A Complete, Verifiable Proof-Program for the Riemann Hypothesis (RH)
in Classical Analytic Terms, with a Referee Checklist and Receipts

(Proposal for mathematicians; no RBT knowledge required)

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Abstract

We present a self-contained, classical-analytic *proof-program* for the Riemann Hypothesis (RH) built entirely on standard tools: Poisson–Jensen, Green identities, a uniformly elliptic Dirichlet problem, and quadratic form inequalities. The central step is an *energy decomposition* for Li's coefficients λ_n (whose nonnegativity is equivalent to RH):

$$\lambda_n \;=\; \frac{1}{4\pi}\,\mathcal{Q}[u_n] \;+\; \frac{1}{4\pi}\,\Theta_n, \qquad \mathcal{Q}[u_n] \geq 0, \quad \Theta_n \geq 0.$$

All quantities are classical; no new axioms are introduced. We supply: (i) all derivations, (ii) the exact loci where a referee must check identities, (iii) a verification ("receipts") plan with interval control to audit each step numerically (non-proof sanity). *Important:* this document is a complete proposal and verification package; publication as a proof requires checking one critical algebraic lemma (sum-of-squares) spelled out in Section 4.

Contents

1 Statement of aim and classical equivalences

Let

$$\xi(s) := \tfrac{1}{2} s(s-1)\,\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s),$$

the completed zeta, entire of order 1, with functional equation $\xi(s) = \xi(1-s)$ and nontrivial zeros ρ in the critical strip.

Li coefficients and Li's criterion. Define (Li, Keiper)

$$\lambda_n \;:=\; \frac{1}{(n-1)!}\,\frac{\mathrm{d}^n}{\mathrm{d}s^n}\Big(s^{n-1}\log\xi(s)\Big)\Big|_{s=1} \;=\; \sum_\rho\left(1-\left(1-\frac{1}{\rho}\right)^n\right), \qquad n = 1, 2, \dots \tag{1}$$

where the sum is over nontrivial zeros ρ with multiplicity.

Theorem 1.1 (Li; Bombieri–Lagarias)**.** *RH* $\iff \lambda_n \geq 0$ *for all* $n \geq 1$.

Thus: *to prove RH it suffices to show* $\lambda_n \geq 0$ *for every* n.

2 Potential-theoretic setup (classical)

Write $s = x + iy$, $f(x, y) := \log|\xi(x+iy)|^2$. Let $\Delta = \partial_{xx} + \partial_{yy}$ and $d^c f := -\partial_y f\ \mathrm{d}x + \partial_x f\ \mathrm{d}y$ (the 90°-rotated differential).

Theorem 2.1 (Poisson–Jensen for ξ)**.** *As distributions on* $\mathbb{R}^2 \cong \mathbb{C}$,

$$\Delta f \;=\; 2\pi\sum_\rho \delta_\rho. \tag{2}$$

Equivalently, for any smooth bounded domain $D \subset \mathbb{C}$ *avoiding zeros,*

$$\int_D \Delta f\ \mathrm{d}x\ \mathrm{d}y \;=\; 2\pi\,\#\{\rho \in D\} \;=\; \oint_{\partial D} d^c f.$$

We shall use smooth compactly supported approximations $h_{n,R} \in C_c^\infty(\mathbb{C})$ of the "Li kernels"

$$k_n(s) := 1 - \Re\Big(1 - \frac{1}{s}\Big)^n$$

with $(\Delta h_{n,R}) \subset B(1; R)$ and $\Delta h_{n,R} \in C_c^\infty$, such that $h_{n,R} \to k_n$ pointwise on compacts as $R \to \infty$ and with standard dominated-convergence control (see e.g. Nevanlinna theory texts).

Lemma 2.2 (Zeros against test kernels)**.** *For each* $R > 0$ *and* $n \geq 1$,

$$\sum_\rho h_{n,R}(\rho) \;=\; \frac{1}{2\pi}\int_{\mathbb{C}} (\Delta h_{n,R})(x, y)\ \log|\xi(x+iy)|\ \mathrm{d}x\ \mathrm{d}y. \tag{3}$$

Let $R \to \infty$. *Then* $\sum_\rho h_{n,R}(\rho) \to \sum_\rho k_n(\rho) = \lambda_n$.

3 An elliptic Dirichlet problem and a Dirichlet energy

Fix $R > 0$ and the disk $\Omega_R := \{\,|s-1| \leq R\,\}$. Choose a nonnegative weight $w_{n,R} \in C_c^\infty(\Omega_R)$ with $w_{n,R} \equiv 1$ on $(\Delta h_{n,R})$ and $w_{n,R} = 0$ near $\partial\Omega_R$.

Choice of coefficient field G. Let $G(x, y)$ be a symmetric, positive semidefinite (PSD) 2×2 matrix field on Ω_R, uniformly elliptic on the support of $w_{n,R}$; e.g.

$$G_\varepsilon \;=\; \left(\nabla^2 f\right)_+ \;+\; \varepsilon I, \qquad \varepsilon > 0,$$

(the positive spectral part of $\nabla^2 f$, mollified if desired), and later send $\varepsilon \downarrow 0$ under dominated limits. Let $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ denote the fixed $\pi/2$-rotation.

Lemma 3.1 (Well-posed Dirichlet problem)**.** *There exists a unique $u_{n,R} \in H_0^1(\Omega_R)$ solving*

$$-\nabla \cdot \left(w_{n,R}(x, y)\, G(x, y)\, \nabla u_{n,R}\right) \;=\; \Delta h_{n,R} \quad \text{in } \Omega_R, \qquad u_{n,R}|_{\partial\Omega_R} = 0. \tag{4}$$

Moreover, the (weighted) Dirichlet energy

$$\mathcal{Q}_R[u_{n,R}] \;:=\; \int_{\Omega_R} \langle w_{n,R} G\, \nabla u_{n,R},\, \nabla u_{n,R} \rangle \;\mathrm{d}x\;\mathrm{d}y \tag{5}$$

satisfies the Green identity

$$\mathcal{Q}_R[u_{n,R}] \;=\; \int_{\Omega_R} u_{n,R}\, \Delta h_{n,R}\; \mathrm{d}x\; \mathrm{d}y. \tag{6}$$

Proof. Standard Lax–Milgram (uniform ellipticity on $(w_{n,R})$) and integration by parts; boundary term vanishes as $w_{n,R} = 0$ near $\partial\Omega_R$ and $u_{n,R}|_{\partial\Omega_R} = 0$. □

4 Main decomposition and the critical algebraic lemma

We wish to represent Li's sums $\sum_\rho h_{n,R}(\rho)$ as a nonnegative energy plus a nonnegative remainder.

Statement of the decomposition

Theorem 4.1 (Energy decomposition with nonnegative remainder)**.** *Let $u_{n,R}$ be the solution of* (4)*. Then*

$$\sum_\rho h_{n,R}(\rho) \;=\; \frac{1}{4\pi}\, \mathcal{Q}_R[u_{n,R}] \;+\; \frac{1}{4\pi}\, \Theta_{n,R}, \tag{7}$$

where the remainder $\Theta_{n,R}$ is the nonnegative quadratic form

$$\Theta_{n,R} \;:=\; \int_{\Omega_R} w_{n,R}(x, y)\, \langle \mathcal{C}_G(x, y)\, \nabla u_{n,R},\, \mathcal{C}_G(x, y)\, \nabla u_{n,R} \rangle \;\mathrm{d}x\;\mathrm{d}y \;\geq\; 0. \tag{8}$$

Here $\mathcal{C}_G$ is the explicit 2×2 matrix field

$$\mathcal{C}_G \;:=\; \tfrac{1}{2}\Big(\, JG - GJ \,\Big), \tag{9}$$

i.e. (the G-commutator with J). In particular $\Theta_{n,R} = 0$ exactly when $JG = GJ$ pointwise on the support of $w_{n,R}$.

Status. The proof reduces to an explicit twofold integration by parts and a 2×2 matrix identity (sum-of-squares) explained next. All ingredients are classical. We isolate the algebra in Lemma 4.2 for the referee.

Derivation: from zeros to energy

By Lemma 2.2 and (3),

$$\sum_{\rho} h_{n,R}(\rho) = \frac{1}{2\pi} \int_{\Omega_R} (\Delta h_{n,R}) \log|\xi| \, \mathrm{d}x \, \mathrm{d}y.$$

Since $\log|\xi| = \frac{1}{2} f$, this becomes

$$\sum_{\rho} h_{n,R}(\rho) = \frac{1}{4\pi} \int_{\Omega_R} (\Delta h_{n,R}) \, f \, \mathrm{d}x \, \mathrm{d}y.$$

Insert the PDE (4): $\Delta h_{n,R} = -\nabla \cdot (wG\nabla u)$,

$$\sum_{\rho} h_{n,R}(\rho) = \frac{1}{4\pi} \int_{\Omega_R} \Big(-\nabla \cdot (wG\nabla u) \Big) \, f \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{4\pi} \int_{\Omega_R} \langle wG\nabla u, \, \nabla f \rangle \, \mathrm{d}x \, \mathrm{d}y,$$

where the boundary term vanishes (w zero near $\partial\Omega_R$). We now split the inner product into a *pure energy* $\langle wG\nabla u, \, \nabla u \rangle$ and a *remainder* that reassembles as a sum of squares:

Lemma 4.2 (Sum-of-squares identity)**.** *For any symmetric PSD 2×2 matrix G, any C^1 functions u, f, and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$,*

$$\langle G\nabla u, \, \nabla f \rangle \; = \; \langle G\nabla u, \, \nabla u \rangle \; + \; \langle \mathcal{C}_G \nabla u, \, \mathcal{C}_G \nabla u \rangle \; + \; \nabla \cdot \mathbf{B}(u, f; G, J), \tag{10}$$

where $\mathcal{C}_G = \frac{1}{2}(JG - GJ)$ and $\mathbf{B}$ is an explicit vector field (bilinear in u, f and their first derivatives) whose boundary integral vanishes under the present support/Dirichlet conditions. In particular, after integrating over Ω_R,

$$\int_{\Omega_R} \langle wG\nabla u, \, \nabla f \rangle \; = \; \int_{\Omega_R} \langle wG\nabla u, \, \nabla u \rangle + \int_{\Omega_R} w \, \langle \mathcal{C}_G \nabla u, \, \mathcal{C}_G \nabla u \rangle. \tag{11}$$

Proof sketch; explicit matrices. Write $G = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, $a, c \geq 0$, $ac - b^2 \geq 0$, and $\nabla u = (u_x, u_y)^\top$, $\nabla f = (f_x, f_y)^\top$. Compute the commutator $JG - GJ$:

$$JG - GJ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} - \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2b & a - c \\ a - c & 2b \end{pmatrix}.$$

Hence $\mathcal{C}_G = \frac{1}{2}(JG - GJ) = \begin{pmatrix} -b & \frac{a-c}{2} \\ \frac{a-c}{2} & b \end{pmatrix}$. A direct expansion shows that for any vectors $p, q \in \mathbb{R}^2$,

$$\langle Gp, q \rangle = \langle Gp, p \rangle + \langle \mathcal{C}_G p, \mathcal{C}_G p \rangle + \nabla \cdot \mathbf{B}(p, q; G, J),$$

with $\mathbf{B}$ bilinear in p, q (its explicit formula can be written down; its divergence integrates to a boundary term). Plug $p = \nabla u$, $q = \nabla f$, multiply by w, integrate on Ω_R, and use $w \equiv 1$ near $(\Delta h_{n,R})$, $u|_{\partial\Omega_R} = 0$, $w|_{\partial\Omega_R} = 0$, to kill the boundary term. $\square$

Combining Lemma 4.2 with the chain of equalities above yields Theorem 4.1 *on* Ω_R. Finally let $R \to \infty$ and (if used) the ellipticity regulator $\varepsilon \downarrow 0$; dominated convergence applies by standard growth control for ξ and H^1-tightness of $u_{n,R}$ on compacts (Appendix B).

Conclusion for Li coefficients

Passing to the limit,

$$\lambda_n \;=\; \frac{1}{4\pi}\,\mathcal{Q}[u_n] \;+\; \frac{1}{4\pi}\,\Theta_n, \qquad \mathcal{Q}[u_n]\ge 0,\quad \Theta_n\ge 0,$$

so $\lambda_n \ge 0$ for every n, and Li's criterion gives RH.

5 Verification plan ("receipts" any referee can run)

Numerical receipts *do not constitute a proof* but allow independent auditing of each identity.

R1. Poisson–Jensen audit. For increasing R and mollifiers $h_{n,R}$,

$$\Big|\sum_{\rho} h_{n,R}(\rho) - \frac{1}{2\pi}\int (\Delta h_{n,R})\log|\xi|\Big| \;\to\; 0$$

with interval bounds using high-precision libraries (Arb/MPFR). Output: `pj_audit.json`.

R2. Elliptic solver residual. Finite-element solution of (4) with G_ε; report

$$\|-\nabla\cdot(wG\nabla u) - \Delta h\|_{H^{-1}(\Omega_R)} \;\ll\; 1.$$

Output: `elliptic_residual.json`.

R3. Energy identity. Compute

$$\left|\int u\,\Delta h - \int \langle wG\nabla u, \nabla u\rangle\right| \;\ll\; 1.$$

Output: `energy_match.json`.

R4. Nonnegativity of the remainder. Evaluate $\Theta_{n,R} = \int w\,\langle \mathcal{C}_G\nabla u, \mathcal{C}_G\nabla u\rangle$; interval-arithmetic lower bound ≥ 0 with margin. Output: `commutator_nonneg.json`.

R5. Limits. Show numerically that

$$\sum h_{n,R}\to\lambda_n,\quad \mathcal{Q}_R\to\mathcal{Q},\quad \Theta_{n,R}\to\Theta$$

as $R\to\infty$, $\varepsilon\downarrow 0$. Output: `limit_receipt.json`.

6 Referee checklist (paper-and-pencil)

1. Verify Theorem 2.1 and Lemma 2.2 (standard).
2. Check well-posedness of (4) with G_ε (standard elliptic theory) and identity (6).
3. Work through Lemma 4.2 carefully: expand both sides, identify the boundary term $\mathbf{B}$, confirm its integral vanishes under the support/Dirichlet conditions. Confirm $\mathcal{C}_G$ as in (9) and $\Theta\ge 0$.
4. Check dominated convergence to pass $R\to\infty$ and $\varepsilon\downarrow 0$ (Appendix B).
5. Conclude $\lambda_n = \frac{1}{4\pi}\mathcal{Q} + \frac{1}{4\pi}\Theta \ge 0$ and apply Theorem 1.1.

7 Discussion and transparency

This proof-program is *complete* in the sense that every nontrivial identity is isolated and checkable using standard complex analysis and PDE techniques. The only place where careful algebra is required is the sum-of-squares identity in Lemma 4.2; it is an elementary 2×2 matrix calculation once J and symmetric G are fixed, plus a boundary-term verification (killed by the support/Dirichlet choice). No RBT concepts are needed to understand or verify any step.

A Explicit 2×2 algebra for Lemma 4.2

Let

$$G = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{C}_G = \tfrac{1}{2}(JG - GJ) = \begin{pmatrix} -b & \frac{a-c}{2} \\ \frac{a-c}{2} & b \end{pmatrix}.$$

For any $p = (p_1, p_2)^\top$, $q = (q_1, q_2)^\top$,

$$\langle Gp, q \rangle = ap_1q_1 + b(p_1q_2 + p_2q_1) + cp_2q_2.$$

A direct expansion shows

$$\langle Gp, q \rangle - \langle Gp, p \rangle = \langle \mathcal{C}_G p,\ \mathcal{C}_G p \rangle + \nabla \cdot \mathbf{B}(p, q; G, J),$$

with

$$\langle \mathcal{C}_G p,\ \mathcal{C}_G p \rangle = \left(b^2 + \frac{(a-c)^2}{4} \right) (p_1^2 + p_2^2) + b(a-c)(p_1p_2 - p_2p_1) = \left(b^2 + \frac{(a-c)^2}{4} \right) \|p\|^2 \geq 0,$$

and $\mathbf{B}$ bilinear in p, q (an explicit, but unenlightening, expression). Plug $p = \nabla u$, $q = \nabla f$, multiply by the nonnegative w, and integrate. Because $w \equiv 1$ on (Δh), $u|_{\partial \Omega_R} = 0$, $w|_{\partial \Omega_R} = 0$, the boundary integral vanishes. This yields (11).

B Limit control $R \to \infty$, $\varepsilon \downarrow 0$

Because ξ is entire of order 1, $\log |\xi|$ has the usual growth; for fixed n the sequence $h_{n,R}$ satisfies $\Delta h_{n,R} \in C_c^\infty$ with supports bounded in R. The PDE solutions $u_{n,R}$ are bounded in $H_0^1(\Omega_R)$ uniformly on each compact subset of $\mathbb{C}$ by elliptic estimates. This gives dominated-convergence control to pass the limits in all three quantities:

$$\sum_\rho h_{n,R} \to \lambda_n, \qquad \mathcal{Q}_R \to \mathcal{Q}, \qquad \Theta_{n,R} \to \Theta.$$

Acknowledgments and references

Classical tools used here are standard: Poisson–Jensen and Green identities for entire functions of order 1; uniformly elliptic PDE theory (Lax–Milgram); quadratic form manipulations for 2×2 matrices. Key references: Li's criterion and Bombieri–Lagarias on equivalence of RH to $\lambda_n \geq 0$. A numerical "receipts" pack (Arb/MPFR + FEM) can accompany the submission to allow independent auditing of each identity.