Solution

i) we have to show product of kernels is a kernel

Let $k_1(x,y)$ be a valid kernel

$k_2(x,y)$ be a valid kernel

then $k_1(x,y) = k_1(y,x)$ &

$k_2(x,y) = k_2(y,x)$

let $x_1, \ldots x_n \in \mathbb{R}^n$

we construct C such that $c_{ij} = k_1(x_i, x_j)$

$\therefore$ since it is a valid kernel C matrix must be P.S.D

Similarly we construct matrix D from kernel $k_2(x,y)$

such that $d_{ij} = k_2(x_i, x_j)$

similarly D is also positive semi definite

let

$k(x,y) = k_1(x,y)\, k_2(x,y)$ (product of kernels)

$k(x,y) = k_1(x,y)\, k_2(x,y) = k_1(y,x)\, k_2(y,x)$

$= k(y,x)$

$\therefore$ it is symmetric

we have to show matrix formed by taking their product is a PSD

we have to show matrix E' formed by

$e_{ij} = c_{ij} d_{ij}$ is PSD

since C is PSD it can be written as

$A^T A$

$A = (a_1 \ldots a_n) \quad \forall a_i \in \mathbb{R}^n$

$c_{ij} = a_i^T a_j = \sum_k a_{ik} a_{jk}$

similarly $D = B^T B$

similarly $B = (b_1, \ldots, b_n) \quad \forall b_i \in \mathbb{R}^n$

$d_{ij} = \sum_k b_{ik} b_{jk}$

$$x^T E x = \sum_{i,j} x_i^T x_j c_{ij} d_{ij}$$

$$= \sum_{i,j} x_i^T x_j \sum_k a_{ik} a_{jk} \sum_l b_{il} b_{jl}$$

$$= \cancel{\sum_{i,j} x_i^T x_j \sum_k a_{ki}^T a_{jk} \sum_l b_{li}^T} b_{jl}$$

$$= \cancel{\sum\sum_{k,l}} \sum_{k,l} \sum_{i,j} x_i x_j a_{ik} a_{jk} b_{il} b_{jl}$$

$$= \sum_{k,l} \left(\sum_i u_i a_{ik} b_{il}\right) \left(\sum_j u_j a_{jk} b_{jl}\right)$$

$$= \sum_{k,l} \left(\sum_i u_i a_{ik} b_{il}\right)^2 \geq 0$$

$\therefore x^T E x \geq 0$ which means P.S.D

& $k(x,y) = k(y,x)$ which represents symmetric

from above two we conclude that $k(x,y)$ is a kernel

$\therefore$ product of kernels is a kernel.

2a) given $k(x,y) = (\langle x,y\rangle + c)^n$

we have to show $\langle x,y\rangle$ is a valid kernel

let $k'(x,x') = \langle x,x'\rangle$
$= \sum_{k=1}^{N} x_{ik} \cdot x'_{jk}$

let $\{x_1 \cdots x_k\} \in \mathbb{R}^n$

positive definite function is $\sum_{i,j} c_i c_j k'(x_i, x_j)$

$$= \sum_{i=1}^{k}\sum_{j=1}^{k} c_i c_j \sum_{a=1}^{N} x_{ia} \cdot x_{ja}$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{a=1}^{N} c_i c_j x_{ia} x_{ja}$$

$$= \sum_{a=1}^{N}\left(\sum_{i=1}^{k} c_i x_{ia}\right)\left(\sum_{j=1}^{k} c_j x_{ja}\right)$$

$$= \sum_{a=1}^{N}\left(\sum_{i=1}^{k} c_i x_{ia}\right)^2 \geq 0$$

$\therefore \sum_{i,j} c_i c_j k'(x_i, x_j) \geq 0$

innerproduct is symmetric (from innerproduct definition)

i.e $\langle x, x'\rangle = \langle x', x\rangle$

$\therefore$ innerproduct $(\langle x,x'\rangle)$ is a valid kernel

$k(x,y) = (k'(x,y) + c)^n$

$= {}^nc_0 (k'(x,y))^n c^0 + \cdots + {}^nc_r (k'(x,y))^{n-r} c^r + \cdots$

$+ {}^nc_n c^n$

which means summation of product of kernels with certain scalars

which is a valid kernel

$\therefore$ sum of kernels is a kernel

Illy product of kernels is a kernel.

$\therefore$ $k(x,y) = (\langle x,y\rangle + c)^n$ is a valid kernel.

2b) $k(x,y) = \exp(\langle x,y\rangle)$

by taylor series expansion

$$k(x,y) = 1 + \frac{\langle x,y\rangle}{1!} + \frac{\langle x,y\rangle^2}{2!} + \dots + \frac{\langle x,y\rangle^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\langle x,y\rangle^n}{n!}$$

$$= \sum_{n=0}^{\infty} a_n (\langle x,y\rangle)^n \qquad a_n = \frac{1}{n!} > 0$$

Already we have $k(x,x') = \sum_{n=1}^{\infty} a_n (\langle x,x'\rangle)^n$ is a valid kernel. with $a_n > 0$

hence $k(x,y)$ is a valid kernel

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3) Moore Aronszain theorem:-

let $k : X \times X \to \mathbb{R}$ is positive definite there is a unique RKHS $H \subset \mathbb{R}^X$ with reproducing kernel k

from k construct pre RKHS $H_0 \subset \mathbb{R}^X$ with properties

i) evaluation functionals δ_x are continous on H_0

ii) Any H_0 cauchy sequence f_n which converges pointwise to 0 also converges in H_0-norm to 0

$H_0 = \text{span}\{k(\cdot, x) \mid x \in X\}$ will be taken set of functions

$$f(x) = \sum_{i=1}^{n} \alpha_i k(x, x_i)$$

space H_0 is associated with innerproduct

$$\langle f, g \rangle_{H_0} = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \, k(x_i, y_j)$$

f, g are valid pre RKHS

Define H to be the set of functions $f \in \mathbb{R}^X$ for which there exists an H_0 - cauchy sequence (f_n) converges pointwise to f the H is an RKHS

Steps to Construct RKHS

1) the inner product define between $f, g \in H$ as the limit of an innerproduct of the H_0 - cauchy sequences f_n, g_n converging to f & g is well defined i.e independent of sequences

the inner product space must satisfy

$\langle f, f \rangle_H$ iff $f = 0$

3) the evaluation functionals δ_x are continous on H

4) H should be complete i.e. every H-cauchy sequence should converge

If the above 4 are satisfied then H is RKHS