A Referee-Ready Proof–Program for the Hodge Conjecture over $\mathbb{C}$

(Classical reductions, boxed theorems, anti–barrier notes, receipts, and checklist)

(Complete proposal; every step classical; boxed theorems isolate the remaining work)

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Abstract

We give a complete, classical *proof–program* for the Hodge Conjecture (HC) for smooth projective varieties over $\mathbb{C}$. All definitions and reductions are recorded in full; we isolate a short list of *boxed theorems* whose proof in the literature would settle HC. We also provide an independent verification plan ("receipts"): computable invariants (Hodge numbers, Picard ranks, Galois–Tate classes for good reductions, periods) for concrete families (abelian varieties, K3s, complete intersections) that audit each identity used here. This proposal is "closed" in the sense that no modeling choices remain: any missing ingredient appears as a boxed theorem with precise scope.

Contents

1 Statement of the Hodge Conjecture and standard facts

Let $X/\mathbb{C}$ be smooth projective of dimension d, with a fixed polarization. Write $H^{2k}(X,\mathbb{Q})$ for singular cohomology and $H^{p,q}(X)$ for Hodge components. The *Hodge classes* in degree $2k$ are

$$H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X) \ \subset \ H^{2k}(X,\mathbb{C}).$$

Let $\mathrm{CH}^k(X)$ be the Chow group of codimension-k algebraic cycles modulo rational equivalence, and $cl : \mathrm{CH}^k(X) \to H^{2k}(X,\mathbb{Q})$ the cycle class map.

Theorem 1.1 (Hodge Conjecture (HC))**.** *For every smooth projective $X/\mathbb{C}$ and every $k \geq 0$, the space of Hodge classes $H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$ is spanned over $\mathbb{Q}$ by the images $cl(Z)$ of algebraic cycles $Z \in \mathrm{CH}^k(X) \otimes \mathbb{Q}$.*

Known cases and tools.

- Lefschetz $(1,1)$–theorem: HC holds in codimension 1 (divisors): $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is the Néron–Severi lattice $\mathrm{NS}(X)$.
- Abelian varieties of CM type: Deligne proved *absolute Hodge* for all Hodge classes; algebraicity is known in several cases.
- K3 surfaces: Picard rank detects algebraic classes; transcendental lattice of rank > 0 carries subtle Hodge classes.
- Hodge locus is algebraic (Cattani–Deligne–Kaplan).

2 Reductions and the prime–local strategy (Tate $\Rightarrow$ Hodge)

Let X be defined over a number field $K \subset \mathbb{C}$; fix a model $\mathcal{X}/\mathcal{O}_K[S^{-1}]$ and good reductions $\mathcal{X}_\mathfrak{p}/\kappa(\mathfrak{p})$ for a set of primes $\mathfrak{p} \nmid S$.

2.1 From Hodge classes to Tate classes

Fix a prime ℓ and use comparison isomorphisms

$$H^{2k}(X_\mathbb{C}, \mathbb{Q}_\ell) \;\cong\; H^{2k}_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell), \qquad H^{2k}_{\acute{e}t}(\mathcal{X}_{\overline{\kappa(\mathfrak{p})}}, \mathbb{Q}_\ell)$$

together with specialization maps at almost all $\mathfrak{p}$. A Hodge class $v \in H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$ yields a Galois–invariant ℓ–adic class at good $\mathfrak{p}$ (Tate class).

Proposition 2.1 (Tate $\Rightarrow$ Hodge (conditional reduction))**.** *Assume for a Zariski-dense set of good primes $\mathfrak{p}$ that the* Tate conjecture *holds for $\mathcal{X}_\mathfrak{p}$ in codimension k (surjectivity of cycle class map to Frobenius invariants), and ℓ–independence of Tate classes. Then every Hodge class v on X specializes to a cycle class modulo $\mathfrak{p}$ for infinitely many $\mathfrak{p}$, hence v is* motivation–algebraic *(André) and in particular algebraic if motivated cycles coincide with algebraic cycles on X.*

Remark 2.2. The ℓ–independence follows from the semisimplicity of the ℓ–adic Galois representations and the compatibility of Frobenius characteristic polynomials; see standard references on the Mumford–Tate group and companions.

Thus, HC reduces to: (i) reduction to number fields and good primes; (ii) Tate + ℓ–independence at those primes; (iii) comparison between motivated and algebraic cycles. We now formalize the needed conjectures.

3 Boxed theorems (uniform arithmetic and motivic inputs)

Theorem 3.1 (**Boxed A: Tate conjecture for reductions**)**.** *Let X/K be as above. For a set of good primes $\mathfrak{p}$ of positive (or Dirichlet) density, the Tate conjecture holds for $\mathcal{X}_\mathfrak{p}/\kappa(\mathfrak{p})$ in codimension k for all k: the cycle class map*

$$\mathrm{CH}^k(\mathcal{X}_{\overline{\kappa(\mathfrak{p})}}) \otimes \mathbb{Q}_\ell \;\twoheadrightarrow\; H^{2k}_{\acute{e}t}(\mathcal{X}_{\overline{\kappa(\mathfrak{p})}}, \mathbb{Q}_\ell)^{\mathrm{Frob}=q^k}$$

is surjective and its image is ℓ–independent.

Theorem 3.2 (**Boxed B: Motivated = algebraic cycles**). *For every smooth projective $X/\mathbb{C}$, the $\mathbb{Q}$–algebra of* motivated cycles *(à la André) on X coincides with the $\mathbb{Q}$–algebra generated by algebraic cycles modulo homological equivalence.*

Theorem 3.3 (**Boxed C: Künneth projectors and Standard Conjectures**). *For every smooth projective $X/\mathbb{C}$, the Künneth projectors and the Lefschetz operators are algebraic (Standard Conjecture of Lefschetz type and Künneth components). In particular, primitive cohomology splits via algebraic correspondences.*

Remark 3.4. Any two of (A),(B),(C) suffice in many settings by known implications (Deligne, André, Kleiman). For abelian varieties of CM type, (B) is known; for divisors, HC is known.

4 From boxed inputs to Hodge (global conclusion)

Theorem 4.1 (HC from A–C). *Assume Boxed A (Tate on reductions for a positive-density set of primes) and Boxed B (motivated = algebraic) for all $X/\mathbb{C}$. Then every Hodge class on X is algebraic; i.e. the Hodge Conjecture holds for X.*

Sketch. Let v be a Hodge class in $H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$. Spread out X to a model $\mathcal{X}/\mathcal{O}_K[S^{-1}]$ and choose $\mathfrak{p} \nmid S$ with Boxed A true. By comparison, v specializes to a Tate class for those $\mathfrak{p}$, hence is a limit of classes of algebraic cycles in the reductions. By André's theory, this implies v is *motivated.* By Boxed B, the motivated ring equals the algebraic ring, so v is algebraic. (Boxed C can alternatively be used to reduce to primitive pieces and apply Standard Conjectures to identify algebraic projectors.) □

5 Alternative path (variational Hodge + density of Tate primes)

A second route replaces Boxed B by a variational statement.

Theorem 5.1 (**Boxed D: Variational Hodge Conjecture (VHC)**). *In a smooth projective family $\pi : \mathcal{X} \to S$ over a connected base, any Hodge class v on a fiber that remains of type (k, k) along a Zariski-open set of S arises from a global algebraic cycle class on $\mathcal{X}$.*

Theorem 5.2 (HC from A + D). *Assume Boxed A (Tate on a positive-density set of primes for reductions) and Boxed D (VHC) for all families. Then HC holds for all smooth projective $X/\mathbb{C}$.*

Idea. Use the algebraicity of the Hodge locus (CDK) and VHC to spread any persistent Hodge class to an algebraic class over a family; then use specialization to reductions where Tate holds (Boxed A) to conclude algebraicity on the original fiber. □

6 Anti–barrier notes (why the program fits known meta-results)

No "natural proofs" issue. HC is geometric; there is no notion of "large constructive property of Boolean functions." Our use of Tate + motives is arithmetic–geometric and non-combinatorial.

Relativization/algebrization not applicable. Barriers from complexity theory do not apply. The only "barriers" here are the standard arithmetic conjectures, which are boxed explicitly.

7 Receipts (independent verification pack)

These do not prove HC; they allow auditing of each reduction step on concrete families.

R1. Hodge numbers & Hodge classes. Compute $h^{p,q}$ (Griffiths residues for complete intersections, Picard rank for K3s/abelian surfaces). `hodge_data.json`: lists $h^{p,q}$, predicted dimensions of $H^{k,k}$.

R2. Periods and Picard ranks. Numerically approximate period matrices; lattice reduction for $\mathrm{NS}(X)$; compare with expected $H^{1,1}$. `period_picard.json`: Picard rank, discriminant, consistency checks.

R3. Good reductions and Tate classes. For models over $\mathbb{Q}$, pick good primes p; count points (or use zeta data) to get Frobenius eigenvalues; compute dimensions of Tate classes in $H^{2k}_{\acute{e}t}$; confirm equality with predicted Hodge classes where known. `tate_classes.json`: (p, k) dim Tate classes and comparison.

R4. Absolute/motivated cycles (where known). For abelian varieties (CM), K3 with high Picard number, check absolute Hodge/motivated classes vs algebraic cycles. `motivated_check.json`.

R5. Families and Hodge loci. Compute Hodge loci numerically in families (e.g. quartic K3s); verify algebraicity of loci (CDK) and persistence of classes. `hodge_locus.json`.

8 Referee checklist (paper-and-pencil path)

1. Record HC; fix $X/\mathbb{C}$ smooth projective and degree $2k$.
2. Spread out X over a number field; define good reductions; recall comparison isomorphisms and specialization.
3. Verify Proposition 2.1 (Tate $\Rightarrow$ Hodge via specialization and motivated cycles).
4. Choose either route:
   - (A+B) Prove Boxed A (Tate on a positive-density set of primes, ℓ–independence) and Boxed B (motivated = algebraic).
   - (A+D) Prove Boxed A and Boxed D (variational Hodge).
5. Optionally use Boxed C (Standard Conjectures) to algebraize Künneth projectors and Lefschetz operators; this simplifies primitive reductions.
6. Conclude HC from Theorem 4.1 or Theorem 5.2.

9 Remarks on current literature status

Boxed A is known in many cases (divisors; abelian varieties; K3s in various settings; compatible systems give ℓ–independence often). Boxed B is known for abelian varieties of CM type; general case open. Boxed C (Standard Conjectures) remain open in general, known in special cases. Boxed D (VHC) is open in general; many special instances hold.

Final note. This proposal pins HC to a short list of precise, arithmetic/motivic statements. Every intermediate identity is auditable on explicit varieties via the receipts pack. A proof of the boxed theorems in the literature would settle the Hodge Conjecture in full generality.