} = (7)(2) - 4(-3)$ Definition of determinant

$= 14 - (-12)$ Multiply.

$= 26$ Simplify.

Study Tip

Reading Math
The term *determinant* is often used to mean the *value* of the determinant.

Resource Manager

Workbook and Reproducible Masters

Chapter 4 Resource Masters

- Study Guide and Intervention, pp. 193–194
- Skills Practice, p. 195
- Practice, p. 196
- Reading to Learn Mathematics, p. 197
- Enrichment, p. 198

Graphing Calculator and Spreadsheet Masters, p. 34

School-to-Career Masters, p. 7

Transparencies

5-Minute Check Transparency 4-5
Answer Key Transparencies

Technology

Interactive Chalkboard

DETERMINANTS OF 3 × 3 MATRICES

Determinants of 3 × 3 matrices are called **third-order determinants**. One method of evaluating third-order determinants is **expansion by minors**. The **minor** of an element is the determinant formed when the row and column containing that element are deleted.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{The minor of } a_1 \text{ is } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{The minor of } b_1 \text{ is } \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{The minor of } c_1 \text{ is } \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

To use expansion by minors with third-order determinants, each member of one row is multiplied by its minor and its *position sign*, and the results are added together. The position signs alternate between positive and negative, beginning with a positive sign in the first row, first column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Key Concept — *Third-Order Determinant*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The definition of third-order determinants shows an expansion using the elements in the first row of the determinant. However, any row can be used.

Example 2 *Expansion by Minors*

Evaluate $\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix}$ **using expansion by minors.**

Decide which row of elements to use for the expansion. For this example, we will use the first row.

$$\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix} = 2\begin{vmatrix} 5 & -4 \\ 9 & 0 \end{vmatrix} - 7\begin{vmatrix} -1 & -4 \\ 6 & 0 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 5 \\ 6 & 9 \end{vmatrix} \quad \text{Expansion by minors}$$

$$= 2(0 - (-36)) - 7(0 - (-24)) - 3(-9 - 30) \quad \text{Evaluate } 2 \times 2 \text{ determinants.}$$

$$= 2(36) - 7(24) - 3(-39)$$

$$= 72 - 168 + 117 \quad \text{Multiply.}$$

$$= 21 \quad \text{Simplify.}$$

Another method for evaluating a third-order determinant is by using diagonals.

Step 1 Begin by writing the first two columns on the right side of the determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \Rightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

(continued on the next page)

www.algebra2.com/extra_examples

2 Teach

DETERMINANTS OF 2 × 2 MATRICES

In-Class Example Power Point®

1 Find the value of each determinant.

a. $\begin{vmatrix} 6 & 4 \\ -1 & 0 \end{vmatrix}$ **4**

b. $\begin{vmatrix} -6 & 7 \\ -9 & 3 \end{vmatrix}$ **45**

DETERMINANTS OF 3 × 3 MATRICES

In-Class Example Power Point®

Teaching Tip When using expansion by minors for the first time, urge students to write down each step in the procedure. Then have them compare their work with a classmate to find any errors in either their calculations or their use of the procedure.

2 Evaluate $\begin{vmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 4 & -2 & -3 \end{vmatrix}$ using expansion by minors. **9**

DAILY INTERVENTION — Unlocking Misconceptions

Not All Matrices Have a Determinant Discuss with students what the dimensions of a matrix must be in order for it to have a determinant. Stress that not every matrix has a determinant. Compare these restrictions to those relating to whether or not two matrices can be multiplied.

In-Class Example

PowerPoint®

3 Evaluate $\begin{vmatrix} 3 & -2 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & -3 \end{vmatrix}$ using diagonals. **−8**

Step 2 Next, draw diagonals from each element of the top row of the determinant downward to the right. Find the product of the elements on each diagonal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \Rightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \quad aei \quad bfg \quad cdh$$

Then, draw diagonals from the elements in the third row of the determinant upward to the right. Find the product of the elements on each diagonal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \Rightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \quad gec \quad hfa \quad idb$$

Step 3 To find the value of the determinant, add the products of the first set of diagonals and then subtract the products of the second set of diagonals. The sum is $aei + bfg + cdh - gec - hfa - idb$.

Example 3 Use Diagonals

Evaluate $\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix}$ **using diagonals.**

Step 1 Rewrite the first two columns to the right of the determinant.

$$\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ 4 & -2 \\ 0 & -5 \end{matrix}$$

Step 2 Find the products of the elements of the diagonals.

$$\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ 4 & -2 \\ 0 & -5 \end{matrix} \quad 4 \quad 0 \quad 60$$

$$0 \quad -5 \quad 24 \quad \begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ 4 & -2 \\ 0 & -5 \end{matrix}$$

Step 3 Add the bottom products and subtract the top products.
$4 + 0 + 60 - 0 - (-5) - 24 = 45$

The value of the determinant is 45.

One very useful application of determinants is finding the areas of polygons. The formula below shows how determinants can be used to find the area of a triangle using the coordinates of the vertices.

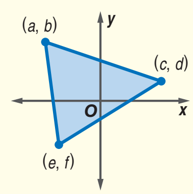
Study Tip

Area Formula
Notice that it is necessary to use the absolute value of A to guarantee a nonnegative value for the area.

Key Concept — Area of a Triangle

The area of a triangle having vertices at (a, b), (c, d), and (e, f) is $|A|$, where

$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}.$$

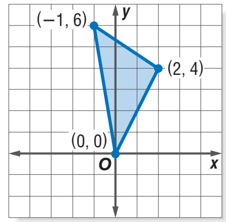


DAILY INTERVENTION — Differentiated Instruction

Interpersonal Organize students into small groups so that those for whom the kind of detailed work involved in finding determinants is easier can work with those who are having trouble carrying through the process to get the correct answer. Ask students to act as coaches to those who are having difficulties, helping them find the sources of their difficulties.

Example 4 Area of a Triangle

GEOMETRY **Find the area of a triangle whose vertices are located at (−1, 6), (2, 4), and (0, 0).**



Assign values to a, b, c, d, e, and f and substitute them into the Area Formula. Then evaluate.

$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix} \quad \text{Area Formula}$$

$$= \frac{1}{2}\begin{vmatrix} -1 & 6 & 1 \\ 2 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} \quad (a, b) = (-1, 6), (c, d) = (2, 4), (e, f) = (0, 0)$$

$$= \frac{1}{2}\left[-1\begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} - 6\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix}\right] \quad \text{Expansion by minors}$$

$$= \frac{1}{2}[-1(4 - 0) - 6(2 - 0) + 1(0 - 0)] \quad \text{Evaluate } 2 \times 2 \text{ determinants.}$$

$$= \frac{1}{2}[-4 - 12 + 0] \quad \text{Multiply.}$$

$$= \frac{1}{2}[-16] \text{ or } -8 \quad \text{Simplify.}$$

Remember that the area of a triangle is the absolute value of A. Thus, the area is $|-8|$ or 8 square units.

In-Class Example

Power Point®

4 Find the area of a triangle whose vertices are located at (0, −1), (−2, −6), and (3, −2). **8.5 units^2**

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms in this lesson to their Vocabulary Builder worksheets for Chapter 4.
- write the formula for evaluating a second-order determinant.
- include their own examples for both methods of finding the determinant of a 3 × 3 matrix.
- write the determinant formula for the area of a triangle.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

Check for Understanding

Concept Check

1. **OPEN ENDED** Write a matrix whose determinant is zero.
2. **FIND THE ERROR** Khalid and Erica are finding the determinant of $\begin{bmatrix} 8 & 3 \\ -5 & 2 \end{bmatrix}$.

Khalid	Erica
$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - (-15)$ $= 31$	$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - 15$ $= 1$

Who is correct? Explain your reasoning.

3. **Explain** why $\begin{bmatrix} 2 & 1 & 7 \\ 3 & -5 & 0 \end{bmatrix}$ does not have a determinant.
4. **Find a counterexample** to disprove the following statement.
   Two different matrices can never have the same determinant.
5. **Describe** how to find the minor of 6 in $\begin{bmatrix} 5 & 11 & 7 \\ -1 & 3 & 8 \\ 6 & 0 & -2 \end{bmatrix}$. **See margin.**
6. **Show** that the value of $\begin{vmatrix} -2 & 3 & 5 \\ 0 & -1 & 4 \\ 9 & 7 & 2 \end{vmatrix}$ is the same whether you use expansion by minors or diagonals. **See margin.**

1. Sample answer: $\begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix}$

2. Khalid; the value of the determinant is the difference of the products of the diagonals.

3. It is not a square matrix.

4. Sample answer: $\begin{bmatrix} 3 & 1 \\ 6 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix}$

Guided Practice

Find the value of each determinant.

7. $\begin{vmatrix} 7 & 8 \\ 3 & -2 \end{vmatrix}$ **−38**
8. $\begin{vmatrix} -3 & -6 \\ 4 & 8 \end{vmatrix}$ **0**
9. $\begin{vmatrix} 0 & 8 \\ 5 & 9 \end{vmatrix}$ **−40**

DAILY INTERVENTION

FIND THE ERROR

To avoid the mistake made by Erica, suggest that students begin finding a determinant by identifying the values of a, b, c, and d in the formula for a second-order determinant. Stress that the value of b is negative in this problem.

Answers

5. Cross out the column and row that contains 6. The minor is the remaining 2 × 2 matrix.

6. $\begin{vmatrix} -2 & 3 & 5 \\ 0 & -1 & 4 \\ 9 & 7 & 2 \end{vmatrix} = -2\begin{vmatrix} -1 & 4 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ 9 & 2 \end{vmatrix} + 5\begin{vmatrix} 0 & -1 \\ 9 & 7 \end{vmatrix}$

$= -2(-2 - 28) - 3(0 - 36) + 5(0 - (-9))$

$= -2(-30) - 3(-36) + 5(9)$

$= 60 + 108 + 45$

$= 213$

$\begin{vmatrix} -2 & 3 & 5 \\ 0 & -1 & 4 \\ 9 & 7 & 2 \end{vmatrix}\begin{matrix} -2 & 3 \\ 0 & -1 \\ 9 & 7 \end{matrix}$ with products 4, 108, 0

$\begin{vmatrix} -2 & 3 & 5 \\ 0 & -1 & 4 \\ 9 & 7 & 2 \end{vmatrix}\begin{matrix} -2 & 3 \\ 0 & -1 \\ 9 & 7 \end{matrix}$ with products −45, −56, 0

$4 + 108 + 0 - (-45) - (-56) - 0 = 213$

About the Exercises…

Organization by Objective

- **Determinants of 2 × 2 Matrices:** 15–26, 39
- **Determinants of 3 × 3 Matrices:** 27–38, 40–45

Odd/Even Assignments

Exercises 15–46 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Alert! Exercises 50–55 require a graphing calculator.

Assignment Guide

Basic: 15–23 odd, 27–35 odd, 45–49, 56–75

Average: 15–43 odd, 45–49, 56–75 (optional: 50–55)

Advanced: 16–44 even, 45–69 (optional: 70–75)

Answers

46. Multiply each member in the top row by its minor and position sign. In this case the minor is a 3 × 3 matrix. Evaluate the 3 × 3 matrix using expansion by minors again.

47. If you know the coordinates of the vertices of a triangle, you can use a determinant to find the area. This is convenient since you don't need to know any additional information such as the measure of the angles. Answers should include the following.
    - You could place a coordinate grid over a map of the Bermuda Triangle with one vertex at the origin. By using the scale of the map, you could determine coordinates to represent the other two vertices and use a determinant to estimate the area.
    - The determinant method is advantageous since you don't need to physically measure the lengths of each side or the measure of the angles between the vertices.

GUIDED PRACTICE KEY	
Exercises	Examples
7–9	1
10, 11	2
12, 13	3
14	4

Evaluate each determinant using expansion by minors.

10. $\begin{vmatrix} 0 & -4 & 0 \\ 3 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix}$ **−28**

11. $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & 2 & 8 \end{vmatrix}$ **−43**

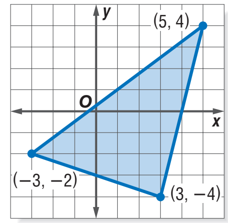
Evaluate each determinant using diagonals.

12. $\begin{vmatrix} 1 & 6 & 4 \\ -2 & 3 & 1 \\ 1 & 6 & 4 \end{vmatrix}$ **0**

13. $\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix}$ **45**

Application

14. GEOMETRY Find the area of the triangle shown at the right. **26 units^2**



★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
15–26, 39	1
27–32, 40	2
33–38	3
41–44	4

Extra Practice

See page 835.

Find the value of each determinant.

15. $\begin{vmatrix} 10 & 6 \\ 5 & 5 \end{vmatrix}$ **20**

16. $\begin{vmatrix} 8 & 5 \\ 6 & 1 \end{vmatrix}$ **−22**

17. $\begin{vmatrix} -7 & 3 \\ -9 & 7 \end{vmatrix}$ **−22**

18. $\begin{vmatrix} -2 & 4 \\ 3 & -6 \end{vmatrix}$ **0**

19. $\begin{vmatrix} 2 & -7 \\ -5 & 3 \end{vmatrix}$ **−29**

20. $\begin{vmatrix} -6 & -2 \\ 8 & 5 \end{vmatrix}$ **−14**

21. $\begin{vmatrix} -9 & 0 \\ -12 & -7 \end{vmatrix}$ **63**

22. $\begin{vmatrix} 6 & 14 \\ -3 & -8 \end{vmatrix}$ **−6**

23. $\begin{vmatrix} 15 & 11 \\ 23 & 19 \end{vmatrix}$ **32**

24. $\begin{vmatrix} 21 & 43 \\ 16 & 31 \end{vmatrix}$ **−37**

★ **25.** $\begin{vmatrix} 7 & 5.2 \\ -4 & 1.6 \end{vmatrix}$ **32**

★ **26.** $\begin{vmatrix} -3.2 & -5.8 \\ 4.1 & 3.9 \end{vmatrix}$ **11.3**

Evaluate each determinant using expansion by minors.

27. $\begin{vmatrix} 3 & 1 & 2 \\ 0 & 6 & 4 \\ 2 & 5 & 1 \end{vmatrix}$ **−58**

28. $\begin{vmatrix} 7 & 3 & -4 \\ -2 & 9 & 6 \\ 0 & 0 & 0 \end{vmatrix}$ **0**

29. $\begin{vmatrix} -2 & 7 & -2 \\ 4 & 5 & 2 \\ 1 & 0 & -1 \end{vmatrix}$ **62**

30. $\begin{vmatrix} -3 & 0 & 6 \\ 6 & 5 & -2 \\ 1 & 4 & 2 \end{vmatrix}$ **60**

31. $\begin{vmatrix} 1 & 5 & -4 \\ -7 & 3 & 2 \\ 6 & 3 & -1 \end{vmatrix}$ **172**

32. $\begin{vmatrix} 3 & 7 & 6 \\ -1 & 6 & 2 \\ 8 & -3 & -5 \end{vmatrix}$ **−265**

Evaluate each determinant using diagonals.

33. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 9 & 5 \\ 8 & 7 & 4 \end{vmatrix}$ **−22**

34. $\begin{vmatrix} 1 & 5 & 2 \\ -6 & -7 & 8 \\ 5 & 9 & -3 \end{vmatrix}$ **21**

35. $\begin{vmatrix} 8 & -9 & 0 \\ 1 & 5 & 4 \\ 6 & -2 & 3 \end{vmatrix}$ **−5**

36. $\begin{vmatrix} 4 & 10 & 7 \\ 3 & 3 & 1 \\ 0 & 5 & 2 \end{vmatrix}$ **49**

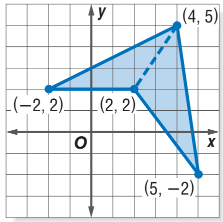
★ **37.** $\begin{vmatrix} 2 & -3 & 4 \\ -2 & 1 & 5 \\ 5 & 3 & -2 \end{vmatrix}$ **−141**

★ **38.** $\begin{vmatrix} 4 & -2 & 3 \\ -2 & 3 & 4 \\ 3 & 4 & 2 \end{vmatrix}$ **−123**

★ **39.** Solve for x if $\det \begin{bmatrix} 2 & x \\ 5 & -3 \end{bmatrix} = 24$. **−6**

★ **40.** Solve $\det \begin{bmatrix} 4 & x & -2 \\ -x & -3 & 1 \\ -6 & 2 & 3 \end{bmatrix} = -3$ for x. **$\frac{5}{3}$, −1**

★ **41.** **GEOMETRY** Find the area of the polygon shown at the right. **14.5 units^2**



★ **42.** **GEOMETRY** Find the value of x such that the area of a triangle whose vertices have coordinates (6, 5), (8, 2), and (x, 11) is 15 square units. **12**

Career Choices

Archaeologist

Archaeologists attempt to reconstruct past ways of life by examining preserved bones, the ruins of buildings, and artifacts such as tools, pottery, and jewelry.

Source: www.encarta.msn.com

Online Research
For information about a career as an archaeologist, visit: www.algebra2.com/careers

★ **43.** **ARCHAEOLOGY** During an archaeological dig, a coordinate grid is laid over the site to identify the location of artifacts as they are excavated. During a dig, three corners of a rectangular building have been partially unearthed at (−1, 6), (4, 5), and (−3, −4). If each square on the grid measures one square foot, estimate the area of the floor of the building. **about 52 ft^2**

★ **44.** **GEOGRAPHY** Mr. Cardona is a regional sales manager for a company in Florida. Tampa, Orlando, and Ocala outline his region. If a coordinate grid in which 1 unit = 10 miles is placed over the map of Florida with Tampa at the origin, the coordinates of the three cities are (0, 0), (7, 5), and (2.5, 10). Use a determinant to estimate the area of his sales territory. **2875 mi^2**

45. **CRITICAL THINKING** Find a third-order determinant in which no element is 0, but for which the determinant is 0. **Sample answer:** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

46. **CRITICAL THINKING** Make a conjecture about how you could find the determinant of a 4 × 4 matrix using the expansion by minors method. Use a diagram in your explanation. **See margin.**

47. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How are determinants used to find areas of polygons?

Include the following in your answer:

- an explanation of how you could use a coordinate grid to estimate the area of the Bermuda Triangle, and
- some advantages of using this method in this situation.

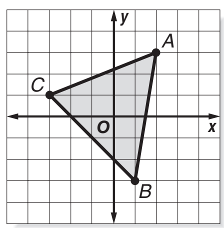
Standardized Test Practice

48. Find the value of det A if $A = \begin{bmatrix} 0 & 3 & -2 \\ -4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$. **C**

(A) 0 (B) 12 (C) 25 (D) 36

49. Find the area of triangle ABC. **C**

(A) 10 units^2
(B) 12 units^2
(C) 14 units^2
(D) 16 units^2
(E) none of these



www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 193 (shown) and p. 194

Determinants of 2 × 2 Matrices

Second-Order Determinant	For the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Example Find the value of each determinant.

a. $\begin{vmatrix} 6 & 3 \\ -8 & 5 \end{vmatrix}$

$\begin{vmatrix} 6 & 3 \\ -8 & 5 \end{vmatrix} = 6(5) - 3(-8)$
$= 30 - (-24)$ or 54

b. $\begin{vmatrix} 11 & -5 \\ 9 & 3 \end{vmatrix}$

$\begin{vmatrix} 11 & -5 \\ 9 & 3 \end{vmatrix} = 11(-3) - (-5)(9)$
$= -33 - (-45)$ or 12

Exercises

Find the value of each determinant.

1. $\begin{vmatrix} 6 & -2 \\ 5 & 7 \end{vmatrix}$ **52** **2.** $\begin{vmatrix} -8 & 3 \\ -2 & 1 \end{vmatrix}$ **−2** **3.** $\begin{vmatrix} 3 & 9 \\ 4 & 6 \end{vmatrix}$ **−18**

4. $\begin{vmatrix} 5 & 12 \\ -7 & -4 \end{vmatrix}$ **64** **5.** $\begin{vmatrix} -6 & -3 \\ -4 & -1 \end{vmatrix}$ **−6** **6.** $\begin{vmatrix} 4 & 7 \\ 5 & 9 \end{vmatrix}$ **1**

7. $\begin{vmatrix} 14 & 8 \\ 9 & -3 \end{vmatrix}$ **−114** **8.** $\begin{vmatrix} 15 & 12 \\ 23 & 28 \end{vmatrix}$ **144** **9.** $\begin{vmatrix} -8 & 35 \\ 5 & 20 \end{vmatrix}$ **−335**

10. $\begin{vmatrix} 10 & 16 \\ 22 & 40 \end{vmatrix}$ **48** **11.** $\begin{vmatrix} 24 & -8 \\ 7 & -3 \end{vmatrix}$ **−16** **12.** $\begin{vmatrix} 13 & 62 \\ -4 & 19 \end{vmatrix}$ **495**

13. $\begin{vmatrix} 0.2 & 8 \\ -1.5 & 15 \end{vmatrix}$ **15** **14.** $\begin{vmatrix} 8.6 & 0.5 \\ 14 & 5 \end{vmatrix}$ **36** **15.** $\begin{vmatrix} 20 & 110 \\ 0.1 & 1.4 \end{vmatrix}$ **17**

16. $\begin{vmatrix} 4.8 & 2.1 \\ 3.4 & 5.3 \end{vmatrix}$ **18.3** **17.** $\begin{vmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{6} & \frac{1}{5} \end{vmatrix}$ $\mathbf{\frac{13}{60}}$ **18.** $\begin{vmatrix} 6.8 & 15 \\ -0.2 & 5 \end{vmatrix}$ **37**

Skills Practice, p. 195 and Practice, p. 196 (shown)

Find the value of each determinant.

1. $\begin{vmatrix} 1 & 6 \\ 2 & 7 \end{vmatrix}$ **−5** **2.** $\begin{vmatrix} 9 & 6 \\ 3 & 2 \end{vmatrix}$ **0** **3.** $\begin{vmatrix} 4 & 1 \\ -2 & -5 \end{vmatrix}$ **−18**

4. $\begin{vmatrix} -14 & -3 \\ 2 & -2 \end{vmatrix}$ **34** **5.** $\begin{vmatrix} 4 & -3 \\ -12 & 4 \end{vmatrix}$ **−20** **6.** $\begin{vmatrix} 2 & -5 \\ 5 & -11 \end{vmatrix}$ **3**

7. $\begin{vmatrix} 4 & 0 \\ -2 & 9 \end{vmatrix}$ **36** **8.** $\begin{vmatrix} 3 & -4 \\ 7 & 9 \end{vmatrix}$ **55** **9.** $\begin{vmatrix} -1 & -11 \\ 10 & -2 \end{vmatrix}$ **112**

10. $\begin{vmatrix} 3 & -4 \\ 3.75 & 5 \end{vmatrix}$ **30** **11.** $\begin{vmatrix} 2 & -1 \\ 3 & -9.5 \end{vmatrix}$ **−16** **12.** $\begin{vmatrix} 0.5 & -0.7 \\ 0.4 & -0.3 \end{vmatrix}$ **0.13**

Evaluate each determinant using expansion by minors.

13. $\begin{vmatrix} -2 & 3 & 1 \\ 0 & 4 & -3 \\ 2 & 5 & -1 \end{vmatrix}$ **−48** **14.** $\begin{vmatrix} 2 & -4 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 7 \end{vmatrix}$ **45** **15.** $\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & -1 \end{vmatrix}$ **7**

16. $\begin{vmatrix} 0 & -4 & 0 \\ 2 & -1 & 1 \\ 3 & -2 & 5 \end{vmatrix}$ **28** **17.** $\begin{vmatrix} 2 & 7 & -6 \\ 8 & 4 & 0 \\ 1 & -1 & 3 \end{vmatrix}$ **−72** **18.** $\begin{vmatrix} -12 & 0 & 3 \\ 7 & 5 & -1 \\ 4 & 2 & -6 \end{vmatrix}$ **318**

Evaluate each determinant using diagonals.

19. $\begin{vmatrix} -4 & 3 & -1 \\ 2 & 1 & -2 \\ 4 & 1 & -4 \end{vmatrix}$ **10** **20.** $\begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & 1 \end{vmatrix}$ **12** **21.** $\begin{vmatrix} 1 & -4 & -1 \\ 1 & -6 & -2 \\ 2 & 3 & 1 \end{vmatrix}$ **5**

22. $\begin{vmatrix} 1 & 2 & -4 \\ 1 & 4 & -6 \\ 2 & 3 & 3 \end{vmatrix}$ **20** **23.** $\begin{vmatrix} 2 & -1 & -2 \\ 4 & 0 & -2 \\ 0 & 3 & 2 \end{vmatrix}$ **−4** **24.** $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 8 & 0 \\ 0 & 5 & -1 \end{vmatrix}$ **0**

25. **GEOMETRY** Find the area of a triangle whose vertices have coordinates (3, 5), (6, −5), and (−4, 10). **27.5 units^2**

26. **LAND MANAGEMENT** A fish and wildlife management organization uses a GIS (geographic information system) to store and analyze data for the parcels of land it manages. All of the parcels are mapped on a grid in which 1 unit represents 1 acre. If the coordinates of the corners of a parcel are (−8, 10), (6, 17), and (2, −4), how many acres is the parcel? **133 acres**

Reading to Learn Mathematics, p. 197 ELL

Pre-Activity **How are determinants used to find areas of polygons?**

Read the introduction to Lesson 4-5 at the top of page 182 in your textbook.

In this lesson, you will learn how to find the area of a triangle if you know the coordinates of its vertices using determinants. Describe a method you already know for finding the area of the Bermuda Triangle. **Sample answer: Use the map. Choose any side of the triangle as the base, and measure this side with a ruler. Multiply this length by the scale factor for the map. Next, draw a segment from the opposite vertex perpendicular to the base. Measure this segment, and multiply its length by the scale for the map. Finally, find the area by using the formula $A = \frac{1}{2}bh$.**

Reading the Lesson

1. Indicate whether each of the following statements is *true* or *false*.

a. Every matrix has a determinant. **false**

b. If both rows of a 2 × 2 matrix are identical, the determinant of the matrix will be 0. **true**

c. Every element of a 3 × 3 matrix has a minor. **true**

d. In order to evaluate a third-order determinant by expansion by minors it is necessary to find the minor of every element of the matrix. **false**

e. If you evaluate a third-order determinant by expansion about the second row, the position signs you will use are − + −. **true**

2. Suppose that triangle RST has vertices $R(-2, 5)$, $S(4, 1)$, and $T(0, 6)$.

a. Write a determinant that could be used in finding the area of triangle RST.
$\begin{vmatrix} -2 & 5 & 1 \\ 4 & 1 & 1 \\ 0 & 6 & 1 \end{vmatrix}$

b. Explain how you would use the determinant you wrote in part **a** to find the area of the triangle. **Sample answer: Evaluate the determinant and multiply the result by $\frac{1}{2}$. Then take the absolute value to make sure the final answer is positive.**

Helping You Remember

3. A good way to remember a complicated procedure is to break it down into steps. Write a list of steps for evaluating a third-order determinant using expansion by minors. **Sample answer: 1. Choose a row of the matrix. 2. Find the position signs for the row you have chosen. 3. Find the minor of each element in that row. 4. Multiply each element by its position sign and by its minor. 5. Add the results.**

Enrichment, p. 198

Matrix Multiplication

A furniture manufacturer makes upholstered chairs and wood tables. Matrix A shows the number of hours spent on each item by three different workers. One day the factory receives an order for 10 chairs and 3 tables. This is shown in matrix B.

$$\begin{array}{l} \\ \text{chair} \\ \text{table} \end{array} \begin{array}{c} \text{hours} \\ \begin{array}{ccc} \text{woodworker} & \text{finsher} & \text{upholsterer} \end{array} \\ \begin{bmatrix} 4 & 2 & 12 \\ 18 & 15 & 0 \end{bmatrix} \end{array} = A \qquad \text{number ordered} \begin{array}{c} \begin{array}{cc} \text{chair} & \text{table} \end{array} \\ \begin{bmatrix} 10 & 3 \end{bmatrix} \end{array} = B$$

$$[10 \quad 3]\begin{bmatrix} 4 & 2 & 12 \\ 18 & 15 & 0 \end{bmatrix} = [10(4) + 3(18) \quad 10(2) + 3(15) \quad 10(12) + 3(0)] = [94 \quad 65 \quad 120]$$

The product of the two matrices shows the number of hours needed for each type of worker to complete the order: 94 hours for woodworking, 65 hours for finishing, and 120 hours for upholstering.

To find the total labor cost, multiply by a matrix that shows the hourly rate for each worker: $15 for woodworking, $9 for finishing, and $12 for upholstering.

Lesson 4-5 Determinants 187

4 Assess

Open-Ended Assessment

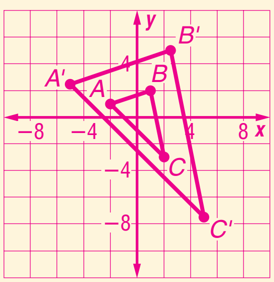
Modeling Have students make posters that show the two processes for finding the determinant of a 3×3 matrix, using colored markers to clearly identify the minors or the diagonals in each procedure.

Getting Ready for Lesson 4-6

PREREQUISITE SKILL Lesson 4-6 presents Cramer's Rule for solving systems of two linear equations. Students will use their familiarity with previous methods for solving systems of equations as they learn this new method. Exercises 70–75 should be used to determine your students' familiarity with solving systems of two equations in two variables.

Answer

58.



Graphing Calculator

MATRX FUNCTION You can use a TI-83 Plus to find determinants of square matrices using the MATRX functions. Enter the matrix under the NAMES menu. Then use the arrow keys to highlight the MATH menu. Choose det(, which is option 1, to calculate the determinant.

Use a graphing calculator to find the value of each determinant.

50. $\begin{vmatrix} 3 & -6.5 \\ 8 & 3.75 \end{vmatrix}$ **63.25**

51. $\begin{vmatrix} 1.3 & 7.2 \\ 6.1 & 5.4 \end{vmatrix}$ **−36.9**

52. $\begin{vmatrix} 6.1 & 4.8 \\ 9.7 & 3.5 \end{vmatrix}$ **−25.21**

53. $\begin{vmatrix} 8 & 6 & -5 \\ 10 & -7 & 3 \\ 9 & 14 & -6 \end{vmatrix}$ **−493**

54. $\begin{vmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{vmatrix}$ **0**

55. $\begin{vmatrix} 10 & 12 & 4 \\ -3 & 18 & -9 \\ 16 & -2 & -1 \end{vmatrix}$ **−3252**

Maintain Your Skills

Mixed Review

For Exercises 56–58, use the following information.
The vertices of $\triangle ABC$ are $A(-2, 1)$, $B(1, 2)$ and $C(2, -3)$. The triangle is dilated so that its perimeter is $2\frac{1}{2}$ times the original perimeter. *(Lesson 4-4)*

56. Write the coordinates of $\triangle ABC$ in a vertex matrix. $\begin{bmatrix} -2 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}$

57. Find the coordinates of $\triangle A'B'C'$. **$A'(-5, 2.5)$, $B'(2.5, 5)$, $C'(5, -7.5)$**

58. Graph $\triangle ABC$ and $\triangle A'B'C'$. **See margin.**

Find each product, if possible. *(Lesson 4-3)*

59. $[5 \quad 2] \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **$[-4]$**

60. $\begin{bmatrix} 2 & 4 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 9 \\ -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 26 \\ -9 & -12 \end{bmatrix}$

61. $\begin{bmatrix} 5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ -4 & 2 \end{bmatrix}$ **undefined**

62. $\begin{bmatrix} 7 & 4 \\ -1 & 2 \\ -3 & 5 \end{bmatrix} \cdot [3 \quad 5]$ **undefined**

63. $[4 \quad 2 \quad 0] \cdot \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 5 & 6 \end{bmatrix}$ **$[14 \quad -8]$**

64. $\begin{bmatrix} 7 & -5 & 4 \\ 6 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ -2 & -8 \\ 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 7 & 69 \\ -5 & 16 \end{bmatrix}$

65. RUNNING The length of a marathon was determined by the first marathon in the 1908 Olympic Games in London, England. The race began at Windsor Castle and ended in front of the royal box at London's Olympic Stadium, which was a distance of 26 miles 385 yards. Determine how many feet the marathon covers using the formula $f(m, y) = 5280m + 3y$, where m is the number of miles and y is the number of yards. *(Lesson 3-4)* **138,435 ft**

Write an equation in slope-intercept form for the line that satisfies each set of conditions. *(Lesson 2-4)* **66. $y = x - 2$ 67. $y = -\frac{4}{3}x$**

66. slope 1 passes through (5, 3)

67. slope $-\frac{4}{3}$ passes through (6, −8)

68. passes through (3, 7) and (−2, −3) **$y = 2x + 1$**

69. passes through (0, 5) and (10, 10) **$y = \frac{1}{2}x + 5$**

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each system of equations.**
*(To review **solving systems of equations**, see Lesson 3-2.)*

70. $x + y = -3$, $3x + 4y = -12$ **(0, −3)**

71. $x + y = 10$, $2x + y = 11$ **(1, 9)**

72. $2x + y = 5$, $4x + y = 9$ **(2, 1)**

73. $3x + 5y = 2$, $2x - y = -3$ **(−1, 1)**

74. $6x + 2y = 22$, $3x + 7y = 41$ **(2, 5)**

75. $3x - 2y = -2$, $4x + 7y = 65$ **(4, 7)**

188 Chapter 4 Matrices

4-6 Cramer's Rule

What You'll Learn

- Solve systems of two linear equations by using Cramer's Rule.
- Solve systems of three linear equations by using Cramer's Rule.

How is Cramer's Rule used to solve systems of equations?

Two sides of a triangle are contained in lines whose equations are $1.4x + 3.8y = 3.4$ and $2.5x - 1.7y = -10.9$. To find the coordinates of the vertex of the triangle between these two sides, you must solve the system of equations. However, solving this system by using substitution or elimination would require many calculations. Another method for solving systems of equations is Cramer's Rule.

Vocabulary

- Cramer's Rule

SYSTEMS OF TWO LINEAR EQUATIONS

Cramer's Rule uses determinants to solve systems of equations. Consider the following system.

$$ax + by = e \quad \text{a, b, c, d, e, and f represent constants, \textit{not} variables.}$$
$$cx + dy = f$$

Solve for x by using elimination.

$$\begin{aligned} adx + bdy &= de && \text{Multiply the first equation by } d.\\ (-)\ bcx + bdy &= bf && \text{Multiply the second equation by } b.\\ adx - bcx &= de - bf && \text{Subtract.}\\ (ad - bc)x &= de - bf && \text{Factor.}\\ x &= \frac{de - bf}{ad - bc} && \text{Notice that } ad - bc \text{ must not be zero.} \end{aligned}$$

Study Tip

Look Back
To review **solving systems of equations**, see Lesson 3-2.

Solving for y in the same way produces the following expression.

$$y = \frac{af - ce}{ad - bc}$$

So the solution of the system of equations $ax + by = e$ and $cx + dy = f$ is $\left(\frac{de - bf}{ad - bc}, \frac{af - ce}{ad - bc}\right)$.

Notice that the denominators for each expression are the same. It can be written using a determinant. The numerators can also be written as determinants.

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad de - bf = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \qquad af - ce = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

Key Concept — *Cramer's Rule for Two Variables*

The solution of the system of linear equations
$$ax + by = e$$
$$cx + dy = f$$
is (x, y), where $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

4-6 Lesson Notes

1 Focus

5-Minute Check Transparency 4-6 Use as a quiz or review of Lesson 4-5.

Mathematical Background notes are available for this lesson on p. 152D.

Building on Prior Knowledge

Students will use their knowledge of determinants from Lesson 4-5 when using Cramer's Rule to solve systems of equations in this lesson.

How is Cramer's rule used to solve systems of equations?

Ask students:

- Why does the text say that the sides of the triangle "are contained in" the two lines whose equations are given? **because lines extend forever while the sides of a triangle are line segments**
- What makes solving the system of these two equations more difficult using substitution or elimination? **The coefficients in the equations are decimals rather than integers.**

Resource Manager

Workbook and Reproducible Masters

Chapter 4 Resource Masters
- Study Guide and Intervention, pp. 199–200
- Skills Practice, p. 201
- Practice, p. 202
- Reading to Learn Mathematics, p. 203
- Enrichment, p. 204
- Assessment, p. 232

Transparencies

5-Minute Check Transparency 4-6
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 7
Interactive Chalkboard

2 Teach

SYSTEMS OF TWO LINEAR EQUATIONS

In-Class Examples Power Point®

1 Use Cramer's Rule to solve the system of equations.
$5x + 4y = 28$
$3x - 2y = 8$ **(4, 2)**

2 **VOTING** In a vote for the school colors of a new high school, blue-and-gold received 440 votes from 10th and 11th graders and red-and-black received 210 votes from 10th and 11th graders. In the 10th grade, blue-and-gold received 72% of the votes, and red-and-black received 28% of the votes. In the 11th grade, blue-and-gold received 64% of the votes and red-and-black received 36%.

a. Write a system of equations that represents the total number of votes cast for each set of colors in these two grades. Let t represent the total number of 10th grade votes and let e represent the total number of 11th grade votes.
$0.72t + 0.64e = 440$
$0.28t + 0.36e = 210$

b. Find the total number of votes cast in the 10th and in the 11th grades. **There were 300 votes cast by 10th graders and 350 votes cast by 11th graders.**

More About. . .

Elections

In 1936, Franklin D. Roosevelt received a record 523 electoral college votes to Alfred M. Landon's 8 votes. This is the largest electoral college majority.

Source: *The Guinness Book of Records*

Example 1 System of Two Equations

Use Cramer's Rule to solve the system of equations.

$5x + 7y = 13$
$2x - 5y = 13$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{Cramer's Rule} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 13 & 7 \\ 13 & -5 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}} \qquad a = 5, b = 7, c = 2, d = -5, e = 13, \text{ and } f = 13 \qquad = \frac{\begin{vmatrix} 5 & 13 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}}$$

$$= \frac{13(-5) - 13(7)}{5(-5) - 2(7)} \qquad \text{Evaluate each determinant.} \qquad = \frac{5(13) - 2(13)}{5(-5) - 2(7)}$$

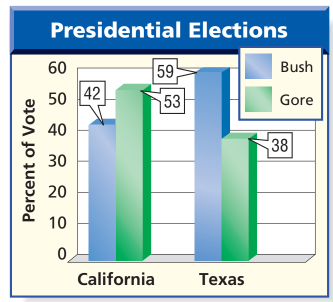
$$= \frac{-156}{-39} \text{ or } 4 \qquad \text{Simplify.} \qquad = \frac{39}{-39} \text{ or } -1$$

The solution is $(4, -1)$.

Cramer's Rule is especially useful when the coefficients are large or involve fractions or decimals.

Example 2 Use Cramer's Rule

ELECTIONS **In the 2000 presidential election, George W. Bush received about 8,400,000 votes in California and Texas while Al Gore received about 8,300,000 votes in those states. The graph shows the percent of the popular vote that each candidate received in those states.**



Source: States' Elections Offices

a. Write a system of equations that represents the total number of votes cast for each candidate in these two states.

Let x represent the total number of votes in California.

Let y represent the total number of votes in Texas.

$0.42x + 0.59y = 8{,}400{,}000$ Votes for Bush

$0.53x + 0.38y = 8{,}300{,}000$ Votes for Gore

b. Find the total number of popular votes cast in California and in Texas.

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{Cramer's Rule} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 8{,}400{,}000 & 0.59 \\ 8{,}300{,}000 & 0.38 \end{vmatrix}}{\begin{vmatrix} 0.42 & 0.59 \\ 0.53 & 0.38 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 0.42 & 8{,}400{,}000 \\ 0.53 & 8{,}300{,}000 \end{vmatrix}}{\begin{vmatrix} 0.42 & 0.59 \\ 0.53 & 0.38 \end{vmatrix}}$$

$$= \frac{8{,}400{,}000(0.38) - 8{,}300{,}000(0.59)}{0.42(0.38) - 0.53(0.59)} \qquad = \frac{0.42(8{,}300{,}000) - 0.53(8{,}400{,}000)}{0.42(0.38) - 0.53(0.59)}$$

$$= \frac{-1{,}705{,}000}{-0.1531} \qquad = \frac{-966{,}000}{-0.1531}$$

$$\approx 11{,}136{,}512.08 \qquad \approx 6{,}309{,}601.57$$

The solution of the system is about (11,136,512.08, 6,309,601.57).

So, there were about 11,100,000 popular votes cast in California and about 6,300,000 popular votes cast in Texas.

SYSTEMS OF THREE LINEAR EQUATIONS

You can also use Cramer's Rule to solve a system of three equations in three variables.

TEACHING TIP

Point out that the numerator for x is found by replacing the coefficients of x with the constant terms of the system. Similarly, the numerators for y and z are found by replacing the coefficients of y and z, respectively, with the constant terms of the system.

Key Concept — *Cramer's Rule for Three Variables*

The solution of the system whose equations are

$$ax + by + cz = j$$
$$dx + ey + fz = k$$
$$gx + hy + iz = \ell$$

is (x, y, z), where $x = \dfrac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $z = \dfrac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, and $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0$.

Example 3 System of Three Equations

Use Cramer's Rule to solve the system of equations.

$$3x + y + z = -1$$
$$-6x + 5y + 3z = -9$$
$$9x - 2y - z = 5$$

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -1 & 1 & 1 \\ -9 & 5 & 3 \\ 5 & -2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 3 & -1 & 1 \\ -6 & -9 & 3 \\ 9 & 5 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 3 & 1 & -1 \\ -6 & 5 & -9 \\ 9 & -2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}}$$

Use a calculator to evaluate each determinant.

$x = \frac{-2}{-9}$ or $\frac{2}{9}$ $\qquad$ $y = \frac{12}{-9}$ or $-\frac{4}{3}$ $\qquad$ $z = \frac{3}{-9}$ or $-\frac{1}{3}$

The solution is $\left(\frac{2}{9}, -\frac{4}{3}, -\frac{1}{3}\right)$.

www.algebra2.com/extra_examples

SYSTEMS OF THREE LINEAR EQUATIONS

In-Class Example Power Point®

3 Use Cramer's Rule to solve the system of equations.

$$2x - 3y + z = 5$$
$$x + 2y + z = -1$$
$$x - 3y + 2z = 1 \quad \left(\frac{9}{4}, -\frac{3}{4}, -\frac{7}{4}\right)$$

Teaching Tip Ask students to compare this method for speed and ease of use to other methods they could use.

DAILY INTERVENTION

Differentiated Instruction

Intrapersonal Ask students to write a brief reflection on their reaction to the various methods they have learned involving the manipulation of matrices. Ask them to comment on what aspects they found efficient and helpful, and what aspects they found to be difficult or confusing.

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms in this lesson to their Vocabulary Builder worksheets for Chapter 4.
- write Cramer's Rule for two variables and for three variables, including examples of how to use both rules.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises…

Organization by Objective
- **Systems of Two Linear Equations:** 12–25, 32–35
- **Systems of Three Linear Equations:** 26–31, 36, 37

Odd/Even Assignments
Exercises 12–31 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 13–19 odd, 25, 27, 29, 32, 33, 38–54

Average: 13–31 odd, 32–35, 38–54

Advanced: 12–30 even, 34–51 (optional: 52–54)

All: Practice Quiz 2 (1–10)

Check for Understanding

Concept Check

1. The determinant of the coefficient matrix cannot be zero.

1. **Describe** the condition that must be met in order to use Cramer's Rule.
2. **OPEN ENDED** Write a system of equations that *cannot* be solved using Cramer's Rule. **Sample answer: $2x + y = 5$ and $6x + 3y = 8$**
3. **Write** a system of equations whose solution is $x = \dfrac{\begin{vmatrix} -6 & 5 \\ 30 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} 3 & -6 \\ 4 & 30 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}}$. **$3x + 5y = -6$, $4x - 2y = 30$**

Guided Practice

GUIDED PRACTICE KEY

Exercises	Examples
4–6	1
7–9	3
10, 11	2

Use Cramer's Rule to solve each system of equations.

4. $x - 4y = 1$, $2x + 3y = 13$ **(5, 1)**
5. $0.2a = 0.3b$, $0.4a - 0.2b = 0.2$ **(0.75, 0.5)**
6. $\frac{1}{2}r - \frac{2}{3}s = 2\frac{1}{3}$, $\frac{3}{5}r + \frac{4}{5}s = -10$ **(−6, −8)**
7. $2x - y + 3z = 5$, $3x + 2y - 5z = 4$, $x - 4y + 11z = 3$ **no solution**
8. $a + 9b - 2c = 2$, $-a - 3b + 4c = 1$, $2a + 3b - 6c = -5$
9. $r + 4s + 3t = 10$, $2r - 2s + t = 15$, $r + 2s - 3t = -1$

Application

8. $\left(-5, \frac{2}{3}, -\frac{1}{2}\right)$
9. $\left(6, -\frac{1}{2}, 2\right)$

INVESTING **For Exercises 10 and 11, use the following information.**
Jarrod Wright has \$4000 he would like to invest so that he can earn some interest on it. He has discovered that he could put it in a savings account paying 6.5% interest annually, or in a certificate of deposit with an annual rate of 8%. He wants his interest for the year to be \$297.50, because earning more than this would put him into a higher tax bracket.

10. Write a system of equations, in which the unknowns s and d stand for the amounts of money Jarrod should deposit in the savings account and the certificate of deposit, respectively. **$s + d = 4000$, $0.065s + 0.08d = 297.50$**
11. How much should he put in a savings account, and how much should he put in the certificate of deposit? **savings account, \$1500; certificate of deposit, \$2500**

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
12–25	1
26–31	3
32–37	2

Extra Practice
See page 835.

Use Cramer's Rule to solve each system of equations.

12. $5x + 2y = 8$, $2x - 3y = 7$ **(2, −1)**
13. $2m + 7n = 4$, $m - 2n = -20$ **(−12, 4)**
14. $2r - s = 1$, $3r + 2s = 19$ **(3, 5)**
15. $3a + 5b = 33$, $5a + 7b = 51$ **(6, 3)**
16. $2m - 4n = -1$, $3n - 4m = -5$ **(2.3, 1.4)**
17. $4x + 3y = 6$, $8x - y = -9$ **(−0.75, 3)**
18. $0.5r - s = -1$, $0.75r + 0.5s = -0.25$ **(−0.75, 0.625)**
19. $1.5m - 0.7n = 0.5$, $2.2m - 0.6n = -7.4$ **(−8.5625, −19.0625)**

★ 20. $3x - 2y = 4$, $\frac{1}{2}x - \frac{2}{3}y = 1$ **$\left(\frac{2}{3}, -1\right)$**

★ 21. $2a + 3b = -16$, $\frac{3}{4}a - \frac{7}{8}b = 10$ **(4, −8)**

★ 22. $\frac{1}{3}r + \frac{2}{5}s = 5$, $\frac{2}{3}r - \frac{1}{2}s = -3$ **(3, 10)**

★ 23. $\frac{3}{4}x + \frac{1}{2}y = \frac{11}{12}$, $\frac{1}{2}x - \frac{1}{4}y = \frac{1}{8}$ **$\left(\frac{2}{3}, \frac{5}{6}\right)$**

24. **GEOMETRY** The two sides of an angle are contained in lines whose equations are $4x + y = -4$ and $2x - 3y = -9$. Find the coordinates of the vertex of the angle. **(−1.5, 2)**
25. **GEOMETRY** Two sides of a parallelogram are contained in the lines whose equations are $2.3x + 1.2y = 2.1$ and $4.1x - 0.5y = 14.3$. Find the coordinates of a vertex of the parallelogram. **(3, −4)**

WebQuest
You can use Cramer's Rule to compare home loans. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

192 Chapter 4 Matrices

Answers

38. If the determinant is zero, there is no unique solution to the system. There is either no solution or there are infinitely many solutions. Sample answer: $2x + y = 4$ and $4x + 2y = 8$ has a det $= 0$; there are infinitely many solutions of this system. $2x + y = 4$ and $4x + 2y = 10$ has a det $= 0$; there are no solutions of this system.

39. Cramer's Rule is a formula for the variables x and y where (x, y) is a solution for a system of equations. Answers should include the following.
- Cramer's Rule uses determinants composed of the coefficients and constants in a system of linear equations to solve the system.
- Cramer's Rule is convenient when coefficients are large or involve fractions or decimals. Finding the value of the determinant is sometimes easier than trying to find a greatest common factor if you are solving by using elimination or substituting complicated numbers.

26. $(-1, 3, 4)$
27. $(2, -1, 3)$
28. $\left(-\frac{11}{19}, \frac{39}{19}, -\frac{14}{19}\right)$
29. $\left(\frac{141}{29}, -\frac{102}{29}, \frac{244}{29}\right)$
31. $\left(-\frac{155}{28}, \frac{143}{70}, \frac{673}{140}\right)$

Use Cramer's Rule to solve each system of equations.

26. $x + y + z = 6$
$2x + y - 4z = -15$
$5x - 3y + z = -10$

27. $a - 2b + c = 7$
$6a + 2b - 2c = 4$
$4a + 6b + 4c = 14$

28. $r - 2s - 5t = -1$
$r + 2s - 2t = 5$
$4r + s + t = -1$

29. $3a + c = 23$
$4a + 7b - 2c = -22$
$8a - b - c = 34$

★ 30. $4x + 2y - 3z = -32$
$-x - 3y + z = 54$
$2y + 8z = 78$
$(11, -17, 14)$

★ 31. $2r + 25s = 40$
$10r + 12s + 6t = -2$
$36r - 25s + 50t = -10$

GAMES **For Exercises 32 and 33, use the following information.**
Marcus purchased a game card to play virtual games at the arcade. His favorite games are the race car simulator, which costs 7 points for each play, and the snowboard simulator, which costs 5 points for each play. Marcus came with enough money to buy a 50-point card, and he has time to play 8 games.

32. Write a system of equations. $r + s = 8$, $7r + 5s = 50$

33. Solve the system using Cramer's Rule to find the number of times Marcus can play race car simulator and snowboard simulator.
race car, 5 plays; snowboard, 3 plays

INTERIOR DESIGN **For Exercises 34 and 35, use the following information.**
An interior designer is preparing invoices for two of her clients. She has ordered silk dupioni and cotton damask fabric for both of them.

Client	Fabric	Yards	Total Cost
Harada	silk cotton	8 13	\$604.79
Martina	silk cotton	$5\frac{1}{2}$ 14	\$542.30

34. Write a system of two equations using the information given. $8s + 13c = 604.79$, $5\frac{1}{2}s + 14c = 542.30$

35. Find the price per yard of each fabric. silk, \$34.99; cotton, \$24.99

PRICING **For Exercises 36 and 37, use the following information.**
The Harvest Nut Company sells made-to-order trail mixes. Santito's favorite mix contains peanuts, raisins, and carob-coated pretzels. Peanuts sell for \$3.20 per pound, raisins are \$2.40 per pound, and the carob-coated pretzels are \$4.00 per pound. Santito chooses to have twice as many pounds of pretzels as raisins, wants 5 pounds of mix, and can afford \$16.80.

36. Write a system of three equations using the information given.

37. How many pounds of peanuts, raisins, and carob-coated pretzels can Santito buy? peanuts, 2 lb; raisins, 1 lb; pretzels, 2 lb

36. $p + r + c = 5$, $2r - p = 0$, $3.2p + 2.4r + 4c = 16.8$

38. **CRITICAL THINKING** In Cramer's Rule, if the value of the determinant is zero, what must be true of the graph of the system of equations represented by the determinant? Give examples to support your answer. See margin.

39. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. See margin.

How is Cramer's Rule used to solve systems of equations?

Include the following in your answer.
- an explanation of how Cramer's rule uses determinants, and
- a situation where Cramer's rule would be easier to solve a system of equations than substitution or elimination and why.

www.algebra2.com/self_check_quiz

Career Choices

Interior Designer

Interior designers work closely with architects and clients to develop a design that is not only aesthetic, but also functional, within budget, and meets all building codes.

Source: www.uwec.edu

Online Research

For information about a career as an interior designer, visit: www.algebra2.com/careers

Study Guide and Intervention, p. 199 (shown) and p. 200

Systems of Two Linear Equations Determinants provide a way for solving systems of equations.

Cramer's Rule for Two-Variable Systems	The solution of the linear system of equations $ax + by = e$, $cx + dy = f$ is (x, y) where $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Example **Use Cramer's Rule to solve the system of equations.** $5x - 10y = 8$, $10x + 25y = -2$

$x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ Cramer's Rule $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$

$= \dfrac{\begin{vmatrix} 8 & -10 \\ -2 & 25 \end{vmatrix}}{\begin{vmatrix} 5 & -10 \\ 10 & 25 \end{vmatrix}}$ $a = 5, b = -10, c = 10, d = 25, e = 8, f = -2$ $= \dfrac{\begin{vmatrix} 5 & 8 \\ 10 & -2 \end{vmatrix}}{\begin{vmatrix} 5 & -10 \\ 10 & 25 \end{vmatrix}}$

$= \dfrac{8(25) - (-2)(-10)}{5(25) - (-10)(10)}$ Evaluate each determinant. $= \dfrac{5(-2) - 8(10)}{5(25) - (-10)(10)}$

$= \dfrac{180}{225}$ or $\dfrac{4}{5}$ Simplify. $= -\dfrac{90}{225}$ or $-\dfrac{2}{5}$

The solution is $\left(\frac{4}{5}, -\frac{2}{5}\right)$.

Exercises

Use Cramer's Rule to solve each system of equations.

1. $3x - 2y = 7$, $2x + 7y = 38$ $(5, 4)$
2. $x - 4y = 17$, $3x - y = 29$ $(9, -2)$
3. $2x - y = -2$, $4x - y = 4$ $(3, 8)$
4. $2x - y = 1$, $5x + 2y = -29$ $(-3, -7)$
5. $4x + 2y = 1$, $5x - 4y = 24$ $\left(2, -\frac{7}{2}\right)$
6. $6x - 3y = -3$, $2x + y = 21$ $(5, 11)$
7. $2x + 7y = 16$, $x - 2y = 30$ $(22, -4)$
8. $2x - 3y = -2$, $3x - 4y = 9$ $(35, 24)$
9. $\frac{x}{3} + \frac{y}{5} = 2$, $\frac{x}{4} - \frac{y}{6} = -8$ $(-12, 30)$
10. $6x - 9y = -1$, $3x + 18y = 12$ $\left(\frac{2}{3}, \frac{5}{9}\right)$
11. $3x - 12y = -14$, $9x + 6y = -7$ $\left(-\frac{4}{3}, \frac{5}{6}\right)$
12. $8x + 2y = \frac{3}{7}$, $5x - 4y = -\frac{27}{7}$ $\left(-\frac{1}{7}, \frac{11}{14}\right)$

Skills Practice, p. 201 and Practice, p. 202 (shown)

Use Cramer's Rule to solve each system of equations.

1. $2x + y = 0$, $3x + 2y = -2$ $(2, -4)$
2. $5c + 9d = 19$, $2c - d = -20$ $(-7, 6)$
3. $2x + 3y = 5$, $3x - 2y = 1$ $(1, 1)$
4. $20m - 3n = 28$, $2m + 3n = 16$ $(2, 4)$
5. $x - 3y = 6$, $3x + y = -22$ $(-6, -4)$
6. $5x - 6y = -45$, $9x + 8y = 13$ $(-3, 5)$
7. $-2e + f = 4$, $-3e + 5f = -15$ $(-5, -6)$
8. $2x - y = -1$, $2x - 4y = 8$ $(-2, -3)$
9. $8a + 3b = 24$, $2a + b = 4$ $(6, -8)$
10. $-3x + 15y = 45$, $-2x + 7y = 18$ $(5, 4)$
11. $3u - 5v = 11$, $6u + 7v = -12$ $\left(\frac{1}{3}, -2\right)$
12. $-6g + h = -10$, $-3g - 4h = 4$ $\left(\frac{4}{3}, -2\right)$
13. $x - 3y = 8$, $x - 0.5y = 3$ $(2, -2)$
14. $0.2x - 0.5y = -1$, $0.6x - 3y = -9$ $(5, 4)$
15. $0.3d - 0.6g = 1.8$, $0.2d + 0.3g = 0.5$ $(4, -1)$

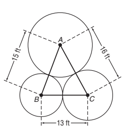
16. **GEOMETRY** The two sides of an angle are contained in the lines whose equations are $x - \frac{4}{3}y = 6$ and $2x + y = 1$. Find the coordinates of the vertex of the angle. $(2, -3)$

17. **GEOMETRY** Two sides of a parallelogram are contained in the lines whose equations are $0.2x - 0.5y = 1$ and $0.02x - 0.3y = -0.9$. Find the coordinates of a vertex of the parallelogram. $(15, 4)$

Use Cramer's Rule to solve each system of equations.

18. $x + 3y + 3z = 4$, $-x + 2y + z = -1$, $4x + y - 2z = -1$ $(1, -1, 2)$
19. $-5a + b - 4c = 7$, $-3a + 2b - c = 0$, $2a + 3b - c = 17$ $(3, 2, -5)$
20. $2x + y - 3z = -5$, $5x + 2y - 2z = 8$, $3x - 3y + 5z = 17$ $(2, 3, 4)$
21. $2c + 3d - e = 17$, $4c + d + 5e = -9$, $c + 2d - e = 12$ $(2, 3, -4)$
22. $2j + k - 3m = -3$, $3j + 2k + 4m = 5$, $-4j - k + 2m = 4$ $(-1, 2, 1)$
23. $3x - 2y + 5z = 3$, $2x + 2y - 4z = 3$, $-5x + 10y + 7z = -3$ $(1.2, 0.3, 0)$

24. **LANDSCAPING** A memorial garden being planted in front of a municipal library will contain three circular beds that are tangent to each other. A landscape architect has prepared a sketch of the design for the garden using CAD (computer-aided drafting) software, as shown at the right. The centers of the three circular beds are represented by points A, B, and C. The distance from A to B is 15 feet, the distance from B to C is 13 feet, and the distance from A to C is 16 feet. What is the radius of each of the circular beds? circle A: 9 ft, circle B: 6 ft, circle C: 7 ft



Reading to Learn Mathematics, p. 203 ELL

Pre-Activity **How is Cramer's Rule used to solve systems of equations?**

Read the introduction to Lesson 4-6 at the top of page 189 in your textbook.

A triangle is bounded by the x-axis, the line $y = \frac{1}{2}x$, and the line $y = -2x + 10$. Write three systems of equations that you could use to find the three vertices of the triangle. (Do not actually find the vertices.)
$x = 0, y = \frac{1}{2}x$; $x = 0, y = -2x + 10$; $y = \frac{1}{2}x, y = -2x + 10$

Reading the Lesson

1. Suppose that you are asked to solve the following system of equations by Cramer's Rule.

$3x + 2y = 7$
$2x - 3y = 22$

Without actually evaluating any determinants, indicate which of the following ratios of determinants gives the correct value for x. B

A. $\dfrac{\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 22 & -3 \end{vmatrix}}$ B. $\dfrac{\begin{vmatrix} 7 & 2 \\ 22 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix}}$ C. $\dfrac{\begin{vmatrix} 3 & 7 \\ 2 & 22 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix}}$

2. In your textbook, the statements of Cramer's Rule for two variables and three variables specify that the determinant formed from the coefficients of the variables cannot be 0. If the determinant is zero, what do you know about the system and its solutions?
The system could be a dependent system and have infinitely many solutions, or it could be an inconsistent system and have no solutions.

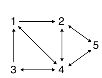
Helping You Remember

3. Some students have trouble remembering how to arrange the determinants that are used in solving a system of two linear equations by Cramer's Rule. What is a good way to remember this?
Sample answer: Let D be the determinant of the coefficients. Let D_x be the determinant formed by replacing the first column of D with the constants from the right-hand side of the system, and let D_y be the determinant formed by replacing the second column of D with the constants. Then $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$.

Enrichment, p. 204

Communications Networks

The diagram at the right represents a communications network linking five computer remote stations. The arrows indicate the direction in which signals can be transmitted and received by each computer. We can generate a matrix to describe this network.



$$A = \begin{array}{r} \\ \text{from computer } i \end{array} \overset{\text{to computer } j}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}}$$

The entry in position a_{ij} represents the number of ways to send a message from computer i to computer j directly. Compare the entries of matrix A to the diagram to verify the entries. For example, there is one way to send a message from computer 3 to computer 4, so $A_{3,4} = 1$. A computer cannot send a message to itself, so $A_{1,1} = 0$.

Matrix A is a communications network for direct communication. Suppose you want to send a message from one computer to another using exactly one other computer as a relay point. It can be shown that the entries of matrix A^2 represent the number of ways to send a message from one point to another by going through a third station. For example, a message may be sent from station 1 to station 5 by going through station 2 or station 4 on the way. Therefore, $A^2{}_{1,5} = 2$.

4 Assess

Open-Ended Assessment

Speaking Have students share tips and techniques for remembering the correct order of the calculations they must perform in the multi-step processes discussed in this lesson.

Getting Ready for Lesson 4-7

PREREQUISITE SKILL Lesson 4-7 shows how to determine if two matrices are inverses and how to find the inverse of a 2×2 matrix. Students will use their familiarity with multiplying matrices as they work with inverses. Exercises 52–54 should be used to determine your students' familiarity with multiplying matrices.

Assessment Options

Practice Quiz 2 The quiz provides students with a brief review of the concepts and skills in Lessons 4-4 through 4-6. Lesson numbers are given to the right of exercises or instruction lines so students can review concepts not yet mastered.

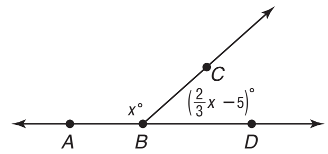
Quiz (Lessons 4-5 and 4-6) is available on p. 232 of the *Chapter 4 Resource Masters*.

Standardized Test Practice

40. Use Cramer's Rule to solve the system of equations $3x + 8y = 28$ and $5x - 7y = -55$. **B**

(A) $(3, 5)$ (B) $(-4, 5)$ (C) $(4, 2)$ (D) $(-5, 4)$

41. SHORT RESPONSE Find the measures of $\angle ABC$ and $\angle CBD$. **111°; 69°**



Maintain Your Skills

Mixed Review

Find the value of each determinant. *(Lesson 4-5)*

42. $\begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix}$ **16** **43.** $\begin{vmatrix} 8 & 6 \\ 4 & 8 \end{vmatrix}$ **40** **44.** $\begin{vmatrix} -5 & 2 \\ 4 & 9 \end{vmatrix}$ **−53**

For Exercises 45–47, use the following information.
Triangle ABC with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-2, -4)$ is translated 1 unit right and 3 units up. *(Lesson 4-4)*

45. Write the translation matrix. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}$

46. Find the coordinates of $\triangle A'B'C'$. **$A'(1, 5)$, $B'(-2, 2)$, $C'(-1, -1)$**

47. Graph the preimage and the image. **See margin.**

48–50. See margin for graphs.

Solve each system of equations by graphing. *(Lesson 3-1)*

48. $y = 3x + 5$ **(−2, −1)**
$y = -2x - 5$

49. $x + y = 7$ **(4, 3)**
$\frac{1}{2}x - y = -1$

50. $x - 2y = 10$ **no solution**
$2x - 4y = 12$

51. BUSINESS The Friendly Fix-It Company charges a base fee of \$35 for any in-home repair. In addition, the technician charges \$10 per hour. Write an equation for the cost c of an in-home repair of h hours. *(Lesson 1-3)* **$c = 10h + 35$**

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each product, if possible.**
*(To review **multiplying matrices**, see Lesson 4-3.)*

52. $\begin{bmatrix} 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$ **$\begin{bmatrix} -4 & 32 \end{bmatrix}$**

53. $\begin{bmatrix} 0 & 9 \\ 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -6 \\ 8 & 1 \end{bmatrix}$ **$\begin{bmatrix} 72 & 9 \\ 66 & -23 \end{bmatrix}$**

54. $\begin{bmatrix} 5 & -4 \\ 8 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ **$\begin{bmatrix} 21 \\ 43 \end{bmatrix}$**

Practice Quiz 2 — Lessons 4-4 through 4-6

For Exercises 1–3, reflect square $ABCD$ with vertices $A(1, 2)$, $B(4, -1)$, $C(1, -4)$, and $D(-2, -1)$ over the y-axis. *(Lesson 4-4)*

1. Write the coordinates in a vertex matrix. $\begin{bmatrix} 1 & 4 & 1 & -2 \\ 2 & -1 & -4 & -1 \end{bmatrix}$
2. Find the coordinates of $A'B'C'D'$. **$A'(-1, 2)$, $B'(-4, -1)$, $C'(-1, -4)$, $D'(2, -1)$**
3. Graph $ABCD$ and $A'B'C'D'$. **See pp. 217A–217B.**

Find the value of each determinant. *(Lesson 4-5)*

4. $\begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix}$ **22** **5.** $\begin{vmatrix} -8 & 3 \\ 6 & 5 \end{vmatrix}$ **−58** **6.** $\begin{vmatrix} 1 & 3 & -2 \\ 7 & 0 & 4 \\ -3 & 5 & -1 \end{vmatrix}$ **−105** **7.** $\begin{vmatrix} 3 & 4 & 4 \\ 2 & 1 & 5 \\ 0 & -8 & 6 \end{vmatrix}$ **26**

Use Cramer's Rule to solve each system of equations. *(Lesson 4-6)*

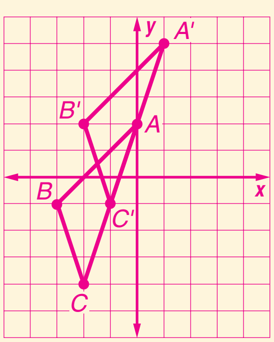
8. $3x - 2y = 7$ **(1, −2)**
$4x - y = 6$

9. $7r + 5s = 3$ **(4, −5)**
$3r - 2s = 22$

10. $3a - 5b + 2c = -5$ **(1, 2, 1)**
$4a + b + 3c = 9$
$2a - c = 1$

Answers

47.



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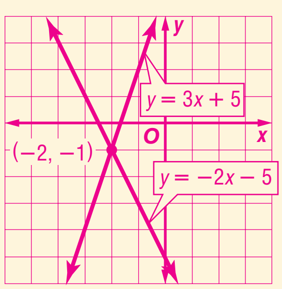

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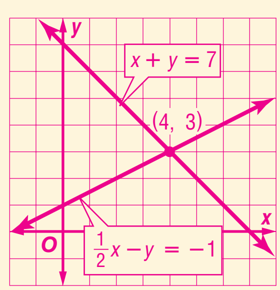

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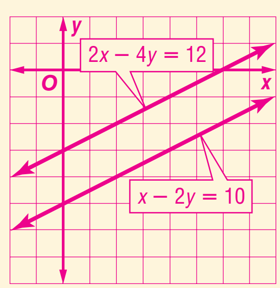


4-7 Identity and Inverse Matrices

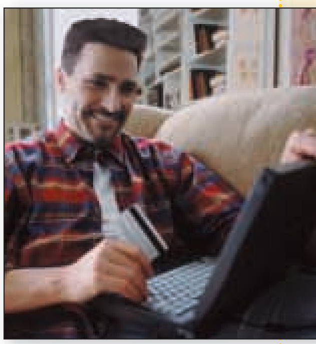
What You'll Learn

- Determine whether two matrices are inverses.
- Find the inverse of a 2×2 matrix.

Vocabulary

- identity matrix
- inverse

How are inverse matrices used in cryptography?



With the rise of Internet shopping, ensuring the privacy of the user's personal information has become an important priority. Companies protect their computers by using codes. Cryptography is a method of preparing coded messages that can only be deciphered by using the "key" to the message.

The following technique is a simplified version of how cryptography works.

- First, assign a number to each letter of the alphabet.
- Convert your message into a matrix and multiply it by the coding matrix. The message is now unreadable to anyone who does not have the key to the code.
- To decode the message, the recipient of the coded message would multiply by the opposite, or inverse, of the coding matrix.

Code

_ 0	**A** 1	**B** 2	**C** 3	**D** 4	**E** 5	**F** 6	**G** 7	**H** 8
I 9	**J** 10	**K** 11	**L** 12	**M** 13	**N** 14	**O** 15	**P** 16	**Q** 17
R 18	**S** 19	**T** 20	**U** 21	**V** 22	**W** 23	**X** 24	**Y** 25	**Z** 26

IDENTITY AND INVERSE MATRICES Recall that in real numbers, two numbers are inverses if their product is the identity, 1. Similarly, for matrices, the **identity matrix** is a square matrix that, when multiplied by another matrix, equals that same matrix. If A is any $n \times n$ matrix and I is the $n \times n$ identity matrix, then $A \cdot I = A$ and $I \cdot A = A$.

2×2 Identity Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3×3 Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Key Concept — *Identity Matrix for Multiplication*

- **Words** The identity matrix for multiplication I is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions. For any square matrix A of the same dimension as I, $A \cdot I = I \cdot A = A$.
- **Symbols** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Two $n \times n$ matrices are **inverses** of each other if their product is the identity matrix. If matrix A has an inverse symbolized by A^{-1}, then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

4-7 Lesson Notes

1 Focus

5-Minute Check Transparency 4-7 Use as a quiz or review of Lesson 4-6.

Mathematical Background notes are available for this lesson on p. 152D.

How are inverse matrices used in cryptography?

Ask students:

- Using just the simple code given in the table, what word is given by the numbers 13, 1, 20, 8? **MATH**
- The decoding process mentions the "inverse of the coding matrix." What other inverses have you seen in mathematics? **multiplicative inverses (or reciprocals), and additive inverses (or opposites)**

Resource Manager

Workbook and Reproducible Masters

Chapter 4 Resource Masters

- Study Guide and Intervention, pp. 205–206
- Skills Practice, p. 207
- Practice, p. 208
- Reading to Learn Mathematics, p. 209
- Enrichment, p. 210

Transparencies

5-Minute Check Transparency 4-7
Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

IDENTITY AND INVERSE MATRICES

In-Class Example

Power Point®

1 Determine whether each pair of matrices are inverses.

a. $X = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ **yes**

b. $P = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ **no**

Example 1 Verify Inverse Matrices

Determine whether each pair of matrices are inverses.

a. $X = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$ **and** $Y = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$

Check to see if $X \cdot Y = I$.

$$X \cdot Y = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 1-2 & 1+\frac{1}{2} \\ -\frac{1}{2}+(-4) & -\frac{1}{2}+1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 1\frac{1}{2} \\ -4\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{Matrix multiplication}$$

Since $X \cdot Y \neq I$, they are *not* inverses.

b. $P = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

Find $P \cdot Q$.

$$P \cdot Q = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 3-2 & -6+6 \\ 1-1 & -2+3 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Matrix multiplication}$$

Now find $Q \cdot P$.

$$Q \cdot P = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 3-2 & 4-4 \\ -\frac{3}{2}+\frac{3}{2} & -2+3 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Matrix multiplication}$$

Since $P \cdot Q = Q \cdot P = I$, P and Q are inverses.

Study Tip

Verifying Inverses
Since multiplication of matrices is not commutative, it is necessary to check the products in both orders.

FIND INVERSE MATRICES Some matrices do not have an inverse. You can determine whether a matrix has an inverse by using the determinant.

Key Concept — *Inverse of a 2 × 2 Matrix*

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Notice that $ad - bc$ is the value of det A. Therefore, if the value of the determinant of a matrix is 0, the matrix cannot have an inverse.

Teacher to Teacher

Beatrice Moore-Harris — Educational Specialist, League City, TX

"Have students make a three-column table. List the terms dilation, dimension, element, image, minor, reflection, rotation, transformation, and translation in one column. Have them provide a real-world definition, example, and/or illustration of each term in another column and the mathematical definition in a third column. Finally, ask students to compare and contrast how the terms are used outside and inside of the math classroom. Are any definitions the same?"

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Example 2 Find the Inverse of a Matrix

Find the inverse of each matrix, if it exists.

a. $R = \begin{bmatrix} -4 & -3 \\ 8 & 6 \end{bmatrix}$

Find the value of the determinant.

$$\begin{vmatrix} -4 & -3 \\ 8 & 6 \end{vmatrix} = -24 - (-24) = 0$$

Since the determinant equals 0, R^{-1} does not exist.

b. $P = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

Find the value of the determinant.

$$\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6 - 5 \text{ or } 1$$

Since the determinant does not equal 0, P^{-1} exists.

$$P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{Definition of inverse}$$

$$= \frac{1}{3(2) - 1(5)}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad a = 3, b = 1, c = 5, d = 1$$

$$= 1\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad \text{Simplify.}$$

CHECK $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 - 5 & 2 - 2 \\ -15 + 15 & -5 + 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

More About. . .

Cryptography

The Enigma was a German coding machine used in World War II. Its code was considered to be unbreakable. However, the code was eventually solved by a group of Polish mathematicians.

Source: www.bletchleypark.org.uk

Matrices can be used to code messages by placing the message in a $2 \times n$ matrix.

Example 3 Use Inverses to Solve a Problem

a. CRYPTOGRAPHY Use the table at the beginning of the lesson to assign a number to each letter in the message GO_TONIGHT. Then code the message with the matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

Convert the message to numbers using the table.

G O _ T O N I G H T

7 | 15 | 0 | 20 | 15 | 14 | 9 | 7 | 8 | 20

Write the message in matrix form. Then multiply the message matrix B by the coding matrix A.

$$BA = \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 14 + 60 & 7 + 45 \\ 0 + 80 & 0 + 60 \\ 30 + 56 & 15 + 42 \\ 18 + 28 & 9 + 21 \\ 16 + 80 & 8 + 60 \end{bmatrix} \quad \text{Matrix multiplication}$$

(continued on the next page)

www.algebra2.com/extra_examples

FIND INVERSE MATRICES

In-Class Examples

Power Point®

2 Find the inverse of each matrix, if it exists.

a. $S = \begin{bmatrix} -1 & 0 \\ 8 & -2 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 \\ -4 & -\frac{1}{2} \end{bmatrix}$

b. $T = \begin{bmatrix} -4 & 6 \\ -2 & 3 \end{bmatrix}$ **No inverse exists.**

3

a. Use the table at the beginning of the lesson to assign a number to each letter in the message ALWAYS_SMILE. Then code the message with the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

The coded message is 13 | 38 | 24 | 49 | 44 | 107 | 19 | 57 | 22 | 53 | 17 | 39.

b. Use the inverse matrix A^{-1} to decode the message in In-Class Example 3a.

1 | 12 | 23 | 1 | 25 | 19 | 0 | 19 | 13 | 9 | 12 | 5; ALWAYS_SMILE

Teaching Tip Remind students to add a zero at the end of the message when it contains an odd number of letters.

Lesson 4-7 Identity and Inverse Matrices 197

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms in this lesson to their Vocabulary Builder worksheets for Chapter 4.
- write the 2×2 identity matrix, and the 3×3 identity matrix.
- write the formula for the inverse of a 2×2 matrix, including an example showing how to find the inverse of such a matrix.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

Answers

1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. Exchange the values for a and d in the first diagonal in the matrix. Multiply the values for b and c by -1 in the second diagonal in the matrix. Find the determinant of the original matrix. Multiply the negative reciprocal of the determinant by the matrix with the above mentioned changes.

3. Sample answer: $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$

Study Tip

Messages

If there is an odd number of letters to be coded, add a 0 at the end of the message.

$$= \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix} \quad \text{Simplify.}$$

The coded message is 74 | 52 | 80 | 60 | 86 | 57 | 46 | 30 | 96 | 68.

b. Use the inverse matrix A^{-1} to decode the message in Example 3a.

First find the inverse matrix of $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{Definition of inverse}$$

$$= \frac{1}{2(3) - (1)(4)}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \quad a = 2, b = 1, c = 4, d = 3$$

$$= \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} \quad \text{Simplify.}$$

Next, decode the message by multiplying the coded matrix C by A^{-1}.

$$CA^{-1} = \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 111 - 104 & -37 + 52 \\ 120 - 120 & -40 + 60 \\ 129 - 114 & -43 + 57 \\ 69 - 60 & -23 + 30 \\ 144 - 136 & -48 + 68 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix}$$

Use the table again to convert the numbers to letters. You can now read the message.

7 | 15 | 0 | 20 | 15 | 14 | 9 | 7 | 8 | 20

G O _ T O N I G H T

Check for Understanding

Concept Check

1. **Write** the 4×4 identity matrix. **1–3. See margin.**
2. **Explain** how to find the inverse of a 2×2 matrix.
3. **OPEN ENDED** Create a square matrix that does not have an inverse.

DAILY INTERVENTION

Differentiated Instruction

Logical Ask students to write a comparison of the inverse of a matrix to the multiplicative and additive inverses of a number, discussing what is the same about these inverses, and what is different.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4, 5	1
6–8	2
9	3

Determine whether each pair of matrices are inverses.

4. $A = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$ **no**

5. $X = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, Y = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ **yes**

Find the inverse of each matrix, if it exists.

6. $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

7. $\begin{bmatrix} 4 & -8 \\ -1 & 2 \end{bmatrix}$ **no inverse exists**

8. $\begin{bmatrix} -5 & 1 \\ 7 & 4 \end{bmatrix}$ $-\frac{1}{27}\begin{bmatrix} 4 & -1 \\ -7 & -5 \end{bmatrix}$

Application

9. **CRYPTOGRAPHY** Select a headline from a newspaper or the title of a magazine article and code it using your own coding matrix. Give your message and the coding matrix to a friend to decode. (*Hint*: Use a coding matrix whose determinant is 1 and that has all positive elements.) **See students' work.**

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
10–19, 32, 33	1
20–31	2
34–41	3

Extra Practice

See page 836.

Determine whether each pair of matrices are inverses.

10. $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ **yes**

11. $R = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, S = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$ **yes**

12. $A = \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -\frac{5}{2} & -3 \end{bmatrix}$ **no**

13. $X = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}, Y = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ **no**

14. $C = \begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix}, D = \begin{bmatrix} \frac{2}{7} & \frac{5}{7} \\ \frac{1}{7} & -\frac{1}{7} \end{bmatrix}$ **yes**

★ 15. $J = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}, K = \begin{bmatrix} -\frac{5}{4} & \frac{1}{4} & \frac{7}{4} \\ \frac{3}{4} & \frac{1}{4} & -\frac{5}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$ **yes**

Determine whether each statement is *true* or *false*.

16. Only square matrices have multiplicative identities. **true**

17. Only square matrices have multiplicative inverses. **true**

18. Some square matrices do not have multiplicative inverses. **true**

19. Some square matrices do not have multiplicative identities. **false**

Find the inverse of each matrix, if it exists.

20. $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ $\frac{1}{5}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

21. $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$ **no inverse exists**

22. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $-\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$ $\frac{1}{7}\begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}$

24. $\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$ **no inverse exists**

25. $\begin{bmatrix} -3 & 7 \\ 2 & -6 \end{bmatrix}$

26. $\begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix}$ $\frac{1}{34}\begin{bmatrix} 7 & 3 \\ -2 & 4 \end{bmatrix}$

27. $\begin{bmatrix} -2 & 0 \\ 5 & 6 \end{bmatrix}$ $-\frac{1}{12}\begin{bmatrix} 6 & 0 \\ -5 & -2 \end{bmatrix}$

28. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$ **no inverse exists**

29. $\begin{bmatrix} 2 & -5 \\ 6 & 1 \end{bmatrix}$ $\frac{1}{32}\begin{bmatrix} 1 & 5 \\ -6 & 2 \end{bmatrix}$

★ 30. $\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix}$

★ 31. $\begin{bmatrix} \frac{3}{10} & \frac{5}{8} \\ \frac{1}{5} & \frac{3}{4} \end{bmatrix}$

25. $\frac{1}{4}\begin{bmatrix} -6 & -7 \\ -2 & -3 \end{bmatrix}$

30. $4\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix}$

31. $10\begin{bmatrix} \frac{3}{4} & -\frac{5}{8} \\ -\frac{1}{5} & \frac{3}{10} \end{bmatrix}$

www.algebra2.com/self_check_quiz

About the Exercises…

Organization by Objective

- **Identity and Inverse Matrices:** 10–17, 32, 33
- **Find Inverse Matrices:** 18–31, 34–41

Odd/Even Assignments

Exercises 10–33 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Alert! Exercise 42 involves research on the Internet or other reference materials. Exercises 47–52 require a graphing calculator.

Assignment Guide

Basic: 11, 13, 17–29 odd, 33, 43–46, 53–77

Average: 11–33 odd, 34–38, 43–46, 53–77 (optional: 47–52)

Advanced: 10–32 even, 34–71 (optional: 72–77)

Study Guide and Intervention, p. 205 (shown) and p. 206

Identity and Inverse Matrices The identity matrix for matrix multiplication is a square matrix with 1s for every element of the main diagonal and zeros elsewhere.

Identity Matrix for Multiplication	If A is an $n \times n$ matrix and I is the identity matrix, then $A \cdot I = A$ and $I \cdot A = A$.

If an $n \times n$ matrix A has an inverse A^{-1}, then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

Example **Determine whether $X = \begin{bmatrix} 7 & 4 \\ 10 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & -2 \\ -5 & \frac{7}{2} \end{bmatrix}$ are inverse matrices.**

Find $X \cdot Y$.

$$X \cdot Y = \begin{bmatrix} 7 & 4 \\ 10 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ -5 & \frac{7}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 21 - 20 & -14 + 14 \\ 30 - 30 & -20 + 21 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find $Y \cdot X$.

$$Y \cdot X = \begin{bmatrix} 3 & -2 \\ -5 & \frac{7}{2} \end{bmatrix} \cdot \begin{bmatrix} 7 & 4 \\ 10 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 21 - 20 & 12 - 12 \\ -35 + 35 & -20 + 21 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $X \cdot Y = Y \cdot X = I$, X and Y are inverse matrices.

Exercises

Determine whether each pair of matrices are inverses.

1. $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$ **yes**
2. $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$ **yes**
3. $\begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ **no**
4. $\begin{bmatrix} 8 & 11 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} -4 & 11 \\ 3 & -8 \end{bmatrix}$ **yes**
5. $\begin{bmatrix} 4 & -1 \\ 5 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ **no**
6. $\begin{bmatrix} 5 & 2 \\ 11 & 4 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 \\ \frac{11}{2} & -\frac{5}{2} \end{bmatrix}$ **yes**
7. $\begin{bmatrix} 4 & 2 \\ 6 & -2 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{5} & -\frac{1}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$ **no**
8. $\begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}$ and $\begin{bmatrix} -3 & 4 \\ 2 & -\frac{5}{2} \end{bmatrix}$ **yes**
9. $\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$ and $\begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 1 & -2 \end{bmatrix}$ **no**
10. $\begin{bmatrix} 3 & 2 \\ 4 & -6 \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$ **no**
11. $\begin{bmatrix} 7 & 2 \\ 17 & 5 \end{bmatrix}$ and $\begin{bmatrix} 5 & -2 \\ -17 & 7 \end{bmatrix}$ **yes**
12. $\begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}$ and $\begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix}$ **yes**

Skills Practice, p. 207 and Practice, p. 208 (shown)

Determine whether each pair of matrices are inverses.

1. $M = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $N = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$ **no**
2. $X = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$, $Y = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ **yes**
3. $A = \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{5} & -\frac{1}{10} \\ \frac{2}{5} & \frac{3}{10} \end{bmatrix}$ **yes**
4. $P = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{3}{14} & \frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$ **yes**

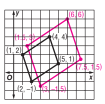
Determine whether each statement is *true* or *false*.

5. All square matrices have multiplicative inverses. **false**
6. All square matrices have multiplicative identities. **true**

Find the inverse of each matrix, if it exists.

7. $\begin{bmatrix} 4 & 5 \\ -4 & -3 \end{bmatrix}$ $\frac{1}{8}\begin{bmatrix} -3 & -5 \\ 4 & 4 \end{bmatrix}$
8. $\begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix}$ $\frac{1}{10}\begin{bmatrix} 5 & 0 \\ -3 & 2 \end{bmatrix}$
9. $\begin{bmatrix} -1 & 3 \\ 4 & -7 \end{bmatrix}$ $-\frac{1}{5}\begin{bmatrix} -7 & -3 \\ -4 & -1 \end{bmatrix}$
10. $\begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$ $\frac{1}{11}\begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}$
11. $\begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix}$ $\frac{1}{17}\begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix}$
12. $\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ **no inverse exists**

GEOMETRY For Exercises 13–16, use the figure at the right.



13. Write the vertex matrix A for the rectangle. $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 4 & 1 & -1 \end{bmatrix}$
14. Use matrix multiplication to find BA if $B = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$. $\begin{bmatrix} 1.5 & 6 & 7.5 & 3 \\ 3 & 6 & 1.5 & -1.5 \end{bmatrix}$
15. Graph the vertices of the transformed triangle on the previous graph. Describe the transformation. **dilation by a scale factor of 1.5**
16. Make a conjecture about what transformation B^{-1} describes on a coordinate plane. **dilation by a scale factor of $\frac{2}{3}$**
17. **CODES** Use the alphabet table below and the inverse of coding matrix $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ to decode this message:
19 | 14 | 11 | 13 | 11 | 22 | 55 | 65 | 57 | 60 | 2 | 1 | 52 | 47 | 33 | 51 | 56 | 55.

CHECK_YOUR_ANSWERS

CODE						
A 1	B 2	C 3	D 4	E 5	F 6	G 7
H 8	I 9	J 10	K 11	L 12	M 13	N 14
O 15	P 16	Q 17	R 18	S 19	T 20	U 21
V 22	W 23	X 24	Y 25	Z 26	– 0	

Reading to Learn Mathematics, p. 209 ELL

Pre-Activity **How are inverse matrices used in cryptography?**

Read the introduction to Lesson 4-7 at the top of page 195 in your textbook.

Refer to the code table given in the introduction to this lesson. Suppose that you receive a message coded by this system as follows:

16 12 5 1 19 5   2 5   13 25   6 18 9 5 14 4.

Decode the message. **Please be my friend.**

Reading the Lesson

1. Indicate whether each of the following statements is *true* or *false*.
   a. Every element of an identity matrix is 1. **false**
   b. There is a 3×2 identity matrix. **false**
   c. Two matrices are inverses of each other if their product is the identity matrix. **true**
   d. If M is a matrix, M^{-1} represents the reciprocal of M. **false**
   e. No 3×2 matrix has an inverse. **true**
   f. Every square matrix has an inverse. **false**
   g. If the two columns of a 2×2 matrix are identical, the matrix does not have an inverse. **true**
2. Explain how to find the inverse of a 2×2 matrix. Do not use any special mathematical symbols in your explanation.
   Sample answer: First find the determinant of the matrix. If it is zero, then the matrix has no inverse. If the determinant is not zero, form a new matrix as follows. Interchange the top left and bottom right elements. Change the signs but not the positions of the other two elements. Multiply the resulting matrix by the reciprocal of the determinant of the original matrix. The resulting matrix is the inverse of the original matrix.

Helping You Remember

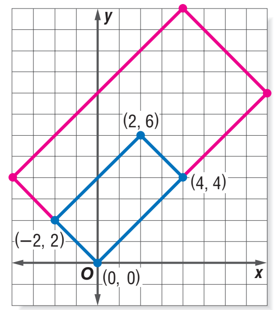
3. One way to remember something is to explain it to another person. Suppose that you are studying with a classmate who is having trouble remembering how to find the inverse of a 2×2 matrix. He remembers how to move elements and change signs in the matrix, but thinks that he should multiply by the determinant of the original matrix. How can you help him remember that he must multiply by the *reciprocal* of this determinant?
   Sample answer: If the determinant of the matrix is 0, its reciprocal is undefined. This agrees with the fact that if the determinant of a matrix is 0, the matrix does not have an inverse.

---

32. Compare the matrix used to reflect a figure over the x-axis to the matrix used to reflect a figure over the y-axis.
    a. Are they inverses? **no**
    b. Does your answer make sense based on the geometry? Use a drawing to support your answer. **See margin.**

33. The matrix used to rotate a figure 270° counterclockwise about the origin is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Compare this matrix with the matrix used to rotate a figure 90° counterclockwise about the origin.
    a. Are they inverses? **yes**
    b. Does your answer make sense based on the geometry? Use a drawing to support your answer. **See margin.**

GEOMETRY **For Exercises 34–38, use the figure below.**



34. Write the vertex matrix A for the rectangle. **34.** $\begin{bmatrix} 0 & -2 & 2 & 4 \\ 0 & 2 & 6 & 4 \end{bmatrix}$
35. Use matrix multiplication to find BA if $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. $\begin{bmatrix} 0 & -4 & 4 & 8 \\ 0 & 4 & 12 & 8 \end{bmatrix}$
36. Graph the vertices of the transformed rectangle. Describe the transformation. **36. dilation by a scale factor of 2**
37. Make a conjecture about what transformation B^{-1} describes on a coordinate plane. **37. dilation by a scale factor of $\frac{1}{2}$**
38. Test your conjecture. Find B^{-1} and multiply it by the result of BA. Make a drawing to verify your conjecture. **38.** $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$; **the graph of the inverse transformation is the original figure.**

CRYPTOGRAPHY **For Exercises 39–41, use the alphabet table below.**
Your friend has sent you a series of messages that were coded with the coding matrix $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Use the inverse of matrix C to decode each message.

Code		
A 26	J 17	S 8
B 25	K 16	T 7
C 24	L 15	U 6
D 23	M 14	V 5
E 22	N 13	W 4
F 21	O 12	X 3
G 20	P 11	Y 2
H 19	Q 10	Z 1
I 18	R 9	_ 0

39. 50 | 36 | 51 | 29 | 18 | 18 | 26 | 13 | 33 | 26 | 44 | 22 | 48 | 33 | 59 | 34 | 61 | 35 | 4 | 2 **39. MEET_IN_THE_LIBRARY**
40. 59 | 33 | 8 | 8 | 39 | 21 | 7 | 7 | 56 | 37 | 25 | 16 | 4 | 2 **AT_SIX_THIRTY**
41. 59 | 34 | 49 | 31 | 40 | 20 | 16 | 14 | 21 | 15 | 25 | 25 | 36 | 24 | 32 | 16 **BRING_YOUR_BOOK**

42. **RESEARCH** Use the Internet or other reference to find examples of codes used throughout history. Explain how messages were coded. **See students' work.**

43. **CRITICAL THINKING** For which values of a, b, c, and d will $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^{-1}$? $a = \pm 1$, $d = \pm 1$, $b = c = 0$

44. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson. **See margin.**

    How are inverse matrices used in cryptography?

    Include the following in your answer:
    - an explanation of why the inverse matrix works in decoding a message, and
    - a description of the conditions you must consider when writing a message in matrix form.

Enrichment, p. 210

Permutation Matrices

A permutation matrix is a square matrix in which each row and each column has one entry that is 1. All the other entries are 0. Find the inverse of a permutation matrix interchanging the rows and columns. For example, row 1 is interchanged with column 1, row 2 is interchanged with column 2.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

P is a 4×4 permutation matrix. P^{-1} is the inverse of P.

Solve each problem.

1. There is just one 2×2 permutation matrix that is not also an identity matrix. Write this matrix. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
2. Find the inverse of the matrix you wrote in Exercise 1. What do you notice? $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **The two matrices are the same.**
3. Show th[illegible] two matrices in E[illegible]ercises 1 and 2 are inverses.

Standardized Test Practice

45. What is the inverse of $\begin{bmatrix} 4 & 1 \\ 10 & 2 \end{bmatrix}$? **A**

(A) $\begin{bmatrix} -1 & \frac{1}{2} \\ 5 & -2 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & -1 \\ -10 & 4 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 5 \\ \frac{1}{2} & 2 \end{bmatrix}$ (D) $\begin{bmatrix} -2 & \frac{1}{2} \\ 5 & -1 \end{bmatrix}$

46. Which matrix does *not* have an inverse? **D**

(A) $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} -3 & -3 \\ 6 & -6 \end{bmatrix}$ (D) $\begin{bmatrix} -10 & -5 \\ 8 & 4 \end{bmatrix}$

Graphing Calculator

INVERSE FUNCTION The x^{-1} key on a TI-83 Plus is used to find the inverse of a matrix. If you get a SINGULAR MATRIX error on the screen, then the matrix has no inverse.

Use a graphing calculator to find the inverse of each matrix.

47. $\begin{bmatrix} -11 & 9 \\ 6 & -5 \end{bmatrix}$ $\begin{bmatrix} -5 & -9 \\ -6 & -11 \end{bmatrix}$

48. $\begin{bmatrix} 12 & 4 \\ 15 & 5 \end{bmatrix}$ **no inverse exists**

49. $\begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$

50. $\begin{bmatrix} 25 & -4 \\ -35 & 6 \end{bmatrix}$

51. $\begin{bmatrix} 2 & 5 & 2 \\ 1 & 4 & 1 \\ 6 & 3 & 3 \end{bmatrix}$

52. $\begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 5 & 2 \end{bmatrix}$

49. $\begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$

50. $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ 3\frac{1}{2} & 2\frac{1}{2} \end{bmatrix}$

Maintain Your Skills

Mixed Review

Use Cramer's Rule to solve each system of equations. *(Lesson 4-6)*

53. $3x + 2y = -2$, $x - 3y = 14$ **(2, −4)**

54. $2x + 5y = 35$, $7x - 4y = -28$ **(0, 7)**

55. $4x - 3z = -23$, $-2x - 5y + z = -9$, $y - z = 3$ **(−5, 4, 1)**

Evaluate each determinant by using diagonals or expansion by minors. *(Lesson 4-5)*

56. $\begin{vmatrix} 2 & 8 & -6 \\ 4 & 5 & 2 \\ -3 & -6 & -1 \end{vmatrix}$ **52**

57. $\begin{vmatrix} -3 & -3 & 1 \\ -9 & -2 & 3 \\ 5 & -2 & -1 \end{vmatrix}$ **−14**

58. $\begin{vmatrix} 5 & -7 & 3 \\ -1 & 2 & -9 \\ 5 & -7 & 3 \end{vmatrix}$ **0**

Find the slope of the line that passes through each pair of points. *(Lesson 2-3)*

59. (2, 5), (6, 9) **1**

60. (1, 0), (−2, 9) **−3**

61. (−5, 4), (−3, −6) **−5**

62. (−2, 2), (−5, 1) $\frac{1}{3}$

63. (0, 3), (−2, −2) $\frac{5}{2}$

64. (−8, 9), (0, 6) $-\frac{3}{8}$

65. **OCEANOGRAPHY** The deepest point in any ocean, the bottom of the Mariana Trench in the Pacific Ocean, is 6.8 miles below sea level. Water pressure in the ocean is represented by the function $f(x) = 1.15x$, where x is the depth in miles and $f(x)$ is the pressure in tons per square inch. Find the water pressure at the deepest point in the Mariana Trench. *(Lesson 2-1)* **7.82 tons/in²**

Evaluate each expression. *(Lesson 1-1)*

66. $3(2^3 + 1)$ **27**

67. $7 - 5 \div 2 + 1$ $5\frac{1}{2}$

68. $\frac{9 - 4 \cdot 3}{6}$ $-\frac{1}{2}$

69. $[40 - (7 + 9)] \div 8$ **3**

70. $[(-2 + 8)6 + 1]8$ **296**

71. $(4 - 1)(8 + 2)^2$ **300**

51. $\begin{bmatrix} -1 & 1 & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{7}{3} & -\frac{8}{3} & -\frac{1}{3} \end{bmatrix}$

52. $\begin{bmatrix} \frac{5}{16} & -\frac{1}{8} & -\frac{1}{16} \\ -\frac{1}{4} & 0 & \frac{1}{4} \\ \frac{5}{32} & \frac{3}{16} & \frac{1}{32} \end{bmatrix}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation.**

*(To review **solving multi-step equations**, see Lesson 1-3.)*

72. $3k + 8 = 5$ **−1**

73. $12 = -5h + 2$ **−2**

74. $7z - 4 = 5z + 8$ **6**

75. $\frac{x}{2} + 5 = 7$ **4**

76. $\frac{3 + n}{6} = -4$ **−27**

77. $6 = \frac{s - 8}{-7}$ **−34**

44. A matrix can be used to code a message. The key to the message is the inverse of the matrix. Answers should include the following.

- **The inverse matrix undoes the work of the matrix. So if you multiply a numeric message by a matrix it changes the message. When you multiply the changed message by the inverse matrix, the result is the original numeric message.**
- **You must consider the dimensions of the coding matrix so that you can write the numeric message in a matrix with dimensions that can be multiplied by the coding matrix.**

4 Assess

Open-Ended Assessment

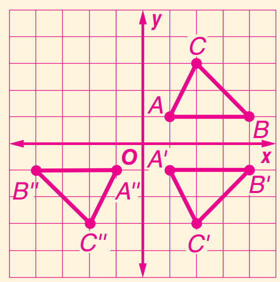
Writing Have students write a one-sentence message and assign a number to each letter using the chart on p. 195. Then have them write a 2×2 matrix of their own that has an inverse, and use it to code their message. Students should exchange their coded message with a classmate along with their coding matrix. The recipient should then find the inverse of the coding matrix they receive, use it to decode the numbers, and read the original message.

Getting Ready for Lesson 4-8

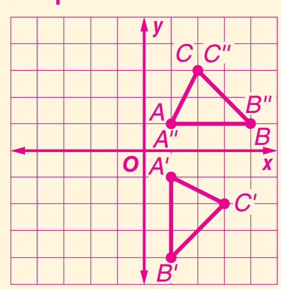
PREREQUISITE SKILL Lesson 4-8 shows how to use matrices to solve systems of equations. Students will use their familiarity with solving linear equations as they solve systems using matrices. Exercises 72–77 should be used to determine your students' familiarity with solving multi-step equations.

Answers

32b. Sample answer:



33b. Sample answer:



4-8 Lesson Notes

1 Focus

5-Minute Check Transparency 4-8 Use as a quiz or review of Lesson 4-7.

Building on Prior Knowledge

To solve a system of equations using a matrix equation, students will use the matrix multiplication skills they acquired in Lesson 4-3.

Mathematical Background notes are available for this lesson on p. 152D.

How can matrices be used in population ecology?

Ask students:

- Which of the two species of birds needs more pounds of food per nesting pair during its nesting season? **Species A**
- Which of the two species needs more territory? **Species A**

4-8 Using Matrices to Solve Systems of Equations

What You'll Learn

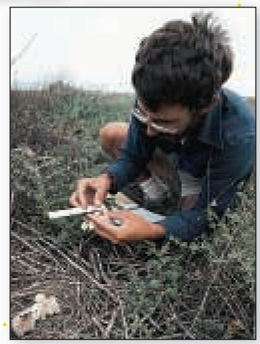
- Write matrix equations for systems of equations.
- Solve systems of equations using matrix equations.

Vocabulary

- matrix equation

How can matrices be used in population ecology?

Population ecology is the study of a species or a group of species that inhabits the same area. A biologist is studying two species of birds that compete for food and territory. He estimates that a particular region with an area of 14.25 acres (approximately 69,000 square yards) can supply 20,000 pounds of food for the birds during their nesting season. Species A needs 140 pounds of food and has a territory of 500 square yards per nesting pair. Species B needs 120 pounds of food and has a territory of 400 square yards per nesting pair. The biologist can use this information to find the number of birds of each species that the area can support.



WRITE MATRIX EQUATIONS The situation above can be represented using a system of equations that can be solved using matrices. Consider the system of equations below. You can write this system with matrices by using the left and right sides of the equations.

$$\begin{array}{l} 5x + 7y = 11 \\ 3x + 8y = 18 \end{array} \rightarrow \begin{bmatrix} 5x + 7y \\ 3x + 8y \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \end{bmatrix}$$

Write the matrix on the left as the product of the coefficients and the variables.

$$\underbrace{\begin{bmatrix} 5 & 7 \\ 3 & 8 \end{bmatrix}}_{\text{coefficient matrix}}^{A} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{variable matrix}}^{X} = \underbrace{\begin{bmatrix} 11 \\ 18 \end{bmatrix}}_{\text{constant matrix}}^{B}$$

The system of equations is now expressed as a **matrix equation**.

Example 1 *Two-Variable Matrix Equation*

Write a matrix equation for the system of equations.
$5x - 6y = -47$
$3x + 2y = -17$

Determine the coefficient, variable, and constant matrices.

$$\begin{array}{l} 5x - 6y = -47 \\ 3x + 2y = -17 \end{array} \rightarrow \begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} -47 \\ -17 \end{bmatrix}$$

Write the matrix equation.

$$\overset{A}{\begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix}} \cdot \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} -47 \\ -17 \end{bmatrix}}$$

Resource Manager

Workbook and Reproducible Masters

Chapter 4 Resource Masters
- Study Guide and Intervention, pp. 211–212
- Skills Practice, p. 213
- Practice, p. 214
- Reading to Learn Mathematics, p. 215
- Enrichment, p. 216
- Assessment, p. 232

School-to-Career Masters, p. 8

Transparencies

5-Minute Check Transparency 4-8
Answer Key Transparencies

Technology

Interactive Chalkboard

You can use a matrix equation to determine the weight of an atom of an element.

Example 2 Solve a Problem Using a Matrix Equation

CHEMISTRY The molecular formula for glucose is $C_6H_{12}O_6$, which represents that a molecule of glucose has 6 carbon (C) atoms, 12 hydrogen (H) atoms, and 6 oxygen (O) atoms. One molecule of glucose weighs 180 atomic mass units (amu), and one oxygen atom weighs 16 atomic mass units. The formulas and weights for glucose and another sugar, sucrose, are listed below.

Sugar	Formula	Atomic Weight (amu)
glucose	$C_6H_{12}O_6$	180
sucrose	$C_{12}H_{22}O_{11}$	342

a. Write a system of equations that represents the weight of each atom.

Let c represent the weight of a carbon atom.
Let h represent the weight of a hydrogen atom.

Write an equation for the weight of each sugar. The subscript represents how many atoms of each element are in the molecule.

Glucose: $6c + 12h + 6(16) = 180$ Equation for glucose
$6c + 12h + 96 = 180$ Simplify.
$6c + 12h = 84$ Subtract 96 from each side.

Sucrose: $12c + 22h + 11(16) = 342$ Equation for sucrose
$12c + 22h + 176 = 342$ Simplify.
$12c + 22h = 166$ Subtract 176 from each side.

b. Write a matrix equation for the system of equations.

Determine the coefficient, variable, and constant matrices.

$$\begin{aligned} 6c + 12h &= 84 \\ 12c + 22h &= 166 \end{aligned} \rightarrow \begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix} \quad \begin{bmatrix} c \\ h \end{bmatrix} \quad \begin{bmatrix} 84 \\ 166 \end{bmatrix}$$

Write the matrix equation.

$$\underset{A}{\begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} c \\ h \end{bmatrix}} = \underset{B}{\begin{bmatrix} 84 \\ 166 \end{bmatrix}}$$

You will solve this matrix equation in Exercise 11.

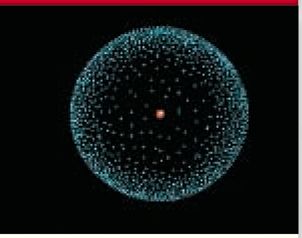
SOLVE SYSTEMS OF EQUATIONS You can solve a system of linear equations by solving a matrix equation. A matrix equation in the form $AX = B$, where A is a coefficient matrix, X is a variable matrix, and B is a constant matrix, can be solved in a similar manner as a linear equation of the form $ax = b$.

$ax = b$	Write the equation.	$AX = B$
$\left(\frac{1}{a}\right)ax = \left(\frac{1}{a}\right)b$	Multiply each side by the inverse of the coefficient, if it exists.	$A^{-1}AX = A^{-1}B$
$1x = \left(\frac{1}{a}\right)b$	$\left(\frac{1}{a}\right)a = 1$, $A^{-1}A = I$	$IX = A^{-1}B$
$x = \left(\frac{1}{a}\right)b$	$1x = x$, $IX = X$	$X = A^{-1}B$

Notice that the solution of the matrix equation is the product of the inverse of the coefficient matrix and the constant matrix.

www.algebra2.com/extra_examples

More About. . .



Chemistry

Atomic mass units are relative units of weight because they were compared to the weight of a hydrogen atom. So a molecule of nitrogen, whose weight is 14.0 amu, weighs 14 times as much as a hydrogen atom.

Source: www.sizes.com

2 Teach

WRITE MATRIX EQUATIONS

In-Class Examples Power Point®

1 Write a matrix equation for the system of equations.
$x + 3y = 3$
$x + 2y = 7$

$$\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

2 **FABRICS** The table below shows the composition of three types of fabrics and the cost per yard of each type.

Type	Wool	Silk	Cotton	Cost
R	10%	20%	70%	$7
S	20%	30%	50%	$8
T	20%	50%	30%	$10

a. Write a system of equations that represents the total cost for each of the three fabric components.
$0.1w + 0.2s + 0.7c = 7$
$0.2w + 0.3s + 0.5c = 8$
$0.2w + 0.5s + 0.3c = 10$

b. Write a matrix equation for the system of equations.

$$\begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.3 & 0.5 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \cdot \begin{bmatrix} w \\ s \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}$$

Lesson 4-8 Using Matrices to Solve Systems of Equations 203

SOLVE SYSTEMS OF EQUATIONS

In-Class Examples Power Point®

3 Use a matrix equation to solve the system of equations.
$5x + 3y = 13$
$4x + 7y = -8$ **(5, −4)**

4 Use a matrix equation to solve the system of equations.
$10x + 5y = 15$
$6x + 3y = -6$
There is no unique solution of this system.

Example 3 Solve a System of Equations

Use a matrix equation to solve the system of equations.
$\mathbf{6x + 2y = 11}$
$\mathbf{3x - 8y = 1}$

The matrix equation is $\begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$, when $A = \begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$.

Step 1 Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-48 - 6}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \text{ or } -\frac{1}{54}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix}$$

Step 2 Multiply each side of the matrix equation by the inverse matrix.

$$-\frac{1}{54}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 1 \end{bmatrix} \quad \text{Multiply each side by } A^{-1}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54}\begin{bmatrix} -90 \\ -27 \end{bmatrix} \quad \text{Multiply matrices.}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The solution is $\left(\frac{5}{3}, \frac{1}{2}\right)$. Check this solution in the original equation.

Study Tip

Identity Matrix
The identity matrix on the left verifies that the inverse matrix has been calculated correctly.

Example 4 System of Equations with No Solution

Use a matrix equation to solve the system of equations.
$\mathbf{6a - 9b = -18}$
$\mathbf{8a - 12b = 24}$

The matrix equation is $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$, when $A = \begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \end{bmatrix}$, and $B = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$.

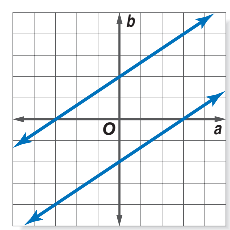
Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-72 + 72}\begin{bmatrix} -12 & 9 \\ -8 & 6 \end{bmatrix}$$

The determinant of the coefficient matrix $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$ is 0, so A^{-1} does not exist.

There is no unique solution of this system.

Graph the system of equations. Since the lines are parallel, this system has no solution. Therefore, the system is inconsistent.



Study Tip

Look Back
To review **inconsistent systems of equations**, see Lesson 3-1.

To solve a system of equations with three variables, you can use the 3×3 identity matrix. However, finding the inverse of a 3×3 matrix may be tedious. Graphing calculators and computer programs offer fast and accurate methods for performing the necessary calculations.

DAILY INTERVENTION

Differentiated Instruction

Logical Ask students to list those matrix operations that they know how to perform using their graphing calculator. Ask those students who are adept with the calculator to explain each series of keystrokes and their purpose to a classmate who is experiencing difficulty.

204 Chapter 4 Matrices

Graphing Calculator Investigation

Systems of Three Equations in Three Variables

You can use a graphing calculator and a matrix equation to solve systems of equations. Consider the system of equations below.

$3x - 2y + z = 0$
$2x + 3y - z = 17$
$5x - y + 4z = -7$

Think and Discuss

1. Write a matrix equation for the system of equations.
2. Enter the coefficient matrix as matrix A and the constant matrix as matrix B in the graphing calculator. Find the product of A^{-1} and B. Recall that the $\boxed{x^{-1}}$ key is used to find A^{-1}.
3. How is the result related to the solution? **(3, 2, −5)**

1. $\begin{bmatrix} 3 & -2 & 1 \\ 2 & 3 & -1 \\ 5 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 17 \\ -7 \end{bmatrix}$

2. $\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

Check for Understanding

Concept Check

1. **Write** the matrix equation $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ as a system of linear equations.
2. **OPEN ENDED** Write a system of equations that does not have a unique solution. **Sample answer: $x + 3y = 8$ and $2x + 6y = 16$**
3. **FIND THE ERROR** Tommy and Laura are solving a system of equations. They find that $A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -7 \\ -9 \end{bmatrix}$, and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

Tommy

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ -9 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Laura

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 42 \\ 31 \end{bmatrix}$

Who is correct? Explain your reasoning.

1. $2r - 3s = 4$, $r + 4s = -2$

3. Tommy; a 2×1 matrix cannot be multiplied by a 2×2 matrix.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–6	1
7–10	3, 4
11	2

Write a matrix equation for each system of equations. **4–6. See pp. 217A–217B.**

4. $x - y = -3$
   $x + 3y = 5$
5. $2g + 3h = 8$
   $-4g - 7h = -5$
6. $3a - 5b + 2c = 9$
   $4a + 7b + c = 3$
   $2a - c = 12$

Solve each matrix equation or system of equations by using inverse matrices.

7. $\begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 24 \end{bmatrix}$ **(5, −2)**
8. $\begin{bmatrix} 8 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 16 \\ -9 \end{bmatrix}$ **(1.5, −4)**
9. $5x - 3y = -30$ **(−3, 5)**
   $8x + 5y = 1$
10. $5s + 4t = 12$ **(1, 1.75)**
    $4s - 3t = -1.25$

Application

11. **CHEMISTRY** Refer to Example 2 on page 203. Solve the system of equations to find the weight of a carbon, hydrogen, and oxygen atom. **$h = 1$, $c = 12$**

3 Practice/Apply

Study Notebook

Have students—
- complete the definitions/examples for the remaining terms on their Vocabulary Builder worksheets for Chapter 4.
- show the conversion of a system of equations to a matrix equation.
- write an explanation of the procedure for solving a matrix equation.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

DAILY INTERVENTION

FIND THE ERROR Emphasize that multiplication of the constant matrix B by the inverse matrix A^{-1} is "multiplication on the left." That is, $X = A^{-1}B$ and not $X = BA^{-1}$.

About the Exercises…

Organization by Objective
- **Write Matrix Equations:** 12–19
- **Solve Systems of Equations:** 20–34

Odd/Even Assignments

Exercises 12–33 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Alert! Exercises 39–41 require a graphing calculator.

Assignment Guide

Basic: 13–33 odd, 35–38, 42–51

Average: 13–33 odd, 35–38, 42–51 (optional: 39–41)

Advanced: 12–34 even, 35–51

Graphing Calculator Investigation

On the TI-83 calculator, to find A^{-1} once you have entered the elements of matrix A and have a clear screen, students should press MATRX 1 (to choose Matrix A, the first in the list), and then the $\boxed{x^{-1}}$ key followed by ENTER. Also, stress that matrix B must be multiplied *on the left* by A^{-1}.

Study Guide and Intervention, p. 211 (shown) and p. 212

Write Matrix Equations A **matrix equation** for a system of equations consists of the product of the coefficient and variable matrices on the left and the constant matrix on the right of the equals sign.

Example **Write a matrix equation for each system of equations.**

a. $3x - 7y = 12$
$x + 5y = -8$

Determine the coefficient, variable, and constant matrices.

$\begin{bmatrix} 3 & -7 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$

b. $2x - y + 3z = -7$
$x + 3y - 4z = 15$
$7x + 2y + z = -28$

$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & -4 \\ 7 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 15 \\ -28 \end{bmatrix}$

Exercises

Write a matrix equation for each system of equations.

1. $2x + y = 8$, $5x - 3y = -12$ $\begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \end{bmatrix}$

2. $4x - 3y = 18$, $x + 2y = 12$ $\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \end{bmatrix}$

3. $7x - 2y = 15$, $3x + y = -10$ $\begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \end{bmatrix}$

4. $4x - 6y = 20$, $3x + y + 8 = 0$ $\begin{bmatrix} 4 & -6 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ -8 \end{bmatrix}$

5. $5x + 2y = 18$, $x = -4y + 25$ $\begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 25 \end{bmatrix}$

6. $3x - y = 24$, $3y = 80 - 2x$ $\begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24 \\ 80 \end{bmatrix}$

7. $2x + y + 7z = 12$, $5x - y + 3z = 15$, $x + 2y - 6z = 25$ $\begin{bmatrix} 2 & 1 & 7 \\ 5 & -1 & 3 \\ 1 & 2 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 25 \end{bmatrix}$

8. $5x - y + 7z = 32$, $x + 3y - 2z = -18$, $2x + 4y - 3z = 12$ $\begin{bmatrix} 5 & -1 & 7 \\ 1 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ -18 \\ 12 \end{bmatrix}$

9. $4x - 3y - z = -100$, $2x + y - 3z = -64$, $5x + 3y - 2z = 8$ $\begin{bmatrix} 4 & -3 & -1 \\ 2 & 1 & -3 \\ 5 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -100 \\ -64 \\ 8 \end{bmatrix}$

10. $x - 3y + 7z = 27$, $2x + y - 5z = 48$, $4x - 2y + 3z = 72$ $\begin{bmatrix} 1 & -3 & 7 \\ 2 & 1 & -5 \\ 4 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 27 \\ 48 \\ 72 \end{bmatrix}$

11. $2x + 3y - 9z = -108$, $x + 5z = 40 + 2y$, $3x + 5y = 89 + 4z$ $\begin{bmatrix} 2 & 3 & -9 \\ 1 & -2 & 5 \\ 3 & 5 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -108 \\ 40 \\ 89 \end{bmatrix}$

12. $z = 45 - 3x + 2y$, $2x + 3y - z = 60$, $x = 4y - 2z + 120$ $\begin{bmatrix} 3 & -2 & 1 \\ 2 & 3 & -1 \\ 1 & -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 60 \\ 120 \end{bmatrix}$

Skills Practice, p. 213 and Practice, p. 214 (shown)

Write a matrix equation for each system of equations.

1. $-3x + 2y = 9$, $5x - 3y = -13$ $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -13 \end{bmatrix}$

2. $6x - 2y = -2$, $3x + 3y = 10$ $\begin{bmatrix} 6 & -2 \\ 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

3. $2a + b = 0$, $3a + 2b = -2$ $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

4. $r + 5s = 10$, $2r - 3s = 7$ $\begin{bmatrix} 1 & 5 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$

5. $3x - 2y + 5z = 3$, $x + y - 4z = 2$, $-2x + 2y + 7z = -5$ $\begin{bmatrix} 3 & -2 & 5 \\ 1 & 1 & -4 \\ -2 & 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

6. $2m + n - 3p = -5$, $5m + 2n - 2p = 8$, $3m - 3n + 5p = 17$ $\begin{bmatrix} 2 & 1 & -3 \\ 5 & 2 & -2 \\ 3 & -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \\ p \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 17 \end{bmatrix}$

Solve each matrix equation or system of equations by using inverse matrices.

7. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ **(2, −4)**

8. $\begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 10 \end{bmatrix}$ **(5, 1)**

9. $\begin{bmatrix} -1 & -3 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 12 \\ -11 \end{bmatrix}$ **(3, −5)**

10. $\begin{bmatrix} -5 & 3 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 16 \\ 34 \end{bmatrix}$ **(1, 7)**

11. $\begin{bmatrix} -4 & 2 \\ 7 & 4 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 17 \\ -26 \end{bmatrix}$ $\left(-4, \frac{1}{2}\right)$

12. $\begin{bmatrix} 8 & 3 \\ 12 & 6 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\left(-\frac{1}{4}, \frac{1}{3}\right)$

13. $2x + 3y = 5$, $3x - 2y = 1$ **(1, 1)**

14. $8d + 9f = 13$, $-6d + 5f = -45$ **(5, −3)**

15. $5m + 9n = 19$, $2m - n = -20$ **(−7, 6)**

16. $-4j + 9k = -8$, $6j + 12k = -5$ $\left(\frac{1}{2}, -\frac{2}{3}\right)$

17. AIRLINE TICKETS Last Monday at 7:30 A.M., an airline flew 89 passengers on a commuter flight from Boston to New York. Some of the passengers paid $120 for their tickets and the rest paid $230 for their tickets. The total cost of all of the tickets was $14,200. How many passengers bought $120 tickets? How many bought $230 tickets?
57; 32

18. NUTRITION A single dose of a dietary supplement contains 0.2 gram of calcium and 0.2 gram of vitamin C. A single dose of a second dietary supplement contains 0.1 gram of calcium and 0.4 gram of vitamin C. If a person wants to take 0.6 gram of calcium and 1.2 grams of vitamin C, how many doses of each supplement should she take?
2 doses of the first supplement and 2 doses of the second supplement

Reading to Learn Mathematics, p. 215 ELL

Pre-Activity **How are inverse matrices used in population ecology?**

Read the introduction to Lesson 4-8 at the top of page 202 in your textbook.

Write a 2×2 matrix that summarizes the information given in the introduction about the food and territory requirements for the two species.

$\begin{bmatrix} 140 & 500 \\ 120 & 400 \end{bmatrix}$

Reading the Lesson

1. a. Write a matrix equation for the following system of equations.
$3x + 5y = 10$
$2x - 4y = -7$
$\begin{bmatrix} 3 & 5 \\ 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \end{bmatrix}$

b. Explain how to use the matrix equation you wrote above to solve the system. Use as few mathematical symbols in your explanation as you can. Do not actually solve the system.

Sample answer: Find the inverse of the 2 × 2 matrix of coefficients. Multiply this inverse by the 2 × 1 matrix of constants, with the 2 × 2 matrix on the left. The product will be a 2 × 1 matrix. The number in the first row will be the value of *x*, and the number in the second row will be the value of *y*.

2. Write a system of equations that corresponds to the following matrix equation.
$\begin{bmatrix} 3 & 2 & -4 \\ 2 & -1 & 0 \\ 0 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ -4 \end{bmatrix}$ $3x + 2y - 4z = -2$, $2x - y = 6$, $5y + 6z = -4$

Helping You Remember

3. Some students have trouble remembering how to set up a matrix equation to solve a system of linear equations. What is an easy way to remember the order in which to write the three matrices that make up the equation?

Sample answer: Just remember "CVC" for "coefficients, variables, constants." The variable matrix is on the left side of the equals sign, just as the variables are in the system of linear equations. The constant matrix is on the right side of the equals sign, just as the constants are in the system of linear equations.

Practice and Apply

Homework Help

For Exercises	See Examples
12–19	1
20–31	3, 4
32–34	2

Extra Practice

See page 836.

Write a matrix equation for each system of equations. **12–19. See margin.**

12. $3x - y = 0$
$x + 2y = -21$

13. $4x - 7y = 2$
$3x + 5y = 9$

14. $5a - 6b = -47$
$3a + 2b = -17$

15. $3m - 7n = -43$
$6m + 5n = -10$

16. $2a + 3b - 5c = 1$
$7a + 3c = 7$
$3a - 6b + c = -5$

17. $3x - 5y + 2z = 9$
$x - 7y + 3z = 11$
$4x - 3z = -1$

18. $x - y = 8$
$-2x - 5y - 6z = -27$
$9x + 10y - z = 54$

19. $3r - 5s + 6t = 21$
$11r - 12s + 16t = 15$
$-5r + 8s - 3t = -7$

Solve each matrix equation or system of equations by using inverse matrices.

20. $\begin{bmatrix} 7 & -3 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 41 \\ 0 \end{bmatrix}$ **(5, −2)**

21. $\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \end{bmatrix}$ **(3, 4)**

22. $\begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -17 \\ -4 \end{bmatrix}$ **(−2, 3)**

23. $\begin{bmatrix} 7 & 1 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 43 \\ 10 \end{bmatrix}$ **(6, 1)**

24. $\begin{bmatrix} 2 & -9 \\ 6 & 5 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 28 \\ -12 \end{bmatrix}$ $\left(\frac{1}{2}, -3\right)$

25. $\begin{bmatrix} 6 & 5 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 18 \\ 7 \end{bmatrix}$ $\left(-\frac{1}{3}, 4\right)$

26. $6r + s = 9$ **(2, −3)**
$3r = -2s$

27. $5a + 9b = -28$ **(−2, −2)**
$2a - b = -2$

28. $p - 2q = 1$ **(7, 3)**
$p + 5q = 22$

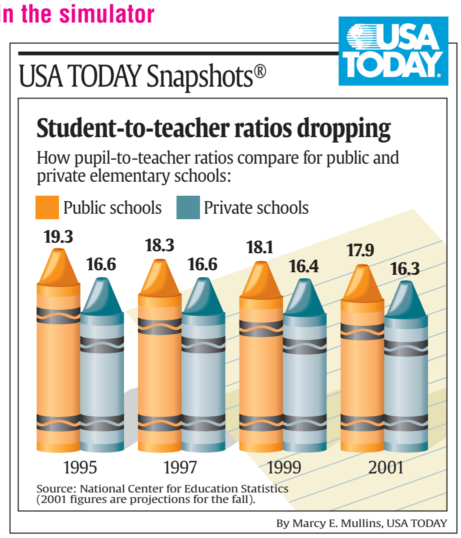
29. $4m - 7n = -63$ **(0, 9)**
$3m + 2n = 18$

30. $x + 2y = 8$
$3x + 2y = 6$ $\left(-1, \frac{9}{2}\right)$

31. $4x - 3y = 5$
$2x + 9y = 6$ $\left(\frac{3}{2}, \frac{1}{3}\right)$

32. PILOT TRAINING Hai-Ling is training for his pilot's license. Flight instruction costs $105 per hour, and the simulator costs $45 per hour. The school requires students to spend 4 more hours in airplane training than in the simulator. If Hai-Ling can afford to spend $3870 on training, how many hours can he spend training in an airplane and in a simulator?
27 h of flight instruction and 23 h in the simulator

33. SCHOOLS The graphic shows that student-to-teacher ratios are dropping in both public and private schools. If these rates of change remain constant, predict when the student-to-teacher ratios for private and public schools will be the same. **2010**



34. CHEMISTRY Cara is preparing an acid solution. She needs 200 milliliters of 48% concentration solution. Cara has 60% and 40% concentration solutions in her lab. How many milliliters of 40% acid solution should be mixed with 60% acid solution to make the required amount of 48% acid solution? **80 mL of the 60% solution and 120 mL of the 40% solution**

Enrichment, p. 216

Properties of Matrices

Computing with matrices is different from computing with real numbers. Stated below are some properties of the real number system. Are these also true for matrices? In the problems on this page, you will investigate this question.

For all real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Multiplication is commutative. For all real numbers a and b, $ab = ba$.

Multiplication is associative. For all real numbers a, b, and c, $a(bc) = (ab)c$.

Use the matrices A, B, and C for the problems. Write whether each statement is true. Assume that a 2-by-2 matrix is the 0 matrix if and only if all of its elements are zero.

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

1. $AB = 0$ **no**

2. $AC = 0$ **no**

3. $BC = 0$ **yes**

35. The solution set is the empty set or all real numbers.

35. **CRITICAL THINKING** Describe the solution set of a system of equations if the coefficient matrix does not have an inverse.

36. ***WRITING IN MATH*** Answer the question that was posed at the beginning of the lesson. See pp. 217A–217B.

How can matrices be used in population ecology?

Include the following in your answer:

- a system of equations that can be used to find the number of each species the region can support, and
- a solution of the problem using matrices.

Standardized Test Practice

37. Solve the system of equations $6a + 8b = 5$ and $10a - 12b = 2$. **D**

Ⓐ $\left(\frac{3}{4}, \frac{1}{2}\right)$ Ⓑ $\left(\frac{1}{2}, \frac{3}{4}\right)$ Ⓒ $\left(\frac{1}{2}, -\frac{1}{2}\right)$ Ⓓ $\left(\frac{1}{2}, \frac{1}{4}\right)$

38. **SHORT RESPONSE** The Yogurt Shoppe sells cones in three sizes: small \$0.89; medium, \$1.19; and large, \$1.39. One day Scott sold 52 cones. He sold seven more medium cones than small cones. If he sold \$58.98 in cones, how many of each size did he sell? **17 small, 24 medium, 11 large**

Graphing Calculator

INVERSE MATRICES **Use a graphing calculator to solve each system of equations using inverse matrices.**

39. $2a - b + 4c = 6$
$a + 5b - 2c = -6$
$3a - 2b + 6c = 8$
(−6, 2, 5)

40. $3x - 5y + 2z = 22$
$2x + 3y - z = -9$
$4x + 3y + 3z = 1$
(1, −3, 2)

41. $2q + r + s = 2$
$-q - r + 2s = 7$
$-3q + 2r + 3s = 7$
(0, −1, 3)

Maintain Your Skills

Mixed Review

Find the inverse of each matrix, if it exists. *(Lesson 4-7)*

42. $\begin{bmatrix} 4 & 4 \\ 2 & 3 \end{bmatrix}$ **42.** $\begin{bmatrix} \frac{3}{4} & -1 \\ -\frac{1}{2} & 1 \end{bmatrix}$

43. $\begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$ $\begin{bmatrix} 4 & -5 \\ -7 & 9 \end{bmatrix}$

44. $\begin{bmatrix} -3 & -6 \\ 5 & 10 \end{bmatrix}$ **no inverse exists**

Use Cramer's Rule to solve each system of equations. *(Lesson 4-6)*

45. $6x + 7y = 10$
$3x - 4y = 20$ **(4, −2)**

46. $6a + 7b = -10.15$
$9.2a - 6b = 69.944$ **46. (4.27, −5.11)**

47. $\frac{x}{2} - \frac{2y}{3} = 2\frac{1}{3}$
$3x + 4y = -50$ **(−6, −8)**

48. **ECOLOGY** If you recycle a $3\frac{1}{2}$-foot stack of newspapers, one less 20-foot loblolly pine tree will be needed for paper. Use a prediction equation to determine how many feet of loblolly pine trees will *not* be needed for paper if you recycle a pile of newspapers 20 feet tall. *(Lesson 2-5)* **about 114.3 ft**

Solve each equation. Check your solutions. *(Lesson 1-4)*

49. $|x - 3| = 7$ **{−4, 10}**

50. $-4|d + 2| = -12$ **{−5, 1}**

51. $5|k - 4| = k + 8$ **{2, 7}**

WebQuest Internet Project

Lessons in Home Buying, Selling

It is time to complete your project. Use the information and data you have gathered about home buying and selling to prepare a portfolio or Web page. Be sure to include your tables, graphs, and calculations. You may also wish to include additional data, information, or pictures.

www.algebra2.com/webquest

www.algebra2.com/self_check_quiz

4 Assess

Open-Ended Assessment

Speaking Have students describe how matrices can be used to solve systems of two equations in two variables.

Assessment Options

Quiz (Lessons 4-7 and 4-8) is available on p. 232 of the *Chapter 4 Resource Masters*.

Answers

12. $\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -21 \end{bmatrix}$

13. $\begin{bmatrix} 4 & -7 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

14. $\begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -47 \\ -17 \end{bmatrix}$

15. $\begin{bmatrix} 3 & -7 \\ 6 & 5 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} -43 \\ -10 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 3 & -5 \\ 7 & 0 & 3 \\ 3 & -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ -5 \end{bmatrix}$

17. $\begin{bmatrix} 3 & -5 & 2 \\ 1 & -7 & 3 \\ 4 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \\ -1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & -5 & -6 \\ 9 & 10 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -27 \\ 54 \end{bmatrix}$

19. $\begin{bmatrix} 3 & -5 & 6 \\ 11 & -12 & 16 \\ -5 & 8 & -3 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 21 \\ 15 \\ -7 \end{bmatrix}$

Online Lesson Plans

USA TODAY Education's Online site offers resources and interactive features connected to each day's newspaper. *Experience TODAY*, USA TODAY's daily lesson plan, is available on the site and delivered daily to subscribers. This plan provides instruction for integrating USA TODAY graphics and key editorial features into your mathematics classroom. Log on to www.education.usatoday.com.

Graphing Calculator Investigation

A Follow-Up of Lesson 4-8

Getting Started

Know Your Calculator The TI-83 Plus allows you to perform row operations on matrices. These row operations are items C through F on the MATRX Math menu. Successive use of row operations allows you to transform a matrix to reduced row echelon form. The rref(function performs all the steps at once, thereby saving a great deal of time.

Teach

- Point out that if one of the variables is absent from an equation in a system of equations, then its coefficient is zero. In Exercise 6, students may find it helpful to rewrite the equation showing 0 as the coefficient of the missing variables in the second and third equations in order to determine the correct augmented matrix.
- Have students complete Exercises 1–6.

Assess

Ask students which method they prefer for solving systems of two equations in two variables, the graphing calculator method shown in this investigation or the method presented in Lesson 4-8. Have them choose a preferred method for solving systems of three equations in three variables. Have them explain their choices.

Graphing Calculator Investigation

A Follow-Up of Lesson 4-8

Augmented Matrices

Using a TI-83 Plus, you can solve a system of linear equations using the MATRX function. An **augmented matrix** contains the coefficient matrix with an extra column containing the constant terms. The reduced row echelon function of a graphing calculator reduces the augmented matrix so that the solution of the system of equations can be easily determined.

Write an augmented matrix for the following system of equations. Then solve the system by using the reduced row echelon form on the graphing calculator.

$3x + y + 3z = 2$
$2x + y + 2z = 1$
$4x + 2y + 5z = 5$

Step 1 Write the augmented matrix and enter it into a calculator.

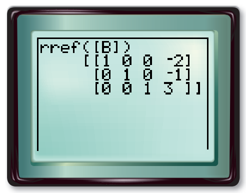
The augmented matrix $B = \left[\begin{array}{ccc|c} 3 & 1 & 3 & 2 \\ 2 & 1 & 2 & 1 \\ 4 & 2 & 5 & 5 \end{array}\right]$.

Begin by entering the matrix.

KEYSTROKES: *Review matrices on page 163.*

Step 2 Find the reduced row echelon form (rref) using the graphing calculator.

KEYSTROKES: 2nd [MATRX] ▶ ALPHA [B] 2nd [MATRX] 2) ENTER



Study the reduced echelon matrix. The first three columns are the same as a 3 × 3 identity matrix. The first row represents $x = -2$, the second row represents $y = -1$, and the third row represents $z = 3$. The solution is $(-2, -1, 3)$.

Exercises

Write an augmented matrix for each system of equations. Then solve with a graphing calculator. 1–6. See margin for matrices.

1. $x - 3y = 5$
$2x + y = 1$ **(1.14, −1.29)**

2. $15x + 11y = 36$
$4x - 3y = -26$ **(−2, 6)**

3. $2x + y = 5$
$2x - 3y = 1$ **(2, 1)**

4. $3x - y = 0$
$2x - 3y = 1$ **(−0.14, −0.43)**

5. $3x - 2y + z = -2$
$x - y + 3z = 5$
$-x + y + z = -1$ **(−7, −9, 1)**

6. $x - y + z = 2$
$x - z = 1$
$y + 2z = 0$ **(1.25, −0.5, 0.25)**

www.algebra2.com/other_calculator_keystrokes

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Answers

1. $A = \left[\begin{array}{cc|c} 1 & -3 & 5 \\ 2 & 1 & 1 \end{array}\right]$

2. $A = \left[\begin{array}{cc|c} 15 & 11 & 36 \\ 4 & -3 & -26 \end{array}\right]$

3. $A = \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 2 & -3 & 1 \end{array}\right]$

4. $A = \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 2 & -3 & 1 \end{array}\right]$

5. $A = \left[\begin{array}{ccc|c} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{array}\right]$

6. $A = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right]$

Study Guide and Review

Vocabulary and Concept Check

column matrix (p. 155)	expansion by minors (p. 183)	minor (p. 183)	second-order determinant (p. 182)
Cramer's Rule (p. 189)	identity matrix (p. 195)	preimage (p. 175)	square matrix (p. 155)
determinant (p. 182)	image (p. 175)	reflection (p. 177)	third-order determinant (p. 183)
dilation (p. 176)	inverse (p. 195)	rotation (p. 178)	transformation (p. 175)
dimension (p. 155)	isometry (p. 175)	row matrix (p. 155)	translation (p. 175)
element (p. 155)	matrix (p. 154)	scalar (p. 162)	vertex matrix (p. 175)
equal matrices (p. 155)	matrix equation (p. 202)	scalar multiplication (p. 162)	zero matrix (p. 155)

Choose the correct term to complete each sentence. **3. Scalar multiplication**

determinant
dilation
dimensions
equal matrices
identity matrix
isometry
matrix equation
rotation
scalar multiplication
translation

1. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a(n) _______ for multiplication. **identity matrix**
2. When an image and a preimage are congruent, then the transformation is called a(n) _______. **isometry**
3. _______ is the process of multiplying a matrix by a constant.
4. A(n) **rotation** is when a figure is moved around a center point.
5. The _______ of $\begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}$ is -5. **determinant**
6. A(n) _______ is the product of the coefficient matrix and the variable matrix equal to the constant matrix. **matrix equation**
7. The _______ of a matrix tell how many rows and columns are in the matrix. **dimensions**
8. A(n) _______ occurs when a figure is moved from one location to another on the coordinate plane. **translation**
9. The matrices $\begin{bmatrix} 3x \\ x + 2y \end{bmatrix}$ and $\begin{bmatrix} y \\ 7 \end{bmatrix}$ are _______ if $x = 1$ and $y = 3$. **equal matrices**
10. A(n) **dilation** is when a geometric figure is enlarged or reduced.

Lesson-by-Lesson Review

4-1 Introduction to Matrices

See pages 154–158.

Concept Summary

- A matrix is a rectangular array of variables or constants in horizontal rows and vertical columns.
- Equal matrices have the same dimensions and corresponding elements equal.

Example **Solve $\begin{bmatrix} 2x \\ y \end{bmatrix} = \begin{bmatrix} 32 + 6y \\ 7 - x \end{bmatrix}$ for x and y.**

Since the matrices are equal, corresponding elements are equal. You can write two linear equations.

$2x = 32 + 6y$
$y = 7 - x$

(continued on the next page)

www.algebra2.com/vocabulary_review

Study Guide and Review

Vocabulary and Concept Check

- This alphabetical list of vocabulary terms in Chapter 4 includes a page reference where each term was introduced.
- **Assessment** A vocabulary test/review for Chapter 4 is available on p. 230 of the *Chapter 4 Resource Masters*.

Lesson-by-Lesson Review

For each lesson,

- the main ideas are summarized,
- additional examples review concepts, and
- practice exercises are provided.

Vocabulary PuzzleMaker

ELL The Vocabulary PuzzleMaker software improves students' mathematics vocabulary using four puzzle formats—crossword, scramble, word search using a word list, and word search using clues. Students can work on a computer screen or from a printed handout.

MindJogger Videoquizzes

ELL MindJogger Videoquizzes provide an alternative review of concepts presented in this chapter. Students work in teams in a game show format to gain points for correct answers. The questions are presented in three rounds.

Round 1 Concepts (5 questions)
Round 2 Skills (4 questions)
Round 3 Problem Solving (4 questions)

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Ask students to check over their notes and descriptions about matrices to see if they wish to add any further information, either about the definitions of operations with matrices or about ways that matrices are applied to systems of equations.

Encourage students to refer to their Foldables while completing the Study Guide and Review and to use them in preparing for the Chapter Test.

Solve the system of equations.

$2x = 32 + 6y$	First equation
$2x = 32 + 6(7 - x)$	Substitute $7 - x$ for y.
$2x = 32 + 42 - 6x$	Distributive Property
$8x = 74$	Add $6x$ to each side.
$x = 9.25$	Divide each side by 8.

The solution is (9.25, −2.25).

To find the value for y, substitute 9.25 for x in either equation.

$y = 7 - x$	Second equation
$= 7 - 9.25$	Substitute 9.25 for x.
$= -2.25$	Simplify.

Exercises **Solve each equation.** *See Example 3 on pages 155 and 156.*

11. $\begin{bmatrix} 2y - x \\ x \end{bmatrix} = \begin{bmatrix} 3 \\ 4y - 1 \end{bmatrix}$ **(−5, −1)**

12. $\begin{bmatrix} 7x \\ x + y \end{bmatrix} = \begin{bmatrix} 5 + 2y \\ 11 \end{bmatrix}$ **(3, 8)**

13. $\begin{bmatrix} 3x + y \\ x - 3y \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ **(−1, 0)**

14. $\begin{bmatrix} 2x - y \\ 6x - y \end{bmatrix} = \begin{bmatrix} 2 \\ 22 \end{bmatrix}$ **(5, 8)**

4-2 Operations with Matrices

See pages 160–166.

Concept Summary

- Matrices can be added or subtracted if they have the same dimensions. Add or subtract corresponding elements.
- To multiply a matrix by a scalar k, multiply each element in the matrix by k.

Examples

1 **Find $A - B$ if $A = \begin{bmatrix} 3 & 8 \\ -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 6 \\ 1 & 9 \end{bmatrix}$.**

$A - B = \begin{bmatrix} 3 & 8 \\ -5 & 2 \end{bmatrix} - \begin{bmatrix} -4 & 6 \\ 1 & 9 \end{bmatrix}$ Definition of matrix subtraction

$= \begin{bmatrix} 3 - (-4) & 8 - 6 \\ -5 - 1 & 2 - 9 \end{bmatrix}$ Subtract corresponding elements.

$= \begin{bmatrix} 7 & 2 \\ -6 & -7 \end{bmatrix}$ Simplify.

2 **If $X = \begin{bmatrix} 3 & 2 & -1 \\ 4 & -6 & 0 \end{bmatrix}$, find $4X$.**

$4X = 4\begin{bmatrix} 3 & 2 & -1 \\ 4 & -6 & 0 \end{bmatrix}$

$= \begin{bmatrix} 4(3) & 4(2) & 4(-1) \\ 4(4) & 4(-6) & 4(0) \end{bmatrix}$ or $\begin{bmatrix} 12 & 8 & -4 \\ 16 & -24 & 0 \end{bmatrix}$ Multiply each element by 4.

Exercises **Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.** *See Examples 1, 2, and 4 on pages 160–162.*

15. $\begin{bmatrix} -4 & 3 \\ -5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 3 & -8 \end{bmatrix}$ $\begin{bmatrix} -3 & 0 \\ -2 & -6 \end{bmatrix}$

16. $[0.2 \quad 1.3 \quad -0.4] - [2 \quad 1.7 \quad 2.6]$ $[-1.8 \quad -0.4 \quad -3]$

17. $\begin{bmatrix} 1 & -5 \\ -2 & 3 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 0 & 4 \\ -16 & 8 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 \\ -14 & 9 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 & -3 \\ 4 & -5 & 2 \end{bmatrix} - 2\begin{bmatrix} -2 & 3 & 5 \\ -3 & -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 5 & -6 & -13 \\ 10 & -3 & -2 \end{bmatrix}$

4-3 Multiplying Matrices

See pages 167–174.

Concept Summary

- Two matrices can be multiplied if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Example **Find XY if $X = [6 \quad 4 \quad 1]$ and $Y = \begin{bmatrix} 2 & 5 \\ -3 & 0 \\ -1 & 3 \end{bmatrix}$.**

$XY = [6 \quad 4 \quad 1] \cdot \begin{bmatrix} 2 & 5 \\ -3 & 0 \\ -1 & 3 \end{bmatrix}$ Write an equation.

$= [6(2) + 4(-3) + 1(-1) \quad 6(5) + 4(0) + 1(3)]$ Multiply columns by rows.

$= [-1 \quad 33]$ Simplify.

Exercises **Find each product, if possible.** *See Example 2 on page 168.*

19. $[2 \quad 7] \cdot \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ $[-18]$

20. $\begin{bmatrix} 8 & -3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 1 & -5 \end{bmatrix}$ $\begin{bmatrix} 13 & -9 \\ 13 & -23 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 5 \\ 3 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ not possible

22. $\begin{bmatrix} 3 & 0 & -1 \\ 4 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 7 & 1 \\ 6 & -3 \\ 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 19 & 2 \\ 22 & 13 \end{bmatrix}$

4-4 Transformations with Matrices

See pages 175–181.

Concept Summary

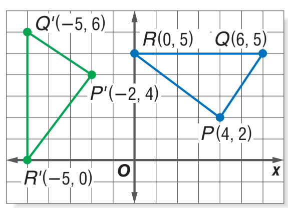
- Use matrix addition and a translation matrix to find the coordinates of a translated figure.
- Use scalar multiplication to perform dilations.
- To reflect a figure, multiply the vertex matrix on the left by a reflection matrix.

reflection over x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflection over y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflection over line $y = x$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- To rotate a figure counterclockwise about the origin, multiply the vertex matrix on the left by a rotation matrix.

$90°$ rotation: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $180°$ rotation: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $270°$ rotation: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Example **Find the coordinates of the vertices of the image of $\triangle PQR$ with $P(4, 2)$, $Q(6, 5)$, and $R(0, 5)$ after it is rotated 90° counterclockwise about the origin.**



Write the ordered pairs in a vertex matrix. Then multiply the vertex matrix by the rotation matrix.

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 & 0 \\ 2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -5 & -5 \\ 4 & 6 & 0 \end{bmatrix}$

The coordinates of the vertices of $\triangle P'Q'R'$ are $P'(-2, 4)$, $Q'(-5, 6)$, and $R'(-5, 0)$.

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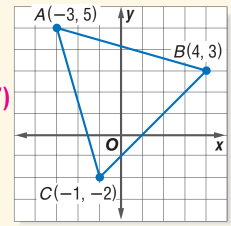
Exercises **For Exercises 23–26, use the figure at the right.**
See Examples 1–5 on pages 175–178.



23. Find the coordinates of the image after a translation 4 units right and 5 units down. **$A'(1, 0), B'(8, -2), C'(3, -7)$**

24. Find the coordinates of the image of the figure after a dilation by a scale factor of 2.

25. Find the coordinates of the image after a reflection over the y-axis. **$A'(3, 5)$, $B'(-4, 3)$, $C'(1, -2)$**

26. Find the coordinates of the image of the figure after a rotation of 180°. **$A'(3, -5)$, $B'(-4, -3)$, $C'(1, 2)$**

24. $A'(-6, 10)$, $B'(8, 6)$, $C'(-2, -4)$

4-5 Determinants

See pages 182–188.

Concept Summary

- Determinant of a 2×2 matrix: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- Determinant of a 3×3 matrix: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$
- Area of a triangle with vertices at (a, b), (c, d), and (e, f):

  $|A|$ where $A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$

Examples

1 **Find the value of $\begin{vmatrix} 3 & 6 \\ -4 & 2 \end{vmatrix}$.**

$$\begin{vmatrix} 3 & 6 \\ -4 & 2 \end{vmatrix} = 3(2) - (-4)(6) \quad \text{Definition of determinant}$$

$$= 6 - (-24) \text{ or } 30 \quad \text{Simplify.}$$

2 **Evaluate $\begin{vmatrix} 3 & 1 & 5 \\ 1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$ using expansion by minors.**

$$\begin{vmatrix} 3 & 1 & 5 \\ 1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 5\begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} \quad \text{Expansion by minors}$$

$$= 3(-4 - (-1)) - 1(2 - 0) + 5(-1 - 0) \quad \text{Evaluate } 2 \times 2 \text{ determinants.}$$

$$= -9 - 2 - 5 \text{ or } -16 \quad \text{Simplify.}$$

Exercises **Find the value of each determinant.** *See Examples 1–3 on pages 182–184.*

27. $\begin{vmatrix} 4 & 11 \\ -7 & 8 \end{vmatrix}$ **109**

28. $\begin{vmatrix} 6 & -7 \\ 5 & 3 \end{vmatrix}$ **53**

29. $\begin{vmatrix} 12 & 8 \\ 9 & 6 \end{vmatrix}$ **0**

30. $\begin{vmatrix} 2 & -3 & 1 \\ 0 & 7 & 8 \\ 2 & 1 & 3 \end{vmatrix}$ **−36**

31. $\begin{vmatrix} 7 & -4 & 5 \\ 1 & 3 & -6 \\ 5 & -1 & -2 \end{vmatrix}$ **−52**

32. $\begin{vmatrix} 6 & 3 & -2 \\ -4 & 2 & 5 \\ -3 & -1 & 0 \end{vmatrix}$ **−35**

4-6 Cramer's Rule

See pages 189–194.

Concept Summary

- Cramer's Rule for two variables:
  The solution of the system of equations $ax + by = e$ and $cx + dy = f$ is (x, y), where $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

- Cramer's Rule for three variables:
  The solution of the system whose equations are $ax + by + cz = j$, $dx + ey + fz = k$, $gx + hy + iz = \ell$ is (x, y, z), where

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}},\ y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}},\ z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}},\ \text{and} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0.$$

Example **Use Cramer's Rule to solve each system of equations $5a - 3b = 7$ and $3a + 9b = -3$.**

$$a = \frac{\begin{vmatrix} 7 & -3 \\ -3 & 9 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 3 & 9 \end{vmatrix}} \quad \text{Cramer's Rule} \qquad b = \frac{\begin{vmatrix} 5 & 7 \\ 3 & -3 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 3 & 9 \end{vmatrix}}$$

$$= \frac{63 - 9}{45 + 9} \quad \text{Evaluate each determinant.} \qquad = \frac{-15 - 21}{45 + 9}$$

$$= \frac{54}{54} \text{ or } 1 \quad \text{Simplify.} \qquad = \frac{-36}{54} \text{ or } -\frac{2}{3}$$

The solution is $\left(1, -\frac{2}{3}\right)$.

Exercises **Use Cramer's Rule to solve each system of equations.**

See Examples 1 and 3 on pages 190 and 191.

33. $9a - b = 1$, $3a + 2b = 12$ $\left(\frac{2}{3}, 5\right)$

34. $x + 5y = 14$, $-2x + 6y = 4$ $(4, 2)$

35. $3x + 4y = -15$, $2x - 7y = 19$ $(-1, -3)$

36. $8a + 5b = 2$, $-6a - 4b = -1$ $\left(\frac{3}{2}, -2\right)$

37. $6x - 7z = 13$, $8y + 2z = 14$, $7x + z = 6$ $(1, 2, -1)$

38. $2a - b - 3c = -20$, $4a + 2b + c = 6$, $2a + b - c = -6$ $\left(-\frac{1}{2}, 1, 6\right)$

4-7 Identity and Inverse Matrices

See pages 195–201.

Concept Summary

- An identity matrix is a square matrix with ones on the diagonal and zeros in the other positions.
- Two matrices are inverses of each other if their product is the identity matrix.
- The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

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Chapter 4 For More ...
- Extra Practice, see pages 834–836.
- Mixed Problem Solving, see page 865.

Example **Find the inverse of $S = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$.**

Find the value of the determinant.

$$\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = 3 - (-8) \text{ or } 11$$

Use the formula for the inverse matrix.

$$S^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$$

Exercises **Find the inverse of each matrix, if it exists.** *See Example 2 on page 197.*

39. $\begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix}$ $-\frac{1}{14}\begin{bmatrix} -2 & -2 \\ -4 & 3 \end{bmatrix}$
40. $\begin{bmatrix} 8 & 6 \\ 9 & 7 \end{bmatrix}$ $\frac{1}{2}\begin{bmatrix} 7 & -6 \\ -9 & 8 \end{bmatrix}$
41. $\begin{bmatrix} 2 & 4 \\ -3 & 6 \end{bmatrix}$ $\frac{1}{24}\begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix}$

42. $\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}$ no inverse exists
43. $\begin{bmatrix} 0 & 2 \\ 5 & -4 \end{bmatrix}$ $-\frac{1}{10}\begin{bmatrix} -4 & -2 \\ -5 & 0 \end{bmatrix}$
44. $\begin{bmatrix} 6 & -1 & 0 \\ 5 & 8 & -2 \end{bmatrix}$ no inverse exists

4-8 Using Matrices to Solve Systems of Equations

See pages 202–207.

Concept Summary

- A system of equations can be written as a matrix equation in the form $A \cdot X = B$.

$$\begin{matrix} 2x + 3y = 12 \\ x - 4y = 6 \end{matrix} \rightarrow \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

- To solve a matrix equation, find the inverse of the coefficient matrix. Then multiply each side by the inverse matrix, so $X = A^{-1}B$.

Example **Solve $\begin{bmatrix} 4 & 8 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \end{bmatrix}$.**

Step 1 Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-12 - 16}\begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \text{ or } -\frac{1}{28}\begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix}$$

Step 2 Multiply each side by the inverse matrix.

$$-\frac{1}{28}\begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 8 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{28}\begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{28}\begin{bmatrix} -140 \\ 28 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

The solution is $(5, -1)$.

Exercises **Solve each matrix equation or system of equations by using inverse matrices.** *See Example 3 on page 204.*

45. $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$ **(4, 2)**
46. $\begin{bmatrix} 4 & 1 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ **(2, 1)**

47. $3x + 8 = -y$
$4x - 2y = -14$ **(−3, 1)**

48. $3x - 5y = -13$
$4x + 3y = 2$ **(−1, 2)**

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Chapter 4 Practice Test

Vocabulary and Concepts

Choose the letter that best matches each description.

1. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ **b**
2. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ **c**
3. $\dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ **a**

a. inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

b. determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

c. matrix equation for $ax + by = e$ and $cx + dy = f$

Skills and Applications

Solve each equation.

4. $\begin{bmatrix} 3x + 1 \\ 2y \end{bmatrix} = \begin{bmatrix} 10 \\ 4 + y \end{bmatrix}$ **(3, 4)**
5. $\begin{bmatrix} 2x & y + 1 \\ 13 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -7 \\ 13 & z - 8 \end{bmatrix}$ **(−8, −8, 6)**

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

6. $\begin{bmatrix} 2 & -4 & 1 \\ 3 & 8 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 7 \end{bmatrix}$ $\begin{bmatrix} 0 & -8 & 9 \\ 7 & 2 & -16 \end{bmatrix}$
7. $\begin{bmatrix} 1 & 6 & 7 \\ 1 & -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 3 \\ -1 & -2 \\ 2 & 5 \end{bmatrix}$ $\begin{bmatrix} 4 & 26 \\ -9 & -11 \end{bmatrix}$

Find the value of each determinant.

8. $\begin{vmatrix} -1 & 4 \\ -6 & 3 \end{vmatrix}$ **21**
9. $\begin{vmatrix} 5 & -3 & 2 \\ -6 & 1 & 3 \\ -1 & 4 & -7 \end{vmatrix}$ **−6**

Find the inverse of each matrix, if it exists.

10. $\begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix}$ $-\dfrac{1}{17}\begin{bmatrix} 1 & -5 \\ -3 & -2 \end{bmatrix}$
11. $\begin{bmatrix} -6 & -3 \\ 8 & 4 \end{bmatrix}$ **no inverse exists**
12. $\begin{bmatrix} 5 & -2 \\ 6 & 3 \end{bmatrix}$ $\dfrac{1}{27}\begin{bmatrix} 3 & 2 \\ -6 & 5 \end{bmatrix}$

Solve each matrix equation or system of equations by using inverse matrices.

13. $\begin{bmatrix} 1 & 8 \\ 2 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -17 \end{bmatrix}$ $\left(-7, \dfrac{1}{2}\right)$
14. $\begin{bmatrix} 5 & 7 \\ -9 & 3 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 41 \\ -105 \end{bmatrix}$ **(11, −2)**
15. $5a + 2b = -49$
    $2a + 9b = 5$ **(−11, 3)**

For Exercises 16–18, use $\triangle ABC$ whose vertices have coordinates $A(6, 3)$, $B(1, 5)$, and $C(-1, 4)$.

16. Use the determinant to find the area of $\triangle ABC$. **4.5 units^2**
17. Translate $\triangle ABC$ so that the coordinates of B' are (3, 1). What are the coordinates of A' and C'? **$A'(8, -1)$, $C'(1, 0)$**
18. Find the coordinates of the vertices of a similar triangle whose perimeter is five times that of $\triangle ABC$. **$A'(30, 15)$, $B'(5, 25)$, $C'(-5, 20)$**
19. **RETAIL SALES** Brittany is preparing boxes of assorted chocolates. Chocolate-covered peanuts cost \$7 per pound. Chocolate-covered caramels cost \$6.50 per pound. The boxes of assorted candies contain five more pounds of peanut candies than caramel candies. If the total amount sold was \$575, how many pounds of each candy were needed to make the boxes? **40 lb caramel, 45 lb peanut**
20. **STANDARDIZED TEST PRACTICE** If $\begin{bmatrix} 43 & z \\ 7x - 2 & 2x + 3 \end{bmatrix} = \begin{bmatrix} z + 3 & 2m + 5 \\ y & 37 \end{bmatrix}$, then $y =$ **B**

(A) 120. (B) 117. (C) 22. (D) not enough information

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Chapter 4 Practice Test

Assessment Options

Vocabulary Test A vocabulary test/review for Chapter 4 can be found on p. 230 of the *Chapter 4 Resource Masters*.

Chapter Tests There are six Chapter 4 Tests and an Open-Ended Assessment task available in the *Chapter 4 Resource Masters*.

Chapter 4 Tests			
Form	**Type**	**Level**	**Pages**
1	MC	basic	217–218
2A	MC	average	219–220
2B	MC	average	221–222
2C	FR	average	223–224
2D	FR	average	225–226
3	FR	advanced	227–228

MC = multiple-choice questions
FR = free-response questions

Open-Ended Assessment Performance tasks for Chapter 4 can be found on p. 229 of the *Chapter 4 Resource Masters*. A sample scoring rubric for these tasks appears on p. A31.

Unit 1 Test A unit test/review can be found on pp. 237–238 of the *Chapter 4 Resource Masters*.

TestCheck and Worksheet Builder

This **networkable software** has three modules for assessment.

- **Worksheet Builder** to make worksheets and tests.
- **Student Module** to take tests on-screen.
- **Management System** to keep student records.

Portfolio Suggestion

Introduction The Associative, Commutative, and Distributive Properties are familiar to students. These properties can also be used, with a few differences, in operations with matrices.

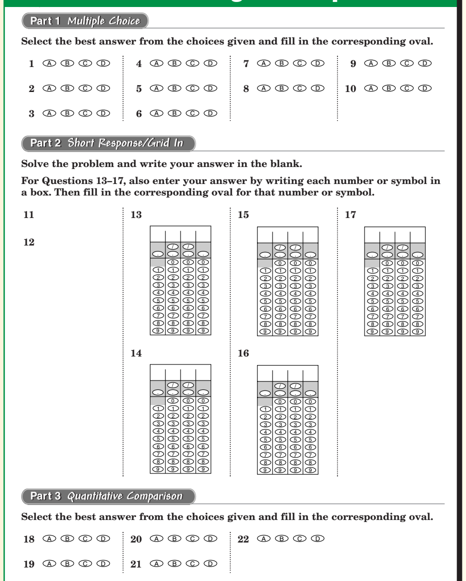
Ask Students Using matrix addition, subtraction, and multiplication, determine to what extent these three properties apply to matrix operations. Write a convincing argument for each property.

Chapter 4 Standardized Test Practice

These two pages contain practice questions in the various formats that can be found on the most frequently given standardized tests.

A practice answer sheet for these two pages can be found on p. A1 of the *Chapter 4 Resource Masters*.

Standardized Test Practice Student Recording Sheet, p. A1


Part 1 Multiple Choice

Select the best answer from the choices given and fill in the corresponding oval.

1 Ⓐ Ⓑ Ⓒ Ⓓ 4 Ⓐ Ⓑ Ⓒ Ⓓ 7 Ⓐ Ⓑ Ⓒ Ⓓ 9 Ⓐ Ⓑ Ⓒ Ⓓ
2 Ⓐ Ⓑ Ⓒ Ⓓ 5 Ⓐ Ⓑ Ⓒ Ⓓ 8 Ⓐ Ⓑ Ⓒ Ⓓ 10 Ⓐ Ⓑ Ⓒ Ⓓ
3 Ⓐ Ⓑ Ⓒ Ⓓ 6 Ⓐ Ⓑ Ⓒ Ⓓ

Part 2 Short Response/Grid In

Solve the problem and write your answer in the blank.

For Questions 13–17, also enter your answer by writing each number or symbol in a box. Then fill in the corresponding oval for that number or symbol.

11 13 15 17

12

14 16

Part 3 Quantitative Comparison

Select the best answer from the choices given and fill in the corresponding oval.

18 Ⓐ Ⓑ Ⓒ Ⓓ 20 Ⓐ Ⓑ Ⓒ Ⓓ 22 Ⓐ Ⓑ Ⓒ Ⓓ
19 Ⓐ Ⓑ Ⓒ Ⓓ 21 Ⓐ Ⓑ Ⓒ Ⓓ

Teaching Tip In Questions 4 and 6, students may want to write down the basic formulas for the circumference and area of a circle before they begin their calculations.

Additional Practice

See pp. 235–236 in the *Chapter 4 Resource Masters* for additional standardized test practice.

Chapter 4 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

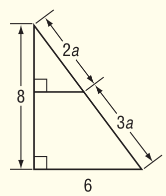
1. If the average (arithmetic mean) of ten numbers is 18 and the average of six of these numbers is 12, what is the average of the other four numbers? **C**

Ⓐ 15 Ⓑ 18
Ⓒ 27 Ⓓ 28

2. A car travels 65 miles per hour for 2 hours. A truck travels 60 miles per hour for 1.5 hours. What is the difference between the number of miles traveled by the car and the number of miles traveled by the truck? **B**

Ⓐ 31.25 Ⓑ 40
Ⓒ 70 Ⓓ 220

3. In the figure, $a =$ **B**



Ⓐ 1.
Ⓑ 2.
Ⓒ 3.
Ⓓ 4.

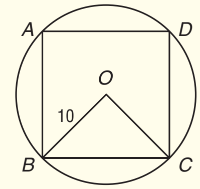
4. If the circumference of a circle is $\frac{4\pi}{3}$, then what is half of its area? **A**

Ⓐ $\frac{2\pi}{9}$ Ⓑ $\frac{4\pi}{9}$
Ⓒ $\frac{8\pi}{9}$ Ⓓ $\frac{2\pi^2}{9}$

5. A line is represented by the equation $x = 6$. What is the slope of the line? **D**

Ⓐ 0 Ⓑ $\frac{5}{6}$
Ⓒ 6 Ⓓ undefined

6. In the figure, $ABCD$ is a square inscribed in the circle centered at O. If $\overline{OB}$ is 10 units long, how many units long is minor arc BC? **B**



Ⓐ $\frac{5}{2}\pi$ units
Ⓑ 5π units
Ⓒ 10π units
Ⓓ 20π units

7. If $3 < x < 5 < y < 10$, then which of the following best defines $\frac{x}{y}$? **A**

Ⓐ $\frac{3}{10} < \frac{x}{y} < 1$
Ⓑ $\frac{3}{10} < \frac{x}{y} < \frac{1}{2}$
Ⓒ $\frac{3}{5} < \frac{x}{y} < \frac{1}{2}$
Ⓓ $\frac{3}{5} < \frac{x}{y} < 1$

8. If $x + 3y = 12$ and $\frac{2}{3}x - y = 5$, then $x =$ **C**

Ⓐ 1. Ⓑ 8.
Ⓒ 9. Ⓓ 13.5.

9. At what point do the two lines with the equations $7x - 3y = 13$ and $y = 2x - 3$ intersect? **C**

Ⓐ $(-4, -11)$ Ⓑ $(4, 11)$
Ⓒ $(4, 5)$ Ⓓ $(5, 4)$

10. If $N = \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix}$ and $M = \begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix}$, find $N - M$. **C**

Ⓐ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Ⓒ $\begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$ Ⓓ $\begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$

Log On **for Test Practice**

The Princeton Review offers additional test-taking tips and practice problems at their web site. Visit www.princetonreview.com or www.review.com

TestCheck and Worksheet Builder

Special banks of standardized test questions similar to those on the SAT, ACT, TIMSS 8, NAEP 8, and Algebra 1 End-of-Course tests can be found on this CD-ROM.

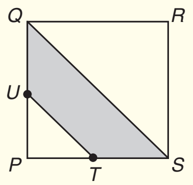
Aligned and verified by The Princeton Review

Part 2 Short Response/Grid In

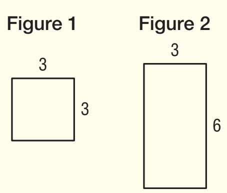
Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. A computer manufacturer reduced the price of its Model X computer by 3%. If the new price of the Model X computer is \$2489, then how much did the computer cost, in dollars, before its price was reduced? (Round to the nearest dollar.) **\$2566**

12. In square $PQRS$, $PQ = 4$, $PU = UQ$, and $PT = TS$. What is the area of the shaded region? **6 units^2**



13. A rectangular solid has two faces the same size and shape as Figure 1 and four faces the same size and shape as Figure 2. What is the volume of the solid in cubic units? **54**



14. If the average (arithmetic mean) of three different positive integers is 60, what is the greatest possible value of one of the integers? **177**

15. The perimeter of a triangle is 15. The lengths of the sides are integers. If the length of one side is 6, what is the shortest possible length of another side of the triangle? **2**

The Princeton Review Test-Taking Tip

Questions 14, 15 Watch for the phrases "greatest possible" or "least possible." Think logically about the conditions that make an expression greatest or least. Notice what types of numbers are used—positive, even, prime, integers.

www.algebra2.com/standardized_test

16. In this sequence below, each term after the first term is $\frac{1}{4}$ of the term preceding it. What is the sixth term of this sequence? **5/16**

320, 80, 20, …

17. If the sum of two numbers is 5 and their difference is 2, what is their product? **5.25**

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- **(A) the quantity in Column A is greater,**
- **(B) the quantity in Column B is greater,**
- **(C) the two quantities are equal, or**
- **(D) the relationship cannot be determined from the information given.**

18. $xy = 0$

Column A	Column B
y	0

D

19. 4, 8, 16, 18

Column A	Column B
the greatest of the numbers listed above which is the sum of two equal even integers	the greatest of the numbers listed above which is the sum of two equal odd integers

B

20.

Column A	Column B
the volume of a cube with edges 4 inches long	the sum of the volumes of eight cubes each having edges 2 units long

C

21. Point P with coordinates (x, y) is exactly 4 units from the origin.

Column A	Column B
x	y

D

22. $r + s + t = 30$
$r + s - t = 8$

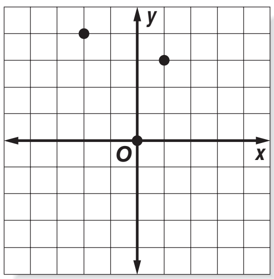
Column A	Column B
t	11

C

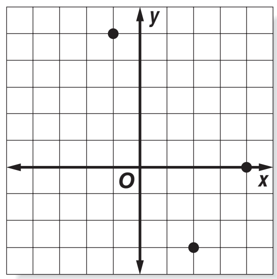
Chapter 4 Standardized Test Practice 217

Page 153, Chapter 4 Getting Started

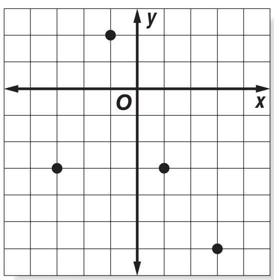
15.



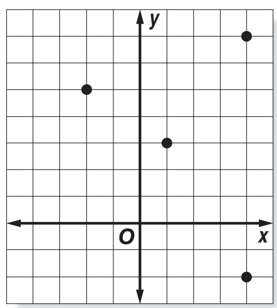
16.



17.



18.



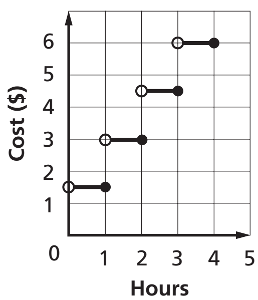
Pages 157–158, Lesson 4-1

28.

$$\begin{array}{l} \\ \text{Catalina Grill} \\ \text{Oyster Club} \\ \text{Casa di Pasta} \\ \text{Mason's Steakhouse} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Cost} & \text{Service} & \text{Atmosphere} & \text{Location} \end{array} \\ \begin{bmatrix} \star\star & \star & \star & \star \\ \star\star\star & \star\star & \star & \star\star \\ \star\star\star\star & \star\star\star & \star\star\star & \star\star\star \\ \star\star & \star\star\star\star & \star\star\star\star & \star\star\star \end{bmatrix} \end{array}$$

29. Sample answer: Mason's Steakhouse; it was given the highest rating possible for service and atmosphere, location was given one of the highest ratings, and it is moderately priced.

43.



Page 159, Follow-Up of Lesson 4-1 Spreadsheet Investigation

1.

	A	B	C	D	E	F
1		Base Price	Horse-power	Towing Capacity (lb)	Cargo Capacity (ft^3)	Fuel Economy (mpg)
2	Large	$32,450	285	12,000	46	17
3	Standard	$29,115	275	8700	16	17.5
4	Mid-Size	$27,975	190	5700	34	20
5	Compact	$18,180	127	3000	15	26.5

Page 166, Lesson 4-2

41. You can use matrices to track dietary requirements and add them to find the total each day or each week. Answers should include the following.

- Breakfast $= \begin{bmatrix} 566 & 18 & 7 \\ 482 & 12 & 17 \\ 530 & 10 & 11 \end{bmatrix}$, Lunch $= \begin{bmatrix} 785 & 22 & 19 \\ 622 & 23 & 20 \\ 710 & 26 & 12 \end{bmatrix}$,

  Dinner $= \begin{bmatrix} 1257 & 40 & 26 \\ 987 & 32 & 45 \\ 1380 & 29 & 38 \end{bmatrix}$

- Add the three matrices: $\begin{bmatrix} 2608 & 80 & 52 \\ 2091 & 67 & 82 \\ 2620 & 65 & 61 \end{bmatrix}$.

Pages 172–174, Lesson 4-3

27. $$AC + BC = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 9 \\ 26 & -8 \end{bmatrix} + \begin{bmatrix} -21 & -13 \\ 26 & -8 \end{bmatrix}$$
$$= \begin{bmatrix} -20 & -4 \\ 52 & -16 \end{bmatrix}$$

$$(A + B)C = \left(\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}\right) \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 0 \\ 8 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} -20 & -4 \\ 52 & -16 \end{bmatrix}$$

28. $$c(AB) = 3\left(\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}\right)$$
$$= 3\begin{bmatrix} -13 & -4 \\ -8 & 17 \end{bmatrix} = \begin{bmatrix} -39 & -12 \\ -24 & 51 \end{bmatrix}$$

$$A(cB) = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \cdot \left(3\begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} -15 & 6 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} -39 & -12 \\ -24 & 51 \end{bmatrix}$$

29. $$C(A + B) = \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 0 \\ 8 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} -12 & 6 \\ -40 & -24 \end{bmatrix}$$

$$AC + BC = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 9 \\ 26 & -8 \end{bmatrix} + \begin{bmatrix} -21 & -13 \\ 26 & -8 \end{bmatrix}$$
$$= \begin{bmatrix} -20 & -4 \\ 52 & -16 \end{bmatrix}$$

30. $ABC = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$

$= \begin{bmatrix} -13 & -4 \\ -8 & 17 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$

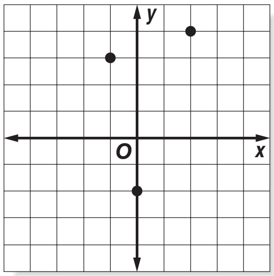
$= \begin{bmatrix} -73 & 3 \\ -6 & -76 \end{bmatrix}$

$CBA = \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} -21 & 13 \\ -26 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 31 & 81 \\ -58 & 28 \end{bmatrix}$

57.



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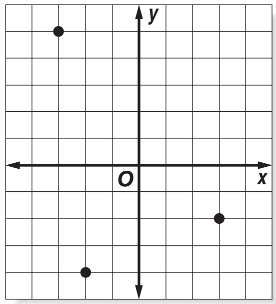
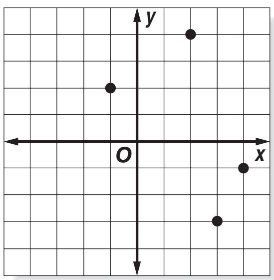
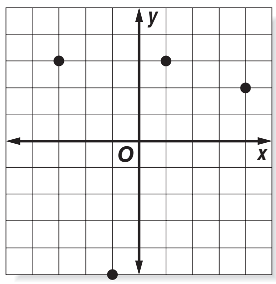


59.



60.



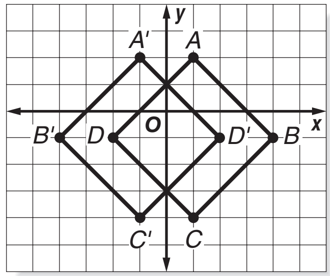
Page 181, Lesson 4-4

43. Transformations are used in computer graphics to create special effects. You can simulate the movement of an object, like in space, which you wouldn't be able to recreate otherwise. Answers should include the following.

- A figure with points (a, b), (c, d), (e, f), (g, h), and (i, j) could be written in a 2×5 matrix $\begin{bmatrix} a & c & e & g & i \\ b & d & f & h & j \end{bmatrix}$ and multiplied on the left by the 2×2 rotation matrix.
- The object would get smaller and appear to be moving away from you.

Page 194, Practice Quiz 2

3.



Pages 205–207, Lesson 4-8

4. $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 3 \\ -4 & -7 \end{bmatrix} \cdot \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$

6. $\begin{bmatrix} 3 & -5 & 2 \\ 4 & 7 & 1 \\ 2 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 12 \end{bmatrix}$

36. The food and territory that two species of birds require form a system of equations. Any independent system of equations can be solved using a matrix equation. Answers should include the following.

- Let a represent the number of nesting pairs of Species A and let b represent the number of nesting pairs of Species B. Then, $140a + 120b = 20{,}000$ and $500a + 400b = 69{,}000$.
- $\begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{4000} \begin{bmatrix} 400 & -120 \\ -500 & 140 \end{bmatrix} \cdot \begin{bmatrix} 20{,}000 \\ 69{,}000 \end{bmatrix}$; $a = 70$ and $b = 85$, so the area can support 70 pairs of Species A and 85 pairs of Species B.