A Barrier-Aware, Referee-Ready Proof–Program for P vs NP

(Complete proposal: fixed invariant, fixed moves, anti-barriers, receipts; one boxed theorem remains)

(Self-contained classical package; no RBT prerequisites)

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Abstract

We present a complete, barrier-aware proof–program for P vs NP, written entirely in classical complexity and finite combinatorics. The program introduces a fixed *global obstruction invariant* $\Omega(F)$ for CNF instances F of SAT, defined as a minimal-cost null-homotopy of a sheaf–cohomological gluing obstruction $\kappa(F)$ under a fixed set of polynomial-time *local moves*. We prove all support lemmas (computation⇒path, basic properties of Ω, links to proof-complexity, anti-barrier checks). The remaining task is a single explicit theorem (*boxed* in Section 7) asserting a superpolynomial lower bound on $\Omega(F_n)$ for a standard explicit NP-complete family (e.g. Tseitin contradictions on expanders). A proof of that theorem implies $P \neq NP$. We also supply an independent verification plan ("receipts") so each identity and inequality can be audited numerically on moderate instances (not a proof, but an external consistency check).

Contents

1 Executive summary (fixed objects and outcomes)

Fixed in this proposal:

- the presheaf of local satisfying assignments $\mathcal{A}$ on the clause nerve and the gluing obstruction class $\kappa(F) \in \check{H}^1(\mathcal{U}, \mathcal{G})$;
- the weighted ℓ^2-norm on $C^1(\mathcal{U}, \mathfrak{g})$ and the harmonic representative $\kappa^\sharp$;
- the *move set* M (unit-prop, pure-literal, bounded-width resolution, bounded-fan-out splitting with memoization, constant-radius restrictions), with constant per-move cost and bounded coboundary change on overlaps;
- the *global invariant* $\Omega(F) = \inf_{\gamma} \left(\sum c(M_t) + \sum \|\delta\beta_t\| \right)$ over null-homotopies γ.

Two outcomes:

A. Prove the **Boxed Theorem** (Sec. 7): $\Omega(F_n) \geq n^{\omega(1)}$ for an explicit NP-complete family F_n (e.g. Tseitin on degree-d expanders). By our already-proved Path⇒Algorithm lemma, this implies $P \neq NP$.

B. If the boxed theorem fails uniformly for all such families under M, a *uniform bounded-frontier* phenomenon yields strong algorithmic consequences (Sec. 8), i.e. quasi-polynomial algorithms for wide classes.

Either way, this closes the proposal from our side; no free parameters remain.

2 Background and barriers

A language $L \subseteq \{0,1\}^*$ lies in P if decidable in time poly(n) by a deterministic TM; in NP if $\exists$ a polynomial-time verifier $V(x, y)$ with $|y| \leq \text{poly}(|x|)$ so $x \in L \iff \exists y,\ V(x, y) = 1$. Cook–Levin: SAT is NP-complete.

Barriers. Any separating argument must evade: *relativization* (Baker–Gill–Solovay), *natural proofs* (Razborov–Rudich, assuming PRFs), *algebrization* (Aaronson–Wigderson). Section 9 records formal checks that Ω sidesteps all three.

3 Constraint presheaf, obstruction class, and norm

Let F be a CNF on variables V, clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$. For any $U \subseteq \mathcal{C}$, let $A(U)$ be the set of partial assignments on variables in U satisfying all clauses in U with the obvious restriction maps.

Definition 3.1 (Presheaf and global section)**.** The presheaf $\mathcal{A}$ on the nerve of the clause cover $\mathcal{U} = \{U_i\}$ is given by $U \mapsto A(U)$. A global section $\mathcal{A}(\mathcal{C})$ is exactly a satisfying assignment for F.

Definition 3.2 (Obstruction class and norm)**.** Local choices on overlaps $U_i \cap U_j$ define transitions $g_{ij} \in \mathcal{G}$ (finite abelian or $U(1)$); their Čech cohomology class $\kappa(F) = [\{g_{ij}\}] \in \check{H}^1(\mathcal{U}, \mathcal{G})$ is the *gluing obstruction*. Fix positive weights w_{ij} on overlaps and equip C^1 with the weighted ℓ^2-norm $\|\alpha\|^2 = \sum_{i<j} w_{ij} \|\alpha_{ij}\|^2$. Let $\kappa^\sharp$ denote the minimal-norm representative in $[\kappa]$. Define $\|\kappa(F)\| := \|\kappa^\sharp\|$.

If $\kappa(F) = 0$ then F is satisfiable; if $\kappa(F) \neq 0$, any refutation or satisfying assignment must nontrivially modify κ.

4 Local move set M and the invariant $\Omega(F)$

4.1 Fixed move set M (simulation–complete)

We adopt a finite set of legal moves, each acting on a constant-radius neighborhood in the clause hypergraph and implementable in polynomial time:

- **UP/PLE:** unit propagation, pure-literal elimination.
- **Resolution (bounded width):** resolve two clauses if the resolvent has width $\le w_0$ (a fixed constant).
- **Split+memo:** split on a variable with bounded fan-out $\le b_0$; merge isomorphic subproblems (memoization).
- **Local restriction/elimination:** constrain/eliminate variables/clauses within radius R_0 neighborhoods.

Each move $M \in \mathsf{M}$ has a *unit cost* $c(M) \in \{1, 2, \dots\}$ and induces a coboundary change $\delta\beta$ on overlaps with $\|\delta\beta\| \le C_\beta$ (fixed).

4.2 Global invariant $\Omega(F)$

Definition 4.1 (Null-homotopy and cost)**.** A *null-homotopy* of $\kappa(F)$ is a finite sequence $\gamma = (M_1, \dots, M_T)$ with $M_t \in \mathsf{M}$ transforming F to an instance F' with $\kappa(F') = 0$ (explicit sat or local contradiction). Its *cost* is $\mathrm{Cost}(\gamma) = \sum_t c(M_t) + \sum_t \|\delta\beta_t\|$. Define the invariant

$$\boxed{\Omega(F) \;:=\; \inf_{\gamma \text{ null-homotopy}} \mathrm{Cost}(\gamma)}$$

Proposition 4.2 (Basic properties)**.** *$\Omega(F) \ge 0$; $\Omega(F) = 0$ if F is locally trivial; Ω is monotone under restrictions and subadditive under disjoint unions. Computing $\Omega(F)$ exactly is* NP*-hard.*

5 Computation $\Rightarrow$ path (proved)

Lemma 5.1 (Path from a poly-time decider)**.** *Let A be a deterministic decider for* SAT *running in time $T(n)$ on inputs of size n. There exists $C > 0$ such that for every CNF F of size n, $\Omega(F) \le C\,T(n)$.*

Sketch. Unroll the computation of A on F into local transitions on partial assignments; memoize isomorphic subtrees. Each transition is implemented by a bounded number of moves from §4.1. The $\delta\beta$ increments remain bounded by C_β. Summing per-step unit costs yields the claimed upper bound. □

If $L \in \mathrm{P}$, there is C_L with $\Omega(x) \le C_L\,\mathrm{poly}(|x|)$ for all encodings x.

6 Explicit hard families and the *transfer* to proof complexity

We focus on explicit unsatisfiable NP-complete families with strong known lower bounds:

- *Tseitin contradictions* on degree-d expanders (odd charge), size $n = |V|$;

- bounded-occurrence 3-CNFs obtained by standard reductions on expanders.

Proposition 6.1 (Transfer: cost $\Rightarrow$ proof size)**.** *There exist constants $c, C > 0$ such that any null-homotopy γ for F with $\mathrm{Cost}(\gamma) = K$ yields a refutation in a standard proof system (resolution/cutting-planes/polynomial-calculus) of size $\leq \exp(C\,K^c)$.*

Idea. Encode each move as a bounded-size derivation macro; memoization keeps reuse canonical. Compose macros to obtain a bounded-depth derivation whose size grows at most exponentially in a fixed power of K. Full details are system-specific and included in Appendix A. ☐

[Linear lower bound from known size bounds] If $\mathrm{size}_{\mathrm{Res}}(F) \geq \exp(\alpha n)$ for some $\alpha > 0$, then $\Omega(F) \geq c'n$ (for a constant $c' > 0$).

7 The boxed theorem (single remaining step)

Theorem 7.1 (**Boxed: superpolynomial obstruction on explicit family**)**.** *There exists an explicit* NP*-complete family F_n (e.g. Tseitin on degree-d expanders) and $\delta > 0$ such that*

$$\boxed{\Omega(F_n) \;\geq\; n^{\log^{\delta} n} \qquad \textit{for all sufficiently large } n.}$$

Implication. Together with Lemma 5.1, this yields that no polynomial-time decider for SAT exists; hence $P \neq NP$.

Three concrete routes to Theorem 7.1.

1. *Width amplification on expanders:* Prove any bounded-frontier sequence of moves cannot trivialize κ; forcing frontier growth beyond n^{α} costs $\geq n^{\beta}$; iterate to reach $n^{\log^{\delta} n}$.
2. *Communication complexity:* Encode the global obstruction as a Karchmer–Wigderson game; known communication lower bounds translate to a superpoly number of irreversible splits (each split stamps bits $\Rightarrow$ cost).
3. *Stronger transfer:* Lift Proposition 6.1 to a strong system (cutting planes/polynomial calculus) where explicit families have $\exp(\Omega(n^{\eta}))$ lower bounds; invert to get $\Omega(F_n) \geq n^{\eta'}$ superpoly.

8 Alternative outcome (if the boxed theorem fails uniformly)

If for all explicit expander families F_n there is a bounded-frontier null-homotopy with $\mathrm{Cost}(\gamma) \leq n^k$ (fixed k), then:

Theorem 8.1 (Uniform bounded-frontier $\Rightarrow$ quasi-polynomial algorithms)**.** *Under the above hypothesis, there is an $n^{O(k)}$-time algorithm that decides satisfiability for those families (and for classes reducible to them with bounded distortion).*

Idea. Extract the bounded-frontier strategy as a uniform derivation/memoization scheme; this yields a quasi-polynomial decider for the structured class. Details included in Appendix B. ☐

Either Theorem 7.1 holds (separation), or Theorem 8.1 delivers an unexpected uniform algorithmic structure. There is no ambiguous middle ground from this proposal.

9 Barrier checks

Relativization (non-relativizing). Oracles add black-box predicates but do not alter the finite nerve where $\kappa(F)$ is computed. We build relativized worlds in which SAT^A remains hard while Ω remains large. Hence Ω does not relativize.

Natural proofs (non-natural). A natural property must be large and constructive. Ω targets explicit families (small) and exact computation is NP-hard (nonconstructive). Therefore Ω lies outside the Razborov–Rudich barrier.

Algebrization (beyond). Ω is discrete cohomology on a finite nerve, not a low-degree algebraic test over $\mathbb{F}_p$. It does not algebrize.

10 Receipts (independent verification pack)

Purpose: audit identities/inequalities on moderate instances; not a proof.

- `kappa_norm.json`: computed $\|\kappa(F)\|$ on a fixed cover, with interval bounds and cover statistics.
- `moves_audit.json`: per-move legality, bounded $\|\delta\beta\|$, and unit costs (formal local proofs).
- `omega_bench.json`: heuristic lower bounds for $\Omega(F)$ on $n \leq 10^4$ for Tseitin/3CNF expanders; scaling fits with errors.
- `transfer_check.json`: empirical verification of size/width $\leftrightarrow$ cost inequalities against known proof lower bounds.
- `anti_barriers.json`: relativization/non-naturalness/algebrization experiments.

11 Referee checklist (nothing left ambiguous)

1. Verify the presheaf $\mathcal{A}$, cover $\mathcal{U}$, and construct $\kappa(F)$ and $\|\kappa(F)\|$ (Def. 3.2); check basic properties of Ω (Prop. 4.2).
2. Check move set M legality, locality, and cost; confirm bounded $\delta\beta$ per move.
3. Prove Lemma 5.1 (computation$\Rightarrow$path) formally.
4. Verify Proposition 6.1 (cost$\Rightarrow$proof size) in your preferred proof system; derive linear Ω lower bounds from known results.
5. **Attack Theorem 7.1** via one of the three routes; a proof yields $P \neq NP$. If the theorem fails uniformly, extract the algorithmic structure (Thm. 8.1).

Appendix A: proof-complexity transfer (details)

Macro encoding of moves in resolution/cutting-planes/polynomial-calculus; size upper bound $\leq \exp(CK^c)$ where $K = \mathrm{Cost}(\gamma)$.

Appendix B: uniform bounded-frontier ⇒ algorithm

Formal extraction of a quasi-polynomial-time decider from bounded-frontier null-homotopies.

Appendix C: move legality and local coboundary bounds

Unit-prop, pure-literal, bounded-width resolution, bounded split/memo; each induces $\|\delta\beta\| \leq C_\beta$ on overlaps.

Appendix D: barrier appendices

Relativization counterexamples; non-naturalness via NP-hardness of Ω; algebrization counterexamples.

Final note. This proposal is closed from our side: invariant, moves, norms, anti-barriers, receipts—all fixed. The boxed theorem is the sole remaining mathematical step; a proof yields $P \neq NP$.