

# INSTRUCTOR'S SOLUTIONS MANUAL

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*with art by*

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# LINEAR ALGEBRA WITH APPLICATIONS

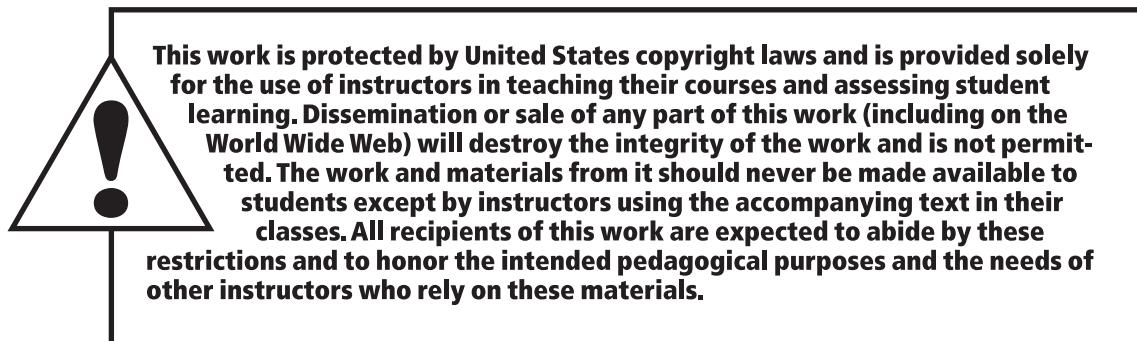
FOURTH EDITION

Otto Bretscher

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ISBN-13: 978-0-13-600928-3  
ISBN-10: 0-13-600928-X

1 2 3 4 5 6 OPM 10 09 08

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# Chapter 1

## Section 1.1

1.1.1  $\begin{bmatrix} x+2y=1 \\ 2x+3y=1 \end{bmatrix}$   $-2 \times 1\text{st equation} \rightarrow \begin{bmatrix} x+2y=1 \\ -y=-1 \end{bmatrix} \div(-1) \rightarrow$   
 $\begin{bmatrix} x+2y=1 \\ y=1 \end{bmatrix} \quad -2 \times 2\text{nd equation} \rightarrow \begin{bmatrix} x=-1 \\ y=1 \end{bmatrix}$ , so that  $(x, y) = (-1, 1)$ .

1.1.2  $\begin{bmatrix} 4x+3y=2 \\ 7x+5y=3 \end{bmatrix} \div 4 \rightarrow \begin{bmatrix} x+\frac{3}{4}y=\frac{1}{2} \\ 7x+5y=3 \end{bmatrix} \quad -7 \times 1\text{st equation} \rightarrow$   
 $\begin{bmatrix} x+\frac{3}{4}y=\frac{1}{2} \\ -\frac{1}{4}y=-\frac{1}{2} \end{bmatrix} \times(-4) \rightarrow \begin{bmatrix} x+\frac{3}{4}y=\frac{1}{2} \\ y=2 \end{bmatrix} \quad -\frac{3}{4} \times 2\text{nd equation} \rightarrow \begin{bmatrix} x=-1 \\ y=2 \end{bmatrix}$ ,  
 so that  $(x, y) = (-1, 2)$ .

1.1.3  $\begin{bmatrix} 2x+4y=3 \\ 3x+6y=2 \end{bmatrix} \div 2 \rightarrow \begin{bmatrix} x+2y=\frac{3}{2} \\ 3x+6y=2 \end{bmatrix} \quad -3 \times 1\text{st equation} \rightarrow \begin{bmatrix} x+2y=\frac{3}{2} \\ 0=-\frac{5}{2} \end{bmatrix}$ .

So there is no solution.

1.1.4  $\begin{bmatrix} 2x+4y=2 \\ 3x+6y=3 \end{bmatrix} \div 2 \rightarrow \begin{bmatrix} x+2y=1 \\ 3x+6y=3 \end{bmatrix} \quad -3 \times 1\text{st equation} \rightarrow \begin{bmatrix} x+2y=1 \\ 0=0 \end{bmatrix}$

This system has infinitely many solutions: if we choose  $y = t$ , an arbitrary real number, then the equation  $x + 2y = 1$  gives us  $x = 1 - 2y = 1 - 2t$ . Therefore the general solution is  $(x, y) = (1 - 2t, t)$ , where  $t$  is an arbitrary real number.

1.1.5  $\begin{bmatrix} 2x+3y=0 \\ 4x+5y=0 \end{bmatrix} \div 2 \rightarrow \begin{bmatrix} x+\frac{3}{2}y=0 \\ 4x+5y=0 \end{bmatrix} \quad -4 \times 1\text{st equation} \rightarrow$   
 $\begin{bmatrix} x+\frac{3}{2}y=0 \\ -y=0 \end{bmatrix} \div(-1) \rightarrow \begin{bmatrix} x+\frac{3}{2}y=0 \\ y=0 \end{bmatrix} \quad -\frac{3}{2} \times 2\text{nd equation} \rightarrow \begin{bmatrix} x=0 \\ y=0 \end{bmatrix}$ ,

so that  $(x, y) = (0, 0)$ .

1.1.6  $\begin{bmatrix} x+2y+3z=8 \\ x+3y+3z=10 \\ x+2y+4z=9 \end{bmatrix} \quad -I \rightarrow \begin{bmatrix} x+2y+3z=8 \\ y=2 \\ z=1 \end{bmatrix} \quad -2(II) \rightarrow$   
 $\begin{bmatrix} x+3z=4 \\ y=2 \\ z=1 \end{bmatrix} \quad -3(III) \rightarrow \begin{bmatrix} x=1 \\ y=2 \\ z=1 \end{bmatrix}$ , so that  $(x, y, z) = (1, 2, 1)$ .

1.1.7  $\begin{bmatrix} x+2y+3z=1 \\ x+3y+4z=3 \\ x+4y+5z=4 \end{bmatrix} \quad -I \rightarrow \begin{bmatrix} x+2y+3z=1 \\ y+z=2 \\ 2y+2z=3 \end{bmatrix} \quad -2(II) \rightarrow \begin{bmatrix} x+z=-3 \\ y+z=2 \\ 0=-1 \end{bmatrix} \quad -2(II) \rightarrow \begin{bmatrix} x+z=-3 \\ y+z=2 \\ 0=-1 \end{bmatrix}$

This system has no solution.

$$1.1.8 \quad \begin{bmatrix} x + 2y + 3z & = 0 \\ 4x + 5y + 6z & = 0 \\ 7x + 8y + 10z & = 0 \end{bmatrix} \begin{array}{l} -4(I) \\ -7(I) \end{array} \rightarrow \begin{bmatrix} x + 2y + 3z & = 0 \\ -3y - 6z & = 0 \\ -6y - 11z & = 0 \end{bmatrix} \div (-3) \rightarrow$$

$$\begin{bmatrix} x + 2y + 3z & = 0 \\ y + 2z & = 0 \\ -6y - 11z & = 0 \end{bmatrix} \begin{array}{l} -2(II) \\ +6(II) \end{array} \rightarrow \begin{bmatrix} x - z & = 0 \\ y + 2z & = 0 \\ z & = 0 \end{bmatrix} \begin{array}{l} +III \\ -2(III) \end{array} \rightarrow \begin{bmatrix} x & = 0 \\ y & = 0 \\ z & = 0 \end{bmatrix},$$

so that  $(x, y, z) = (0, 0, 0)$ .

$$1.1.9 \quad \begin{bmatrix} x + 2y + 3z & = 1 \\ 3x + 2y + z & = 1 \\ 7x + 2y - 3z & = 1 \end{bmatrix} \begin{array}{l} -3(I) \\ -7(I) \end{array} \rightarrow \begin{bmatrix} x + 2y + 3z & = 1 \\ -4y - 8z & = -2 \\ -12y - 24z & = -6 \end{bmatrix} \div (-4) \rightarrow$$

$$\begin{bmatrix} x + 2y + 3z & = 1 \\ y + 2z & = \frac{1}{2} \\ -12y - 24z & = -6 \end{bmatrix} \begin{array}{l} -2(II) \\ +12(II) \end{array} \rightarrow \begin{bmatrix} x - z & = 0 \\ y + 2z & = \frac{1}{2} \\ 0 & = 0 \end{bmatrix}$$

This system has infinitely many solutions: if we choose  $z = t$ , an arbitrary real number, then we get  $x = z = t$  and  $y = \frac{1}{2} - 2z = \frac{1}{2} - 2t$ . Therefore, the general solution is  $(x, y, z) = (t, \frac{1}{2} - 2t, t)$ , where  $t$  is an arbitrary real number.

$$1.1.10 \quad \begin{bmatrix} x + 2y + 3z & = 1 \\ 2x + 4y + 7z & = 2 \\ 3x + 7y + 11z & = 8 \end{bmatrix} \begin{array}{l} -2(I) \\ -3(I) \end{array} \rightarrow \begin{bmatrix} x + 2y + 3z & = 1 \\ z & = 0 \\ y + 2z & = 5 \end{bmatrix} \begin{array}{l} Swap: \\ II \leftrightarrow III \end{array} \rightarrow$$

$$\begin{bmatrix} x + 2y + 3z & = 1 \\ y + 2z & = 5 \\ z & = 0 \end{bmatrix} \begin{array}{l} -2(II) \\ +III \end{array} \rightarrow \begin{bmatrix} x - z & = -9 \\ y + 2z & = 5 \\ z & = 0 \end{bmatrix} \begin{array}{l} -2(III) \end{array} \rightarrow \begin{bmatrix} x & = -9 \\ y & = 5 \\ z & = 0 \end{bmatrix},$$

so that  $(x, y, z) = (-9, 5, 0)$ .

$$1.1.11 \quad \begin{bmatrix} x - 2y & = 2 \\ 3x + 5y & = 17 \end{bmatrix} \begin{array}{l} -3(I) \end{array} \rightarrow \begin{bmatrix} x - 2y & = 2 \\ 11y & = 11 \end{bmatrix} \div 11 \rightarrow \begin{bmatrix} x - 2y & = 2 \\ y & = 1 \end{bmatrix} \begin{array}{l} +2(II) \end{array} \rightarrow \begin{bmatrix} x & = 4 \\ y & = 1 \end{bmatrix},$$

so that  $(x, y) = (4, 1)$ . See Figure 1.1.

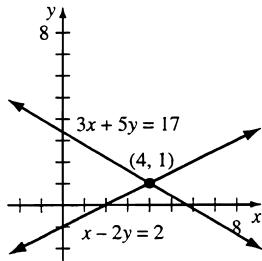


Figure 1.1: for Problem 1.1.11.

$$1.1.12 \quad \begin{bmatrix} x - 2y & = 3 \\ 2x - 4y & = 6 \end{bmatrix} \begin{array}{l} -2(I) \end{array} \rightarrow \begin{bmatrix} x - 2y & = 3 \\ 0 & = 0 \end{bmatrix}$$

This system has infinitely many solutions: If we choose  $y = t$ , an arbitrary real number, then the equation  $x - 2y = 3$  gives us  $x = 3 + 2y = 3 + 2t$ . Therefore the general solution is  $(x, y) = (3 + 2t, t)$ , where  $t$  is an arbitrary real number. (See Figure 1.2.)

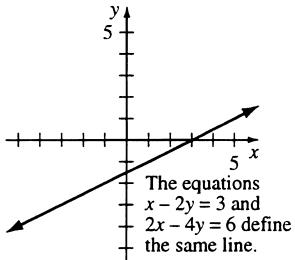


Figure 1.2: for Problem 1.1.12.

1.1.13  $\begin{bmatrix} x - 2y &= 3 \\ 2x - 4y &= 8 \end{bmatrix} \xrightarrow{-2(I)} \begin{bmatrix} x - 2y &= 3 \\ 0 &= 2 \end{bmatrix}$ , which has no solutions. (See Figure 1.3.)

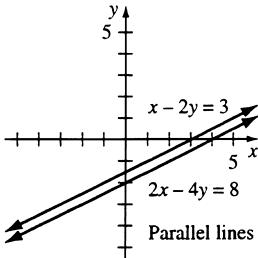


Figure 1.3: for Problem 1.1.13.

1.1.14 The system reduces to  $\begin{bmatrix} x + 5z &= 0 \\ y - z &= 0 \\ 0 &= 1 \end{bmatrix}$ , so that there is no solution; no point in space belongs to all three planes.

Compare with Figure 2b.

1.1.15 The system reduces to  $\begin{bmatrix} x &= 0 \\ y &= 0 \\ z &= 0 \end{bmatrix}$  so the unique solution is  $(x, y, z) = (0, 0, 0)$ . The three planes intersect at the origin.

1.1.16 The system reduces to  $\begin{bmatrix} x + 5z &= 0 \\ y - z &= 0 \\ 0 &= 0 \end{bmatrix}$ , so the solutions are of the form  $(x, y, z) = (-5t, t, t)$ , where  $t$  is an arbitrary number. The three planes intersect in a line; compare with Figure 2a.

1.1.17  $\begin{bmatrix} x + 2y &= a \\ 3x + 5y &= b \end{bmatrix} \xrightarrow{-3(I)} \begin{bmatrix} x + 2y &= a \\ -y &= -3a + b \end{bmatrix} \xrightarrow{\div(-1)} \begin{bmatrix} x + 2y &= a \\ y &= 3a - b \end{bmatrix} \xrightarrow{-2(II)} \dots$

$\begin{bmatrix} x & = -5a + 2b \\ y & = 3a - b \end{bmatrix}$ , so that  $(x, y) = (-5a + 2b, 3a - b)$ .

$$\begin{aligned} 1.1.18 \quad & \begin{bmatrix} x + 2y + 3z & = a \\ x + 3y + 8z & = b \\ x + 2y + 2z & = c \end{bmatrix} -I \rightarrow \begin{bmatrix} x + 2y + 3z & = a \\ y + 5z & = -a + b \\ -z & = -a + c \end{bmatrix} -2(II) \rightarrow \\ & \begin{bmatrix} x - 7z & = 3a - 2b \\ y + 5z & = -a + b \\ -z & = -a + c \end{bmatrix} \div(-1) \rightarrow \begin{bmatrix} x - 7z & = 3a - 2b \\ y + 5z & = -a + b \\ z & = a - c \end{bmatrix} +7(III) \rightarrow \begin{bmatrix} x & = 10a - 2b - 7c \\ y & = -6a + b + 5c \\ z & = a - c \end{bmatrix}, \end{aligned}$$

so that  $(x, y, z) = (10a - 2b - 7c, -6a + b + 5c, a - c)$ .

1.1.19 a Note that the demand  $D_1$  for product 1 increases with the increase of price  $P_2$ ; likewise the demand  $D_2$  for product 2 increases with the increase of price  $P_1$ . This indicates that the two products are competing; some people will switch if one of the products gets more expensive.

b Setting  $D_1 = S_1$  and  $D_2 = S_2$  we obtain the system  $\begin{bmatrix} 70 - 2P_1 + P_2 & = -14 + 3P_1 \\ 105 + P_1 - P_2 & = -7 + 2P_2 \end{bmatrix}$ , or  $\begin{bmatrix} -5P_1 + P_2 & = -84 \\ P_1 - 3P_2 & = 112 \end{bmatrix}$ , which yields the unique solution  $P_1 = 26$  and  $P_2 = 46$ .

1.1.20 The total demand for the product of Industry A is 1000 (the consumer demand) plus 0.1b (the demand from Industry B). The output  $a$  must meet this demand:  $a = 1000 + 0.1b$ .

Setting up a similar equation for Industry B we obtain the system  $\begin{bmatrix} a & = 1000 + 0.1b \\ b & = 780 + 0.2a \end{bmatrix}$  or  $\begin{bmatrix} a - 0.1b & = 1000 \\ -0.2a + b & = 780 \end{bmatrix}$ , which yields the unique solution  $a = 1100$  and  $b = 1000$ .

1.1.21 The total demand for the products of Industry A is 310 (the consumer demand) plus 0.3b (the demand from Industry B). The output  $a$  must meet this demand:  $a = 310 + 0.3b$ .

Setting up a similar equation for Industry B we obtain the system  $\begin{bmatrix} a & = 310 + 0.3b \\ b & = 100 + 0.5a \end{bmatrix}$  or  $\begin{bmatrix} a - 0.3b & = 310 \\ -0.5a + b & = 100 \end{bmatrix}$ , which yields the solution  $a = 400$  and  $b = 300$ .

1.1.22 Since  $x(t) = a \sin(t) + b \cos(t)$  we can compute  $\frac{dx}{dt} = a \cos(t) - b \sin(t)$  and  $\frac{d^2x}{dt^2} = -a \sin(t) - b \cos(t)$ . Substituting these expressions into the equation  $\frac{d^2x}{dt^2} - \frac{dx}{dt} - x = \cos(t)$  and simplifying gives  $(b - 2a) \sin(t) + (-a - 2b) \cos(t) = \cos(t)$ . Comparing the coefficients of  $\sin(t)$  and  $\cos(t)$  on both sides of the equation then yields the system  $\begin{bmatrix} -2a + b & = 0 \\ -a - 2b & = 1 \end{bmatrix}$ , so that  $a = -\frac{1}{5}$  and  $b = -\frac{2}{5}$ . See Figure 1.4.

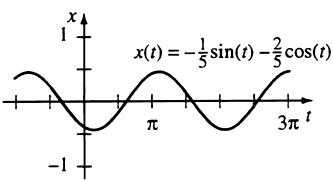


Figure 1.4: for Problem 1.1.22.

**1.1.23 a** Substituting  $\lambda = 5$  yields the system

$$\begin{bmatrix} 7x - y & = 5x \\ -6x + 8y & = 5y \end{bmatrix} \text{ or } \begin{bmatrix} 2x - y & = 0 \\ -6x + 3y & = 0 \end{bmatrix} \text{ or } \begin{bmatrix} 2x - y & = 0 \\ 0 & = 0 \end{bmatrix}.$$

There are infinitely many solutions, of the form  $(x, y) = (\frac{t}{2}, t)$ , where  $t$  is an arbitrary real number.

**b** Proceeding as in part (a), we find  $(x, y) = (-\frac{1}{3}t, t)$ .

**c** Proceedings as in part (a), we find only the solution  $(0, 0)$ .

**1.1.24** Let  $v$  be the speed of the boat relative to the water, and  $s$  be the speed of the stream; then the speed of the boat relative to the land is  $v+s$  downstream and  $v-s$  upstream. Using the fact that (distance) = (speed)(time), we obtain the system

$$\begin{bmatrix} 8 & = (v+s)\frac{1}{3} \\ 8 & = (v-s)\frac{2}{3} \end{bmatrix} \quad \leftarrow \begin{array}{l} \text{downstream} \\ \text{upstream} \end{array}$$

The solution is  $v = 18$  and  $s = 6$ .

**1.1.25** The system reduces to  $\begin{bmatrix} x+z & = 1 \\ y-2z & = -3 \\ 0 & = k-7 \end{bmatrix}$ .

- a. The system has solutions if  $k-7=0$ , or  $k=7$ .
- b. If  $k=7$  then the system has infinitely many solutions.
- c. If  $k=7$  then we can choose  $z=t$  freely and obtain the solutions

$$(x, y, z) = (1-t, -3+2t, t).$$

**1.1.26** The system reduces to  $\begin{bmatrix} x-3z & = 1 \\ y+2z & = 1 \\ (k^2-4)z & = k-2 \end{bmatrix}$

This system has a unique solution if  $k^2-4 \neq 0$ , that is, if  $k \neq \pm 2$ .

If  $k=2$ , then the last equation is  $0=0$ , and there will be infinitely many solutions.

If  $k=-2$ , then the last equation is  $0=-4$ , and there will be no solutions.

**1.1.27** Let  $x =$  the number of male children and  $y =$  the number of female children.

Then the statement “Emile has twice as many sisters as brothers” translates into

$y = 2(x-1)$  and “Gertrude has as many brothers as sisters” translates into

$$x = y - 1.$$

Solving the system  $\begin{bmatrix} -2x + y & = -2 \\ x - y & = -1 \end{bmatrix}$  gives  $x = 3$  and  $y = 4$ .

There are seven children in this family.

## Chapter 1

---

**1.1.28** The thermal equilibrium condition requires that  $T_1 = \frac{T_2+200+0+0}{4}$ ,  $T_2 = \frac{T_1+T_3+200+0}{4}$ , and  $T_3 = \frac{T_2+400+0+0}{4}$ .

We can rewrite this system as  $\begin{bmatrix} -4T_1 + T_2 & = -200 \\ T_1 - 4T_2 + T_3 & = -200 \\ T_2 - 4T_3 & = -400 \end{bmatrix}$

The solution is  $(T_1, T_2, T_3) = (75, 100, 125)$ .

**1.1.29** To assure that the graph goes through the point  $(1, -1)$ , we substitute  $t = 1$  and  $f(t) = -1$  into the equation  $f(t) = a + bt + ct^2$  to give  $-1 = a + b + c$ .

Proceeding likewise for the two other points, we obtain the system  $\begin{bmatrix} a + b + c & = -1 \\ a + 2b + 4c & = 3 \\ a + 3b + 9c & = 13 \end{bmatrix}$ .

The solution is  $a = 1$ ,  $b = -5$ , and  $c = 3$ , and the polynomial is  $f(t) = 1 - 5t + 3t^2$ . (See Figure 1.5.)

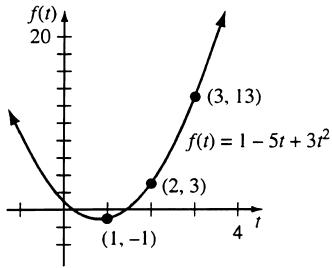


Figure 1.5: for Problem 1.1.29.

**1.1.30** Proceeding as in the previous exercise, we obtain the system  $\begin{bmatrix} a + b + c & = p \\ a + 2b + 4c & = q \\ a + 3b + 9c & = r \end{bmatrix}$ .

The unique solution is  $\begin{bmatrix} a & = 3p - 3q + r \\ b & = -2.5p + 4q - 1.5r \\ c & = 0.5p - q + 0.5r \end{bmatrix}$ .

Only one polynomial of degree 2 goes through the three given points, namely,

$$f(t) = 3p - 3q + r + (-2.5p + 4q - 1.5r)t + (0.5p - q + 0.5r)t^2.$$

**1.1.31**  $f(t)$  is of the form  $at^2 + bt + c$ . So  $f(1) = a(1^2) + b(1) + c = 3$ , and  $f(2) = a(2^2) + b(2) + c = 6$ . Also,  $f'(t) = 2at + b$ , meaning that  $f'(1) = 2a + b = 1$ .

So we have a system of equations:  $\begin{bmatrix} a + b + c = 3 \\ 4a + 2b + c = 6 \\ 2a + b = 1 \end{bmatrix}$

which reduces to  $\begin{bmatrix} a = 2 \\ b = -3 \\ c = 4 \end{bmatrix}$ .

Thus,  $f(t) = 2t^2 - 3t + 4$  is the only solution.

**1.1.32**  $f(t)$  is of the form  $at^2 + bt + c$ . So,  $f(1) = a(1^2) + b(1) + c = 1$  and  $f(2) = 4a + 2b + c = 0$ . Also,

$$\int_1^2 f(t)dt = \int_1^2 (at^2 + bt + c)dt$$

$$\begin{aligned} &= \frac{a}{3}t^3 + \frac{b}{2}t^2 + ct|_1^2 \\ &= \frac{8}{3}a + 2b + 2c - (\frac{a}{3} + \frac{b}{2} + c) \\ &= \frac{7}{3}a + \frac{3}{2}b + c = -1. \end{aligned}$$

So we have a system of equations:  $\begin{bmatrix} a + b + c = 1 \\ 4a + 2b + c = 0 \\ \frac{7}{3}a + \frac{3}{2}b + c = -1 \end{bmatrix}$

which reduces to  $\begin{bmatrix} a = 9 \\ b = -28 \\ c = 20 \end{bmatrix}$ .

Thus,  $f(t) = 9t^2 - 28t + 20$  is the only solution.

**1.1.33**  $f(t)$  is of the form  $at^2 + bt + c$ .  $f(1) = a + b + c = 1$ ,  $f(3) = 9a + 3b + c = 3$ , and  $f'(t) = 2at + b$ , so  $f'(2) = 4a + b = 1$ .

Now we set up our system to be  $\begin{bmatrix} a + b + c = 1 \\ 9a + 3b + c = 3 \\ 4a + b = 1 \end{bmatrix}$ .

This reduces to  $\begin{bmatrix} a - \frac{c}{3} = 0 \\ b + \frac{4}{3}c = 1 \\ 0 = 0 \end{bmatrix}$ .

We write everything in terms of  $a$ , revealing  $c = 3a$  and  $b = 1 - 4a$ .

So,  $f(t) = at^2 + (1 - 4a)t + 3a$  for an arbitrary  $a$ .

**1.1.34**  $f(t) = at^2 + bt + c$ , so  $f(1) = a + b + c = 1$ ,  $f(3) = 9a + 3b + c = 3$ . Also,  $f'(2) = 3$ , so  $2(2)a + b = 4a + b = 3$ .

Thus, our system is  $\begin{bmatrix} a + b + c = 1 \\ 9a + 3b + c = 3 \\ 4a + b = 3 \end{bmatrix}$ .

When we reduce this, however, our last equation becomes  $0 = 2$ , meaning that this system is inconsistent.

**1.1.35**  $f(t) = ae^{3t} + be^{2t}$ , so  $f(0) = a + b = 1$  and  $f'(t) = 3ae^{3t} + 2be^{2t}$ , so  $f'(0) = 3a + 2b = 4$ .

Thus we obtain the system  $\begin{bmatrix} a + b = 1 \\ 3a + 2b = 4 \end{bmatrix}$ ,

which reveals  $\begin{bmatrix} a = 2 \\ b = -1 \end{bmatrix}$ .

So  $f(t) = 2e^{3t} - e^{2t}$ .

**1.1.36**  $f(t) = a \cos(2t) + b \sin(2t)$  and  $3f(t) + 2f'(t) + f''(t) = 17 \cos(2t)$ .

$$f'(t) = 2b \cos(2t) - 2a \sin(2t) \text{ and } f''(t) = -4b \sin(2t) - 4a \cos(2t).$$

$$\text{So, } 17 \cos(2t) = 3(a \cos(2t) + b \sin(2t)) + 2(2b \cos(2t) - 2a \sin(2t)) + (-4b \sin(2t) - 4a \cos(2t)) = (-4a + 4b + 3a) \cos(2t) + (-4b - 4a + 3b) \sin(2t) = (-a + 4b) \cos(2t) + (-4a - b) \sin(2t).$$

$$\text{So, our system is: } \begin{bmatrix} -a + 4b = 17 \\ -4a - b = 0 \end{bmatrix}.$$

$$\text{This reduces to: } \begin{bmatrix} a = -1 \\ b = 4 \end{bmatrix}.$$

So our function is  $f(t) = -\cos(2t) + 4 \sin(2t)$ .

**1.1.37** Plugging the three points  $(x, y)$  into the equation  $a + bx + cy + x^2 + y^2 = 0$ , leads to a system of linear equations for the three unknowns  $(a, b, c)$ .

$$\begin{aligned} a + 5b + 25 + 25 &= 0 \\ a + 4b + 6c + 16 + 36 &= 0 \\ a + 6b + 2c + 36 + 4 &= 0. \end{aligned}$$

The solution is  $a = -20, b = -2, c = -4$ .  $-20 - 2x - 4y + x^2 + y^2 = 0$  is a circle of radius 5 centered at  $(1, 2)$ .

**1.1.38** Plug the three points into the equation  $ax^2 + bxy + cy^2 = 1$ . We obtain a system of linear equations

$$\begin{aligned} a + 2b + 4c &= 1 \\ 4a + 4b + 4c &= 1 \\ 9a + 3b + c &= 1. \end{aligned}$$

The solution is  $a = 3/20, b = -9/40, c = 13/40$ . This is the ellipse  $(3/20)x^2 - (9/40)xy + (13/40)y^2 = 1$ .

**1.1.39** The given system reduces to  $\begin{bmatrix} x - z &= \frac{-5a+2b}{3} \\ y + 2z &= \frac{4a-b}{3} \\ 0 &= a - 2b + c \end{bmatrix}$ .

This system has solutions (in fact infinitely many) if  $a - 2b + c = 0$ .

The points  $(a, b, c)$  with this property form a plane through the origin.

**1.1.40 a**  $x_1 = -3$

$$x_2 = 14 + 3x_1 = 14 + 3(-3) = 5$$

$$x_3 = 9 - x_1 - 2x_2 = 9 + 3 - 10 = 2$$

$$x_4 = 33 + x_1 - 8x_2 + 5x_3 - x_4 = 33 - 3 - 40 + 10 = 0,$$

so that  $(x_1, x_2, x_3, x_4) = (-3, 5, 2, 0)$ .

b  $x_4 = 0$

$$x_3 = 2 - 2x_4 = 2$$

$$x_2 = 5 - 3x_3 - 7x_4 = 5 - 6 = -1$$

$$x_1 = -3 - 2x_2 + x_3 - 4x_4 = -3 + 2 + 2 = 1,$$

so that  $(x_1, x_2, x_3, x_4) = (1, -1, 2, 0)$ .

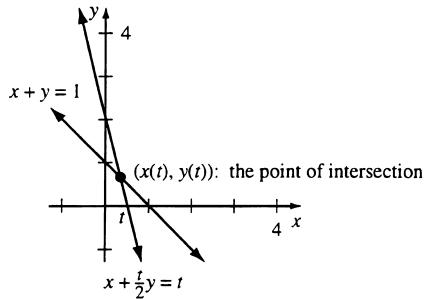


Figure 1.6: for Problem 1.1.41a.

1.1.41 a The two lines intersect unless  $t = 2$  (in which case both lines have slope  $-1$ ).

To draw a rough sketch of  $x(t)$ , note that

$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow -\infty} x(t) = -1$  (the line  $x + \frac{t}{2}y = t$  becomes almost horizontal) and

$\lim_{t \rightarrow 2^-} x(t) = \infty$ ,  $\lim_{t \rightarrow 2^+} x(t) = -\infty$ .

Also note that  $x(t)$  is positive if  $t$  is between 0 and 2, and negative otherwise.

Apply similar reasoning to  $y(t)$ . (See Figures 1.6 and 1.7.)

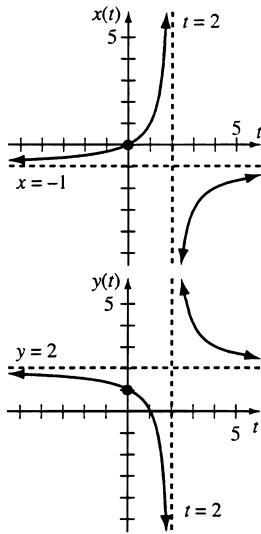


Figure 1.7: for Problem 1.1.41a.

b  $x(t) = \frac{-t}{t-2}$ , and  $y(t) = \frac{2t-2}{t-2}$ .

**1.1.42** We can think of the line through the points  $(1, 1, 1)$  and  $(3, 5, 0)$  as the intersection of any two planes through these two points; each of these planes will be defined by an equation of the form  $ax + by + cz = d$ . It is required that  $1a + 1b + 1c = d$  and  $3a + 5b + 0c = d$ .

Now the system  $\begin{bmatrix} a & +b & +c & -d & = & 0 \\ 3a & +5b & & -d & = & 0 \end{bmatrix}$  reduces to

$$\begin{bmatrix} a & +\frac{5}{2}c & -2d & = & 0 \\ b & -\frac{3}{2}c & +d & = & 0 \end{bmatrix}.$$

We can choose arbitrary real numbers for  $c$  and  $d$ ; then  $a = -\frac{5}{2}c + 2d$  and  $b = \frac{3}{2}c - d$ . For example, if we choose  $c = 2$  and  $d = 0$ , then  $a = -5$  and  $b = 3$ , leading to the equation  $-5x + 3y + 2z = 0$ . If we choose  $c = 0$  and  $d = 1$ , then  $a = 2$  and  $b = -1$ , giving the equation  $2x - y = 1$ .

We have found one possible answer:  $\begin{bmatrix} -5x & +3y & +2z & = & 0 \\ 2x & -y & & = & 1 \end{bmatrix}$ .

**1.1.43** To eliminate the arbitrary constant  $t$ , we can solve the last equation for  $t$  to give  $t = z - 2$ , and substitute  $z - 2$  for  $t$  in the first two equations, obtaining  $\begin{bmatrix} x & = 6 + 5(z - 2) \\ y & = 4 + 3(z - 2) \end{bmatrix}$  or  $\begin{bmatrix} x - 5z & = -4 \\ y - 3z & = -2 \end{bmatrix}$ .

This system does the job.

**1.1.44** Let  $b$  = Boris' money,  $m$  = Marina's money, and  $c$  = cost of a chocolate bar.

We are told that  $\begin{bmatrix} b + \frac{1}{2}m & = c \\ \frac{1}{2}b + m & = 2c \end{bmatrix}$ , with solution  $(b, m) = (0, 2c)$ .

Boris has no money.

**1.1.45** Let us start by reducing the system:

$$\begin{bmatrix} x + 2y + 3z & = 39 \\ x + 3y + 2z & = 34 \\ 3x + 2y + z & = 26 \end{bmatrix} \xrightarrow{-I} \begin{bmatrix} x + 2y + 3z & = 39 \\ y - z & = -5 \\ -4y - 8z & = -91 \end{bmatrix}$$

Note that the last two equations are exactly those we get when we substitute

$$x = 39 - 2y - 3z: \text{ either way, we end up with the system } \begin{bmatrix} y - z & = -5 \\ -4y - 8z & = -91 \end{bmatrix}.$$

**1.1.46 a** We set up two equations here, with our variables:  $x_1$  = servings of rice,  $x_2$  = servings of yogurt.

So our system is:  $\begin{bmatrix} 3x_1 & +12x_2 & = 60 \\ 30x_1 & +20x_2 & = 300 \end{bmatrix}$ .

Solving this system reveals that  $x_1 = 8, x_2 = 3$ .

b Again, we set up our equations:  $\begin{bmatrix} 3x_1 & +12x_2 & = P \\ 30x_1 & +20x_2 & = C \end{bmatrix}$ ,

and reduce them to find that  $x_1 = -\frac{P}{15} + \frac{C}{25}$ , while  $x_2 = \frac{P}{10} - \frac{C}{100}$ .

**1.1.47** Let  $x_1$  = number of one-dollar bills,  $x_2$  = the number of five-dollar bills, and  $x_3$  = the number of ten-dollar bills. Then our system looks like:  $\begin{bmatrix} x_1 + x_2 + x_3 & = 32 \\ x_1 + 5x_2 + 10x_3 & = 100 \end{bmatrix}$ ,

which reduces to give us solutions that fit:  $x_1 = 15 + \frac{5}{4}x_3$ ,  $x_2 = 17 - \frac{9}{4}x_3$ , where  $x_3$  can be chosen freely. Now let's keep in mind that  $x_1$ ,  $x_2$ , and  $x_3$  must be positive integers and see what conditions this imposes on the variable  $x_3$ . We see that since  $x_1$  and  $x_2$  must be integers,  $x_3$  must be a multiple of 4. Furthermore,  $x_3$  must be positive, and  $x_2 = 17 - \frac{9}{4}x_3$  must be positive as well, meaning that  $x_3 < \frac{68}{9}$ . These constraints leave us with only one possibility,  $x_3 = 4$ , and we can compute the corresponding values  $x_1 = 15 + \frac{5}{4}x_3 = 20$  and  $x_2 = 17 - \frac{9}{4}x_3 = 8$ .

Thus, we have 20 one-dollar bills, 8 five-dollar bills, and 4 ten-dollar bills.

**1.1.48** Let  $x_1, x_2, x_3$  be the number of 20 cent, 50 cent, and 2 Euro coins, respectively. Then we need solutions to the system:  $\begin{bmatrix} x_1 & +x_2 & +x_3 & = 1000 \\ .2x_1 & +.5x_2 & +2x_3 & = 1000 \end{bmatrix}$

this system reduces to:  $\begin{bmatrix} x_1 & -5x_3 & = -\frac{5000}{3} \\ x_2 & +6x_3 & = \frac{8000}{3} \end{bmatrix}$ .

Our solutions are then of the form  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 - \frac{5000}{3} \\ -6x_3 + \frac{8000}{3} \\ x_3 \end{bmatrix}$ . Unfortunately for the meter maids, there are no integer solutions to this problem. If  $x_3$  is an integer, then neither  $x_1$  nor  $x_2$  will be an integer, and no one will ever claim the Ferrari.

## Section 1.2

$$1.2.1 \quad \begin{bmatrix} 1 & 1 & -2: & 5 \\ 2 & 3 & 4: & 2 \end{bmatrix} \xrightarrow{-2(I)} \begin{bmatrix} 1 & 1 & -2: & 5 \\ 0 & 1 & 8: & -8 \end{bmatrix} \xrightarrow{-II} \begin{bmatrix} 1 & 0 & -10: & 13 \\ 0 & 1 & 8: & -8 \end{bmatrix}$$

$$\begin{bmatrix} x - 10z & = 13 \\ y + 8z & = -8 \end{bmatrix} \longrightarrow \begin{bmatrix} x & = 13 + 10z \\ y & = -8 - 8z \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 + 10t \\ -8 - 8t \\ t \end{bmatrix}, \text{ where } t \text{ is an arbitrary real number.}$$

$$1.2.2 \quad \begin{bmatrix} 3 & 4 & -1: & 8 \\ 6 & 8 & -2: & 3 \end{bmatrix} \xrightarrow{\div 3} \begin{bmatrix} 1 & \frac{4}{3} & -\frac{1}{3}: & \frac{8}{3} \\ 6 & 8 & -2: & 3 \end{bmatrix} \xrightarrow{-6(I)} \begin{bmatrix} 1 & \frac{4}{3} & -\frac{1}{3}: & \frac{8}{3} \\ 0 & 0 & 0: & -13 \end{bmatrix}$$

This system has no solutions, since the last row represents the equation  $0 = -13$ .

$$1.2.3 \quad x = 4 - 2y - 3z$$

$y$  and  $z$  are free variables; let  $y = s$  and  $z = t$ .

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 - 2s - 3t \\ s \\ t \end{bmatrix}, \text{ where } s \text{ and } t \text{ are arbitrary real numbers.}$$

$$1.2.4 \quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 3 & 4 & 2 \end{bmatrix} \xrightarrow{-2(I)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{\div(-3)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-II} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so that } \begin{array}{l} x = 2 \\ y = -1 \end{array}.$$

$$1.2.5 \quad \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[I \leftrightarrow III]{\text{swap}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{-I} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[+II]{-II}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[-III]{+III} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & + & x_4 & = & 0 \\ x_2 & - & x_4 & = & 0 \\ x_3 & + & x_4 & = & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 & = & -x_4 \\ x_2 & = & x_4 \\ x_3 & = & -x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -t \\ t \\ \end{bmatrix}, \text{ where } t \text{ is an arbitrary real number.}$$

1.2.6 The system is in rref already.

$$\begin{bmatrix} x_1 & = 3 + 7x_2 - x_5 \\ x_3 & = 2 + 2x_5 \\ x_4 & = 1 - x_5 \end{bmatrix}$$

Let  $x_2 = t$  and  $x_5 = r$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 + 7t - r \\ t \\ 2 + 2r \\ 1 - r \\ r \end{bmatrix}$$

$$1.2.7 \quad \begin{bmatrix} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{-II} \begin{bmatrix} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[-3(III)]{-2(III)} \begin{bmatrix} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 11 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{-9(IV)} \left[ \begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{-11(IV)} \left[ \begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{+3(IV)} \left[ \begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} x_1 + 2x_2 & = 0 \\ x_3 & = 0 \\ x_4 & = 0 \\ x_5 & = 0 \end{array} \right] \longrightarrow \left[ \begin{array}{cc|c} x_1 & = -2x_2 \\ x_3 & = 0 \\ x_4 & = 0 \\ x_5 & = 0 \end{array} \right]$$

Let  $x_2 = t$ .  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix}$ , where  $t$  is an arbitrary real number.

$$1.2.8 \quad \left[ \begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 & 8 & 0 \end{array} \right] \div 4 \rightarrow \left[ \begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{-2(II)} \left[ \begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} x_2 - x_5 & = 0 \\ x_4 + 2x_5 & = 0 \end{array} \right] \longrightarrow \left[ \begin{array}{cc|c} x_2 & = x_5 \\ x_4 & = -2x_5 \end{array} \right]$$

Let  $x_1 = r$ ,  $x_3 = s$ ,  $x_5 = t$ .

$$\left[ \begin{array}{ccccc|c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \right] = \left[ \begin{array}{c} r \\ t \\ s \\ -2t \\ t \end{array} \right], \text{ where } r, t \text{ and } s \text{ are arbitrary real numbers.}$$

$$1.2.9 \quad \left[ \begin{array}{cccccc|c} 0 & 0 & 0 & 1 & 2 & -1 & 2 \\ 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 1 & 2 & 2 & 0 & -1 & 1 & 2 \end{array} \right] \xrightarrow{\text{swap: } I \leftrightarrow II} \left[ \begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \\ 1 & 2 & 2 & 0 & -2 & 1 & 2 \end{array} \right] \xrightarrow{-I} \left[ \begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \\ 1 & 2 & 2 & 0 & -2 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 2 & 0 & -2 & 2 & 2 \end{array} \right] \xrightarrow{\text{swap: } II \leftrightarrow III} \left[ \begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \end{array} \right] \div 2 \rightarrow$$

$$\left[ \begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} x_1 + 2x_2 + x_5 - x_6 & = 0 \\ x_3 - x_5 + x_6 & = 1 \\ x_4 + 2x_5 - x_6 & = 2 \end{array} \right] \longrightarrow \left[ \begin{array}{cc|c} x_1 & = -2x_2 - x_5 + x_6 \\ x_3 & = 1 + x_5 - x_6 \\ x_4 & = 2 - 2x_5 + x_6 \end{array} \right]$$

Let  $x_2 = r$ ,  $x_5 = s$ , and  $x_6 = t$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2r - s + t \\ r \\ 1 + s - t \\ 2 - 2s + t \\ s \\ t \end{bmatrix}, \text{ where } r, s \text{ and } t \text{ are arbitrary real numbers.}$$

1.2.10 The system reduces to  $\begin{bmatrix} x_1 & & + & x_4 & = & 1 \\ & x_2 & - & 3x_4 & = & 2 \\ & & x_3 & + & 2x_4 & = & -3 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 & = 1 - x_4 \\ x_2 & = 2 + 3x_4 \\ x_3 & = -3 - 2x_4 \end{bmatrix}$

Let  $x_4 = t$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - t \\ 2 + 3t \\ -3 - 2t \\ t \end{bmatrix}, \text{ where } t \text{ is an arbitrary real number.}$$

1.2.11 The system reduces to  $\begin{bmatrix} x_1 & & + & 2x_3 & = & 0 \\ & x_2 & - & 3x_3 & = & 4 \\ & & & x_4 & = & -2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 & = -2x_3 \\ x_2 & = 4 + 3x_3 \\ x_4 & = -2 \end{bmatrix}.$

Let  $x_3 = t$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ 4 + 3t \\ t \\ -2 \end{bmatrix}$$

1.2.12 The system reduces to  $\begin{bmatrix} x_1 & & + & 3.5x_5 & + & x_6 & = & 0 \\ & x_2 & + & x_5 & = & 0 \\ & & x_3 & - & \frac{5}{3}x_6 & = & 0 \\ & & x_4 & + & 3x_5 & + & x_6 & = & 0 \end{bmatrix} \rightarrow$

$$\begin{bmatrix} x_1 & = -3.5x_5 - x_6 \\ x_2 & = -x_5 \\ x_3 & = \frac{5}{3}x_6 \\ x_4 & = -3x_5 - x_6 \end{bmatrix}.$$

Let  $x_5 = r$  and  $x_6 = t$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3.5r - t \\ -r \\ \frac{5}{3}t \\ -3r - t \\ r \\ t \end{bmatrix}$$

1.2.13 The system reduces to  $\begin{bmatrix} x & - & z & = & 0 \\ y & + & 2z & = & 0 \\ & & 0 & = & 1 \end{bmatrix}.$

There are no solutions.

1.2.14 The system reduces to  $\begin{bmatrix} x & + & 2y & = & -2 \\ & & z & = & 2 \end{bmatrix} \rightarrow \begin{bmatrix} x & = & -2 - 2y \\ z & = & 2 \end{bmatrix}$ .

Let  $y = t$ .

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 - 2t \\ t \\ 2 \end{bmatrix}$$

1.2.15 The system reduces to  $\begin{bmatrix} x & & = & 4 \\ y & & = & 2 \\ z & & = & 1 \end{bmatrix}$ .

1.2.16 The system reduces to  $\begin{bmatrix} x_1 + 2x_2 + 3x_3 & & +5x_5 & = & 6 \\ x_4 & & +2x_5 & = & 7 \end{bmatrix} \rightarrow$

$$\begin{bmatrix} x_1 & = & 6 - 2x_2 - 3x_3 - 5x_5 \\ x_4 & = & 7 - 2x_5 \end{bmatrix}.$$

Let  $x_2 = r$ ,  $x_3 = s$ , and  $x_5 = t$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 - 2r - 3s - 5t \\ r \\ s \\ 7 - 2t \\ t \end{bmatrix}$$

1.2.17 The system reduces to  $\begin{bmatrix} x_1 & & = & -\frac{8221}{4340} \\ x_2 & & = & \frac{8591}{8680} \\ x_3 & & = & \frac{4695}{434} \\ x_4 & & = & -\frac{459}{434} \\ x_5 & & = & \frac{699}{434} \end{bmatrix}$ .

1.2.18 a No, since the third column contains two leading ones.

b Yes

c No, since the third row contains a leading one, but the second row does not.

d Yes

1.2.19  $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

1.2.20 Four, namely  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  ( $k$  is an arbitrary constant.)

**1.2.21** Four, namely  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & k \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$  ( $k$  is an arbitrary constant.)

**1.2.22** Seven, namely  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & e \end{bmatrix}, \begin{bmatrix} 1 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

Here,  $a, b, \dots, f$  are arbitrary constants.

**1.2.23** The conditions a, b, and c for the reduced row-echelon form correspond to the properties P1, P2, and P3 given on Page 13. The Gauss-Jordan algorithm, summarized on Page 15, guarantees that those properties are satisfied.

**1.2.24** Yes; each elementary row operation is reversible, that is, it can be “undone.” For example, the operation of row swapping can be undone by swapping the same rows again. The operation of dividing a row by a scalar can be reversed by multiplying the same row by the same scalar.

**1.2.25** Yes; if  $A$  is transformed into  $B$  by a sequence of elementary row operations, then we can recover  $A$  from  $B$  by applying the inverse operations in the reversed order (compare with Exercise 24).

**1.2.26** Yes, by Exercise 25, since  $\text{rref}(A)$  is obtained from  $A$  by a sequence of elementary row operations.

**1.2.27** No; whatever elementary row operations you apply to  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ , you cannot make the last column equal to zero.

**1.2.28** Suppose  $(c_1, c_2, \dots, c_n)$  is a solution of the system  $\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \dots \dots \end{bmatrix}$ .

To keep the notation simple, suppose we add  $k$  times the first equation to the second; then the second equation of the new system will be  $(a_{21} + ka_{11})x_1 + \dots + (a_{2n} + ka_{1n})x_n = b_2 + kb_1$ .

We have to verify that  $(c_1, c_2, \dots, c_n)$  is a solution of this new equation. Indeed,  $(a_{21} + ka_{11})c_1 + \dots + (a_{2n} + ka_{1n})c_n = a_{21}c_1 + \dots + a_{2n}c_n + k(a_{11}c_1 + \dots + a_{1n}c_n) = b_2 + kb_1$ .

We have shown that any solution of the “old” system is also a solution of the “new.” To see that, conversely, any solution of the new system is also a solution of the old system, note that elementary row operations are reversible (compare with Exercise 24); we can obtain the old system by subtracting  $k$  times the first equation from the second equation of the new system.

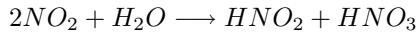
**1.2.29** Since the number of oxygen atoms remains constant, we must have  $2a + b = 2c + 3d$ .

Considering hydrogen and nitrogen as well, we obtain the system  $\begin{bmatrix} 2a & + & b & = & 2c & + & 3d \\ & & 2b & = & c & + & d \\ a & & & & = & c & + & d \end{bmatrix}$  or

$\begin{bmatrix} 2a & + & b & - & 2c & - & 3d & = & 0 \\ & & 2b & - & c & - & d & = & 0 \\ a & & - & c & - & d & = & 0 \end{bmatrix}$ , which reduces to  $\begin{bmatrix} a & - & 2d & = & 0 \\ b & - & d & = & 0 \\ c & - & d & = & 0 \end{bmatrix}$ .

The solutions are  $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix}$ .

To get the smallest positive integers, we set  $t = 1$ :



**1.2.30** Plugging the points into  $f(t)$ , we obtain the system

$$\begin{bmatrix} a & & & & & = & 1 \\ a + b + c + d & = & 0 \\ a - b + c - d & = & 0 \\ a + 2b + 4c + 8d & = & -15 \end{bmatrix}$$

with unique solution  $a = 1$ ,  $b = 2$ ,  $c = -1$ , and  $d = -2$ , so that  $f(t) = 1 + 2t - t^2 - 2t^3$ . (See Figure 1.8.)

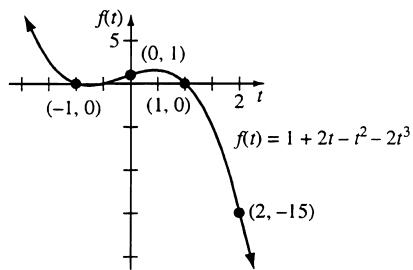


Figure 1.8: for Problem 1.2.30.

**1.2.31** Let  $f(t) = a + bt + ct^2 + dt^3 + et^4$ . Substituting the points in, we get

$$\begin{bmatrix} a + b + c + d + e & = & 1 \\ a + 2b + 4c + 8d + 16e & = & -1 \\ a + 3b + 9c + 27d + 81e & = & -59 \\ a - b + c - d + e & = & 5 \\ a - 2b + 4c - 8d + 16e & = & -29 \end{bmatrix}$$

This system has the unique solution  $a = 1$ ,  $b = -5$ ,  $c = 4$ ,  $d = 3$ , and  $e = -2$ , so that  $f(t) = 1 - 5t + 4t^2 + 3t^3 - 2t^4$ . (See Figure 1.9.)

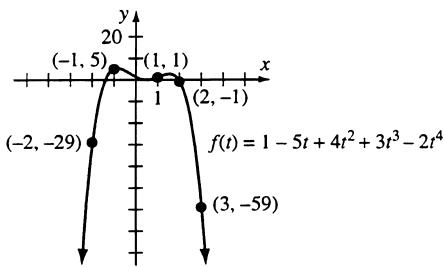


Figure 1.9: for Problem 1.2.31.

**1.2.32** The requirement  $f'_i(a_i) = f'_{i+1}(a_i)$  and  $f''_i(a_i) = f''_{i+1}(a_i)$  ensure that at each junction two different cubics fit “into” one another in a “smooth” way, since they must have the same slope and be equally curved. The requirement that  $f'_1(a_0) = f'_n(a_n) = 0$  ensures that the track is horizontal at the beginning and at the end. How many unknowns are there? There are  $n$  pieces to be fit, and each one is a cubic of the form  $f(t) = p + qt + rt^2 + st^3$ , with  $p, q, r$ , and  $s$  to be determined; therefore, there are  $4n$  unknowns. How many equations are there?

$$\begin{array}{lll} f_i(a_i) = b_i & \text{for } i = 1, 2, \dots, n & \text{gives } n \text{ equations} \\ f_i(a_{i-1}) = b_{i-1} & \text{for } i = 1, 2, \dots, n & \text{gives } n \text{ equations} \\ f'_i(a_i) = f'_{i+1}(a_i) & \text{for } i = 1, 2, \dots, n-1 & \text{gives } n-1 \text{ equations} \\ f''_i(a_i) = f''_{i+1}(a_i) & \text{for } i = 1, 2, \dots, n-1 & \text{gives } n-1 \text{ equations} \\ f'_1(a_0) = 0, f'_n(a_n) = 0 & & \text{gives 2 equations} \end{array}$$

Altogether, we have  $4n$  equations; convince yourself that all these equations are linear.

**1.2.33** Let  $f(t) = a + bt + ct^2 + dt^3$ , so that  $f'(t) = b + 2ct + 3dt^2$ .

Substituting the given points into  $f(t)$  and  $f'(t)$  we obtain the system

$$\begin{bmatrix} a + b + c + d & = 1 \\ a + 2b + 4c + 8d & = 5 \\ b + 2c + 3d & = 2 \\ b + 4c + 12d & = 9 \end{bmatrix}$$

This system has the unique solution  $a = -5, b = 13, c = -10$ , and  $d = 3$ , so that  $f(t) = -5 + 13t - 10t^2 + 3t^3$ . (See Figure 1.10.)

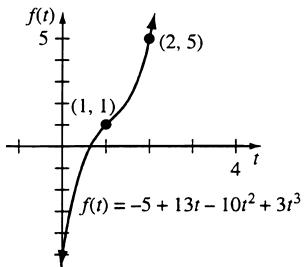


Figure 1.10: for Problem 1.2.33.

**1.2.34** We want all vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  in  $\mathbb{R}^3$  such that  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = x + 3y - z = 0$ . The endpoints of these vectors form a plane.

These vectors are of the form  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3r + t \\ r \\ t \end{bmatrix}$ , where  $r$  and  $t$  are arbitrary real numbers.

**1.2.35** We need to solve the system  $\begin{bmatrix} x_1 + x_2 + x_3 + x_4 & = 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 & = 0 \\ x_1 + 9x_2 + 9x_3 + 7x_4 & = 0 \end{bmatrix}$ ,

which reduces to  $\begin{bmatrix} x_1 & + & 0.25x_4 & = & 0 \\ x_2 & - & 1.5x_4 & = & 0 \\ x_3 & + & 2.25x_4 & = & 0 \end{bmatrix}$ .

The solutions are of the form  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -0.25t \\ 1.5t \\ -2.25t \\ t \end{bmatrix}$ , where  $t$  is an arbitrary real number.

**1.2.36** Writing the equation  $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$  in terms of its components, we obtain the system

$$\begin{bmatrix} x_1 + 2x_2 + 4x_3 & = -8 \\ 4x_1 + 5x_2 + 6x_3 & = -1 \\ 7x_1 + 8x_2 + 9x_3 & = 9 \\ 5x_1 + 3x_2 + x_3 & = 15 \end{bmatrix}$$

The system has the unique solution  $x_1 = 2$ ,  $x_2 = 3$ , and  $x_3 = -4$ .

**1.2.37** Compare with the solution of Exercise 1.1.21.

The diagram tells us that  $\begin{bmatrix} x_1 & = 0.2x_2 + 0.3x_3 + 320 \\ x_2 & = 0.1x_1 + 0.4x_3 + 90 \\ x_3 & = 0.2x_1 + 0.5x_2 + 150 \end{bmatrix}$  or  $\begin{bmatrix} x_1 - 0.2x_2 - 0.3x_3 & = 320 \\ -0.1x_1 + x_2 - 0.4x_3 & = 90 \\ -0.2x_1 - 0.5x_2 + x_3 & = 150 \end{bmatrix}$ .

This system has the unique solution  $x_1 = 500$ ,  $x_2 = 300$ , and  $x_3 = 400$ .

**1.2.38 a**  $\vec{v}_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0.2 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 0.2 \\ 0 \\ 0.5 \end{bmatrix}$ ,  $\vec{v}_3 = \begin{bmatrix} 0.3 \\ 0.4 \\ 0 \end{bmatrix}$ ,  $\vec{b} = \begin{bmatrix} 320 \\ 90 \\ 150 \end{bmatrix}$

b Recall that  $x_j$  is the output of industry  $I_j$ , and the  $i$ th component  $a_{ij}$  of  $\vec{v}_j$  is the demand of Industry  $I_j$  on industry  $I_i$  for each dollar of output of industry  $I_j$ .

Therefore, the product  $x_j a_{ij}$  (that is, the  $i$ th component of  $x_j \vec{v}_j$ ), represents the total demand of industry  $I_j$  on Industry  $I_i$  (in dollars).

c  $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n + \vec{b}$  is the vector whose  $i$ th component represents the total demand on industry  $I_i$  (consumer demand and interindustry demand combined).

d The  $i$ th component of the equation  $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n + \vec{b} = \vec{x}$  expresses the requirement that the output  $x_i$  of industry  $I_i$  equal the total demand on that industry.

**1.2.39 a** These components are zero because neither manufacturing nor the energy sector directly require agricultural products.

b We have to solve the system  $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + \vec{b} = \vec{x}$  or

$$\begin{bmatrix} 0.707x_1 & = 13.2 \\ -0.014x_1 + 0.793x_2 - 0.017x_3 & = 17.6 \\ -0.044x_1 + 0.01x_2 + 0.784x_3 & = 1.8 \end{bmatrix}$$

The unique solution is approximately  $x_1 = 18.67$ ,  $x_2 = 22.60$ , and  $x_3 = 3.63$ .

**1.2.40** We want to find  $m_1, m_2, m_3$  such that  $m_1 + m_2 + m_3 = 1$  and

$\frac{1}{1} \left( m_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + m_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + m_3 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ , that is, we have to solve the system

$$\begin{bmatrix} m_1 + m_2 + m_3 & = 1 \\ m_1 + 2m_2 + 4m_3 & = 2 \\ 2m_1 + 3m_2 + m_3 & = 2 \end{bmatrix}.$$

The unique solution is  $m_1 = \frac{1}{2}$ ,  $m_2 = \frac{1}{4}$ , and  $m_3 = \frac{1}{4}$ .

We will put  $\frac{1}{2}$  kg at the point  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\frac{1}{4}$  kg at each of the two other vertices.

1.2.41 We know that  $m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{w}_1 + m_2 \vec{w}_2$  or  $m_1(\vec{v}_1 - \vec{w}_1) + m_2(\vec{v}_2 - \vec{w}_2) = \vec{0}$

or  $\begin{bmatrix} -3m_1 + 2m_2 & = 0 \\ -6m_1 + 4m_2 & = 0 \\ -3m_1 + 2m_2 & = 0 \end{bmatrix}.$

We can conclude that  $m_1 = \frac{2}{3}m_2$ .

1.2.42 Let  $x_1, x_2, x_3$ , and  $x_4$  be the traffic volume at the four locations indicated in Figure 1.11.

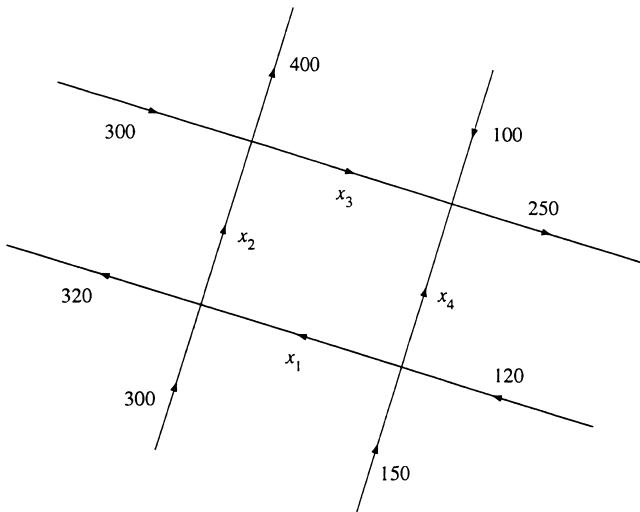


Figure 1.11: for Problem 1.2.42.

We are told that the number of cars coming into each intersection is the same as the number of cars coming out:

$$\begin{bmatrix} x_1 + 300 & = 320 + x_2 \\ x_2 + 300 & = 400 + x_3 \\ x_3 + x_4 + 100 & = 250 \\ 150 + 120 & = x_1 + x_4 \end{bmatrix} \text{ or } \begin{bmatrix} x_1 - x_2 & = 20 \\ x_2 - x_3 & = 100 \\ x_3 + x_4 & = 150 \\ x_1 + x_4 & = 270 \end{bmatrix}$$

The solutions are of the form  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 270 - t \\ 250 - t \\ 150 - t \\ t \end{bmatrix}$ .

Since the  $x_i$  must be positive integers (or zero),  $t$  must be an integer with  $0 \leq t \leq 150$ .

The lowest possible values are  $x_1 = 120$ ,  $x_2 = 100$ ,  $x_3 = 0$ , and  $x_4 = 0$ , while the highest possible values are  $x_1 = 270$ ,  $x_2 = 250$ ,  $x_3 = 150$ , and  $x_4 = 150$ .

**1.2.43** Plugging the data into the function  $S(t)$  we obtain the system

$$\begin{bmatrix} a + b \cos\left(\frac{2\pi \cdot 47}{365}\right) + c \sin\left(\frac{2\pi \cdot 47}{365}\right) & = 11.5 \\ a + b \cos\left(\frac{2\pi \cdot 74}{365}\right) + c \sin\left(\frac{2\pi \cdot 74}{365}\right) & = 12 \\ a + b \cos\left(\frac{2\pi \cdot 273}{365}\right) + c \sin\left(\frac{2\pi \cdot 273}{365}\right) & = 12 \end{bmatrix}$$

The unique solution is approximately  $a = 12.17$ ,  $b = -1.15$ , and  $c = 0.18$ , so that

$$S(t) = 12.17 - 1.15 \cos\left(\frac{2\pi t}{365}\right) + 0.18 \sin\left(\frac{2\pi t}{365}\right).$$

The longest day is about 13.3 hours. (See Figure 1.12.)

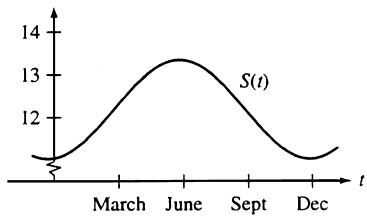


Figure 1.12: for Problem 1.2.43.

**1.2.44** Kyle first must solve the following system:  $\begin{bmatrix} x_1 & +x_2 & +x_3 & = 24 \\ 3x_1 & +2x_2 & +\frac{1}{2}x_3 & = 24 \end{bmatrix}$ .

This system reduces to  $\begin{bmatrix} x_1 & -1.5x_3 & = -24 \\ x_2 & +2.5x_3 & = 48 \end{bmatrix}$ .

Thus, our solutions will be of the form  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.5x_3 - 24 \\ -2.5x_3 + 48 \\ x_3 \end{bmatrix}$ . Since all of our values must be non-negative integers (and  $x_3$  must be even), we find the following solutions for  $\begin{bmatrix} lilies \\ roses \\ daisies \end{bmatrix}$ :  $\begin{bmatrix} 0 \\ 8 \\ 16 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 3 \\ 18 \end{bmatrix}$ . Since Olivia loves lilies, Kyle spends his 24 dollars on 3 lilies, 3 roses and 18 daisies.

**1.2.45 a** When  $k \neq 1$  and  $k \neq 2$ , we can see that this will continue to reduce to a consistent system with a unique solution.

**b** When  $k = 1$ , our bottom row reveals the inconsistency  $0 = 2$ .

**c** When  $k = 2$ , the second row and third row both represent the equation  $z = 2$ , meaning that the third row will be replaced with the equation  $0 = 0$  during further reduction. This reveals that we will have an infinite number of solutions.

1.2.46 a We reduce our matrix in the following steps:  $\left[ \begin{array}{cccc|c} 0 & 1 & 2k & \vdots & 0 \\ 1 & 2 & 6 & \vdots & 2 \\ k & 0 & 2 & \vdots & 1 \end{array} \right] \xrightarrow[I \leftrightarrow II]{swap} \rightarrow$

$$\left[ \begin{array}{cccc|c} 1 & 2 & 6 & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ k & 0 & 2 & \vdots & 1 \end{array} \right] \xrightarrow{-k(I)} \left[ \begin{array}{cccc|c} 1 & 2 & 6 & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ 0 & -2k & 2-6k & \vdots & 1-2k \end{array} \right] \xrightarrow{-2(II)} \rightarrow$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 6-4k & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ 0 & 0 & 2-6k+4k^2 & \vdots & 1-2k \end{array} \right] \xrightarrow{\quad} \left[ \begin{array}{cccc|c} 1 & 0 & 6-4k & \vdots & 2 \\ 0 & 1 & 2k & \vdots & 0 \\ 0 & 0 & 2(2k-1)(k-1) & \vdots & -(2k-1) \end{array} \right].$$

We see that there will be a unique solution when the  $2(2k-1)(k-1)$  term is not equal to zero, when  $2k-1 \neq 0$  and  $k-1 \neq 0$ , or  $k \neq \frac{1}{2}$  and  $k \neq 1$ .

- b We will have no solutions when the term  $2(2k-1)(k-1)$  is equal to zero, but the term  $-(2k-1)$  is not. This occurs only when  $k = 1$ .
- c We will have infinitely many solutions when the last row represents the equation  $0 = 0$ . This occurs when  $2k-1 = 0$ , or  $k = \frac{1}{2}$ .

1.2.47 a So  $-\frac{1}{2}x_1 + x_2 - \frac{1}{2}x_3 = 0$  and  $-\frac{1}{2}x_2 + x_3 - \frac{1}{2}x_4 = 0$ .

After reduction of the system, we find that our solutions are all of the form

$$\left[ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right] = s \left[ \begin{array}{c} -2 \\ -1 \\ 0 \\ 1 \end{array} \right] + t \left[ \begin{array}{c} 3 \\ 2 \\ 1 \\ 0 \end{array} \right].$$

- b Yes, from our solution in part (a), if we plug in 1 for  $x_1$  and 13 for  $x_4$ , we obtain  $3t - 2s = 1$  and  $s = 13$ , which leads to  $t = 9$ , and  $x_2 = 5, x_3 = 9$ .

So we have the solution:  $x_1 = 1, x_2 = 5, x_3 = 9$  and  $x_4 = 13$ , which is an arithmetic progression.

1.2.48 It is required that  $x_k = \frac{1}{2}(x_{k-1} + x_{k+1})$ , or  $2x_k = x_{k-1} + x_{k+1}$ , or  $x_k - x_{k-1} = x_{k+1} - x_k$ . This means that the difference of any two consecutive terms must be the same; we are looking at the finite arithmetic sequences. Thus the solutions are of the form  $(x_1, x_2, x_3, \dots, x_n) = (t, t+r, t+2r, \dots, t+(n-1)r)$ , where  $t$  and  $r$  are arbitrary constants.

$$\left[ \begin{array}{ccc|c} 2 & 1 & 0 & C \\ 0 & 3 & 1 & C \\ 1 & 0 & 4 & C \end{array} \right]$$

1.2.49 We begin by solving the system. Our augmented matrix begins as:

and is reduced to  $\begin{bmatrix} 1 & 0 & 0 & : & \frac{9}{25}C \\ 0 & 1 & 0 & : & \frac{7}{25}C \\ 0 & 0 & 1 & : & \frac{4}{25}C \end{bmatrix}$ . In order for  $x$ ,  $y$  and  $z$  to be integers,  $C$  must be a multiple of 25. We want the smallest positive choice, so  $C = 25$ .

**1.2.50**  $f(t) = a + bt + ct^2 + dt^3$  and we learn that  $f(0) = a = 3$ ,  $f(1) = a + b + c + d = 2$ ,  $f(2) = a + 2b + 4c + 8d = 0$ .  
Also,

$$\int_0^2 f(t)dt = at + \frac{1}{2}bt^2 + \frac{1}{3}ct^3 + \frac{1}{4}dt^4|_0^2 = 2a + 2b + \frac{8}{3}c + 4d = 4.$$

Now, we set up our matrix,  $\begin{bmatrix} 1 & 0 & 0 & 0 & : & 3 \\ 1 & 1 & 1 & 1 & : & 2 \\ 1 & 2 & 4 & 8 & : & 0 \\ 2 & 2 & \frac{8}{3} & 4 & : & 4 \end{bmatrix}$ . However, when we reduce this, the last line becomes  $0 = 1$ , meaning that the system is inconsistent.

In introductory calculus you may have seen the approximation formula:

$$\int_a^b f(t)dt \approx \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b)),$$

the simplest form of Simpson's Rule. For polynomials  $f(t)$  of degree  $\leq 3$ , Simpson's Rule gives the exact value of the integral. Thus, for the  $f(t)$  in our problem,

$$\int_0^2 f(t)dt = \frac{2}{6}(f(0) + 4f(1) + f(2)) = \frac{1}{3}(3 + 8 + 0) = \frac{11}{3}.$$

Thus it is impossible to find such a cubic with

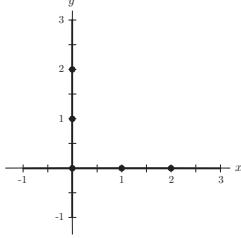
$$\int_0^2 f(t)dt = 4,$$

as required.

**1.2.51** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + c_3 + c_6 &= 0 \\ c_1 + 2c_3 + 4c_6 &= 0 \end{aligned}$$

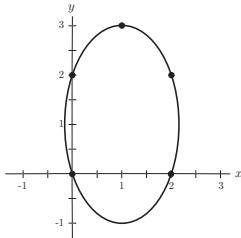
It has the solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, 0, 0, 0, c_5, 0)$ . This is the conic  $xy = 0$ .



**1.2.52** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + 2c_3 + 4c_6 &= 0 \\ c_1 + 2c_2 + 2c_3 + 4c_4 + 4c_5 + 4c_6 &= 0 \\ c_1 + c_2 + 3c_3 + c_4 + 3c_5 + 9c_6 &= 0 \end{aligned}$$

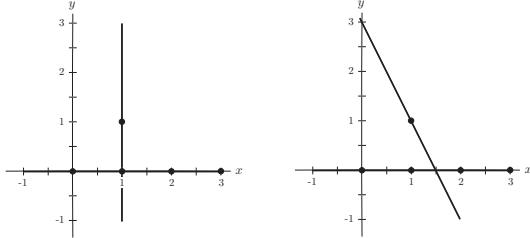
It has the solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, -6c_6, -2c_6, 3c_6, 0, c_6)$ . This is the conic  $-6x - 2y + 3x^2 + y^2 = 0$  or  $3(x - 1)^2 + (y - 1)^2 = 4$ , the ellipse centered at  $(1, 1)$  with semiaxis  $2/\sqrt{3}$  and 2.



**1.2.53** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + 3c_2 + 9c_4 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 + c_6 &= 0 \end{aligned}$$

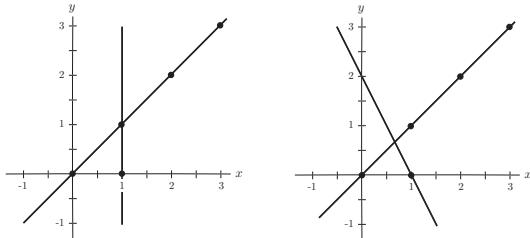
This system has the solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, 0, -c_5 - c_6, 0, c_5, c_6)$ . Setting  $c_5 = a$  and  $c_6 = b$ , we get the family of conics  $a(xy - y) + b(y^2 - y) = ay(x - 1) + by(y - 1) = 0$  where  $a \neq 0$  or  $b \neq 0$ . Each such conic is the union of the  $x$  axis with some line through the point  $(1, 1)$ . Two sample solutions are shown in the accompanying figures.



**1.2.54** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 + c_6 &= 0 \\ c_1 + 2c_2 + 2c_3 + 4c_4 + 4c_5 + 4c_6 &= 0 \\ c_1 + 3c_2 + 3c_3 + 9c_4 + 9c_5 + 9c_6 &= 0 \\ c_1 + c_2 + c_4 &= 0 \end{aligned}$$

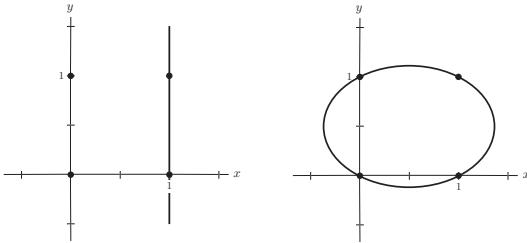
It has the solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, c_5 + c_6, -c_5 - c_6, -c_5 - c_6, c_5, c_6)$ . Setting  $c_5 = a$  and  $c_6 = b$ , we get the family of conics  $a(x - y + xy - x^2) + b(x - y - x^2 + y^2) = a(x - y)(1 - x) + b(x - y)(1 - x - y) = 0$  where  $a \neq 0$  or  $b \neq 0$ . Each such conic is the union of the line  $y = x$  with some line through the point  $(1, 0)$ . Two sample solutions are shown in the accompanying figures.



**1.2.55** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + c_3 + c_6 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 + c_6 &= 0 \end{aligned}$$

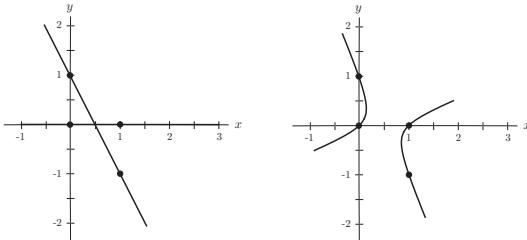
It has the solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, -c_4, -c_6, c_4, 0, c_6)$ . Setting  $c_4 = a$  and  $c_6 = b$ , we get the family of conics  $a(x^2 - x) + b(y^2 - y) = ax(x - 1) + by(y - 1) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Two sample solutions are shown in the accompanying figures.



**1.2.56** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + c_3 + c_6 &= 0 \\ c_1 + c_2 - c_3 + c_4 - c_5 + c_6 &= 0 \end{aligned}$$

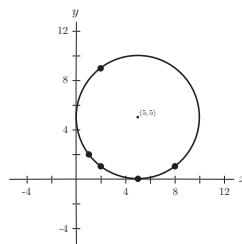
It has the solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, -c_4, -c_6, c_4, 2c_6, c_6)$ . Setting  $c_4 = a$  and  $c_6 = b$ , we get the family of conics  $a(x^2 - x) + b(y^2 + 2xy - y) = ax(x-1) + by(y+2x-1) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Two sample solutions are shown in the accompanying figures.



**1.2.57** The system of linear equations is

$$\begin{aligned} c_1 + 5c_2 + 25c_4 &= 0 \\ c_1 + c_2 + 2c_3 + c_4 + 2c_5 + 4c_6 &= 0 \\ c_1 + 2c_2 + c_3 + 4c_4 + 2c_5 + c_6 &= 0 \\ c_1 + 8c_2 + c_3 + 64c_4 + 8c_5 + c_6 &= 0 \\ c_1 + 2c_2 + 9c_3 + 4c_4 + 18c_5 + 81c_6 &= 0 \end{aligned}$$

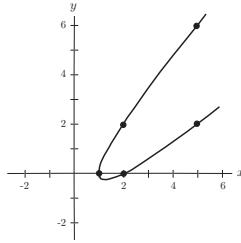
It has the solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (25c_6, -10c_6, -10c_6, c_6, 0, c_6)$ . This is the conic  $25 - 10x - 10y + x^2 + y^2 = 0$ , a circle of radius 5 centered at  $(5, 5)$ .



**1.2.58** The system of linear equations is

$$\begin{aligned} c_1 + c_2 + c_4 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + 2c_2 + 2c_3 + 4c_4 + 4c_5 + 4c_6 &= 0 \\ c_1 + 5c_2 + 2c_3 + 25c_4 + 10c_5 + 4c_6 &= 0 \\ c_1 + 5c_2 + 6c_3 + 25c_4 + 30c_5 + 36c_6 &= 0 \end{aligned}$$

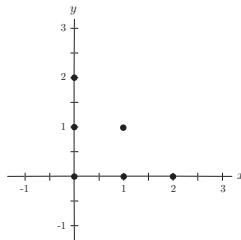
It has the solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (2c_6, -3c_6, 2c_6, c_6, -2c_6, c_6)$ . This is the conic  $2-3x+2y+x^2-2xy+y^2 = 0$ .



**1.2.59** The system of linear equations is

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_4 &= 0 \\ c_1 + 2c_2 + 4c_4 &= 0 \\ c_1 + c_3 + c_6 &= 0 \\ c_1 + 2c_3 + 4c_6 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 + c_6 &= 0 \end{aligned}$$

It has the only solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, 0, 0, 0, 0, 0)$ . There is no conic which goes through these points. Alternatively, note that the only conic through the first five points is  $xy = 0$ , according to Exercise 51. But that conic fails to run through the point  $(1, 1)$ , so that there is no conic through all six points.

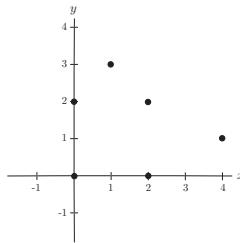


**1.2.60** The system of linear equations is

$$c_1 = 0$$

$$\begin{aligned}
 c_1 + 2c_2 + 4c_4 &= 0 \\
 c_1 + 2c_3 + 4c_6 &= 0 \\
 c_1 + 2c_2 + 2c_3 + 4c_4 + 4c_5 + 4c_6 &= 0 \\
 c_1 + c_2 + 3c_3 + c_4 + 3c_5 + 9c_6 &= 0 \\
 c_1 + 4c_2 + c_3 + 16c_4 + 4c_5 + c_6 &= 0
 \end{aligned}$$

It has the solution  $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, 0, 0, 0, 0, 0)$ . There is no conic which goes through these points.



**1.2.61** Let  $x_1$  be the cost of the environmental statistics book,  $x_2$  be the cost of the set theory text and  $x_3$  be the cost

of the educational psychology book. Then, from the problem, we deduce the augmented matrix  $\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 178 \\ 2 & 1 & 1 & 319 \\ 0 & 1 & 1 & 147 \end{array} \right]$ .

We can reduce this matrix to  $\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 86 \\ 0 & 1 & 0 & 92 \\ 0 & 0 & 1 & 55 \end{array} \right]$ , revealing that  $x_1 = 86$ ,  $x_2 = 92$  and  $x_3 = 55$ . Thus, the environmental statistics book costs \$ 86, the set theory book costs \$ 92 and the educational psychology book is only priced at \$ 55.

**1.2.62** Let our vectors  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  represent the numbers of the books  $\begin{bmatrix} \text{grammar} \\ \text{Werther} \\ \text{LinearAlg.} \end{bmatrix}$ . Then we can set up the matrix

$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 64 \\ 1 & 0 & 1 & 98 \\ 0 & 1 & 1 & 76 \end{array} \right]$ . This system yields one solution,  $\begin{bmatrix} 43 \\ 21 \\ 55 \end{bmatrix}$ , meaning that the grammar book costs 43 Euro, the novel costs 21 Euro, and the linear algebra text costs 55 Euro.

**1.2.63** The difficult part of this problem lies in setting up a system from which we can derive our matrix. We will define  $x_1$  to be the number of “liberal” students at the beginning of the class, and  $x_2$  to be the number of “conservative” students at the beginning. Thus, since there are 260 students in total,  $x_1 + x_2 = 260$ . We need one more equation involving  $x_1$  and  $x_2$  in order to set up a useful system. Since we know that the number of conservative students at the end of the semester is equal to the number of liberal student initially, we obtain the equation  $\frac{3}{10}x_1 + \frac{6}{10}x_2 = x_1$ , or  $-\frac{7}{10}x_1 + \frac{6}{10}x_2 = 0$ .

We then use  $\begin{bmatrix} 1 & 1 & 260 \\ -\frac{7}{10} & \frac{6}{10} & 0 \end{bmatrix}$  to obtain  $\begin{bmatrix} 1 & 0 & 120 \\ 0 & 1 & 140 \end{bmatrix}$ .

Thus, there are initially 120 liberal students, and 140 conservative students. Since the number of liberal students initially is the same as the number of conservative students in the end, the class ends with 120 conservative students and 140 liberal students.

**1.2.64** Let  $x_1$  and  $x_2$  be the initial number of students in Sections A and B, respectively. Then, since there are 55 students total,  $x_1 + x_2 = 55$ . Also, interpreting the change of students from the perspective of Section B, we gain  $0.2x_1$ , lose  $0.3x_2$ , and in the process, lose 4 students. Thus,  $0.2x_1 - 0.3x_2 = -4.0$ . Our matrix becomes  $\begin{bmatrix} 1 & 1 & : & 55 \\ .2 & -.3 & : & -4 \end{bmatrix}$ , which reduces to  $\begin{bmatrix} 1 & 0 & : & 25 \\ 0 & 1 & : & 30 \end{bmatrix}$ . This reveals that there are initially 25 students in Section A and 30 students in Section B.

**1.2.65** We are told that five cows and two sheep cost ten liang, and two cows and five sheep cost eight liang of silver. So, we let  $C$  be the cost of a cow, and  $S$  be the cost of a sheep. From this we derive  $\begin{bmatrix} 5C & +2S & = 10 \\ 2C & +5S & = 8 \end{bmatrix}$ .

This reduces to  $\begin{bmatrix} C & = \frac{34}{21} \\ S & = \frac{20}{21} \end{bmatrix}$  which gives the prices:  $\frac{34}{21}$  liang silver for a cow, and  $\frac{20}{21}$  liang silver for a sheep.

**1.2.66** Letting  $x_1$ ,  $x_2$ , and  $x_3$  be the prize, in coins, of cows, sheep and pigs, respectively, we can represent the system in a matrix:  $\begin{bmatrix} 2 & 5 & -13 & : & 1000 \\ 3 & -9 & 3 & : & 0 \\ -5 & 6 & 8 & : & -600 \end{bmatrix}$ . We reduce this matrix to  $\begin{bmatrix} 1 & 0 & 0 & : & 1200 \\ 0 & 1 & 0 & : & 500 \\ 0 & 0 & 1 & : & 300 \end{bmatrix}$ . The prize of a cow, a sheep, and a pig is 1200, 500 and 300 coins, respectively.

**1.2.67** The second measurement in the problem tells us that 4 sparrows and 1 swallow weigh as much as 1 sparrow and 5 swallows. We will immediately interpret this as 3 sparrows weighing the same as 4 swallows. The other measurement we use is that all the birds together weigh 16 liang. Setting  $x_1$  to be the weight of a sparrow, and  $x_2$  to be the weight of a swallow, we find the augmented matrix  $\begin{bmatrix} 3 & -4 & : & 0 \\ 5 & 6 & : & 16 \end{bmatrix}$  representing these two equations.

We reduce this to  $\begin{bmatrix} 1 & 0 & : & \frac{32}{19} \\ 0 & 1 & : & \frac{24}{19} \end{bmatrix}$ , meaning that each sparrow weighs  $\frac{32}{19}$  liang, and each swallow weighs  $\frac{24}{19}$  liang.

**1.2.68** This problem gives us three different combinations of horses that can pull exactly 40 *dan* up a hill. We condense the statements to fit our needs, saying that, One military horse and one ordinary horse can pull 40 *dan*, two ordinary and one weak horse can pull 40 *dan* and one military and three weak horses can also pull 40 *dan*.

With this information, we set up our matrix:  $\begin{bmatrix} 1 & 1 & 0 & : & 40 \\ 0 & 2 & 1 & : & 40 \\ 1 & 0 & 3 & : & 40 \end{bmatrix}$ , which reduces to  $\begin{bmatrix} 1 & 0 & 0 & : & \frac{40}{7} \\ 0 & 1 & 0 & : & \frac{120}{7} \\ 0 & 0 & 1 & : & \frac{40}{7} \end{bmatrix}$ .

Thus, the military horses can pull  $\frac{40}{7}$  *dan*, the ordinary horses can pull  $\frac{120}{7}$  *dan* and the weak horses can pull  $\frac{40}{7}$  *dan* each.

**1.2.69** Here, let  $W$  be the depth of the well.

Then our system becomes

$$\begin{array}{ccccccccc} 2A & +B & & -W & = & 0 \\ 3B & +C & & -W & = & 0 \\ & 4C & +D & -W & = & 0 \\ & & 5D & +E & -W & = & 0 \\ A & & & +6E & -W & = & 0 \end{array}.$$

We transform this system into an augmented matrix, then perform a prolonged reduction to reveal

$$\left[ \begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -\frac{265}{721} & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{191}{721} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{148}{721} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{129}{721} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{76}{721} & 0 \end{array} \right]. \text{ Thus, } A = \frac{265}{721}W, B = \frac{191}{721}W, C = \frac{148}{721}W, D = \frac{129}{721}W \text{ and } E = \frac{76}{721}W.$$

If we choose 721 to be the depth of the well, then  $A = 265$ ,  $B = 191$ ,  $C = 148$ ,  $D = 129$  and  $E = 76$ .

**1.2.70** We let  $x_1, x_2$  and  $x_3$  be the numbers of roosters, hens and chicks respectively. Then, since we buy a total of a hundred birds, and spend a hundred coins on them, we find the equations  $x_1 + x_2 + x_3 = 100$  and  $5x_1 + 3x_2 + \frac{1}{3}x_3 = 100$ .

We fit these into our matrix,

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & : 100 \\ 5 & 3 & \frac{1}{3} & : 100 \end{array} \right],$$

which reduces to

$$\left[ \begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & : -100 \\ 0 & 1 & \frac{7}{3} & : 200 \end{array} \right].$$

So,  $x_1 - \frac{4}{3}x_3 = -100$ , and  $x_2 + \frac{7}{3}x_3 = 200$ . Now, we can write our solution vectors in terms of  $x_3$ :

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 - 100 \\ -\frac{7}{3}x_3 + 200 \\ x_3 \end{bmatrix}.$$

Since all of our values must be non-negative,  $x_1$  must be greater than or equal to zero, or  $\frac{4}{3}x_3 - 100 \geq 0$ , which means that  $x_3 \geq 75$ .

Also,  $x_3$  must be greater than or equal to zero, meaning that  $-\frac{7}{3}x_3 + 200 \geq 0$  or  $x_3 \leq \frac{600}{7}$ . Since  $x_3$  must be an integer, this forces  $x_3 \leq 85$ . Thus, we are looking for solutions where  $75 \leq x_3 \leq 85$ . We notice, however, that  $x_1$  and  $x_2$  are only integers when  $x_3$  is a multiple of 3. Thus, the possible values for  $x_3$  are 75, 78, 81 and 84.

Now the possible solutions for

$$\begin{bmatrix} \text{roosters} \\ \text{hens} \\ \text{chicks} \end{bmatrix} \text{ are } \begin{bmatrix} 0 \\ 25 \\ 75 \end{bmatrix}, \begin{bmatrix} 4 \\ 18 \\ 78 \end{bmatrix}, \begin{bmatrix} 8 \\ 11 \\ 81 \end{bmatrix}, \text{ and } \begin{bmatrix} 12 \\ 4 \\ 84 \end{bmatrix}.$$

**1.2.71** We let  $x_1, x_2, x_3$  and  $x_4$  be the numbers of pigeons, sarasabirds, swans and peacocks respectively. We first determine the cost of each bird. Each pigeon costs  $\frac{3}{5}$  panas, each sarasabird costs  $\frac{5}{7}$  panas, the swans cost  $\frac{7}{9}$  panas apiece and each peacock costs 3 panas. We use these numbers to set up our system, but we must remember to make sure we are buying the proper amount of each to qualify for these deals when we find our solutions (for example, the number of sarasabirds we buy must be a multiple of 7).

Our matrix then is  $\begin{bmatrix} 1 & 1 & 1 & 1 & : & 100 \\ \frac{3}{5} & \frac{5}{7} & \frac{7}{9} & 3 & : & 100 \end{bmatrix}$

which reduces to  $\begin{bmatrix} 1 & 0 & -\frac{5}{9} & -20 & : & -250 \\ 0 & 1 & \frac{14}{9} & 21 & : & 350 \end{bmatrix}.$

Thus,  $x_1 = \frac{5}{9}x_3 + 20x_4 - 250$  and  $x_2 = -\frac{14}{9}x_3 - 21x_4 + 350$ .

Then our solutions are of the form  $\begin{bmatrix} \frac{5}{9}x_3 + 20x_4 - 250 \\ -\frac{14}{9}x_3 - 21x_4 + 350 \\ x_3 \\ x_4 \end{bmatrix}.$

We determine the possible solutions by choosing combinations of  $x_3$  and  $x_4$  of the correct multiples (9 for  $x_3$ , 3 for  $x_4$ ) that give us non-negative integer solutions for  $x_1$  and  $x_2$ . Thus it is required that  $x_1 = \frac{5}{9}x_3 + 20x_4 - 250 \geq 0$  and  $x_2 = -\frac{14}{9}x_3 - 21x_4 + 350 \geq 0$ .

Solving for  $x_3$  we find that  $225 - \frac{27}{2}x_4 \geq x_3 \geq 450 - 36x_4$ .

To find all the solutions, we can begin by letting  $x_4 = 0$ , and finding all corresponding values of  $x_3$ . Then we can increase  $x_4$  in increments of 3, and find the corresponding  $x_3$  values in each case, until we are through.

For  $x_4 = 0$  we have the inequality  $225 \geq x_3 \geq 450$ , so that there aren't any solutions for  $x_3$ . Likewise, there are no feasible  $x_3$  values for  $x_4 = 3, 6$  and  $9$ , since  $450 - 36x_4$  exceeds 100 in these cases.

In the case of  $x_4 = 12$  our inequality becomes  $63 \geq x_3 \geq 18$ , so that  $x_3$  could be 18, 27, 36, 45, 54 or 63.

In the next case,  $x_4 = 15$ , we have  $\frac{45}{2} \geq x_3 \geq -90$ , so that the non-negative solutions are 0, 9 and 18.

If  $x_4$  is 18 or more, then the term  $225 - \frac{27}{2}x_4$  becomes negative, so that there are only negative solutions for  $x_3$ . (Recall that it is required that  $225 - \frac{27}{2}x_4 \geq x_3$ .)

We have found nine solutions. If we compute the corresponding values of

$x_1 = \frac{5}{9}x_3 + 20x_4 - 250$  and  $x_2 = -\frac{14}{9}x_3 - 21x_4 + 350$ , we end up with the following vectors for:  $\begin{bmatrix} \text{number of pigeons} \\ \text{number of sarasabirds} \\ \text{number of swans} \\ \text{number of peacocks} \end{bmatrix}$   
 to be:  $\begin{bmatrix} 0 \\ 70 \\ 18 \\ 12 \end{bmatrix}, \begin{bmatrix} 5 \\ 56 \\ 27 \\ 12 \end{bmatrix}, \begin{bmatrix} 10 \\ 42 \\ 36 \\ 12 \end{bmatrix}, \begin{bmatrix} 15 \\ 28 \\ 45 \\ 12 \end{bmatrix}, \begin{bmatrix} 20 \\ 14 \\ 54 \\ 12 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ 63 \\ 12 \end{bmatrix}, \begin{bmatrix} 50 \\ 35 \\ 0 \\ 15 \end{bmatrix}, \begin{bmatrix} 55 \\ 21 \\ 9 \\ 15 \end{bmatrix}, \begin{bmatrix} 60 \\ 7 \\ 18 \\ 15 \end{bmatrix}.$

**1.2.72** We follow the outline of Exercise 70 to find the matrix  $\begin{bmatrix} 1 & 1 & 1 & : & 100 \\ 4 & \frac{1}{5} & 1 & : & 100 \end{bmatrix}$ , which reduces to  $\begin{bmatrix} 1 & 0 & \frac{4}{19} & : & \frac{400}{19} \\ 0 & 1 & \frac{15}{19} & : & \frac{1500}{19} \end{bmatrix}.$

Thus, our solutions are of the form  $\begin{bmatrix} \frac{400-4x_3}{19} \\ \frac{1500-15x_3}{19} \\ x_3 \end{bmatrix}$ . We find that our solutions are bound by  $0 \leq x_3 \leq 100$ . However, since both  $\frac{400-4x_3}{19} = 4 \frac{100-x_3}{19}$  and  $\frac{1500-15x_3}{19} = 15 \frac{100-x_3}{19}$  must be non-negative integers, the quantity  $\frac{100-x_3}{19}$  must be a non-negative integer,  $k$ , so that  $x_3 = 100 - 19k$ . The condition  $x_3 \geq 0$  now leaves us with the possibilities  $k = 0, 1, 2, 3, 4, 5$ .

Thus, we find our solutions for  $\begin{bmatrix} \text{ducks} \\ \text{sparrows} \\ \text{roosters} \end{bmatrix} : \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}, \begin{bmatrix} 4 \\ 15 \\ 81 \end{bmatrix}, \begin{bmatrix} 8 \\ 30 \\ 62 \end{bmatrix}, \begin{bmatrix} 12 \\ 45 \\ 43 \end{bmatrix}, \begin{bmatrix} 16 \\ 60 \\ 24 \end{bmatrix}$  and  $\begin{bmatrix} 20 \\ 75 \\ 5 \end{bmatrix}$ .

**1.2.73** We let  $x_1$  be the number of sheep,  $x_2$  be the number of goats, and  $x_3$  be the number of hogs. We can then use the two equations  $\frac{1}{2}x_1 + \frac{4}{3}x_2 + \frac{7}{2}x_3 = 100$  and  $x_1 + x_2 + x_3 = 100$  to generate the following augmented matrix:

$$\left[ \begin{array}{ccc|c} \frac{1}{2} & \frac{4}{3} & \frac{7}{2} & 100 \\ 1 & 1 & 1 & 100 \end{array} \right]$$

then reduce it to  $\left[ \begin{array}{ccc|c} 1 & 0 & -\frac{13}{5} & 40 \\ 0 & 1 & \frac{18}{5} & 60 \end{array} \right]$ .

With this, we see that our solutions will be of the form  $\begin{bmatrix} 40 + \frac{13}{5}s \\ 60 - \frac{18}{5}s \\ s \end{bmatrix}$ . Now all three components of this vector

must be non-negative integers, meaning that  $s$  must be a non-negative multiple of 5 (that is,  $s = 0, 5, 10, \dots$ ) such that  $60 - \frac{18}{5}s \geq 0$ , or,  $s \leq \frac{50}{3}$ . This leaves the possible solutions  $x_3 = s = 0, 5, 10$  and  $15$ , and we can compute the corresponding values of  $x_1 = 40 + \frac{13}{5}s$  and  $x_2 = 60 - \frac{18}{5}s$  in each case.

So we find the following solutions:  $\begin{bmatrix} 40 \\ 60 \\ 0 \end{bmatrix}, \begin{bmatrix} 53 \\ 42 \\ 5 \end{bmatrix}, \begin{bmatrix} 66 \\ 24 \\ 10 \end{bmatrix}$  and  $\begin{bmatrix} 79 \\ 6 \\ 15 \end{bmatrix}$ .

**1.2.74** This problem is similar in nature to Exercise 70, and we will follow that example, revealing the ma-

trix:  $\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 3 & 2 & \frac{1}{2} & 100 \end{array} \right]$ . We reduce this to  $\left[ \begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & -100 \\ 0 & 1 & \frac{5}{2} & 200 \end{array} \right]$ , which yields solutions of the form  $\begin{bmatrix} \frac{3}{2}x_3 - 100 \\ -\frac{5}{2}x_3 + 200 \\ x_3 \end{bmatrix}$ . Since all the values must be positive (there are at least one man, one woman and one child),

we see that  $66 < x_3 < 80$ , and  $x_3$  must be even. From this, we use  $x_3$  to find our solutions:

$$\left[ \begin{array}{c} 2 \\ 30 \\ 68 \end{array} \right], \left[ \begin{array}{c} 5 \\ 25 \\ 70 \end{array} \right], \left[ \begin{array}{c} 8 \\ 20 \\ 72 \end{array} \right], \left[ \begin{array}{c} 11 \\ 15 \\ 74 \end{array} \right], \left[ \begin{array}{c} 14 \\ 10 \\ 76 \end{array} \right] \text{ and } \left[ \begin{array}{c} 17 \\ 5 \\ 78 \end{array} \right].$$

**1.2.75** Rather than setting up a huge system, here we will reason this out logically. Since there are 30 barrels, each son will get 10 of them. If we use the content of a full barrel as our unit for wine, we see that each brother will get  $\frac{15}{3} = 5$  barrel-fulls of wine. Thus, the ten barrels received by each son will, on average, be half full, meaning that for every full barrel a son receives, he also receives an empty one.

Now let  $x_1$ ,  $x_2$ , and  $x_3$  be the numbers of half-full barrels received by each of the three sons. The first son, receiving  $x_1$  half-full barrels will also gain  $10 - x_1$  other barrels, half of which must be full and half of which must be empty, each equal to the quantity  $\frac{10-x_1}{2}$ . Thus,  $x_1$  must be even. The same works for  $x_2$  and  $x_3$ . Since  $x_1 + x_2 + x_3 = 10$ , we have boiled down our problem to simply finding lists of three non-negative even numbers that add up to 10. We find our solutions by inspection:

$$\left[ \begin{array}{c} 10 \\ 0 \\ 0 \end{array} \right], \left[ \begin{array}{c} 8 \\ 2 \\ 0 \end{array} \right], \left[ \begin{array}{c} 8 \\ 0 \\ 2 \end{array} \right], \left[ \begin{array}{c} 6 \\ 4 \\ 0 \end{array} \right], \left[ \begin{array}{c} 6 \\ 2 \\ 2 \end{array} \right], \left[ \begin{array}{c} 6 \\ 0 \\ 4 \end{array} \right], \left[ \begin{array}{c} 4 \\ 6 \\ 0 \end{array} \right], \left[ \begin{array}{c} 4 \\ 4 \\ 2 \end{array} \right], \left[ \begin{array}{c} 4 \\ 2 \\ 4 \end{array} \right], \left[ \begin{array}{c} 4 \\ 0 \\ 6 \end{array} \right],$$

$$\begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 8 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}.$$

As we stated before, the number of full and empty barrels is dependent on the number of half-full barrels. Thus, each solution here translates into exactly one solution for the overall problem. Here we list those solutions, for  $\begin{bmatrix} \text{first son} \\ \text{second son} \\ \text{third son} \end{bmatrix}$ , using triples of the form (full barrels, half-full barrels, empty barrels) as our entries:

$$\begin{bmatrix} (0, 10, 0) \\ (5, 0, 5) \\ (5, 0, 5) \end{bmatrix}, \begin{bmatrix} (1, 8, 1) \\ (4, 2, 4) \\ (5, 0, 5) \end{bmatrix}, \begin{bmatrix} (1, 8, 1) \\ (5, 0, 5) \\ (4, 2, 4) \end{bmatrix}, \begin{bmatrix} (2, 6, 2) \\ (3, 4, 3) \\ (5, 0, 5) \end{bmatrix}, \begin{bmatrix} (2, 6, 2) \\ (4, 2, 4) \\ (5, 0, 5) \end{bmatrix}, \begin{bmatrix} (2, 6, 2) \\ (5, 0, 5) \\ (4, 2, 4) \end{bmatrix},$$

$$\begin{bmatrix} (3, 4, 3) \\ (2, 6, 2) \\ (5, 0, 5) \end{bmatrix}, \begin{bmatrix} (3, 4, 3) \\ (3, 4, 3) \\ (4, 2, 4) \end{bmatrix}, \begin{bmatrix} (3, 4, 3) \\ (4, 2, 4) \\ (3, 4, 3) \end{bmatrix}, \begin{bmatrix} (3, 4, 3) \\ (5, 0, 5) \\ (2, 6, 2) \end{bmatrix}, \begin{bmatrix} (4, 2, 4) \\ (1, 8, 1) \\ (5, 0, 5) \end{bmatrix}, \begin{bmatrix} (4, 2, 4) \\ (2, 6, 2) \\ (5, 0, 5) \end{bmatrix},$$

$$\begin{bmatrix} (4, 2, 4) \\ (3, 4, 3) \\ (3, 4, 3) \end{bmatrix}, \begin{bmatrix} (4, 2, 4) \\ (4, 2, 4) \\ (2, 6, 2) \end{bmatrix}, \begin{bmatrix} (4, 2, 4) \\ (5, 0, 5) \\ (1, 8, 1) \end{bmatrix}, \begin{bmatrix} (5, 0, 5) \\ (0, 10, 0) \\ (5, 0, 5) \end{bmatrix}, \begin{bmatrix} (5, 0, 5) \\ (1, 8, 1) \\ (4, 2, 4) \end{bmatrix}, \begin{bmatrix} (5, 0, 5) \\ (2, 6, 2) \\ (3, 4, 3) \end{bmatrix},$$

$$\begin{bmatrix} (5, 0, 5) \\ (3, 4, 3) \\ (2, 6, 2) \end{bmatrix}, \begin{bmatrix} (5, 0, 5) \\ (4, 2, 4) \\ (1, 8, 1) \end{bmatrix} \text{ and } \begin{bmatrix} (5, 0, 5) \\ (5, 0, 5) \\ (0, 10, 0) \end{bmatrix}.$$

**1.2.76** We let  $x_1$  be the amount of gold in the crown,  $x_2$  be the amount of bronze,  $x_3$  be the amount of tin and  $x_4$  be the amount of iron. Then, for example, since the first requirement in the problem is: “Let the gold and bronze together form two-thirds,” we will interpret this as  $x_1 + x_2 = \frac{2}{3}(60)$ . We do this for all three requirements, and use the fact that all combined will be the total weight of the crown as our fourth. So we find the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & : & \frac{2}{3}(60) \\ 1 & 0 & 1 & 0 & : & \frac{3}{4}(60) \\ 1 & 0 & 0 & 1 & : & \frac{3}{5}(60) \\ 1 & 1 & 1 & 1 & : & 60 \end{bmatrix}, \text{ which has the solution } \begin{bmatrix} \text{gold} \\ \text{bronze} \\ \text{tin} \\ \text{iron} \end{bmatrix} = \begin{bmatrix} 30.5 \\ 9.5 \\ 14.5 \\ 5.5 \end{bmatrix}.$$

**1.2.77** Let  $x_i$  be the number of coins the  $i^{th}$  merchant has. We interpret the statement of the first merchant, “If I keep the purse, I shall have twice as much money as the two of you together” as  $x_1 + 60 = 2(x_2 + x_3)$ , or  $-x_1 + 2x_2 + 2x_3 = 60$ . We interpret the other statements in a similar fashion, translating this into the augmented

$$\text{matrix, } \begin{bmatrix} -1 & 2 & 2 & : & 60 \\ 3 & -1 & 3 & : & 60 \\ 5 & 5 & -1 & : & 60 \end{bmatrix}.$$

The reduced row echelon form of this matrix is  $\begin{bmatrix} 1 & 0 & 0 & : & 4 \\ 0 & 1 & 0 & : & 12 \\ 0 & 0 & 1 & : & 20 \end{bmatrix}$ . Thus we deduce that the first merchant has 4 coins, the second has 12, and the third is the richest, with 20 coins.

**1.2.78** For each of the three statements, we set up an equation of the form

(initial amount of grass) + (grass growth) = (grass consumed by cows), or

$$(\#of fields)x + (\#of fields)(\#of days)y = (\#of cows)(\#of days)z.$$

For the first statement, this produces the equation  $x + 2y = 6z$ , or  $x + 2y - 6z = 0$ . Similarly, we obtain the equations  $4x + 16y - 28z = 0$  and  $2x + 10y - 15z = 0$  for the other two statements. From this information, we

write the matrix  $\begin{bmatrix} 1 & 2 & -6 & : & 0 \\ 4 & 16 & -28 & : & 0 \\ 2 & 10 & -15 & : & 0 \end{bmatrix}$ , which reduces to  $\begin{bmatrix} 1 & 0 & -5 & : & 0 \\ 0 & 1 & -\frac{1}{2} & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$ . Thus our solutions are of the form  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5t \\ \frac{1}{2}t \\ t \end{bmatrix}$ , where  $t$  is an arbitrary positive real number.

## Section 1.3

**1.3.1** a No solution, since the last row indicates  $0 = 1$ .

b The unique solution is  $x = 5$ ,  $y = 6$ .

c Infinitely many solutions; the first variable can be chosen freely.

**1.3.2** The rank is 3 since each row contains a leading one.

**1.3.3** This matrix has rank 1 since its rref is  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

**1.3.4** This matrix has rank 2 since its rref is  $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

**1.3.5** a  $x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

b The solution of the system in part (a) is  $x = 3$ ,  $y = 2$ . (See Figure 1.13.)

**1.3.6** No solution, since any linear combination  $x\vec{v}_1 + y\vec{v}_2$  of  $\vec{v}_1$  and  $\vec{v}_2$  will be parallel to  $\vec{v}_1$  and  $\vec{v}_2$ .

**1.3.7** A unique solution, since there is only one parallelogram with sides along  $\vec{v}_1$  and  $\vec{v}_2$  and one vertex at the tip of  $\vec{v}_3$ .

**1.3.8** Infinitely many solution. There are at least two obvious solutions. Write  $\vec{v}_4$  as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$  alone or as a linear combination of  $\vec{v}_3$  and  $\vec{v}_2$  alone. Therefore, this linear system has infinitely many solutions, by Theorem 1.3.1.

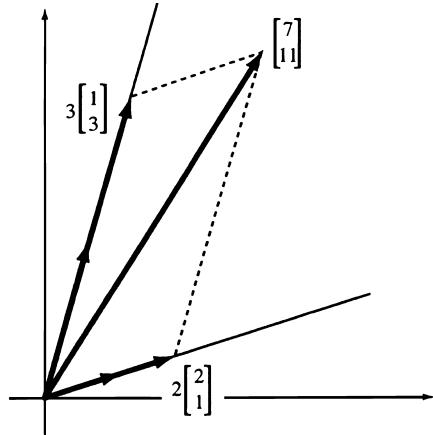


Figure 1.13: for Problem 1.3.5.

$$1.3.9 \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

$$1.3.10 \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 \cdot 1 + 2 \cdot (-2) + 3 \cdot 1 = 0$$

1.3.11 Undefined since the two vectors do not have the same number of components.

$$1.3.12 \quad [1 \ 2 \ 3 \ 4] \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 70$$

$$1.3.13 \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 11 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 29 \\ 65 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 11 \\ 3 \cdot 7 + 4 \cdot 11 \end{bmatrix} = \begin{bmatrix} 29 \\ 65 \end{bmatrix}$$

$$1.3.14 \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-1) + 2 \cdot 2 + 3 \cdot 1 \\ 2 \cdot (-1) + 3 \cdot 2 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$1.3.15 \quad [1 \ 2 \ 3 \ 4] \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 5 \cdot 1 + 6 \cdot 2 + 7 \cdot 3 + 4 \cdot 8 = 70 \text{ either way.}$$

$$1.3.16 \quad \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \cdot 2 + 1 \cdot (-3) \\ 3 \cdot 2 + 2 \cdot (-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

**1.3.17** Undefined, since the matrix has three columns, but the vector has only two components.

$$1.3.18 \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

$$1.3.19 \quad \begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1.3.20 \text{ a } \begin{bmatrix} 9 & 8 \\ 7 & 6 \\ 6 & 6 \end{bmatrix}$$

$$\text{b } \begin{bmatrix} 9 & -9 & 18 \\ 27 & 36 & 45 \end{bmatrix}$$

$$1.3.21 \quad \begin{bmatrix} 158 \\ 70 \\ 81 \\ 123 \end{bmatrix}$$

$$1.3.22 \quad \text{By Theorem 1.3.4, the rref is } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$1.3.23 \quad \text{All variables are leading, that is, there is a leading one in each column of the rref: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$1.3.24 \quad \text{By Theorem 1.3.4, rref } (A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

**1.3.25** In this case,  $\text{rref}(A)$  has a row of zeros, so that  $\text{rank}(A) < 4$ ; there will be a free variable. The system  $A\vec{x} = \vec{c}$  could have infinitely many solutions (for example, when  $\vec{c} = \vec{0}$ ) or no solutions (for example, when  $\vec{c} = \vec{b}$ ), but it cannot have a unique solution, by Theorem 1.3.4.

$$1.3.26 \quad \text{From Example 3d we know that } \text{rank}(A) = 3, \text{ so that } \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since all variables are leading, the system  $A\vec{x} = \vec{c}$  cannot have infinitely many solutions, but it could have a unique solution (for example, if  $\vec{c} = \vec{b}$ ) or no solutions at all (compare with Example 3c).

**1.3.27** By Theorem 1.3.4,  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

**1.3.28** There must be a leading one in each column:  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

**1.3.29**  $A$  is of the form  $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$$\text{and } \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 5a \\ 3b \\ -9c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{So } a = \frac{2}{5}, b = 0 \text{ and } c = -\frac{1}{9}, \text{ and } A = \begin{bmatrix} \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}$$

**1.3.30** We must satisfy the equation  $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ . Thus,  $5a + 3b - 9c = 2$ ,  $5d + 3e - 9f = 0$ , and  $5g + 3h - 9i = 1$ . One way to force our matrix to have rank 1 is to make all the entries in the second and third columns zero, meaning that  $a = \frac{5}{2}$ ,  $d = 0$ , and  $g = \frac{1}{5}$ . Thus, one possible matrix is  $\begin{bmatrix} \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{5} & 0 & 0 \end{bmatrix}$ .

**1.3.31**  $A$  is of the form  $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

$$\text{and } \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 5a + 3b - 9c \\ 3d - 9e \\ -9f \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly,  $f$  must equal  $-\frac{1}{9}$ . Then, since  $3d = 9e$ , we can choose any non-zero value for the free variable  $e$ , and  $d$  will be  $3e$ . So, if we choose 1 for  $e$ , then  $d = 3e = 3$ . Lastly, we must resolve  $5a + 3b - 9c = 2$ . Here,  $b$  and  $c$  are the free variables, and  $a = \frac{2-3b+9c}{5}$ . If we let  $b = c = 1$ . Then,  $a = \frac{2-3(1)+9(1)}{5} = \frac{8}{5}$ .

$$\text{So, in our example, } A = \begin{bmatrix} \frac{8}{5} & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}$$

**1.3.32** For this problem, we set up the same three equations as in Exercise 30. However, here, we must enforce that our matrix,  $A$ , contains no zero entries. One possible solution to this problem is the matrix  $\begin{bmatrix} -2 & -2 & -2 \\ 3 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$ .

**1.3.33** The  $i$ th component of  $A\vec{x}$  is  $[0 \ 0 \ \dots \ 1 \ \dots \ 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = x_i$ . (The 1 is in the  $i$ th position.)

Therefore,  $A\vec{x} = \vec{x}$ .

**1.3.34 a**  $A\vec{e}_1 = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$ ,  $A\vec{e}_2 = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$ , and  $A\vec{e}_3 = \begin{bmatrix} c \\ f \\ k \end{bmatrix}$ .

$$\text{b } B\vec{e}_1 = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{v}_1.$$

Likewise,  $B\vec{e}_2 = \vec{v}_2$  and  $B\vec{e}_3 = \vec{v}_3$ .

**1.3.35** Write  $A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_i \ \dots \ \vec{v}_m]$ , then

$$A\vec{e}_i = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_i \ \dots \ \vec{v}_m] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 1\vec{v}_i + \dots + 0\vec{v}_m = \vec{v}_i = \text{ith column of } A.$$

**1.3.36** By Exercise 35, the  $i$ th column of  $A$  is  $A\vec{e}_i$ , for  $i = 1, 2, 3$ . Therefore,  $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ .

**1.3.37** We have to solve the system  $\begin{array}{rcl} x_1 & + & 2x_2 \\ & & x_3 = 1 \end{array}$  or  $\begin{bmatrix} x_1 & = 2 - 2x_2 \\ x_3 & = 1 \end{bmatrix}$ .

Let  $x_2 = t$ . Then the solutions are of the form  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 2t \\ t \\ 1 \end{bmatrix}$ , where  $t$  is an arbitrary real number.

**1.3.38** We will illustrate our reasoning with an example. We generate the “random”  $3 \times 3$  matrix

$$A = \begin{bmatrix} 0.141 & 0.592 & 0.653 \\ 0.589 & 0.793 & 0.238 \\ 0.462 & 0.643 & 0.383 \end{bmatrix}.$$

Since the entries of this matrix are chosen from a large pool of numbers (in our case 1000, from 0.000 to 0.999), it is unlikely that any of the entries will be zero (and even less likely that the whole first column will consist of zeros). This means that we will usually be able to apply the Gauss-Jordan algorithm to turn the first column

into  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ; this is indeed possible in our example:  $\begin{bmatrix} 0.141 & 0.592 & 0.653 \\ 0.589 & 0.793 & 0.238 \\ 0.462 & 0.643 & 0.383 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4.199 & 4.631 \\ 0 & -1.680 & -2.490 \\ 0 & -1.297 & -1.757 \end{bmatrix}$ .

Again, it is unlikely that any entries in the second column of the new matrix will be zero. Therefore, we can turn the second column into  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .

Likewise, we will be able to clear up the third column, so that  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

We summarize:

As we apply Gauss-Jordan elimination to a random matrix  $A$  (of any size), it is unlikely that we will ever encounter a zero on the diagonal. Therefore,  $\text{rref}(A)$  is likely to have all ones along the diagonal.

**1.3.39** We will usually get  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$ , where  $a, b$ , and  $c$  are arbitrary.

**1.3.40** We will usually have  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .

(Compare with the summary to Exercise 38.)

**1.3.41** If  $A\vec{x} = \vec{b}$  is a “random” system, then  $\text{rref}(A)$  will usually be  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , so that we will have a unique solution.

**1.3.42** If  $A\vec{x} = \vec{b}$  is a “random” system of three equations with four unknowns, then  $\text{rref}(A)$  will usually be

$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$  (by Exercise 39), so that the system will have infinitely many solutions ( $x_4$  is a free variable).

**1.3.43** If  $A\vec{x} = \vec{b}$  is a “random” system of equations with three unknowns, then  $\text{rref}[A:\vec{b}]$  will usually be

$\begin{bmatrix} 1 & 0 & 0 & :0 \\ 0 & 1 & 0 & :0 \\ 0 & 0 & 1 & :0 \\ 0 & 0 & 0 & :1 \end{bmatrix}$ , so that the system is inconsistent.

**1.3.44** Let  $E = \text{rref}(A)$ , and note that all the entries in the last row of  $E$  must be zero, by the definition of rref. If  $\vec{c}$  is any vector in  $\mathbb{R}^n$  whose last component isn’t zero, then the system  $E\vec{x} = \vec{c}$  will be inconsistent. Now consider the elementary row operations that transform  $A$  into  $E$ , and apply the opposite operations, in reversed order, to the augmented matrix  $[E : \vec{c}]$ . You end up with an augmented matrix  $[A : \vec{b}]$  that represents an inconsistent system  $A\vec{x} = \vec{b}$ , as required.

**1.3.45** Write  $A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_m]$  and  $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix}$ . Then  $A(k\vec{x}) = [\vec{v}_1 \ \dots \ \vec{v}_m] \begin{bmatrix} kx_1 \\ \dots \\ kx_m \end{bmatrix} = kx_1\vec{v}_1 + \dots + kx_m\vec{v}_m$  and  $k(A\vec{x}) = k(x_1\vec{v}_1 + \dots + x_m\vec{v}_m) = kx_1\vec{v}_1 + \dots + kx_m\vec{v}_m$ . The two results agree, as claimed.

**1.3.46** Since  $a, d$ , and  $f$  are all nonzero, we can divide the first row by  $a$ , the second row by  $d$ , and the third row by  $f$  to obtain

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \\ 0 & 1 & \frac{e}{d} \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that the rank of the matrix is 3.

**1.3.47** a  $\vec{x} = \vec{0}$  is a solution.

b This holds by part (a) and Theorem 1.3.3.

c If  $\vec{x}_1$  and  $\vec{x}_2$  are solutions, then  $A\vec{x}_1 = \vec{0}$  and  $A\vec{x}_2 = \vec{0}$ .

Therefore,  $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$ , so that  $\vec{x}_1 + \vec{x}_2$  is a solution as well. Note that we have used Theorem 1.3.10a.

d  $A(k\vec{x}) = k(A\vec{x}) = k\vec{0} = \vec{0}$

We have used Theorem 1.3.10b.

**1.3.48** The fact that  $\vec{x}_1$  is a solution of  $A\vec{x} = \vec{b}$  means that  $A\vec{x}_1 = \vec{b}$ .

a.  $A(\vec{x}_1 + \vec{x}_h) = A\vec{x}_1 + A\vec{x}_h = \vec{b} + \vec{0} = \vec{b}$

b.  $A(\vec{x}_2 - \vec{x}_1) = A\vec{x}_2 - A\vec{x}_1 = \vec{b} - \vec{b} = \vec{0}$

c. Parts (a) and (b) show that the solutions of  $A\vec{x} = \vec{b}$  are exactly the vectors of the form  $\vec{x}_1 + \vec{x}_h$ , where  $\vec{x}_h$  is a solution of  $A\vec{x} = \vec{0}$ ; indeed if  $\vec{x}_2$  is a solution of  $A\vec{x} = \vec{b}$ , we can write  $\vec{x}_2 = \vec{x}_1 + (\vec{x}_2 - \vec{x}_1)$ , and  $\vec{x}_2 - \vec{x}_1$  will be a solution of  $A\vec{x} = \vec{0}$ , by part (b).

Geometrically, the vectors of the form  $\vec{x}_1 + \vec{x}_h$  are those whose tips are on the line  $L$  in Figure 1.14; the line  $L$  runs through the tip of  $\vec{x}_1$  and is parallel to the given line consisting of the solutions of  $A\vec{x} = \vec{0}$ .

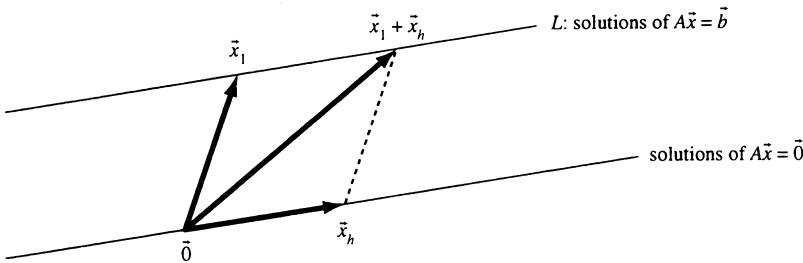


Figure 1.14: for Problem 1.3.48c.

**1.3.49 a** This system has either infinitely many solutions (if the right-most column of  $\text{rref}[A:\vec{b}]$  does not contain a leading one), or no solutions (if the right-most column *does* contain a leading one).

b This system has either a unique solution (if  $\text{rank}[A:\vec{b}] = 3]$ , or no solution (if  $\text{rank}[A:\vec{b}] = 4]$ ).

c The right-most column of  $\text{rref}[A:\vec{b}]$  must contain a leading one, so that the system has no solutions.

d This system has infinitely many solutions, since there is one free variable.

**1.3.50** The right-most column of  $\text{rref}[A:\vec{b}]$  must contain a leading one, so that the system has no solutions.

**1.3.51** For  $B\vec{x}$  to be defined, the number of columns of  $B$ , which is  $s$ , must equal the number of components of  $\vec{x}$ , which is  $p$ , so that we must have  $s = p$ . Then  $B\vec{x}$  will be a vector in  $\mathbb{R}^r$ ; for  $A(B\vec{x})$  to be defined we must have  $m = r$ . Summary: We must have  $s = p$  and  $m = r$ .

$$1.3.52 \quad A(B\vec{x}) = A \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & -x_2 \\ 2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\text{so that } C = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}.$$

$$1.3.53 \quad \text{Yes; write } A = [\vec{v}_1 \dots \vec{v}_m], \quad B = [\vec{w}_1 \dots \vec{w}_m], \quad \text{and } \vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix}.$$

$$\text{Then } (A + B)\vec{x} = [\vec{v}_1 + \vec{w}_1 \dots \vec{v}_m + \vec{w}_m] \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = x_1(\vec{v}_1 + \vec{w}_1) + \dots + x_m(\vec{v}_m + \vec{w}_m) \text{ and}$$

$$A\vec{x} + B\vec{x} = [\vec{v}_1 \dots \vec{v}_m] \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} + [\vec{w}_1 \dots \vec{w}_m] \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m + x_1\vec{w}_1 + \dots + x_m\vec{w}_m.$$

The two results agree, as claimed.

**1.3.54** The vectors of the form  $c_1\vec{v}_1 + c_2\vec{v}_2$  form a plane through the origin containing  $\vec{v}_1$  and  $\vec{v}_2$ ; in Figure 1.15 we draw a typical vector in this plane.

$$1.3.55 \quad \text{We are looking for constants } a \text{ and } b \text{ such that } a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

The resulting system  $\begin{bmatrix} a + 4b & = 7 \\ 2a + 5b & = 8 \\ 3a + 6b & = 9 \end{bmatrix}$  has the unique solution  $a = -1, b = 2$ , so that  $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$  is indeed a linear combination of the vector  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ .

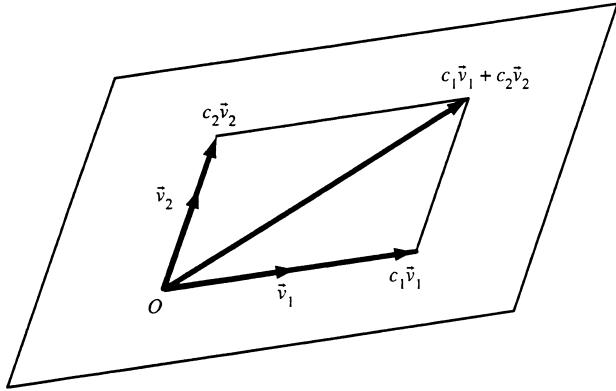


Figure 1.15: for Problem 1.3.54.

**1.3.56** We can use technology to determine that the system  $\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}$  is inconsistent; therefore, the vector  $\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$  fails to be a linear combination of the other four vectors.

**1.3.57** Pick a vector on each line, say  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  on  $y = \frac{x}{2}$  and  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  on  $y = 3x$ .

Then write  $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$  as a linear combination of  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ :  $a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ .

The unique solution is  $a = 2$ ,  $b = 3$ , so that the desired representation is  $\begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ .

$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$  is on the line  $y = \frac{x}{2}$ ;  $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$  is on line  $y = 3x$ .

**1.3.58** We want  $\begin{bmatrix} 3 \\ b \\ c \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + k_3 \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$ , for some  $k_1, k_2$  and  $k_3$ .

Note that we can rewrite this right-hand side as  $k_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 2k_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - k_3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$= (k_1 + 2k_2 - k_3) \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ . It follows that  $k_1 + 2k_2 - k_3 = 3$ , so that  $b = 9$  and  $c = 6$ .

$$1.3.59 \quad \begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} a+b \\ a+2b \\ a+3b \\ a+4b \end{bmatrix}.$$

So we have a small system:  $\begin{bmatrix} a & +b & = & 5 \\ a & +2b & = & 7 \end{bmatrix}$ , which we quickly solve to find  $a = 3$  and  $b = 2$ . Then,  $c = a + 3b = 3 + 6 = 9$  and  $d = a + 4b = 3 + 8 = 11$ .

$$1.3.60 \quad \text{We need } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} k_2 + 2k_3 \\ 0 \\ 3k_1 + 4k_2 + 5k_3 \\ 6k_3 \end{bmatrix}. \text{ From this we see that } a, c \text{ and } d \text{ can be any value, while } b \text{ must equal zero.}$$

1.3.61 We need to solve the system

$$\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

with augmented matrix

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & c & \\ 4 & 9 & c^2 & \end{array} \right].$$

The matrix reduces to

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & c-2 & \\ 0 & 0 & c^2 - 5c + 6 & \end{array} \right].$$

This system is consistent if and only if  $c = 2$  or  $c = 3$ . Thus the vector is a linear combination if  $c = 2$  or  $c = 3$ .

1.3.62 We need to solve the system

$$\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix} = x \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix} + y \begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$$

with augmented matrix

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ a & b & c & \\ a^2 & b^2 & c^2 & \end{array} \right].$$

The matrix reduces to

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & \\ 0 & 0 & (c-a)(c-b) & \end{array} \right].$$

This system is consistent if and only if  $c = a$  or  $c = b$ . Thus the vector is a linear combination if  $c = a$  or  $c = b$ .

**1.3.63** This is the line parallel to  $\vec{w}$  which goes through the end point of the vector  $\vec{v}$ .

**1.3.64** This is the line segment connecting the head of the vector  $\vec{v}$  to the head of the vector  $\vec{v} + \vec{w}$ .

**1.3.65** This is the full parallelogram spanned by the two vectors  $\vec{v}$  and  $\vec{w}$ .

**1.3.66** Write  $b = 1 - a$  and  $a\vec{v} + b\vec{w} = a\vec{v} + (1 - a)\vec{w} = \vec{w} + a(\vec{v} - \vec{w})$  to see that this is the line segment connecting the head of the vector  $\vec{v}$  to the head of the vector  $\vec{w}$ .

**1.3.67** This is the full triangle with its vertices at the origin and at the heads of the vectors  $\vec{v}$  and  $\vec{w}$ .

**1.3.68** Writing  $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$  as  $\vec{u} \cdot (\vec{v} - \vec{w}) = 0$ , we see that this is the line perpendicular to the vector  $\vec{v} - \vec{w}$ .

**1.3.69** We write out the augmented matrix: 
$$\left[ \begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 1 & 1 & 0 & c \end{array} \right]$$
 and reduce it to 
$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & \frac{-a+b+c}{2} \\ 0 & 1 & 0 & \frac{a-b+c}{2} \\ 0 & 0 & 1 & \frac{a+b-c}{2} \end{array} \right]$$
.

So  $x = \frac{-a+b+c}{2}$ ,  $y = \frac{a-b+c}{2}$  and  $z = \frac{a+b-c}{2}$ .

**1.3.70** We find it useful to let  $s = x_1 + x_2 + \dots + x_n$ . Adding up all  $n$  equations of the system, and realizing that the term  $x_i$  is missing from the  $i^{th}$  equation, we see that  $(n-1)s = b_1 + \dots + b_n$ , or,  $s = \frac{b_1 + \dots + b_n}{n-1}$ . Now the  $i^{th}$  equation of the system can be written as  $s - x_i = b_i$ , so that  $x_i = s - b_i = \frac{b_1 + \dots + b_n}{n-1} - b_i$ .

## True or False

Ch 1.TF.1 F, by Example 3a of Section 1.3

Ch 1.TF.2 T, by Definition 1.3.7

Ch 1.TF.3 T, by Theorem 1.3.4

Ch 1.TF.4 F, by Theorem 1.3.1

Ch 1.TF.5 F, by Theorem 1.3.4

Ch 1.TF.6 F; As a counter-example, consider the zero matrix.

Ch 1.TF.7 T, by Theorem 1.3.8

Ch 1.TF.8 T, by Definition 1.3.9

Ch 1.TF.9 T, by Definition.

Ch 1.TF.10 F; Consider the equation  $x + y + z = 0$ , repeated four times.

Ch 1.TF.11 T; Find rref.

Ch 1.TF.12 T; Find rref

Ch 1.TF.13 F; Consider the  $4 \times 3$  matrix  $A$  that contains all zeroes, except for a 1 in the lower left corner.

Ch 1.TF.14 F; Note that  $A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  for all  $2 \times 2$  matrices  $A$ .

Ch 1.TF.15 F; The rank is 1.

Ch 1.TF.16 F; The product on the left-hand side has two components.

Ch 1.TF.17 T; Let  $A = \begin{bmatrix} -3 & 0 \\ -5 & 0 \\ -7 & 0 \end{bmatrix}$ , for example.

Ch 1.TF.18 T; We have  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ .

Ch 1.TF.19 T; The last component of the left-hand side is zero for all vectors  $\vec{x}$ .

Ch 1.TF.20 T;  $A = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$ , for example.

Ch 1.TF.21 F; Let  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , for example. We can apply elementary row operations to  $A$  all we want, we will always end up with a matrix that has all zeros in the first column.

Ch 1.TF.22 T; If  $\vec{u} = a\vec{v} + b\vec{w}$  and  $\vec{v} = c\vec{p} + d\vec{q} + e\vec{r}$ , then  $\vec{u} = ac\vec{p} + ad\vec{q} + ae\vec{r} + b\vec{w}$ .

Ch 1.TF.23 F; The system  $x = 2$ ,  $y = 3$ ,  $x + y = 5$  has a unique solution.

Ch 1.TF.24 F; Let  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , for example.

Ch 1.TF.25 F; Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$  and  $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ , for example.

Ch 1.TF.26 T, by Exercise 1.3.44.

Ch 1.TF.27 F; Find rref to see that the rank is always 2.

Ch 1.TF.28 T; Note that  $\vec{v} = 1\vec{v} + 0\vec{w}$ .

Ch 1.TF.29 F; Let  $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , for example.

Ch 1.TF.30 T; Note that  $\vec{0} = 0\vec{v} + 0\vec{w}$

Ch 1.TF.31 F; If  $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{0}$ , then  $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  is a solution to  $[A:\vec{0}]$ . However, since  $\text{rank}(A) = 3$ ,  $\text{rref}[A:\vec{0}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ , meaning that only  $\vec{0}$  is a solution to  $A\vec{x} = \vec{0}$ .

Ch 1.TF.32 F; If  $\vec{b} = \vec{0}$ , then having a row of zeroes in  $\text{rref}(A)$  does not force the system to be inconsistent.

Ch 1.TF.33 T; By Example 3c of Section 1.3, the equation  $A\vec{x} = \vec{0}$  has the unique solution  $\vec{x} = \vec{0}$ . Now note that  $A(\vec{v} - \vec{w}) = \vec{0}$ , so that  $\vec{v} - \vec{w} = \vec{0}$  and  $\vec{v} = \vec{w}$ .

Ch 1.TF.34 T; Note that  $\text{rank}(A) = 4$ , by Theorem 1.3.4

Ch 1.TF.35 F; Let  $\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , for example..

Ch 1.TF.36 T; We use rref to solve the system  $A\vec{x} = \vec{0}$  and find  $\vec{x} = \begin{bmatrix} -2t \\ -3t \\ t \end{bmatrix}$ , where  $t$  is an arbitrary constant.

Letting  $t = 1$ , we find  $[\vec{u} \ \vec{v} \ \vec{w}] \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = -2\vec{u} - 3\vec{v} + \vec{w} = \vec{0}$ , so that  $\vec{w} = 2\vec{u} + 3\vec{v}$ .

Ch 1.TF.37 F; Let  $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , for example.

Ch 1.TF.38 T; Matrices  $A$  and  $B$  can both be transformed into  $I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . Running the elementary operations backwards, we can transform  $I$  into  $B$ . Thus we can first transform  $A$  into  $I$  and then  $I$  into  $B$ .

Ch 1.TF.39 T; If  $\vec{v} = a\vec{u} + b\vec{w}$ , then  $A\vec{v} = A(a\vec{u} + b\vec{w}) = A(a\vec{u}) + A(b\vec{w}) = aA\vec{u} + bA\vec{w}$ .

Ch 1.TF.40 T; check that the three defining properties of a matrix in rref still hold. F; If  $\vec{b} = \vec{0}$ , then having a row of zeroes in  $\text{rref}(A)$  does not force the system to be inconsistent.

Ch 1.TF.41 T;  $A\vec{x} = \vec{b}$  is inconsistent if and only if  $\text{rank}[A:\vec{b}] = \text{rank}(A)+1$ , since there will be an extra leading one in the last column of the augmented matrix: (See Figure 1.16.)

$$\left[ \begin{array}{cccc} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 \end{array} \right]$$

Figure 1.16: for Problem T/F 41.

Ch 1.TF.42 T; The system  $A\vec{x} = \vec{b}$  is consistent, by Example 3b, and there are, in fact, infinitely many solutions, by Theorem 1.3.3. Note that  $A\vec{x} = \vec{b}$  is a system of three equations with four unknowns.

Ch 1.TF.43 T; Recall that we use  $\text{rref}[A:\vec{0}]$  to solve the system  $A\vec{x} = \vec{0}$ . Now,  $\text{rref}[A:\vec{0}] = [\text{rref}(A):\vec{0}] = [\text{rref}(B):\vec{0}] = \text{rref}[B:\vec{0}]$ . Then, since  $[\text{rref}(A):\vec{0}] = [\text{rref}(B):\vec{0}]$ , they must have the same solutions.

Ch 1.TF.44 F; Consider  $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ . If we remove the first column, then the remaining matrix fails to be in rref.

Ch 1.TF.45 T; First we list all possible matrices  $\text{rref}(M)$ , where  $M$  is a  $2 \times 2$  matrix, and show the corresponding solutions for  $M\vec{x} = \vec{0}$ :

$\text{rref}(M)$	solutions of $M\vec{x} = \vec{0}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\{\vec{0}\}$
$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -at \\ t \end{bmatrix}$ , for an arbitrary $t$
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} t \\ 0 \end{bmatrix}$ , for an arbitrary $t$
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\mathbb{R}^2$

Now, we see that if  $\text{rref}(A) \neq \text{rref}(B)$ , then the systems  $A\vec{x} = \vec{0}$  and  $B\vec{x} = \vec{0}$  must have different solutions. Thus, it must be that if the two systems have the same solutions, then  $\text{rref}(A) = \text{rref}(B)$ .

# Chapter 2

## Section 2.1

2.1.1 Not a linear transformation, since  $y_2 = x_2 + 2$  is not linear in our sense.

2.1.2 Linear, with matrix  $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$

2.1.3 Not linear, since  $y_2 = x_1 x_3$  is nonlinear.

2.1.4  $A = \begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}$

2.1.5 By Theorem 2.1.2, the three columns of the  $2 \times 3$  matrix  $A$  are  $T(\vec{e}_1)$ ,  $T(\vec{e}_2)$ , and  $T(\vec{e}_3)$ , so that

$$A = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}.$$

2.1.6 Note that  $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ , so that  $T$  is indeed linear, with matrix  $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ .

2.1.7 Note that  $x_1 \vec{v}_1 + \dots + x_m \vec{v}_m = [\vec{v}_1 \dots \vec{v}_m] \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ , so that  $T$  is indeed linear, with matrix  $[\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_m]$ .

2.1.8 Reducing the system  $\begin{bmatrix} x_1 + 7x_2 & = y_1 \\ 3x_1 + 20x_2 & = y_2 \end{bmatrix}$ , we obtain  $\begin{bmatrix} x_1 & = -20y_1 + 7y_2 \\ x_2 & = 3y_1 - y_2 \end{bmatrix}$ .

2.1.9 We have to attempt to solve the equation  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  for  $x_1$  and  $x_2$ . Reducing the system  $\begin{bmatrix} 2x_1 + 3x_2 & = y_1 \\ 6x_1 + 9x_2 & = y_2 \end{bmatrix}$  we obtain  $\begin{bmatrix} x_1 + 1.5x_2 & = 0.5y_1 \\ 0 & = -3y_1 + y_2 \end{bmatrix}$ .

No unique solution  $(x_1, x_2)$  can be found for a given  $(y_1, y_2)$ ; the matrix is noninvertible.

2.1.10 We have to attempt to solve the equation  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  for  $x_1$  and  $x_2$ . Reducing the system  $\begin{bmatrix} x_1 + 2x_2 & = y_1 \\ 4x_1 + 9x_2 & = y_2 \end{bmatrix}$  we find that  $\begin{bmatrix} x_1 & = 9y_1 + 2y_2 \\ x_2 & = -4y_1 + y_2 \end{bmatrix}$  or  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ .

The inverse matrix is  $\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$ .

**2.1.11** We have to attempt to solve the equation  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  for  $x_1$  and  $x_2$ . Reducing the system  $\begin{bmatrix} x_1 + 2x_2 = y_1 \\ 3x_1 + 9x_2 = y_2 \end{bmatrix}$  we find that  $\begin{bmatrix} x_1 = 3y_1 - \frac{2}{3}y_2 \\ x_2 = -y_1 + \frac{1}{3}y_2 \end{bmatrix}$ . The inverse matrix is  $\begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$ .

**2.1.12** Reducing the system  $\begin{bmatrix} x_1 + kx_2 = y_1 \\ x_2 = y_2 \end{bmatrix}$  we find that  $\begin{bmatrix} x_1 = y_1 - ky_2 \\ x_2 = y_2 \end{bmatrix}$ . The inverse matrix is  $\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ .

**2.1.13 a** First suppose that  $a \neq 0$ . We have to attempt to solve the equation  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  for  $x_1$  and  $x_2$ .

$$\begin{aligned} \left[ \begin{array}{cc|c} ax_1 + bx_2 & = & y_1 \\ cx_1 + dx_2 & = & y_2 \end{array} \right] \div a &\rightarrow \left[ \begin{array}{cc|c} x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ cx_1 + dx_2 & = & y_2 \end{array} \right] - c(I) \rightarrow \\ \left[ \begin{array}{cc|c} x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ (d - \frac{bc}{a})x_2 & = & -\frac{c}{a}y_1 + y_2 \end{array} \right] &\rightarrow \\ \left[ \begin{array}{cc|c} x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ (\frac{ad-bc}{a})x_2 & = & -\frac{c}{a}y_1 + y_2 \end{array} \right] \end{aligned}$$

We can solve this system for  $x_1$  and  $x_2$  if (and only if)  $ad - bc \neq 0$ , as claimed.

If  $a = 0$ , then we have to consider the system

$$\left[ \begin{array}{cc|c} bx_2 & = & y_1 \\ cx_1 + dx_2 & = & y_2 \end{array} \right] \text{swap : } I \leftrightarrow II \left[ \begin{array}{cc|c} cx_1 + dx_2 & = & y_2 \\ bx_2 & = & y_1 \end{array} \right]$$

We can solve for  $x_1$  and  $x_2$  provided that both  $b$  and  $c$  are nonzero, that is if  $bc \neq 0$ . Since  $a = 0$ , this means that  $ad - bc \neq 0$ , as claimed.

**b** First suppose that  $ad - bc \neq 0$  and  $a \neq 0$ . Let  $D = ad - bc$  for simplicity. We continue our work in part (a):

$$\begin{aligned} \left[ \begin{array}{cc|c} x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ \frac{D}{a}x_2 & = & -\frac{c}{a}y_1 + y_2 \end{array} \right] \cdot \frac{a}{D} &\rightarrow \\ \left[ \begin{array}{cc|c} x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right] - \frac{b}{a}(II) &\rightarrow \\ \left[ \begin{array}{cc|c} x_1 & = & (\frac{1}{a} + \frac{bc}{aD})y_1 - \frac{b}{D}y_2 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right] \\ \left[ \begin{array}{cc|c} x_1 & = & \frac{d}{D}y_1 - \frac{b}{D}y_2 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right] \end{aligned}$$

(Note that  $\frac{1}{a} + \frac{bc}{aD} = \frac{D+bc}{aD} = \frac{ad}{aD} = \frac{d}{D}$ .)

It follows that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ , as claimed. If  $ad - bc \neq 0$  and  $a = 0$ , then we have to solve the system

$$\begin{aligned} \left[ \begin{array}{cc|c} cx_1 + dx_2 & = y_2 \\ bx_2 & = y_1 \end{array} \right] &\div c \\ \left[ \begin{array}{cc|c} x_1 + \frac{d}{c}x_2 & = \frac{1}{c}y_2 \\ x_2 & = \frac{1}{b}y_1 \end{array} \right] &\div \frac{d}{c} (II) \\ \left[ \begin{array}{cc|c} x_1 & = -\frac{d}{bc}y_1 + \frac{1}{c}y_2 \\ x_2 & = \frac{1}{b}y_1 \end{array} \right] \end{aligned}$$

It follows that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$  (recall that  $a = 0$ ), as claimed.

**2.1.14 a** By Exercise 13a,  $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$  is invertible if (and only if)  $2k - 15 \neq 0$ , or  $k \neq 7.5$ .

b By Exercise 13b,  $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15} \begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix}$ .

If all entries of this inverse are integers, then  $\frac{3}{2k-15} - \frac{2}{2k-15} = \frac{1}{2k-15}$  is a (nonzero) integer  $n$ , so that  $2k-15 = \frac{1}{n}$  or  $k = 7.5 + \frac{1}{2n}$ . Since  $\frac{k}{2k-15} = kn = 7.5n + \frac{1}{2}$  is an integer as well,  $n$  must be odd.

We have shown: If all entries of the inverse are integers, then  $k = 7.5 + \frac{1}{2n}$ , where  $n$  is an odd integer. The converse is true as well: If  $k$  is chosen in this way, then the entries of  $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$  will be integers.

**2.1.15** By Exercise 13a, the matrix  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  is invertible if (and only if)  $a^2 + b^2 \neq 0$ , which is the case unless  $a = b = 0$ . If  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  is invertible, then its inverse is  $\frac{1}{a^2+b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ , by Exercise 13b.

**2.1.16** If  $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ , then  $A\vec{x} = 3\vec{x}$  for all  $\vec{x}$  in  $\mathbb{R}^2$ , so that  $A$  represents a scaling by a factor of 3. Its inverse is a scaling by a factor of  $\frac{1}{3}$ :  $A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ . (See Figure 2.1.)

**2.1.17** If  $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ , then  $A\vec{x} = -\vec{x}$  for all  $\vec{x}$  in  $\mathbb{R}^2$ , so that  $A$  represents a reflection about the origin.

This transformation is its own inverse:  $A^{-1} = A$ . (See Figure 2.2.)

**2.1.18** Compare with Exercise 16: This matrix represents a scaling by the factor of  $\frac{1}{2}$ ; the inverse is a scaling by 2. (See Figure 2.3.)

**2.1.19** If  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , then  $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ , so that  $A$  represents the orthogonal projection onto the  $\vec{e}_1$  axis. (See Figure 2.1.) This transformation is not invertible, since the equation  $A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  has infinitely many solutions  $\vec{x}$ . (See Figure 2.4.)

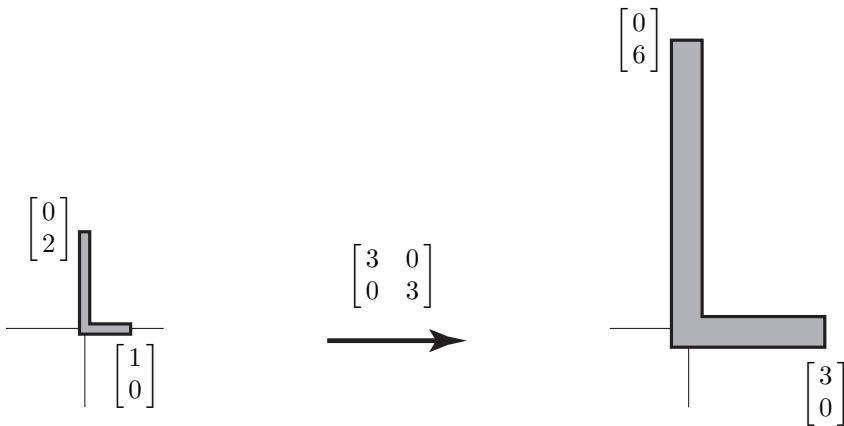


Figure 2.1: for Problem 2.1.16.

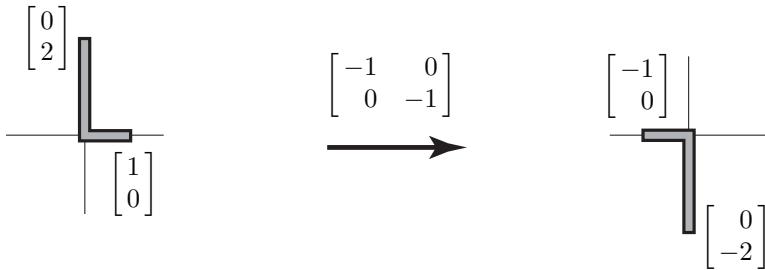


Figure 2.2: for Problem 2.1.17.

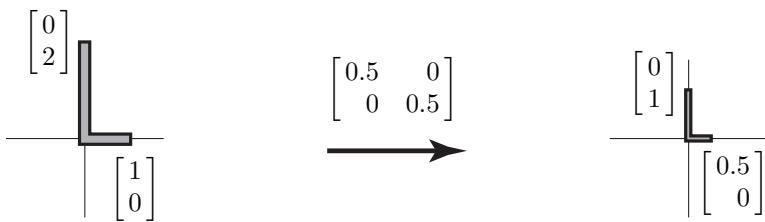


Figure 2.3: for Problem 2.1.18.

**2.1.20** If  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , then  $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ , so that  $A$  represents the reflection about the line  $x_2 = x_1$ . This transformation is its own inverse:  $A^{-1} = A$ . (See Figure 2.5.)

**2.1.21** Compare with Example 5.

If  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ , then  $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$ . Note that the vectors  $\vec{x}$  and  $A\vec{x}$  are perpendicular and have the same

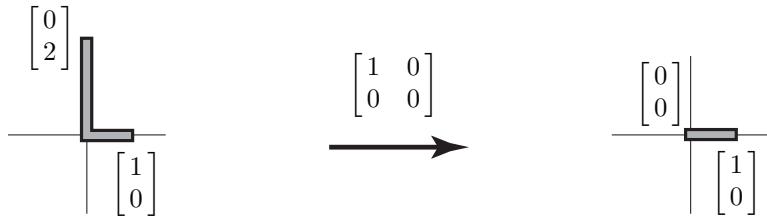


Figure 2.4: for Problem 2.1.19.

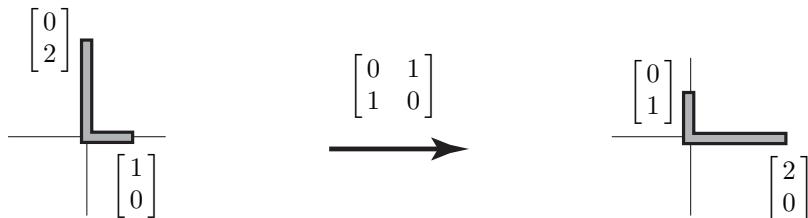


Figure 2.5: for Problem 2.1.20.

length. If  $\vec{x}$  is in the first quadrant, then  $A\vec{x}$  is in the fourth. Therefore,  $A$  represents the rotation through an angle of  $90^\circ$  in the clockwise direction. (See Figure 2.6.) The inverse  $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  represents the rotation through  $90^\circ$  in the counterclockwise direction.

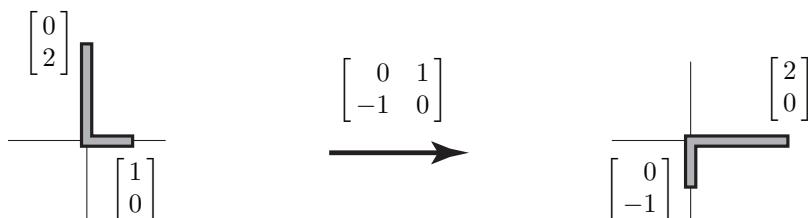


Figure 2.6: for Problem 2.1.21.

**2.1.22** If  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , then  $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$ , so that  $A$  represents the reflection about the  $e_1$  axis. This transformation is its own inverse:  $A^{-1} = A$ . (See Figure 2.7.)

**2.1.23** Compare with Exercise 21.

Note that  $A = 2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ , so that  $A$  represents a rotation through an angle of  $90^\circ$  in the clockwise direction, followed by a scaling by the factor of 2.

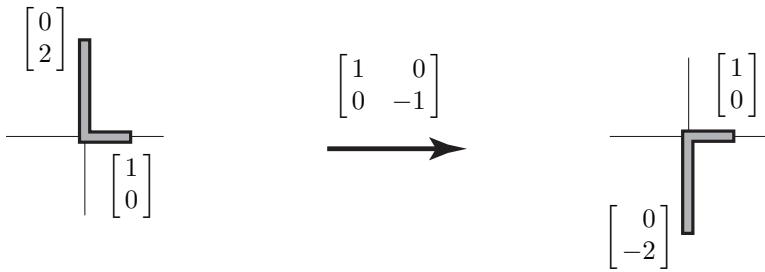


Figure 2.7: for Problem 2.1.22.

The inverse  $A^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$  represents a rotation through an angle of  $90^\circ$  in the counterclockwise direction, followed by a scaling by the factor of  $\frac{1}{2}$ . (See Figure 2.8.)

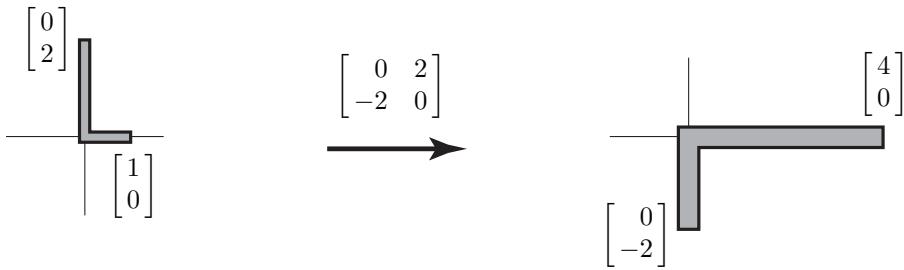


Figure 2.8: for Problem 2.1.23.

**2.1.24** Compare with Example 5. (See Figure 2.9.)

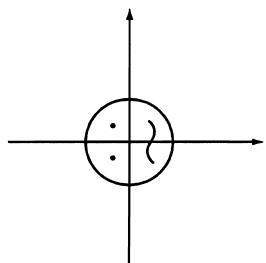


Figure 2.9: for Problem 2.1.24.

**2.1.25** The matrix represents a scaling by the factor of 2. (See Figure 2.10.)

**2.1.26** This matrix represents a reflection about the line  $x_2 = x_1$ . (See Figure 2.11.)

**2.1.27** This matrix represents a reflection about the  $\vec{e}_1$  axis. (See Figure 2.12.)

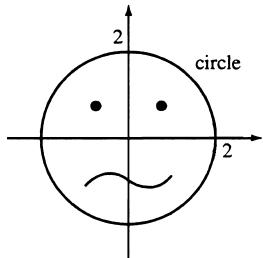


Figure 2.10: for Problem 2.1.25.

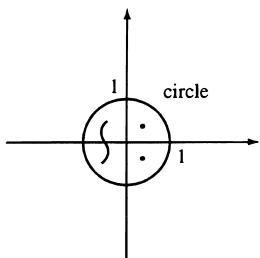


Figure 2.11: for Problem 2.1.26.

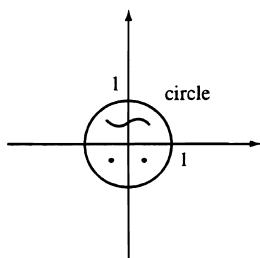


Figure 2.12: for Problem 2.1.27.

**2.1.28** If  $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ , then  $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$ , so that the  $x_2$  component is multiplied by 2, while the  $x_1$  component remains unchanged. (See Figure 2.13.)

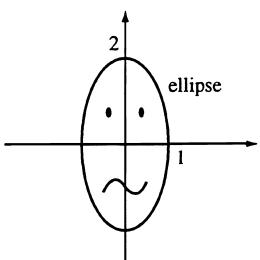


Figure 2.13: for Problem 2.1.28.

**2.1.29** This matrix represents a reflection about the origin. Compare with Exercise 17. (See Figure 2.14.)

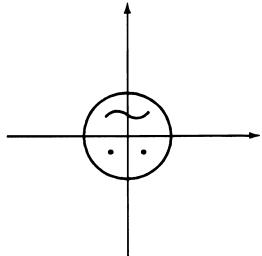


Figure 2.14: for Problem 2.1.29.

**2.1.30** If  $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , then  $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ , so that  $A$  represents the projection onto the  $\vec{e}_2$  axis. (See Figure 2.15.)

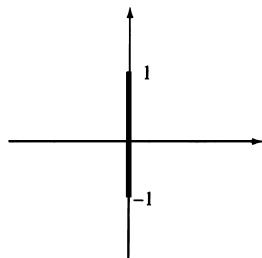


Figure 2.15: for Problem 2.1.30.

**2.1.31** The image must be reflected about the  $\vec{e}_2$  axis, that is  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  must be transformed into  $\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ : This can be accomplished by means of the linear transformation  $T(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$ .

**2.1.32** Using Theorem 2.1.2, we find  $A = \begin{bmatrix} 3 & 0 & \cdots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}$ . This matrix has 3's on the diagonal and 0's everywhere else.

**2.1.33** By Theorem 2.1.2,  $A = \left[ T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$ . (See Figure 2.16.)

$$\text{Therefore, } A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

**2.1.34** As in Exercise 2.1.33, we find  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$ ; then by Theorem 2.1.2,  $A = [T(\vec{e}_1) \ T(\vec{e}_2)]$ . (See Figure 2.17.)

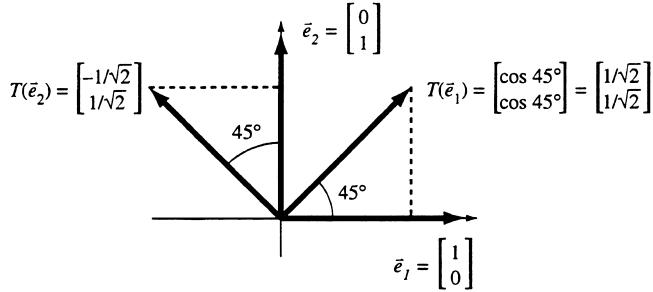


Figure 2.16: for Problem 2.1.33.

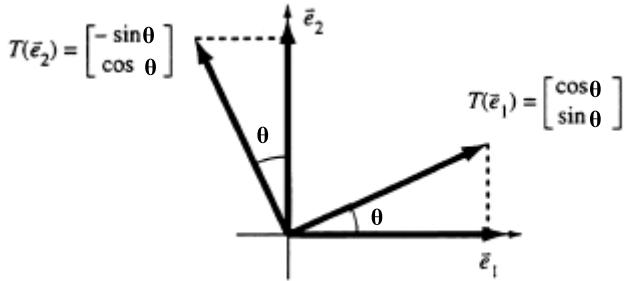


Figure 2.17: for Problem 2.1.34.

Therefore,  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ .

**2.1.35** We want to find a matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $A \begin{bmatrix} 5 \\ 42 \end{bmatrix} = \begin{bmatrix} 89 \\ 52 \end{bmatrix}$  and  $A \begin{bmatrix} 6 \\ 41 \end{bmatrix} = \begin{bmatrix} 88 \\ 53 \end{bmatrix}$ . This amounts to

solving the system  $\begin{bmatrix} 5a + 42b & = 89 \\ 6a + 41b & = 88 \\ 5c + 42d & = 52 \\ 6c + 41d & = 53 \end{bmatrix}$ .

(Here we really have two systems with two unknowns each.)

The unique solution is  $a = 1$ ,  $b = 2$ ,  $c = 2$ , and  $d = 1$ , so that  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ .

**2.1.36** First we draw  $\vec{w}$  in terms of  $\vec{v}_1$  and  $\vec{v}_2$  so that  $\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2$  for some  $c_1$  and  $c_2$ . Then, we scale the  $\vec{v}_2$ -component by 3, so our new vector equals  $c_1 \vec{v}_1 + 3c_2 \vec{v}_2$ .

**2.1.37** Since  $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$ , we have  $T(\vec{x}) = T(\vec{v} + k(\vec{w} - \vec{v})) = T(\vec{v}) + k(T(\vec{w}) - T(\vec{v}))$ , by Theorem 2.1.3

Since  $k$  is between 0 and 1, the tip of this vector  $T(\vec{x})$  is on the line segment connecting the tips of  $T(\vec{v})$  and  $T(\vec{w})$ . (See Figure 2.18.)

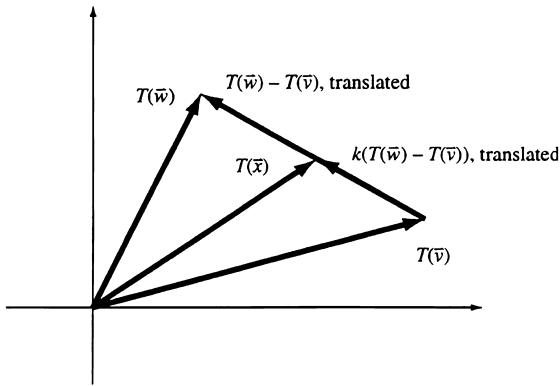


Figure 2.18: for Problem 2.1.37.

2.1.38  $T \begin{bmatrix} 2 \\ -1 \end{bmatrix} = [\vec{v}_1 \quad \vec{v}_2] \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\vec{v}_1 - \vec{v}_2 = 2\vec{v}_1 + (-\vec{v}_2)$ . (See Figure 2.19.)

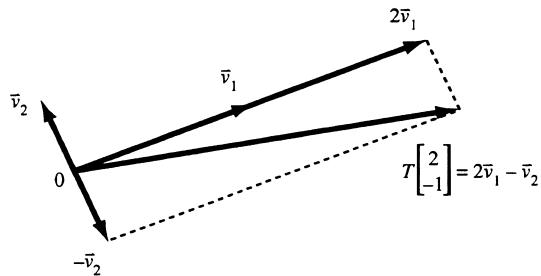


Figure 2.19: for Problem 2.1.38.

2.1.39 By Theorem 2.1.2, we have  $T \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & \dots & T(\vec{e}_m) \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = x_1 T(\vec{e}_1) + \dots + x_m T(\vec{e}_m)$ .

2.1.40 These linear transformations are of the form  $[y] = [a][x]$ , or  $y = ax$ . The graph of such a function is a line through the origin.

2.1.41 These linear transformations are of the form  $[y] = [a \ b] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ , or  $y = ax_1 + bx_2$ . The graph of such a function is a plane through the origin.

2.1.42 a See Figure 2.20.

b The image of the point  $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$  is the origin,  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

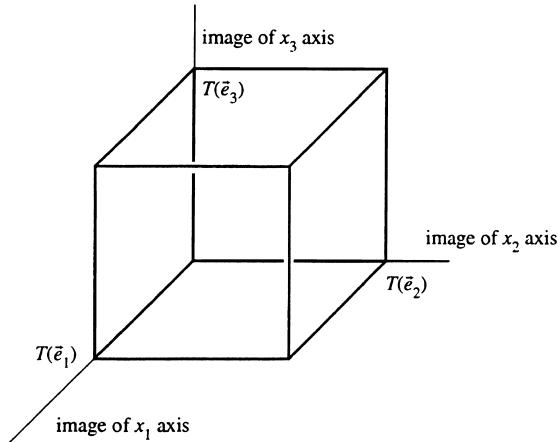


Figure 2.20: for Problem 2.1.42.

c Solve the equation  $\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , or  $\begin{bmatrix} -\frac{1}{2}x_1 + x_2 & = 0 \\ -\frac{1}{2}x_1 + x_3 & = 0 \end{bmatrix}$ . (See Figure 2.16.)

The solutions are of the form  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix}$ , where  $t$  is an arbitrary real number. For example, for  $t = \frac{1}{2}$ , we

find the point  $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$  considered in part b. These points are on the line through the origin and the observer's eye.

$$2.1.43 \text{ a } T(\vec{x}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3 = [2 \ 3 \ 4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The transformation is indeed linear, with matrix  $[2 \ 3 \ 4]$ .

b If  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ , then  $T$  is linear with matrix  $[v_1 \ v_2 \ v_3]$ , as in part (a).

c Let  $[a \ b \ c]$  be the matrix of  $T$ . Then  $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [a \ b \ c] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ax_1 + bx_2 + cx_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ , so that  $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  does the job.

$$2.1.44 \quad T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ so that } T \text{ is linear, with matrix } \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

**2.1.45** Yes,  $\vec{z} = L(T(\vec{x}))$  is also linear, which we will verify using Theorem 2.1.3. Part a holds, since  $L(T(\vec{v} + \vec{w})) = L(T(\vec{v}) + T(\vec{w})) = L(T(\vec{v})) + L(T(\vec{w}))$ , and part b also works, because  $L(T(k\vec{v})) = L(kT(\vec{v})) = kL(T(\vec{v}))$ .

$$2.1.46 \quad T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = B \left( A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = B \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} pa + qc \\ ra + sc \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B \left( A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = B \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} pb + qd \\ rb + sd \end{bmatrix}$$

$$\text{So, } T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \left( T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + x_2 \left( T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} pb + qd \\ rb + sd \end{bmatrix}$$

**2.1.47** Write  $\vec{w}$  as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ :  $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$ . (See Figure 2.21.)

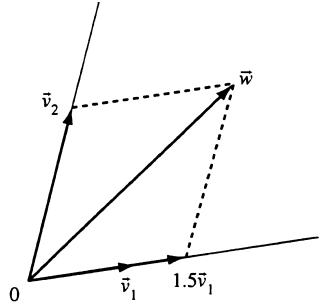


Figure 2.21: for Problem 2.1.47.

Measurements show that we have *roughly*  $\vec{w} = 1.5\vec{v}_1 + \vec{v}_2$ .

Therefore, by linearity,  $T(\vec{w}) = T(1.5\vec{v}_1 + \vec{v}_2) = 1.5T(\vec{v}_1) + T(\vec{v}_2)$ . (See Figure 2.22.)

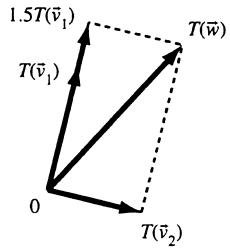


Figure 2.22: for Problem 2.1.47.

**2.1.48** Let  $\vec{x}$  be some vector in  $\mathbb{R}^2$ . Since  $\vec{v}_1$  and  $\vec{v}_2$  are not parallel, we can write  $\vec{x}$  in terms of components of  $\vec{v}_1$  and  $\vec{v}_2$ . So, let  $c_1$  and  $c_2$  be scalars such that  $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ . Then, by Theorem 2.1.3,  $T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = c_1L(\vec{v}_1) + c_2L(\vec{v}_2) = L(c_1\vec{v}_1 + c_2\vec{v}_2) = L(\vec{x})$ . So  $T(\vec{x}) = L(\vec{x})$  for all  $\vec{x}$  in  $\mathbb{R}^2$ .

**2.1.49** a Let  $x_1$  be the number of 2 Franc coins, and  $x_2$  be the number of 5 Franc coins. Then  $\begin{bmatrix} 2x_1 & +5x_2 & = & 144 \\ x_1 & +x_2 & = & 51 \end{bmatrix}$ .

From this we easily find our solution vector to be  $\begin{bmatrix} 37 \\ 14 \end{bmatrix}$ .

$$\text{b } \begin{bmatrix} \text{total value of coins} \\ \text{total number of coins} \end{bmatrix} = \begin{bmatrix} 2x_1 & +5x_2 \\ x_1 & +x_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$\text{So, } A = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}.$$

c By Exercise 13, matrix  $A$  is invertible (since  $ad - bc = -3 \neq 0$ ), and  $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix}$ .

Then  $-\frac{1}{3} \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 144 \\ 51 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 144 & -5(51) \\ -144 & +2(51) \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -111 \\ -42 \end{bmatrix} = \begin{bmatrix} 37 \\ 14 \end{bmatrix}$ , which was the vector we found in part a.

**2.1.50 a** Let  $\begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{mass of the platinum alloy} \\ \text{mass of the silver alloy} \end{bmatrix}$ . Using the definition density = mass/volume, or volume = mass/density, we can set up the system:

$\begin{bmatrix} p & +s & = & 5,000 \\ \frac{p}{20} & +\frac{s}{10} & = & 370 \end{bmatrix}$ , with the solution  $p = 2,600$  and  $s = 2,400$ . We see that the platinum alloy makes up only 52 percent of the crown; this gold smith is a crook!

b We seek the matrix  $A$  such that  $A \begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} p+s \\ \frac{p}{20} + \frac{s}{10} \end{bmatrix}$ . Thus  $A = \begin{bmatrix} 1 & 1 \\ \frac{1}{20} & \frac{1}{10} \end{bmatrix}$ .

c Yes. By Exercise 13,  $A^{-1} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}$ . Applied to the case considered in part a, we find that  $\begin{bmatrix} p \\ s \end{bmatrix} = A^{-1} \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix} \begin{bmatrix} 5,000 \\ 370 \end{bmatrix} = \begin{bmatrix} 2,600 \\ 2,400 \end{bmatrix}$ , confirming our answer in part a.

$$\text{2.1.51 a } \begin{bmatrix} C \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}(F - 32) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}F - \frac{160}{9} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F \\ 1 \end{bmatrix}.$$

$$\text{So } A = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix}.$$

b Using Exercise 13, we find  $\frac{5}{9}(1) - (-\frac{160}{9})0 = \frac{5}{9} \neq 0$ , so  $A$  is invertible.

$$A^{-1} = \frac{9}{5} \begin{bmatrix} 1 & \frac{160}{9} \\ 0 & \frac{5}{9} \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & 32 \\ 0 & 1 \end{bmatrix}. \text{ So, } F = \frac{9}{5}C + 32.$$

**2.1.52 a**  $A\vec{x} = \begin{bmatrix} 300 \\ 2,400 \end{bmatrix}$ , meaning that the total value of our money is C\$300, or, equivalently, ZAR2400.

b From Exercise 13, we test the value  $ad - bc$  and find it to be zero. Thus  $A$  is not invertible. To determine when  $A$  is consistent, we begin to compute rref  $[A:\vec{b}]$ :

$$\begin{bmatrix} 1 & \frac{1}{8} & \vdots & b_1 \\ 8 & 1 & \vdots & b_2 \end{bmatrix} - 8I \rightarrow \begin{bmatrix} 1 & \frac{1}{8} & \vdots & b_1 \\ 0 & 0 & \vdots & b_2 - 8b_1 \end{bmatrix}.$$

Thus, the system is consistent only when  $b_2 = 8b_1$ . This makes sense, since  $b_2$  is the total value of our money in terms of Rand, while  $b_1$  is the value in terms of Canadian dollars. Consider the example in part a. If the system  $A\vec{x} = \vec{b}$  is consistent, then there will be infinitely many solutions  $\vec{x}$ , representing various compositions of our portfolio in terms of Rand and Canadian dollars, all representing the same total value.

- 2.1.53** All four entries along the diagonal must be 1: they represent the process of converting a currency to itself. We also know that  $a_{ij} = 1/a_{ji}$  for all  $i, j$  because converting from one currency  $i$  to currency  $j$  is the inverse

to converting currency  $j$  to currency  $i$ . This gives us 3 more entries:  $\begin{bmatrix} 1 & 5/8 & 1/170 & * \\ 8/5 & 1 & * & 2 \\ 170 & * & 1 & * \\ * & 1/2 & * & 1 \end{bmatrix}$ . Next, let's

find the entry  $a_{41}$ , giving the value of one Euro expressed in Pounds. Now  $E1 = \$/(8/5) = \$1.60$  and  $\$1 = £(1/2) = £(0.50)$  so that  $E1 = £(1/2)(8/5) = £(4/5) = £0.80$ . We have found that  $a_{41} = a_{42}a_{21} = 4/5$  and

the matrix is  $\begin{bmatrix} 1 & 5/8 & 1/170 & 5/4 \\ 8/5 & 1 & * & 2 \\ 170 & * & 1 & * \\ 4/5 & 1/2 & * & 1 \end{bmatrix}$ . Similarly, we have  $a_{ij} = a_{ik}a_{kj}$  for all indices  $i, j, k = 1, 2, 3, 4$ . This

gives  $a_{32} = a_{31}a_{12} = 170 * 5/8 = 425/4$  and  $a_{43} = a_{41}a_{13} = (4/5)(1/170) = 2/425$ . Using the fact that  $a_{ij} = a_{ji}^{-1}$ ,

we can complete the matrix:  $\begin{bmatrix} 1 & 5/8 & 1/170 & 5/4 \\ 8/5 & 1 & 4/425 & 2 \\ 170 & 425/4 & 1 & 425/2 \\ 4/5 & 1/2 & 2/425 & 1 \end{bmatrix}$ .

- 2.1.54** a 1: this represents converting a currency to itself.

b  $a_{ij}$  is the reciprocal of  $a_{ji}$ , meaning that  $a_{ij}a_{ji} = 1$ . This represents converting on currency to another, then converting it back.

c Note that  $a_{ik}$  is the conversion factor from currency  $k$  to currency  $i$  meaning that

$$(1 \text{ unit of currency } k) = (a_{ik} \text{ units of currency } i)$$

Likewise,

$$(1 \text{ unit of currency } j) = (a_{kj} \text{ units of currency } k).$$

It follows that

$$(1 \text{ unit of currency } j) = (a_{kj}a_{ik} \text{ units of currency } i) = (a_{ij} \text{ units of currency } i), \text{ so that } a_{ik}a_{kj} = a_{ij}.$$

d The rank of  $A$  is only 1, because every row is simply a scalar multiple of the top row. More precisely, since  $a_{ij} = a_{i1}a_{1j}$ , by part c, the  $i^{th}$  row is  $a_{i1}$  times the top row. When we compute the rref, every row but the top will be removed in the first step. Thus,  $\text{rref}(A)$  is a matrix with the top row of  $A$  and zeroes for all other entries.

## Section 2.2

- 2.2.1** The standard L is transformed into a distorted L whose foot is the vector  $T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

Meanwhile, the back becomes the vector  $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .

**2.2.2** By Theorem 2.2.3, this matrix is  $\begin{bmatrix} \cos(60^\circ) & -\sin(60^\circ) \\ \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ .

**2.2.3** If  $\vec{x}$  is in the unit square in  $\mathbb{R}^2$ , then  $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$  with  $0 \leq x_1, x_2 \leq 1$ , so that

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2).$$

The image of the unit square is a parallelogram in  $\mathbb{R}^3$ ; two of its sides are  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$ , and the origin is one of its vertices. (See Figure 2.23.)

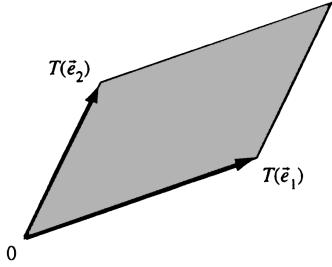


Figure 2.23: for Problem 2.2.3.

**2.2.4** By Theorem 2.2.4, this is a rotation combined with a scaling. The transformation rotates 45 degrees counter-clockwise, and has a scaling factor of  $\sqrt{2}$ .

**2.2.5** Note that  $\cos(\theta) = -0.8$ , so that  $\theta = \arccos(-0.8) \approx 2.498$ .

**2.2.6** By Theorem 2.2.1,  $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left( \vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \vec{u}$ , where  $\vec{u}$  is a unit vector on  $L$ . To get  $\vec{u}$ , we normalize  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ :

$$\vec{u} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \text{ so that } \text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{3} \cdot \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix}.$$

**2.2.7** According to the discussion in the text,  $\text{ref}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 \left( \vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \vec{u} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , where  $\vec{u}$  is a unit vector on  $L$ . To get  $\vec{u}$ , we normalize  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ :  $\vec{u} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ , so that  $\text{ref}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 \left( \frac{5}{3} \right) \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{9} \\ \frac{1}{9} \\ \frac{11}{9} \end{bmatrix}$ .

**2.2.8** From Definition 2.2.2, we can see that this is a reflection about the line  $x_1 = -x_2$ .

**2.2.9** By Theorem 2.2.5, this is a vertical shear.

**2.2.10** By Theorem 2.2.1,  $\text{proj}_L \vec{x} = (\vec{u} \cdot \vec{x})\vec{u}$ , where  $\vec{u}$  is a unit vector on  $L$ . We can choose  $\vec{u} = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.6 \\ 0.12 \end{bmatrix}$ . Then  $\text{proj}_L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left( \begin{bmatrix} 0.8 \\ 0.6 \\ 0.12 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{bmatrix} 0.8 \\ 0.6 \\ 0.12 \end{bmatrix} = (0.8x_1 + 0.6x_2) \begin{bmatrix} 0.8 \\ 0.6 \\ 0.12 \end{bmatrix} = \begin{bmatrix} 0.64x_1 + 0.48x_2 \\ 0.48x_1 + 0.36x_2 \\ 0.096x_1 + 0.072x_2 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ . The matrix is  $A = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}$ .

**2.2.11** In Exercise 10 we found the matrix  $A = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}$  of the projection onto the line  $L$ . By Theorem 2.2.2,  $\text{ref}_L \vec{x} = 2(\text{proj}_L \vec{x}) - \vec{x} = 2A\vec{x} - \vec{x} = (2A - I_2)\vec{x}$ , so that the matrix of the reflection is  $2A - I_2 = \begin{bmatrix} 0.28 & 0.96 \\ 0.96 & -0.28 \end{bmatrix}$ .

**2.2.12** Let  $\vec{u} = (1/\|\vec{w}\|)\vec{w}$  be the unit vector in the direction of  $\vec{w}$ . It has the components  $u_1 = w_1/\sqrt{w_1^2 + w_2^2}$  and  $u_2 = w_2/\sqrt{w_1^2 + w_2^2}$ . On Pages 57/58, we see that the matrix representing the projection is

$$\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}.$$

This can be written as

$$\frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix},$$

as claimed.

**2.2.13** By Theorem 2.2.2,

$$\begin{aligned} \text{ref}_L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 2 \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2(u_1x_1 + u_2x_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (2u_1^2 - 1)x_1 + 2u_1u_2x_2 \\ 2u_1u_2x_1 + (2u_2^2 - 1)x_2 \end{bmatrix}. \end{aligned}$$

The matrix is  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$ . Note that the sum of the diagonal entries is  $a + d = 2(u_1^2 + u_2^2) - 2 = 0$ , since  $\vec{u}$  is a unit vector. It follows that  $d = -a$ . Since  $c = b$ ,  $A$  is of the form  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ . Also,  $a^2 + b^2 = (2u_1^2 - 1)^2 + 4u_1^2u_2^2 = 4u_1^4 - 4u_1^2 + 1 + 4u_1^2(1 - u_1^2) = 1$ , as claimed.

**2.2.14 a** Proceeding as on Page 57/58 in the text, we find that  $A$  is the matrix whose  $ij$ th entry is  $u_iu_j$ :

$$A = \begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2^2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3^2 \end{bmatrix}$$

**b** The sum of the diagonal entries is  $u_1^2 + u_2^2 + u_3^2 = 1$ , since  $\vec{u}$  is a unit vector.

**2.2.15** According to the discussion on Page 60 in the text,  $\text{ref}_L(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$

$$\begin{aligned}
 &= 2(x_1 u_1 + x_2 u_2 + x_3 u_3) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
 &= \begin{bmatrix} 2x_1 u_1^2 & +2x_2 u_2 u_1 & +2x_3 u_3 u_1 & -x_1 \\ 2x_1 u_1 u_2 & +2x_2 u_2^2 & +2x_3 u_3 u_2 & -x_2 \\ 2x_1 u_1 u_3 & +2x_2 u_2 u_3 & +2x_3 u_3^2 & -x_3 \end{bmatrix} = \begin{bmatrix} (2u_1^2 - 1)x_1 & +2u_2 u_1 x_2 & +2u_1 u_3 x_3 \\ 2u_1 u_2 x_1 & +(2u_2^2 - 1)x_2 & +2u_2 u_3 x_3 \\ 2u_1 u_3 x_1 & +2u_2 u_3 x_2 & +(2u_3^2 - 1)x_3 \end{bmatrix}. \\
 \text{So } A = & \begin{bmatrix} (2u_1^2 - 1) & 2u_2 u_1 & 2u_1 u_3 \\ 2u_1 u_2 & (2u_2^2 - 1) & 2u_2 u_3 \\ 2u_1 u_3 & 2u_2 u_3 & (2u_3^2 - 1) \end{bmatrix}.
 \end{aligned}$$

**2.2.16 a** See Figure 2.24.

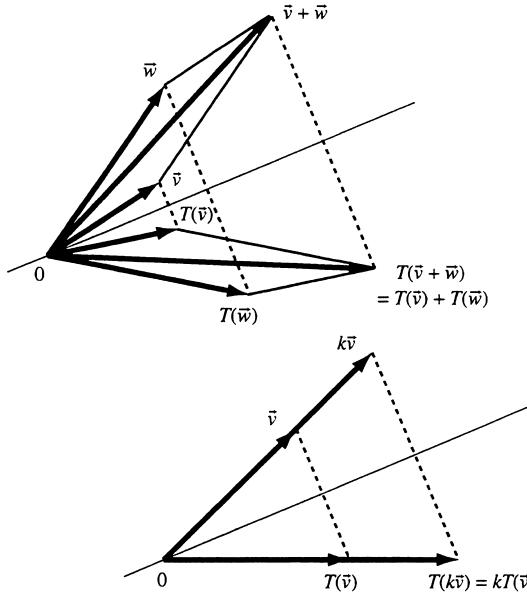


Figure 2.24: for Problem 2.2.16a.

b By Theorem 2.1.2, the matrix of  $T$  is  $[T(\vec{e}_1) \ T(\vec{e}_2)]$ .

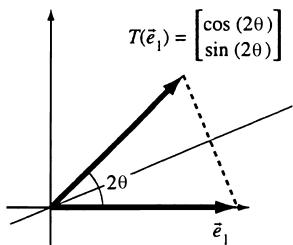


Figure 2.25: for Problem 2.2.16b.

$T(\vec{e}_2)$  is the unit vector in the fourth quadrant perpendicular to  $T(\vec{e}_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$ , so that

$T(\vec{e}_2) = \begin{bmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{bmatrix}$ . The matrix of  $T$  is therefore  $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ .

Alternatively, we can use the result of Exercise 13, with  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$  to find the matrix  $\begin{bmatrix} 2\cos^2 \theta - 1 & 2\cos \theta \sin \theta \\ 2\cos \theta \sin \theta & 2\sin^2 \theta - 1 \end{bmatrix}$ .

You can use trigonometric identities to show that the two results agree. (See Figure 2.25.)

**2.2.17** We want,  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 & +bv_2 \\ bv_1 & -av_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ .

Now,  $(a-1)v_1 + bv_2 = 0$  and  $bv_1 - (a+1)v_2 = 0$ , which is a system with solutions of the form  $\begin{bmatrix} bt \\ (1-a)t \end{bmatrix}$ , where  $t$  is an arbitrary constant.

Let's choose  $t = 1$ , making  $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix}$ .

Similarly, we want  $A\vec{w} = -\vec{v}$ . We perform a computation as above to reveal  $\vec{w} = \begin{bmatrix} a-1 \\ b \end{bmatrix}$  as a possible choice. A quick check of  $\vec{v} \cdot \vec{w} = 0$  reveals that they are indeed perpendicular.

Now, any vector  $\vec{x}$  in  $\mathbb{R}^2$  can be written in terms of components with respect to  $L = \text{span}(\vec{v})$  as  $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp} = c\vec{v} + d\vec{w}$ . Then,  $T(\vec{x}) = A\vec{x} = A(c\vec{v} + d\vec{w}) = A(c\vec{v}) + A(d\vec{w}) = cA\vec{v} + dA\vec{w} = c\vec{v} - d\vec{w} = \vec{x}^{\parallel} - \vec{x}^{\perp} = \text{ref}_L(\vec{x})$ , by Definition 2.2.2.

(The vectors  $\vec{v}$  and  $\vec{w}$  constructed above are both zero in the special case that  $a = 1$  and  $b = 0$ . In that case, we can let  $\vec{v} = \vec{e}_1$  and  $\vec{w} = \vec{e}_2$  instead.)

**2.2.18** From Exercise 17, we know that the reflection is about the line parallel to  $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.4 \end{bmatrix} = 0.4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .

So, every point on this line can be described as  $\begin{bmatrix} x \\ y \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ . So,  $y = k = \frac{1}{2}x$ , and  $y = \frac{1}{2}x$  is the line we are looking for.

**2.2.19**  $T(\vec{e}_1) = \vec{e}_1$ ,  $T(\vec{e}_2) = \vec{e}_2$ , and  $T(\vec{e}_3) = \vec{0}$ , so that the matrix is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

**2.2.20**  $T(\vec{e}_1) = \vec{e}_1$ ,  $T(\vec{e}_2) = -\vec{e}_2$ , and  $T(\vec{e}_3) = \vec{e}_3$ , so that the matrix is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

**2.2.21**  $T(\vec{e}_1) = \vec{e}_2$ ,  $T(\vec{e}_2) = -\vec{e}_1$ , and  $T(\vec{e}_3) = \vec{e}_3$ , so that the matrix is  $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . (See Figure 2.26.)

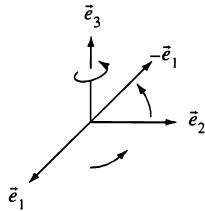


Figure 2.26: for Problem 2.2.21.

**2.2.22** Sketch the  $\vec{e}_1 - \vec{e}_3$  plane, as viewed from the positive  $\vec{e}_2$  axis.

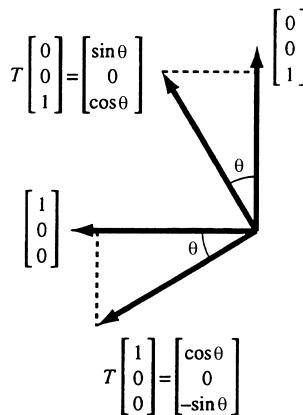


Figure 2.27: for Problem 2.2.22.

Since  $T(\vec{e}_2) = \vec{e}_2$ , the matrix is  $\begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$ . (See Figure 2.27.)

**2.2.23**  $T(\vec{e}_1) = \vec{e}_3$ ,  $T(\vec{e}_2) = \vec{e}_2$ , and  $T(\vec{e}_3) = \vec{e}_1$ , so that the matrix is  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ . (See Figure 2.28.)

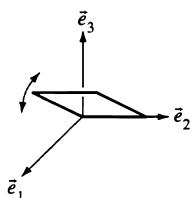


Figure 2.28: for Problem 2.2.23.

**2.2.24 a**  $A = [\vec{v} \quad \vec{w}]$ , so  $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}$  and  $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{w}$ . Since  $A$  preserves length, both  $\vec{v}$  and  $\vec{w}$  must be unit vectors. Furthermore, since  $A$  preserves angles and  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are clearly perpendicular,  $\vec{v}$  and  $\vec{w}$  must also

be perpendicular.

- b Since  $\vec{w}$  is a unit vector perpendicular to  $\vec{v}$ , it can be obtained by rotating  $\vec{v}$  through 90 degrees, either in the counterclockwise or in the clockwise direction. Using the corresponding rotation matrices, we see that  $\vec{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} -b \\ a \end{bmatrix}$  or  $\vec{w} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} b \\ -a \end{bmatrix}$ .

- c Following part b,  $A$  is either of the form  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , representing a rotation, or  $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , representing a reflection.

- 2.2.25** The matrix  $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$  represents a horizontal shear, and its inverse  $A^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$  represents such a shear as well, but “the other way.”

- 2.2.26** a  $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$ . So  $k = 4$  and  $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ .

- b This is the orthogonal projection onto the horizontal axis, with matrix  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .

- c  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -5b \\ 5a \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ . So  $a = \frac{4}{5}$ ,  $b = -\frac{3}{5}$ , and  $C = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$ . Note that  $a^2 + b^2 = 1$ , as required for a rotation matrix.

- d Since the  $x_1$  term is being modified, this must be a horizontal shear.

$$\text{Then } \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 + 3k \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}. \text{ So } k = 2 \text{ and } D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

- e  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7a + b \\ 7b - a \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$ . So  $a = -\frac{4}{5}$ ,  $b = \frac{3}{5}$ , and  $E = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ . Note that  $a^2 + b^2 = 1$ , as required for a reflection matrix.

- 2.2.27** Matrix  $B$  clearly represents a scaling.

Matrix  $C$  represents a projection, by Definition 2.2.1, with  $u_1 = 0.6$  and  $u_2 = 0.8$ .

Matrix  $E$  represents a shear, by Theorem 2.2.5.

Matrix  $A$  represents a reflection, by Definition 2.2.2.

Matrix  $D$  represents a rotation, by Definition 2.2.3.

- 2.2.28** a  $D$  is a scaling, being of the form  $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ .

- b  $E$  is the shear, since it is the only matrix which has the proper form (Theorem 2.2.5).

- c  $C$  is the rotation, since it fits Theorem 2.2.3.

d  $A$  is the projection, following the form given in Definition 2.2.1.

e  $F$  is the reflection, using Definition 2.2.2.

**2.2.29** To check that  $L$  is linear, we verify the two parts of Theorem 2.1.3:

a) Use the hint and apply  $L$  to both sides of the equation  $\vec{x} + \vec{y} = T(L(\vec{x}) + L(\vec{y}))$ :

$$L(\vec{x} + \vec{y}) = L(T(L(\vec{x}) + L(\vec{y}))) = L(\vec{x}) + L(\vec{y}) \text{ as claimed.}$$

$$\begin{array}{c} b) L(k\vec{x}) = L(kT(L(\vec{x}))) = L(T(kL(\vec{x}))) = kL(\vec{x}), \text{ as claimed} \\ \uparrow \quad \uparrow \\ \vec{x} = T(L(\vec{x})) \quad T \text{ is linear} \end{array}$$

**2.2.30** Write  $A = [\vec{v}_1 \quad \vec{v}_2]$ ; then  $A\vec{x} = [\vec{v}_1 \quad \vec{v}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2$ . We must choose  $\vec{v}_1$  and  $\vec{v}_2$  in such a way that  $x_1\vec{v}_1 + x_2\vec{v}_2$  is a scalar multiple of the vector  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , for all  $x_1$  and  $x_2$ . This is the case if (and only if) both  $\vec{v}_1$  and  $\vec{v}_2$  are scalar multiples of  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

For example, choose  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , so that  $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ .

**2.2.31** Write  $A = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$ ; then  $A\vec{x} = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$ .

We must choose  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$  in such a way that  $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$  is perpendicular to  $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  for all  $x_1, x_2$ , and  $x_3$ . This is the case if (and only if) all the vectors  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$  are perpendicular to  $\vec{w}$ , that is, if  $\vec{v}_1 \cdot \vec{w} = \vec{v}_2 \cdot \vec{w} = \vec{v}_3 \cdot \vec{w} = 0$ .

For example, we can choose  $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$  and  $\vec{v}_2 = \vec{v}_3 = \vec{0}$ , so that  $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

**2.2.32** a See Figure 2.29.

b Compute  $D\vec{v} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta \\ \sin \beta \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta \end{bmatrix}$ .

Comparing this result with our finding in part (a), we get the addition theorems

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

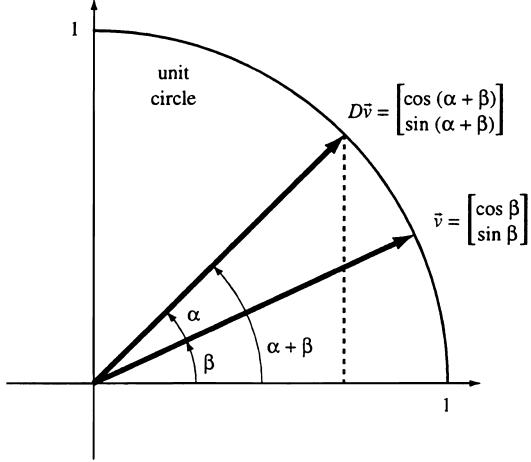


Figure 2.29: for Problem 2.2.32a.

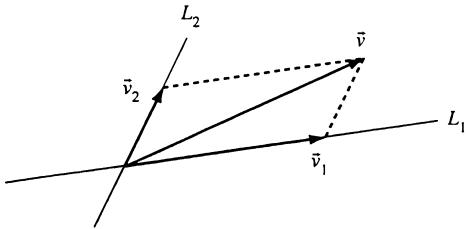


Figure 2.30: for Problem 2.2.33.

**2.2.33** Geometrically, we can find the representation  $\vec{v} = \vec{v}_1 + \vec{v}_2$  by means of a parallelogram, as shown in Figure 2.30.

To show the existence and uniqueness of this representation algebraically, choose a nonzero vector  $\vec{w}_1$  in  $L_1$  and a nonzero  $\vec{w}_2$  in  $L_2$ . Then the system  $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{0}$  or  $[\vec{w}_1 \ \vec{w}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$  has only the solution  $x_1 = x_2 = 0$  (if  $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{0}$  then  $x_1\vec{w}_1 = -x_2\vec{w}_2$  is both in  $L_1$  and in  $L_2$ , so that it must be the zero vector).

Therefore, the system  $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{v}$  or  $[\vec{w}_1 \ \vec{w}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{v}$  has a unique solution  $x_1, x_2$  for all  $\vec{v}$  in  $\mathbb{R}^2$  (by Theorem 1.3.4). Now set  $\vec{v}_1 = x_1\vec{w}_1$  and  $\vec{v}_2 = x_2\vec{w}_2$  to obtain the desired representation  $\vec{v} = \vec{v}_1 + \vec{v}_2$ . (Compare with Exercise 1.3.57.)

To show that the transformation  $T(\vec{v}) = \vec{v}_1$  is linear, we will verify the two parts of Theorem 2.1.3.

Let  $\vec{v} = \vec{v}_1 + \vec{v}_2$ ,  $\vec{w} = \vec{w}_1 + \vec{w}_2$ , so that  $\vec{v} + \vec{w} = (\vec{v}_1 + \vec{w}_1) + (\vec{v}_2 + \vec{w}_2)$  and  $k\vec{v} = k\vec{v}_1 + k\vec{v}_2$ .

$$\begin{array}{ccccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \text{in } L_1 & \text{in } L_2 & \text{in } L_1 & \text{in } L_2 & \text{in } L_1 & \text{in } L_2 & \text{in } L_1 \end{array}$$

a.  $T(\vec{v} + \vec{w}) = \vec{v}_1 + \vec{w}_1 = T(\vec{v}) + T(\vec{w})$ , and

b.  $T(k\vec{v}) = k\vec{v}_1 = kT(\vec{v})$ , as claimed.

**2.2.34** Keep in mind that the columns of the matrix of a linear transformation  $T$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  are  $T(\vec{e}_1)$ ,  $T(\vec{e}_2)$ , and  $T(\vec{e}_3)$ .

If  $T$  is the orthogonal projection onto a line  $L$ , then  $T(\vec{x})$  will be on  $L$  for all  $\vec{x}$  in  $\mathbb{R}^3$ ; in particular, the three columns of the matrix of  $T$  will be on  $L$ , and therefore pairwise parallel. This is the case only for matrix  $B$ :  $B$  represents an orthogonal projection onto a line.

A reflection transforms orthogonal vectors into orthogonal vectors; therefore, the three columns of its matrix must be pairwise orthogonal. This is the case only for matrix  $E$ :  $E$  represents the reflection about a line.

**2.2.35** If the vectors  $\vec{v}_1$  and  $\vec{v}_2$  are defined as shown in Figure 2.27, then the parallelogram  $P$  consists of all vectors of the form  $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ , where  $0 \leq c_1, c_2 \leq 1$ .

The image of  $P$  consists of all vectors of the form  $T(\vec{v}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$ .

These vectors form the parallelogram shown in Figure 2.31 on the right.

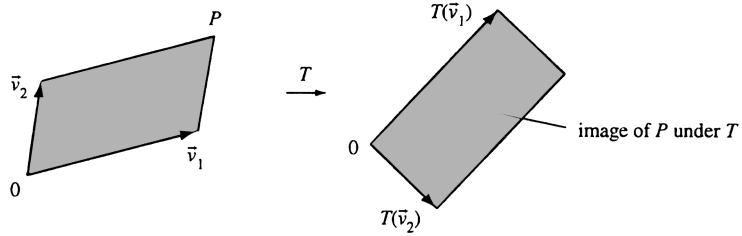


Figure 2.31: for Problem 2.2.35.

**2.2.36** If the vectors  $\vec{v}_0, \vec{v}_1$ , and  $\vec{v}_2$  are defined as shown in Figure 2.28, then the parallelogram  $P$  consists of all vectors  $\vec{v}$  of the form  $\vec{v} = \vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2$ , where  $0 \leq c_1, c_2 \leq 1$ .

The image of  $P$  consists of all vectors of the form  $T(\vec{v}) = T(\vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2) = T(\vec{v}_0) + c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$ .

These vectors form the parallelogram shown in Figure 2.32 on the right.

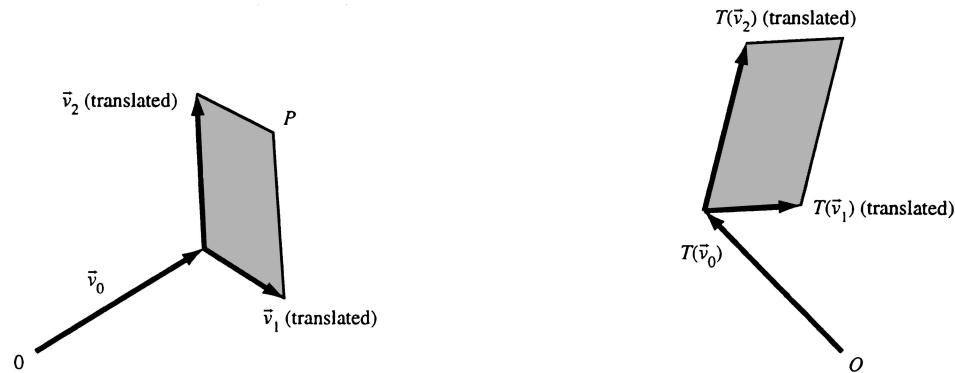


Figure 2.32: for Problem 2.2.36.

**2.2.37 a** By Definition 2.2.1, a projection has a matrix of the form  $\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$ , where  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  is a unit vector.

So the trace is  $u_1^2 + u_2^2 = 1$ .

- b By Definition 2.2.2, reflection matrices look like  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , so the trace is  $a - a = 0$ .
- c According to Theorem 2.2.3, a rotation matrix has the form  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ , so the trace is  $\cos \theta + \cos \theta = 2 \cos \theta$  for some  $\theta$ . Thus, the trace is in the interval  $[-2, 2]$ .
- d By Theorem 2.2.5, the matrix of a shear appears as either  $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$  or  $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ , depending on whether it represents a vertical or horizontal shear. In both cases, however, the trace is  $1 + 1 = 2$ .

**2.2.38** a  $A = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$ , so  $\det(A) = u_1^2 u_2^2 - u_1 u_2 u_1 u_2 = 0$ .

b  $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , so  $\det(A) = -a^2 - b^2 = -(a^2 + b^2) = -1$ .

c  $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , so  $\det(A) = a^2 - (-b^2) = a^2 + b^2 = 1$ .

d  $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$  or  $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ , both of which have determinant equal to  $1^2 - 0 = 1$ .

**2.2.39** a Note that  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ . The matrix  $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$  represents an orthogonal projection (Definition 2.2.1), with  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ . So,  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  represents a projection combined with a scaling by a factor of 2.

b This looks similar to a shear, with the one zero off the diagonal. Since the two diagonal entries are identical, we can write  $\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$ , showing that this matrix represents a vertical shear combined with a scaling by a factor of 3.

c We are asked to write  $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = k \begin{bmatrix} \frac{3}{k} & \frac{4}{k} \\ \frac{4}{k} & -\frac{3}{k} \end{bmatrix}$ , with our scaling factor  $k$  yet to be determined. This matrix,  $\begin{bmatrix} \frac{3}{k} & \frac{4}{k} \\ \frac{4}{k} & -\frac{3}{k} \end{bmatrix}$  has the form of a reflection matrix  $\left( \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \right)$ . This form further requires that  $1 = a^2 + b^2 = (\frac{3}{k})^2 + (\frac{4}{k})^2$ , or  $k = 5$ . Thus, the matrix represents a reflection combined with a scaling by a factor of 5.

**2.2.40**  $\vec{x} = \text{proj}_P \vec{x} + \text{proj}_Q \vec{x}$ , as illustrated in Figure 2.33.

**2.2.41**  $\text{ref}_Q \vec{x} = -\text{ref}_P \vec{x}$  since  $\text{ref}_Q \vec{x}$ ,  $\text{ref}_P \vec{x}$ , and  $\vec{x}$  all have the same length, and  $\text{ref}_Q \vec{x}$  and  $\text{ref}_P \vec{x}$  enclose an angle of  $2\alpha + 2\beta = 2(\alpha + \beta) = \pi$ . (See Figure 2.34.)

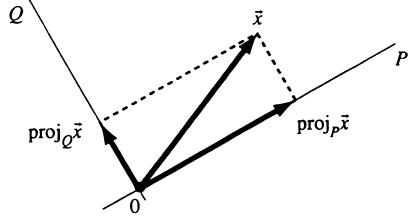


Figure 2.33: for Problem 2.2.40.

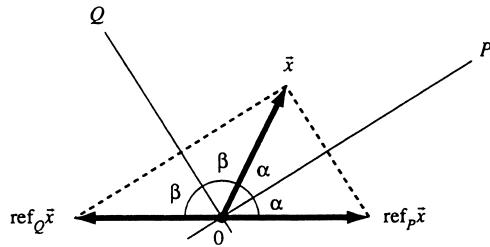


Figure 2.34: for Problem 2.2.41.

**2.2.42**  $T(\vec{x}) = T(T(\vec{x}))$  since  $T(\vec{x})$  is on  $L$  hence the projection of  $T(\vec{x})$  onto  $L$  is  $T(\vec{x})$  itself.

**2.2.43** Since  $\vec{y} = A\vec{x}$  is obtained from  $\vec{x}$  by a rotation through  $\theta$  in the counterclockwise direction,  $\vec{x}$  is obtained from  $\vec{y}$  by a rotation through  $\theta$  in the *clockwise* direction, that is, a rotation through  $-\theta$ . (See Figure 2.35.)

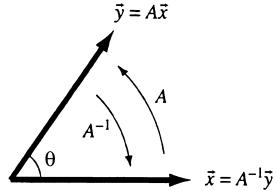


Figure 2.35: for Problem 2.2.43.

Therefore, the matrix of the inverse transformation is  $A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ . You can use the formula in Exercise 2.1.13b to check this result.

**2.2.44** By Exercise 1.1.13b,  $A^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2+b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ .

If  $A$  represents a rotation through  $\theta$  followed by a scaling by  $r$ , then  $A^{-1}$  represents a rotation through  $-\theta$  followed by a scaling by  $\frac{1}{r}$ . (See Figure 2.36.)

**2.2.45** By Exercise 2.1.13,  $A^{-1} = \frac{1}{a^2+b^2} \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \frac{1}{-(a^2+b^2)} \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = -1 \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ .

So  $A^{-1} = A$ , which makes sense. Reflecting a vector twice about the same line will return it to its original state.

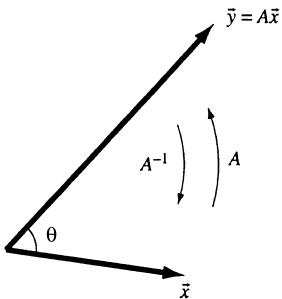


Figure 2.36: for Problem 2.2.44.

**2.2.46** We want to write  $A = k \begin{bmatrix} \frac{a}{k} & \frac{b}{k} \\ \frac{b}{k} & -\frac{a}{k} \end{bmatrix}$ , where the matrix  $B = \begin{bmatrix} \frac{a}{k} & \frac{b}{k} \\ \frac{b}{k} & -\frac{a}{k} \end{bmatrix}$  represents a reflection. It is required that  $(\frac{a}{k})^2 + (\frac{b}{k})^2 = 1$ , meaning that  $a^2 + b^2 = k^2$ , or,  $k = \sqrt{a^2 + b^2}$ . Now  $A^{-1} = \frac{1}{a^2+b^2} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \frac{1}{k^2} A = \frac{1}{k} B$ , for the reflection matrix  $B$  and the scaling factor  $k$  introduced above. In summary: If  $\tilde{A}$  represents a reflection combined with a scaling by  $k$ , then  $A^{-1}$  represents the same reflection combined with a scaling by  $\frac{1}{k}$ .

**2.2.47** Write  $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$ .

$$\begin{aligned} \text{a. } f(t) &= \left( T \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \right) \cdot \left( T \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \right) = \begin{bmatrix} a \cos t + b \sin t \\ c \cos t + d \sin t \end{bmatrix} \cdot \begin{bmatrix} -a \sin t + b \cos t \\ -c \sin t + d \cos t \end{bmatrix} \\ &= (a \cos t + b \sin t)(-a \sin t + b \cos t) + (c \cos t + d \sin t)(-c \sin t + d \cos t) \end{aligned}$$

This function  $f(t)$  is continuous, since  $\cos(t)$ ,  $\sin(t)$ , and constant functions are continuous, and sums and products of continuous functions are continuous.

$$\text{b. } f(\frac{\pi}{2}) = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T \begin{bmatrix} -1 \\ 0 \end{bmatrix} = - \left( T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \text{ since } T \text{ is linear.}$$

$$f(0) = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \text{ The claim follows.}$$

c. By part (b), the numbers  $f(0)$  and  $f(\frac{\pi}{2})$  have different signs (one is positive and the other negative), or they are both zero. Since  $f(t)$  is continuous, by part (a), we can apply the intermediate value theorem. (See Figure 2.37.)

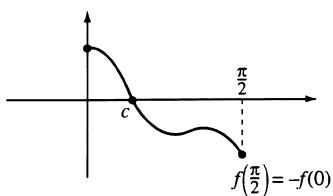


Figure 2.37: for Problem 2.2.47c.

d. Note that  $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$  and  $\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$  are perpendicular unit vectors, for any  $t$ . If we set

## Chapter 2

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$\vec{v}_1 = \begin{bmatrix} \cos(c) \\ \sin(c) \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} -\sin(c) \\ \cos(c) \end{bmatrix}$ , with the number  $c$  we found in part (c), then  $f(c) = T(\vec{v}_1) \cdot T(\vec{v}_2) = 0$ , so that  $T(\vec{v}_1)$  and  $T(\vec{v}_2)$  are perpendicular, as claimed. Note that  $T(\vec{v}_1)$  or  $T(\vec{v}_2)$  may be zero.

**2.2.48** We find

$$\begin{aligned} f(t) &= \left( \begin{bmatrix} 0 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \right) \cdot \left( \begin{bmatrix} 0 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \right) \\ &= \begin{bmatrix} 4 \sin(t) \\ 5 \cos(t) - 3 \sin(t) \end{bmatrix} \cdot \begin{bmatrix} 4 \cos(t) \\ -5 \sin(t) - 3 \cos(t) \end{bmatrix} \\ &= 15(\sin^2 t - \cos^2 t) = 15(2 \sin^2 t - 1). \text{ See Figure 2.38.} \end{aligned}$$

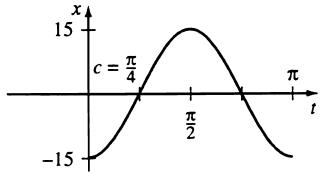


Figure 2.38: for Problem 2.2.48.

The only zero of  $f(t)$  between 0 and  $\frac{\pi}{2}$  is at  $t = \frac{\pi}{4}$ .

Therefore,  $\vec{v}_1 = \begin{bmatrix} \cos(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} -\sin(\frac{\pi}{4}) \\ \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$  work. Note that  $T(\vec{v}_1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

and  $T(\vec{v}_2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 \\ -8 \end{bmatrix}$  are indeed perpendicular. See Figure 2.39.

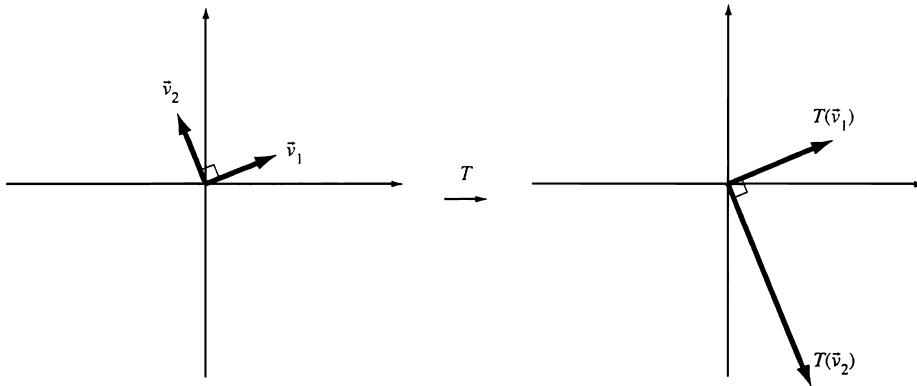


Figure 2.39: for Problem 2.2.48.

**2.2.49** If  $\vec{x} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$  then  $T(\vec{x}) = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} 5 \cos(t) \\ 2 \sin(t) \end{bmatrix} = \cos(t) \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \sin(t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ .

These vectors form an ellipse; consider the characterization of an ellipse given in the footnote on Page 69, with  $\vec{w}_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$  and  $\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ . (See Figure 2.40.)

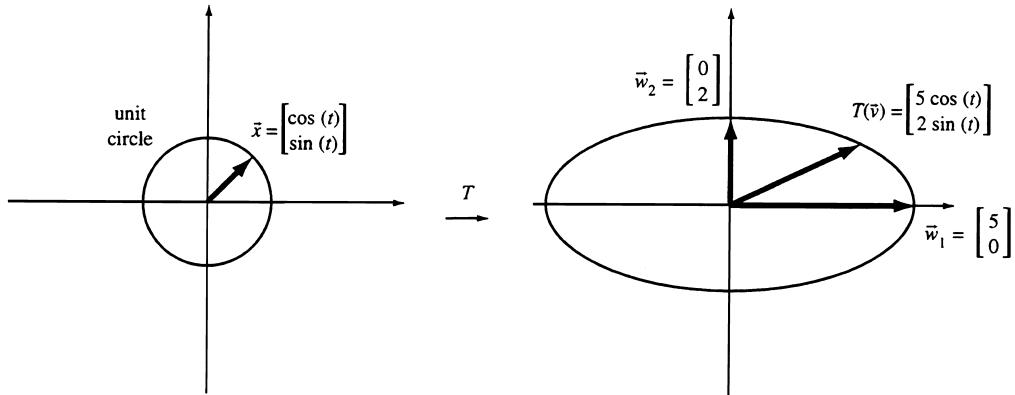


Figure 2.40: for Problem 2.2.49.

**2.2.50** Use the hint: Since the vectors on the unit circle are of the form  $\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2$ , the image of the unit circle consists of the vectors of the form  $T(\vec{v}) = T(\cos(t)\vec{v}_1 + \sin(t)\vec{v}_2) = \cos(t)T(\vec{v}_1) + \sin(t)T(\vec{v}_2)$ .

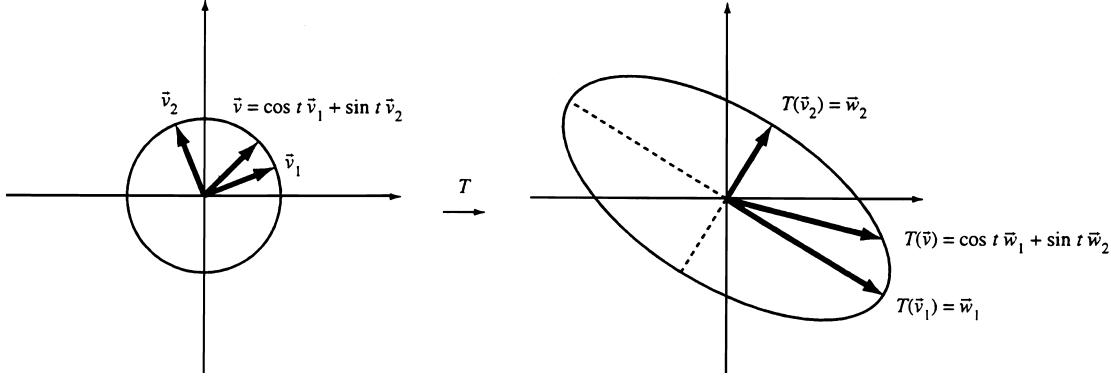


Figure 2.41: for Problem 2.2.50.

These vectors form an ellipse: Consider the characterization of an ellipse given in the footnote, with  $\vec{w}_1 = T(\vec{v}_1)$  and  $\vec{w}_2 = T(\vec{v}_2)$ . The key point is that  $T(\vec{v}_1)$  and  $T(\vec{v}_2)$  are perpendicular. See Figure 2.41.

**2.2.51** Consider the linear transformation  $T$  with matrix  $A = [\vec{w}_1 \quad \vec{w}_2]$ , that is,

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\vec{w}_1 \quad \vec{w}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \vec{w}_1 + x_2 \vec{w}_2.$$

The curve  $C$  is the image of the unit circle under the transformation  $T$ : if  $\vec{v} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$  is on the unit circle, then  $T(\vec{v}) = \cos(t)\vec{w}_1 + \sin(t)\vec{w}_2$  is on the curve  $C$ . Therefore,  $C$  is an ellipse, by Exercise 50. (See Figure 2.42.)

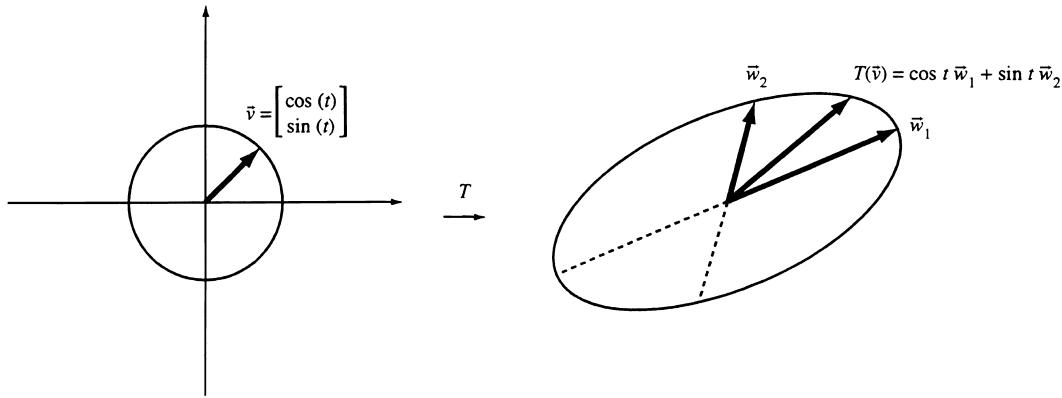


Figure 2.42: for Problem 2.2.51.

**2.2.52** By definition, the vectors  $\vec{v}$  on an ellipse  $E$  are of the form  $\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2$ , for some perpendicular vectors  $\vec{v}_1$  and  $\vec{v}_2$ . Then the vectors on the image  $C$  of  $E$  are of the form  $T(\vec{v}) = \cos(t)T(\vec{v}_1) + \sin(t)T(\vec{v}_2)$ . These vectors form an ellipse, by Exercise 51 (with  $\vec{w}_1 = T(\vec{v}_1)$  and  $\vec{w}_2 = T(\vec{v}_2)$ ). See Figure 2.43.

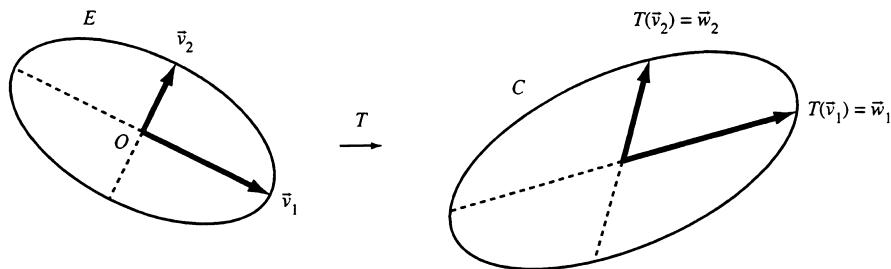


Figure 2.43: for Problem 2.2.52.

## Section 2.3

2.3.1  $\begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$

2.3.2  $\begin{bmatrix} 4 & 4 \\ -8 & -8 \end{bmatrix}$

2.3.3 Undefined

2.3.4  $\begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$

2.3.5  $\begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix}$

2.3.6  $\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$

2.3.7  $\begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}$

2.3.8  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.3.9  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.3.10  $[0 \ 1]$

2.3.11  $[10]$

2.3.12  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

2.3.13  $[h]$

2.3.14  $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ ,  $BC = [14 \ 8 \ 2]$ ,  $BD = [6]$ ,  $C^2 = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$ ,  $CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ ,  $DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ ,

$$DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, EB = [5 \ 10 \ 15], E^2 = [25]$$

2.3.15  $\left[ \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [3]}{[1 \ 3] \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [4][3]} \middle| \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [4]}{[1 \ 3] \begin{bmatrix} 0 \\ 0 \end{bmatrix} + [4][4]} \right] = \left[ \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}}{[19] \begin{bmatrix} 0 \\ 16 \end{bmatrix}} \right] = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 19 & 16 \end{bmatrix}$

2.3.16  $\left[ \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \middle| \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \right] = \left[ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} \middle| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \right] = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$

2.3.17 We must find all  $S$  such that  $SA = AS$ , or  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

So  $\begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$ , meaning that  $b = 2b$  and  $c = 2c$ , so  $b$  and  $c$  must be zero.

We see that all diagonal matrices (those of the form  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ ) commute with  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ .

2.3.18 As in Exercise 2.3.17, we let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Now we want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

So,  $\begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$ , revealing that  $c = 0$  (since  $a+2c = a$ ) and  $a = d$  (since  $b+2d = 2a+b$ ).

Thus  $B$  is any matrix of the form  $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ .

2.3.19 Again, let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . We want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Thus,  $\begin{bmatrix} 2b & -2a \\ 2d & -2c \end{bmatrix} = \begin{bmatrix} -2c & -2d \\ 2a & 2b \end{bmatrix}$ , meaning that  $c = -b$  and  $d = a$ .

We see that all matrices of the form  $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$  commute with  $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ .

**2.3.20** Following the form of Exercise 17, we let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Now we want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

So,  $\begin{bmatrix} 2a - 3b & 3a + 2b \\ 2c - 3d & 3c + 2d \end{bmatrix} = \begin{bmatrix} 2a + 3c & 2b + 3d \\ -3a + 2c & -3b + 2d \end{bmatrix}$ , revealing that  $a = d$  (since  $3a + 2b = 2b + 3d$ ) and  $-b = c$  (since  $2a + 3c = 2a - 3b$ ).

Thus  $B$  is any matrix of the form  $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ .

**2.3.21** Now we want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Thus,  $\begin{bmatrix} a + 2b & 2a - b \\ c + 2d & 2c - d \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ 2a - c & 2b - d \end{bmatrix}$ . So  $a + 2b = a + 2c$ , or  $c = b$ , and  $2a - b = b + 2d$ , revealing  $d = a - b$ . (The other two equations are redundant.)

All matrices of the form  $\begin{bmatrix} a & b \\ b & a - b \end{bmatrix}$  commute with  $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ .

**2.3.22** As in Exercise 17, we let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Now we want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

So,  $\begin{bmatrix} a + b & a + b \\ c + d & c + d \end{bmatrix} = \begin{bmatrix} a + c & b + d \\ a + c & b + d \end{bmatrix}$ , revealing that  $a = d$  (since  $a + b = b + d$ ) and  $b = c$  (since  $a + c = a + b$ ).

Thus  $B$  is any matrix of the form  $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ .

**2.3.23** We want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Then,  $\begin{bmatrix} a + 2b & 3a + 6b \\ c + 2d & 3c + 6d \end{bmatrix} = \begin{bmatrix} a + 3c & b + 3d \\ 2a + 6c & 2b + 6d \end{bmatrix}$ . So  $a + 2b = a + 3c$ , or  $c = \frac{2}{3}b$ , and  $3a + 6b = b + 3d$ , revealing  $d = a + \frac{5}{3}b$ . The other two equations are redundant.

Thus all matrices of the form  $\begin{bmatrix} a & b \\ \frac{2}{3}b & a + \frac{5}{3}b \end{bmatrix}$  commute with  $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ .

**2.3.24** Following the form of Exercise 2.3.17, we let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ .

Now we want  $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ .

So,  $\begin{bmatrix} 2a & 3b & 4c \\ 2d & 3e & 4f \\ 2g & 3h & 4i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{bmatrix}$ , which forces  $b, c, d, f, g$  and  $h$  to be zero.  $a, e$  and  $i$ , however, can be chosen freely.

Thus  $B$  is any matrix of the form  $\begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}$ .

**2.3.25** Now we want  $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ ,

or,  $\begin{bmatrix} 2a & 3b & 2c \\ 2d & 3e & 2f \\ 2g & 3h & 2i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 2g & 2h & 2i \end{bmatrix}$ . So,  $3b = 2b$ ,  $2d = 3d$ ,  $3f = 2f$  and  $3h = 2h$ , meaning that  $b, d, f$  and  $h$  must all be zero.

Thus all matrices of the form  $\begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{bmatrix}$  commute with  $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ .

**2.3.26** Following the form of Exercise 2.3.17, we let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ .

Then we want  $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ .

So,  $\begin{bmatrix} 2a & 2b & 3c \\ 2d & 2e & 3f \\ 2g & 2h & 3i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 3g & 3h & 3i \end{bmatrix}$ . Thus  $c, f, g$  and  $h$  must be zero, leaving  $B$  to be any matrix of the form  $\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & i \end{bmatrix}$ .

**2.3.27** We will prove that  $A(C + D) = AC + AD$ , repeatedly using Theorem 1.3.10a:  $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ .

Write  $B = [\vec{v}_1 \dots \vec{v}_m]$  and  $C = [\vec{w}_1 \dots \vec{w}_m]$ . Then

$$A(C + D) = A[\vec{v}_1 + \vec{w}_1 \dots \vec{v}_m + \vec{w}_m] = [A\vec{v}_1 + A\vec{w}_1 \dots A\vec{v}_m + A\vec{w}_m], \text{ and}$$

$$AC + AD = A[\vec{v}_1 \dots \vec{v}_m] + A[\vec{w}_1 \dots \vec{w}_m] = [A\vec{v}_1 + A\vec{w}_1 \dots A\vec{v}_m + A\vec{w}_m].$$

The results agree.

**2.3.28** The  $ij$ th entries of the three matrices are

$$\sum_{h=1}^p (ka_{ih})b_{hj}, \sum_{h=1}^p a_{ih}(kb_{hj}), \text{ and } k \left( \sum_{h=1}^p a_{ih}b_{hj} \right)$$

The three results agree.

**2.3.29 a**  $D_\alpha D_\beta$  and  $D_\beta D_\alpha$  are the same transformation, namely, a rotation through  $\alpha + \beta$ .

$$\begin{aligned} b \quad D_\alpha D_\beta &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} \end{aligned}$$

$D_\beta D_\alpha$  yields the same answer.

**2.3.30 a** See Figure 2.44.

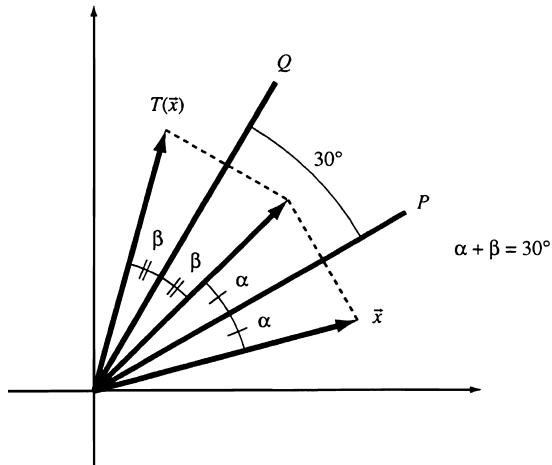


Figure 2.44: for Problem 2.4.30.

The vectors  $\vec{x}$  and  $T(\vec{x})$  have the same length (since reflections leave the length unchanged), and they enclose an angle of  $2(\alpha + \beta) = 2 \cdot 30^\circ = 60^\circ$

b Based on the answer in part (a), we conclude that  $T$  is a rotation through  $60^\circ$ .

c The matrix of  $T$  is  $\begin{bmatrix} \cos(60^\circ) & -\sin(60^\circ) \\ \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ .

**2.3.31** Write  $A$  in terms of its rows:  $A = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \dots \\ \vec{w}_n \end{bmatrix}$  (suppose  $A$  is  $n \times m$ ).

We can think of this as a partition into  $n$

$$1 \times m \text{ matrices. Now } AB = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \dots \\ \vec{w}_n \end{bmatrix} B = \begin{bmatrix} \vec{w}_1 B \\ \vec{w}_2 B \\ \dots \\ \vec{w}_n B \end{bmatrix} \text{ (a product of partitioned matrices).}$$

We see that the  $i$ th row of  $AB$  is the product of the  $i$ th row of  $A$  and the matrix  $B$ .

**2.3.32** Let  $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then we want  $X \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X$ , or  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , or  $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ , meaning that  $b = c = 0$ . Also, we want  $X \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X$ , or  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ , or  $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$  so  $a = d$ . Thus,  $X = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI_2$  must be a multiple of the identity matrix. ( $X$  will then commute with any  $2 \times 2$  matrix  $M$ , since  $XM = aM = MX$ .)

**2.3.33**  $A^2 = I_2$ ,  $A^3 = A$ ,  $A^4 = I_2$ . The power  $A^n$  alternates between  $A = -I_2$  and  $I_2$ . The matrix  $A$  describes a reflection about the origin. Alternatively one can say  $A$  represents a rotation by  $180^\circ = \pi$ . Since  $A^2$  is the identity,  $A^{1000}$  is the identity and  $A^{1001} = A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ .

**2.3.34**  $A^2 = I_2$ ,  $A^3 = A$ ,  $A^4 = I_2$ . The power  $A^n$  alternates between  $A$  and  $I_2$ . The matrix  $A$  describes a reflection about the  $x$  axis. Because  $A^2$  is the identity,  $A^{1000}$  is the identity and  $A^{1001} = A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

**2.3.35**  $A^2 = I_2$ ,  $A^3 = A$ ,  $A^4 = I_2$ . The power  $A^n$  alternates between  $A$  and  $I_2$ . The matrix  $A$  describes a reflection about the diagonal  $x = y$ . Because  $A^2$  is the identity,  $A^{1000}$  is the identity and  $A^{1001} = A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

**2.3.36**  $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ ,  $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$  and  $A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ . The power  $A^n$  represents a horizontal shear along the  $x$ -axis. The shear strength increases linearly in  $n$ . We have  $A^{1001} = \begin{bmatrix} 1 & 1001 \\ 0 & 1 \end{bmatrix}$ .

**2.3.37**  $A^2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ ,  $A^3 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$  and  $A^4 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$ . The power  $A^n$  represents a vertical shear along the  $y$  axis. The shear magnitude increases linearly in  $n$ . We have  $A^{1001} = \begin{bmatrix} 1 & 0 \\ -1001 & 1 \end{bmatrix}$ .

**2.3.38**  $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $A^3 = -A$ ,  $A^4 = I_2$ . The matrix  $A$  represents the rotation through  $\pi/2$  in the counterclockwise direction. Since  $A^4$  is the identity matrix, we know that  $A^{1000}$  is the identity matrix and  $A^{1001} = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

**2.3.39**  $A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ,  $A^3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ ,  $A^4 = -I_2$ . The matrix  $A$  describes a rotation by  $\pi/4$  in the clockwise direction. Because  $A^8$  is the identity matrix, we know that  $A^{1000}$  is the identity matrix and  $A^{1001} =$

$$A = (1/\sqrt{2}) \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

**2.3.40**  $A^2 = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$ ,  $A^3 = I_2$ ,  $A^4 = A$ . The matrix  $A$  describes a rotation by  $120^\circ = 2\pi/3$  in the counterclockwise direction. Because  $A^3$  is the identity matrix, we know that  $A^{999}$  is the identity matrix and  $A^{1001} = A^2 = A^{-1} = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$ .

**2.3.41**  $A^2 = I_2$ ,  $A^3 = A$ ,  $A^4 = I_2$ . The power  $A^n$  alternates between  $I_2$  for even  $n$  and  $A$  for odd  $n$ . Therefore  $A^{1001} = A$ . The matrix represents a reflection about a line.

**2.3.42**  $A^n = A$ . The matrix  $A$  represents a projection on the line  $x = y$  spanned by the vector  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . We have  $A^{1001} = A = (1/2) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .

**2.3.43** An example is  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , representing the reflection about the horizontal axis.

**2.3.44** A rotation by  $\pi/2$  given by the matrix  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

**2.3.45** For example,  $A = (1/2) \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ , the rotation through  $2\pi/3$ . See Problem 2.3.40.

**2.3.46** For example,  $A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ , the orthogonal projection onto the line spanned by  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**2.3.47** For example,  $A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ , the orthogonal projection onto the line spanned by  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**2.3.48** For example, the shear  $A = \begin{bmatrix} 1 & 1/10 \\ 0 & 1 \end{bmatrix}$ .

**2.3.49**  $AF = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  represents the reflection about the  $x$ -axis, while  $FA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  represents the reflection about the  $y$ -axis. (See Figure 2.45.)

**2.3.50**  $CG = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  represents a reflection about the line  $x = y$ , while  $GC = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$  represents a reflection about the line  $x = -y$ . (See Figure 2.46.)

**2.3.51**  $FJ = JF = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$  both represent a rotation through  $3\pi/4$  combined with a scaling by  $\sqrt{2}$ . (See Figure 2.47.)

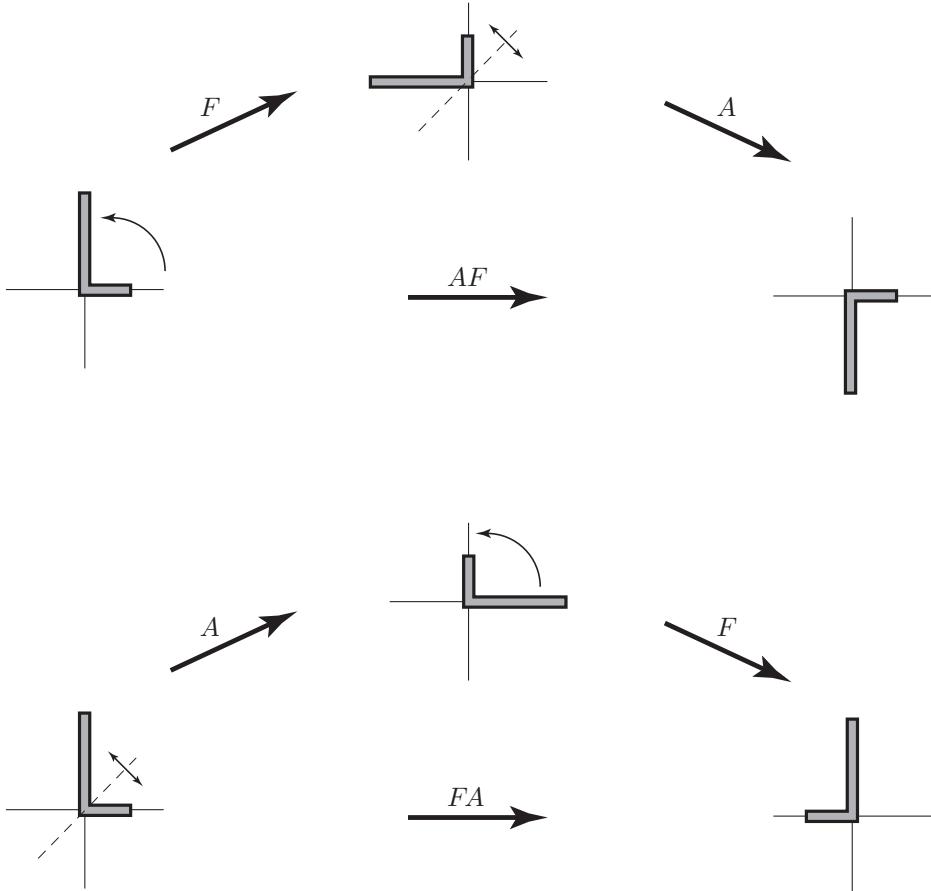


Figure 2.45: for Problem 2.3.49.

**2.3.52**  $JH = HJ = \begin{bmatrix} 0.2 & -1.4 \\ 1.4 & 0.2 \end{bmatrix}$ . Since  $H$  represents a rotation and  $J$  represents a rotation through  $\pi/4$  combined with a scaling by  $\sqrt{2}$ , the products in either order will be the same, representing a rotation combined with a scaling by  $\sqrt{2}$ . (See Figure 2.48.)

**2.3.53**  $CD = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  represents the rotation through  $\pi/2$ , while  $DC = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  represents the rotation through  $-\pi/2$ . (See Figure 2.49.)

**2.3.54**  $BE = \begin{bmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{bmatrix}$  represents the rotation through the angle  $\theta = \arccos(-0.6) \approx 2.21$ , while  $EB = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$  represents the rotation through  $-\theta$ . (See Figure 2.50.)

**2.3.55** We need to solve the matrix equation

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

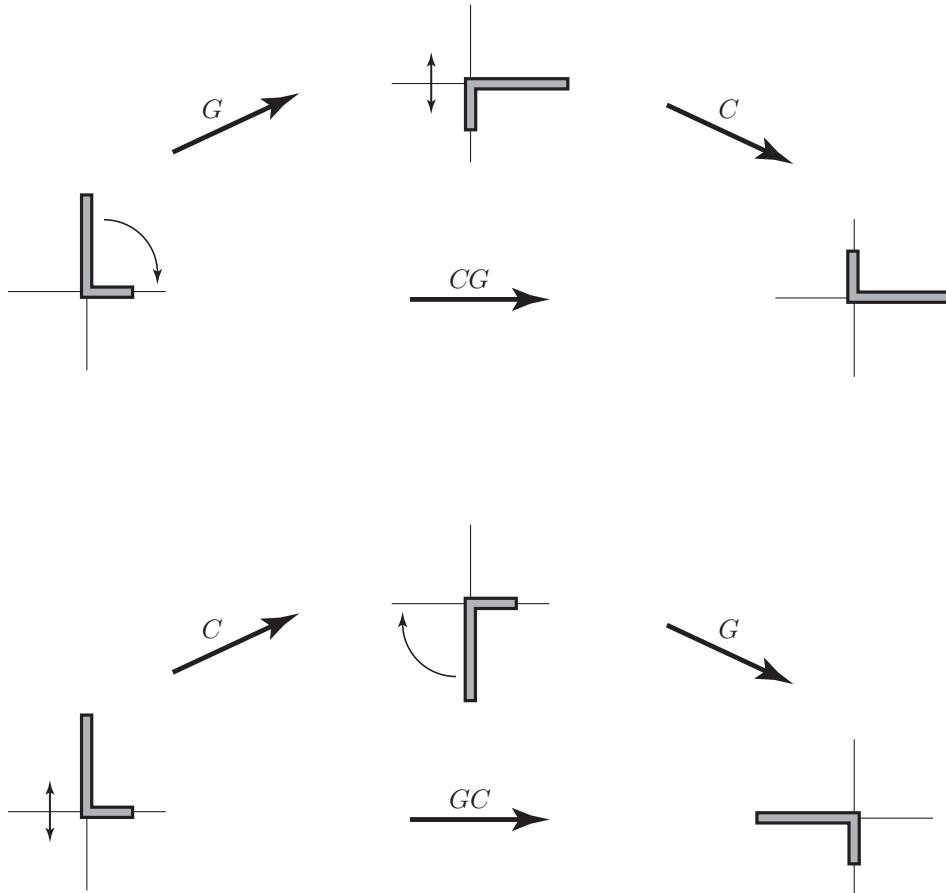


Figure 2.46: for Problem 2.3.50.

which amounts to solving the system  $a + 2c = 0$ ,  $2a + 4c = 0$ ,  $b + 2d = 0$  and  $2b + 4d = 0$ . The solutions are of the form  $a = -2c$  and  $b = -2d$ . Thus  $X = \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix}$ , where  $c, d$  are arbitrary constants.

**2.3.56** Proceeding as in Exercise 55, we find  $X = \begin{bmatrix} -2b & b \\ -2d & d \end{bmatrix}$ , where  $b$  and  $d$  are arbitrary.

**2.3.57** We need to solve the matrix equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which amounts to solving the system  $a + 2c = 1$ ,  $3a + 5c = 0$ ,  $b + 2d = 0$  and  $3b + 5d = 1$ . The solution is  $X = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ .

**2.3.58** Proceeding as in Exercise 57, we find  $X = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ .

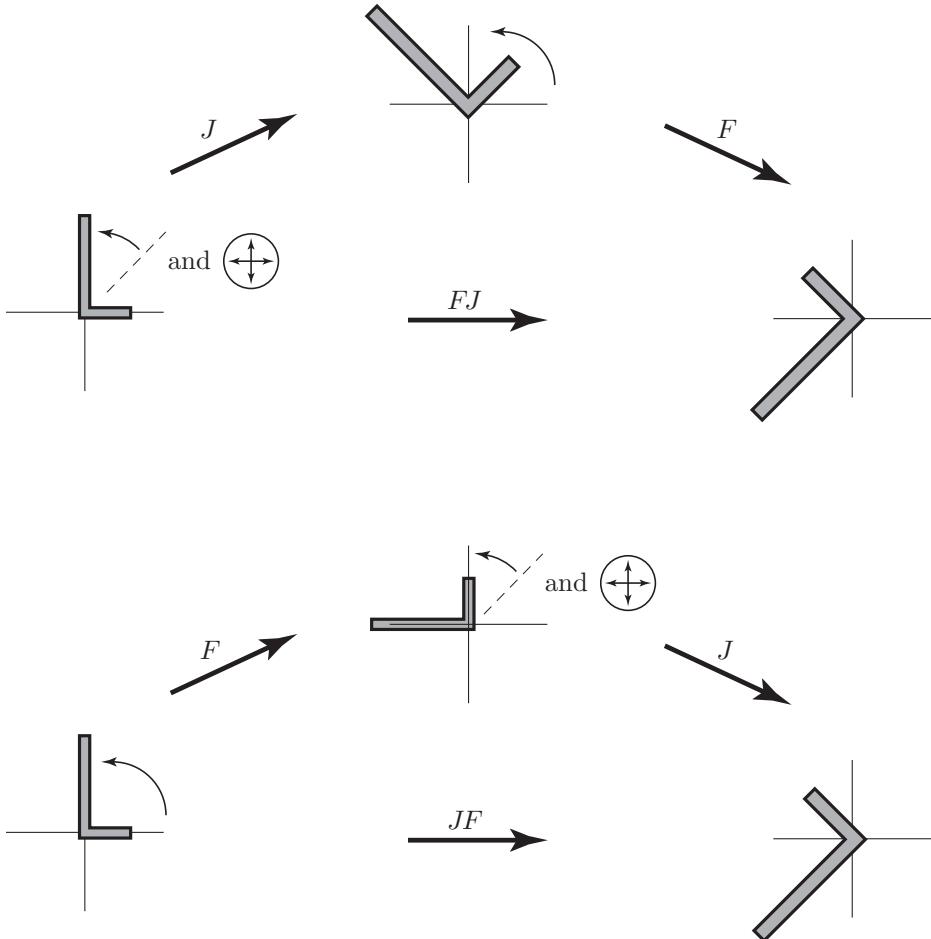


Figure 2.47: for Problem 2.3.51.

**2.3.59** The matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has no solutions, since we have the inconsistent equations  $2a + 4b = 1$  and  $a + 2b = 0$ .

**2.3.60** Proceeding as in Exercise 59, we find that this equation has no solutions.

**2.3.61** We need to solve the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which amounts to solving the system  $a + 2c + 3e = 0$ ,  $c + 2e = 0$ ,  $b + 2d + 3f = 0$  and  $d + 2f = 1$ . The solutions are of the form  $X = \begin{bmatrix} e+1 & f-2 \\ -2e & 1-2f \\ e & f \end{bmatrix}$ , where  $e, f$  are arbitrary constants.

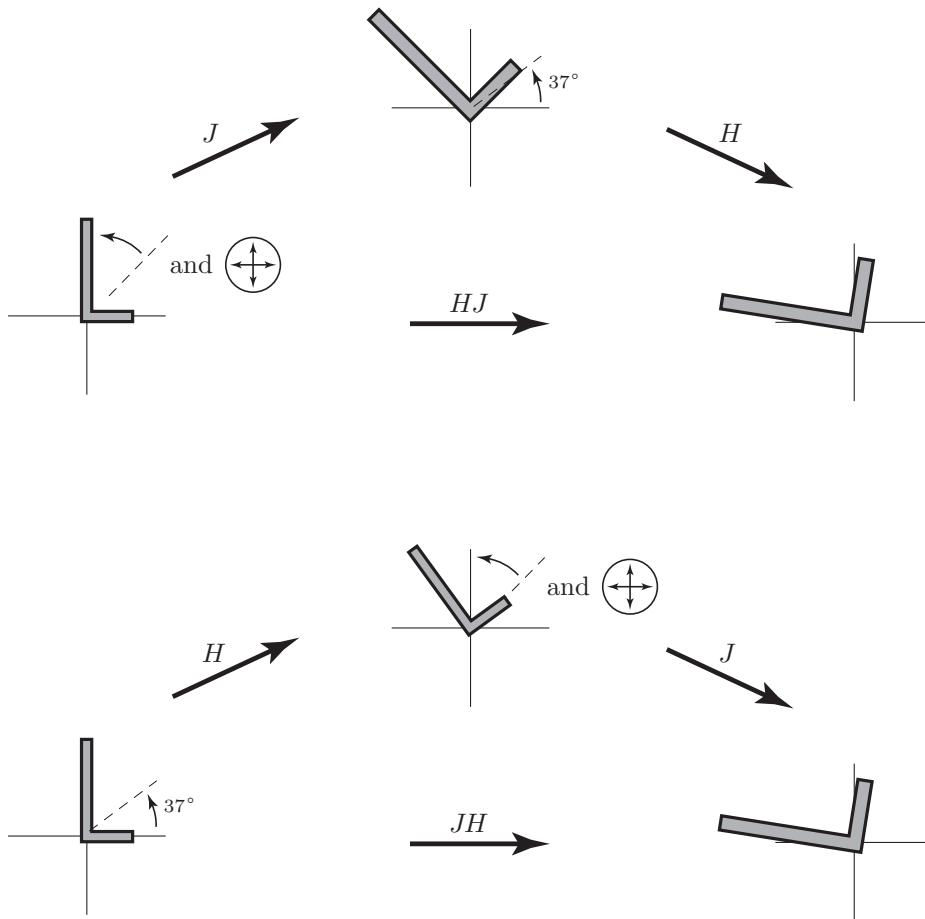


Figure 2.48: for Problem 2.3.52.

**2.3.62** Proceeding as in Exercise 61, we find  $X = \begin{bmatrix} e - 5/3 & f + 2/3 \\ -2e + 4/3 & -2f - 1/3 \\ e & f \end{bmatrix}$ , where  $e, f$  are arbitrary constants.

**2.3.63** The matrix equation

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has no solutions, since we have the inconsistent equations  $a + 4d = 1$ ,  $2a + 5d = 0$ , and  $3a + 6d = 0$ .

**2.3.64** The matrix equation

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has no solutions, since we have the inconsistent equations  $a = 1$ ,  $2a + d = 0$  and  $3a + 2d = 0$ .

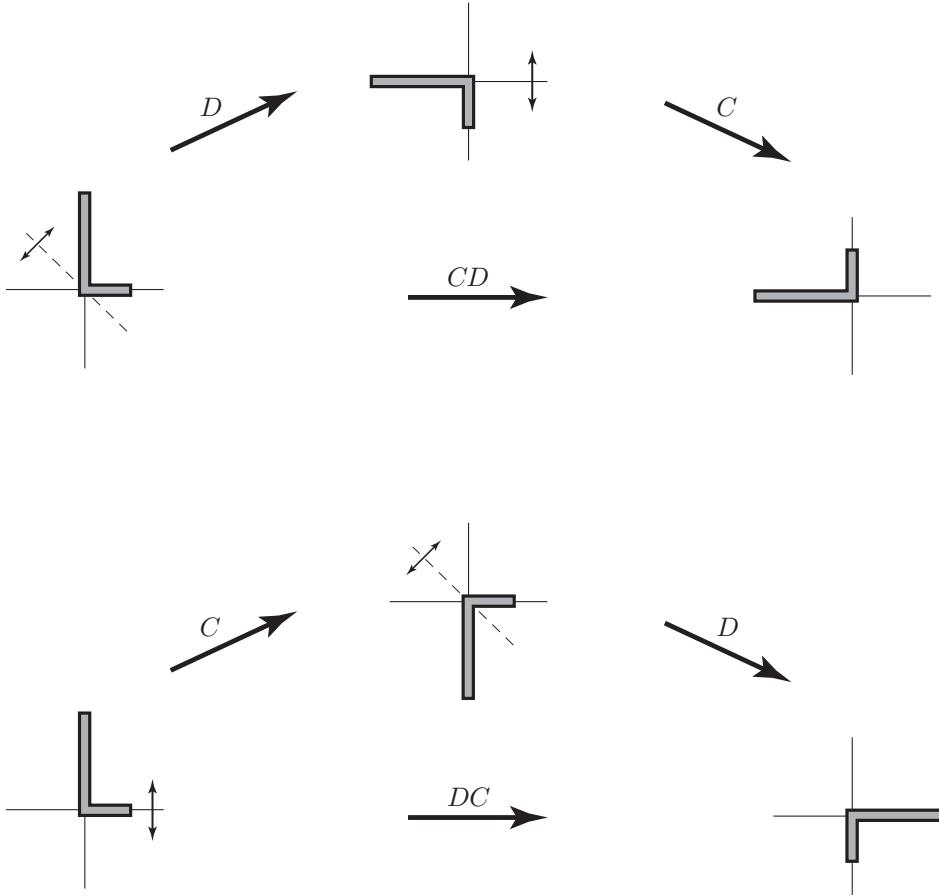


Figure 2.49: for Problem 2.3.53.

**2.3.65** With  $X = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ , we have to solve  $X^2 = \begin{bmatrix} a^2 & ab + bc \\ 0 & c^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . This means  $a = 0, c = 0$  and  $b$  can be arbitrary. The general solution is  $X = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ .

**2.3.66** If  $X = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$  then the diagonal entries of  $X^3$  will be  $a^3, c^3$ , and  $f^3$ . Since we want  $X^3 = 0$ , we must have  $a = c = f = 0$ . If  $X = \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{bmatrix}$ , then a direct computation shows that  $X^3 = 0$ . Thus the solutions are of the form  $X = \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{bmatrix}$ , where  $b, d, e$  are arbitrary.

**2.3.67** For a horizontal shear,  $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ , we have  $(A - I_2)^2 = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . Note that  $A\vec{x} - \vec{x} =$

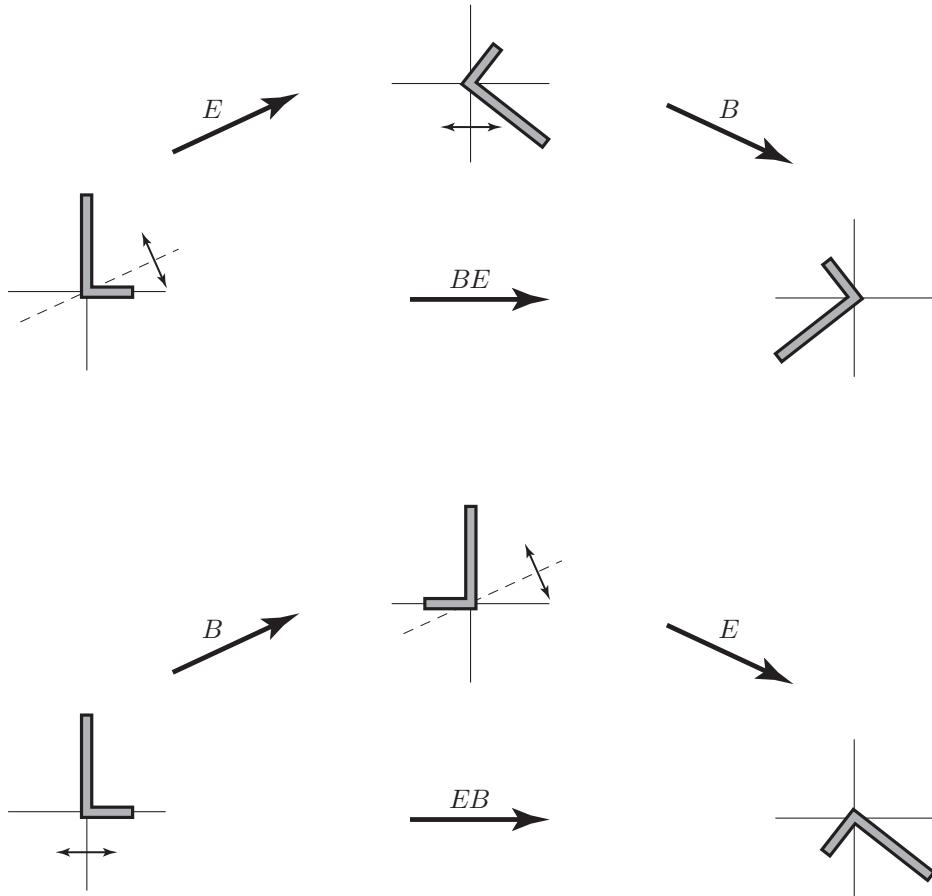
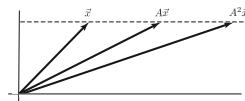


Figure 2.50: for Problem 2.3.54.

$A^2\vec{x} - A\vec{x}$  for all vectors  $\vec{x}$ , as illustrated in the accompanying figure. This equation means that  $A^2\vec{x} - 2A\vec{x} + \vec{x} = (A - I_2)^2\vec{x} = \vec{0}$ . Analogous results hold for vertical shears.



**2.3.68** Let  $\vec{v}_1, \dots, \vec{v}_n$  be the columns of the matrix  $X$ . Solving the matrix equation  $AX = I_n$  amounts to solving the linear systems  $A\vec{v}_i = \vec{e}_i$  for  $i = 1, \dots, n$ . Since  $A$  is a  $n \times m$  matrix of rank  $n$ , all these systems are consistent, so that the matrix equation  $AX = I_n$  does have at least one solution. If  $n < m$ , then each of the systems  $A\vec{v}_i = \vec{e}_i$  has infinitely many solutions, so that the matrix equation  $AX = I_n$  has infinitely many solutions as well. See the examples in Exercises 2.3.57, 2.3.61 and 2.3.62.

**2.3.69** Let  $\vec{v}_1, \dots, \vec{v}_n$  be the columns of the matrix  $X$ . Solving the matrix equation  $AX = I_n$  amounts to solving the linear systems  $A\vec{v}_i = \vec{e}_i$  for  $i = 1, \dots, n$ . Since  $A$  is an  $n \times n$  matrix of rank  $n$ , all these systems have a unique solution, by Theorem 1.3.4, so that the matrix equation  $AX = I_n$  has a unique solution as well.

## Section 2.4

2.4.1 rref  $\left[ \begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 5 & 8 & 0 & 1 \end{array} \right] = \left[ \begin{array}{cc|cc} 1 & 0 & 8 & -3 \\ 0 & 1 & -5 & 2 \end{array} \right]$ , so that  $\left[ \begin{array}{cc} 2 & 3 \\ 5 & 8 \end{array} \right]^{-1} = \left[ \begin{array}{cc} 8 & -3 \\ -5 & 2 \end{array} \right]$ .

2.4.2 rref  $\left[ \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] = \left[ \begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right]$ , so that  $\left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$  fails to be invertible.

2.4.3 rref  $\left[ \begin{array}{cc|cc} 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] = \left[ \begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{array} \right]$ , so that  $\left[ \begin{array}{cc} 0 & 2 \\ 1 & 1 \end{array} \right]^{-1} = \left[ \begin{array}{cc} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{array} \right]$ .

2.4.4 Use Theorem 2.4.5; the inverse is  $\left[ \begin{array}{ccc} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{array} \right]$ .

2.4.5 rref  $\left[ \begin{array}{ccc} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$ , so that the matrix fails to be invertible, by Theorem 2.4.3.

2.4.6 Use Theorem 2.4.5; the inverse is  $\left[ \begin{array}{ccc} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array} \right]$ .

2.4.7 rref  $\left[ \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$ , so that the matrix fails to be invertible, by Theorem 2.4.3.

2.4.8 Use Theorem 2.4.5; the inverse is  $\left[ \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right]$ .

2.4.9 rref  $\left[ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$ , so that the matrix fails to be invertible, by Theorem 2.4.3.

2.4.10 Use Theorem 2.4.5; the inverse is  $\left[ \begin{array}{ccc} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{array} \right]$ .

2.4.11 Use Theorem 2.4.5; the inverse is  $\left[ \begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$ .

2.4.12 Use Theorem 2.4.5; the inverse is  $\begin{bmatrix} 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 \\ -2 & 6 & 1 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix}$

2.4.13 Use Theorem 2.4.5; the inverse is  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$ .

2.4.14 Use Theorem 2.4.5; the inverse is  $\begin{bmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$ .

2.4.15 Use Theorem 2.4.5; the inverse is  $\begin{bmatrix} -6 & 9 & -5 & 1 \\ 9 & -1 & -5 & 2 \\ -5 & -5 & 9 & -3 \\ 1 & 2 & -3 & 1 \end{bmatrix}$

2.4.16 Solving for  $x_1$  and  $x_2$  in terms of  $y_1$  and  $y_2$  we find that

$$\begin{aligned} x_1 &= -8y_1 + 5y_2 \\ x_2 &= 5y_1 - 3y_2 \end{aligned}$$

2.4.17 We make an attempt to solve for  $x_1$  and  $x_2$  in terms of  $y_1$  and  $y_2$ :

$$\left[ \begin{array}{cc|c} x_1 + 2x_2 & = & y_1 \\ 4x_1 + 8x_2 & = & y_2 \end{array} \right] \xrightarrow{-4(I)} \left[ \begin{array}{cc|c} x_1 + 2x_2 & = & y_1 \\ 0 & = & -4y_1 + y_2 \end{array} \right].$$

This system has no solutions  $(x_1, x_2)$  for some  $(y_1, y_2)$ , and infinitely many solutions for others; the transformation fails to be invertible.

2.4.18 Solving for  $x_1, x_2$ , and  $x_3$  in terms of  $y_1, y_2$ , and  $y_3$  we find that

$$\begin{aligned} x_1 &= y_3 \\ x_2 &= y_1 \\ x_3 &= y_2 \end{aligned}$$

2.4.19 Solving for  $x_1, x_2$ , and  $x_3$  in terms of  $y_1, y_2$ , and  $y_3$ , we find that

$$\begin{aligned} x_1 &= 3y_1 - \frac{5}{2}y_2 + \frac{1}{2}y_3 \\ x_2 &= -3y_1 + 4y_2 - y_3 \\ x_3 &= y_1 - \frac{3}{2}y_2 + \frac{1}{2}y_3 \end{aligned}$$

2.4.20 Solving for  $x_1, x_2$ , and  $x_3$  in terms of  $y_1, y_2$ , and  $y_3$  we find that

$$\begin{aligned} x_1 &= -8y_1 - 15y_2 + 12y_3 \\ x_2 &= 4y_1 + 6y_2 - 5y_3 \\ x_3 &= -y_1 - y_2 + y_3 \end{aligned}$$

2.4.21  $f(x) = x^2$  fails to be invertible, since the equation  $f(x) = x^2 = 1$  has two solutions,  $x = \pm 1$ .

**2.4.22**  $f(x) = 2^x$  fails to be invertible, since the equation  $f(x) = 2^x = 0$  has no solution  $x$ .

**2.4.23** Note that  $f'(x) = 3x^2 + 1$  is always positive; this implies that the function  $f(x) = x^3 + x$  is increasing throughout. Therefore, the equation  $f(x) = b$  has at most one solution  $x$  for all  $b$ . (See Figure 2.51.)

Now observe that  $\lim_{x \rightarrow \infty} f(x) = \infty$  and  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ ; this implies that the equation  $f(x) = b$  has at least one solution  $x$  for a given  $b$  (for a careful proof, use the intermediate value theorem; compare with Exercise 2.2.47c).

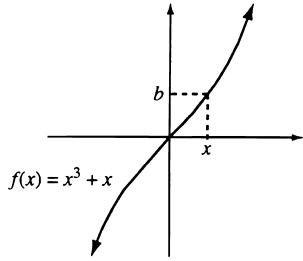


Figure 2.51: for Problem 2.3.23.

**2.4.24** We can write  $f(x) = x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$ .

The equation  $f(x) = 0$  has three solutions,  $x = 0, 1, -1$ , so that  $f(x)$  fails to be invertible.

**2.4.25** Invertible, with inverse  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt[3]{y_1} \\ y_2 \end{bmatrix}$

**2.4.26** Invertible, with inverse  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt[3]{y_2 - y_1} \\ y_1 \end{bmatrix}$

**2.4.27** This transformation fails to be invertible, since the equation  $\begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  has no solution.

**2.4.28** We are asked to find the inverse of the matrix  $A = \begin{bmatrix} 22 & 13 & 8 & 3 \\ -16 & -3 & -2 & -2 \\ 8 & 9 & 7 & 2 \\ 5 & 4 & 3 & 1 \end{bmatrix}$ .

We find that  $A^{-1} = \begin{bmatrix} 1 & -2 & 9 & -25 \\ -2 & 5 & -22 & 60 \\ 4 & -9 & 41 & -112 \\ -9 & 17 & 80 & 222 \end{bmatrix}$ .

$T^{-1}$  is the transformation from  $\mathbb{R}^4$  to  $\mathbb{R}^4$  with matrix  $A^{-1}$ .

**2.4.29** Use Theorem 2.4.3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix} \xrightarrow{-I} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 3 & k^2-1 \end{bmatrix} \xrightarrow{-II} \begin{bmatrix} 1 & 0 & 2-k \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-3k+2 \end{bmatrix} \xrightarrow{-3(II)}$$

The matrix is invertible if (and only if)  $k^2 - 3k + 2 = (k-2)(k-1) \neq 0$ , in which case we can further reduce it to  $I_3$ . Therefore, the matrix is invertible if  $k \neq 1$  and  $k \neq 2$ .

**2.4.30** Use Theorem 2.4.3:

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix} \xrightarrow{I \leftrightarrow II} \begin{bmatrix} -1 & 0 & c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\div(-1)} \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{+b(I) + c(II)} \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix fails to be invertible, regardless of the values of  $b$  and  $c$ .

**2.4.31** Use Theorem 2.4.3; first assume that  $a \neq 0$ .

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\text{swap}: I \leftrightarrow II} \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\div(-a)} \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{+b(I)} \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & a & b \\ 0 & -c & -\frac{bc}{a} \end{bmatrix} \xrightarrow{\div a} \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & 1 & \frac{b}{a} \\ 0 & -c & -\frac{bc}{a} \end{bmatrix} \xrightarrow{+c(II)} \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & 0 \end{bmatrix}$$

Now consider the case when  $a = 0$ :

$$\begin{bmatrix} 0 & 0 & b \\ 0 & 0 & c \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\text{swap}: I \leftrightarrow III} \begin{bmatrix} -b & -c & 0 \\ 0 & 0 & c \\ 0 & 0 & b \end{bmatrix} : \text{The second entry on the diagonal of rref will be 0.}$$

It follows that the matrix  $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$  fails to be invertible, regardless of the values of  $a, b$ , and  $c$ .

**2.4.32** Use Theorem 2.4.9.

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is a matrix such that  $ad - bc = 1$  and  $A^{-1} = A$ , then

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ so that } b = 0, c = 0, \text{ and } a = d.$$

The condition  $ad - bc = a^2 = 1$  now implies that  $a = d = 1$  or  $a = d = -1$ .

This leaves only two matrices  $A$ , namely,  $I_2$  and  $-I_2$ . Check that these two matrices do indeed satisfy the given requirements.

**2.4.33** Use Theorem 2.4.9.

The requirement  $A^{-1} = A$  means that  $-\frac{1}{a^2+b^2} \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ . This is the case if (and only if)  $a^2 + b^2 = 1$ .

**2.4.34** a By Theorem 2.4.3,  $A$  is invertible if (and only if)  $a, b$ , and  $c$  are all nonzero. In this case,  $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$ .

b In general, a diagonal matrix is invertible if (and only if) all of its diagonal entries are nonzero.

**2.4.35** a  $A$  is invertible if (and only if) all its diagonal entries,  $a, d$ , and  $f$ , are nonzero.

b As in part (a): if all the diagonal entries are nonzero.

c Yes,  $A^{-1}$  will be upper triangular as well; as you construct  $\text{rref}[A : I_n]$ , you will perform only the following row operations:

- divide rows by scalars
- subtract a multiple of the  $j$ th row from the  $i$ th row, where  $j > i$ .

Applying these operations to  $I_n$ , you end up with an upper triangular matrix.

d As in part (b): if all diagonal entries are nonzero.

**2.4.36** If a matrix  $A$  can be transformed into  $B$  by elementary row operations, then  $A$  is invertible if (and only if)  $B$  is invertible. The claim now follows from Exercise 35, where we show that a triangular matrix is invertible if (and only if) its diagonal entries are nonzero.

**2.4.37** Make an attempt to solve the linear equation  $\vec{y} = (cA)\vec{x} = c(A\vec{x})$  for  $\vec{x}$ :

$$A\vec{x} = \frac{1}{c}\vec{y}, \text{ so that } \vec{x} = A^{-1}\left(\frac{1}{c}\vec{y}\right) = \left(\frac{1}{c}A^{-1}\right)\vec{y}.$$

This shows that  $cA$  is indeed invertible, with  $(cA)^{-1} = \frac{1}{c}A^{-1}$ .

**2.4.38** Use Theorem 2.4.9;  $A^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & -k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} (= A)$ .

**2.4.39** Suppose the  $ij$ th entry of  $M$  is  $k$ , and all other entries are as in the identity matrix. Then we can find  $\text{rref}[M : I_n]$  by subtracting  $k$  times the  $j$ th row from the  $i$ th row. Therefore,  $M$  is indeed invertible, and  $M^{-1}$  differs from the identity matrix only at the  $ij$ th entry; that entry is  $-k$ . (See Figure 2.52.)

**2.4.40** If you apply an elementary row operation to a matrix with two equal columns, then the resulting matrix will also have two equal columns. Therefore,  $\text{rref}(A)$  has two equal columns, so that  $\text{rref}(A) \neq I_n$ . Now use Theorem 2.4.3.

**2.4.41** a Invertible: the transformation is its own inverse.

b Not invertible: the equation  $T(\vec{x}) = \vec{b}$  has infinitely many solutions if  $\vec{b}$  is on the plane, and none otherwise.

The diagram illustrates the Gauss-Jordan elimination process. It shows two matrices side-by-side. The left matrix is  $[M : I_n]$ , where  $M$  is a  $n \times n$  matrix with entries  $1, k, 0, -1$  along the main diagonal, and  $k$  in the  $(2,1)$  position. The right matrix is  $[I_n : M^{-1}]$ , where  $M^{-1}$  has entries  $1, -k, 0, -1$  along the main diagonal, and  $-k$  in the  $(2,1)$  position. Arrows indicate row operations: a horizontal arrow labeled  $-k(j)$  points from the first matrix to the second. Within each matrix, arrows show the steps of row reduction:  $i^{\text{th}}$  and  $j^{\text{th}}$  rows are swapped, and scalar multiples of the  $i^{\text{th}}$  row are added to the  $j^{\text{th}}$  row.

Figure 2.52: for Problem 2.3.39.

c Invertible: The inverse is a scaling by  $\frac{1}{5}$  (that is, a contraction by 5). If  $\vec{y} = 5\vec{x}$ , then  $\vec{x} = \frac{1}{5}\vec{y}$ .

d Invertible: The inverse is a rotation about the same axis through the same angle in the opposite direction.

**2.4.42** Permutation matrices are invertible since they row reduce to  $I_n$  in an obvious way, just by row swaps. The inverse of a permutation matrix  $A$  is also a permutation matrix since  $\text{rref}[A:I_n] = [I_n:A^{-1}]$  is obtained from  $[A:I_n]$  by a sequence of row swaps.

**2.4.43** We make an attempt to solve the equation  $\vec{y} = A(B\vec{x})$  for  $\vec{x}$ :

$B\vec{x} = A^{-1}\vec{y}$ , so that  $\vec{x} = B^{-1}(A^{-1}\vec{y})$ .

**2.4.44 a**  $\text{rref}(M_4) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ , so that  $\text{rank}(M_4) = 2$ .

b To simplify the notation, we introduce the row vectors  $\vec{v} = [1 \ 1 \ \dots \ 1]$  and  $\vec{w} = [0 \ n \ 2n \ \dots \ (n-1)n]$  with  $n$  components.

Then we can write  $M_n$  in terms of its rows as  $M_n = \begin{bmatrix} \vec{v} + \vec{w} \\ 2\vec{v} + \vec{w} \\ \dots \\ n\vec{v} + \vec{w} \end{bmatrix} \xrightarrow[-2(I)]{} \begin{bmatrix} \vec{v} + \vec{w} \\ -\vec{w} \\ -2\vec{w} \\ \dots \\ -(n-1)\vec{w} \end{bmatrix}$ .

Applying the Gauss-Jordan algorithm to the first column we get  $\begin{bmatrix} \vec{v} + \vec{w} \\ -\vec{w} \\ -2\vec{w} \\ \dots \\ -(n-1)\vec{w} \end{bmatrix}$ .

All the rows below the second are scalar multiples of the second; therefore,  $\text{rank}(M_n) = 2$ .

c By part (b), the matrix  $M_n$  is invertible only if  $n = 1$  or  $n = 2$ .

**2.4.45 a** Each of the three row divisions requires three multiplicative operations, and each of the six row subtractions requires three multiplicative operations as well; altogether, we have  $3 \cdot 3 + 6 \cdot 3 = 9 \cdot 3 = 3^3 = 27$  operations.

b Suppose we have already taken care of the first  $m$  columns:  $[A:I_n]$  has been reduced the matrix in Figure 2.53.

$$\left[ \begin{array}{c|cc|cc} 1 & * & * & 0 \\ -1 & & & & \\ 0 & & & & \\ 1 & & & & \\ \hline 0 & * & * & 0 \\ 0 & * & * & 0 \\ \hline \end{array} \right] \quad \text{with } \underbrace{\begin{array}{c} 1 \\ -1 \\ 0 \\ 1 \end{array}}_{m}, \underbrace{\begin{array}{c} * \\ * \\ * \\ * \end{array}}_{n-m}, \underbrace{\begin{array}{c} * \\ * \\ * \\ * \end{array}}_m, \underbrace{\begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \end{array}}_{n-m}$$

Figure 2.53: for Problem 2.3.45b.

Here, the stars represent arbitrary entries.

Suppose the  $(m+1)$ th entry on the diagonal is  $k$ . Dividing the  $(m+1)$ th row by  $k$  requires  $n$  operations:  $n-m-1$  to the left of the dotted line (not counting the computation  $\frac{k}{k} = 1$ ), and  $m+1$  to the right of the dotted line (including  $\frac{1}{k}$ ). Now the matrix has the form shown in Figure 2.54.

$$\left[ \begin{array}{c|cc|cc} 1 & * & * & 0 \\ -1 & & & & \\ 0 & * & * & 0 \\ 1 & & & \\ \hline 0 & * & * & 0 \\ 0 & * & * & 0 \\ \hline \end{array} \right] \quad \text{with } \underbrace{\begin{array}{c} 1 \\ -1 \\ 0 \\ 1 \end{array}}_{m}, \underbrace{\begin{array}{c} * \\ * \\ * \\ * \end{array}}_{n-m}, \underbrace{\begin{array}{c} * \\ * \\ * \\ * \end{array}}_m, \underbrace{\begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \end{array}}_{n-m}$$

Figure 2.54: for Problem 2.4.45b.

Eliminating each of the other  $n-1$  components of the  $(m+1)$ th column now requires  $n$  multiplicative operations ( $n-m-1$  to the left of the dotted line, and  $m+1$  to the right). Altogether, it requires  $n + (n-1)n = n^2$  operations to process the  $m$ th column. To process all  $n$  columns requires  $n \cdot n^2 = n^3$  operations.

c The inversion of a  $12 \times 12$  matrix requires  $12^3 = 4^3 \cdot 3^3 = 64 \cdot 3^3$  operations, that is, 64 times as much as the inversion of a  $3 \times 3$  matrix. If the inversion of a  $3 \times 3$  matrix takes one second, then the inversion of a  $12 \times 12$  matrix takes 64 seconds.

**2.4.46** Computing  $A^{-1}\vec{b}$  requires  $n^3 + n^2$  operations: First, we need  $n^3$  operations to find  $A^{-1}$  (see Exercise 45b) and then  $n^2$  operations to compute  $A^{-1}\vec{b}$  ( $n$  multiplications for each component).

How many operations are required to perform Gauss-Jordan eliminations on  $[A:\vec{b}]$ ? Let us count these operations “column by column.” If  $m$  columns of the coefficient matrix are left, then processing the next column requires  $nm$  operations (compare with Exercise 45b). To process all the columns requires

$$n \cdot n + n(n-1) + \cdots + n \cdot 2 + n \cdot 1 = n(n+n-1+\cdots+2+1) = n \frac{n(n+1)}{2} = \frac{n^3+n^2}{2} \text{ operations.}$$

only half of what was required to compute  $A^{-1}\vec{b}$ .

We mention in passing that one can reduce the number of operations further (by about 50% for large matrices) by performing the steps of the row reduction in a different order.

**2.4.47** Let  $f(x) = x^2$ ; the equation  $f(x) = 0$  has the unique solution  $x = 0$ .

**2.4.48** Consider the linear system  $A\vec{x} = \vec{0}$ . The equation  $A\vec{x} = \vec{0}$  implies that  $BA\vec{x} = \vec{0}$ , so  $\vec{x} = \vec{0}$  since  $BA = I_m$ . Thus the system  $A\vec{x} = \vec{0}$  has the unique solution  $\vec{x} = \vec{0}$ . This implies  $m \leq n$ , by Theorem 1.3.3. Likewise the linear system  $B\vec{y} = \vec{0}$  has the unique solution  $\vec{y} = \vec{0}$ , implying that  $n \leq m$ . It follows that  $n = m$ , as claimed.

$$\text{2.4.49 a } A = \begin{bmatrix} 0.293 & 0 & 0 \\ 0.014 & 0.207 & 0.017 \\ 0.044 & 0.01 & 0.216 \end{bmatrix}, \quad I_3 - A = \begin{bmatrix} 0.707 & 0 & 0 \\ -0.014 & 0.793 & -0.017 \\ -0.044 & -0.01 & 0.784 \end{bmatrix}$$

$$(I_3 - A)^{-1} = \begin{bmatrix} 1.41 & 0 & 0 \\ 0.0267 & 1.26 & 0.0274 \\ 0.0797 & 0.0161 & 1.28 \end{bmatrix}$$

b We have  $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ , so that  $\vec{x} = (I_3 - A)^{-1}\vec{e}_1$  = first column of  $(I_3 - A)^{-1} \approx \begin{bmatrix} 1.41 \\ 0.0267 \\ 0.0797 \end{bmatrix}$ .

c As illustrated in part (b), the  $i$ th column of  $(I_3 - A)^{-1}$  gives the output vector required to satisfy a consumer demand of 1 unit on industry  $i$ , in the absence of any other consumer demands. In particular, the  $i$ th diagonal entry of  $(I_3 - A)^{-1}$  gives the output of industry  $i$  required to satisfy this demand. Since industry  $i$  has to satisfy the consumer demand of 1 as well as the interindustry demand, its total output will be at least 1.

d Suppose the consumer demand increases from  $\vec{b}$  to  $\vec{b} + \vec{e}_2$  (that is, the demand on manufacturing increases by one unit). Then the output must change from  $(I_3 - A)^{-1}\vec{b}$  to

$$(I_3 - A)^{-1}(\vec{b} + \vec{e}_2) = (I_3 - A)^{-1}\vec{b} + (I_3 - A)^{-1}\vec{e}_2 = (I_3 - A)^{-1}\vec{b} + (\text{second column of } (I_3 - A)^{-1}).$$

The components of the second column of  $(I_3 - A)^{-1}$  tells us by how much each industry has to increase its output.

e The  $ij$ th entry of  $(I_n - A)^{-1}$  gives the required increase of the output  $x_i$  of industry  $i$  to satisfy an increase of the consumer demand  $b_j$  on industry  $j$  by one unit. In the language of multivariable calculus, this quantity is  $\frac{\partial x_i}{\partial b_j}$ .

**2.4.50** Recall that  $1 + k + k^2 + \dots = \frac{1}{1-k}$ .

The top left entry of  $I_3 - A$  is  $I - k$ , and the top left entry of  $(I_3 - A)^{-1}$  will therefore be  $\frac{1}{1-k}$ , as claimed:

$$\left[ \begin{array}{ccc|ccc} 1-k & 0 & 0 & \vdots & 1 & 0 & 0 \\ * & * & * & \vdots & 0 & 1 & 0 \\ * & * & * & \vdots & 0 & 0 & 1 \end{array} \right] \xrightarrow{\div(1-k)} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \vdots & \frac{1}{1-k} & 0 & 0 \\ * & * & * & \vdots & 0 & 1 & 0 \\ * & * & * & \vdots & 0 & 0 & 1 \end{array} \right]$$

$\rightarrow \dots$  (first row will remain unchanged).

In terms of economics, we can explain this fact as follows: The top left entry of  $(I_3 - A)^{-1}$  is the output of industry 1 (Agriculture) required to satisfy a consumer demand of 1 unit on industry 1. Producing this one unit to satisfy the consumer demand will generate an extra demand of  $k = 0.293$  units on industry 1. Producing these  $k$  units in turn will generate an extra demand of  $k \cdot k = k^2$  units, and so forth. We are faced with an infinite series of (ever smaller) demands,  $1 + k + k^2 + \dots$ .

**2.4.51 a** Since  $\text{rank}(A) < n$ , the matrix  $E = \text{rref}(A)$  will not have a leading one in the last row, and all entries in the last row of  $E$  will be zero.

Let  $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ . Then the last equation of the system  $E\vec{x} = \vec{c}$  reads  $0 = 1$ , so this system is inconsistent.

Now, we can “rebuild”  $\vec{b}$  from  $\vec{c}$  by performing the reverse row-operations in the opposite order on  $[E:\vec{c}]$  until we reach  $[A:\vec{b}]$ . Since  $E\vec{x} = \vec{c}$  is inconsistent,  $A\vec{x} = \vec{b}$  is inconsistent as well.

**b** Since  $\text{rank}(A) \leq \min(n, m)$ , and  $m < n$ ,  $\text{rank}(A) < n$  also. Thus, by part a, there is a  $\vec{b}$  such that  $A\vec{x} = \vec{b}$  is inconsistent.

**2.4.52** Let  $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ . Then  $[A:\vec{b}] = \begin{bmatrix} 0 & 1 & 2 & \vdots & 0 \\ 0 & 2 & 4 & \vdots & 0 \\ 0 & 3 & 6 & \vdots & 1 \\ 1 & 4 & 8 & \vdots & 0 \end{bmatrix}$ . We find that  $\text{rref } [A:\vec{b}] = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$ , which has an inconsistency in the third row.

**2.4.53 a**  $A - \lambda I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 3 & 5 - \lambda \end{bmatrix}$ .

This fails to be invertible when  $(3 - \lambda)(5 - \lambda) - 3 = 0$ ,

or  $15 - 8\lambda + \lambda^2 - 3 = 0$ ,

or  $12 - 8\lambda + \lambda^2 = 0$

or  $(6 - \lambda)(2 - \lambda) = 0$ . So  $\lambda = 6$  or  $\lambda = 2$ .

**b** For  $\lambda = 6$ ,  $A - \lambda I_2 = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$ .

The system  $(A - 6I_2)\vec{x} = \vec{0}$  has the solutions  $\begin{bmatrix} t \\ 3t \end{bmatrix}$ , where  $t$  is an arbitrary constant. Pick  $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ , for example.

For  $\lambda = 2$ ,  $A - \lambda I_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ .

The system  $(A - 2I_2)\vec{x} = \vec{0}$  has the solutions  $\begin{bmatrix} t \\ -t \end{bmatrix}$ , where  $t$  is an arbitrary constant. Pick  $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , for example.

c For  $\lambda = 6$ ,  $A\vec{x} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ .

For  $\lambda = 2$ ,  $A\vec{x} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

**2.4.54**  $A - \lambda I_2 = \begin{bmatrix} 1 - \lambda & 10 \\ -3 & 12 - \lambda \end{bmatrix}$ . This fails to be invertible when  $\det(A - \lambda I_2) = 0$ ,

so  $0 = (1 - \lambda)(12 - \lambda) + 30 = 12 - 13\lambda + \lambda^2 + 30 = \lambda^2 - 13\lambda + 42 = (\lambda - 6)(\lambda - 7)$ . In order for this to be zero,  $\lambda$  must be 6 or 7.

If  $\lambda = 6$ , then  $A - 6I_2 = \begin{bmatrix} -5 & 10 \\ -3 & 6 \end{bmatrix}$ . We solve the system  $(A - 6I_2)\vec{x} = \vec{0}$  and find that the solutions are of the form  $\vec{x} = \begin{bmatrix} 2t \\ t \end{bmatrix}$ . For example, when  $t = 1$ , we find  $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .

If  $\lambda = 7$ , then  $A - 7I_2 = \begin{bmatrix} -6 & 10 \\ -3 & 5 \end{bmatrix}$ . Here we solve the system  $(A - 7I_2)\vec{x} = \vec{0}$ , this time finding that our solutions are of the form  $\vec{x} = \begin{bmatrix} 5t \\ 3t \end{bmatrix}$ . For example, for  $t = 1$ , we find  $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ .

**2.4.55** The determinant of  $A$  is equal to 4 and  $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ . The linear transformation defined by  $A$  is a scaling by a factor 2 and  $A^{-1}$  defines a scaling by 1/2. The determinant of  $A$  is the area of the square spanned by  $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ . The angle  $\theta$  from  $\vec{v}$  to  $\vec{w}$  is  $\pi/2$ . (See Figure 2.55.)

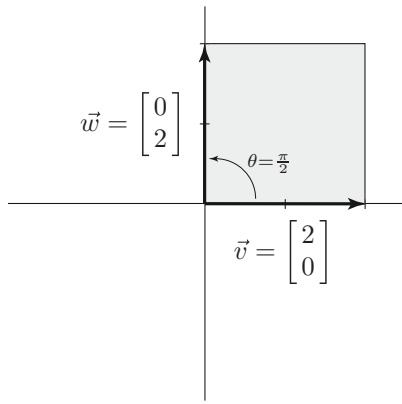


Figure 2.55: for Problem 2.4.55.

**2.4.56** The determinant of  $A$  is 9. The matrix is invertible with inverse  $A^{-1} = \begin{bmatrix} -3^{-1} & 0 \\ 0 & -3^{-1} \end{bmatrix}$ . The linear transformation defined by  $A$  is a reflection about the origin combined with a scaling by a factor 3. The inverse

defines a reflection about the origin combined with a scaling by a factor  $1/3$ . The determinant is the area of the square spanned by  $\vec{v} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ . The angle  $\theta$  from  $\vec{v}$  to  $\vec{w}$  is  $\pi/2$ . (See Figure 2.56.)

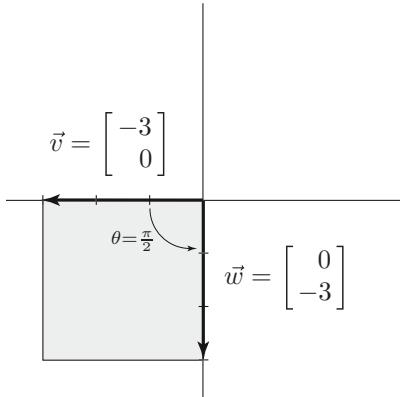


Figure 2.56: for Problem 2.4.56.

**2.4.57** The determinant of  $A$  is  $-1$ . Matrix  $A$  is invertible, with  $A^{-1} = A$ . Matrices  $A$  and  $A^{-1}$  define reflection about the line spanned by the  $\vec{v} = \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{bmatrix}$ . The absolute value of the determinant of  $A$  is the area of the unit square spanned by  $\vec{v} = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} \sin(\alpha) \\ -\cos(\alpha) \end{bmatrix}$ . The angle  $\theta$  from  $\vec{v}$  to  $\vec{w}$  is  $-\pi/2$ . (See Figure 2.57.)

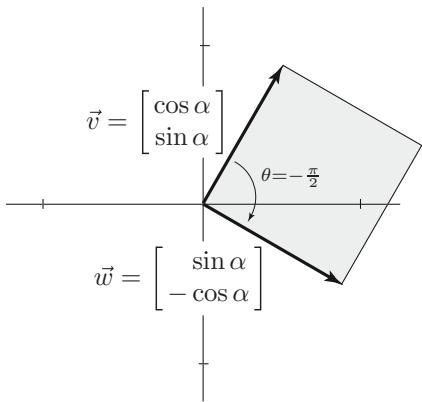


Figure 2.57: for Problem 2.4.57.

**2.4.58** The determinant of  $A$  is  $1$ . The matrix is invertible with inverse  $A^{-1} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$ . The linear transformation defined by  $A$  is a rotation by angle  $\alpha$  in the counterclockwise direction. The inverse represents a rotation by the angle  $\alpha$  in the clockwise direction. The determinant of  $A$  is the area of the unit square spanned

by  $\vec{v} = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{bmatrix}$ . The angle  $\theta$  from  $\vec{v}$  to  $\vec{w}$  is  $\pi/2$ . (See Figure 2.58.)

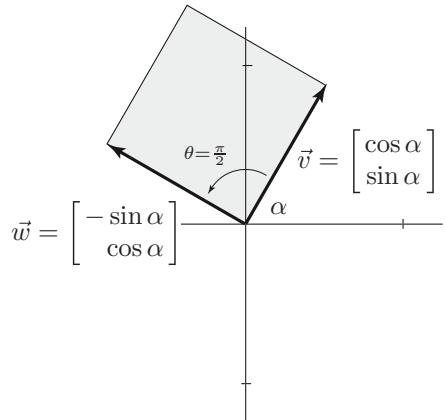


Figure 2.58: for Problem 2.4.58.

**2.4.59** The determinant of  $A$  is 1. The matrix  $A$  is invertible with inverse  $A^{-1} = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$ . The matrix  $A$  represents the rotation through the angle  $\alpha = \arccos(0.6)$ . Its inverse represents a rotation by the same angle in the clockwise direction. The determinant of  $A$  is the area of the unit square spanned by  $\vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}$ . The angle  $\theta$  from  $\vec{v}$  to  $\vec{w}$  is  $\pi/2$ . (See Figure 2.59.)

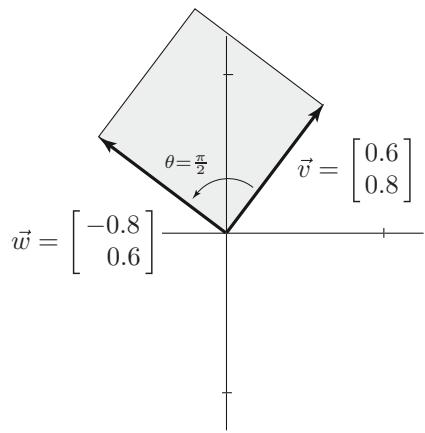


Figure 2.59: for Problem 2.4.59.

**2.4.60** The determinant of  $A$  is  $-1$ . The matrix  $A$  is invertible with inverse  $A^{-1} = A$ . Matrices  $A$  and  $A^{-1}$  define the reflection about the line spanned by  $\vec{v} = \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{bmatrix}$ , where  $\alpha = \arccos(-0.8)$ . The absolute value of the determinant of  $A$  is the area of the unit square spanned by  $\vec{v} = \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$ . The angle  $\theta$  from  $v$

to  $w$  is  $-\pi/2$ . (See Figure 2.60.)

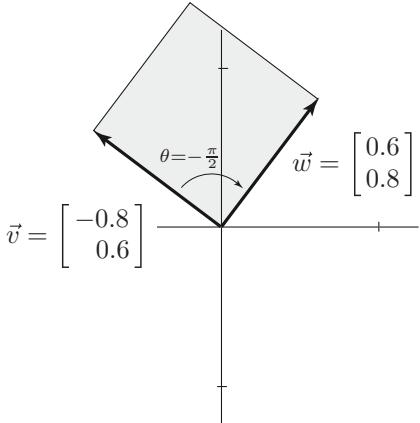


Figure 2.60: for Problem 2.4.60.

**2.4.61** The determinant of  $A$  is 2 and  $A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ . The matrix  $A$  represents a rotation through the angle  $-\pi/4$  combined with scaling by  $\sqrt{2}$ . describes a rotation through  $\pi/4$  and scaling by  $1/\sqrt{2}$ . The determinant of  $A$  is the area of the square spanned by  $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  with side length  $\sqrt{2}$ . The angle  $\theta$  from  $\vec{v}$  to  $\vec{w}$  is  $\pi/2$ . (See Figure 2.61.)

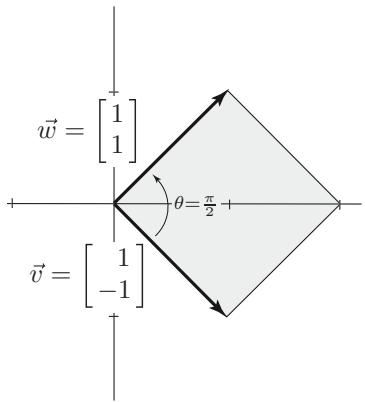


Figure 2.61: for Problem 2.4.61.

**2.4.62** The determinant of  $A$  is 25. The matrix  $A$  is a rotation dilation matrix with scaling factor 5 and rotation by an angle  $\arccos(0.6)$  in the clockwise direction. The inverse  $A^{-1} = (1/25) \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$  is a rotation dilation too with a scaling factor  $1/5$  and rotation angle  $\arccos(0.6)$ . The determinant of  $A$  is the area of the parallelogram spanned by  $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$  with side length 5. The angle from  $\vec{v}$  to  $\vec{w}$  is  $\pi/2$ . (See Figure 2.62.)

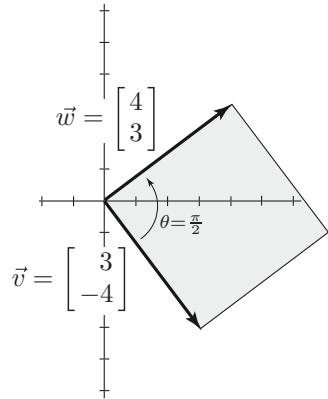


Figure 2.62: for Problem 2.4.62.

**2.4.63** The determinant of  $A$  is  $-25$  and  $A^{-1} = (1/25) \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} = \frac{1}{25}A$ . The matrix  $A$  represents a reflection about a line combined with a scaling by  $5$  while  $A^{-1}$  represents a reflection about the same line combined with a scaling by  $1/5$ . The absolute value of the determinant of  $A$  is the area of the square spanned by  $\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$  with side length  $5$ . The angle from  $\vec{v}$  to  $\vec{w}$  is  $-\pi/2$ . (See Figure 2.63.)

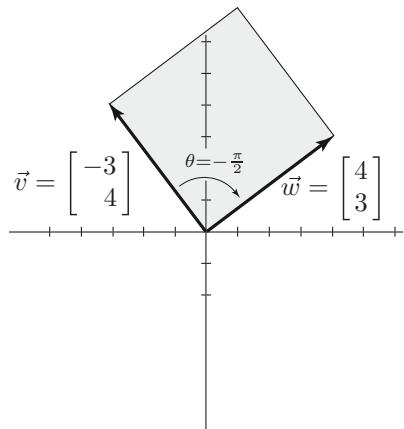


Figure 2.63: for Problem 2.4.63.

**2.4.64** The determinant of  $A$  is  $1$  and  $A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . Both  $A$  and  $A^{-1}$  represent horizontal shears. The determinant of  $A$  is the area of the parallelogram spanned by  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ . The angle from  $\vec{v}$  to  $\vec{w}$  is  $3\pi/4$ . (See Figure 2.64.)

**2.4.65** The determinant of  $A$  is  $1$  and  $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ . Both  $A$  and  $A^{-1}$  represent vertical shears. The determinant

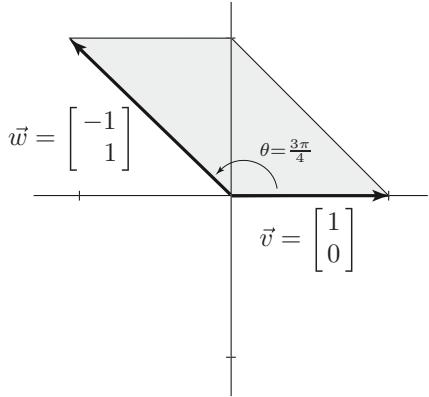


Figure 2.64: for Problem 2.4.64.

of  $A$  is the area of the parallelogram spanned by  $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . The angle from  $\vec{v}$  to  $\vec{w}$  is  $\pi/4$ . (See Figure 2.65.)

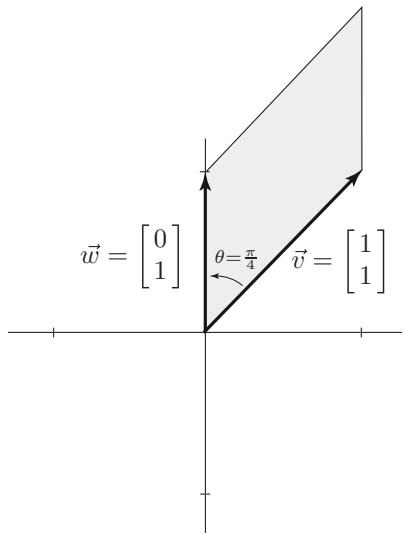


Figure 2.65: for Problem 2.4.65.

**2.4.66** We can write  $AB(AB)^{-1} = A(B(AB)^{-1}) = I_n$  and  $(AB)^{-1}AB = ((AB)^{-1}A)B = I_n$ .

By Theorem 2.4.8,  $A$  and  $B$  are invertible.

**2.4.67** Not necessarily true;  $(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$  if  $AB \neq BA$ .

**2.4.68** True; apply Theorem 2.4.7 to  $B = A$ .

**2.4.69** Not necessarily true; consider the case  $A = I_n$  and  $B = -I_n$ .

**2.4.70** Not necessarily true;  $(A - B)(A + B) = A^2 + AB - BA - B^2 \neq A^2 - B^2$  if  $AB \neq BA$ .

**2.4.71** True;  $ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$ .

**2.4.72** Not necessarily true; the equation  $ABA^{-1} = B$  is equivalent to  $AB = BA$  (multiply by  $A$  from the right), which is not true in general.

**2.4.73** True;  $(ABA^{-1})^3 = ABA^{-1}ABA^{-1}ABA^{-1} = AB^3A^{-1}$ .

**2.4.74** True;  $(I_n + A)(I_n + A^{-1}) = I_n^2 + A + A^{-1} + AA^{-1} = 2I_n + A + A^{-1}$ .

**2.4.75** True;  $(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A$  (use Theorem 2.4.7).

**2.4.76** We want  $A$  such that  $A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ , so that  $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 8 & -3 \\ -1 & 1 \end{bmatrix}$ .

**2.4.77** We want  $A$  such that  $A\vec{v}_i = \vec{w}_i$ , for  $i = 1, 2, \dots, m$ , or  $A[\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_m] = [\vec{w}_1 \ \vec{w}_2 \ \dots \ \vec{w}_m]$ , or  $AS = B$ .

Multiplying by  $S^{-1}$  from the right we find the unique solution  $A = BS^{-1}$ .

**2.4.78** Use the result of Exercise 2.4.77, with  $S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$ ;

$$A = BS^{-1} = \begin{bmatrix} 33 & -13 \\ 21 & -8 \\ 9 & -3 \end{bmatrix}$$

**2.4.79** Use the result of Exercise 2.4.77, with  $S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 6 & 3 \\ 2 & 6 \end{bmatrix}$ ;

$$A = BS^{-1} = \frac{1}{5} \begin{bmatrix} 9 & 3 \\ -2 & 16 \end{bmatrix}.$$

**2.4.80**  $P_0 \xrightarrow{T} P_1$ ,  $P_1 \xrightarrow{T} P_3$ ,  $P_2 \xrightarrow{T} P_2$ ,  $P_3 \xrightarrow{T} P_0$

$P_0 \xrightarrow{L} P_0$ ,  $P_1 \xrightarrow{L} P_2$ ,  $P_2 \xrightarrow{L} P_1$ ,  $P_3 \xrightarrow{L} P_3$

a.  $T^{-1}$  is the rotation about the axis through 0 and  $P_2$  that transforms  $P_3$  into  $P_1$ .

b.  $L^{-1} = L$

c.  $T^2 = T^{-1}$  (See part (a).)

d.  $P_0 \xrightarrow{T \circ L} P_1$      $P_0 \xrightarrow{L \circ T} P_2$     The transformations  $T \circ L$  and  $L \circ T$  are not the same.

$$\begin{array}{ll} P_1 \longrightarrow P_2 & P_1 \longrightarrow P_3 \\ P_2 \longrightarrow P_3 & P_2 \longrightarrow P_1 \\ P_3 \longrightarrow P_0 & P_3 \longrightarrow P_0 \end{array}$$

e.

$$\begin{array}{l} P_0 \xrightarrow{L \circ T \circ L} P_2 \\ P_1 \longrightarrow P_1 \\ P_2 \longrightarrow P_3 \\ P_3 \longrightarrow P_0 \end{array}$$

This is the rotation about the axis through 0 and  $P_1$  that sends  $P_0$  to  $P_2$ .

**2.4.81** Let  $A$  be the matrix of  $T$  and  $C$  the matrix of  $L$ . We want that  $AP_0 = P_1$ ,  $AP_1 = P_3$ , and  $AP_2 = P_2$ . We can use the result of Exercise 77, with  $S = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ .

$$\text{Then } A = BS^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

$$\text{Using an analogous approach, we find that } C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{2.4.82 a } EA = \begin{bmatrix} a & b & c \\ d-3a & e-3b & f-3c \\ g & h & k \end{bmatrix}$$

The matrix  $EA$  is obtained from  $A$  by an elementary row operation: subtract three times the first row from the second.

$$\text{b } EA = \begin{bmatrix} a & b & c \\ \frac{1}{4}d & \frac{1}{4}e & \frac{1}{4}f \\ g & h & k \end{bmatrix}$$

The matrix  $EA$  is obtained from  $A$  by dividing the second row of  $A$  by 4 (an elementary row operation).

$$\text{c If we set } E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ then } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & k \\ d & e & f \end{bmatrix}, \text{ as desired.}$$

d An elementary  $n \times n$  matrix  $E$  has the same form as  $I_n$  except that either

- $e_{ij} = k (\neq 0)$  for some  $i \neq j$  [as in part (a)], or
- $e_{ii} = k (\neq 0, 1)$  for some  $i$  [as in part (b)], or
- $e_{ij} = e_{ji} = 1$ ,  $e_{ii} = e_{jj} = 0$  for some  $i \neq j$  [as in part (c)].

**2.4.83** Let  $E$  be an elementary  $n \times n$  matrix (obtained from  $I_n$  by a certain elementary row operation), and let  $F$  be the elementary matrix obtained from  $I_n$  by the reversed row operation. Our work in Exercise 2.4.82 [parts (a) through (c)] shows that  $EF = I_n$ , so that  $E$  is indeed invertible, and  $E^{-1} = F$  is an elementary matrix as well.

**2.4.84 a** The matrix  $\text{rref}(A)$  is obtained from  $A$  by performing a sequence of  $p$  elementary row operations. By Exercise 2.4.82 [parts (a) through (c)] each of these operations can be represented by the left multiplication with an elementary matrix, so that  $\text{rref}(A) = E_1 E_2 \dots E_p A$ .

b  $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$  swap rows 1 and 2, represented by  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

 $\downarrow$ 

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \div 2, \text{ represented by } \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

 $\downarrow$ 

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} -3(II), \text{ represented by } \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

 $\downarrow$ 

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Therefore, rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = E_1 E_2 E_3 A.$$

**2.4.85 a** Let  $S = E_1 E_2 \dots E_p$  in Exercise 2.4.84a.

By Exercise 2.4.83, the elementary matrices  $E_i$  are invertible: now use Theorem 2.4.7 repeatedly to see that  $S$  is invertible.

b  $A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \div 2$ , represented by  $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} -4(I), \text{ represented by } \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\text{Therefore, rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = E_1 E_2 A = SA, \text{ where}$$

$$S = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -2 & 1 \end{bmatrix}.$$

(There are other correct answers.)

**2.4.86 a** By Exercise 2.4.84a,  $I_n = \text{rref}(A) = E_1 E_2 \dots E_p A$ , for some elementary matrices  $E_1, \dots, E_p$ . By Exercise 2.4.83, the  $E_i$  are invertible and their inverses are elementary as well. Therefore,

$$A = (E_1 E_2 \dots E_p)^{-1} = E_p^{-1} \dots E_2^{-1} E_1^{-1} \text{ expresses } A \text{ as a product of elementary matrices.}$$

b We can use our work in Exercise 2.4.84 b:

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = \left( \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

**2.4.87**  $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$  represents a horizontal shear,  $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$  represents a vertical shear,

$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$  represents a “scaling in  $\vec{e}_1$  direction” (leaving the  $\vec{e}_2$  component unchanged),

$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$  represents a “scaling in  $\vec{e}_2$  direction” (leaving the  $\vec{e}_1$  component unchanged), and

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  represents the reflection about the line spanned by  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**2.4.88** Performing a sequence of  $p$  elementary row operations on a matrix  $A$  amounts to multiplying  $A$  with  $E_1 E_2 \dots E_p$  from the left, where the  $E_i$  are elementary matrices. If  $I_n = E_1 E_2 \dots E_p A$ , then  $E_1 E_2 \dots E_p = A^{-1}$ , so that

- a.  $E_1 E_2 \dots E_p AB = B$ , and
- b.  $E_1 E_2 \dots E_p I_n = A^{-1}$ .

**2.4.89** Let  $A$  and  $B$  be two lower triangular  $n \times n$  matrices. We need to show that the  $ij$ th entry of  $AB$  is 0 whenever  $i < j$ .

This entry is the dot product of the  $i$ th row of  $A$  and the  $j$ th column of  $B$ ,

$$[a_{i1} \ a_{i2} \ \dots \ a_{ii} \ 0 \ \dots \ 0] \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{jj} \\ \vdots \\ b_{nj} \end{bmatrix}, \text{ which is indeed } 0 \text{ if } i < j.$$

**2.4.90 a**  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix} \xrightarrow{-2I} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \xrightarrow{-2I}$ , represented by

↓

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{+II} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \text{ so that}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix}$$

↑

$U$

↑

$E_3$

↑

$E_2$

↑

$E_1$

↑

$A$

b  $A = (E_3 E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} E_3^{-1} U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$\overset{\uparrow}{M_1} \quad \overset{\uparrow}{M_2} \quad \overset{\uparrow}{M_3} \quad \overset{\uparrow}{U}$

c Let  $L = M_1 M_2 M_3$  in part (b); we compute  $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$ .

Then  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$\overset{\uparrow}{A} \quad \overset{\uparrow}{L} \quad \overset{\uparrow}{U}$

d We can use the matrix  $L$  we found in part (c), but  $U$  needs to be modified. Let  $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ .

(Take the diagonal entries of the matrix  $U$  in part (c)).

Then  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$

$\overset{\uparrow}{A} \quad \overset{\uparrow}{L} \quad \overset{\uparrow}{D} \quad \overset{\uparrow}{U}$

**2.4.91 a** Write the system  $L\vec{y} = \vec{b}$  in components:

$$\begin{bmatrix} y_1 & & & = -3 \\ -3y_1 + y_2 & & & = 14 \\ y_1 + 2y_2 + y_3 & & & = 9 \\ -y_1 + 8y_2 - 5y_3 + y_4 & = 33 \end{bmatrix}, \text{ so that } y_1 = -3, y_2 = 14 + 3y_1 = 5,$$

$$y_3 = 9 - y_1 - 2y_2 = 2, \text{ and } y_4 = 33 + y_1 - 8y_2 + 5y_3 = 0:$$

$$\vec{y} = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 0 \end{bmatrix}.$$

b Proceeding as in part (a) we find that  $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ .

**2.4.92** We try to find matrices  $L = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$  and  $U = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$  such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae \\ bd & be + cf \end{bmatrix}.$$

Note that the equations  $ad = 0$ ,  $ae = 1$ , and  $bd = 1$  cannot be solved simultaneously: If  $ad = 0$  then  $a$  or  $d$  is 0 so that  $ae$  or  $bd$  is zero.

Therefore, the matrix  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  does not have an  $LU$  factorization.

**2.4.93 a** Write  $L = \begin{bmatrix} L^{(m)} & 0 \\ L_3 & L_4 \end{bmatrix}$  and  $U = \begin{bmatrix} U^{(m)} & U_2 \\ 0 & U_4 \end{bmatrix}$ .

Then  $A = LU = \begin{bmatrix} L^{(m)}U^{(m)} & L^{(m)}U_2 \\ L_3U^{(m)} & L_3U_2 + L_4U_4 \end{bmatrix}$ , so that  $A^{(m)} = L^{(m)}U^{(m)}$ , as claimed.

b By Exercise 2.4.66, the matrices  $L$  and  $U$  are both invertible. By Exercise 2.4.35, the diagonal entries of  $L$  and  $U$  are all nonzero. For any  $m$ , the matrices  $L^{(m)}$  and  $U^{(m)}$  are triangular, with nonzero diagonal entries, so that they are invertible. By Theorem 2.4.7, the matrix  $A^{(m)} = L^{(m)}U^{(m)}$  is invertible as well.

c Using the hint, we write  $A = \begin{bmatrix} A^{(n-1)} & \vec{v} \\ \vec{w} & k \end{bmatrix} = \begin{bmatrix} L' & 0 \\ \vec{x} & t \end{bmatrix} \begin{bmatrix} U' & \vec{y} \\ 0 & s \end{bmatrix}$ .

We are looking for a column vector  $\vec{y}$ , a row vector  $\vec{x}$ , and scalars  $t$  and  $s$  satisfying these equations. The following equations need to be satisfied:  $\vec{v} = L'\vec{y}$ ,  $\vec{w} = \vec{x}U'$ , and  $k = \vec{x}\vec{y} + ts$ .

We find that  $\vec{y} = (L')^{-1}\vec{v}$ ,  $\vec{x} = \vec{w}(U')^{-1}$ , and  $ts = k - \vec{w}(U')^{-1}(L')^{-1}\vec{v}$ .

We can choose, for example,  $s = 1$  and  $t = k - \vec{w}(U')^{-1}(L')^{-1}\vec{v}$ , proving that  $A$  does indeed have an  $LU$  factorization.

Alternatively, one can show that if all principal submatrices are invertible then no row swaps are required in the Gauss-Jordan Algorithm. In this case, we can find an  $LU$ -factorization as outlined in Exercise 2.4.90.

**2.4.94 a** If  $A = LU$  is an  $LU$  factorization, then the diagonal entries of  $L$  and  $U$  are nonzero (compare with Exercise 2.4.93). Let  $D_1$  and  $D_2$  be the diagonal matrices whose diagonal entries are the same as those of  $L$  and  $U$ , respectively.

Then  $A = (LD_1^{-1})(D_1D_2)(D_2^{-1}U)$  is the desired factorization

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{new } L & D & \text{new } U \end{array}$$

(verify that  $LD_1^{-1}$  and  $D_2^{-1}U$  are of the required form).

b If  $A = L_1D_1U_1 = L_2D_2U_2$  and  $A$  is invertible, then  $L_1, D_1, U_1, L_2, D_2, U_2$  are all invertible, so that we can multiply the above equation by  $D_2^{-1}L_2^{-1}$  from the left and by  $U_1^{-1}$  from the right:

$$D_2^{-1}L_2^{-1}L_1D_1 = U_2U_1^{-1}.$$

Since products and inverses of upper triangular matrices are upper triangular (and likewise for lower triangular matrices), the matrix  $D_2^{-1}L_2^{-1}L_1D_1 = U_2U_1^{-1}$  is both upper and lower triangular, that is, it is diagonal. Since the diagonal entries of  $U_2$  and  $U_1$  are all 1, so are the diagonal entries of  $U_2U_1^{-1}$ , that is  $U_2U_1^{-1} = I_n$ , and thus  $U_2 = U_1$ .

Now  $L_1 D_1 = L_2 D_2$ , so that  $L_2^{-1} L_1 = D_2 D_1^{-1}$  is diagonal. As above, we have in fact  $L_2^{-1} L_1 = I_n$  and therefore  $L_2 = L_1$ .

**2.4.95** Suppose  $A_{11}$  is a  $p \times p$  matrix and  $A_{22}$  is a  $q \times q$  matrix. For  $B$  to be the inverse of  $A$  we must have  $AB = I_{p+q}$ . Let us partition  $B$  the same way as  $A$ :

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \text{ where } B_{11} \text{ is } p \times p \text{ and } B_{22} \text{ is } q \times q.$$

$$\text{Then } AB = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} \text{ means that}$$

$$A_{11}B_{11} = I_p, \quad A_{22}B_{22} = I_q, \quad A_{11}B_{12} = 0, \quad A_{22}B_{21} = 0.$$

This implies that  $A_{11}$  and  $A_{22}$  are invertible, and  $B_{11} = A_{11}^{-1}$ ,  $B_{22} = A_{22}^{-1}$ .

This in turn implies that  $B_{12} = 0$  and  $B_{21} = 0$ .

We summarize:  $A$  is invertible if (and only if) both  $A_{11}$  and  $A_{22}$  are invertible; in this case

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

**2.4.96** This exercise is very similar to Example 7 in the text. We outline the solution:

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} \text{ means that}$$

$$A_{11}B_{11} = I_q, \quad A_{11}B_{12} = 0, \quad A_{21}B_{11} + A_{22}B_{21} = 0, \quad A_{21}B_{12} + A_{22}B_{22} = I_q.$$

This implies that  $A_{11}$  is invertible, and  $B_{11} = A_{11}^{-1}$ . Multiplying the second equation with  $A_{11}^{-1}$ , we conclude that  $B_{12} = 0$ . Then the last equation simplifies to  $A_{22}B_{22} = I_q$ , so that  $B_{22} = A_{22}^{-1}$ .

$$\text{Finally, } B_{21} = -A_{22}^{-1}A_{21}B_{11} = -A_{22}^{-1}A_{21}A_{11}^{-1}.$$

We summarize:  $A$  is invertible if (and only if) both  $A_{11}$  and  $A_{22}$  are invertible. In this case,

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}.$$

**2.4.97** Suppose  $A_{11}$  is a  $p \times p$  matrix. Since  $A_{11}$  is invertible,  $\text{rref}(A) = \begin{bmatrix} I_p & A_{12} & * \\ 0 & 0 & \text{rref}(A_{23}) \end{bmatrix}$ , so that

$$\text{rank}(A) = p + \text{rank}(A_{23}) = \text{rank}(A_{11}) + \text{rank}(A_{23}).$$

**2.4.98** Try to find a matrix  $B = \begin{bmatrix} X & \vec{x} \\ \vec{y} & t \end{bmatrix}$  (where  $X$  is  $n \times n$ ) such that

$$AB = \begin{bmatrix} I_n & \vec{v} \\ \vec{w} & 1 \end{bmatrix} \begin{bmatrix} X & \vec{x} \\ \vec{y} & t \end{bmatrix} = \begin{bmatrix} X + \vec{v}\vec{y} & \vec{x} + t\vec{v} \\ \vec{w}X + \vec{y} & \vec{w}\vec{x} + t \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}.$$

We want  $X + \vec{v}\vec{y} = I_n$ ,  $\vec{x} + t\vec{v} = \vec{0}$ ,  $\vec{w}X + \vec{y} = \vec{0}$ , and  $\vec{w}\vec{x} + t = 1$ .

Substituting  $\vec{x} = -t\vec{v}$  into the last equation we find  $-t\vec{w}\vec{v} + t = 1$  or  $t(1 - \vec{w}\vec{v}) = 1$ .

This equation can be solved only if  $\vec{w}\vec{v} \neq 1$ , in which case  $t = \frac{1}{1-\vec{w}\vec{v}}$ . Now substituting  $X = I_n - \vec{v}\vec{y}$  into the third equation, we find  $\vec{w} - \vec{w}\vec{v}\vec{y} + \vec{y} = \vec{0}$  or  $\vec{y} = -\frac{1}{1-\vec{w}\vec{v}}\vec{w} = -t\vec{w}$ .

We summarize:  $A$  is invertible if (and only if)  $\vec{w}\vec{v} \neq 1$ . In this case,  $A^{-1} = \begin{bmatrix} I_n + t\vec{w}\vec{w} & -t\vec{v} \\ -t\vec{w} & t \end{bmatrix}$ , where  $t = \frac{1}{1-\vec{w}\vec{v}}$ .

The same result can be found (perhaps more easily) by working with  $\text{rref}[A : I_{n+1}]$ , rather than partitioned matrices.

**2.4.99** Multiplying both sides with  $A^{-1}$  we find that  $A = I_n$ : The identity matrix is the only invertible matrix with this property.

**2.4.100** Suppose the entries of  $A$  are all  $a$ , where  $a \neq 0$ . Then the entries of  $A^2$  are all  $na^2$ . The equation  $na^2 = a$

is satisfied if  $a = \frac{1}{n}$ . Thus the solution is  $A = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \vdots & & & \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{bmatrix}$ .

**2.4.101** The  $ij$ th entry of  $AB$  is

$$\sum_{k=1}^n a_{ik}b_{kj}.$$

Then

$$\sum_{k=1}^n a_{ik}b_{kj} \leq \sum_{k=1}^n sb_{kj} = s \left( \sum_{k=1}^n b_{kj} \right) \leq sr.$$

$\uparrow$   $\uparrow$   
 since  $a_{ik} \leq s$  this is  $\leq r$ , as it is the  
 $j^{th}$  column sum of  $B$ .

**2.4.102 a** We proceed by induction on  $m$ . Since the column sums of  $A$  are  $\leq r$ , the entries of  $A^1 = A$  are also  $\leq r^1 = r$ , so that the claim holds for  $m = 1$ . Suppose the claim holds for some fixed  $m$ . Now write  $A^{m+1} = A^m A$ ; since the entries of  $A^m$  are  $\leq r^m$  and the column sums of  $A$  are  $\leq r$ , we can conclude that the entries of  $A^{m+1}$  are  $\leq r^m r = r^{m+1}$ , by Exercise 101.

b For a fixed  $i$  and  $j$ , let  $b_m$  be the  $ij$ th entry of  $A^m$ . In part (a) we have seen that  $0 \leq b_m \leq r^m$ .

Note that  $\lim_{m \rightarrow \infty} r^m = 0$  (since  $r < 1$ ), so that  $\lim_{m \rightarrow \infty} b_m = 0$  as well (this follows from what some calculus texts call the “squeeze theorem”).

c For a fixed  $i$  and  $j$ , let  $c_m$  be the  $ij$ th entry of the matrix  $I_n + A + A^2 + \dots + A^m$ . By part (a),

$$c_m \leq 1 + r + r^2 + \dots + r^m < \frac{1}{1-r}.$$

Since the  $c_m$  form an increasing bounded sequence,  $\lim_{m \rightarrow \infty} c_m$  exists (this is a fundamental fact of calculus).

$$\begin{aligned} d \quad & (I_n - A)(I_n + A + A^2 + \cdots + A^m) = I_n + A + A^2 + \cdots + A^m - A - A^2 - \cdots - A^m - A^{m+1} \\ & = I_n - A^{m+1} \end{aligned}$$

Now let  $m$  go to infinity; use parts (b) and (c).  $(I_n - A)(I_n + A + A^2 + \cdots + A^m + \cdots) = I_n$ , so that

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \cdots.$$

**2.4.103 a** The components of the  $j$ th column of the technology matrix  $A$  give the demands industry  $J_j$  makes on the other industries, per unit output of  $J_j$ . The fact that the  $j$ th column sum is less than 1 means that industry  $J_j$  adds value to the products it produces.

b A productive economy can satisfy any consumer demand  $\vec{b}$ , since the equation

$$(I_n - A)\vec{x} = \vec{b} \text{ can be solved for the output vector } \vec{x} : \vec{x} = (I_n - A)^{-1}\vec{b} \text{ (compare with Exercise 2.4.49).}$$

c The output  $\vec{x}$  required to satisfy a consumer demand  $\vec{b}$  is

$$\vec{x} = (I_n - A)^{-1}\vec{b} = (I_n + A + A^2 + \cdots + A^m + \cdots)\vec{b} = \vec{b} + A\vec{b} + A^2\vec{b} + \cdots + A^m\vec{b} + \cdots.$$

To interpret the terms in this series, keep in mind that whatever output  $\vec{v}$  the industries produce generates an interindustry demand of  $A\vec{v}$ .

The industries first need to satisfy the consumer demand,  $\vec{b}$ . Producing the output  $\vec{b}$  will generate an interindustry demand,  $A\vec{b}$ . Producing  $A\vec{b}$  in turn generates an extra interindustry demand,  $A(A\vec{b}) = A^2\vec{b}$ , and so forth.

For a simple example, see Exercise 2.4.50; also read the discussion of “chains of interindustry demands” in the footnote to Exercise 2.4.49.

**2.4.104 a** We write our three equations below:

$$\begin{aligned} I &= \frac{1}{3}R + \frac{1}{3}G + \frac{1}{3}B \\ L &= R - G \\ S &= -\frac{1}{2}R - \frac{1}{2}G + B \end{aligned}, \text{ so that the matrix is } P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

$$b \quad \begin{bmatrix} R \\ G \\ B \end{bmatrix} \text{ is transformed into } \begin{bmatrix} R \\ G \\ 0 \end{bmatrix}, \text{ with matrix } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$c \quad \text{This matrix is } PA = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \text{ (we apply first } A, \text{ then } P\text{.)}$$

$$d \quad \text{See Figure 2.66. A “diagram chase” shows that } M = PAP^{-1} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{9} \\ 0 & 1 & 0 \\ -1 & 0 & \frac{1}{3} \end{bmatrix}.$$

$$2.4.105 a \quad A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

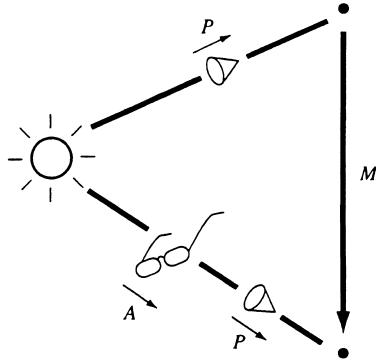


Figure 2.66: for Problem 2.4.104d.

Matrix  $A^{-1}$  transforms a wife's clan into her husband's clan, and  $B^{-1}$  transforms a child's clan into the mother's clan.

- b  $B^2$  transforms a woman's clan into the clan of a child of her daughter.
- c  $AB$  transforms a woman's clan into the clan of her daughter-in-law (her son's wife), while  $BA$  transforms a man's clan into the clan of his children. The two transformations are different. (See Figure 2.67.)

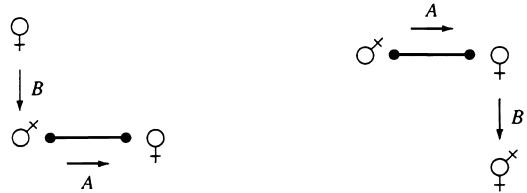


Figure 2.67: for Problem 2.4.105c.

- d The matrices for the four given diagrams (in the same order) are  $BB^{-1} = I_3$ ,

$$BAB^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B(BA)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad BA(BA)^{-1} = I_3.$$

- e Yes; since  $BAB^{-1} = A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ , in the second case in part (d) the cousin belongs to Bueya's husband's clan.

**2.4.106** a We need 8 multiplications: 2 to compute each of the four entries of the product.

- b We need  $n$  multiplications to compute each of the  $mp$  entries of the product,  $mnp$  multiplications altogether.

**2.4.107**  $g(f(x)) = x$ , for all  $x$ , so that  $g \circ f$  is the identity, but  $f(g(x)) = \begin{cases} x & \text{if } x \text{ is even} \\ x+1 & \text{if } x \text{ is odd} \end{cases}$ .

2.4.108 a The formula  $\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 - Rk & L + R - kLR \\ -k & 1 - kL \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}$  is given, which implies that

$$y = (1 - Rk)x + (L + R - kLR)m.$$

In order for  $y$  to be independent of  $x$  it is required that  $1 - Rk = 0$ , or  $k = \frac{1}{R} = 40$  (diopters).

$\frac{1}{k}$  then equals  $R$ , which is the distance between the plane of the lens and the plane on which parallel incoming rays focus at a point; thus the term “focal length” for  $\frac{1}{k}$ .

b Now we want  $y$  to be independent of the slope  $m$  (it must depend on  $x$  alone). In view of the formula above, this is the case if  $L + R - kLR = 0$ , or  $k = \frac{L + R}{LR} = \frac{1}{R} + \frac{1}{L} = 40 + \frac{10}{3} \approx 43.3$  (diopters).

c Here the transformation is

$$\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 - k_1 D & D \\ k_1 k_2 D - k_1 - k_2 & 1 - k_2 D \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

We want the slope  $n$  of the outgoing rays to depend on the slope  $m$  of the incoming rays alone, and not on  $x$ ; this forces  $k_1 k_2 D - k_1 - k_2 = 0$ , or,  $D = \frac{k_1 + k_2}{k_1 k_2} = \frac{1}{k_1} + \frac{1}{k_2}$ , the sum of the focal lengths of the two lenses. See Figure 2.68.

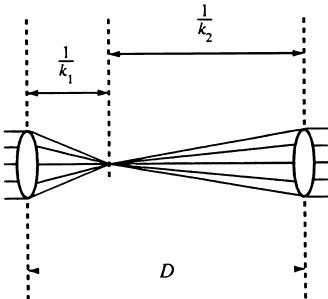


Figure 2.68: for Problem 2.4.108c.

## True or False

Ch 2.TF.1 T, by Theorem 2.4.3.

Ch 2.TF.2 T; Let  $A = B$  in Theorem 2.4.7.

Ch 2.TF.3 F, by Theorem 2.3.3.

Ch 2.TF.4 T, by Theorem 2.4.8.

Ch 2.TF.5 F; Matrix  $AB$  will be  $3 \times 5$ , by Definition 2.3.1b.

Ch 2.TF.6 F; Note that  $T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . A linear transformation transforms  $\vec{0}$  into  $\vec{0}$ .

## Chapter 2

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Ch 2.TF.7 T, by Theorem 2.2.4.

Ch 2.TF.8 T, by Theorem 2.4.6.

Ch 2.TF.9 T; The matrix is  $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ .

Ch 2.TF.10 F; The columns of a rotation matrix are unit vectors; see Theorem 2.2.3.

Ch 2.TF.11 F; Note that  $\det(A) = (k-2)^2 + 9$  is always positive, so that  $A$  is invertible for all values of  $k$ .

Ch 2.TF.12 T; Note that the columns are unit vectors, since  $(-0.6)^2 + (\pm 0.8)^2 = 1$ . The matrix has the form presented in Theorem 2.2.3.

Ch 2.TF.13 F; Consider  $A = I_2$  (or any other invertible  $2 \times 2$  matrix).

Ch 2.TF.14 T; Note that  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1}$  is the unique solution.

Ch 2.TF.15 F, by Theorem 2.4.9. Note that the determinant is 0.

Ch 2.TF.16 T, by Theorem 2.4.3.

Ch 2.TF.17 T; The shear matrix  $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$  works.

Ch 2.TF.18 T; Simplify to see that  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4y \\ -12x \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ .

Ch 2.TF.19 T; The equation  $\det(A) = k^2 - 6k + 10 = 0$  has no real solution.

Ch 2.TF.20 T; The matrix fails to be invertible for  $k = 5$  and  $k = -1$ , since the determinant  $\det A = k^2 - 4k - 5 = (k-5)(k+1)$  is 0 for these values of  $k$ .

Ch 2.TF.21 T; The product is  $\det(A)I_2$ .

Ch 2.TF.22 T; Writing an upper triangular matrix  $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$  and solving the equation  $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  we find that  $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ , where  $b$  is any nonzero constant.

Ch 2.TF.23 T; Note that the matrix  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  represents a rotation through  $\pi/2$ . Thus  $n = 4$  (or any multiple of 4) works.

Ch 2.TF.24 F; If a matrix  $A$  is invertible, then so is  $A^{-1}$ . But  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  fails to be invertible.

Ch 2.TF.25 F; If matrix  $A$  has two identical rows, then so does  $AB$ , for any matrix  $B$ . Thus  $AB$  cannot be  $I_n$ , so that  $A$  fails to be invertible.

Ch 2.TF.26 T, by Theorem 2.4.8. Note that  $A^{-1} = A$  in this case.

Ch 2.TF.27 F; For any  $2 \times 2$  matrix  $A$ , the two columns of  $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  will be identical.

Ch 2.TF.28 T; One solution is  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ .

Ch 2.TF.29 F; A reflection matrix is of the form  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a^2 + b^2 = 1$ . Here,  $a^2 + b^2 = 1 + 1 = 2$ .

Ch 2.TF.30 T; Just multiply it out.

Ch 2.TF.31 F; Consider matrix  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ , for example.

Ch 2.TF.32 T; Apply Theorem 2.4.8 to the equation  $(A^2)^{-1}AA = I_n$ , with  $B = (A^2)^{-1}A$ .

Ch 2.TF.33 F; Consider the matrix  $A$  that represents a rotation through the angle  $2\pi/17$ .

Ch 2.TF.34 F; Consider the reflection matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

Ch 2.TF.35 T; We have  $(5A)^{-1} = \frac{1}{5}A^{-1}$ .

Ch 2.TF.36 T; The equation  $A\vec{e}_i = B\vec{e}_i$  means that the  $i$ th columns of  $A$  and  $B$  are identical. This observation applies to all the columns.

Ch 2.TF.37 T; Note that  $A^2B = AAB = ABA = BAA = BA^2$ .

Ch 2.TF.38 T; Multiply both sides of the equation  $A^2 = A$  with  $A^{-1}$ .

Ch 2.TF.39 F; Consider  $A = I_2$  and  $B = -I_2$ .

Ch 2.TF.40 T; Since  $A\vec{x}$  is on the line onto which we project, the vector  $A\vec{x}$  remains unchanged when we project again:  $A(A\vec{x}) = A\vec{x}$ , or  $A^2\vec{x} = A\vec{x}$ , for all  $\vec{x}$ . Thus  $A^2 = A$ .

Ch 2.TF.41 T; If you reflect twice in a row (about the same line), you will get the original vector back:  $A(A\vec{x}) = \vec{x}$ , or,  $A^2\vec{x} = \vec{x} = I_2\vec{x}$ . Thus  $A^2 = I_2$  and  $A^{-1} = A$ .

Ch 2.TF.42 F; Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , for example.

Ch 2.TF.43 T; Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ , for example.

Ch 2.TF.44 F; By Theorem 1.3.3, there is a nonzero vector  $\vec{x}$  such that  $B\vec{x} = \vec{0}$ , so that  $AB\vec{x} = \vec{0}$  as well. But  $I_3\vec{x} = \vec{x} \neq \vec{0}$ , so that  $AB \neq I_3$ .

Ch 2.TF.45 T; We can rewrite the given equation as  $A^2 + 3A = -4I_3$  and  $-\frac{1}{4}(A + 3I_3)A = I_3$ . By Theorem 2.4.8, the matrix  $A$  is invertible, with  $A^{-1} = -\frac{1}{4}(A + 3I_3)$ .

Ch 2.TF.46 T; Note that  $(I_n + A)(I_n - A) = I_n^2 - A^2 = I_n$ , so that  $(I_n + A)^{-1} = I_n - A$ .

Ch 2.TF.47 F;  $A$  and  $C$  can be two matrices which fail to commute, and  $B$  could be  $I_n$ , which commutes with anything.

Ch 2.TF.48 F; Consider  $T(\vec{x}) = 2\vec{x}$ ,  $\vec{v} = \vec{e}_1$ , and  $\vec{w} = \vec{e}_2$ .

Ch 2.TF.49 F; Since there are only eight entries that are not 1, there will be at least two rows that contain only ones. Having two identical rows, the matrix fails to be invertible.

Ch 2.TF.50 F; Let  $A = B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , for example.

Ch 2.TF.51 F; We will show that  $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S$  fails to be diagonal, for an arbitrary invertible matrix  $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Now,  $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$ . Since  $c$  and  $d$  cannot both be zero (as  $S$  must be invertible), at least one of the off-diagonal entries ( $-c^2$  and  $d^2$ ) is nonzero, proving the claim.

Ch 2.TF.52 T; Consider an  $\vec{x}$  such that  $A^2\vec{x} = \vec{b}$ , and let  $\vec{x}_0 = A\vec{x}$ . Then  $A\vec{x}_0 = A(A\vec{x}) = A^2\vec{x} = \vec{b}$ , as required.

Ch 2.TF.53 T; Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Now we want  $A^{-1} = -A$ , or  $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ . This holds if  $ad - bc = 1$  and  $d = -a$ . These equations have many solutions: for example,  $a = d = 0, b = 1, c = -1$ . More generally, we can choose an arbitrary  $a$  and an arbitrary nonzero  $b$ . Then,  $d = -a$  and  $c = -\frac{1+a^2}{b}$ .

Ch 2.TF.54 F; Consider a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . We make an attempt to solve the equation  $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . Now the equation  $b(a+d) = 0$  implies that  $b = 0$  or  $d = -a$ .

If  $b = 0$ , then the equation  $d^2 + bc = -1$  cannot be solved.

If  $d = -a$ , then the two diagonal entries of  $A^2$ ,  $a^2 + bc$  and  $d^2 + bc$ , will be equal, so that the equations  $a^2 + bc = 1$  and  $d^2 + bc = -1$  cannot be solved simultaneously.

In summary, the equation  $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  cannot be solved.

---

Ch 2.TF.55 T; Recall from Definition 2.2.1 that a projection matrix has the form  $\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$ , where  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  is a unit vector. Thus,  $a^2 + b^2 + c^2 + d^2 = u_1^4 + (u_1 u_2)^2 + (u_1 u_2)^2 + u_2^4 = u_1^4 + 2(u_1 u_2)^2 + u_2^4 = (u_1^2 + u_2^2)^2 = 1^2 = 1$ .

Ch 2.TF.56 T; We observe that the systems  $AB\vec{x} = 0$  and  $B\vec{x} = 0$  have the same solutions (multiply with  $A^{-1}$  and  $A$ , respectively, to obtain one system from the other). Then, by True or False Exercise 45 in Chapter 1,  $\text{rref}(AB) = \text{rref}(B)$ .

## Chapter 3

### Section 3.1

3.1.1 Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ :

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ so that } x_1 = x_2 = 0.$$

$$\ker(A) = \{\vec{0}\}.$$

3.1.2 Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ :

$$\begin{bmatrix} 2 & 3 & 0 \\ 6 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so that } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix}$$

$$\text{Setting } t = 2 \text{ we find } \ker(A) = \text{span} \left[ \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right].$$

3.1.3 Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ ; note that all  $\vec{x}$  in  $\mathbb{R}^2$  satisfy the equation, so that  $\ker(A) = \mathbb{R}^2 = \text{span}(\vec{e}_1, \vec{e}_2)$ .

3.1.4 Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ , or  $x_1 + 2x_2 + 3x_3 = 0$ .

$$\text{The solutions are of the form } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t - 3r \\ t \\ r \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \text{ so that}$$

$$\ker(A) = \text{span} \left( \left[ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right] \right).$$

3.1.5 Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ .

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \begin{array}{l} x_1 = x_3 \\ x_2 = -2x_3 \end{array}; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

$$\ker(A) = \text{span} \left[ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right].$$

3.1.6 Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ .

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; x_1 + x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r-t \\ r \\ t \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\ker(A) = \text{span} \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

**3.1.7** Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ . Since  $\text{rref}(A) = I_3$  we have  $\ker(A) = \{\vec{0}\}$ .

**3.1.8** Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ . Solving this system yields  $\ker(A) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

**3.1.9** Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ . Solving this system yields  $\ker(A) = \{\vec{0}\}$ .

**3.1.10** Solving the system  $A\vec{x} = \vec{0}$  we find that  $\ker(A) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ .

**3.1.11** Solving the system  $A\vec{x} = \vec{0}$  we find that  $\ker(A) = \text{span} \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ .

**3.1.12** Solving the system  $A\vec{x} = \vec{0}$  we find that  $\ker(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right)$ .

**3.1.13** Solving the system  $A\vec{x} = \vec{0}$  we find that  $\ker(A) = \text{span} \left( \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$ .

**3.1.14** By Theorem 3.1.3, the image of  $A$  is the span of the column vectors of  $A$ :

$$\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right).$$

**3.1.15** By Theorem 3.1.3, the image of  $A$  is the span of the columns of  $A$ :

$$\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right).$$

Since any two of these vectors span all of  $\mathbb{R}^2$  already, we can write

$$\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right).$$

**3.1.16** By Theorem 3.1.3, the image of  $A$  is the span of the column vectors of  $A$ :

$$\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right).$$

Since these three vectors are parallel, we need only one of them to span the image:

$$\text{im}(A) = \text{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

**3.1.17** By Theorem 3.1.3,  $\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) = \mathbb{R}^2$  (the whole plane).

**3.1.18** By Theorem 3.1.3,  $\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 12 \end{bmatrix} \right) = \text{span} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  (a line in  $\mathbb{R}^2$ ).

**3.1.19** Since the four column vectors of  $A$  are parallel, we have  $\text{im}(A) = \text{span} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ , a line in  $\mathbb{R}^2$ .

**3.1.20** Since the three column vectors of  $A$  are parallel, we have  $\text{im}(A) = \text{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , a line in  $\mathbb{R}^3$ .

**3.1.21** By Theorem 3.1.3,  $\text{im}(A) = \text{span} \left( \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix} \right)$ .

We must simply find out how many of the column vectors are not redundant to determine a basis of the image. We can determine this by taking the rref of the matrix:

$$\left[ \begin{array}{ccc|c} 4 & 7 & 3 \\ 1 & 9 & 2 \\ 5 & 6 & 8 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right], \text{ which shows us that all three column vectors are independent: the span is all of } \mathbb{R}^3.$$

**3.1.22** Compare with the solution to Exercise 21.

$$\left[ \begin{array}{ccc|c} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{array} \right] \xrightarrow{\text{rref}} \left[ \begin{array}{ccc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$$

This computation shows that the third column vector of  $A$ ,  $\vec{v}_3$ , is a linear combination of the first two, Thus, only the first two vectors are independent, and the image is a plane in  $\mathbb{R}^3$ .

**3.1.23**  $\text{im}(T) = \mathbb{R}^2$  and  $\ker(T) = \{\vec{0}\}$ , since  $T$  is invertible (see Summary 3.1.8).

**3.1.24**  $\text{im}(T)$  is the plane  $x + 2y + 3z = 0$ , and  $\ker(T)$  is the line perpendicular to this plane, spanned by the vector  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  (compare with Examples 5 and 9).

**3.1.25**  $\text{im}(T) = \mathbb{R}^2$  and  $\ker(T) = \{\vec{0}\}$ , since  $T$  is invertible (see Summary 3.1.8).

**3.1.26** Since  $\lim_{t \rightarrow \infty} f(t) = \infty$  and  $\lim_{t \rightarrow -\infty} f(t) = -\infty$ , we have  $\text{im}(f) = \mathbb{R}$ .

A careful proof involves the intermediate value theorem (see Exercise 2.2.47c),

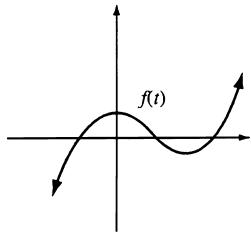


Figure 3.1: for Problem 3.1.26.

Any horizontal line intersects this graph at least once (compare with Example 3 and see Figure 3.1).

**3.1.27** Let  $f(x) = x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$ .

Then  $\text{im}(f) = \mathbb{R}$ , since

$$\lim_{x \rightarrow \infty} f(x) = \infty \text{ and } \lim_{x \rightarrow -\infty} f(x) = -\infty$$

but the function fails to be invertible since the equation  $f(x) = 0$  has three solutions,  $x = 0, 1$ , and  $-1$ .

**3.1.28** This ellipse can be obtained from the unit circle by means of the linear transformation with matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ , as shown in Figure 3.2 (compare with Exercise 2.2.49).

We obtain the parametrization  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$  for the ellipse.

We can check that  $x^2 + \frac{y^2}{4} = \cos^2(t) + \frac{4\sin^2(t)}{4} = 1$ .

**3.1.29** Use spherical coordinates (see any good text on multivariable calculus):  $f \begin{bmatrix} \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{bmatrix}$

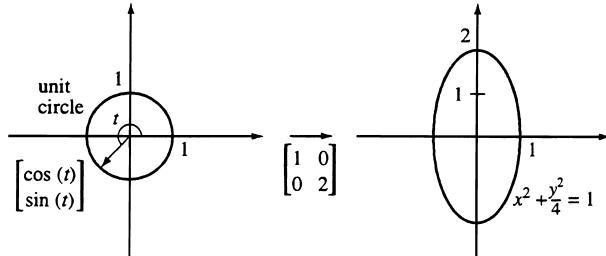


Figure 3.2: for Problem 3.1.28.

**3.1.30** By Theorem 3.1.3,  $A = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$  does the job. There are many other possible answers: any nonzero  $2 \times n$  matrix  $A$  whose column vectors are scalar multiples of vector  $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ .

**3.1.31** The plane  $x + 3y + 2z = 0$  is spanned by the two vectors  $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ , for example. Therefore,

$A = \begin{bmatrix} -2 & -3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$  does the job. There are many other correct answers.

**3.1.32** By Theorem 3.1.3,  $A = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$  does the job. There are many other correct answers: any nonzero  $3 \times n$  matrix  $A$  whose column vectors are scalar multiples of  $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ .

**3.1.33** The plane is the kernel of the linear transformation  $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 2y + 3z$  from  $\mathbb{R}^3$  to  $\mathbb{R}$ .

**3.1.34** To describe a subset of  $\mathbb{R}^3$  as a kernel means to describe it as an intersection of planes (think about it). By inspection, the given line is the intersection of the planes

$$\begin{aligned} x + y &= 0 & \text{and} \\ 2x + z &= 0. \end{aligned}$$

This means that the line is the kernel of the linear transformation  $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + z \end{bmatrix}$  from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

**3.1.35**  $\ker(T) = \{\vec{x} : T(\vec{x}) = \vec{v} \cdot \vec{x} = 0\} =$  the plane with normal vector  $\vec{v}$ .

$\text{im}(T) = \mathbb{R}$ , since for every real number  $k$  there is a vector  $\vec{x}$  such that  $T(\vec{x}) = k$ , for example,  $\vec{x} = \frac{k}{\vec{v} \cdot \vec{v}} \vec{v}$ .

**3.1.36**  $\ker(T) = \{\vec{x} : T(\vec{x}) = \vec{v} \times \vec{x} = \vec{0}\} =$  the line spanned by  $\vec{v}$

(see Theorem A.10d in the Appendix)

$\text{im}(T) = \text{the plane with normal vector } \vec{v}$

By Definition A.9,  $T(\vec{x}) = \vec{v} \times \vec{x}$  is in this plane, for all  $\vec{x}$  in  $\mathbb{R}^3$ . Conversely, for every vector  $\vec{w}$  in this plane there is an  $\vec{x}$  in  $\mathbb{R}^3$  such that  $T(\vec{x}) = \vec{w}$ , namely  $\vec{x} = -\frac{1}{\vec{v} \cdot \vec{v}} T(\vec{w})$  (verify this!).

$$3.1.37 \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so that}$$

$\ker(A) = \text{span}(\vec{e}_1)$ ,  $\ker(A^2) = \text{span}(\vec{e}_1, \vec{e}_2)$ ,  $\ker(A^3) = \mathbb{R}^3$ , and

$\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2)$ ,  $\text{im}(A^2) = \text{span}(\vec{e}_1)$ ,  $\text{im}(A^3) = \{\vec{0}\}$ .

3.1.38 a If a vector  $\vec{x}$  is in  $\ker(A^k)$ , that is,  $A^k \vec{x} = \vec{0}$ , then  $\vec{x}$  is also in  $\ker(A^{k+1})$ , since  $A^{k+1} \vec{x} = AA^k \vec{x} = A\vec{0} = \vec{0}$ .

Therefore,  $\ker(A) \subseteq \ker(A^2) \subseteq \ker(A^3) \subseteq \dots$

Exercise 37 shows that these kernels need not be equal.

b If a vector  $\vec{y}$  is in  $\text{im}(A^{k+1})$ , that is,  $\vec{y} = A^{k+1} \vec{x}$  for some  $\vec{x}$ , then  $\vec{y}$  is also in  $\text{im}(A^k)$ , since we can write  $\vec{y} = A^k(A\vec{x})$ . Therefore,  $\text{im}(A) \supseteq \text{im}(A^2) \supseteq \text{im}(A^3) \supseteq \dots$

Exercise 37 shows that these images need not be equal.

3.1.39 a If a vector  $\vec{x}$  is in  $\ker(B)$ , that is,  $B\vec{x} = \vec{0}$ , then  $\vec{x}$  is also in  $\ker(AB)$ , since  $AB(\vec{x}) = A(B\vec{x}) = A\vec{0} = \vec{0}$ :

$\ker(B) \subseteq \ker(AB)$ .

Exercise 37 (with  $A = B$ ) illustrates that these kernels need not be equal.

b If a vector  $\vec{y}$  is in  $\text{im}(AB)$ , that is,  $\vec{y} = AB\vec{x}$  for some  $\vec{x}$ , then  $\vec{y}$  is also in  $\text{im}(A)$ , since we can write

$\vec{y} = A(B\vec{x})$ :

$\text{im}(AB) \subseteq \text{im}(A)$ .

Exercise 37 (with  $A = B$ ) illustrates that these images need not be equal.

3.1.40 For any  $\vec{x}$  in  $\mathbb{R}^m$ , the vector  $B\vec{x}$  is in  $\text{im}(B) = \ker(A)$ , so that  $AB\vec{x} = \vec{0}$ . If we apply this fact to  $\vec{x} = \vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$ , we find that all the columns of the matrix  $AB$  are zero, so that  $AB = 0$ .

$$3.1.41 \quad \text{a rref}(A) = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{bmatrix}, \text{ so that } \ker(A) = \text{span} \left[ \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right].$$

$$\text{im}(A) = \text{span} \left[ \begin{bmatrix} 0.36 \\ 0.48 \end{bmatrix} \right] = \text{span} \left[ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right].$$

Note that  $\text{im}(A)$  and  $\ker(A)$  are perpendicular lines.

$$\text{b } A^2 = A$$

If  $\vec{v}$  is in  $\text{im}(A)$ , with  $\vec{v} = A\vec{x}$ , then  $A\vec{v} = A^2\vec{x} = A\vec{x} = \vec{v}$ .

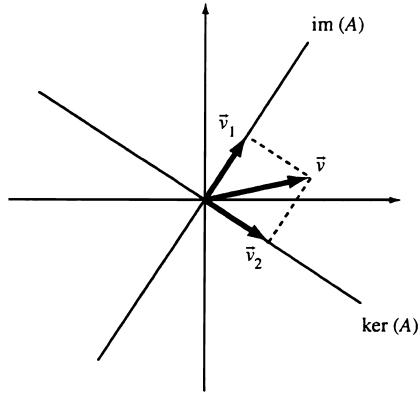


Figure 3.3: for Problem 3.1.41c.

- c Any vector  $\vec{v}$  in  $\mathbb{R}^2$  can be written uniquely as  $\vec{v} = \vec{v}_1 + \vec{v}_2$ , where  $\vec{v}_1$  is in  $\text{im}(A)$  and  $\vec{v}_2$  is in  $\ker(A)$ . (See Figure 3.3.) Then  $A\vec{v} = A\vec{v}_1 + A\vec{v}_2 = \vec{v}_1$  ( $A\vec{v}_1 = \vec{v}_1$  by part b,  $A\vec{v}_2 = \vec{0}$  since  $\vec{v}_2$  is in  $\ker(A)$ ), so that  $A$  represents the *orthogonal projection* onto  $\text{im}(A) = \text{span}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ .

- 3.1.42 Using the hint, we see that the vector  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$  is in the image of  $A$  if

$$\begin{array}{rrrr} y_1 & -3y_3 & +2y_4 & = 0 \\ y_2 & -2y_3 & +y_4 & = 0. \end{array} \quad \text{and}$$

This means that  $\text{im}(A)$  is the kernel of the matrix  $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$ .

- 3.1.43 Using our work in Exercise 42 as a guide, we come up with the following procedure to express the image of an  $n \times m$  matrix  $A$  as the kernel of a matrix  $B$ :

If  $\text{rank}(A) = n$ , let  $B$  be the  $n \times n$  zero matrix.

If  $r = \text{rank}(A) < n$ , let  $B$  be the  $(n-r) \times n$  matrix obtained by omitting the first  $r$  rows and the first  $m$  columns of  $\text{rref}[A : I_n]$ .

- 3.1.44 a Yes; by construction of the echelon form, the systems  $A\vec{x} = \vec{0}$  and  $B\vec{x} = \vec{0}$  have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).

- b No; as a counterexample, consider  $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ , with  $\text{im}(A) = \text{span}(\vec{e}_2)$ , but  $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , with  $\text{im}(B) = \text{span}(\vec{e}_1)$ .

- 3.1.45 As we solve the system  $A\vec{x} = \vec{0}$ , we obtain  $r$  leading variables and  $m - r$  free variables. The “general vector” in  $\ker(A)$  can be written as a linear combination of  $m - r$  vectors, with the free variables as coefficients. (See Example 11, where  $m - r = 5 - 3 = 2$ .)

**3.1.46** If  $\text{rank}(A) = r$ , then  $\text{im}(A) = \text{span}(\vec{e}_1, \dots, \vec{e}_r)$ . See Figure 3.4.

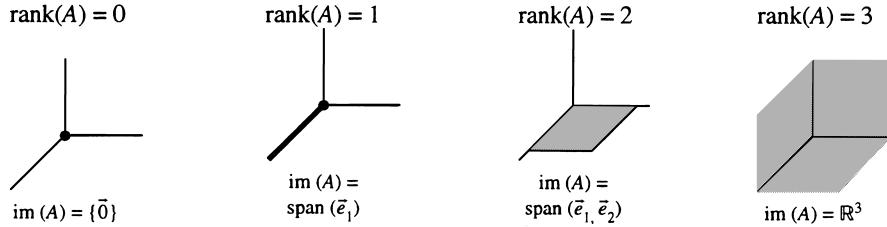


Figure 3.4: for Problem 3.1.46.

**3.1.47**  $\text{im}(T) = L_2$  and  $\ker(T) = L_1$ .

**3.1.48 a**  $\vec{w} = A\vec{x}$ , for some  $\vec{x}$ , so that  $A\vec{w} = A^2\vec{x} = A\vec{x} = \vec{w}$ .

b If  $\text{rank}(A) = 2$ , then  $A$  is invertible, and the equation  $A^2 = A$  implies that  $A = I_2$  (multiply by  $A^{-1}$ ).

$$\text{If } \text{rank}(A) = 0 \text{ then } A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

c First note that  $\text{im}(A)$  and  $\ker(A)$  are lines (there is one nonleading variable).

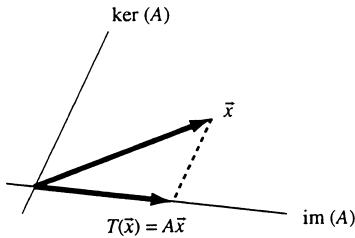


Figure 3.5: for Problem 3.1.48c.

By definition of a projection, we need to verify that  $\vec{x} - A\vec{x}$  is in  $\ker(A)$ . This is indeed the case, since

$$A(\vec{x} - A\vec{x}) = A\vec{x} - A^2\vec{x} = A\vec{x} - A\vec{x} = \vec{0} \quad (\text{we are told that } A^2 = A).$$

**3.1.49** If  $\vec{v}$  and  $\vec{w}$  are in  $\ker(T)$ , then  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{0} + \vec{0} = \vec{0}$ , so that  $\vec{v} + \vec{w}$  is in  $\ker(T)$  as well.

If  $\vec{v}$  is in  $\ker(T)$  and  $k$  is an arbitrary scalar, then  $T(k\vec{v}) = kT(\vec{v}) = k\vec{0} = \vec{0}$ , so that  $k\vec{v}$  is in  $\ker(T)$  as well.

**3.1.50** From Exercise 38 we know that  $\ker(A^3) \subseteq \ker(A^4)$ . Conversely, if  $\vec{x}$  is in  $\ker(A^4)$ , then  $A^4\vec{x} = A^3(A\vec{x}) = \vec{0}$ , so that  $A\vec{x}$  is in  $\ker(A^3) = \ker(A^2)$ , which implies that  $A^2(A\vec{x}) = A^3\vec{x} = \vec{0}$ , that is,  $\vec{x}$  is in  $\ker(A^3)$ . We have shown that  $\ker(A^3) = \ker(A^4)$ .

**3.1.51** We need to find all  $\vec{x}$  such that  $AB\vec{x} = \vec{0}$ . If  $AB\vec{x} = \vec{0}$ , then  $B\vec{x}$  is in  $\ker(A) = \{\vec{0}\}$ , so that  $B\vec{x} = \vec{0}$ .

Since  $\ker(B) = \{\vec{0}\}$ , we can conclude that  $\vec{x} = \vec{0}$ . It follows that  $\ker(AB) = \{\vec{0}\}$ .

**3.1.52** Since  $C\vec{x} = \begin{bmatrix} A \\ B \end{bmatrix}\vec{x} = \begin{bmatrix} A\vec{x} \\ B\vec{x} \end{bmatrix}$ , we can conclude that  $C\vec{x} = \vec{0}$  if (and only if) both  $A\vec{x} = \vec{0}$  and  $B\vec{x} = \vec{0}$ . It follows that  $\ker(C)$  is the intersection of  $\ker(A)$  and  $\ker(B)$ :  $\ker(C) = \ker(A) \cap \ker(B)$ .

**3.1.53 a** Using the equation  $1 + 1 = 0$  (or  $-1 = 1$ ), we can write the general vector  $\vec{x}$  in  $\ker(H)$  as

$$\begin{aligned} \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} p+r+s \\ p+q+s \\ p+q+r \\ p \\ q \\ r \\ s \end{bmatrix} \\ &= p \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &\quad \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array} \end{aligned}$$

b  $\ker(H) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$  by part (a), and  $\text{im}(M) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$  by Theorem 3.1.3, so that  $\text{im}(M) = \ker(H)$ .  $M\vec{x}$  is in  $\text{im}(M) = \ker(H)$ , so that  $H(M\vec{x}) = \vec{0}$ .

**3.1.54 a** If no error occurred, then  $\vec{w} = \vec{v} = M\vec{u}$ , and  $H\vec{w} = H(M\vec{u}) = \vec{0}$ , by Exercise 53b.

If an error occurred in the  $i$ th component, then  $\vec{w} = \vec{v} + \vec{e}_i = M\vec{u} + \vec{e}_i$ , so that

$$H\vec{w} = H(M\vec{u}) + H\vec{e}_i = \text{ith column of } H.$$

Since the columns of  $H$  are all different, this method allows us to find out where an error occurred.

b  $H\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  = seventh column of  $H$ : an error occurred in the seventh component of  $\vec{v}$ .

$$\text{Therefore } \vec{v} = \vec{w} + \vec{e}_7 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

## Section 3.2

**3.2.1** Not a subspace, since  $W$  does not contain the zero vector.

**3.2.2** Not a subspace, since  $W$  contains the vector  $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  but not the vector  $(-1)\vec{v} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ .

**3.2.3**  $W = \text{im} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$  is a subspace of  $\mathbb{R}^3$ , by Theorem 3.2.2.

**3.2.4**  $\text{span}(\vec{v}_1, \dots, \vec{v}_m) = \text{im}[\vec{v}_1 \dots \vec{v}_m]$  is a subspace of  $\mathbb{R}^n$ , by Theorem 3.2.2.

**3.2.5** We have subspaces  $\{\vec{0}\}$ ,  $\mathbb{R}^3$ , and all lines and planes (through the origin). To prove this, mimic the reasoning in Example 2.

**3.2.6** a Yes!

- The zero vector is in  $V \cap W$ , since  $\vec{0}$  is in both  $V$  and  $W$ .
- If  $\vec{x}$  and  $\vec{y}$  are in  $V \cap W$ , then both  $\vec{x}$  and  $\vec{y}$  are in  $V$ , so that  $\vec{x} + \vec{y}$  is in  $V$  as well, since  $V$  is a subspace of  $\mathbb{R}^n$ . Likewise,  $\vec{x} + \vec{y}$  is in  $W$ , so that  $\vec{x} + \vec{y}$  is in  $V \cap W$ .
- If  $\vec{x}$  is in  $V \cap W$  and  $k$  is an arbitrary scalar, then  $k\vec{x}$  is in both  $V$  and  $W$ , since they are subspaces of  $\mathbb{R}^n$ . Therefore,  $k\vec{x}$  is in  $V \cap W$ .

b No; as a counterexample consider  $V = \text{span}(\vec{e}_1)$  and  $W = \text{span}(\vec{e}_2)$  in  $\mathbb{R}^2$ .

**3.2.7** Yes; we need to show that  $W$  contains the zero vector. We are told that  $W$  is nonempty, so that it contains some vector  $\vec{v}$ . Since  $W$  is closed under scalar multiplication, it will contain the vector  $0\vec{v} = \vec{0}$ , as claimed.

**3.2.8** We need to solve the system  $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

The general solution is  $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$ .

Picking  $t = 1$  we find the nontrivial relation  $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

**3.2.9** These vectors are linearly dependent, since  $\vec{v}_m = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_{m-1}$ .

**3.2.10** Linearly dependent, since  $\begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ . Thus, the vector  $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$  is redundant.

**3.2.11** Linearly independent, since the two vectors are not parallel, and therefore not redundant.

**3.2.12** Linearly dependent, since  $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ . Thus, the vector  $\vec{0}$  is redundant.

**3.2.13** Linearly dependent, since the second vector is redundant ( $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ).

**3.2.14** Linearly independent, since  $\text{ref} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = I_3$  (use Theorem 3.2.6).

**3.2.15** Linearly dependent. By Theorem 3.2.8, since we have three vectors in  $\mathbb{R}^2$ , at least one must be redundant. We can perform a straightforward computation to reveal that  $\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2$ .

**3.2.16** Certainly  $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$  is not a multiple of  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , so it is not redundant. However,  $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ , so  $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$  is redundant. Thus, these vectors are linearly dependent.

**3.2.17** Linearly independent. The first two vectors are clearly not redundant, and since  $\text{rref} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = I_3$ , the last vector is also not redundant. Thus, the three vectors turn out to be linearly independent.

**3.2.18** Linearly dependent, since  $\text{rref} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 4 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . So, we find that the vector  $\begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$  turns out to be redundant.

**3.2.19** Linearly dependent. First we see that  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$  is not redundant, because it is first, and non-zero. However,

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ so it is redundant.}$$

$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$  are clearly not redundant, but  $\begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ , so it is redundant.

**3.2.20**  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  is redundant, simply because it is the zero vector.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is our first non-zero vector, and thus, is not redundant.

$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and is redundant.

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is not a multiple of  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and is not redundant.

$\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  and is redundant.

Similarly,  $\begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  and is also redundant.

However, by inspection,  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  is not a linear combination of  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , meaning that this last vector is not redundant. Thus, the seven vectors are linearly dependent.

**3.2.21** Certainly, since the second vector equals the first, the second is redundant. So  $\vec{v}_1 = \vec{v}_2$ ,  $1\vec{v}_1 - 1\vec{v}_2 = \vec{0}$ , revealing that  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is in  $\ker(A)$ .

**3.2.22**  $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$  is redundant, because  $\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . So,  $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \vec{0}$ . Thus,  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$  is in the kernel of  $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ .

**3.2.23** The first column is  $\vec{0}$ , so it is redundant.  $1\vec{v}_1 = \vec{0}$ , so  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is in  $\ker(A)$ .

**3.2.24**  $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$  is redundant, because  $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ . Thus,  $3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \vec{0}$  and  $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$  is in the kernel of  $\begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$ .

**3.2.25** The third column equals the first, so it is redundant and  $\vec{v}_1 = \vec{v}_3$ , or  $1\vec{v}_1 + 0\vec{v}_2 - 1\vec{v}_3 = \vec{0}$ . Thus,  $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$  is in  $\ker(A)$ .

**3.2.26**  $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , so  $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$  is redundant. Now,  $2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$ , revealing that  $\begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \end{bmatrix}$  is in the kernel of  $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

**3.2.27** A basis of  $\text{im}(A)$  is  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , by Theorem 3.2.4.

**3.2.28** The three column vectors are linearly independent, since  $\text{rref} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} = I_3$ .

Therefore, the three columns form a basis of  $\text{im}(A)(=\mathbb{R}^3)$ :

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}.$$

Another sensible choice for a basis of  $\text{im}(A)$  is  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ .

**3.2.29** The three column vectors of  $A$  span all of  $\mathbb{R}^2$ , so that  $\text{im}(A) = \mathbb{R}^2$ . We can choose any two of the columns of  $A$  to form a basis of  $\text{im}(A)$ ; another sensible choice is  $\vec{e}_1, \vec{e}_2$ .

**3.2.30**  $\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2)$

We can choose  $\vec{e}_1, \vec{e}_2$  as a basis of  $\text{im}(A)$ .

**3.2.31** The two column vectors of the given matrix  $A$  are linearly independent (they are not parallel), so that they form a basis of  $\text{im}(A)$ .

**3.2.32** By inspection, the first, third and sixth columns are redundant. Thus, a basis of the image consists of the remaining column vectors:  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ .

**3.2.33**  $\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ , so that  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  is a basis of  $\text{im}(A)$ .

**3.2.34** The fact that  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$  is in  $\ker(A)$  means that

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0}, \text{ so that } \vec{v}_4 = -\frac{1}{4}\vec{v}_1 - \frac{1}{2}\vec{v}_2 - \frac{3}{4}\vec{v}_3.$$

**3.2.35** If  $\vec{v}_i$  is a linear combination of the other vectors in the list,  $\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n$ , then we can subtract  $\vec{v}_i$  from both sides to generate a nontrivial relation (the coefficient of  $\vec{v}_i$  will be -1).

Conversely, if there is a nontrivial relation  $c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_n\vec{v}_n = \vec{0}$ , with  $c_i \neq 0$ , then we can solve for vector  $\vec{v}_i$  and thus express  $\vec{v}_i$  as a linear combination of the other vectors in the list.

**3.2.36** Yes; we know that there is a nontrivial relation  $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}$ .

Now apply the transformation  $T$  to the vectors on both sides, and use linearity:

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = T(\vec{0}), \text{ so that } c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_mT(\vec{v}_m) = \vec{0}.$$

This is a nontrivial relation among the vectors  $T(\vec{v}_1), \dots, T(\vec{v}_m)$ , so that these vectors are linearly dependent, as claimed.

**3.2.37** No; as a counterexample, consider the extreme case when  $T$  is the zero transformation, that is,  $T(\vec{x}) = \vec{0}$  for all  $\vec{x}$ . Then the vectors  $T(\vec{v}_1), \dots, T(\vec{v}_m)$  will all be zero, so that they are linearly dependent.

**3.2.38 a** Using the terminology introduced in the exercise, we need to show that any vector  $\vec{v}$  in  $V$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_m$ . Choose a specific vector  $\vec{v}$  in  $V$ . Since we can find no more than  $m$  linearly independent vectors in  $V$ , the  $m+1$  vectors  $\vec{v}_1, \dots, \vec{v}_m, \vec{v}$  will be linearly dependent. Since the vectors  $\vec{v}_1, \dots, \vec{v}_m$  are independent,  $\vec{v}$  must be redundant, meaning that  $\vec{v}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_m$ , as claimed.

b With the terminology introduced in part a, we can let  $V = \text{im} [\vec{v}_1 \ \cdots \ \vec{v}_m]$ .

**3.2.39** Yes; the vectors are linearly independent. The vectors in the list  $\vec{v}_1, \dots, \vec{v}_m$  are linearly independent (and therefore non-redundant), and  $\vec{v}$  is non-redundant since it fails to be in the span of  $\vec{v}_1, \dots, \vec{v}_m$ .

**3.2.40** Yes; by Theorem 3.2.8,  $\ker(A) = \{\vec{0}\}$  and  $\ker(B) = \{\vec{0}\}$ . Then  $\ker(AB) = \{\vec{0}\}$  by Exercise 3.1.51, so that the columns of  $AB$  are linearly independent, by Theorem 3.2.8.

**3.2.41** To show that the columns of  $B$  are linearly independent, we show that  $\ker(B) = \{\vec{0}\}$ . Indeed, if  $B\vec{x} = \vec{0}$ , then  $AB\vec{x} = A\vec{0} = \vec{0}$ , so that  $\vec{x} = \vec{0}$  (since  $AB = I_m$ ).

By Theorem 3.2.8,  $\text{rank}(B) = \# \text{ columns} = m$ , so that  $m \leq n$  and in fact  $m < n$  (we are told that  $m \neq n$ ). This implies that the rank of the  $m \times n$  matrix  $A$  is less than  $n$ , so that the columns of  $A$  are linearly dependent (by Theorem 3.2.8).

**3.2.42** We can use the hint and form the dot product of  $\vec{v}_i$  and both sides of the relation

$$c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m = \vec{0};$$

$$(c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m) \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i, \text{ so that } c_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_m(\vec{v}_m \cdot \vec{v}_i) = 0.$$

Since  $\vec{v}_i$  is perpendicular to all the other  $\vec{v}_j$ , we will have  $\vec{v}_i \cdot \vec{v}_j = 0$  whenever  $j \neq i$ ; since  $\vec{v}_i$  is a unit vector, we will have  $\vec{v}_i \cdot \vec{v}_i = 1$ . Therefore, the equation above simplifies to  $c_i = 0$ .

Since this reasoning applies to all  $i = 1, \dots, m$ , we have only the trivial relation among the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ , so that these vectors are linearly independent, as claimed.

**3.2.43** Consider a linear relation  $c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}$ , or,  $(c_1 + c_2 + c_3)\vec{v}_1 + (c_2 + c_3)\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ . Since there is only the trivial relation among the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ , we must have  $c_1 + c_2 + c_3 = c_2 + c_3 = c_3 = 0$ , so that  $c_3 = 0$  and then  $c_2 = 0$  and then  $c_1 = 0$ , as claimed.

**3.2.44** Yes; this is a special case of Exercise 40 (recall that  $\ker(A) = \{\vec{0}\}$ , by Theorem 3.1.7b).

**3.2.45** Yes; if  $A$  is invertible, then  $\ker(A) = \{\vec{0}\}$ , so that the columns of  $A$  are linearly independent, by Theorem 3.2.8.

**3.2.46** Solve the system  $\begin{bmatrix} x_1 & + & 2x_2 & & & + & 3x_4 & + & 5x_5 & = 0 \\ & & & x_3 & + & 4x_4 & + & 6x_5 & = 0 \end{bmatrix}$ .

The solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 3t - 5r \\ s \\ -4t - 6r \\ t \\ r \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}$  span the kernel, by construction, and they are linearly independent, by Theorem 3.2.5. Therefore, the three vectors form a basis of the kernel.

**3.2.47** By Theorem 3.2.8, the rank of  $A$  is 3. Thus,  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .

**3.2.48** We can write  $3x_1 + 4x_2 + 5x_3 = [3 \ 4 \ 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ , so that  $V = \ker[3 \ 4 \ 5]$ .

To express  $V$  as an image, choose a basis of  $V$ , for example,  $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix}$ .

Then,  $V = \text{im} \begin{bmatrix} 4 & 0 \\ -3 & 5 \\ 0 & -4 \end{bmatrix}$ .

There are other solutions.

**3.2.49**  $L = \text{im} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

To write  $L$  as a kernel, think of  $L$  as the intersection of the planes  $x = y$  and  $y = z$ , that is, as the solution set of the system  $\begin{bmatrix} x & -y & & = 0 \\ & y & -z & = 0 \end{bmatrix}$ .

Therefore,  $L = \ker \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ .

There are other solutions.

**3.2.50** The verification of the three properties listed in Definition 3.2.1 is straightforward. Alternatively, we can choose a basis  $\vec{v}_1, \dots, \vec{v}_p$  of  $V$  and a basis  $\vec{w}_1, \dots, \vec{w}_q$  of  $W$  (see Exercise 38a) and show that  $V + W = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q)$  (compare with Exercise 4).

Indeed, if  $\vec{v} + \vec{w}$  is in  $V + W$ , then  $\vec{v}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_p$  and  $\vec{w}$  is a linear combination of  $\vec{w}_1, \dots, \vec{w}_q$ , so that  $\vec{v} + \vec{w}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ . Conversely, if  $\vec{x}$  is in  $\text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q)$ , then  $\vec{x} = (c_1 \vec{v}_1 + \dots + c_p \vec{v}_p) + (d_1 \vec{w}_1 + \dots + d_q \vec{w}_q)$ , so that  $\vec{x}$  is in  $V + W$ .

If  $V$  and  $W$  are distinct lines in  $\mathbb{R}^3$  (spanned by  $\vec{v}$  and  $\vec{w}$ , respectively), then  $V + W$  is the plane spanned by  $\vec{v}$  and  $\vec{w}$ .

**3.2.51 a** Consider a relation  $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p + d_1\vec{w}_1 + \cdots + d_q\vec{w}_q = \vec{0}$ .

Then the vector  $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = -d_1\vec{w}_1 - \cdots - d_q\vec{w}_q$  is both in  $V$  and in  $W$ , so that this vector is  $\vec{0}$ :  $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$  and  $d_1\vec{w}_1 + \cdots + d_q\vec{w}_q = \vec{0}$ .

Now the  $c_i$  are all zero (since the  $\vec{v}_i$  are linearly independent) and the  $d_j$  are zero (since the  $\vec{w}_j$  are linearly independent).

Since there is only the trivial relation among the vectors  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ , they are linearly independent.

**b** In Exercise 50 we show that  $V + W = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q)$ , and in part (a) we show that these vectors are linearly independent.

**3.2.52** If  $a, c$  and  $f$  are nonzero, then rref  $\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ , and the three vectors are linearly independent,

by Theorem 3.2.6. If at least one of the constants  $a, c$  or  $f$  is zero, then at least one column of rref will not contain a leading one, so that the three vectors are linearly dependent.

**3.2.53** The zero vector is in  $V^\perp$ , since  $\vec{0} \cdot \vec{v} = 0$  for all  $\vec{v}$  in  $V$ .

If  $\vec{w}_1$  and  $\vec{w}_2$  are both in  $V^\perp$ , then  $(\vec{w}_1 + \vec{w}_2) \cdot \vec{v} = \vec{w}_1 \cdot \vec{v} + \vec{w}_2 \cdot \vec{v} = 0 + 0 = 0$  for all  $\vec{v}$  in  $V$ , so that  $\vec{w}_1 + \vec{w}_2$  is in  $V^\perp$  as well.

If  $\vec{w}$  is in  $V^\perp$  and  $k$  is an arbitrary constant, then  $(k\vec{w}) \cdot \vec{v} = k(\vec{w} \cdot \vec{v}) = k0 = 0$  for all  $\vec{v}$  in  $V$ , so that  $k\vec{w}$  is in  $V^\perp$  as well.

**3.2.54** We need to find all vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  in  $\mathbb{R}^3$  such that  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x + 2y + 3z = 0$ .

These vectors have the form  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ .

Therefore,  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$  is a basis of  $L^\perp$ .

**3.2.55** We need to find all vectors  $\vec{x}$  in  $\mathbb{R}^5$  such that  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0$ .

These vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2a - 3b - 4c - 5d \\ a \\ b \\ c \\ d \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The four vectors to the right form a basis of  $L^\perp$ ; they span  $L^\perp$ , by construction, and they are linearly independent, by Theorem 3.2.5.

**3.2.56** Consider a linear relation  $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$  among the four given vectors. The last component of the vector on the left hand side is  $c_3$ , so that  $c_3 = 0$ . Now the fifth component on the left is  $c_1$ , so that  $c_1 = 0$ . The third component is now  $c_4$ , so  $c_4 = 0$ . It follows that  $c_2 = 0$  as well.

We have shown that there is only the trivial relation among the given vectors, so that they are linearly independent, regardless of the values of the constants  $a, b, \dots, m$ .

**3.2.57** We will begin to go through the possibilities for  $j$  until we see a pattern:

$$j = 1: \text{Yes, because } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ is in } \ker(A) \text{ (the first column is } \vec{0}\text{).}$$

$j = 2$ : No, this would just be a multiple of the second column, and only  $\vec{0}$  if the  $j$ th component is zero.

$$j = 3: \text{Yes, since } \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ is in } \ker(A).$$

At this point, we realize that we are choosing the redundant columns. Thus,  $j$  can also be 6 and 7, because

$$\begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ are in } \ker(A).$$

**3.2.58** This occurs for each column,  $j$ , that is redundant. If  $\vec{x}$  is in the kernel, and the  $j^{th}$  component of  $\vec{x}$  is the last non-zero component, then

$$x_1\vec{v}_1 + \cdots + x_j\vec{v}_j + x_{j+1}\vec{v}_{j+1} + \cdots + x_m\vec{v}_m = \vec{0}, \text{ but } x_{j+1} = \cdots = x_m = 0, \text{ so}$$

$$x_1\vec{v}_1 + \cdots + x_j\vec{v}_j = \vec{0}.$$

Thus, since  $x_j \neq 0$ ,  $\vec{v}_j = -\frac{x_1\vec{v}_1 + \cdots + x_{j-1}\vec{v}_{j-1}}{x_j}$  and  $\vec{v}_j$  is redundant. Conversely, if  $\vec{v}_j$  is redundant, with  $\vec{v}_j = c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1}$ , then the vector  $\vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_{j-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$  is in the kernel of  $A$ . The last non-zero component of  $\vec{x}$  is the  $j^{th}$ , as required.

## Section 3.3

**3.3.1** Clearly the second column is just three times the first, and thus is redundant. Applying the notion of Kyle Numbers, we see:

$\begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 2 & 6 \end{bmatrix}$ , so  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$  is in the  $\ker(A)$ . No other vectors belong in our list, so a basis of the kernel is  $\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$ , and a basis of the image is  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .

**3.3.2** Using Kyle Numbers, we see that the second column is redundant:

$\begin{bmatrix} 4 & -1 \\ 1 & 4 \\ 2 & 8 \end{bmatrix}$ , so  $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$  is in the  $\ker(A)$ . No other vectors belong in our list, so a basis of the kernel is  $\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right)$ , and a basis of the image is  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .

**3.3.3** The two columns here are independent, so there are no redundant vectors. Thus,  $\emptyset$  is a basis of the kernel, and the two columns form a basis of the image:  $\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$ .

**3.3.4** The first column is redundant. We use the following Kyle Numbers:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$ , so  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is in the  $\ker(A)$ . No other vectors belong in our list, so a basis of the kernel is  $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ , and a basis of the image is  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .

**3.3.5** The first two vectors are non-redundant, but the third is a multiple of the first. We see:

$\begin{bmatrix} 3 & 0 & -1 \\ 1 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ , so a basis of the kernel is  $\left(\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}\right)$ , and a basis of the image consists of the non-redundant columns, or  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}\right)$ .

**3.3.6** The first two vectors are non-redundant, but the third is a combination of the first two:

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \text{ so a basis of the kernel is } \left( \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right), \text{ and a basis of the image is } \left( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right).$$

**3.3.7** We immediately see fitting Kyle numbers for one relation:

$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ . Now, since the second column is redundant, we remove it from further inspection and keep a zero above it:

$\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}$ . However, in this case, there are no more redundant vectors. Thus, a basis of the kernel is  $\left( \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right)$ , and a basis of the image is  $\left( \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right)$ .

**3.3.8** Here the second column is redundant, with Kyle Numbers as:

$\begin{bmatrix} 3 & 1 \\ 1 & -3 \\ 2 & -6 \\ 3 & -9 \end{bmatrix}$ . This reveals a basis of our kernel as  $\left( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$  and a basis of the image to be  $\left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$ .

**3.3.9** The second column is redundant, and we can choose Kyle numbers as follows:

$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ , but the third column is non-redundant. Thus, a basis of the kernel is  $\left( \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right)$ , while a basis of the image is  $\left( \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right)$ .

**3.3.10** The first column is redundant, and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is in the kernel:

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ . No other columns are redundant, however, meaning that a basis of the kernel is  $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$ , while a basis of the image is  $\left( \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right)$ .

**3.3.11** Here it is clear that only the third column is redundant, since it is equal to the first. Thus, a basis of the kernel is  $\left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$ , and  $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$  is a basis of the image.

**3.3.12** The third vector is the only redundant vector here, shown by:

$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ . No other columns are redundant, however, meaning that a basis of the kernel is  $\left( \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$ , while a basis of the image is  $\left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ .

**3.3.13** Here we first see  $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ , then  $\begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ ,

so both the second and third columns are redundant, and a basis of the kernel is  $\left( \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right)$ . This leaves ([1]) to be a basis of the image.

**3.3.14** The first and the third columns are redundant, as the Kyle Numbers show us:

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ , then  $\begin{bmatrix} 0 & 2 & -1 \\ \emptyset & 1 & 2 \end{bmatrix}$ , so that a basis of the kernel is  $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right)$ . This leaves ([1]) to be a basis of the image.

**3.3.15** We quickly find that the third column is redundant, with the Kyle numbers

$\begin{bmatrix} 2 & 2 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ , then see that the fourth column is also redundant,

$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ .

Thus, a basis of our kernel is  $\left( \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$ , while  $\left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$  is a basis of our image.

**3.3.16** The third column is redundant, as we find with

$\begin{bmatrix} 3 & 2 & -1 & 0 \\ 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ . The fourth column, however, fails to be redundant.

Thus, a basis of our kernel is  $\left(\begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}\right)$ , while  $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right)$  is a basis of our image.

**3.3.17** For this problem, we again successively use Kyle Numbers to find our kernel, investigating the columns from left to right. We initially see that the first column is redundant:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}, \text{ then the third column: } \begin{bmatrix} 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}, \text{ followed by the fifth column: } \begin{bmatrix} 0 & 3 & 0 & 4 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

Thus,  $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \\ -1 \end{bmatrix}\right)$  is a basis of the kernel, and  $\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$  is a basis of the image.

**3.3.18** This matrix is already in rref, and we see that there are two columns without leading ones. These will be our redundant columns. Thus we see

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 & 0 & 5 & -1 & 0 \\ 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then  $\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 5 \\ -1 \\ 0 \end{bmatrix}\right)$  is a basis of the kernel, and  $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$  is a basis of the image.

**3.3.19** We see that the third column is redundant, and choose Kyle numbers as follows:

$$\begin{bmatrix} 5 & 4 & -1 & 0 & 0 \\ 1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then we see that the fourth column is also redundant,}$$

$$\begin{bmatrix} 3 & 2 & 0 & -1 & 0 \\ 1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,  $\left(\begin{bmatrix} 5 \\ 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right)$  is a basis of the kernel, and  $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$  is a basis of the image.

**3.3.20** Although this matrix is not quite in rref, we can still quickly see that columns 2, 3, and 5 are the redundant columns:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 & -1 & 0 & 0 \\ 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -12 & 0 & 0 & 3 & -1 \\ 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So,  $\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -12 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right)$  is a basis of the kernel, and  $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$  is a basis of the image.

**3.3.21** rref  $\begin{bmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ , which we can use to “spot” a vector in the kernel:  $\begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$ . Since the third column is the only redundant one, this forms a basis of the kernel, and implies that the third column of  $A$  is also redundant. Thus, a basis of  $\text{im}(A)$  is  $\left(\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}\right)$ .

**3.3.22** rref  $\begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ . It is clear that the third vector is redundant, and we quickly see that the vector  $\begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}$  is in the kernel. Since this is the only redundant column,  $\left(\begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}\right)$  is a basis of the kernel. Thus, a basis of  $\text{im}(A)$  is  $\left(\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}\right)$ .

**3.3.23** rref  $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ .

Using the method of Exercises 17 and 19, we find the kernel:

$$\begin{bmatrix} 2 & -3 & -1 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then } \begin{bmatrix} 4 & -1 & 0 & -1 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So a basis of  $\ker(A)$  is  $\left(\begin{bmatrix} 2 \\ -3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \end{bmatrix}\right)$ . The non-redundant column vectors of  $A$  form a basis of  $\text{im}(A)$ :  $\left(\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}\right)$ .

**3.3.24** rref  $\begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ . Here our kernel is the span of only one vector:

$\left( \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$ , while a basis of the image of  $A$  is  $\left( \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 10 \\ 0 \\ 0 \end{bmatrix} \right)$ .

**3.3.25** rref  $\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ . We will emulate Exercise 23 to find the

kernel:  $\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ , then  $\begin{bmatrix} 5 & 0 & -1 & -1 & 0 \\ 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ .

So a basis of  $\ker(A)$  is  $\left( \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$  and a basis of  $\text{im}(A)$  is  $\left( \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 9 \\ 2 \end{bmatrix} \right)$ .

**3.3.26** a We notice that each of the six matrices has two identical columns. In matrices  $C$  and  $L$ , the second column is identical to the third, so that  $\ker(C) = \ker(L) = \text{span} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ . In matrices  $H, T, X$  and  $Y$ , the first column is identical to the third, so that  $\ker(H) = \ker(T) = \ker(X) = \ker(Y) = \text{span} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ . Thus, only  $L$  has the same kernel as  $C$ .

b We observe that each of the six matrices in the list has two identical rows. For example, the first and the last row of matrix  $C$  are identical, so that any vector  $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$  in  $\text{im}(C)$  will satisfy the equation  $y_1 = y_3$ . We can conclude that  $\text{im}(C) = \text{im}(H) = \text{im}(X) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_3 \right\}$ ,  $\text{im}(L) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_2 \right\}$ , and  $\text{im}(T) = \text{im}(Y) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_2 = y_3 \right\}$ .

c Our discussion in part b shows that the answer is matrix  $L$ .

**3.3.27** Form a  $4 \times 4$  matrix  $A$  with the given vectors as its columns. We find that  $\text{rref}(A) = I_4$ , so that the vectors

do indeed form a basis of  $\mathbb{R}^4$ , by Summary 3.3.10.

**3.3.28** Form a  $4 \times 4$  matrix  $A$  with the given vectors as its columns. The matrix  $A$  reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k-29 \end{bmatrix}.$$

This matrix can be reduced further to  $I_4$  if (and only if)  $k-29 \neq 0$ , that is, if  $k \neq 29$ .

By Summary 3.3.10, the four given vectors form a basis of  $\mathbb{R}^4$  unless  $k = 29$ .

**3.3.29**  $x_1 = -\frac{3}{2}x_2 - \frac{1}{2}x_3$ ; let  $x_2 = s$  and  $x_3 = t$ . Then the solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s - \frac{1}{2}t \\ s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Multiplying the two vectors by 2 to simplify, we obtain the basis  $\begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$ .

**3.3.30** Proceeding as in Exercise 29, we find the basis  $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ .

**3.3.31** Proceeding as in Exercise 29, we can find the following basis of  $V$ :  $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ .

Now let  $A$  be the  $4 \times 3$  matrix with these three vectors as its columns. Then  $\text{im}(A) = V$  by Theorem 3.1.3, and  $\ker(A) = \{\vec{0}\}$  by Theorem 3.2.8, so that  $A$  does the job.

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**3.3.32** We need to find all vectors  $\vec{x}$  in  $\mathbb{R}^4$  such that  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0$  and  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$ .

This amounts to solving the system  $\begin{bmatrix} x_1 & -x_3 & +x_4 & = 0 \\ x_2 & +2x_3 & +3x_4 & = 0 \end{bmatrix}$ , which in turn amounts to finding the kernel of  $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ .

Using Kyle Numbers, we find the basis  $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ .

**3.3.33** We can write  $V = \ker(A)$ , where  $A$  is the  $1 \times n$  matrix  $A = [c_1 \ c_2 \ \cdots \ c_n]$ .

Since at least one of the  $c_i$  is nonzero, the rank of  $A$  is 1, so that  $\dim(V) = \dim(\ker(A)) = n - \text{rank}(A) = n - 1$ , by Theorem 3.3.7.

A “hyperplane” in  $\mathbb{R}^2$  is a line, and a “hyperplane” in  $\mathbb{R}^3$  is just a plane.

**3.3.34** We can write  $V = \ker(A)$ , where  $A$  is the  $n \times m$  matrix with entries  $a_{ij}$ . Note that  $\text{rank}(A) \leq n$ . Therefore,  $\dim(V) = \dim(\ker(A)) = m - \text{rank}(A) \geq m - n$ , by Theorem 3.3.7.

**3.3.35** We need to find all vectors  $\vec{x}$  in  $\mathbb{R}^n$  such that  $\vec{v} \cdot \vec{x} = 0$ , or  $v_1x_1 + v_2x_2 + \cdots + v_nx_n = 0$ , where the  $v_i$  are the components of the vector  $\vec{v}$ . These vectors form a hyperplane in  $\mathbb{R}^n$  (see Exercise 33), so that the dimension of the space is  $n - 1$ .

**3.3.36** No; if  $\text{im}(A) = \ker(A)$  for an  $n \times n$  matrix  $A$ , then  $n = \dim(\ker(A)) + \dim(\text{im}(A)) = 2\dim(\text{im}(A))$ , so that  $n$  is an even number.

**3.3.37** Since  $\dim(\ker(A)) = 5 - \text{rank}(A)$ , any  $4 \times 5$  matrix with rank 2 will do; for example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

**3.3.38** a The rank of a  $3 \times 5$  matrix  $A$  is 0,1,2, or 3, so that  $\dim(\ker(A)) = 5 - \text{rank}(A)$  is 2,3,4, or 5.

b The rank of a  $7 \times 4$  matrix  $A$  is at most 4, so that  $\dim(\text{im}(A)) = \text{rank}(A)$  is 0,1,2,3, or 4.

**3.3.39** Note that  $\ker(C) \neq \{\vec{0}\}$ , by Theorem 3.1.7a, and  $\ker(C) \subseteq \ker(A)$ . Therefore,  $\ker(A) \neq \{\vec{0}\}$ , so that  $A$  is not invertible.

**3.3.40** Substituting a point  $(x, y) = (x_i, y_i)$  into the equation  $c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 = 0$  produces a linear equation in the six unknowns  $c_1, \dots, c_6$ . Fitting a conic through  $m$  points  $P_i(x_i, y_i)$ , for  $i = 1, \dots, m$ , amounts to solving a system of  $m$  homogenous linear equations in six unknowns. This system can be written in matrix form as  $A\vec{x} = \vec{0}$ , where  $A$  is an  $m \times 6$  matrix. The  $i^{th}$  row of  $A$  is  $[1 \ x_i \ y_i \ x_i^2 \ x_iy_i \ y_i^2]$ .

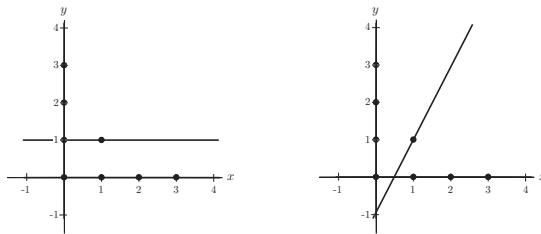
**3.3.41** The kernel of a  $4 \times 6$  matrix is at least two-dimensional. Since every one-dimensional subspace of this kernel defines a conic through the four given points, there will be infinitely many such conics.

**3.3.42** The kernel of a  $5 \times 6$  matrix is at least one-dimensional, so that there is at least one conic passing through five given points. Exercise 1.2.52 provides an example with exactly one solution, while there are infinitely many solutions in Exercise 1.2.54.

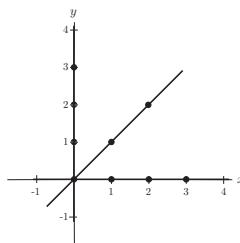
**3.3.43** Here, “anything can happen”:

1. If  $\ker A = \{\vec{0}\}$ , then there is no solution, as in Exercise 1.2.59
2. If  $\dim \ker A = 1$ , then there is a unique solution. For example, the only conic through the six points  $(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (0, 2)$  is  $xy = 0$ .
3. If  $\dim \ker A > 1$ , then there are infinitely many solutions. For example, any conic consisting of the  $x$  axis and some other line runs through the six points  $(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0)$ .

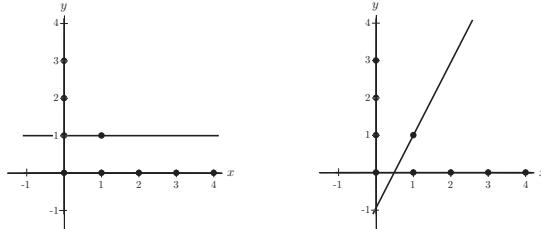
**3.3.44** To run through the points  $(0, 0), (1, 0), (2, 0)$ , and  $(3, 0)$ , the cubic must satisfy the equations  $c_1 = 0$ ,  $c_1 + c_2 + c_4 + c_7 = 0$ ,  $c_1 + 2c_2 + 4c_4 + 8c_7 = 0$ , and  $c_1 + 3c_2 + 9c_4 + 27c_7 = 0$ . This means that  $c_1 = c_2 = c_4 = c_7 = 0$ . Likewise, the cubic runs through the points  $(0, 0), (0, 1), (0, 2)$  and  $(0, 3)$  if (and only if)  $c_1 = c_3 = c_6 = c_{10} = 0$ . Therefore, to run through the first seven given points, the equation must be of the form  $c_5xy + c_8x^2y + c_9xy^2 = 0$ . The last point,  $(1, 1)$  imposes the condition  $c_5 + c_8 + c_9 = 0$ , or  $c_5 = -c_8 - c_9$ . Setting  $c_8 = a$  and  $c_9 = b$ , we find the family of cubics  $(-a - b)xy + ax^2y + bxy^2 = xy(ax + by - a - b) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Such a cubic is the union of the  $x$  axis, the  $y$  axis, and some line through the point  $(1, 1)$ . Two sample solutions are shown in the accompanying figures.



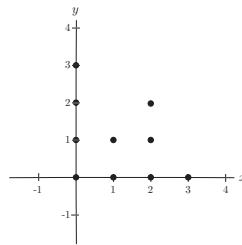
**3.3.45** To run through the points  $(0, 0), (1, 0), (2, 0)$ , and  $(3, 0)$ , the cubic must satisfy the equations  $c_1 = 0$ ,  $c_1 + c_2 + c_4 + c_7 = 0$ ,  $c_1 + 2c_2 + 4c_4 + 8c_7 = 0$ , and  $c_1 + 3c_2 + 9c_4 + 27c_7 = 0$ . This means that  $c_1 = c_2 = c_4 = c_7 = 0$ . Likewise, the cubic runs through the points  $(0, 0), (0, 1), (0, 2)$  and  $(0, 3)$  if (and only if)  $c_1 = c_3 = c_6 = c_{10} = 0$ . Therefore, to run through the first seven given points, the equation must be of the form  $c_5xy + c_8x^2y + c_9xy^2 = 0$ . The last two points,  $(1, 1)$  and  $(2, 2)$ , impose the conditions  $c_5 + c_8 + c_9 = 0$  and  $4c_5 + 8c_8 + 8c_9 = 0$ , implying that  $c_5 = 0$  and  $c_8 = -c_9$ . Letting  $c_9 = 1$ , we find that there is one such cubic,  $-x^2y + xy^2 = xy(y - x) = 0$ , the union of the  $x$  axis, the  $y$  axis, and the line  $y = x$ .



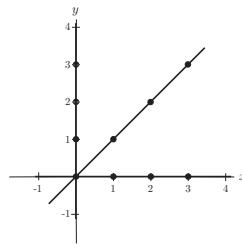
**3.3.46** Here we add the point  $(4, 0)$  to the list of points in Exercise 44. In Exercise 44 we found the family of cubics  $xy(ax + by - a - b) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Since all these cubics run through the point  $(4, 0)$ , we find the same solutions here:  $xy(ax + by - a - b) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Two sample solutions are shown in the accompanying figures.



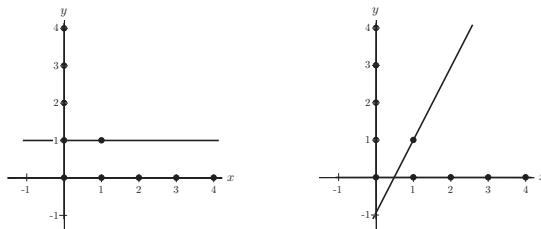
**3.3.47** Here we add the point  $(2, 1)$  to the list of points in Exercise 45 . In Exercise 45 we found the cubic  $xy(y - x) = 0$ . Since this cubic fails to run through the point  $(2, 1)$ , there are no solutions here.



**3.3.48** Here we add the point  $(3, 3)$  to the list of points in Exercise 45 . In Exercise 45 we found the cubic  $xy(y - x) = 0$ . Since this cubic runs though the point  $(3, 3)$ , we get the same solution here,  $xy(y - x) = 0$ .

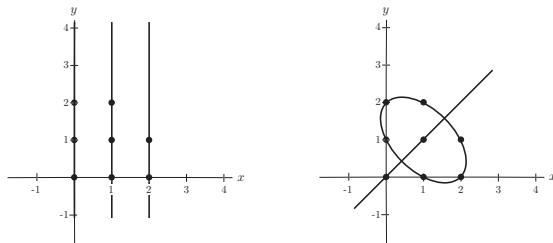


**3.3.49** Here we add the point  $(0, 4)$  to the list of points in Exercise 46 . In Exercise 46 we found the family of cubics  $xy(ax + by - a - b) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Since all these cubics run through the point  $(0, 4)$ , we find the same solutions here:  $xy(ax + by - a - b) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Each such cubic is the union of the  $x$  axis, the  $y$  axis, and some line through the point  $(1,1)$ . Two sample solutions are shown in the accompanying figures.

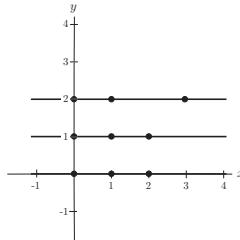


**3.3.50** To run through the points  $(0, 0)$ ,  $(1, 0)$ , and  $(2, 0)$ , the cubic must satisfy the equations  $c_1 = 0$ ,  $c_1 + c_2 + c_4 + c_7 = 0$ , and  $c_1 + 2c_2 + 4c_4 + 8c_7 = 0$ . This means that  $c_1 = 0$ ,  $c_2 = 2c_7$  and  $c_4 = -3c_7$ . Likewise, the cubic runs

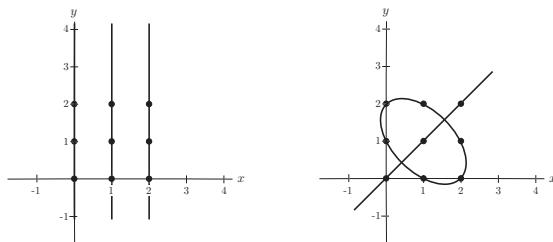
through the points  $(0, 0)$ ,  $(0, 1)$ , and  $(0, 2)$  if (and only if)  $c_1 = 0$ ,  $c_3 = 2c_{10}$  and  $c_6 = -3c_{10}$ . Therefore, to run through the points  $(0, 0)$ ,  $(1, 0)$ ,  $(2, 0)$ ,  $(0, 1)$ ,  $(0, 2)$  the cubic must be of the form  $c_5xy + c_7(x^3 - 3x^2 + 2x) + c_8x^2y + c_9xy^2 + c_{10}(y^3 - 3y^2 + 2y) = 0$ . The three additional points,  $(1, 1)$ ,  $(2, 1)$ , and  $(1, 2)$  impose the conditions  $c_5 + c_8 + c_9 = 0$ ,  $2c_5 + 4c_8 + 2c_9 = 0$  and  $2c_5 + 2c_8 + 4c_9 = 0$ , implying that  $c_5 = c_8 = c_9 = 0$ . Letting  $c_7 = a$  and  $c_{10} = b$ , we find the family of cubics  $a(x^3 - 3x^2 + 2x) + b(y^3 - 3y^2 + 2y) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Two sample solutions are shown in the accompanying figures.



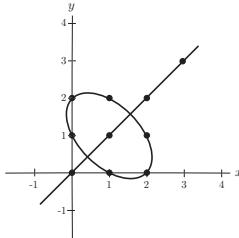
**3.3.51** Here we add the point  $(3, 2)$  to the list of points in Exercise 50. In Exercise 50 we found the family of cubics  $a(x^3 - 3x^2 + 2x) + b(y^3 - 3y^2 + 2y) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . The point  $(3, 2)$  now imposed the condition  $a = 0$ , so that the only solution here is  $y^3 - 3y^2 + 2y = 0$ , the union of the lines  $y = 0$ ,  $y = 1$ , and  $y = 2$ .



**3.3.52** Here we add the point  $(2, 2)$  to the list of points in Exercise 50. In Exercise 50 we found the family of cubics  $a(x^3 - 3x^2 + 2x) + b(y^3 - 3y^2 + 2y) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Since all these cubics run through the point  $(2, 2)$ , we find the same solutions here:  $a(x^3 - 3x^2 + 2x) + b(y^3 - 3y^2 + 2y) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . Two sample solutions are shown in the accompanying figures.



**3.3.53** Here we add the point  $(3, 3)$  to the list of points in Exercise 52. In Exercise 52 we found the family of cubics  $a(x^3 - 3x^2 + 2x) + b(y^3 - 3y^2 + 2y) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . The point  $(3, 3)$  imposes the condition  $a = -b$ , so that the only cubic here (with  $b = 1$ ) is  $-2x + 2y + 3x^2 - 3y^2 - x^3 + y^3 = 0$ , or  $x(x-1)(x-2) = y(y-1)(y-2)$ .



**3.3.54** Substituting a point  $(x, y) = (x_i, y_i)$  into the equation  $c_1 + c_2x + \dots + c_9xy^2 + c_{10}y^3 = 0$  produces a linear equation in the ten unknowns  $c_1, \dots, c_{10}$ . Fitting a cubic through  $m$  points  $P_i(x_i, y_i)$ , for  $i = 1, \dots, m$ , amounts to solving a system of  $m$  homogenous linear equations in ten unknowns. This system can be written in matrix form as  $A\vec{x} = \vec{0}$ , where  $A$  is an  $m \times 10$  matrix. The  $i^{th}$  row of  $A$  is  $[1 \ x_i \ y_i \ x_i^2 \ x_iy_i \ y_i^2 \ x_i^3 \ x_i^2y_i \ x_iy_i^2 \ y_i^3]$

**3.3.55** The kernel of a  $8 \times 10$  matrix is at least two-dimensional. Since every one-dimensional subspace of this kernel defines a cubic through the eight given points, there will be infinitely many such cubics.

**3.3.56** The kernel of a  $9 \times 10$  matrix is at least one-dimensional, so that there is at least one cubic passing through nine given points. Exercise 51 provides an example with exactly one solution, while there are infinitely many solutions in Exercise 52.

**3.3.57** Here, “anything can happen”:

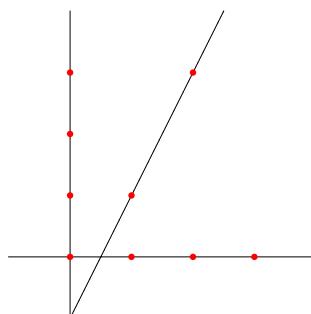
If  $\ker A = \{\vec{0}\}$ , then there is no solution, as in Exercise 47

If  $\dim \ker A = 1$ , then there is a unique solution, as in Exercise 48

If  $\dim \ker A > 1$ , then there are infinitely many solutions, as in Exercise 49

**3.3.58** It is not true that “Any nine distinct points determine a unique cubic,” as illustrated in Exercises 46 and 52, although “most” sets of nine distinct points do determine a unique cubic.

**3.3.59** Here we add the point  $P$  to the list of points in Exercise 44. In Exercise 44 we found the family of cubics  $xy(ax + by - a - b) = 0$ , where  $a \neq 0$  or  $b \neq 0$ . As long as we choose a point  $P$  on the  $x$  or on the  $y$  axis (with  $y = 0$  or  $x = 0$ ), all these cubics will run through  $P$ , so that there are infinitely many cubics through the nine given points. (Another, “cheap” solution is  $P = (1, 1)$ ). However, if  $P$  is neither  $(1, 1)$  nor located on one of the axes, then the only cubic through the nine points is the union of the two axes and the line through  $(1, 1)$  and  $P$ .



**3.3.60** We can choose a basis  $\vec{v}_1, \dots, \vec{v}_p$  in  $V$ , where  $p = \dim(V)$ . Then  $\vec{v}_1, \dots, \vec{v}_p$  are linearly independent vectors in  $W$ , so that  $\dim(V) = p \leq \dim(W)$ , by Theorem 3.3.4a, as claimed.

**3.3.61** We can choose a basis  $\vec{v}_1, \dots, \vec{v}_p$  of  $V$ , where  $p = \dim(V) = \dim(W)$ . Then  $\vec{v}_1, \dots, \vec{v}_p$  is a basis of  $W$  as well, by Theorem 3.3.4c, so that  $V = W = \text{span}(\vec{v}_1, \dots, \vec{v}_p)$ , as claimed.

**3.3.62** Consider a basis  $\vec{v}_1, \dots, \vec{v}_n$  of  $V$ . Since the  $\vec{v}_i$  are  $n$  linearly independent vectors in  $\mathbb{R}^n$ , they form a basis of  $\mathbb{R}^n$  (by parts (vii) and (ix) of Summary 3.3.10), so that  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_n) = \mathbb{R}^n$ , as claimed. (Note that Exercise 3.3.62 is a special case of Exercise 3.3.61.)

**3.3.63**  $\dim(V + W) = \dim(V) + \dim(W)$ , by Exercise 3.2.51b.

**3.3.64** Suppose that  $V \cap W = \{\vec{0}\}$  and  $\dim(V) + \dim(W) = n$ .

Choose a basis  $\vec{v}_1, \dots, \vec{v}_p$  of  $V$  and a basis  $\vec{w}_1, \dots, \vec{w}_q$  in  $W$ ; note that  $p + q = n$ . By Exercise 3.2.51b, the  $n$  vectors  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  in  $\mathbb{R}^n$  are linearly independent, so that they form a basis of  $\mathbb{R}^n$  (by parts (vii) and (ix) of Summary 3.3.10). By Theorem 3.2.10, any vector  $\vec{x}$  can be written uniquely as

$\vec{x} = (c_1\vec{v}_1 + \dots + c_p\vec{v}_p) + (d_1\vec{w}_1 + \dots + d_q\vec{w}_q)$ , with  $\vec{v} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p$  in  $V$  and  $\vec{w} = d_1\vec{w}_1 + \dots + d_q\vec{w}_q$  in  $W$ , which gives the desired representation.

Conversely, suppose  $V$  and  $W$  are complements. Let us first show that  $V \cap W = \{\vec{0}\}$  in this case.

Indeed, if  $\vec{x}$  is in  $V \cap W$ , then we can write  $\vec{x} = \vec{x} + \vec{0} = \vec{0} + \vec{x}$

$$\begin{array}{ccccccc} & \uparrow & \uparrow & \uparrow & \uparrow \\ & \text{in} & \text{in} & \text{in} & \text{in} \\ V & W & V & W \end{array}$$

Since this representation is unique (by definition of complements), we must have  $\vec{x} = \vec{0}$ , so that  $V \cap W = \{\vec{0}\}$ . By definition of complements, we have  $\mathbb{R}^n = V + W$ , so that  $n = \dim(V + W) = \dim(V) + \dim(W)$ , by Exercise 63.

**3.3.65** Note that  $\text{im}(A) = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q) = V$ , since the  $\vec{w}_j$  alone span  $V$ .

To find a basis of  $V = \text{im}(A)$ , we omit the redundant vectors from the list  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ , by Theorem 3.2.4. Since the vectors  $\vec{v}_1, \dots, \vec{v}_p$  are linearly independent, none of them are redundant, so that our basis of  $V$  contains all vectors  $\vec{v}_1, \dots, \vec{v}_p$  and some of the vectors from the list  $\vec{w}_1, \dots, \vec{w}_q$ .

**3.3.66** Use Exercise 65 with  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ , and  $\vec{w}_i = \vec{e}_i$  for  $i = 1, 2, 3, 4$ .

$$\text{Now rref } \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{3}{4} \end{bmatrix}.$$

Picking the non-redundant columns gives the basis  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ .

**3.3.67** Using the terminology suggested in the hint, we need to show that

$$\vec{u}_1, \dots, \vec{u}_m, \vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$$

is a basis of  $V + W$ . Then  $\dim(V + W) + \dim(V \cap W) = (m + p + q) + m = (m + p) + (m + q) = \dim(V) + \dim(W)$ , as claimed. Any vector  $\vec{x}$  in  $V + W$  can be written as  $\vec{x} = \vec{v} + \vec{w}$ , where  $\vec{v}$  is in  $V$  and  $\vec{w}$  is in  $W$ . Since  $\vec{v}$  is a linear combination of the  $\vec{u}_i$  and the  $\vec{v}_j$ , and  $\vec{w}$  is a linear combination of the  $\vec{u}_i$  and  $\vec{w}_j$ ,  $\vec{x}$  will be a linear combination of the  $\vec{u}_i, \vec{v}_j$ , and  $\vec{w}_k$ ; this shows that the vectors  $\vec{u}_1, \dots, \vec{u}_m, \vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  span  $V + W$ .

To show linear independence, consider the relation  $a_1\vec{u}_1 + \dots + a_m\vec{u}_m + b_1\vec{v}_1 + \dots + b_p\vec{v}_p + c_1\vec{w}_1 + \dots + c_q\vec{w}_q = \vec{0}$ . Then the vector  $a_1\vec{u}_1 + \dots + a_m\vec{u}_m + b_1\vec{v}_1 + \dots + b_p\vec{v}_p = -c_1\vec{w}_1 - \dots - c_q\vec{w}_q$  is in  $V \cap W$ , so that it can be expressed uniquely as a linear combination of  $\vec{u}_1, \dots, \vec{u}_m$  alone; this implies that the  $b_i$  are all zero. Now our relation simplifies to  $a_1\vec{u}_1 + \dots + a_m\vec{u}_m + c_1\vec{w}_1 + \dots + c_q\vec{w}_q = \vec{0}$ , which implies that the  $a_i$  and the  $c_j$  are zero as well (since the vectors  $\vec{u}_1, \dots, \vec{u}_m, \vec{w}_1, \dots, \vec{w}_q$  are linearly independent).

**3.3.68** By Exercise 3.3.67,  $\dim(V \cap W) = \dim(V) + \dim(W) - \dim(V + W) = 13 - \dim(V + W)$ .

The dimension of  $V + W$  is at least 7 (since  $W \subseteq V + W$ ) and at most 10 (since  $V + W \subseteq \mathbb{R}^{10}$ ); therefore the dimension of  $V \cap W$  is at least 3 and at most 6.

**3.3.69** The nonzero rows of  $E$  span the row space, and they are linearly independent (consider the leading ones), so that they form a basis of the row space:  $[0 \ 1 \ 0 \ 2 \ 0]$ ,  $[0 \ 0 \ 1 \ 3 \ 0]$ ,  $[0 \ 0 \ 0 \ 0 \ 1]$ .

**3.3.70** As in Exercise 3.3.69, we observe that the nonzero rows of  $E$  form a basis of the row space, so that  $\dim(\text{row space of } E) = \text{rank}(E)$ .

**3.3.71 a** All elementary row operations leave the row space unchanged, so that  $A$  and  $\text{rref}(A)$  have the same row space.

b By part (a) and Exercise 3.3.70,  $\dim(\text{row space of } A) = \dim(\text{row space of } \text{rref}(A)) = \text{rank}(\text{rref}(A)) = \text{rank}(A)$ .

$$3.3.72 \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Exercises 3.3.70 and 3.3.71a,  $[1 \ 0 \ -1 \ -2]$ ,  $[0 \ 1 \ 2 \ 3]$  is a basis of the row space of  $A$ .

**3.3.73** Using the terminology suggested in the hint, we observe that the vectors  $\vec{v}, A\vec{v}, \dots, A^n\vec{v}$  are linearly dependent (by Theorem 3.2.8), so that there is a nontrivial relation  $c_0\vec{v} + c_1A\vec{v} + \dots + c_nA^n\vec{v} = \vec{0}$ .

We can rewrite this relation in the form  $(c_0I_n + c_1A + \dots + c_nA^n)\vec{v} = \vec{0}$ .

The nonzero vector  $\vec{v}$  is in the kernel of the matrix  $c_0I_n + c_1A + \dots + c_nA^n$ , so that this matrix fails to be invertible.

**3.3.74** We can use the approach outlined in Exercise 3.3.73, with  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , say.

$$\text{Then } \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ and } A^2\vec{v} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

We find the relation  $5\vec{v} - 2A\vec{v} + A^2\vec{v} = \vec{0}$ , so that the matrix  $5I_2 - 2A + A^2$  does the job.

**3.3.75** If  $\text{rank}(A) = n$ , then the  $n$  non-redundant columns of  $A$  form a basis of  $\text{im}(A) = \mathbb{R}^n$ , so that the matrix formed by the non-redundant columns is invertible (by Summary 3.3.10).

Conversely, if  $A$  has an invertible  $n \times n$  submatrix  $B$ , then the columns of  $B$  form a basis of  $\mathbb{R}^n$  (again by Summary 3.3.10), so that  $\text{im}(A) = \mathbb{R}^n$  and therefore  $\text{rank}(A) = \dim(\text{im}(A)) = n$ .

**3.3.76** Using the terminology suggested in the Exercise, we multiply the relation  $c_0\vec{v} + c_1A\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$  with  $A^{m-1}$  and obtain  $c_0A^{m-1}\vec{v} = \vec{0}$  (all other terms vanish since  $A^m = 0$ ).

Since the vector  $A^{m-1}\vec{v}$  is nonzero (by construction), the scalar  $c_0$  must be zero, and our relation simplifies to  $c_1A\vec{v} + c_2A^2\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$ .

Now we multiply both sides with  $A^{m-2}$  and obtain  $c_1A^{m-1}\vec{v} = \vec{0}$ , so that  $c_1 = 0$  as above. Continuing like this we conclude that all the  $c_i$  must be zero, as claimed.

**3.3.77** As in Exercise 76, let  $m$  be the smallest positive integer such that  $A^m = 0$ . In Exercise 76 we construct  $m$  linearly independent vectors  $\vec{v}, A\vec{v}, \dots, A^{m-1}\vec{v}$  in  $\mathbb{R}^n$ ; now  $m \leq n$  by Theorem 3.2.8.

Therefore  $A^n = A^m A^{n-m} = 0A^{n-m} = 0$ , as claimed.

**3.3.78** If the vectors  $\vec{w}_1, \dots, \vec{w}_q$  span an  $m$ -dimensional space  $V$  (with basis  $\vec{v}_1, \dots, \vec{v}_m$ ), then  $m \leq q$  by Theorem 3.3.1 (since the vectors  $\vec{v}_i$  are linearly independent).

**3.3.79** Prove Theorem 3.3.4d: If  $m$  vectors  $\vec{v}_1, \dots, \vec{v}_m$  span an  $m$ -dimensional space  $V$ , then they form a basis of  $V$ . We need to show that the vectors  $\vec{v}_i$  are linearly independent. We will argue indirectly, assuming that the vectors are linearly dependent; this means that at least one of the vectors  $\vec{v}_i$  is redundant, say  $\vec{v}_p$ .

But then  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \dots, \vec{v}_m) = \text{span}(\vec{v}_1, \dots, \vec{v}_{p-1}, \vec{v}_{p+1}, \dots, \vec{v}_m)$ , contradicting Theorem 3.3.4b.

**3.3.80**  $\text{im}(A)$  is the plane onto which we project, so that  $\text{rank}(A) = \dim(\text{im}(A)) = 2$ .

**3.3.81 a** Note that  $\text{rank}(B) \leq 2$ , so that  $\dim(\ker(B)) = 5 - \text{rank}(B) \geq 3$  and  $\dim(\ker(AB)) \geq 3$  since  $\ker(B) \subseteq \ker(AB)$ . Since  $\ker(AB)$  is a subspace of  $\mathbb{R}^5$ ,  $\dim(\ker(AB))$  could be 3, 4, or 5. It is easy to give an example

for each case; for example, if  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ , then  $AB = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$  and  $\dim(\ker(AB)) = 3$ .

b Since  $\dim(\text{im}(AB)) = 5 - \dim(\ker(AB))$ , the possible values of  $\dim(\text{im}(AB))$  are 0, 1, and 2, by part a.

**3.3.82** Write  $A = [\vec{v}_1 \dots \vec{v}_m]$  and  $B = [\vec{w}_1 \dots \vec{w}_m]$ , so that  $A+B = [\vec{v}_1 + \vec{w}_1 \dots \vec{v}_m + \vec{w}_m]$ . Any linear combination of the columns of  $A+B$ ,  $\vec{y} = c_1(\vec{v}_1 + \vec{w}_1) + \cdots + c_m(\vec{v}_m + \vec{w}_m)$ , can be written as

$$\vec{y} = \underbrace{(c_1\vec{v}_1 + \cdots + c_m\vec{v}_m)}_{\text{in } \text{im}(A)} + \underbrace{(c_1\vec{w}_1 + \cdots + c_m\vec{w}_m)}_{\text{in } \text{im}(B)}$$

so that  $\text{im}(A+B) \subseteq \text{im}(A) + \text{im}(B)$  (see Exercise 3.2.50). Since  $\dim(V+W) \leq \dim(V) + \dim(W)$ , by Exercise 3.3.67, we can conclude that  $\text{rank}(A+B) = \dim(\text{im}(A+B)) \leq \dim(\text{im}(A)) + \dim(\text{im}(B)) = \text{rank}(A) + \text{rank}(B)$ .

Summary:  $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$ .

**3.3.83 a** By Exercise 3.1.39b,  $\text{im}(AB) \subseteq \text{im}(A)$ , and therefore  $\text{rank}(AB) \leq \text{rank}(A)$ .

b Write  $B = [\vec{v}_1 \cdots \vec{v}_m]$  and  $AB = [A\vec{v}_1 \cdots A\vec{v}_m]$ . If  $r = \text{rank}(B)$ , then the  $r$  non-redundant columns of  $B$  will span  $\text{im}(B)$ , and the corresponding  $r$  columns of  $AB$  will span  $\text{im}(AB)$ , by linearity of  $A$ . By Theorem 3.3.4b,  $\text{rank}(AB) = \dim(\text{im}(AB)) \leq r = \text{rank}(B)$ .

Summary:  $\text{rank}(AB) \leq \text{rank}(A)$ , and  $\text{rank}(AB) \leq \text{rank}(B)$ .

**3.3.84** Same answer as Exercise 3.3.85.

**3.3.85** Let  $\vec{v}_1, \dots, \vec{v}_6$  be the columns of matrix  $A$ . Following the hint, we observe that  $\vec{v}_5 = 4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_4$ , which gives the relation  $4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_4 - \vec{v}_5 = \vec{0}$ . Thus the vector

$$\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 0 \\ 6 \\ -1 \\ 0 \end{bmatrix}$$

is in the kernel of matrix  $A$ . Since  $\vec{x}$  fails to be in the kernel of matrix  $B$ , the two kernels are different, as claimed.

**3.3.86** We will freely use the terminology introduced in the hint. First we need to show that at least one of the column vectors  $\vec{a}_k$  and  $\vec{b}_k$  fails to contain a leading 1. If  $\text{rank}[\vec{a}_1 \cdots \vec{a}_{k-1}] = \text{rank}[\vec{b}_1 \cdots \vec{b}_{k-1}] = r$ , and if  $\vec{a}_k$  contains a leading 1, then  $\vec{a}_k$  is the standard vector  $\vec{e}_{r+1}$ ; likewise for  $\vec{b}_k$ . Since  $\vec{a}_k$  and  $\vec{b}_k$  are different vectors, they cannot both contain a leading 1. Without loss of generality, we can assume that  $\vec{a}_k$  fails to contain a leading 1, so that  $\vec{a}_k$  is redundant: We can write  $\vec{a}_k = c_1\vec{a}_1 + \cdots + c_{k-1}\vec{a}_{k-1}$ . Then the vector

$$\vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is in the kernel of  $A$ . We will show that  $\vec{x}$  fails to be in the kernel of matrix  $B$ , so that  $\ker(A) \neq \ker(B)$ , as claimed. Indeed,  $B\vec{x} = c_1\vec{b}_1 + \cdots + c_{k-1}\vec{b}_{k-1} - \vec{b}_k = c_1\vec{a}_1 + \cdots + c_{k-1}\vec{a}_{k-1} - \vec{b}_k = \vec{a}_k - \vec{b}_k \neq \vec{0}$ . We have used the fact that the first  $k-1$  columns of  $B$  are identical to those of  $A$ , while  $\vec{b}_k \neq \vec{a}_k$ .

**3.3.87** Exercise 86 shows that if two matrices  $A$  and  $B$  of the same size are both in rref and have the same kernel, then  $A = B$ . Apply this fact to  $A$  and  $B = \text{rref}(M)$ .

## Section 3.4

**3.4.1**  $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

**3.4.2**  $\begin{bmatrix} 3 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ .

3.4.3  $\begin{bmatrix} 31 \\ 37 \end{bmatrix} = 0 \begin{bmatrix} 23 \\ 29 \end{bmatrix} + 1 \begin{bmatrix} 31 \\ 37 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

3.4.4  $\begin{bmatrix} 23 \\ 29 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 46 \\ 58 \end{bmatrix} + 0 \begin{bmatrix} 61 \\ 67 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ .

3.4.5  $\begin{bmatrix} 7 \\ 16 \end{bmatrix} = -4 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ .

This may not be as obvious as Exercises 1 and 3, but we can find our coefficients simply by reducing the matrix  $\begin{bmatrix} 2 & 5 & : & 7 \\ 5 & 12 & : & 16 \end{bmatrix}$ .

3.4.6  $\begin{bmatrix} -4 \\ 4 \end{bmatrix} = 11 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$ .

We arrive at this solution by reducing the matrix  $\begin{bmatrix} 1 & 5 & : & -4 \\ 2 & 6 & : & 4 \end{bmatrix}$ .

3.4.7 We need to find the scalars  $c_1$  and  $c_2$  such that  $\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ . Solving a linear system gives  $c_1 = 3$ ,  $c_2 = 4$ . Thus  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ .

3.4.8 We need to find the scalars  $c_1$  and  $c_2$  such that  $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ . Attempting to solve the linear system reveals an inconsistency;  $\vec{x}$  is not in the span of  $\vec{v}_1$  and  $\vec{v}_2$ .

3.4.9 We can solve this by inspection: Note that our first coefficient must be 3 because of the first terms of the vectors. Also, the second coefficient must be 2 due to the last terms.

However,  $3\vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ . Thus, we reason that  $\vec{x}$  is not in the span of  $\vec{v}_1$  and  $\vec{v}_2$ .

We can also see this by attempting to solve  $\begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & : 3 \\ 0 & 2 & 4 \end{bmatrix}$ , which turns out to be inconsistent. Thus,  $\vec{x}$  is not in  $V$ .

3.4.10 Proceeding as in Example 1, we find  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

3.4.11 Proceeding as in Example 1, we find  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ .

3.4.12 Proceeding as in Example 1, we find  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ .

**3.4.13** Here, we quickly see that since  $x_1 = 1 = 1c_1 + 0c_2 + 0c_3$ ,  $c_1$  must equal 1. We find  $c_2 = -1$  similarly, since  $x_2 = 1 = 2(1) + 1c_2 + 0c_3$ . Finally, now that  $x_3 = 3(1) + 2(-1) + 1c_3$ ,  $c_3$  must be zero.

$$\text{So } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

**3.4.14** We proceed by inspection here, noting that we need  $c_1 = 3$ , then see that  $c_2$  must be 4. Finally,  $c_3$  must be 6.

$$\text{Thus, } \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}.$$

**3.4.15** This may be a bit too difficult to do by inspection. Instead we reduce  $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 0 \\ 1 & 4 & 8 & 0 \end{bmatrix}$  to  $\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ ,

$$\text{revealing that } \vec{x} = 8\vec{v}_1 - 12\vec{v}_2 + 5\vec{v}_3, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -12 \\ 5 \end{bmatrix}.$$

**3.4.16** We reduce  $\begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 3 \end{bmatrix}$  to  $\begin{bmatrix} 1 & 0 & 0 & 21 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 8 \end{bmatrix}$ ,

$$\text{revealing that } \vec{x} = 21\vec{v}_1 - 22\vec{v}_2 + 8\vec{v}_3, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 21 \\ -22 \\ 8 \end{bmatrix}.$$

**3.4.17** By inspection, we see that in order for  $\vec{x}$  to be in  $V$ ,  $\vec{x} = 1\vec{v}_1 + 1\vec{v}_2 - 1\vec{v}_3$  (by paying attention to the first, second, and fourth terms). Now we need to verify that the third terms “work out”. So,  $1(2) + 1(3) - 1(4) = 5 - 4 = 1 = x_3$ .

$$\text{Thus } \vec{x} \text{ is in } V, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

**3.4.18** Here,  $\vec{x}$  is not in  $V$ , as we find an inconsistency while attempting to solve the system.

**3.4.19 a**  $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

b Our commutative diagram:

$$\begin{array}{ccc}
 \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \xrightarrow{T} & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
 & & = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
 \downarrow & & \downarrow \\
 [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{B} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix}
 \end{array}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

$$c \quad B = [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[ \left[ \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathcal{B}} \right] \right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

3.4.20 a  $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

b Our commutative diagram:

$$\begin{array}{ccc}
 \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \xrightarrow{T} & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
 & & = c_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
 \downarrow & & \downarrow \\
 [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{B} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 2c_1 \\ 0 \end{bmatrix}
 \end{array}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ 0 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ .

$$c \quad B = [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[ \left[ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right] \right] = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

3.4.21 a  $S = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\frac{1}{7} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7 & 14 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 49 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}.$$

b Our commutative diagram:

$$\begin{aligned}\vec{x} &= c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \xrightarrow{\overline{T}} T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 A \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 7 \\ 21 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 7c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ \downarrow & \quad \quad \quad \downarrow \\ [\vec{x}]_{\mathcal{B}} &= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \xrightarrow{\overline{B}} [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 7c_1 \\ 0 \end{bmatrix}\end{aligned}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7c_1 \\ 0 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$ .

$$\begin{aligned}c \ B &= [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[ \begin{bmatrix} 7 \\ 21 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

3.4.22 a  $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 10 \\ 10 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.$$

b Our commutative diagram:

$$\begin{aligned}\vec{x} &= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \xrightarrow{\overline{T}} T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 A \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 5 \\ 10 \end{bmatrix} + c_2 \begin{bmatrix} 10 \\ -5 \end{bmatrix} = 5c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 5c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ \downarrow & \quad \quad \quad \downarrow \\ [\vec{x}]_{\mathcal{B}} &= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \xrightarrow{\overline{B}} [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 5c_1 \\ -5c_2 \end{bmatrix}\end{aligned}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5c_1 \\ -5c_2 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$ .

$$\begin{aligned}c \ B &= [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[ \begin{bmatrix} 5 \\ 10 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 10 \\ -5 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.\end{aligned}$$

3.4.23 a  $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

b Our commutative diagram:

$$\begin{array}{ccc} \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \xrightarrow{T} & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ & & = c_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = 2c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{B} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 2c_1 \\ -c_2 \end{bmatrix} \end{array}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ -c_2 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ .

$$\begin{aligned} c \quad B &= [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[ \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} -1 \\ -2 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

**3.4.24 a**  $S = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -15 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

b Our commutative diagram:

$$\begin{array}{ccc} \vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \xrightarrow{T} & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 5 \\ 3 \end{bmatrix} \\ & & = c_1 \begin{bmatrix} 6 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 3c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{B} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 3c_1 \\ c_2 \end{bmatrix} \end{array}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3c_1 \\ c_2 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ .

$$\begin{aligned} c \quad B &= [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[ \begin{bmatrix} 6 \\ 3 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

**3.4.25** We will use the commutative diagram method here (though any method suffices).

$$\begin{aligned}
 \vec{x} &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{\overline{T}} T(\vec{x}) = A\vec{x} = c_1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
 &\quad = c_1 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 11 \end{bmatrix} \\
 &\quad = c_1 \left( -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) + c_2 \left( -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \\
 &\quad = (-c_1 - c_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (4c_1 + 6c_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
 \downarrow & \qquad \qquad \qquad \downarrow \\
 [\vec{x}]_{\mathcal{B}} &= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \xrightarrow{\overline{B}} [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} -c_1 - c_2 \\ 4c_1 + 6c_2 \end{bmatrix} \\
 B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} -c_1 - c_2 \\ 4c_1 + 6c_2 \end{bmatrix}, \text{ so } B = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix}.
 \end{aligned}$$

**3.4.26** Let's build  $B$  “column-by-column”:

$$\begin{aligned}
 B &= [[T(\vec{v}_1)]_{\mathcal{B}} | [T(\vec{v}_2)]_{\mathcal{B}}] \\
 &= \left[ \left[ \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right] \Big| \left[ \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] \right]_{\mathcal{B}} \\
 &= \left[ \begin{bmatrix} 2 \\ 8 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 6 & 4 \\ -4 & -3 \end{bmatrix}.
 \end{aligned}$$

**3.4.27** We use a commutative diagram:

$$\begin{aligned}
 \vec{x} &= c_1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\overline{T}} T(\vec{x}) = A\vec{x} \\
 &\quad = c_1 A \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + c_2 A \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_3 A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
 &\quad = c_1 \begin{bmatrix} 18 \\ 9 \\ -18 \end{bmatrix} + \vec{0} + \vec{0} = 9c_1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
 \downarrow & \qquad \qquad \qquad \downarrow \\
 [\vec{x}]_{\mathcal{B}} &= \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \xrightarrow{\overline{B}} [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 9c_1 \\ 0 \\ 0 \end{bmatrix} \\
 B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \begin{bmatrix} 9c_1 \\ 0 \\ 0 \end{bmatrix}, \text{ so } B = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

**3.4.28** Let's build  $B$  “column-by-column”:

$$B = [[T(\vec{v}_1)]_{\mathcal{B}} | [T(\vec{v}_2)]_{\mathcal{B}} | [T(\vec{v}_3)]_{\mathcal{B}}]$$

$$\begin{aligned}
 &= \left[ \left[ \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right]_{\mathcal{B}} \right] \\
 &= \left[ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 9 \\ -9 \\ 0 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 0 \\ 9 \\ -18 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}.
 \end{aligned}$$

**3.4.29** Let's build  $B$  “column-by-column”:

$$\begin{aligned}
 B &= [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}} [T(\vec{v}_3)]_{\mathcal{B}}] \\
 &= \left[ \left[ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right]_{\mathcal{B}} \right] \\
 &= \left[ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.
 \end{aligned}$$

**3.4.30** Let's build  $B$  “column-by-column”:

$$\begin{aligned}
 B &= [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}} [T(\vec{v}_3)]_{\mathcal{B}}] \\
 &= \left[ \left[ \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right]_{\mathcal{B}} \right] \\
 &= \left[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

**3.4.31** We can use a commutative diagram to see how this works:

$$\begin{array}{ccc}
 \vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 & \xrightarrow{\overline{T}} & T(\vec{x}) = \vec{v}_2 \times \vec{x} = c_1(\vec{v}_2 \times \vec{v}_1) + c_2(\vec{v}_2 \times \vec{v}_2) + c_3(\vec{v}_2 \times \vec{v}_3) \\
 & & = c_1(-\vec{v}_3) + c_2(\vec{0}) + c_3(\vec{v}_1) = c_3 \vec{v}_1 - c_1 \vec{v}_3 \\
 \downarrow & & \downarrow \\
 [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} & \xrightarrow{\overline{B}} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_3 \\ 0 \\ -c_1 \end{bmatrix} \\
 B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_3 \\ 0 \\ -c_1 \end{bmatrix}, \text{ so } B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.
 \end{array}$$

**3.4.32** Here we will build  $B$  column-by-column:

$$\begin{aligned}
 B &= [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}] \\
 &= [[\vec{v}_1 \times \vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_2 \times \vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_3 \times \vec{v}_3]_{\mathcal{B}}] = [[-\vec{v}_2]_{\mathcal{B}} \quad [\vec{v}_1]_{\mathcal{B}} \quad \vec{0}], \text{ since all three are perpendicular unit vectors.}
 \end{aligned}$$

$$\text{So, } B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**3.4.33** Here we will build  $B$  column-by-column:

$$\begin{aligned} B &= [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}] \\ &= [[(\vec{v}_2 \cdot \vec{v}_1)\vec{v}_2]_{\mathcal{B}} \quad [(\vec{v}_2 \cdot \vec{v}_2)\vec{v}_2]_{\mathcal{B}} \quad [(\vec{v}_2 \cdot \vec{v}_3)\vec{v}_2]_{\mathcal{B}}] = [\vec{0} \quad [1\vec{v}_2]_{\mathcal{B}} \quad \vec{0}], \text{ since all three are perpendicular unit vectors.} \\ \text{So, } B &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

**3.4.34** Here we will build  $B$  column-by-column:

$$\begin{aligned} B &= [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}] \\ &= [[\vec{v}_1 - 2(\vec{v}_3 \cdot \vec{v}_1)\vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_2 - 2(\vec{v}_3 \cdot \vec{v}_2)\vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_3 - 2(\vec{v}_3 \cdot \vec{v}_3)\vec{v}_3]_{\mathcal{B}}] = [[\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_2]_{\mathcal{B}} \quad [-\vec{v}_3]_{\mathcal{B}}]. \\ \text{So, } B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \text{ This is the reflection about the plane spanned by } \vec{v}_1 \text{ and } \vec{v}_2. \end{aligned}$$

**3.4.35** Using another commutative diagram:

$$\begin{array}{ccc} \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 & \xrightarrow{\overline{T}} & T(\vec{x}) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) \\ & & = c_1(\vec{v}_1 - 2(\vec{v}_1 \cdot \vec{v}_1)\vec{v}_2) + c_2(\vec{v}_2 - 2(\vec{v}_1 \cdot \vec{v}_2)\vec{v}_2) + \\ & & \quad c_3(\vec{v}_3 - 2(\vec{v}_1 \cdot \vec{v}_3)\vec{v}_2) \\ & & = c_1(\vec{v}_1 - 2\vec{v}_2) + c_2(\vec{v}_2 - \vec{0}) + c_3(\vec{v}_3 - \vec{0}) \\ & & = c_1\vec{v}_1 + (-2c_1 + c_2)\vec{v}_2 + c_3\vec{v}_3 \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} & = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ -2c_1 + c_2 \\ c_3 \end{bmatrix} \end{array}$$

So  $B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . This is a shear along the second term.

**3.4.36** Here we will build  $B$  column-by-column:

$$\begin{aligned} B &= [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}] \\ &= [[\vec{v}_1 \times \vec{v}_1 + (\vec{v}_1 \cdot \vec{v}_1)\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_1 \times \vec{v}_2 + (\vec{v}_1 \cdot \vec{v}_2)\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_1 \times \vec{v}_3 + (\vec{v}_1 \cdot \vec{v}_3)\vec{v}_1]_{\mathcal{B}}] \\ &= [[\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_3]_{\mathcal{B}} \quad [-\vec{v}_2]_{\mathcal{B}}]. \end{aligned}$$

$$\text{So, } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \text{ This is a 90-degree rotation about the line spanned by } \vec{v}_1. \text{ The rotation is counterclockwise when looking from the positive } \vec{v}_1 \text{ direction.}$$

**3.4.37** We want a basis  $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$  such that  $T(\vec{v}_1) = a\vec{v}_1$  and  $T(\vec{v}_2) = b\vec{v}_2$  for some scalars  $a$  and  $b$ . Then the  $\mathcal{B}$ -matrix of  $T$  will be  $B = [[T(\vec{v}_1)]_{\mathcal{B}} \ [T(\vec{v}_2)]_{\mathcal{B}}] = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ , which is a diagonal matrix as required. Note that  $T(\vec{v}) = \vec{v} = 1\vec{v}$  for vectors parallel to the line  $L$  onto which we project, and  $T(\vec{w}) = \vec{0} = 0\vec{w}$  for vectors perpendicular to  $L$ . Thus, we can pick a basis where  $\vec{v}_1$  is parallel to  $L$  and  $\vec{v}_2$  is perpendicular, for example,  $\mathcal{B} = \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$ .

**3.4.38** We want a basis  $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$  such that  $T(\vec{v}_1) = a\vec{v}_1$  and  $T(\vec{v}_2) = b\vec{v}_2$  for some scalars  $a$  and  $b$ . Then the  $\mathcal{B}$ -matrix of  $T$  will be  $B = [[T(\vec{v}_1)]_{\mathcal{B}} \ [T(\vec{v}_2)]_{\mathcal{B}}] = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ , which is a diagonal matrix as required. Note that  $T(\vec{v}) = \vec{v} = 1\vec{v}$  for vectors parallel to the line  $L$  about which we reflect, and  $T(\vec{w}) = -\vec{w} = (-1)\vec{w}$  for vectors perpendicular to  $L$ . Thus, we can pick a basis where  $\vec{v}_1$  is parallel to  $L$  and  $\vec{v}_2$  is perpendicular, for example,  $\mathcal{B} = \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right)$ .

**3.4.39** Using the same approach as in Exercise 37, we want a basis,  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  such that  $T(\vec{v}_1) = a\vec{v}_1, T(\vec{v}_2) = b\vec{v}_2$  and  $T(\vec{v}_3) = c\vec{v}_3$ . First we see that if  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , then  $T(\vec{v}_1) = \vec{v}_1$ . Next we notice that if  $\vec{v}_2$  and  $\vec{v}_3$  are perpendicular to  $\vec{v}_1$ , then  $T(\vec{v}_2) = -\vec{v}_2$  and  $T(\vec{v}_3) = -\vec{v}_3$ . So we can pick  $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$  and  $\vec{v}_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ , for example.

**3.4.40** From Exercise 37, we see that we want one of our basis vectors to be parallel to the line, while the others must be perpendicular the line. We can easily find such a basis:  $\mathcal{B} = \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$ .

**3.4.41** We will use the same approach as in Exercises 37 and 39. Any basis with 2 vectors in the plane and one perpendicular to it will work nicely here! So, let  $\vec{v}_1, \vec{v}_2$  be in the plane.  $\vec{v}_1$  can be  $\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$ , and  $\vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$  (note that these must be independent). Then  $\vec{v}_3$  should be perpendicular to the plane. We will use  $\vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ —the coefficient vector. This is perpendicular to the plane because all vectors perpendicular to  $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$  lie in the plane.

So, our basis is:  $\left( \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right)$ .

**3.4.42** From Exercise 38, we deduce that one of our vectors should be perpendicular to this plane, while two should fall inside it. Finding the perpendicular is not difficult: we simply take the coefficient vector:  $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ . Then

we add two linearly independent vectors on the plane,  $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ , for instance. These three vectors form one possible basis.

$$3.4.43 \text{ By definition of coordinates (Definition 3.4.1), } \vec{x} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}.$$

$$3.4.44 \text{ By definition of coordinates, } \vec{x} = 2 \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix}.$$

3.4.45 If  $\vec{v}_1, \vec{v}_2$  is a basis with the desired property, then  $\vec{x} = 2\vec{v}_1 + 3\vec{v}_2$ , or  $\vec{v}_2 = \frac{1}{3}\vec{x} - \frac{2}{3}\vec{v}_1$ . Thus we can make  $\vec{v}_1$  any vector in the plane that is not parallel to  $\vec{x}$ , and then let  $\vec{v}_2 = \frac{1}{3}\vec{x} - \frac{2}{3}\vec{v}_1$ . For example, if we choose  $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ , then  $\vec{v}_2 = \frac{1}{3} \begin{bmatrix} -4 \\ -4 \\ -1 \end{bmatrix}$ .

3.4.46 As in Exercise 3.4.45, we can make  $\vec{v}_1$  any vector in the plane that is not parallel to  $\vec{x}$ , and then let  $\vec{v}_2 = 2\vec{v}_1 - \vec{x}$ . For example, if we choose  $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ , then  $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ .

$$3.4.47 \text{ By Theorem 3.4.4, we have } A = SBS^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

$$3.4.48 [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ means that } \vec{x} = -\vec{v} + 2\vec{w}. \text{ See Figure 3.6}$$

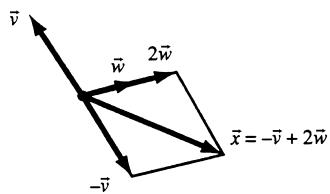


Figure 3.6: for Problem 3.4.48.

$$3.4.49 \vec{u} + \vec{v} = -\vec{w}, \text{ so that } \vec{w} = -\vec{u} - \vec{v}, \text{ i.e., } [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

$$3.4.50 \text{ a } \overrightarrow{OP} = \vec{w} + 2\vec{v}, \text{ so that } [\overrightarrow{OP}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \overrightarrow{OQ} = \vec{v} + 2\vec{w}, \text{ so that } [\overrightarrow{OQ}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

b  $\overrightarrow{OR} = 3\vec{v} + 2\vec{w}$ . See Figure 3.7.

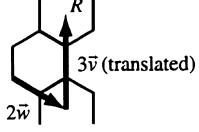


Figure 3.7: for Problem 3.4.50.

c If the tip of  $\vec{u}$  is a vertex, then so is the tip of  $\vec{u} + 3\vec{v}$  and also the tip of  $\vec{u} + 3\vec{w}$  (draw a sketch!). We know that the tip  $P$  of  $2\vec{v} + \vec{w}$  is a vertex (see part a.). Therefore, the tip  $S$  of  $\vec{OS} = 17\vec{v} + 13\vec{w} = (2\vec{v} + \vec{w}) + 5(3\vec{v}) + 4(3\vec{w})$  is a vertex as well.

**3.4.51** Let  $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m)$ . Then, let  $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m$  and  $\vec{y} = b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_m\vec{v}_m$ . Then  $[\vec{x} + \vec{y}]_{\mathcal{B}} = [a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m + b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_m\vec{v}_m]_{\mathcal{B}} = [(a_1 + b_1)\vec{v}_1 + (a_2 + b_2)\vec{v}_2 + \dots + (a_m + b_m)\vec{v}_m]_{\mathcal{B}}$

$$= \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$$

**3.4.52** Yes;  $T(\vec{x}) = [\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}$ , so that  $T$  is “given by a matrix.” (See Definition 2.1.1.)

**3.4.53** By Definition 3.4.1, we have  $\vec{x} = S[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 40 \\ 58 \end{bmatrix}$ .

**3.4.54** Let  $Q$  be the matrix whose columns are the vectors of the basis  $\mathcal{T}$ . Then  $[[\vec{v}_1]_{\mathcal{T}} \dots [\vec{v}_n]_{\mathcal{T}}] = [Q^{-1}\vec{v}_1 \dots Q^{-1}\vec{v}_n] = Q^{-1}[\vec{v}_1 \dots \vec{v}_n]$  is an invertible matrix, so that the vectors  $[\vec{v}_1]_{\mathcal{T}} \dots [\vec{v}_n]_{\mathcal{T}}$  form a basis of  $\mathbb{R}^n$ .

**3.4.55** By Definition 3.4.1, we have  $\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_{\mathcal{B}}$  and  $\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [\vec{x}]_{\mathcal{R}}$ , so that  $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [\vec{x}]_{\mathcal{R}}$  and  $[\vec{x}]_{\mathcal{R}} = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_P [\vec{x}]_{\mathcal{B}}$ , i.e.,  $P = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$ .

**3.4.56** Let  $S = [\vec{v}_1 \vec{v}_2]$  where  $\vec{v}_1, \vec{v}_2$  is the desired basis. Then by Theorem 3.4.1,  $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = S \begin{bmatrix} 3 \\ 5 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = S \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ , i.e.  $S \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ . Hence  $S = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 12 & -7 \\ 14 & -8 \end{bmatrix}$ . The desired basis is  $\begin{bmatrix} 12 \\ 14 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \end{bmatrix}$ .

**3.4.57** If we can find a basis  $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$  such that the  $\mathcal{B}$ -matrix of  $A$  is

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ , then  $A$  must be similar to  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ . Because of the entries in the matrix  $B$ , it is required that  $A\vec{v}_1 = \vec{v}_1, A\vec{v}_2 = \vec{v}_2$  and  $A\vec{v}_3 = -\vec{v}_3$ . So, all we need for our basis is to pick independent  $\vec{v}_1, \vec{v}_2$  in the plane, and  $\vec{v}_3$  perpendicular to the plane.

**3.4.58 a** Consider a linear relation  $c_1A^2\vec{v} + c_2A\vec{v} + c_3\vec{v} = \vec{0}$ .

Multiplying  $A^2$  with the vectors on both sides and using that  $A^3\vec{v} = \vec{0}$  and  $A^4\vec{v} = \vec{0}$ , we find that  $c_3A^2\vec{v} = \vec{0}$  and therefore  $c_3 = 0$ , since  $A^2\vec{v} \neq \vec{0}$ .

Therefore, our relation simplifies to  $c_1A^2\vec{v} + c_2A\vec{v} = \vec{0}$ .

Multiplying  $A$  with the vectors on both sides we find that  $c_2A^2\vec{v} = \vec{0}$  and therefore  $c_2 = 0$ . Then  $c_1 = 0$  as well. We have shown that there is only the trivial relation among the vectors  $A^2\vec{v}, A\vec{v}$ , and  $\vec{v}$ , so that these three vectors from a basis of  $\mathbb{R}^3$ , as claimed.

$$\text{b } T(A^2\vec{v}) = A^3\vec{v} = \vec{0} \text{ so } [T(A^2\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$T(A\vec{v}) = A^2\vec{v} \text{ so } [T(A\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$T(\vec{v}) = A\vec{v} \text{ so } [T(\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence, by Theorem 3.4.3, the desired matrix is  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .

**3.4.59** First we find the matrices  $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ , or

$\begin{bmatrix} 2x & 2y \\ 3z & 3t \end{bmatrix} = \begin{bmatrix} 2x & x+3y \\ 2z & z+3t \end{bmatrix}$ . The solutions are of the form  $S = \begin{bmatrix} -y & y \\ 0 & t \end{bmatrix}$ , where  $y$  and  $t$  are arbitrary constants.

Since there are *invertible* solutions  $S$  (for example, let  $y = t = 1$ ), the matrices  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  and  $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$  are indeed similar.

**3.4.60** First we find the matrices  $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , or,  $\begin{bmatrix} x & y \\ -z & -t \end{bmatrix} = \begin{bmatrix} y & x \\ t & z \end{bmatrix}$ . The solutions are of the form  $S = \begin{bmatrix} y & y \\ -t & t \end{bmatrix}$ , where  $y$  and  $t$  are arbitrary constants. Since there are *invertible* solutions  $S$  (for example, let  $y = t = 1$ ), the matrices  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  are indeed similar.

**3.4.61** We seek a basis  $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$  such that the matrix  $S = [\vec{v}_1 \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  satisfies the equation  $\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . Solving the ensuing linear system gives  $S = \begin{bmatrix} -\frac{3z}{2} & \frac{z}{4} - \frac{3t}{2} \\ z & t \end{bmatrix}$ .

We need to choose  $z$  and  $t$  so that  $S$  will be invertible. For example, if we let  $z = 6$  and  $t = 1$ , then  $S = \begin{bmatrix} -9 & 0 \\ 6 & 1 \end{bmatrix}$ , so that  $\vec{v}_1 = \begin{bmatrix} -9 \\ 6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

**3.4.62** We seek a basis  $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$  such that the matrix  $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  satisfies the equation  $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ . Solving the ensuing linear system gives  $S = \begin{bmatrix} \frac{z}{2} & -t \\ z & t \end{bmatrix}$ . We need to choose both  $z$  and  $t$  nonzero to make  $S$  invertible. For example, if we let  $z = 2$  and  $t = 1$ , then  $S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ , so that  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

**3.4.63** First we find the matrices  $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$ , or,  $\begin{bmatrix} px - qz & py - qt \\ qx + pz & qy + pt \end{bmatrix} = \begin{bmatrix} px - qy & qx + py \\ pz - qt & qz + pt \end{bmatrix}$ . If  $q \neq 0$ , then the solutions are of the form  $S = \begin{bmatrix} -t & z \\ z & t \end{bmatrix}$ , where  $z$  and  $t$  are arbitrary constants. Since there are *invertible* solutions  $S$  (for example, let  $z = t = 1$ ), the matrices  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$  and  $\begin{bmatrix} p & q \\ -q & p \end{bmatrix}$  are indeed similar. (If  $q = 0$ , then the two matrices are equal.)

**3.4.64** If  $b$  and  $c$  are both zero, then the given matrices are equal, so that they are similar, by Theorem 3.4.6.a. Let's now assume that at least one of the scalars  $b$  and  $c$  is nonzero; reversing the roles of  $b$  and  $c$  if necessary, we can assume that  $c \neq 0$ .

Let's find the matrices  $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ , or

$\begin{bmatrix} ax + bz & ay + bt \\ cx + bz & cy + dt \end{bmatrix} = \begin{bmatrix} ax + by & cx + dy \\ az + bt & cz + dt \end{bmatrix}$ . The solutions are of the form

$S = \begin{bmatrix} \frac{(a-d)z+b}{c} & z \\ z & t \end{bmatrix}$ , where  $z$  and  $t$  are arbitrary constants. Since there are *invertible* solutions  $S$  (for example, let  $z = 1, t = 0$ ), the matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$  are indeed similar.

**3.4.65 a** If  $S = I_n$ , then  $S^{-1}AS = A$ .

b If  $S^{-1}AS = B$ , then  $SBS^{-1} = A$ . If we let  $R = S^{-1}$ , then  $R^{-1}BR = A$ , showing that  $B$  is similar to  $A$ .

**3.4.66** We build  $B$  “column-by-column”:

$$\begin{aligned} B &= \left[ \left[ T \begin{bmatrix} b \\ 1-a \end{bmatrix} \right] \right]_{\mathcal{B}} \left[ T \begin{bmatrix} a-1 \\ b \end{bmatrix} \right]_{\mathcal{B}} = \left[ \begin{bmatrix} ab + b - ba \\ b^2 + a^2 - a \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} a^2 + b^2 - a \\ ba - b - ab \end{bmatrix}_{\mathcal{B}} \right] \\ &= \left[ \begin{bmatrix} b \\ 1-a \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 1-a \\ -b \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Thus, this matrix represents the reflection about the line spanned by  $\begin{bmatrix} b \\ 1-a \end{bmatrix}$ . Note that the two vectors  $\begin{bmatrix} b \\ 1-a \end{bmatrix}$  and  $\begin{bmatrix} a-1 \\ b \end{bmatrix}$  are perpendicular.

**3.4.67** The matrix we seek is  $\left[ \begin{bmatrix} T & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} T & \begin{bmatrix} a \\ c \end{bmatrix} \end{bmatrix}_{\mathcal{B}} \right] = \left[ \begin{bmatrix} a \\ c \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} a^2 + bc \\ ac + cd \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 0 & bc - ad \\ 1 & a + d \end{bmatrix}$ .

**3.4.68** Using Exercise 67 as a guide, consider the basis  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ , and let  $S = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ .

**3.4.69** The matrix of the transformation  $T(\vec{x}) = A\vec{x}$  with respect to the basis  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is  $D = \left[ \begin{bmatrix} 3 \\ 6 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} -2 \\ -1 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ . Thus  $S^{-1}AS = D$  for  $S = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ .

**3.4.70** Suppose such a basis  $\vec{v}_1, \vec{v}_2$  exists. If  $B = [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}]$  is upper triangular, of the form  $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ , then  $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ , so that  $T(\vec{v}_1) = a\vec{v}_1$ , that is,  $T(\vec{v}_1)$  is parallel to  $\vec{v}_1$ . But this is impossible, since  $T$  is a rotation through  $\frac{\pi}{2}$ .

**3.4.71** a Note that  $AS = SB$ . If  $\vec{x}$  is in  $\ker(B)$ , then  $A(S\vec{x}) = SB\vec{x} = S\vec{0} = \vec{0}$ , so that  $S\vec{x}$  is in  $\ker(A)$ , as claimed.

b We use the hint and observe that  $\text{nullity}(B) = \dim(\ker B) = p \leq \dim(\ker A) = \text{nullity}(A)$ , since  $S\vec{v}_1, \dots, S\vec{v}_p$  are  $p$  linearly independent vectors in  $\ker(A)$ . Reversing the roles of  $A$  and  $B$  shows that, conversely,  $\text{nullity}(A) \leq \text{nullity}(B)$ , so that the equation  $\text{nullity}(A) = \text{nullity}(B)$  holds, as claimed.

**3.4.72** If  $A$  and  $B$  are similar  $n \times n$  matrices, then  $\text{rank}(A) = n - \text{nullity}(A) = n - \text{nullity}(B) = \text{rank}(B)$ , by Exercise 71 and the rank nullity theorem (Theorem 3.3.7).

**3.4.73** a By inspection, we can find an orthonormal basis  $\vec{v}_1 = \vec{v}, \vec{v}_2, \vec{v}_3$  of  $\mathbb{R}^3$ ,  $\vec{v}_1 = \vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ,  $\vec{v}_3 = \begin{bmatrix} 0.8 \\ -0.6 \\ 0 \end{bmatrix}$ .

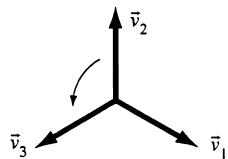


Figure 3.8: for Problem 3.4.73b.

b Now  $T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = \vec{v}_3$  and  $T(\vec{v}_3) = -\vec{v}_2$  (see Figure 3.8), so that the matrix  $B$  of  $T$  with respect to the basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ . Then  $A = SBS^{-1} = \begin{bmatrix} 0.36 & 0.48 & 0.8 \\ 0.48 & 0.64 & -0.6 \\ -0.8 & 0.6 & 0 \end{bmatrix}$ .

3.4.74 a  $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .

b If  $\mathcal{B}$  is the basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ , then  $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$  (by part a) so  $\vec{v}_0 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3$ , i.e.  $[\vec{v}_0]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ .

c  $T(\vec{v}_2) = T(-\vec{v}_0 - \vec{v}_1 - \vec{v}_3) = -T(\vec{v}_0) - T(\vec{v}_1) - T(\vec{v}_3) = -\vec{v}_3 - \vec{v}_0 - \vec{v}_1 = \vec{v}_2$

Hence,  $T$  is a rotation through  $120^\circ$  about the line spanned by  $\vec{v}_2$ . Its matrix,  $B$ , is given by

$[[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}} [T(\vec{v}_3)]_{\mathcal{B}}$  where

$$T(\vec{v}_1) = \vec{v}_0 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3 \text{ so } [T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$T(\vec{v}_2) = \vec{v}_2 \text{ so } [T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{v}_3) = \vec{v}_1 \text{ so } [T(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$B^3 = I_3$  since if the tetrahedron rotates through  $120^\circ$  three times, it returns to the original position.

3.4.75  $B = S^{-1}AS$ , where  $S = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ . Thus  $B = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

3.4.76  $B = S^{-1}AS$ , where  $S = A = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$ . Thus  $B = A = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$ .

3.4.77 Let  $S$  be the  $n \times n$  matrix whose columns are  $\vec{e}_n, \vec{e}_{n-1}, \dots, \vec{e}_1$ . Note that  $S$  has all 1's on “the other diagonal” and 0's elsewhere:

$$s_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1 \\ 0 & \text{otherwise} \end{cases}$$

Also,  $S^{-1} = S$ .

Now,  $B = S^{-1}AS = SAS$ ; the entries of  $B$  are  $b_{ij} = s_{i,n+1-i}a_{n+1-i,n+1-j}s_{n+1-j,j} = a_{n+1-i,n+1-j}$ .

Answer:  $b_{ij} = a_{n+1-i,n+1-j}$

$B$  is obtained from  $A$  by reversing the order of the rows and of the columns.

**3.4.78** Note first that the diagonal entry  $s_{ij}$  of  $S$  gives the unit price of good  $i$ .

If  $a_{ij}$  tells us how many dollars' worth of good  $i$  are required to produce one dollar's worth of good  $j$ , then  $a_{ij}s_{jj}$  tells us how many dollars' worth of good  $i$  are required to produce one unit of good  $j$ , and  $s_{ii}^{-1}a_{ij}s_{jj}$  is the number of units of good  $i$  required to produce one unit of good  $j$ . Thus  $b_{ij} = s_{ii}^{-1}a_{ij}s_{jj}$ , and  $B = S^{-1}AS$ .

## True or False

Ch 3.TF.1 T, by Summary 3.3.10.

Ch 3.TF.2 F, by Theorem 3.3.7.

Ch 3.TF.3 T, by Summary 3.3.10.

Ch 3.TF.4 F; The identity matrix is similar only to itself.

Ch 3.TF.5 T; We have the nontrivial relation  $3\vec{u} + 3\vec{v} + 3\vec{w} = \vec{0}$ .

Ch 3.TF.6 F; The columns could be  $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$  in  $\mathbb{R}^5$ , for example.

Ch 3.TF.7 T, by Theorem 3.3.2.

Ch 3.TF.8 F; The nullity is  $6 - 4 = 2$ , by Theorem 3.3.7.

Ch 3.TF.9 F; It's a subspace of  $\mathbb{R}^3$ .

Ch 3.TF.10 T; by Definition 3.1.2.

Ch 3.TF.11 T, by Definition 3.2.3.

Ch 3.TF.12 T, by Definition 3.2.1.

Ch 3.TF.13 T; Check that  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

Ch 3.TF.14 T, by Theorem 3.3.9.

Ch 3.TF.15 T, by Theorem 3.2.8.

Ch 3.TF.16 T, by Summary 3.3.10.

Ch 3.TF.17 F; The number  $n$  may exceed 4.

Ch 3.TF.18 T, by Definition 3.2.1 ( $V$  is closed under linear combinations)

Ch 3.TF.19 T, by Theorem 3.4.6, parts b and c.

Ch 3.TF.20 F; Let  $V = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  in  $\mathbb{R}^2$ , for example.

Ch 3.TF.21 T, since  $AB\vec{v} = A\vec{0} = \vec{0}$ .

Ch 3.TF.22 T, by Definition 3.2.3.

Ch 3.TF.23 F; Suppose  $\vec{v}_2 = 2\vec{v}_1$ . Then  $T(\vec{v}_2) = 2T(\vec{v}_1) = 2\vec{e}_1$  cannot be  $\vec{e}_2$ .

Ch 3.TF.24 F; Consider  $\vec{u} = \vec{e}_1$ ,  $\vec{v} = 2\vec{e}_1$ , and  $\vec{w} = \vec{e}_2$ .

Ch 3.TF.25 T, since  $A^{-1}(AB)A = BA$ .

Ch 3.TF.26 T, since both kernels consist of the zero vector alone.

Ch 3.TF.27 F; There is no invertible matrix  $S$  as required in the definition of similarity.

Ch 3.TF.28 F; Five vectors in  $\mathbb{R}^4$  must be dependent, by Theorem 3.2.8.

Ch 3.TF.29 T, by Definition 3.2.1 (all vectors in  $\mathbb{R}^3$  are linear combinations of  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ ).

Ch 3.TF.30 T; Use a basis with one vector on the line and the other perpendicular to it.

Ch 3.TF.31 F; Note that  $\mathbb{R}^2$  isn't even a subset of  $\mathbb{R}^3$ . A vector in  $\mathbb{R}^2$ , with two components, does not belong to  $\mathbb{R}^3$ .

Ch 3.TF.32 T; If  $B = S^{-1}AS$ , then  $B + 7I_n = S^{-1}(A + 7I_n)S$ .

Ch 3.TF.33 T; for any  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  of  $V$  also  $k\vec{v}_1, \vec{v}_2, \vec{v}_3$  is a basis too, for any nonzero scalar  $k$ .

Ch 3.TF.34 F; The identity matrix is similar only to itself.

Ch 3.TF.35 F; Consider  $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , but  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ .

Ch 3.TF.36 F; Let  $A = I_2$ ,  $B = -I_2$  and  $\vec{v} = \vec{e}_1$ , for example.

Ch 3.TF.37 F; Let  $V = \text{span}(\vec{e}_1)$  and  $W = \text{span}(\vec{e}_2)$  in  $\mathbb{R}^2$ , for example.

Ch 3.TF.38 T; If  $A\vec{v} = A\vec{w}$ , then  $A(\vec{v} - \vec{w}) = \vec{0}$ , so that  $\vec{v} - \vec{w} = \vec{0}$  and  $\vec{v} = \vec{w}$ .

Ch 3.TF.39 T; Consider the linear transformation with matrix  $[\vec{w}_1 \dots \vec{w}_n][\vec{v}_1 \dots \vec{v}_n]^{-1}$ .

Ch 3.TF.40 F; Suppose  $A$  were similar to  $B$ . Then  $A^4 = I_2$  were similar to  $B^4 = -I_2$ , by Example 7 of Section 3.4. But this isn't the case:  $I_2$  is similar only to itself.

Ch 3.TF.41 T; Let  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , for example, with  $\ker(A) = \text{im}(A) = \text{span}(\vec{e}_1)$ .

Ch 3.TF.42 F; Consider  $I_n$  and  $2I_n$ , for example.

Ch 3.TF.43 T; Matrix  $B = S^{-1}AS$  is invertible, being the product of invertible matrices.

Ch 3.TF.44 T; Note that  $\text{im}(A)$  is a subspace of  $\ker(A)$ , so that

$$\dim(\text{im } A) = \text{rank}(A) \leq \dim(\ker A) = 10 - \text{rank}(A).$$

Ch 3.TF.45 T; Pick three vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  that span  $V$ . Then  $V = \text{im}[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ .

Ch 3.TF.46 T; Check that  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is similar to  $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ .

Ch 3.TF.47 T; Pick a vector  $\vec{v}$  that is neither on the line nor perpendicular to it. Then the matrix of the linear transformation  $T(\vec{x}) = R\vec{x}$  with respect to the basis  $\vec{v}$ ,  $R\vec{v}$  is  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , since  $R(R\vec{v}) = \vec{v}$ .

Ch 3.TF.48 F; If  $B = S^{-1}AS$ , then  $B = (2S)^{-1}A(2S)$  as well.

Ch 3.TF.49 T; Note that  $A(B - C) = 0$ , so that all the columns of matrix  $B - C$  are in the kernel of  $A$ . Thus  $B - C = 0$  and  $B = C$ , as claimed.

Ch 3.TF.50 T; Suppose  $\vec{v}$  is in both  $\ker(A)$  and  $\text{im}(A)$ , so that  $\vec{v} = A\vec{w}$  for some vector  $\vec{w}$ . Then  $\vec{0} = A\vec{v} = A^2\vec{w} = A\vec{w} = \vec{v}$ , as claimed.

Ch 3.TF.51 F; Suppose such a matrix  $A$  exists. Then there is a vector  $\vec{v}$  in  $\mathbb{R}^2$  such that  $A^2\vec{v} \neq \vec{0}$  but  $A^3\vec{v} = \vec{0}$ . As in Exercise 3.4.58a we can show that vectors  $\vec{v}, A\vec{v}, A^2\vec{v}$  are linearly independent, a contradiction (we are looking at three vectors in  $\mathbb{R}^2$ ).

Ch 3.TF.52 T; The  $i$ th column  $\vec{a}_i$  of  $A$ , being in the image of  $A$ , is also in the image of  $B$ , so that  $\vec{a}_i = B\vec{c}_i$  for some  $\vec{c}_i$  in  $\mathbb{R}^m$ . If we let  $C = [\vec{c}_1 \ \cdots \ \vec{c}_m]$ , then  $BC = [B\vec{c}_1 \ \cdots \ B\vec{c}_m] = [\vec{a}_1 \ \cdots \ \vec{a}_m] = A$ , as required.

Ch 3.TF.53 F; Think about this problem in terms of “building” such an invertible matrix column by column. If we wish the matrix to be invertible, then the first column can be any column other than  $\vec{0}$  (7 choices). Then the second column can be any column other than  $\vec{0}$  or the first column (6 choices). For the third column, we have at most 5 choices (not  $\vec{0}$  or the first or second columns, as well as possibly some other columns). For some choices of the first two columns there will be other columns we have to exclude (the sum or difference of the first two), but not for others. Thus, in total, fewer than  $7 \times 6 \times 5 = 210$  matrices are invertible, out of a total  $2^9 = 512$  matrices. Thus, most are not invertible.

## Chapter 4

### Section 4.1

**4.1.1** Not a subspace since it does not contain the neutral element, that is, the function  $f(t) = 0$ , for all  $t$ .

**4.1.2** This subset  $V$  is a subspace of  $P_2$ :

- The neutral element  $f(t) = 0$  (for all  $t$ ) is in  $V$ .
- If  $f$  and  $g$  are in  $V$  (so that  $f(2) = g(2) = 0$ ), then  $(f + g)(2) = f(2) + g(2) = 0 + 0 = 0$ , so that  $f + g$  is in  $V$ .
- If  $f$  is in  $V$  (so that  $f(2) = 0$ ), and  $k$  is any constant, then  $(kf)(2) = kf(2) = k \cdot 0 = 0$ , so that  $kf$  is in  $V$ .

A polynomial  $f(t) = a + bt + ct^2$  is in  $V$  if  $f(2) = a + 2b + 4c = 0$ , or  $a = -2b - 4c$ . The general element of  $V$  is of the form  $f(t) = (-2b - 4c) + bt + ct^2 = b(t - 2) + c(t^2 - 4)$ , so that  $t - 2, t^2 - 4$  is a basis of  $V$ .

**4.1.3** This subset  $V$  is a subspace of  $P_2$ :

- The neutral element  $f(t) = 0$  (for all  $t$ ) is in  $V$  since  $f'(1) = f(2) = 0$ .
- If  $f$  and  $g$  are in  $V$  (so that  $f'(1) = f(2)$  and  $g'(1) = g(2)$ ), then  $(f + g)'(1) = (f' + g')(1) = f'(1) + g'(1) = f(2) + g(2) = (f + g)(2)$ , so that  $f + g$  is in  $V$ .
- If  $f$  is in  $V$  (so that  $f'(1) = f(2)$ ) and  $k$  is any constant, then  $(kf)'(1) = (kf')(1) = kf'(1) = kf(2) = (kf)(2)$ , so that  $kf$  is in  $V$ .

If  $f(t) = a + bt + ct^2$  then  $f'(t) = b + 2ct$ , and  $f$  is in  $V$  if

$f'(1) = b + 2c = a + 2b + 4c = f(2)$ , or  $a + b + 2c = 0$ . The general element of  $V$  is of the form  $f(t) = (-b - 2c) + bt + ct^2 = b(t - 1) + c(t^2 - 2)$ , so that  $t - 1, t^2 - 2$  is a basis of  $V$ .

**4.1.4** This subset  $V$  is a subspace of  $P_2$ :

- The neutral element  $f(t) = 0$  (for all  $t$ ) is in  $V$  since  $\int_0^1 0 dt = 0$ .
- If  $f$  and  $g$  are in  $V$  (so that  $\int_0^1 f = \int_0^1 g = 0$ ) then  $\int_0^1 (f + g) = \int_0^1 f + \int_0^1 g = 0$ , so that  $f + g$  is in  $V$ .
- If  $f$  is in  $V$  (so that  $\int_0^1 f = 0$ ) and  $k$  is any constant, then  $\int_0^1 kf = k \int_0^1 f = k \cdot 0 = 0$ , so that  $kf$  is in  $V$ .

If  $f(t) = a + bt + ct^2$  then  $\int_0^1 f(t)dt = \left[ at + \frac{b}{2}t^2 + \frac{c}{3}t^3 \right]_0^1 = a + \frac{b}{2} + \frac{c}{3} = 0$  if  $a = -\frac{b}{2} - \frac{c}{3}$ .

The general element of  $V$  is  $f(t) = \left( -\frac{b}{2} - \frac{c}{3} \right) + bt + ct^2 = b \left( t - \frac{1}{2} \right) + c \left( t^2 - \frac{1}{3} \right)$ , so that  $t - \frac{1}{2}, t^2 - \frac{1}{3}$  is a basis of  $V$ .

**4.1.5** If  $p(t) = a + bt + ct^2$  then  $p(-t) = a - bt + ct^2$  and  $-p(-t) = -a + bt - ct^2$ .

Comparing coefficients we see that  $p(t) = -p(-t)$  for all  $t$  if (and only if)  $a = c = 0$ .

The general element of the subset is of the form  $p(t) = bt$ .

These polynomials form a subspace of  $P_2$ , with basis  $t$ .

**4.1.6** Not a subspace, since  $I_3$  and  $-I_3$  are invertible, but their sum is not.

**4.1.7 a** The set  $V$  of diagonal  $3 \times 3$  matrices is a subspace of  $\mathbb{R}^{3 \times 3}$ :

b The zero matrix  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  is in  $V$ ,

c If  $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$  and  $B = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}$  are in  $V$ , then so is their sum

$$A + B = \begin{bmatrix} a+p & 0 & 0 \\ 0 & b+q & 0 \\ 0 & 0 & c+r \end{bmatrix}.$$

d If  $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$  is in  $V$ , then so is  $kA = \begin{bmatrix} ka & 0 & 0 \\ 0 & kb & 0 \\ 0 & 0 & kc \end{bmatrix}$ , for all constants  $k$ .

**4.1.8** This is a subspace; the justification is analogous to Exercise 7.

**4.1.9** Not a subspace; consider multiplication with a negative scalar.  $I_3$  belongs to the set, but  $-I_3$  doesn't.

**4.1.10 a** Let  $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . Let  $V$  be the set of all  $3 \times 3$  matrices  $A$  such that  $A\vec{v} = \vec{0}$ . Then  $V$  is a subspace of  $\mathbb{R}^{3 \times 3}$ :

b The zero matrix 0 is in  $V$ , since  $0\vec{v} = \vec{0}$ .

c If  $A$  and  $B$  are in  $V$ , then so is  $A + B$ , since  $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \vec{0} + \vec{0} = \vec{0}$ .

d If  $A$  is in  $V$ , then so is  $kA$  for all scalars  $k$ , since  $(kA)\vec{v} = k(A\vec{v}) = k\vec{0} = \vec{0}$ .

**4.1.11** Not a subspace:  $I_3$  is in rref, but the scalar multiple  $2I_3$  isn't.

**4.1.12** Yes, the set  $W$  of all arithmetic sequences is a subspace of  $V$ . Use the fact that a sequence  $(x_0, x_1, x_2, \dots)$  is arithmetic if  $x_n = x_0 + kn$  for some constant  $k$ .

- The sequence  $(0, 0, 0, \dots)$  is an arithmetic sequence, with  $k = 0$ .

- If  $(x_n)$  and  $(y_n)$  are arithmetic sequences (with  $x_n = x_0 + pn$  and  $y_n = y_0 + qn$ ), then  $x_n + y_n = x_0 + y_0 + (p+q)n$ , so that  $(x_n + y_n)$  is an arithmetic sequence as well.
- If  $(x_n)$  is an arithmetic sequence (with  $x_n = x_0 + pn$ ) and  $k$  is an arbitrary constant, then  $kx_n = kx_0 + (kp)n$ , so that  $(kx_n)$  is an arithmetic sequence as well.

**4.1.13** Not a subspace:  $(1, 2, 4, 8, \dots)$  and  $(1, 1, 1, 1, \dots)$  are both geometric sequences, but their sum  $(2, 3, 5, 9, \dots)$  is not, since the ratios of consecutive terms fail to be equal, for example,  $\frac{3}{2} \neq \frac{5}{3}$ .

**4.1.14** Yes

- $(0, 0, 0, \dots, 0, \dots)$  converges to 0.
- If  $\lim_{n \rightarrow \infty} x_n = 0$  and  $\lim_{n \rightarrow \infty} y_n = 0$ , then  $\lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n = 0$ .
- If  $\lim_{n \rightarrow \infty} x_n = 0$  and  $k$  is any constant, then  $\lim_{n \rightarrow \infty} (kx_n) = k \lim_{n \rightarrow \infty} x_n = 0$ .

**4.1.15** The set  $W$  of all square-summable sequences is a subspace of  $V$ :

- The sequence  $(0, 0, 0, \dots)$  is in  $W$ .
- Suppose  $(x_n)$  and  $(y_n)$  are in  $W$ . Note that the inequality  $(x_n + y_n)^2 \leq 2x_n^2 + 2y_n^2$  holds for all  $n$ , since  $2x_n^2 + 2y_n^2 - (x_n + y_n)^2 = x_n^2 + y_n^2 - 2x_n y_n = (x_n - y_n)^2 \geq 0$ . Thus  $\sum_{n=1}^{\infty} (x_n + y_n)^2 \leq 2\sum_{n=1}^{\infty} x_n^2 + 2\sum_{n=1}^{\infty} y_n^2$  converges, so that the sequence  $(x_n + y_n)$  is in  $W$  as well.
- If  $(x_n)$  is in  $W$  (so that  $\sum_{n=1}^{\infty} x_n^2$  converges), then  $(kx_n)$  is in  $W$  as well, for any constant  $k$ , since

$$\sum_{n=1}^{\infty} (kx_n)^2 = k^2 \sum_{n=1}^{\infty} x_n^2$$

will converge.

$$4.1.16 \quad \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + e \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrices  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  form a basis of  $\mathbb{R}^{3 \times 2}$ , so that  $\dim(\mathbb{R}^{3 \times 2}) = 6$ .

**4.1.17** Let  $E_{ij}$  be the  $n \times m$  matrix with a 1 as its  $ij$ th entry, and zeros everywhere else. Any  $A$  in  $\mathbb{R}^{n \times m}$  can be written as the sum of all  $a_{ij}E_{ij}$ , and the  $E_{ij}$  are linearly independent, so that they form a basis of  $\mathbb{R}^{n \times m}$ . Thus  $\dim(\mathbb{R}^{n \times m}) = nm$ .

**4.1.18** Any  $f$  in  $P_n$  can be written as a linear combination of  $1, t, t^2, \dots, t^n$ , by definition of  $P_n$ . Also,  $1, t, \dots, t^n$  are linearly independent; to see this consider a relation  $c_0 + c_1t + \dots + c_nt^n = 0$ ; since the polynomial  $c_0 + c_1t + \dots + c_nt^n$  has more than  $n$  zeros, we must have  $c_0 = c_1 = \dots = c_n = 0$ , as claimed. Thus,  $\dim(P_n) = n + 1$ .

4.1.19  $\begin{bmatrix} a+bi \\ c+di \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} i \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ i \end{bmatrix}$

The vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}$  form a basis of  $\mathbb{C}^2$  as a *real* linear space, so that  $\dim(\mathbb{C}^2) = 4$ .

4.1.20 We use Summary 4.1.6. We have  $a = -d$ , so that the general element of the subspace is  $\begin{bmatrix} -d & b \\ c & d \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ .

Thus  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  is a basis of the subspace; the dimension is 3.

4.1.21 Use Summary 4.1.6. The general element of the subspace is

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \text{ Thus } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ is a basis of the subspace; the dimension is 2.}$$

4.1.22 Using Exercise 21 as a guide, we find the basis  $E_{11}, E_{22}, \dots, E_{nn}$ , where  $E_{ii}$  is the  $n \times n$  matrix with all 0 entries, except for a 1 at the  $i$ th place on the diagonal. The dimension of this space is  $n$ .

4.1.23 Proceeding as in Exercise 21, we find the basis  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ ; the dimension is 3.

4.1.24 Proceeding as in Exercise 21, we find the basis  $E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}$ . Here  $E_{ij}$  is the  $3 \times 3$  matrix with all 0 entries, except for a 1 in the  $i$ th component of the  $j$ th column; the dimension is 6.

4.1.25 A polynomial  $f(t) = a + bt + ct^2$  is in this subspace if  $f(1) = a + b + c = 0$ , or  $a = -b - c$ .

The polynomials in the subspace are of the form  $f(t) = (-b - c) + bt + ct^2 = b(t - 1) + c(t^2 - 1)$ , so that  $t - 1, t^2 - 1$  is a basis of the subspace, whose dimension is 2.

4.1.26 Denote the subspace by  $W$ . A polynomial  $f(t) = a + bt + ct^2 + dt^3$  is in  $W$  if  $f(1) = a + b + c + d = 0$  and

$$\int_{-1}^1 f(t) dt = \left[ at + \frac{b}{2}t^2 + \frac{c}{3}t^3 + \frac{d}{4}t^4 \right]_{-1}^1 = 2a + \frac{2}{3}c = 0.$$

The system  $\begin{bmatrix} a+b+c+d & =0 \\ 2a+\frac{2}{3}c & =0 \end{bmatrix}$  reduces to  $\begin{bmatrix} a+\frac{1}{3}c & =0 \\ b+\frac{2}{3}c+d & =0 \end{bmatrix}$ , with general solution  $\begin{bmatrix} -\frac{1}{3}c \\ -\frac{2}{3}c-d \\ c \\ d \end{bmatrix}$ .

The polynomials in  $W$  are of the form  $f(t) = -\frac{1}{3}c - (\frac{2}{3}c + d)t + ct^2 + dt^3 = c(t^2 - \frac{2}{3}t - \frac{1}{3}) + d(t^3 - t)$ , so that  $t^2 - \frac{2}{3}t - \frac{1}{3}, t^3 - t$  is a basis of  $W$ , and  $\dim(W) = 2$ .

4.1.27  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is in the subspace if  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix}$  equals  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$ , which is

the case if  $b = c = 0$ . The matrices in the subspace are of the form  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , so that  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  is a basis, and the dimension is 2.

**4.1.28**  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is in the subspace if  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$  equals  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$  which is the case if  $c = 0$  and  $a = d$ .

The matrices in the subspace are of the form  $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , so that  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is a basis, and the dimension is 2.

**4.1.29** We are looking for the matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . It is required that  $a = -b$  and  $c = -d$ . The general element is  $\begin{bmatrix} -b & b \\ -d & d \end{bmatrix} = b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ . Thus  $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$  is a basis, and the dimension is 2.

**4.1.30** We are looking for the matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+6c & 3b+6d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . It is required that  $a = -2c$  and  $b = -2d$ . Thus the general element is  $\begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix} = c \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$ .

Thus  $\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$  is a basis, and the dimension is 2.

**4.1.31** We are looking for the matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , or,  $\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}$ . It is required that  $a = c$  and  $b = -d$ .

The general element is  $\begin{bmatrix} c & -d \\ c & d \end{bmatrix} = c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ . Thus  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$  is a basis, and the dimension is 2.

**4.1.32** We are looking for the matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ , or,  $\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 2c & 0 \end{bmatrix}$ . It is required that  $a = c$  and  $b = -d$ .

The general element is  $\begin{bmatrix} a & b \\ a & -b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ . Thus  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$  is a basis, and the dimension is 2.

**4.1.33** Let  $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , meaning

$\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . So  $a+c=a$ ,  $b+d=b$ ,  $a+c=c$  and  $b+d=d$ . These imply, respectively, that  $c=0$ ,  $d=0$ ,  $a=0$  and  $b=0$ .

Thus,  $S$  can only equal  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , and the basis is  $\emptyset$ .

**4.1.34** Let  $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . We want  $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , meaning

$\begin{bmatrix} 3a+2c & 3b+2d \\ 4a+5c & 4b+5d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . So  $3a+2c=a$ ,  $3b+2d=b$ ,  $4a+5c=c$  and  $4b+5d=d$ . These imply that  $a=-c$  and  $b=-d$ .

So the general element is  $\begin{bmatrix} a & b \\ -a & -b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ . Thus  $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$  is a basis, and the dimension is 2.

**4.1.35** Let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ . We want  $AB = BA$ , or

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} 2a & 3b & 4c \\ 2d & 3e & 4f \\ 2g & 3h & 4i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{bmatrix}.$$

We note that  $b, c, d, f, g, h$  must be zero, but  $a, e$ , and  $i$  are chosen freely. So, our space,  $V$ , consists of all matrices

$$\text{of the form } \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is a basis of  $V$ , and  $\dim(V) = 3$ .

**4.1.36** Let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ . We want  $AB = BA$ , or

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} 2a & 3b & 3c \\ 2d & 3e & 3f \\ 2g & 3h & 3i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 3g & 3h & 3i \end{bmatrix}.$$

We note that  $b, c, d, g$  must be zero, but  $a, e, f, h$  and  $i$  are chosen freely. So, our space,  $V$ , consists of all matrices

$$\text{of the form } \begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is a basis of  $V$ , and  $\dim(V) = 5$ .

**4.1.37** If all diagonal entries of  $B$  are different, then  $\dim(V) = 3$ , as in Exercise 35. If two of the diagonal entries are equal, then  $\dim(V) = 5$ , as in Example 36. If all three entries are equal, then  $B$  is a scalar multiple of  $I_3$ , and will commute with all  $3 \times 3$  matrices, so that  $\dim(V) = \dim(\mathbb{R}^{3 \times 3}) = 9$ . So, in summary,  $\dim(V)$  could be 3, 5 or 9.

**4.1.38** We look at similar cases mentioned in Exercise 37, and see that the different possibilities occur when all four entries are different ( $\dim(V) = 4$ ), when exactly two are the same, but the other two are different ( $\dim(V) = 6$ ), when exactly three are the same ( $\dim(V) = 10$ ), when all four are the same ( $\dim(V) = 16$ ) and when two of the terms of  $B$  are equal, and the other two diagonal terms of  $B$  are also equal, but different from the first pair ( $\dim(V) = 8$ ).

**4.1.39** An upper-triangular matrix has the form: 
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{aligned} &= \left( a_{11} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + a_{1n} \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right) \\ &\quad + \left( a_{22} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + a_{2n} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right) + \cdots + \left( a_{nn} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right). \end{aligned}$$

So the dimension of this space is  $n + (n - 1) + \cdots + 1 = \frac{n(n+1)}{2}$ .

**4.1.40** Let  $V$  be the space of all  $n \times n$  matrices  $A$  such that  $A\vec{v} = \vec{0}$ . We look at some possibilities for  $\vec{v}$ . If  $\vec{v} = \vec{0}$ , then any matrix  $A$  will work, and  $\dim(V) = n^2$ . Now assume that  $\vec{v} \neq \vec{0}$ , and suppose that the  $i^{th}$  component  $v_i$  is nonzero. If we denote the columns of  $A$  by  $\vec{w}_1, \dots, \vec{w}_n$ , then the condition  $A\vec{v} = \vec{0}$  means that  $v_1\vec{w}_1 + \cdots + v_i\vec{w}_i + \cdots + v_n\vec{w}_n = \vec{0}$ . We can solve this relation for  $\vec{w}_i$  and express  $\vec{w}_i$  in terms of the other  $n - 1$  column vectors of  $A$ . This means that we can choose  $n - 1$  columns (or  $n(n - 1)$  entries) of  $A$  freely; the column  $\vec{w}_i$  is then determined by these choices. Thus  $\dim(V) = n(n - 1)$  in this case. In summary, we have  $\dim(V) = n^2$  if  $\vec{v} = \vec{0}$ , and  $\dim(V) = n(n - 1)$  otherwise.

**4.1.41** Let  $V$  be the space of all matrices  $A$  such that  $BA = 0$ . Note that  $A$  is in  $V$  if (and only if) all the columns of  $A$  are in the kernel of  $B$ . Since the columns of  $A$  can be chosen independently, it is plausible that  $\dim(V) = 3\dim(\ker(B))$ . We show this more clearly by investigating the different possibilities for  $\ker(B)$ .

In the case where  $\ker(B) \neq \{\vec{0}\}$ , let  $\vec{v}_1, \dots, \vec{v}_n$  be a basis of the kernel of  $B$ , and note that  $n = \text{nullity}(B)$  can be either 1, 2 or 3. Then, the general element of  $V$  is of the form:  $[a_1\vec{v}_1 + \dots + a_n\vec{v}_n \ b_1\vec{v}_1 + \dots + b_n\vec{v}_n \ c_1\vec{v}_1 + \dots + c_n\vec{v}_n]$ , which has the  $3n$  arbitrary constants  $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n$ . Using Summary 4.1.6, we can construct a basis of  $V$  with  $3n$  elements, proving our claim that  $\dim(V) = 3\dim(\ker(B))$ . Thus,  $\dim(V)$  can be 0, 3, 6 or 9.

**4.1.42** Let  $B$  be a matrix such that  $\dim(\ker(B)) = k$ . Then, it is required that the columns of  $A$  contain only vectors in the kernel of  $B$ . Thus, each column of  $A$  can be written as:  $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$ , where the vectors  $\vec{v}_i$  form a basis of the kernel of  $B$ . Thus, each of the  $n$  columns in  $A$  involves  $k$  arbitrary constants, and matrix  $A$  involves  $nk$  arbitrary constants overall. The space of matrices  $A$  has dimension  $nk$ , where  $k$  is an integer in the range  $[0, n]$ .

**4.1.43** Let  $V$  be the space of all matrices  $S$  such that  $AS = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . Using the terminology introduced in the hint, we want  $A[\vec{v} \ \vec{w}] = [\vec{v} \ \vec{w}] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  or  $[A\vec{v} \ A\vec{w}] = [\vec{v} \ -\vec{w}]$ . Thus,  $\vec{v}$  must be parallel to  $L$ , and  $\vec{w}$  must be perpendicular to  $L$ . If  $\vec{v}_1$  is a nonzero vector parallel to  $L$ , and  $\vec{v}_2$  a nonzero vector perpendicular to  $L$ , then the general element of  $V$  is of the form  $S = [\vec{v} \ \vec{w}] = [a\vec{v}_1 \ b\vec{v}_2] = a[\vec{v}_1 \ 0] + b[0 \ \vec{v}_2]$ , where  $a$  and  $b$  are arbitrary constants. We see that  $[\vec{v}_1 \ 0], [0 \ \vec{v}_2]$  is a basis of  $V$  and  $\dim(V) = 2$ .

**4.1.44** Let  $V$  be the space of all matrices  $S$  such that  $AS = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Let's denote the column vectors of  $S$

by  $\vec{u}, \vec{v}$  and  $\vec{w}$ . The condition  $AS = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  means that  $A\vec{u} = \vec{u}, A\vec{v} = \vec{v}$  and  $A\vec{w} = \vec{0}$ . This in turn means

that the vectors  $\vec{u}$  and  $\vec{v}$  have to be on the plane  $V$ , while  $\vec{w}$  is perpendicular to  $V$ . If we choose a basis  $\vec{v}_1, \vec{v}_2$  of  $V$  and a nonzero vector  $\vec{v}_3$  perpendicular to  $V$ , then we can write  $\vec{u} = a\vec{v}_1 + b\vec{v}_2, \vec{v} = c\vec{v}_1 + d\vec{v}_2, \vec{w} = e\vec{v}_3$ , and

$$S = [\vec{u} \ \vec{v} \ \vec{w}] = [a\vec{v}_1 + b\vec{v}_2 \ c\vec{v}_1 + d\vec{v}_2 \ e\vec{v}_3] = a[\vec{v}_1 \ \vec{0} \ \vec{0}] + b[\vec{v}_2 \ \vec{0} \ \vec{0}] + c[\vec{0} \ \vec{v}_1 \ \vec{0}] + d[\vec{0} \ \vec{v}_2 \ \vec{0}] + e[\vec{0} \ \vec{0} \ \vec{v}_3].$$

Thus  $\dim(V) = 5$ ; the five matrices in the linear combination above form a basis of  $V$ .

**4.1.45** Let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ . We want  $AB = BA$ , or

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}.$$

So  $d, g, h = 0$ ,  $a = e = i$  and  $f = b$ . Our space  $V$  consists of all matrices of the form  $\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$

$$b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus,  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  is a basis of  $V$  and  $\dim(V) = 3$ .

**4.1.46** The arithmetic sequences are of the form  $(a, a+k, a+2k, a+3k, \dots) = a(1, 1, 1, 1, \dots) + k(0, 1, 2, 3, \dots)$ , so that the sequences  $(1, 1, 1, \dots)$  (all 1's) and  $(0, 1, 2, 3, \dots)$  (the  $n$ th entry is  $n$ ) form a basis of this space, whose dimension is 2.

**4.1.47** We show that the set of all even functions is a subspace of  $F(\mathbb{R}, \mathbb{R})$ :

- If  $f(t) = 0$  for all  $t$ , then  $f(-t) = f(t) = 0$  for all  $t$ .
- If  $f$  and  $g$  are even (that is,  $f(-t) = f(t)$  and  $g(-t) = g(t)$  for all  $t$ ), then  $(f+g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f+g)(t)$ , so that  $f+g$  is even as well.
- If  $f$  is even and  $k$  is any constant, then  $(kf)(-t) = kf(-t) = kf(t) = (kf)(t)$ , so that  $kf$  is even as well.

An analogous proof shows that the odd functions form a subspace of  $F(\mathbb{R}, \mathbb{R})$ .

**4.1.48 a**  $f$  is even if  $f(-t) = f(t)$  for all  $t$ . Comparing coefficients we find that  $b = d = 0$ , so that  $f(t)$  is of the form  $f(t) = a + ct^2 + et^4$ , with basis  $1, t^2, t^4$ . The dimension is 3.

**b**  $f$  is odd if  $f(-t) = -f(t)$ , which is the case if  $a = c = e = 0$ . The odd polynomials are of the form  $f(t) = bt + dt^3$ , with basis  $t, t^3$  and dimension 2.

**4.1.49** We show that  $L(\mathbb{R}^m, \mathbb{R}^n)$  is a subspace of  $F(\mathbb{R}^m, \mathbb{R}^n)$ :

- The zero transformation  $T(\vec{x}) = \vec{0}$  (for all  $\vec{x}$ ) is linear, represented by the zero matrix.
- If  $T$  and  $S$  are linear transformations from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  (with  $T(\vec{x}) = A\vec{x}$  and  $S(\vec{x}) = B\vec{x}$  for some  $n \times m$  matrices  $A$  and  $B$ ), then  $(T+S)(\vec{x}) = T(\vec{x}) + S(\vec{x}) = A\vec{x} + B\vec{x} = (A+B)\vec{x}$ , so that  $T+S$  is linear as well, given by the matrix  $A+B$ .
- If  $T$  is a linear transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  (with  $T(\vec{x}) = A\vec{x}$ ) and  $k$  is any constant, then  $(kT)(\vec{x}) = kT(\vec{x}) = kA\vec{x} = (kA)\vec{x}$ , so that  $kT$  is linear as well, with matrix  $kA$ .

**4.1.50** Using Example 18 as a guide, we first look for solutions of the form  $f(x) = e^{kx}$ . It is required that  $f''(x) + 8f'(x) - 20f(x) = k^2e^{kx} + 8ke^{kx} - 20e^{kx} = 0$  for all  $x$ , or  $k^2 + 8k - 20 = (k-2)(k+10) = 0$ . Thus  $k = 2$  or  $k = -10$ . By Theorem 4.1.5, the solutions of the differential equation are of the form  $f(x) = c_1e^{2x} + c_2e^{-10x}$ , where  $c_1$  and  $c_2$  are arbitrary constants.

**4.1.51** Using Example 18 as a guide, we first look for solutions of the form  $f(x) = e^{kx}$ . It is required that  $f''(x) - 7f'(x) + 12f(x) = k^2e^{kx} - 7ke^{kx} + 12e^{kx} = 0$  for all  $x$ , or  $k^2 - 7k + 12 = (k-3)(k-4) = 0$ . Thus  $k = 3$  or  $k = 4$ . By Theorem 4.1.7, the solutions of the differential equation are of the form  $f(x) = c_1e^{3x} + c_2e^{4x}$ , where  $c_1$  and  $c_2$  are arbitrary constants.

**4.1.52** We have to find constants  $a$  and  $b$  such that the functions  $e^{-x}$  and  $e^{-5x}$  are solutions of the differential equation  $f''(x) + af'(x) + bf(x) = 0$ . Thus it is required that  $e^{-x} - ae^{-x} + be^{-x} = 0$ , or  $1 - a + b = 0$ , and also that  $25 - 5a + b = 0$ . The solution of this system of two equations in two unknowns is  $a = 6, b = 5$ , so that the desired differential equation is  $f''(x) + 6f'(x) + 5f(x) = 0$ .

**4.1.53** Let  $\mathcal{B} = (f_1, \dots, f_n)$  be a basis of  $V$  and suppose that the elements  $g_1, \dots, g_m$  in  $V$  are linearly independent. In the proof of Theorem 4.1.5, we show that the coordinate vectors  $[g_1]_{\mathcal{B}}, \dots, [g_m]_{\mathcal{B}}$  in  $\mathbb{R}^n$  are linearly independent, so that  $m \leq n$  by Theorem 3.2.8.

**4.1.54** We can adapt the answer to Exercise 3.2.38a. Let  $m$  be the largest number of linearly independent elements we can find in  $W$ ; note that  $m \leq n$ , by Exercise 53. Choose linearly independent elements  $f_1, \dots, f_m$  in  $W$ . We claim that the elements  $f_1, \dots, f_m$  span  $W$ . Indeed, if  $f$  is any element of  $W$ , then the  $m+1$  elements  $f_1, \dots, f_m, f$  are linearly dependent, so that  $f$  is redundant:  $f$  is a linear combination of  $f_1, \dots, f_m$ . It follows that  $f_1, \dots, f_m$  is a basis of  $W$ , so that  $\dim(W) = m \leq n = \dim(V)$ , as claimed.

**4.1.55** We will argue indirectly, assuming that  $F(\mathbb{R}, \mathbb{R})$  is  $n$ -dimensional for some  $n$ . Now, the  $n+1$  polynomials  $1, x, x^2, \dots, x^n$  in  $F(\mathbb{R}, \mathbb{R})$  are linearly independent, contradicting the fact that we can find at most  $n$  linearly independent elements in an  $n$ -dimensional space (see Exercise 53). We conclude that  $F(\mathbb{R}, \mathbb{R})$  is infinite dimensional, as claimed.

**4.1.56** Argue indirectly and assume that the space  $V$  of infinite sequences is finite-dimensional, with  $\dim(V) = n$ . According to the solution to Exercise 57, there can be at most  $n$  linearly independent elements in  $V$ . But here is our contradiction: It is easy to give  $n+1$  linearly independent infinite sequences, namely,

$(1, 0, 0, 0, \dots), (0, 1, 0, 0, \dots), (0, 0, 1, 0, \dots), \dots, (0, 0, 0, \dots, 0, 1, 0, \dots)$ ; in the last sequence the 1 is in the  $(n+1)$ th place.

**4.1.57** We can construct a basis of  $V$  by omitting the redundant elements from the list  $g_1, \dots, g_m$ . It follows that  $V$  is finite-dimensional, and, in fact,  $\dim(V) \leq m$ , since our basis is a “sub-list” of the original list  $g_1, \dots, g_m$ .

**4.1.58 a** Let  $g(x)$  be a function in  $V$ . Thus,  $g''(x) = -g(x)$ . Now, if  $f(x) = g(x)^2 + g'(x)^2$ , then  $f'(x) = 2(g(x))g'(x) + 2(g'(x))g''(x) = 2(g(x))g'(x) - 2(g(x))g'(x) = 0$ . So,  $f(x) = g(x)^2 + g'(x)^2$  is a constant function.

**b** Let  $g(x)$  be a function in  $V$  such that  $g(0) = g'(0) = 0$ . From part a we know that  $g(x)^2 + g'(x)^2 = k$ , a constant. Now  $g(0)^2 + g'(0)^2 = 0^2 + 0^2 = 0$ , so that  $k = 0$ . The equation  $g(x)^2 + g'(x)^2 = 0$  means that  $g(x) = g'(x) = 0$  for all  $x$ , as claimed.

**c** First we note that  $g(x) = f(x) - f(0)\cos(x) - f'(0)\sin(x)$  is in  $V$ , since the functions  $f(x)$ ,  $\cos(x)$  and  $\sin(x)$  are all in  $V$ , and  $V$  is a subspace of  $F(\mathbb{R}, \mathbb{R})$ . Note that  $g(0) = f(0) - f(0)\cos(0) - f'(0)\sin(0) = f(0) - f(0) = 0$ . Also,  $g'(x) = f'(x) + f(0)\sin(x) - f'(0)\cos(x)$ , so that  $g'(0) = f'(0) - f'(0) = 0$ .

By part b, we can conclude that  $g(x) = 0$  for all  $x$ , so that  $f(x) = f(0)\cos(x) + f'(0)\sin(x)$ , as claimed.

## Section 4.2

**4.2.1** Fails to be linear, since  $T(A + B) = A + B + I_2$  doesn't equal

$$T(A) + T(B) = A + I_2 + B + I_2 = A + B + 2I_2.$$

**4.2.2** Linear, since  $T(A + B) = 7(A + B) = 7A + 7B$  equals  $T(A) + T(B) = 7A + 7B$ ,

and  $T(kA) = 7kA$  equals  $kT(A) = k(7A) = 7kA$ .

Yes,  $T$  is an isomorphism, with  $T^{-1}(A) = \frac{1}{7}A$ .

**4.2.3** Linear, since  $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = T\begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix} = a+p+d+s$

equals  $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} + T\begin{bmatrix} p & q \\ r & s \end{bmatrix} = a+d+p+s$ , and  $T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = ka+kd$

equals  $kT\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k(a+d) = ka+kd$ .

No,  $T$  fails to be an isomorphism, since  $4 = \dim(\mathbb{R}^{2 \times 2}) \neq \dim(\mathbb{R}) = 1$ ; see Theorem 4.2.4b.

**4.2.4** Fails to be linear, since  $T(2I_2) = \det(2I_2) = 4$  does not equal

$$2T(I_2) = 2\det(I_2) = 2.$$

**4.2.5** Fails to be linear, since  $T(2I_2) = 4I_2$  does not equal  $2T(I_2) = 2I_2$ .

**4.2.6** Let  $P = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ . Transformation  $T$  is linear, since

$T(A + B) = (A + B)P = AP + BP$  equals  $T(A) + T(B) = AP + BP$ , and

$T(kA) = (kA)P = kAP$  equals  $kT(A) = kAP$ .

No,  $T$  isn't an isomorphism, since  $\ker(T) \neq \{0\}$ ; the matrix  $A = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$  is in the kernel of  $T$  (see Theorem 4.2.4a).

**4.2.7** Linear, since  $T(M + N) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}(M + N) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}M + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}N = T(M) + T(N)$ . Also,  $T(kM) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}(kM) = k\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}M = kT(M)$ .

This is also an isomorphism. Solve the equation  $N = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}M$  for  $M$  to find the inverse  $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}N = \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}N$ .

**4.2.8** Not linear, since  $T(kM) = (kM)\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}(kM) = k^2M\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}M = k^2T(M) \neq kT(M)$ , in general.

**4.2.9** Linear, since  $T(A + B) = S^{-1}(A + B)S = S^{-1}AS + S^{-1}BS$  equals

$T(A) + T(B) = S^{-1}AS + S^{-1}BS$ , and  $T(kA) = S^{-1}(kA)S = kS^{-1}AS$  equals

$kT(A) = kS^{-1}AS$ .

Yes,  $T$  is an isomorphism. Solve the equation  $B = S^{-1}AS$  for  $A$  to find the inverse  
 $A = SBS^{-1}$ .

**4.2.10** Linear, since  $T(A + B) = P(A + B)P^{-1} = PAP^{-1} + PBP^{-1}$  equals

$$T(A) + T(B) = PAP^{-1} + PBP^{-1}, \text{ and } T(kA) = P(kA)P^{-1} = kPAP^{-1} \text{ equals}$$

$$kT(A) = kPAP^{-1}.$$

Yes,  $T$  is an isomorphism. Solve the equation  $B = PAP^{-1}$  for  $A$  to find the inverse

$$A = P^{-1}BP.$$

**4.2.11** Linear, since  $T(M + N) = P(M + N)Q = (PM + PN)Q = PMQ + PNQ = T(M) + T(N)$ .

$$\text{Also, } T(kM) = P(kM)Q = kPMQ = kT(M).$$

This is also an isomorphism. Solve the equation  $N = PMQ$  for  $M$  to find the inverse  $M = P^{-1}NQ^{-1}$ .

**4.2.12** Linear. Let  $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ . Then  $T(c + d) = (c + d)A = cA + dA$  equals  $T(c) + T(d) = cA + dA$ , and

$$T(kc) = kcA \text{ equals } kT(c) = kcA.$$

No,  $T$  isn't an isomorphism, since domain and codomain have different dimensions.

**4.2.13** Let  $Q = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ . Transformation  $T$  is linear, since

$$T(M + N) = (M + N)Q - Q(M + N) = MQ + NQ - QM - QN \text{ equals}$$

$$T(M) + T(N) = MQ - QM + NQ - QN, \text{ and } T(kM) = (kM)Q - Q(kM) = kMQ - kQM$$

$$\text{equals } kT(M) = k(MQ - QM) = kMQ - kQM.$$

No,  $T$  isn't an isomorphism, since  $A = I_2$  is in  $\ker(T)$ ; see Theorem 4.2.4a.

**4.2.14** Let  $Q = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ . Transformation  $T$  is linear, since

$$T(M + N) = Q(M + N) - (M + N)Q = QM + QN - MQ - NQ \text{ equals}$$

$$T(M) + T(N) = QM - MQ + QN - NQ, \text{ and } T(kM) = Q(kM) - (kM)Q = kQM - kMQ$$

$$\text{equals } kT(M) = k(QM - MQ) = kQM - kMQ.$$

No,  $T$  isn't an isomorphism, since  $A = I_2$  is in  $\ker(T)$ ; see Theorem 4.2.4a.

**4.2.15** Linear, since  $T(M + N) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}(M + N) - (M + N)\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}M + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}N - M\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} - N\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}M - M\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}N - N\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} = T(M) + T(N)$ .

$$\text{Also, } T(kM) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}(kM) - (kM)\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} = k\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}M - kM\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} = kT(M).$$

This is also an isomorphism. We first see that the dimensions of  $V$  and  $W$  are equal, then show that the kernel contains only zero. Let  $T(M) = T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 3c & 3d \end{bmatrix} - \begin{bmatrix} 4a & 5b \\ 4c & 5d \end{bmatrix} = \begin{bmatrix} -2a & -3b \\ -c & -2d \end{bmatrix}$ . Clearly the only time this matrix will equal zero is when  $a, b, c, d = 0$ . Thus,  $\ker(T) = \{0\}$ .

**4.2.16** Linear, since  $T(M + N) = (M + N) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} (M + N) = M \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + N \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} M - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} N = M \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} M + N \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} N = T(M) + T(N)$ .

Also,  $T(kM) = (kM) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} (kM) = k \left( M \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} M \right) = kT(M)$ .

To determine whether  $T$  is an isomorphism, we find the kernel of  $T$ . Now,  $T(M) = T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 3b \\ 2c & 3d \end{bmatrix} - \begin{bmatrix} 3a & 3b \\ 4c & 4d \end{bmatrix} = \begin{bmatrix} -a & 0 \\ -2c & -d \end{bmatrix}$ . We see that  $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is in the kernel of  $T$  if  $a = c = d = 0$ . Thus, the kernel consists of all matrices  $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ , where  $b$  is arbitrary. Since the kernel is nonzero,  $T$  fails to be an isomorphism.

**4.2.17**  $T$  is linear:  $T((a + ib) + (c + id)) = T(a + c + i(b + d)) = a + c = T(a + ib) + T(c + id)$ , and  $T(k(x + iy)) = T(kx + iky) = kx = kT(x + iy)$ .

However,  $T(5i) = 0$ , so  $T$  fails to be an isomorphism.

**4.2.18** This transformation fails to be linear, since  $2T(3) = 2(9) = 18 \neq T(6) = 36$ .

**4.2.19**  $T$  is linear: if  $w$  and  $z$  are complex numbers, then  $T(w + z) = i(w + z) = iw + iz = T(w) + T(z)$  and  $T(kz) = i(kz) = k(iz) = kT(z)$ .

$T$  is also an isomorphism. Solve the equation  $w = iz$  for  $z$  to find that  $z = \frac{1}{i}w = -iw$ . We have used the fact that  $i^2 = -1$ , so that  $\frac{1}{i} = -i$ .

**4.2.20** Linear, since  $T((x + iy) + (z + it)) = T(x + z + i(y + t)) = x + z - i(y + t)$  equals

$T(x + iy) + T(z + it) = x - iy + z - it = x + z - i(y + t)$  and

$T(k(x + iy)) = T(kx + iky) = kx - iky$  equals  $kT(x + iy) = k(x - iy) = kx - iky$ .

Yes,  $T$  is an isomorphism; it's its own inverse, since  $T(T(x + iy)) = T(x - iy) = x + iy$ .

**4.2.21** Linear, since  $T((x + iy) + (z + it)) = T(x + z + i(y + t)) = y + t + i(x + z)$  equals

$T(x + iy) + T(z + it) = y + ix + t + iz = y + t + i(x + z)$ , and

$T(k(x + iy)) = T(kx + iky) = ky + ikx$  equals  $kT(x + iy) = k(y + ix) = ky + ikx$ .

Yes,  $T$  is an isomorphism; it's its own inverse, since  $T(T(x + iy)) = T(y + ix) = x + iy$ .

**4.2.22** Linear, since  $T(f + g) = \int_{-2}^3 (f + g) = \int_{-2}^3 f + \int_{-2}^3 g$  equals

$$T(f) + T(g) = \int_{-2}^3 f + \int_{-2}^3 g, \text{ and } T(kf) = \int_{-2}^3 kf = k \int_{-2}^3 f \text{ equals } kT(f) = k \int_{-2}^3 f.$$

No,  $T$  isn't an isomorphism, since domain and codomain have different dimensions.

**4.2.23**  $T$  is linear, because  $T(f(t) + g(t)) = f(7) + g(7) = T(f(t)) + T(g(t))$ , and  $T(kf(t)) = kf(7) = kT(f(t))$ . However,  $T$  cannot be an isomorphism, because the dimensions of the domain and codomain fail to be equal.

**4.2.24**  $T$  is not linear.  $2T(t^2) = 2(2(t^2)) = 4t^2 \neq T(2t^2) = 4(2t^2) = 8t^2$ .

**4.2.25** Linear, since  $T(f + g) = (f + g)'' + 4(f + g)' = f'' + g'' + 4f' + 4g'$  equals

$$T(f) + T(g) = f'' + 4f' + g'' + 4g', \text{ and } T(kf) = (kf)'' + 4(kf)' = kf'' + 4kf' \text{ equals}$$

$kT(f) = k(f'' + 4f') = kf'' + 4kf'$ . No, it isn't an isomorphism, since the constant function  $f(x) = 1$  is in  $\ker(T)$ .

**4.2.26** Linear, since  $T(f(t) + g(t)) = f(-t) + g(-t)$  equals

$$T(f(t)) + T(g(t)) = f(-t) + g(-t), \text{ and } T(kf(t)) = kf(-t) \text{ equals } kT(f(t)) = kf(-t).$$

Yes,  $T$  is an isomorphism; it's its own inverse, since  $T(T(f(t))) = T(f(-t)) = f(t)$ .

**4.2.27** Linear, since  $T(f(t) + g(t)) = f(2t) + g(2t)$  equals

$$T(f(t)) + T(g(t)) = f(2t) + g(2t), \text{ and } T(kf(t)) = kf(2t) \text{ equals } kT(f(t)) = kf(2t).$$

Yes,  $T$  is an isomorphism; the inverse is  $T^{-1}(g(t)) = g\left(\frac{t}{2}\right)$ .

**4.2.28**  $T$  is linear, since  $T(f(t) + g(t)) = f(2t) + g(2t) - f(t) - g(t) = f(2t) - f(t) + g(2t) - g(t) = T(f(t)) + T(g(t))$ , and  $T(kf(t)) = kf(2t) - kf(t) = k(f(2t) - f(t)) = kT(f(t))$ .

$T$  is not an isomorphism, however, since  $T(3) = 3 - 3 = 0$ .

**4.2.29** Linear, because  $T(f(t) + g(t)) = f'(t) + g'(t) = T(f(t)) + T(g(t))$  and  $T(kf(t)) = kf'(t) = kT(f(t))$ .

However, since  $T(5) = 0$ , the kernel is nonzero, and  $T$  fails to be an isomorphism.

**4.2.30** Linear, since  $T(f(t) + g(t)) = t(f'(t) + g'(t)) = t(f'(t)) + t(g'(t))$  equals

$$T(f(t)) + T(g(t)) = t(f'(t)) + t(g'(t)), \text{ and } T(kf(t)) = t(kf'(t)) = kt(f'(t))$$

equals  $kT(f(t)) = kt(f'(t))$ .

No,  $T$  isn't an isomorphism, since the constant function  $f(t) = 1$  is in  $\ker(T)$ .

**4.2.31**  $T$  is linear:  $T(f(t) + g(t)) = \begin{bmatrix} f(0) + g(0) & f(1) + g(1) \\ f(2) + g(2) & f(3) + g(3) \end{bmatrix} = \begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix} + \begin{bmatrix} g(0) & g(1) \\ g(2) & g(3) \end{bmatrix}$

$$= T(f(t)) + T(g(t)), \text{ and } T(kf(t)) = \begin{bmatrix} kf(0) & kf(1) \\ kf(2) & kf(3) \end{bmatrix} = k \begin{bmatrix} f(0) & f(1) \\ f(2) & f(3) \end{bmatrix} = kT(f(t)).$$

However, the dimensions here are different, so that  $T$  fails to be an isomorphism.

**4.2.32**  $T$  is not linear:  $T(2) = 0 + t^2 \neq 2T(1) = 2(0 + t^2)$ .

**4.2.33** Linear, since  $T((x_0, x_1, x_2, \dots) + (y_0, y_1, y_2, \dots)) = T(x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots) = (x_0 + y_0, x_2 + y_2, \dots)$  equals  $T(x_0, x_1, x_2, \dots) + T(y_0, y_1, y_2, \dots) = (x_0, x_2, \dots) + (y_0, y_2, \dots) = (x_0 + y_0, x_2 + y_2, \dots)$ , and  $T(k(x_0, x_1, x_2, \dots)) = T(kx_0, kx_1, kx_2, \dots) = (kx_0, kx_2, \dots)$  equals  $kT(x_0, x_1, x_2, \dots) = k(x_0, x_2, \dots) = (kx_0, kx_2, \dots)$ .

No,  $T$  isn't an isomorphism, since  $(0, 1, 0, 0, 0, \dots)$  is in  $\ker(T)$ .

**4.2.34** Linear, since  $T((x_0, x_1, x_2, \dots) + (y_0, y_1, y_2, \dots)) = T(x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots) =$

$(0, x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots)$  equals

$T(x_0, x_1, x_2, \dots) + T(y_0, y_1, y_2, \dots) = (0, x_0, x_1, x_2, \dots) + (0, y_0, y_1, y_2, \dots) =$

$(0, x_0 + y_0, x_1 + y_1, x_2 + y_2, \dots)$ , and

$T(k(x_0, x_1, x_2, \dots)) = T(kx_0, kx_1, kx_2, \dots) = (0, kx_0, kx_1, kx_2, \dots)$  equals

$kT(x_0, x_1, x_2, \dots) = k(0, x_0, x_1, x_2, \dots) = (0, kx_0, kx_1, kx_2, \dots)$ .

No,  $T$  isn't an isomorphism, since  $(1, 0, 0, 0, \dots)$  isn't in  $\text{im}(T)$ .

**4.2.35** Linear, since  $T(f(t) + g(t)) = (f(0) + g(0), f'(0) + g'(0), \dots)$

$= (f(0), f'(0), \dots) + (g(0), g'(0), \dots) = T(f(t)) + T(g(t))$  and  $T(kf(t))$

$= (kf(0), kf'(0), \dots) = k(f(0), f'(0), \dots) = kT(f(t))$ .

$T$  fails to be an isomorphism. Note that the sequences in the image of  $T$  have only finitely many nonzero entries, so that a sequence like  $(1, 1, 1, 1, \dots)$ , with all 1's, fails to be in the image of  $T$ . Now use Theorem 4.2.4a.

**4.2.36**  $T$  is linear:  $T(f(t) + g(t)) = (f(0) + g(0), f(1) + g(1), \dots) = (f(0), f(1), \dots) + (g(0), g(1), \dots) = T(f(t)) + T(g(t))$ , and  $T(kf(t)) = (kf(0), kf(1), \dots) = k(f(0), f(1), \dots) = kT(f(t))$ .

We will show that the image of  $T$  isn't all of  $V$ , so that  $T$  fails to be an isomorphism. More specifically, we claim that the sequence  $(1, 0, 0, 0, \dots)$ , a 1 followed by all 0's, fails to be in the image. We make an attempt to find a polynomial  $f(t)$  such that  $T(f(t)) = (f(0), f(1), f(2), \dots) = (1, 0, 0, \dots)$ . This polynomial  $f(t)$  is required to have infinitely many zeros, at  $t = 1, 2, 3, \dots$ , so that  $f(t)$  must be the zero polynomial, and the equation  $f(0) = 1$  isn't satisfied. Thus there is no polynomial  $f(t)$  such that  $T(f(t)) = (1, 0, 0, 0, \dots)$ .

**4.2.37** Linear, since  $T(f+g) = f+g+f'+g' = f+f'+g+g' = T(f)+T(g)$  and  $T(kf) = kf+kf' = k(f+f') = kT(f)$ .

However,  $T(e^{-x}) = e^{-x} - e^{-x} = 0$ , so  $T$  fails to be an isomorphism.

**4.2.38** Linear, just as in Exercise 37.

$T$  is not an isomorphism, since  $T(\sin(x)) = \sin(x) - \sin(x) = 0$ .

**4.2.39** Linear; the proof is analogous to Exercise 25.

No,  $T$  isn't an isomorphism, since the kernel of  $T$  is two-dimensional, by Theorem 4.1.7.

**4.2.40** Same answer as in Exercise 39.

**4.2.41** Not linear, because  $T(f(t) + g(t)) = f(t) + g(t) + f''(t) + g''(t) + \sin(t)$  does not equal  $T(f(t)) + T(g(t)) = f(t) + f''(t) + \sin(t) + g(t) + g''(t) + \sin(t)$ .

**4.2.42** Linear, since  $T(f(t) + g(t)) = \begin{bmatrix} f(7) + g(7) \\ f(11) + g(11) \end{bmatrix}$  equals  $T(f(t)) + T(g(t)) =$

$$\begin{bmatrix} f(7) \\ f(11) \end{bmatrix} + \begin{bmatrix} g(7) \\ g(11) \end{bmatrix} = \begin{bmatrix} f(7) + g(7) \\ f(11) + g(11) \end{bmatrix}, \text{ and } T(kf(t)) = \begin{bmatrix} kf(7) \\ kf(11) \end{bmatrix} \text{ equals}$$

$$kT(f(t)) = k \begin{bmatrix} f(7) \\ f(11) \end{bmatrix} = \begin{bmatrix} kf(7) \\ kf(11) \end{bmatrix}.$$

Not an isomorphism, since domain and codomain have different dimensions.

**4.2.43** Linear, since  $T(f(t) + g(t)) = \begin{bmatrix} f(5) + g(5) \\ f(7) + g(7) \\ f(11) + g(11) \end{bmatrix} = \begin{bmatrix} f(5) \\ f(7) \\ f(11) \end{bmatrix} + \begin{bmatrix} g(5) \\ g(7) \\ g(11) \end{bmatrix} = T(f(t)) + T(g(t))$ , and  $T(kf(t)) = \begin{bmatrix} kf(5) \\ kf(7) \\ kf(11) \end{bmatrix} = k \begin{bmatrix} f(5) \\ f(7) \\ f(11) \end{bmatrix} = kT(f(t))$ .

$T$  is an isomorphism; the proof is analogous to Example 6b.

**4.2.44** Linear, since  $T(f(t) + g(t)) = \begin{bmatrix} f(1) + g(1) \\ f'(2) + g'(2) \\ f(3) + g(3) \end{bmatrix} = \begin{bmatrix} f(1) \\ f'(2) \\ f(3) \end{bmatrix} + \begin{bmatrix} g(1) \\ g'(2) \\ g(3) \end{bmatrix} = T(f(t)) + T(g(t))$ , and

$$T(kf(t)) = \begin{bmatrix} kf(1) \\ kf'(2) \\ kf(3) \end{bmatrix} = k \begin{bmatrix} f(1) \\ f'(2) \\ f(3) \end{bmatrix} = kT(f(t)).$$

Not an isomorphism, since  $T((t-1)(t-3)) = T(t^2 - 4t + 3) = \vec{0}$ .

**4.2.45** Linear, since  $T(f(t) + g(t)) = t(f(t) + g(t)) = tf(t) + tg(t) = T(f(t)) + T(g(t))$  and  $T(kf(t)) = kt f(t) = kT(f(t))$ .

This is not an isomorphism, since the constant function  $f(t) = 1$  isn't in the image.

**4.2.46** Linear, since  $T(f(t) + g(t)) = (t-1)(f(t) + g(t)) = (t-1)f(t) + (t-1)g(t) = T(f(t)) + T(g(t))$  and  $T(kf(t)) = (t-1)(kf(t)) = k(t-1)f(t) = kT(f(t))$ .

This is not an isomorphism, since the constant function  $f(t) = 1$  isn't in the image.

**4.2.47**  $T$  is linear, because  $T(f(t) + g(t)) = \int_0^t (f(x) + g(x)) dx = \int_0^t f(x) dx + \int_0^t g(x) dx = T(f(t)) + T(g(t))$ . Also,  $T(kf(t)) = \int_0^t kf(x) dx = k \int_0^t f(x) dx = kT(f(t))$ .

However, there is no polynomial  $f(t)$  such that  $T(f(t)) = \int_0^t f(x) = 6$ . Thus,  $T$  is not an isomorphism.

**4.2.48** Linear, since  $T(f(t)+g(t)) = (f+g)'(t) = f'(t)+g'(t) = T(f(t))+T(g(t))$  and  $T(kf(t)) = kf'(t) = kT(f(t))$ .

This is not an isomorphism, however, since  $T(5) = 0$ .

**4.2.49** Linear, since  $T(f(t)+g(t)) = f(t^2)+g(t^2) = T(f(t))+T(g(t))$  and  $T(kf(t)) = kf(t^2) = kT(f(t))$ . However, there is no  $f(t)$  in  $P$  such that  $T(f(t)) = f(t^2) = t$ . Thus, the image of  $T$  fails to be all of  $P$ , and  $T$  fails to be an isomorphism.

**4.2.50** Linear.  $T(f(t)+g(t)) = \frac{f(t+2)+g(t+2)-f(t)-g(t)}{2} = \frac{f(t+2)-f(t)}{2} + \frac{g(t+2)-g(t)}{2} = T(f(t))+T(g(t))$ , and  $T(kf(t)) = \frac{kf(t+2)-kf(t)}{2} = k\frac{f(t+2)-f(t)}{2} = kT(f(t))$ .

This is not an isomorphism, however, since  $T(5) = \frac{5-5}{2} = 0$ .

**4.2.51** We need to find the matrices  $M = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  that commute with  $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ , that is,

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix}, \text{ or, } \begin{bmatrix} x & 2x+y \\ z & 2z+t \end{bmatrix} = \begin{bmatrix} x+2z & y+2t \\ z & t \end{bmatrix}.$$

It follows that  $z = 0$  and  $x = t$ , so that the kernel of  $T$  consists of all matrices of the form  $\begin{bmatrix} t & y \\ 0 & t \end{bmatrix}$ . The nullity (i.e., the dimension of the kernel of  $T$ ) is 2.

**4.2.52** We need to find the matrices  $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} x+3y & 2x+6y \\ z+3t & 2z+6t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ It is required that } x = -3y \text{ and } z = -3t, \text{ so that the kernel of } T \text{ consists of all matrices of the form } \begin{bmatrix} -3y & y \\ -3t & t \end{bmatrix}.$$

The nullity (i.e., the dimension of the kernel of  $T$ ) is 2.

**4.2.53** Note that  $T(a + bt + ct^2) = 2c + 4b + 8ct$ . Thus the kernel consists of all constant polynomials  $f(t) = a$  (when  $b = c = 0$ ), and the nullity is 1. The image consists of all linear polynomials  $f(t) = p + qt$ , and the rank is 2.

**4.2.54** Use calculus to see that  $T(a + bt + ct^2) = 5a + \frac{5}{2}b + \frac{35}{3}c$ . The image is all of

$\mathbb{R}$ , so that the rank is 1. The kernel consists of all polynomials of the form

$$f(t) = -\frac{1}{2}b - \frac{7}{3}c + bt + ct^2, \text{ and thus the nullity is 2.}$$

**4.2.55** The kernel consists of all infinite sequences  $(x_0, x_1, x_2, x_3, \dots)$  such that

$$T(x_0, x_1, x_2, x_3, x_4, \dots) = (x_0, x_2, x_4, \dots) = (0, 0, 0, \dots), \text{ that is, all terms } x_k \text{ with even } k \text{ must be 0.}$$

Thus the kernel consists of all sequences of the form  $(0, x_1, 0, x_3, 0, \dots)$ .

The image consists of all infinite sequences  $(y_0, y_1, y_2, \dots)$ , since  $(y_0, y_1, y_2, \dots) =$

$T(y_0, 0, y_1, 0, y_2, 0, \dots)$  for example.

**4.2.56** Note that  $T(a + bt + ct^2) = bt + 2ct^2$ . Thus the kernel consists of all constant polynomials  $f(t) = a$  (when  $b = c = 0$ ), and the nullity is 1. The image consists of all polynomials of the form  $f(t) = pt + qt^2$ , and the rank is 2.

**4.2.57** The kernel consists of the solutions of the differential equation  $f''(t) - 5f'(t) + 6f(t) = 0$ . Using the approach outlined in Example 18 of Section 4.1 (involving a trial solution  $f(t) = e^{kt}$ ), we find the general solution  $f(t) = c_1e^{2t} + c_2e^{3t}$ ; thus the nullity is 2.

**4.2.58** The kernel consists of all infinite sequences such that

$$T(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots) = (0, 0, 0, 0, \dots), \text{ that is, all terms } x_k \text{ must be 0.}$$

Thus the kernel consists of the zero sequence  $(0, 0, 0, \dots)$  alone. The image

consists of all infinite sequences of the form  $(0, x_0, x_1, x_2, \dots)$ .

**4.2.59** To find the kernel, we solve the equation  $T(f(t)) = T(a + bt + ct^2) = a + 7b + 49c = 0$ . It follows that  $a = -7b - 49c$ , and the general element of the kernel is  $(-7b - 49c) + bt + ct^2 = b(-7 + t) + c(-49 + t^2)$ . Then a basis of the kernel is  $-7 + t, -49 + t^2$ , and the nullity of  $T$  is 2. Now the rank of  $T$  must be 1, and the image is all of  $\mathbb{R}$ .

**4.2.60** Note that  $T(a + bt + ct^2) = \begin{bmatrix} a + 7b + 49c \\ a + 11b + 121c \end{bmatrix}$ . To find the kernel, solve the linear system  $\begin{bmatrix} a + 7b + 49c \\ a + 11b + 121c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . The solution is  $a = 77c$ ,  $b = -18c$ , so that the kernel consists of all polynomials of the form  $f(t) = c(77 - 18t + t^2) = c(t - 11)(t - 7)$ . You can also see directly that the quadratic polynomials  $f(t)$  with  $f(7) = f(11) = 0$  are of this form. The nullity is 1. The image consists of all of  $\mathbb{R}^2$ , so that the rank is 2.

**4.2.61** The kernel consists of all polynomials  $f(t)$  such that  $t(f(t)) = 0$  for all  $t$ , that is, the zero polynomial  $f(t) = 0$  alone. The image consists of all polynomials  $g(t)$  that can be written as  $g(t) = t(f(t))$ , meaning that we can factor out a  $t$ . These are the polynomials with constant term 0, of the form  $g(t) = a_1t + a_2t^2 + \dots + a_nt^n$ .

**4.2.62** The image of this transformation consists of all polynomials, since any polynomial is the derivative of another. The kernel of this transformation consists of all constant functions, or the span of the function  $f(t) = 1$ .

**4.2.63** This is impossible, since  $\dim(P_3) = 4$  and  $\dim(\mathbb{R}^3) = 3$ . See Theorem 4.2.4b.

**4.2.64** Consider  $T(a + bt + ct^2 + dt^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , for example.

**4.2.65 a** First, we need to show that  $T(A + B) = T(A) + T(B)$  for all  $n \times m$  matrices  $A$

and  $B$ , that is  $(T(A + B))(\vec{v}) = (T(A) + T(B))(\vec{v})$  for all  $\vec{v}$  in  $\mathbb{R}^m$ . Indeed

$$(T(A + B))(\vec{v}) = (A + B)\vec{v} = A\vec{v} + B\vec{v} \text{ equals}$$

$$(T(A) + T(B))(\vec{v}) = T(A)(\vec{v}) + T(B)(\vec{v}) = A\vec{v} + B\vec{v}. \text{ Also}$$

$$(T(kA))(\vec{v}) = (kA)\vec{v} = kA\vec{v} \text{ equals } (kT(A))(\vec{v}) = k(T(A))(\vec{v}) = kA\vec{v}.$$

- b The kernel of  $T$  consists of all  $n \times m$  matrices  $A$  such that  $T(A) = 0$ , that is  $(T(A))(\vec{v}) = A\vec{v} = \vec{0}$  for all  $\vec{v}$  in  $\mathbb{R}^m$ . This holds for the zero matrix only. Thus  $\ker(T) = \{0\}$ .
- c This is true by definition of a linear transformation (Definition 2.1.1).

- d Note that  $T$  gives an isomorphism from  $\mathbb{R}^{n \times m}$  to  $L(\mathbb{R}^m, \mathbb{R}^n)$ , by parts a, b, and c.

Since  $\dim(\mathbb{R}^{n \times m}) = nm$ , by Exercise 17 of Section 4.1, we have  $\dim(L(\mathbb{R}^m, \mathbb{R}^n)) = nm$ , by Theorem 4.2.4b.

**4.2.66** The kernel of  $T$  consists of all smooth functions  $f(t)$  such that

$T(f(t)) = f(t) - f'(t) = 0$ , or  $f'(t) = f(t)$ . As you may recall from a

discussion of exponential functions in calculus, those are the functions of the form  $f(t) = Ce^t$ , where  $C$  is a constant. Thus the nullity of  $T$  is 1.

**4.2.67** To show that  $T$  is linear, proceed as in Exercise 15. Now let  $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

$$\begin{aligned} \text{Then } T(M) &= \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & k \end{bmatrix} \\ &= \begin{bmatrix} 2a + 3c & 2b + 3d \\ 4c & 4d \end{bmatrix} - \begin{bmatrix} 3a & kb \\ 3c & kd \end{bmatrix} = \begin{bmatrix} -a + 3c & (2 - k)b + 3d \\ c & (4 - k)d \end{bmatrix}. \end{aligned}$$

The matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is in the kernel of  $T$  if  $a = c = 0$ ,  $(2 - k)b + 3d = 0$  and  $(4 - k)d = 0$ .

If  $k$  is neither 2 nor 4, then the equation  $(4 - k)d = 0$  implies that  $d = 0$ , and  $(2 - k)b + 3d = 0$  then implies that  $b = 0$ . Thus the kernel of  $T$  is 0, and  $T$  is an isomorphism.

If  $k$  is 2, then  $b$  is arbitrary (while  $a = c = d = 0$ ), and  $T$  fails to be an isomorphism. A nonzero matrix in the kernel is  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  in this case.

If  $k$  is 4, then  $b$  is arbitrary, and  $d = \frac{2}{3}b$ , so that, again, the kernel contains nonzero matrices (for example,  $\begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$ ), and  $T$  fails to be an isomorphism.

In summary,  $T$  is an isomorphism if (and only if)  $k$  is neither 2 nor 4.

**4.2.68** We find the constants  $k$  which make the kernel of  $T$  nonzero. Let  $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then  $T(M) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5a & b \\ 5c & d \end{bmatrix} - \begin{bmatrix} 2a & 2b \\ kc & kd \end{bmatrix} = \begin{bmatrix} 3a & -b \\ (5 - k)c & (1 - k)d \end{bmatrix}$ . We see that if  $k = 1$ ,  $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . Similarly, if  $k = 5$ ,  $T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . Thus, for  $k = 1, 5$ , the kernel of  $T$  is non-zero, and  $T$  is not an isomorphism. For all other values of  $k$ , however,  $T$  is an isomorphism.

**4.2.69** No, since  $A$  is similar to  $B$ , there exists an invertible  $S$  such that  $AS = SB$ . Now  $T(S) = AS - SB = 0$ , so that the nonzero matrix  $S$  is in the kernel of  $T$ .

**4.2.70** Since the dimensions of the domain and codomain are equal, it suffices to examine when the kernel is zero. Now  $f(t)$  is in the kernel of  $T$  if  $f(c_0) = f(c_1) = \dots = f(c_n) = 0$ .

Recall that a nonzero polynomial of degree  $\leq n$  has at most  $n$  zeros. If the  $n + 1$  numbers  $c_0, c_1, \dots, c_n$  are all different, then the only polynomial in  $P_n$  with  $f(c_0) = f(c_1) = \dots = f(c_n) = 0$  is the zero polynomial, so that the kernel of  $T$  is zero, and  $T$  is an isomorphism.

However, if some of the numbers are equal, for example,  $c_p = c_q$ , then there will be nonzero polynomials in the kernel (for example, the product of all  $(t - c_i)$  where  $i \neq q$ ), so that  $T$  fails to be an isomorphism.

**4.2.71** Exercise 70 tells us that  $T$  is an isomorphism so long as  $c_0, c_1, \dots, c_n$  are all different. This condition is met

here, so  $T(f(t)) = \begin{bmatrix} f(2) \\ f(3) \\ f(5) \\ f(7) \\ f(11) \end{bmatrix}$  is an isomorphism from  $P_4$  to  $\mathbb{R}^5$ . Thus, there is exactly one polynomial  $f(t)$  with the required properties.

**4.2.72** We satisfy all the requirements of Definition 4.1.2. Clearly 0 is an element of  $Z_n$ . Let  $h = f + g$ , where  $f$  and  $g$  are elements of  $Z_n$ . Then  $h(0) = f(0) + g(0) = 0 + 0 = 0$ . Also, if  $f$  is in  $Z_n$ , then  $kf(0) = k(0) = 0$ . We notice that the space  $Z_n$  has the basis,  $t, t^2, \dots, t^n$ . Thus, the dimension of  $Z_n$  is  $n$ .

**4.2.73** Yes.  $T$  is an isomorphism; the inverse transformation is  $D(f(t)) = f'(t) = \frac{df}{dt}$ , the derivative. We will check that the composite of  $T$  with  $D$  is the identity, in either order. Indeed

$$D(T(f(t))) = \frac{d}{dt}(T(f(t))) = \frac{d}{dt} \int_0^t f(x)dx = f(t)$$

by what is sometimes called the first fundamental theorem. And

$$T(D(f(t))) = T(f'(t)) = \int_0^t f'(x)dx = f(t) - f(0) = f(t) \text{ for all } f(t) \text{ in } Z_n$$

by the "second" fundamental theorem.

**4.2.74** Let  $T(f(t)) = f'(t)$ . This is an isomorphism, since the spaces have the same dimension, and the kernel consists only of the zero polynomial.

**4.2.75** To see that  $0 + 0 = 0$ , apply the formula  $f + 0 = f$  to  $f = 0$  (see Definition 4.1.1). Then  $0 = k0 - k0 = k(0 + 0) - k0 = k0 + k0 - k0 = k0$ .

**4.2.76**  $0_W = T(0_V) - T(0_V) = T(0_V + 0_V) - T(0_V) = T(0_V) + T(0_V) - T(0_V) = T(0_V)$

We have used the equation  $0_V + 0_V = 0_V$  derived in Exercise 41.

**4.2.77** If  $T$  and  $L$  are linear, then

$$(L \circ T)(f + g) = L(T(f + g)) = L(T(f) + T(g)) = L(T(f)) + L(T(g))$$

$$= (L \circ T)(f) + (L \circ T)(g),$$

and  $(L \circ T)(kf) = L(T(kf)) = L(kT(f)) = kL(T(f)) = k(L \circ T)(f)$ , so that  $L \circ T$  is linear as well.

If  $T$  and  $L$  are isomorphisms, then  $L \circ T$  is an isomorphism as well, since the composite of invertible functions is invertible. (See Figure 4.1.)

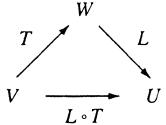


Figure 4.1: for Problem 4.2.77.

**4.2.78 a** Check all the conditions in Definition 4.1.1. A basis is 2.

b  $T(x \oplus y) = T(xy) = \ln(xy) = \ln(x) + \ln(y) = T(x) + T(y)$  and

$$T(k \odot x) = T(x^k) = \ln(x^k) = k \ln(x) = kT(x).$$

The inverse of  $T$  is  $L(y) = e^y$ , so that  $T$  is indeed an isomorphism.

**4.2.79** Yes; let  $T$  be an invertible function from  $\mathbb{R}^2$  to  $\mathbb{R}$ . On  $\mathbb{R}^2$  we can define the “exotic” operations  $\vec{v} \oplus \vec{w} = T^{-1}(T(\vec{v}) + T(\vec{w}))$  and  $k \odot \vec{v} = T^{-1}(kT(\vec{v}))$  (check that the conditions of Definition 4.1.1 hold). Then  $T$  is a linear transformation from  $\mathbb{R}^2$  (with these exotic operations) to  $\mathbb{R}$  (with the usual operations):  $T(\vec{v} \oplus \vec{w}) = T(T^{-1}(T(\vec{v}) + T(\vec{w}))) = T(\vec{v}) + T(\vec{w})$ , and  $T(k \odot v) = kT(v)$ . Now  $T$  is an isomorphism, so that the dimension of  $\mathbb{R}^2$  (with our exotic operations) equals the dimension of the “usual”  $\mathbb{R}$ , which is 1.

**4.2.80** If a real linear space  $V$  has more than one element, then it is infinite; to see this, note that the scalar multiples  $kf$  of a nonzero  $f$  in  $V$  are all distinct (think about it!). Since there is more than one student in your class, but the number of students is finite, we cannot make the set  $X$  into a real linear space.

**4.2.81 a**  $\ker(T)$  is a subspace of  $V$ , so by Exercise 4.1.54,  $\ker(T)$  must also be finite-dimensional. Also,  $\text{im}(T)$  is finite-dimensional, because it is finitely-generated by some elements  $T(f_1), T(f_2), \dots, T(f_k)$ , where  $f_1, f_2, \dots, f_k$  is a basis of  $V$ .

b Following the hint:  $T(c_1u_1 + \dots + c_ru_r + d_1v_1 + \dots + d_nv_n) = T(0)$ , so  $c_1T(u_1) + \dots + c_rT(u_r) + d_1T(v_1) + \dots + d_nT(v_n) = c_1w_1 + \dots + c_rw_r + 0 + \dots + 0 = 0$ . So, since  $w_1, \dots, w_r$  is a basis and must be linearly independent,  $c_1, \dots, c_r$  must all be zero. Now,  $d_1v_1 + \dots + d_nv_n = 0$ , but  $v_1, \dots, v_n$  is also a basis, so  $d_1, \dots, d_n$  also must all equal zero.

Thus, the elements  $u_1, \dots, u_r, v_1, \dots, v_n$  are linearly independent.

c The hint guides us right along here. Let  $v$  be in  $V$ . Then  $T(v)$  is in  $\text{im}(T)$ , so  $T(v) = d_1w_1 + \dots + d_rw_r$ . Now,  $T(v - d_1u_1 - \dots - d_ru_r) = T(v) - d_1T(u_1) - \dots - d_rT(u_r) = d_1w_1 + \dots + d_rw_r - d_1w_1 - \dots - d_rw_r = 0$ . Thus,  $v - d_1u_1 - \dots - d_ru_r$  is in the kernel of  $T$ , and there are some  $c_1, \dots, c_n$  such that  $v - d_1u_1 - \dots - d_ru_r = c_1v_1 + \dots + c_nv_n$ . Thus,  $v = d_1u_1 + \dots + d_ru_r + c_1v_1 + \dots + c_nv_n$ . Thus the elements  $u_1, \dots, u_r, v_1, \dots, v_n$  span  $V$ .

**4.2.82** Consider a basis  $v_1, \dots, v_n$  of  $\ker T$  and a basis  $w_1, \dots, w_r$  of  $\text{im } T$ . Consider elements  $u_1, \dots, u_r$  in  $V$  such that  $T(u_i) = w_i$  for  $i = 1, \dots, r$ . In Exercise 81, parts b and c, we prove that the elements  $v_1, \dots, v_n, u_1, \dots, u_r$  form a basis of  $V$ , proving our claim.

**4.2.83** The transformation  $T$  induces a transformation  $\tilde{T}$  from  $\ker(L \circ T)$  to  $\ker L$ , with  $\ker \tilde{T} = \ker T$ . Since  $\ker L$  is assumed to be finite-dimensional, its subspace  $\text{im } \tilde{T}$  will be finite-dimensional as well, with  $\dim \text{im } \tilde{T} \leq \dim \ker L$ . Thus we can apply the rank-nullity-theorem as presented in Exercise 82 to  $\tilde{T}$ , finding that  $\ker(L \circ T)$  is finite-dimensional, with  $\dim \ker(L \circ T) = \dim \ker \tilde{T} + \dim \text{im } \tilde{T} \leq \dim \ker T + \dim \ker L$ , as claimed.

**4.2.84** Using the terminology and the results introduced in Exercise 83, we observe that  $\text{im } \tilde{T} = \ker L$ . Indeed, if  $w$  is in  $\ker L$ , then  $w = T(v)$  for some  $v$  in  $V$ , since  $\text{im } T = W$ . But this  $v$  will be in  $\ker(L \circ T)$  since  $L(T(v)) = L(w) = 0$ . Thus  $w$  is in  $\text{im } \tilde{T}$ . Therefore  $\dim \ker(L \circ T) = \dim \ker \tilde{T} + \dim \text{im } \tilde{T} = \dim \ker T + \dim \ker L$ , as claimed.

## Section 4.3

**4.3.1** Let  $\mathcal{B}$  be the standard basis of  $P_2 : 1, t, t^2$ . Then the coordinates of the given polynomials with respect to  $\mathcal{B}$  are  $[f]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$ ,  $[g]_{\mathcal{B}} = \begin{bmatrix} 9 \\ 9 \\ 4 \end{bmatrix}$ ,  $[h]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ . Finding  $\text{rref} \begin{bmatrix} 7 & 9 & 3 \\ 3 & 9 & 2 \\ 1 & 4 & 1 \end{bmatrix} = I_3$ , we conclude that  $[f]_{\mathcal{B}}, [g]_{\mathcal{B}}, [h]_{\mathcal{B}}$  are linearly independent, hence so are  $f, g, h$ , since the coordinate transformation is an isomorphism.

**4.3.2** Let  $\mathcal{B}$  be the basis  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  of  $\mathbb{R}^{2 \times 2}$ . Then the coordinates of the given matrices with respect to  $\mathcal{B}$  are  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ . Finding  $\text{rref} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 1 & 4 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq I_4$ , we conclude that the four vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$  are linearly dependent, and so are the four given matrices. In fact  $\begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 4 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ .

**4.3.3** We proceed as in Exercise 1. Since  $\text{rref} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 7 & 8 & 8 \\ 9 & 0 & 1 & 4 \\ 1 & 7 & 5 & 8 \end{bmatrix} = I_4$ , the four given polynomials do form a basis of  $P_3$ .

**4.3.4** Consider the coordinate vectors of the 3 given polynomials with respect to the standard basis of  $P_2$ :  $1, t, t^2$ .

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2k \\ 2+k \\ 1 \end{bmatrix}$$

Since matrix  $\begin{bmatrix} 1 & 0 & 2k \\ 1 & 1 & 2+k \\ 0 & 1 & 1 \end{bmatrix}$  reduces to  $\begin{bmatrix} 1 & 0 & 2k \\ 0 & 1 & 2-k \\ 0 & 0 & k-1 \end{bmatrix}$ , these three vectors form a basis of  $\mathbb{R}^3$  unless  $k = 1$ .

Therefore, the three polynomials  $f(t), tf(t), g(t)$  form a basis of  $P_2$  unless  $k = 1$ .

4.3.5 Use a diagram:

$$\begin{array}{ccc} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} & \xrightarrow{T} & \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b+2c \\ 0 & 3c \end{bmatrix} \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} a \\ b+2c \\ 3c \end{bmatrix} \end{array}$$

The matrix  $A$  that transforms  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  into  $\begin{bmatrix} a \\ b+2c \\ 3c \end{bmatrix}$  is  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ .

$A$  is invertible, so  $T$  is an isomorphism.

4.3.6 Use Theorem 4.3.2. to construct matrix  $B$  column by column:

$$\begin{aligned} B &= \left[ \left[ T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ T \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned}$$

This matrix is invertible, so  $T$  is an isomorphism.

4.3.7 Use Theorem 4.3.2 to construct  $B$  column by column:

$$\begin{aligned} B &= \left[ \left[ T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}} \right] \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ is a basis of the kernel of } B, \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ is a basis of the image of } B. \text{ This implies that} \\ &\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ is a basis of the kernel of } T, \text{ and } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ is a basis of the image of } T. \text{ Thus, } T \text{ fails to be an isomorphism.} \end{aligned}$$

4.3.8 Use a diagram:

$$\begin{array}{ccc} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} & \xrightarrow{T} & \begin{bmatrix} 0 & 2a-2c \\ 0 & 0 \end{bmatrix} \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} 0 \\ 2a-2c \\ 0 \end{bmatrix} \end{array}$$

The matrix that transforms  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  into  $\begin{bmatrix} 0 \\ 2a - 2c \\ 0 \end{bmatrix}$  is  $A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ .

We see that a basis of the kernel of  $A$  is  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  and a basis of the image of  $A$  is  $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ . Thus, a basis of the kernel of  $T$  is  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, I_2$  while a basis of the image is  $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ . Thus, the rank of  $T$  is 1 and  $T$  is not an isomorphism.

**4.3.9** Use a diagram:

$$\begin{array}{ccc} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} & \xrightarrow{\overline{T}} & \begin{bmatrix} a & 2b \\ 0 & c \end{bmatrix} \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{\overline{A}} & \begin{bmatrix} a \\ 2b \\ c \end{bmatrix} \end{array}$$

The matrix  $A$  that transforms  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  into  $\begin{bmatrix} a \\ 2b \\ c \end{bmatrix}$  is  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

$A$  is invertible, so  $T$  is an isomorphism.

**4.3.10** Use a diagram:

$$\begin{array}{ccc} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} & \xrightarrow{\overline{T}} & \frac{1}{3} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} a & 2a + 3b - 2c \\ 0 & c \end{bmatrix} \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{\overline{A}} & \begin{bmatrix} a \\ 2a + 3b - 2c \\ c \end{bmatrix} \end{array}$$

The matrix is  $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ .

Since  $A$  is invertible,  $T$  is an isomorphism.

**4.3.11** Use Theorem 4.3.2. to construct matrix  $B$  column by column:

$$\begin{aligned} B &= \left[ \left[ T \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ T \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}} \right] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned}$$

$A$  is invertible, so  $T$  is an isomorphism.

**4.3.12** Use a diagram as in Definition 4.3.1:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 2a & 3b \\ 2c & 3d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 2a \\ 3b \\ 2c \\ 3d \end{bmatrix}$$

$$\text{The matrix is } A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Since  $A$  is invertible,  $T$  is an isomorphism.

**4.3.13** Use a diagram:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} \begin{bmatrix} a+c & b+d \\ 2a+2c & 2b+2d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xrightarrow{A} \begin{bmatrix} a+c \\ b+d \\ 2a+2c \\ 2b+2d \end{bmatrix}$$

$$\text{The matrix is } A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}.$$

We find  $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$  to be a basis of the kernel of  $A$ , and  $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$  to be a basis of the image of  $A$ . Thus, a basis of the kernel of  $T$  is  $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ , a basis of the image of  $T$  is  $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ , and  $T$  fails to be an isomorphism.

**4.3.14** Use Theorem 4.3.2 to construct matrix  $B$  column by column:

$$B = \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_B \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_B \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix}_B \begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix}_B \right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

We find bases of the kernel and image of  $B$ , and use them to find that a basis of the kernel of  $T$  is  $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ , and a basis of the image is  $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ . Thus the rank of  $T$  is 2, and  $T$  is not an isomorphism.

**4.3.15** We use a diagram again:

$$\begin{array}{ccc} x+iy & \xrightarrow{\overline{T}} & x-iy \\ \downarrow & & \downarrow \\ \begin{bmatrix} x \\ y \end{bmatrix} & \xrightarrow{\overline{A}} & \begin{bmatrix} x \\ -y \end{bmatrix} \end{array}$$

$$\text{Thus } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$A$  is invertible, so  $T$  is an isomorphism.

**4.3.16**  $B = [[1-i]_{\mathcal{B}} [1+i]_{\mathcal{B}}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . Since  $B$  is invertible,  $T$  must be an isomorphism.

**4.3.17** Another diagram:

$$\begin{array}{ccc} x+iy & \xrightarrow{\overline{T}} & -y+ix \\ \downarrow & & \downarrow \\ \begin{bmatrix} x \\ y \end{bmatrix} & \xrightarrow{\overline{A}} & \begin{bmatrix} -y \\ x \end{bmatrix} \end{array}$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$A$  is invertible, so  $T$  is an isomorphism.

**4.3.18**

$$\begin{array}{ccc} x+iy & \xrightarrow{\overline{T}} & 2x-3y+i(3x+2y) \\ \downarrow & & \downarrow \\ \begin{bmatrix} x \\ y \end{bmatrix} & \xrightarrow{\overline{A}} & \begin{bmatrix} 2x-3y \\ 3x+2y \end{bmatrix} \end{array}.$$

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}. \text{ Since } A \text{ is invertible, } T \text{ must be an isomorphism.}$$

**4.3.19** We use a diagram to show our work:

$$\begin{array}{ccc} x+iy & \xrightarrow{\overline{T}} & px-qy+i(qx+py) \\ \downarrow & & \downarrow \\ \begin{bmatrix} x \\ y \end{bmatrix} & \xrightarrow{\overline{A}} & \begin{bmatrix} px-qy \\ qx+py \end{bmatrix} \end{array}$$

$A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ . If  $p = q = 0$ , then  $T(z) = 0$  for all  $z$ , so that the kernel is all of  $\mathbb{C}$ , while the image is  $\{0\}$ . Otherwise,  $T$  is an isomorphism.

$$4.3.20 \quad \begin{array}{ccc} a + bt + ct^2 & \xrightarrow{T} & b + 2ct \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix} \end{array}$$

Thus  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ . We see that a basis of the kernel of  $A$  is  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and a basis of the image of  $A$  is  $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ . Thus, a basis of the kernel of  $T$  is 1, while a basis of the image of  $T$  is  $1, 2t$ . Thus, the rank of  $T$  is 2, and  $T$  is not an isomorphism.

4.3.21 We use a diagram to show our work:

$$\begin{array}{ccc} a + bt + ct^2 & \xrightarrow{T} & b - 3a + (2c - 3b)t - 3ct^2 \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} b - 3a \\ 2c - 3b \\ -3c \end{bmatrix} \end{array}$$

Thus  $A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & -3 \end{bmatrix}$ .

$A$  is invertible, so  $T$  is an isomorphism.

$$4.3.22 \quad \begin{array}{ccc} a + bt + ct^2 & \xrightarrow{T} & 4b + 2c + 8ct \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} 4b + 2c \\ 8c \\ 0 \end{bmatrix} \end{array}$$

Thus  $A = \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$ . We find a basis of the kernel of  $A$  to be  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ , a basis of the image of  $A$  to be  $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix}$ . From this we see that a basis of kernel of  $T$  is 1, while a basis of the image of  $T$  is  $4, 2 + 8t$ . Thus,  $T$  is not an isomorphism, since the rank of  $T$  is only 2.

4.3.23 A diagram shows our work:

$$\begin{array}{ccc} a + bt + ct^2 & \xrightarrow{T} & a + 3b + 9c \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} a + 3b + 9c \\ 0 \\ 0 \end{bmatrix} \end{array}$$

Thus  $A = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

We find  $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 9 \\ 0 \\ -1 \end{bmatrix}$  to be a basis of the kernel of  $A$ , and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  to be a basis of the image of  $A$ . Thus, a basis of the kernel of  $T$  is  $3-t, 9-t^2$ , a basis of the image of  $T$  is 1 (the image is  $\mathbb{R}$ ), and  $T$  fails to be an isomorphism.

**4.3.24**  $B = [[1]_{\mathcal{B}} \ [0]_{\mathcal{B}} \ [0]_{\mathcal{B}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . A basis of the kernel and image of  $B$  are obvious,  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  respectively.

So we find a basis of the kernel of  $T$  to be  $t-3, (t-3)^2$ , and a basis of the image of  $T$  to be 1. Thus  $T$  has a rank of 1, and is not an isomorphism.

**4.3.25** We use the following diagram:

$$\begin{array}{ccc} a+bt+ct^2 & \xrightarrow{T} & a-bt+ct^2 \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} a \\ -b \\ c \end{bmatrix} \end{array}$$

Thus  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

$A$  is invertible, so  $T$  is an isomorphism.

**4.3.26**

$$\begin{array}{ccc} a+bt+ct^2 & \xrightarrow{T} & a+2bt+4ct^2 \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} a \\ 2b \\ 4c \end{bmatrix} \end{array}$$

Thus  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ . Since  $A$  is invertible,  $T$  must be an isomorphism.

**4.3.27** We use a diagram:

$$\begin{array}{ccc} a+bt+ct^2 & \xrightarrow{T} & a+b(2t-1)+c(2t-1)^2 \\ & & = a-b+c+(2b-4c)t+4ct^2 \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} a-b+c \\ 2b-4c \\ 4c \end{bmatrix} \end{array}$$

Thus  $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$ .

$A$  is invertible, so  $T$  is an isomorphism.

4.3.28  $B = [[1]_{\mathcal{B}} \ [2t - 2]_{\mathcal{B}} \ [4(t - 1)^2]_{\mathcal{B}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ . Since  $B$  is invertible,  $T$  will be an isomorphism.

4.3.29 This diagram shows our work:

$$\begin{array}{ccc} a + bt + ct^2 & \xrightarrow{T} & 2a + 2b + 8c/3 \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} 2a + 2b + 8c/3 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$\text{Thus } A = \begin{bmatrix} 2 & 2 & 8/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We find  $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{3} \\ 0 \\ -1 \end{bmatrix}$  to be a basis of the kernel of  $A$ , and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  to be a basis of the image of  $A$ . Thus, a basis of the kernel of  $T$  is  $1-t, \frac{4}{3}-t^2$ , a basis of the image of  $T$  is  $1$  (the image is  $\mathbb{R}$ ), and  $T$  fails to be an isomorphism.

4.3.30

$$\begin{array}{ccc} a + bt + ct^2 & \xrightarrow{T} & \frac{a+b(t+h)+c(t+h)^2-a-bt-ct^2}{h} \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} b+ch \\ 2c \\ 0 \end{bmatrix} \end{array}$$

Thus  $A = \begin{bmatrix} 0 & 1 & h \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ . A basis of the kernel of  $A$  is  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ , while a basis of the image of  $A$  is  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 2 \\ 0 \end{bmatrix}$ . Thus, a basis of the kernel of  $T$  is  $1$  while a basis of the image of  $T$  is  $1, h + 2t$ . So the rank of  $T$  is  $2$ , and it is not an isomorphism.

Recall that in calculus  $\frac{f(t+h)-f(t)}{h}$  is called the difference quotient of function  $f(t)$ .

Geometrically, it represents the slope of the secant through points  $(t, f(t))$  and

$(t + h, f(t + h))$ . The limit of the difference quotient, as  $h$  approaches  $0$ , is the derivative

$f'(t)$ . Compare with Exercise 20.

$$\begin{array}{ccc}
 a + bt + ct^2 & \xrightarrow{T} & \frac{a+b(t+h)+c(t+h)^2 - a-b(t-h)-c(t-h)^2}{2h} \\
 & & = b + 2ct \\
 4.3.31 & \downarrow & \downarrow \\
 \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}
 \end{array}$$

Thus  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ .

We find  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  to be a basis of the kernel of  $A$ , and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$  to be a basis of the image of  $A$ . Thus, a basis of the kernel of  $T$  is 1, a basis of the image of  $T$  is  $1, 2t$ , and  $T$  fails to be an isomorphism.

Geometrically,  $\frac{f(t+h)-f(t-h)}{2h}$  is the slope of the secant through points  $(t-h, f(t-h))$  and  $(t+h, f(t+h))$ . This exercise shows that for a quadratic polynomial  $f(t)$  this secant is parallel to the tangent at the midpoint  $(t, f(t))$ , a fact that was known to Archimedes. Draw a sketch!

Thus  $T(f(t)) = \frac{f(t+h)-f(t-h)}{2h} = f'(t)$  in this case. Compare with Exercise 20.

$$\begin{array}{ccc}
 a + bt + ct^2 & \xrightarrow{T} & a + b + c + (b + 2c)(t - 1) \\
 & & = a - c + (b + 2c)t \\
 4.3.32 & \downarrow & \downarrow \\
 \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} a - c \\ b + 2c \\ 0 \end{bmatrix}
 \end{array}$$

Thus  $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ . We find a basis of the kernel of  $A$  to be  $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ , while a basis of the image of  $A$  is  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . Thus, a basis of the kernel of  $T$  is  $-1 + 2t - t^2$  and a basis of the image of  $T$  is  $1, t$ . The rank of  $T$  is only 2, and  $T$  is not an isomorphism.

Note that  $T(f(t)) = f(1) + f'(1)(t - 1)$  is the equation of the tangent to the graph of  $f(t)$  at the point  $(1, f(1))$ .

$$4.3.33 \quad B = [[1]_{\mathcal{B}} \ [t-1]_{\mathcal{B}} \ [0]_{\mathcal{B}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We find  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  to be a basis of the kernel of  $B$ , and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  to be a basis of the image of  $B$ . Thus, a basis of the kernel of  $T$  is  $(t - 1)^2$ , a basis of the image of  $T$  is  $1, t - 1$ , and  $T$  fails to be an isomorphism.

$$4.3.34 \quad A = \left[ \left[ T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ T \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right]_{\mathcal{A}} \right]$$

$$= \left[ \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{A}} \right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We find  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  to be a basis of the kernel of  $A$ , and  $\begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$  to be a basis of the image of  $A$ . Thus, a basis of the kernel of  $T$  is  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , a basis of the image of  $T$  is  $\begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$ , meaning that  $T$  has a rank of 2 and  $T$  fails to be an isomorphism.

4.3.35 Again, we use a diagram to show our work:

$$\begin{array}{ccc} M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \xrightarrow{T} & T(M) = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix} = c \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (d-a) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \downarrow & & \downarrow \\ [M]_{\mathcal{A}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} & \xrightarrow{A} & [T(M)]_{\mathcal{A}} = \begin{bmatrix} c \\ d-a \\ 0 \\ -c \end{bmatrix} \end{array}$$

$$\text{Thus, } A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

By inspection, a basis of the kernel and image of this matrix are  $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  and

$\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ , respectively. Thus, we see that a basis of  $\ker(T)$  is  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,

and a basis of  $\text{im}(T)$  is  $\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . Thus,  $T$  fails to be an isomorphism.

4.3.36 We will build  $A$  column-by-column:

$$\begin{aligned} A &= \left[ \left[ T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ T \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right]_{\mathcal{A}} \right] \\ &= \left[ \left[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right]_{\mathcal{A}} \left[ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right]_{\mathcal{A}} \left[ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right]_{\mathcal{A}} \right] = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}. \end{aligned}$$

We find  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$  to be a basis of the kernel of  $A$ , and  $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  to be a basis of the image of  $A$ .

Thus, a basis of the kernel of  $T$  is  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ , a basis of the image of  $T$  is  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ , the rank of  $T$  is 2 and  $T$  fails to be an isomorphism.

**4.3.37** We will construct our matrix  $B$  column-by-column:

$$B = \left[ \left[ 0 - \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} - 0 \right]_{\mathcal{B}} [0 - 0]_{\mathcal{B}} [0 - 0]_{\mathcal{B}} \right] = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We find  $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  to be a basis of the kernel of  $B$ , and  $\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$  to be a basis of the image of  $B$ . Thus, a

basis of the kernel of  $T$  is  $I_2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , and a basis of the image of  $T$  is  $\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ , and  $T$  fails to be an isomorphism.

**4.3.38** We will construct our matrix  $B$  column-by-column:

$$B = \left[ [0]_{\mathcal{B}} \left[ \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \right]_{\mathcal{B}} [0]_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We find  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  to be a basis of the kernel of  $B$ , and  $\begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$  to be a basis of the image of  $B$ . From this

we conclude that a basis of  $\ker(T)$  is  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ , a basis of  $\text{im}(T)$  is  $\begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$  and the rank of  $T$  is 2. So,  $T$  fails to be an isomorphism.

**4.3.39** We use a diagram to show our work:

$$\begin{array}{ccc} M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \xrightarrow{T} & T(M) = \begin{bmatrix} c-a & d-b \\ a-c & b-d \end{bmatrix} \\ \downarrow & & \downarrow \\ \text{Let } [M]_{\mathcal{A}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} & \xrightarrow{A} & [T(M)]_{\mathcal{A}} = \begin{bmatrix} c-a \\ d-b \\ a-c \\ b-d \end{bmatrix} \end{array}$$

Then our  $\mathcal{A}$ -matrix of  $T$  is:  $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ .

Thus,  $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$  is a basis of the kernel, and  $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$  is a basis of the image. From this we conclude that a basis of  $\ker(T)$  is  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$  and a basis of  $\text{im}(T)$  is  $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ . So,  $T$  fails to be an isomorphism.

**4.3.40** We will construct our matrix  $A$  column-by-column:

$$A = \left[ \begin{bmatrix} -4 & 0 \\ 4 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} \right]_{\mathcal{A}} \left[ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} \right]_{\mathcal{A}} \left[ \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} \right]_{\mathcal{A}} = \begin{bmatrix} -4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 4 & 0 & -2 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix}.$$

We find  $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$  to be a basis of the kernel of  $A$ , and  $\begin{bmatrix} -4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \end{bmatrix}$  to be a basis of the image of  $A$ .

From this we conclude that a basis of  $\ker(T)$  is  $\begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$  and a basis of  $\text{im}(T)$  is  $\begin{bmatrix} -4 & 0 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$ . So, the rank of  $T$  is 2 and  $T$  fails to be an isomorphism.

**4.3.41 a** In Exercise 5 we consider the standard basis  $\mathcal{A}$  of  $U^{2 \times 2}$ , and in Exercise 6 we work with the alternative basis  $\mathcal{B}$  consisting of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

$$\text{Now } S_{\mathcal{B} \rightarrow \mathcal{A}} = \left[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{\mathcal{A}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{\mathcal{A}} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}_{\mathcal{A}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{b Check that } AS = SB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\text{c } S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

**4.3.42 a** If  $\mathcal{A}$  is the standard basis considered in Exercise 8 and  $\mathcal{B}$  is the basis in Exercise 7, then

$$S = \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{\mathcal{A}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{\mathcal{A}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{\mathcal{A}} \right] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

$$\text{b Check that } AS = SB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$c \quad S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

4.3.43 a If  $\mathcal{A}$  is the standard basis considered in Exercise 10 and  $\mathcal{B}$  is the basis in Exercise 11, then

$$S_{\mathcal{B} \rightarrow \mathcal{A}} = \left[ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}_{\mathcal{A}} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}_{\mathcal{A}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{\mathcal{A}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$b \quad \text{Check that } AS = SB = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$c \quad S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

4.3.44 a If  $\mathcal{A}$  is the standard basis considered in Exercise 13 and  $\mathcal{B}$  is the basis in Exercise 14, then

$$S = \left[ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}_{\mathcal{A}} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}_{\mathcal{A}} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}_{\mathcal{A}} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}_{\mathcal{A}} \right] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}.$$

$$b \quad \text{Check that } AS = SB = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

4.3.45 a If  $\mathcal{A}$  is the standard basis considered in Exercise 15 and  $\mathcal{B}$  is the basis in Exercise 16, then

$$S_{\mathcal{B} \rightarrow \mathcal{A}} = [[1+i]_{\mathcal{A}} \ [1-i]_{\mathcal{A}}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

b In Exercises 16 and 15 we found  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  and  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , respectively. Now check that  $AS = SB = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ .

$$c \quad S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

4.3.46 a If  $\mathcal{A}$  is the standard basis considered in Exercise 23 and  $\mathcal{B}$  is the basis in Exercise 24, then

$$S = [[1]_{\mathcal{A}} \ [-3+t]_{\mathcal{A}} \ [9-6t+t^2]_{\mathcal{A}}] = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$

b Check that  $AS = SB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

c  $S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \begin{bmatrix} 1 & 3 & -9 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ .

4.3.47 a If  $\mathcal{A}$  is the standard basis considered in Exercise 27 and  $\mathcal{B}$  is the basis in Exercise 28, then

$$S_{\mathcal{B} \rightarrow \mathcal{A}} = [[1]_{\mathcal{A}} \ [ -1 + t ]_{\mathcal{A}} \ [ 1 - 2t + t^2 ]_{\mathcal{A}}] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

b In Exercises 28 and 27 we found  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$  and  $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$ , respectively. Now check that

$$AS = SB = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & -8 \\ 0 & 0 & 4 \end{bmatrix}.$$

c  $S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ .

4.3.48 Use a diagram:

$$\begin{array}{ccc} a \cos(t) + b \sin(t) & \xrightarrow{\overline{T}} & b \cos(t) - a \sin(t) \\ \downarrow & & \downarrow \\ \begin{bmatrix} a \\ b \end{bmatrix} & \xrightarrow{\overline{B}} & \begin{bmatrix} b \\ -a \end{bmatrix} \end{array}$$

$$\text{Thus } B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

4.3.49  $B = [[2 \cos(t) - 2 \sin(t)]_{\mathcal{B}} \ [2 \cos(t) + 2 \sin(t)]_{\mathcal{B}}] = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ . Yes,  $T$  is an isomorphism, since matrix  $B$  is invertible.

4.3.50  $B = [(b-1) \cos(t) - a \sin(t)]_{\mathcal{B}} \ [a \cos(t) + (b-1) \sin(t)]_{\mathcal{B}} = \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}$ .

4.3.51 Note that  $\cos(t - \pi/2) = \sin(t)$  and  $\sin(t - \pi/2) = -\cos(t)$ . Thus

$$B = [[\cos(t - \pi/2)]_{\mathcal{B}} \ [\sin(t - \pi/2)]_{\mathcal{B}}] = [[\sin(t)]_{\mathcal{B}} \ [-\cos(t)]_{\mathcal{B}}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \text{ Yes, } T \text{ is an isomorphism.}$$

4.3.52 Recall that  $\cos(t - \delta) = \cos(\delta) \cos(t) + \sin(\delta) \sin(t)$  and

$\sin(t - \delta) = \cos(\delta)\sin(t) - \sin(\delta)\cos(t)$ . Also,  $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$ .

Thus

$$\begin{aligned} B &= [[\cos(t - \pi/4)]_{\mathcal{B}} \quad [\sin(t - \pi/4)]_{\mathcal{B}}] \\ &= \left[ \left[ \frac{\sqrt{2}}{2} \cos(t) + \frac{\sqrt{2}}{2} \sin(t) \right]_{\mathcal{B}} \quad \left[ -\frac{\sqrt{2}}{2} \cos(t) + \frac{\sqrt{2}}{2} \sin(t) \right]_{\mathcal{B}} \right] = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

**4.3.53** Recall that  $\cos(t - \theta) = \cos(\theta)\cos(t) + \sin(\theta)\sin(t)$  and  $\sin(t - \theta) = \cos(\theta)\sin(t) - \sin(\theta)\cos(t)$ .

$$\text{Thus } B = [[\cos(t - \theta)]_{\mathcal{B}} \quad [\sin(t - \theta)]_{\mathcal{B}}]$$

$$= [[\cos(\theta)\cos(t) + \sin(\theta)\sin(t)]_{\mathcal{B}} \quad [-\sin(\theta)\cos(t) + \cos(\theta)\sin(t)]_{\mathcal{B}}] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}. \text{ Yes, } T \text{ is an isomorphism.}$$

Note that  $B$  is a rotation matrix.

**4.3.54** Note that the two basis vectors  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$  and  $\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$  are perpendicular. Thus  $T \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$  and  $T \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ . Now  $B = \left[ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .

**4.3.55** Let  $\vec{u} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$  be the unit vector in the direction of  $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

$$\text{Now } T \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{6} \left( \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ and}$$

$$T \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \frac{1}{6} \left( \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{7}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \text{ by Theorem 2.2.1.}$$

$$\text{Also note that } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = -\frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}, \text{ so that } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}_{\mathcal{B}} = \frac{1}{3} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$$\text{Now } B = \left[ \left[ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \frac{1}{9} \begin{bmatrix} 2 & -14 \\ -1 & 7 \end{bmatrix}.$$

**4.3.56**  $T \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix}$  and  $T \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \\ -14 \end{bmatrix}$ . Thus

$$B = \left[ \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 14 \\ 14 \\ -14 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 14 \\ -1 & 0 \end{bmatrix}.$$

4.3.57  $T \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$  and  $T \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ .

Thus  $B = \left[ \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} -1 & 3 \\ -1 & 0 \end{bmatrix}$ .

4.3.58  $T \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$  and  $T \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ . Thus  $B = \left[ \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$ .

4.3.59  $T(f) = t \cdot f$  is linear and  $\ker(T) = \{0\}$ , but  $T$  is not an isomorphism since the constant function 1 is not in the image of  $T$ .

4.3.60 a We will use Theorem 4.3.3.  $S_{\mathcal{B} \rightarrow \mathcal{A}} = \left[ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}_{\mathcal{A}} \quad \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}_{\mathcal{A}} \right] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

b Here, we know  $S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ .

c Theorem 4.3.4 reveals that  $[\vec{b}_1 \quad \vec{b}_2] = [\vec{a}_1 \quad \vec{a}_2] S_{\mathcal{B} \rightarrow \mathcal{A}}$

4.3.61 a We will use Theorem 4.3.3.  $S_{\mathcal{B} \rightarrow \mathcal{A}} = \left[ \begin{bmatrix} 5 \\ -10 \end{bmatrix}_{\mathcal{A}} \quad \begin{bmatrix} 10 \\ 5 \end{bmatrix}_{\mathcal{A}} \right] = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} = 5 \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ , and is thus a reflection combined with a scaling.

b Here, we know  $S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \frac{1}{-9-16} \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix} = -\frac{1}{25} \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$ .

c Theorem 4.3.4 reveals that  $[\vec{b}_1 \quad \vec{b}_2] = [\vec{a}_1 \quad \vec{a}_2] S_{\mathcal{B} \rightarrow \mathcal{A}}$

4.3.62 a Finding this basis is equivalent to finding a basis of the kernel of  $\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$  that does not contain any zeroes. We can quickly spot the vectors  $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$ , so  $\mathcal{B} = \left( \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \right)$ , for example.

b From Theorem 4.3.3,  $S_{\mathcal{B} \rightarrow \mathcal{A}} = \left[ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{A}} \quad \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{A}} \right] = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ .

c  $S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ .

d Theorem 4.3.4 reveals that  $[\vec{b}_1 \quad \vec{b}_2] = [\vec{a}_1 \quad \vec{a}_2] S_{\mathcal{B} \rightarrow \mathcal{A}}$

**4.3.63 a** Finding this basis is equivalent to finding a basis of the kernel of  $[1 \ 3 \ -2]$ . We can quickly spot the vectors  $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ , so  $\mathcal{B} = \left( \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right)$ , for example.

b From Theorem 4.3.3,  $S_{\mathcal{B} \rightarrow \mathcal{A}} = \begin{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{A}} & \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$ .

c  $S_{\mathcal{A} \rightarrow \mathcal{B}} = S_{\mathcal{B} \rightarrow \mathcal{A}}^{-1} = \frac{1}{-3+1} \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$ .

d Theorem 4.3.4 reveals that  $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = [\vec{a}_1 \ \vec{a}_2] S_{\mathcal{B} \rightarrow \mathcal{A}}$

**4.3.64 a** We find  $A$  column-by-column:  $A = [[I_2]_{\mathcal{B}} \ [P]_{\mathcal{B}} \ [P^2]_{\mathcal{B}}] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & 9 \end{bmatrix}$ .

b We find a basis of the kernel of  $A$  to be  $\begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$ , and a basis of the image of  $A$  to be  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . Thus,  $\begin{bmatrix} -3 & 4 \\ 0 & -1 \end{bmatrix}$  is a basis of the kernel of  $T$ , and  $I_2, P$  is a basis of the image of  $T$ .

**4.3.65 a**  $P^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + bc & (a+d)b \\ (a+d)c & bc + d^2 \end{bmatrix} = (a+d) \begin{bmatrix} a & b \\ c & d \end{bmatrix} + (bc - ad) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . So  $[P^2]_{\mathcal{B}} = \begin{bmatrix} bc - ad \\ a + d \end{bmatrix}$ .

b We will do this column-by-column:  $B = [[T(I_2)]_{\mathcal{B}} \ [T(P)]_{\mathcal{B}}] = [[P]_{\mathcal{B}} \ [P^2]_{\mathcal{B}}]$

$$= \begin{bmatrix} 0 & bc - ad \\ 1 & a + d \end{bmatrix}, \text{ by part a. } T \text{ is an isomorphism if } bc - ad \neq 0, \text{ that is, if } P \text{ is invertible.}$$

c Assume that  $bc - ad = 0$ . Then the  $\mathcal{B}$ -matrix of  $T$  is  $B = \begin{bmatrix} 0 & 0 \\ 1 & a + d \end{bmatrix}$ . So,  $\ker(B) = \text{span} \begin{bmatrix} a + d \\ -1 \end{bmatrix}$  and  $\text{im}(B) = \text{span} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Thus a basis of  $\ker(T)$  is  $(a+d)I_2 - P = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ , a basis of  $\text{im}(T)$  is  $P$ , and the rank of  $T$  is 1.

**4.3.66 a**  $B = [[T(x_1^2)]_{\mathcal{B}} \ [T(x_1x_2)]_{\mathcal{B}} \ [T(x_2^2)]_{\mathcal{B}}] = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$ .

b Note that  $\text{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Now  $\text{im}(B) = \text{span} \left( \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$  and

$\ker(B) = \text{span} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ . Thus  $2x_1x_2, -x_1^2 + x_2^2$  is a basis of  $\text{im}(T)$  and  $x_1^2 + x_2^2$  is a basis of  $\ker(T)$ .

4.3.67 a  $[T(\cos t)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \vec{0}$   
 $[T(\sin t)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \vec{0}$

$$[T(t \cos(t))]_{\mathcal{B}} = [-2 \sin(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(t \sin(t))]_{\mathcal{B}} = [2 \cos(t)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{so } M = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

b The equation  $T(f) = \cos(t)$  corresponds to  $M\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ , with solutions  $\vec{x} = \begin{bmatrix} p \\ q \\ 0 \\ \frac{1}{2} \end{bmatrix}$ , where  $p$  and  $q$  are arbitrary.

Thus  $f(t) = p \cos(t) + q \sin(t) + \frac{1}{2}t \sin(t)$ . In Figure 4.2 we graph  $f(t)$  for  $p = q = 0$ ; note that  $p \cos(t) + q \sin(t)$  is just a sinusoidal function

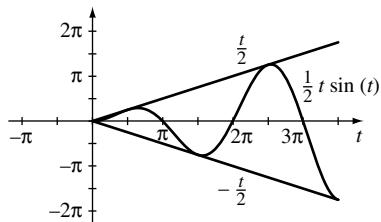


Figure 4.2: for Problem 4.3.67b.

4.3.68 a The sequence  $(0, 0, 0, \dots)$  is in  $W$ .

- If the sequences  $(x_n)$  and  $(y_n)$  are in  $W$  (that is,  $x_{n+2} = x_{n+1} + 6x_n$  and  $y_{n+2} = y_{n+1} + 6y_n$  for all  $n$ ), then  $x_{n+2} + y_{n+2} = x_{n+1} + y_{n+1} + 6(x_n + y_n)$  so that the sequence  $(x_n + y_n)$  is in  $W$  as well.
- If the sequence  $(x_n)$  is in  $W$  and  $k$  is any constant, then  $kx_{n+2} = kx_{n+1} + 6kx_n$ , so that the sequence  $(kx_n)$  is in  $W$  as well.

b A sequence in  $W$  is determined by its first two components ( $a$  and  $b$ , say), which we can choose freely. All the later components can then be expressed in terms of  $a$  and  $b$ , since  $x_{n+2} = x_{n+1} + 6x_n$ :  $(a, b, b+6a, 6a+7b, 42a+13b, \dots) = a(1, 0, 6, 6, 42, \dots) + b(0, 1, 1, 7, 13, \dots)$ . The two sequences  $(1, 0, 6, 6, 42, \dots)$  and  $(0, 1, 1, 7, 13, \dots)$  form a basis of  $W$ , so that  $\dim(W) = 2$ .

c It is required that  $c^{n+2} = c^{n+1} + 6c^n$  for all  $n \geq 0$

$$\text{or } c^2 = c + 6$$

$$\text{or } c^2 - c - 6 = 0$$

$$\text{or } (c - 3)(c + 2) = 0$$

The solution are  $c_1 = 3$  and  $c_2 = -2$ . The geometric sequences  $(1, 3, 9, 27, 81, \dots)$  and  $(1, -2, 4, -8, 16, \dots)$  are in  $W$ .

d Yes, the two sequences we found in part c do the job, since  $\dim(W) = 2$ .

e  $(x_0, x_1, x_2, x_3, x_4, \dots) = (0, 1, 1, 7, 13, \dots)$ .

We are looking for constants  $p$  and  $q$  such that

$$(x_0, x_1, x_2, x_3, \dots) = (0, 1, 1, 7, \dots) = p(1, 3, 9, 27, \dots) + q(1, -2, 4, -8, \dots).$$

To find  $p$  and  $q$ , it suffices to consider the first two components:  $\begin{bmatrix} 0 = p + q \\ 1 = 3p - 2q \end{bmatrix}$ , so that  $p = \frac{1}{5}$  and  $q = -\frac{1}{5}$ .

$$\text{Thus } x_n = \frac{1}{5}3^n - \frac{1}{5}(-2)^n.$$

**4.3.69** As the hint suggests, we find the kernel of  $M = \begin{bmatrix} f_1(a_1) & f_2(a_1) & \cdots & f_n(a_1) \\ f_1(a_2) & f_2(a_2) & \cdots & f_n(a_2) \\ \vdots & \vdots & & \vdots \\ f_1(a_n) & f_2(a_n) & \cdots & f_n(a_n) \end{bmatrix}$ .

$$\text{If } M \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f_1(a_1) & f_2(a_1) & \cdots & f_n(a_1) \\ f_1(a_2) & f_2(a_2) & \cdots & f_n(a_2) \\ \vdots & \vdots & & \vdots \\ f_1(a_n) & f_2(a_n) & \cdots & f_n(a_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_1 f_1(a_1) + \cdots + c_n f_n(a_1) \\ c_1 f_1(a_2) + \cdots + c_n f_n(a_2) \\ \vdots \\ c_1 f_1(a_n) + \cdots + c_n f_n(a_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

then the polynomial  $c_1 f_1 + \cdots + c_n f_n$  in  $P_{n-1}$  has at least  $n$  zeros, namely,  $a_1, a_2, \dots, a_n$ . It follows that  $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$  and therefore  $c_1 = c_2 = \cdots = c_n = 0$  since the  $f_i$  are linearly independent. We have shown that  $\ker(M) = \{\vec{0}\}$ , so that  $M$  is invertible.

**4.3.70** Yes; suppose  $\dim(V) = \dim(W) = n$ , and let  $\mathcal{B}$  and  $\mathcal{C}$  be bases of  $V$  and  $W$ , respectively. Then the coordinate transformation  $T_{\mathcal{B}}$  and  $T_{\mathcal{C}}$  define isomorphisms from  $V$  to  $\mathbb{R}^n$  and from  $W$  to  $\mathbb{R}^n$ , respectively.

$T_{\mathcal{C}}^{-1} \circ T_{\mathcal{B}}$  is an isomorphism from  $V$  to  $W$  (see Exercise 4.2.77).

**4.3.71** We need to show that there are constants  $w_1, \dots, w_n$  such that

$$w_1 f_1(a_1) + w_2 f_1(a_2) + \cdots + w_n f_1(a_n) = \int_{-1}^1 f_1$$

$$w_1 f_2(a_1) + w_2 f_2(a_2) + \cdots + w_n f_2(a_n) = \int_{-1}^1 f_2$$

$$\vdots \quad \vdots$$

$$w_1 f_n(a_1) + w_2 f_n(a_2) + \cdots + w_n f_n(a_n) = \int_{-1}^1 f_n.$$

In order to be able to use the matrix  $M$  introduced in Exercise 69, as the hint suggests, we write this system in a somewhat unusual way, as

$$[w_1 \ w_2 \ \dots \ w_n] \underbrace{\begin{bmatrix} f_1(a_1) & f_2(a_1) & \cdots & f_n(a_1) \\ f_1(a_2) & f_2(a_2) & \cdots & f_n(a_2) \\ \dots & \dots & \cdots & \dots \\ f_1(a_n) & f_2(a_n) & \cdots & f_n(a_n) \end{bmatrix}}_M = \left[ \int_{-1}^1 f_1 \ \int_{-1}^1 f_2 \ \cdots \ \int_{-1}^1 f_n \right]$$

Since  $M$  is invertible (Exercise 69), we have the unique solution

$$[w_1 \ w_2 \ \dots \ w_n] = \left[ \int_{-1}^1 f_1 \ \int_{-1}^1 f_2 \ \cdots \ \int_{-1}^1 f_n \right] M^{-1}.$$

Now if  $f$  is any polynomial in  $P_{n-1}$  and  $f = \sum_{j=1}^n c_j f_j$ , then

$$\int_{-1}^1 f = \sum_{j=1}^n c_j \int_{-1}^1 f_j = \sum_{j=1}^n c_j \sum_{i=1}^n w_i f_j(a_i) = \sum_{i=1}^n w_i \sum_{j=1}^n c_j f_j(a_i) = \sum_{i=1}^n w_i f(a_i),$$

as claimed.

**4.3.72** If we work with the basis  $f_1(t) = 1$ ,  $f_2(t) = t$ , and  $f_3(t) = t^2$  of  $P_2$ , then we have to solve the system

$$\begin{bmatrix} w_1 + w_2 + w_3 & = 2 \\ -w_1 + w_2 & = 0 \\ w_1 + w_2 & = \frac{2}{3} \end{bmatrix} \text{ (see Exercise 71). The solution is } w_1 = w_3 = \frac{1}{3} \text{ and } w_2 = \frac{4}{3}, \text{ so that}$$

$\int_{-1}^1 f = \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)$  for  $f$  in  $P_2$ . This is what you get when you apply Simpson's Rule (with two subintervals) to  $f$ ; note that Simpson's Rule gives the exact value of the integral in this case.

## True or False

Ch 4.TF.1 F; A basis of  $\mathbb{R}^{2 \times 3}$  is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ,

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , so it has a dimension of 6.

Ch 4.TF.2 T; check with Definition 4.1.3c.

Ch 4.TF.3 T; The linear transformation  $T(ax + b) = a + ib$  is an isomorphism from  $P_1$  to  $\mathbb{C}$ , with the inverse  $T^{-1}(a + ib) = ax + b$ .

Ch 4.TF.4 T; by Theorem 4.2.4c.

Ch 4.TF.5 T; This fits all properties of Definition 4.1.2.

Ch 4.TF.6 F; The transformation  $T$  could be:  $T(f) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , in which case the kernel would be all of  $P_6$  and the dimension of the kernel would be 7.

Ch 4.TF.7 T; We are looking at  $P_6$ , with a basis  $1, t, t^2, t^3, t^4, t^5, t^6$ , which has seven elements.

Ch 4.TF.8 True, we can check both requirements of Definition 4.2.1.

Ch 4.TF.9 T; check the three properties listed in Definition 4.1.2.

Ch 4.TF.10 T; by Definition 4.2.1.

Ch 4.TF.11 F;  $T(\sin(x)) = \sin(x) - \sin(x) = 0$ .

Ch 4.TF.12 F;  $T(f) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  is not an isomorphism.

Ch 4.TF.13 F; Let  $V = \mathbb{R}^2$ ,  $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ . Now  $\text{im}(A) = \ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ .

Ch 4.TF.14 T; the dimensions of both spaces are the same: 10.

Ch 4.TF.15 F;  $\dim(P_3) = 4$ , so the three given polynomials cannot span  $P_3$ .

Ch 4.TF.16 T; We can construct a basis of  $V$  by omitting the redundant elements from a list of ten elements that span  $V$ . Thus  $\dim(V) \leq 10$ .

Ch 4.TF.17 F;  $\det\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \neq \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ .

Ch 4.TF.18 F; For any matrix  $A$ , the space of matrices commuting with  $A$  is at least two-dimensional. Indeed, if  $A$  is a scalar multiple of  $I_2$ , then  $A$  commutes with all  $2 \times 2$  matrices, and if  $A$  fails to be a scalar multiple of  $I_2$ , then  $A$  commutes with the linearly independent matrices  $A$  and  $I_2$ .

Ch 4.TF.19 F;  $t^3, t^3 + t^2, t^3 + t, t^3 + 1$  is a basis of  $P_3$ .

Ch 4.TF.20 T; If  $T$  is linear and invertible, then  $T^{-1}$  will be linear and invertible as well.

Ch 4.TF.21 F;  $T(f(t)) = f'(t)$  is not an isomorphism.

Ch 4.TF.22 T; We need only show that either the new list contains no redundant elements, or spans the whole space. The latter is slightly easier to show. Since  $f_1, f_2, f_3$  form a basis of  $V$ , it suffices to show that these three elements are in the span of  $f_1, f_1 + f_2, f_1 + f_2 + f_3$ . This is simple to demonstrate:  $f_2 = (f_1 + f_2) - f_1$ , and  $f_3 = (f_1 + f_2 + f_3) - (f_1 + f_2)$ .

Ch 4.TF.23 T; We show that none of the polynomials is redundant; let's call them  $f(x), g(x)$  and  $h(x)$ . Now  $g(x)$  isn't a multiple of  $f(x)$  since  $f(b) = 0$ , but  $g(b) \neq 0$ . Likewise,  $h(x)$  isn't a linear combination of  $f(x)$  and  $g(x)$  since  $f(c) = g(c) = 0$ , but  $h(c) \neq 0$ .

Ch 4.TF.24 T; Make the substitution  $4t - 3 = s$  to see that the inverse is  $T^{-1}(g(s)) = g(\frac{s+3}{4})$ .

Ch 4.TF.25 F;  $T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+6c & 3b+6d \end{bmatrix}$   
 $= (a+2c) \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} + (b+2d) \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$ . So the image is the span of  $\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$ , and  $\text{rank}(T) = 2$ .

Ch 4.TF.26 T; If the basis  $\mathcal{B}$  we consider is  $f_1, f_2$ , then the given matrix tells us that  $T(f_1) = 3f_1$  and  $T(f_2) = 5f_1 + 4f_2$ . Thus  $f = f_1$  does the job.

Ch 4.TF.27 T; If  $T(f(t)) = f(t^2) = 0$ ,  $f(t)$  must also be zero.

Ch 4.TF.28 T; The inverse is  $T^{-1}(N) = S^{-1}NS^{-1}$ .

Ch 4.TF.29 T; Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ . Then we want  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ , or  
 $\begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$ . Thus,  $c = 0$  and  $a = b + d$ . So our space is the span of  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ .

Ch 4.TF.30 T; Let our basis be  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . Each matrix here is invertible, and also clearly none are redundant.

Ch 4.TF.31 F;  $P_2$  is a subspace of  $P$ , and  $P$  is infinite dimensional.

Ch 4.TF.32 T; Let  $T \left( \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = \begin{bmatrix} b & c \\ 0 & f \end{bmatrix}$ . We can easily see that the kernel and image of this transformation are exactly as required.

Ch 4.TF.33 F; The space spanned by  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  contains no invertible matrices.

Ch 4.TF.34 F; This is the change of basis matrix from  $\mathcal{B}$  to  $\mathcal{A}$ . The change of basis matrix we are looking for is:  
 $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ .

Ch 4.TF.35 F; Let  $\mathcal{B} = (f, g)$  and  $\mathcal{C} = (g, f)$ . The fact that  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  is the  $\mathcal{B}$ -matrix of  $T$  implies that  $[T(f)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ , or  $T(f) = f + 3g$ . But then  $[T(f)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ , meaning that the second column of the  $\mathcal{C}$ -matrix of  $T$  is  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . This shows that the matrix  $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$  fails to be the  $\mathcal{C}$ -matrix of  $T$ .

Ch 4.TF.36 T; The image of  $T$  is  $P_{n-1}$ , so that  $\text{rank}(T) = \dim(\text{im } T) = \dim(P_{n-1}) = n$ .

Ch 4.TF.37 T; because the matrix is invertible.

Ch 4.TF.38 T; The dimension of  $P_9$  is 10, and the dimension of  $\mathbb{R}^{3 \times 4}$  is 12. Thus, any 10-dimensional subspace of  $\mathbb{R}^{3 \times 4}$  will be acceptable. For example, we can consider the space of all  $3 \times 4$  matrices  $A$  with  $a_{11} = a_{12} = 0$ .

Ch 4.TF.39 T; let  $W_1$  be  $\{\vec{0}\}$ . Then any other subspace  $W_2$  unioned with  $W_1$  will simply be  $W_2$  again, which we know is a subspace.

Ch 4.TF.40 T; Let  $T(a_0 + a_1t + a_2t^2 + \dots + a_5t^5 + \dots) = a_0 + a_1t + a_2t^2 + \dots + a_5t^5$ . The image of this transformation is clearly all of  $P_5$ , and  $T$  satisfies the requirements of Definition 4.2.1.

Ch 4.TF.41 T; there will be no redundant elements in this list.

Ch 4.TF.42 F; The kernel of  $T$  consists of all constant functions.

Ch 4.TF.43 T; We apply the rank-nullity theorem:  $\dim(W) = \dim(\text{im}(T)) = \dim(P_4) - \dim(\ker(T)) = 5 - \dim(\ker(T)) \leq 5$ .

Ch 4.TF.44 F; We can construct as many linearly independent elements in  $\ker(T)$  as we want, for example, the polynomials  $f(t) = t^n - \frac{1}{n+1}$ , for all positive integers  $n$ .

Ch 4.TF.45 T; 0 is in our set, and if  $f$  and  $g$  are in our set, then  $T(f + g) = T(f) + T(g) = f + g$  so that  $f + g$  is in  $V$  as well. Also, if  $f$  is in  $V$  and  $k$  is an arbitrary scalar, then  $T(kf) = kT(f) = kf$ , so  $kf$  is in  $V$  as well.

Ch 4.TF.46 F; Consider the transformation that drops the constant term:  $T(c_0 + c_1t + \dots + c_6t^6) = c_1t + \dots + c_6t^6$ .

Ch 4.TF.47 T; Let  $P = I_2$ ,  $Q = -I_2$ . Then  $T(M) = I_2M - M(-I_2) = 2M$ , which is an isomorphism.

Ch 4.TF.48 F; We use dimension arithmetic here to show that this cannot happen. Any transformation  $T$  from  $P_6$  to  $\mathbb{C}$  must have a kernel of at least 5 dimensions, since  $P_6$  is 7-dimensional and  $\mathbb{C}$  is only a 2-dimensional space. Thus, any such kernel cannot be isomorphic to  $\mathbb{R}^{2 \times 2}$ , which is a 4-dimensional space.

Ch 4.TF.49 F; If  $f = -f_1$ , then 0 is a member of the list!

Ch 4.TF.50 T; Consider the space of all matrices of the form  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , for example.

Ch 4.TF.51 T; note that  $\dim(P_{11}) = 12 = \dim(\mathbb{R}^{3 \times 4})$ . The linear spaces  $P_{11}$  and  $\mathbb{R}^{3 \times 4}$  are both isomorphic to  $\mathbb{R}^{12}$ , via the coordinate transformation, and thus they are isomorphic to each other.

Ch 4.TF.52 F; Consider the linear transformation  $T(f(t)) = f(t)$  from  $P_2$  to  $P$ , for example.

Ch 4.TF.53 T; We use the rank-nullity theorem:  $\dim(V) = \dim(\text{im}(T)) + \dim(\ker(T)) = \dim(\text{im}(T)) \leq \dim(\mathbb{R}^{2 \times 2}) = 4$ .

Ch 4.TF.54 T; Using the fundamental theorem of calculus, we can write  $g(t) = T(f(t)) = 3f(3t + 4)$ . Make the substitution  $3t + 4 = s$  to see that the inverse is  $T^{-1}(g(s)) = g((s - 4)/3)/3$ .

Ch 4.TF.55 T; Using a coordinate transformation, it suffices to show this for  $\mathbb{R}^4$ . For every real number  $k$ , we define the three dimensional subspace  $V_k$  of  $\mathbb{R}^4$  consisting of all vectors  $\vec{x}$  such that  $x_4 = kx_3$ . If  $c$  is different from  $k$ ,

then  $V_c$  and  $V_k$  will be different subspaces of  $\mathbb{R}^4$ , since  $V_k$  contains the vector  $\begin{bmatrix} 0 \\ 0 \\ 1 \\ k \end{bmatrix}$ , but  $V_c$  does not. Thus we have generated infinitely many distinct three-dimensional subspaces  $V_k$  of  $\mathbb{R}^4$ , one for every real number  $k$ .

Ch 4.TF.56 T; If the basis  $\mathcal{B}$  we consider is  $f_1, f_2$ , then the given matrix tells us that  $T(f_1) = 3f_1$  and  $T(f_2) = 5f_1 + 4f_2$ . We are looking for a nonzero  $f = af_1 + bf_2$  such that  $T(f) = 4f$ . Now  $T(f) = aT(f_1) + bT(f_2) = 3af_1 + 5bf_1 + 4bf_2 = (3a + 5b)f_1 + 4bf_2$  must be equal to  $4f = 4af_1 + 4bf_2$ . Thus it is required that  $3a + 5b = 4a$ , or  $a = 5b$ . For example,  $f = 5f_1 + f_2$  does the job.

Ch 4.TF.57 T; This is logically equivalent to the following statement: If the domain of  $T$  is finite dimensional, then so is the image of  $T$ . Compare with Exercises 4.2.81a and 4.1.57.

Ch 4.TF.58 F; If  $A$  is a scalar multiple of  $I_2$ , then all  $2 \times 2$  matrices commute with  $A$ , so that the space of commuting matrices is 4 - dimensional. If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  fails to be a scalar multiple of  $I_2$ , consider the equation

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , which amounts to the system  $cy - bz = 0, bx + (d - a)y - bt = 0, cx + (d - a)z - ct = 0$ . If  $b \neq 0$ , then the first two equations are independent; if  $c \neq 0$ , then the first and the third equation are independent; and if  $a \neq d$ , then the second and the third equation are independent. Thus the rank of the system is at least two and the solution space is at most two-dimensional. (The solution space is in fact two -dimensional, since  $A$  and  $I_2$  are independent solutions.)

Ch 4.TF.59 T; If  $A = 0$ , then we are done. If  $\text{rank}(A) = 1$ , then the image of the linear transformation  $T(M) = AM$  from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$  is two dimensional (if  $\vec{v}$  is a basis of  $\text{im}(A)$ , then  $\begin{bmatrix} \vec{v} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \vec{v} \end{bmatrix}$  is a basis of  $\text{im}(T)$ ). Since the three matrices  $AB = T(B)$ ,  $AC = T(C)$  , and  $AD = T(D)$  are all in  $\text{im}(T)$ , they must be linearly dependent.

Ch 4.TF.60 F; Consider two distinct three-dimensional subspaces  $W_1$  and  $W_2$  of  $P_4$ . Since the spaces  $W_1$  and  $W_2$  are distinct, neither of them is a subspace of the other, so that we can find a polynomial  $f_1$  that is in  $W_1$  but not in  $W_2$  as well as an  $f_2$  that is in  $W_2$  but not in  $W_1$ . Then  $f_1$  and  $f_2$  are both in the union of  $W_1$  and  $W_2$ , but  $f_1 + f_2$  isn't.

Ch 4.TF.61 T; Pick the *first* redundant element  $f_k$  in the list. Since the elements  $f_1, \dots, f_{k-1}$  are linearly independent, the representation of  $f_k$  as a linear combination of the preceding elements will be unique.

Ch 4.TF.62 F;  $T(I_3) = P - P = 0$ , and  $T$  can never be an isomorphism.

Ch 4.TF.63 T; Let  $W = \text{span}(f_1, f_2, f_3, f_4, f_5) = \text{span}(f_2, f_4, f_5, f_1, f_3)$ . If we omit the two redundant elements from the first list,  $f_1, f_2, f_3, f_4, f_5$ , we end up with a basis of  $W$  with three elements, so that  $\dim(W) = 3$ . If we omit the redundant elements from the second list,  $f_2, f_4, f_5, f_1, f_3$ , we end up with a (possibly different) basis of  $W$ , but that basis must consist of 3 elements as well. Thus there must be two redundant elements in the second list.

Ch 4.TF.64 F; The dimensions of the kernel and image would have to be equal, and both add up to the dimension of  $P_6$ , which is the odd number 7.

Ch 4.TF.65 T; Consider the proof of the rank nullity theorem outlined in Exercise 4.2.81. In the proof, we use bases of  $\ker(T)$  and  $\text{im}(T)$  to construct a basis of the domain.

Ch 4.TF.66 F; If the basis  $\mathcal{B}$  we consider is  $f_1, f_2$ , then the given matrix tells us that  $T(f_1) = 3f_1$  and  $T(f_2) = 5f_1 + 4f_2$ . We are looking for a nonzero  $f = af_1 + bf_2$  such that  $T(f) = 5f$ . Now  $T(f) = aT(f_1) + bT(f_2) = 3af_1 + 5bf_1 + 4bf_2 = (3a + 5b)f_1 + 4bf_2$  must be equal to  $5f = 5af_1 + 5bf_2$ . Thus it is required that  $3a + 5b = 5a$  and  $4b = 5b$ , implying that  $a = b = 0$ . We are unable to find a nonzero  $f$  with the desired property.

Ch 4.TF.67 T; Consider a 3-dimensional subspace  $W$  of  $\mathbb{R}^{2 \times 2}$ . The matrices  $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$  in  $W$  can be described by a single linear equation  $ax + by + cz + dt = 0$ , where at least one of the coefficients is nonzero. Suppose  $x$  is the leading variable (meaning that  $a \neq 0$ ), and  $y, z$  and  $t$  are the free variables. We can choose  $y = z = 1$  and  $t = 0$ , and the resulting matrix  $\begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix}$  in  $W$  will be invertible. We represent  $x$  by a star, since its value does not affect the invertibility. If  $y$  is the leading variable and the other three are the free variables, then we can construct the invertible matrix  $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$  in  $W$ . If  $z$  is the leading variable, we have the invertible matrix  $\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$ . Finally, for the leading variable  $t$  we have  $\begin{bmatrix} 0 & 1 \\ 1 & * \end{bmatrix}$ .

# Chapter 5

## Section 5.1

5.1.1  $\|\vec{v}\| = \sqrt{7^2 + 11^2} = \sqrt{49 + 121} = \sqrt{170} \approx 13.04$

5.1.2  $\|\vec{v}\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29} \approx 5.39$

5.1.3  $\|\vec{v}\| = \sqrt{2^2 + 3^2 + 4^2 + 5^2} = \sqrt{4 + 9 + 16 + 25} = \sqrt{54} \approx 7.35$

5.1.4  $\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \arccos \frac{7+11}{\sqrt{2} \sqrt{170}} = \arccos \frac{18}{\sqrt{340}} \approx 0.219$  (radians)

5.1.5  $\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \arccos \frac{2+6+12}{\sqrt{14} \sqrt{29}} \approx 0.122$  (radians)

5.1.6  $\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \arccos \frac{2-3+8-10}{\sqrt{10} \sqrt{54}} \approx 1.700$  (radians)

5.1.7 Use the fact that  $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$ , so that the angle is acute if  $\vec{u} \cdot \vec{v} > 0$ , and obtuse if  $\vec{u} \cdot \vec{v} < 0$ . Since  $\vec{u} \cdot \vec{v} = 10 - 12 = -2$ , the angle is obtuse.

5.1.8 Since  $\vec{u} \cdot \vec{v} = 4 - 24 + 20 = 0$ , the two vectors enclose a right angle.

5.1.9 Since  $\vec{u} \cdot \vec{v} = 3 - 4 + 5 - 3 = 1$ , the angle is acute (see Exercise 7).

5.1.10  $\vec{u} \cdot \vec{v} = 2 + 3k + 4 = 6 + 3k$ . The two vectors enclose a right angle if  $\vec{u} \cdot \vec{v} = 6 + 3k = 0$ , that is, if  $k = -2$ .

5.1.11 a  $\theta_n = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \arccos \frac{1}{\sqrt{n}}$

$$\theta_2 = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4} (= 45^\circ)$$

$$\theta_3 = \arccos \frac{1}{\sqrt{3}} \approx 0.955$$
 (radians)

$$\theta_4 = \arccos \frac{1}{2} = \frac{\pi}{3} (= 60^\circ)$$

b Since  $y = \arccos(x)$  is a continuous function,

$$\lim_{n \rightarrow \infty} \theta_n = \arccos \left( \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \right) = \arccos(0) = \frac{\pi}{2} (= 90^\circ)$$

5.1.12  $\|\vec{v} + \vec{w}\|^2 = (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w})$  (by hint)

$$= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2(\vec{v} \cdot \vec{w})$$
 (by definition of length)

$$\leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\| \|\vec{w}\|$$
 (by Cauchy-Schwarz)

$$= (\|\vec{v}\| + \|\vec{w}\|)^2, \text{ so that}$$

$$\|\vec{v} + \vec{w}\|^2 \leq (\|\vec{v}\| + \|\vec{w}\|)^2$$

Taking square roots of both sides, we find that  $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ , as claimed.

**5.1.13** Figure 5.1 shows that  $\|\vec{F}_2 + \vec{F}_3\| = 2 \cos(\frac{\theta}{2}) \|\vec{F}_2\| = 20 \cos(\frac{\theta}{2})$ .

It is required that  $\|\vec{F}_2 + \vec{F}_3\| = 16$ , so that  $20 \cos(\frac{\theta}{2}) = 16$ , or  $\theta = 2 \arccos(0.8) \approx 74^\circ$ .

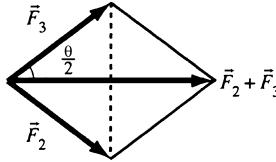


Figure 5.1: for Problem 5.1.13.

**5.1.14** The horizontal components of  $\vec{F}_1$  and  $\vec{F}_2$  are  $-\|\vec{F}_1\| \sin \beta$  and  $\|\vec{F}_2\| \sin \alpha$ , respectively (the horizontal component of  $\vec{F}_3$  is zero).

Since the system is at rest, the horizontal components must add up to 0, so that  $-\|\vec{F}_1\| \sin \beta + \|\vec{F}_2\| \sin \alpha = 0$  or  $\|\vec{F}_1\| \sin \beta = \|\vec{F}_2\| \sin \alpha$  or  $\frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} = \frac{\sin \alpha}{\sin \beta}$ .

To find  $\frac{\overline{EA}}{\overline{EB}}$ , note that  $\overline{EA} = \overline{ED} \tan \alpha$  and  $\overline{EB} = \overline{ED} \tan \beta$  so that  $\frac{\overline{EA}}{\overline{EB}} = \frac{\tan \alpha}{\tan \beta} = \frac{\sin \alpha}{\sin \beta} \cdot \frac{\cos \beta}{\cos \alpha} = \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} \frac{\cos \beta}{\cos \alpha}$ . Since  $\alpha$  and  $\beta$  are two distinct acute angles, it follows that  $\frac{\overline{EA}}{\overline{EB}} \neq \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|}$ , so that Leonardo was mistaken.

**5.1.15** The subspace consists of all vectors  $\vec{x}$  in  $\mathbb{R}^4$  such that

$$\vec{x} \cdot \vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = x_1 + 2x_2 + 3x_3 + 4x_4 = 0.$$

These are vectors of the form  $\begin{bmatrix} -2r & -3s & -4t \\ r & s & t \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$ .

The three vectors to the right form a basis.

**5.1.16** You may be able to find the solutions by educated guessing. Here is the systematic approach: we first find all vectors  $\vec{x}$  that are orthogonal to  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$ , then we identify the unit vectors among them.

Finding the vectors  $\vec{x}$  with  $\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \vec{x} \cdot \vec{v}_3 = 0$  amounts to solving the system

$$\begin{bmatrix} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{bmatrix}$$

(we can omit all the coefficients  $\frac{1}{2}$ ).

The solutions are of the form  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$ .

Since  $\|\vec{x}\| = 2|t|$ , we have a unit vector if  $t = \frac{1}{2}$  or  $t = -\frac{1}{2}$ . Thus there are two possible choices for  $\vec{v}_4$ :

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

**5.1.17** The orthogonal complement  $W^\perp$  of  $W$  consists of the vectors  $\vec{x}$  in  $\mathbb{R}^4$  such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0.$$

Finding these vectors amounts to solving the system  $\begin{bmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 = 0 \end{bmatrix}$ .

The solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s+2t \\ -2s-3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

The two vectors to the right form a basis of  $W^\perp$ .

**5.1.18 a**  $\|\vec{x}\|^2 = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$  (use the formula for a geometric series, with  $a = \frac{1}{4}$ ), so that  $\|\vec{x}\| = \frac{2}{\sqrt{3}} \approx 1.155$ .

b If we let  $\vec{u} = (1, 0, 0, \dots)$  and  $\vec{v} = (1, \frac{1}{2}, \frac{1}{4}, \dots)$ , then

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \arccos \frac{\frac{1}{2}}{\sqrt{3}} = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6} (= 30^\circ).$$

c  $\vec{x} = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots, \frac{1}{\sqrt{n}}, \dots\right)$  does the job, since the harmonic series  $1 + \frac{1}{2} + \frac{1}{3} + \dots$  diverges (a fact discussed in introductory calculus classes).

d If we let  $\vec{v} = (1, 0, 0, \dots)$ ,  $\vec{x} = (1, \frac{1}{2}, \frac{1}{4}, \dots)$  and  $\vec{u} = \frac{\vec{x}}{\|\vec{x}\|} = \frac{\sqrt{3}}{2} (1, \frac{1}{2}, \frac{1}{4}, \dots)$  then

$$\text{proj}_L \vec{v} = (\vec{u} \cdot \vec{v}) \vec{u} = \frac{3}{4} (1, \frac{1}{2}, \frac{1}{4}, \dots).$$

**5.1.19** See Figure 5.2.

**5.1.20** On the line  $L$  spanned by  $\vec{x}$  we want to find the vector  $m\vec{x}$  closest to  $\vec{y}$  (that is, we want  $\|m\vec{x} - \vec{y}\|$  to be minimal). We want  $m\vec{x} - \vec{y}$  to be perpendicular to  $L$  (that is, to  $\vec{x}$ ), which means that  $\vec{x} \cdot (m\vec{x} - \vec{y}) = 0$  or  $m(\vec{x} \cdot \vec{x}) - \vec{x} \cdot \vec{y} = 0$  or  $m = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}} \approx \frac{4182.9}{198.53^2} \approx 0.106$ .

Recall that the correlation coefficient  $r$  is  $r = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$ , so that  $m = \frac{\|\vec{y}\|}{\|\vec{x}\|} r$ . See Figure 5.3.

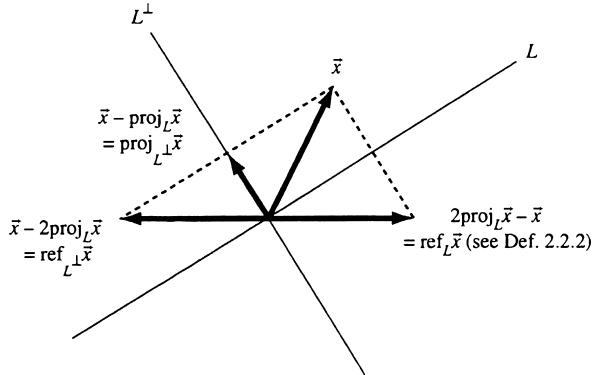


Figure 5.2: for Problem 5.1.19.

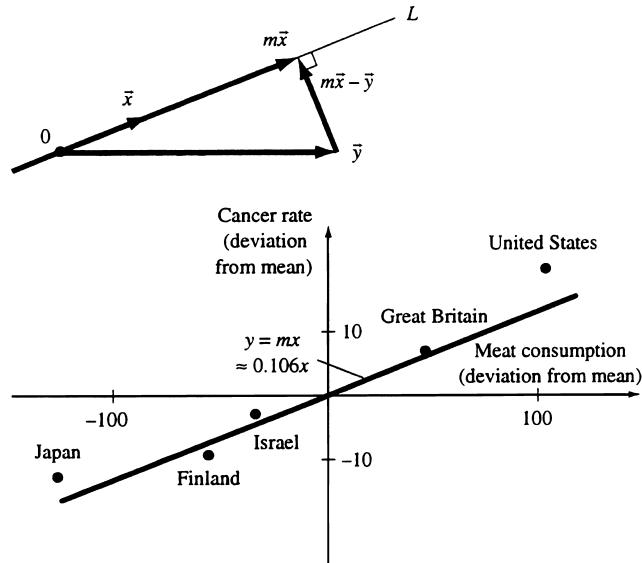


Figure 5.3: for Problem 5.1.20.

**5.1.21** Call the three given vectors  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$ . Since  $\vec{v}_2$  is required to be a unit vector, we must have  $b = g = 0$ . Now  $\vec{v}_1 \cdot \vec{v}_2 = d$  must be zero, so that  $d = 0$ .

Likewise,  $\vec{v}_2 \cdot \vec{v}_3 = e$  must be zero, so that  $e = 0$ .

Since  $\vec{v}_3$  must be a unit vector, we have  $\|\vec{v}_3\|^2 = c^2 + \frac{1}{4} = 1$ , so that  $c = \pm \frac{\sqrt{3}}{2}$ .

Since we are asked to find just one solution, let us pick  $c = \frac{\sqrt{3}}{2}$ .

The condition  $\vec{v}_1 \cdot \vec{v}_3 = 0$  now implies that  $\frac{\sqrt{3}}{2}a + \frac{1}{2}f = 0$ , or  $f = -\sqrt{3}a$ .

Finally, it is required that  $\|\vec{v}_1\|^2 = a^2 + f^2 = a^2 + 3a^2 = 4a^2 = 1$ , so that  $a = \pm \frac{1}{2}$ .

Let us pick  $a = \frac{1}{2}$ , so that  $f = -\frac{\sqrt{3}}{2}$ .

Summary:

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

There are other solutions; some components will have different signs.

**5.1.22** Let  $W = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{x} \cdot \vec{v}_i = 0 \text{ for all } i = 1, \dots, m\}$ . We are asked to show that  $V^\perp = W$ , that is, any  $\vec{x}$  in  $V^\perp$  is in  $W$ , and vice versa.

If  $\vec{x}$  is in  $V^\perp$ , then  $\vec{x} \cdot \vec{v} = 0$  for all  $\vec{v}$  in  $V$ ; in particular,  $x \cdot \vec{v}_i = 0$  for all  $i$  (since the  $\vec{v}_i$  are in  $V$ ), so that  $\vec{x}$  is in  $W$ .

Conversely, consider a vector  $\vec{x}$  in  $W$ . To show that  $\vec{x}$  is in  $V^\perp$ , we have to verify that  $\vec{x} \cdot \vec{v} = 0$  for all  $\vec{v}$  in  $V$ . Pick a particular  $\vec{v}$  in  $V$ . Since the  $\vec{v}_i$  span  $V$ , we can write  $\vec{v} = c_1 \vec{v}_1 + \dots + c_m \vec{v}_m$ , for some scalars  $c_i$ . Then  $\vec{x} \cdot \vec{v} = c_1(\vec{x} \cdot \vec{v}_1) + \dots + c_m(\vec{x} \cdot \vec{v}_m) = 0$ , as claimed.

**5.1.23** We will follow the hint. Let  $\vec{v}$  be a vector in  $V$ . Then  $\vec{v} \cdot \vec{x} = 0$  for all  $\vec{x}$  in  $V^\perp$ . Since  $(V^\perp)^\perp$  contains all vectors  $\vec{y}$  such that  $\vec{y} \cdot \vec{x} = 0$ ,  $\vec{v}$  is in  $(V^\perp)^\perp$ . So  $V$  is a subspace of  $(V^\perp)^\perp$ .

Then, by Theorem 5.1.8c,  $\dim(V) + \dim(V^\perp) = n$  and  $\dim(V^\perp) + \dim((V^\perp)^\perp) = n$ , so  $\dim(V) + \dim(V^\perp) = \dim(V^\perp) + \dim((V^\perp)^\perp)$  and  $\dim(V) = \dim((V^\perp)^\perp)$ . Since  $V$  is a subspace of  $(V^\perp)^\perp$ , it follows that  $V = (V^\perp)^\perp$ , by Exercise 3.3.61.

**5.1.24** Write  $T(\vec{x}) = \text{proj}_V(\vec{x})$  for simplicity.

To prove the linearity of  $T$  we will use the definition of a projection:  $T(\vec{x})$  is in  $V$ , and  $\vec{x} - T(\vec{x})$  is in  $V^\perp$ .

To show that  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ , note that  $T(\vec{x}) + T(\vec{y})$  is in  $V$  (since  $V$  is a subspace), and  $\vec{x} + \vec{y} - (T(\vec{x}) + T(\vec{y})) = (\vec{x} - T(\vec{x})) + (\vec{y} - T(\vec{y}))$  is in  $V^\perp$  (since  $V^\perp$  is a subspace, by Theorem 5.1.8a).

To show that  $T(k\vec{x}) = kT(\vec{x})$ , note that  $kT(\vec{x})$  is in  $V$  (since  $V$  is a subspace), and  $k\vec{x} - kT(\vec{x}) = k(\vec{x} - T(\vec{x}))$  is in  $V^\perp$  (since  $V^\perp$  is a subspace).

**5.1.25 a**  $\|k\vec{v}\|^2 = (k\vec{v}) \cdot (k\vec{v}) = k^2(\vec{v} \cdot \vec{v}) = k^2\|\vec{v}\|^2$

Now take square roots of both sides; note that  $\sqrt{k^2} = |k|$ , the absolute value of  $k$  (think about the case when  $k$  is negative).  $\|k\vec{v}\| = |k|\|\vec{v}\|$ , as claimed.

b  $\|\vec{u}\| = \left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \frac{1}{\|\vec{v}\|} \|\vec{v}\| = 1$ , as claimed.

↑

by part a

**5.1.26** The two given vectors spanning the subspace are orthogonal, but they are not unit vectors: both have length 7. To obtain an orthonormal basis  $\vec{u}_1, \vec{u}_2$  of the subspace, we divide by 7:

$$\vec{u}_1 = \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \vec{u}_2 = \frac{1}{7} \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

Now we can use Theorem 5.1.5, with  $\vec{x} = \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$ :

$$\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 = 11 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}.$$

**5.1.27** Since the two given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{u}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Now we can use Theorem 5.1.5, with  $\vec{x} = 9\vec{e}_1$ :  $\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2$

$$= 2 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 2 \\ -2 \end{bmatrix}.$$

**5.1.28** Since the three given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \vec{u}_3 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Now we can use Theorem 5.1.5, with  $\vec{x} = \vec{e}_1$ :  $\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + (\vec{u}_3 \cdot \vec{x})\vec{u}_3 = \frac{1}{4} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$

**5.1.29** By the Pythagorean theorem (Theorem 5.1.9),

$$\begin{aligned} \|\vec{x}\|^2 &= \|7\vec{u}_1 - 3\vec{u}_2 + 2\vec{u}_3 + \vec{u}_4 - \vec{u}_5\|^2 \\ &= \|7\vec{u}_1\|^2 + \|3\vec{u}_2\|^2 + \|2\vec{u}_3\|^2 + \|\vec{u}_4\|^2 + \|\vec{u}_5\|^2 \\ &= 49 + 9 + 4 + 1 + 1 \\ &= 64, \text{ so that } \|\vec{x}\| = 8. \end{aligned}$$

**5.1.30** Since  $\vec{y} = \text{proj}_V \vec{x}$ , the vector  $\vec{x} - \vec{y}$  is orthogonal to  $\vec{y}$ , by definition of an orthogonal projection (see Theorem 5.1.4):  $(\vec{x} - \vec{y}) \cdot \vec{y} = 0$  or  $\vec{x} \cdot \vec{y} - \|\vec{y}\|^2 = 0$  or  $\vec{x} \cdot \vec{y} = \|\vec{y}\|^2$ . See Figure 5.4.

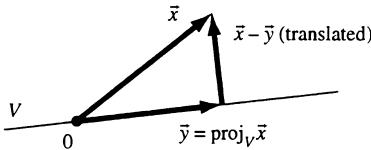


Figure 5.4: for Problem 5.1.30.

**5.1.31** If  $V = \text{span}(\vec{u}_1, \dots, \vec{u}_m)$ , then  $\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$ , by Theorem 5.1.5, and  $\|\text{proj}_V \vec{x}\|^2 = (\vec{u}_1 \cdot \vec{x})^2 + \dots + (\vec{u}_m \cdot \vec{x})^2 = p$ , by the Pythagorean theorem (Theorem 5.1.9). Therefore  $p \leq \|\vec{x}\|^2$ , by Theorem 5.1.10. The two quantities are equal if (and only if)  $\vec{x}$  is in  $V$ .

**5.1.32** By Theorem 2.4.9a, the matrix  $G$  is invertible if (and only if)  $(\vec{v}_1 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{v}_2) - (\vec{v}_1 \cdot \vec{v}_2)^2 = \|\vec{v}_1\|^2 \|\vec{v}_2\|^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 \neq 0$ . The Cauchy-Schwarz inequality (Theorem 5.1.11) tells us that  $\|\vec{v}_1\|^2 \|\vec{v}_2\|^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 \geq 0$ ; equality holds if (and only if)  $\vec{v}_1$  and  $\vec{v}_2$  are parallel (that is, linearly dependent).

**5.1.33** Let  $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  be a vector in  $\mathbb{R}^n$  whose components add up to 1, that is,  $x_1 + \dots + x_n = 1$ . Let  $\vec{y} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$  (all  $n$  components are 1). The Cauchy-Schwarz inequality (Theorem 5.1.11) tells us that  $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$ , or,  $|x_1 + \dots + x_n| \leq \|\vec{x}\| \sqrt{n}$ , or  $\|\vec{x}\| \geq \frac{1}{\sqrt{n}}$ . By Theorem 5.1.11, the equation  $\|\vec{x}\| = \frac{1}{\sqrt{n}}$  holds if (and only if) the vectors  $\vec{x}$  and  $\vec{y}$  are parallel, that is,  $x_1 = x_2 = \dots = x_n = \frac{1}{n}$ . Thus the vector of minimal length is  $\vec{x} = \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$  (all components are  $\frac{1}{n}$ ).

Figure 5.5 illustrates the case  $n = 2$ .

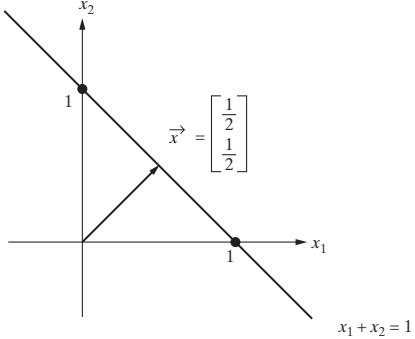


Figure 5.5: for Problem 5.1.33.

**5.1.34** Let  $\vec{x}$  be a unit vector in  $\mathbb{R}^n$ , that is,  $\|\vec{x}\| = 1$ . Let  $\vec{y} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$  (all  $n$  components are 1). The Cauchy-Schwarz inequality (Theorem 5.1.11) tells us that  $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$ , or,  $|x_1 + \dots + x_n| \leq \|\vec{x}\| \sqrt{n} = \sqrt{n}$ . By Theorem 5.1.11, the equation  $x_1 + \dots + x_n = \sqrt{n}$  holds if  $\vec{x} = k\vec{y}$  for positive  $k$ . Thus  $\vec{x}$  must be a unit vector of the form  $\vec{x} = \begin{bmatrix} k \\ \vdots \\ k \end{bmatrix}$  for some positive  $k$ . It is required that  $nk^2 = 1$ , or,  $k = \frac{1}{\sqrt{n}}$ . Thus  $\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix}$  (all components are  $\frac{1}{\sqrt{n}}$ ).

Figure 5.6 illustrates the case  $n = 2$ .

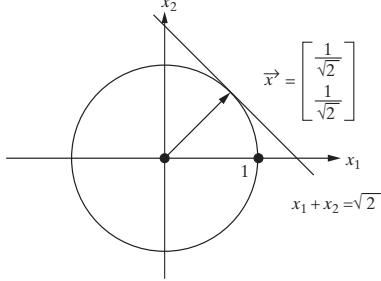


Figure 5.6: for Problem 5.1.34.

**5.1.35** Applying the Cauchy-Schwarz inequality to  $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  gives  $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$ , or  $|x + 2y + 3z| \leq \sqrt{14}$ . The minimal value  $x + 2y + 3z = -\sqrt{14}$  is attained when  $\vec{u} = k\vec{v}$  for negative  $k$ . Thus  $\vec{u}$  must be a unit vector of the form  $\vec{u} = \begin{bmatrix} k \\ 2k \\ 3k \end{bmatrix}$ , for negative  $k$ . It is required that  $14k^2 = 1$ , or,  $k = -\frac{1}{\sqrt{14}}$ . Thus  $\vec{u} = \begin{bmatrix} -\frac{1}{\sqrt{14}} \\ -\frac{2}{\sqrt{14}} \\ -\frac{3}{\sqrt{14}} \end{bmatrix}$ .

**5.1.36** Let  $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$ . It is required that  $\vec{x} \cdot \vec{y} = 0.2a + 0.3b + 0.5c = 76$ . Our goal is to minimize quantity  $\vec{x} \cdot \vec{x} = a^2 + b^2 + c^2$ . The Cauchy-Schwarz inequality (squared) tells us that  $(\vec{x} \cdot \vec{y})^2 \leq \|\vec{x}\|^2 \|\vec{y}\|^2$ , or  $76^2 \leq (a^2 + b^2 + c^2)(0.2^2 + 0.3^2 + 0.5^2)$  or  $a^2 + b^2 + c^2 \geq \frac{76^2}{0.38}$ . The quantity  $a^2 + b^2 + c^2$  is minimal when  $a^2 + b^2 + c^2 = \frac{76^2}{0.38}$ . This is the case when  $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.2k \\ 0.3k \\ 0.5k \end{bmatrix}$  for some positive constant  $k$ . It is required that  $0.2a + 0.3b + 0.5c = (0.2)^2k + (0.3)^2k + (0.5)^2k = 0.38k = 76$ , so that  $k = 200$ . Thus  $a = 40, b = 60, c = 100$ : The student must study 40 hours for the first exam, 60 hours for the second, and 100 hours for the third.

**5.1.37** Using Definition 2.2.2 as a guide, we find that  $\text{ref}_V \vec{x} = 2(\text{proj}_V \vec{x}) - \vec{x} = 2(\vec{u}_1 \cdot \vec{x})\vec{u}_1 + 2(\vec{u}_2 \cdot \vec{x})\vec{u}_2 - \vec{x}$ .

**5.1.38** Since  $\vec{v}_1$  and  $\vec{v}_2$  are unit vectors, the condition  $\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \|\vec{v}_2\| \cos(\alpha) = \cos(\alpha) = \frac{1}{2}$  implies that  $\vec{v}_1$  and  $\vec{v}_2$  enclose an angle of  $60^\circ$  ( $= \frac{\pi}{3}$ ). The vectors  $\vec{v}_1$  and  $\vec{v}_3$  enclose an angle of  $60^\circ$  as well.

In the case  $n = 2$  there are two possible scenarios: either  $\vec{v}_2 = \vec{v}_3$ , or  $\vec{v}_2$  and  $\vec{v}_3$  enclose an angle of  $120^\circ$ . Therefore, either  $\vec{v}_2 \cdot \vec{v}_3 = 1$  or  $\vec{v}_2 \cdot \vec{v}_3 = \cos(120^\circ) = -\frac{1}{2}$ .

In the case  $n = 3$ , the vectors  $\vec{v}_2$  and  $\vec{v}_3$  could enclose any angle between  $0^\circ$  (if  $\vec{v}_2 = \vec{v}_3$ ) and  $120^\circ$ , as illustrated in Figure 5.7. We have  $-\frac{1}{2} \leq \vec{v}_2 \cdot \vec{v}_3 \leq 1$ .

For example, consider  $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$ ,  $\vec{v}_3 = \begin{bmatrix} \left(\frac{\sqrt{3}}{2}\right) \cos \theta \\ \left(\frac{\sqrt{3}}{2}\right) \sin \theta \\ \frac{1}{2} \end{bmatrix}$

Note that  $\vec{v}_2 \cdot \vec{v}_3 = \left(\frac{3}{4}\right) \sin \theta + \frac{1}{4}$  could be anything between  $-\frac{1}{2}$  (when  $\sin \theta = -1$ ) and  $1$  (when  $\sin \theta = 1$ ), as claimed.

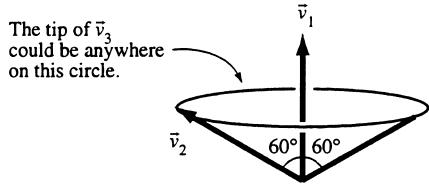


Figure 5.7: for Problem 5.1.38.

If  $n$  exceeds three, we can consider the orthogonal projection  $\vec{w}$  of  $\vec{v}_3$  onto the plane  $E$  spanned by  $\vec{v}_1$  and  $\vec{v}_2$ .

Since  $\text{proj}_{\vec{v}_1} \vec{w} = (\vec{v}_1 \cdot \vec{w})\vec{v}_1 = \frac{1}{2}\vec{v}_1$ , and since  $\|\vec{w}\| \leq \|\vec{v}_3\| = 1$ , (by Theorem 5.1.10), the tip of  $\vec{w}$  will be on the line segment in Figure 5.8. Note that the angle  $\phi$  enclosed by the vectors  $\vec{v}_2$  and  $\vec{w}$  is between  $0^\circ$  and  $120^\circ$ , so that  $\cos \phi$  is between  $-\frac{1}{2}$  and 1.

Therefore,  $\vec{v}_2 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{w} = \|\vec{w}\| \cos \phi$  is between  $-\frac{1}{2}$  and 1.

This implies that  $\angle(\vec{v}_2, \vec{v}_3)$  is between  $0^\circ$  and  $120^\circ$  as well. To see that all these values are attained, add  $(n - 3)$  zeros to the three vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^3$  given above.

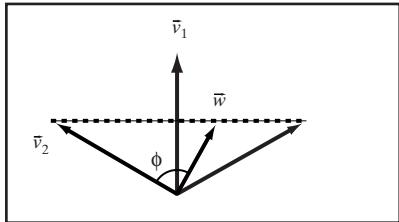


Figure 5.8: for Problem 5.1.38.

**5.1.39** No! By definition of a projection, the vector  $\vec{x} - \text{proj}_L \vec{x}$  is perpendicular to  $\text{proj}_L \vec{x}$ , so that  $(\vec{x} - \text{proj}_L \vec{x}) \cdot (\text{proj}_L \vec{x}) = \vec{x} \cdot \text{proj}_L \vec{x} - \|\text{proj}_L \vec{x}\|^2 = 0$  and  $\vec{x} \cdot \text{proj}_L \vec{x} = \|\text{proj}_L \vec{x}\|^2 \geq 0$ . (See Figure 5.9.)

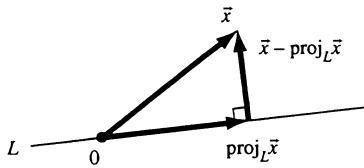


Figure 5.9: for Problem 5.1.39.

**5.1.40**  $\|\vec{v}_2\| = \sqrt{\vec{v}_2 \cdot \vec{v}_2} = \sqrt{a_{22}} = 3$ .

**5.1.41**  $\theta = \arccos\left(\frac{\vec{v}_2 \cdot \vec{v}_3}{\|\vec{v}_2\| \|\vec{v}_3\|}\right) = \arccos\left(\frac{a_{23}}{\sqrt{a_{22}} \sqrt{a_{33}}}\right) = \arccos\left(\frac{20}{21}\right) \approx 0.31$  radians.

**5.1.42**  $\|\vec{v}_1 + \vec{v}_2\| = \sqrt{(\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 + \vec{v}_2)} = \sqrt{a_{11} + 2a_{12} + a_{22}} = \sqrt{22}$ .

**5.1.43** Let  $\vec{u} = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{\vec{v}_2}{3}$ . Then,  $\vec{u}$  is an orthonormal basis for  $\text{span}(\vec{v}_2)$ . Using Theorem 5.1.5,  $\text{proj}_{\vec{v}_2}(\vec{v}_1) = (\vec{u} \cdot \vec{v}_1)\vec{u} = \left(\frac{\vec{v}_2}{3} \cdot \vec{v}_1\right)\frac{\vec{v}_2}{3} = \frac{1}{3}(\vec{v}_2 \cdot \vec{v}_1)\frac{\vec{v}_2}{3} = \frac{1}{3}(a_{12})\frac{\vec{v}_2}{3} = \frac{5}{9}\vec{v}_2$ .

**5.1.44** One method to solve this is to take  $\vec{v} = \vec{v}_2 - \text{proj}_{\vec{v}_3}\vec{v}_2 = \vec{v}_2 - \frac{20}{49}\vec{v}_3$ .

**5.1.45** Write the projection as a linear combination of  $\vec{v}_2$  and  $\vec{v}_3$ ,  $c_2\vec{v}_2 + c_3\vec{v}_3$ . Now you want  $\vec{v}_1 - c_2\vec{v}_2 - c_3\vec{v}_3$  to be perpendicular to  $V$ , that is, perpendicular to both  $\vec{v}_2$  and  $\vec{v}_3$ . Using dot products, this boils down to two linear equations in two unknowns,  $9c_2 + 20c_3 = 5$ , and  $20c_2 + 49c_3 = 11$ , with the solution  $c_2 = \frac{25}{41}$  and  $c_3 = -\frac{1}{41}$ . Thus the answer is  $\frac{25}{41}\vec{v}_2 - \frac{1}{41}\vec{v}_3$ .

**5.1.46** Write the projection as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ :  $c_1\vec{v}_1 + c_2\vec{v}_2$ . Now we want  $\vec{v}_3 - c_1\vec{v}_1 - c_2\vec{v}_2$  to be perpendicular to  $V$ , that is, perpendicular to both  $\vec{v}_1$  and  $\vec{v}_2$ . Using dot products, this boils down to two linear equations in two unknowns,  $11 = 3c_1 + 5c_2$  and  $20 = 5c_1 + 9c_2$ , with the solution  $c_1 = -\frac{1}{2}$ ,  $c_2 = \frac{5}{2}$ . Thus, the answer is  $-\frac{1}{2}\vec{v}_1 + \frac{5}{2}\vec{v}_2$ .

## Section 5.2

In Exercises 1–14, we will refer to the given vectors as  $\vec{v}_1, \dots, \vec{v}_m$ , where  $m = 1, 2$ , or  $3$ .

$$5.2.1 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$5.2.2 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{7} \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{7} \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

Note that  $\vec{u}_1 \cdot \vec{v}_2 = 0$ .

$$5.2.3 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{5} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$5.2.4 \quad \vec{u}_1 = \frac{1}{5} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \text{ and } \vec{u}_2 = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \text{ as in Exercise 3.}$$

Since  $\vec{v}_3$  is orthogonal to  $\vec{u}_1$  and  $\vec{u}_2$ ,  $\vec{u}_3 = \frac{1}{\|\vec{v}_3\|} \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ .

$$5.2.5 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

$$5.2.6 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{e}_1$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_2$$

$$\vec{u}_3 = \frac{\vec{v}_3^\perp}{\|\vec{v}_3^\perp\|} = \frac{\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2}{\|\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_3$$

$$5.2.7 \quad \text{Note that } \vec{v}_1 \text{ and } \vec{v}_2 \text{ are orthogonal, so that } \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ and } \vec{u}_2 = \frac{1}{\|\vec{v}_2\|} \vec{v}_2 = \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}. \quad \text{Then}$$

$$\vec{u}_3 = \frac{\vec{v}_3^\perp}{\|\vec{v}_3^\perp\|} = \frac{\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2}{\|\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2\|} = \frac{1}{\sqrt{36}} \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

$$5.2.8 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{7} \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2\|} \vec{v}_2 = \frac{1}{7} \begin{bmatrix} -2 \\ 2 \\ 5 \\ -4 \end{bmatrix}$$

$$5.2.9 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{10} \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}$$

$$5.2.10 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$5.2.11 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{5} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{225}} \begin{bmatrix} -3 \\ 2 \\ 14 \\ 4 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} -3 \\ 2 \\ 14 \\ 4 \end{bmatrix}$$

$$5.2.12 \quad \vec{u}_1 = \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{3} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$$5.2.13 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3^\perp}{\|\vec{v}_3^\perp\|} = \frac{\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2}{\|\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2\|} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$5.2.14 \quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{10} \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3^\perp}{\|\vec{v}_3^\perp\|} = \frac{\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2}{\|\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

In Exercises 15–28, we will use the results of Exercises 1–14 (note that Exercise  $k$ , where  $k = 1, \dots, 14$ , gives the  $QR$  factorization of the matrix in Exercise  $(k + 14)$ ). We can set  $Q = [\vec{u}_1 \dots \vec{u}_m]$ ; the entries of  $R$  are

$$\begin{aligned} r_{11} &= \|\vec{v}_1\| \\ r_{22} &= \|\vec{v}_2^\perp\| = \|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\| \\ r_{33} &= \|\vec{v}_3^\perp\| = \|\vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2\| \\ r_{ij} &= \vec{u}_i \cdot \vec{v}_j, \text{ where } i < j. \end{aligned}$$

$$5.2.15 \quad Q = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, R = [3]$$

$$5.2.16 \quad Q = \frac{1}{7} \begin{bmatrix} 6 & 2 \\ 5 & -6 \\ 2 & 3 \end{bmatrix}, R = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$5.2.17 \quad Q = \frac{1}{5} \begin{bmatrix} 4 & 3 \\ 0 & 0 \\ 5 & -4 \end{bmatrix}, R = \begin{bmatrix} 5 & 5 \\ 0 & 35 \end{bmatrix}$$

$$5.2.18 \quad Q = \frac{1}{5} \begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & -5 \\ 5 & -4 & 0 \end{bmatrix}, R = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$5.2.19 \quad Q = \frac{1}{3} \begin{bmatrix} 2 & -\frac{1}{\sqrt{2}} \\ 2 & -\frac{1}{\sqrt{2}} \\ 1 & \frac{4}{\sqrt{2}} \end{bmatrix}, R = 3 \begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$5.2.20 \quad Q = I_3, R = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3] = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$$

$$5.2.21 \quad Q = \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}, R = \begin{bmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{bmatrix}$$

$$5.2.22 \quad Q = \frac{1}{7} \begin{bmatrix} 5 & -2 \\ 4 & 2 \\ 2 & 5 \\ 2 & -4 \end{bmatrix}, R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$$

5.2.23  $Q = \begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 0.7 \\ 0.5 & -0.7 \\ 0.5 & 0.1 \end{bmatrix}, R = \begin{bmatrix} 2 & 4 \\ 0 & 10 \end{bmatrix}$

5.2.24  $Q = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, R = \begin{bmatrix} 2 & 10 \\ 0 & 2 \end{bmatrix}$

5.2.25  $Q = \frac{1}{15} \begin{bmatrix} 12 & -3 \\ 0 & 2 \\ 0 & 14 \\ 9 & 4 \end{bmatrix}, R = \begin{bmatrix} 5 & 10 \\ 0 & 15 \end{bmatrix}$

5.2.26  $Q = \begin{bmatrix} \frac{2}{7} & 0 \\ \frac{3}{7} & -\frac{2}{3} \\ 0 & \frac{2}{3} \\ \frac{6}{7} & \frac{1}{3} \end{bmatrix}, R = \begin{bmatrix} 7 & 14 \\ 0 & 3 \end{bmatrix}$

5.2.27  $Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

5.2.28  $Q = \begin{bmatrix} \frac{1}{10} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{10} & \frac{1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, R = \begin{bmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

5.2.29  $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \quad \vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1\|} = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix}. \text{ (See Figure 5.10.)}$

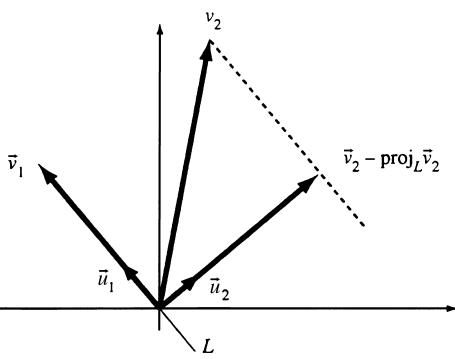


Figure 5.10: for Problem 5.2.29.

5.2.30 See Figure 5.11.

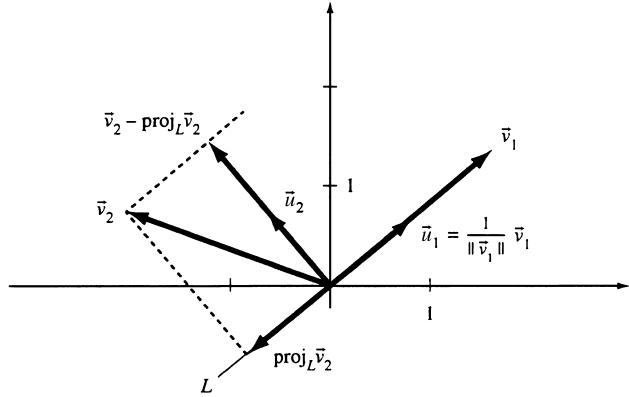


Figure 5.11: for Problem 5.2.30.

5.2.31  $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{e}_1$

$$\vec{v}_2^\perp = \vec{v}_2 - \text{proj}_{V_1} \vec{v}_2 = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix} - \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix}, \text{ so that } \vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_2$$

Here  $V_1 = \text{span}(\vec{e}_1) = x$  axis.

$$\vec{v}_3^\perp = \vec{v}_3 - \text{proj}_{V_2} \vec{v}_3 = \begin{bmatrix} d \\ e \\ f \end{bmatrix} - \begin{bmatrix} d \\ e \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}, \text{ so that } \vec{u}_3 = \frac{\vec{v}_3^\perp}{\|\vec{v}_3^\perp\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_3.$$

Here  $V_2 = \text{span}(\vec{e}_1, \vec{e}_2) = x-y$  plane. (See Figure 5.12.)

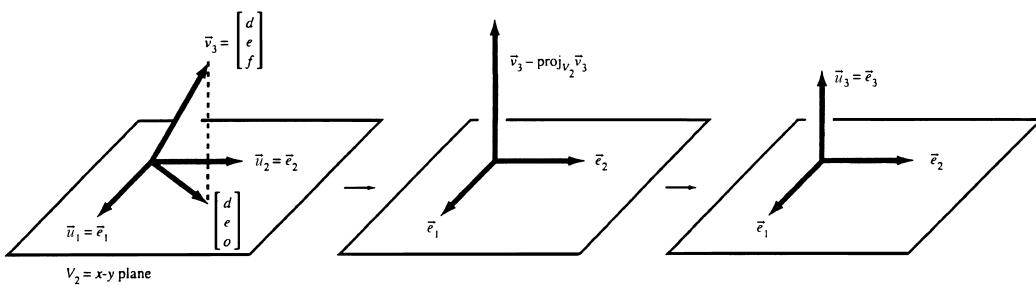


Figure 5.12: for Problem 5.2.31.

5.2.32 A basis of the plane is  $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ .

Now apply the Gram-Schmidt process.

$$\begin{aligned}\vec{u}_1 &= \frac{1}{\|\vec{v}_1\|} \vec{v}_1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ \vec{u}_2 &= \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\end{aligned}$$

Your solution may be different if you start with a different basis  $\vec{v}_1, \vec{v}_2$  of the plane.

$$5.2.33 \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\text{A basis of } \ker(A) \text{ is } \vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Since  $\vec{v}_1$  and  $\vec{v}_2$  are orthogonal already, we obtain  $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ,  $\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ .

$$5.2.34 \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\text{A basis of } \ker(A) \text{ is } \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

We apply the Gram-Schmidt process and obtain

$$\begin{aligned}\vec{u}_1 &= \frac{1}{\|\vec{v}_1\|} \vec{v}_1 &= \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \\ \vec{u}_2 &= \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1\|} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}\end{aligned}$$

$$5.2.35 \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

The non-redundant columns of  $A$  give us a basis of  $\text{im}(A)$ :

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Since  $\vec{v}_1$  and  $\vec{v}_2$  are orthogonal already, we obtain  $\vec{u}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ ,  $\vec{u}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ .

5.2.36 Write  $M = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$

$$\begin{array}{c} \uparrow \\ Q_0 \end{array} \quad \begin{array}{c} \uparrow \\ R_0 \end{array}$$

This is almost the  $QR$  factorization of  $M$ : the matrix  $Q_0$  has orthonormal columns and  $R_0$  is upper triangular; the only problem is the entry  $-4$  on the diagonal of  $R_0$ . Keeping in mind how matrices are multiplied, we can change all the signs in the second column of  $Q_0$  and in the second row of  $R_0$  to fix this problem:

$$M = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\begin{array}{c} \uparrow \\ Q \end{array} \quad \begin{array}{c} \uparrow \\ R \end{array}$$

5.2.37 Write  $M = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{array}{c} \uparrow \\ Q_0 \end{array} \quad \begin{array}{c} \uparrow \\ R_0 \end{array}$$

Note that the last two columns of  $Q_0$  and the last two rows of  $R_0$  have no effect on the product  $Q_0 R_0$ ; if we drop them, we have the  $QR$  factorization of  $M$ :

$$M = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$$

$$\begin{array}{c} \uparrow \\ Q \end{array} \quad \begin{array}{c} \uparrow \\ R \end{array}$$

5.2.38 Since  $\vec{v}_1 = 2\vec{e}_3$ ,  $\vec{v}_2 = -3\vec{e}_1$  and  $\vec{v}_3 = 4\vec{e}_4$  are orthogonal, we have

$$Q = \begin{bmatrix} \frac{\vec{v}_1}{\|\vec{v}_1\|} & \frac{\vec{v}_2}{\|\vec{v}_2\|} & \frac{\vec{v}_3}{\|\vec{v}_3\|} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } R = \begin{bmatrix} \|\vec{v}_1\| & 0 & 0 \\ 0 & \|\vec{v}_2\| & 0 \\ 0 & 0 & \|\vec{v}_3\| \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

5.2.39  $\vec{u}_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 = \vec{u}_1 \times \vec{u}_2 = \frac{1}{\sqrt{42}} \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix}$

5.2.40 If  $\vec{v}_1, \dots, \vec{v}_n$  are the columns of  $A$ , then  $Q = [\frac{\vec{v}_1}{\|\vec{v}_1\|} \cdots \frac{\vec{v}_n}{\|\vec{v}_n\|}]$  and  $R = \begin{bmatrix} \|\vec{v}_1\| & & 0 \\ & \ddots & \\ 0 & & \|\vec{v}_n\| \end{bmatrix}$ .

(See Exercise 38 as an example.)

5.2.41 If all diagonal entries of  $A$  are positive, then we have  $Q = I_n$  and  $R = A$ . A small modification is necessary if  $A$  has negative entries on the diagonal: if  $a_{ii} < 0$  we let  $r_{ij} = -a_{ij}$  for all  $j$ , and we let  $q_{ii} = -1$ ; if  $a_{ii} > 0$  we let  $r_{ij} = a_{ij}$  and  $q_{ii} = 1$ . Furthermore,  $q_{ij} = 0$  if  $i \neq j$  (that is,  $Q$  is diagonal).

For example,  $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ A & Q & R \end{array}$$

5.2.42 We have  $r_{11} = \|\vec{v}_1\|$  and  $r_{22} = \|\vec{v}_2^\perp\| = \|\vec{v}_2 - \text{proj}_L \vec{v}_2\|$ , so that  $r_{11}r_{22}$  is the area of the parallelogram defined by  $\vec{v}_1$  and  $\vec{v}_2$ . See Figure 5.13.

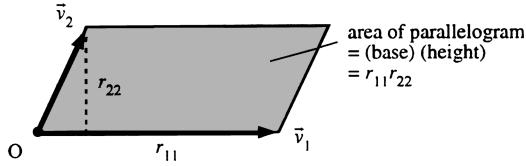


Figure 5.13: for Problem 5.2.42.

5.2.43 Partition the matrices  $Q$  and  $R$  in the  $QR$  factorization of  $A$  as follows:

$$[A_1 \ A_2] = A = QR = [Q_1 \ Q_2] \begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix} = [Q_1 R_1 \ Q_1 R_2 + Q_2 R_3],$$

where  $Q_1$  is  $n \times m_1$ ,  $Q_2$  is  $n \times m_2$ ,  $R_1$  is  $m_1 \times m_1$ , and  $R_3$  is  $m_2 \times m_2$ .

Then,  $A_1 = Q_1 R_1$  is the  $QR$  factorization of  $A_1$ : note that the columns of  $A_1$  are orthonormal, and  $R_1$  is upper triangular with positive diagonal entries.

5.2.44 No! If  $m$  exceeds  $n$ , then there is no  $n \times m$  matrix  $Q$  with orthonormal columns (if the columns of a matrix are orthonormal, then they are linearly independent).

5.2.45 Yes. Let  $A = [\vec{v}_1 \ \cdots \ \vec{v}_m]$ . The idea is to perform the Gram-Schmidt process in reversed order, starting with  $\vec{u}_m = \frac{1}{\|\vec{v}_m\|} \vec{v}_m$ .

Then we can express  $\vec{v}_j$  as a linear combination of  $\vec{u}_j, \dots, \vec{u}_m$ , so that  $[\vec{v}_1 \ \cdots \ \vec{v}_j \ \cdots \ \vec{v}_m] = [\vec{u}_1 \ \cdots \ \vec{u}_j \ \cdots \ \vec{u}_m] L$  for some lower triangular matrix  $L$ , with

$$\vec{v}_j = [\vec{u}_1 \quad \cdots \quad \vec{u}_j \quad \cdots \quad \vec{u}_m] \begin{bmatrix} l_{1j} \\ \vdots \\ l_{jj} \\ \vdots \\ l_{mj} \end{bmatrix} = l_{jj}\vec{u}_j + \cdots + l_{mj}\vec{u}_m.$$

## Section 5.3

**5.3.1** Not orthogonal, the column vectors fail to be perpendicular to each other.

**5.3.2** This matrix is orthogonal. Check that the column vectors are unit vectors, and that they are perpendicular to each other.

**5.3.3** This matrix is orthogonal. Check that the column vectors are unit vectors, and that they are perpendicular to each other.

**5.3.4** Not orthogonal, the first and third column vectors fail to be perpendicular to each other.

**5.3.5**  $3A$  will not be orthogonal, because the length of the column vectors will be 3 instead of 1, and they will fail to be unit vectors.

**5.3.6**  $-B$  will certainly be orthogonal, since the columns will be perpendicular unit vectors.

**5.3.7**  $AB$  is orthogonal by Theorem 5.3.4a.

**5.3.8**  $A + B$  will not necessarily be orthogonal, because the columns may not be unit vectors. For example, if  $A = B = I_n$ , then  $A + B = 2I_n$ , which is not orthogonal.

**5.3.9**  $B^{-1}$  is orthogonal by Theorem 5.3.4b.

**5.3.10** This matrix will be orthogonal, by Theorem 5.3.4.

**5.3.11**  $A^T$  is orthogonal.  $A^T = A^{-1}$ , by Theorem 5.3.7, and  $A^{-1}$  is orthogonal by Theorem 5.3.4b.

**5.3.13**  $3A$  is symmetric, since  $(3A)^T = 3A^T = 3A$ .

**5.3.14**  $-B$  is symmetric, since  $(-B)^T = -B^T = -B$ .

**5.3.15**  $AB$  is not necessarily symmetric, since  $(AB)^T = B^T A^T = BA$ , which is not necessarily the same as  $AB$ . (Here we used Theorem 5.3.9a.)

**5.3.16**  $A + B$  is symmetric, since  $(A + B)^T = A^T + B^T = A + B$ .

**5.3.17**  $B^{-1}$  is symmetric, because  $(B^{-1})^T = (B^T)^{-1} = B^{-1}$ . In the first step we have used 5.3.9b.

**5.3.18**  $A^{10}$  is symmetric, since  $(A^{10})^T = (A^T)^{10} = A^{10}$ .

**5.3.19** This matrix is symmetric. First note that  $(A^2)^T = (A^T)^2 = A^2$  for a symmetric matrix  $A$ . Now we can use the linearity of the transpose,  $(2I_n + 3A - 4A^2)^T = 2I_n^T + 3A^T - (4A^2)^T = 2I_n + 3A - 4(A^T)^2 = 2I_n + 3A - 4A^2$ .

**5.3.20**  $AB^2A$  is symmetric, since  $(AB^2A)^T = (ABBA)^T = (BA)^T(AB)^T = A^TB^TB^TA^T = AB^2A$ .

**5.3.21** Symmetric.  $(A^TA)^T = A^T(A^T)^T = A^TA$ .

**5.3.22**  $BB^T$  is symmetric:  $(BB^T)^T = (B^T)^T B^T = BB^T$ .

**5.3.23** Not necessarily symmetric.  $(A - A^T)^T = A^T - A = -(A - A^T)$ .

**5.3.24** Not necessarily symmetric.  $(A^TBA)^T = A^T(A^TB)^T = A^TB^TA$ .

**5.3.25** Symmetric, because  $(A^TB^TBA)^T = A^TB^T(B^T)^T(A^T)^T = A^TB^TBA$ .

**5.3.26** Symmetric, since  $(B(A + A^T)B^T)^T = ((A + A^T)B^T)^T B^T = B(A + A^T)^T B^T$   
 $= B(A^T + A)^T B^T = B((A^T)^T + A^T)B^T = B(A + A^T)B^T$ .

**5.3.27** Using Theorems 5.3.6 and 5.3.9a, we find that  $(A\vec{v}) \cdot \vec{w} = (A\vec{v})^T \vec{w} = \vec{v}^T A^T \vec{w} = \vec{v} \cdot (A^T \vec{w})$ , as claimed.

**5.3.28** Write  $L(\vec{x}) = A\vec{x}$ ; by Definition 5.3.1,  $A$  is an orthogonal  $n \times n$  matrix, so that  $A^TA = I_n$ , by Theorem 5.3.7. Now  $L(\vec{v}) \cdot L(\vec{w}) = (A\vec{v}) \cdot (A\vec{w}) = (A\vec{v})^T A\vec{w} = \vec{v}^T A^T A\vec{w} = \vec{v}^T I_n \vec{w} = \vec{v}^T \vec{w} = \vec{v} \cdot \vec{w}$ , as claimed. Note that we have used Theorems 5.3.6 and 5.3.9a.

**5.3.29** We will use the fact that  $L$  preserves length (by Definition 5.3.1) and the dot product (by Exercise 28):

$$\angle(L(\vec{v}), L(\vec{w})) = \arccos \frac{L(\vec{v}) \cdot L(\vec{w})}{\|L(\vec{v})\| \|L(\vec{w})\|} = \arccos \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} = \angle(\vec{v}, \vec{w}).$$

**5.3.30** If  $L(\vec{x}) = \vec{0}$ , then  $\|L(\vec{x})\| = \|\vec{x}\| = 0$ , so that  $\vec{x} = \vec{0}$ . Therefore,  $\ker(L) = \{\vec{0}\}$ .

By Theorem 3.3.7,  $\dim(\text{im}(L)) = m - \dim(\ker(L)) = m$ .

Since  $\mathbb{R}^n$  has an  $m$ -dimensional subspace (namely,  $\text{im}(L)$ ), the inequality  $m \leq n$  holds.

The transformation  $L$  preserves right angles (the proof of Theorem 5.3.2 applies), so that the columns of  $A$  are orthonormal (since they are  $L(\vec{e}_1), \dots, L(\vec{e}_m)$ ).

Therefore, we have  $A^TA = I_m$  (the proof of Theorem 5.3.7 applies).

Since the vectors  $\vec{v}_1, \dots, \vec{v}_m$  form an orthonormal basis of  $\text{im}(A)$ , the matrix  $AA^T$  represents the orthogonal projection onto  $\text{im}(A)$ , by Theorem 5.3.10.

A simple example of such a transformation is  $L(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$ , that is,  $L \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ .

**5.3.31** Yes! If  $A$  is orthogonal, then so is  $A^T$ , by Exercise 11. Since the columns of  $A^T$  are orthogonal, so are the rows of  $A$ .

**5.3.32** a No! As a counterexample, consider  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$  (see Exercise 30).

b Yes! More generally, if  $A$  and  $B$  are  $n \times n$  matrices such that  $BA = I_n$ , then  $AB = I_n$ , by Theorem 2.4.8c.

**5.3.33** Write  $A = [\vec{v}_1 \ \vec{v}_2]$ . The unit vector  $\vec{v}_1$  can be expressed as  $\vec{v} = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$ , for some  $\phi$ . Then  $\vec{v}_2$  will be one of the two unit vectors orthogonal to  $\vec{v}_1$ :  $\vec{v}_2 = \begin{bmatrix} -\sin(\phi) \\ \cos(\phi) \end{bmatrix}$  or  $\vec{v}_2 = \begin{bmatrix} \sin(\phi) \\ -\cos(\phi) \end{bmatrix}$ . (See Figure 5.7.)

Therefore, an orthogonal  $2 \times 2$  matrix is either of the form  $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$  or  $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ \sin(\phi) & -\cos(\phi) \end{bmatrix}$ , representing a rotation or a reflection. Compare with Exercise 2.2.24. See Figure 5.14.

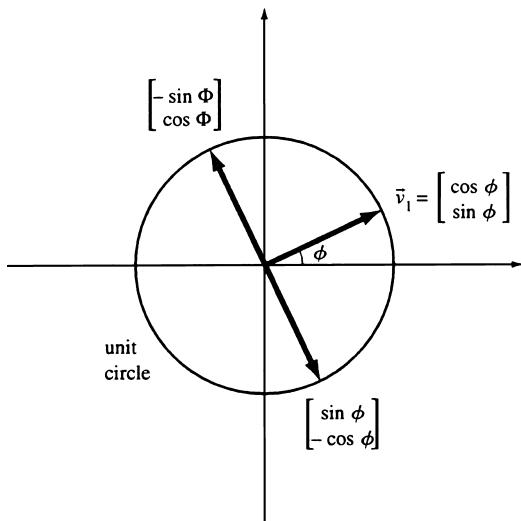


Figure 5.14: for Problem 5.3.33.

**5.3.34** Since the first two columns are orthogonal to the third, we have  $c = d = 0$ . Then  $\begin{bmatrix} a & b \\ e & f \end{bmatrix}$  is an orthogonal  $2 \times 2$  matrix; By Exercise 33, the  $3 \times 3$  matrix  $A$  is either of the form  $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ 0 & 0 & 1 \\ \sin(\phi) & \cos(\phi) & 0 \end{bmatrix}$  or  $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ 0 & 0 & 1 \\ \sin(\phi) & -\cos(\phi) & 0 \end{bmatrix}$ .

$$A = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ 0 & 0 & 1 \\ \sin(\phi) & -\cos(\phi) & 0 \end{bmatrix}.$$

**5.3.35** Let us first think about the inverse  $L = T^{-1}$  of  $T$ .

Write  $L(\vec{x}) = A\vec{x} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]\vec{x}$ . It is required that  $L(\vec{e}_3) = \vec{v}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ .

Furthermore, the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  must form an orthonormal basis of  $\mathbb{R}^3$ . By inspection, we find  $\vec{v}_1 = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ .

Then  $\vec{v}_2 = \vec{v}_1 \times \vec{v}_3 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{3}{3} \end{bmatrix}$  does the job. In summary, we have  $L(\vec{x}) = \frac{1}{3} \begin{bmatrix} -2 & -1 & 2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix} \vec{x}$ .

Since the matrix of  $L$  is orthogonal, the matrix of  $T = L^{-1}$  is the transpose of the matrix of  $L$ :

$$T(\vec{x}) = \frac{1}{3} \begin{bmatrix} -2 & 1 & 2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{bmatrix} \vec{x}.$$

There are many other answers (since there are many choices for the vector  $\vec{v}_1$  above).

**5.3.36** Let the third column be the cross product of the first two:  $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{3} & 0 & -\frac{4}{\sqrt{18}} \end{bmatrix}$ .

There is another solution, with the signs in the last column reversed.

**5.3.37** No, since the vectors  $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$  are orthogonal, whereas  $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$  are not (see Theorem 5.3.2).

**5.3.38 a** The general form of a skew-symmetric  $3 \times 3$  matrix is  $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ , with

$$A^2 = \begin{bmatrix} -a^2 - b^2 & -bc & ac \\ -bc & -a^2 - c^2 & -ab \\ ac & -ab & -b^2 - c^2 \end{bmatrix}, \text{ a symmetric matrix.}$$

b By Theorem 5.3.9.a,  $(A^2)^T = (A^T)^2 = (-A)^2 = A^2$ , so that  $A^2$  is symmetric.

**5.3.39** By Theorem 5.3.10, the matrix of the projection is  $\vec{u}\vec{u}^T$ ; the  $ij$ th entry of this matrix is  $u_i u_j$ .

**5.3.40** An orthonormal basis of  $W$  is  $\vec{u}_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$ ,  $\vec{u}_2 = \begin{bmatrix} -0.1 \\ 0.7 \\ -0.7 \\ 0.1 \end{bmatrix}$  (see Exercise 5.2.9).

By Theorem 5.3.10, the matrix of the projection onto  $W$  is  $QQ^T$ , where  $Q = [\vec{u}_1 \quad \vec{u}_2]$ .

$$QQ^T = \frac{1}{100} \begin{bmatrix} 26 & 18 & 32 & 24 \\ 18 & 74 & -24 & 32 \\ 32 & -24 & 74 & 18 \\ 24 & 32 & 18 & 26 \end{bmatrix}$$

**5.3.41** A unit vector on the line is  $\vec{u} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ .

The matrix of the orthogonal projection is  $\vec{u}\vec{u}^T$ , the  $n \times n$  matrix whose entries are all  $\frac{1}{n}$  (compare with Exercise 39).

**5.3.42 a** Suppose we are projecting onto a subspace  $W$  of  $\mathbb{R}^n$ . Since  $A\vec{x}$  is in  $W$  already, the orthogonal projection of  $A\vec{x}$  onto  $W$  is just  $A\vec{x}$  itself:  $A(A\vec{x}) = A\vec{x}$ , or  $A^2\vec{x} = A\vec{x}$ .

Since this equation holds for all  $\vec{x}$ , we have  $A^2 = A$ .

**b**  $A = QQ^T$ , for some matrix  $Q$  with orthonormal columns  $\vec{u}_1, \dots, \vec{u}_m$ . Note that  $Q^TQ = I_m$ , since the  $ij$ th entry of  $Q^TQ$  is  $\vec{u}_i \cdot \vec{u}_j$ . Then  $A^2 = QQ^TQQ^T = Q(Q^TQ)Q^T = QI_mQ^T = QQ^T = A$ .

**5.3.43** Examine how  $A$  acts on  $\vec{u}$ , and on a vector  $\vec{v}$  orthogonal to  $\vec{u}$ :

$$A\vec{u} = (2\vec{u}\vec{u}^T - I_3)\vec{u} = 2\vec{u}\vec{u}^T\vec{u} - \vec{u} = \vec{u}, \text{ since } \vec{u}^T\vec{u} = \vec{u} \cdot \vec{u} = \|\vec{u}\|^2 = 1.$$

$$A\vec{v} = (2\vec{u}\vec{u}^T - I_3)\vec{v} = 2\vec{u}\vec{u}^T\vec{v} - \vec{v} = -\vec{v}, \text{ since } \vec{u}^T\vec{v} = \vec{u} \cdot \vec{v} = 0.$$

Since  $A$  leaves the vectors in  $L = \text{span}(\vec{u})$  unchanged and reverses the vectors in  $V = L^\perp$ , it represents the *reflection about  $L$* .

Note that  $B = -A$ , so that  $B$  reverses the vectors in  $L$  and leaves the vectors in  $V$  unchanged; that is,  $B$  represents the reflection about  $V$ .

**5.3.44** Note that  $A^T$  is an  $m \times n$  matrix. By Theorems 3.3.7 and 5.3.9c we have

$$\dim(\ker(A^T)) = n - \text{rank}(A^T) = n - \text{rank}(A).$$

By Theorem 3.3.6,  $\dim(\text{im}(A)) = \text{rank}(A)$ , so that  $\dim(\text{im}(A)) + \dim(\ker(A^T)) = n$ .

**5.3.45** Note that  $A^T$  is an  $m \times n$  matrix. By Theorems 3.3.7 and 5.3.9c, we have

$$\dim(\ker(A)) = m - \text{rank}(A) \text{ and } \dim(\ker(A^T)) = n - \text{rank}(A^T) = n - \text{rank}(A),$$

so that  $\dim(\ker(A)) = \dim(\ker(A^T))$  if (and only if)  $A$  is a square matrix.

**5.3.46** By Theorem 5.2.2, the columns  $\vec{u}_1, \dots, \vec{u}_m$  of  $Q$  are orthonormal. Therefore,  $Q^TQ = I_m$ , since the  $ij$ th entry of  $Q^TQ$  is  $\vec{u}_i \cdot \vec{u}_j$ .

If we multiply the equation  $M = QR$  by  $Q^T$  from the left then  $Q^TM = Q^TQR = R$ , as claimed.

**5.3.47** By Theorem 5.2.2, the columns  $\vec{u}_1, \dots, \vec{u}_m$  of  $Q$  are orthonormal. Therefore,  $Q^TQ = I_m$ , since the  $ij$ th entry of  $Q^TQ$  is  $\vec{u}_i \cdot \vec{u}_j$ .

By Theorem 5.3.9a, we now have  $A^TA = (QR)^TQR = R^TQ^TQR = R^TR$ .

**5.3.48** As suggested, we consider the  $QR$  factorization

$$A^T = PR$$

of  $A^T$ , where  $P$  is orthogonal and  $R$  is upper triangular with positive diagonal entries.

By Theorem 5.3.9a,  $A = (PR)^T = R^T P^T$ .

Note that  $L = R^T$  is lower triangular and  $Q = P^T$  is orthogonal.

**5.3.49** Yes! By Exercise 5.2.45, we can write  $A^T = PL$ , where  $P$  is orthogonal and  $L$  is lower triangular.

By Theorem 5.3.9a,  $A = (PL)^T = L^T P^T$ .

Note that  $R = L^T$  is upper triangular, and  $Q = P^T$  is orthogonal (by Exercise 11).

**5.3.50 a** If an  $n \times n$  matrix  $A$  is orthogonal and upper triangular, then  $A^{-1}$  is both lower triangular (since  $A^{-1} = A^T$ ) and upper triangular (being the inverse of an upper triangular matrix; compare with Exercise 2.4.35c).

Therefore,  $A^{-1} = A^T$  is a diagonal matrix, and so is  $A$  itself. Since  $A$  is orthogonal with positive diagonal entries, all the diagonal entries must be 1, so that  $A = I_n$ .

**b** Using the terminology suggested in the hint, we observe that  $Q_2^{-1} Q_1$  is orthogonal (by Theorem 5.3.4) and  $R_2 R_1^{-1}$  is upper triangular with positive diagonal entries. By part a, the matrix  $Q_2^{-1} Q_1 = R_2 R_1^{-1}$  is  $I_n$ , so that  $Q_1 = Q_2$  and  $R_1 = R_2$ , as claimed.

**5.3.51 a** Using the terminology suggested in the hint, we observe that  $I_m = Q_1^T Q_1 = (Q_2 S)^T Q_2 S = S^T Q_2^T Q_2 S = S^T S$ , so that  $S$  is orthogonal, by Theorem 5.3.7.

**b** Using the terminology suggested in the hint, we observe that  $R_2 R_1^{-1}$  is both orthogonal (let  $S = R_2 R_1^{-1}$  in part a) and upper triangular, with positive diagonal entries. By Exercise 50a, we have  $R_2 R_1^{-1} = I_m$ , so that  $R_1 = R_2$ . Then  $Q_1 = Q_2 R_2 R_1^{-1} = Q_2$ , as claimed.

**5.3.52** Applying the strategy outlined in Summary 4.1.6 to the general element  $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$  of  $V$ , we find the basis  $\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right], \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right], \left[ \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right], \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \cdot \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right], \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right], \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]$ , so that  $\dim(V) = 6$ .

**5.3.53** Applying the strategy outlined in Summary 4.1.6 to the general element  $\begin{bmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{bmatrix}$  of  $V$ , we find the basis  $\left[ \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right], \left[ \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right], \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{array} \right]$ , so that  $\dim(V) = 3$ .

**5.3.54** To write the general form of a skew-symmetric  $n \times n$  matrix  $A$ , we can place arbitrary constants above the diagonal, the opposite entries below the diagonal ( $a_{ij} = -a_{ji}$ ), and zeros on the diagonal (since  $a_{ii} = -a_{ii}$ ). See Exercise 53 for the case  $n = 3$ . Thus the dimension of the space equals the number of entries above the diagonal of an  $n \times n$  matrix. In Exercise 55 we will see that there are  $(n^2 - n)/2$  such entries. Thus  $\dim(V) = (n^2 - n)/2$ .

**5.3.55** To write the general form of a symmetric  $n \times n$  matrix  $A$ , we can place arbitrary constants on and above the diagonal, and then write the corresponding entries below the diagonal ( $a_{ij} = a_{ji}$ ). See Exercise 52 for the

case  $n = 3$ . Thus the dimension of the space equals the number of entries on and above the diagonal of an  $n \times n$  matrix. Now there are  $n^2$  entries in the matrix,  $n^2 - n$  off the diagonal, and half of them,  $(n^2 - n)/2$ , above the diagonal. Since there are  $n$  entries on the diagonal, we have  $\dim(V) = (n^2 - n)/2 + n = (n^2 + n)/2$ .

**5.3.56** Yes and yes (see Exercise 57).

**5.3.57** Yes,  $L$  is linear, since  $L(A+B) = (A+B)^T = A^T + B^T = L(A) + L(B)$  and  $L(kA) = (kA)^T = kA^T = kL(A)$ . Yes,  $L$  is an isomorphism; the inverse is the transformation  $R(A) = A^T$  from  $\mathbb{R}^{n \times m}$  to  $\mathbb{R}^{m \times n}$ .

**5.3.58** Adapting the solution of Exercise 59, we see that the kernel consists of all skew-symmetric matrices, and the image consists of all symmetric matrices.

**5.3.59** The kernel consists of all matrixes  $A$  such that  $L(A) = \frac{1}{2}(A - A^T) = 0$ , that is,  $A^T = A$ ; those are the symmetric matrices.

Following the hint, let's apply  $L$  to a skew-symmetric matrix  $A$ , with  $A^T = -A$ . Then  $L(A) = (1/2)(A - A^T) = (1/2)2A = A$ , so that  $A$  is in the image of  $L$ . Conversely, if  $A$  is any  $2 \times 2$  matrix, then  $L(A)$  will be skew-symmetric, since  $(L(A))^T = (1/2)(A - A^T)^T = (1/2)(A^T - A) = -L(A)$ . In conclusion: The kernel of  $L$  consists of all symmetric matrices, and the image consists of all skew-symmetric matrices.

**5.3.60** Using Theorem 4.3.2, we find the matrix  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ .

**5.3.61** Note that the first three matrices of the given basis  $\mathcal{B}$  are symmetric, so that  $L(A) = A - A^T = 0$ , and the coordinate vector  $[L(A)]_{\mathcal{B}}$  is  $\vec{0}$  for all three of them. The last matrix of the basis is skew-symmetric, so that

$L(A) = 2A$ , and  $[L(A)]_{\mathcal{B}} = 2\vec{e}_4$ . Using Theorem 4.3.2, we find that the  $\mathcal{B}$ -matrix of  $L$  is  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ .

**5.3.62** By Theorem 5.3.9a,  $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

**5.3.63** By Exercise 2.4.94b, the given LDU factorization of  $A$  is unique.

By Theorem 5.3.9a,  $A = A^T = (LDU)^T = U^T D^T L^T = U^T D L^T$  is another way to write the LDU factorization of  $A$  (since  $U^T$  is lower triangular and  $L^T$  is upper triangular). By the uniqueness of the LDU factorization, we have  $U = L^T$  (and  $L = U^T$ ), as claimed.

**5.3.64 a**  $\begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix} + \begin{bmatrix} C & -D^T \\ D & C^T \end{bmatrix} = \begin{bmatrix} A+C & -B^T - D^T \\ B+D & A^T + C^T \end{bmatrix}$   
 $= \begin{bmatrix} A+C & -(B+D)^T \\ B+D & (A+C)^T \end{bmatrix}$  is of the required form.

**b**  $k \begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix} = \begin{bmatrix} kA & -kB^T \\ kB & kA^T \end{bmatrix} = \begin{bmatrix} kA & -(kB)^T \\ kB & (kA)^T \end{bmatrix}$  is of the required form.

c The general element of  $H$  is  $M = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$ , with four arbitrary constants,  $r,s,p$ , and  $q$ . Thus  $\dim(H) = 4$ ; use the strategy outlined in Summary 4.1.6 to construct a basis.

d  $\begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix} \begin{bmatrix} C & -D^T \\ D & C^T \end{bmatrix} = \begin{bmatrix} AC - B^T D & -AD^T - B^T C^T \\ BC + A^T D & -BD^T + A^T C^T \end{bmatrix} = \begin{bmatrix} AC - B^T D & -(BC + A^T D)^T \\ BC + A^T D & (AC - B^T D)^T \end{bmatrix}$  is of the required form.

Note that  $A, B, C, D$ , and their transposes are rotation-dilation matrices, so that they all commute.

e  $\begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix}^T = \begin{bmatrix} A^T & B^T \\ -B & (A^T)^T \end{bmatrix}$  is of the required form.

f Note that the columns  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  or  $M$  are orthogonal, and they all have length  $\sqrt{p^2 + q^2 + r^2 + s^2}$ . Now  $M^T M$  is the  $4 \times 4$  matrix whose  $ij$ th entry is  $\vec{v}_i \cdot \vec{v}_j$ , so that  $M^T M = (p^2 + q^2 + r^2 + s^2)I_4$ .

g If  $M \neq 0$ , then  $k = p^2 + q^2 + r^2 + s^2 > 0$ , and  $(\frac{1}{k}M^T)M = I_4$ , so that  $M$  is invertible, with

$$M^{-1} = \frac{1}{p^2+q^2+r^2+s^2} M^T.$$

By parts b and e,  $M^{-1}$  is in  $H$  as well.

h No!  $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$  do not commute ( $AB = -BA$ ).

**5.3.65** Write  $10A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ; it is required that  $a, b, c$  and  $d$  be integers. Now  $A = \begin{bmatrix} \frac{a}{10} & \frac{b}{10} \\ \frac{c}{10} & \frac{d}{10} \end{bmatrix}$  must be an orthogonal matrix, implying that  $(\frac{a}{10})^2 + (\frac{c}{10})^2 = 1$ , or  $a^2 + c^2 = 100$ . Checking the squares of all integers from 1 to 9, we see that there are only two ways to write 100 as a sum of two positive perfect squares:  $100 = 36 + 64 = 64 + 36$ . Since  $a$  and  $c$  are required to be positive, we have either  $a = 6$  and  $c = 8$  or  $a = 8$  and  $c = 6$ . In each case we have two options for the second column of  $A$ , namely, the two unit vectors perpendicular to the first column vector. Thus we end up with four solutions:

$$A = \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}, \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}, \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix} \text{ or } \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}.$$

**5.3.66** One approach is to take one of the solutions from Exercise 65, say, the rotation matrix  $B = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ , and then let  $A = B^2 = \begin{bmatrix} 0.28 & -0.96 \\ 0.96 & 0.28 \end{bmatrix}$ . Matrix  $A$  is orthogonal by Theorems 5.3.4a.

**5.3.67 a** We need to show that  $A^T A \vec{c} = A^T \vec{x}$ , or, equivalently, that  $A^T(\vec{x} - A \vec{c}) = \vec{0}$ . But  $A^T(\vec{x} - A \vec{c}) = A^T(\vec{x} - c_1 \vec{v}_1 - \cdots - c_m \vec{v}_m)$  is the vector whose  $i^{th}$  component is

$$(\vec{v}_i)^T(\vec{x} - c_1 \vec{v}_1 - \cdots - c_m \vec{v}_m) = \vec{v}_i \cdot (\vec{x} - c_1 \vec{v}_1 - \cdots - c_m \vec{v}_m), \text{ which we know to be zero.}$$

- b The system  $A^T A \vec{c} = A^T \vec{x}$  has a unique solution  $\vec{c}$  for a given  $\vec{x}$ , since  $\vec{c}$  is the coordinate vector of  $\text{proj}_V \vec{x}$  with respect to the basis  $\vec{v}_1, \dots, \vec{v}_m$ . Thus the coefficient matrix  $A^T A$  must be invertible, so that we can solve for  $\vec{c}$  and write  $\vec{c} = (A^T A)^{-1} A^T \vec{x}$ . Then  $\text{proj}_V \vec{x} = c_1 \vec{v}_1 + \dots + c_m \vec{v}_m = A \vec{c} = A(A^T A)^{-1} A^T \vec{x}$ .

**5.3.68** If  $A = QR$ , then  $A(A^T A)^{-1} A^T = QR(R^T Q^T QR)^{-1} R^T Q^T = QR(R^T R)^{-1} R^T Q^T = QRR^{-1}(R^T)^{-1} R^T Q^T = QQ^T$ , as in Theorem 5.3.10. The equation  $Q^T Q = I_m$  holds since the columns of  $Q$  are orthonormal.

## Section 5.4

- 5.4.1** A basis of  $\ker(A^T)$  is  $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . (See Figure 5.15.)

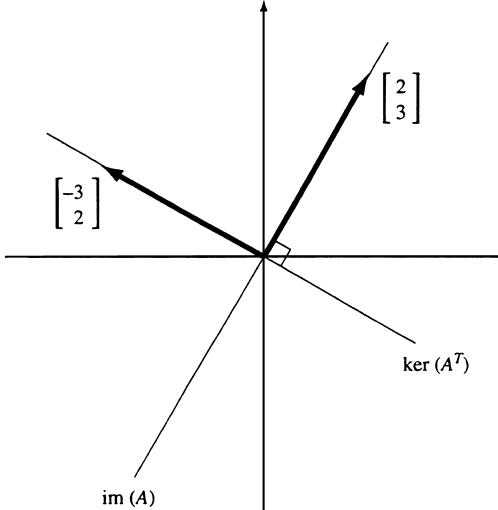


Figure 5.15: for Problem 5.4.1.

- 5.4.2** A basis of  $\ker(A^T)$  is  $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .  $\text{im}(A)$  is the plane perpendicular to this line.

**5.4.3** We will first show that the vectors  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  span  $\mathbb{R}^n$ . Any vector  $\vec{v}$  in  $\mathbb{R}^n$  can be written as  $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$ , where  $\vec{v}_{\parallel}$  is in  $V$  and  $\vec{v}_{\perp}$  is in  $V^{\perp}$  (by definition of orthogonal projection, Theorem 5.1.4).

Now  $\vec{v}_{\parallel}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_p$ , and  $\vec{v}_{\perp}$  is a linear combination of  $\vec{w}_1, \dots, \vec{w}_q$ , showing that the vectors  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  span  $\mathbb{R}^n$ .

Note that  $p + q = n$ , by Theorem 5.1.8c; therefore, the vectors  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  form a basis of  $\mathbb{R}^n$ , by Theorem 3.3.4d.

- 5.4.4** By Theorem 5.4.1, the equation  $(\text{im } B)^{\perp} = \ker(B^T)$  holds for any matrix  $B$ . Now let  $B = A^T$ . Then  $(\text{im}(A^T))^{\perp} = \ker(A)$ . Taking transposes of both sides and using Theorem 5.1.8d we obtain  $\text{im}(A^T) = (\ker A)^{\perp}$ , as claimed.

5.4.5  $V = \ker(A)$ , where  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 \end{bmatrix}$ .

Then  $V^\perp = (\ker A)^\perp = \text{im}(A^T)$ , by Exercise 4.

The two columns of  $A^T$  form a basis of  $V^\perp$ :

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \end{bmatrix}$$

5.4.6 Yes! For any matrix  $A$ ,

$$\begin{array}{cccccc} \text{im}(A) & = & (\ker(A^T))^\perp & = & (\ker(AA^T))^\perp & = & (\ker((AA^T)^T))^\perp & = & \text{im}(AA^T). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \end{array}$$

Theorem 5.4.1

Theorem 5.4.2a

Theorems 5.4.1 and 5.1.8d.

5.4.7  $\text{im}(A)$  and  $\ker(A)$  are orthogonal complements by Theorem 5.4.1:

$$(\text{im}A)^\perp = \ker(A^T) = \ker(A)$$

5.4.8 a By Theorem 5.4.6,  $L^+(\vec{y}) = (A^T A)^{-1} A^T \vec{y}$ .

The transformation  $L^+$  is linear since it is “given by a matrix,” by Definition 2.1.1.

b If  $L$  (and therefore  $A$ ) is invertible, then  $L^+(\vec{y}) = A^{-1}(A^T)^{-1} A^T \vec{y} = A^{-1} \vec{y} = L^{-1} \vec{y}$ , so that  $L^+ = L^{-1}$ .

c  $L^+(L(\vec{x})) = (\text{the unique least-squares solution } \vec{u} \text{ of } L(\vec{u}) = L(\vec{x})) = \vec{x}$ .

d  $L(L^+(\vec{y})) = A(A^T A)^{-1} A^T \vec{y} = \text{proj}_V \vec{y}$ , where  $V = \text{im}(A)$ , by Theorem 5.4.7.

e Here  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Then  $L^+(\vec{y}) = (A^T A)^{-1} A^T \vec{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vec{y}$ .

5.4.9  $\vec{x}_0$  is the shortest of all the vectors in  $S$ . (See Figure 5.16.)

5.4.10 a If  $\vec{x}$  is an arbitrary solution of the system  $A\vec{x} = \vec{b}$ , let  $\vec{x}_h = \text{proj}_V \vec{x}$ , where  $V = \ker(A)$ , and  $\vec{x}_0 = \vec{x} - \text{proj}_V \vec{x}$ . Note that  $\vec{b} = A\vec{x} = A(\vec{x}_h + \vec{x}_0) = A\vec{x}_h + A\vec{x}_0 = A\vec{x}_0$ , since  $\vec{x}_h$  is in  $\ker(A)$ .

b If  $\vec{x}_0$  and  $\vec{x}_1$  are two solutions of the system  $A\vec{x} = \vec{b}$ , both from  $(\ker A)^\perp$ , then  $\vec{x}_1 - \vec{x}_0$  is in the subspace  $(\ker A)^\perp$  as well. Also,  $A(\vec{x}_1 - \vec{x}_0) = A\vec{x}_1 - A\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}$ , so that  $\vec{x}_1 - \vec{x}_0$  is in  $\ker(A)$ . By Theorem 5.1.8b, it follows that  $\vec{x}_1 - \vec{x}_0 = \vec{0}$ , or  $\vec{x}_1 = \vec{x}_0$ , as claimed.

c Write  $\vec{x}_1 = \vec{x}_h + \vec{x}_0$  as in part a; note that  $\vec{x}_h$  is orthogonal to  $\vec{x}_0$ . The claim now follows from the Pythagorean Theorem (Theorem 5.1.9).

5.4.11 a Note that  $L^+(\vec{y}) = A^T(AA^T)^{-1}\vec{y}$ ; indeed, this vector is in  $\text{im}(A^T) = (\ker A)^\perp$ , and it is a solution of  $L(\vec{x}) = A\vec{x} = \vec{y}$ .

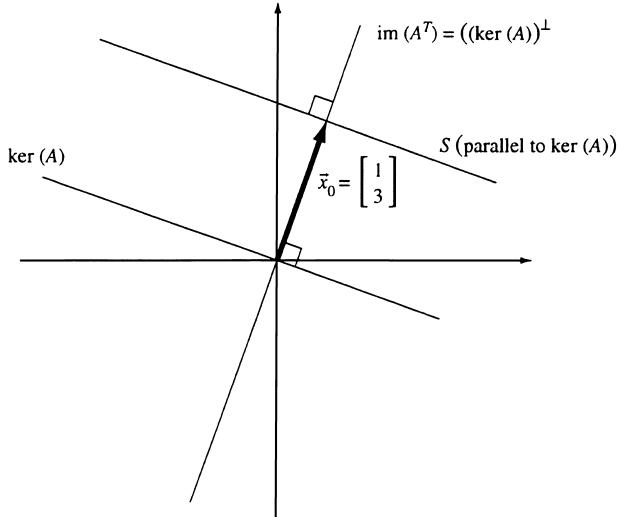


Figure 5.16: for Problem 5.4.9.

b  $L(L^+(\vec{y})) = \vec{y}$ , by definition of  $L^+$ .

c  $L^+(L(\vec{x})) = A^T(AA^T)^{-1}A\vec{x} = \text{proj}_V \vec{x}$ , where  $V = \text{im}(A^T) = (\ker A)^\perp$ , by Theorem 5.4.7.

d  $\text{im}(L^+) = \text{im}(A^T)$ , by part c, and  $\ker(L^+) = \{\vec{0}\}$  (if  $\vec{y}$  is in  $\ker(L^+)$ , then  $\vec{y} = L(L^+(\vec{y})) = L(\vec{0}) = \vec{0}$ , by part b).

e Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ; then the matrix of  $L^+$  is  $A^T(AA^T)^{-1} = A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

**5.4.12** By Theorem 5.4.5, the least-squares solutions of the linear system  $A\vec{x} = \vec{b}$  are the exact solutions of the (consistent) system  $A^T A\vec{x} = A^T \vec{b}$ . The minimal solution of this normal equation (in the sense of Exercise 10) is called the minimal least-squares solution of the system  $A\vec{x} = \vec{b}$ .

Equivalently, the minimal least-squares solution of  $A\vec{x} = \vec{b}$  can be defined as the minimal solution of the consistent system  $A\vec{x} = \text{proj}_V \vec{b}$ , where  $V = \text{im}(A)$ .

**5.4.13** a Suppose that  $L^+(\vec{y}_1) = \vec{x}_1$  and  $L^+(\vec{y}_2) = \vec{x}_2$ ; this means that  $\vec{x}_1$  and  $\vec{x}_2$  are both in  $(\ker A)^\perp = \text{im}(A^T)$ ,  $A^T A\vec{x}_1 = A^T \vec{y}_1$ , and  $A^T A\vec{x}_2 = A^T \vec{y}_2$ . Then  $\vec{x}_1 + \vec{x}_2$  is in  $\text{im}(A^T)$  as well, and  $A^T A(\vec{x}_1 + \vec{x}_2) = A^T(\vec{y}_1 + \vec{y}_2)$ , so that  $L^+(\vec{y}_1 + \vec{y}_2) = \vec{x}_1 + \vec{x}_2$ .

The verification of the property  $L^+(k\vec{y}) = kL^+(\vec{y})$  is analogous.

b  $L^+(L(\vec{x}))$  is the orthogonal projection of  $\vec{x}$  onto  $(\ker A)^\perp = \text{im}(A^T)$ .

c  $L(L^+(\vec{y}))$  is the orthogonal projection of  $\vec{y}$  onto  $\text{im}(A) = (\ker A^T)^\perp$ .

d  $\text{im}(L^+) = \text{im}(A^T)$  and  $\ker(L^+) = \ker(A^T)$ , by parts b and c.

e  $L^+ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{y_1}{2} \\ 0 \\ 0 \end{bmatrix}$ , so that the matrix of  $L^+$  is  $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ .

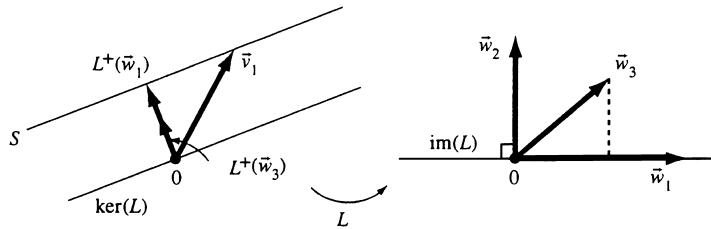


Figure 5.17: for Problem 5.4.14.

**5.4.14**  $L^+(\vec{w}_1)$  is the minimal solution of the system  $L(\vec{x}) = \vec{w}_1$ . The line  $S$  in Figure 5.17 shows all solutions of the system

$L(\vec{x}) = \vec{w}_1$  (compare with Exercise 9). The minimal solution,  $L^+(\vec{w}_1)$ , is perpendicular to  $\ker(L)$ .

$$L^+(\vec{w}_2) = L^+(\text{proj}_{\text{im}(L)} \vec{w}_2) = L^+(\vec{0}) = \vec{0}$$

$$L^+(\vec{w}_3) = L^+(\text{proj}_{\text{im}(L)} \vec{w}_3) \approx L^+(0.55\vec{w}_1) = 0.55L^+(\vec{w}_1)$$

**5.4.15** Note that  $(A^T A)^{-1} A^T A = I_n$ ; let  $B = (A^T A)^{-1} A^T$ .

**5.4.16** If  $A$  is an  $m \times n$  matrix, then

$$\dim(\text{im}A)^\perp = m - \dim(\text{im}A) = m - \text{rank}(A)$$

↑                              ↑

Theorem 5.1.8c

Theorem 3.3.6

$$\text{and } \dim(\ker(A^T)) = m - \text{rank}(A^T).$$

↑

Theorem 3.3.7

It follows that  $\text{rank}(A) = \text{rank}(A^T)$ , as claimed.

**5.4.17** Yes! By Theorem 5.4.2,  $\ker(A) = \ker(A^T A)$ . Taking dimensions of both sides and using Theorem 3.3.7, we find that  $n - \text{rank}(A) = n - \text{rank}(A^T A)$ ; the claim follows.

**5.4.18** Yes! By Exercise 17,  $\text{rank}(A) = \text{rank}(A^T A)$ . Substituting  $A^T$  for  $A$  in Exercise 17 and using Theorem 5.3.9c, we find that  $\text{rank}(A) = \text{rank}(A^T) = \text{rank}(AA^T)$ . The claim follows.

**5.4.19**  $\vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , by Theorem 5.4.6.

**5.4.20** Using Theorem 5.4.6, we find  $\vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$  and  $\vec{b} - A\vec{x}^* = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ .

Note that  $\vec{b} - A\vec{x}^*$  is perpendicular to the two columns of  $A$ .

**5.4.21** Using Theorem 5.4.6, we find  $\vec{x}^* = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$  and  $\vec{b} - A\vec{x}^* = \begin{bmatrix} -12 \\ 36 \\ -18 \end{bmatrix}$ , so that  $\|\vec{b} - A\vec{x}^*\| = 42$ .

**5.4.22** Using Theorem 5.4.6, we find  $\vec{x}^* = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$  and  $\vec{b} - A\vec{x}^* = \vec{0}$ . This system is in fact consistent and  $\vec{x}^*$  is the exact solution; the error  $\|\vec{b} - A\vec{x}^*\|$  is 0.

**5.4.23** Using Theorem 5.4.6, we find  $\vec{x}^* = \vec{0}$ ; here  $\vec{b}$  is perpendicular to  $\text{im}(A)$ .

**5.4.24** Using Theorem 5.4.6, we find  $\vec{x}^* = [2]$ .

**5.4.25** In this case, the normal equation  $A^T A \vec{x} = A^T \vec{b}$  is  $\begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$ , which simplifies to  $x_1 + 3x_2 = 1$ , or  $x_1 = 1 - 3x_2$ . The solutions are of the form  $\vec{x}^* = \begin{bmatrix} 1 - 3t \\ t \end{bmatrix}$ , where  $t$  is an arbitrary constant.

**5.4.26** Here, the normal equation  $A^T A \vec{x} = A^T \vec{b}$  is  $\begin{bmatrix} 66 & 78 & 90 \\ 78 & 93 & 108 \\ 90 & 108 & 126 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , with solutions  $\vec{x}^* = \begin{bmatrix} t - \frac{7}{6} \\ 1 - 2t \\ t \end{bmatrix}$ , where  $t$  is an arbitrary constant.

**5.4.27** The least-squares solutions of the system  $SA\vec{x} = S\vec{b}$  are the exact solutions of the normal equation  $(SA)^T SA\vec{x} = (SA)^T S\vec{b}$ .

Note that  $S^T S = I_n$ , since  $S$  is orthogonal; therefore, the normal equation simplifies as follows:  $(SA)^T SA\vec{x} = A^T S^T SA\vec{x} = A^T A\vec{x}$  and  $(SA)^T S\vec{b} = A^T S^T S\vec{b} = A^T \vec{b}$ , so that the normal equation is  $A^T A\vec{x} = A^T \vec{b}$ , the same as the normal equation of the system  $A\vec{x} = \vec{b}$ . Therefore, the systems  $A\vec{x} = \vec{b}$  and  $SA\vec{x} = S\vec{b}$  have the same least-squares solution,  $\vec{x}^* = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ .

**5.4.28** The least-squares solutions of the system  $A\vec{x} = \vec{u}_n$  are the exact solutions of  $A\vec{x} = \text{proj}_{\text{im}(A)} \vec{u}_n$ . Note that  $\vec{u}_n$  is orthogonal to  $\text{im}(A)$ , so that  $\text{proj}_{\text{im}(A)} \vec{u}_n = \vec{0}$ , and the unique least-squares solution is  $\vec{x}^* = \vec{0}$ .

**5.4.29** By Theorem 5.4.6,  $\vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 1 + \varepsilon & 1 \\ 1 & 1 + \varepsilon \end{bmatrix}^{-1} \begin{bmatrix} 1 + \varepsilon \\ 1 + \varepsilon \end{bmatrix} = \frac{1}{2+\varepsilon} \begin{bmatrix} 1 + \varepsilon \\ 1 + \varepsilon \end{bmatrix} \approx \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ , where  $\varepsilon = 10^{-20}$ .

If we use a hand-held calculator, due to roundoff errors we find the normal equation  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , with infinitely many solutions.

**5.4.30** We attempt to solve the system

$$\begin{array}{l} c_0 + 0c_1 = 0 \\ c_0 + 0c_1 = 1, \text{ or} \\ c_0 + 1c_1 = 1 \end{array} \quad \left[ \begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{array} \right] \left[ \begin{array}{c} c_0 \\ c_1 \end{array} \right] = \left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right].$$

This system cannot be solved exactly; the least-squares solution is  $\left[ \begin{array}{c} c_0^* \\ c_1^* \end{array} \right] = \left[ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right]$ . The line that fits the data points best is  $f^*(t) = \frac{1}{2} + \frac{1}{2}t$ .

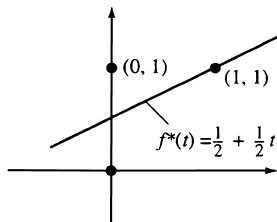


Figure 5.18: for Problem 5.4.30.

The line goes through the point  $(1, 1)$  and “splits the difference” between  $(0, 0)$  and  $(0, 1)$ . See Figure 5.18.

**5.4.31** We want  $\left[ \begin{array}{c} c_0 \\ c_1 \end{array} \right]$  such that

$$\begin{array}{l} 3 = c_0 + 0c_1 \\ 3 = c_0 + 1c_1 \text{ or} \\ 6 = c_0 + 1c_1 \end{array} \quad \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{array} \right] \left[ \begin{array}{c} c_0 \\ c_1 \end{array} \right] = \left[ \begin{array}{c} 3 \\ 3 \\ 6 \end{array} \right].$$

Since  $\ker \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{array} \right] = \{\vec{0}\}$ ,  $\left[ \begin{array}{c} c_0 \\ c_1 \end{array} \right]^* = \left( \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] \right)^{-1} \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right] \left[ \begin{array}{c} 3 \\ 3 \\ 6 \end{array} \right]$   
 $= \left[ \begin{array}{cc} 3 & 2 \\ 2 & 2 \end{array} \right]^{-1} \left[ \begin{array}{c} 12 \\ 9 \end{array} \right] = \left[ \begin{array}{c} 3 \\ \frac{3}{2} \end{array} \right]$  so  $f^*(t) = 3 + \frac{3}{2}t$ . (See Figure 5.19.)

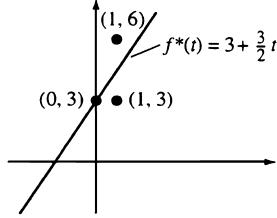


Figure 5.19: for Problem 5.4.31.

**5.4.32** We want  $\left[ \begin{array}{c} c_0 \\ c_1 \\ c_2 \end{array} \right]$  of  $f(t) = c_0 + c_1t + c_2t^2$  such that

$$\begin{aligned} 27 &= c_0 + 0c_1 + 0c_2 \\ 0 &= c_0 + 1c_1 + 1c_2 \text{ or} \\ 0 &= c_0 + 2c_1 + 4c_2 \\ 0 &= c_0 + 3c_1 + 9c_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If we call the coefficient matrix  $A$ , we notice that  $\ker(A) = \{\vec{0}\}$  so

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}^* = (A^T A)^{-1} A^T \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25.65 \\ -28.35 \\ 6.75 \end{bmatrix} \text{ so } f^*(t) = 25.65 - 28.35t + 6.75t^2.$$

**5.4.33** We want  $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$  such that

$$\begin{aligned} 0 &= c_0 + \sin(0)c_1 + \cos(0)c_2 \\ 1 &= c_0 + \sin(1)c_1 + \cos(1)c_2 \text{ or} \\ 2 &= c_0 + \sin(2)c_1 + \cos(2)c_2 \\ 3 &= c_0 + \sin(3)c_1 + \cos(3)c_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & \sin(1) & \cos(1) \\ 1 & \sin(2) & \cos(2) \\ 1 & \sin(3) & \cos(3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Since the coefficient matrix has kernel  $\{\vec{0}\}$ , we compute  $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}^*$  using Theorem 5.4.6, obtaining

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}^* \approx \begin{bmatrix} 1.5 \\ 0.1 \\ -1.41 \end{bmatrix} \text{ so } f^*(t) \approx 1.5 + 0.1 \sin t - 1.41 \cos t.$$

**5.4.34** We want  $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$  such that

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & \sin(0.5) & \cos(0.5) & \sin(1) & \cos(1) \\ 1 & \sin(1) & \cos(1) & \sin(2) & \cos(2) \\ 1 & \sin(1.5) & \cos(1.5) & \sin(3) & \cos(3) \\ 1 & \sin(2) & \cos(2) & \sin(4) & \cos(4) \\ 1 & \sin(2.5) & \cos(2.5) & \sin(5) & \cos(5) \\ 1 & \sin(3) & \cos(3) & \sin(6) & \cos(6) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 1 \\ 1.5 \\ 2 \\ 2.5 \\ 3 \end{bmatrix}$$

Since the columns of the coefficient matrix are linearly independent, its kernel is  $\{\vec{0}\}$ . We can use Theorem 5.4.6

to compute  $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}^* \approx \begin{bmatrix} 1.5 \\ 0.109 \\ -1.537 \\ 0.303 \\ 0.043 \end{bmatrix}$  so  $f^*(t) \approx 1.5 + 0.109 \sin(t) - 1.537 \cos(t) + 0.303 \sin(2t) + 0.043 \cos(2t)$ .

**5.4.35 a** The  $ij$ th entry of  $A_n^T A_n$  is the dot product of the  $i$ th row of  $A_n^T$  and the  $j$ th column of  $A_n$ , i.e.

$$A_n^T A_n = \begin{bmatrix} n & \sum_{i=1}^n \sin a_i & \sum_{i=1}^n \cos a_i \\ \sum_{i=1}^n \sin a_i & \sum_{i=1}^n \sin^2 a_i & \sum_{i=1}^n \sin a_i \cos a_i \\ \sum_{i=1}^n \cos a_i & \sum_{i=1}^n \sin a_i \cos a_i & \sum_{i=1}^n \cos^2 a_i \end{bmatrix} \text{ and } A_n^T \vec{b} = \begin{bmatrix} \sum_{i=1}^n g(a_i) \\ \sum_{i=1}^n g(a_i) \sin a_i \\ \sum_{i=1}^n g(a_i) \cos a_i \end{bmatrix}.$$

$$\text{b) } \lim_{n \rightarrow \infty} \frac{2\pi}{n} A_n^T A_n = \begin{bmatrix} 2\pi & \int_0^{2\pi} \sin t dt & \int_0^{2\pi} \cos t dt \\ \int_0^{2\pi} \sin t dt & \int_0^{2\pi} \sin^2 t dt & \int_0^{2\pi} \sin t \cos t dt \\ \int_0^{2\pi} \cos t dt & \int_0^{2\pi} \sin t \cos t dt & \int_0^{2\pi} \cos^2 t dt \end{bmatrix} = \begin{bmatrix} 2\pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{bmatrix}$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{2\pi}{n} A_n^T \vec{b} = \begin{bmatrix} \int_0^{2\pi} g(t) dt \\ \int_0^{2\pi} g(t) \sin t dt \\ \int_0^{2\pi} g(t) \cos t dt \end{bmatrix}$$

(Here  $\frac{2\pi}{n} = \Delta t$  so  $\lim_{n \rightarrow \infty} \frac{2\pi}{n} \sum_{i=1}^n \cos(t_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \cos(t_i) \Delta t = \int_0^{2\pi} \cos t dt$  for instance.)

$$c) \begin{bmatrix} c \\ p \\ q \end{bmatrix} = \lim_{n \rightarrow \infty} \begin{bmatrix} c_n \\ p_n \\ q_n \end{bmatrix} = \begin{bmatrix} 2\pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{bmatrix}^{-1} \begin{bmatrix} \int_0^{2\pi} g(t) dt \\ \int_0^{2\pi} g(t) \sin t dt \\ \int_0^{2\pi} g(t) \cos t dt \end{bmatrix} = \begin{bmatrix} \frac{1}{2\pi} \int_0^{2\pi} g(t) dt \\ \frac{1}{\pi} \int_0^{2\pi} g(t) \sin t dt \\ \frac{1}{\pi} \int_0^{2\pi} g(t) \cos t dt \end{bmatrix} \text{ and } f(t) = c + p \sin t + q \cos t,$$

where  $c, p, q$  are given above.

**5.4.36** We want  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  such that

$$\begin{aligned} a + b \sin\left(\frac{2\pi}{366} 32\right) + c \cos\left(\frac{2\pi}{366} 32\right) &= 10 \\ a + b \sin\left(\frac{2\pi}{366} 77\right) + c \cos\left(\frac{2\pi}{366} 77\right) &= 12 \\ a + b \sin\left(\frac{2\pi}{366} 121\right) + c \cos\left(\frac{2\pi}{366} 121\right) &= 14 \end{aligned}$$

$$a + b \sin\left(\frac{2\pi}{366} 152\right) + c \cos\left(\frac{2\pi}{366} 152\right) = 15$$

Using  $A = \begin{bmatrix} 1 & \sin\left(\frac{2\pi}{366} 32\right) & \cos\left(\frac{2\pi}{366} 32\right) \\ 1 & \sin\left(\frac{2\pi}{366} 77\right) & \cos\left(\frac{2\pi}{366} 77\right) \\ 1 & \sin\left(\frac{2\pi}{366} 121\right) & \cos\left(\frac{2\pi}{366} 121\right) \\ 1 & \sin\left(\frac{2\pi}{366} 152\right) & \cos\left(\frac{2\pi}{366} 152\right) \end{bmatrix}$  and  $\vec{b} = \begin{bmatrix} 10 \\ 12 \\ 14 \\ 15 \end{bmatrix}$ , we compute  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}^*$

$$= (A^T A)^{-1} A^T \vec{b} \approx \begin{bmatrix} 12.26 \\ 0.431 \\ -2.899 \end{bmatrix} \text{ and } f^*(t) \approx 12.26 + 0.431 \sin\left(\frac{2\pi}{366} t\right) - 2.899 \cos\left(\frac{2\pi}{366} t\right).$$

**5.4.37 a** We want  $c_0, c_1$  such that

$$\begin{array}{l} c_0 + c_1(35) = \log 35 \\ c_0 + c_1(46) = \log 46 \\ c_0 + c_1(59) = \log 77 \\ c_0 + c_1(69) = \log 133 \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & 35 \\ 1 & 46 \\ 1 & 59 \\ 1 & 69 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \log 35 \\ \log 46 \\ \log 77 \\ \log 133 \end{bmatrix}$$

$$\begin{array}{c} \uparrow \\ A \end{array} \quad \begin{array}{c} \uparrow \\ \vec{b} \end{array}$$

$$\text{so } \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}^* = (A^T A)^{-1} A^T \vec{b} \approx \begin{bmatrix} 0.915 \\ 0.017 \end{bmatrix} \text{ so } \log(d) \approx 0.915 + 0.017t.$$

$$\text{b } d \approx 10^{0.915} \cdot 10^{0.017t} \approx 8.22 \cdot 10^{0.017t}$$

c If  $t = 88$  then  $d \approx 258$ . Since the Airbus has only 93 displays, new technologies must have rendered the old trends obsolete.

**5.4.38** We want  $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$  such that

$$\begin{array}{l} 110 = c_0 + 2c_1 + c_2 \\ 180 = c_0 + 12c_1 + 0c_2 \\ 120 = c_0 + 5c_1 + c_2 \\ 160 = c_0 + 11c_1 + c_2 \\ 160 = c_0 + 6c_1 + 0c_2 \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 12 & 0 \\ 1 & 5 & 1 \\ 1 & 11 & 1 \\ 1 & 6 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 110 \\ 180 \\ 120 \\ 160 \\ 160 \end{bmatrix}.$$

The least-squares solution is  $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}^* = \begin{bmatrix} 125 \\ 5 \\ -25 \end{bmatrix}$ , so that  $w^* = 125 + 5h - 25g$ .

For a general population, we expect  $c_0$  and  $c_1$  to be positive, since  $c_0$  gives the weight of a 5' male, and increased height should contribute positively to the weight. We expect  $c_2$  to be negative, since females tend to be lighter than males of equal height.

**5.4.39 a** We want  $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$  such that

$$\log(250) = c_0 + c_1 \log(600,000)$$

$$\log(60) = c_0 + c_1 \log(200,000)$$

$$\log(25) = c_0 + c_1 \log(60,000)$$

$$\log(12) = c_0 + c_1 \log(10,000)$$

$$\log(5) = c_0 + c_1 \log(2500)$$

The least-squares solution to the above system is  $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}^* \approx \begin{bmatrix} -1.616 \\ 0.664 \end{bmatrix}$  so  $\log z \approx -1.616 + 0.664 \log g$ .

b Exponentiating both sides of the answer to a, we get  $z \approx 10^{-1.616} \cdot g^{0.664} \approx 0.0242 \cdot g^{0.664}$ .

c This model is close since  $\sqrt{g} = g^{0.5}$ .

**5.4.40** First we look for  $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$  such that  $\log D = c_0 + c_1 \log a$ .

Proceeding as in Exercise 39, we get  $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}^* \approx \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$ , i.e.  $\log D \approx 1.5 \log a$ , hence  $D \approx 10^{1.5 \log a} = a^{1.5}$ .

Note that the formula  $D = a^{1.5}$  is Kepler's third law of planetary motion.

**5.4.41 a** We want  $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$  such that  $\log D = c_0 + c_1 t$  ( $t$  in years since 1975), i.e.

$$\begin{aligned} \log 533 &= c_0 + c_1(0) \\ \log 1,823 &= c_0 + c_1(10) \\ \log 4,974 &= c_0 + c_1(20) \\ \log 7,933 &= c_0 + c_1(30) \end{aligned}$$

The least-squares solution to the system is  $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}^* \approx \begin{bmatrix} 2.8 \\ 0.040 \end{bmatrix}$ , i.e.  $\log D \approx 2.8 + 0.04t$  or  $D \approx 10^{2.8} \cdot 10^{0.040t}$ .

b In the year 2015, we have  $t = 40$  and  $D \approx 24,200$ . The formula predicts a debt of about US\$24 trillion.

**5.4.42** Clearly,  $L$  is a linear transformation. We will use Theorem 4.2.4a and show that  $\ker(L) = \{\vec{0}\}$  and  $\text{im}(L) = \text{im}(A)$ . Now  $\ker(L) = \ker(A) \cap \text{im}(A^T) = \{\vec{0}\}$ , by Theorems 5.4.1 and 5.1.8b. Also,  $\text{im}(L) = \{A\vec{v} : \vec{v} \text{ in } \text{im}(A^T)\} = \text{im}(AA^T) = \text{im}(A)$ , by Exercise 6.

## Section 5.5

**5.5.1** Since  $f$  is nonzero, there is a  $c$  on  $[a, b]$  such that  $f(c) = d \neq 0$ . By continuity, there is an  $\varepsilon > 0$  such that  $|f(x)| > \frac{|d|}{2}$  for all  $x$  on the open interval  $(c - \varepsilon, c + \varepsilon)$  where  $f$  is defined (see any good Calculus text).

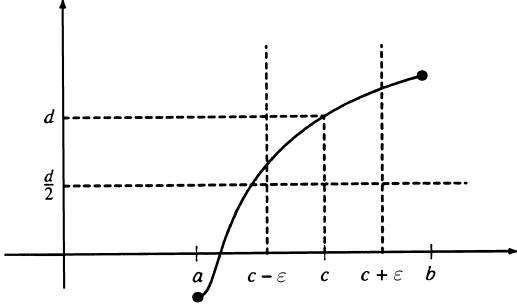


Figure 5.20: for Problem 5.5.1.

We assume that  $c - \varepsilon \geq a$  and  $c + \varepsilon \leq b$  and leave the other cases to the reader. Then  $\langle f, f \rangle = \int_a^b (f(t))^2 dt \geq \int_{c-\varepsilon}^{c+\varepsilon} \left(\frac{d}{2}\right)^2 dt = \frac{d^2 \varepsilon}{2} > 0$ , as claimed. (See Figure 5.20.)

**5.5.2** We perform the following operations:

$$\begin{array}{ccccccccc} \langle f, g+h \rangle & = & \langle g+h, f \rangle & = & \langle g, f \rangle + \langle h, f \rangle & = & \langle f, g \rangle + \langle f, h \rangle \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{Definition} & & \text{Definition} & & \text{Definition} & & \\ 5.5.1a & & 5.5.1b & & 5.5.1a & & \end{array}$$

**5.5.3 a** Note that  $\langle \vec{x}, \vec{y} \rangle = (S\vec{x})^T S\vec{y} = S\vec{x} \cdot S\vec{y}$ . We will check the four parts of Definition 5.5.1

- α.  $\langle \vec{x}, \vec{y} \rangle = S\vec{x} \cdot S\vec{y} = S\vec{y} \cdot S\vec{x} = \langle \vec{y}, \vec{x} \rangle$
- β.  $\langle \vec{x} + \vec{y}, \vec{z} \rangle = S(\vec{x} + \vec{y}) \cdot S\vec{z} = (S\vec{x} + S\vec{y}) \cdot S\vec{z} = (S\vec{x} \cdot S\vec{z}) + (S\vec{y} \cdot S\vec{z}) = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$
- γ.  $\langle c\vec{x}, \vec{y} \rangle = S(c\vec{x}) \cdot S\vec{y} = c(S\vec{x}) \cdot S\vec{y} = c\langle \vec{x}, \vec{y} \rangle$
- δ. If  $\vec{x} \neq 0$ , then  $\langle \vec{x}, \vec{x} \rangle = S\vec{x} \cdot S\vec{x} = \|S\vec{x}\|^2$  is positive if  $S\vec{x} \neq \vec{0}$ , that is, if  $\vec{x}$  is not in the kernel of  $S$ . It is required that  $S\vec{x} \neq \vec{0}$  whenever  $\vec{x} \neq \vec{0}$ , that is,  $\ker(S) = \{\vec{0}\}$ .

*Answer:*  $S$  must be invertible.

b It is required that  $\langle \vec{x}, \vec{y} \rangle = (S\vec{x})^T S\vec{y} = \vec{x}^T S^T S\vec{y}$  equal  $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$  for all  $\vec{x}$  and  $\vec{y}$ . This is the case if and only if  $S^T S = I_n$ , that is,  $S$  is orthogonal.

**5.5.4 a** For column vectors  $\vec{v}, \vec{w}$ , we have  $\langle \vec{v}, \vec{w} \rangle = \text{trace}(\vec{v}^T \vec{w}) = \text{trace}(\vec{v} \cdot \vec{w}) = \vec{v} \cdot \vec{w}$ , the dot product.

b For row vectors  $\vec{v}, \vec{w}$ , the  $ij$ th entry of  $\vec{v}^T \vec{w}$  is  $v_i w_j$ , so that  $\langle \vec{v}, \vec{w} \rangle = \text{trace}(\vec{v}^T \vec{w}) = \sum_{i=1}^m v_i w_i = \vec{v} \cdot \vec{w}$ , again the dot product.

**5.5.5 a**  $\langle \langle A, B \rangle \rangle = \text{tr}(AB^T) = \text{tr}((AB^T)^T) = \text{tr}(BA^T) = \langle \langle B, A \rangle \rangle$  In the second step we have used the fact that  $\text{tr}(M) = \text{tr}(M^T)$ , for any square matrix  $M$ .

$$\begin{aligned} \text{b } \langle\langle A + B, C \rangle\rangle &= \text{tr}((A + B)C^T) = \text{tr}(AC^T + BC^T) = \text{tr}(AC^T) + \text{tr}(BC^T) \\ &= \langle\langle A, C \rangle\rangle + \langle\langle B, C \rangle\rangle \end{aligned}$$

$$\text{c } \langle\langle cA, B \rangle\rangle = \text{tr}(cAB^T) = c\text{tr}(AB^T) = c\langle\langle A, B \rangle\rangle$$

$$\text{d } \langle\langle A, A \rangle\rangle = \text{tr}(AA^T) = \sum_{i=1}^n \|\vec{v}_i\|^2 > 0 \text{ if } A \neq 0, \text{ where } \vec{v}_i \text{ is the } i\text{th row of } A.$$

We have shown that  $\langle\langle \cdot, \cdot \rangle\rangle$  does indeed define an inner product.

**5.5.6 a** The  $i$ th entry of  $PQ$  is  $\sum_{k=1}^m p_{ik}q_{ki}$ , so that  $\text{tr}(PQ) = \sum_{i=1}^n \sum_{k=1}^m p_{ik}q_{ki}$ .

$$\text{Likewise, } \text{tr}(QP) = \sum_{i=1}^m \sum_{k=1}^n q_{ik}p_{ki}.$$

Reversing the roles of  $i$  and  $k$  and the order of summation we see that  $\text{tr}(PQ) = \text{tr}(QP)$ , as claimed.

**b** Using part a and the fact that  $\text{tr}(M) = \text{tr}(M^T)$  for any square matrix  $M$ , we find that  $\langle A, B \rangle = \text{tr}(A^T B) = \text{tr}(BA^T) = \text{tr}((BA^T)^T) = \text{tr}(AB^T) = \langle\langle A, B \rangle\rangle$

**5.5.7** Axioms *a*, *b*, and *c* hold for any choice of  $k$  (check this!). Also, it is required that  $\langle\langle v, v \rangle\rangle = k\langle v, v \rangle$  be positive for nonzero  $v$ . Since  $\langle v, v \rangle$  is positive, this is the case if (and only if)  $k$  is *positive*.

**5.5.8** By parts b and c of Definition 5.5.1, we have  $T(u + v) = \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = T(u) + T(v)$  and  $T(cv) = \langle cv, w \rangle = c\langle v, w \rangle = cT(v)$ , so that  $T$  is linear. If  $w = 0$ , then  $\text{im}(T) = \{0\}$  and  $\ker(T) = V$ . If  $w \neq 0$ , then  $\text{im}(T) = \mathbb{R}$  and  $\ker(T)$  consists of all  $v$  perpendicular to  $w$ .

**5.5.9** If  $f$  is even and  $g$  is odd, then  $fg$  is odd, so that  $\langle f, g \rangle = \int_{-1}^1 fg = 0$ .

**5.5.10** A function  $g(t) = a + bt + ct^2$  is orthogonal to  $f(t) = t$  if

$$\langle f, g \rangle = \int_{-1}^1 (at + bt^2 + ct^3) dt = [\frac{a}{2}t^2 + \frac{b}{3}t^3 + \frac{c}{4}t^4]_{-1}^1 = \frac{2}{3}b = 0, \text{ that is, if } b = 0.$$

Thus, the functions 1 and  $t^2$  form a basis of the space of all functions in  $P_2$  orthogonal to  $f(t) = t$ . To find an *orthonormal* basis  $g_1(t), g_2(t)$ , we apply Gram-Schmidt. Now  $\|1\| = \frac{1}{2} \int_{-1}^1 1 dt = 1$ , so that we can let  $g_1(t) = 1$ .

$$\text{Then } g_2(t) = \frac{t^2 - \langle 1, t^2 \rangle 1}{\|t^2 - \langle 1, t^2 \rangle 1\|} = \frac{t^2 - \frac{1}{3}}{\|t^2 - \frac{1}{3}\|} = \frac{\sqrt{5}}{2}(3t^2 - 1)$$

$$\text{Answer: } g_1(t) = 1, g_2(t) = \frac{\sqrt{5}}{2}(3t^2 - 1)$$

**5.5.11**  $\langle f, g \rangle = \langle \cos(t), \cos(t + \delta) \rangle = \langle \cos(t), \cos(t)\cos(\delta) - \sin(t)\sin(\delta) \rangle = \cos(\delta)\langle \cos(t), \cos(t) \rangle - \sin(\delta)\langle \cos(t), \sin(t) \rangle = \cos(\delta)$ , by Theorem 5.5.4.

Also,  $\langle g, g \rangle = 1$  (left to reader).

Thus,  $\angle(f, g) = \arccos\left(\frac{\langle f, g \rangle}{\|f\|\|g\|}\right) = \arccos(\cos \delta) = \delta$ .

**5.5.12** By Theorem 5.5.5

$$a_0 = \left\langle |t|, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} |t| dt = \frac{\pi}{\sqrt{2}}, b_k = \langle |t|, \sin(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \sin(kt) dt = 0, \text{ since the integrand is an odd function.}$$

$$\begin{aligned} c_k &= \langle |t|, \cos(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \cos(kt) dt = \frac{2}{\pi} \int_0^{\pi} t \cos(kt) dt \\ &= \frac{2}{\pi} \left\{ \left[ \frac{1}{k} t \sin(kt) \right]_0^\pi - \int_0^\pi \frac{1}{k} \sin(kt) dt \right\} \\ &= \frac{2}{\pi} \frac{1}{k^2} [\cos(kt)]_0^\pi = \frac{2}{\pi k^2} (\cos(k\pi) - 1) = \begin{cases} 0 & \text{if } k \text{ is even} \\ -\frac{4}{k^2\pi} & \text{if } k \text{ is odd} \end{cases} \end{aligned}$$

In summary:

$$\begin{aligned} a_0 &= 0 \\ b_k &= 0 \end{aligned}$$

$$c_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ -\frac{4}{k^2\pi} & \text{if } k \text{ is odd} \end{cases}$$

**5.5.13** The sequence  $(a_0, b_1, c_1, b_2, c_2, \dots)$  is “square-summable” by Theorem 5.5.6, so that it is in  $\ell_2$ . Also,  $\|(a_0, b_1, c_1, b_2, c_2, \dots)\| = \sqrt{a_0^2 + b_1^2 + c_1^2 + b_2^2 + c_2^2 + \dots} = \|f\|^2$ , by Theorem 5.5.6, so that the two norms are equal.

**5.5.14 a** This is not an inner product since there are nonzero polynomials  $f(t)$  in  $P_2$  with  $f(1) = f(2) = 0$ , so that  $\langle f, f \rangle = (f(1))^2 + (f(2))^2 = 0$ . (For example, let  $f(t) = (t-1)(t-2)$ .)

b This is an inner product. We leave it to the reader to check axioms a to c. As for d: A nonzero polynomial  $f$  in  $P_2$  has at most two zeros, so that  $f(1) \neq 0$  or  $f(2) \neq 0$  or  $f(3) \neq 0$ , and  $\langle f, f \rangle = (f(1))^2 + (f(2))^2 + (f(3))^2 > 0$ .

**5.5.15** First note that  $b = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = c$ , by part a of Definition 5.5.1, so that  $b = c$ . Check that if  $b = c$  then  $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$  for all  $\vec{v}, \vec{w}$  in  $\mathbb{R}^2$ . Check that parts (b) and (c) of Definition 5.5.1 are satisfied for all values of constants  $c, d$ . Next, we need to worry about part (d) of that definition. It is required that  $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = x_1^2 + 2bx_1x_2 + dx_2^2$  be positive for all nonzero  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ . We can complete the square and write  $x_1^2 + 2bx_1x_2 + dx_2^2 = (x_1 + bx_2)^2 + (d - b^2)x_2^2$ ; this quantity is positive for all nonzero  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  if (and only if)  $d - b^2 > 0$ , or,  $d > b^2$ . In summary, the function is an inner product if (and only if)  $b = c$  and  $d > b^2$ .

**5.5.16 a** We start with the standard basis  $1, t$  and use the Gram-Schmidt process to construct an *orthonormal* basis  $g_1(t), g_2(t)$ .

$$\|1\| = \sqrt{\int_0^1 dt} = 1, \text{ so that we can let } g_1(t) = 1. \text{ Then } g_2(t) = \frac{t - \langle 1, t \rangle 1}{\|t - \langle 1, t \rangle 1\|} = \frac{t - \frac{1}{2}}{\|t - \frac{1}{2}\|} = \sqrt{3}(2t - 1).$$

*Summary:*  $g_1(t) = 1$  and  $g_2(t) = \sqrt{3}(2t - 1)$  is an orthonormal basis.

b We are looking for  $\text{proj}_{P_1}(t^2) = \langle g_1(t), t^2 \rangle g_1(t) + \langle g_2(t), t^2 \rangle g_2(t)$ , by Theorem 5.5.3.

We find that  $\langle g_1(t), t^2 \rangle = \int_0^1 t^2 dt = \frac{1}{3}$  and  $\langle g_2(t), t^2 \rangle = \sqrt{3} \int_0^1 (2t^3 - t^2) dt = \frac{\sqrt{3}}{6}$ , so that  $\text{proj}_{P_1} t^2 = \frac{1}{3} + \frac{1}{2}(2t - 1) = t - \frac{1}{6}$ . See Figure 5.21.

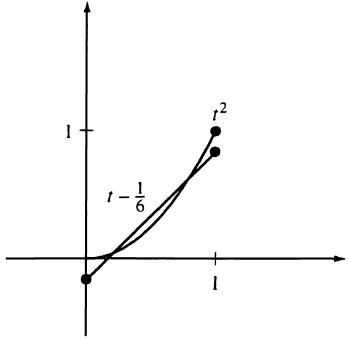


Figure 5.21: for Problem 5.5.16b.

**5.5.17** We leave it to the reader to check that the first three axioms are satisfied for any such  $T$ . As for axiom d: It is required that  $\langle v, v \rangle = T(v) \cdot T(v) = \|T(v)\|^2$  be positive for any nonzero  $v$ , that is,  $T(v) \neq \vec{0}$ . This means that the *kernel of  $T$  must be  $\{0\}$* .

**5.5.18** Let the orthonormal basis be  $f_1, \dots, f_n$  and  $f = c_1 f_1 + \dots + c_n f_n$ ; then  $[f]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ . The Pythagorean

Theorem tells us that  $\|f\|^2 = c_1^2 + \dots + c_n^2 = \|[f]_{\mathcal{B}}\|^2$ , so that  $\|f\| = \|[f]_{\mathcal{B}}\|$ .

**5.5.19** If we write  $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ , and  $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ , then  $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle$

$= [x_1 x_2] \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = px_1y_1 + qx_1y_2 + rx_2y_1 + sx_2y_2$ . Note that in Exercise 15 we considered the special case  $p = 1$ . First it is required that  $p = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$  be positive.

Now we can write  $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = p \left[ x_1y_1 + \frac{q}{p}x_1y_2 + \frac{r}{p}x_2y_1 + \frac{s}{p}x_2y_2 \right]$  and use our work in Exercise 15 (with  $b = \frac{q}{p}$ ,  $c = \frac{r}{p}$ ) to see that the conditions  $q = r$  and  $q^2 < ps$  must hold. In summary, the function is an inner product if (and only if) the entries of matrix  $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$  satisfy the conditions  $p > 0$ ,  $q = r$  and  $\det(A) = ps - q^2 > 0$ .

**5.5.20** a  $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = [x_1 x_2] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1 + 2x_2 = 0$  when  $x_1 = -2x_2$ . This is the line spanned by vector  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ .

b Since vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$  are orthogonal, we merely have to multiply each of them with the recipro-

cal of its norm. Now  $\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|^2 = [1 \ 0] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$ , so that  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is a unit vector, and  $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\|^2 = [-2 \ 1] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 4$ , so that  $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\| = 2$ . Thus  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$  is an orthonormal basis.

**5.5.21** Note that  $\langle \vec{v}, \vec{w} \rangle = (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) - \vec{v} \cdot \vec{v} - \vec{w} \cdot \vec{w} = 2(\vec{v} \cdot \vec{w})$ , double the dot product. Thus it's an inner product, by Exercise 7.

**5.5.22** Apply the Cauchy-Schwarz inequality to  $f(t)$  and  $g(t) = 1$ ; note that  $\|g\| = 1$ :

$$|\langle f, g \rangle| \leq \|f\| \|g\| = \|f\| \text{ or } \langle f, g \rangle^2 \leq \|f\|^2 \text{ or } \left( \int_0^1 f(t) dt \right)^2 \leq \int_0^1 (f(t))^2 dt.$$

**5.5.23** We start with the standard basis  $1, t$  of  $P_1$  and use the Gram-Schmidt process to construct an *orthonormal basis*  $g_1(t), g_2(t)$ .

$$\|1\| = \sqrt{\frac{1}{2}(1 \cdot 1) + 1 \cdot 1} = 1, \text{ so that we can let } g_1(t) = 1. \text{ Then } g_2(t) = \frac{t - \langle 1, t \rangle 1}{\|t - \langle 1, t \rangle 1\|} = \frac{t - \frac{1}{2}}{\|t - \frac{1}{2}\|} = 2t - 1.$$

Summary:  $g_1(t) = 1$  and  $g_2(t) = 2t - 1$  is an orthonormal basis.

**5.5.24 a**  $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle = 0 + 8 = 8$

$$\text{b } \|g + h\| = \sqrt{\langle g + h, g + h \rangle} = \sqrt{\langle g, g \rangle + 2\langle g, h \rangle + \langle h, h \rangle} = \sqrt{1 + 6 + 50} = \sqrt{57}$$

c Since  $\langle f, g \rangle = 0$ ,  $\|g\| = 1$ , and  $\|f\| = 2$ , we know that  $\frac{f}{2}, g$  is an orthonormal basis of  $\text{span}(f, g)$ .

$$\text{Now } \text{proj}_E h = \left\langle \frac{f}{2}, h \right\rangle \frac{f}{2} + \langle g, h \rangle g = \frac{1}{4} \langle f, h \rangle f + \langle g, h \rangle g = 2f + 3g.$$

d From part c we know that  $\frac{1}{2}f, g$  are orthonormal, so we apply Theorem 5.2.1 to obtain the third polynomial in an orthonormal basis of  $\text{span}(f, g, h)$ :

$$\frac{h - \text{proj}_E h}{\|h - \text{proj}_E h\|} = \frac{h - 2f - 3g}{\|h - 2f - 3g\|} = \frac{h - 2f - 3g}{\frac{1}{5}} = -\frac{2}{5}f - \frac{3}{5}g + \frac{1}{5}h$$

$$\text{Orthonormal basis: } \frac{1}{2}f, g, -\frac{2}{5}f - \frac{3}{5}g + \frac{1}{5}h$$

**5.5.25** Using the inner product defined in Example 2, we find that

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots} = \sqrt{\frac{\pi^2}{6}} = \frac{\pi}{\sqrt{6}} \text{ (see the text right after Theorem 5.5.6).}$$

$$\text{5.5.26 } a_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) dt = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt = \frac{1}{\pi} \left\{ - \int_{-\pi}^0 \sin(kt) dt + \int_0^{\pi} \sin(kt) dt \right\} = \frac{2}{\pi} \int_0^{\pi} \sin(kt) dt$$

$$= -\frac{2}{k\pi} [\cos(kt)]_0^\pi = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{4}{\pi k} & \text{if } k \text{ is odd} \end{cases}$$

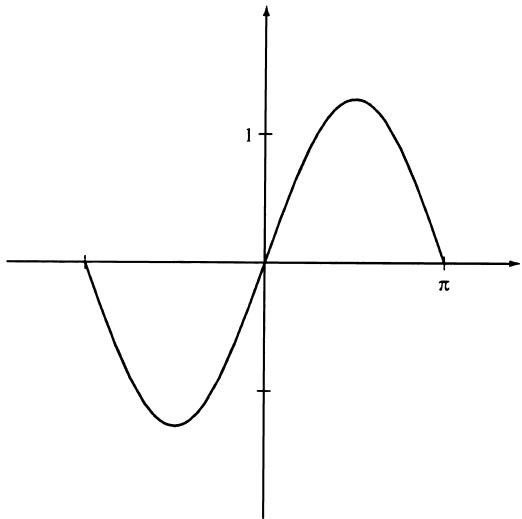


Figure 5.22: for Problem 5.5.26.

$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt = 0$ , since the integrand is odd.

$f_1(t) = f_2(t) = \frac{4}{\pi} \sin(t)$ . See Figure 5.22.

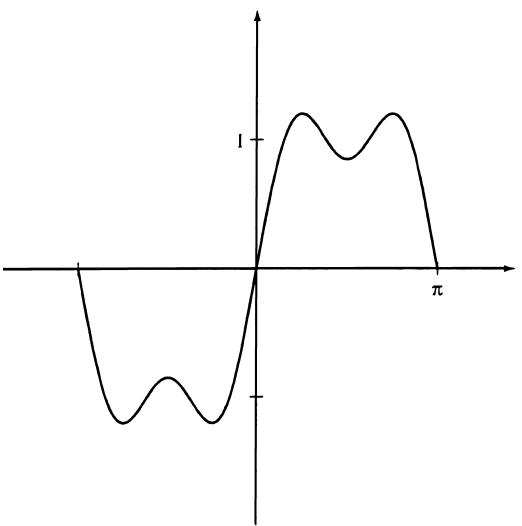


Figure 5.23: for Problem 5.5.26.

$f_3(t) = f_4(t) = \frac{4}{\pi} \sin(t) + \frac{4}{3\pi} \sin(3t)$ . See Figure 5.23.

$f_5(t) = f_6(t) = \frac{4}{\pi} \sin(t) + \frac{4}{3\pi} \sin(3t) + \frac{4}{5\pi} \sin(5t)$ . See Figure 5.24.

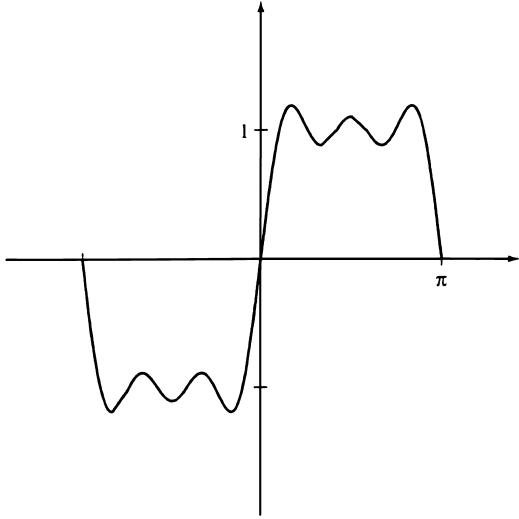


Figure 5.24: for Problem 5.5.26.

$$5.5.27 \quad a_0 = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{\sqrt{2}}.$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt = \frac{1}{\pi} \int_0^{\pi} \sin(kt) dt = -\frac{1}{k\pi} [\cos(kt)]_0^\pi = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt = \frac{1}{\pi} \int_0^{\pi} \cos(kt) dt = \frac{1}{k\pi} [\sin(kt)]_0^\pi = 0$$

$$f_1(t) = f_2(t) = \frac{1}{2} + \frac{2}{\pi} \sin(t), \quad f_3(t) = f_4(t) = \frac{1}{2} + \frac{2}{\pi} \sin(t) + \frac{2}{3\pi} \sin(3t)$$

$$\vdots$$

$$5.5.28 \quad \|f\|^2 = \langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(t))^2 dt = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 dt = 2$$

Now Theorem 5.5.6 tells us that  $\frac{16}{\pi^2} + \frac{16}{9\pi^2} + \frac{16}{25\pi^2} + \dots = \frac{16}{\pi^2} \left( \sum_{k \text{ odd}} \frac{1}{k^2} \right) = 2$ , or  $\sum_{k \text{ odd}} \frac{1}{k^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots = \frac{\pi^2}{8}$ .

$$5.5.29 \quad \|f\|^2 = \langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(t))^2 dt = \frac{1}{\pi} \int_0^{\pi} 1 dt = 1$$

Now Theorem 5.5.6 tells us that  $\frac{1}{2} + \frac{4}{\pi^2} + \frac{4}{9\pi^2} + \frac{4}{25\pi^2} + \dots = 1$ , or  $\frac{1}{2} + \frac{4}{\pi^2} \left( \sum_{k \text{ odd}} \frac{1}{k^2} \right) = 1$  or  $\sum_{k \text{ odd}} \frac{1}{k^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots = \frac{\frac{1}{2}}{\frac{4}{\pi^2}} = \frac{\pi^2}{8}$ .

5.5.30 There is an invertible linear transformation  $T(\vec{x}) = A\vec{x}$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that transforms  $E$  into the unit circle. (If  $E$  is parametrized by  $\cos(t)\vec{w}_1 + \sin(t)\vec{w}_2$ , let  $A = [\vec{w}_1 \ \vec{w}_2]^{-1}$ . Compare with Exercises 2.2.50 and 2.2.51.)

Now  $\vec{x}$  is on  $E$  if  $\|T(\vec{x})\| = 1$ , or  $T(\vec{x}) \cdot T(\vec{x}) = 1$ . This means that the inner product  $\langle \vec{x}, \vec{y} \rangle = T(\vec{x}) \cdot T(\vec{y})$  does the job (see Exercise 17). (There are other, very different approaches to this problem.)

**5.5.31** An orthonormal basis of  $P_2$  of the desired form is  $f_0(t) = \frac{1}{\sqrt{2}}$ ,  $f_1(t) = \sqrt{\frac{3}{2}}t$ ,  $f_2(t) = \frac{1}{2}\sqrt{\frac{5}{2}}(3t^2 - 1)$  (compare with Exercise 10), and the zeros of  $f_2(t)$  are  $a_{1,2} = \pm\frac{1}{\sqrt{3}}$ .

Next we find the weights  $w_1, w_2$  such that  $\int_{-1}^1 f(t) dt = \sum_{i=1}^2 w_i f(a_i)$  for all  $f$  in  $P_1$ . We need to make sure that the equation holds for 1 and  $t$ .

$$\begin{bmatrix} 2 = w_1 + w_2 \\ 0 = \frac{1}{\sqrt{3}}w_1 - \frac{1}{\sqrt{3}}w_2 \end{bmatrix}, \text{ with solution } w_1 = w_2 = 1.$$

It follows that the equation  $\int_{-1}^1 f(t) dt = f(a_1) + f(a_2) = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$  holds for all polynomials  $f$  in  $P_1$ . We can check that it holds for  $t^2$  and  $t^3$  as well, that is, it holds in fact for all cubic polynomials.

$$\int_{-1}^1 t^2 dt = \frac{2}{3} \text{ equals } \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2 = \frac{2}{3}, \text{ and } \int_{-1}^1 t^3 dt = 0 \text{ equals } \left(\frac{1}{\sqrt{3}}\right)^3 + \left(-\frac{1}{\sqrt{3}}\right)^3 = 0.$$

**5.5.32** a.  $\langle t^n, t^m \rangle = \frac{1}{2} \int_{-1}^1 t^{n+m} dt = \frac{1}{2} \left[ \frac{t^{n+m+1}}{n+m+1} \right]_{-1}^1 = \begin{cases} \frac{1}{n+m+1} & \text{if } n+m \text{ is even} \\ 0 & \text{if } n+m \text{ is odd} \end{cases}$

b.  $\|t^n\| = \sqrt{\langle t^n, t^n \rangle} = \sqrt{\frac{1}{2n+1}}$  by part a.

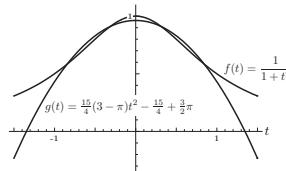
c. It helps to observe that 1 and  $t^2$  are orthogonal to  $t$  and  $t^3$ . Accordingly, many terms in the Gram-Schmidt formulas vanish.

$$g_0(t) = \frac{1}{\|1\|} = 1, g_1(t) = \frac{t}{\|t\|} = \sqrt{3}t, g_2(t) = \frac{t^2 - \langle t^2, 1 \rangle 1}{\|t^2 - \langle t^2, 1 \rangle 1\|} = \frac{\sqrt{5}}{2}(3t^2 - 1)$$

$$g_3(t) = \frac{t^3 - \langle t^3, \sqrt{3}t \rangle \sqrt{3}t}{\|t^3 - \langle t^3, \sqrt{3}t \rangle \sqrt{3}t\|} = \frac{t^3 - 3t/5}{\|t^3 - 3t/5\|} = \frac{\sqrt{7}}{2}(5t^3 - 3t)$$

d. The first few Legendre Polynomials are  $\frac{g_0(t)}{g_1(1)} = 1$ ,  $\frac{g_1(t)}{g_1(1)} = t$ ,  $\frac{g_2(t)}{g_2(1)} = (3t^2 - 1)/2$ ,  $\frac{g_3(t)}{g_3(1)} = (5t^3 - 3t)/2$ .

e. Since  $f(t) = \frac{1}{1+t^2}$  is an even function, we need to consider  $g_0(t)$  and  $g_2(t)$  only. The solution is  $\langle f(t), g_0(t) \rangle g_0(t) + \langle f(t), g_2(t) \rangle g_2(t) = \langle f(t), 1 \rangle 1 + \frac{5}{4} \langle f(t), 3t^2 - 1 \rangle (3t^2 - 1) = \frac{\pi}{4} + \frac{5}{8} (6 - 2\pi) (3t^2 - 1) = \frac{15}{4} (3 - \pi) t^2 - \frac{15}{4} + \frac{3\pi}{2}$ .



**5.5.33** a. The first property of an inner product,  $\langle f, g \rangle = \langle g, f \rangle$ , follows from the fact that  $f(t)g(t) = g(t)f(t)$ . The property  $\langle f + h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$  follows from the sum rule for integrals, and the property  $\langle cf, g \rangle = c\langle f, g \rangle$  follows from the constant multiple rule for integrals. Proceed as in Exercise 1 to show that  $\langle f, f \rangle > 0$  for all nonzero  $f$ .

b. If  $f(t) = 1$  then  $\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b w(t) \cdot 1^2 dt} = \sqrt{\int_a^b w(t) dt} = 1$ .

**5.5.34** a. Note that the graph of  $\sqrt{1-t^2}$  is the upper half of the unit circle centered at the origin. Thus  $\int_{-1}^1 w(t) dt = \frac{2}{\pi} \int_{-1}^1 \sqrt{1-t^2} dt = \frac{2}{\pi}$  (half the area of the unit circle) = 1

b. If  $f(t) = 1$  then  $\|f\| = \sqrt{\langle f, f \rangle} = 1$  by part a and Exercise 33b.

c. If  $n+m$  is odd then  $\langle t^n, t^m \rangle = \frac{2}{\pi} \int_{-1}^1 \sqrt{1-t^2} t^{n+m} dt = 0$  since the integrand is an odd function.

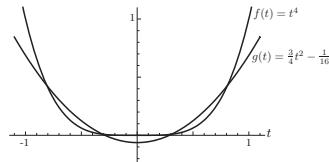
d. It follows from the definition of this inner product that  $\langle t, t \rangle = \langle 1, t^2 \rangle = 1/4$  and  $\langle t^2, t^2 \rangle = \langle t, t^3 \rangle = 1/8$  so that  $\|t\| = \sqrt{1/4} = 1/2$  and  $\|t^2\| = \sqrt{1/8} = \sqrt{2}/4$ .

e. It helps to observe that 1 and  $t^2$  are orthogonal to  $t$  and  $t^3$ . Accordingly, many terms in the Gram-Schmidt formulas vanish. The first few Chebyshev Polynomials of the Second Kind are

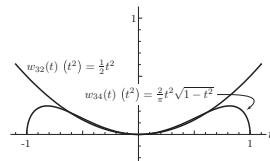
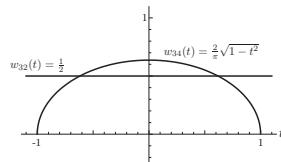
$$g_0(t) = \frac{1}{\|1\|} = 1, g_1(t) = \frac{t}{\|t\|} = 2t, g_2(t) = \frac{t^2 - \langle t^2, 1 \rangle 1}{\|t^2 - \langle t^2, 1 \rangle 1\|} = \frac{t^2 - 1/4}{\sqrt{1/16}} = 4t^2 - 1$$

$$g_3(t) = \frac{t^3 - \langle t^3, 2t \rangle 2t}{\|t^3 - \langle t^3, 2t \rangle 2t\|} = \frac{t^3 - t/2}{\sqrt{1/64}} = 8t^3 - 4t.$$

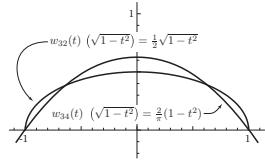
f. Since  $f(t) = t^4$  is an even function, we need to consider  $g_0(t)$  and  $g_2(t)$  only. The solution is  $\langle f(t), g_0(t) \rangle g_0(t) + \langle f(t), g_2(t) \rangle g_2(t) = \langle t^4, 1 \rangle 1 + \langle t^4, 4t^2 - 1 \rangle (4t^2 - 1) = \frac{1}{8} + (\frac{5}{16} - \frac{1}{8}) (4t^2 - 1) = \frac{3}{4}t^2 - \frac{1}{16}$ .



**5.5.35** a.  $\|t\|_{32} = \sqrt{\frac{1}{2} \int_{-1}^1 t^2 dt} = \sqrt{1/3}$  and  $\|t\|_{34} = \sqrt{\langle t, t \rangle_{34}} = \sqrt{\langle 1, t^2 \rangle_{34}} = 1/2$ , so that  $\|t\|_{32} > \|t\|_{34}$ . Note that  $\langle 1, t^2 \rangle_{34} = 1/4$  is given in Exercise 34.



b. For  $f(t) = \sqrt{1-t^2}$  we have  $\|f\|_{32} = \sqrt{\frac{1}{2} \int_{-1}^1 (1-t^2) dt} = \sqrt{2/3}$  and  $\|f\|_{34} = \sqrt{\langle \sqrt{1-t^2}, \sqrt{1-t^2} \rangle_{34}} = \sqrt{\langle 1, 1-t^2 \rangle_{34}} = \sqrt{1-1/4} = \sqrt{3/4}$ .



## True or False

Ch 5.TF.1 T, since  $(A+B)^T = A^T + B^T = A + B$

Ch 5.TF.2 T, by Theorem 5.3.4

Ch 5.TF.3 F. Consider  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .

Ch 5.TF.4 T. First note that  $A^T = A^{-1}$ , by Theorem 2.4.8. Thus  $A$  is orthogonal, by Theorem 5.3.7.

Ch 5.TF.5 F. The correct formula is  $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$ , by Definition 2.2.1.

Ch 5.TF.6 T, since  $(7A)^T = 7A^T = 7A$ .

Ch 5.TF.7 F. Consider  $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$ .

Ch 5.TF.8 T, by Theorem 5.3.9.b

Ch 5.TF.9 T, by Theorem 5.3.4a

Ch 5.TF.10 F. We have  $(AB)^T = B^T A^T$ , by Theorem 5.3.9a.

Ch 5.TF.11 T. If  $A$  is orthogonal, then  $A^T = A^{-1}$ , and  $A^{-1}$  is orthogonal by Theorem 5.3.4b.

Ch 5.TF.12 F. Consider  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

Ch 5.TF.13 F. Consider  $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Then  $AB^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  isn't equal to  $B^T A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ .

Ch 5.TF.14 F. It is required that the columns of  $A$  be orthonormal (Theorem 5.3.10). As a counterexample, consider  $A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ , with  $AA^T = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ .

Ch 5.TF.15 T, since  $(ABBA)^T = A^T B^T B^T A^T = ABBA$ , by Theorem 5.3.9a

Ch 5.TF.16 T, since  $A^T B^T = (BA)^T = (AB)^T = B^T A^T$ , by Theorem 5.3.9a

Ch 5.TF.17 F.  $\dim(V) + \dim(V^\perp) = 5$ , by Theorem 5.1.8c. Thus one of the dimensions is even and the other odd.

Ch 5.TF.18 T. Consider the  $QR$  factorization (Theorem 5.2.2)

Ch 5.TF.19 F. The Pythagorean Theorem holds for orthogonal vectors  $\vec{x}, \vec{y}$  only (Theorem 5.1.9)

Ch 5.TF.20 T.  $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Ch 5.TF.21 F. As a counterexample, consider  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

Ch 5.TF.22 T, by Theorem 5.4.1.

Ch 5.TF.23 T, by Theorem 5.4.2a.

Ch 5.TF.24 F. Consider  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ , or any other symmetric matrix that fails to be orthogonal.

Ch 5.TF.25 F.  $\det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -1 - 0 = -1$ , yet  $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  is orthogonal.

Ch 5.TF.26 T.  $[\frac{1}{2}(A - A^T)]^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -[\frac{1}{2}(A - A^T)]$ .

Ch 5.TF.27 T, since the columns are unit vectors.

Ch 5.TF.28 T. Use the Gram-Schmidt process to construct such a basis (Theorem 5.2.1)

Ch 5.TF.29 F. The columns fail to be unit vectors (use Theorem 5.3.3b)

Ch 5.TF.30 T, by definition of an orthogonal projection (Theorem 5.1.4).

Ch 5.TF.31 T. Try  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ , so that  $A + B = \begin{bmatrix} 1 + \cos \theta & -\sin \theta \\ \sin \theta & 1 + \cos \theta \end{bmatrix}$ . It is required that  $\begin{bmatrix} 1 + \cos \theta \\ \sin \theta \end{bmatrix}$  and  $\begin{bmatrix} -\sin \theta \\ 1 + \cos \theta \end{bmatrix}$  be unit vectors, meaning that  $1 + 2\cos \theta + \cos^2 \theta + \sin^2 \theta = 2 + 2\cos \theta = 1$ , or  $\cos \theta = -\frac{1}{2}$ , and  $\sin \theta = \pm \frac{\sqrt{3}}{2}$ . Thus  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$  is a solution.

Ch 5.TF.32 F. Consider  $A = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ , for example, representing a rotation combined with a scaling.

Ch 5.TF.33 F. Consider  $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ .

Ch 5.TF.34 T. By Definition 5.1.12, quantity  $\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$  is positive, so that  $\theta$  is an acute angle.

Ch 5.TF.35 T. In Theorem 5.4.1, let  $A = B^T$  to see that  $(\text{im}(B^T))^\perp = \ker(B)$ . Now take the orthogonal complements of both sides and use Theorem 5.1.8d.

Ch 5.TF.36 T, since  $(A^T A)^T = A^T (A^T)^T = A^T A$ , by Theorem 5.3.9a.

Ch 5.TF.37 F. Verify that matrices  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$  are similar.

Ch 5.TF.38 F. Consider  $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ . The correct formula  $\text{im}(B) = \text{im}(BB^T)$  follows from Theorems 5.4.1 and 5.4.2.

Ch 5.TF.39 T. We know that  $A^T = A$  and  $S^{-1} = S^T$ . Now  $(S^{-1}AS)^T = S^T A^T (S^{-1})^T = S^{-1}AS$ , by Theorem 5.3.9a.

Ch 5.TF.40 T. By Theorem 5.4.2, we have  $\ker(A) = \ker(A^T A)$ . Replacing  $A$  by  $A^T$  in this formula, we find that  $\ker(A^T) = \ker(AA^T)$ . Now  $\ker(A) = \ker(A^T A) = \ker(AA^T) = \ker(A^T)$ .

Ch 5.TF.41 T. We attempt to write  $A = S + Q$ , where  $S$  is symmetric and  $Q$  is skew-symmetric. Then  $A^T = S^T + Q^T = S - Q$ . Adding the equations  $A = S + Q$  and  $A^T = S - Q$  together gives  $2S = A + A^T$  and  $S = \frac{1}{2}(A + A^T)$ . Similarly we find  $Q = \frac{1}{2}(A - A^T)$ . Check that the decomposition  $A = S + Q = (\frac{1}{2}(A + A^T)) + (\frac{1}{2}(A - A^T))$  does the job.

Ch 5.TF.42 T. Apply the Cauchy-Schwarz inequality (squared),  $(\vec{x} \cdot \vec{y})^2 \leq \|\vec{x}\|^2 \|\vec{y}\|^2$ , to  $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix}$  (all  $n$  entries are 1).

Ch 5.TF.43 T. Let  $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ . We know that  $AA^T = A^2$ , or  $\begin{bmatrix} x^2 + y^2 & xz + yt \\ xz + yt & z^2 + t^2 \end{bmatrix} = \begin{bmatrix} x^2 + yz & xy + yt \\ zx + tz & yz + t^2 \end{bmatrix}$ . We need to show that  $y = z$ . If  $y \neq 0$ , this follows from the equation  $x^2 + y^2 = x^2 + yz$ ; if  $z \neq 0$ , it follows from  $z^2 + t^2 = yz + t^2$ ; if both  $y$  and  $z$  are zero, we are all set.

Ch 5.TF.44 T, since  $\vec{x} \cdot (\text{proj}_V \vec{x}) = (\text{proj}_V \vec{x} + (\vec{x} - \text{proj}_V \vec{x})) \cdot \text{proj}_V \vec{x} = \|\text{proj}_V \vec{x}\|^2 \geq 0$ . Note that  $\vec{x} - \text{proj}_V \vec{x}$  is orthogonal to  $\text{proj}_V \vec{x}$ , by the definition of a projection.

Ch 5.TF.45 T. Note that  $1 = \|A(\frac{1}{\|\vec{x}\|}\vec{x})\| = \left\| \frac{1}{\|\vec{x}\|} A\vec{x} \right\| = \frac{1}{\|\vec{x}\|} \|A\vec{x}\|$  for all nonzero  $\vec{x}$ , so that  $\|A\vec{x}\| = \|\vec{x}\|$ . See Definition 5.3.1.

Ch 5.TF.46 T. If  $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$  is a symmetric matrix, then  $A - xI_2 = \begin{bmatrix} a-x & b \\ b & c-x \end{bmatrix}$ . This matrix fails to be invertible if (and only if)  $\det(A - xI_2) = (a-x)(c-x) - b^2 = 0$ . We use the quadratic formula to find the (real)

solutions  $x = \frac{a+c \pm \sqrt{(a+c)^2 - 4ac + 4b^2}}{2} = \frac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$ . Note that the discriminant  $(a - c)^2 + 4b^2$  is positive or zero.

Ch 5.TF.47 T; one basis is:  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

Ch 5.TF.48 F; A direct computation or a geometrical argument shows that  $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ , representing a reflection, not a rotation.

Ch 5.TF.49 F;  $\dim(\mathbb{R}^{3 \times 3}) = 9$ ,  $\dim(\mathbb{R}^{2 \times 2}) = 4$ , so  $\dim(\ker(L)) \geq 5$ , but the space of all  $3 \times 3$  skew-symmetric matrices has dimension of 3.

$\left( \text{A basis is } \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \right)$

Ch 5.TF.50 T; Consider an orthonormal basis  $\vec{v}_1, \vec{v}_2$  of  $V$ , and a unit vector  $\vec{v}_3$  perpendicular to  $V$ , and form the orthogonal matrix  $S = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ . Now  $AS = [\vec{v}_1 \ \vec{v}_2 \ \vec{0}] = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Since  $S$  is orthogonal, we have

$S^T AS = S^{-1} AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , a diagonal matrix.

## Chapter 6

### Section 6.1

6.1.1 Fails to be invertible; since  $\det \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = 6 - 6 = 0$ .

6.1.2 Invertible; since  $\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 10 - 12 = -2$ .

6.1.3 Invertible; since  $\det \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix} = 33 - 35 = -2$ .

6.1.4 Fails to be invertible; since  $\det \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = 8 - 8 = 0$ .

6.1.5 Invertible; since  $\det \begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix} = 2 \cdot 11 \cdot 5 + 0 + 0 - 0 - 0 - 0 = 110$ .

6.1.6 Invertible; since  $\det \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} = 6 \cdot 4 \cdot 1 + 0 + 0 - 0 - 0 - 0 = 24$ .

6.1.7 This matrix is clearly not invertible, so the determinant must be zero.

6.1.8 This matrix fails to be invertible, since the  $\det(A) = 0$ .

6.1.9 Invertible; since  $\det \begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{bmatrix} = 0 + 3 \cdot 6 + 2 \cdot 7 \cdot 5 - 7 \cdot 4 - 2 \cdot 8 \cdot 6 = -36$ .

6.1.10 Invertible; since  $\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = 1 \cdot 2 \cdot 6 + 1 \cdot 3 \cdot 1 + 1 \cdot 1 \cdot 3 - 3 \cdot 3 \cdot 1 - 2 \cdot 1 \cdot 1 - 6 \cdot 1 \cdot 1 = 1$ .

6.1.11  $\det \begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix} \neq 0$  when  $4k \neq 6$ , or  $k \neq \frac{3}{2}$ .

6.1.12  $\det \begin{bmatrix} 1 & k \\ k & 4 \end{bmatrix} \neq 0$  when  $k^2 \neq 4$ , or  $k \neq 2, -2$ .

6.1.13  $\det \begin{bmatrix} k & 3 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix} = 8k$ , so  $k \neq 0$  will ensure that this matrix is invertible.

6.1.14  $\det \begin{bmatrix} 4 & 0 & 0 \\ 3 & k & 0 \\ 2 & 1 & 0 \end{bmatrix} = 0$ , so the matrix will never be invertible, no matter which  $k$  is chosen.

6.1.15  $\det \begin{bmatrix} 0 & k & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} = 6k - 3$ . This matrix is invertible when  $k \neq \frac{1}{2}$ .

6.1.16  $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & k & 5 \\ 6 & 7 & 8 \end{bmatrix} = 60 + 84 + 8k - 18k - 35 - 64 = 45 - 10k$ . So this matrix is invertible when  $k \neq 4.5$ .

6.1.17  $\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & k & -1 \\ 1 & k^2 & 1 \end{bmatrix} = 2k^2 - 2 = 2(k^2 - 1) = 2(k - 1)(k + 1)$ . So  $k$  cannot be 1 or -1.

6.1.18  $\det \begin{bmatrix} 0 & 1 & k \\ 3 & 2k & 5 \\ 9 & 7 & 5 \end{bmatrix} = 30 + 21k - 18k^2 = -3(k - 2)(6k + 5)$ . So  $k$  cannot be 2 or  $-\frac{5}{6}$ .

6.1.19  $\det \begin{bmatrix} 1 & 1 & k \\ 1 & k & k \\ k & k & k \end{bmatrix} = -k^3 + 2k^2 - k = -k(k - 1)^2$ . So  $k$  cannot be 0 or 1.

6.1.20  $\det \begin{bmatrix} 1 & k & 1 \\ 1 & k + 1 & k + 2 \\ 1 & k + 2 & 2k + 4 \end{bmatrix} = (k + 1)(2k + 4) + k(k + 2) + (k + 2) - (k + 1) - k(2k + 4) - (k + 2)(k + 2) = (k + 1)(3k + 6) - (3k^2 + 9k + 5) = 1$ . Thus,  $A$  will always be invertible, no matter the value of  $k$ , meaning that  $k$  can have any value.

6.1.21  $\det \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix} = k^3 - 3k + 2 = (k - 1)^2(k + 2)$ . So  $k$  cannot be -2 or 1.

6.1.22  $\det \begin{bmatrix} \cos k & 1 & -\sin k \\ 0 & 2 & 0 \\ \sin k & 0 & \cos k \end{bmatrix} = 2\cos^2 k + 2\sin^2 k = 2$ . So  $k$  can have any value.

6.1.23  $\det(A - \lambda I_2) = \det \begin{bmatrix} 1 - \lambda & 2 \\ 0 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) = 0$  if  $\lambda$  is 1 or 4.

6.1.24  $\det(A - \lambda I_2) = \det \begin{bmatrix} 2 - \lambda & 0 \\ 1 & 0 - \lambda \end{bmatrix} = (2 - \lambda)(-\lambda) = 0$  if  $\lambda$  is 2 or 0.

6.1.25  $\det(A - \lambda I_2) = \det \begin{bmatrix} 4 - \lambda & 2 \\ 4 & 6 - \lambda \end{bmatrix} = (4 - \lambda)(6 - \lambda) - 8 = (\lambda - 8)(\lambda - 2) = 0$  if  $\lambda$  is 2 or 8.

6.1.26  $\det(A - \lambda I_2) = \det \begin{bmatrix} 4-\lambda & 2 \\ 2 & 7-\lambda \end{bmatrix} = (4-\lambda)(7-\lambda) - 4 = (\lambda-8)(\lambda-3) = 0$  if  $\lambda$  is 3 or 8.

6.1.27  $A - \lambda I_3$  is a lower triangular matrix with the diagonal entries  $(2-\lambda)$ ,  $(3-\lambda)$  and  $(4-\lambda)$ . Now,  $\det(A - \lambda I_3) = (2-\lambda)(3-\lambda)(4-\lambda) = 0$  if  $\lambda$  is 2, 3 or 4.

6.1.28  $A - \lambda I_3$  is an upper triangular matrix with the diagonal entries  $(2-\lambda)$ ,  $(3-\lambda)$  and  $(5-\lambda)$ . Now,  $\det(A - \lambda I_3) = (2-\lambda)(3-\lambda)(5-\lambda) = 0$  if  $\lambda$  is 2, 3 or 5.

6.1.29  $\det(A - \lambda I_3) = \det \begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix} = (3-\lambda)(\lambda-8)(\lambda-3) = 0$  if  $\lambda$  is 3 or 8.

6.1.30  $\det(A - \lambda I_3) = \det \begin{bmatrix} 4-\lambda & 2 & 0 \\ 4 & 6-\lambda & 0 \\ 5 & 2 & 3-\lambda \end{bmatrix} = (4-\lambda)(6-\lambda)(3-\lambda) - 8(3-\lambda)$   
 $= (3-\lambda)(8-\lambda)(2-\lambda) = 0$  if  $\lambda$  is 3, 8 or 2.

6.1.31 This matrix is upper triangular, so the determinant is the product of the diagonal entries, which is 24.

6.1.32 This matrix is upper triangular, so the determinant is the product of the diagonal entries, which is 210.

6.1.33 The determinant of this block matrix is  $\det \begin{bmatrix} 1 & 2 \\ 8 & 7 \end{bmatrix} \det \begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix} = (7-16)(10-21) = 99$ , by Theorem 6.1.5.

6.1.34 The determinant of this block matrix is  $\det \begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix} \det \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = (24-15)(3-8) = -45$ , by Theorem 6.1.5.

6.1.35 There are two patterns with a nonzero product,  $(a_{12}, a_{23}, a_{31}, a_{44}) = (3, 2, 6, 4)$ , with two inversions, and  $(a_{12}, a_{23}, a_{34}, a_{41}) = (3, 2, 3, 7)$ , with 3 inversions. Thus  $\det A = 3 \cdot 2 \cdot 6 \cdot 4 - 3 \cdot 2 \cdot 3 \cdot 7 = 18$ .

6.1.36 There is one pattern with a nonzero product, containing all the 1's, with six inversions. Thus  $\det A = 1$ .

6.1.37 The determinant of this block matrix is

$$\det \begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix} \det \begin{bmatrix} 5 & 6 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = (35-24)(5 \cdot 1 \cdot 1) = 55, \text{ by Theorem 6.1.5.}$$

6.1.38 The determinant of this block matrix is

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 2 & 1 & 2 \end{bmatrix} \det \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} = (2 \cdot 4 \cdot 2 + 3 \cdot 3 \cdot 1 - 1 \cdot 4 \cdot 1 - 2 \cdot 3 \cdot 2)(6 \cdot 6 - 5 \cdot 5) = 99, \text{ by Theorem 6.1.5.}$$

6.1.39 There is only one pattern with a nonzero product, containing all the nonzero entries of the matrix, with eight inversions. Thus  $\det A = 1 \cdot 2 \cdot 4 \cdot 3 \cdot 5 = 120$ .

**6.1.40** There is only one pattern with a nonzero product, containing all the nonzero entries of the matrix, with seven inversions. Thus  $\det A = -3 \cdot 2 \cdot 4 \cdot 1 \cdot 5 = -120$ .

**6.1.41** There are two patterns with a nonzero product,  $(a_{15}, a_{24}, a_{32}, a_{41}, a_{53}) = (2, 2, 3, 2, 3)$ , with eight inversions, and  $(a_{13}, a_{24}, a_{32}, a_{41}, a_{55}) = (1, 2, 3, 2, 4)$ , with five inversions. Thus  $\det A = 2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 - 1 \cdot 2 \cdot 3 \cdot 2 \cdot 4 = 24$ .

**6.1.42** There is only one pattern with a nonzero product,  $(a_{13}, a_{24}, a_{32}, a_{45}, a_{51}) = (2, 2, 9, 5, 3)$ , with six inversions. Thus  $\det A = 2 \cdot 2 \cdot 9 \cdot 5 \cdot 3 = 540$ .

**6.1.43** For each pattern  $P$  in  $A$ , consider the corresponding pattern  $P_{opp}$  in  $-A$ , with all the  $n$  entries being opposites. Then  $\text{prod}(P_{opp}) = (-1)^n \text{prod}(P)$  and  $\text{sgn}(P_{opp}) = \text{sgn}(P)$ , so that  $\det(-A) = (-1)^n \det A$ .

**6.1.44** For each pattern  $P$  in  $A$ , consider the corresponding pattern  $P_m$  in  $kA$ , with all the  $n$  entries being multiplied by the scalar  $k$ . Then  $\text{prod}(P_m) = k^n \text{prod}(P)$  and  $\text{sgn}(P_m) = \text{sgn}(P)$ , so that  $\det(kA) = k^n \det A$ .

**6.1.45** If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $\det(A^T) = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - cb = \det(A)$ . It turns out that  $\det(A^T) = \det(A)$ .

**6.1.46** Let  $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ . If  $a_1a_4 - a_2a_3 \neq 0$ , then  $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix}$ .

By Exercise 44,  $\det(A^{-1}) = \left(\frac{1}{\det(A)}\right)^2 (a_1a_4 - a_2a_3) = \left(\frac{1}{\det(A)}\right)^2 \cdot \det(A)$  so  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

**6.1.47** We have  $\det(A) = (ah - cf)k + bef + cdg - aeg - bdh$ . Thus matrix  $A$  is invertible for all  $k$  if (and only if) the coefficient  $(ah - cf)$  of  $k$  is 0, while the sum  $bef + cdg - aeg - bdh$  is nonzero. A numerical example is  $a = c = d = f = h = g = 1$  and  $b = e = 2$ , but there are infinitely many other solutions as well.

**6.1.48** Consider  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ ,  $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  so  $\det(A) = \det(B) = \det(C) = \det(D) = 0$  hence  $\det(A)\det(D) - \det(B)\det(C) = 0$  but  $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = -1$ .

**6.1.49** The kernel of  $T$  consists of all vectors  $\vec{x}$  such that the matrix  $[\vec{x} \ \vec{v} \ \vec{w}]$  fails to be invertible. This is the case if  $\vec{x}$  is a linear combination of  $\vec{v}$  and  $\vec{w}$  as discussed on Pages 249 and 250. Thus  $\ker(T) = \text{span}(\vec{v}, \vec{w})$ . The image of  $T$  isn't  $\{0\}$ , since  $T(\vec{v} \times \vec{w}) \neq 0$ , for example. Being a subspace of  $\mathbb{R}$ , the image must be all of  $\mathbb{R}$ .

**6.1.50** Theorem 6.1.1 tells us that  $\det[\vec{u} \ \vec{v} \ \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}) = \|\vec{u}\| \cos(\theta) \|\vec{v} \times \vec{w}\| = \|\vec{u}\| \cos(\theta) \|\vec{v}\| \sin(\alpha) \|\vec{w}\| = \cos(\theta) \sin(\alpha)$ , where  $\theta$  is the angle enclosed by vectors  $\vec{u}$  and  $\vec{v} \times \vec{w}$ , and  $\alpha$  is the angle between  $\vec{v}$  and  $\vec{w}$ . Thus  $\det[\vec{u} \ \vec{v} \ \vec{w}]$  can be any number on the closed interval  $[-1, 1]$ .

**6.1.51** Let  $a_{ii}$  be the first entry on the diagonal that fails to belong to the pattern. The pattern must contain an entry in the  $i^{th}$  row to the right of  $a_{ii}$ , above the diagonal, and also an entry in the  $i^{th}$  column below  $a_{ii}$ , below the diagonal.

**6.1.52** By Definition 6.1.1, we have  $\det \begin{bmatrix} \vec{v} \times \vec{w} & \vec{v} & \vec{w} \end{bmatrix} = (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w}) = \|\vec{v} \times \vec{w}\|^2$ .

**6.1.53** There is one pattern with a nonzero product, containing all the 1's. We have  $n^2$  inversions, since each of the 1's in the lower left block forms an inversion with each of the 1's in the upper right block. Thus  $\det A = (-1)^{n^2} = (-1)^n$ .

**6.1.54** The pattern containing all the 1000's has 4 inversions so it contributes  $(1000)^5 = 10^{15}$  to the determinant. There are  $5! - 1 = 119$  other patterns with at most 3 entries being 1000, the others being  $\leq 9$ . Thus the product associated with each of those patterns is less than  $(1000)^3 (10)^2 = 10^{11}$ . Now  $\det A > 10^{15} - 119 \cdot 10^{11} > 0$ .

**6.1.55** By Exercise 2.4.93, a square matrix admits an LU factorization if (and only if) all its principal submatrices are invertible. Now

$$A^{(1)} = [7], A^{(2)} = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}, A^{(3)} = A = \begin{bmatrix} 7 & 4 & 2 \\ 5 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix},$$

with  $\det(A^{(1)}) = 7, \det(A^{(2)}) = 1, \det(A^{(3)}) = 1$ .

Since all principal submatrices turn out to be invertible, the matrix  $A$  does indeed admit an *LU* factorization.

**6.1.56** There is only one pattern with a nonzero product, containing all the 1's. The number of inversions is  $(n-1) + (n-2) + \dots + 2 + 1 = \sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}$ . This number is even if either  $n$  or  $n-1$  is divisible by 4, that is, for  $n = 4, 5, 8, 9, 12, 13, \dots$

- a.  $\det M_4 = \det M_5 = 1, \det M_2 = \det M_3 = \det M_6 = \det M_7 = -1$ .
- b.  $\det M_n = (-1)^{n(n-1)/2}$

**6.1.57** In a permutation matrix  $P$ , there is only one pattern with a nonzero product, containing all the 1's. Depending on the number of inversions in that pattern, we have  $\det P = 1$  or  $\det P = -1$ .

**6.1.58** a If  $a, b, c, d$  are distinct prime numbers, then  $ad \neq bc$ , since the prime factorization of a positive integer is unique. Thus  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$ : No matrix of the required form exists.

b We are looking for a noninvertible matrix  $A = [\vec{u} \ \vec{v} \ \vec{w}]$  whose entries are nine distinct prime numbers. The last column vector,  $\vec{w}$ , must be redundant; to keep things simple, we will make  $\vec{w} = \vec{u} + 2\vec{v}$ . Now we have to pick six distinct prime entries for the first two columns,  $\vec{u}$  and  $\vec{v}$ , such that the entries of  $\vec{w} = \vec{u} + 2\vec{v}$  are prime as well.

This can be done in many different ways; one solution is  $A = \begin{bmatrix} 7 & 2 & 11 \\ 17 & 3 & 23 \\ 19 & 5 & 29 \end{bmatrix}$ .

**6.1.59**  $F \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = ab + cd$

- a. Yes,  $F$  is linear in both columns. To prove linearity in the second column, observe that  $ab + cd$  is a linear combination of the variables  $b$  and  $d$ , with the constant coefficients  $a$  and  $c$ . An analogous argument proves linearity in the first column.
- b. No, since  $ab + cd$  fails to be a linear combination of  $a$  and  $b$ .
- c. No,  $F [\vec{w} \ \vec{v}] = F [\vec{v} \ \vec{w}] = \vec{v} \cdot \vec{w}$ : swapping the columns leaves  $F$  unchanged.

**6.1.60** The functions in parts a, b, and d are linear in both columns; the functions in parts a, c, and d are linear in both rows; and  $F(A) = -\det A$  in part d is alternating on the columns. For example, the function  $F(A) = cd$  in part b is linear in the first column since  $cd$  is a linear combination of the entries  $a$  and  $c$  in the first column. However,  $F(A) = cd$  fails to be linear in the second row since  $cd$  fails to be a linear combination of the entries  $c$  and  $d$  in the second row. Furthermore,  $F \begin{bmatrix} a & b \\ c & d \end{bmatrix} = F \begin{bmatrix} b & a \\ d & c \end{bmatrix} = cd$ , showing that  $F$  fails to be alternating on the columns.

**6.1.61** The function  $F(A) = bfg$  is linear in all three columns and in all three rows since the product  $bfg$  contains exactly one factor from each row and from each column; it is the product associated with a pattern. For example,  $F$  is linear in the second row since  $bfg$  is a scalar multiple of  $f$  and thus a linear combination of the entries  $d, e, f$  in the second row.  $F$  fails to be alternating; for example,  $F \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 1$  but  $F \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 0$  after swapping the first two columns.

**6.1.62** If  $A = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$ , then  $D(A) = D \begin{bmatrix} a & a \\ c & c \end{bmatrix} \underset{\substack{\text{swap} \\ \text{columns}}}{=} -D \begin{bmatrix} a & a \\ c & c \end{bmatrix} = -D(A)$ , so that  $D(A) = 0$ .

**6.1.63**  $D \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \underset{\text{Step 1}}{=} D \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + D \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \underset{\text{Step 2}}{=} ab D \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + ad D \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{\text{Step 3}}{=} ad$ . In Step 1, we write  $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix}$  and use linearity in the second column. In Step 2, we write  $\begin{bmatrix} a \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  etc. and use linearity in both columns. In Step 3, we use Exercise 62 for the first summand and the given property  $D(I_2) = 1$  for the second summand.

**6.1.64** Writing  $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$  and repeatedly using linearity in the columns, we find

$$\begin{aligned} D \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= D \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + D \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \underset{\text{Step 2}}{=} ad + D \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} + D \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \\ &= ad + bc D \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + cd D \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \underset{\text{Step 4}}{=} ad - bc D \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = ad - bc = \det A \end{aligned}$$

See the analogous computations in Exercise 63. In Step 2, we are using the result  $D \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = ad$  from Exercise 63. In Step 4, we swap the columns of the matrix in the second summand and we apply Exercise 62 to the third summand.

**6.1.65** Freely using the linearity and alternating properties in the columns, and omitting terms with two equal columns (see Exercise 62), we find

$$\begin{aligned} D \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} &= a D \begin{bmatrix} 1 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} + d D \begin{bmatrix} 0 & b & c \\ 1 & e & f \\ 0 & h & i \end{bmatrix} + g D \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 1 & h & i \end{bmatrix} \\ &= ae D \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & i \end{bmatrix} + ah D \begin{bmatrix} 1 & 0 & c \\ 0 & 0 & f \\ 0 & 1 & i \end{bmatrix} + db D \begin{bmatrix} 0 & 1 & c \\ 1 & 0 & f \\ 0 & 0 & i \end{bmatrix} + dh D \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & f \\ 0 & 1 & i \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
& +gb D \begin{bmatrix} 0 & 1 & c \\ 0 & 0 & f \\ 1 & 0 & i \end{bmatrix} + ge D \begin{bmatrix} 0 & 0 & c \\ 0 & 1 & f \\ 1 & 0 & i \end{bmatrix} = aei D \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + ahf D \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
& dbi D \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + dhc D \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + gbf D \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + gec D \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
& = aei - ahf - dbi + dhc + gbf - gec = \det A
\end{aligned}$$

In the penultimate step, we perform up to two column swaps on each of the six matrices to make them  $I_3$ , recalling that  $D(I_3) = 1$ .

- 6.1.66** a. Proceeding as in Exercises 63 and 64, we can see that  $F \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ab F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + ad F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + cb F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + cd F \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$  for all functions  $F$  in  $V$ . Thus the functions  $ab$ ,  $ad$ ,  $cb$ , and  $cd$  span  $V$ . Apply these functions to the matrices  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , and  $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$  to see that the four functions are linearly independent. Thus the functions  $ab$ ,  $ad$ ,  $cb$ , and  $cd$  form a basis of  $V$ , and  $\dim V = 4$ .
- b. In Exercise 64 we see that the formula  $F \begin{bmatrix} a & b \\ c & d \end{bmatrix} = F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (ad - bc) = F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\det A)$  holds for all functions  $F$  in  $W$ . Thus  $W$  is one-dimensional, with  $\det A$  as a basis.

## Section 6.2

**6.2.1**  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{bmatrix} \xrightarrow{-I} B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ . Now  $\det(A) = \det(B) = 6$ , by Algorithm 6.2.5b.

**6.2.2**  $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 6 & 8 \\ -2 & -4 & 0 \end{bmatrix} \xrightarrow{-I} B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ . Now  $\det(A) = \det(B) = 24$ , by Algorithm 6.2.5b.

**6.2.3**  $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{bmatrix} \xrightarrow{-I} B = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ . Now  $\det(A) = \det(B) = -24$ , by Algorithm 6.2.5b.

**6.2.4**  $A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{bmatrix} \xrightarrow{-2I} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 10 & 14 \\ 0 & 4 & 14 & 29 \end{bmatrix} \xrightarrow{+I} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 10 & 14 \\ 0 & 4 & 14 & 29 \end{bmatrix} \xrightarrow{-3II} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 14 \\ 0 & 4 & 14 & 29 \end{bmatrix} \xrightarrow{-4II} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 14 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 13 \end{bmatrix} \xrightarrow{-2III} B = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix}. \text{ Now } \det(A) = \det(B) = 9, \text{ by Algorithm 6.2.5b.}$$

6.2.5 After three row swaps, we end up with  $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ . Now, by Algorithm 6.2.5b,  $\det(A) = (-1)^3 \det(B) = -24$ .

$$6.2.6 A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix} \xrightarrow{-I} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & -2 & 7 & -9 \end{bmatrix}$$

$$\xrightarrow{\text{swap: } II \leftrightarrow III} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{bmatrix}$$

$$\xrightarrow{\div -2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{bmatrix} \xrightarrow{+2II}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{bmatrix} \xrightarrow{-2III} B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$$\det(A) = -\frac{1}{2}(-1) \det(B) = -72, \text{ by Algorithm 6.2.5b.}$$

6.2.7 After two row swaps, we end up with an upper triangular matrix  $B$  with all 1's along the diagonal. Now  $\det(A) = (-1)^2 \det(B) = 1$ , by Algorithm 6.2.5b.

6.2.8 After four row swaps, we end up with an upper triangular matrix  $B$  with all 1's along the diagonal, except for a 2 in the bottom right corner. Now  $\det(A) = (-1)^4 \det(B) = 2$ , by Algorithm 6.2.5b.

6.2.9 If we subtract the first row from every other row, then we have an upper triangular matrix  $B$ , with diagonal entries 1, 1, 2, 3 and 4. Then  $\det(A) = \det(B) = 24$  by Algorithm 6.2.5b.

$$6.2.10 A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix} \xrightarrow{-I} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 & 4 \\ 0 & 0 & 3 & 10 & 10 \\ 0 & 0 & 0 & 20 & 20 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{bmatrix} \xrightarrow{-2II} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{bmatrix} \xrightarrow{-3II}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{bmatrix} \xrightarrow{-6III}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 17 \end{bmatrix} \xrightarrow{-4IV} B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now  $\det(A) = \det(B) = 1$  by Algorithm 6.2.5b.

**6.2.11** By Theorem 6.2.3a, the desired determinant is  $(-9)(8) = -72$ .

**6.2.12** By Theorem 6.2.3b, the desired determinant is  $-8$ .

**6.2.13** By Theorem 6.2.3b, applied twice, since there are two row swaps, the desired determinant is  $(-1)(-1)(8) = 8$ .

**6.2.14** By Theorem 6.2.3c, the desired determinant is  $8$ .

**6.2.15** By Theorem 6.2.3c, the desired determinant is  $8$ .

**6.2.16** This determinant is  $0$ , since the first row is twice the last.

**6.2.17** The standard matrix of  $T$  is  $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ , so that  $\det(T) = \det(A) = 8$ .

**6.2.18** The standard matrix of  $T$  is  $A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -12 \\ 0 & 0 & 9 \end{bmatrix}$ , so that  $\det(T) = \det(A) = 27$ .

**6.2.19** The standard matrix of  $T$  is  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , so that  $\det(T) = \det(A) = -1$ .

**6.2.20** The standard matrix of  $L$  is  $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ , so that  $\det(L) = \det(M) = -1$ .

**6.2.21** The standard matrix of  $T$  is  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ , so that  $\det(T) = \det(A) = 1$ .

**6.2.22** Using Exercises 19 and 21 as a guide, we observe that the standard matrix  $A$  of  $T$  is diagonal, of size  $(n+1) \times (n+1)$ , with diagonal entries  $(-1)^0, (-1)^1, (-1)^2, \dots, (-1)^n$ . Thus  $\det(T) = \det(A) = (-1)^{1+2+\dots+n} = (-1)^{n(n+1)/2}$ .

**6.2.23** Consider the matrix  $M$  of  $T$  with respect to a basis consisting of  $n(n+1)/2$  symmetric matrices and  $n(n-1)/2$  skew-symmetric matrices (see Exercises 54 and 55 or Section 5.3). Matrix  $M$  will be diagonal, with  $n(n+1)/2$  entries 1 and  $n(n-1)/2$  entries -1 on the diagonal. Thus,  $\det(T) = \det(M) = (-1)^{n(n-1)/2}$ .

**6.2.24** The standard matrix of  $T$  is  $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ , so that  $\det(T) = \det(A) = 13$ .

**6.2.25** The standard matrix of  $T$  is  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ , so that  $\det(T) = \det(A) = 16$ .

**6.2.26** The matrix of  $T$  with respect to the basis  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  is  $A = \begin{bmatrix} 2 & 4 & 0 \\ 2 & 4 & 2 \\ 0 & 4 & 6 \end{bmatrix}$ , so that  $\det(T) = \det(A) = -16$ .

**6.2.27** The matrix of  $T$  with respect to the basis  $\cos(x), \sin(x)$  is  $A = \begin{bmatrix} -b & a \\ -a & -b \end{bmatrix}$ , so that  $\det(T) = \det(A) = a^2 + b^2$ .

**6.2.28** The matrix of  $T$  with respect to the basis  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$  is  $A = \begin{bmatrix} -6 & -10 \\ 5 & 6 \end{bmatrix}$ , so that  $\det(T) = \det(A) = 14$ .

**6.2.29** Expand down the first column, realizing that all but the first contribution are zero, since  $a_{21} = 0$  and  $A_{i1}$  has two equal rows for all  $i > 2$ . Therefore,  $\det(P_n) = \det(P_{n-1})$ .

Since  $\det(P_1) = 1$ , we can conclude that  $\det(P_n) = 1$ , for all  $n$ .

**6.2.30 a**  $f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix} = (ab^2 - a^2b) + (a^2 - b^2)t + (b - a)t^2$  so  $f(t)$  is a quadratic function of  $t$ . The coefficient of  $t^2$  is  $(b - a)$ .

**b** In the cases  $t = a$  and  $t = b$  the matrix has two identical columns. It follows that  $f(t) = k(t - a)(t - b)$  with  $k = \text{coefficient of } t^2 = (b - a)$ .

**c** The matrix is invertible for the values of  $t$  for which  $f(t) \neq 0$ , i.e., for  $t \neq a, t \neq b$ .

**6.2.31 a** If  $n = 1$ , then  $A = \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix}$ , so  $\det(A) = a_1 - a_0$  (and the product formula holds).

b Expanding the given determinant down the right-most column, we see that the coefficient  $k$  of  $t^n$  is the  $n - 1$  Vandermonde determinant which we assume is

$$\prod_{n-1 \geq i > j} (a_i - a_j).$$

Now  $f(a_0) = f(a_1) = \dots = f(a_{n-1}) = 0$ , since in each case the given matrix has two identical columns, hence its determinant equals zero. Therefore

$$f(t) = \left( \prod_{n-1 \geq i > j} (a_i - a_j) \right) (t - a_0)(t - a_1) \cdots (t - a_{n-1})$$

and

$$\det(A) = f(a_n) = \prod_{n \geq i > j} (a_i - a_j),$$

as required.

**6.2.32** By Exercise 31, we need to compute  $\prod_{i>j} (a_i - a_j)$  where  $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5$  so

$$\prod_{i>j} (a_i - a_j) = (2 - 1)(3 - 1)(3 - 2)(4 - 1)(4 - 2)(4 - 3)(5 - 1)(5 - 2)(5 - 3)(5 - 4) = 288.$$

**6.2.33** Think of the  $i^{th}$  column of the given matrix as  $a_i \begin{bmatrix} 1 \\ a_i \\ a_i^2 \\ \vdots \\ a_i^{n-1} \end{bmatrix}$ , so, by Theorem 6.2.3a, the determinant can

be written as  $(a_1 a_2 \cdots a_n) \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}$ . The new determinant is a Vandermonde determinant

(see Exercise 6.2.31), and we get

$$\prod_{i=1}^n a_i \prod_{i>j} (a_i - a_j).$$

**6.2.34 a** The hint pretty much gives it away. Since the columns of matrix  $\begin{bmatrix} B \\ -I_n \end{bmatrix}$  are in the kernel of  $[I_n \ M]$ , we have  $[I_n \ M] \begin{bmatrix} B \\ -I_n \end{bmatrix} = B - M = 0$ , and  $M = B$ , as claimed.

b If  $B = A^{-1}$  we get  $\text{rref}[A : I_n] = [I_n : A^{-1}]$  which tells us how to compute  $A^{-1}$  (see Theorem 2.4.5).

**6.2.35**  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  must satisfy  $\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & a_1 & b_1 \\ x_2 & a_2 & b_2 \end{bmatrix} = 0$ , i.e., must satisfy the linear equation

$$(a_1 b_2 - a_2 b_1) - x_1(b_2 - a_2) + x_2(b_1 - a_1) = 0.$$

We can see that  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$  and  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  satisfy this equation, since the matrix has two identical columns in these cases.

**6.2.36** Expanding down the first column we see that the equation has the form

$A - Bx_1 + Cx_2 - D(x_1^2 + x_2^2) = 0$ . If  $D \neq 0$  this equation defines a circle; otherwise it is a line. From Exercise 35 we know that  $D = 0$  if and only if the three given points  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  are collinear. Note that the circle or line runs through the three given points.

**6.2.37** Applying Theorem 6.2.6 to the equation  $AA^{-1} = I_n$  we see that  $\det(A)\det(A^{-1}) = 1$ . The only way the product of the two integers  $\det(A)$  and  $\det(A^{-1})$  can be 1 is that they are both 1 or both -1. Therefore,  $\det(A) = 1$  or  $\det(A) = -1$ .

$$6.2.38 \quad \det(A^T A) = \det(A^T) \det(A) = [\det(A)]^2 = 9$$

$$\begin{array}{c} \uparrow \\ \text{Theorem 6.2.6} \end{array} \quad \begin{array}{c} \uparrow \\ \text{Theorem 6.2.1} \end{array}$$

$$6.2.39 \quad \det(A^T A) = \det(A^T) \det(A) = [\det(A)]^2 > 0$$

$$\begin{array}{c} \uparrow \\ \text{Theorem 6.2.6} \end{array} \quad \begin{array}{c} \uparrow \\ \text{Theorem 6.2.1} \end{array}$$

**6.2.40** By Exercise 38,  $\det(A^T A) = [\det(A)]^2$ . Since  $A$  is orthogonal,  $A^T A = I_n$  so that

$$1 = \det(I_n) = \det(A^T A) = [\det(A)]^2 \text{ and } \det(A) = \pm 1.$$

**6.2.41**  $\det(A) = \det(A^T) = \det(-A) = (-1)^n(\det A) = -\det(A)$ , so that  $\det(A) = 0$ . We have used Theorems 6.2.1 and 6.2.3a.

$$\begin{aligned} 6.2.42 \quad \det(A^T A) &\stackrel{\text{Step 1}}{=} \det((QR)^T QR) = \det(R^T Q^T QR) \stackrel{\text{Step 3}}{=} \det(R^T I_m R) \\ &= \det(R^T R) \stackrel{\text{Step 5}}{=} \det(R^T) \det(R) \stackrel{\text{Step 6}}{=} (\det R)^2 \stackrel{\text{Step 7}}{=} (\prod_{i=1}^m r_{ii})^2 > 0. \end{aligned}$$

In Step 1 we are using the definition of matrix  $A$ . Equation 3 holds since the column vectors of matrix  $Q$  are orthonormal. In Steps 5 and 6 we are using Theorems 6.2.6 and 6.2.1, respectively. Finally, Equation 7 holds since matrix  $R$  is triangular.

$$\begin{aligned} 6.2.43 \quad \det(A^T A) &= \det\left(\begin{bmatrix} \vec{v}^T \\ \vec{w}^T \end{bmatrix} [\vec{v} \ \vec{w}]\right) = \det\begin{bmatrix} \vec{v} \cdot \vec{v} & \vec{v} \cdot \vec{w} \\ \vec{v} \cdot \vec{w} & \vec{w} \cdot \vec{w} \end{bmatrix} = \det\begin{bmatrix} \|\vec{v}\|^2 & \vec{v} \cdot \vec{w} \\ \vec{v} \cdot \vec{w} & \|\vec{w}\|^2 \end{bmatrix} \\ &= \|\vec{v}\|^2 \|\vec{w}\|^2 - (\vec{v} \cdot \vec{w})^2 \geq 0 \text{ by the Cauchy-Schwarz inequality (Theorem 5.1.11).} \end{aligned}$$

**6.2.44 a** We claim that  $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \neq \vec{0}$  if and only if the vectors  $\vec{v}_2, \dots, \vec{v}_n$  are linearly independent. If the vectors  $\vec{v}_2, \dots, \vec{v}_n$  are linearly independent, then we can find a basis  $\vec{x}, \vec{v}_2, \dots, \vec{v}_n$  of  $\mathbb{R}^n$  (any vector  $\vec{x}$  that is not in span  $(\vec{v}_2, \dots, \vec{v}_n)$  will do). Then  $\vec{x} \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) = \det[\vec{x} \ \vec{v}_2 \cdots \vec{v}_n] \neq 0$ , so that  $\vec{v}_2 \times \cdots \times \vec{v}_n \neq \vec{0}$ . Conversely, suppose that

$\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \neq 0$ ; say the  $i$ th component of this vector is nonzero. Then  $0 \neq \vec{e}_i \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) = \det[\vec{e}_i \vec{v}_2 \cdots \vec{v}_n]$ , so that the vectors  $\vec{v}_2, \dots, \vec{v}_n$  are linearly independent (being columns of an invertible matrix).

b  $i$ th component of  $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \det \begin{bmatrix} 1 & 1 & & 1 \\ \vec{e}_i & \vec{e}_2 & \cdots & \vec{e}_n \\ 1 & 1 & & 1 \end{bmatrix} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$

so  $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n = \vec{e}_1$ .

c  $\vec{v}_i \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{v}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix} = 0$

for any  $2 \leq i \leq n$  since the above matrix has two identical columns.

d Compare the  $i$ th components of the two vectors:

$$\det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix} \text{ and } \det \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vec{e}_i & \vec{v}_3 & \vec{v}_2 & \cdots & \vec{v}_n \\ 1 & 1 & 1 & & 1 \end{bmatrix}.$$

The two determinants differ by a factor of  $-1$  by Theorem 6.2.3b, so that

$$\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n = -\vec{v}_3 \times \vec{v}_2 \times \cdots \times \vec{v}_n.$$

e  $\det[\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \vec{v}_2 \vec{v}_3 \cdots \vec{v}_n] = (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \|\vec{v}_2 \times \cdots \times \vec{v}_n\|^2$

f In Definition 6.1.1 we saw that the “old” cross product satisfies the defining equation of the “new” cross product:  
 $\vec{x} \cdot (\vec{v}_2 \times \vec{v}_3) = \det[\vec{x} \vec{v}_2 \vec{v}_3]$ .

**6.2.45**  $f(x)$  is a linear function, so  $f'(x)$  is the coefficient of  $x$  (the slope). Expanding down the first column, we see

that the coefficient of  $x$  is  $-\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} = -24$ , so  $f'(x) = -24$ .

**6.2.46** a  $\det \begin{bmatrix} a & 3 & d \\ b & 3 & e \\ c & 3 & f \end{bmatrix} = 3 \det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 21$ . We have used Theorem 6.2.3.

b  $\det \begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix} = \det \begin{bmatrix} a & 2(1)+1 & d \\ b & 2(2)+1 & e \\ c & 2(3)+1 & f \end{bmatrix} = \det \begin{bmatrix} a & 2(1) & d \\ b & 2(2) & e \\ c & 2(3) & f \end{bmatrix} + \det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 2 \det \begin{bmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{bmatrix} + \det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 2 \cdot 11 + 7 = 29$ . We have used Theorem 6.2.2 and Theorem 6.2.3a.

**6.2.47** Yes! For example,  $T \begin{bmatrix} x & b \\ y & d \end{bmatrix} = dx + by$  is given by the matrix  $[d \ b]$ , so that  $T$  is linear in the first column.

**6.2.48** Since  $\vec{v}_2, \dots, \vec{v}_n$  are linearly independent,  $T(\vec{x}) = 0$  only if  $\vec{x}$  is a linear combination of the  $\vec{v}_i$  ‘s, (otherwise the matrix  $[\vec{x} \vec{v}_2 \cdots \vec{v}_n]$  is invertible, and  $T(\vec{x}) \neq 0$ ). Hence, the kernel of  $T$  is the span of  $\vec{v}_2, \dots, \vec{v}_n$ , an  $(n-1)$ -dimensional subspace of  $\mathbb{R}^n$ . The image of  $T$  is the real line  $\mathbb{R}$  (since it must be 1-dimensional).

**6.2.49** For example, we can start with an upper triangular matrix  $B$  with  $\det(B) = 13$ , such as  $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 13 \end{bmatrix}$ .

Adding the first row of  $B$  to both the second and the third to make all entries nonzero, we end up with  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 14 \end{bmatrix}$ . Note that  $\det(A) = \det(B) = 13$ .

**6.2.50** There are many ways to do this problem; here is one possible approach:

Subtracting the second to last row from the last, we can make the last row into

$$[0 \ 0 \ \cdots \ 0 \ 1].$$

Now expanding along the last row we see that  $\det(M_n) = \det(M_{n-1})$ .

Since  $\det(M_1) = 1$  we can conclude that  $\det(M_n) = 1$  for all  $n$ .

**6.2.51** Notice that it takes  $n$  row swaps (swap row  $i$  with  $n + i$  for each  $i$  between 1 and  $n$ ) to turn  $A$  into  $I_{2n}$ . So,  $\det(A) = (-1)^n \det(I_{2n}) = (-1)^n$ .

**6.2.52** a We build  $B$  column-by-column:

$$\left[ T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \left[ \begin{bmatrix} d \\ -c \end{bmatrix} \ \begin{bmatrix} -b \\ a \end{bmatrix} \right] = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

b  $\det(A) = ad - bc = \det(B)$ . The two determinants are equal.

$$c \ BA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & 0 \\ 0 & -cb + ad \end{bmatrix} = (ad - bc)I_2.$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & -cb + da \end{bmatrix} = (ad - bc)I_2 \text{ also.}$$

d Any vector  $\vec{u}$  in the image of  $A$  will be of the form  $c_1 \begin{bmatrix} a \\ c \end{bmatrix} + c_2 \begin{bmatrix} b \\ d \end{bmatrix}$ . We note that  $B \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} ad - bc \\ -ca + ac \end{bmatrix} = \vec{0}$ . The same is true of  $B \begin{bmatrix} b \\ d \end{bmatrix}$ . Thus, anything in the image of  $A$  will be in the kernel of  $B$ . Since both matrices have a rank of 1, the dimensions of the kernel and image of each will be exactly 1. So, it must be that  $\text{im}(A) = \ker(B)$ .

Also, any vector  $\vec{u}$  in the image of  $B$  will be of the form  $c_1 \begin{bmatrix} d \\ -c \end{bmatrix} + c_2 \begin{bmatrix} -b \\ a \end{bmatrix}$ . However, we see that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -ab + ba \\ -bc + ad \end{bmatrix} = \vec{0}$ . The same is true for  $A \begin{bmatrix} d \\ -c \end{bmatrix}$ . Thus, by the same reasoning as above, the image of  $B$  will equal the kernel of  $A$ .

$$e \ A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} B.$$

**6.2.53** a See Exercise 37.

b If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$  has integer entries.

$$6.2.54 \quad f(t) = (\det(A + tB))^2 - 1 = \left( \det \begin{bmatrix} a_1 + tb_1 & a_2 + tb_2 \\ a_3 + tb_3 & a_4 + tb_4 \end{bmatrix} \right)^2 - 1 \text{ assuming } A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

Then the determinant above is a polynomial of degree  $\leq 2$  so its square is a polynomial of degree  $\leq 4$ . Hence  $f(t)$  is a polynomial of degree  $\leq 4$ .

Since  $A, A + B, A + 2B, A + 3B, A + 4B$  are invertible and their inverses have integer entries, by Exercise 53a, it follows that their determinants are  $\pm 1$ . Hence  $f(0) = f(1) = f(2) = f(3) = f(4) = 0$ .

Since  $f$  is a polynomial of degree  $\leq 4$  with at least 5 roots, it follows that  $f(t) = 0$  for all  $t$ , in particular for  $t = 5$ , so  $\det(A + 5B) = \pm 1$ . Hence  $A + 5B$  is an invertible  $2 \times 2$  matrix whose inverse has integer entries by Exercise 53b.

**6.2.55** We start out with a preliminary remark: If a square matrix  $A$  has two equal rows, then  $D(A) = 0$ . Indeed, if we swap the two equal rows and call the resulting matrix  $B$ , then  $B = A$ , so that  $D(A) = D(B) = -D(A)$ , by property b, and  $D(A) = 0$  as claimed.

Next we need to understand how the elementary row operations affect  $D$ . Properties a and b tell us about how row multiplications and row swaps, but we still need to think about row additions.

We will show that if  $B$  is obtained from  $A$  by adding  $k$  times the  $i^{th}$  row to the  $j^{th}$ , then  $D(B) = D(A)$ . Let's label the row vectors of  $A$  by  $\vec{v}_1, \dots, \vec{v}_n$ . By linearity of  $D$  in the  $j^{th}$  row (property c) we have

$$D(B) = D \left( \begin{bmatrix} \cdots & \vdots & \cdots \\ \cdots & \vec{v}_i & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \vec{v}_j + k\vec{v}_i & \cdots \\ \cdots & \vdots & \cdots \end{bmatrix} \right) = D(A) + kD \left( \begin{bmatrix} \cdots & \vdots & \cdots \\ \cdots & \vec{v}_i & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \vec{v}_i & \cdots \\ \cdots & \vdots & \cdots \end{bmatrix} \right) = D(A).$$

Note that in the last step we have used the preliminary remark. Now, using the terminology introduced on Page 265, we can write  $D(A) = (-1)^s k_1 k_2 \cdots k_r D(\text{rref } A)$ .

Next we observe that  $D(\text{rref } A) = \det(\text{rref } A)$  for all square matrices  $A$ . Indeed, if  $A$  is invertible, then  $\text{rref}(A) = I_n$ , and  $D(I_n) = 1 = \det(I_n)$  by property c of function  $D$ . If  $A$  fails to be invertible, then  $D(\text{rref } A) = 0 = \det(\text{rref } A)$  by linearity in the last row.

It follows that  $D(A) = (-1)^s k_1 k_2 \cdots k_r D(\text{rref } A) = (-1)^s k_1 k_2 \cdots k_r \det(\text{rref } A) = \det(A)$  for all square matrices, as claimed.

**6.2.56 a** We show first that  $D$  is linear in the  $i$ th row.

$$D \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vdots \\ \vec{v}_n \end{bmatrix} = \frac{1}{\det M} \det \begin{bmatrix} \vec{v}_1 M \\ \vdots \\ \vec{x} M \\ \vdots \\ \vec{v}_n M \end{bmatrix}$$

$\uparrow \quad \uparrow$

$A \quad AM$

The entries in the  $i$ th row of  $AM$  are linear combinations of the components  $x_i$  of the vector  $\vec{x}$ , while the other entries of  $AM$  are constants. Therefore,  $\det(AM)$  is a linear combination of the  $x_i$  (expand along the  $i^{th}$  row).

Since  $\frac{1}{\det M}$  is a constant, we have  $D \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vdots \\ \vec{v}_n \end{bmatrix} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$  for some constants  $c_i$ , as claimed.

c The property  $D(I_n) = 1$  is obvious.

It now follows from Exercise 41 that  $\det(A) = D(A) = \frac{\det(AM)}{\det(M)}$  and therefore  $\det(AM) = \det(A) \det(M)$ .

**6.2.57** Note that matrix  $A_1$  is invertible, since  $\det(A_1) \neq 0$ . Now

$$T \begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix} = [A_1 \ A_2] \begin{bmatrix} \vec{y} \\ \vec{x} \end{bmatrix} = A_1\vec{y} + A_2\vec{x} = \vec{0} \text{ when } A_1\vec{y} = -A_2\vec{x}, \text{ or,}$$

$\vec{y} = -A_1^{-1}A_2\vec{x}$ . This shows that for every  $\vec{x}$  there is a unique  $\vec{y}$  (that is,  $\vec{y}$  is a function of  $\vec{x}$ ); furthermore, this function is linear, with matrix  $M = -A_1^{-1}A_2$ .

**6.2.58** Using the approach of Exercise 57, we have  $A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ ,

and  $M = -A_1^{-1}A_2 = \begin{bmatrix} 1 & -8 \\ -1 & 3 \end{bmatrix}$ . The function is  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ .

Alternatively, we can solve the linear system

$$\begin{aligned} y_1 + 2y_2 + x_1 + 2x_2 &= 0 \\ 3y_1 + 7y_2 + 4x_1 + 3x_2 &= 0 \end{aligned}$$

Gaussian Elimination gives

$$\begin{aligned} y_1 - x_1 + 8x_2 &= 0 & y_1 &= x_1 - 8x_2 \\ \text{and} \\ y_2 + x_1 - 3x_2 &= 0 & y_2 &= -x_1 + 3x_2 \end{aligned}$$

**6.2.59**  $\det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 = \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , but these matrices fail to be similar.

**6.2.60** We argue using induction on  $n$ . The base case ( $n = 2$ ) is discussed in the text. Now we assume that  $B$  is obtained from the  $n \times n$  matrix  $A$  by adding  $k$  times the  $p^{th}$  row to the  $q^{th}$  row.

We will evaluate the determinant of  $B$  by expanding across the  $i^{th}$  row (where  $i$  is neither  $p$  nor  $q$ ).

$$\begin{aligned} \det(B) &= \sum_{j=1}^n (-1)^{i+j} b_{ij} \det(B_{ij}) \\ &= \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(B_{ij}) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}) = \det(A) \end{aligned}$$

Note that the  $(n - 1) \times (n - 1)$  matrix  $B_{ij}$  is obtained from  $A_{ij}$  by adding  $k$  times some row to another row. Now,  $\det(B_{ij}) = \det(A_{ij})$  by induction hypothesis.

6.2.61 We follow the hint:  $\begin{bmatrix} I_n & 0 \\ -C & A \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ -CA + AC & -CB + AD \end{bmatrix}$   
 $= \begin{bmatrix} A & B \\ 0 & AD - CB \end{bmatrix}$ . So,  $\det\left(\begin{bmatrix} I_n & 0 \\ -C & A \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A) \det(AD - CB)$ .

Thus,  $\det(I_n) \det(A) \det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A) \det(AD - CB)$ , which leads to

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(AD - CB), \text{ since } \det(A) \neq 0.$$

6.2.62 a We compute  $\begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & -CA^{-1}B + D \end{bmatrix}$ . Since the matrix  $\begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix}$  is invertible (its determinant is 1), the product  $\begin{bmatrix} A & B \\ 0 & -CA^{-1}B + D \end{bmatrix}$  will have the same rank as  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ , namely,  $n$ . With  $A$  being invertible, this implies that  $-CA^{-1}B + D = 0$ , or  $CA^{-1}B = D$ , as claimed.

- b Take determinants on both sides of the equation  $D = CA^{-1}B$  from part (a) to find that  $\det(D) = \det(C)(\det A)^{-1} \det(B)$ , or  $\det(A) \det(D) - \det(B) \det(C) = 0$ , proving the claim.

6.2.63 Let  $M_n$  be the number of multiplications required to compute the determinant of an  $n \times n$  matrix by Laplace expansion. We will use induction on  $n$  to prove that  $M_n > n!$ , for  $n \geq 3$ .

In the lowest applicable case,  $n = 3$ , we can check that  $M_3 = 9$  and  $3! = 6$ .

Now let's do the induction step. If  $A$  is an  $n \times n$  matrix, then  $\det(A) = a_{11} \det(A_{11}) + \cdots + (-1)^{n+1} a_{n1} \det(A_{n1})$ , by Theorem 6.2.10. We need to compute  $n$  determinants of  $(n - 1) \times (n - 1)$  matrices, and then do  $n$  extra multiplications  $a_{i1} \det(A_{i1})$ , so that  $M_n = nM_{n-1} + n$ . If  $n > 3$ , then  $M_{n-1} > (n - 1)!$ , by induction hypothesis, so that  $M_n > n(n - 1)! + n > n!$ , as claimed.

6.2.64 To compute  $\det(A)$  for an  $n \times n$  matrix  $A$  by Laplace expansion,  $\det(A) = a_{11} \det(A_{11}) - a_{21} \det(A_{21}) + \cdots + (-1)^{n+1} a_{n1} \det(A_{n1})$ , we first need to compute the  $n$  minors, which requires  $nL_{n-1}$  operations; then we compute the  $n$  products  $a_{i1} \det(A_{i1})$ ; and finally we have to do  $n - 1$  additions. Altogether,

$$L_n = nL_{n-1} + n + (n - 1) = nL_{n-1} + 2n - 1.$$

Now we can prove the formula  $\frac{L_n}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!}$  by induction on  $n$ .

For  $n = 2$ , the formula gives  $\frac{L_2}{2!} = 1 + 1 - \frac{1}{2}$ , or  $L_2 = 3$ , which is correct: We have to perform 2 multiplications and 1 addition to compute the determinant of a  $2 \times 2$  matrix.

For an  $n \times n$  matrix  $A$  we can use the recursive formula derived above to see that  $\frac{L_n}{n!} = \frac{nL_{n-1} + 2n - 1}{n!} = \frac{L_{n-1}}{(n-1)!} + \frac{2}{(n-1)!} - \frac{1}{n!}$ . Applying the induction hypothesis to the first summand, we find that  $\frac{L_n}{n!} = \frac{L_{n-1}}{(n-1)!} + \frac{2}{(n-1)!} - \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!}$ , as claimed.

Now recall from the theory of Taylor series in calculus that  $e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!} + \cdots$ . Thus  $L_n = (1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!})n! < en!$ , as claimed.

**6.2.65** a. Using Laplace expansion along the first row, we find  $d_n = \det(M_n) = \det(M_{n-1}) - \det \begin{bmatrix} -1 & * \\ 0 & M_{n-2} \end{bmatrix} = \det(M_{n-1}) + \det(M_{n-2}) = d_{n-1} + d_{n-2}$ .

b.  $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 5, \dots, d_{10} = 89$

c. Since  $d_n > 0$  for all positive integers  $n$ , the matrix  $M_n$  is always invertible.

**6.2.66** a. Using Laplace expansion along the first row, we find  $d_n = \det(M_n) = \det(M_{n-1}) - \det \begin{bmatrix} 1 & * \\ 0 & M_{n-2} \end{bmatrix} = \det(M_{n-1}) - \det(M_{n-2}) = d_{n-1} - d_{n-2}$ .

b.  $d_1 = 1, d_2 = 0, d_3 = -1, d_4 = -1, d_5 = 0, d_6 = 1, d_7 = 1, d_8 = 0$ .

c. Since  $d_4 = -d_1$  and  $d_5 = -d_2$ , the formula  $d_{n+3} = -d_n$  holds for all positive integers  $n$ . (Give a formal proof by induction.) Now  $d_{n+6} = -d_{n+3} = -(-d_n) = d_n$  for all positive integers  $n$ , meaning that the sequence  $d_n$  has a period of six.

**6.2.67** Let  $k$  be the number of pattern entries to the left and above  $a_{ij}$ . Then the number of pattern entries to the right and above  $a_{ij}$  is  $i - 1 - k$ , since there are  $i - 1$  rows above  $a_{ij}$ , each of which contains exactly one pattern entry. Likewise, the number of pattern entries to the left and below  $a_{ij}$  is  $j - 1 - k$ . Now the number of inversions involving  $a_{ij}$  is  $(i - 1 - k) + (j - 1 - k) = i + j - 2 - 2k$ . It follows that  $(-1)^{\#(\text{inversions involving } a_{ij})} = (-1)^{i+j-2-2k} = (-1)^{i+j}$ . This means that the number of inversions involving  $a_{ij}$  is even if (and only if)  $i + j$  is even, as claimed.

**6.2.68** Using Exercise 67 and the terminology introduced in the proof of Theorem 6.2.10, we have

$$\text{sgn}P = (-1)^{\#(\text{inversions in } P)} = (-1)^{\#(\text{inversions involving } a_{ij})} (-1)^{\#(\text{inversions not involving } a_{ij})}$$

$$= (-1)^{i+j} (-1)^{\#(\text{inversions in } P_{ij})} = (-1)^{i+j} \text{sgn}(P_{ij}).$$

## Section 6.3

**6.3.1** By Theorem 2.4.10, the area equals  $|\det \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}| = |-50| = 50$ .

**6.3.2** By Theorem 2.4.10, Area =  $\frac{1}{2} \left[ \det \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} \right] = \frac{1}{2} |-50| = 25$

**6.3.3** Area of triangle =  $\frac{1}{2} \left| \det \begin{bmatrix} 6 & 1 \\ -2 & 4 \end{bmatrix} \right| = 13$  (See Figure 6.1.)

**6.3.4** Note that area of triangle =  $\frac{1}{2} \left[ \det \begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix} \right]$ . (See Figure 6.2.)

On the other hand,  $\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} a_1 & b_1 - a_1 & c_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 \\ 1 & 0 & 0 \end{bmatrix}$ , by subtracting the first column from the second and third.

This, in turn, equals  $\det \begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix}$ , by expanding across the bottom row.

Therefore, area of triangle =  $\frac{1}{2} \left[ \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} \right]$ .

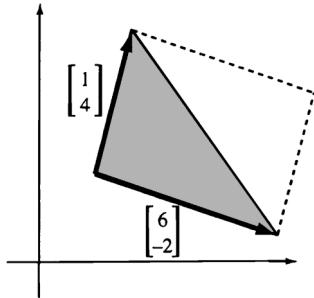


Figure 6.1: for Problem 6.3.3.

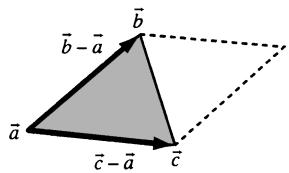


Figure 6.2: for Problem 6.3.4.

**6.3.5** The volume of the tetrahedron  $T_0$  defined by  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  is  $\frac{1}{3}(\text{base})(\text{height}) = \frac{1}{6}$ .

Here we are using the formula for the volume of a pyramid. (See Figure 6.3.)

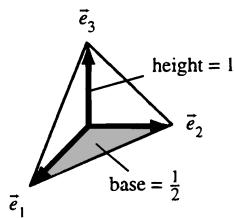


Figure 6.3: for Problem 6.3.5.

The tetrahedron  $T$  defined by  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  can be obtained by applying the linear transformation with matrix  $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$  to  $T_0$ .

Now we have  $\text{vol}(T) = |\det[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]| \text{vol}(T_0) = \frac{1}{6} |\det[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]| = \frac{1}{6} V(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ .

↑

Theorem 6.3.7 and Page 283

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Theorem 6.3.4 and Page 280

**6.3.6** From Exercise 5 we know that volume of tetrahedron  $= \frac{1}{6} \left[ \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} \right]$ , and Exercise 4 tells us that

area of triangle  $= \frac{1}{2} \left[ \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} \right]$ , so that area of tetrahedron  $= \frac{1}{3}$  (area of triangle).

We can see this result more directly if we think of the tetrahedron as an inverted pyramid whose base is the triangle and whose height is 1. (See Figure 6.4.)

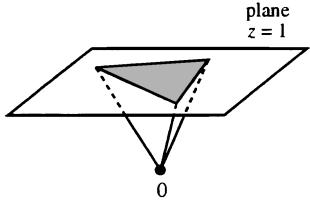


Figure 6.4: for Problem 6.3.6.

The three vertices of the shaded triangle are  $\begin{bmatrix} a_1 \\ a_2 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix}$ .

6.3.7 Area =  $\frac{1}{2} \left[ \det \begin{bmatrix} 10 & -2 \\ 11 & 13 \end{bmatrix} \right] + \frac{1}{2} \left[ \det \begin{bmatrix} 8 & 10 \\ 2 & 11 \end{bmatrix} \right] = 110$ . (See Figure 6.5.)

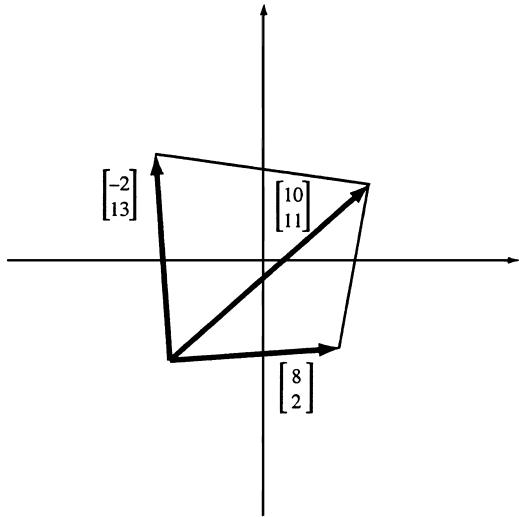


Figure 6.5: for Problem 6.3.7.

6.3.8 We need to show that both sides of the equation in Theorem 6.3.3 give zero.

$|\det(A)| = 0$  since  $A$  is not invertible. On the other hand, since  $A$  is not invertible, the  $\vec{v}_i$  will be linearly dependent, i.e., one of the  $\vec{v}_i$  will be redundant. This implies that  $\vec{v}_i^{\parallel} = \vec{v}_i$  and  $\vec{v}_i^{\perp} = \vec{0}$ , so that the right-hand side of the equation is 0, as claimed.

6.3.9 Using linearity in the second column, we find that  $\det [\vec{v}_1 \vec{v}_2] = \det [\vec{v}_1 (\vec{v}_2^{\parallel} + \vec{v}_2^{\perp})] = \det [\underbrace{\vec{v}_1 \vec{v}_2^{\parallel}}_{\text{noninvertible}}] + \det [\vec{v}_1 \vec{v}_2^{\perp}]$ . Thus the two determinants are equal.

**6.3.10**  $|\det(A)| \leq \|\vec{v}_1\| \|\vec{v}_2\| \cdots \|\vec{v}_n\|$  since  $|\det(A)| = \|\vec{v}_1\| \|\vec{v}_2^\perp\| \cdots \|\vec{v}_n^\perp\|$  and  $\|\vec{v}_i\| \geq \|\vec{v}_i^\perp\|$ . The equality holds if  $\|\vec{v}_i\| = \|\vec{v}_i^\perp\|$  for all  $i$ , that is, if the  $\vec{v}_i$ 's are mutually perpendicular.

**6.3.11** The matrix of the transformation  $T$  with respect to the basis  $\vec{v}_1, \vec{v}_2$  is  $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ , so that  $\det(A) = \det(B) = 12$ , by Theorem 6.2.7.

**6.3.12** Denote the columns by  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ . By Theorem 6.3.3 and Exercise 6.3.8 we know that  $|\det(A)| \leq \|\vec{v}_1\| \|\vec{v}_2\| \|\vec{v}_3\| \|\vec{v}_4\|$ ; equality holds if the columns are orthogonal. Since the entries of the  $\vec{v}_i$  are 0, 1, and  $-1$ , we have  $\|\vec{v}_i\| \leq \sqrt{1+1+1+1}=2$ . Therefore,  $|\det A| \leq 16$ .

To build an example where  $\det(A) = 16$  we want all 1's and  $-1$ 's as entries, and the columns need to be orthogonal. A little experimentation produces  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$  (there are other solutions). Note that we need to *check* that  $\det(A) = 16$  (and not  $-16$ ).

**6.3.13** By Theorem 6.3.6, the desired 2-volume is

$$\sqrt{\det \left( \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \right)} = \sqrt{\det \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}} = \sqrt{20}.$$

**6.3.14** By Theorem 6.3.6, the desired 3-volume is

$$\sqrt{\det \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} \right)} = \sqrt{\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{bmatrix}} = \sqrt{6}.$$

**6.3.15** If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  are linearly dependent and if  $A = [\vec{v}_1 \cdots \vec{v}_m]$ , then  $\det(A^T A) = 0$  since  $A^T A$  and  $A$  have equal and nonzero kernels (by Theorem 5.4.2), hence  $A^T A$  fails to be invertible.

On the other hand, since the  $\vec{v}_i$  are linearly dependent, at least one of them will be redundant. For such a redundant  $\vec{v}_i$ , we will have  $\vec{v}_i = \vec{v}_i^\parallel$  and  $\vec{v}_i^\perp = \vec{0}$ , so that  $V(\vec{v}_1, \dots, \vec{v}_m) = 0$ , by Definition 6.3.5. This discussion shows that  $V(\vec{v}_1, \dots, \vec{v}_m) = 0 = \sqrt{\det(A^T A)}$  if the vectors  $\vec{v}_1, \dots, \vec{v}_m$  are linearly dependent.

**6.3.16** False

If  $T$  is given by  $A = 2I_3$  then  $|\det(A)| = 8$ . But if  $\Omega$  is the square defined by  $\vec{e}_1, \vec{e}_2$  in  $\mathbb{R}^3$  (of area 1), then  $T(\Omega)$  is the square defined by  $2\vec{e}_1, 2\vec{e}_2$  and the area of  $T(\Omega)$  is 4.

**6.3.17 a** Let  $\vec{w} = \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3$ . Note that  $\vec{w}$  is orthogonal to  $\vec{v}_1, \vec{v}_2$  and  $\vec{v}_3$ , by Exercise 6.2.44c. Then  $V(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}) = V(\vec{v}_1, \vec{v}_2, \vec{v}_3) \|\vec{w}\| = V(\vec{v}_1, \vec{v}_2, \vec{v}_3) \|\vec{w}\|$ .

↑

by Definition 6.3.5.

b By Exercise 6.2.44e,

$$V(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3) = |\det[\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_1 \times \vec{v}_2 \times \vec{v}_3]| = |\det[\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3 \quad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]| = \|\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3\|^2.$$

c By parts a and b,  $V(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \|\vec{v}_1 \times \vec{v}_2 \times \vec{v}_3\|$ . If the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are linearly dependent, then both sides of the equation are 0, by Exercise 15 and Exercise 6.2.44a.

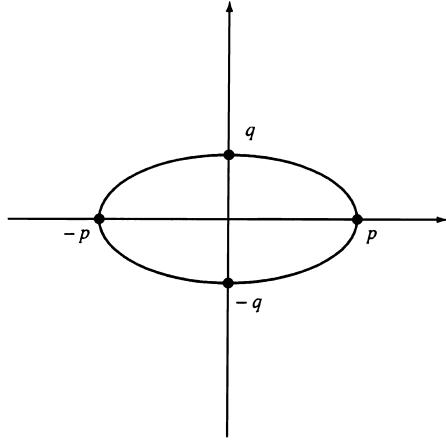


Figure 6.6: for Problem 6.3.18a.

6.3.18 a (See Figure 6.6.)  $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} p \cdot \cos(t) \\ q \cdot \sin(t) \end{bmatrix}$ , the ellipse with semi-axis  $\pm \begin{bmatrix} p \\ 0 \end{bmatrix}$  and  $\pm \begin{bmatrix} 0 \\ q \end{bmatrix}$ .

$$(\text{area of the ellipse}) = |\det(A)|(\text{area of the unit circle}) = pq\pi$$

b By Theorem 6.3.7,  $|\det(A)| = \frac{\text{area of the ellipse}}{\text{area of the unit circle}} = \frac{ab\pi}{\pi} = ab$  so  $|\det(A)| = ab$ .

c The unit circle consists of all vectors of the form  $\vec{x} = \cos(t)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sin(t)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ; its image is the ellipse consisting of all vectors

$$T(\vec{x}) = \underbrace{\cos(t) 2\sqrt{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{semi-major axis}} + \underbrace{\sin(t) \sqrt{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{semi-minor axis}}. \quad (\text{See Figure 6.7.})$$

6.3.19  $\det[\vec{v}_1 \vec{v}_2 \vec{v}_3] = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = \|\vec{v}_1\| \|\vec{v}_2 \times \vec{v}_3\| \cos \theta$  where  $\theta$  is the angle between  $\vec{v}_1$  and  $\vec{v}_2 \times \vec{v}_3$  so  $\det[\vec{v}_1 \vec{v}_2 \vec{v}_3] > 0$  if and only if  $\cos \theta > 0$ , i.e., if and only if  $\theta$  is acute ( $0 \leq \theta \leq \frac{\pi}{2}$ ). (See Figure 6.8.)

6.3.20 By Exercise 19,  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  constitute a positively oriented basis if and only if  $\det[\vec{v}_1 \vec{v}_2 \vec{v}_3] > 0$ . Assume that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is such a basis. We want to show that  $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$  is positively oriented if and only if  $\det(A) > 0$ . We have  $\det[A\vec{v}_1 A\vec{v}_2 A\vec{v}_3] = \det(A[\vec{v}_1 \vec{v}_2 \vec{v}_3]) = \det(A) \det[\vec{v}_1 \vec{v}_2 \vec{v}_3]$  so since  $\det[\vec{v}_1 \vec{v}_2 \vec{v}_3] > 0$  by assumption,  $\det[A\vec{v}_1 A\vec{v}_2 A\vec{v}_3] > 0$  if and only if  $\det(A) > 0$ . Hence  $A$  is orientation preserving if and only if  $\det(A) > 0$ .

6.3.21 a Reverses

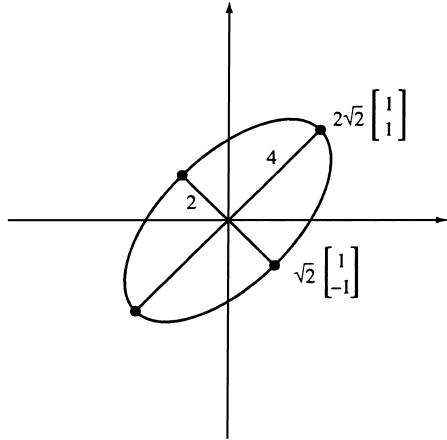


Figure 6.7: for Problem 6.3.18c.

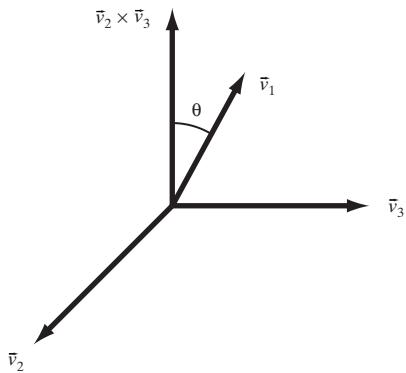


Figure 6.8: for Problem 6.3.19.

Consider  $\vec{v}_2$  and  $\vec{v}_3$  in the plane (not parallel), and let  $\vec{v}_1 = \vec{v}_2 \times \vec{v}_3$ ; then  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is a positively oriented basis, but  $T(\vec{v}_1) = -\vec{v}_1, T(\vec{v}_2) = \vec{v}_2, T(\vec{v}_3) = \vec{v}_3$  is negatively oriented.

b Preserves

Consider  $\vec{v}_2$  and  $\vec{v}_3$  orthogonal to the line (not parallel), and let  $\vec{v}_1 = \vec{v}_2 \times \vec{v}_3$ ; then  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is a positively oriented basis, and  $T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = -\vec{v}_2, T(\vec{v}_3) = -\vec{v}_3$  is positively oriented as well.

c Reverses

The standard basis  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  is positively oriented, but  $T(\vec{e}_1) = -\vec{e}_1, T(\vec{e}_2) = -\vec{e}_2, T(\vec{e}_3) = -\vec{e}_3$  is negatively oriented.

**6.3.22** Here  $A = \begin{bmatrix} 3 & 7 \\ 4 & 11 \end{bmatrix}$ ,  $\det(A) = 5$ ,  $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ , so by Theorem 6.3.8

$$x = \frac{\det \begin{bmatrix} 1 & 7 \\ 3 & 11 \end{bmatrix}}{5} = -2, y = \frac{\det \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}}{5} = 1.$$

**6.3.23** Here  $A = \begin{bmatrix} 5 & -3 \\ -6 & 7 \end{bmatrix}$ ,  $\det(A) = 17$ ,  $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , so by Theorem 6.3.8

$$x_1 = \frac{\det \begin{bmatrix} 1 & -3 \\ 0 & 7 \end{bmatrix}}{17} = \frac{7}{17}, x_2 = \frac{\det \begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}}{17} = \frac{6}{17}.$$

**6.3.24** Here  $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}$ ,  $\det(A) = 146$ ,  $\vec{b} = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$ , so by Theorem 6.3.8,

$$x = \frac{\det \begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{bmatrix}}{146} = 1, y = \frac{\det \begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{bmatrix}}{146} = 2, z = \frac{\det \begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{bmatrix}}{146} = -1.$$

**6.3.25** By Theorem 6.3.9, the  $ij^{th}$  entry of  $\text{adj}(A)$  is given by  $(-1)^{i+j} \det(A_{ji})$ , so since

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$  for  $i = 1, j = 1$ , we get  $(-1)^2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ , and for  $i = 1, j = 2$  we get  $(-1)^3 \det \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$ , and so forth.

Completing this process gives  $\text{adj}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ , hence by Theorem 6.3.9,

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{-1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

**6.3.26** By Theorem 6.3.9,  $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$ , so if  $\det(A) = 1, A^{-1} = \text{adj}(A)$ . If  $A$  has integer entries then  $(-1)^{i+j} \det(A_{ji})$  will be an integer for all  $1 \leq i, j \leq n$ , hence  $\text{adj}(A)$  will have integer entries. Therefore,  $A^{-1}$  will also have integer entries.

**6.3.27** By Theorem 6.3.8, using  $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ ,  $\det(A) = a^2 + b^2$ ,  $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , we get

$$x = \det \begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix} \left( \frac{1}{a^2 + b^2} \right) = \frac{a}{a^2 + b^2}, y = \det \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \left( \frac{1}{a^2 + b^2} \right) = \frac{-b}{a^2 + b^2},$$

so  $x$  is positive,  $y$  is negative (since  $a, b > 0$ ), and  $x$  decreases as  $b$  increases.

**6.3.28** Here  $A = \begin{bmatrix} s & a \\ m & -h \end{bmatrix}$ ,  $\det(A) = -sh - ma$ ,  $\vec{b} = \begin{bmatrix} I^\circ + G \\ M_s + M^\circ \end{bmatrix}$  so, by Theorem 6.3.8

$$Y = \frac{\det \begin{bmatrix} I^\circ + G & a \\ M_s - M^\circ & -h \end{bmatrix}}{-sh - ma} = \frac{-h(I^\circ + G) - a(M_s - M^\circ)}{-sh - ma} = \frac{h(I^\circ + G) + a(M_s - M^\circ)}{sh + ma},$$

$$r = \frac{\det \begin{bmatrix} s & I^\circ + G \\ m & M_s - M^\circ \end{bmatrix}}{-sh - ma} = \frac{s(M_s - M^\circ) - m(I^\circ + G)}{-sh - ma} = \frac{m(I^\circ + G) - s(M_s - M^\circ)}{sh + ma}.$$

**6.3.29** By Theorem 6.3.8,

$$dx_1 = \frac{\det \begin{bmatrix} 0 & R_1 & -(1-\alpha) \\ 0 & 1-\alpha & -(1-\alpha)^2 \\ -R_2 de_2 & -R_2 & -\frac{(1-\alpha)^2}{\alpha} \end{bmatrix}}{D} = \frac{R_1 R_2 (1-\alpha)^2 de_2 - R_2 (1-\alpha)^2 de_2}{D}$$

$$dy_1 = \frac{\det \begin{bmatrix} -R_1 & 0 & -(1-\alpha) \\ \alpha & 0 & -(1-\alpha)^2 \\ R_2 & -R_2 de_2 & -\frac{(1-\alpha)^2}{\alpha} \end{bmatrix}}{D} = \frac{R_2 de_2 (R_1 (1-\alpha)^2 + \alpha (1-\alpha))}{D} > 0$$

$$dp = \frac{\det \begin{bmatrix} -R_1 & R_1 & 0 \\ \alpha & 1-\alpha & 0 \\ R_2 & -R_2 & -R_2 de_2 \end{bmatrix}}{D} = \frac{R_1 R_2 de_2}{D} > 0.$$

**6.3.30** Using the procedure outlined in Exercise 25, we find  $\text{adj}(A) = \begin{bmatrix} 18 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 3 \end{bmatrix}$ .

**6.3.31** Using the procedure outlined in Exercise 25, we find  $\text{adj}(A) = \begin{bmatrix} -6 & 0 & 1 \\ -3 & 5 & -2 \\ 4 & -5 & 1 \end{bmatrix}$ .

**6.3.32** Using the procedure outlined in Exercise 25, we find that  $\text{adj}(A) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ .

**6.3.33** Using the procedure outlined in Exercise 25, we find that  $\text{adj}(A) = \begin{bmatrix} 24 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ . Note that the matrix  $\text{adj}(A)$  is diagonal, and the  $i^{th}$  diagonal entry of  $\text{adj}(A)$  is the product of all  $a_{jj}$  where  $j \neq i$ .

**6.3.34** For an invertible  $n \times n$  matrix  $A$ ,  $A \text{adj}(A) = A(\det(A)A^{-1}) = \det(A)AA^{-1} = \det(A)I_n$ . The same is true for  $\text{adj}(A)A$ .

**6.3.35**  $\det(\text{adj}(A)) = \det(\det(A)A^{-1})$ . Taking the product  $\det(A)A^{-1}$  amounts to multiplying each row of  $A^{-1}$  by  $\det(A)$ , so that  $\det(\text{adj}(A)) = (\det A)^n \det(A^{-1}) = (\det A)^n \frac{1}{\det(A)} = (\det A)^{n-1}$ .

**6.3.36**  $\text{adj}(\text{adj}A) = \text{adj}(\det(A)A^{-1})$

$$\begin{aligned} &= \det(\det(A)A^{-1})(\det(A)A^{-1})^{-1} = (\det A)^n \det(A^{-1})(\det(A)A^{-1})^{-1} \\ &= (\det A)^{n-1}(\det(A)A^{-1})^{-1} = (\det A)^{n-1} \frac{1}{\det(A)}(A^{-1})^{-1} \\ &= (\det A)^{n-2}A. \end{aligned}$$

**6.3.37**  $\text{adj}(A^{-1}) = \det(A^{-1})(A^{-1})^{-1} = (\det A)^{-1}(A^{-1})^{-1} = (\text{adj}A)^{-1}$ .

$$\begin{aligned}
6.3.38 \quad & \text{adj}(AB) = \det(AB)(AB)^{-1} \\
&= \det(A)(\det(B)B^{-1})A^{-1} \\
&= \det(B)B^{-1}(\det(A)A^{-1}) \\
&= \text{adj}(B)\text{adj}(A).
\end{aligned}$$

6.3.39 Yes, let  $S$  be an invertible matrix such that  $AS = SB$ , or  $SB^{-1} = A^{-1}S$ . Multiplying both sides by  $\det(A) = \det(B)$ , we find that  $S(\det(B)B^{-1}) = (\det(A)A^{-1})S$ , or,  $S(\text{adj}B) = (\text{adj}A)S$ , as claimed.

6.3.40 The  $ij^{th}$  entry of the matrix  $B$  of  $T$  is ( $i^{th}$  component of  $T(\vec{e}_j)$ ) =  $\det(A_{\vec{e}_j, i})$ . Expand down the  $i^{th}$  column to see that this is the  $ij^{th}$  entry of  $\text{adj}(A)$ . Thus  $B = \text{adj}(A)$ . See Theorem 6.3.9.

6.3.41 If  $A$  has a nonzero minor  $\det(A_{ij})$ , then the  $n - 1$  columns of the invertible matrix  $A_{ij}$  will be independent, so that the  $n - 1$  columns of  $A$ , minus the  $j^{th}$ , will be independent as well. Thus, the rank of  $A$  (the dimension of the image) is at least  $n - 1$ .

Conversely, if  $\text{rank}(A) \geq n - 1$ , then we can find  $n - 1$  independent columns of  $A$ . The  $n \times (n - 1)$  matrix consisting of those  $n - 1$  columns will have rank  $n - 1$ , so that there will be exactly one redundant row (compare with Exercises 3.3.69 through 3.3.71). Omitting this redundant row produces an invertible  $(n - 1) \times (n - 1)$  submatrix of  $A$ , giving us a nonzero minor of  $A$ .

6.3.42 By Theorem 6.3.9,  $\text{adj}(A) = 0$  if (and only if) all the minors  $A_{ji}$  of  $A$  are zero. By Exercise 41, this is the case if (and only if)  $\text{rank}(A) \leq n - 2$ .

6.3.43 A direct computation shows that  $A(\text{adj}A) = (\text{adj}A)A = (\det A)(I_n)$  for all square matrices. Thus we have  $A(\text{adj}A) = (\text{adj}A)A = 0$  for noninvertible matrices, as claimed.

Let's write  $B = \text{adj}(A)$ , and let's verify the equation  $AB = (\det A)(I_n)$  for the diagonal entries; the verification for the off-diagonal entries is analogous. The  $i^{th}$  diagonal entry of  $AB$  is

$$[i^{th} \text{ row of } A] \begin{bmatrix} i^{th} \\ \text{column} \\ \text{of } B \end{bmatrix} = a_{i1}b_{1i} + \cdots + a_{in}b_{ni} = \sum_{j=1}^n a_{ij}b_{ji}.$$

Since  $B$  is the adjunct of  $A$ ,  $b_{ji} = (-1)^{j+i} \det(A_{ij})$ .

So, our summation equals

$$\sum_{j=1}^n a_{ij}(-1)^{i+j} \det(A_{ij})$$

which is our formula for Laplace expansion across the  $i^{th}$  row, and equals  $\det(A)$ , proving our claim for the diagonal entries.

6.3.44 The equation  $A(\text{adj}A) = 0$  from Exercise 43 means that  $\text{im}(\text{adj}A)$  is a subspace of  $\ker(A)$ . Thus  $\text{rank}(\text{adj}A) = \dim(\text{im}(\text{adj}A)) \leq \dim(\ker A) = n - \text{rank}(A) = n - (n - 1) = 1$ , implying that  $\text{rank}(\text{adj}A) \leq 1$ . Since  $\text{adj}(A) \neq 0$ , by Exercise 42, we can conclude that  $\text{rank}(\text{adj}A) = 1$ .

**6.3.45** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . We want  $A^T = \text{adj}(A)$ , or  $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ . So,  $a = d$  and  $b = -c$ . Thus, the equation  $A^T = \text{adj}(A)$  holds for all matrices of the form  $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ .

**6.3.46** In the simple case when  $f(x, y) = 1$  we have  $\int_{\Omega_2} f(x, y)dA = \int_{\Omega_2} dA = \text{area of } \Omega_2 = |\det M|$  and  $\int_{\Omega_1} g(u, v)dA = \int_{\Omega_1} dA = \text{area of } \Omega_1 = 1$ , so that  $\int_{\Omega_2} f(x, y)dA = |\det M| \cdot \int_{\Omega_1} g(u, v)dA$ .

This formula holds, in fact, for any continuous function  $f(x, y)$ ; see an introductory text in multivariable calculus for a justification.

**6.3.47** Note that  $\frac{1}{2} \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$  is the area of the triangle  $OP_1P_2$ , where  $O$  denotes the origin. This is likewise true for one-half the second matrix. See Theorem 2.4.10. However, because of the reversal in orientation,  $\frac{1}{2} \det \begin{bmatrix} x_3 & x_4 \\ y_3 & y_4 \end{bmatrix}$  is negative the area of the triangle  $OP_3P_4$ ; likewise for the last matrix. Finally, note that the area of the quadrilateral  $P_1P_2P_3P_4$  is equal to:

$$\begin{aligned} & \text{the area of triangle } OP_1P_2 + \text{ area of triangle } OP_2P_3 \\ & - \text{ area of triangle } OP_3P_4 - \text{ area of triangle } OP_4P_1. \end{aligned}$$

**6.3.48** In what follows, we will freely use the fact that an invertible linear transformation  $L$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  maps an ellipse into an ellipse (see Exercise 2.2.52).

Now consider a linear transformation  $L$  that transforms our 3-4-5 right triangle  $R$  into an equilateral triangle  $T$ . If we place the vertices of the right triangle  $R$  at the points  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ , and the vertices of the equilateral triangle  $T$  at  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ , then the transformation  $L$  has the matrix  $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$ , with  $\det(A) = \frac{1}{2\sqrt{3}}$ .

According to the hint,  $L$  will map the largest ellipse  $E$  inscribed into  $R$  into the circle  $C$  inscribed into  $T$ . The Figure 6.9 illustrates that the radius of  $C$  is  $\tan(\pi/6) = 1/\sqrt{3}$ , so that the area of  $C$  is  $\pi/3$ . Using the interpretation of the determinant as an expansion fact, we find that  $(\text{area of } C) = (\det A)(\text{area of } E)$ , or  $(\text{area of } E) = \frac{\text{area of } C}{\det(A)} = \frac{2\pi}{\sqrt{3}} \approx 3.6$

**6.3.49** We will use the terminology introduced in the solution of Exercise 48 throughout. Note that the transformation  $L^{-1}$ , with matrix  $A^{-1} = \begin{bmatrix} 2 & -2/\sqrt{3} \\ 0 & \sqrt{3} \end{bmatrix}$ , maps the circle  $C$  (with radius  $1/\sqrt{3}$ ) into the ellipse  $E$ . Now consider a radial vector  $\vec{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$  of  $C$ , and find the maximal value  $M$  and the minimal value  $m$  of  $\|A^{-1}\vec{v}\|^2 = \frac{4}{3} + \frac{1}{9} \sin^2 \theta - \frac{8}{3\sqrt{3}} (\sin \theta)(\cos \theta) = \frac{25}{18} - \frac{1}{18} (\cos 2\theta) - \frac{4}{3\sqrt{3}} (\sin 2\theta)$  (we are taking the square to facilitate the computations). Then  $\sqrt{M}$  and  $\sqrt{m}$  will be the lengths of the semi-axis of  $E$ . The function above is sinusoidal with average value  $\frac{25}{18}$  and amplitude  $\sqrt{\frac{1}{18^2} + \frac{16}{27}} = \frac{\sqrt{193}}{18}$ . Thus  $M = \frac{25+\sqrt{193}}{18}$  and  $m = \frac{25-\sqrt{193}}{18}$ , so that the length of the semi-major axis of  $E$  is

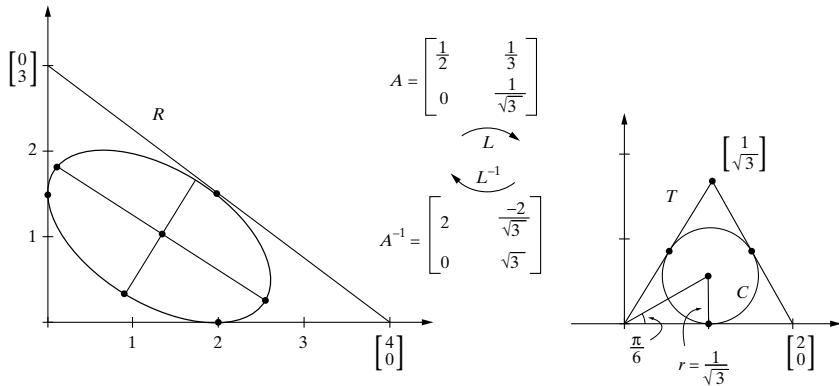


Figure 6.9: for Problem 6.3.48 and Problem 6.3.49.

$\sqrt{M} = \sqrt{\frac{25+\sqrt{193}}{18}} \approx 1.47$ , and for the semi-minor axis we get

$$\sqrt{m} = \sqrt{\frac{25-\sqrt{193}}{18}} \approx 0.79.$$

## True or False

Ch 6.TF.1 T, by Definition 6.1.1

Ch 6.TF.2 F; We have  $\det(4A) = 4^4 \det(A)$ , by Theorem 6.2.3a.

Ch 6.TF.3 F; Let  $A = B = I_5$ , for example

Ch 6.TF.4 T; We have  $\det(-A) = (-1)^6 \det(A) = \det(A)$ , by Theorem 6.2.3a.

Ch 6.TF.5 F; In fact,  $\det(A) = 0$ , since  $A$  fails to be invertible

Ch 6.TF.6 F; The matrix  $A$  fails to be invertible if  $\det(A) = 0$  by Theorem 6.2.4.

Ch 6.TF.7 T, by Theorem 6.2.3a, applied to the columns.

Ch 6.TF.8 T, by Theorem 6.2.6.

Ch 6.TF.9 T, By theorem 6.1.4, a diagonal matrix is triangular as well.

Ch 6.TF.10 T, by Theorem 6.2.3b.

Ch 6.TF.11 T. Without computing its exact value, we will show that the determinant is positive. The pattern that contains all the entries 100 has a product of  $100^4 = 10^8$ , with two inversions. Each of the other  $4! - 1 = 23$  patterns contains at most two entries 100, with the other entries being less than 10, so that the product of each of these patterns is less than  $100^2 \cdot 10^2 = 10^6$ . Thus the determinant is more than  $10^8 - 23 \cdot 10^6 > 0$ , so that the matrix is invertible.

## Chapter 6

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Ch 6.TF.12 F; The correct formula is  $\det(A^{-1}) = \frac{1}{\det(A^T)}$ , by Theorems 6.2.1 and 6.2.8.

Ch 6.TF.13 T; The matrix  $A$  is invertible.

Ch 6.TF.14 T; Any nonzero noninvertible matrix  $A$  will do.

Ch 6.TF.15 T, by Theorem 6.2.7.

Ch 6.TF.16 F, by Theorem 6.3.1. The determinant can be  $-1$ .

Ch 6.TF.17 T, by Theorem 6.2.6.

Ch 6.TF.18 F; The second and the fourth column are linearly dependent.

Ch 6.TF.19 T; The determinant is 0 for  $k = -1$  or  $k = -2$ , so that the matrix is invertible for all *positive*  $k$ .

Ch 6.TF.20 F. There is only one pattern with a nonzero product, containing all the 1's. Since there are three inversions in this pattern,  $\det A = -1$ .

Ch 6.TF.21 T; Let  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ . The column vectors of  $A$  are orthogonal and they all have length 2.

Ch 6.TF.22 F; Let  $A = \begin{bmatrix} 8 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , for example.

Ch 6.TF.23 F; In fact,  $\det(A) = \det[\vec{u} \ \vec{v} \ \vec{w}] = -\det[\vec{v} \ \vec{u} \ \vec{w}] = -\vec{v} \cdot (\vec{u} \times \vec{w})$ . We have used Theorem 6.2.3b and Definition 6.1.1.

Ch 6.TF.24 T; Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , for example.

Ch 6.TF.25 F; Note that  $\det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = 2$ .

Ch 6.TF.26 T, by Theorem 6.3.9.

Ch 6.TF.27 T, by Theorem 6.3.3, since  $\|\vec{v}_i^\perp\| \leq \|\vec{v}_i\| = 1$  for all column vectors  $\vec{v}_i$ .

Ch 6.TF.28 T; We have  $\det(A) = \det(\text{rref } A) = 0$ .

Ch 6.TF.29 F; Let  $A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ , for example. See Theorem 6.2.10.

Ch 6.TF.30 F; Let  $A = 2I_2$ , for example

Ch 6.TF.31 F; Note that  $\det(S^{-1}AS) = \det(A)$  but  $\det(2A) = 2^3(\det A) = 8(\det A)$ .

Ch 6.TF.32 F; Note that  $\det(S^TAS) = (\det S)^2(\det A)$  and  $\det(-A) = -(\det A)$  have opposite signs.

Ch 6.TF.33 F; Let  $A = 2I_2$ , for example.

Ch 6.TF.34 F; Let  $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ , for example.

Ch 6.TF.35 F; Let  $A = I_2$  and  $B = -I_2$ , for example.

Ch 6.TF.36 T; Note that  $\det(B) = -\det(A) < \det(A)$ , so that  $\det(A) > 0$ .

Ch 6.TF.37 T; Let's do Laplace expansion along the first row, for example (see Theorem 6.2.10).

Then  $\det(A) = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det(A_{1j}) \neq 0$ . Thus  $\det(A_{1j}) \neq 0$  for at least one  $j$ , so that  $A_{1j}$  is invertible.

Ch 6.TF.38 T; Note that  $\det(A)$  and  $\det(A^{-1})$  are both integers, and  $(\det A)(\det A^{-1}) = 1$ . This leaves only the possibilities  $\det(A) = \det(A^{-1}) = 1$  and  $\det(A) = \det(A^{-1}) = -1$ .

Ch 6.TF.39 T, since  $\text{adj}(A) = (\det A)(A^{-1})$ , by Theorem 6.3.9.

Ch 6.TF.40 F; Note that  $\det(A^2) = (\det A)^2$  cannot be negative, but  $\det(-I_3) = -1$ .

Ch 6.TF.41 T; The product associated with the diagonal pattern is odd, while the products associated with all other patterns are even. Thus the determinant of  $A$  is odd, so that  $A$  is invertible, as claimed.

Ch 6.TF.42 F; Let  $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$ , for example

Ch 6.TF.43 T; Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If  $a \neq 0$ , let  $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ ; if  $b \neq 0$ , let  $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ ; if  $c \neq 0$ , let  $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , and if  $d \neq 0$ , let  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .

Ch 6.TF.44 T; Use Gaussian elimination for the first column only to transform  $A$  into a matrix of the form

$$B = \begin{bmatrix} 1 & \pm 1 & \pm 1 & \pm 1 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

Note that  $\det(B) = \det(A)$  or  $\det(B) = -(\det A)$ . The stars in matrix  $B$  all represent numbers  $(\pm 1) \pm (\pm 1)$ , so that they are 2, 0, or -2. Thus the determinant of the  $3 \times 3$  matrix  $M$  containing the stars is divisible by 8,

since each of the 6 terms in Sarrus' rule is 8, 0 or -8. Now perform Laplace expansion down the first column of  $B$  to see that  $\det(M) = \det(B) = +/ - \det(A)$ .

Ch 6.TF.45 T;  $A(\text{adj}A) = A(\det(A)A^{-1}) = \det(A)I_n = \det(A)A^{-1}A = \text{adj}(A)A$ .

Ch 6.TF.46 T; Laplace expansion along the second row gives  $\det(A) = -k \det \begin{bmatrix} 1 & 2 & 4 \\ 8 & 9 & 7 \\ 0 & 0 & 5 \end{bmatrix} + C = 35k + C$ , for some constant  $C$  (we need not compute that  $C = -259$ ). Thus  $A$  is invertible except for  $k = \frac{-C}{35}$  (which turns out to be  $\frac{259}{35} = \frac{37}{5} = 7.4$ ).

Ch 6.TF.47 F;  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  are both orthogonal and  $\det(A) = \det(B) = 1$ . However,  $AB \neq BA$ .

## Chapter 7

### Section 7.1

**7.1.1** If  $\vec{v}$  is an eigenvector of  $A$ , then  $A\vec{v} = \lambda\vec{v}$ .

Hence  $A^3\vec{v} = A^2(A\vec{v}) = A^2(\lambda\vec{v}) = A(A\lambda\vec{v}) = A(\lambda A\vec{v}) = A(\lambda^2\vec{v}) = \lambda^2 A\vec{v} = \lambda^3\vec{v}$ , so  $\vec{v}$  is an eigenvector of  $A^3$  with eigenvalue  $\lambda^3$ .

**7.1.2** We know  $A\vec{v} = \lambda\vec{v}$  so  $\vec{v} = A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v}$ , so  $\vec{v} = \lambda A^{-1}\vec{v}$  or  $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$ .

Hence  $\vec{v}$  is an eigenvector of  $A^{-1}$  with eigenvalue  $\frac{1}{\lambda}$ .

**7.1.3** We know  $A\vec{v} = \lambda\vec{v}$ , so  $(A + 2I_n)\vec{v} = A\vec{v} + 2I_n\vec{v} = \lambda\vec{v} + 2\vec{v} = (\lambda + 2)\vec{v}$ , hence  $\vec{v}$  is an eigenvector of  $(A + 2I_n)$  with eigenvalue  $\lambda + 2$ .

**7.1.4** We know  $A\vec{v} = \lambda\vec{v}$ , so  $7A\vec{v} = 7\lambda\vec{v}$ , hence  $\vec{v}$  is an eigenvector of  $7A$  with eigenvalue  $7\lambda$ .

**7.1.5** Assume  $A\vec{v} = \lambda\vec{v}$  and  $B\vec{v} = \beta\vec{v}$  for some eigenvalues  $\lambda, \beta$ . Then  $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \lambda\vec{v} + \beta\vec{v} = (\lambda + \beta)\vec{v}$  so  $\vec{v}$  is an eigenvector of  $A + B$  with eigenvalue  $\lambda + \beta$ .

**7.1.6** Yes. If  $A\vec{v} = \lambda\vec{v}$  and  $B\vec{v} = \mu\vec{v}$ , then  $AB\vec{v} = A(\mu\vec{v}) = \mu(A\vec{v}) = \mu\lambda\vec{v}$

**7.1.7** We know  $A\vec{v} = \lambda\vec{v}$  so  $(A - \lambda I_n)\vec{v} = A\vec{v} - \lambda I_n\vec{v} = \lambda\vec{v} - \lambda\vec{v} = \vec{0}$  so a nonzero vector  $\vec{v}$  is in the kernel of  $(A - \lambda I_n)$  so  $\ker(A - \lambda I_n) \neq \{\vec{0}\}$  and  $A - \lambda I_n$  is not invertible.

**7.1.8** We want all  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  hence  $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ , i.e. the desired matrices must have the form  $\begin{bmatrix} 5 & b \\ 0 & d \end{bmatrix}$ .

**7.1.9** We want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  for any  $\lambda$ . Hence  $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}$ , i.e., the desired matrices must have the form  $\begin{bmatrix} \lambda & b \\ 0 & d \end{bmatrix}$ , they must be upper triangular.

**7.1.10** We want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , i.e. the desired matrices must have the form  $\begin{bmatrix} 5 - 2b & b \\ 10 - 2d & d \end{bmatrix}$ .

**7.1.11** We want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ . So,  $2a + 3b = -2$  and  $2c + 3d = -3$ . Thus,  $b = \frac{-2-2a}{3}$ , and  $d = \frac{-3-2c}{3}$ . So all matrices of the form  $\begin{bmatrix} a & \frac{-2-2a}{3} \\ c & \frac{-3-2c}{3} \end{bmatrix}$  will fit.

**7.1.12** Solving  $\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  we get  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ -\frac{3}{2}t \end{bmatrix}$  (with  $t \neq 0$ ) and

solving  $\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  we get  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$  (with  $t \neq 0$ ).

**7.1.13** Solving  $\begin{bmatrix} -6 & 6 \\ -15 & 13 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ , we get  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5}t \\ t \end{bmatrix}$  (with  $t \neq 0$ ).

**7.1.14** We want to find all  $4 \times 4$  matrices  $A$  such that  $A\vec{e}_2 = \lambda\vec{e}_2$ , i.e. the second column of  $A$  must be of the form  $\begin{bmatrix} 0 \\ \lambda \\ 0 \\ 0 \end{bmatrix}$ , so  $A = \begin{bmatrix} a & 0 & c & d \\ e & \lambda & f & g \\ h & 0 & i & j \\ k & 0 & l & m \end{bmatrix}$ .

**7.1.15** Any vector on  $L$  is unaffected by the reflection, so that a nonzero vector on  $L$  is an eigenvector with eigenvalue 1. Any vector on  $L^\perp$  is flipped about  $L$ , so that a nonzero vector on  $L^\perp$  is an eigenvector with eigenvalue  $-1$ . Picking a nonzero vector from  $L$  and one from  $L^\perp$ , we obtain a basis consisting of eigenvectors.

**7.1.16** Rotation by  $180^\circ$  is a flip about the origin so every nonzero vector is an eigenvector with the eigenvalue  $-1$ . Any basis for  $\mathbb{R}^2$  consists of eigenvectors.

**7.1.17** No (real) eigenvalues

**7.1.18** Any nonzero vector in the plane is unchanged, hence is an eigenvector with the eigenvalue 1. Since any nonzero vector in  $V^\perp$  is flipped about the origin, it is an eigenvector with eigenvalue  $-1$ . Pick any two non-collinear vectors from  $V$  and one from  $V^\perp$  to form a basis consisting of eigenvectors.

**7.1.19** Any nonzero vector in  $L$  is an eigenvector with eigenvalue 1, and any nonzero vector in the plane  $L^\perp$  is an eigenvector with eigenvalue 0. Form a basis consisting of eigenvectors by picking any nonzero vector in  $L$  and any two nonparallel vectors in  $L^\perp$ .

**7.1.20** Any nonzero vector along the  $\vec{e}_3$ -axis is unchanged, hence is an eigenvector with eigenvalue 1. No other (real) eigenvalues can be found.

**7.1.21** Any nonzero vector in  $\mathbb{R}^3$  is an eigenvector with eigenvalue 5. Any basis for  $\mathbb{R}^3$  consists of eigenvectors.

**7.1.22** Any nonzero scalar multiple of  $\vec{v}$  is an eigenvector with eigenvalue 1.

**7.1.23 a** Since  $S = [\vec{v}_1 \cdots \vec{v}_n]$ ,  $S^{-1}\vec{v}_i = S^{-1}(S\vec{e}_i) = \vec{e}_i$ .

b  $i$ th column of  $S^{-1}AS$

$$\begin{aligned} &= S^{-1}AS\vec{e}_i \\ &= S^{-1}A\vec{v}_i \text{ (by definition of } S) \\ &= S^{-1}\lambda_i\vec{v}_i \text{ (since } \vec{v}_i \text{ is an eigenvector)} \\ &= \lambda_i S^{-1}\vec{v}_i \\ &= \lambda_i \vec{e}_i \text{ (by part a)} \end{aligned}$$

hence  $S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ .

7.1.24 See Figure 7.1.

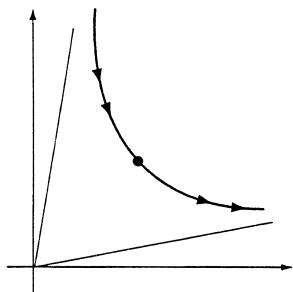


Figure 7.1: for Problem 7.1.24.

7.1.25 See Figure 7.2.

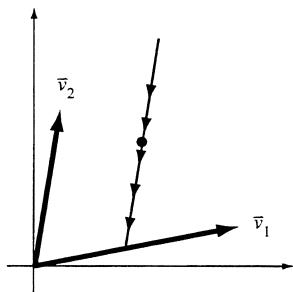


Figure 7.2: for Problem 7.1.25.

7.1.26 See Figure 7.3.

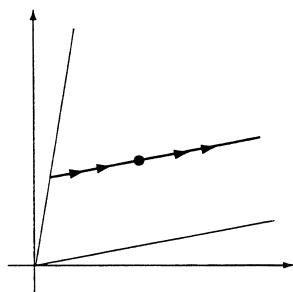


Figure 7.3: for Problem 7.1.26.

7.1.27 See Figure 7.4.

7.1.28 See Figure 7.5.

7.1.29 See Figure 7.6.

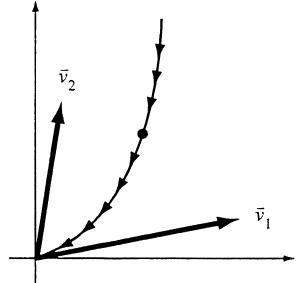


Figure 7.4: for Problem 7.1.27.

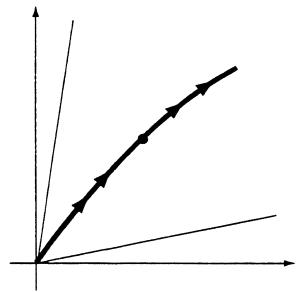


Figure 7.5: for Problem 7.1.28.

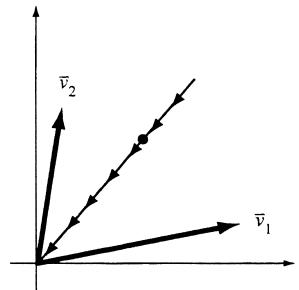


Figure 7.6: for Problem 7.1.29.

**7.1.30** Since the matrix is diagonal,  $\vec{e}_1$  and  $\vec{e}_2$  are eigenvectors. See Figure 7.7.

**7.1.31** See Figure 7.8.

**7.1.32** Since the matrix is diagonal,  $\vec{e}_1$  and  $\vec{e}_2$  are eigenvectors. See Figure 7.9.

**7.1.33** We are given that  $\vec{x}(t) = 2^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 6^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ , hence we know that the eigenvalues are 2 and 6 with corresponding eigenvectors  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$  respectively (see Theorem 7.1.3), so we want a matrix  $A$  such that  $A \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 2 & 6 \end{bmatrix}$ . Multiplying on the right by  $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$ , we get  $A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$ .

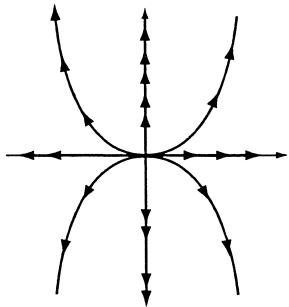


Figure 7.7: for Problem 7.1.30.

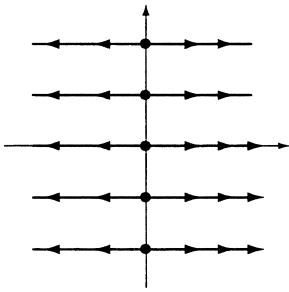


Figure 7.8: for Problem 7.1.31.

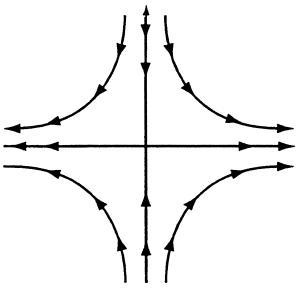


Figure 7.9: for Problem 7.1.32.

**7.1.34**  $(A^2 + 2A + 3I_n)\vec{v} = A^2\vec{v} + 2A\vec{v} + 3I_n\vec{v} = 4^2\vec{v} + 2 \cdot 4\vec{v} + 3\vec{v} = (16 + 8 + 3)\vec{v} = 27\vec{v}$  so  $\vec{v}$  is an eigenvector of  $A^2 + 2A + 3I_n$  with eigenvalue 27.

**7.1.35** Let  $\lambda$  be an eigenvalue of  $S^{-1}AS$ . Then for some nonzero vector  $\vec{v}$ ,  $S^{-1}AS\vec{v} = \lambda\vec{v}$ , i.e.,  $AS\vec{v} = S\lambda\vec{v} = \lambda S\vec{v}$  so  $\lambda$  is an eigenvalue of  $A$  with eigenvector  $S\vec{v}$ .

Conversely, if  $\alpha$  is an eigenvalue of  $A$  with eigenvector  $\vec{w}$ , then  $A\vec{w} = \alpha\vec{w}$ , for some nonzero  $\vec{w}$ .

Therefore,  $S^{-1}AS(S^{-1}\vec{w}) = S^{-1}A\vec{w} = S^{-1}\alpha\vec{w} = \alpha S^{-1}\vec{w}$ , so  $S^{-1}\vec{w}$  is an eigenvector of  $S^{-1}AS$  with eigenvalue  $\alpha$ .

**7.1.36** We want  $A$  such that  $A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$  and  $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$ , i.e.  $A \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix}$ , so

$$A = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 \\ -2 & 11 \end{bmatrix}.$$

**7.1.37 a**  $A = 5 \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$  is a scalar multiple of an orthogonal matrix. By Theorem 7.1.2, the possible eigenvalues of the orthogonal matrix are  $\pm 1$ , so that the possible eigenvalues of  $A$  are  $\pm 5$ . In part b we see that both are indeed eigenvalues.

b Solve  $A\vec{v} = \pm 5\vec{v}$  to get  $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ .

**7.1.38**  $\begin{bmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ . The associated eigenvalue is 2.

**7.1.39** We want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$ . So  $b = 0$ , and  $d = \lambda$  (for any  $\lambda$ ). Thus, we need matrices of the form  $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ .

So,  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$  is a basis of  $V$ , and  $\dim(V) = 3$ .

**7.1.40** We need all matrices  $A$  such that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \lambda \\ -3\lambda \end{bmatrix}$ .

Thus,  $a - 3b = \lambda$  and  $c - 3d = -3\lambda$ . Thus,  $c - 3d = -3(a - 3b) = -3a + 9b$ , or  $c = -3a + 9b + 3d$ .

So  $A$  must be of the form  $\begin{bmatrix} a & b \\ -3a + 9b + 3d & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$ . Thus, a basis of  $V$  is  $\begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$ , and the dimension of  $V$  is 3.

**7.1.41** We want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . So,  $a + b = \lambda_1 = c + d$  and  $a + 2b = \lambda_2$  and  $2\lambda_2 = c + 2d$ .

So  $(a+2b)-(a+b) = \lambda_2 - \lambda_1 = b$ ,  $a = \lambda_1 - b = 2\lambda_1 - \lambda_2$ . Also,  $(c+2d)-(c+d) = 2\lambda_2 - \lambda_1 = d$ ,  $c = \lambda_1 - d = 2\lambda_1 - 2\lambda_2$ .

So  $A$  must be of the form:  $\begin{bmatrix} 2\lambda_1 - \lambda_2 & \lambda_2 - \lambda_1 \\ 2\lambda_1 - 2\lambda_2 & 2\lambda_2 - \lambda_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ .

So a basis of  $V$  is  $\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ , and  $\dim(V) = 2$ .

**7.1.42** We will do this in a slightly simpler manner than Exercise 40. Since  $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is simply the first column of  $A$ , the first column must be a multiple of  $\vec{e}_1$ . Similarly, the third column must be a multiple of  $\vec{e}_3$ . There are no

other restrictions on the form of  $A$ , meaning it can be any matrix of the form  $\begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & e \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

Thus, a basis of  $V$  is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and the dimension of  $V$  is 5.

**7.1.43**  $A = AI_n = A[\vec{e}_1 \dots \vec{e}_n] = [\lambda_1 \vec{e}_1 \dots \lambda_n \vec{e}_n]$ , where the eigenvalues  $\lambda_1, \dots, \lambda_n$  are arbitrary. Thus  $A$  can be any diagonal matrix, and  $\dim(V) = n$ .

**7.1.44** We see that each of the columns 1 through  $m$  of  $A$  will have to be a multiple of its respective vector  $\vec{e}_i$ . Thus, there will be  $m$  free variables in the first  $m$  columns. The remaining  $n - m$  columns will each have  $n$  free variables. Thus, in total, the dimension of  $V$  is  $m + (n - m)n = m + n^2 - nm$ .

**7.1.45** Consider a vector  $\vec{w}$  that is not parallel to  $\vec{v}$ . We want  $A[\vec{v} \ \vec{w}] = [\lambda \vec{v} \ a\vec{v} + b\vec{w}]$ , where  $\lambda, a$  and  $b$  are arbitrary constants. Thus the matrices  $A$  in  $V$  are of the form  $A = [\lambda \vec{v} \ a\vec{v} + b\vec{w}] [\vec{v} \ \vec{w}]^{-1}$ . Using Summary 4.1.6, we see that  $[\vec{v} \ \vec{0}] [\vec{v} \ \vec{w}]^{-1}, [\vec{0} \ \vec{v}] [\vec{v} \ \vec{w}]^{-1}, [\vec{0} \ \vec{w}] [\vec{v} \ \vec{w}]^{-1}$  is a basis of  $V$ , so that  $\dim(V) = 3$ .

**7.1.46 a** We need all matrices  $A$  such that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} k \\ 2k \end{bmatrix}$ .

Thus,  $a + 2b = k$  and  $c + 2d = 2k$ . So,  $c + 2d = 2a + 4b$ , or  $c = -2d + 2a + 4b$  and  $A$  must be of the form  $\begin{bmatrix} a & b \\ -2d + 2a + 4b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$ . So a basis of  $V$  is  $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$ , and the dimension of  $V$  is 3.

**b** Clearly  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is a basis of the image of  $T$  by definition of  $V$ , so that the rank of  $T$  is 1. The kernel of  $T$  consists of all matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{0}$ , or  $a + 2b = 0, c + 2d = 0$ . These are the matrices of the form  $\begin{bmatrix} -2b & b \\ -2d & d \end{bmatrix} = b \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$ . Thus a basis of the kernel of  $T$  is  $\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$ .

**c** Let's find the kernel of  $L$  first. In part (a) we saw that the matrices in  $V$  are of the form  $A = \begin{bmatrix} a & b \\ -2d + 2a + 4b & d \end{bmatrix}$ . A matrix  $A$  in  $V$  is in the kernel of  $L$  if  $\begin{bmatrix} a & b \\ -2d + 2a + 4b & d \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \vec{0}$ , or  $a + 3b = 0, 2a + 4b + d = 0$ . This system simplifies to  $a = -3b$  and  $d = 2b$ , so that the matrices in the kernel of  $L$  are of the form  $\begin{bmatrix} -3b & b \\ -6b & 2b \end{bmatrix} = b \begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix}$ . The matrix  $\begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix}$  forms a basis of the kernel of  $L$ . By the rank-nullity theorem, the rank of  $L$  is  $\dim(V) - \dim(\ker L) = 3 - 1 = 2$ , and the image of  $L$  is all of  $\mathbb{R}^2$ .

**7.1.47** Suppose  $V$  is a one-dimensional  $A$ -invariant subspace of  $\mathbb{R}^n$ , and  $\vec{v}$  is a non-zero vector in  $V$ . Then  $A\vec{v}$  will be in  $V$ , so that  $A\vec{v} = \lambda\vec{v}$  for some  $\lambda$ , and  $\vec{v}$  is an eigenvector of  $A$ . Conversely, if  $\vec{v}$  is any eigenvector of  $A$ , then

$V = \text{span}(\vec{v})$  will be a one-dimensional  $A$ -invariant subspace. Thus the one-dimensional  $A$ -invariant subspaces  $V$  are of the form  $V = \text{span}(\vec{v})$ , where  $\vec{v}$  is an eigenvector of  $A$ .

**7.1.48 a** Since  $\text{span}(\vec{e}_1)$  is an  $A$ -invariant subspace of  $\mathbb{R}^3$ , it must be that  $\vec{e}_1$  is an eigenvector of  $A$ , as revealed in

Exercise 47. Thus, the first column of  $A$  must be of the form  $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ . Since  $\text{span}(\vec{e}_1, \vec{e}_2)$  is also an  $A$ -invariant

subspace, it must be that  $A\vec{e}_2$  is in  $\text{span}(\vec{e}_1, \vec{e}_2)$ . Thus, the second column of  $A$  must have the form  $\begin{bmatrix} b \\ c \\ 0 \end{bmatrix}$ . The

third column may be any vector in  $\mathbb{R}^3$ . Thus, we can choose  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$  to maximize the number of non-zero entries.

**b** We see, from our construction above, that upper-triangular matrices fit this description. This space,  $V$  consists

of all matrices of the form  $\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$  and has a dimension of 6.

**7.1.49** The eigenvalues of the system are  $\lambda_1 = 1.1$ , and  $\lambda_2 = 0.9$  and corresponding eigenvectors are  $\vec{v}_1 = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$

and  $\vec{v}_2 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ , respectively. So if  $\vec{x}_0 = \begin{bmatrix} 100 \\ 800 \end{bmatrix}$ , we can see that  $\vec{x}_0 = 3\vec{v}_1 - \vec{v}_2$ . Therefore, by Theorem 7.1.3, we have  $\vec{x}(t) = 3(1.1)^t \begin{bmatrix} 100 \\ 300 \end{bmatrix} - (0.9)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ , i.e.  $c(t) = 300(1.1)^t - 200(0.9)^t$  and  $r(t) = 900(1.1)^t - 100(0.9)^t$ .

**7.1.50 a**  $\vec{v}(0) = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ , and we see that  $A\vec{v}(0) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 200 \\ 200 \end{bmatrix} = 2 \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ . So,  $\vec{v}(t) = A^t \vec{v}(0) = A^t \begin{bmatrix} 100 \\ 100 \end{bmatrix} = 2^t \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ .

So  $c(t) = r(t) = 100(2)^t$ .

**b**  $\vec{v}(0) = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ , and we see that  $A\vec{v}(0) = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 600 \\ 300 \end{bmatrix} = 3 \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ . So,  $\vec{v}(t) = A^t \vec{v}(0) = A^t \begin{bmatrix} 200 \\ 100 \end{bmatrix} = 3^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ .

So  $c(t) = 200(3)^t$  and  $r(t) = 100(3)^t$ .

**c**  $\vec{v}(0) = \begin{bmatrix} 600 \\ 500 \end{bmatrix}$ . We can write this in terms of the previous eigenvectors as  $\vec{v}(0) = 4 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ . So,  $\vec{v}(t) = A^t \vec{v}(0) = A^t 4 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + A^t \begin{bmatrix} 200 \\ 100 \end{bmatrix} = 4(2)^t \begin{bmatrix} 100 \\ 100 \end{bmatrix} + (3)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ .

So  $c(t) = 400(2)^t + 200(3)^t$  and  $r(t) = 400(2)^t + 100(3)^t$ .

**7.1.51** a  $\vec{v}(0) = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ , and we see that  $A\vec{v}(0) = \begin{bmatrix} 0 & .75 \\ -1.5 & 2.25 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 150 \\ 300 \end{bmatrix} = 1.5 \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ . So,  $\vec{v}(t) = A^t \vec{v}(0) = A^t \begin{bmatrix} 100 \\ 200 \end{bmatrix} = (1.5)^t \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ .

So  $c(t) = 100(1.5)^t$  and  $r(t) = 200(1.5)^t$ .

b  $\vec{v}(0) = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ , and we see that  $A\vec{v}(0) = \begin{bmatrix} 0 & .75 \\ -1.5 & 2.25 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 75 \\ 75 \end{bmatrix} = .75 \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ . So,  $\vec{v}(t) = A^t \vec{v}(0) = A^t \begin{bmatrix} 100 \\ 100 \end{bmatrix} = (.75)^t \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ .

So  $c(t) = 100(.75)^t$  and  $r(t) = 100(.75)^t$ .

c  $\vec{v}(0) = \begin{bmatrix} 500 \\ 700 \end{bmatrix}$ . We can write this in terms of the previous eigenvectors as  $\vec{v}(0) = 3 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + 2 \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ . So,  $\vec{v}(t) = A^t \vec{v}(0) = A^t 3 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + A^t 2 \begin{bmatrix} 100 \\ 200 \end{bmatrix} = 3(.75)^t \begin{bmatrix} 100 \\ 100 \end{bmatrix} + 2(1.5)^t \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ .

So  $c(t) = 300(.75)^t + 200(1.5)^t$  and  $r(t) = 300(.75)^t + 400(1.5)^t$ .

**7.1.52** a  $\begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.99 \\ 1.98 \end{bmatrix} = 0.99 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ ,

and  $\begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2.94 \\ -0.98 \end{bmatrix} = 0.98 \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ . The eigenvalues are  $\lambda_1 = 0.99$  and  $\lambda_2 = 0.98$ .

b  $\vec{x}_0 = \begin{bmatrix} g_0 \\ l_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} = 20 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 40 \begin{bmatrix} 3 \\ -1 \end{bmatrix}$  so  $\vec{x}(t) = 20(0.99)^t \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 40(0.98)^t \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ , hence

$g(t) = -20(0.99)^t + 120(0.98)^t$  and  $h(t) = 40(0.99)^t - 40(0.98)^t$ .

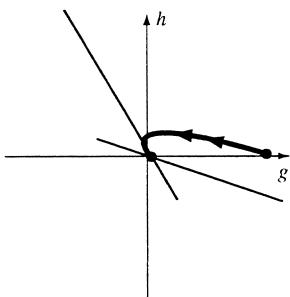


Figure 7.10: for Problem 7.1.52b.

$h(t)$  first rises, then falls back to zero.  $g(t)$  falls a little below zero, then goes back up to zero. See Figure 7.10.

c We set  $g(t) = -20(0.99)^t + 120(0.98)^t = 0$ .

Solving for  $t$  we get that  $g(t) = 0$  for  $t \approx 176$  minutes. (After  $t = 176$ ,  $g(t) < 0$ ).

7.1.53 a  $\vec{v}(0) = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ .

$$\begin{aligned} \text{So, } \vec{v}(t) &= A^t \vec{v}(0) = A^t \left( 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) \\ &= 3A^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2A^t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + A^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 3\lambda_1^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\lambda_2^t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \lambda_3^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2(-\frac{1}{2})^t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (-\frac{1}{2})^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \end{aligned}$$

So  $a(t) = 3 + 3(-\frac{1}{2})^t$ ,  $b(t) = 3 - 2(-\frac{1}{2})^t$  and  $c(t) = 3 - (-\frac{1}{2})^t$ .

b  $a(365) = 3 + 3(-\frac{1}{2})^{365} = 3 - \frac{3}{2^{365}}$ ,  $b(365) = 3 - 2(-\frac{1}{2})^{365} = 3 + \frac{1}{2^{364}}$  and  
 $c(365) = 3 - (-\frac{1}{2})^{365} = 3 + \frac{1}{2^{365}}$ . So, Benjamin will have the most gold.

7.1.54 a We are given that

$$n(t+1) = 2a(t)$$

$$a(t+1) = n(t) + a(t),$$

so that the matrix is  $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ .

b  $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $A \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ , hence 2 and -1 are the eigenvalues associated with  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$  respectively.

c We are given  $\vec{x}_0 = \begin{bmatrix} n_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  so  $\vec{x}_0 = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ , and  $\vec{x}(t) = \frac{1}{3} 2^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3} (-1)^t \begin{bmatrix} -2 \\ -1 \end{bmatrix}$  (by Theorem 7.1.3), hence  $n(t) = \frac{1}{3} 2^t + \frac{2}{3} (-1)^t$  and  $a(t) = \frac{1}{3} 2^t - \frac{1}{3} (-1)^t$ .

## Section 7.2

7.2.1  $\lambda_1 = 1$ ,  $\lambda_2 = 3$  by Theorem 7.2.2.

7.2.2  $\lambda_1 = 2$  (Algebraic multiplicity 2)

$\lambda_2 = 1$  (Algebraic multiplicity 2), by Theorem 7.2.2.

7.2.3  $\det(A - \lambda I_2) = \det \begin{bmatrix} 5 - \lambda & -4 \\ 2 & -1 - \lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$  so  $\lambda_1 = 1$ ,  $\lambda_2 = 3$ .

7.2.4  $\det(A - \lambda I_2) = \det \begin{bmatrix} -\lambda & 4 \\ -1 & 4 - \lambda \end{bmatrix} = -\lambda(4 - \lambda) + 4 = (\lambda - 2)^2 = 0$  so  $\lambda = 2$  with algebraic multiplicity 2.

7.2.5  $\det(A - \lambda I_2) = \det \begin{bmatrix} 11 - \lambda & -15 \\ 6 & -7 - \lambda \end{bmatrix} = \lambda^2 - 4\lambda + 13 = 0$  so  $\det(A - \lambda I_2) = 0$  for no real  $\lambda$ .

7.2.6  $\det(A - \lambda I_2) = \det \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix} = \lambda^2 - 5\lambda - 2 = 0$  so  $\lambda_{1,2} = \frac{5 \pm \sqrt{33}}{2}$ .

7.2.7  $\lambda = 1$  with algebraic multiplicity 3, by Theorem 7.2.2.

7.2.8  $f_A(\lambda) = -\lambda^2(\lambda + 3)$  so

$$\lambda_1 = 0 \text{ (Algebraic multiplicity 2)}$$

$$\lambda_2 = -3.$$

7.2.9  $f_A(\lambda) = -(\lambda - 2)^2(\lambda - 1)$  so

$$\lambda_1 = 2 \text{ (Algebraic multiplicity 2)}$$

$$\lambda_2 = 1.$$

7.2.10  $f_A(\lambda) = (1 + \lambda)^2(1 - \lambda)$  so  $\lambda_1 = -1$  (Algebraic multiplicity 2),  $\lambda_2 = 1$ .

7.2.11  $f_A(\lambda) = -\lambda^3 - \lambda^2 - \lambda - 1 = -(\lambda + 1)(\lambda^2 + 1) = 0$

$$\lambda = -1 \text{ (Algebraic multiplicity 1).}$$

7.2.12  $f_A(\lambda) = \lambda(\lambda + 1)(\lambda - 1)^2$  so  $\lambda_1 = 0$ ,  $\lambda_2 = -1$ ,  $\lambda_3 = 1$  (Algebraic multiplicity 2).

7.2.13  $f_A(\lambda) = -\lambda^3 + 1 = -(\lambda - 1)(\lambda^2 + \lambda + 1)$  so  $\lambda = 1$  (Algebraic multiplicity 1).

7.2.14  $f_A(\lambda) = \det(B - \lambda I_2) \det(D - \lambda I_2)$  (see Theorem 6.1.5).

The eigenvalues of  $A$  are the eigenvalues of  $B$  and  $D$ . The eigenvalues of  $C$  are irrelevant.

7.2.15  $f_A(\lambda) = \lambda^2 - 2\lambda + (1 - k) = 0$  if  $\lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4(1 - k)}}{2} = 1 \pm \sqrt{k}$

The matrix  $A$  has 2 distinct real eigenvalues when  $k > 0$ , no real eigenvalues when  $k < 0$ .

7.2.16  $f_A(\lambda) = \lambda^2 - (a + c)\lambda + (ac - b^2)$

The discriminant of this quadratic equation is  $(a + c)^2 - 4(ac - b^2) = a^2 + 2ac + c^2 - 4ac + 4b^2 = (a - c)^2 + 4b^2$ ; this quantity is always positive (since  $b \neq 0$ ). There will always be two distinct real eigenvalues.

7.2.17  $f_A(\lambda) = \lambda^2 - a^2 - b^2 = 0$  so  $\lambda_{1,2} = \pm\sqrt{a^2 + b^2}$ .

The matrix  $A$  represents a reflection about a line followed by a scaling by  $\sqrt{a^2 + b^2}$ , hence the eigenvalues.

**7.2.18**  $f_A(\lambda) = \lambda^2 - 2a\lambda + a^2 - b^2$  so  $\lambda_{1,2} = \frac{+2a \pm \sqrt{4a^2 - 4(a^2 - b^2)}}{2} = a \pm b$ .

Hence the eigenvalues are  $a \pm b$ .

**7.2.19** True, since  $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$  and the discriminant  $[\text{tr}(A)]^2 - 4\det(A)$  is positive if  $\det(A)$  is negative.

**7.2.20** The characteristic polynomial of  $A$  is  $f_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$ . But from Theorem 7.2.4 we know that  $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$ . Comparing the coefficient of  $\lambda$ , we see that  $\lambda_1 + \lambda_2 = \text{tr}(A)$ , as claimed.

**7.2.21** If  $A$  has  $n$  eigenvalues, then  $f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$ . Then  $f_A(\lambda) = (-\lambda)^n + (\lambda_1 + \lambda_2 + \cdots + \lambda_n)(-\lambda)^{n-1} + \cdots + (\lambda_1\lambda_2 \cdots \lambda_n)$ . But, by Theorem 7.2.5, the coefficient of  $(-\lambda)^{n-1}$  is  $\text{tr}(A)$ . So,  $\text{tr}(A) = \lambda_1 + \cdots + \lambda_n$ .

**7.2.22** By Theorem 6.2.1,  $f_A(\lambda) = \det(A - \lambda I_n) = \det(A - \lambda I_n)^T = \det(A^T - \lambda I_n) = f_{A^T}(\lambda)$ . Since the characteristic polynomials of  $A$  and  $A^T$  are identical, the two matrices have the same eigenvalues, with the same algebraic multiplicities.

$$\begin{aligned} \text{7.2.23 } f_B(\lambda) &= \det(B - \lambda I_n) = \det(S^{-1}AS - \lambda I_n) \\ &= \det(S^{-1}AS - \lambda S^{-1}I_nS) \\ &= \det(S^{-1}(A - \lambda I_n)S) = \det(S^{-1}) \det(A - \lambda I_n) \det(S) \\ &= (\det S)^{-1} \det(A - \lambda I_n) \det(S) = \det(A - \lambda I_n) = f_A(\lambda) \end{aligned}$$

Hence, since  $f_A(\lambda) = f_B(\lambda)$ ,  $A$  and  $B$  have the same eigenvalues.

**7.2.24**  $\lambda_1 = 0.25$ ,  $\lambda_2 = 1$

**7.2.25**  $A \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} ab + cb \\ cb + cd \end{bmatrix} = \begin{bmatrix} (a+c)b \\ (b+d)c \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$  since  $a+c = b+d = 1$ ; therefore,  $\begin{bmatrix} b \\ c \end{bmatrix}$  is an eigenvector with eigenvalue  $\lambda_1 = 1$ .

Also,  $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix} = (a-b) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  since  $a-b = -(c-d)$ ; therefore,  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is an eigenvector with eigenvalue  $\lambda_2 = a-b$ . Note that  $|a-b| < 1$ ; a possible phase portrait is shown in Figure 7.11.

**7.2.26** Here  $\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$  with  $\lambda_1 = 1$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  with  $\lambda_2 = a-b = 0.25$ . See Figure 7.12.

**7.2.27 a** We know  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\lambda_1 = 1$  and  $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ,  $\lambda_2 = \frac{1}{4}$ . If  $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  then  $\vec{x}_0 = \frac{1}{3}\vec{v}_1 + \frac{2}{3}\vec{v}_2$ , so by Theorem 7.1.3,

$$x_1(t) = \frac{1}{3} + \frac{2}{3} \left( \frac{1}{4} \right)^t$$

$$x_2(t) = \frac{2}{3} - \frac{2}{3} \left( \frac{1}{4} \right)^t.$$

If  $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  then  $\vec{x}_0 = \frac{1}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2$ , so by Theorem 7.1.3,

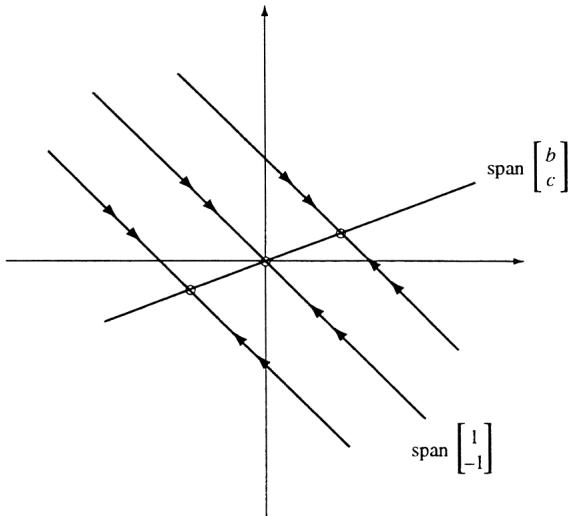


Figure 7.11: for Problem 7.2.25.

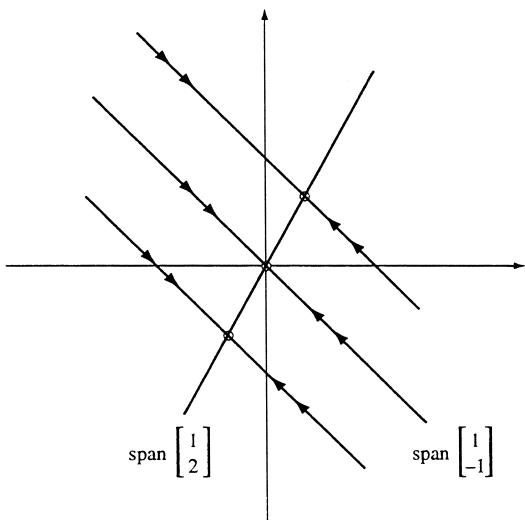


Figure 7.12: for Problem 7.2.26.

$$x_1(t) = \frac{1}{3} - \frac{1}{3} \left(\frac{1}{4}\right)^t$$

$$x_2(t) = \frac{2}{3} + \frac{1}{3} \left(\frac{1}{4}\right)^t. \text{ See Figure 7.13.}$$

b  $A^t$  approaches  $\frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ , as  $t \rightarrow \infty$ . See part c for a justification.

c Let us think about the first column of  $A^t$ , which is  $A^t \vec{e}_1$ . We can use Theorem 7.1.3 to compute  $A^t \vec{e}_1$ .

Start by writing  $\vec{e}_1 = c_1 \begin{bmatrix} b \\ c \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ; a straightforward computation shows that  $c_1 = \frac{1}{b+c}$  and  $c_2 = \frac{c}{b+c}$ .

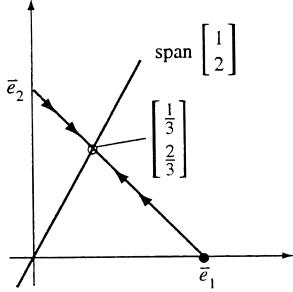


Figure 7.13: for Problem 7.2.27a.

Now  $A^t \vec{e}_1 = \frac{1}{b+c} \begin{bmatrix} b \\ c \end{bmatrix} + \frac{c}{b+c} (\lambda_2)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , where  $\lambda_2 = a - b$ .

Since  $|\lambda_2| < 1$ , the second summand goes to zero, so that  $\lim_{t \rightarrow \infty} (A^t \vec{e}_1) = \frac{1}{b+c} \begin{bmatrix} b \\ c \end{bmatrix}$ .

Likewise,  $\lim_{t \rightarrow \infty} (A^t \vec{e}_2) = \frac{1}{b+c} \begin{bmatrix} b \\ c \end{bmatrix}$ , so that  $\lim_{t \rightarrow \infty} A^t = \frac{1}{b+c} \begin{bmatrix} b & b \\ c & c \end{bmatrix}$ .

$$7.2.28 \text{ a } w(t+1) = 0.8w(t) + 0.1m(t)$$

$$m(t+1) = 0.2w(t) + 0.9m(t)$$

so  $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$  which is a regular transition matrix since its columns sum to 1 and its entries are positive.

b The eigenvectors of  $A$  are  $\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$  or  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  with  $\lambda_1 = 1$ , and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  with  $\lambda_2 = 0.7$ .

$$\vec{x}_0 = \begin{bmatrix} 1200 \\ 0 \end{bmatrix} = 400 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 800 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ so } \vec{x}(t) = 400 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 800(0.7)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ or}$$

$$w(t) = 400 + 800(0.7)^t$$

$$m(t) = 800 - 800(0.7)^t.$$

c As  $t \rightarrow \infty$ ,  $w(t) \rightarrow 400$  so Wipfs won't have to close the store.

7.2.29 The  $i$ th entry of  $A\vec{e}$  is  $[a_{i1} a_{i2} \cdots a_{in}] \vec{e} = \sum_{j=1}^n a_{ij} = 1$ , so  $A\vec{e} = \vec{e}$  and  $\lambda = 1$  is an eigenvalue of  $A$ , corresponding to the eigenvector  $\vec{e}$ .

**7.2.30 a** Let  $v_i$  be the largest component of the vector  $\vec{v}$ , that is,  $v_i \geq v_j$  for  $j = 1, \dots, n$ . Then the  $i$ th component of

$$A \vec{v} \text{ is } \lambda v_i = \sum_{j=1}^n a_{ij} v_j \leq \sum_{j=1}^n a_{ij} v_i = \left( \sum_{j=1}^n a_{ij} \right) v_i = v_i$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ v_j \leq v_i & & \sum_{j=1}^n a_{ij} = 1 \end{array}$$

We can conclude that  $\lambda v_i \leq v_i$ , and therefore  $\lambda \leq 1$ , as claimed. Also note that if  $\vec{v}$  is not a multiple of the eigenvector  $\vec{e}$  discussed in Exercise 29, then  $v_j < v_i$  for some index  $j$ , so that  $\sum_{j=1}^n a_{ij} v_j < \sum_{j=1}^n a_{ij} v_i$  and therefore  $\lambda < 1$ .

**b** Let  $v_i$  be the component of  $\vec{v}$  with the largest absolute value, that is,  $|v_i| \geq |v_j|$  for  $j = 1, 2, \dots, n$ . Then the absolute value of the  $i$ th component of  $A\vec{v}$  is  $|\lambda||v_i| = \left[ \sum_{j=1}^n a_{ij} v_j \right] \leq \sum_{j=1}^n a_{ij} |v_j| \leq \sum_{j=1}^n a_{ij} |v_i| = \left( \sum_{j=1}^n a_{ij} \right) |v_i| = |v_i|$  so that  $|\lambda||v_i| \leq |v_i|$  and  $|\lambda| \leq 1$ , as claimed.

**7.2.31** Since  $A$  and  $A^T$  have the same eigenvalues (by Exercise 22), Exercise 29 states that  $\lambda = 1$  is an eigenvalue of  $A$ , and Exercise 30 says that  $|\lambda| \leq 1$  for all eigenvalues  $\lambda$ . Vector  $\vec{e}$  need not be an eigenvector of  $A$ ; consider  $A = \begin{bmatrix} 0.9 & 0.9 \\ 0.1 & 0.1 \end{bmatrix}$ .

**7.2.32**  $f_A(\lambda) = -\lambda^3 + 3\lambda + k$ . The eigenvalues of  $A$  are the solutions of the equation  $-\lambda^3 + 3\lambda + k = 0$ , or,  $\lambda^3 - 3\lambda = k$ . Following the hint, we graph the function  $g(\lambda) = \lambda^3 - 3\lambda$  as shown in Figure 7.14. We use the derivative  $f'(\lambda) = 3\lambda^2 - 3$  to see that  $g(\lambda)$  has a global minimum at  $(1, -2)$  and a global maximum at  $(-1, 2)$ .

To count the eigenvalues of  $A$ , we need to find out how many times the horizontal line  $y = k$  intersects the graph of  $g(\lambda)$ . In Figure 7.14, we see that there are three solutions if  $k$  satisfies the inequality  $2 > k > -2$ , two solutions if  $k = 2$  or  $k = -2$ , and one solution if  $|k| > 2$ .

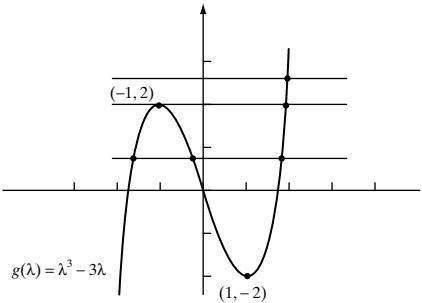


Figure 7.14: for Problem 7.2.32.

**7.2.33 a**  $f_A(\lambda) = \det(A - \lambda I_3) = -\lambda^3 + c\lambda^2 + b\lambda + a$

b By part a, we have  $c = 17$ ,  $b = -5$  and  $a = \pi$ , so  $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pi & -5 & 17 \end{bmatrix}$ .

**7.2.34** Consider the possible graphs of  $f_A(\lambda)$  assuming that it has 2 distinct real roots.

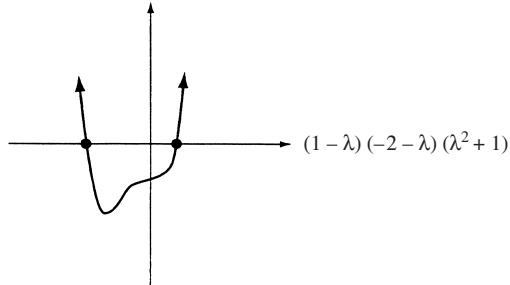


Figure 7.15: for Problem 7.2.34.

Algebraic multiplicity of each eigenvalue is 1. Example:  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ . See Figure 7.15.

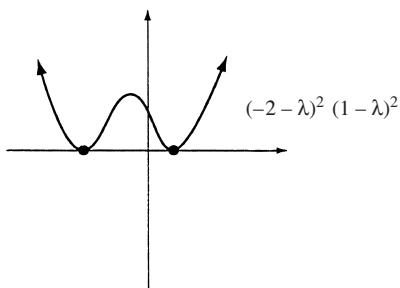


Figure 7.16: for Problem 7.2.34.

Algebraic multiplicity of each eigenvalue is 2. Example:  $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . See Figure 7.16.

Algebraic multiplicity of  $\lambda_1$  is 1, and of  $\lambda_2$  is 3.

Example:  $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . See Figure 7.17.

**7.2.35**  $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ , with  $f_A(\lambda) = (\lambda^2 + 1)^2$

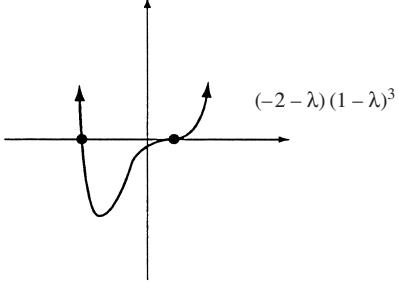


Figure 7.17: for Problem 7.2.34.

**7.2.36** Let  $A = \begin{bmatrix} B & & 0 \\ & B & \\ & & \ddots & \\ 0 & & & B \end{bmatrix}$  where  $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ,  $f_A(\lambda) = (\lambda^2 + 1)^n$ .

**7.2.37** We can write  $f_A(\lambda) = (\lambda - \lambda_0)^2 g(\lambda)$ , for some polynomial  $g$ . The product rule for derivatives tells us that  $f'_A(\lambda) = 2(\lambda - \lambda_0)g(\lambda) + (\lambda - \lambda_0)^2 g'(\lambda)$ , so that  $f'_A(\lambda_0) = 0$ , as claimed.

**7.2.38** By Theorem 7.2.4, the characteristic polynomial of  $A$  is  $f_A(\lambda) = \lambda^2 - 5\lambda - 14 = (\lambda - 7)(\lambda + 2)$ , so that the eigenvalues are 7 and -2.

$$7.2.39 \quad \text{tr}(AB) = \text{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \text{tr}\left(\begin{bmatrix} ae + bg & \cdots \\ \cdots & cf + dh \end{bmatrix}\right) = ae + bg + cf + dh.$$

$$\text{tr}(BA) = \text{tr}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \text{tr}\left(\begin{bmatrix} ea + fc & \cdots \\ \cdots & gb + hd \end{bmatrix}\right) = ea + fc + gb + hd. \text{ So they are equal.}$$

**7.2.40** Let the entries of  $A$  be  $a_{ij}$  and the entries of  $B$  be  $b_{ij}$ .

Now,  $\text{tr}(AB) = (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) + (a_{21}b_{12} + \cdots + a_{2n}b_{n2}) + \cdots + (a_{n1}b_{1n} + \cdots + a_{nn}b_{nn})$ . This is the sum of all products of the form  $a_{ij}b_{ji}$ .

We see that  $\text{tr}(BA) = (b_{11}a_{11} + \cdots + b_{1n}a_{n1}) + \cdots + (b_{n1}a_{1n} + \cdots + b_{nn}a_{nn})$ , which also is the sum of all products of the form  $b_{ji}a_{ij} = a_{ij}b_{ji}$ . Thus,  $\text{tr}(AB) = \text{tr}(BA)$ .

**7.2.41** So there exists an invertible  $S$  such that  $B = S^{-1}AS$ , and  $\text{tr}(B) = \text{tr}(S^{-1}AS) = \text{tr}((S^{-1}A)S)$ . By Exercise 40, this equals  $\text{tr}(S(S^{-1}A)) = \text{tr}(A)$ .

**7.2.42**  $\text{tr}((A + B)^2) = \text{tr}(A^2 + AB + BA + B^2) = \text{tr}(A^2) + \text{tr}(AB) + \text{tr}(BA) + \text{tr}(B^2)$ . By Exercise 40,  $\text{tr}(AB) = \text{tr}(BA)$ . Thus,  $\text{tr}((A + B)^2) = \text{tr}(A^2) + 2\text{tr}(BA) + \text{tr}(B^2) = \text{tr}(A^2) + \text{tr}(B^2)$ , since  $BA = 0$ .

**7.2.43**  $\text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = \text{tr}(AB) - \text{tr}(AB) = 0$ , but  $\text{tr}(I_n) = n$ , so no such  $A, B$  exist. We have used Exercise 40.

**7.2.44** No, there are no such matrices  $A$  and  $B$ . We will argue indirectly, assuming that invertible matrices  $A$  and

$B$  with  $AB - BA = A$  do exist. Then  $AB = BA + A = (B + I_n)A$ , and  $ABA^{-1} = B + I_n$ . Using Exercise 41, we see that  $\text{tr}(B) = \text{tr}(ABA^{-1}) = \text{tr}(B + I_n) = \text{tr}(B) + n$ , a contradiction.

**7.2.45**  $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + (-3 - 4k)$ . We want  $f_A(5) = 25 - 10 - 3 - 4k = 0$ , or,  $12 - 4k = 0$ , or  $k = 3$ .

**7.2.46** a  $\lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = (\text{tr}A)^2 - 2\det(A) = (a+d)^2 - 2(ad-bc) = a^2 + d^2 + 2bc$ .

b Based on part (a), we need to show that  $a^2 + d^2 + 2bc \leq a^2 + b^2 + c^2 + d^2$ , or  $2bc \leq b^2 + c^2$ , or  $0 \leq (b-c)^2$ . But the last inequality is obvious.

c By parts (a) and (b), the equality  $\lambda_1^2 + \lambda_2^2 = a^2 + b^2 + c^2 + d^2$  holds if (and only if)  $0 = (b-c)^2$ , or  $b = c$ . Thus equality holds for symmetric matrices  $A$ .

**7.2.47** Let  $M = [\vec{v}_1 \quad \vec{v}_2]$ . We want  $A[\vec{v}_1 \quad \vec{v}_2] = [\vec{v}_1 \quad \vec{v}_2] \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ , or,  $[A\vec{v}_1 \quad A\vec{v}_2] = [2\vec{v}_1 \quad 3\vec{v}_2]$ . Since  $\vec{v}_1$  or  $\vec{v}_2$  must be nonzero, 2 or 3 must be an eigenvalue of  $A$ .

**7.2.48** Let  $S = [\vec{v}_1 \vec{v}_2]$ . Then  $AS = [A\vec{v}_1 \ A\vec{v}_2]$  and  $SD = [2\vec{v}_1 \ 3\vec{v}_2]$ , so that  $\vec{v}_1$  must be an eigenvector with eigenvalue 2, and  $\vec{v}_2$  must be an eigenvector with eigenvalue 3. Thus, both 2 and 3 must be eigenvalues of  $A$ .

**7.2.49** As in problem 47, such an  $M$  will exist if  $A$  has an eigenvalue 2, 3 or 4.

**7.2.50** a If  $f(x) = x^3 + 6x - 20$  then  $f'(x) = 3x^2 + 6$  so  $f'(x) > 0$  for all  $x$ , i.e.  $f$  is always increasing, hence has only one real root.

b If  $v^3 - u^3 = 20$  and  $vu = 2$  then

$$\begin{aligned} (v-u)^3 + 6(v-u) &= v^3 - 3v^2u + 3vu^2 - u^3 + 6(v-u) = v^3 - u^3 - 3vu(v-u) + 6(v-u) \\ &= 20 - 6(v-u) + 6(v-u) = 20 \end{aligned}$$

Hence  $x = v - u$  satisfies the equation  $x^3 + 6x = 20$ .

c The second equation tells us that  $u = \frac{2}{v}$  or  $u^3 = \frac{8}{v^3}$ . Substituting into the first equation we find that

$$v^3 - \frac{8}{v^3} = 20, \text{ or, } (v^3)^2 - 8 = 20v^3 \text{ or } (v^3)^2 - 20v^3 - 8 = 0, \text{ with solutions}$$

$$v^3 = \frac{20 \pm \sqrt{400+32}}{2} = 10 \pm \sqrt{108} = 10 \pm 6\sqrt{3} \text{ and } v = \sqrt[3]{10 \pm \sqrt{108}}$$

Now  $u^3 = v^3 - 20 = -10 \pm \sqrt{108}$  and  $u = \sqrt[3]{-10 \pm \sqrt{108}}$ .

d Let  $v = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$  and  $u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$ .

$$\text{Then } v^3 - u^3 = q \text{ and } vu = \sqrt[3]{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2} = \frac{p}{3}.$$

Since  $x = v - u$  we have

$$x^3 + px = v^3 - 3v^2u + 3vu^2 - u^3 + p(v-u) = v^3 - u^3 - 3vu(v-u) + p(v-u)$$

$= q - p(v - u) + p(v - u) = q$ , as claimed.

If  $p$  is negative, the expression  $(\frac{q}{2})^2 + (\frac{p}{3})^3$  may be negative. Also, the equation  $x^3 + px = q$  may have more than one solution in this case.

- e Setting  $x = t - (\frac{a}{3})$  we get  $(t - \frac{a}{3})^3 + a(t - \frac{a}{3})^2 + b(t - \frac{a}{3}) + c = 0$  or  $t^3 - at^2 + at^2 + (\text{linear and constant terms}) = 0$  or  $t^3 + (\text{linear and constant terms}) = 0$ , as claimed (bring the constant terms to the right-hand side).

## Section 7.3

7.3.1  $\lambda_1 = 7, \lambda_2 = 9, E_7 = \ker \begin{bmatrix} 0 & 8 \\ 0 & 2 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, E_9 = \ker \begin{bmatrix} -2 & 8 \\ 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Eigenbasis:  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

7.3.2  $\lambda_1 = 2, \lambda_2 = 0, E_2 = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_0 = \text{span} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenbasis:  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

7.3.3  $\lambda_1 = 4, \lambda_2 = 9, E_4 = \text{span} \begin{bmatrix} 3 \\ -2 \end{bmatrix}, E_9 = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenbasis:  $\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7.3.4  $\lambda_1 = \lambda_2 = 1, E_1 = \text{span} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

No eigenbasis

7.3.5 No real eigenvalues as  $f_A(\lambda) = \lambda^2 - 2\lambda + 2$ .

7.3.6  $\lambda_{1,2} = \frac{7 \pm \sqrt{57}}{2}$

Eigenbasis:  $\begin{bmatrix} 3 \\ \lambda_1 - 2 \end{bmatrix} \approx \begin{bmatrix} 3 \\ 5.27 \end{bmatrix}, \begin{bmatrix} 3 \\ \lambda_2 - 2 \end{bmatrix} \approx \begin{bmatrix} 3 \\ -2.27 \end{bmatrix}$

7.3.7  $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ , eigenbasis:  $\vec{e}_1, \vec{e}_2, \vec{e}_3$

7.3.8  $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ , eigenbasis:  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

7.3.9  $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$ , eigenbasis:  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

7.3.10  $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0, E_1 = \text{span } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, E_0 = \text{span } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

No eigenbasis

7.3.11  $\lambda_1 = \lambda_2 = 0, \lambda_3 = 3$ , eigenbasis:  $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

7.3.12  $\lambda_1 = \lambda_2 = \lambda_3 = 1, E_1 = \text{span } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ , no eigenbasis

7.3.13  $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1$ , eigenbasis:  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

7.3.14  $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$ , eigenbasis:  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

7.3.15  $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1, E_0 = \text{span } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . We can use Kyle Numbers to see that

$$E_1 = \ker \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ -3 & -1 & 1 \\ -4 & 0 & 2 \end{bmatrix} = \text{span } \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

There is no eigenbasis since the eigenvalue 1 has algebraic multiplicity 2, but the geometric multiplicity is only 1.

7.3.16  $\lambda_1 = 0$  (no other real eigenvalues), with eigenvector  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

No real eigenbasis

7.3.17  $\lambda_1 = \lambda_2 = 0, \lambda_3 = \lambda_4 = 1$

with eigenbasis  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

**7.3.18**  $\lambda_1 = \lambda_2 = 0, \lambda_3 = \lambda_4 = 1, E_0 = \text{span}(\vec{e}_1, \vec{e}_3), E_1 = \text{span}(\vec{e}_2)$

No eigenbasis

**7.3.19** Since 1 is the only eigenvalue, with algebraic multiplicity 3, there exists an eigenbasis for  $A$  if (and only if) the geometric multiplicity of the eigenvalue 1 is 3 as well, that is, if  $E_1 = \mathbb{R}^3$ . Now  $E_1 = \ker \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$  is  $\mathbb{R}^3$  if (and only if)  $a = b = c = 0$ .

If  $a = b = c = 0$  then  $E_1$  is 3-dimensional with eigenbasis  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ .

If  $a \neq 0$  and  $c \neq 0$  then  $E_1$  is 1-dimensional and otherwise  $E_1$  is 2-dimensional. The geometric multiplicity of the eigenvalue 1 is  $\dim(E_1)$ .

**7.3.20** For  $\lambda_1 = 1, E_1 = \ker \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  so if  $a = 0$  then  $E_1$  is 2-dimensional, otherwise it is 1-dimensional.

For  $\lambda_2 = 2, E_2 = \ker \begin{bmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{bmatrix}$  so  $E_2$  is 1-dimensional.

Hence, there is an eigenbasis if  $a = 0$ .

**7.3.21** We want  $A$  such that  $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $A \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ , i.e.  $A \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$  so  $A = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$ .

The answer is unique.

**7.3.22** We want  $A$  such that  $A\vec{e}_1 = 7\vec{e}_1$  and  $A\vec{e}_2 = 7\vec{e}_2$  hence  $A = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ .

**7.3.23**  $\lambda_1 = \lambda_2 = 1$  and  $E_1 = \text{span}(\vec{e}_1)$ , hence there is no eigenbasis. The matrix represents a shear parallel to the  $x$ -axis.

**7.3.24** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . First we want  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ , or  $2a + b = 2, 2c + d = 1$ . This condition is satisfied by all matrices of the form  $A = \begin{bmatrix} a & 2 - 2a \\ c & 1 - 2c \end{bmatrix}$ . Next, we want there to be no other eigenvalue, besides 1, so that 1 must have an algebraic multiplicity of 2.

We want the characteristic polynomial to be  $(\lambda - 1)^2 = \lambda^2 - 2\lambda + 1$ , so that the trace must be 2, and  $a + (1 - 2c) = 2$ , or,  $a = 1 + 2c$ . Thus we want a matrix of the form  $A = \begin{bmatrix} 1 + 2c & -4c \\ c & 1 - 2c \end{bmatrix}$ .

Finally, we have to make sure the  $E_1 = \text{span} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  instead of  $E_1 = \mathbb{R}^2$ . This means that we must exclude the case  $A = I_2$ . In order to ensure this, we state simply that  $A = \begin{bmatrix} 1 + 2c & -4c \\ c & 1 - 2c \end{bmatrix}$ , where  $c$  is any nonzero constant.

**7.3.25** If  $\lambda$  is an eigenvalue of  $A$ , then  $E_\lambda = \ker(A - \lambda I_3) = \ker \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ a & b & c - \lambda \end{bmatrix}$ .

The second and third columns of the above matrix aren't parallel, hence  $E_\lambda$  is always 1-dimensional, i.e., the geometric multiplicity of  $\lambda$  is 1.

**7.3.26** Note that  $f_A(0) = \det(A - 0I_6) = \det(A)$  is negative. Since  $\lim_{\lambda \rightarrow \infty} f_A(\lambda) = \infty$ , there must be a positive root, by the Intermediate Value Theorem (see Exercise 2.2.47c). Therefore, the matrix  $A$  has a positive eigenvalue. See Figure 7.18.

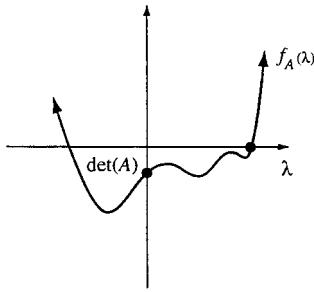


Figure 7.18: for Problem 7.3.26.

**7.3.27** By Theorem 7.2.4, we have  $f_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$  so  $\lambda_1 = 2$ ,  $\lambda_2 = 3$ .

**7.3.28** Since  $J_n(k)$  is triangular, its eigenvalues are its diagonal entries, hence its only eigenvalue is  $k$ . Moreover,

$$E_k = \ker(J_n(k) - kI_n) = \ker \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & 0 & & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix} = \text{span } (\vec{e}_1).$$

The geometric multiplicity of  $k$  is 1 while its algebraic multiplicity is  $n$ .

**7.3.29** Note that  $r$  is the number of nonzero diagonal entries of  $A$ , since the nonzero columns of  $A$  form a basis of  $\text{im}(A)$ . Therefore, there are  $n - r$  zeros on the diagonal, so that the algebraic multiplicity of the eigenvalue 0 is  $n - r$ . It is true for any  $n \times n$  matrix  $A$  that the geometric multiplicity of the eigenvalue 0 is  $\dim(\ker(A)) = n - \text{rank}(A) = n - r$ .

**7.3.30** Since  $A$  is triangular,  $f_A(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{mm} - \lambda)(0 - \lambda)^{n-m}$ .

Hence the algebraic multiplicity of  $\lambda = 0$  is  $(n - m)$ .

Also note that the rank of  $A$  is at least  $m$ , since the first  $m$  columns of  $A$  are linearly independent. Therefore, the geometric multiplicity of the eigenvalue 0 is  $\dim(\ker(A)) = n - \text{rank}(A) \leq n - m$ .

**7.3.31** They must be the same. For if they are not, by Theorem 7.3.7, the geometric multiplicities would not add up to  $n$ .

**7.3.32** Recall that a matrix and its transpose have the same rank (Theorem 5.3.9c). The geometric multiplicity of  $\lambda$  as an eigenvalue of  $A$  is  $\dim(\ker(A - \lambda I_n)) = n - \text{rank}(A - \lambda I_n)$ .

$$\begin{aligned} \text{The geometric multiplicity of } \lambda \text{ as an eigenvalue of } A^T &= \dim(\ker(A^T - \lambda I_n)) \\ &= \dim(\ker(A - \lambda I_n)^T) = n - \text{rank}(A - \lambda I_n)^T = n - \text{rank}(A - \lambda I_n). \end{aligned}$$

We can see that the two multiplicities are the same.

**7.3.33** If  $S^{-1}AS = B$ , then

$$S^{-1}(A - \lambda I_n)S = S^{-1}(AS - \lambda S) = S^{-1}AS - \lambda S^{-1}S = B - \lambda I_n.$$

**7.3.34** a If  $\vec{x}$  is in the kernel of  $B$ , then  $AS\vec{x} = SB\vec{x} = S\vec{0} = \vec{0}$ , so that  $S\vec{x}$  is in  $\ker(A)$ .

b  $T$  is clearly linear, and the transformation  $R(\vec{x}) = S^{-1}\vec{x}$  is the inverse of  $T$  (if  $\vec{x}$  is in the kernel of  $B$ , then  $S^{-1}\vec{x}$  is in the kernel of  $A$ , by part (a)).

c The equation  $\text{nullity}(A) = \text{nullity}(B)$  follows from part (b); the equation  $\text{rank}(A) = \text{rank}(B)$  then follows from the rank-nullity theorem (Theorem 3.3.7).

**7.3.35** No, since the two matrices have different eigenvalues (see Theorem 7.3.6c).

**7.3.36** No, since the two matrices have different traces (see Theorem 7.3.6.d)

**7.3.37** a  $A\vec{v} \cdot \vec{w} = (A\vec{v})^T \vec{w} = (\vec{v}^T A^T) \vec{w} = (\vec{v}^T A) \vec{w} = v^T (A\vec{w}) = \vec{v} \cdot A\vec{w}$

↑

$A$  symmetric

b Assume  $A\vec{v} = \lambda\vec{v}$  and  $A\vec{w} = \alpha\vec{w}$  for  $\lambda \neq \alpha$ , then  $(A\vec{v}) \cdot \vec{w} = (\lambda\vec{v}) \cdot \vec{w} = \lambda(\vec{v} \cdot \vec{w})$ , and  $\vec{v} \cdot A\vec{w} = \vec{v} \cdot \alpha\vec{w} = \alpha(\vec{v} \cdot \vec{w})$ .

By part a,  $\lambda(\vec{v} \cdot \vec{w}) = \alpha(\vec{v} \cdot \vec{w})$  i.e.,  $(\lambda - \alpha)(\vec{v} \cdot \vec{w}) = 0$ .

Since  $\lambda \neq \alpha$ , it must be that  $\vec{v} \cdot \vec{w} = 0$ , i.e.,  $\vec{v}$  and  $\vec{w}$  are perpendicular.

**7.3.38** Note that  $f_A(0) = \det(A - 0I_3) = \det(A) = 1$ .

Since  $\lim_{\lambda \rightarrow \infty} f_A(\lambda) = -\infty$ , the polynomial  $f_A(\lambda)$  must have a positive root  $\lambda_0$ , by the Intermediate Value Theorem in single variable calculus. In other words, the matrix  $A$  will have a positive eigenvalue  $\lambda_0$ . Since  $A$  is orthogonal, this eigenvalue  $\lambda_0$  will be 1, by Theorem 7.1.2. This means that there is a nonzero vector  $\vec{v}$  in  $\mathbb{R}^3$  such that  $A\vec{v} = 1\vec{v} = \vec{v}$ , as claimed. See Figure 7.19.

**7.3.39** a There are two eigenvalues,  $\lambda_1 = 1$  (with  $E_1 = V$ ) and  $\lambda_2 = 0$  (with  $E_0 = V^\perp$ ).

Now geometric multiplicity(1) =  $\dim(E_1) = \dim(V) = m$ , and

geometric multiplicity(0) =  $\dim(E_0) = \dim(V^\perp) = n - \dim(V) = n - m$ .

Since geometric multiplicity( $\lambda$ )  $\leq$  algebraic multiplicity( $\lambda$ ), by Theorem 7.3.7, and the algebraic multiplicities cannot add up to more than  $n$ , the geometric and algebraic multiplicities of the eigenvalues are the same here.

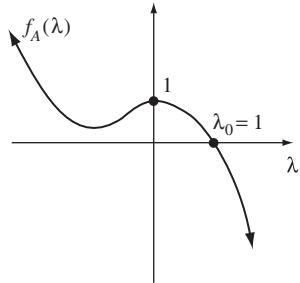


Figure 7.19: for Problem 7.3.38.

b Analogous to part a:  $E_1 = V$  and  $E_{-1} = V^\perp$ .

geometric multiplicity(1) = algebraic multiplicity(1) =  $\dim(V) = m$ , and

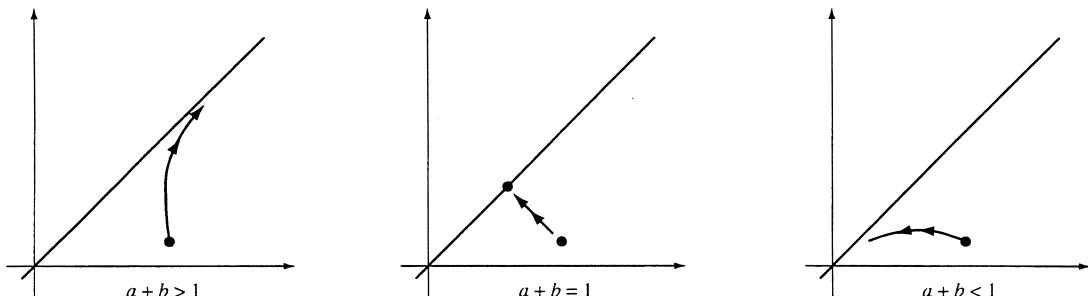
geometric multiplicity(-1) = algebraic multiplicity(-1) =  $\dim(V^\perp) = n - m$ .

**7.3.40** The matrix of the dynamical system is  $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$  so  $f_A(\lambda) = (a - \lambda)^2 - b^2$ .

Hence,  $\lambda_{1,2} = a \pm b$ , and the respective eigenvectors are  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

Since  $\vec{x}(0) = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix} = \frac{7}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{4} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ , by Theorem 7.1.3,  $\vec{x}(t) = \frac{7}{4}(a+b)^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{5}{4}(a-b)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

Note that  $a - b$  is between 0 and 1, so that the second summand in the formula above goes to  $\vec{0}$  as  $t$  goes to infinity. Qualitatively different outcomes occur depending on whether  $a + b$  exceeds 1, equals 1, or is less than 1. See Figure 7.20.



The spending of both nations grows: an arms race

The spending of both nations stabilizes at an equal level of \$b 1.75.

The spending of both nations decreases to zero: disarmament

Figure 7.20: for Problem 7.3.40.

**7.3.41** The eigenvalues of  $A$  are  $1.2, -0.8, -0.4$  with eigenvectors  $\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ .

Since  $\vec{x}_0 = 50 \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} + 50 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + 50 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$  we have  $\vec{x}(t) = 50(1.2)^t \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} + 50(-0.8)^t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + 50(-0.4)^t \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ , so, as  $t$  goes to infinity,  $j(t) : n(t) : a(t)$  approaches the proportion  $9 : 6 : 2$ .

**7.3.42**  $C(t+1) = 0.8C(t) + 10$  so if  $A \begin{bmatrix} C(t) \\ 1 \end{bmatrix} = \begin{bmatrix} C(t+1) \\ 1 \end{bmatrix}$ ,  $A = \begin{bmatrix} 0.8 & 10 \\ 0 & 1 \end{bmatrix}$ .  $A$  has eigenvectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 50 \\ 1 \end{bmatrix}$  corresponding to  $\lambda_1 = 0.8$  and  $\lambda_2 = 1$ . Since  $\begin{bmatrix} C(0) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -50 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 50 \\ 1 \end{bmatrix}$ , we have  $C(t) = -50(0.9)^t + 50$ , hence in the long run, there will be 50 spectators. The graph of  $C(t)$  looks similar to the graph in Figure 7.21.

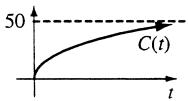


Figure 7.21: for Problem 7.3.42.

**7.3.43 a**  $A = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

b After 10 rounds, we have  $A^{10} \begin{bmatrix} 7 \\ 11 \\ 5 \end{bmatrix} \approx \begin{bmatrix} 7.6660156 \\ 7.6699219 \\ 7.6640625 \end{bmatrix}$ .

After 50 rounds, we have  $A^{50} \begin{bmatrix} 7 \\ 11 \\ 5 \end{bmatrix} \approx \begin{bmatrix} 7.66666666667 \\ 7.66666666667 \\ 7.66666666667 \end{bmatrix}$ .

c The eigenvalues of  $A$  are 1 and  $-\frac{1}{2}$  with

$$E_1 = \text{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } E_{-\frac{1}{2}} = \text{span} \left( \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right)$$

$$\text{so } \vec{x}(t) = \left(1 + \frac{c_0}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(-\frac{1}{2}\right)^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \left(-\frac{1}{2}\right)^t \frac{c_0}{3} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

After 1001 rounds, Alberich will be ahead of Brunnhilde  $\left(\text{by } \left(\frac{1}{2}\right)^{1001}\right)$ , so that Carl needs to beat Alberich to win the game. A straightforward computation shows that  $c(1001) - a(1001) = \left(\frac{1}{2}\right)^{1001} (1 - c_0)$ ; Carl wins if this quantity is positive, which is the case if  $c_0$  is less than 1.

Alternatively, observe that the ranking of the players is reversed in each round: whoever is first will be last after the next round. Since the total number of rounds is odd (1001), Carl wants to be last initially to win the game; he wants to choose a smaller number than both Alberich and Brunnhilde.

**7.3.44 a**  $a_{11} = 0.7$  means that only 70% of the pollutant present in Lake Sils at a given time is still there a week later; some is carried down to Lake Silvaplana by the river Inn, and some is absorbed or evaporates. The other diagonal entries can be interpreted analogously:  $a_{21} = 0.1$  means that 10% of the pollutant present in Lake Sils at any given time can be found in Lake Silvaplana a week later, carried down by the river Inn. The significance of the coefficient  $a_{32} = 0.2$  is analogous;  $a_{31} = 0$  means that no pollutant is carried down from Lake Sils to Lake St. Moritz in just one week. The matrix is lower triangular since no pollutant is carried from Lake Silvaplana to Lake Sils. The river Inn would have to flow the other way.

b The eigenvalues of  $A$  are 0.8, 0.6, 0.7, with corresponding eigenvectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

$$\vec{x}(0) = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = 100 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 100 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 100 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

$$\text{so } \vec{x}(t) = 100(0.8)^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 100(0.6)^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 100(0.7)^t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \text{ or}$$

$$x_1(t) = 100(0.7)^t$$

$$x_2(t) = 100(0.7)^t - 100(0.6)^t$$

$$x_3(t) = 100(0.8)^t + 100(0.6)^t - 200(0.7)^t. \text{ See Figure 7.22.}$$

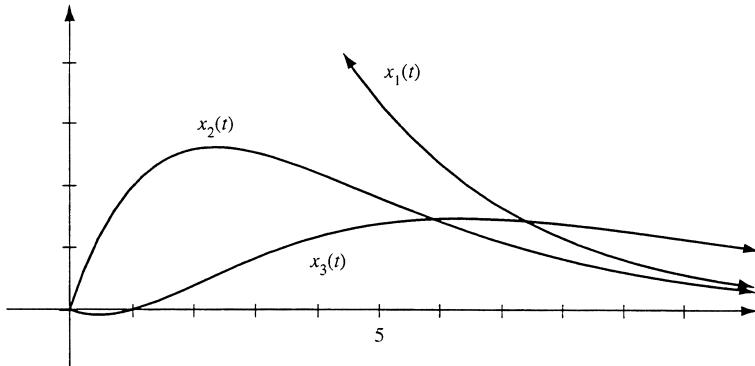


Figure 7.22: for Problem 7.3.44b.

Using calculus, we find that the function  $x_2(t) = 100(0.7)^t - 100(0.6)^t$  reaches its maximum at  $t \approx 2.33$ . Keep in mind, however, that our model holds for integer  $t$  only.

**7.3.45 a**  $A = \begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b  $B = \begin{bmatrix} A & \vec{b} \\ 0 & 1 \end{bmatrix}$

c The eigenvalues of  $A$  are 0.5 and  $-0.1$  with associated eigenvectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

The eigenvalues of  $B$  are 0.5,  $-0.1$ , and 1. If  $A\vec{v} = \lambda\vec{v}$  then  $B\begin{bmatrix} \vec{v} \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} \vec{v} \\ 0 \end{bmatrix}$  so  $\begin{bmatrix} \vec{v} \\ 0 \end{bmatrix}$  is an eigenvector of  $B$ .

Furthermore,  $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$  is an eigenvector of  $B$  corresponding to the eigenvalue 1. Note that this vector is  $\begin{bmatrix} -(A - I_2)^{-1}\vec{b} \\ 1 \end{bmatrix}$ .

d Write  $\vec{y}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ .

Note that  $c_3 = 1$ .

Now  $\vec{y}(t) = c_1(0.5)^t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2(-0.1)^t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \xrightarrow{t \rightarrow \infty} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$  so that  $\vec{x}(t) \xrightarrow{t \rightarrow \infty} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .

$$7.3.46 \text{ a } T_1(t+1) = 0.6T_1(t) + 0.1T_2(t) + 20$$

$$T_2(t+1) = 0.1T_1(t) + 0.6T_2(t) + 0.1T_3(t) + 20$$

$$T_3(t+1) = 0.1T_2(t) + 0.6T_3(t) + 40$$

$$\text{so } A = \begin{bmatrix} 0.6 & 0.1 & 0 \\ 0.1 & 0.6 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 20 \\ 20 \\ 40 \end{bmatrix}$$

$$\text{b } B = \begin{bmatrix} A & \vec{b} \\ 0 & 1 \end{bmatrix}$$

$$\text{c } \vec{y}(10) = B^{10} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 70.86 \\ 93.95 \\ 120.56 \\ 1 \end{bmatrix}$$

$$\vec{y}(30) = B^{30} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 74.989 \\ 99.985 \\ 124.989 \\ 1 \end{bmatrix}$$

$$\vec{y}(t) \text{ seems to approach } \begin{bmatrix} 75 \\ 100 \\ 125 \\ 1 \end{bmatrix} \text{ as } t \rightarrow \infty$$

d The eigenvalues of  $A$  are  $\lambda_1 \approx 0.45858$ ,  $\lambda_2 = 0.6$ ,  $\lambda_3 \approx 0.74142$  so the eigenvalues of  $B$  are  $\lambda_1 \approx 0.45858$ ,  $\lambda_2 = 0.6$ ,  $\lambda_3 \approx 0.74142$ ,  $\lambda_4 = 1$ .

If  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are eigenvectors of  $A$  (with  $A\vec{v}_i = \lambda_i \vec{v}_i$ ), then  $\begin{bmatrix} \vec{v}_1 \\ 0 \end{bmatrix}, \begin{bmatrix} \vec{v}_2 \\ 0 \end{bmatrix}, \begin{bmatrix} \vec{v}_3 \\ 0 \end{bmatrix}$  are corresponding eigenvectors of  $B$ . Furthermore,  $\begin{bmatrix} 75 \\ 100 \\ 125 \\ 1 \end{bmatrix}$  is an eigenvector of  $B$  with eigenvalue 1. Since  $\lambda_1, \lambda_2, \lambda_3$  are all less than 1,  $\lim_{t \rightarrow \infty} \vec{x}(t) = \begin{bmatrix} 75 \\ 100 \\ 125 \end{bmatrix}$ , as in Exercise 45.

**7.3.47 a** If  $\vec{x}(t) = \begin{bmatrix} r(t) \\ p(t) \\ w(t) \end{bmatrix}$ , then  $\vec{x}(t+1) = A\vec{x}(t)$  with  $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$ .

The eigenvalues of  $A$  are  $0, \frac{1}{2}, 1$  with eigenvectors  $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ .

Since  $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ,  $\vec{x}(t) = \frac{1}{2} \left(\frac{1}{2}\right)^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  for  $t > 0$ .

b As  $t \rightarrow \infty$  the ratio is  $1 : 2 : 1$  (since the first term of  $\vec{x}(t)$  drops out).

**7.3.48 a** We are told that

$$a(t+1) = a(t) + j(t)$$

$$j(t+1) = a(t), \text{ so that } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

b  $f_A(\lambda) = \lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1$  so  $\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$  with eigenvectors  $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ .

Since  $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$  we have  $\vec{x}(t) = \frac{1}{\sqrt{5}} (\lambda_1)^t \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}} (\lambda_2)^t \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ , i.e.

$$a(t) = \frac{1}{\sqrt{5}} ((\lambda_1)^{t+1} - (\lambda_2)^{t+1})$$

$$j(t) = \frac{1}{\sqrt{5}} ((\lambda_1)^t - (\lambda_2)^t).$$

c As  $t \rightarrow \infty$ ,  $\frac{a(t)}{j(t)} \rightarrow \lambda_1 = \frac{1+\sqrt{5}}{2}$ , since  $|\lambda_2| < 1$ .

**7.3.49** This “random” matrix  $A = [\vec{v}_0 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$  is unlikely to have any zeros above the diagonal. In this case, the columns  $\vec{v}_2, \dots, \vec{v}_n$  will be linearly independent (none of them is redundant), so that  $\text{rank}(A) = n - 1$  and  $\text{geometric multiplicity}(0) = \dim(\ker(A)) = n - \text{rank}(A) = 1$ . Alternatively, you can argue in terms of  $\text{rref}(A)$ .

**7.3.50 a**  $f_A(\lambda) = (2 - \lambda)^2 - 3$  so  $\lambda_{1,2} = 2 \pm \sqrt{3}$  (or approximately 3.73 and 0.27) with eigenvectors  $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$ .

See Figure 7.23.

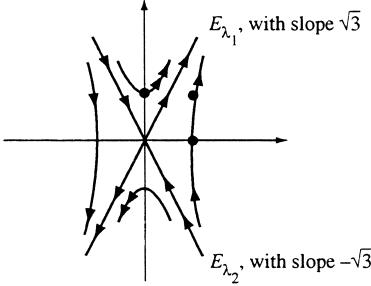


Figure 7.23: for Problem 7.3.50a.

b The trajectory starting at  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  is above the line  $E_{\lambda_1}$ , so that  $A^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$

(second column of  $A^t$ ) has a slope of more than  $\sqrt{3}$ , for all  $t$ . Applying this to  $t = 6$  gives the estimate  $\sqrt{3} < \frac{1351}{780}$ .

Likewise, the trajectory starting at  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is below  $E_{\lambda_1}$ , so that  $A^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$

(sum of the two columns of  $A^t$ ) has a slope of less than  $\sqrt{3}$ . Applying this to  $t = 4$  gives  $\frac{265}{153} < \sqrt{3}$ .

c  $\det(A^6) = (\det A)^6 = 1$  and  $\det(A^6) = 1351^2 - 780 \cdot 2340$ , so that  $1351^2 - 780 \cdot 2340 = 1$ .

Dividing both sides by  $1351 \cdot 780$  we obtain  $\frac{1351}{780} - \frac{2340}{1351} = \frac{1}{780 \cdot 1351} < 10^{-6}$ .

Now note that  $\frac{2340}{1351}$  is the slope of  $A^6 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , which is less than  $\sqrt{3}$ .

Therefore  $\frac{1351}{780} - \sqrt{3} < \frac{1351}{780} - \frac{2340}{1351} < 10^{-6}$ , as claimed.

d The slope of  $A^6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2131 \\ 3691 \end{bmatrix}$  is less than  $\sqrt{3}$ , i.e.  $\frac{3691}{2131} < \sqrt{3}$ .

$$7.3.51 \quad f_A(\lambda) = \det(A - \lambda I_3) = \det \begin{bmatrix} -\lambda & 0 & a \\ 1 & -\lambda & b \\ 0 & 1 & c - \lambda \end{bmatrix} = -\lambda^3 + c\lambda^2 + b\lambda + a.$$

7.3.52 The characteristic polynomial is  $f_A(\lambda) = (-1)^n (\lambda^n - a_{n-1}\lambda^{n-1} - \dots - a_1\lambda - a_0)$ . We can prove this by induction, using Laplace expansion along the first row:

$$f_A(\lambda) = \det(A - \lambda I_n) = -\lambda \det(A - \lambda I_n)_{11} + (-1)^{n+1} a_0 \det(A - \lambda I_n)_{1n}$$

$$= -\lambda (-1)^{n-1} (\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \dots - a_2\lambda - a_1) + (-1)^{n+1} a_0$$

$= (-1)^n (\lambda^n - a_{n-1}\lambda^{n-1} - \dots - a_1\lambda - a_0)$ . Note that  $(A - \lambda I_n)_{1n}$  is upper triangular with all 1's on the diagonal, so that  $\det(A - \lambda I_n)_{1n} = 1$ .

7.3.53 a.  $B = \begin{bmatrix} 0 & 0 & a & * & * \\ 1 & 0 & b & * & * \\ 0 & 1 & c & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$ .

b. Let  $B_{11} = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}$  and  $B_{22}$  be the diagonal blocks of the matrix  $B$  we found in part (a). Since  $A$  is similar to  $B$ , we have  $f_A(\lambda) = f_B(\lambda) = f_{B_{22}}(\lambda) f_{B_{11}}(\lambda) = f_{B_{22}}(\lambda)(-\lambda^3 + c\lambda^2 + b\lambda + a)$ , by Exercise 51. Now we observe that  $h(\lambda) = f_{B_{22}}(\lambda)$  is a quadratic polynomial, completing our proof.

c. From part (b) we know that  $f_A(A) = h(A)(-A^3 + cA^2 + bA + aI_5)$ . The given equation  $A^3\vec{v} = a\vec{v} + bA\vec{v} + cA^2\vec{v}$  can be written as  $(-A^3 + cA^2 + bA + aI_5)\vec{v} = \vec{0}$ , implying that  $f_A(A)\vec{v} = h(A)(-A^3 + cA^2 + bA + aI_5)\vec{v} = \vec{0}$ , as claimed.

7.3.54 a.  $B_{11} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & 0 & \dots & 0 & a_1 \\ 0 & 1 & 0 & \dots & 0 & a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & a_{m-2} \\ 0 & 0 & 0 & \dots & 1 & a_{m-1} \end{bmatrix}, B_{21} = 0$ .

b. We find  $f_A(\lambda) = f_B(\lambda) = f_{B_{22}}(\lambda) f_{B_{11}}(\lambda)$  as in part (b) of Exercise 53. Now matrix  $B_{11}$  has the form discussed in Exercise 52, where we show that

$$\det(B_{11}) = (-1)^m (\lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_1\lambda - a_0).$$

It follows that  $f_A(\lambda) = (-1)^m f_{B_{22}}(\lambda)(\lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_1\lambda - a_0)$ , as claimed.

c. Using the same approach as in Exercise 53, part (c), we can write the given equation for  $A^m\vec{v}$  as  $(A^m - a_{m-1}A^{m-1} - \dots - a_1A - a_0I_n)\vec{v} = \vec{0}$ . Thus

$$f_A(A)\vec{v} = (-1)^m f_{B_{22}}(A)(A^m - a_{m-1}A^{m-1} - \dots - a_1A - a_0I_n)\vec{v} = \vec{0}$$
, as claimed.

d. Our work in parts (a) through (c) shows that  $f_A(A)\vec{v} = \vec{0}$  for all  $\vec{v}$  in  $\mathbb{R}^n$ , meaning that  $f_A(A) = 0$ , the zero matrix.

## Section 7.4

7.4.1 Matrix  $A$  is diagonal already, so it's certainly diagonalizable. Let  $S = I_2$ .

7.4.2 Diagonalizable. The eigenvalues are 2,3, with associated eigenvectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ .

**7.4.3** Diagonalizable. The eigenvalues are 0,3, with associated eigenvectors  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ .

**7.4.4** Diagonalizable. The eigenvalues are 0,7, with associated eigenvectors  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$ .

**7.4.5** Fails to be diagonalizable. There is only one eigenvalue, 1, with a one-dimensional eigenspace.

**7.4.6** Fails to be diagonalizable. There is only one eigenvalue, 2, with a one-dimensional eigenspace.

**7.4.7** Diagonalizable. The eigenvalues are 2,-3, with associated eigenvectors  $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ .

**7.4.8** Diagonalizable. The eigenvalues are 4,-2, with associated eigenvectors  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ .

**7.4.9** Fails to be diagonalizable. There is only one eigenvalue, 1, with a one-dimensional eigenspace.

**7.4.10** Fails to be diagonalizable. There is only one eigenvalue, 1, with a one-dimensional eigenspace.

**7.4.11** Fails to be diagonalizable. The eigenvalues are 1,2,1, and the eigenspace

$$E_1 = \ker(A - I_3) = \text{span}(\vec{e}_1)$$

is only one-dimensional.

**7.4.12** Diagonalizable. The eigenvalues are 2,1,1, with associated eigenvectors  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

**7.4.13** Diagonalizable. The eigenvalues are 1,2,3, with associated eigenvectors  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ .

**7.4.14** Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ . If we let

$$S = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**7.4.15** Diagonalizable. The eigenvalues are 1, -1, 1, with associated eigenvectors  $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

**7.4.16** Diagonalizable. The eigenvalues are 3, 2, 1, with associated eigenvectors  $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

**7.4.17** Diagonalizable.

The eigenvalues are 0, 3, 0, with associated eigenvectors  $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

**7.4.18** Diagonalizable. The eigenvalues are 0, 2, 1, with associated eigenvectors  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

**7.4.19** Fails to be diagonalizable. The eigenvalues are 1, 0, 1, and the eigenspace  $E_1 = \ker(A - I_3) = \text{span}(\vec{e}_1)$  is only one-dimensional.

**7.4.20** Diagonalizable. The eigenvalues are 1, 2, 0, with associated eigenvectors  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

**7.4.21** Diagonalizable for all values of  $a$ , since there are always two distinct eigenvalues, 1 and 2. See Theorem 7.4.3.

**7.4.22** Diagonalizable except if  $b = 1$  and  $a \neq 0$ . (In that case we have only one eigenvalue, 1, with a one-dimensional eigenspace.).

**7.4.23** Diagonalizable for positive  $a$ . The characteristic polynomial is  $(\lambda - 1)^2 - a$ , so that the eigenvalues are  $\lambda = 1 \pm \sqrt{a}$ . If  $a$  is positive, then we have two distinct real eigenvalues, so that the matrix is diagonalizable. If  $a$  is negative, then there are no real eigenvalues. If  $a$  is 0, then 1 is the only eigenvalue, with a one-dimensional eigenspace.

**7.4.24** Diagonalizable for all values of  $a$ ,  $b$ , and  $c$ . The characteristic polynomial is  $\lambda^2 - (a+c)\lambda + ac - b^2$ , so that the eigenvalues are  $\lambda = \frac{a+c \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} = \frac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$ . Note that the expression whose square root we take (the “discriminant”) is always positive or 0, since it is the sum of two squares. If the discriminant is positive, then we have two distinct real eigenvalues, and everything is fine. The discriminant is 0 only if  $a = c$  and  $b = 0$ . In that case the matrix is diagonal already, and certainly diagonalizable as well.

**7.4.25** Diagonalizable for all values of  $a$ ,  $b$ , and  $c$ , since we have three distinct eigenvalues, 1, 2, and 3.

**7.4.26** The eigenvalues are 1, 2, 1, and the matrix is diagonalizable if (and only if) the eigenspace  $E_1$  is two-dimensional. Now  $E_1 = \ker(A - I_3) = \ker \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & c \\ 0 & 0 & b-ac \\ 0 & 0 & 0 \end{bmatrix}$  is two-dimensional if (and only if)  $b - ac = 0$ . Thus the matrix is diagonalizable if and only if  $b = ac$ .

**7.4.27** Diagonalizable only if  $a = b = c = 0$ . Since 1 is the only eigenvalue, it is required that  $E_1 = \mathbb{R}^3$ , that is, the matrix must be the identity matrix.

**7.4.28** Diagonalizable for positive values of  $a$ . The characteristic polynomial is  $-\lambda^3 + a\lambda = -\lambda(\lambda^2 - a)$ . If  $a$  is positive, then we have three distinct real eigenvalues,  $0, \pm\sqrt{a}$ , so that the matrix will be diagonalizable. If  $a$  is negative or 0, then 0 is the only real eigenvalue, and the matrix fails to be diagonalizable.

**7.4.29** Not diagonalizable for any  $a$ . The characteristic polynomial is  $-\lambda^3 + a$ , so that there is only one real eigenvalue,  $\sqrt[3]{a}$ , for all  $a$ . Since the corresponding eigenspace isn’t all of  $\mathbb{R}^3$ , the matrix fails to be diagonalizable.

**7.4.30** First we observe that all the eigenspaces of  $A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$  are one-dimensional, regardless of the value of  $a$ , since  $\text{rref}(A - \lambda I_3)$  is of the form  $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix}$  for all  $\lambda$ . Thus  $A$  is diagonalizable if and only if there are three distinct real eigenvalues. The characteristic polynomial of  $A$  is  $-\lambda^3 + 3\lambda + a$ . Thus the eigenvalues of  $A$  are the solutions of the equation  $\lambda^3 - 3\lambda = a$ . See Figure 7.24 with the function  $f(\lambda) = \lambda^3 - 3\lambda$ ; using calculus, we find the local maximum  $f(-1) = 2$  and the local minimum  $f(1) = -2$ . To count the distinct eigenvalues of  $A$ , we have to examine how many times the horizontal line  $y = a$  intersects the graph of  $f(\lambda)$ . The answer is three if  $|a| < 2$ , two if  $a = \pm 2$ , and one if  $|a| > 2$ . Thus  $A$  is diagonalizable if and only if  $|a| < 2$ , that is,  $-2 < a < 2$ .

**7.4.31** In Example 2 of Section 7.3 we see that the eigenvalues of  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$  are  $-1$  and  $5$ , with associated eigenvectors  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$ .

$$\begin{aligned} \text{Thus } A = SDS^{-1} \text{ and } A^t = SD^tS^{-1} &= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-1)^t & 0 \\ 0 & 5^5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2(-1)^t + 5^t & (-1)^{t+1} + 5^t \\ 2(5^t) - 2(-1)^t & 2(5^t) + (-1)^t \end{bmatrix} \end{aligned}$$

**7.4.32** The eigenvalues of  $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$  are 3 and 2, with associated eigenvectors  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ . Thus  $A = SDS^{-1}$  and  $A^t = SD^tS^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 2^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(3^t) - 2^t & 2^{t+1} - 2(3^t) \\ 3^t - 2^t & 2^{t+1} - 3^t \end{bmatrix}$ .

**7.4.33** The eigenvalues of  $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$  are 0 and 7, with associated eigenvectors  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$ . Thus  $A = SDS^{-1}$  and  $A^t = SD^tS^{-1} = \frac{1}{7} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 7^t \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7^t & 2(7^t) \\ 3(7^t) & 6(7^t) \end{bmatrix} = 7^{t-1}A$ . We can find the same result more directly by observing that  $A^2 = 7A$ .

**7.4.34** The eigenvalues of  $A$  are  $1/4$  and 1, with associated eigenvectors  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . If we let  $S = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$ , then  $S^{-1}AS = D = \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}$ . Thus  $A = SDS^{-1}$  and  $A^t = SD^tS^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (1/4)^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 + 2(1/4)^t & 1 - (1/4)^t \\ 2 - 2(1/4)^t & 2 + (1/4)^t \end{bmatrix}$ .

**7.4.35** Matrix  $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$  has the eigenvalues 3 and 2. If  $\vec{v}$  and  $\vec{w}$  are associated eigenvectors, and if we let  $S = [\vec{v} \ \vec{w}]$ , then  $S^{-1} \begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix} S = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ , so that matrix  $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$  is indeed similar to  $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ .

**7.4.36** Yes. The matrices  $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$  both have the eigenvalues 3 and 2, so that each of them is similar to the diagonal matrix  $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ , by Algorithm 7.4.4. Thus  $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$  is similar to  $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ , by parts b and c of Theorem 3.4.6.

**7.4.37** Yes. Matrices  $A$  and  $B$  have the same characteristic polynomial,  $\lambda^2 - 7\lambda + 7$ , so that they have the same two distinct real eigenvalues  $\lambda_{1,2} = \frac{7 \pm \sqrt{21}}{2}$ . Thus both  $A$  and  $B$  are similar to the diagonal matrix  $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ , by Algorithm 7.4.4. Therefore  $A$  is similar to  $B$ , by parts b and c of Theorem 3.4.6.

**7.4.38** No. As a counterexample, consider  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ .

**7.4.39** The eigenfunctions with eigenvalue  $\lambda$  are the nonzero functions  $f(x)$  such that  $T(f(x)) = f'(x) - f(x) = \lambda f(x)$ , or  $f'(x) = (\lambda + 1)f(x)$ . From calculus we recall that those are the exponential functions of the form

$f(x) = Ce^{(\lambda+1)x}$ , where  $C$  is a nonzero constant. Thus all real numbers are eigenvalues of  $T$ , and the eigenspace  $E_\lambda$  is one-dimensional, spanned by  $e^{(\lambda+1)x}$ .

**7.4.40** The eigenfunctions with eigenvalue  $\lambda$  are the nonzero functions  $f(x)$  such that  $T(f(x)) = 5f'(x) - 3f(x) = \lambda f(x)$ , or  $f'(x) = \frac{\lambda+3}{5}f(x)$ . From calculus we recall that those are the exponential functions of the form  $f(x) = Ce^{(\lambda+3)x/5}$ , where  $C$  is a nonzero constant. Thus all real numbers are eigenvalues of  $T$ , and the eigenspace  $E_\lambda$  is one-dimensional, spanned by  $e^{(\lambda+3)x/5}$ .

**7.4.41** The nonzero symmetric matrices are eigenmatrices with eigenvalue 2, since  $L(A) = A + A^T = 2A$  in this case. The nonzero skew-symmetric matrices have eigenvalue 0, since  $L(A) = A + A^T = A - A = 0$ . Yes,  $L$  is diagonalizable, since we have the eigenbasis  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  (three symmetric matrices, and one skew-symmetric one).

**7.4.42** The nonzero symmetric matrices are eigenmatrices with eigenvalue 0, since  $L(A) = A - A^T = A - A = 0$  in this case. The nonzero skew-symmetric matrices have eigenvalue 2, since  $L(A) = A - A^T = A + A = 2A$ . Yes,  $L$  is diagonalizable, since we have the eigenbasis  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  (three symmetric matrices, and one skew-symmetric one).

**7.4.43** The nonzero real numbers are “eigenvectors” with eigenvalue 1, and the nonzero imaginary numbers (of the form  $iy$ ) are “eigenvectors” with eigenvalue  $-1$ . Yes,  $T$  is diagonalizable, since we have the eigenbasis  $1, i$ .

**7.4.44** The nonzero sequence  $(x_0, x_1, x_2, \dots)$  is an eigensequence with eigenvalue  $\lambda$  if

$T(x_0, x_1, x_2, \dots) = (x_2, x_3, x_4, \dots) = \lambda(x_0, x_1, x_2, \dots) = (\lambda x_0, \lambda x_1, \lambda x_2, \dots)$ . This means that  $x_2 = \lambda x_0, x_3 = \lambda x_1, \dots, x_{n+2} = \lambda x_n, \dots$ . These are the sequences of the form  $(a, b, \lambda a, \lambda b, \lambda^2 a, \lambda^2 b, \dots)$ , where at least one of the first two terms,  $a$  and  $b$ , is nonzero.

Thus all real numbers  $\lambda$  are eigenvalues of  $T$ , and the eigenspace  $E_\lambda$  is two-dimensional, with basis  $(1, 0, \lambda, 0, \lambda^2, 0, \dots), (0, 1, 0, \lambda, 0, \lambda^2, \dots)$ .

**7.4.45** The nonzero sequence  $(x_0, x_1, x_2, \dots)$  is an eigensequence with eigenvalue  $\lambda$  if  $T(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots) = \lambda(x_0, x_1, x_2, \dots) = (\lambda x_0, \lambda x_1, \lambda x_2, \dots)$ . This means that  $0 = \lambda x_0, x_0 = \lambda x_1, x_1 = \lambda x_2, \dots, x_n = \lambda x_{n+1}, \dots$ . If  $\lambda$  is nonzero, then these equations imply that  $x_0 = \frac{1}{\lambda}0 = 0, x_1 = \frac{1}{\lambda}x_0 = 0, x_2 = \frac{1}{\lambda}x_1 = 0, \dots$ , so that there are no eigensequences in this case. If  $\lambda = 0$ , then we have  $x_0 = \lambda x_1 = 0, x_1 = \lambda x_2 = 0, x_2 = \lambda x_3 = 0, \dots$ , so that there aren't any eigensequences either. In summary: There are no eigenvalues and eigensequences for  $T$ .

**7.4.46** The nonzero sequence  $(x_0, x_1, x_2, \dots)$  is an eigensequence with eigenvalue  $\lambda$  if

$T(x_0, x_1, x_2, \dots) = (x_0, x_2, x_4, \dots) = \lambda(x_0, x_1, x_2, \dots) = (\lambda x_0, \lambda x_1, \lambda x_2, \dots)$ . This means that  $x_0 = \lambda x_0, x_2 = \lambda x_1, x_4 = \lambda x_2, \dots, x_{2n} = \lambda x_n, \dots$ . For each  $\lambda$ , there are lots of eigensequences: we can choose the terms  $x_k$  for odd  $k$  freely and then fix the  $x_k$  for even  $k$  according to the formula  $x_{2n} = \lambda x_n$ . For example, eigenspace  $E_3$  consists of the sequences of the form  $(x_0 = 0, x_1, x_2 = 3x_1, x_3, x_4 = 9x_1, x_5, x_6 = 3x_3, x_7, x_8 = 27x_1, x_9, \dots)$ , where  $x_1, x_3, x_5, x_7, x_9, \dots$  are arbitrary. Note that all the eigenspaces are infinite-dimensional. The condition  $x_0 = \lambda x_0$  implies that  $x_0 = 0$ , except for  $\lambda = 1$ , in which case  $x_0$  is arbitrary.

**7.4.47** The nonzero even functions, of the form  $f(x) = a + cx^2$ , are eigenfunctions with eigenvalue 1, and the nonzero odd functions, of the form  $f(x) = bx$ , have eigenvalue  $-1$ . Yes,  $T$  is diagonalizable, since the standard basis,

$1, x, x^2$ , is an eigenbasis for  $T$ .

**7.4.48** Apply  $T$  to the standard basis:  $T(1) = 1$ ,  $T(x) = 2x$ , and  $T(x^2) = (2x)^2 = 4x^2$ . This gives the eigenvalues 1, 2, and 4, with corresponding eigenfunctions  $1, x, x^2$ . Yes,  $T$  is diagonalizable, since the standard basis is an eigenbasis for  $T$ .

**7.4.49** The matrix of  $T$  with respect to the standard basis  $1, x, x^2$  is  $B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -6 \\ 0 & 0 & 9 \end{bmatrix}$ . The eigenvalues of  $B$  are 1, 3, 9, with corresponding eigenvectors  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$ . The eigenvalues of  $T$  are 1, 3, 9, with corresponding eigenfunctions  $1, 2x - 1, 4x^2 - 4x + 1 = (2x - 1)^2$ . Yes,  $T$  is diagonalizable, since the functions  $1, 2x - 1, (2x - 1)^2$  from an eigenbasis.

**7.4.50** The matrix of  $T$  with respect to the standard basis  $1, x, x^2$  is  $B = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$ . The only eigenvalue of  $B$  is 1, with corresponding eigenvector  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ . The only eigenvalue of  $T$  is 1 as well, with corresponding eigenfunction  $f(x) = 1$ .  $T$  fails to be diagonalizable, since there is only one eigenvalue, with a one-dimensional eigenspace.

**7.4.51** The nonzero constant functions  $f(x) = b$  are the eigenfunctions with eigenvalue 0. If  $f(x)$  is a polynomial of degree  $\geq 1$ , then the degree of  $f(x)$  exceeds the degree of  $f'(x)$  by 1 (by the power rule of calculus), so that  $f'(x)$  cannot be a scalar multiple of  $f(x)$ . Thus 0 is the only eigenvalue of  $T$ , and the eigenspace  $E_0$  consists of the constant functions.

**7.4.52** Let  $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ , with  $a_n \neq 0$ , be an eigenfunction of  $T$  with eigenvalue  $\lambda$ . Then  $T(f(x)) = x(a_1 + 2a_2x + \cdots + na_nx^{n-1}) = a_1x + 2a_2x^2 + \cdots + na_nx^n = \lambda(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = \lambda a_0 + \lambda a_1x + \lambda a_2x^2 + \cdots + \lambda a_nx^n$ . This means that  $\lambda a_0 = 0$ ,  $\lambda a_1 = a_1$ ,  $\lambda a_2 = 2a_2, \dots, \lambda a_n = na_n$ . Since we assumed that  $a_n \neq 0$ , we can conclude that  $\lambda = n$ . Now it follows that  $a_0 = a_1 = \cdots = a_{n-1} = 0$ , so that the eigenfunctions with eigenvalue  $n$  are the nonzero scalar multiples of  $x^n$ , of the form  $f(x) = a_nx^n$ . This makes good sense, since  $T(x^n) = x(nx^{n-1}) = nx^n$ . In summary: The eigenvalues are the integers  $n = 0, 1, 2, \dots$ , and the eigenspace  $E_n$  is  $\text{span}(x^n)$ .

**7.4.53** Suppose basis  $\mathcal{D}$  consists of  $f_1, \dots, f_n$ . We are told that the  $\mathcal{D}$ -matrix  $D$  of  $T$  is diagonal; let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the diagonal entries of  $D$ . By Theorem 4.3.3., we know that  $[T(f_i)]_{\mathcal{D}} = (\text{i}th \text{ column of } D) = \lambda_i \vec{e}_i$ , for  $i = 1, 2, \dots, n$ , so that  $T(f_i) = \lambda_i f_i$ , by definition of coordinates. Thus  $f_1, \dots, f_n$  is an eigenbasis for  $T$ , as claimed.

**7.4.54** Note that  $A^2 = 0$ , but  $B^2 \neq 0$ . Since  $A^2$  fails to be similar to  $B^2$ , matrix  $A$  isn't similar to  $B$  (see Example 7 of Section 3.4).

**7.4.55** Let  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , for example.

**7.4.56** The hint shows that matrix  $M = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$  is similar to  $N = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$ ; thus matrices  $M$  and  $N$  have the

same characteristic polynomial, by Theorem 7.3.6a. Now  $f_M(\lambda) = \det \begin{bmatrix} AB - \lambda I_n & 0 \\ B & -\lambda I_n \end{bmatrix} = (-\lambda)^n \det(AB - \lambda I_n) = (-\lambda)^n f_{AB}(\lambda)$ . To understand the second equality, consider Theorem 6.1.5. Likewise,  $f_N(\lambda) = (-\lambda)^n f_{BA}(\lambda)$ . It follows that  $(-\lambda)^n f_{AB}(\lambda) = (-\lambda)^n f_{BA}(\lambda)$  and therefore  $f_{AB}(\lambda) = f_{BA}(\lambda)$ , as claimed.

**7.4.57** Modifying the hint in Exercise 56 slightly, we can write  $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$ . Thus matrix  $M = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$  is similar to  $N = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$ . By Theorem 7.3.6a, matrices  $M$  and  $N$  have the same characteristic polynomial.

Now  $f_M(\lambda) = \det \begin{bmatrix} AB - \lambda I_m & 0 \\ B & -\lambda I_n \end{bmatrix} = (-\lambda)^n \det(AB - \lambda I_m) = (-\lambda)^n f_{AB}(\lambda)$ . To understand the second equality, consider Theorem 6.1.5. Likewise,  $f_N(\lambda)$

$$= \det \begin{bmatrix} -\lambda I_m & 0 \\ B & BA - \lambda I_n \end{bmatrix} = (-\lambda)^m f_{BA}(\lambda).$$

It follows that  $(-\lambda)^n f_{AB}(\lambda) = (-\lambda)^m f_{BA}(\lambda)$ . Thus matrices  $AB$  and  $BA$  have the same *nonzero* eigenvalues, with the same algebraic multiplicities.

If  $\text{mult}(AB)$  and  $\text{mult}(BA)$  are the algebraic multiplicities of 0 as an eigenvalue of  $AB$  and  $BA$ , respectively, then the equation  $(-\lambda)^n f_{AB}(\lambda) = (-\lambda)^m f_{BA}(\lambda)$  implies that

$$n + \text{mult}(AB) = m + \text{mult}(BA).$$

**7.4.58** a. If  $\vec{v}$  is in the image of  $A$ , then  $\vec{v} = A\vec{w}$  for some vector  $\vec{w}$ . Now  $A\vec{v} = A^2\vec{w} = \vec{0}$ , showing that  $\vec{v}$  is in the kernel of  $A$ .

b. From part (a) we know that  $\dim \text{im } A \leq \dim \ker A$ . Also,  $\dim \text{im } A > 0$  since  $A$  is nonzero, and  $\dim \ker A + \dim \text{im } A = 3$  by the rank-nullity theorem. This leaves us with only one possibility, namely,  $\dim \text{im } A = 1$  and  $\dim \ker A = 2$ .

c. Consider a relation  $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ . Multiplying both sides with  $A$  from the left, keeping in mind that  $\vec{v}_1$  and  $\vec{v}_3$  are in the kernel, we find  $c_2A\vec{v}_2 = c_2\vec{v}_1 = \vec{0}$ , so  $c_2 = 0$ . Now  $c_1 = c_3 = 0$  since  $\vec{v}_1$  and  $\vec{v}_3$  are independent by construction.

d. We have  $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  since  $A\vec{v}_1 = \vec{0}$ ,  $A\vec{v}_2 = \vec{v}_1$  and  $A\vec{v}_3 = \vec{0}$ .

**7.4.59** Yes,  $A$  is similar to  $B$ , since both  $A$  and  $B$  are similar to  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , by Exercise 58.

**7.4.60** We will use the method outlined in Exercise 58. We start with the vector  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  in the image of  $A$ .

Since  $\vec{v}_1 = (\text{first column of } A) = A\vec{e}_1$ , we can let  $\vec{v}_2 = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ . Finally, we let  $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ , another vector in

the kernel (use Kyle numbers!). Thus  $S = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$  will do the job, but there are other answers.

**7.4.61** First we need to verify that the vectors  $\vec{v}_1, \dots, \vec{v}_r, \vec{w}_1, \dots, \vec{w}_r, \vec{u}_1, \dots, \vec{u}_m$  are linearly independent. Consider a relation  $a_1\vec{v}_1 + \dots + a_r\vec{v}_r + b_1\vec{w}_1 + \dots + b_r\vec{w}_r + c_1\vec{u}_1 + \dots + c_m\vec{u}_m = \vec{0}$ . Multiplying both sides with  $A$ , we find  $b_1\vec{v}_1 + \dots + b_r\vec{v}_r = \vec{0}$ , so that  $b_1 = \dots = b_r = 0$  since the  $\vec{v}_i$  are independent by construction. Now the  $a_i$  and the  $c_i$  must be 0 as well, since the vectors  $\vec{v}_1, \dots, \vec{v}_r, \vec{u}_1, \dots, \vec{u}_m$  are independent by construction.

Secondly, we need to show that  $2r + m = n$ , meaning that we have the right number of vectors to form a basis of  $\mathbb{R}^n$ . Indeed,  $n = \dim \ker A + \dim \text{im } A = (r + m) + r = 2r + m$ . Each of the  $r$  pairs  $\vec{v}_i, \vec{w}_i$  gives us a block  $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  along the diagonal of  $B$ , since  $A\vec{v}_i = \vec{0}$  and  $A\vec{w}_i = \vec{v}_i$ , while all other entries of matrix  $B$  are zero.

**7.4.62** A nonzero function  $f$  is an eigenfunction of  $T$  with eigenvalue  $\lambda$  if  $T(f) = f'' + af' + bf = \lambda f$ , or,  $f'' + af' + (b - \lambda)f = 0$ . By Theorem 4.1.7, this differential equation has a two-dimensional solution space. Thus all real numbers are eigenvalues of  $T$ , and all the eigenspaces are two-dimensional.

**7.4.63 a** We need to solve the differential equation  $f''(x) = f(x)$ . As in Example 18 of Section 4.1, we will look for *exponential* solutions. The function  $f(x) = e^{kx}$  is a solution if  $k^2 = 1$ , or  $k = \pm 1$ . Thus the eigenspace  $E_1$  is the span of functions  $e^x$  and  $e^{-x}$ .

**b** We need to solve the differential equation  $f''(x) = 0$ . Integration gives  $f'(x) = C$ , a constant. If we integrate again, we find  $f(x) = Cx + c$ , where  $c$  is another arbitrary constant. Thus  $E_0 = \text{span}(1, x)$ .

**c** The solutions of the differential equation  $f''(x) = -f(x)$  are the functions  $f(x) = a \cos(x) + b \sin(x)$ , so that  $E_{-1} = \text{span}(\cos x, \sin x)$ . See the introductory example of Section 4.1 and Exercise 4.1.58.

**d** Modifying part c, we see that the solutions of the differential equation  $f''(x) = -4f(x)$  are the functions  $f(x) = a \cos(2x) + b \sin(2x)$ , so that  $E_{-4} = \text{span}(\cos(2x), \sin(2x))$ .

**7.4.64** The eigenvalues of  $A$  are 1 and 3, with associated eigenvectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Arguing as in Exercise 65, we find the basis  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$  for  $V$ , so that  $\dim(V) = 2$ .

**7.4.65** Let's write  $S$  in terms of its columns, as  $S = [\vec{v} \ \vec{w}]$ .

We want  $A[\vec{v} \ \vec{w}] = [\vec{v} \ \vec{w}] \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ , or,  $[A\vec{v} \ A\vec{w}] = [5\vec{v} \ -\vec{w}]$ , that is, we want  $\vec{v}$  to be in the eigenspace  $E_5$ , and  $\vec{w}$  in  $E_{-1}$ . We find that  $E_5 = \text{span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $E_{-1} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , so that  $S$  must be of the form  $\begin{bmatrix} a \begin{bmatrix} 1 \\ 2 \end{bmatrix} & b \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ . Thus, a basis of the space  $V$  is  $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ , and  $\dim(V) = 2$ .

**7.4.66** For  $A$  we find the eigenspaces  $E_1 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$  and  $E_2 = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$ . If we write  $S = [\vec{u} \ \vec{v} \ \vec{w}]$ ,

then we want  $A[\vec{u} \ \vec{v} \ \vec{w}] = [\vec{u} \ \vec{v} \ \vec{w}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ , or  $[A\vec{u} \ A\vec{v} \ A\vec{w}] = [\vec{u} \ \vec{v} \ 2\vec{w}]$ , that is,  $\vec{u}$  and  $\vec{v}$  must be in  $E_1$ , and

$\vec{w}$  must be in  $E_2$ . The matrices  $S$  we seek are of the form  $S = \begin{bmatrix} a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} & c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} & e \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix}$ ,

and a basis of  $V$  is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . The dimension of  $V$  is five.

**7.4.67** Let  $E_{\lambda_1} = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$  and  $E_{\lambda_2} = \text{span}(\vec{w}_1, \vec{w}_2)$ . As in Exercise 65, we can see that  $S$  must be of the form  $[\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3 \ \vec{x}_4 \ \vec{x}_5]$  where  $\vec{x}_1, \vec{x}_2$  and  $\vec{x}_3$  are in  $E_{\lambda_1}$  and  $\vec{x}_4$  and  $\vec{x}_5$  are in  $E_{\lambda_2}$ . Thus, we can write  $\vec{x}_1 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ , for example, or  $\vec{x}_5 = d_1\vec{w}_1 + d_2\vec{w}_2$ .

Using Summary 4.1.6, we find a basis:  $[\vec{v}_1 \ \vec{0} \ \vec{0} \ \vec{0} \ \vec{0}], [\vec{v}_2 \ \vec{0} \ \vec{0} \ \vec{0} \ \vec{0}], [\vec{v}_3 \ \vec{0} \ \vec{0} \ \vec{0} \ \vec{0}], [\vec{0} \ \vec{v}_1 \ \vec{0} \ \vec{0} \ \vec{0}], [\vec{0} \ \vec{v}_2 \ \vec{0} \ \vec{0} \ \vec{0}], [\vec{0} \ \vec{v}_3 \ \vec{0} \ \vec{0} \ \vec{0}], [\vec{0} \ \vec{0} \ \vec{v}_1 \ \vec{0} \ \vec{0}], [\vec{0} \ \vec{0} \ \vec{v}_2 \ \vec{0} \ \vec{0}], [\vec{0} \ \vec{0} \ \vec{v}_3 \ \vec{0} \ \vec{0}], [\vec{0} \ \vec{0} \ \vec{0} \ \vec{w}_1 \ \vec{0}], [\vec{0} \ \vec{0} \ \vec{0} \ \vec{w}_2 \ \vec{0}], [\vec{0} \ \vec{0} \ \vec{0} \ \vec{0} \ \vec{w}_1], [\vec{0} \ \vec{0} \ \vec{0} \ \vec{0} \ \vec{w}_2]$ .

Thus, the dimension of the space of matrices  $S$  is  $3 + 3 + 3 + 2 + 2 = 13$ .

**7.4.68** Let  $\vec{v}_1, \dots, \vec{v}_n$  be an eigenbasis for  $A$ , with  $A\vec{v}_i = \lambda_i\vec{v}_i$ . Arguing as in Exercises 64 through 67, we see that the  $i^{\text{th}}$  column of  $S$  must be in  $E_{\lambda_i}$ , so that it must be of the form  $c_i\vec{v}_i$  for some scalar  $c_i$ . The matrices  $S$  we seek are of the form  $S = [c_1\vec{v}_1 \ \dots \ c_n\vec{v}_n]$ , involving the  $n$  arbitrary constants  $c_1, \dots, c_n$ , so that the dimension of  $V$  is  $n$ .

**7.4.69** a  $B$  is diagonalizable since it has three distinct eigenvalues, so that  $S^{-1}BS$  is diagonal for some invertible  $S$ . But  $S^{-1}AS = S^{-1}I_3S = I_3$  is diagonal as well. Thus  $A$  and  $B$  are indeed simultaneously diagonalizable.

b There is an invertible  $S$  such that  $S^{-1}AS = D_1$  and  $S^{-1}BS = D_2$  are both diagonal. Then  $A = SD_1S^{-1}$  and  $B = SD_2S^{-1}$ , so that  $AB = (SD_1S^{-1})(SD_2S^{-1}) = SD_1D_2S^{-1}$  and  $BA = (SD_2S^{-1})(SD_1S^{-1}) = SD_2D_1S^{-1}$ . These two results agree, since  $D_1D_2 = D_2D_1$  for the diagonal matrices  $D_1$  and  $D_2$ .

c Let  $A$  be  $I_n$  and  $B$  a nondiagonalizable  $n \times n$  matrix, for example,  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

d Suppose  $BD = DB$  for a diagonal  $D$  with distinct diagonal entries. The  $ij$ -th entry of the matrix  $BD = DB$  is  $b_{ij}d_{jj} = d_{ii}b_{ij}$ . For  $i \neq j$  this implies that  $b_{ij} = 0$ . Thus  $B$  must be diagonal.

e Since  $A$  has  $n$  distinct eigenvalues,  $A$  is diagonalizable, that is, there is an invertible  $S$  such that  $S^{-1}AS = D$  is a diagonal matrix with  $n$  distinct diagonal entries. We claim that  $S^{-1}BS$  is diagonal as well; by part d it suffices to show that  $S^{-1}BS$  commutes with  $D = S^{-1}AS$ . This is easy to verify:

$$(S^{-1}BS)D = (S^{-1}BS)(S^{-1}AS) = S^{-1}BAS = S^{-1}ABS = (S^{-1}AS)(S^{-1}BS) = D(S^{-1}BS).$$

**7.4.70** Consider an  $n \times n$  matrix  $A$  with  $m$  distinct eigenvalues  $\lambda_1, \dots, \lambda_m$ .

If  $\vec{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda_m$ , then  $(A - \lambda_m I_n) \vec{v} = \vec{0}$ , so that  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) \vec{v} = \vec{0}$ . Since the factors in the product  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n)$  commute, we can conclude that  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) \vec{v} = \vec{0}$  for all eigenvectors  $\vec{v}$  of  $A$ .

Now assume that  $A$  is diagonalizable. Then there exists a basis consisting of eigenvectors, so that  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) \vec{v} = \vec{0}$  for all vectors in  $\mathbb{R}^n$ . We can conclude that  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) = 0$ .

Conversely, assume that  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) = 0$ . Then

$$n = \dim \ker((A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n))$$

$\leq \dim \ker(A - \lambda_1 I_n) + \dots + \dim \ker(A - \lambda_m I_n)$  by Exercise 4.2.83. This means that the geometric multiplicities of the eigenvalues add up to  $n$  (we know that this sum cannot exceed  $n$ ), so that  $A$  is diagonalizable, by Theorem 7.3.4b.

**7.4.71** The eigenvalues are 1 and 2, and  $(A - I_3)(A - 2I_3) = 0$ . Thus  $A$  is diagonalizable.

**7.4.72** The eigenvalues of this upper triangular matrix are 0 and 1. Now  $A(A - I_3) = \begin{bmatrix} 0 & 0 & b+ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Matrix  $A$  is diagonalizable if (and only if)  $b = -ac$ .

**7.4.73** If an  $n \times n$  matrix  $A$  has  $m$  distinct eigenvalues  $\lambda_1, \dots, \lambda_m$ , then we can write its characteristic polynomial as  $f_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_m)g(\lambda)$  for some polynomial  $g(\lambda)$  of degree  $n - m$ . Now  $f_A(A) = (A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n)g(A) = 0$  for a diagonalizable matrix  $A$  since  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_m I_n) = 0$ , as we have seen in Exercise 70. (In Exercise 7.3.54 we prove the Cayley-Hamilton theorem for all  $n \times n$  matrices  $A$ , but that proof is a bit longer.)

**7.4.74 a** For a diagonalizable  $n \times n$  matrix  $A$  with only two distinct eigenvalues,  $\lambda_1$  and  $\lambda_2$ , we have  $(A - \lambda_1 I_n)(A - \lambda_2 I_n) = 0$ , by Exercise 70. Thus the column vectors of  $A - \lambda_2 I_n$  are in the kernel of  $A - \lambda_1 I_n$ , that is, they are eigenvectors of  $A$  with eigenvalue  $\lambda_1$  (or else they are  $\vec{0}$ ). Conversely, the column vectors of  $A - \lambda_1 I_n$  are eigenvectors of  $A$  with eigenvalue  $\lambda_2$  (or else they are  $\vec{0}$ ).

**b** If  $A$  is a  $2 \times 2$  matrix with distinct eigenvalues  $\lambda_1$  and  $\lambda_2$ , then the nonzero columns of  $A - \lambda_1 I_2$  are eigenvectors of  $A$  with eigenvalue  $\lambda_2$ , as we observed in part (a). Since the matrices  $A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$  and  $A - \lambda_1 I_2$  have the same first column, the first column of  $A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$  will be an eigenvector of  $A$  with eigenvalue  $\lambda_2$  as well (or it is zero). Likewise, the second column of  $A - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$  will be an eigenvector of  $A$  with eigenvalue  $\lambda_1$  (or it is zero).

## Section 7.5

**7.5.1**  $z = 3 - 3i$  so  $|z| = \sqrt{3^2 + (-3)^2} = \sqrt{18}$  and  $\arg(z) = -\frac{\pi}{4}$ ,

$$\text{so } z = \sqrt{18} \left( \cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right).$$

**7.5.2** If  $z = r(\cos \theta + i \sin \theta)$  then  $z^4 = r^4(\cos 4\theta + i \sin 4\theta)$ .

$z^4 = 1$  if  $r = 1, \cos 4\theta = 1$  and  $\sin 4\theta = 0$  so  $4\theta = 2k\pi$  for an integer  $k$ , and  $\theta = \frac{k\pi}{2}$ ,

i.e.  $z = \cos\left(\frac{k\pi}{2}\right) + i \sin\left(\frac{k\pi}{2}\right)$ ,  $k = 0, 1, 2, 3$ . Thus  $z = 1, i, -1, -i$ . See Figure 7.24.

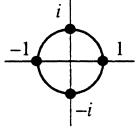


Figure 7.24: for Problem 7.5.2.

**7.5.3** If  $z = r(\cos \theta + i \sin \theta)$ , then  $z^n = r^n(\cos(n\theta) + i \sin(n\theta))$ .

$z^n = 1$  if  $r = 1, \cos(n\theta) = 1, \sin(n\theta) = 0$  so  $n\theta = 2k\pi$  for an integer  $k$ , and  $\theta = \frac{2k\pi}{n}$ ,

i.e.  $z = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$ ,  $k = 0, 1, 2, \dots, n-1$ . See Figure 7.25.

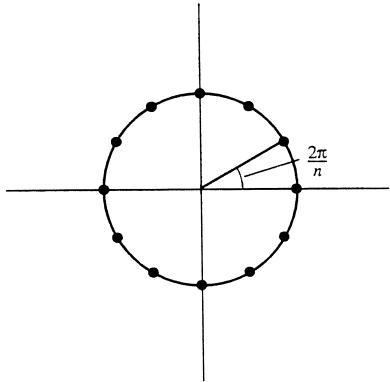


Figure 7.25: for Problem 7.5.3.

**7.5.4** Let  $z = r(\cos \theta + i \sin \theta)$  then  $w = \sqrt[r]{r} \left( \cos\left(\frac{\theta+2\pi k}{2}\right) + i \sin\left(\frac{\theta+2\pi k}{2}\right) \right)$ ,  $k = 0, 1$ .

**7.5.5** Let  $z = r(\cos \theta + i \sin \theta)$  then  $w = \sqrt[n]{r} \left( \cos\left(\frac{\theta+2\pi k}{n}\right) + i \sin\left(\frac{\theta+2\pi k}{n}\right) \right)$ ,  $k = 0, 1, 2, \dots, n-1$ .

**7.5.6** If we have  $z = r(\cos \theta + i \sin \theta)$  then  $\frac{1}{z}$  must have the property that  $z \cdot \frac{1}{z} = 1 = \cos 0 + i \sin 0$

i.e.  $|z| \cdot \left[\frac{1}{z}\right] = 1$  and  $\arg(z \cdot \frac{1}{z}) = \arg(z) + \arg\left(\frac{1}{z}\right) = 0$  so  $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i \sin(-\theta)) = \frac{1}{r}(\cos \theta - i \sin \theta)$  (since cosine is even, sine odd). Hence  $\frac{1}{z}$  is a real scalar multiple of  $\bar{z}$ . See Figure 7.26.

**7.5.7**  $|T(z)| = |z|\sqrt{2}$  and  $\arg(T(z)) = \arg(1-i) + \arg(z) = -\frac{\pi}{4} + \arg(z)$  so  $T$  is a clockwise rotation by  $\frac{\pi}{4}$  followed by a scaling of  $\sqrt{2}$ .

**7.5.8** By Theorem 7.5.1,  $(\cos 3\theta + i \sin 3\theta) = (\cos \theta + i \sin \theta)^3$ , i.e.

$$\cos 3\theta + i \sin 3\theta = \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \phi$$

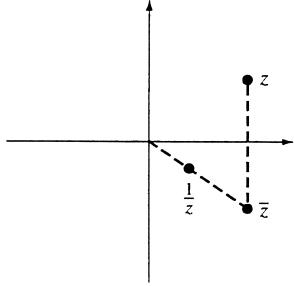


Figure 7.26: for Problem 7.5.6.

$$= (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta).$$

Equating real and imaginary parts, we get

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$$

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

**7.5.9**  $|z| = \sqrt{0.8^2 + 0.7^2} = \sqrt{1.15}$ ,  $\arg(z) = \arctan(-\frac{0.7}{0.8}) \approx -0.72$ . See Figure 7.27.

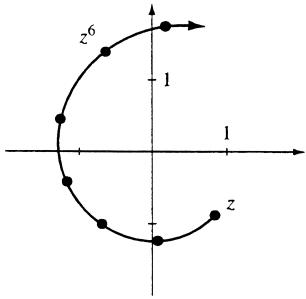


Figure 7.27: for Problem 7.5.9.

The trajectory spirals outward, in the clockwise direction.

**7.5.10** Let  $p(x) = ax^3 + bx^2 + cx + d$ , where  $a \neq 0$ . Since  $p$  must have a real root, say  $\lambda_1$ , we can write  $p(x) = a(x - \lambda_1)g(x)$  where  $g(x)$  is of the form  $g(x) = x^2 + px + q$ . On page 350, we see that  $g(x) = (x - \lambda_2)(x - \lambda_3)$ , so that  $p(x) = a(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$ , as claimed.

**7.5.11** Notice that  $f(1) = 0$  so  $\lambda = 1$  is a root of  $f(\lambda)$ . Hence  $f(\lambda) = (\lambda - 1)g(\lambda)$ , where  $g(\lambda) = \frac{f(\lambda)}{\lambda - 1} = \lambda^2 - 2\lambda + 5$ . Setting  $g(\lambda) = 0$  we get  $\lambda = 1 \pm 2i$  so that  $f(\lambda) = (\lambda - 1)(\lambda - 1 - 2i)(\lambda - 1 + 2i)$ .

**7.5.12** We will use the facts:

i)  $\overline{z + w} = \bar{z} + \bar{w}$  and

ii)  $(\overline{z^n}) = \bar{z}^n$

which are easy to check. Assume  $\lambda_0$  is a complex root of  $f(\lambda) = a_n\lambda^n + \dots + a_1\lambda + a_0$ , where the coefficients  $a_i$  are real. Since  $\lambda_0$  is a root of  $f$ , we have  $a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \dots + a_1\lambda_0 + a_0 = 0$ .

Taking the conjugate of both sides we get  $\overline{a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0} = \bar{0}$  so by fact i), and factoring the real constants we get  $a_n\bar{\lambda}_0^n + a_{n-1}\bar{\lambda}_0^{n-1} + \cdots + a_1\bar{\lambda}_0 + a_0 = 0$ .

Now, by fact ii),  $a_n(\bar{\lambda}_0)^n + a_{n-1}(\bar{\lambda}_0)^{n-1} + \cdots + a_1\bar{\lambda}_0 + a_0 = 0$ , i.e.  $\bar{\lambda}_0$  is also a root of  $f$ , as claimed.

7.5.13  $\begin{bmatrix} 2i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{w}}$  is an eigenvector with eigenvalue  $2i$ , so that we can let  $S = [\vec{w} \quad \vec{v}] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ , with  $S^{-1}AS = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ .

7.5.14  $\begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{w}}$  is an eigenvector with eigenvalue  $i$ , so that we can let  $S = [\vec{w} \quad \vec{v}] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , with  $S^{-1}AS = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

7.5.15  $\begin{bmatrix} 1 \\ 2+i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{w}}$  is an eigenvector with eigenvalue  $2+i$ , so that we can let  $S = [\vec{w} \quad \vec{v}] = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ , with  $S^{-1}AS = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ .

7.5.16  $\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{w}}$  is an eigenvector with eigenvalue  $4+i$ , so that we can let  $S = [\vec{w} \quad \vec{v}] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ , with  $S^{-1}AS = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$ .

7.5.17  $\begin{bmatrix} 2 \\ -1+2i \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{\vec{w}}$  is an eigenvector with eigenvalue  $3+4i$ , so that we can let  $S = [\vec{w} \quad \vec{v}] = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}$ , with  $S^{-1}AS = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ .

7.5.18 Let  $\vec{v}_1, \vec{v}_2$  be two eigenvectors of  $A$ . They define a parallelogram of area  $S = |\det[\vec{v}_1 \vec{v}_2]|$ . Now  $A\vec{v}_1 = \lambda_1\vec{v}_1$  and  $A\vec{v}_2 = \lambda_2\vec{v}_2$  define a parallelogram of area  $S_1 = |\det[\lambda_1\vec{v}_1 \lambda_2\vec{v}_2]| = |\lambda_1\lambda_2 \det[\vec{v}_1 \vec{v}_2]|$  so  $\frac{S_1}{S} = |\lambda_1\lambda_2| = |\det(A)|$ . Hence  $|\det(A)| = |\lambda_1\lambda_2|$ , as claimed. In  $\mathbb{R}^3$ , a similar argument holds if we replace areas by volumes. See Figure 7.28.

7.5.19 a Since  $A$  has eigenvalues 1 and 0 associated with  $V$  and  $V^\perp$  respectively and since  $V$  is the eigenspace of  $\lambda = 1$ , by Theorem 7.5.5,  $\text{tr}(A) = m$ ,  $\det(A) = 0$ .

b Since  $B$  has eigenvalues 1 and  $-1$  associated with  $V$  and  $V^\perp$  respectively and since  $V$  is the eigenspace associated with  $\lambda = 1$ ,  $\text{tr}(A) = m - (n-m) = 2m-n$ ,  $\det B = (-1)^{n-m}$ .

7.5.20  $f_A(\lambda) = (3-\lambda)(-3-\lambda) + 10 = \lambda^2 + 1$  so  $\lambda_{1,2} = \pm i$ .

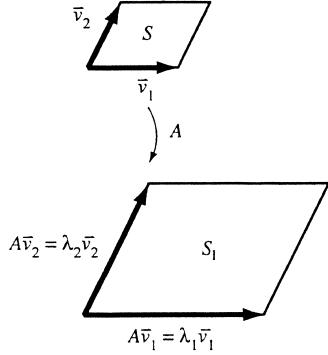


Figure 7.28: for Problem 7.5.18.

**7.5.21**  $f_A(\lambda) = (11 - \lambda)(-7 - \lambda) + 90 = \lambda^2 - 4\lambda + 13$  so  $\lambda_{1,2} = 2 \pm 3i$ .

**7.5.22**  $f_A(\lambda) = (1 - \lambda)(10 - \lambda) + 12 = \lambda^2 - 11\lambda + 22$  so  $\lambda_{1,2} = \frac{11 \pm \sqrt{33}}{2}$ .

**7.5.23**  $f_A(\lambda) = -\lambda^3 + 1 = -(\lambda - 1)(\lambda^2 + \lambda + 1)$  so  $\lambda_1 = 1, \lambda_{2,3} = \frac{-1 \pm \sqrt{3}i}{2}$ .

**7.5.24**  $f_A(\lambda) = -\lambda^3 + 3\lambda^2 - 7\lambda + 5$  so  $\lambda_1 = 1, \lambda_{2,3} = 1 \pm 2i$ . (See Exercise 11.)

**7.5.25**  $f_A(\lambda) = \lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1) = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i)$  so  $\lambda_{1,2} = \pm 1$  and  $\lambda_{3,4} = \pm i$

**7.5.26**  $f_A(\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda^2 - 2\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda - 2)\lambda = 0$ , so  $\lambda_{1,2} = 1 \pm i, \lambda_3 = 2, \lambda_4 = 0$ .

**7.5.27** By Theorem 7.5.5,  $\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$ ,  $\det(A) = \lambda_1 \lambda_2 \lambda_3$  but  $\lambda_1 = \lambda_2 \neq \lambda_3$  by assumption, so  $\text{tr}(A) = 1 = 2\lambda_2 + \lambda_3$  and  $\det(A) = 3 = \lambda_2^2 \lambda_3$ .

Solving for  $\lambda_2, \lambda_3$  we get  $-1, 3$  hence  $\lambda_1 = \lambda_2 = -1$  and  $\lambda_3 = 3$ . (Note that the eigenvalues must be real; why?)

**7.5.28** Suppose the complex eigenvalues are  $z = a + ib$  and  $\bar{z} = a - ib$ . By Theorem 7.5.5, we have  $\text{tr}(A) = 2 + z + \bar{z} = 2 + 2a = 8$ , so that  $a = 3$ . Furthermore,  $\det(A) = 2z\bar{z} = 2(a^2 + b^2) = 2(9 + b^2) = 50$ , so that  $b = 4$ . Hence the complex eigenvalues are  $3 \pm 4i$ .

**7.5.29**  $\text{tr}(A) = 0$  so  $\lambda_1 + \lambda_2 + \lambda_3 = 0$ .

Also, we can compute  $\det(A) = bcd > 0$  since  $b, c, d > 0$ . Therefore,  $\lambda_1 \lambda_2 \lambda_3 > 0$ .

Hence two of the eigenvalues must be negative, and the largest one (in absolute value) must be positive.

**7.5.30 a** The  $i$ th entry of  $A\vec{x}$  is  $\sum_{k=1}^n a_{ik}x_k$ , so that the sum of all the entries of  $A\vec{x}$  is

$$\sum_{i=1}^n \sum_{k=1}^n a_{ik}x_k = \sum_{k=1}^n \sum_{i=1}^n a_{ik}x_k = \sum_{k=1}^n \left( \sum_{i=1}^n a_{ik} \right) x_k = \sum_{k=1}^n x_k = 1.$$

↑

- b As we do some computer experiments,  $A^t$  appears to approach a matrix with identical columns, with column sum 1. Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  be an eigenbasis with  $\lambda_1 = 1$  and  $|\lambda_j| < 1$  for  $j = 2, \dots, n$ . For a fixed  $i$ , write  $\vec{e}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ , so that (ith column of  $A^t$ ) =  $A^t\vec{e}_i = c_1\vec{v}_1 + [c_2\lambda_2^t\vec{v}_2 + \dots + c_n\lambda_n^t\vec{v}_n]$ .

(The term in square brackets goes to zero as  $t$  goes to infinity.)

Therefore,  $\lim_{t \rightarrow \infty} (\text{ith column of } A^t) = \lim_{t \rightarrow \infty} (A^t\vec{e}_i) = c_1\vec{v}_1$ .

Furthermore, the entries of  $A^t\vec{e}_i$  add up to 1, for all  $t$ , by part a. Therefore, the same is true for the limit (since the limit of a sum is the sum of the limits).

It follows that  $\lim_{t \rightarrow \infty} (A^t)$  exists and has identical columns, with column sum 1, as claimed.

- 7.5.31** No matter how we choose  $A$ ,  $\frac{1}{15}A$  is a regular transition matrix, so that  $\lim_{t \rightarrow \infty} \left(\frac{1}{15}A\right)^t$  is a matrix with identical columns by Exercise 30. Therefore, the columns of  $A^t$  “become more and more alike” as  $t$  approaches infinity, in the sense that  $\lim_{t \rightarrow \infty} \frac{\text{ijth entry of } A^t}{\text{ikth entry of } A^t} = 1$  for all  $i, j, k$ .

**7.5.32** a  $\vec{x}(t) = \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} 0.6a(t) + 0.1m(t) + 0.5s(t) \\ 0.2a(t) + 0.7m(t) + 0.1s(t) \\ 0.2a(t) + 0.2m(t) + 0.4s(t) \end{bmatrix}$  so  $A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$ .

Note that  $A$  is a regular transition matrix.

- b By Exercise 30,  $\lim_{t \rightarrow \infty} (A^t) = [\vec{v} \vec{v} \vec{v}]$ , where  $\vec{v}$  is the unique eigenvector of  $A$  with eigenvalue 1 and column sum 1.

We find that  $\vec{v} = \begin{bmatrix} 0.4 \\ 0.35 \\ 0.25 \end{bmatrix}$ .

Now  $\lim_{t \rightarrow \infty} \vec{x}(t) = \lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \left( \lim_{t \rightarrow \infty} A^t \right) \vec{x}_0 = [\vec{v} \vec{v} \vec{v}] \vec{x}_0 = \vec{v}$ , since the components of  $\vec{x}_0$  add up to 1. The market shares approach 40%, 35%, and 25%, respectively, regardless of the initial shares.

- 7.5.33** a  $C$  is obtained from  $B$  by dividing each column of  $B$  by its first component. Thus, the first row of  $C$  will consist of 1's.

- b We observe that the columns of  $C$  are almost identical, so that the columns of  $B$  are “almost parallel” (that is, almost scalar multiples of each other).

- c Let  $\lambda_1, \lambda_2, \dots, \lambda_5$  be the eigenvalues. Assume  $\lambda_1$  real and positive and  $\lambda_1 > |\lambda_j|$  for  $2 \leq j \leq 5$ .

Let  $\vec{v}_1, \dots, \vec{v}_5$  be corresponding eigenvectors. For a fixed  $i$ , write  $\vec{e}_i = \sum_{j=1}^5 c_j \vec{v}_j$ ; then

(ith column of  $A^t$ ) =  $A^t \vec{e}_i = c_1 \lambda_1^t \vec{v}_1 + \dots + c_5 \lambda_5^t \vec{v}_5$ .

But in the last expression, for large  $t$ , the first term is dominant, so the  $i$ th column of  $A^t$  is almost parallel to  $\vec{v}_1$ , the eigenvector corresponding to the dominant eigenvalue.

d By part c, the columns of  $B$  and  $C$  are almost eigenvectors of  $A$  associated with the largest eigenvalue,  $\lambda_1$ . Since the first row of  $C$  consists of 1's, the entries in the first row of  $AC$  will be close to  $\lambda_1$ .

**7.5.34 a** The eigenvalues of  $A - \lambda I_n$  are  $\lambda_1 - \lambda, \lambda_2 - \lambda, \dots, \lambda_n - \lambda$ , and we were told that  $|\lambda_1 - \lambda| < |\lambda_i - \lambda|$  for  $i = 2, \dots, n$ . We may assume that  $\lambda_1 \neq \lambda$  (otherwise we are done).

The eigenvalues of  $(A - \lambda I_n)^{-1}$  are  $(\lambda_1 - \lambda)^{-1}, (\lambda_2 - \lambda)^{-1}, \dots, (\lambda_n - \lambda)^{-1}$ , and  $(\lambda_1 - \lambda)^{-1}$  has the largest modulus. The matrices  $A$ ,  $A - \lambda I_n$ , and  $(A - \lambda I_n)^{-1}$  have the same eigenvectors.

For large  $t$ , the columns of the  $t$ th power of  $(A - \lambda I_n)^{-1}$  will be almost eigenvectors of  $A$ . If  $\vec{v}$  is such a column, compare  $\vec{v}$  and  $A\vec{v}$  to find an approximation of  $\lambda_1$ .

b See Figure 7.29.

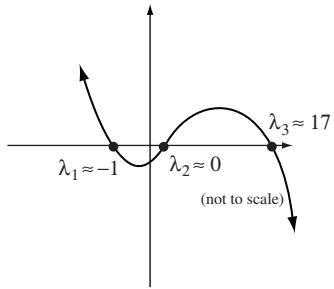


Figure 7.29: for Problem 7.5.34b.

Let  $\lambda = -1$ .

$$A - \lambda I_3 = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 6 & 6 \\ 7 & 8 & 11 \end{bmatrix}, N = (A - \lambda I_3)^{-1} = \begin{bmatrix} 9 & 1 & -3 \\ -1 & 0.5 & 0 \\ -5 & -1 & 2 \end{bmatrix}, \text{ and } B = N^{20}.$$

Obtain  $C$  from  $B$  as in Exercise 33:

$$C \approx \begin{bmatrix} 1 & 1 & 1 \\ -0.098922005729 & -0.098922005729 & -0.098922005729 \\ -0.569298722688 & -0.569298722688 & -0.569298722688 \end{bmatrix}$$

$$AC \approx \begin{bmatrix} -0.905740179522 & -0.905740179522 & -0.905740179522 \\ * & * & * \\ * & * & * \end{bmatrix}$$

The entries in the first row of  $AC$  give us a good approximation for  $\lambda_1$ , and the columns of  $C$  give us a good approximation for a corresponding eigenvector.

**7.5.35** We have  $f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$

$= (-\lambda)^n + (\lambda_1 + \lambda_2 + \cdots + \lambda_n)(-\lambda)^{n-1} + \cdots + (\lambda_1 \lambda_2 \cdots \lambda_n)$ . But, by Theorem 7.2.5, the coefficient of  $(-\lambda)^{n-1}$  is  $\text{tr}(A)$ . So,  $\text{tr}(A) = \lambda_1 + \cdots + \lambda_n$ .

**7.5.36 a** The entries in the first row are age-specific birth rates and the entries just below the diagonal are age-specific survival rates. For example, the entry 1.6 in the first row tells us that during the next 15 years the people who

are 15–30 years old today will *on average* have 1.6 children (3.2 per couple) who will survive to the next census. The entry 0.53 tells us that 53% of those in the age group 45–60 today will still be alive in 15 years (they will then be in the age group 60–75).

- b Using technology, we find the largest eigenvalue  $\lambda_1 = 1.908$  with associated eigenvector

$$\vec{v}_1 \approx \begin{bmatrix} 0.574 \\ 0.247 \\ 0.115 \\ 0.047 \\ 0.014 \\ 0.002 \end{bmatrix}.$$

The components of  $\vec{v}_1$  give the distribution of the population among the age groups in the long run, assuming that current trends continue.  $\lambda_1$  gives the factor by which the population will grow in the long run in a period of 15 years; this translates to an annual growth factor of  $\sqrt[15]{1.908} \approx 1.044$ , or an annual growth of about 4.4%.

- 7.5.37** a Use that  $\bar{w} + \bar{z} = \overline{w+z}$  and  $\overline{wz} = \overline{w}\overline{z}$ .

$$\begin{bmatrix} w_1 & -\bar{z}_1 \\ z_1 & \bar{w}_1 \end{bmatrix} + \begin{bmatrix} w_2 & -\bar{z}_2 \\ z_2 & \bar{w}_2 \end{bmatrix} = \begin{bmatrix} w_1 + w_2 & -(\bar{z}_1 + z_2) \\ z_1 + z_2 & \frac{\bar{w}_1 + w_2}{w_1 + w_2} \end{bmatrix} \text{ is in } \mathbb{H}.$$

$$\begin{bmatrix} w_1 & -\bar{z}_1 \\ z_1 & \bar{w}_1 \end{bmatrix} \begin{bmatrix} w_2 & -\bar{z}_2 \\ z_2 & \bar{w}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \bar{z}_1 z_2 & -(\bar{z}_1 w_2 + \bar{w}_1 z_2) \\ z_1 w_2 + \bar{w}_1 z_2 & \frac{w_1 w_2 - \bar{z}_1 z_2}{w_1 w_2 - \bar{z}_1 z_2} \end{bmatrix} \text{ is in } \mathbb{H}.$$

- b If  $A$  in  $\mathbb{H}$  is nonzero, then  $\det(A) = w\bar{w} + z\bar{z} = |w|^2 + |z|^2 > 0$ , so that  $A$  is invertible.

- c Yes; if  $A = \begin{bmatrix} w & -\bar{z} \\ z & \bar{w} \end{bmatrix}$ , then  $A^{-1} = \frac{1}{|w|^2 + |z|^2} \begin{bmatrix} \bar{w} & \bar{z} \\ -z & w \end{bmatrix}$  is in  $\mathbb{H}$ .

- d For example, if  $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , then  $AB = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$  and  $BA = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ .

- 7.5.38** a  $C_4^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ ,  $C_4^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ ,  $C_4^4 = I_4$ , then  $C_4^{4+k} = C_4^k$ .

Figure 7.30 illustrates how  $C_4$  acts on the basis vectors  $\vec{e}_i$ .

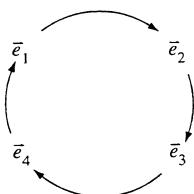


Figure 7.30: for Problem 7.5.38a.

- b The eigenvalues are  $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i$ , and  $\lambda_4 = -i$ , and for each eigenvalue  $\lambda_k$ ,  $\vec{v}_k = \begin{bmatrix} \lambda_k^3 \\ \lambda_k^2 \\ \lambda_k \\ 1 \end{bmatrix}$  is an associated eigenvector.

c  $M = aI_4 + bC_4 + cC_4^2 + dC_4^3$

If  $\vec{v}$  is an eigenvector of  $C_4$  with eigenvalue  $\lambda$ , then  $M\vec{v} = a\vec{v} + b\lambda\vec{v} + c\lambda^2\vec{v} + d\lambda^3\vec{v} = (a + b\lambda + c\lambda^2 + d\lambda^3)\vec{v}$ , so that  $\vec{v}$  is an eigenvector of  $M$  as well, with eigenvalue  $a + b\lambda + c\lambda^2 + d\lambda^3$ .

The eigenbasis for  $C_4$  we found in part b is an eigenbasis for all circulant  $4 \times 4$  matrices.

**7.5.39** Figure 7.31 illustrates how  $C_n$  acts on the standard basis vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$  of  $\mathbb{R}^n$ .

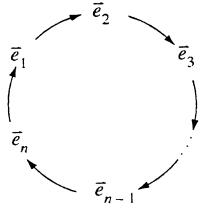


Figure 7.31: for Problem 7.5.39.

**7.5.39 a** Based on Figure 7.31, we see that  $C_n^k$  takes  $\vec{e}_i$  to  $\vec{e}_{i+k}$  “modulo  $n$ ,” that is, if  $i+k$  exceeds  $n$  then  $C_n^k$  takes  $\vec{e}_i$  to  $\vec{e}_{i+k-n}$  (for  $k = 1, \dots, n-1$ ).

To put it differently:  $C_n^k$  is the matrix whose  $i$ th column is  $\vec{e}_{i+k}$  if  $i+k \leq n$  and  $\vec{e}_{i+k-n}$  if  $i+k > n$  (for  $k = 1, \dots, n-1$ ).

- b The characteristic polynomial is  $1 - \lambda^n$ , so that the eigenvalues are the  $n$  distinct solutions of the equation  $\lambda^n = 1$  (the so-called  $n$ th roots of unity), equally spaced points along the unit circle,  $\lambda_k = \cos(\frac{2\pi k}{n}) + i \sin(\frac{2\pi k}{n})$ , for  $k = 0, 1, \dots, n-1$  (compare with Exercise 5 and Figure 7.7.). For each eigenvalue  $\lambda_k$ ,

$$\vec{v}_k = \begin{bmatrix} \lambda_k^{n-1} \\ \vdots \\ \lambda_k^2 \\ \lambda_k \\ 1 \end{bmatrix} \text{ is an associated eigenvector.}$$

c The eigenbasis  $\vec{v}_0, \vec{v}_1, \dots, \vec{v}_{n-1}$  for  $C_n$  we found in part b is in fact an eigenbasis for all circulant  $n \times n$  matrices.

**7.5.40** In Exercise 7.2.50 we derived the formula  $x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$

for the solution of the equation  $x^3 + px = q$ . Here  $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$  is negative, and we can write

$$x = \sqrt[3]{\frac{q}{2} + i\sqrt{-\left(\frac{q}{2}\right)^2 + \left(\frac{-p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - i\sqrt{-\left(\frac{q}{2}\right)^2 + \left(\frac{-p}{3}\right)^3}}.$$

Let us write this solution in polar coordinates:

$$\begin{aligned} x &= \sqrt[3]{\left(-\frac{p}{3}\right)^{3/2}} (\cos \alpha + i \sin \alpha) + \sqrt[3]{\left(-\frac{p}{3}\right)^{3/2}} (\cos \alpha - i \sin \alpha) \\ &= \sqrt{-\frac{p}{3}} \left( \cos \frac{\alpha+2\pi k}{3} + i \sin \frac{\alpha+2\pi k}{3} \right) + \sqrt{-\frac{p}{3}} \left( \cos \frac{\alpha+2\pi k}{3} - i \sin \frac{\alpha+2\pi k}{3} \right) \\ &= 2\sqrt{-\frac{p}{3}} \cos \left( \frac{\alpha+2\pi k}{3} \right), k = 0, 1, 2. \text{ See Figure 7.32.} \end{aligned}$$

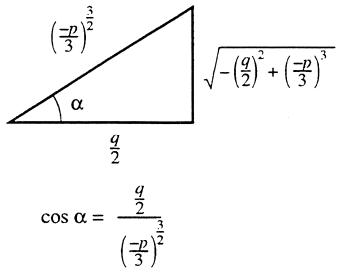


Figure 7.32: for Problem 7.5.40.

*Answer:*

$$x_{1,2,3} = 2\sqrt{-\frac{p}{3}} \cos \left( \frac{\alpha+2\pi k}{3} \right), k = 0, 1, 2, \text{ where } \alpha = \arccos \left( \frac{\frac{q}{2}}{\left(\frac{-p}{3}\right)^{3/2}} \right).$$

Note that  $x$  is on the interval  $\left(\sqrt{-\frac{p}{3}}, 2\sqrt{-\frac{p}{3}}\right)$  when  $k = 0$ , on  $\left(-2\sqrt{-\frac{p}{3}}, -\sqrt{-\frac{p}{3}}\right)$  when  $k = 1$  and on  $\left(-\sqrt{-\frac{p}{3}}, \sqrt{-\frac{p}{3}}\right)$  when  $k = -1$  (Think about it!).

**7.5.41** Substitute  $\rho = \frac{1}{x}$  into  $14\rho^2 + 12\rho^3 - 1 = 0$ ;

$$\frac{14}{x^2} + \frac{12}{x^3} - 1 = 0$$

$$14x + 12 - x^3 = 0$$

$$x^3 - 14x = 12$$

Now use the formula derived in Exercise 40 to find  $x$ , with  $p = -14$  and  $q = 12$ . There is only one positive solution,  $x \approx 4.114$ , so that  $\rho = \frac{1}{x} \approx 0.243$ .

**7.5.42 a** We will use the fact that for any two complex numbers  $z$  and  $w$ ,  $\overline{z+w} = \overline{z} + \overline{w}$  and  $\overline{zw} = \overline{z}\overline{w}$ .

The  $ij$ th entry of  $\overline{AB}$  is  $\sum_{k=1}^p \overline{a_{ik}b_{kj}} = \sum_{k=1}^p \overline{a_{ik}} \overline{b_{kj}} = \sum_{k=1}^p \overline{a_{ik}b_{kj}}$ , which is the  $ij$ th entry of  $\overline{AB}$ , as claimed.

b Use part a, where  $B$  is the  $n \times 1$  matrix  $\vec{v} + i\vec{w}$ . We are told that  $AB = \lambda B$ , where  $\lambda = p + iq$ . Then  $A\overline{B} = \bar{A}\bar{B} = \overline{AB} = \bar{\lambda}\bar{B} = \bar{\lambda}\bar{B}$ , or  $A(\vec{v} - iq\vec{w}) = (p - iq)(\vec{v} - iq\vec{w})$ .

**7.5.43** Note that  $f(z)$  is not the zero polynomial, since  $f(i) = \det(S_1 + iS_2) = \det(S) \neq 0$ , as  $S$  is invertible. A nonzero polynomial has only finitely many zeros, so that there is a real number  $x$  such that  $f(x) = \det(S_1 + xS_2) \neq 0$ .

0, that is,  $S_1 + xS_2$  is invertible. Now  $SB = AS$  or  $(S_1 + iS_2)B = A(S_1 + iS_2)$ . Considering the real and the imaginary part, we can conclude that  $S_1B = AS_1$  and  $S_2B = AS_2$  and therefore  $(S_1 + xS_2)B = A(S_1 + xS_2)$ . Since  $S_1 + xS_2$  is invertible, we have  $B = (S_1 + xS_2)^{-1}A(S_1 + xS_2)$ , as claimed.

**7.5.44** Let  $A$  be a complex  $2 \times 2$  matrix. Let  $\lambda$  be a complex eigenvalue of  $A$ , and consider an associated eigenvector  $\vec{v}$ , so that  $A\vec{v} = \lambda\vec{v}$ . Now let  $P$  be an invertible  $2 \times 2$  matrix of the form  $P = [\vec{v} \vec{w}]$  (the first column of  $P$  is our eigenvector  $\vec{v}$ ). Then  $P^{-1}AP$  will be of the form  $\begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}$ , so that we have found an upper triangular matrix similar to  $A$  (compare with the proof of Theorem 7.4.1).

Yes, any complex square matrix is similar to an upper triangular matrix, although the proof is challenging at this stage of the course. Following the hint, we will assume that the claim holds for  $n \times n$  matrices, and we will prove it for an  $(n+1) \times (n+1)$  matrix  $A$ . As in the case of a  $2 \times 2$  matrix discussed above, we can find an invertible  $P$  such that  $P^{-1}AP$  is of the form  $\begin{bmatrix} \lambda & \vec{w} \\ 0 & B \end{bmatrix}$  for some scalar  $\lambda$ , a row vector  $\vec{w}$  with  $n$  components, and an  $n \times n$  matrix  $B$  (just make the first column of  $P$  an eigenvector of  $A$ ). By the induction hypothesis,  $B$  is similar to some upper triangular matrix  $T$ , that is,  $R^{-1}BR = T$  for some invertible  $R$ .

Now let  $S = P \begin{bmatrix} 1 & \vec{0} \\ 0 & R \end{bmatrix}$ , an invertible  $(n+1) \times (n+1)$  matrix. Then

$$S^{-1}AS = \begin{bmatrix} 1 & \vec{0} \\ 0 & R^{-1} \end{bmatrix} P^{-1}AP \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} 1 & \vec{0} \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} \lambda & \vec{w} \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} \lambda & \vec{w}R \\ 0 & T \end{bmatrix},$$

an upper triangular matrix, showing that  $A$  is indeed similar to an upper triangular matrix. You will see an analogous proof in Section 8.1 (proof of Theorem 8.1.1, Page 371).

**7.5.45** If  $a \neq 0$ , then there are two distinct eigenvalues,  $1 \pm \sqrt{a}$ , so that the matrix is diagonalizable. If  $a = 0$ , then  $\begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  fails to be diagonalizable.

**7.5.46** If  $a \neq 0$ , then there are two distinct eigenvalues,  $\pm ia$ , so that the matrix is diagonalizable. If  $a = 0$ , then  $\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  is diagonalizable as well. Thus the matrix is diagonalizable for all  $a$ .

**7.5.47** If  $a \neq 0$ , then there are three distinct eigenvalues,  $0, \pm\sqrt{a}$ , so that the matrix is diagonalizable. If  $a = 0$ , then  $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & a \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  fails to be diagonalizable.

**7.5.48** The characteristic polynomial is  $f(\lambda) = -\lambda^3 + 3\lambda + a$ . We need to find the values  $a$  such that this polynomial has multiple roots. Now  $\lambda$  is a multiple root if (and only if)  $f(\lambda) = f'(\lambda) = 0$  (see Exercise 7.2.37). Since  $f'(\lambda) = -3\lambda^2 + 3 = -3(\lambda - 1)(\lambda + 1)$ , the only possible multiple roots are 1 and -1. Now 1 is a multiple root if  $f(1) = 2 + a = 0$ , or,  $a = -2$ , and -1 is a multiple root if  $a = 2$ . Thus, if  $a$  is neither 2 nor -2, then the matrix is diagonalizable. Conversely, if  $a = 2$  or  $a = -2$ , then the matrix fails to be diagonalizable, since all the eigenspaces will be one-dimensional (verify this!).

**7.5.49** The eigenvalues are  $0, 1, a - 1$ . If  $a$  is neither 1 nor 2, then there are three distinct eigenvalues, so that the matrix is diagonalizable. Conversely, if  $a = 1$  or  $a = 2$ , then the matrix fails to be diagonalizable, since all the eigenspaces will be one-dimensional (verify this!).

**7.5.50** The eigenvalues are 0, 0, 1. Since the kernel is always two-dimensional, with basis  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ , the matrix is diagonalizable for all values of constant  $a$ .

**7.5.51** Yes,  $\mathbb{Q}$  is a field. Check the axioms on page 350.

**7.5.52** No,  $\mathbb{Z}$  is not a field since multiplicative inverses do not exist, i.e. division within  $\mathbb{Z}$  is not possible. (Axiom 8 does not hold).

**7.5.53** Yes, check the axioms on page 350. (additive identity  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , multiplicative identity  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ).

**7.5.54** Yes, check the axioms on page 350. (additive identity  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , multiplicative identity  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ). Also notice that rotation-scaling matrices commute when multiplied.

**7.5.55** No, since multiplication is not commutative; Axiom 5 does not hold.

## Section 7.6

**7.6.1**  $\lambda_1 = 0.9, \lambda_2 = 0.8$ , so, by Theorem 7.6.2,  $\vec{0}$  is a stable equilibrium.

**7.6.2**  $\lambda_1 = -1.1, \lambda_2 = 0.9$ , so by Theorem 7.6.2,  $\vec{0}$  is not a stable equilibrium. ( $|\lambda_1| > 1$ )

**7.6.3**  $\lambda_{1,2} = 0.8 \pm (0.7)i$  so  $|\lambda_1| = |\lambda_2| = \sqrt{0.64 + 0.49} > 1$  so  $\vec{0}$  is not a stable equilibrium.

**7.6.4**  $\lambda_{1,2} = -0.9 \pm (0.4)i$  so  $|\lambda_1| = |\lambda_2| = \sqrt{0.81 + 0.16} < 1$  so  $\vec{0}$  is a stable equilibrium.

**7.6.5**  $\lambda_1 = 0.8, \lambda_2 = 1.1$  so  $\vec{0}$  is not a stable equilibrium.

**7.6.6**  $\lambda_{1,2} = 0.8 \pm (0.6)i$  so  $|\lambda_1| = |\lambda_2| = \sqrt{0.64 + 0.36} = 1$  and  $\vec{0}$  is not a stable equilibrium.

**7.6.7**  $\lambda_{1,2} = 0.9 \pm (0.5)i$  so  $|\lambda_1| = |\lambda_2| = \sqrt{0.81 + 0.25} > 1$  and  $\vec{0}$  is not a stable equilibrium.

**7.6.8**  $\lambda_1 = 0.9, \lambda_2 = 0.8$  so  $\vec{0}$  is a stable equilibrium.

**7.6.9**  $\lambda_{1,2} = 0.8 \pm (0.6)i, \lambda_3 = 0.7$ , so  $|\lambda_1| = |\lambda_2| = 1$  and  $\vec{0}$  is not a stable equilibrium.

**7.6.10**  $\lambda_{1,2} = 0, \lambda_3 = 0.9$  so  $\vec{0}$  is a stable equilibrium.

**7.6.11**  $\lambda_1 = k, \lambda_2 = 0.9$  so  $\vec{0}$  is a stable equilibrium if  $|k| < 1$ .

**7.6.12**  $\lambda_{1,2} = 0.6 \pm ik$  so  $\vec{0}$  is a stable equilibrium if  $|\lambda_1| = |\lambda_2| = \sqrt{0.36 + k^2} < 1$  i.e. if  $k^2 < 0.64$  or  $|k| < 0.8$ .

**7.6.13** Since  $\lambda_1 = 0.7, \lambda_2 = -0.9, \vec{0}$  is a stable equilibrium regardless of the value of  $k$ .

**7.6.14**  $\lambda_1 = 0, \lambda_2 = 2k$  so  $\vec{0}$  is a stable equilibrium if  $|2k| < 1$  or  $|k| < \frac{1}{2}$ .

**7.6.15**  $\lambda_{1,2} = 1 \pm \frac{1}{10}\sqrt{k}$

If  $k \geq 0$  then  $\lambda_1 = 1 + \frac{1}{10}\sqrt{k} \geq 1$ . If  $k < 0$  then  $|\lambda_1| = |\lambda_2| > 1$ . Thus, the zero state isn't a stable equilibrium for any real  $k$ .

**7.6.16**  $\lambda_{1,2} = \frac{2 \pm \sqrt{1+30k}}{10}$  so  $|2 \pm \sqrt{1+30k}|$  must be less than 10.  $\lambda_{1,2}$  are *real* if  $k \geq -\frac{1}{30}$ . In this case it is required that  $2 + \sqrt{1+30k} < 10$  and  $-10 < 2 - \sqrt{1+30k}$ , which means that  $\sqrt{1+30k} < 8$  or  $k < \frac{21}{10}$ .

$\lambda_{1,2}$  are *complex* if  $k < -\frac{1}{30}$ . Here it is required that  $4 + (-1 - 30k) < 100$  or  $k > -\frac{97}{30}$ . Overall,  $\vec{0}$  is a stable equilibrium if  $-\frac{97}{30} < k < \frac{21}{10}$ .

**7.6.17**  $\lambda_{1,2} = 0.6 \pm (0.8)i = 1(\cos \theta \pm i \sin \theta)$ , where

$$\theta = \arctan\left(\frac{0.8}{0.6}\right) = \arctan\left(\frac{0.8}{0.6}\right) = \arctan\left(\frac{4}{3}\right) \approx 0.927.$$

$$E_{\lambda_1} = \ker \begin{bmatrix} -0.8i & -0.8 \\ 0.8 & -0.8i \end{bmatrix} = \text{span} \begin{bmatrix} -1 \\ i \end{bmatrix} \text{ so } \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

$$\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\vec{w} + 0\vec{v}, \text{ so } a = 1 \text{ and } b = 0. \text{ Now we use Theorem 7.6.3:}$$

$$\vec{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta t \\ \sin \theta t \end{bmatrix} = \begin{bmatrix} -\sin \theta t \\ \cos \theta t \end{bmatrix}, \text{ where}$$

$$\theta = \arctan\left(\frac{4}{3}\right) \approx 0.927.$$

The trajectory is the circle shown in Figure 7.33.

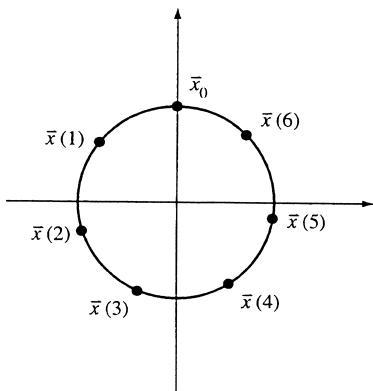


Figure 7.33: for Problem 7.6.17.

**7.6.18**  $\lambda_{1,2} = \frac{-4 \pm 2\sqrt{3}i}{5} = r(\cos \theta \pm i \sin \theta)$ , where  $r \approx 1.058$  and  $\theta = \pi - \arctan\left(\frac{\frac{2\sqrt{3}}{5}}{\frac{4}{5}}\right) \approx 2.428$  (second quadrant).

$$E_{\lambda_1} = \text{span} \left[ \begin{bmatrix} \frac{\sqrt{3}}{2} \\ i \end{bmatrix} \right], \text{ so } \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}.$$

$\vec{x}_0 = 1\vec{w} + 0\vec{v}$ , so  $a = 1, b = 0$ .

$$\begin{aligned} \vec{x}(t) &= r^t [\vec{w} \quad \vec{v}] \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx (1.058)^t \begin{bmatrix} 0 & 0.866 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta t) \\ \sin(\theta t) \end{bmatrix} \\ &\approx (1.058)^t \begin{bmatrix} 0.866 \cdot \sin(2.428t) \\ \cos(2.428t) \end{bmatrix}. \text{ See Figure 7.34.} \end{aligned}$$

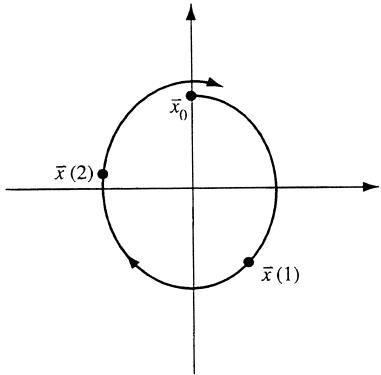


Figure 7.34: for Problem 7.6.18.

Spirals slowly outwards (plot the first few points).

**7.6.19**  $\lambda_{1,2} = 2 \pm 3i$ ,  $r = \sqrt{13}$ , and  $\theta = \arctan(\frac{3}{2}) \approx 0.98$ , so

$$\begin{aligned} \lambda_1 &\approx \sqrt{13}(\cos(0.98) + i \sin(0.98)), [\vec{w} \quad \vec{v}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{x}(t) \\ &\approx \sqrt{13}^t \begin{bmatrix} -\sin(0.98t) \\ \cos(0.98t) \end{bmatrix}. \end{aligned}$$

The trajectory spirals outwards; see Figure 7.35.

**7.6.20**  $\lambda_{1,2} = 4 \pm 3i$ ,  $r = 5$ ,  $\theta = \arctan(\frac{3}{4}) \approx 0.64$ , so  $\lambda_1 \approx 5(\cos(0.64) + i \sin(0.64))$ ,  $[\vec{w} \quad \vec{v}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\vec{x}(t) \approx 5^t \begin{bmatrix} \sin(0.64t) \\ \cos(0.64t) \end{bmatrix}$ . See Figure 7.36.

Spirals outwards (rotation-dilation).

**7.6.21**  $\lambda_{1,2} = 4 \pm i$ ,  $r = \sqrt{17}$ ,  $\theta = \arctan(\frac{1}{4}) \approx 0.245$  so

$$\begin{aligned} \lambda_1 &\approx \sqrt{17}(\cos(0.245) + i \sin(0.245)), [\vec{w} \quad \vec{v}] = \begin{bmatrix} 0 & 5 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \text{and } \vec{x}(t) &\approx \sqrt{17}^t \begin{bmatrix} 5 \sin(0.245t) \\ \cos(0.245t) + 3 \sin(0.245t) \end{bmatrix} \end{aligned}$$

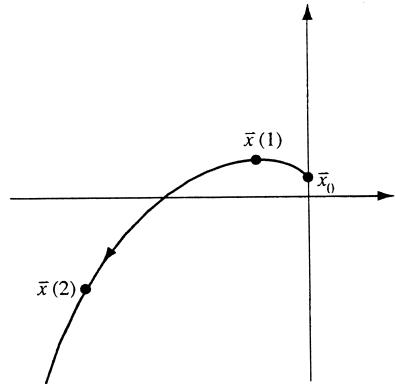


Figure 7.35: for Problem 7.6.19.

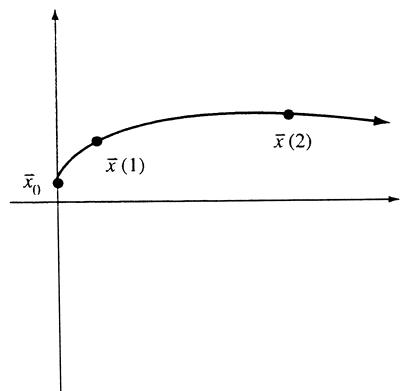


Figure 7.36: for Problem 7.6.20.

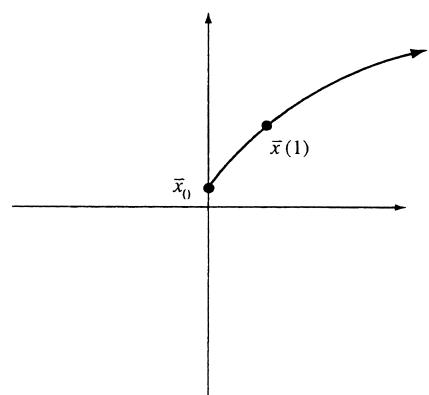


Figure 7.37: for Problem 7.6.21.

The trajectory spirals outwards; see Figure 7.37.

**7.6.22**  $\lambda_{1,2} = -2 \pm 3i, r = \sqrt{13}, \theta \approx 2.16$  (in second quadrant)

$$[\vec{w} \ \vec{v}] = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ so } \vec{x}(t) = \sqrt{13}^t \begin{bmatrix} -5 \sin(\theta t) \\ \cos(\theta t) - 3 \sin(\theta t) \end{bmatrix}, \text{ where } \theta \approx 2.16.$$

Spirals outwards, as in Figure 7.38.

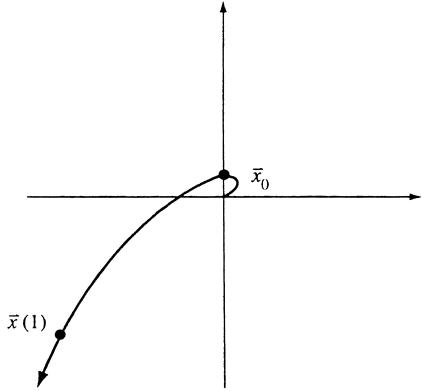


Figure 7.38: for Problem 7.6.22.

$$7.6.23 \quad \lambda_{1,2} = 0.4 \pm 0.3i, r = \frac{1}{2}, \theta = \arctan\left(\frac{0.3}{0.4}\right) \approx 0.643$$

$$[\vec{w} \ \vec{v}] = \begin{bmatrix} 0 & 5 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ so } \vec{x}(t) = \left(\frac{1}{2}\right)^t \begin{bmatrix} 5 \sin(\theta t) \\ \cos(\theta t) + 3 \sin(\theta t) \end{bmatrix}.$$

The trajectory spirals inwards as shown in Figure 7.39.

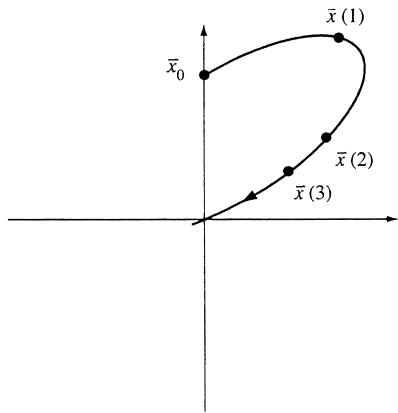


Figure 7.39: for Problem 7.6.23.

$$7.6.24 \quad \lambda_{1,2} = -0.8 \pm 0.6i, r = 1, \theta = \pi - \arctan\left(\frac{6}{8}\right) \approx 2.5 \text{ (second quadrant)}$$

$$[\vec{w} \ \vec{v}] = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ so } \vec{x}(t) = \begin{bmatrix} -5 \sin(\theta t) \\ \cos(\theta t) - 3 \sin(\theta t) \end{bmatrix}, \text{ an ellipse, as shown in Figure 7.40.}$$

7.6.25 Not stable since if  $\lambda$  is an eigenvalue of  $A$ , then  $\frac{1}{\lambda}$  is an eigenvalue of  $A^{-1}$  and  $\left[\frac{1}{\lambda}\right] = \frac{1}{|\lambda|} > 1$ .

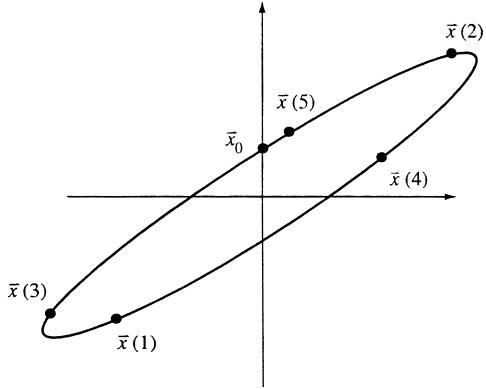


Figure 7.40: for Problem 7.6.24.

**7.6.26** Stable since  $A$  and  $A^T$  have the same eigenvalues.

**7.6.27** Stable since if  $\lambda$  is an eigenvalue of  $-A$ , then  $-\lambda$  is an eigenvalue of  $-A$  and  $|-\lambda| = |\lambda|$ .

**7.6.28** Not stable, since if  $\lambda$  is an eigenvalue of  $A$ , then  $(\lambda - 2)$  is an eigenvalue of  $(A - 2I_n)$  and  $|\lambda - 2| > 1$ .

**7.6.29** Cannot tell; for example, if  $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ , then  $A + I_2$  is  $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$  and the zero state is not stable, but if  $A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$  then  $A + I_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$  and the zero state is stable.

**7.6.30** Consider the dynamical systems  $\vec{x}(t+1) = A^2 \vec{x}(t)$  and  $\vec{y}(t+1) = A \vec{y}(t)$  with equal initial values,  $\vec{x}(0) = \vec{y}(0)$ . Then  $\vec{x}(t) = \vec{y}(2t)$  for all positive integers  $t$ . We know that  $\lim_{t \rightarrow \infty} \vec{y}(t) = \vec{0}$ ; thus  $\lim_{t \rightarrow \infty} \vec{x}(t) = \vec{0}$ , proving that the zero state is a stable equilibrium of the system  $\vec{x}(t+1) = A^2 \vec{x}(t)$ .

**7.6.31** We need to determine for which values of  $\det(A)$  and  $\text{tr}(A)$  the modulus of both eigenvalues is less than 1. We will first think about the border line case and examine when one of the moduli is exactly 1: If one of the eigenvalues is 1 and the other is  $\lambda$ , then  $\text{tr}(A) = \lambda + 1$  and  $\det(A) = \lambda$ , so that  $\det(A) = \text{tr}(A) - 1$ . If one of the eigenvalues is  $-1$  and the other is  $\lambda$ , then  $\text{tr}(A) = \lambda - 1$  and  $\det(A) = -\lambda$ , so that  $\det(A) = -\text{tr}(A) - 1$ . If the eigenvalues are complex conjugates with modulus 1, then  $\det(A) = 1$  and  $|\text{tr}(A)| < 2$  (think about it!). It is convenient to represent these conditions in the  $\text{tr}-\det$  plane, where each  $2 \times 2$  matrix  $A$  is represented by the point  $(\text{tr}A, \det A)$ , as shown in Figure 7.41.

If  $\text{tr}(A) = \det(A) = 0$ , then both eigenvalues of  $A$  are zero. We can conclude that throughout the shaded triangle in Figure 7.42 the modulus of both eigenvalues will be less than 1, since the modulus of the eigenvalues changes continuously with  $\text{tr}(A)$  and  $\det(A)$  (consider the quadratic formula!). Conversely, we can choose sample points to show that in all the other four regions in Figure 7.42 the modulus of at least one of the eigenvalues exceeds one; consider

the matrices  $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ , and  $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ .

$\uparrow$        $\uparrow$        $\uparrow$        $\uparrow$   
 in (I)      in (II)      in (III)      in (IV)

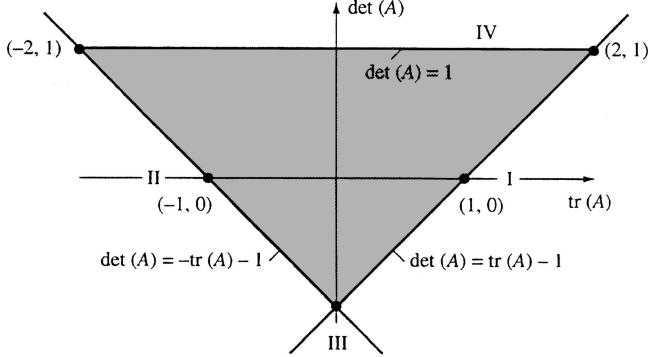


Figure 7.41: for Problem 7.6.31.

It follows that throughout these four regions, (I), (II), (III), and (IV), at least one of the eigenvalues will have a modulus exceeding one.

The point  $(\text{tr}A, \det A)$  is in the shaded triangle if  $\det(A) < 1$ ,  $\det(A) > \text{tr}(A) - 1$  and  $\det(A) > -\text{tr}(A) - 1$ . This means that  $|\text{tr}A| - 1 < \det(A) < 1$ , as claimed.

**7.6.32** Take conjugates of both sides of the equation  $\vec{x}_0 = c_1(\vec{v} + i\vec{w}) + c_2(\vec{v} - i\vec{w})$ :

$$\vec{x}_0 = \bar{\vec{x}}_0 = \overline{c_1(\vec{v} + i\vec{w}) + c_2(\vec{v} - i\vec{w})} = \bar{c}_1(\vec{v} - i\vec{w}) + \bar{c}_2(\vec{v} + i\vec{w}) = \bar{c}_2(\vec{v} + i\vec{w}) + \bar{c}_1(\vec{v} - i\vec{w}).$$

The claim that  $c_2 = \bar{c}_1$  now follows from the fact that the representation of  $\vec{x}_0$  as a linear combination of the linearly independent vectors  $\vec{v} + i\vec{w}$  and  $\vec{v} - i\vec{w}$  is unique.

**7.6.33** Take conjugates of both sides of the equation  $\vec{x}_0 = c_1(\vec{v} + i\vec{w}) + c_2(\vec{v} - i\vec{w})$ :

$$\vec{x}_0 = \bar{\vec{x}}_0 = \overline{c_1(\vec{v} + i\vec{w}) + c_2(\vec{v} - i\vec{w})} = \bar{c}_1(\vec{v} - i\vec{w}) + \bar{c}_2(\vec{v} + i\vec{w}) = \bar{c}_2(\vec{v} + i\vec{w}) + \bar{c}_1(\vec{v} - i\vec{w}).$$

The claim that  $c_2 = \bar{c}_1$  now follows from the fact that the representation of  $\vec{x}_0$  as a linear combination of the linearly independent vectors  $\vec{v} + i\vec{w}$  and  $\vec{v} - i\vec{w}$  is unique.

**7.6.34 a** If  $|\det A| = |\lambda_1 \lambda_2 \cdots \lambda_n| = |\lambda_1 \lambda_2| \cdots |\lambda_n| > 1$  then at least one eigenvalue is greater than one in modulus and the zero state fails to be stable.

b If  $|\det A| = |\lambda_1| |\lambda_2| \cdots |\lambda_n| < 1$  we cannot conclude anything about the stability of  $\vec{0}$ .

$|2\|0.1\| < 1$  and  $|0.2\|0.1\| < 1$  but in the first case we would not have stability, in the second case we would.

**7.6.35 a** Let  $\vec{v}_1, \dots, \vec{v}_n$  be an eigenbasis for  $A$ . Then  $\vec{x}(t) = \sum_{i=1}^n c_i \lambda_i^t \vec{v}_i$  and

$$\|\vec{x}(t)\| = \left| \sum_{i=1}^n c_i \lambda_i^t \vec{v}_i \right| \leq \sum_{i=1}^n \|c_i \lambda_i^t \vec{v}_i\| = \sum_{i=1}^n |\lambda_i|^t \|c_i \vec{v}_i\| \leq \sum_{i=1}^n \|c_i \vec{v}_i\|.$$

$$\begin{matrix} \uparrow \\ \leq 1 \end{matrix}$$

The last quantity,  $\sum_{i=1}^n \|c_i \vec{v}_i\|$ , gives the desired bound  $M$ .

b  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  represents a shear parallel to the  $x$ -axis, with  $A \begin{bmatrix} k \\ 1 \end{bmatrix} = \begin{bmatrix} k+1 \\ 1 \end{bmatrix}$ , so that  $\vec{x}(t) = A^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$  is not bounded. This does not contradict part a, since there is no eigenbasis for  $A$ .

**7.6.36** If the zero state is stable, then  $\lim_{t \rightarrow \infty} (\text{ith column of } A^t) = \lim_{t \rightarrow \infty} (A^t \vec{e}_i) = \vec{0}$ , so that all columns and therefore all entries of  $A^t$  approach 0.

Conversely, if  $\lim_{t \rightarrow \infty} A^t = 0$ , then  $\lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \left( \lim_{t \rightarrow \infty} A^t \right) \vec{x}_0 = \vec{0}$  for all  $\vec{x}_0$  (check the details).

**7.6.37** a Write  $Y(t+1) = Y(t) = Y, C(t+1) = C(t) = C, I(t+1) = I(t) = I$ .

$$\begin{bmatrix} Y = C + I + G_0 \\ C = \gamma Y \\ I = 0 \end{bmatrix} \rightarrow \begin{array}{l} Y = \gamma Y + G_0 \\ Y = \frac{G_0}{1-\gamma} \end{array}$$

$$Y = \frac{G_0}{1-\gamma}, C = \frac{\gamma G_0}{1-\gamma}, I = 0$$

b  $y(t) = Y(t) - \frac{G_0}{1-\gamma}, c(t) = C(t) - \frac{\gamma G_0}{1-\gamma}, i(t) = I(t)$

Substitute to verify the equations.

$$\begin{bmatrix} C(t+1) \\ i(t+1) \end{bmatrix} = \begin{bmatrix} \gamma & \gamma \\ \alpha\gamma - \alpha & \alpha\gamma \end{bmatrix} \begin{bmatrix} c(t) \\ i(t) \end{bmatrix}$$

c  $A = \begin{bmatrix} 0.2 & 0.2 \\ -4 & 1 \end{bmatrix}$  eigenvalues  $0.6 \pm 0.8i$

not stable

d  $A = \begin{bmatrix} \gamma & \gamma \\ \gamma - 1 & \gamma \end{bmatrix}, \text{tr}A = 2\gamma, \det A = \gamma$ , stable (use Exercise 31)

e  $A = \begin{bmatrix} \gamma & \gamma \\ \alpha\gamma - \alpha & \alpha\gamma \end{bmatrix} \text{tr}A = \gamma(1 + \alpha) > 0, \det A = \alpha\gamma$

Use Exercise 31; stable if  $\det(A) = \alpha\gamma < 1$  and  $\text{tr}A - 1 = \alpha\gamma + \gamma - 1 < \alpha\gamma$ .

The second condition is satisfied since  $\gamma < 1$ .

Stable if  $\gamma < \frac{1}{\alpha}$

(eigenvalues are real if  $\gamma \geq \frac{4\alpha}{(1+\alpha)^2}$ )

**7.6.38** a  $T(\vec{v}) = A\vec{v} + \vec{b} = \vec{v}$  if  $\vec{v} - A\vec{v} = \vec{b}$  or  $(I_n - A)\vec{v} = \vec{b}$ .

$I_n - A$  is invertible since 1 is not an eigenvalue of  $A$ . Therefore,  $\vec{v} = (I_n - A)^{-1} \vec{b}$  is the only solution.

b Let  $\vec{y}(t) = \vec{x}(t) - \vec{v}$  be the deviation of  $\vec{x}(t)$  from the equilibrium  $\vec{v}$ .

Then  $\vec{y}(t+1) = \vec{x}(t+1) - \vec{v} = A\vec{x}(t) + \vec{b} - \vec{v} = A(\vec{y}(t) + \vec{v}) + \vec{b} - \vec{v} = A\vec{y}(t) + A\vec{v} + \vec{b} - \vec{v} = A\vec{y}(t)$ , so that  $\vec{y}(t) = A^t\vec{y}(0)$ , or  $\vec{x}(t) = \vec{v} + A^t(\vec{x}_0 - \vec{v})$ .

$\lim_{t \rightarrow \infty} \vec{x}(t) = \vec{v}$  for all  $\vec{x}_0$  if  $\lim_{t \rightarrow \infty} A^t(\vec{x}_0 - \vec{v}) = \vec{0}$ . This is the case if the modulus of all the eigenvalues of  $A$  is less than 1.

7.6.39 Use Exercise 38:  $\vec{v} = (I_2 - A)^{-1}\vec{b} = \begin{bmatrix} 0.9 & -0.2 \\ -0.4 & 0.7 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .

$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$  is a stable equilibrium since the eigenvalues of  $A$  are 0.5 and -0.1.

7.6.40 a  $A^T A = \begin{bmatrix} B^T & C^T \\ -C & B \end{bmatrix} \begin{bmatrix} B & -C^T \\ C & B^T \end{bmatrix} = (p^2 + q^2 + r^2 + s^2)I_4$

b By part a,  $A^{-1} = \frac{1}{p^2 + q^2 + r^2 + s^2} A^T$  if  $A \neq 0$ .

c  $(\det A)^2 = (p^2 + q^2 + r^2 + s^2)^4$ , by part a, so that  $\det A = \pm(p^2 + q^2 + r^2 + s^2)^2$ .

Laplace Expansion along the first row produces the term  $+p^4$ ,

so that  $\det(A) = (p^2 + q^2 + r^2 + s^2)^2$ .

d Consider  $\det(A - \lambda I_4)$ . Note that the matrix  $A - \lambda I_4$  has the same “format” as  $A$ , with  $p$  replaced by  $p - \lambda$  and  $q, r, s$  remaining unchanged. By part c,  $\det(A - \lambda I_4) = ((p - \lambda)^2 + q^2 + r^2 + s^2)^2 = 0$  when

$$(p - \lambda)^2 = -q^2 - r^2 - s^2$$

$$p - \lambda = \pm i\sqrt{q^2 + r^2 + s^2}$$

$$\lambda = p \pm i\sqrt{q^2 + r^2 + s^2}$$

Each of these eigenvalues has algebraic multiplicity 2 (if  $q = r = s = 0$  then  $\lambda = p$  has algebraic multiplicity 4).

e By part a we can write  $A = \sqrt{p^2 + q^2 + r^2 + s^2} \underbrace{\begin{pmatrix} 1 \\ \sqrt{p^2 + q^2 + r^2 + s^2} A \end{pmatrix}}_S$ , where  $S$  is orthogonal.

Therefore,  $\|A\vec{x}\| = \|\sqrt{p^2 + q^2 + r^2 + s^2}(S\vec{x})\| = \sqrt{p^2 + q^2 + r^2 + s^2}\|\vec{x}\|$ .

f Let  $A = \begin{bmatrix} 3 & -3 & -4 & -5 \\ 3 & 3 & 5 & -4 \\ 4 & -5 & 3 & 3 \\ 5 & 4 & -3 & 3 \end{bmatrix}$  and  $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$ ; then  $A\vec{x} = \begin{bmatrix} -39 \\ 13 \\ 18 \\ 13 \end{bmatrix}$ .

By part e,  $\|A\vec{x}\|^2 = (3^2 + 3^2 + 4^2 + 5^2)\|\vec{x}\|^2$ , or

$39^2 + 13^2 + 18^2 + 13^2 = (3^2 + 3^2 + 4^2 + 5^2)(1^2 + 2^2 + 4^2 + 4^2)$ , as desired.

g Any positive integer  $m$  can be written as  $m = p_1 p_2 \dots p_n$ . Using part f repeatedly we see that the numbers  $p_1, p_1 p_2, p_1 p_2 p_3, \dots, p_1 p_2 p_3 \dots p_{n-1}$ , and finally  $m = p_1 \dots p_n$  can be expressed as the sums of four squares.

**7.6.41** Find the  $2 \times 2$  matrix  $A$  that transforms  $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$  into  $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$  and  $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$  into  $\begin{bmatrix} -8 \\ -6 \end{bmatrix}$ :

$$A \begin{bmatrix} 8 & -3 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -8 \\ 4 & -6 \end{bmatrix} \text{ and } A = \begin{bmatrix} -3 & -8 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 8 & -3 \\ 6 & 4 \end{bmatrix}^{-1} = \frac{1}{50} \begin{bmatrix} 36 & -73 \\ 52 & -36 \end{bmatrix}.$$

There are many other correct answers.

**7.6.42 a**  $x(t+1) = x(t) - ky(t)$

$$y(t+1) = kx(t) + y(t) = kx(t) + (1 - k^2)y(t) \text{ so } \begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = \begin{bmatrix} 1 & -k \\ k & 1 - k^2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

b  $f_A(\lambda) = \lambda^2 - (2 - k^2)\lambda + 1 = 0$

The discriminant is  $(2 - k^2)^2 - 4 = -4k^2 + k^4 = k^2(k^2 - 4)$ , which is negative if  $k$  is a small positive number ( $k < 2$ ). Therefore, the eigenvalues are complex. By Theorem 7.6.4 the trajectory will be an ellipse, since  $\det(A) = 1$ .

## True or False

Ch 7.TF.1 F; If  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , then eigenvalue 1 has geometric multiplicity 1 and algebraic multiplicity 2.

Ch 7.TF.2 T, by Theorem 7.4.3

Ch 7.TF.3 T;  $A = AI_n = A[\vec{e}_1 \dots \vec{e}_n] = [\lambda_1 \vec{e}_1 \dots \lambda_n \vec{e}_n]$  is diagonal.

Ch 7.TF.4 T; If  $A\vec{v} = \lambda\vec{v}$ , then  $A^3\vec{v} = \lambda^3\vec{v}$ .

Ch 7.TF.5 T; Consider a diagonal  $5 \times 5$  matrix with only two distinct diagonal entries.

Ch 7.TF.6 F, by Theorem 7.2.7.

Ch 7.TF.7 T, by Summary 7.1.5

Ch 7.TF.8 T, by Theorem 7.2.4

Ch 7.TF.9 T, by Theorem 7.2.2

Ch 7.TF.10 T, by Definition 7.2.3

Ch 7.TF.11 T, by Example 6 of Section 7.5

Ch 7.TF.12 T; The geometric multiplicity of eigenvalue 0 is  $\dim(\ker A) = n - \text{rank}(A)$ .

Ch 7.TF.13 T; If  $S^{-1}AS = D$ , then  $S^T A^T (S^T)^{-1} = D$ .

Ch 7.TF.14 F; Let  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , for example.

Ch 7.TF.15 F; Consider  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

Ch 7.TF.16 F; Let  $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ ,  $\alpha = 2$ ,  $B = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$ ,  $\beta = 5$ , for example. Then  $\alpha\beta = 10$  isn't an eigenvalue of  $AB = \begin{bmatrix} 8 & 0 \\ 0 & 15 \end{bmatrix}$ .

Ch 7.TF.17 T; If  $A\vec{v} = 3\vec{v}$ , then  $A^2\vec{v} = 9\vec{v}$ .

Ch 7.TF.18 T; Construct an eigenbasis by concatenating a basis of  $V$  with a basis of  $V^\perp$ .

Ch 7.TF.19 T, by Theorem 7.5.5

Ch 7.TF.20 F; Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ , for example.

Ch 7.TF.21 T; If  $S^{-1}AS = D$ , then  $S^{-1}A^{-1}S = D^{-1}$  is diagonal.

Ch 7.TF.22 F; the equation  $\det(A) = \det(A^T)$  holds for all square matrices, by Theorem 6.2.1.

Ch 7.TF.23 T; The sole eigenvalue, 7, must have geometric multiplicity 3.

Ch 7.TF.24 F; Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ , with  $A + B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ , for example.

Ch 7.TF.25 F; Consider the zero matrix.

Ch 7.TF.26 T; If  $A\vec{v} = \alpha\vec{v}$  and  $B\vec{v} = \beta\vec{v}$ , then  $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \alpha\vec{v} + \beta\vec{v} = (\alpha + \beta)\vec{v}$ .

Ch 7.TF.27 F; Consider  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , with  $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .

Ch 7.TF.28 T, by Theorem 7.5.5

Ch 7.TF.29 F; Let  $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ , for example.

Ch 7.TF.30 F; Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ , with  $AB = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ , for example.

## Chapter 7

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Ch 7.TF.31 T; Consider the proof of Theorem 7.3.4a.

Ch 7.TF.32 T; An eigenbasis for  $A$  is an eigenbasis for  $A + 4I_4$  as well.

Ch 7.TF.33 F; Consider the identity matrix.

Ch 7.TF.34 T; Both  $A$  and  $B$  are similar to  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ , by Theorem 7.4.1

Ch 7.TF.35 F; Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , for example.

Ch 7.TF.36 F; Consider  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Ch 7.TF.37 F; Let  $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , and  $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , for example.

Ch 7.TF.38 T; A nonzero vector on  $L$  and a nonzero vector on  $L^\perp$  form an eigenbasis.

Ch 7.TF.39 T; The eigenvalues are 3 and  $-2$ .

Ch 7.TF.40 T; We will us Theorem 7.3.7 throughout: The geometric multiplicity of an eigenvalue is  $\leq$  its algebraic multiplicity.

Now let's show the contrapositive of the given statement: If the geometric multiplicity of some eigenvalue is less than its algebraic multiplicity, then the matrix  $A$  fails to be diagonalizable. Indeed, in this case the sum of the geometric multiplicities of all the eigenvalues is less than the sum of their algebraic multiplicities, which in turn is  $\leq n$  (where  $A$  is an  $n \times n$  matrix). Thus the geometric multiplicities do not add up to  $n$ , so that  $A$  fails to be diagonalizable, by Theorem 7.3.4b.

Ch 7.TF.41 F; Consider a rotation through  $\pi/2$ .

Ch 7.TF.42 T; Suppose  $\begin{bmatrix} A & A \\ 0 & A \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} A(\vec{v} + \vec{w}) \\ A\vec{w} \end{bmatrix} = \begin{bmatrix} \lambda\vec{v} \\ \lambda\vec{w} \end{bmatrix}$  for a nonzero vector  $\begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix}$ . If  $\vec{w}$  is nonzero, then it is an eigenvector of  $A$  with eigenvalue  $\lambda$ ; otherwise  $\vec{v}$  is such an eigenvector.

Ch 7.TF.43 F; Consider  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

Ch 7.TF.44 T; Note that  $S^{-1}AS = D$ , so that  $D^4 = S^{-1}A^4S = S^{-1}0S = 0$ , and therefore  $D = 0$  (since  $D$  is diagonal) and  $A = SDS^{-1} = 0$ .

Ch 7.TF.45 T; There is an eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$ , and we can write  $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ . The vectors  $c_i\vec{v}_i$  are either eigenvectors or zero.

Ch 7.TF.46 T; If  $A\vec{v} = \alpha\vec{v}$  and  $B\vec{v} = \beta\vec{v}$ , then  $AB\vec{v} = \alpha\beta\vec{v}$ .

Ch 7.TF.47 T, by Theorem 7.3.6a

Ch 7.TF.48 F; Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , for example.

Ch 7.TF.49 T; Recall that the rank is the dimension of the image. If  $\vec{v}$  is in the image of  $A$ , then  $A\vec{v}$  is in the image of  $A$  as well, so that  $A\vec{v}$  is parallel to  $\vec{v}$ .

Ch 7.TF.50 F; Consider  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

Ch 7.TF.51 T; If  $A\vec{v} = \lambda\vec{v}$  for a nonzero  $\vec{v}$ , then  $A^4\vec{v} = \lambda^4\vec{v} = \vec{0}$ , so that  $\lambda^4 = 0$  and  $\lambda = 0$ .

Ch 7.TF.52 F; Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ , for example.

Ch 7.TF.53 T; If the eigenvalue associated with  $\vec{v}$  is  $\lambda = 0$ , then  $A\vec{v} = \vec{0}$ , so that  $\vec{v}$  is in the kernel of  $A$ ; otherwise  $\vec{v} = A(\frac{1}{\lambda}\vec{v})$ , so that  $\vec{v}$  is in the image of  $A$ .

Ch 7.TF.54 T; either there are two distinct real eigenvalues, or the matrix is of the form  $kI_2$ .

Ch 7.TF.55 T; Either  $A\vec{u} = 3\vec{u}$  or  $A\vec{u} = 4\vec{u}$ .

Ch 7.TF.56 T; Note that  $(\vec{u}\vec{u}^T)\vec{u} = \|\vec{u}\|^2\vec{u}$ .

Ch 7.TF.57 T; Suppose  $A\vec{v}_i = \alpha_i\vec{v}_i$  and  $B\vec{v}_i = \beta_i\vec{v}_i$ , and let  $S = [\vec{v}_1 \dots \vec{v}_n]$ .  
Then  $ABS = BAS = [\alpha_1\beta_1\vec{v}_1 \dots \alpha_n\beta_n\vec{v}_n]$ , so that  $AB = BA$ .

Ch 7.TF.58 T; Note that a nonzero vector  $\vec{v} = \begin{bmatrix} p \\ q \end{bmatrix}$  is an eigenvector of  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  if (and only if)  $A\vec{v} = \begin{bmatrix} ap + bq \\ cp + dq \end{bmatrix}$  is parallel to  $\vec{v} = \begin{bmatrix} p \\ q \end{bmatrix}$ , that is, if  $\det \begin{bmatrix} p & ap + bq \\ q & cp + dq \end{bmatrix} = 0$ . Check that this is the case if (and only if)  $\vec{v}$  is an eigenvector of  $\text{adj}(A)$  (use the same criterion).

## Chapter 8

### Section 8.1

8.1.1  $\vec{e}_1, \vec{e}_2$  is an orthonormal eigenbasis.

8.1.2  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is an orthonormal eigenbasis.

8.1.3  $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$  is an orthonormal eigenbasis.

8.1.4  $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$  is an orthonormal eigenbasis.

8.1.5 Eigenvalues  $-1, -1, 2$

Choose  $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$  in  $E_{-1}$  and  $\vec{v}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  in  $E_2$  and let  $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ .

8.1.6  $\frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$  is an orthonormal eigenbasis.

8.1.7  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is an orthonormal eigenbasis, so  $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  and  $D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ .

8.1.8  $\frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$  is an orthonormal eigenbasis, with  $\lambda_1 = 4$  and  $\lambda_2 = -6$ , so  $S = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$  and  $D = \begin{bmatrix} 4 & 0 \\ 0 & -6 \end{bmatrix}$ .

8.1.9  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is an orthonormal eigenbasis, with  $\lambda_1 = 3$ ,  $\lambda_2 = -3$ , and  $\lambda_3 = 2$ , so

$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$  and  $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ .

8.1.10  $\lambda_1 = \lambda_2 = 0$  and  $\lambda_3 = 9$ .

$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$  is in  $E_0$  and  $\vec{v}_2 = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$  is in  $E_9$ .

Let  $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix}$ ; then  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is an orthonormal eigenbasis.

$$S = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{3} & -\frac{4}{3\sqrt{5}} \\ 0 & \frac{2}{3} & -\frac{\sqrt{5}}{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

8.1.11  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is an orthonormal eigenbasis, with  $\lambda_1 = 2$ ,  $\lambda_2 = 0$ , and  $\lambda_3 = 1$ , so  $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$  and  $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

8.1.12 a.  $E_1 = \text{span} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$  and  $E_{-1} = (E_1)^\perp$ . An orthonormal eigenbasis is  $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ .

b. Use Theorem 7.4.1:  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ .

c.  $A = SBS^{-1} = \begin{bmatrix} -0.6 & 0 & 0.8 \\ 0 & -1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix}$ , where  $S = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{bmatrix}$ .

8.1.13 Yes; if  $\vec{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ , then  $\vec{v} = I_3 \vec{v} = A^2 \vec{v} = \lambda^2 \vec{v}$ , so that  $\lambda^2 = 1$  and  $\lambda = 1$  or  $\lambda = -1$ . Since  $A$  is symmetric,  $E_1$  and  $E_{-1}$  will be orthogonal complements, so that  $A$  represents the reflection about  $E_1$ .

8.1.14 Let  $S$  be as in Example 3. Then  $S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ .

a. This matrix is  $2A$  so that  $S^{-1}(2A)S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ .

b. This is  $A - 3I_3$ , so that  $S^{-1}(A - 3I_3)S = S^{-1}AS - 3I_3 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

c. This is  $\frac{1}{2}(A - I_3)$ , so that  $S^{-1}(\frac{1}{2}(A - I_3))S = \frac{1}{2}(S^{-1}AS - I_3) = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

**8.1.15** Yes, if  $A\vec{v} = \lambda\vec{v}$ , then  $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$ , so that an orthonormal eigenbasis for  $A$  is also an orthonormal eigenbasis for  $A^{-1}$  (with reciprocal eigenvalues).

**8.1.16** a  $\ker(A)$  is four-dimensional, so that the eigenvalue 0 has multiplicity 4, and the remaining eigenvalue is  $\text{tr}(A) = 5$ .

b  $B = A + 2I_5$ , so that the eigenvalues are 2, 2, 2, 2, 7.

c  $\det(B) = 2^4 \cdot 7 = 112$  (product of eigenvalues)

**8.1.17** If  $A$  is the  $n \times n$  matrix with all 1's, then the eigenvalues of  $A$  are 0 (with multiplicity  $n - 1$ ) and  $n$ . Now  $B = qA + (p - q)I_n$ , so that the eigenvalues of  $B$  are  $p - q$  (with multiplicity  $n - 1$ ) and  $qn + p - q$ . Thus  $\det(B) = (p - q)^{n-1}(qn + p - q)$ .

**8.1.18** By Theorem 6.3.6, the volume is  $|\det A| = \sqrt{\det(A^T A)}$ . Now  $\vec{v}_i \cdot \vec{v}_j = \|\vec{v}_i\| \|\vec{v}_j\| \cos(\theta) = \frac{1}{2}$ , so that  $A^T A$  has all 1's on the diagonal and  $\frac{1}{2}$ 's outside. By Exercise 17 (with  $p = 1$  and  $q = \frac{1}{2}$ ),  $\det(A^T A) = \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{2}n + \frac{1}{2}\right) = \left(\frac{1}{2}\right)^n (n + 1)$ , so that the volume is  $\sqrt{\det(A^T A)} = \left(\frac{1}{2}\right)^{n/2} \sqrt{n + 1}$ .

**8.1.19** Let  $L(\vec{x}) = A\vec{x}$ . Then  $A^T A$  is symmetric, since  $(A^T A)^T = A^T (A^T)^T = A^T A$ , so that there is an orthonormal eigenbasis  $\vec{v}_1, \dots, \vec{v}_m$  for  $A^T A$ . Then the vectors  $A\vec{v}_1, \dots, A\vec{v}_m$  are orthogonal, since  $A\vec{v}_i \cdot A\vec{v}_j = (A\vec{v}_i)^T A\vec{v}_j = \vec{v}_i^T A^T A\vec{v}_j = \vec{v}_i \cdot (A^T A\vec{v}_j) = \vec{v}_i \cdot (\lambda_j \vec{v}_j) = \lambda_j (\vec{v}_i \cdot \vec{v}_j) = 0$  if  $i \neq j$ .

**8.1.20** By Exercise 19, there is an orthonormal basis  $\vec{v}_1, \dots, \vec{v}_m$  of  $\mathbb{R}^m$  such that  $T(\vec{v}_1), \dots, T(\vec{v}_m)$  are orthogonal. Suppose that  $T(\vec{v}_1), \dots, T(\vec{v}_r)$  are nonzero and  $T(\vec{v}_{r+1}), \dots, T(\vec{v}_m)$  are zero. Then let  $\vec{w}_i = \frac{1}{\|T(\vec{v}_i)\|} T(\vec{v}_i)$  for  $i = 1, \dots, r$  and choose an orthonormal basis  $\vec{w}_{r+1}, \dots, \vec{w}_n$  of  $[\text{span}(\vec{w}_1, \dots, \vec{w}_r)]^\perp$ . Then  $\vec{w}_1, \dots, \vec{w}_n$  does the job.

**8.1.21** For each eigenvalue there are two unit eigenvectors:  $\pm\vec{v}_1$ ,  $\pm\vec{v}_2$ , and  $\pm\vec{v}_3$ . We have 6 choices for the first column of  $S$ , 4 choices remaining for the second column, and 2 for the third.

Answer:  $6 \cdot 4 \cdot 2 = 48$ .

**8.1.22** a If we let  $k = 2$  then  $A$  is symmetric and therefore (orthogonally) diagonalizable.

b If we let  $k = 0$  then 0 is the only eigenvalue (but  $A \neq 0$ ), so that  $A$  fails to be diagonalizable.

**8.1.23** The eigenvalues are real (by Theorem 8.1.3), so that the only possible eigenvalues are  $\pm 1$ . Since  $A$  is symmetric,  $E_1$  and  $E_{-1}$  are orthogonal complements. Thus  $A$  represents a *reflection* about  $E_1$ .

**8.1.24** Note that  $A$  is symmetric and orthogonal, so that the eigenvalues are 1 and  $-1$  (see Exercise 23).

$$E_1 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right) \text{ and } E_{-1} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right), \text{ so that}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \text{ is an orthonormal eigenbasis.}$$

**8.1.25** Note that  $A$  is symmetric and orthogonal, so that the eigenvalues of  $A$  are 1 and  $-1$ .

$$E_1 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right), E_{-1} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right)$$

The columns of  $S$  must form an eigenbasis for  $A : S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix}$  is one possible choice.

**8.1.26** Since  $J_n$  is both orthogonal and symmetric, the eigenvalues are 1 and  $-1$ . If  $n$  is even, then both have multiplicity  $\frac{n}{2}$  (as in Exercise 24). If  $n$  is odd, then the multiplicities are  $\frac{n+1}{2}$  for 1 and  $\frac{n-1}{2}$  for  $-1$  (as in Exercise 25). One way to see this is to observe that  $\text{tr}(J_n)$  is 0 for even  $n$ , and 1 for odd  $n$  (recall that the trace is the sum of the eigenvalues).

**8.1.27** If  $n$  is even, then this matrix is  $J_n + I_n$ , for the  $J_n$  introduced in Exercise 26, so that the eigenvalues are 0 and 2, with multiplicity  $\frac{n}{2}$  each.  $E_2$  is the span of all  $\vec{e}_i + \vec{e}_{n+1-i}$ , for  $i = 1, \dots, \frac{n}{2}$ , and  $E_0$  is spanned by all  $\vec{e}_i - \vec{e}_{n+1-i}$ . If  $n$  is odd, then  $E_2$  is spanned by all  $\vec{e}_i + \vec{e}_{n+1-i}$ , for  $i = 1, \dots, \frac{n-1}{2}$ ;  $E_0$  is spanned by all  $\vec{e}_i - \vec{e}_{n+1-i}$ , for  $i = 1, \dots, \frac{n-1}{2}$ , and  $E_1$  is spanned by  $\vec{e}_{\frac{n+1}{2}}$ .

**8.1.28** For  $\lambda \neq 0$

$$\begin{aligned} f_A(\lambda) &= \det \begin{vmatrix} -\lambda & 0 & 1 \\ -\lambda & \ddots & 1 \\ \ddots & -\lambda & 1 \\ 0 & & 1 \\ 1 & 1 & \cdots & 1 & 1-\lambda \end{vmatrix} = \frac{1}{\lambda} \det \begin{vmatrix} -\lambda & 0 & 1 \\ -\lambda & \ddots & 1 \\ \ddots & 0 & -\lambda \\ \lambda & \lambda & \cdots & \lambda & \lambda - \lambda^2 \end{vmatrix} \\ &= \frac{1}{\lambda} \det \begin{vmatrix} -\lambda & 0 & 1 \\ -\lambda & \ddots & 1 \\ \ddots & 0 & -\lambda \\ 0 & 0 & \cdots & 0 & -\lambda^2 + \lambda + 12 \end{vmatrix} \\ &= -\lambda^{11}(\lambda^2 - \lambda - 12) = -\lambda^{11}(\lambda - 4)(\lambda + 3) \end{aligned}$$

Eigenvalues are 0 (with multiplicity 11), 4 and  $-3$ .

Eigenvalues for 0 are  $\vec{e}_1 - \vec{e}_i$  ( $i = 2, \dots, 12$ ),

$$E_4 = \text{span} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 4 \end{bmatrix} \text{ (12 ones)}, E_{-3} = \text{span} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ -3 \end{bmatrix} \text{ (12 ones)}$$

so

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 \end{bmatrix}$$

diagonalizes  $A$ , and  $D = S^{-1}AS$  will have all zeros as entries except  $d_{12},_{12} = 4$  and  $d_{13},_{13} = -3$ .

**8.1.29** By Theorem 5.4.1  $(\text{im } A)^\perp = \ker(A^T) = \ker(A)$ , so that  $\vec{v}$  is orthogonal to  $\vec{w}$ .

**8.1.30** The columns  $\vec{v}, \vec{v}_2, \dots, \vec{v}_n$  of  $R$  form an orthogonal eigenbasis for  $A = \vec{v}\vec{v}^T$ , with eigenvalues  $1, 0, 0, \dots, 0$  ( $n-1$  zeros), since

$A\vec{v} = \vec{v}\vec{v}^T\vec{v} = \vec{v}(\vec{v} \cdot \vec{v}) = \vec{v}$ , (since  $\vec{v} \cdot \vec{v} = 1$ ) and  $A\vec{v}_i = \vec{v}\vec{v}^T\vec{v}_i = \vec{v}(\vec{v} \cdot \vec{v}_i) = \vec{0}$  (since  $\vec{v} \cdot \vec{v}_i = 0$ ).

Therefore we can let  $S = R$ , and  $D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ .

**8.1.31** True;  $A$  is diagonalizable, that is,  $A$  is similar to a diagonal matrix  $D$ ; then  $A^2$  is similar to  $D^2$ . Now  $\text{rank}(D) = \text{rank}(D^2)$  is the number of nonzero entries on the diagonal of  $D$  (and  $D^2$ ). Since similar matrices have the same rank (by Theorem 7.3.6b) we can conclude that  $\text{rank}(A) = \text{rank}(D) = \text{rank}(D^2) = \text{rank}(A^2)$ .

**8.1.32** By Exercise 17,  $\det(A) = (1-q)^{n-1}(qn+1-q)$ .  $A$  is invertible if  $\det(A) \neq 0$ , that is, if  $q \neq 1$  and  $q \neq \frac{1}{1-n}$ .

**8.1.33** The angles must add up to  $2\pi$ , so  $\theta = \frac{2\pi}{3} = 120^\circ$ . (See Figure 8.1.)

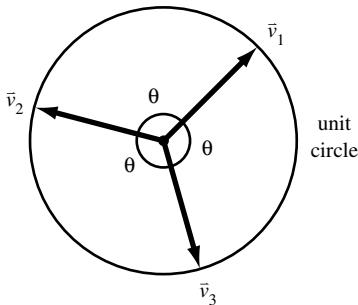


Figure 8.1: for Problem 8.1.33.

Algebraically, we can see this as follows: let  $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ , a  $2 \times 3$  matrix.

Then  $A^T A = \begin{bmatrix} 1 & \cos \theta & \cos \theta \\ \cos \theta & 1 & \cos \theta \\ \cos \theta & \cos \theta & 1 \end{bmatrix}$  is a noninvertible  $3 \times 3$  matrix, so that  $\cos \theta = \frac{1}{1-3} = -\frac{1}{2}$ , by Exercise 32, and  $\theta = \frac{2\pi}{3} = 120^\circ$ .

**8.1.34** Let  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  be such vectors. Form  $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4]$ , a  $3 \times 4$  matrix.

Then  $A^T A = \begin{bmatrix} 1 & \cos \theta & \cos \theta & \cos \theta \\ \cos \theta & 1 & \cos \theta & \cos \theta \\ \cos \theta & \cos \theta & 1 & \cos \theta \\ \cos \theta & \cos \theta & \cos \theta & 1 \end{bmatrix}$  is noninvertible, so that  $\cos \theta = \frac{1}{1-4} = -\frac{1}{3}$ , by Exercise 32, and  $\theta = \arccos(-\frac{1}{3}) \approx 109.5^\circ$ . See Figure 8.2.

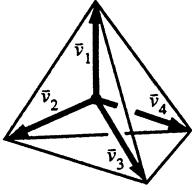


Figure 8.2: for Problem 8.1.34.

The tips of  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  form a regular tetrahedron.

**8.1.35** Let  $\vec{v}_1, \dots, \vec{v}_{n+1}$  be these vectors. Form  $A = [\vec{v}_1 \ \cdots \ \vec{v}_{n+1}]$ , an  $n \times (n+1)$  matrix.

Then  $A^T A = \begin{bmatrix} 1 & \cos \theta & \cdots & \cos \theta \\ \cos \theta & 1 & \cdots & \cos \theta \\ \vdots & & \ddots & \\ \cos \theta & \cdots & & 1 \end{bmatrix}$  is a noninvertible  $(n+1) \times (n+1)$  matrix with 1's on the diagonal and  $\cos \theta$  outside, so that  $\cos \theta = \frac{1}{1-n}$ , by Exercise 32, and  $\theta = \arccos\left(\frac{1}{1-n}\right)$ .

**8.1.36** If  $\vec{v}$  is an eigenvector with eigenvalue  $\lambda$ , then  $\lambda \vec{v} = A\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$ , so that  $\lambda = \lambda^2$  and therefore  $\lambda = 0$  or  $\lambda = 1$ . Since  $A$  is symmetric,  $E_0$  and  $E_1$  are orthogonal complements, so that  $A$  represents the orthogonal projection onto  $E_1$ .

**8.1.37** In Example 4 we see that the image of the unit circle is an ellipse with semi-axes 2 and 3. Thus  $\|A\vec{u}\|$  takes all values in the interval  $[2, 3]$ .

**8.1.38** The spectral theorem tells us that there exists an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2$  for  $A$ , with associated eigenvalues -2 and 3. Consider a unit vector  $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$  in  $\mathbb{R}^2$ , with  $c_1^2 + c_2^2 = 1$ . Then  $\vec{u} \cdot A\vec{u} = (c_1\vec{v}_1 + c_2\vec{v}_2) \cdot (-2c_1\vec{v}_1 + 3c_2\vec{v}_2) = -2c_1^2 + 3c_2^2$ , which takes all values on the interval  $[-2, 3]$  since  $-2 = -2c_1^2 - 2c_2^2 \leq -2c_1^2 + 3c_2^2 \leq 3c_1^2 + 3c_2^2 = 3$ .

**8.1.39** The spectral theorem tells us that there exists an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  for  $A$ , with associated eigenvalues -2, 3 and 4. Consider a unit vector  $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$  in  $\mathbb{R}^3$ , with  $c_1^2 + c_2^2 + c_3^2 = 1$ . Then

$\vec{u} \cdot A\vec{u} = (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \cdot (-2c_1\vec{v}_1 + 3c_2\vec{v}_2 + 4c_3\vec{v}_3) = -2c_1^2 + 3c_2^2 + 4c_3^2$ , which takes all values on the interval  $[-2, 4]$  since  $-2 = -2c_1^2 - 2c_2^2 - 2c_3^2 \leq -2c_1^2 + 3c_2^2 + 4c_3^2 \leq 4c_1^2 + 4c_2^2 + 4c_3^2 = 4$ .

**8.1.40** Using the terminology introduced in Exercise 8.1.39, we have

$\|A\vec{u}\| = \| -2c_1\vec{v}_1 + 3c_2\vec{v}_2 + 4c_3\vec{v}_3 \| = \sqrt{4c_1^2 + 9c_2^2 + 16c_3^2}$ , which takes all values on the interval  $[2, 4]$ . Geometrically, the image of the unit sphere under  $A$  is the ellipsoid with semi-axes 2, 3, and 4.

**8.1.41** The spectral theorem tells us that there exists an orthogonal matrix  $S$  such that  $S^{-1}AS = D$  is diagonal.

Let  $D_1$  be the diagonal matrix such that  $D_1^3 = D$ ; the diagonal entries of  $D_1$  are the cube roots of those of  $D$ .

Now  $B = SD_1S^{-1}$  does the job, since  $B^3 = (SD_1S^{-1})^3 = SD_1^3S^{-1} = SDS^{-1} = A$ .

**8.1.42** We will use the strategy outlined in Exercise 8.1.41. An orthogonal matrix that diagonalizes  $A = \frac{1}{5} \begin{bmatrix} 12 & 14 \\ 14 & 33 \end{bmatrix}$

is  $S = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ , with  $S^{-1}AS = D = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}$ . Now  $D_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$  and  $B = SD_1S^{-1} = \frac{1}{5} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$ .

**8.1.43** There is an orthonormal eigenbasis  $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ,  $\vec{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ ,  $\vec{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  with associated

eigenvalues -9, -9, 24. We are looking for a nonzero vector  $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$  such that  $\vec{v} \cdot A\vec{v} = (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \cdot (-9c_1\vec{v}_1 - 9c_2\vec{v}_2 + 24c_3\vec{v}_3) = -9c_1^2 - 9c_2^2 + 24c_3^2 = 0$  or  $-3c_1^2 - 3c_2^2 + 8c_3^2 = 0$ . One possible

solution is  $c_1 = \sqrt{8} = 2\sqrt{2}$ ,  $c_2 = 0$ ,  $c_3 = \sqrt{3}$ , so that  $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ .

**8.1.44** Use Exercise 8.1.43 as a guide. Consider an orthonormal eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  for  $A$ , with associated eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ , listed in ascending order. If  $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$  is any nonzero vector in  $\mathbb{R}^n$ , then  $\vec{v} \cdot A\vec{v} = (c_1\vec{v}_1 + \dots + c_n\vec{v}_n) \cdot (\lambda_1 c_1\vec{v}_1 + \dots + \lambda_n c_n\vec{v}_n) = \lambda_1 c_1^2 + \dots + \lambda_n c_n^2$ . If all the eigenvalues are positive, then  $\vec{v} \cdot A\vec{v}$  will be positive. Likewise, if all the eigenvalues are negative, then  $\vec{v} \cdot A\vec{v}$  will be negative. However, if  $A$  has positive as well as negative eigenvalues, meaning that  $\lambda_1 < 0 < \lambda_n$  (as in Example 8.1.43), then there exist nonzero vectors  $\vec{v}$  with  $\vec{v} \cdot A\vec{v} = 0$ , for example,  $\vec{v} = \sqrt{\lambda_n}\vec{v}_1 + \sqrt{-\lambda_1}\vec{v}_n$ .

**8.1.45 a** If  $S^{-1}AS$  is upper triangular then the first column of  $S$  is an eigenvector of  $A$ . Therefore, any matrix without real eigenvectors fails to be triangulizable over  $\mathbb{R}$ , for example,  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

**b** Proof by induction on  $n$ : For an  $n \times n$  matrix  $A$  we can choose a complex invertible  $n \times n$  matrix  $P$  whose first column is an eigenvector for  $A$ . Then  $P^{-1}AP = \begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix}$ .  $B$  is triangulizable, by induction hypothesis, that is, there is an invertible  $(n-1) \times (n-1)$  matrix  $Q$  such that  $Q^{-1}BQ = T$  is upper triangular.

Now let  $R = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$ . Then  $R^{-1} \begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix} R = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \lambda & \vec{v}Q \\ 0 & T \end{bmatrix}$  is upper triangular.  $R^{-1} \begin{bmatrix} \lambda & \vec{v} \\ 0 & B \end{bmatrix} R = R^{-1}P^{-1}APR = S^{-1}AS$ , where  $S = PR$ , proving our claim.

**8.1.46 a** By definition of an upper triangular matrix,  $\vec{e}_1$  is in  $\ker U$ ,  $\vec{e}_2$  is in  $\ker(U^2), \dots, \vec{e}_n$  is in  $\ker(U^n)$ , so that all  $\vec{x}$  in  $\mathbb{C}^n$  are in  $\ker(U^n)$ , that is,  $U^n = 0$ .

b By Exercise 45b, there exists an invertible  $S$  such that  $S^{-1}AS = U$  is upper triangular. The diagonal entries of  $U$  are all zero, since  $A$  and  $U$  have the same eigenvalues; therefore  $U^n = 0$  by part a. Now  $A = SUS^{-1}$  and  $A^n = SU^nS^{-1} = 0$ , as claimed.

8.1.47 a For all  $i, j$ ,  $\left[ \sum_{k=1}^n a_{ik}b_{kj} \right] \leq \sum_{k=1}^n |a_{ik}b_{kj}| = \sum_{k=1}^n |a_{ik}||b_{kj}|$

↑

triangle inequality

b By induction on  $t$ :  $|A^t| = |A^{t-1}A| \leq |A^{t-1}||A| \leq |A|^{t-1}|A| = |A|^t$

↑                ↑

part a      by induction hypothesis

8.1.48 If  $t \geq n - 1$  then  $(I_n + U)^t = I_n + \binom{t}{1}U + \binom{t}{2}U^2 + \cdots + \binom{t}{n-1}U^{n-1}$ , since  $U^n = 0$ . Now  $\binom{t}{k} \leq t^n$  for  $k = 1, \dots, n - 1$ , so that  $(I_n + U)^t \leq t^n(I_n + U + \cdots + U^{n-1})$ , as claimed. Check that the formula holds for  $t < n - 1$  as well.

8.1.49 Let  $\lambda$  be the largest  $|r_{ii}|$ ; note that  $\lambda < 1$ . Then  $|R| = \begin{bmatrix} |r_{11}| & * \\ * & \ddots \\ 0 & |r_{nn}| \end{bmatrix} \leq \begin{bmatrix} \lambda & * \\ * & \ddots \\ 0 & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & * \\ * & 1 \end{bmatrix} = \lambda(I_n + U)$ , and  $|R^t| \leq |R|^t \leq \lambda^t(I_n + U)^t \leq \lambda^t t^n(I_n + U + \cdots + U^{n-1})$ .

We learn in Calculus that  $\lim_{t \rightarrow \infty} (\lambda^t t^n) = 0$ , so that  $\lim_{t \rightarrow \infty} (R^t) = 0$ .

8.1.50 a From Exercise 45b we know that there is an invertible  $S$  and an upper triangular  $R$  such that  $S^{-1}AS = R$ , and  $|r_{ii}| < 1$  for all  $i$ , since the diagonal entries of  $R$  are the eigenvalues of  $A$ . Now  $\lim_{t \rightarrow \infty} R^t = 0$  by Exercise 49. Note that  $A = SRS^{-1}$  and  $A^t = SR^tS^{-1}$ , so that  $\lim_{t \rightarrow \infty} A^t = 0$ , as claimed.

b See the remark after Definition 7.6.1.

## Section 8.2

8.2.1 We have  $a_{11} = \text{coefficient of } x_1^2 = 6, a_{22} = \text{coefficient of } x_2^2 = 8, a_{12} = a_{21} = \frac{1}{2}(\text{coefficient of } x_1x_2) = -\frac{7}{2}$ . So,  $A = \begin{bmatrix} 6 & -\frac{7}{2} \\ -\frac{7}{2} & 8 \end{bmatrix}$

8.2.2  $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

8.2.3  $A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 4 & \frac{7}{2} \\ 3 & \frac{7}{2} & 5 \end{bmatrix}$

8.2.4  $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$ , positive definite

8.2.5  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ , indefinite (since  $\det(A) < 0$ )

8.2.6  $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ , indefinite (since  $\det(A) < 0$ )

8.2.7  $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ , indefinite (eigenvalues 2, -2, 3)

8.2.8 If  $S^{-1}AS = D$  is diagonal, then  $S^{-1}A^2S = D^2$ , so that all eigenvalues of  $A^2$  are  $\geq 0$ . So  $A^2$  is positive semi-definite; it is positive definite if and only if  $A$  is invertible.

8.2.9 a  $(A^2)^T = (A^T)^2 = (-A)^2 = A^2$ , so that  $A^2$  is symmetric.

b  $q(\vec{x}) = \vec{x}^T A^2 \vec{x} = \vec{x}^T AA\vec{x} = -\vec{x}^T A^T A\vec{x} = -(A\vec{x}) \cdot (A\vec{x}) = -\|A\vec{x}\|^2 \leq 0$  for all  $\vec{x}$ , so that  $A^2$  is negative semi-definite. The eigenvalues of  $A^2$  will be  $\geq 0$ .

c If  $\vec{v}$  is a complex eigenvector of  $A$  with eigenvalue  $\lambda$ , then  $A^2\vec{v} = \lambda^2\vec{v}$ , and  $\lambda^2 \leq 0$ , by part b. Therefore,  $\lambda$  is *imaginary*, that is,  $\lambda = bi$  for a real  $b$ . Thus, the zero matrix is the only skew-symmetric matrix that is diagonalizable over  $\mathbb{R}$ .

8.2.10  $L(\vec{x}) = (\vec{x} + \vec{v})^T A(\vec{x} + \vec{v}) - \vec{x}^T A\vec{x} - \vec{v}^T A\vec{v} = \vec{x}^T A\vec{x} + \vec{x}^T A\vec{v} + \vec{v}^T A\vec{x} + \vec{v}^T A\vec{v} - \vec{x}^T A\vec{x} - \vec{v}^T A\vec{v} = \vec{x}^T A\vec{v} + \vec{v}^T A\vec{x} = \vec{v}^T A\vec{x} + \vec{v}^T A\vec{x} = (2\vec{v}^T A)\vec{x}$ ,

↑

note that  $\vec{x}^T A \vec{v}$  is a scalar so that  $\vec{x}^T A \vec{v} = (\vec{x}^T A \vec{v})^T = \vec{v}^T A^T \vec{x} = \vec{v}^T A \vec{x}$  if  $A$  is symmetric.

So  $L$  is linear with matrix  $2\vec{v}^T A$ .

8.2.11 The eigenvalues of  $A^{-1}$  are the reciprocals of those of  $A$ , so that  $A$  and  $A^{-1}$  have the same definiteness.

8.2.12  $\det(A)$  is the product of the two (real) eigenvalues.  $q$  is indefinite if and only if those have different signs, that is, their product is negative.

8.2.13  $q(\vec{e}_i) = \vec{e}_i \cdot A \vec{e}_i = a_{ii} > 0$

8.2.14 If  $\det(A)$  is positive then both eigenvalues have the same sign, so that  $A$  is positive definite or negative definite. Since  $\vec{e}_1 \cdot A \vec{e}_1 = a > 0$ ,  $A$  is in fact positive definite.

8.2.15  $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$ ; eigenvalues  $\lambda_1 = 7$  and  $\lambda_2 = 2$

orthonormal eigenbasis  $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda_1 c_1^2 + \lambda_2 c_2^2 = 1$  or  $7c_1^2 + 2c_2^2 = 1$ . (See Figure 8.3.)

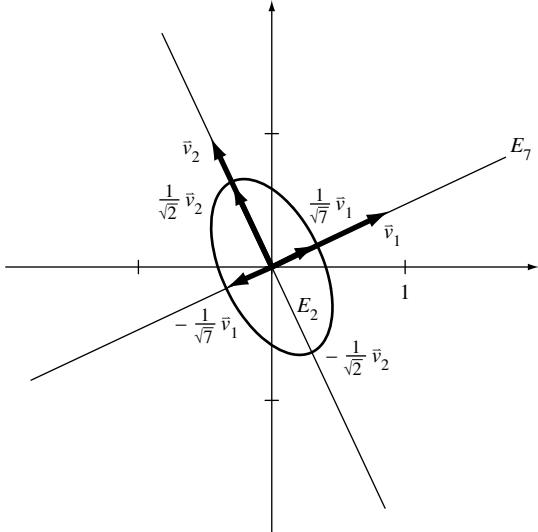


Figure 8.3: for Problem 8.2.15.

8.2.16  $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ ; eigenvalues  $\lambda_1 = \frac{1}{2}$ , and  $\lambda_2 = -\frac{1}{2}$

orthonormal eigenbasis  $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\frac{1}{2}c_1^2 - \frac{1}{2}c_2^2 = 1$ . (See Figure 8.4.)

8.2.17  $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$ , eigenvalues  $\lambda_1 = 4$ ,  $\lambda_2 = -1$

orthonormal eigenbasis  $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$4c_1^2 - c_2^2 = 1$  (hyperbola) (See Figure 8.5.)

8.2.18  $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$ , eigenvalues  $\lambda_1 = 10$ ,  $\lambda_2 = 5$

orthonormal eigenbasis  $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$10c_1^2 + 5c_2^2 = 1$ . This is an ellipse, as shown in Figure 8.6.

8.2.19  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ; eigenvalues  $\lambda_1 = 5$ ,  $\lambda_2 = 0$

eigenvectors  $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

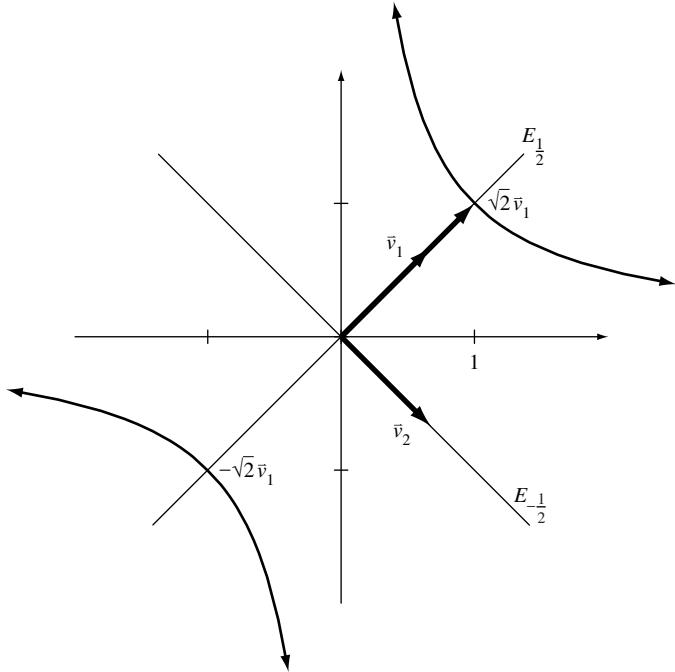


Figure 8.4: for Problem 8.2.16.

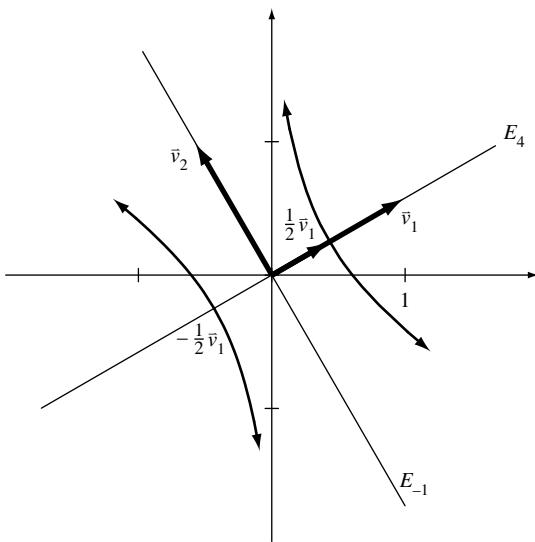


Figure 8.5: for Problem 8.2.17.

$$5c_1^2 = 1 \text{ (a pair of lines) (See Figure 8.7.)}$$

Note that  $(x_1^2 + 4x_1x_2 + 4x_2^2) = (x_1 + 2x_2)^2 = 1$ , so that  $x_1 + 2x_2 = \pm 1$ , and the two lines are  $x_2 = \frac{1-x_1}{2}$  and  $x_2 = \frac{-1-x_1}{2}$ .

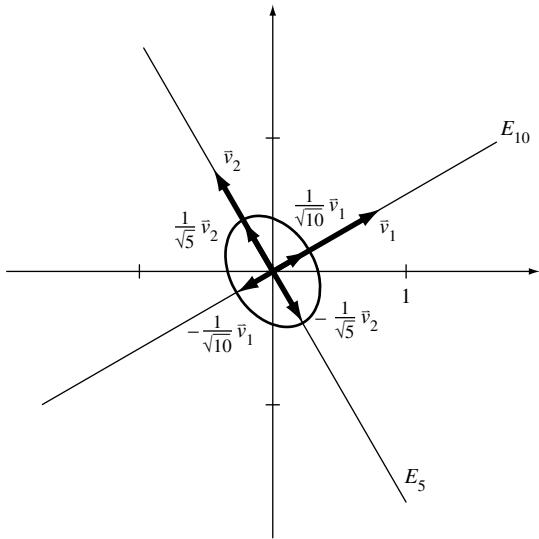


Figure 8.6: for Problem 8.2.18.

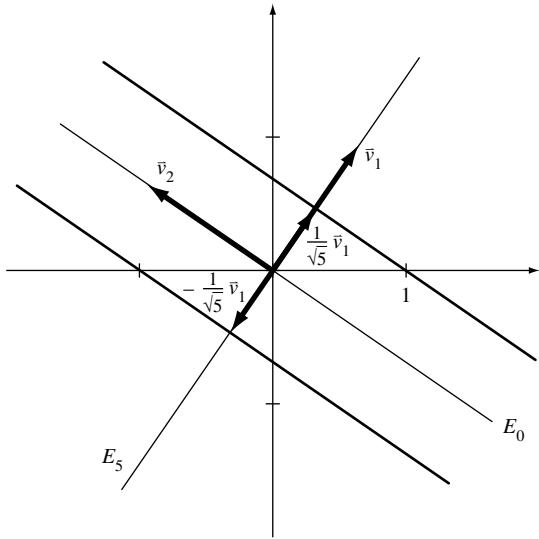


Figure 8.7: for Problem 8.2.19.

8.2.20  $A = \begin{bmatrix} -3 & 3 \\ 3 & 5 \end{bmatrix}$ ; eigenvalues  $\lambda_1 = 6$  and  $\lambda_2 = -4$

orthonormal eigenbasis  $\vec{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ,  $\vec{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$6c_1^2 - 4c_2^2 = 1$ . This is a hyperbola, as shown in Figure 8.8.

8.2.21 a In each case, it is informative to think about the intersections with the three coordinate planes:  $x_1 - x_2$ ,  $x_1 - x_3$ , and  $x_2 - x_3$ .

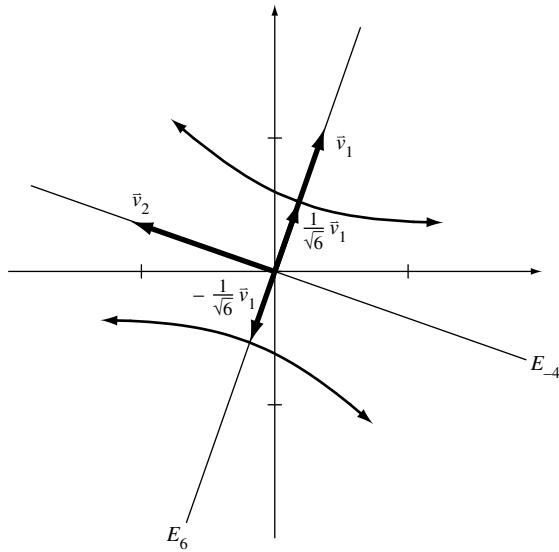
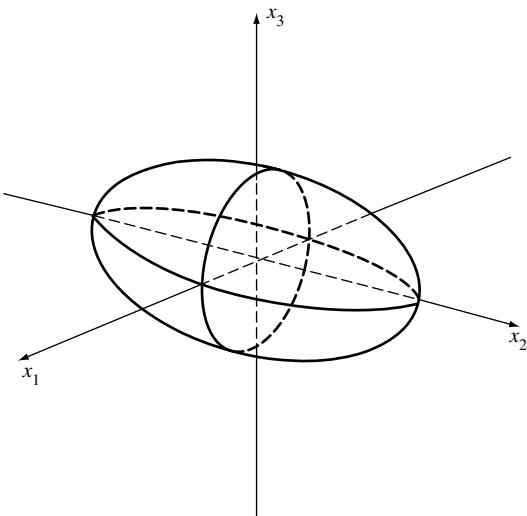


Figure 8.8: for Problem 8.2.20.

- For the surface  $x_1^2 + 4x_2^2 + 9x_3^2 = 1$ , all these intersections are *ellipses*, and the surface itself is an *ellipsoid*.

This surface is connected and bounded; the points closest to the origin are  $\pm \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$ , and those farthest  $\pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ .  
(See Figure 8.9.)

Figure 8.9: for Problem 8.2.21a:  $x_1^2 + 4x_2^2 + 9x_3^2 = 1$ , an *ellipsoid* (not to scale).

- In the case of  $x_1^2 + 4x_2^2 - 9x_3^2 = 1$ , the intersection with the  $x_1 - x_2$  plane is an ellipse, and the two other intersections are hyperbolas. The surface is connected and not bounded; the points closest to the origin are

$$\pm \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}. \text{ (See Figure 8.10.)}$$

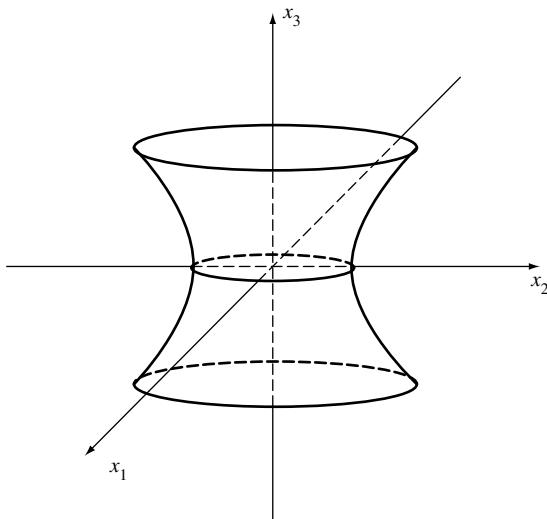


Figure 8.10: for Problem 8.2.21a:  $x_1^2 + 4x_2^2 - 9x_3^2 = 1$ , a *hyperboloid of one sheet* (not to scale).

- In the case  $-x_1^2 - 4x_2^2 + 9x_3^2 = 1$ , the intersection with the  $x_1 - x_2$  plane is empty, and the two other intersections are hyperbolas. The surface consists of two pieces and is unbounded. The points closest to the origin are  $\pm \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$ . (See Figure 8.11.)

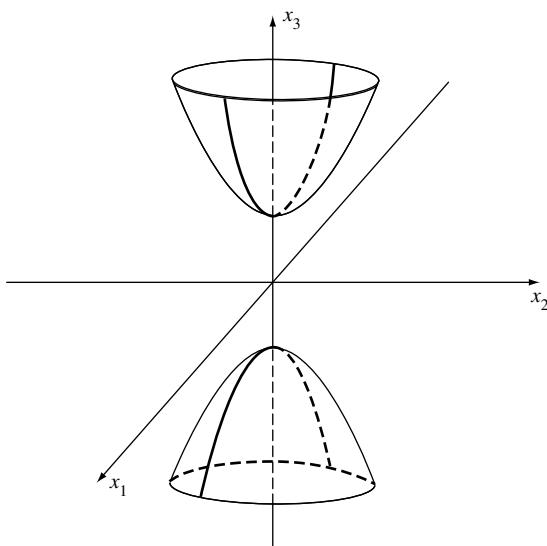


Figure 8.11: for Problem 8.2.21a:  $-x_1^2 - 4x_2^2 + 9x_3^2 = 1$ , a *hyperboloid of two sheets* (not to scale).

b)  $A = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 2 & \frac{3}{2} \\ 1 & \frac{3}{2} & 3 \end{bmatrix}$  is positive definite, with three positive eigenvalues  $\lambda_1, \lambda_2, \lambda_3$ .

The surface is given by  $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$  with respect to the principal axis, an *ellipsoid*. To find the points closest to and farthest from the origin, use technology to find the eigenvalues and eigenvectors:

eigenvalues:  $\lambda_1 \approx 0.56, \lambda_2 \approx 4.44, \lambda_3 = 1$

unit eigenvectors:  $\vec{v}_1 \approx \begin{bmatrix} 0.86 \\ 0.19 \\ -0.47 \end{bmatrix}, \vec{v}_2 \approx \begin{bmatrix} 0.31 \\ 0.54 \\ 0.78 \end{bmatrix}, \vec{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Equation:  $0.56c_1^2 + 4.44c_2^2 + c_3^2 = 1$

Farthest points when  $c_1 = \pm \frac{1}{\sqrt{0.56}}$  and  $c_2 = c_3 = 0$

Closest points when  $c_2 = \pm \frac{1}{\sqrt{4.44}}$  and  $c_1 = c_3 = 0$

Farthest points  $\approx \pm \frac{1}{\sqrt{0.56}} \begin{bmatrix} 0.86 \\ 0.19 \\ -0.47 \end{bmatrix} \approx \pm \begin{bmatrix} 1.15 \\ 0.26 \\ -0.63 \end{bmatrix}$

Closest points  $\approx \pm \frac{1}{\sqrt{4.44}} \begin{bmatrix} 0.31 \\ 0.54 \\ 0.78 \end{bmatrix} \approx \pm \begin{bmatrix} 0.15 \\ 0.26 \\ 0.37 \end{bmatrix}$

8.2.22  $A = \begin{bmatrix} -1 & 0 & 5 \\ 0 & 1 & 0 \\ 5 & 0 & -1 \end{bmatrix}$ ; eigenvalues  $\lambda_1 = 4, \lambda_2 = -6, \lambda_3 = 1$

Equation with respect to principal axis:  $4c_1^2 - 6c_2^2 + c_3^2 = 1$ , a hyperboloid of one sheet (see Figure 8.10).

Closest to origin when  $c_1 = \pm \frac{1}{2}, c_2 = c_3 = 0$ .

A unit eigenvector for eigenvalue 4 is  $\vec{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ , so that the desired points are  $\pm \frac{1}{2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \approx \pm \begin{bmatrix} 0.35 \\ 0 \\ 0.35 \end{bmatrix}$ .

8.2.23 Yes;  $M = \frac{1}{2}(A + A^T)$  is symmetric, and

$$\vec{x}^T M \vec{x} = \frac{1}{2} \vec{x}^T A \vec{x} + \frac{1}{2} \vec{x}^T A^T \vec{x} = \frac{1}{2} \vec{x}^T A \vec{x} + \frac{1}{2} \vec{x}^T A \vec{x} = \vec{x}^T A \vec{x}$$

Note that  $\vec{x}^T A \vec{x}$  is a  $1 \times 1$  matrix, so that  $\vec{x}^T A \vec{x} = (\vec{x}^T A \vec{x})^T = \vec{x}^T A^T \vec{x}$ .

8.2.24  $q(\vec{e}_1) = \vec{e}_1 \cdot A \vec{e}_1 = \vec{e}_1 \cdot (\text{first column of } A) = a_{11}$

8.2.25  $q(\vec{v}) = \vec{v} \cdot A \vec{v} = \vec{v} \cdot (\lambda \vec{v}) = \lambda(\vec{v} \cdot \vec{v}) = \lambda$

↑

$\vec{v}$  is a unit vector

**8.2.26** False; If  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  then  $q \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$ .

**8.2.27** Let  $\vec{v}_1, \dots, \vec{v}_n$  be an orthonormal eigenbasis for  $A$  with  $A\vec{v}_i = \lambda_i \vec{v}_i$ . We know that  $q(\vec{v}_i) = \lambda_i$  (see Exercise 25), so that  $q(\vec{v}_1) = \lambda_1$  and  $q(\vec{v}_n) = \lambda_n$  are in the image.

We claim that all numbers between  $\lambda_n$  and  $\lambda_1$  are in the image as well. To see this, apply the Intermediate Value Theorem to the continuous function  $f(t) = q((\cos t)\vec{v}_n + (\sin t)\vec{v}_1)$  on  $[0, \frac{\pi}{2}]$  (note that  $f(0) = q(\vec{v}_n) = \lambda_n$  and  $f(\frac{\pi}{2}) = q(\vec{v}_1) = \lambda_1$ ). (See Figure 8.12.)

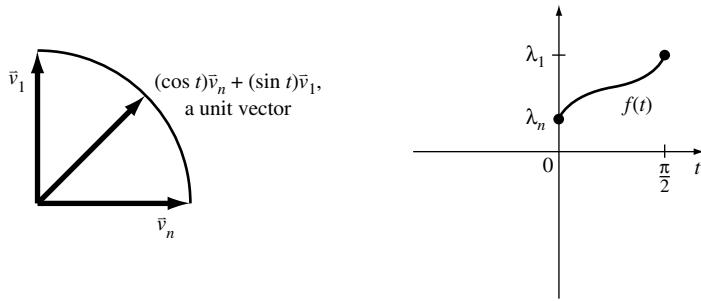


Figure 8.12: for Problem 8.2.27.

The Intermediate Value Theorem tells us that for any  $c$  between  $\lambda_n$  and  $\lambda_1$ , there is a  $t_0$  such that  $f(t_0) = q((\cos t_0)\vec{v}_n + (\sin t_0)\vec{v}_1) = c$ . Note that  $(\cos t_0)\vec{v}_n + (\sin t_0)\vec{v}_1$  is a unit vector. Now we will show that, conversely,  $q(\vec{v})$  is on  $[\lambda_n, \lambda_1]$  for all unit vectors  $\vec{v}$ . Write  $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$  and note that  $\|\vec{v}\|^2 = c_1^2 + \dots + c_n^2 = 1$ . Then  $q(\vec{v}) = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \dots + \lambda_n c_n^2 \leq \lambda_1 c_1^2 + \lambda_1 c_2^2 + \dots + \lambda_1 c_n^2 = \lambda_1$ . Likewise,  $q(\vec{v}) \geq \lambda_n$ . We have shown that the image of  $S^{n-1}$  under  $q$  is the closed interval  $[\lambda_n, \lambda_1]$ .

**8.2.28** The hint almost gives it away. Since  $D$  is a diagonal matrix with positive diagonal entries, we can write  $D = D_1^2$ , where  $D_1$  is diagonal with positive diagonal entries (the square roots of the entries of  $D$ ). Now  $A = SDS^T = SD_1D_1S^T = SD_1(SD_1)^T = BB^T$  where  $B = SD_1$ . The columns of  $B$  are scalar multiples of the corresponding columns of  $S$ , so that they are orthogonal.

**8.2.29** From Example 1 we have  $S = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$  and  $D = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$ . Let  $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$  and  $B = SD_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 6 & 2 \\ -3 & 4 \end{bmatrix}$ .

**8.2.30** Define  $D_1$  as in Exercise 28. Then  $A = SDS^{-1} = SD_1D_1S^{-1} = (SD_1S^{-1})(SD_1S^{-1}) = B^2$ , where  $B = SD_1S^{-1}$ .  $B$  is positive definite, since  $S^{-1}BS = D_1$  is diagonal with positive diagonal entries.

**8.2.31**  $S = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$  and  $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$  (see Exercise 29), so that  $B = SD_1S^{-1} = \begin{bmatrix} 2.8 & -0.4 \\ -0.4 & 2.2 \end{bmatrix}$ .

**8.2.32** Recall that  $a = q(\vec{e}_1) > 0$  and  $\det A = ac - b^2 = \lambda_1 \lambda_2 > 0$ .

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^2 & xy \\ xy & y^2 + z^2 \end{bmatrix} \text{ means that } \begin{array}{l} x^2 = a \\ xy = b \\ y^2 + z^2 = c \end{array}$$

It is required that  $x$  and  $z$  be positive. This system has the unique solution

$$x = \sqrt{a}$$

$$y = \frac{b}{x} = \frac{b}{\sqrt{a}}$$

$$z = \sqrt{c - y^2} = \sqrt{c - \frac{b^2}{a}} = \sqrt{\frac{ac - b^2}{a}}$$

**8.2.33** Use the formulas for  $x, y, z$  derived in Exercise 32.

$$x = \sqrt{a} = \sqrt{8} = 2\sqrt{2}$$

$$y = \frac{b}{\sqrt{a}} = -\frac{2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$z = \sqrt{\frac{ac - b^2}{a}} = \sqrt{\frac{36}{8}} = \frac{3}{\sqrt{2}}, \text{ so that}$$

$$L = \begin{bmatrix} 2\sqrt{2} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix}.$$

**8.2.34** (i) implies (ii): See the hint at the end of the exercise.

(ii) implies (iii):  $\det A^{(m)}$  is the product of the (positive) eigenvalues.

(iii) implies (iv):

$$A = \begin{bmatrix} A^{(n-1)} & \vec{v} \\ \vec{v}^T & k \end{bmatrix} = \begin{bmatrix} B & 0 \\ \vec{x}^T & 1 \end{bmatrix} \begin{bmatrix} B^T & \vec{x} \\ 0 & t \end{bmatrix} = \begin{bmatrix} BB^T & B\vec{x} \\ \vec{x}^T B^T & \vec{x}^T \vec{x} + t \end{bmatrix}$$

The system  $\begin{bmatrix} B\vec{x} = \vec{v} \\ \vec{x}^T \vec{x} + t = k \end{bmatrix}$  has the unique solution

$$\vec{x} = B^{-1}\vec{v}$$

$$t = k - \vec{x}^T \vec{x} = k - \|B^{-1}\vec{v}\|^2.$$

Note that  $t$  is positive since  $0 < \det(A^{(n)}) = \det(A) = \det \begin{bmatrix} B & 0 \\ \vec{x}^T & 1 \end{bmatrix} \det \begin{bmatrix} B^T & \vec{x} \\ 0 & t \end{bmatrix} = (\det B)^2 \cdot t$ .

(iv)  $\Rightarrow$  (i)

$$\vec{x}^T A \vec{x} = \vec{x}^T LL^T \vec{x} = (L^T \vec{x})^T (L^T \vec{x}) = \|L^T \vec{x}\|^2 > 0 \text{ if } \vec{x} \neq \vec{0}, \text{ since } L \text{ is invertible.}$$

**8.2.35** Solve the system  $\begin{bmatrix} 4 & -4 & 8 \\ -4 & 13 & 1 \\ 8 & 1 & 26 \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ y & w & 0 \\ z & t & s \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & w & t \\ 0 & 0 & s \end{bmatrix}$

$$\left. \begin{array}{l} x^2 = 4, \text{ so } x = 2 \\ 2y = -4, \text{ so } y = -2 \\ 2z = 8, \text{ so } z = 4 \\ 4 + w^2 = 13, \text{ so } w = 3 \\ -8 + 3t = 1, \text{ so } t = 3 \\ 16 + 9 + s^2 = 26, \text{ so } s = 1 \end{array} \right\} L = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 3 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

**8.2.36** If  $A = QR$ , then  $A^T A = (QR)^T QR = R^T Q^T QR = R^T R = LL^T$ ,  $L = R^T$ .

**8.2.37**  $\frac{\partial q}{\partial x_1} = 2ax_1 + bx_2$  and  $\frac{\partial q}{\partial x_2} = bx_1 + 2cx_2$ , so that  $q_{11} = \frac{\partial^2 q}{\partial x_1^2} = 2a$ ,  $q_{22} = \frac{\partial^2 q}{\partial x_2^2} = 2c$ , and  $q_{12} = \frac{\partial^2 q}{\partial x_1 \partial x_2} = b$ , and  $D = \det \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} = \det \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} = 4ac - b^2 > 0$ .

The matrix  $A = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}$  of  $q$  is positive definite, since  $a > 0$  and  $\det(A) = \frac{1}{4}D > 0$ . This means, by definition, that  $q$  has a minimum at  $\vec{0}$ , since  $q(\vec{x}) > 0 = q(\vec{0})$  for all  $\vec{x} \neq \vec{0}$ .

**8.2.38** The eigenvalues of  $B$  are  $p - q$  and  $nq + p - q = p + (n - 1)q$ , so that  $B$  is positive definite if  $p - q > 0$  and  $p + (n - 1)q > 0$ .

**8.2.39** If  $\vec{v}_1, \dots, \vec{v}_n$  is such a basis consisting of unit vectors, and we let  $A = [\vec{v}_1 \cdots \vec{v}_n]$ , then

$A^T A = \begin{bmatrix} 1 & \cos \theta & \cdots & \cos \theta \\ \cos \theta & 1 & \ddots & \cos \theta \\ \vdots & \ddots & \ddots & \vdots \\ \cos \theta & \cos \theta & \cdots & 1 \end{bmatrix}$  is positive definite, so that, by Exercise 38,  $1 - \cos \theta > 0$  and  $1 + (n - 1) \cos \theta > 0$  or  $1 > \cos \theta > \frac{1}{1-n}$  or  $0 < \theta < \arccos\left(\frac{1}{1-n}\right)$ .

Conversely, if  $\theta$  is in this range, then the matrix  $\begin{bmatrix} 1 & \cos \theta & \cdots & \cos \theta \\ \cos \theta & 1 & \ddots & \cos \theta \\ \vdots & \ddots & \ddots & \vdots \\ \cos \theta & \cos \theta & \cdots & 1 \end{bmatrix}$  is positive definite, so that it has a Cholesky factorization  $LL^T$ . The columns of  $L^T$  give us a basis with the desired property.

**8.2.40** Let  $\lambda$  be the smallest eigenvalue of  $A$ . If we let  $k = 1 - \lambda$ , then the smallest eigenvalue of the matrix  $A + kI_n$  will be  $\lambda + k = 1$ , so that all the eigenvalues of  $A + kI_n$  will be positive. Thus matrix  $A + kI_n$  will be positive definite, by Theorem 8.2.4.

**8.2.41** The functions  $x_1^2, x_1x_2, x_2^2$  form a basis of  $Q_2$ , so that  $\dim(Q_2) = 3$ .

**8.2.42** The functions  $x_i x_j$  form a basis of  $Q_n$ , where  $1 \leq i \leq j \leq n$ . A little combinatorics shows that there are  $1 + 2 + 3 + \cdots + n = n(n+1)/2$  such functions, so that  $\dim(Q_n) = n(n+1)/2$

**8.2.43** Note that  $T(ax_1^2 + bx_1x_2 + cx_2^2) = ax_1^2$  (we let  $x_2 = 0$ ). Thus  $\text{im}(T) = \text{span}(x_1^2)$ ,  $\text{rank}(T) = 1$ ,  $\ker(T) = \text{span}(x_1x_2, x_2^2)$ ,  $\text{nullity}(T) = 2$ .

**8.2.44** Note that  $T(ax_1^2 + bx_1x_2 + cx_2^2) = ax_1^2 + bx_1 + c$  (we let  $x_2 = 1$ ). Thus  $\text{im}(T) = P_2$ ,  $\text{rank}(T) = 3$ ,  $\ker(T) = \{0\}$ ,  $\text{nullity}(T) = 0$  ( $T$  is an isomorphism).

**8.2.45** Note that  $T(ax_1^2 + bx_2^2 + cx_3^2 + dx_1x_2 + ex_1x_3 + fx_2x_3) = ax_1^2 + b + c + dx_1 + ex_1 + f$  (we let  $x_2 = x_3 = 1$ ). Thus  $\text{im}(T) = P_2$  and  $\text{rank}(T) = 3$ . The kernel of  $T$  consists of the quadratic forms with  $a = 0$ ,  $d + e = 0$ , and  $b + c + f = 0$  (consider the coefficients of  $x_1^2$ ,  $x_1$ , and 1). The general element of the kernel is  $q(x_1, x_2, x_3) = (-c - f)x_2^2 + cx_3^2 - ex_1x_2 + ex_1x_3 + fx_2x_3 = c(x_3^2 - x_2^2) + e(x_1x_3 - x_1x_2) + f(x_2x_3 - x_2^2)$ . Thus  $\ker(T) = \text{span}(x_3^2 - x_2^2, x_1x_3 - x_1x_2, x_2x_3 - x_2^2)$  and  $\text{nullity}(T) = 3$ .

**8.2.46** Note that  $T(ax_1^2 + bx_2^2 + cx_3^2 + dx_1x_2 + ex_1x_3 + fx_2x_3) = ax_1^2 + bx_2^2 + cx_1^2 + dx_1x_2 + ex_1^2 + fx_1x_2$  (we let  $x_3 = x_1$ ). Thus  $\text{im}(T) = Q_2$  and  $\text{rank}(T) = 3$ . The kernel of  $T$  consists of the quadratic forms with

$a + c + e = 0, b = 0$ , and  $d + f = 0$  (consider the coefficients of  $x_1^2, x_2^2$ , and  $x_1x_2$ ). The general element of the kernel is  $q(x_1, x_2, x_3) = (-c - e)x_1^2 + cx_3^2 - fx_1x_2 + ex_1x_3 + fx_2x_3 = c(x_3^2 - x_1^2) + e(x_1x_3 - x_1^2) + f(x_2x_3 - x_1x_2)$ . Thus  $\ker(T) = \text{span}(x_3^2 - x_1^2, x_1x_3 - x_1^2, x_2x_3 - x_1x_2)$  and  $\text{nullity}(T) = 3$ .

8.2.47  $T(A + B)(\vec{x}) = \vec{x}^T(A + B)\vec{x} = \vec{x}^TA\vec{x} + \vec{x}^TB\vec{x}$  equals  $(T(A) + T(B))(\vec{x})$

$= T(A)(\vec{x}) + T(B)(\vec{x}) = \vec{x}^TA\vec{x} + \vec{x}^TB\vec{x}$ . The verification of the second axiom of linearity is analogous.

By definition of a quadratic form,  $\text{im}(T) = Q_n$ : For every quadratic form  $q$  in  $Q_n$  there is a *symmetric*  $n \times n$  matrix  $A$  such that  $q = T(A)$ . Thus, the rank of  $T$  is  $\dim(Q_n) = n(n+1)/2$  (see Exercise 42).

By the rank nullity theorem,

$$\text{nullity}(T) = \dim(\mathbb{R}^{n \times n}) - \text{rank}(T) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

Next, let's think about the kernel of  $T$ . In our solution to Exercise 23 we observed that  $T(A) = T(\frac{1}{2}(A + A^T))$ ; note that matrix  $\frac{1}{2}(A + A^T)$  is symmetric. Now  $\frac{1}{2}(A + A^T) = 0$  if (and only if)  $A^T = -A$ , that is, if  $A$  is skew-symmetric. Thus the skew-symmetric matrices are in the kernel of  $T$ . Since the space of skew-symmetric matrices has the same dimension as  $\ker(T)$ , namely,  $n(n-1)/2$ , we can conclude that  $\ker(T)$  consists of all skew-symmetric  $n \times n$  matrices.

8.2.48 The matrix of  $T$  with respect to the basis  $x_1^2, x_1x_2, x_2^2$  is  $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ , with the eigenvalues 1, 1,  $-1$  and corresponding eigenvectors  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ . Thus  $x_1x_2$  and  $x_1^2 + x_2^2$  are eigenfunctions with eigenvalue 1, and  $x_1^2 - x_2^2$  has eigenvalue  $-1$ . Yes,  $T$  is diagonalizable, since there is an eigenbasis.

8.2.49 The matrix of  $T$  with respect to the basis  $x_1^2, x_1x_2, x_2^2$  is  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ , with the eigenvalues 1, 2, 4 and corresponding eigenvectors  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . Thus  $x_1^2, x_1x_2, x_2^2$  are eigenfunctions with eigenvalues 1, 2, and 4, respectively. Yes,  $T$  is diagonalizable, since there is an eigenbasis.

8.2.50 The matrix of  $T$  with respect to the basis  $x_1^2, x_1x_2, x_2^2$  is  $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ , with the eigenvalues 0, 2,  $-2$  and corresponding eigenvectors  $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

Thus  $x_1^2 - x_2^2, x_1^2 + 2x_1x_2 + x_2^2, x_1^2 - 2x_1x_2 + x_2^2$  are eigenfunctions with eigenvalues 0, 2, and  $-2$ , respectively. Yes,  $T$  is diagonalizable, since there is an eigenbasis.

8.2.51 If  $B$  is negative definite, then  $A = -B$  is positive definite, so that the determinants of all principal submatrices  $A^{(m)}$  are positive. Thus  $\det(B^{(m)}) = \det(-A^{(m)}) = (-1)^m \det(A^{(m)})$  is positive for even  $m$  and negative for odd  $m$ .

**8.2.52** Because  $a_{ij} = \vec{e}_j^T A \vec{e}_i$ , we have  $q(\vec{e}_i) = a_{ii}$ . Further, using linearity  $q(\vec{e}_i + \vec{e}_j) = (\vec{e}_i + \vec{e}_j)^T A (\vec{e}_i + \vec{e}_j) = \vec{e}_i^T A \vec{e}_i + \vec{e}_i^T A \vec{e}_j + \vec{e}_j^T A \vec{e}_i + \vec{e}_j^T A \vec{e}_j = q(\vec{e}_i) + q(\vec{e}_j) + 2a_{ij}$ . Solving for  $a_{ij}$  gives  $a_{ij} = \frac{1}{2}(q(\vec{e}_i + \vec{e}_j) - q(\vec{e}_i) - q(\vec{e}_j))$ .

**8.2.53** a. Because  $p(x, y) = q(x\vec{e}_i + y\vec{e}_j) = (x\vec{e}_i + y\vec{e}_j)^T A (x\vec{e}_i + y\vec{e}_j) = a_{ii}x^2 + a_{ij}xy + a_{ji}yx + a_{jj}y^2$ , this is a quadratic form with matrix  $B$ .

b. If  $q$  is positive definite and  $(x, y) \neq (0, 0)$ , then  $p(x, y) = q(x\vec{e}_i + y\vec{e}_j) > 0$ .

c. If  $q$  is positive semidefinite, then  $p(x, y) = q(x\vec{e}_i + y\vec{e}_j) \geq 0$  for all  $x, y$ .

d. If  $q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$  and we let  $i = 1, j = 2$ , then  $p(x, y, 0) = q(x, y, 0) = x^2 + y^2$  is positive definite.

**8.2.54** The entries  $a_{1j} = a_{j1}$  must all be 0. To see that  $a_{1j} = 0$ , consider the function  $p(x, y) = q(x\vec{e}_1 + y\vec{e}_j)$  defined in Exercise 8.2.53. By Exercise 53a, the symmetric matrix of  $p$  will be  $\begin{bmatrix} a_{11} & a_{1j} \\ a_{j1} & a_{jj} \end{bmatrix} = \begin{bmatrix} 0 & a_{1j} \\ a_{j1} & a_{jj} \end{bmatrix}$ . This matrix is positive semidefinite, by Exercise 53c, implying that  $\det \begin{bmatrix} 0 & a_{1j} \\ a_{j1} & a_{jj} \end{bmatrix} = -a_{1j}^2 \geq 0$ . Thus  $a_{1j} = 0$ , as claimed.

**8.2.55** As the hint suggests, it suffices to prove that  $a_{ij} < a_{ii}$  or  $a_{ij} < a_{jj}$ , implying that for every entry off the diagonal there exists a larger entry on the diagonal. Now  $q(\vec{e}_i - \vec{e}_j) = (\vec{e}_i - \vec{e}_j)^T A (\vec{e}_i - \vec{e}_j) = a_{ii} - 2a_{ij} + a_{jj} > 0$ , or,  $a_{ii} + a_{jj} > 2a_{ij}$ , proving the claim.

**8.2.56** Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  be the eigenvalues of  $A$ . In Exercise 8.2.27 we see that the range of  $q$  on unit vectors is the interval  $[\lambda_n, \lambda_1]$ . Since  $a_{11} = q(\vec{e}_1)$  is in that range, we must have  $a_{11} \leq \lambda_1$ , as claimed.

**8.2.57** Working in coordinates with respect to an orthonormal eigenbasis for  $A$ , we can write the equation  $q(\vec{x}) = 1$  as  $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$ , where the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  are positive. This level surface is an ellipsoid.

**8.2.58** Working in coordinates with respect to an orthonormal eigenbasis of  $A$ , we can write the equation  $q(\vec{x}) = 1$  as  $\lambda_1 c_1^2 + \lambda_2 c_2^2 = 1$ , where the eigenvalues  $\lambda_1$  and  $\lambda_2$  are positive. This level surface is a cylinder.

**8.2.59** Working in coordinates with respect to an orthonormal eigenbasis of  $A$ , we can write the equation  $q(\vec{x}) = 1$  as  $\lambda_1 c_1^2 = 1$ , where the eigenvalue  $\lambda_1$  is positive. This level surface is a pair of parallel planes,  $c_1 = \pm 1/\sqrt{\lambda_1}$ .

**8.2.60** Working in coordinates with respect to an orthonormal eigenbasis of  $A$ , we can write the equation  $q(\vec{x}) = 1$  as  $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$ , where the eigenvalue  $\lambda_1$  is positive, while  $\lambda_2$  and  $\lambda_3$  are negative. This level surface is a hyperboloid of two sheets.

**8.2.61** Working in coordinates with respect to an orthonormal eigenbasis of  $A$ , we can write the equation  $q(\vec{x}) = 1$  as  $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$ , where the eigenvalues  $\lambda_1$  and  $\lambda_2$  are positive, while  $\lambda_3$  is negative. This level surface is a hyperboloid of one sheet.

**8.2.62** Working in coordinates with respect to an orthonormal eigenbasis of  $A$ , we can write the equation  $q(\vec{x}) = 0$  as  $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 0$ , where the eigenvalues  $\lambda_1$  and  $\lambda_2$  are positive, while  $\lambda_3$  is negative. This level surface is a cone.

**8.2.63** Note that  $\vec{w}_i \cdot \vec{w}_i = \frac{1}{\lambda_i}$ . Now  $q(c_1 \vec{w}_1 + \dots + c_n \vec{w}_n) = (c_1 \vec{w}_1 + \dots + c_n \vec{w}_n) \cdot A (c_1 \vec{w}_1 + \dots + c_n \vec{w}_n) = (c_1 \vec{w}_1 + \dots + c_n \vec{w}_n) \cdot (\lambda_1 c_1 \vec{w}_1 + \dots + \lambda_n c_n \vec{w}_n) = \lambda_1 c_1^2 \frac{1}{\lambda_1} + \dots +$

$\lambda_n c_n^2 \frac{1}{\lambda_n} = c_1^2 + \dots + c_n^2$ , as claimed.

**8.2.64** We will use the strategy outlined in Exercise 8.2.63. The symmetric matrix of  $q$  is  $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$ , with an orthonormal eigenbasis  $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ ,  $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , with associated eigenvalues  $\lambda_1 = 9$  and  $\lambda_2 = 4$ . Thus the orthogonal basis  $\vec{w}_1 = \frac{1}{\sqrt{\lambda_1}} \vec{v}_1 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ ,  $\vec{w}_2 = \frac{1}{\sqrt{\lambda_2}} \vec{v}_2 = \frac{1}{2\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  has the required property. (See Figure 8.13.)

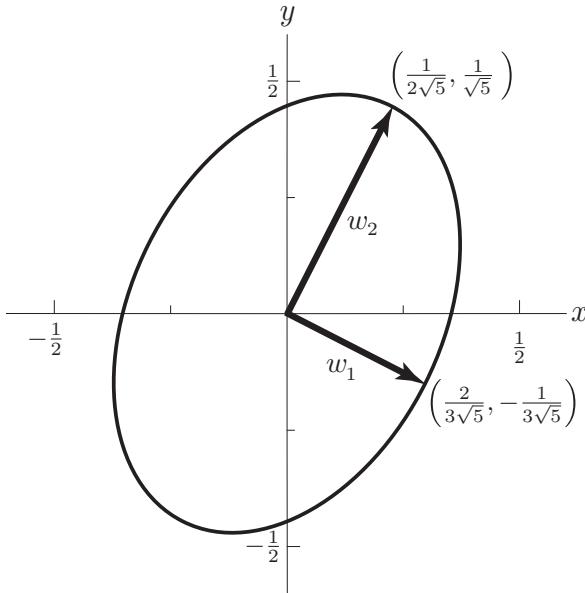


Figure 8.13: for Problem 8.2.64.

**8.2.65** Working in coordinates  $c_1, c_2$  with respect to an orthonormal eigenbasis  $\vec{v}_1, \vec{v}_2$  for  $A$ , we can write  $q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2$ , where the eigenvalue  $\lambda_1$  is positive while  $\lambda_2$  is negative. We define the orthogonal vectors  $\vec{w}_1 = \frac{1}{\sqrt{\lambda_1}} \vec{v}_1$  and  $\vec{w}_2 = \frac{1}{\sqrt{-\lambda_2}} \vec{v}_2$ . Note that  $\vec{w}_1 \cdot \vec{w}_1 = \frac{1}{\lambda_1}$  and  $\vec{w}_2 \cdot \vec{w}_2 = \frac{1}{(-\lambda_2)}$ . Now  $q(c_1 \vec{w}_1 + c_2 \vec{w}_2) = (c_1 \vec{w}_1 + c_2 \vec{w}_2) \cdot (\lambda_1 c_1 \vec{w}_1 + \lambda_2 c_2 \vec{w}_2) = \lambda_1 c_1^2 \frac{1}{\lambda_1} + \lambda_2 c_2^2 \frac{1}{(-\lambda_2)} = c_1^2 - c_2^2$ , as claimed.

**8.2.66** We will use the strategy outlined in Exercise 8.2.65. The symmetric matrix of  $q$  is  $A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}$ , with an orthonormal eigenbasis  $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ,  $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , with associated eigenvalues  $\lambda_1 = 8$  and  $\lambda_2 = -2$ . Thus the orthogonal basis  $\vec{w}_1 = \frac{1}{\sqrt{\lambda_1}} \vec{v}_1 = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ,  $\vec{w}_2 = \frac{1}{\sqrt{-\lambda_2}} \vec{v}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  has the required property that  $q(c_1 \vec{w}_1 + c_2 \vec{w}_2) = c_1^2 - c_2^2$ . (See Figure 8.14.)

**8.2.67** Consider an orthonormal eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  for  $A$  with associated eigenvalues  $\lambda_1, \dots, \lambda_n$ , such that the eigenvalues  $\lambda_1, \dots, \lambda_p$  are positive,  $\lambda_{p+1}, \dots, \lambda_r$  are negative, and the remaining eigenvalues are 0. Define a

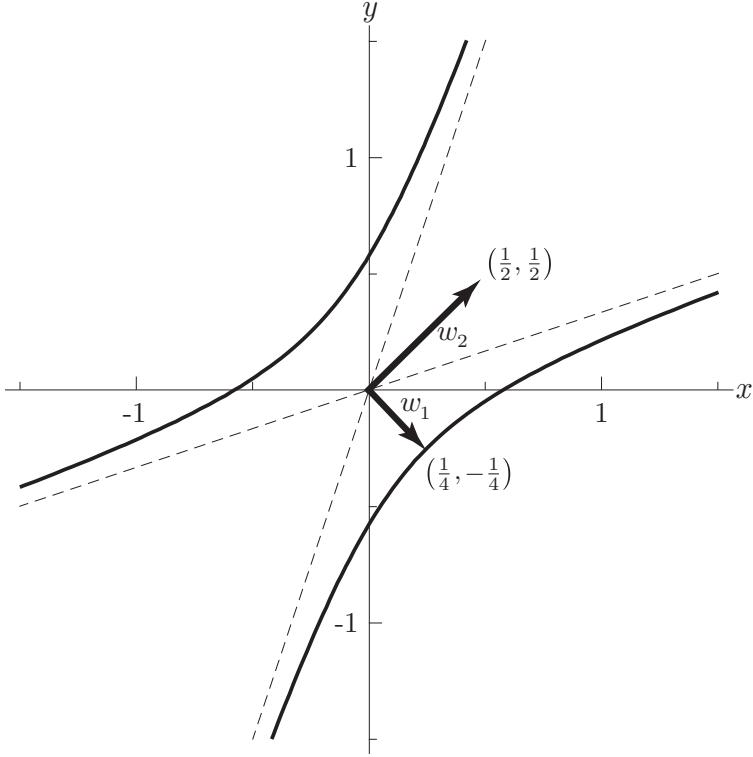


Figure 8.14: for Problem 8.2.66.

new orthogonal eigenbasis  $\vec{w}_1, \dots, \vec{w}_n$  by setting  $\vec{w}_i = \left(1/\sqrt{|\lambda_i|}\right)\vec{v}_i$  for  $i = 1, \dots, r$  and  $\vec{w}_i = \vec{v}_i$  for  $i = r+1, \dots, n$ . Note that  $\vec{w}_i \cdot \vec{w}_i = 1/\lambda_i$  for  $i = 1, \dots, p$  and  $\vec{w}_i \cdot \vec{w}_i = 1/(-\lambda_i)$  for  $i = p+1, \dots, r$ . Now  $q(c_1\vec{w}_1 + \dots + c_p\vec{w}_p + \dots + c_r\vec{w}_r + \dots + c_n\vec{w}_n) = (c_1\vec{w}_1 + \dots + c_p\vec{w}_p + \dots + c_r\vec{w}_r + \dots + c_n\vec{w}_n) \cdot (\lambda_1 c_1 \vec{w}_1 + \dots + \lambda_p c_p \vec{w}_p + \dots + \lambda_r c_r \vec{w}_r) = \lambda_1 c_1^2 \frac{1}{\lambda_1} + \dots + \lambda_p c_p^2 \frac{1}{\lambda_p} - \dots - \lambda_r c_r^2 \frac{1}{\lambda_r} = c_1^2 + \dots + c_p^2 - c_{p+1}^2 - \dots - c_r^2$ , as claimed.

**8.2.68**  $p(\vec{x}) = q(L(\vec{x})) = q(R\vec{x}) = (R\vec{x})^T A (R\vec{x}) = \vec{x}^T (R^T A R) \vec{x}$ , proving that  $p$  is a quadratic form with symmetric matrix  $R^T A R$ .

**8.2.69** If  $A$  is positive definite, then  $\vec{x}^T (R^T A R) \vec{x} = (R\vec{x})^T A (R\vec{x}) \geq 0$  for all  $\vec{x}$ , meaning that  $R^T A R$  is positive semidefinite. If  $A$  is positive definite and  $\ker R = \{\vec{0}\}$ , then  $\vec{x}^T (R^T A R) \vec{x} = (R\vec{x})^T A (R\vec{x}) > 0$  for all nonzero  $\vec{x}$ , meaning that  $R^T A R$  is positive definite. Conclusion:  $R^T A R$  is always positive semidefinite, and  $R^T A R$  is positive definite if (and only if)  $\ker R = \{\vec{0}\}$ , meaning that the rank of  $R$  is  $m$ .

**8.2.70** Since  $A$  is indefinite, there exist vectors  $\vec{v}_1$  and  $\vec{v}_2$  in  $\mathbb{R}^n$  such that  $\vec{v}_1^T A \vec{v}_1 > 0$  and  $\vec{v}_2^T A \vec{v}_2 < 0$ . Since the  $n \times m$  matrix  $R$  has rank  $n$ , we know that the image of  $R$  is all of  $\mathbb{R}^n$ , so that there exist vectors  $\vec{w}_1$  and  $\vec{w}_2$  in  $\mathbb{R}^m$  with  $R\vec{w}_1 = \vec{v}_1$  and  $R\vec{w}_2 = \vec{v}_2$ . Now  $\vec{w}_1^T R^T A R \vec{w}_1 = (R\vec{w}_1)^T A (R\vec{w}_1) = \vec{v}_1^T A \vec{v}_1 > 0$  and  $\vec{w}_2^T R^T A R \vec{w}_2 = \vec{v}_2^T A \vec{v}_2 < 0$ , proving that matrix  $R^T A R$  is indefinite.

**8.2.71** Anything can happen. Consider the example  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $R_1 = I_2$ ,  $R_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $R_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Then

$R_1^T A R_1 = A$  is indefinite,  $R_2^T A R_2 = [1]$  is positive definite, and  $R_3^T A R_3 = [-1]$  is negative definite.

## Section 8.3

8.3.1  $\sigma_1 = 2, \sigma_2 = 1$

8.3.2 The image of the unit circle is the unit circle, since the transformation defined by  $A$  preserves length. Thus  $\sigma_1 = \sigma_2 = 1$  by Theorem 8.3.2.

8.3.3  $A^T A = I_n$ ; the eigenvalues of  $A^T A$  are all 1, so that the singular values of  $A$  are all 1.

8.3.4  $A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ , with eigenvalues  $\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$ . The singular values of  $A$  are  $\sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2} \approx 1.62$  and  $\sigma_2 = \sqrt{\frac{3-\sqrt{5}}{2}} = \frac{-1+\sqrt{5}}{2} \approx 0.62$ .

8.3.5  $A^T A = \begin{bmatrix} p^2 + q^2 & 0 \\ 0 & p^2 + q^2 \end{bmatrix}$ , with eigenvalues  $\lambda_1 = \lambda_2 = p^2 + q^2$ . The singular values of  $A$  are  $\sigma_1 = \sigma_2 = \sqrt{p^2 + q^2}$ .  $A$  represents a rotation combined with a scaling, with a scaling factor of  $\sqrt{p^2 + q^2}$ , so that the image of the unit circle is a circle with radius  $\sqrt{p^2 + q^2}$ .

8.3.6 The eigenvalues of  $A^T A$  are  $\lambda_1 = 25$  and  $\lambda_2 = 0$ , so that the singular values of  $A$  are  $\sigma_1 = 5$  and  $\sigma_2 = 0$  (these are also the eigenvalues of  $A$ ; compare with Exercise 24).

$E_5 = \text{span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , so that  $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  works. The image of the unit circle is the line segment connecting the tips of  $A\vec{v}_1 = 5\vec{v}_1$  and  $A(-\vec{v}_1) = -5\vec{v}_1$ . See Figure 8.15.

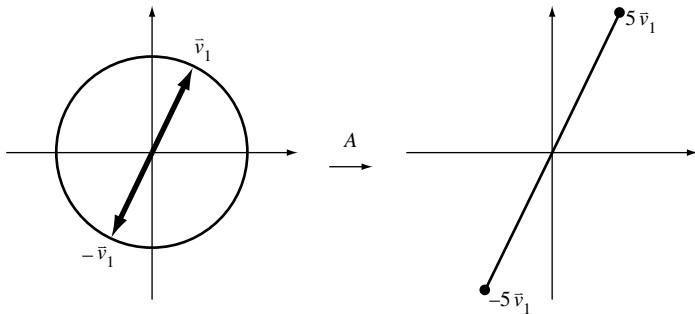


Figure 8.15: for Problem 8.3.6.

8.3.7  $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$\lambda_1 = 4, \lambda_2 = 1; \sigma_1 = 2, \sigma_2 = 1$

eigenvectors of  $A^T A : \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{u}_1 = \frac{1}{\sigma_1}(A\vec{v}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ ,  $\vec{u}_2 = \frac{1}{\sigma_2}(A\vec{v}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , so that  $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ,  $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

$$8.3.8 \quad A^T A = \begin{bmatrix} p^2 + q^2 & 0 \\ 0 & p^2 + q^2 \end{bmatrix}; \lambda_1 = \lambda_2 = p^2 + q^2; \sigma_1 = \sigma_2 = \sqrt{p^2 + q^2}$$

eigenvectors of  $A^T A : \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{p^2+q^2}} \begin{bmatrix} p \\ q \end{bmatrix}$ ,  $\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{\sqrt{p^2+q^2}} \begin{bmatrix} -q \\ p \end{bmatrix}$ , so that  $U = \frac{1}{\sqrt{p^2+q^2}} \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ ,  $\Sigma = (\sqrt{p^2+q^2})I_2$ ,  $V = I_2$ .

$$8.3.9 \quad A^T A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \text{ (See Exercise 6)}$$

$\lambda_1 = 25$ ,  $\lambda_2 = 0$ ;  $\sigma_1 = 5$ ,  $\sigma_2 = 0$ ; eigenvectors of  $A^T A :$

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{u}_2 = \text{a unit vector orthogonal to}$$

$$\vec{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ so that } U = V = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}.$$

8.3.10 In Example 4 we found  $\begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \left( \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \right)$ ; now take the transpose of both sides:

$$\begin{bmatrix} 6 & -7 \\ 2 & 6 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \left( \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \right).$$

$$\begin{array}{c} \uparrow \\ U \end{array} \quad \begin{array}{c} \uparrow \\ \Sigma \end{array} \quad \begin{array}{c} \uparrow \\ V^T \end{array}$$

$$8.3.11 \quad A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}; \lambda_1 = 4, \lambda_2 = 1; \sigma_1 = 2, \sigma_2 = 1 \text{ eigenvectors of } A^T A :$$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

8.3.12 In Example 5 we see that  $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ .

Now take the transpose of both sides.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)$$

$\uparrow \quad \uparrow \quad \uparrow$   
 $U \quad \Sigma \quad V^T$

8.3.13  $A^T A = \begin{bmatrix} 37 & 16 \\ 16 & 13 \end{bmatrix}$ ;  $\lambda_1 = 45$ ,  $\lambda_2 = 5$ ;  $\sigma_1 = 3\sqrt{5}$ ,  $\sigma_2 = \sqrt{5}$  eigenvectors of  $A^T A$ :

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ so that}$$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}, V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

8.3.14  $A^T A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$ ;  $\lambda_1 = 16$ ,  $\lambda_2 = 1$ ;  $\sigma_1 = 4$ ,  $\sigma_2 = 1$

$$\text{eigenvectors of } A^T A : \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ so that}$$

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, V = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

8.3.15 If  $A \vec{v}_1 = \sigma_1 \vec{u}_1$  and  $A \vec{v}_2 = \sigma_2 \vec{u}_2$ , then  $A^{-1} \vec{u}_1 = \frac{1}{\sigma_1} \vec{v}_1$  and  $A^{-1} \vec{u}_2 = \frac{1}{\sigma_2} \vec{v}_2$ , so that the singular values of  $A^{-1}$  are the reciprocals of the singular values of  $A$ .

8.3.16 If  $A = U \Sigma V^T$  then  $A^{-1} = V \Sigma^{-1} U^T$  and  $(A^{-1})^T A^{-1} = U (\Sigma^{-1})^2 U^T$ . Thus  $(A^{-1})^T A^{-1}$  is similar to  $(\Sigma^{-1})^2$ , so that the eigenvalues of  $(A^{-1})^T A^{-1}$  are the squares of the reciprocals of the singular values of  $A$ . It follows that the singular values of  $A^{-1}$  are the reciprocals of those of  $A$ .

8.3.17 We need to check that  $A \left( \frac{\vec{b} \cdot \vec{u}_1}{\sigma_1} \vec{v}_1 + \cdots + \frac{\vec{b} \cdot \vec{u}_m}{\sigma_m} \vec{v}_m \right) = \text{proj}_{\text{im} A} \vec{b}$ . (see Page 223).

$$\text{But } A \left( \frac{\vec{b} \cdot \vec{u}_1}{\sigma_1} \vec{v}_1 + \cdots + \frac{\vec{b} \cdot \vec{u}_m}{\sigma_m} \vec{v}_m \right) = \frac{\vec{b} \cdot \vec{u}_1}{\sigma_1} A \vec{v}_1 + \cdots + \frac{\vec{b} \cdot \vec{u}_m}{\sigma_m} A \vec{v}_m = (\vec{b} \cdot \vec{u}_1) \vec{u}_1 + \cdots + (\vec{b} \cdot \vec{u}_m) \vec{u}_m$$

$$= \text{proj}_{\text{im} A} \vec{b}, \text{ since } \vec{u}_1, \dots, \vec{u}_m \text{ is an orthonormal basis of } \text{im}(A) \text{ (see Theorem 5.1.5).}$$

8.3.18  $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$ ,  $\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ ,  $\vec{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ ,  $\vec{v}_1 = \frac{1}{5} \begin{bmatrix} 3 \\ -4 \\ -4 \end{bmatrix}$ ,  $\vec{v}_2 = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$ ,  $\sigma_1 = 2$ ,  $\sigma_2 = 1$ , so that

$$\vec{x}^* = \frac{\vec{b} \cdot \vec{u}_1}{\sigma_1} \vec{v}_1 + \frac{\vec{b} \cdot \vec{u}_2}{\sigma_2} \vec{v}_2 = \begin{bmatrix} -0.1 \\ -3.2 \end{bmatrix}.$$

**8.3.19**  $\vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$  is a least-squares solution if  $A\vec{x} = c_1A\vec{v}_1 + \cdots + c_mA\vec{v}_m = c_1\sigma_1\vec{u}_1 + \cdots + c_r\sigma_r\vec{u}_r = \text{proj}_{\text{im } A}\vec{b}$ . But  $\text{proj}_{\text{im } A}\vec{b} = (\vec{b} \cdot \vec{u}_1)\vec{u}_1 + \cdots + (\vec{b} \cdot \vec{u}_r)\vec{u}_r$ , since  $\vec{u}_1, \dots, \vec{u}_r$  is an orthonormal basis of  $\text{im}(A)$ . Comparing the coefficients of  $\vec{u}_i$  above we find that it is required that  $c_i\sigma_i = \vec{b} \cdot \vec{u}_i$  or  $c_i = \frac{\vec{b} \cdot \vec{u}_i}{\sigma_i}$ , for  $i = 1, \dots, r$ , while no condition is imposed on  $c_{r+1}, \dots, c_m$ . The least-squares solutions are of the form  $\vec{x}^* = \frac{\vec{b} \cdot \vec{u}_1}{\sigma_1}\vec{v}_1 + \cdots + \frac{\vec{b} \cdot \vec{u}_r}{\sigma_r}\vec{v}_r + c_{r+1}\vec{v}_{r+1} + \cdots + c_m\vec{v}_m$ , where  $c_{r+1}, \dots, c_m$  are arbitrary (see Exercise 17 for a special case).

**8.3.20 a**  $A = U\Sigma V^T = UV^T V\Sigma V^T = QS$ , where  $Q = UV^T$  and  $S = V\Sigma V^T$ . Note that  $Q$  is orthogonal, being the product of orthogonal matrices;  $S$  is symmetric as  $S^T = (V^T)^T \Sigma^T V^T = V\Sigma V^T = S$ ; and  $S$  is similar to  $\Sigma$ , so that the eigenvalues of  $S$  are the (nonnegative) diagonal entries of  $\Sigma$ .

b Yes, write  $A = U\Sigma V^T = U\Sigma U^T UV^T = S_1 Q_1$  where  $S_1 = U\Sigma U^T$  and  $Q_1 = UV^T$ .

$$\begin{aligned} \text{8.3.21 } A &= \underbrace{\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}_U \underbrace{\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}}_\Sigma \underbrace{\left( \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \right)}_{V^T} \\ &= \underbrace{\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}_U \underbrace{\left( \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \right)}_{V^T} \underbrace{\left( \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \right)}_V \underbrace{\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}}_\Sigma \underbrace{\left( \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \right)}_{V^T} \\ &= \underbrace{\frac{1}{5} \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}}_Q \underbrace{\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}}_S \end{aligned}$$

**8.3.22** a.  $T_1$  is the orthogonal projection onto the plane perpendicular to the vector  $\vec{v}$ .  $T_2$  scales by the length of the vector  $\vec{v}$  and  $T_3$  is a rotation about the line through the origin spanned by  $\vec{v}$  by a rotation angle  $\pi/2$ . Because  $Q = A_3$  is orthogonal and  $S = A_2 A_1$  is symmetric this is a polar decomposition:  $A = QS$ .  
 b. Here,  $A_1$  represents the orthogonal projection onto the  $xz$  plane,  $A_2$  represents a scaling by a factor of 2, and  $A_3$  represents a rotation about the  $y$  axis through an angle of  $\pi/2$ , counterclockwise as viewed from the positive  $y$  axis:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Thus  $Q = A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ ,  $S = A_2 A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ , and  $A = QS = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ .

**8.3.23**  $AA^T U = U\Sigma V^T V\Sigma^T U^T U = U\Sigma Sigma^T$ , since  $V^T V = I_m$  and  $U^T U = I_n$ , so that

$$AA^T \vec{u}_i = \begin{cases} \sigma_i^2 \vec{u}_i & \text{for } i = 1, \dots, r \\ \vec{0} & \text{for } i = r+1, \dots, n \end{cases}$$

The nonzero eigenvalues of  $A^T A$  and  $AA^T$  are the same.

**8.3.24** The eigenvalues of  $A^T A = A^2$  are the squares of the eigenvalues of  $A$ , so that the singular values of  $A$  are the absolute values of the eigenvalues of  $A$ .

Chapter 8

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8.3.25 See Figure 8.16.

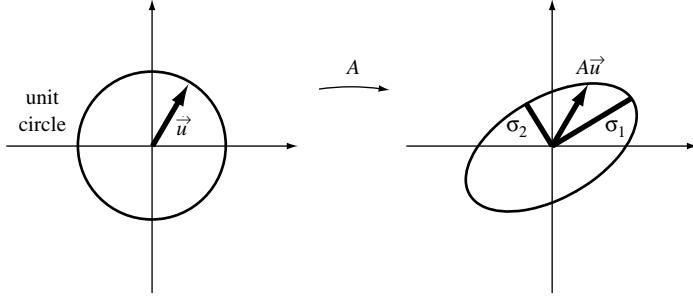


Figure 8.16: for Problem 8.3.25.

Algebraically: Write  $\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2$  and note that  $\|\vec{u}\|^2 = c_1^2 + c_2^2 = 1$ .

Then  $A\vec{u} = c_1\sigma_1\vec{u}_1 + c_2\sigma_2\vec{u}_2$ , so that  $\|A\vec{u}\|^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 \geq c_1^2\sigma_2^2 + c_2^2\sigma_2^2 = \sigma_2^2$  and  $\|A\vec{u}\| \geq \sigma_2$ . Likewise  $\|A\vec{u}\| \leq \sigma_1$ .

8.3.26 Write  $\vec{v} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$  and note that  $\|\vec{v}\|^2 = c_1^2 + \cdots + c_m^2$ . Then  $A\vec{v} = c_1\sigma_1\vec{u}_1 + \cdots + c_r\sigma_r\vec{u}_r$  and  $\|A\vec{v}\|^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_r^2\sigma_r^2 \leq c_1^2\sigma_1^2 + c_2^2\sigma_1^2 + \cdots + c_r^2\sigma_1^2 \leq \sigma_1^2\|\vec{v}\|^2$  so that  $\|A\vec{v}\| \leq \sigma_1\|\vec{v}\|$ . Likewise,  $\|A\vec{v}\| \geq \sigma_m\|\vec{v}\|$ .

8.3.27 Let  $\vec{v}$  be a unit eigenvector with eigenvalue  $\lambda$  and use Exercise 26.

8.3.28 If  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A^T A$ , then  $(\det A)^2 = \det(A^T A) = \lambda_1 \cdots \lambda_n = \sigma_1^2 \cdots \sigma_n^2$ , so that  $|\det A| = \sigma_1 \cdots \sigma_n$ . For a  $2 \times 2$  matrix:

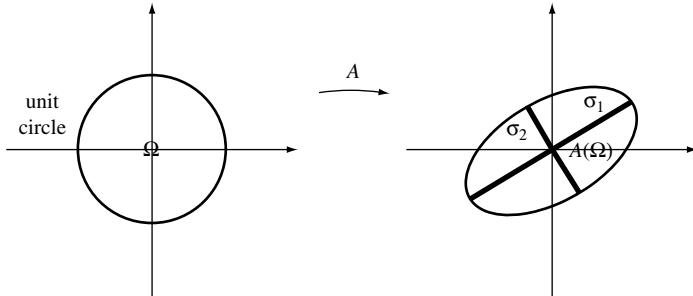


Figure 8.17: for Problem 8.3.28.

$|\det(A)| = \text{expansion factor} = \frac{\text{area of ellipse } A(\Omega)}{\text{area of unit circle } \Omega} = \frac{\pi\sigma_1\sigma_2}{\pi} = \sigma_1\sigma_2$ . See Figure 8.17.

$$8.3.29 A = U\Sigma V^T = [\vec{u}_1 \cdots \vec{u}_r \cdots] \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_r \\ 0 & & & \ddots & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \\ \vdots \end{bmatrix} = [\vec{u}_1 \cdots \vec{u}_r \cdots] \begin{bmatrix} \sigma_1 \vec{v}_1^T \\ \vdots \\ \sigma_r \vec{v}_r^T \\ 0 \end{bmatrix}$$

$$= \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

8.3.30  $\begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = 10 \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \left( \frac{1}{\sqrt{5}} [2 - 1] \right)$   
 $+ 5 \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left( \frac{1}{\sqrt{5}} [1 - 2] \right) = \begin{bmatrix} 4 & -2 \\ -8 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

8.3.31 The formula  $A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$  gives such a representation.

8.3.32  $(SAR)^T SAR = R^T A^T S^T SAR = R^T A^T AR$  is similar to  $A^T A$ , so that the matrices  $A^T A$  and  $(SAR)^T SAR$  have the same eigenvalues. Thus  $A$  and  $SAR$  have the same singular values.

8.3.33 Yes; since  $A^T A$  is diagonalizable and has only 1 as an eigenvalue, we must have  $A^T A = I_n$ .

8.3.34  $A = U\Sigma U^T$  means that  $U^T AU = U^{-1}AU = \Sigma$ , i.e.,  $A$  is orthogonally diagonalizable, with eigenvalues  $\geq 0$ . This is the case if and only if  $A$  is *symmetric* and *positive semidefinite*.

8.3.35 We will freely use the diagram on Page 393 (with  $r = m$ ). We have  $A^T A \vec{v}_i = A^T(\sigma_i \vec{u}_i) = \sigma_i^2 \vec{v}_i$  and therefore  $(A^T A)^{-1} \vec{v}_i = \frac{1}{\sigma_i^2} \vec{v}_i$  for  $i = 1, \dots, m$ . Then  $(A^T A)^{-1} A^T \vec{u}_i = (A^T A)^{-1}(\sigma_i \vec{v}_i) = \frac{1}{\sigma_i} \vec{v}_i$  for  $i = 1, \dots, m$  and  $(A^T A)^{-1} A^T \vec{u}_i = \vec{0}$  for  $i = m+1, \dots, n$  since  $\vec{u}_i$  is in  $\ker(A^T)$  in this case. Note that  $(A^T A)^{-1} A^T \vec{u}_i$  is the least-squares solution of the equation  $A\vec{x} = \vec{u}_i$ ; for  $i = 1, \dots, m$  this is the exact solution since  $\vec{u}_i$  is in  $\text{im}(A)$ .

8.3.36 We will freely use the diagram on Page 393. By construction of the  $\vec{v}_i$  as eigenvectors of  $A^T A$  we have  $A^T A \vec{v}_i = \lambda_i \vec{v}_i = \sigma_i^2 \vec{v}_i$ , or  $(A^T A)^{-1} \vec{v}_i = \frac{1}{\sigma_i^2} \vec{v}_i$ . Then  $A(A^T A)^{-1} A^T \vec{u}_i = A(A^T A)^{-1}(\sigma_i \vec{v}_i) = A\left(\frac{1}{\sigma_i} \vec{v}_i\right) = \frac{1}{\sigma_i} A \vec{v}_i = \vec{u}_i$  for  $i = 1, \dots, m$  and  $A(A^T A)^{-1} A^T \vec{u}_i = \vec{0}$  for  $i = m+1, \dots, n$  since  $\vec{u}_i$  is in  $\ker(A^T)$  in this case. The fact that  $A(A^T A)^{-1} A^T \vec{u}_i = \begin{cases} \vec{u}_i & \text{if } i = 1, \dots, m \\ \vec{0} & \text{if } i = m+1, \dots, n \end{cases}$  means that the matrix  $A(A^T A)^{-1} A^T$  represents the orthogonal projection onto  $\text{im}(A) = \text{span}(\vec{u}_1, \dots, \vec{u}_m)$ .

## True or False

Ch 8.TF.1 F. The orthogonal matrix  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  fails to be diagonalizable (over  $\mathbb{R}$ ).

Ch 8.TF.2 T. If  $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ , then the eigenvalue of  $A^T A = [3 \quad 4] \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [25]$  is  $\lambda = 25$ , so that the singular value of  $A$  is  $\sigma = \sqrt{\lambda} = 5$ .

Ch 8.TF.3 F. The last term,  $5x_2$ , does not have the form required in Definition 8.2.1

Ch 8.TF.4 F. The singular values of  $A$  are *the square roots of* the eigenvalues of  $A^T A$ , by Definition 8.3.1.

Ch 8.TF.5 T, by Theorem 8.2.4.

Ch 8.TF.6 T, by Definition 8.2.1

**Ch 8.TF.7** T. If  $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ , then  $D^T D = D^2 = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix}$ . The eigenvalues of  $D^T D$  are  $\lambda_1^2, \dots, \lambda_n^2$ , and the singular values of  $D$  are  $\sqrt{\lambda_1^2} = |\lambda_1|, \dots, \sqrt{\lambda_n^2} = |\lambda_n|$ .

**Ch 8.TF.8** F, since  $\det \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 3 \end{bmatrix} = -\frac{1}{4} < 0$  (see Theorem 8.2.7).

**Ch 8.TF.9** T, by the spectral theorem (Theorem 8.1.1)

**Ch 8.TF.10** T. Note that  $[1 \ 0] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a > 0$ , by Definition 8.2.3.

**Ch 8.TF.11** T, by Theorem 8.3.2

**Ch 8.TF.12** T. All four eigenvalues are negative, so that their product, the determinant, is positive.

**Ch 8.TF.13** T, by Theorem 8.1.2

**Ch 8.TF.14** F, since the determinant is 0, so that 0 is an eigenvalue.

**Ch 8.TF.15** F. Consider  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

**Ch 8.TF.16** T, since  $AA^T$  is symmetric (use the spectral theorem)

**Ch 8.TF.17** T, by Theorem 8.2.4: all the eigenvalues are positive.

**Ch 8.TF.18** T, since the matrix is symmetric.

**Ch 8.TF.19** F. Consider the shear matrix  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . The unit circle isn't mapped into itself, so that the singular values fail to be 1, 1.

**Ch 8.TF.20** F. In general,  $(A^T A)^T = A^T A \neq AA^T$

**Ch 8.TF.21** F. Let  $A = I_2$ ,  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**Ch 8.TF.22** T. Consider the singular value decomposition  $A = U \sum V^T$ , or  $AV = U \sum$ , where  $V$  is orthogonal (see Theorem 8.3.5). We can let  $S = V$ , since the columns of  $AS = AV = U \sum$  are orthogonal, by construction.

**Ch 8.TF.23** T. By the spectral theorem,  $A$  is diagonalizable:  $S^{-1}AS = D$  for some invertible  $S$  and a diagonal  $D$ . Now  $D^n = S^{-1}A^nS = S^{-1}0S = 0$ , so that  $D = 0$  (since  $D$  is diagonal). Finally,  $A = SDS^{-1} = S0S^{-1} = 0$ , as claimed.

**Ch 8.TF.24** F. If  $k$  is negative, then  $kq(\vec{x})$  will be negative definite.

Ch 8.TF.25 T. The eigenvalues  $\lambda_1, \dots, \lambda_n$  of  $A$  are nonzero, since  $A$  is invertible, so that the eigenvalues  $\lambda_1^2, \dots, \lambda_n^2$  of  $A^2$  are positive. Now use Theorem 8.2.4.

Ch 8.TF.26 T, by Theorem 8.3.2, since  $\vec{v} = A\vec{e}_1$  and  $\vec{w} = A\vec{e}_2$  are the principal semi-axis of the image of the unit circle.

Ch 8.TF.27 F. As a counterexample, consider  $A = S = 2I_n$ .

Ch 8.TF.28 T, since  $\vec{e}_i^T A \vec{e}_i = a_{ii} < 0$ .

Ch 8.TF.29 T. By Theorem 7.3.6, matrices  $A$  and  $B$  have the same eigenvalues. Now use Theorem 8.2.4.

Ch 8.TF.30 T. The spectral theorem guarantees that there is an orthogonal  $R$  such that  $R^T AR$  is diagonal. Now let  $S = R^T$ .

Ch 8.TF.31 F. Let  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ . Then 1 is a singular value of  $BA$  but not of  $AB$ .

Ch 8.TF.32 T, since  $A + A^{-1} = A + A^T$  is symmetric.

Ch 8.TF.33 F. For example,  $(x_1^2)(x_2 x_3)$  fails to be a quadratic form.

Ch 8.TF.34 T. We can write  $q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix} \vec{x}$ .

Ch 8.TF.35 F. Consider  $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , which is indefinite.

Ch 8.TF.36 T, by Definition 8.2.3:  $\vec{x}^T (A + B) \vec{x} = \vec{x}^T A \vec{x} + \vec{x}^T B \vec{x} > 0$  for all nonzero  $\vec{x}$ .

Ch 8.TF.37 T, since  $\vec{x} \cdot A \vec{x}$  is positive, so that  $\cos \theta$  is positive, where  $\theta$  is the angle enclosed by  $\vec{x}$  and  $A \vec{x}$ .

Ch 8.TF.38 T. Preliminary remark: If  $\sigma$  is the largest singular value of an  $n \times m$  matrix  $M$ , then  $\|M\vec{v}\| \leq \sigma \|\vec{v}\|$  for all  $\vec{v}$  in  $\mathbb{R}^m$  (see Exercise 8.3.26). Now let  $\sigma_1, \sigma_2$  be the singular values of matrix  $AB$ , with  $\sigma_1 \geq \sigma_2$ , and let  $\vec{v}_1$  be a unit vector in  $\mathbb{R}^2$  such that  $\|AB\vec{v}_1\| = \sigma_1$  (see Theorem 8.3.3). Now  $\sigma_2 \leq \sigma_1 = \|A(B\vec{v}_1)\| \leq 3\|B\vec{v}_1\| \leq 3 \cdot 5\|\vec{v}_1\| = 15$ , proving our claim; note that we have used the preliminary remark twice.

Ch 8.TF.39 F. Consider  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

Ch 8.TF.40 T. If  $\lambda$  is the smallest eigenvalue of  $A$ , let  $k = 1 - \lambda$ . Then the smallest eigenvalue of  $A + kI_n$  is  $\lambda + k = 1$ , so that all the eigenvalues of  $A + kI_n$  are positive. Now use Theorem 8.2.4.

Ch 8.TF.41 T. The quadratic form  $q(x_1, x_2) = [x_1 \ 0 \ x_2] \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix} = ax_1^2 + 2cx_1x_2 + fx_2^2$  is positive

definite. The matrix of this quadratic form is  $A = \begin{bmatrix} a & c \\ c & f \end{bmatrix}$ , and  $\det(A) = af - c^2 > 0$  since  $A$  is positive definite. Thus  $af > c^2$ , as claimed.

Ch 8.TF.42 F. Consider the positive definite matrix  $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ .

Ch 8.TF.43 F. Consider the indefinite matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

Ch 8.TF.44 T. By Theorem 8.3.2., the continuous function  $f(x) = A \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}$  has the global maximum 5 and the global minimum 3. Note that the image of the unit circle consists of all vectors of the form  $A \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}$ . By the intermediate value theorem,  $f(c) = 4$  for some  $c$ . Let  $\vec{u} = \begin{bmatrix} \cos c \\ \sin c \end{bmatrix}$  (draw a sketch!).

Ch 8.TF.45 T, since  $\vec{x}^T A^2 \vec{x} = -\vec{x}^T A^T A \vec{x} = -(A \vec{x})^T A \vec{x} = -\|A \vec{x}\|^2 \leq 0$  for all  $\vec{x}$ .

Ch 8.TF.46 T. If  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A^T A$ , then  $\lambda_1 \lambda_2 \dots \lambda_n = \det(A^T A) = (\det A)^2$ . If

$\sigma_1 = \sqrt{\lambda_1}, \dots, \sigma_n = \sqrt{\lambda_n}$  are the singular values of  $A$ , then

$\sigma_1 \sigma_2 \dots \sigma_n = \sqrt{\lambda_1 \lambda_2 \dots \lambda_n} = |\det A|$ , as claimed.

Ch 8.TF.47 F. Note that the columns of  $S$  must be unit eigenvectors of  $A$ . There are two distinct real eigenvalues,  $\lambda_1, \lambda_2$ , and for each of them there are two unit eigenvectors,  $\pm \vec{v}_1$  (for  $\lambda_1$ ) and  $\pm \vec{v}_2$  (for  $\lambda_2$ ). (Draw a sketch!) Thus there are 8 matrices  $S$ , namely  $S = [\pm \vec{v}_1 \quad \pm \vec{v}_2]$  and  $S = [\pm \vec{v}_2 \quad \pm \vec{v}_1]$

Ch 8.TF.48 T. See the remark following Definition 8.2.1.

Ch 8.TF.49 F. Some eigenvalues of  $A$  may be negative.

Ch 8.TF.50 F. Consider the similar matrices  $A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 4 \\ 0 & 3 \end{bmatrix}$ . Matrix  $A$  has the singular values 0 and 3, while those of  $B$  are 0 and 5.

Ch 8.TF.51 T. Let  $\vec{v}_1, \vec{v}_2$  be an orthonormal eigenbasis, with  $A \vec{v}_1 = \vec{v}_1$  and  $A \vec{v}_2 = 2 \vec{v}_2$ . Consider a nonzero vector  $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$ ; then  $A \vec{x} = c_1 \vec{v}_1 + 2c_2 \vec{v}_2$ . If  $c_1 = 0$ , then  $\vec{x} = c_2 \vec{v}_2$  and  $A \vec{x} = 2c_2 \vec{v}_2$  are parallel, and we are all set. Now consider the case when  $c_1 \neq 0$ . Then the angle between  $\vec{x}$  and  $A \vec{x}$  is  $\arctan(2c_2/c_1) - \arctan(c_2/c_1)$ ; to see this, subtract the angle between  $\vec{v}_1$  and  $\vec{x}$  from the angle between  $\vec{v}_1$  and  $A \vec{x}$  (draw a sketch). Let  $m = c_2/c_1$  and use calculus to see that the function  $f(m) = \arctan(2m) - \arctan(m)$  assumes its global maximum at  $m = \frac{1}{\sqrt{2}}$ . The maximal angle between  $\vec{x}$  and  $A \vec{x}$  is  $\arctan(\sqrt{2}) - \arctan(1/\sqrt{2}) < 0.34 < \pi/6$ .

Ch 8.TF.52 T. Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . By Theorem 8.3.2,  $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \sqrt{a^2 + c^2} < 5$  (since the length of the semi-major axis of the image of the unit circle is less than 5). Thus  $a < 5$  and  $c < 5$ . Likewise,  $b < 5$  and  $d < 5$ .

Ch 8.TF.53 T. We need to show that each entry  $a_{ij} = a_{ji}$  off the diagonal is smaller than some entry on the diagonal. Now  $(\vec{e}_i - \vec{e}_j)^T A (\vec{e}_i - \vec{e}_j) = a_{ii} + a_{jj} - 2a_{ij} > 0$ , so that  $a_{ii} + a_{jj} > 2a_{ij}$ . Thus the larger of the diagonal entries  $a_{ii}$  and  $a_{jj}$  must exceed  $a_{ij}$ .

Ch 8.TF.54 T. Let  $\lambda_1, \dots, \lambda_m$  be the distinct eigenvalues of  $A$ , with the associated eigenspaces  $E_{A,\lambda_i}$ . Since  $A$  is diagonalizable, we know that  $\sum_{k=1}^m \dim(E_{A,\lambda_i}) = n$ . By definition of an eigenvector,  $E_{A,\lambda_i}$  is a subspace of  $E_{A^3,\lambda_i^3}$ . Since  $\sum_{k=1}^m \dim(E_{A^3,\lambda_i^3})$  cannot exceed  $n$ , we must have  $E_{A,\lambda_i} = E_{A^3,\lambda_i^3}$  for all eigenvalues. Applying the same reasoning to  $B$  and  $B^3$ , we can conclude that  $E_{A,\lambda_i} = E_{A^3,\lambda_i^3} = E_{B^3,\lambda_i^3} = E_{B,\lambda_i}$ . Since the diagonalizable matrices  $A$  and  $B$  have the same eigenvectors with the same eigenvalues, they must be equal.

## Chapter 9

### Section 9.1

9.1.1  $x(t) = 7e^{5t}$ , by Theorem 9.1.1.

9.1.2  $x(t) = -e \cdot e^{-0.71t} = -e^{1-0.71t}$ , by Theorem 9.1.1.

9.1.3  $P(t) = 7e^{0.03t}$ , by Theorem 9.1.1.

9.1.4 This is just an antiderivative problem:  $y(t) = 0.8 \frac{t^2}{2} + C = 0.4t^2 + C$ , and  $C = -0.8$ , so that  $y(t) = 0.4t^2 - 0.8$ .

9.1.5  $y(t) = -0.8e^{0.8t}$ , by Theorem 9.1.1.

9.1.6  $x dx = dt$

$$\frac{x^2}{2} = t + C, \text{ and } \frac{1}{2} = 0 + C, \text{ so that } \frac{x^2}{2} = t + \frac{1}{2}$$

$$x^2 = 2t + 1$$

$$x(t) = \sqrt{2t + 1}$$

9.1.7  $x^{-2} dx = dt$

$$-x^{-1} = t + C$$

$$-\frac{1}{x} = t + C, \text{ and } -1 = 0 + C, \text{ so that}$$

$$-\frac{1}{x} = t - 1$$

$$x(t) = \frac{1}{1-t}; \text{ note that } \lim_{x \rightarrow 1^-} x(t) = \infty.$$

9.1.8  $x^{-1/2} dx = dt$

$$2x^{1/2} = t + C, \text{ and } 2\sqrt{4} = 0 + C, \text{ so that } 2x^{1/2} = t + 4.$$

$$x(t) = \left(\frac{t}{2} + 2\right)^2 \text{ for } t \geq -4.$$

9.1.9  $x^{-k} dx = dt$

$$\frac{1}{1-k}x^{1-k} = t + C, \text{ and } \frac{1}{1-k} = C, \text{ so that}$$

$$\frac{1}{1-k}x^{1-k} = t + \frac{1}{1-k}$$

$$x^{1-k} = (1-k)t + 1$$

$$x(t) = ((1-k)t + 1)^{1/(1-k)}.$$

9.1.10  $\cos x dx = dt$

$$\sin x = t + C, \text{ and } C = 0.$$

$$x(t) = \arcsin(t) \text{ for } |t| < 1.$$

9.1.11  $\frac{dx}{1+x^2} = dt$

$\arctan(x) = t + C$  and  $C = 0$ .

$x(t) = \tan(t)$  for  $|t| < \frac{\pi}{2}$ .

9.1.12 We want  $e^{kt} = 3^t$  or  $e^k = 3$  or  $k = \ln(3) : \frac{dx}{dt} = \ln(3)x$ .

9.1.13 a The debt in millions is  $0.45(1.06)^{212} \approx 104,245$ , or about 100 billion dollars.

b The debt in millions is  $0.45e^{0.06 \cdot 212} \approx 150,466$ , or about 150 billion dollars.

9.1.14 a  $x(t) = e^{-\frac{t}{8270}}$ , by Theorem 9.1.1

If  $T$  is the half-life, then  $e^{-\frac{T}{8270}} = \frac{1}{2}$  or  $-\frac{T}{8270} = \ln(\frac{1}{2})$  or  $T = -8270 \ln(\frac{1}{2}) \approx 5732$ .

The half-life is about 5732 years.

b We want to find  $t$  such that  $e^{-\frac{t}{8270}} = 1 - 0.47 = 0.53$  or  $-\frac{t}{8270} = \ln(0.53)$  or  $t = -8270 \ln(0.53) \approx 5250$ . The Iceman died about 5000 years before A.D. 1991, or about 3000 B.C. The Austrian expert was wrong.

9.1.15 If a quantity  $P$  grows at an instantaneous rate of  $k\%$ , then  $P(t) = P_0 e^{kt/100}$ , by Theorem 9.1.1. The doubling time  $T$  is given by  $P(T) = P_0 e^{kT/100} = 2P_0$ , or  $e^{kT/100} = 2$  or  $kT/100 = \ln 2$  or  $k = \frac{100(\ln 2)}{T} \approx \frac{69}{T}$  since  $\ln 2 \approx 0.69$ .

9.1.16 See Figure 9.1.

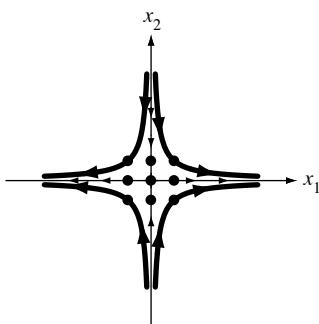


Figure 9.1: for Problem 9.1.16.

9.1.17 See Figure 9.2.

9.1.18 See Figure 9.3.

9.1.19 See Figure 9.4.

9.1.20  $A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ . See Figure 9.5.

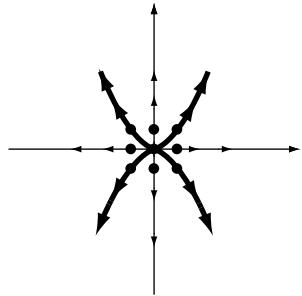


Figure 9.2: for Problem 9.1.17.

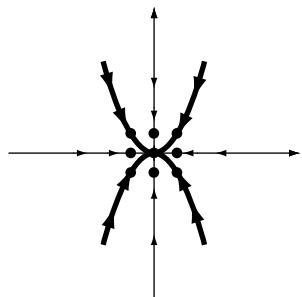


Figure 9.3: for Problem 9.1.18.

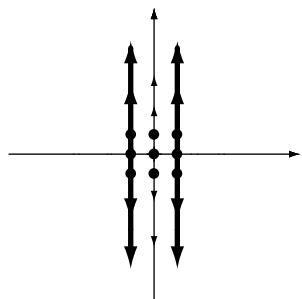


Figure 9.4: for Problem 9.1.19.

It appears that the trajectories will be circles. If we start at  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  we will trace out the unit circle  $\vec{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ .

We can verify that  $\frac{d\vec{x}}{dt} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$  equals  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$ , as claimed.

$$9.1.21 \quad A\vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} \text{ (see Figure 9.6).}$$

The trajectories will be horizontal lines. If we start at  $\begin{bmatrix} p \\ q \end{bmatrix}$ , then the horizontal velocity will be  $q$ , so that  $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} p + qt \\ q \end{bmatrix}$ . We can verify that  $\frac{d\vec{x}}{dt} = \begin{bmatrix} q \\ 0 \end{bmatrix}$  equals  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}(t) = \begin{bmatrix} q \\ 0 \end{bmatrix}$ , as claimed.

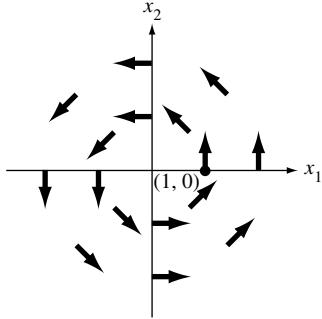


Figure 9.5: for Problem 9.1.20.

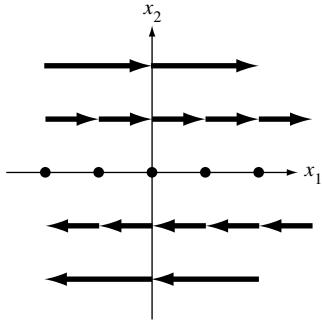


Figure 9.6: for Problem 9.1.21.

**9.1.22** We are told that  $\frac{d\vec{x}_1}{dt} = A\vec{x}_1$  and  $\frac{d\vec{x}_2}{dt} = A\vec{x}_2$ . Let  $\vec{x}(t) = \vec{x}_1(t) + \vec{x}_2(t)$ . Then  $\frac{d\vec{x}}{dt} = \frac{d\vec{x}_1}{dt} + \frac{d\vec{x}_2}{dt} = A\vec{x}_1 + A\vec{x}_2 = A(\vec{x}_1 + \vec{x}_2) = A\vec{x}$ , as claimed.

**9.1.23** We are told that  $\frac{d\vec{x}_1}{dt} = A\vec{x}_1$ . Let  $\vec{x}(t) = k\vec{x}_1(t)$ . Then  $\frac{d\vec{x}}{dt} = \frac{d}{dt}(k\vec{x}_1) = k\frac{d\vec{x}_1}{dt} = kA\vec{x}_1 = A(k\vec{x}_1) = A\vec{x}$ , as claimed.

**9.1.24** We are told that  $\frac{d\vec{x}}{dt} = A\vec{x}$ . Let  $\vec{c}(t) = e^{kt}\vec{x}(t)$ . Then  $\frac{d\vec{c}}{dt} = \frac{d}{dt}(e^{kt}\vec{x}) = (\frac{d}{dt}e^{kt})\vec{x} + e^{kt}\frac{d\vec{x}}{dt} = ke^{kt}\vec{x} + e^{kt}A\vec{x} = (A + kI_n)(e^{kt}\vec{x}) = (A + kI_n)\vec{c}$ , as claimed.

**9.1.25** We are told that  $\frac{d\vec{x}}{dt} = A\vec{x}$ . Let  $\vec{c}(t) = \vec{x}(kt)$ . Using the chain rule we find that  $\frac{d\vec{c}}{dt} = \frac{d}{dt}(\vec{x}(kt)) = k\frac{d\vec{x}}{dt}|_{kt} = kA(\vec{x}(kt)) = kA\vec{c}(t)$ , as claimed.

To get the vector field  $kA\vec{c}$  we scale the vectors of the field  $A\vec{x}$  by  $k$ .

**9.1.26**  $\lambda_1 = 3, \lambda_2 = -2; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, c_1 = 5, c_2 = -1$ , so that  $\vec{x}(t) = 5e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-2t} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ .

**9.1.27** Use Theorem 9.1.3.

The eigenvalues of  $A = \begin{bmatrix} -4 & 3 \\ 2 & -3 \end{bmatrix}$  are  $\lambda_1 = -6$  and  $\lambda_2 = -1$ , with associated eigenvectors  $\vec{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . The coordinates of  $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  with respect to  $\vec{v}_1$  and  $\vec{v}_2$  are  $c_1 = -\frac{1}{5}$  and  $c_2 = \frac{2}{5}$ .

By Theorem 9.1.3 the solution is  $\vec{x}(t) = -\frac{1}{5}e^{-6t} \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{2}{5}e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**9.1.28**  $\lambda_1 = 2, \lambda_2 = 10; \vec{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; c_1 = -\frac{1}{8}, c_2 = \frac{5}{8}$ , so that  $\vec{x}(t) = -\frac{1}{8}e^{2t} \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{5}{8}e^{10t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

**9.1.29**  $\lambda_1 = 0, \lambda_2 = 5; \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; c_1 = -2, c_2 = 1$ ,

so that  $\vec{x}(t) = -2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} + e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

**9.1.30**  $\lambda_1 = 0, \lambda_2 = 5, \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; c_1 = -1, c_2 = 0$ , so that  $\vec{x}(t) = - \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ .

**9.1.31**  $\lambda_1 = 1, \lambda_2 = 6, \lambda_3 = 0; \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ . Since  $\vec{x}(0) = \vec{v}_1$  we need not find  $\vec{v}_2$  and  $\vec{v}_3$ .

$c_1 = 1, c_2 = c_3 = 0$ , so that  $\vec{x}(t) = e^t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

In Exercises 32 to 35, find the eigenvalues and eigenspaces. Then determine the direction of the flow along the eigenspaces (outward if  $\lambda > 0$  and inward if  $\lambda < 0$ ). Use Figure 11 of Section 9.1 as a guide to sketch the other trajectories.

**9.1.32** See Exercise 26 and Figure 9.7.

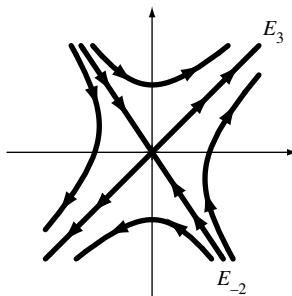


Figure 9.7: for Problem 9.1.32.

**9.1.33** See Exercise 27 and Figure 9.8.

**9.1.34** See Exercise 28 and Figure 9.9.

**9.1.35** See Exercise 29 and Figure 9.10.

In Exercises 36 to 39, find the eigenvalues and eigenspaces (the eigenvalues will always be positive). Then determine the direction of the flow along the eigenspaces (outward if  $\lambda > 1$  and inward if  $1 > \lambda > 0$ ). Use Figure 11 of Section 7.1 as a guide to sketch the other trajectories.

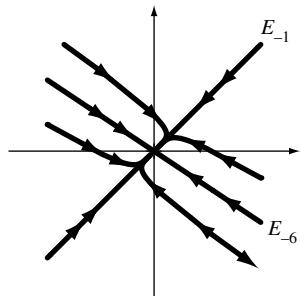


Figure 9.8: for Problem 9.1.33.

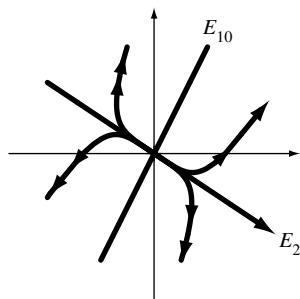


Figure 9.9: for Problem 9.1.34.

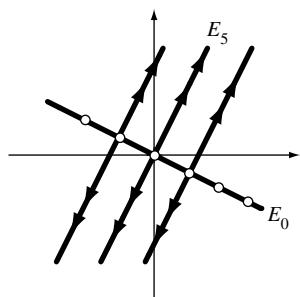


Figure 9.10: for Problem 9.1.35.

**9.1.36** See Figure 9.11.

**9.1.37** See Figure 9.12.

**9.1.38** See Figure 9.13.

**9.1.39** See Figure 9.14.

**9.1.40**  $\vec{x}(t) = e^{2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + e^{3t} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

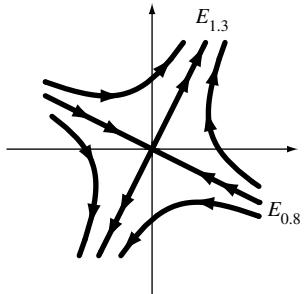


Figure 9.11: for Problem 9.1.36.

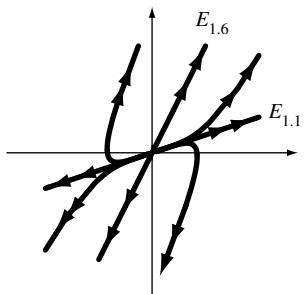


Figure 9.12: for Problem 9.1.37.

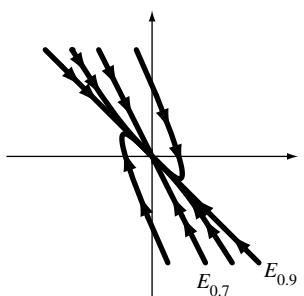


Figure 9.13: for Problem 9.1.38.

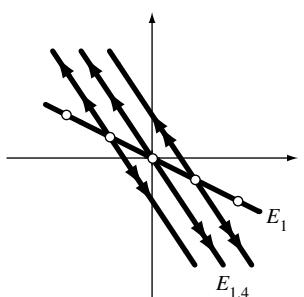


Figure 9.14: for Problem 9.1.39.

We want a  $2 \times 2$  matrix  $A$  with eigenvalues  $\lambda_1 = 2$  and  $\lambda_2 = 3$  and associated eigenvectors  $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ ; that is  $A \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 6 & 12 \end{bmatrix}$  or  $A = \begin{bmatrix} 4 & 9 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 9 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 11 & -6 \\ 12 & -6 \end{bmatrix}$ .

**9.1.41** The trajectories are of the form  $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = c_1 \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$ . See Figure 9.15.

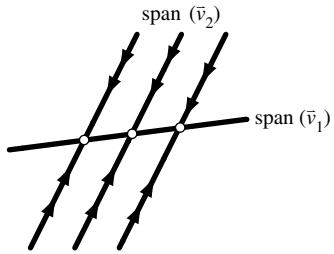


Figure 9.15: for Problem 9.1.41.

**9.1.42 a** The term  $0.8x$  in the second equation indicates that species  $y$  is helped by  $x$ , while species  $x$  is hindered by  $y$  (consider the term  $-1.2y$  in the first equation). Thus  $y$  preys on  $x$ .

b See Figure 9.16.

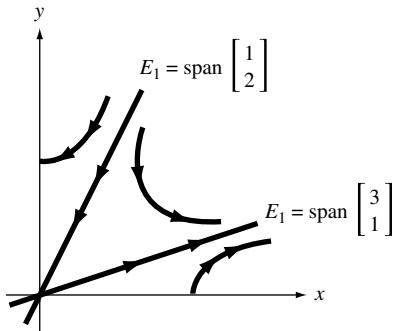


Figure 9.16: for Problem 9.1.42b.

c If  $\frac{y(0)}{x(0)} < 2$  then both species will prosper, and  $\lim_{t \rightarrow \infty} \frac{y(t)}{x(t)} = \frac{1}{3}$ .

If  $\frac{y(0)}{x(0)} \geq 2$  then both species will die out.

**9.1.43 a** These two species are *competing* as each is hindered by the other (consider the terms  $-y$  and  $-2x$ ).

b Although only the first quadrant is relevant for our model, it is useful to consider the phase portrait in the other quadrants as well. See Figure 9.17.

c If  $\frac{y(0)}{x(0)} > 2$  then species  $y$  wins ( $x$  will die out); if  $\frac{y(0)}{x(0)} < 2$  then  $x$  wins. If  $\frac{y(0)}{x(0)} = 2$  then both will prosper and  $\frac{y(t)}{x(t)} = 2$  for all  $t$ .

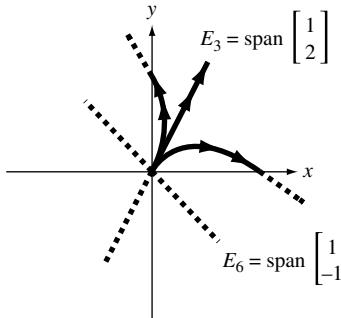


Figure 9.17: for Problem 9.1.43b.

**9.1.44** a The two species are in symbiosis: Each is helped by the other (consider the terms  $4y$  and  $2x$ ).

b See Figure 9.18.

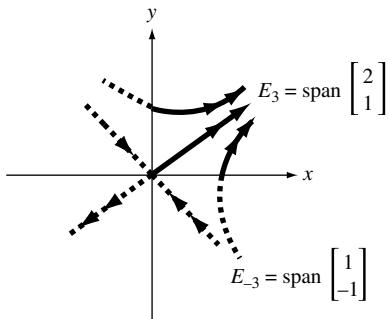


Figure 9.18: for Problem 9.1.44b.

c Both populations will prosper and  $\lim_{t \rightarrow \infty} \frac{y(t)}{x(t)} = \frac{1}{2}$ , regardless of the initial populations.

**9.1.45** a Species  $y$  has the more vicious fighters, since they kill members of species  $x$  at a rate of 4 per time unit, while the fighters of species  $x$  only kill at a rate of 1.

b See Figure 9.19.

c If  $\frac{y(0)}{x(0)} < \frac{1}{2}$  then  $x$  wins; if  $\frac{y(0)}{x(0)} > \frac{1}{2}$  then  $y$  wins; if  $\frac{y(0)}{x(0)} = \frac{1}{2}$  nobody will survive the battle.

**9.1.46** Look at the *phase portrait* in Figure 9.20.

**9.1.47** a The two species are in symbiosis: Each is helped by the other (consider the positive terms  $kx$  and  $ky$ ).

$$b \quad \lambda_{1,2} = \frac{-5 \pm \sqrt{9+4k^2}}{2}$$

Both eigenvalues are negative if  $\sqrt{9+4k^2} < 5$  or  $9+4k^2 < 25$  or  $4k^2 < 16$  or  $k < 2$  (recall that  $k$  is positive).

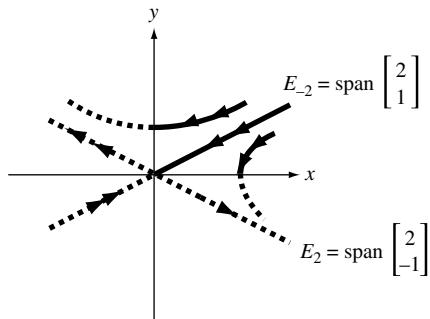


Figure 9.19: for Problem 9.1.45b.

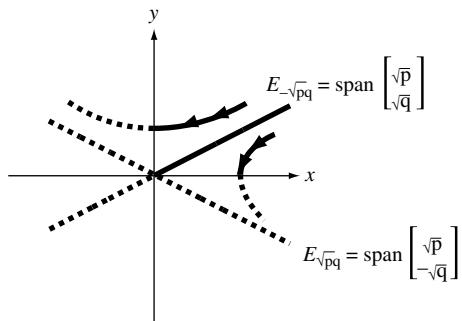


Figure 9.20: for Problem 9.1.46.

If  $k = 2$  then the eigenvalues are  $-5$  and  $0$ .

If  $k > 2$  then there is a positive and a negative eigenvalue.

c See Figure 9.21.

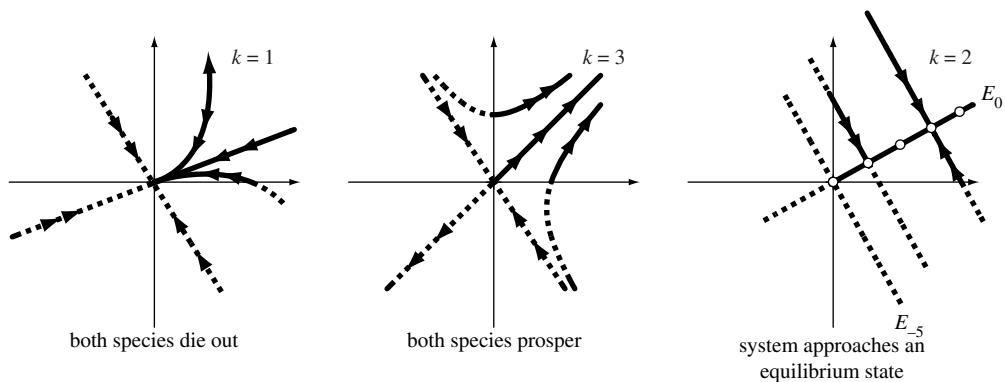


Figure 9.21: for Problem 9.1.47c.

#### 9.1.48 a Symbiosis

b  $\lambda_{1,2} = \frac{-5 \pm \sqrt{9+4k}}{2}$

Both eigenvalues are negative if  $\sqrt{9+4k} < 5$  or  $9+4k < 25$  or  $4k < 16$  or  $k < 4$ .

If  $k = 4$  then the eigenvalues are  $-5$  and  $0$ .

If  $k > 4$  then there is a positive and a negative eigenvalue.

c  $k = 1$ : See corresponding figure in Exercise 47 and Figure 9.22.

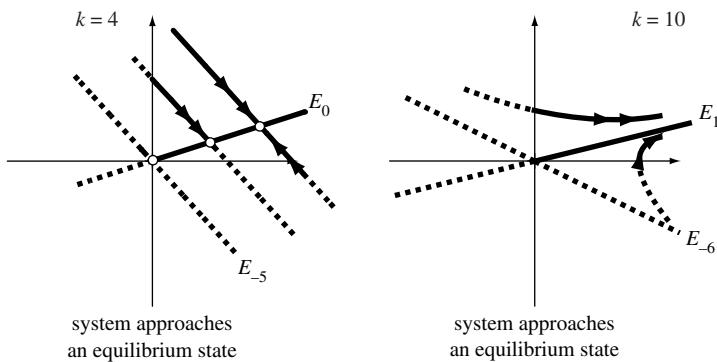


Figure 9.22: for Problem 9.1.48c.

9.1.49  $A = \begin{bmatrix} -1 & -0.2 \\ 0.6 & -0.2 \end{bmatrix}, \lambda_1 = -0.4, \lambda_2 = -0.8$

$$E_{-0.4} = \text{span} \begin{bmatrix} -1 \\ 3 \end{bmatrix}, E_{-0.8} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} g(0) \\ h(0) \end{bmatrix} = 15 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + 45 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{ so that } c_1 = 15, c_2 = 45.$$

$$\begin{bmatrix} g(t) \\ h(t) \end{bmatrix} = 15e^{-0.4t} \begin{bmatrix} -1 \\ 3 \end{bmatrix} + 45e^{-0.8t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{ so that}$$

$$g(t) = -15e^{-0.4t} + 45e^{-0.8t}$$

$$h(t) = 45e^{-0.4t} - 45e^{-0.8t}. \text{ See Figure 9.23.}$$

9.1.50 We want both eigenvalues  $\lambda_1$  and  $\lambda_2$  to be negative, so that  $\text{tr}(A) = \lambda_1 + \lambda_2 < 0$  and  $\det(A) = \lambda_1 \lambda_2 > 0$ .

Conversely, if  $\text{tr}(A) < 0$  and  $\det(A) > 0$ , then the two eigenvalues  $\lambda_{1,2} = \frac{\text{tr}(A) \pm \sqrt{(\text{tr}A)^2 - 4\det(A)}}{2}$  are both negative.

Answer:  $\text{tr}(A) < 0$  and  $\det(A) > 0$ .

9.1.51  $i$ th component of  $\frac{d}{dt}(S\vec{x}) = \frac{d}{dt}(s_{i1}x_1(t) + s_{i2}x_2(t) + \dots + s_{in}x_n(t))$

$$= s_{i1} \frac{dx_1}{dt} + s_{i2} \frac{dx_2}{dt} + \dots + s_{in} \frac{dx_n}{dt}$$

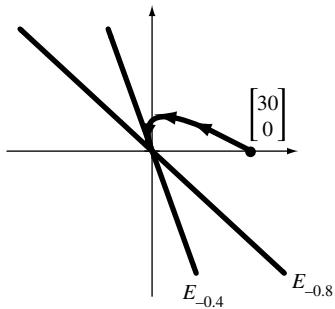


Figure 9.23: for Problem 9.1.49.

$= i$ th component of  $S \frac{d\vec{x}}{dt}$

**9.1.52** The solutions of  $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$  are of the form  $\begin{bmatrix} p + qt \\ q \end{bmatrix}$ , where  $\vec{x}(0) = \begin{bmatrix} p \\ q \end{bmatrix}$ , by Exercise 21. Since  $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \lambda I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , the solutions of the given system are of the form  $\vec{x}(t) = e^{\lambda t} \begin{bmatrix} p + qt \\ q \end{bmatrix}$ , by Exercise 24. The zero state is a stable equilibrium solution if and only if  $\lambda < 0$ . The case  $\lambda = 0$  is discussed in Exercise 21. See Figure 9.24.

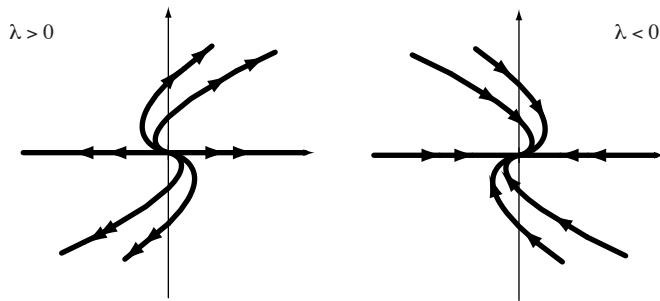


Figure 9.24: for Problem 9.1.52.

**9.1.53** For the initial value  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , the system  $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$  has the solution  $\vec{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ , by Exercise 20; the system  $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & -q \\ q & 0 \end{bmatrix} \vec{x}$  has the solution  $\vec{x}(t) = \begin{bmatrix} \cos(qt) \\ \sin(qt) \end{bmatrix}$ , by Exercise 25; and the system  $\frac{d\vec{x}}{dt} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} \vec{x}$  has the solution  $\vec{x}(t) = e^{pt} \begin{bmatrix} \cos(qt) \\ \sin(qt) \end{bmatrix}$ , by Exercise 24 (write  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix} = pI_2 + \begin{bmatrix} 0 & -q \\ q & 0 \end{bmatrix}$ ). See Figure 9.25.

**9.1.54**  $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ ,  $\lambda_1 = -1$ ,  $\lambda_2 = -2$ ;  $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ . See Figure 9.26.

In the case of trajectory 3 the door will slam: Initially the door is opened just a little ( $\theta$  is small) and given a strong push to close it ( $\omega$  is large negative). More generally, the door will slam if the point  $\begin{bmatrix} \theta(0) \\ \omega(0) \end{bmatrix}$  representing

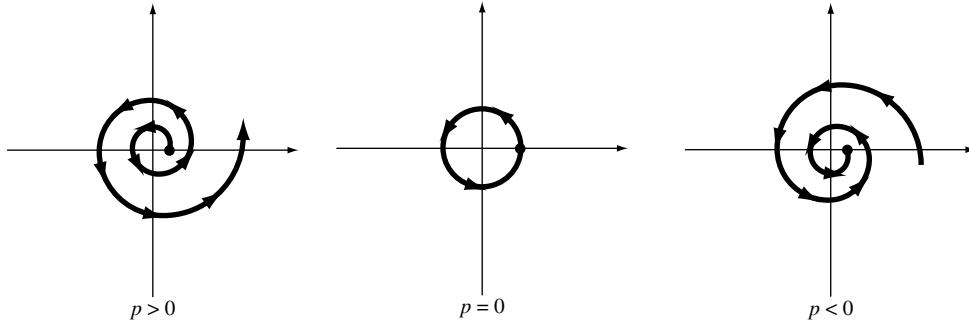


Figure 9.25: for Problem 9.1.53.

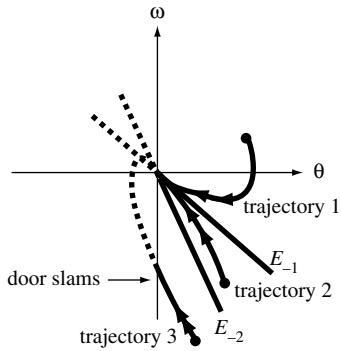


Figure 9.26: for Problem 9.1.54.

the initial state is located below the line  $E_{-2} = \text{span} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ , that is, if  $\frac{\omega(0)}{\theta(0)} < -2$ .

**9.1.55**  $A = \begin{bmatrix} 0 & 1 \\ -p & -q \end{bmatrix}$ ,  $\lambda_{1,2} = \frac{1}{2}(-q \pm \sqrt{q^2 - 4p})$ ; note that both eigenvalues are negative.

$$E_{\lambda_1} = \text{span} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \text{ and } E_{\lambda_2} = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

See Figure 9.27. In the case of trajectory 3 the door will slam: Initially the door is opened just a little ( $\theta$  is small) and given a strong push to close it ( $\omega$  is large negative). More generally, the door will slam if the point  $\begin{bmatrix} \theta(0) \\ \omega(0) \end{bmatrix}$  representing the initial state is located below the line  $E_{\lambda_2} = \text{span} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$ , that is, if  $\frac{\omega(0)}{\theta(0)} < \lambda_2$ .

## Section 9.2

**9.2.1** By Euler's formula (Theorem 9.2.2),  $e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1$ .

**9.2.2** By Euler's formula (Theorem 9.2.2),  $e^{\frac{1}{2}\pi i} = \cos(\frac{\pi}{2}) + i \sin(\frac{\pi}{2}) = i$ .

**9.2.3**  $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

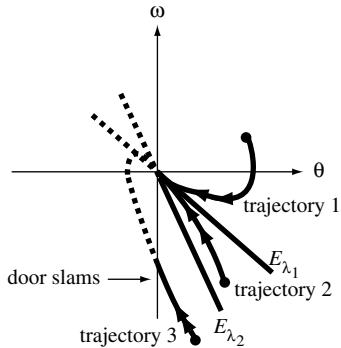


Figure 9.27: for Problem 9.1.55.

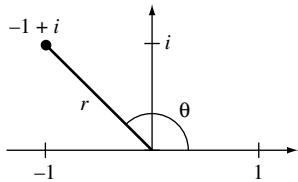


Figure 9.28: for Problem 9.2.3.

$\theta = \frac{3\pi}{4}$ , so that  $z = \sqrt{2}e^{\frac{3}{4}\pi i}$ . See Figure 9.28.

$$9.2.4 \quad e^{3it} = \cos(3t) + i \sin(3t)$$

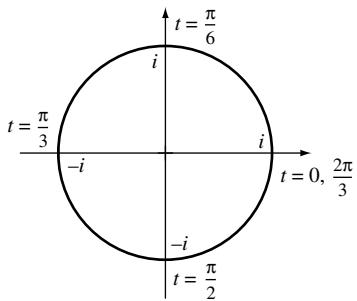


Figure 9.29: for Problem 9.2.4.

Period  $T$  given by  $3T = 2\pi$  or  $T = \frac{2\pi}{3}$ . See Figure 9.29.

$$9.2.5 \quad e^{-0.1t-2it} = e^{-0.1t}e^{-2it} = e^{-0.1t}(\cos(2t) - i \sin(2t)) \text{ spirals inward, in clockwise direction. See Figure 9.30.}$$

$$9.2.6 \quad \lambda_{1,2} = \pm i; E_i = \ker \begin{bmatrix} 3-i & -2 \\ 5 & -3-i \end{bmatrix} = \text{span} \begin{bmatrix} 3+i \\ 5 \end{bmatrix} \text{ and } E_{-i} = \text{span} \begin{bmatrix} 3-i \\ 5 \end{bmatrix}.$$

General solution:

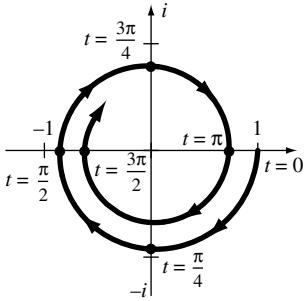


Figure 9.30: for Problem 9.2.5.

$$\vec{x}(t) = c_1 e^{it} \begin{bmatrix} 3+i \\ 5 \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 3-i \\ 5 \end{bmatrix}$$

$$\text{If } c_1 = c_2 = 1 \text{ then } \vec{x}(t) = (\cos(t) + i \sin(t)) \begin{bmatrix} 3+i \\ 5 \end{bmatrix} + (\cos(t) - i \sin(t)) \begin{bmatrix} 3-i \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \cos(t) - 2 \sin(t) \\ 10 \cos(t) \end{bmatrix}.$$

9.2.7  $\det(A) = -2$ , so that the zero state is not stable, by Theorem 9.2.5.

9.2.8 Recall that all eigenvalues of a symmetric matrix are real. The zero state is stable if (and only if) all eigenvalues are negative (by Theorem 9.2.4); this is the case if (and only if) the matrix is negative definite (by Theorem 8.2.2).

9.2.9 The eigenvalues are conjugate complex,  $\lambda_{1,2} = p \pm iq$ , and  $\text{tr}(A) = 2p < 0$ , so that  $p$  is negative. By Theorem 9.2.4, the zero state is stable.

9.2.10 a The matrix of the system is  $\begin{bmatrix} 2a & 2b \\ 2b & 2c \end{bmatrix}$ , which is  $2A$ , so that  $\frac{d\vec{x}}{dt} = \text{grad}(q) = 2A\vec{x}$ .

b By Theorem 8.2.7, the level curves are ellipses. From multivariable calculus we know that  $\text{grad}(q)$  is perpendicular to the level curve, pointing inwards, so that all trajectories will approach the origin: The zero state is stable. See Figure 9.31.

c By Theorem 8.2.7, the level curves are hyperbolas, as shown in Figure 9.32.

$$\lambda_1 > 0 > \lambda_2$$

d The zero state is a stable equilibrium solution of the system  $\frac{d\vec{x}}{dt} = \text{grad}(q) = 2A\vec{x}$  if (and only if) the eigenvalues of  $2A$  (and  $A$ ) are negative. This means that the quadratic form  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$  is negative definite.

9.2.11 a  $q(\vec{x}) = 2a_{11}x_1x_1 + 2a_{12}x_1x_2 + \cdots + a_{ii}x_i^2 + \cdots + 2a_{in}x_ix_n + \text{terms not involving } x_i$ , so that  $\frac{\partial q}{\partial x_i} = 2a_{11}x_1 + 2a_{12}x_2 + \cdots + 2a_{ii}x_i + \cdots + 2a_{in}x_n$  and  $\frac{d\vec{x}}{dt} = \text{grad}(q) = 2A\vec{x}$ .

The matrix of the system is  $B = 2A$ .

b The zero state is a stable equilibrium solution of the system  $\frac{d\vec{x}}{dt} = \text{grad}(q) = 2A\vec{x}$  if (and only if) all the eigenvalues of  $2A$  are negative. This means that the quadratic form  $q(\vec{x}) = \vec{x} \cdot A\vec{x}$  is negative definite.

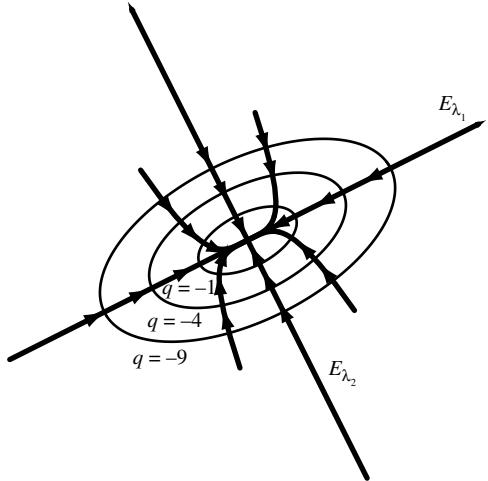


Figure 9.31: for Problem 9.2.10b.

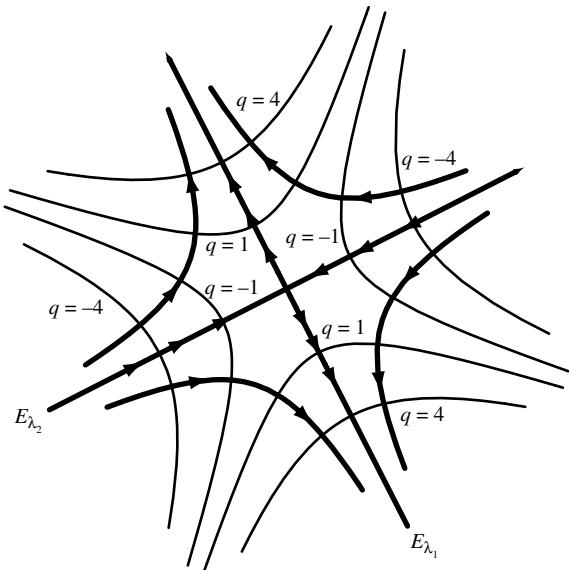


Figure 9.32: for Problem 9.2.10c.

**9.2.12** We will show that the real parts of all the eigenvalues are negative, so that the zero state is a stable equilibrium solution. Now the characteristic polynomial of  $A$  is  $f_A(\lambda) = -\lambda^3 - 2\lambda^2 - \lambda - 1$ . It is convenient to get rid of all these minus signs: The eigenvalues are the solutions of the equation  $g(\lambda) = \lambda^3 + 2\lambda^2 + \lambda + 1 = 0$ . Since  $g(-1) = 1$  and  $g(-2) = -1$ , there will be an eigenvalue  $\lambda_1$  between  $-2$  and  $-1$ . Using calculus (or a graphing calculator), we see that the equation  $g(\lambda) = 0$  has no other real solutions. Thus there must be two complex conjugate eigenvalues  $p \pm iq$ . Now the sum of the eigenvalues is  $\lambda_1 + 2p = \text{tr}(A) = -2$ , and  $p = \frac{-2-\lambda_1}{2}$  will be negative, as claimed. The graph of  $g(\lambda)$  is shown in Figure 9.33.

**9.2.13** Recall that the zero state is stable if (and only if) the real parts of all eigenvalues are negative. Now the eigenvalues of  $A^{-1}$  are the reciprocals of those of  $A$ ; the real parts have the same sign (if  $\lambda = p + iq$ , then  $\frac{1}{\lambda} = \frac{1}{p+iq} = \frac{p-iq}{p^2+q^2}$ ).

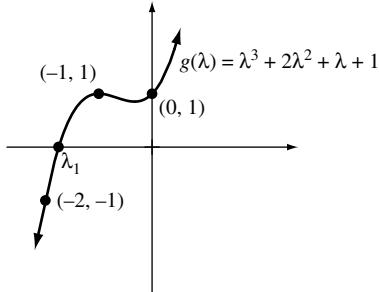


Figure 9.33: for Problem 9.2.12.

**9.2.14 a** For  $i > 1$ ,  $\frac{dx_i}{dt} = -k_i x_i + x_{i-1}$ . This means that in the absence of quantity  $x_{i-1}(t)$ , the quantity  $x_i(t)$  will decay exponentially, but the presence of  $x_{i-1}$  helps  $x_i$  to grow.

For  $i = 1$ , the beginning of the loop,  $\frac{dx_1}{dt} = -k_1 x_1 - b x_n$ , so that the presence of  $x_n$  contributes to the decrease of  $x_1$ .

b If  $n = 2$  then the matrix of the system is  $A = \begin{bmatrix} -k_1 & -b \\ 1 & -k_2 \end{bmatrix}$  with  $\text{tr}(A) = -k_1 - k_2 < 0$  and  $\det(A) = k_1 k_2 + b > 0$ , so that the zero state is stable, by Theorem 9.2.5.

c No; consider the case  $k_1 = k_2 = k_3 = 1$  for simplicity. Then the matrix of the system is  $A = \begin{bmatrix} -1 & 0 & -b \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$  and  $f_A(\lambda) = -(\lambda + 1)^3 - b$ . Matrix  $A$  has the real eigenvalue  $\lambda_1 = -\sqrt[3]{b} - 1$  and the complex eigenvalues  $\lambda_{2,3} = \sqrt[3]{b} e^{\pm \pi i / 3} - 1$ . (Compare with Exercise 7.5.5.) The real part of  $\lambda_2$  and  $\lambda_3$  is  $(\sqrt[3]{b}/2) - 1$ , which is positive if  $b$  exceeds 8. The zero state fails to be stable in that case.

**9.2.15** The eigenvalues are  $\lambda_1 = \text{tr}(A) > 0$  and  $\lambda_2 = 0$ . See Figure 9.34.

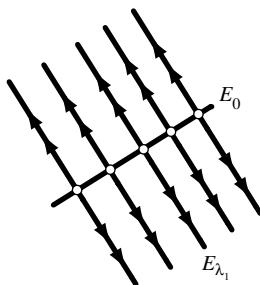


Figure 9.34: for Problem 9.2.15.

**9.2.16** If  $A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$  then  $\text{tr}(A) = b$  and  $\det(A) = -a$ . By Theorem 9.2.5, the zero state is stable if  $a$  and  $b$  are both negative.

**9.2.17** If  $A = \begin{bmatrix} -1 & k \\ k & -1 \end{bmatrix}$  then  $\text{tr}(A) = -2$  and  $\det(A) = 1 - k^2$ . By Theorem 9.2.5, the zero state is stable if  $\det(A) = 1 - k^2 > 0$ , that is, if  $|k| < 1$ .

**9.2.18** If  $\lambda_1, \lambda_2, \lambda_3$  are real and negative, then  $\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 < 0$  and  $\det(A) = \lambda_1 \lambda_2 \lambda_3 < 0$ . If  $\lambda_1$  is real and negative and  $\lambda_{2,3} = p \pm iq$ , where  $p$  is negative, then  $\text{tr}(A) = \lambda_1 + 2p < 0$  and  $\det(A) = \lambda_1(p^2 + q^2) < 0$ . Either way, both trace and determinant are negative.

**9.2.19** False, consider  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ .

**9.2.20** Use Theorem 9.2.6, with  $p = 0, q = \pi; a = 1, b = 0$ .

$$\vec{x}(t) = [\vec{w} \quad \vec{v}] \begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (\cos(\pi t))\vec{w} + (\sin(\pi t))\vec{v}. \text{ See Figure 9.35.}$$

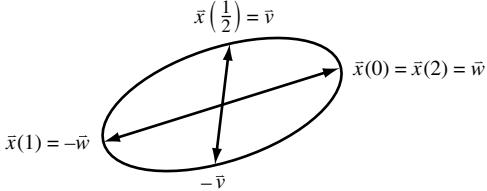


Figure 9.35: for Problem 9.2.20.

**9.2.21 a**  $\begin{bmatrix} \frac{db}{dt} = 0.05b + s \\ \frac{ds}{dt} = 0.07s \end{bmatrix}$  and  $\begin{bmatrix} b(0) \\ s(0) \end{bmatrix} = \begin{bmatrix} 1,000 \\ 1,000 \end{bmatrix}$

b  $\lambda_1 = 0.07, \lambda_2 = 0.05; \vec{v}_1 = \begin{bmatrix} 50 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \vec{x}(0) = 1,000\vec{v}_1 - 49,000\vec{v}_2$ ; so that

$$b(t) = 50,000e^{0.07t} - 49,000e^{0.05t} \text{ and } s(t) = 1,000e^{0.07t}.$$

**9.2.22**  $\lambda_1 = 3, \lambda_2 = 0.5; E_3 = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \text{span} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

System is discrete so choose VII.

**9.2.23**  $\lambda_{1,2} = -\frac{1}{2} \pm i, r > 1$ , so that trajectory spirals outwards. Choose II.

**9.2.24**  $\lambda_1 = 3, \lambda_2 = 0.5, E_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

System is continuous, so choose I.

**9.2.25**  $\lambda_{1,2} = -\frac{1}{2} \pm i$ ; real part is negative so that trajectories spiral inwards in the counterclockwise direction (if  $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  then  $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1.5 \\ 2 \end{bmatrix}$ ). Choose IV.

**9.2.26**  $\lambda_1 = 1, \lambda_2 = -2; E_1 = \text{span} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_{-2} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

System is continuous so choose V.

**9.2.27**  $\lambda_{1,2} = \pm 3i, E_{3i} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$ , so that  $p = 0, q = 3, \vec{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

Now use Theorem 9.2.6:

$$\vec{x}(t) = e^{0t} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sin(3t) & \cos(3t) \\ -\cos(3t) & \sin(3t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

**9.2.28**  $\lambda_{1,2} = \pm 6i, E_{6i} = \text{span} \left( \begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right)$ , so that

$$\vec{x}(t) = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \cos(6t) & -\sin(6t) \\ \sin(6t) & \cos(6t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2\sin(6t) & 2\cos(6t) \\ 3\cos(6t) & -3\sin(6t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

**9.2.29**  $\lambda_{1,2} = 2 \pm 4i, E_{2+4i} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ , so that

$$\vec{x}(t) = e^{2t} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(4t) & -\sin(4t) \\ \sin(4t) & \cos(4t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = e^{2t} \begin{bmatrix} \sin(4t) & \cos(4t) \\ \cos(4t) & -\sin(4t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

**9.2.30**  $\lambda_{1,2} = -2 \pm 3i, E_{-2+3i} = \text{span} \left( \begin{bmatrix} 5 \\ 3 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ , so that

$$\vec{x}(t) = e^{-2t} \begin{bmatrix} 0 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = e^{-2t} \begin{bmatrix} 5\sin(3t) & 5\cos(3t) \\ \cos(3t) + 3\sin(3t) & -\sin(3t) + 3\cos(3t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

**9.2.31**  $\lambda_{1,2} = -1 \pm 2i, E_{-1+2i} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$ , so that  $p = -1, q = 2, \vec{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Now  $\vec{x}(0) = \vec{w} + \vec{v}$ , so that  $a = 1$  and  $b = 1$ .

$$\text{Then } \vec{x}(t) = e^{-t} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} \sin(2t) + \cos(2t) \\ \sin(2t) - \cos(2t) \end{bmatrix}.$$

See Figure 9.36.

**9.2.32**  $\lambda_{1,2} = \pm 2i, E_{2i} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right), \vec{x}(0) = 0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , so that  $a = 0$  and  $b = 1$ .

$$\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}. \text{ See Figure 9.37.}$$

**9.2.33**  $\lambda_{1,2} = \pm i, E_i = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

$$a = 1, b = 0, \text{ so that } \vec{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos(t) & -\cos(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \sin(t) + \cos(t) \end{bmatrix}$$

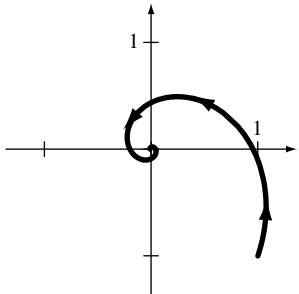


Figure 9.36: for Problem 9.2.31.

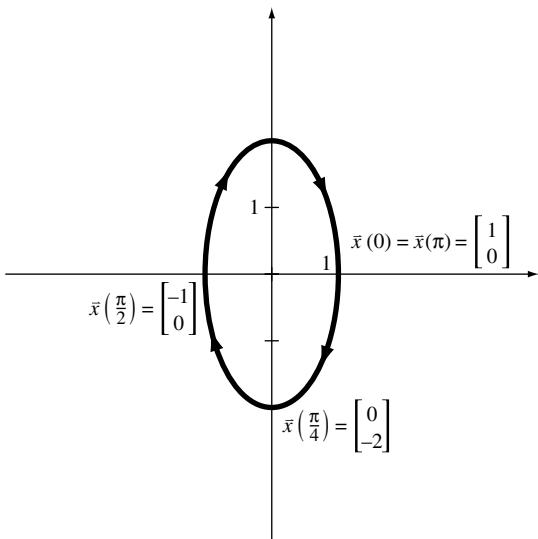


Figure 9.37: for Problem 9.2.32.

$$= \cos(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \text{ See Figure 9.38.}$$

9.2.34  $\lambda_{1,2} = 1 \pm 2i$ ,  $E_{1+2i} = \text{span} \left( \begin{bmatrix} 3 \\ -2 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

$$a = 1, b = 0, \text{ so that } \vec{x}(t) = e^t \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^t \begin{bmatrix} \cos(2t) + 3\sin(2t) \\ -2\sin(2t) \end{bmatrix}. \text{ See Figure 9.39.}$$

9.2.35 If  $z = f + ig$  and  $w = p + iq$  then  $zw = (fp - gq) + i(fq + gp)$ , so that  $(zw)' = (f'p + fp' - g'q - gq') + i(f'q + fq' + g'p + gp')$ .

Also  $z'w = (f' + ig')(p + iq) = (f'p - g'q) + i(f'q + g'p)$  and  $zw' = (f + ig)(p' + iq') = (fp' - gq') + i(gp' + fq')$ . We can see that  $(zw)' = z'w + zw'$ , as claimed.

9.2.36  $A = \begin{bmatrix} 0 & 1 \\ -b & -c \end{bmatrix}$  and  $f_A(\lambda) = \lambda^2 + c\lambda + b$ , with eigenvalues  $\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4b}}{2}$ .

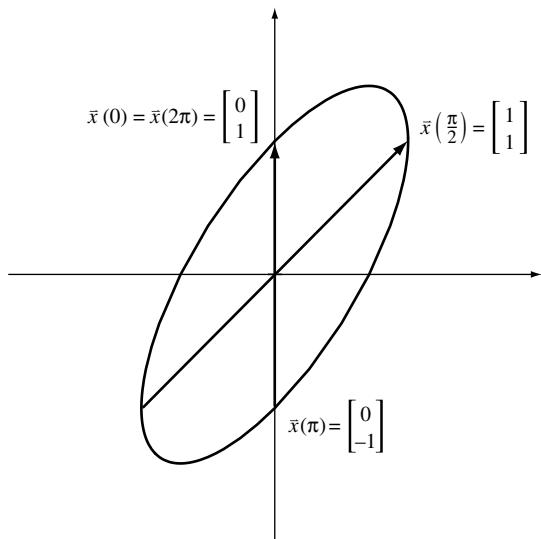


Figure 9.38: for Problem 9.2.33.

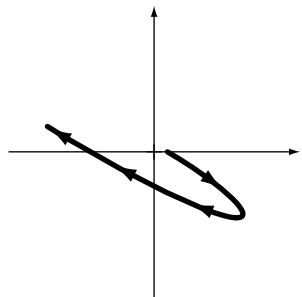


Figure 9.39: for Problem 9.2.34.

**9.2.36 a** If  $c = 0$  then  $\lambda_{1,2} = \pm i\sqrt{b}$ . The trajectories are ellipses. See Figure 9.40.

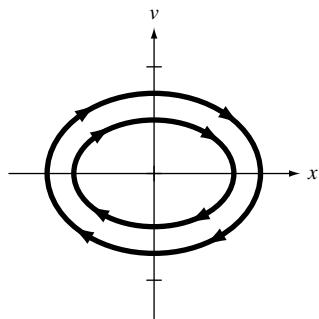


Figure 9.40: for Problem 9.2.36a.

The block *oscillates harmonically*, with period  $\frac{2\pi}{\sqrt{b}}$ . The zero state fails to be asymptotically stable.

b  $\lambda_{1,2} = \frac{-c \pm i\sqrt{4b-c^2}}{2}$

The trajectories spiral inwards, since  $\operatorname{Re}(\lambda_1) = \operatorname{Re}(\lambda_2) = -\frac{c}{2} < 0$ . This is the case of a *damped oscillation*. The zero state is asymptotically stable. See Figure 9.41.

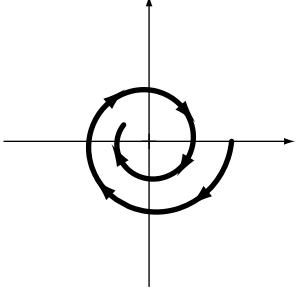


Figure 9.41: for Problem 9.2.36b.

c This case is discussed in Exercise 9.1.55. The zero state is stable here.

**9.2.37 a**  $\frac{1}{z(t)}$  is differentiable when  $z(t) \neq 0$ , since both the real and the imaginary parts are differentiable (if  $z = p + iq$  then  $\frac{1}{z} = \frac{p-iq}{p^2+q^2}$ ). To find  $(\frac{1}{z})'$ , apply the product rule to the equation  $z(\frac{1}{z}) = 1$ :  $z'(\frac{1}{z}) + z(\frac{1}{z})' = 0$ , so that  $(\frac{1}{z})' = -\frac{z'}{z^2}$ .

b  $(\frac{z}{w})' = (z\frac{1}{w})' = z'\frac{1}{w} + z(\frac{1}{w})' = \frac{z'}{w} - \frac{zw'}{w^2} = \frac{z'w - zw'}{w^2}$

**9.2.38 a**  $\left(\frac{z_1}{z_2}\right)' = \frac{z'_1 z_2 - z_1 z'_2}{z_2^2} = \frac{\lambda z_1 z_2 - \lambda z_1 z_2}{z_2^2} = 0$ , so that  $\frac{z_1(t)}{z_2(t)} = k$ , a constant.

Now  $z_1(t) = kz_2(t)$ ; substituting  $t = 0$  gives  $1 = z_1(0) = kz_2(0) = k$ , so that  $z_1(t) = z_2(t)$ , as claimed.

b Let  $z_2(t) = e^{pt}(\cos(qt) + i \sin(qt))$  be the solution constructed in the text (see Page 413). Since  $z_2(t) \neq 0$  for all  $t$ , this is the only solution, by Part a.

**9.2.39** Let  $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ . We first solve the system  $\frac{d\vec{c}}{dt} = (A - \lambda I_3)\vec{c} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\vec{c}$ , or  $\frac{dc_1}{dt} = c_2(t)$ ,  $\frac{dc_2}{dt} = c_3(t)$ ,  $\frac{dc_3}{dt} = 0$ .

$c_3(t) = k_3$ , a constant, so that  $\frac{dc_2}{dt} = k_3$  and  $c_2(t) = k_3 t + k_2$ . Likewise  $c_1(t) = \frac{k_3}{2}t^2 + k_2 t + k_1$ .

Applying Exercise 9.1.24, with  $k = -\lambda$ , we find that  $\vec{c}(t) = e^{-\lambda t}\vec{x}(t)$  or  $\vec{x}(t) = e^{\lambda t}\vec{c}(t)$

$= e^{\lambda t} \begin{bmatrix} k_1 + k_2 t + \frac{k_3}{2}t^2 \\ k_2 + k_3 t \\ k_3 \end{bmatrix}$  where  $k_1, k_2, k_3$  are arbitrary constants. The zero state is stable if (and only if) the real part of  $\lambda$  is negative.

**9.2.40 a**  $B(t) = 1000(1 + 0.05i)^t = 1000(r(\cos \theta + i \sin \theta))^t = 1000r^t(\cos(\theta t) + i \sin(\theta t))$ , where

$r = \sqrt{1 + 0.05^2} > 1$  and  $\theta = \arctan(0.05) \approx 0.05$ . See Figure 9.42.

b  $B(t) = 1000e^{0.05i} = 1000(\cos(0.05t) + i \sin(0.05t))$ . See Figure 9.42.

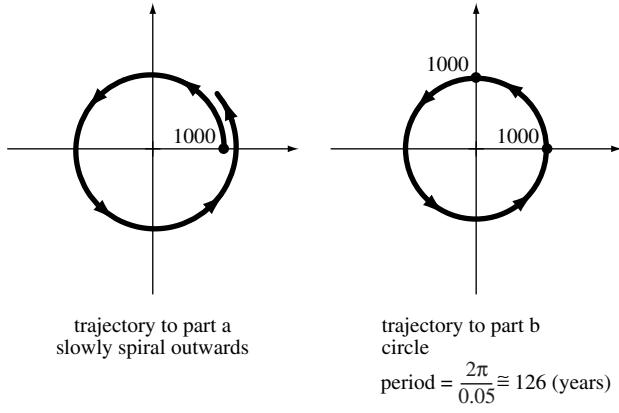


Figure 9.42: for Problem 9.2.40.

c We would choose an account with annual compounding, since the modulus of the balance grows in this case. In the case of continuous compounding the modulus of the balance remains unchanged.

## Section 9.3

**9.3.1** The characteristic polynomial of this differential equation is  $\lambda - 5$ , so that  $\lambda_1 = 5$ . By Theorem 9.3.8 the general solution is  $f(t) = Ce^{5t}$ , where  $C$  is an arbitrary constant.

**9.3.2** The solutions of  $\frac{dx}{dt} + 3x = 0$  are of the form  $x(t) = Ce^{-3t}$ , where  $C$  is an arbitrary constant, and the differential equation  $\frac{dx}{dt} + 3x = 7$  has the particular solution  $x_p(t) = \frac{7}{3}$ , so that the general solution is  $x(t) = Ce^{-3t} + \frac{7}{3}$  (where  $C$  is a constant). Alternatively, we could use Theorem 9.3.13.

**9.3.3** Use Theorem 9.3.13, where  $a = -2$  and  $g(t) = e^{3t}$ :

$$f(t) = e^{-2t} \int e^{2t} e^{3t} dt = e^{-2t} \int e^{5t} dt = e^{-2t} \left( \frac{1}{5} e^{5t} + C \right) = \frac{1}{5} e^{3t} + Ce^{-2t}, \text{ where } C \text{ is a constant.}$$

**9.3.4** We can look for a sinusoidal solution  $x_p(t) = P \cos(3t) + Q \sin(3t)$ , as in Example 7.  $P$  and  $Q$  need to be chosen in such a way that  $-3P \sin(3t) + 3Q \cos(3t) - 2P \cos(3t) - 2Q \sin(3t) = \cos(3t)$  or  $\begin{bmatrix} -2P + 3Q = 1 \\ -3P - 2Q = 0 \end{bmatrix}$  with solution  $P = -\frac{2}{13}$  and  $Q = \frac{3}{13}$ . Since the general solution of  $\frac{dx}{dt} - 2x = 0$  is  $x(t) = Ce^{2t}$ , the general solution of  $\frac{dx}{dt} - 2x = \cos(3t)$  is  $x(t) = Ce^{2t} - \frac{2}{13} \cos(3t) + \frac{3}{13} \sin(3t)$ , where  $C$  is an arbitrary constant.

**9.3.5** Using Theorem 9.3.13,  $f(t) = e^t \int e^{-t} t dt = e^t (-te^{-t} - e^{-t} + C) = Ce^t - t - 1$ , where  $C$  is an arbitrary constant.

**9.3.6** Using Theorem 9.3.13,  $f(t) = e^{2t} \int e^{-2t} e^{2t} dt = e^{2t} \int dt = e^{2t}(t + C)$ , where  $C$  is an arbitrary constant.

**9.3.7** By Definition 9.3.6,  $p_T(\lambda) = \lambda^2 + \lambda - 12 = (\lambda + 4)(\lambda - 3)$ .

Since  $p_T(\lambda)$  has distinct roots  $\lambda_1 = -4$  and  $\lambda_2 = 3$ , the solutions of the differential equation are of the form  $f(t) = c_1 e^{-4t} + c_2 e^{3t}$ , where  $c_1$  and  $c_2$  are arbitrary constants (by Theorem 9.3.8).

**9.3.8**  $p_T(\lambda) = \lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2) = 0$

$x(t) = c_1 e^{-5t} + c_2 e^{2t}$ , where  $c_1, c_2$  are arbitrary constants.

**9.3.9**  $p_T(\lambda) = \lambda^2 - 9 = (\lambda - 3)(\lambda + 3) = 0$

$f(t) = c_1 e^{3t} + c_2 e^{-3t}$ , where  $c_1, c_2$  are arbitrary constants.

**9.3.10**  $p_T(\lambda) = \lambda^2 + 1 = 0$  has roots  $\lambda_{1,2} = \pm i$ . By Theorem 9.3.9,  $f(t) = c_1 \cos(t) + c_2 \sin(t)$ , where  $c_1, c_2$  are arbitrary constants.

**9.3.11**  $p_T(\lambda) = \lambda^2 - 2\lambda + 2 = 0$  has roots  $\lambda_{1,2} = 1 \pm i$ . By Theorem 9.3.9,  $x(t) = e^t(c_1 \cos(t) + c_2 \sin(t))$ , where  $c_1, c_2$  are arbitrary constants.

**9.3.12**  $p_T(\lambda) = \lambda^2 - 4\lambda + 13 = 0$  has roots  $\lambda_{1,2} = 2 \pm 3i$ .

By Theorem 9.3.9,  $f(t) = e^{2t}(c_1 \cos(3t) + c_2 \sin(3t))$ , where  $c_1, c_2$  are arbitrary constants.

**9.3.13**  $p_T(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$  has the double root  $\lambda = -1$ . Following Summary 9.3.15 in the text, we find  $f(t) = e^{-t}(c_1 t + c_2)$ , where  $c_1, c_2$  are arbitrary constants.

**9.3.14**  $p_T(\lambda) = \lambda^2 + 3\lambda = \lambda(\lambda + 3) = 0$  has roots  $\lambda_1 = 0, \lambda_2 = -3$ .

**9.3.15** By integrating twice we find  $f(t) = c_1 + c_2 t$ , where  $c_1, c_2$  are arbitrary constants.

**9.3.16** By Theorem 9.3.10, the differential equation has a particular solution of the form  $f_p(t) = P \cos(t) + Q \sin(t)$ . Plugging  $f_p$  into the equation we find

$$(-P \cos(t) - Q \sin(t)) + 4(-P \sin(t) + Q \cos(t)) + 13(P \cos(t) + Q \sin(t)) = \cos(t) \text{ or}$$

$$\begin{bmatrix} 12P + 4Q = 1 \\ -4P + 12Q = 0 \end{bmatrix}, \text{ so}$$

$$P = \frac{3}{40}$$

$$Q = \frac{1}{40}.$$

Therefore,  $f_p(t) = \frac{3}{40} \cos(t) + \frac{1}{40} \sin(t)$ .

Next we find a basis of the solution space of  $f''(t) + 4f'(t) + 13f(t) = 0$ .  $p_T(\lambda) = \lambda^2 + 4\lambda + 13 = 0$  has roots  $-2 \pm 3i$ . By Theorem 9.3.9,  $f_1(t) = e^{-2t} \cos(3t)$  and  $f_2(t) = e^{-2t} \sin(3t)$  is a basis of the solution space.

By Theorem 9.3.4, the solutions of the original differential equation are of the form  $f(t) = c_1 f_1(t) + c_2 f_2(t) + f_p(t) = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t) + \frac{3}{40} \cos(t) + \frac{1}{40} \sin(t)$ , where  $c_1, c_2$  are arbitrary constants.

**9.3.17** By Theorem 9.3.10, the differential equation has a particular solution of the form  $f_p(t) = P \cos(t) + Q \sin(t)$ .

Plugging  $f_p$  into the equation we find  $(-P \cos(t) - Q \sin(t)) + 2(-P \sin(t) + Q \cos(t)) + P \cos(t) + Q \sin(t) = \sin(t)$  or  $\begin{bmatrix} 2Q = 0 \\ -2P = 1 \end{bmatrix}$ , so  $P = -\frac{1}{2}$ ,  $Q = 0$ .

Therefore,  $f_p(t) = -\frac{1}{2} \cos(t)$ .

Next we find a basis of the solution space of  $f''(t) + 2f'(t) + f(t) = 0$ . In Exercise 13 we see that  $f_1(t) = e^{-t}$ ,  $f_2(t) = te^{-t}$  is such a basis.

By Theorem 9.3.4, the solutions of the original differential equation are of the form  $f(t) = c_1 f_1(t) + c_2 f_2(t) + f_p(t) = c_1 e^{-t} + c_2 t e^{-t} - \frac{1}{2} \cos(t)$ , where  $c_1, c_2$  are arbitrary constants.

**9.3.18** We follow the approach outlined in Exercises 16 and 17.

- Particular solution  $f_p = \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t)$
- Solutions of  $f''(t) + 3f'(t) + 2f(t) = 0$  are  $f_1(t) = e^{-t}$  and  $f_2(t) = e^{-2t}$ .
- The solutions of the original differential equation are of the form  $f(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t)$ , where  $c_1$  and  $c_2$  are arbitrary constants.

**9.3.19** We follow the approach outlined in Exercise 17.

- Particular solution  $x_p(t) = \cos(t)$
- Solutions of  $\frac{d^2x}{dt^2} + 2x = 0$  are  $x_1(t) = \cos(\sqrt{2}t)$  and  $x_2(t) = \sin(\sqrt{2}t)$ .
- The solutions of the original differential equation are of the form  $x(t) = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) + \cos(t)$ , where  $c_1$  and  $c_2$  are arbitrary constants.

**9.3.20**  $p_T(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 1)(\lambda - 2) = 0$  has roots  $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$ .

By Theorem 9.3.8, the general solution is  $f(t) = c_1 + c_2 e^t + c_3 e^{2t}$ , where  $c_1, c_2, c_3$  are arbitrary constants.

**9.3.21**  $p_T(\lambda) = \lambda^3 + 2\lambda^2 - \lambda - 2 = (\lambda - 1)(\lambda + 1)(\lambda + 2) = 0$  has roots  $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -2$ .

By Theorem 9.3.8, the general solution is  $f(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t}$ , where  $c_1, c_2, c_3$  are arbitrary constants.

**9.3.22**  $p_T(\lambda) = \lambda^3 - \lambda^2 - 4\lambda + 4 = (\lambda - 1)(\lambda - 2)(\lambda + 2) = 0$  has roots  $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -2$ .

By Theorem 9.3.8, the general solution is  $f(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{-2t}$ , where  $c_1, c_2, c_3$  are arbitrary constants.

**9.3.23** General solution  $f(t) = Ce^{5t}$

Plug in:  $3 = f(0) = Ce^0 = C$ , so that  $f(t) = 3e^{5t}$ .

**9.3.24** General solution  $x(t) = Ce^{-3t} + \frac{7}{3}$  (see Exercise 2).

Plug in:  $0 = x(0) = C + \frac{7}{3}$ , so that  $C = -\frac{7}{3}$  and  $x(t) = -\frac{7}{3}e^{-3t} + \frac{7}{3}$ .

**9.3.25** General solution  $f(t) = Ce^{-2t}$

Plug in:  $1 = f(1) = Ce^{-2}$ , so that  $C = e^2$  and  $f(t) = e^2 e^{-2t} = e^{2-2t}$ .

**9.3.26** General solution  $f(t) = c_1 e^{3t} + c_2 e^{-3t}$  (see Exercise 9), with  $f'(t) = 3c_1 e^{3t} - 3c_2 e^{-3t}$

Plug in:  $0 = f(0) = c_1 + c_2$  and  $1 = f'(0) = 3c_1 - 3c_2$ , so that  $c_1 = \frac{1}{6}$ ,  $c_2 = -\frac{1}{6}$ , and  $f(t) = \frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}$ .

**9.3.27** General solution  $f(t) = c_1 \cos(3t) + c_2 \sin(3t)$  (Theorem 9.3.9)

Plug in:  $0 = f(0) = c_1$  and  $1 = f(\frac{\pi}{2}) = -c_2$ , so that  $c_1 = 0$ ,  $c_2 = -1$ , and  $f(t) = -\sin(3t)$ .

**9.3.28** General solution  $f(t) = c_1 e^{-4t} + c_2 e^{3t}$ , with  $f'(t) = -4c_1 e^{-4t} + 3c_2 e^{3t}$

Plug in:  $0 = f(0) = c_1 + c_2$  and  $0 = f'(0) = -4c_1 + 3c_2$ , so that  $c_1 = c_2 = 0$  and  $f(t) = 0$ .

**9.3.29** General solution  $f(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{3} \sin(t)$ , so that  $f'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{1}{3} \cos(t)$  (use the approach outlined in Exercise 17)

Plug in:  $0 = f(0) = c_1$  and  $0 = f'(0) = 2c_2 + \frac{1}{3}$ , so that  $c_1 = 0$ ,  $c_2 = -\frac{1}{6}$ , and  $f(t) = -\frac{1}{6} \sin(2t) + \frac{1}{3} \sin(t)$ .

**9.3.30** a  $k$  is a positive constant that depends on the rate of cooling of the coffee (it varies with the material of the cup, for example).

$A$  is the room temperature.

b  $T'(t) + kT(t) = kA$

Constant particular solution:  $T_p(t) = A$

General solution of  $T'(t) + kT(t) = 0$  is  $T(t) = Ce^{-kt}$ .

General solution of the original differential equation:  $T(t) = Ce^{-kt} + A$

Plug in:  $T_0 = T(0) = C + A$ , so that  $C = T_0 - A$  and  $T(t) = (T_0 - A)e^{-kt} + A$ .

**9.3.31**  $\frac{dv}{dt} + \frac{k}{m}v = g$

constant particular solution:  $v_p = \frac{mg}{k}$

General solution of  $\frac{dv}{dt} + \frac{k}{m}v = 0$  is  $v(t) = Ce^{-\frac{k}{m}t}$ .

General solution of the original differential equation:  $v(t) = Ce^{-\frac{k}{m}t} + \frac{mg}{k}$

Plug in:  $0 = v(0) = C + \frac{mg}{k}$ , so that  $C = -\frac{mg}{k}$  and  $v(t) = \frac{mg}{k} \left(1 - e^{-\frac{k}{m}t}\right)$

$\lim_{t \rightarrow \infty} v(t) = \frac{mg}{k}$  (the “terminal velocity”). See Figure 9.43.

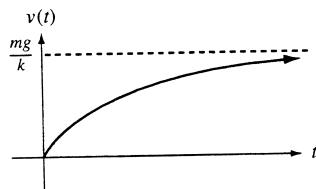


Figure 9.43: for Problem 9.3.31.

## Chapter 9

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**9.3.32**  $\frac{dB}{dt} = kB - r$  or  $\frac{dB}{dt} - kB = -r$

↑      ↑  
interest    withdrawals

constant particular solution  $B_p = \frac{r}{k}$

General solution of  $\frac{dB}{dt} - kB = 0$  is  $B(t) = Ce^{kt}$

General solution of the original differential equation:  $B(t) = Ce^{kt} + \frac{r}{k}$

Plug in:  $B_0 = B(0) = C + \frac{r}{k}$ , so that  $C = B_0 - \frac{r}{k}$  and  $B(t) = (B_0 - \frac{r}{k})e^{kt} + \frac{r}{k}$

if  $B_0 > \frac{r}{k}$  then interest will exceed withdrawals and balance will grow.

if  $B_0 < \frac{r}{k}$  then withdrawals will exceed interest and account will eventually be depleted.

if  $B_0 = \frac{r}{k}$  then the balance will remain the same.

The graph in Figure 9.44 shows the three possible scenarios.

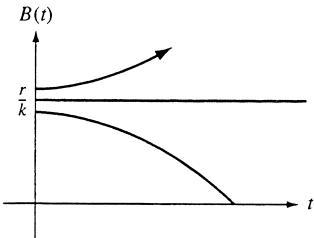


Figure 9.44: for Problem 9.3.32.

**9.3.33** By Theorem 9.3.9,  $x(t) = c_1 \cos(\sqrt{\frac{g}{L}}t) + c_2 \sin(\sqrt{\frac{g}{L}}t)$ , with period  $P = \frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi\sqrt{\frac{L}{g}}$ . It is required that  $2 = P = 2\pi\sqrt{\frac{L}{g}}$  or  $L = \frac{g}{\pi^2} \approx 0.994$  (meters).

**9.3.34 a** We will take downward forces as positive.

Let  $g$  = acceleration due to gravity,

$\rho$  = density of block

$a$  = length of edge of block

Then (weight of block) = (mass of block)  $\cdot g$  = (density of block)(volume of block)  $g$  =  $\rho a^3 g$   
 buoyancy = (weight of displaced water) = (mass of displaced water)  $\cdot g$  =  
 (density of water) (volume of displaced water)  $g$  =  $1a^2 x(t)g = a^2 g x(t)$ .

**b** Newton's Second Law of Motion tells us that

$m \frac{d^2x}{dt^2} = F = \text{weight} - \text{buoyancy} = \rho a^3 g - a^2 g x(t)$ , where  $m = \rho a^3$  is the mass of the block.

$$\rho a^3 \frac{d^2x}{dt^2} = \rho a^3 g - a^2 g x(t)$$

$$\frac{d^2x}{dt^2} = g - \frac{g}{\rho a}x(t)$$

$$\frac{d^2x}{dt^2} + \frac{g}{\rho a}x = g$$

constant solution  $x_p = \rho a$

general solution (use Theorem 9.3.9):  $x(t) = c_1 \cos\left(\sqrt{\frac{g}{\rho a}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{\rho a}}t\right) + \rho a$

Now  $c_2 = 0$  since block is at rest at  $t = 0$ .

Plug in:  $a = x(0) = c_1 + \rho a$ , so that  $c_1 = a - \rho a$  and

$$x(t) = (a - \rho a) \cos\left(\sqrt{\frac{g}{\rho a}}t\right) + \rho a \approx 2 \cos(11t) + 8 \text{ (measured in centimeters)}$$

c The period is  $P = \frac{2\pi}{\sqrt{\frac{g}{\rho a}}} = \frac{2\pi\sqrt{\rho a}}{\sqrt{g}}$ . Thus the period increases as  $\rho$  or  $a$  increases (denser wood or larger block), or as  $g$  decreases (on the moon). The period is independent of the initial state.

**9.3.35** a  $p_T(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$  with roots  $\lambda_1 = -1$  and  $\lambda_2 = -2$ , so  $x(t) = c_1 e^{-t} + c_2 e^{-2t}$ .

b  $x'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$

Plug in:  $1 = x(0) = c_1 + c_2$  and  $0 = x'(0) = -c_1 - 2c_2$ , so that  $c_1 = 2$ ,  $c_2 = -1$  and  $x(t) = 2e^{-t} - e^{-2t}$ . See Figure 9.45.

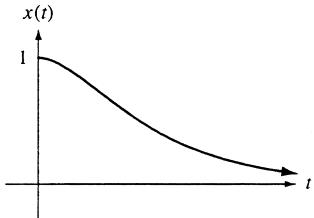


Figure 9.45: for Problem 9.3.35b.

c Plug in:  $1 = x(0) = c_1 + c_2$  and  $-3 = x'(0) = -c_1 - 2c_2$ , so that  $c_1 = -1$ ,  $c_2 = 2$ , and  $x(t) = -e^{-t} + 2e^{-2t}$ . See Figure 9.46.

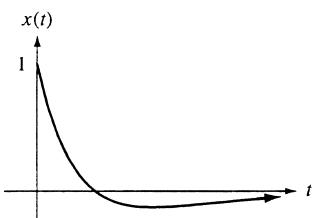


Figure 9.46: for Problem 9.3.35c.

d The oscillator in part (b) never reaches the equilibrium, while the oscillator in part (c) goes through the equilibrium once, at  $t = \ln(2)$ . Take another look at Figures 9.45 and 9.46.

**9.3.36**  $f_T(\lambda) = \lambda^2 + 2\lambda + 101 = 0$  has roots  $\lambda_{1,2} = -1 \pm 20i$ .

By Theorem 9.3.9,  $x(t) = e^{-t}(c_1 \cos(20t) + c_2 \sin(20t))$ .

Any nonzero solution goes through the equilibrium infinitely many times. See Figure 9.47.

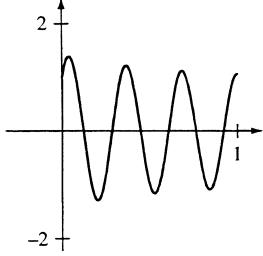


Figure 9.47: for Problem 9.3.36.

**9.3.37**  $f_T(\lambda) = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$  has roots  $\lambda_{1,2} = -3$ .

Following the method of Example 10, we find the general solution  $x(t) = e^{-3t}(c_1 + c_2 t)$  with  $x'(t) = e^{-3t}(c_2 - 3c_1 - 3c_2 t)$ .

Plug in:  $0 = x(0) = c_1$ , and  $1 = x'(0) = c_2 - 3c_1$ , so that  $c_1 = 0$ ,  $c_2 = 1$ , and  $x(t) = te^{-3t}$ . See Figure 9.48.

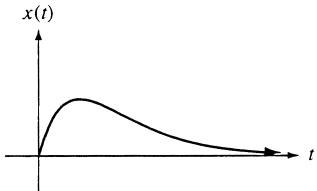


Figure 9.48: for Problem 9.3.37.

The oscillator does not go through the equilibrium at  $t > 0$ .

**9.3.38 a**  $(D - \lambda)(p(t)e^{\lambda t}) = [p(t)e^{\lambda t}]' - \lambda p(t)e^{\lambda t} = p'(t)e^{\lambda t} + \lambda p(t)e^{\lambda t} - \lambda p(t)e^{\lambda t} = p'(t)e^{\lambda t}$ , as claimed.

b Applying the result from part (a)  $m$  times we find  $(D - \lambda)^m(p(t)e^{\lambda t}) = p^{(m)}(t)e^{\lambda t} = 0$ , since  $p^{(m)}(t) = 0$  for a polynomial of degree less than  $m$ .

c By Theorem 9.3.3, we are looking for  $m$  linearly independent functions. By part (b), the functions  $e^{\lambda_1 t}, te^{\lambda_1 t}, t^2 e^{\lambda_1 t}, \dots, t^{m_1-1} e^{\lambda_1 t}$  do the job.

d Note that the kernel of  $(D - \lambda_i)^{m_i}$  is contained in the kernel of  $(D - \lambda_1)^{m_1} \cdots (D - \lambda_r)^{m_r}$ , for any  $1 \leq i \leq r$ . Therefore, we have the following basis:

$$e^{\lambda_1 t}, te^{\lambda_1 t}, \dots, t^{m_1-1} e^{\lambda_1 t},$$

$$e^{\lambda_2 t}, te^{\lambda_2 t}, \dots, t^{m_2-1} e^{\lambda_2 t}, \dots$$

$$e^{\lambda_r t}, te^{\lambda_r t}, \dots, t^{m_r-1} e^{\lambda_r t}.$$

**9.3.39**  $f_T(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$  has roots  $\lambda_{1,2,3} = -1$ . In other words, we can write the differential equation as  $(D + 1)^3 f = 0$ .

By Exercise 38, part (c), the general solution is  $f(t) = e^{-t}(c_1 + c_2t + c_3t^2)$ .

**9.3.40**  $f_T(\lambda) = \lambda^3 + \lambda^2 - \lambda - 1 = (\lambda + 1)^2(\lambda - 1) = 0$  has roots  $\lambda_{1,2} = -1$ ,  $\lambda_3 = 1$ .

In other words, we can write the differential equation as  $(D + 1)^2(D - 1) = 0$ .

By Exercise 38, part (d), the general solution is  $x(t) = e^{-t}(c_1 + c_2t) + c_3e^t$ .

**9.3.41** We are looking for functions  $x$  such that  $T(x) = \lambda x$ , or  $T(x) - \lambda x = 0$ . Now  $T(x) - \lambda x$  is an  $n$ th-order linear differential operator, so that its kernel is  $n$ -dimensional, by Theorem 9.3.3. Thus  $\lambda$  is indeed an eigenvalue of  $T$ , with an  $n$ -dimensional eigenspace.

**9.3.42 a** We need to solve the second-order differential equation  $Tx = D^2x = \frac{d^2x}{dt^2} = \lambda x$ . This differential equation has a two-dimensional solution space  $E_\lambda$  for any  $\lambda$ , so that all  $\lambda$  are eigenvalues of  $T$ .

$$\text{if } \lambda > 0 \text{ then } E_\lambda = \text{span}\left(e^{\sqrt{\lambda}t}, e^{-\sqrt{\lambda}t}\right)$$

$$\text{if } \lambda = 0 \text{ then } E_\lambda = \text{span}(1, t)$$

$$\text{if } \lambda < 0 \text{ then } E_\lambda = \text{span}(\sin(\sqrt{-\lambda}t), \cos(\sqrt{-\lambda}t))$$

**b** Among the eigenfunctions of  $T$  we found in part (a), we seek those of period 1. In the case  $\lambda < 0$  the shortest period is  $P = \frac{2\pi}{\sqrt{-\lambda}}$ . Now 1 is a period if  $P = \frac{2\pi}{\sqrt{-\lambda}} = \frac{1}{k}$  for some positive integer  $k$ , or,  $\lambda = -4\pi^2k^2$ . Then  $E_\lambda = \text{span}(\cos(2\pi kt), (\sin(2\pi kt)))$ .

In the case  $\lambda > 0$  there are no periodic solutions. In the case  $\lambda = 0$  we have the constant solutions, so that  $\lambda = 0$  is an eigenvalue with  $E_0 = \text{span}(1)$ .

*Summary:*

$\lambda = -4\pi^2k^2$  is an eigenvalue, for  $k = 1, 2, 3, \dots$ , with  $E_\lambda = \text{span}(\cos(2\pi kt), (\sin(2\pi kt)))$ .  $\lambda = 0$  is an eigenvalue, with  $E_0 = \text{span}(1)$ .

**9.3.43 a** Using the approach of Exercise 17, we find  $x(t) = c_1e^{-2t} + c_2e^{-3t} + \frac{1}{10}\cos t + \frac{1}{10}\sin t$ .

**b** For large  $t$ ,  $x(t) \approx \frac{1}{10}\cos t + \frac{1}{10}\sin t$ .

**9.3.44 a** Using the approach of Exercises 16 and 17 we find  $x(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) - \frac{1}{40}\cos(3t) + \frac{3}{40}\sin(3t)$ .

**b** For large  $t$ ,  $x(t) \approx -\frac{1}{40}\cos(3t) + \frac{3}{40}\sin(3t)$ .

**9.3.45** We can write the system as  $\begin{bmatrix} \frac{dx_1}{dt} = x_1 + 2x_2 \\ \frac{dx_2}{dt} = x_2 \end{bmatrix}$  with  $x_1(0) = 1, x_2(0) = -1$ .

The solution of the second equation, with the given initial value, is  $x_2(t) = -e^t$ .

Now the first equation takes the form  $\frac{dx_1}{dt} - x_1 = -2e^t$ .

Using Example 9 (with  $a = 1$  and  $c = -2$ ) we find  $x_1(t) = e^t(-2t + C)$ .

plug in:  $1 = x_1(0) = C$ , so that  $x_1(t) = e^t(1 - 2t)$  and  $\vec{x}(t) = e^t \begin{bmatrix} 1 - 2t \\ -1 \end{bmatrix}$ .

**9.3.46** We can write the system as  $\begin{bmatrix} \frac{dx_1}{dt} = & 2x_1 + 3x_2 + x_3 \\ \frac{dx_2}{dt} = & x_2 + 2x_3 \\ \frac{dx_3}{dt} = & x_3 \end{bmatrix}$  with  $x_1(0) = 2, x_2(0) = 1, x_3(0) = -1$ .

We solve for  $x_2$  and  $x_3$  as in Exercise 45:

$$x_2(t) = e^t(1 - 2t)$$

$$x_3(t) = -e^t$$

Now the first equation takes the form  $\frac{dx_1}{dt} - 2x_1 = 3e^t(1 - 2t) - e^t = e^t(2 - 6t), x_1(0) = 2$ .

We use Theorem 9.3.13 to solve this differential equation:

$$x_1(t) = e^{2t} \int e^{-2t} e^t (2 - 6t) dt = e^{2t} \int (2e^{-t} - 6te^{-t}) dt = e^{2t} [-2e^{-t} + 6te^{-t} + 6e^{-t} + C]$$

plug in:  $2 = x_1(0) = (-2 + 6 + c)$ , so that  $c = -2$  and  $x_1(t) = e^{2t}(4e^{-t} + 6te^{-t} - 2) = 4e^t + 6te^t - 2e^{2t}$ .

$$\vec{x}(t) = \begin{bmatrix} 4e^t + 6te^t - 2e^{2t} \\ e^t - 2te^t \\ -e^t \end{bmatrix}$$

**9.3.47 a** We start with a *preliminary remark* that will be useful below: If  $f(t) = p(t)e^{\lambda t}$ , where  $p(t)$  is a polynomial, then  $f(t)$  has an antiderivative of the form  $q(t)e^{\lambda t}$ , where  $q(t)$  is another polynomial. We leave this remark as a calculus exercise.

The function  $x_n(t)$  satisfies the differential equation  $\frac{dx_n}{dt} = a_{nn}x_n$ , so that  $x_n = Ce^{a_{nn}t}$ , which is of the desired form.

Now we will show that  $x_k$  is of the desired form, assuming that  $x_{k+1}, \dots, x_n$  have this form.  $x_k$  satisfies the differential equation  $\frac{dx_k}{dt} = a_{kk}x_k + a_{k,k+1}x_{k+1} + \dots + a_{kn}x_n$  or  $\frac{dx_k}{dt} - a_{kk}x_k = a_{k,k+1}x_{k+1} + \dots + a_{kn}x_n$ .

Note that, by assumption, the function on the right-hand side has the form  $p_1(t)e^{\lambda_1 t} + \dots + p_m(t)e^{\lambda_m t}$ . If we set  $a_{kk} = a$  for simplicity, we can write  $\frac{dx_k}{dt} - ax_k = p_1(t)e^{\lambda_1 t} + \dots + p_m(t)e^{\lambda_m t}$ .

By Theorem 9.3.13, the solution is

$$x_k(t) = e^{at} \int e^{-at} (p_1(t)e^{\lambda_1 t} + \dots + p_m(t)e^{\lambda_m t}) dt$$

$$= e^{at} \int (p_1(t)e^{(\lambda_1-a)t} + \dots + p_m(t)e^{(\lambda_m-a)t}) dt$$

$= e^{at} (q_1(t)e^{(\lambda_1-a)t} + \dots + q_m(t)e^{(\lambda_m-a)t} + C) = q_1(t)e^{\lambda_1 t} + \dots + q_m(t)e^{\lambda_m t} + Ce^{at}$  as claimed (note that  $a$  is one of the  $\lambda_i$ ). The constant  $C$  is determined by  $x_k(0)$ . Note that we used the *preliminary remark* in the second to last step.

**b** It is shown in introductory calculus classes that  $\lim_{t \rightarrow \infty} (t^m e^{\lambda t}) = 0$  if and only if  $\lambda$  is negative (here  $m$  is a fixed positive integer). In light of part (a), this proves the claim.

**9.3.48** a By Exercise 47, the system  $\frac{d\vec{u}}{dt} = B\vec{u}$  has a unique solution  $\vec{u}(t)$ . Then the system  $\frac{d\vec{x}}{dt} = A\vec{x}$  has the unique solution  $\vec{x}(t) = S\vec{u}(t)$ .

b It suffices to note that  $\lim_{t \rightarrow \infty} \vec{x}(t) = \vec{0}$  if and only if  $\lim_{t \rightarrow \infty} \vec{u}(t) = \vec{0}$ , where  $\vec{u} = S^{-1}\vec{x}$ .