Homework #2

(Due: April 19)

GROUP NUMBER: 18

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Collaborating Groups

Group Number	Specifics (e.g., specific group member? specific task/subtask?)
NA	NA

External Resources Used

	Specifics (e.g., cite papers, webpages, etc. and mention why each was used)
1.	NA
2.	
3.	
4.	
5.	

Q1) a

Here we are asked to show that it takes $n - \frac{1}{3}$ comparisons to compute the 7th and 8th steps of the algorithm to obtain a stable partitioning of the array $A[1:n]$.

Firstly let's check what step 7 does.

$$x = median\ of A[q], A[2], A[3]$$

So for our array $q = 1$ as it starts from 1. And Step 8 generates.

$$rearrange\ the\ numbers of A[1:n] such that$$

$$A[t] = x\ for\ some\ t \in [1, n]$$

$$A[i] < x\ for\ each\ t \in [1, t-1]$$

$$A[i] > x\ for\ each\ t \in [t+1, n]$$

This explains to place the median x obtained in Step 7 at a specific position in the original array such that all numbers to the left are smaller and right are greater than x.

Now as per the deterministic Quick Sort partitioning as discussed in the class, we know after fixing on a pivot, we do at maximum $n-1$ comparisons to move the pivot to a position such that all the values before this x in the array are smaller and the to the right are larger.

Now, in this algorithm we have a different approach where instead of taking the first index as pivot, we fix on the pivot by finding the median of the first three elements.

Let's find the average number of comparisons required to find the median of 3 numbers a,b,c. Let's fix on an algorithm where we check if $a < b$ or $a > b$, $b < c$ or $b > c$ and if we can't find median using this, then go to $a < c$ or $a > c$

So in the total 3! permutations of the 3 numbers, there will be only two scenarios where we can find the median in two comparisons. i.e. when

$$a < b\ and\ b < c\ then\ b\ is\ the\ median$$

$$a > b\ and\ b > c\ then\ b\ is\ the\ median$$

So in the rest cases, we need to go to check for the third comparison too to find the median. So in total the number of comparisons for all permutations come as $4 * 3 + 2 * 2 = 16$. And the number of possible permutations is 6. So in total the average number of comparisons performed here are $\frac{16}{6} = \frac{8}{3}$

Where after finding the median, we know for sure that one element which is smaller is on the left after the partitioning and one element is on the right after the partitioning. So there would be more $n-1$ comparisons to perform. This comparisons stay the same what the permutation of original array is provided. So the average would also be $n-3$. So the total average number of comparisons performed is

$$\boxed{\frac{8}{3} + n - 3 = n - \frac{1}{3}}$$

Q1) b

Now t_n be the average number of element comparisons performed by the algorithm on an input array of size n , where the average is taken over all $n!$ permutations of the array and all n possibilities of the value k .

So initially let's take for different values of n, the base conditions are for $n < 1$ we can return 0. And when $n = 1$, we move into the if condition at Line 3. Sorting a 1 length array would be done in 0 comparisons. Similarly when $n = 2$, we move again into the if condition at Line 3. Sorting a 2 length array would also be done in 1 comparison. And when $n = 3$, sorting a 3 length array would take 3 comparisons.

Now our goal is to calculate when $n > 3$ where it enters into the recurrence equations.

As before we calculated the average number of comparisons taken to get a specific k^{th} rank element from an array using the algorithm shown in Figure 1, this is $n - \frac{1}{3}$. And this remains the same for all values of $k \in [1, n]$. So even the average remains the same for partitioning the array w.r.t the median found from the first 3 elements for all possible values of k.

Now let's calculate for the recursion part. Similar to the Quick Sort algorithm discussed in our class, where we took the first element as pivot and moved forward with the recursion, let's take the first three elements fixed now; such that there exists an i^{th} ranked element and we try to partition around it for values of $i \in [1, n]$. So in such case, we can find the average number of comparisons made for all possible values.

So for selecting the first three elements where there exists an i^{th} ranked element, we need to select the other two elements in such a way that one is lesser and the other is bigger. So the total number of ways are $(i-1)*(n-i)$ with 3! permutations and similarly we can take the rest array in $(n-3)!$ permutations.

So finally the equation for $n > 3$ looks like.

$$total\ number\ of\ ways = T = \sum_{k=1}^{n}\sum_{i=1}^{n}(i-1)(n-i)3!(n-3)!([i > k]t_{i-1} + [i < k]t_{n-k})$$

Here the $[i > k]$ and $[i < k]$ are Indicator variables which give 1 when True and 0 when false.

$$T = \sum_{k=1}^{n}\sum_{i=1}^{k}(i-1)(n-i)3!(n-3)!t_{n-i} + \sum_{k=1}^{n}\sum_{i=k+1}^{n}(i-1)(n-i)3!(n-3)!t_{i-1}$$

$$T = 6(n-3)!\sum_{k=1}^{n}\sum_{i=1}^{k}(i-1)(n-i)t_{n-i} + 6(n-3)!\sum_{k=1}^{n}\sum_{i=k+1}^{n}(i-1)(n-i)t_{i-1}$$

Now we will interchange the summations.

$$T = 6(n-3)! \sum_{i=1}^{n} \sum_{k=i}^{n} (i-1)(n-i)t_{n-i} + 6(n-3)! \sum_{i=1}^{n} \sum_{k=1}^{i-1} (i-1)(n-i)t_{i-1}$$

$$T = 6(n-3)! \sum_{i=1}^{n} (i-1)(n-i)^2 t_{n-i} + 6(n-3)! \sum_{i=1}^{n} (i-1)^2(n-i)t_{i-1}$$

And we know, the total number of possible ways are number of possible k^s i.e. n and number of ways to permutate the array $n!$. So total number of ways are $n * n!$. So we will find $\frac{T}{n*n!}$ and add up with $n - \frac{1}{3}$ to find the t_n.

$$t_n = n - \frac{1}{3} + \frac{6(n-3)! \sum_{i=1}^{n} (i-1)(n-i)^2 t_{n-i} + 6(n-3)! \sum_{i=1}^{n} (i-1)^2(n-i)t_{i-1}}{n * n!}$$

$$t_n = n - \frac{1}{3} + \frac{6(n-3)!}{n * n!} \sum_{i=1}^{n} (i-1)(n-i)^2 t_{n-i} + \frac{6(n-3)!}{n * n!} \sum_{i=1}^{n} (i-1)^2(n-i)t_{i-1}$$

$$t_n = n - \frac{1}{3} + \frac{6}{n^2(n-1)(n-2)} \sum_{i=1}^{n} (i-1)(n-i)^2 t_{n-i} + \frac{6}{n^2(n-1)(n-2)} \sum_{i=1}^{n} (i-1)^2(n-i)t_{i-1}$$

$$t_n = n - \frac{1}{3} + \frac{6}{n^2(n-1)(n-2)} \sum_{i=1}^{n} (i-1)(n-i)((n-i)t_{n-i} + (i-1)t_{i-1})$$

So finally we proved that

$$\boxed{t_n = \begin{cases} 0 \; if \;\, n < 2, \\ 1 \; if \;\, n = 2, \\ 3 \; if \;\, n < 3, \\ n - \frac{1}{3} + \frac{6}{n^2(n-1)(n-2)} \sum_{i=1}^{n} (i-1)(n-i)((n-i)t_{n-i} + (i-1)t_{i-1}) \; otherwise. \end{cases}}$$

Q1) c

Let us take the generating function for t_n as

$$T(x) = t_0 + t_1 x + t_2 x^2 + ... + t_n x^n + ...$$

Based on the base conditions, we can take $t_0, t_1 = 0$, $t_2 = 1$. For $n = 3$, it was given as part of the base condition. But this is not required if we take it as part of the recurrence equation. Where we have the median which is middle element in $n - \frac{1}{3}$ comparisons and the recurrence equation will return using the base condition which returns with 0 comparisons as the sub-arrays are of 1 length, if asked for 1^{st} or 3^{rd} ranked element. So keeping the base condition as 3 for $n = 3$ is not the right way and let's pull that into the recursion too.

So the generation function looks like when $t_0, t_1 = 0$, $t_2 = 1$ and solving by differentiation w.r.t n looks like

$$T(x) = x^2 + \sum_{n=3}^{\infty} t_n x^n$$

where for $n > 2$,

$$t_n = n - \frac{1}{3} + \frac{6}{n^2(n-1)(n-2)} \sum_{i=1}^{n} (i-1)(n-i)((n-i)t_{n-i} + (i-1)t_{i-1})$$

So let's move forward to remove the n's in the denominator

$$T^{(1)}(x) = 2x + n \sum_{n=3}^{\infty} ((n - \frac{1}{3})(n) + \frac{6}{n(n-1)(n-2)} \sum_{i=1}^{n} (i-1)(n-i)((n-i)t_{n-i} + (i-1)t_{i-1}))x^{n-1}$$

Multiplying by x on both sides gives

$$xT^{(1)}(x) = 2x^2 + \sum_{n=3}^{\infty} ((n - \frac{1}{3})(n) + \frac{6}{n(n-1)(n-2)} \sum_{i=1}^{n} (i-1)(n-i)((n-i)t_{n-i} + (i-1)t_{i-1}))x^{n}$$

$$xT^{(2)}(x) + T^{(1)}(x) = 4x + \sum_{n=3}^{\infty} ((n - \frac{1}{3})(n^2) + \frac{6}{(n-1)(n-2)} \sum_{i=1}^{n} (i-1)(n-i)((n-i)t_{n-i} + (i-1)t_{i-1}))x^{n-1}$$

$$xT^{(3)}(x) + 2T^{(2)}(x) = 4 + \sum_{n=3}^{\infty} ((n - \frac{1}{3})(n^2)(n-1) + \frac{6}{(n-2)} \sum_{i=1}^{n} (i-1)(n-i)((n-i)t_{n-i} + (i-1)t_{i-1}))x^{n-2}$$

$$xT^{(4)}(x) + 3T^{(3)}(x) = \sum_{n=3}^{\infty} ((n - \frac{1}{3})(n^2)(n-1)(n-2) + 6 \sum_{i=1}^{n} (i-1)(n-i)((n-i)t_{n-i} + (i-1)t_{i-1}))x^{n-3}$$

Now in this equation if we check the two terms in the inner summation. $(n-i)t_{n-i}$ *and* $(i-1)t_{i-1}$. For the values $i \in [1, n]$. They follow the same pattern and their summation remains the same. So instead of both we can take $2 * (i-1)t_{i-1}$

$$xT^{(4)}(x) + 3T^{(3)}(x) = \sum_{n=3}^{\infty} ((n - \frac{1}{3})(n^2)(n-1)(n-2) + 12 \sum_{i=1}^{n} (i-1)(n-i)((i-1)t_{i-1}))x^{n-3}$$

$$xT^{(4)}(x) + 3T^{(3)}(x) = \sum_{n=3}^{\infty}(n - \frac{1}{3})(n^2)(n-1)(n-2)x^{n-3} + 12\sum_{n=3}^{\infty}\sum_{i=1}^{n}(i-1)^2(n-i)t_{i-1}x^{n-3}$$

Now on the right side let's take $\sum_{n=3}^{\infty}(n-\frac{1}{3})(n^2)(n-1)(n-2)x^{n-3}$ as A and $12\sum_{n=3}^{\infty}\sum_{i=1}^{n}(i-1)^2(n-i)t_{i-1}x^{n-3}$ as B.

Let's try to solve B first. $12\sum_{n=3}^{\infty}\sum_{i=1}^{n}(i-1)^2(n-i)t_{i-1}x^{n-3}$. Let's once try to expand the summation to check whether we can write in any other form.

n/i	1	2	3	4
3	0	$t_1 x$	0	0
4	0	$2t_1 x$	$2^2 t_2 x$	0
5	0	$3t_1 x^2$	$2^2 2t_2 x^2$	$3^2 t_3 x^2$
6	0	$4t_1 x^3$	$2^2 3t_2 x^3$	$3^2 2t_3 x^3$

So instead of adding the values from left to right, let add all the values in the column. So the final summation looks like

$$B = 12(t_1(1+2x+3x^2+4x^3+....)+2^2t_2x(1+2x+3x^2+4x^3+....)+3^2t_3x(1+2x+3x^2+4x^3+....)+....)$$

$$B = \frac{1}{(1-x)^2}\sum_{n=1}^{\infty}t_nx^{n-1}n^2$$

We know that

$$T(x) = \sum_{n=0}^{\infty}t_nx^n$$

$$T^{(1)}(x) = \sum_{n=0}^{\infty}nt_nx^{n-1}$$

$$xT^{(1)}(x) = \sum_{n=0}^{\infty}nt_nx^n$$

$$xT^{(2)}(x) + T^{(1)}(x) = \sum_{n=0}^{\infty}n^2t_nx^{n-1}$$

By substituting this in B we get

$$B = \frac{12}{(1-x)^2}(xT^{(2)}(x) + T^{(1)}(x))$$

Now let's solve for A $\sum_{n=3}^{\infty}(n-\frac{1}{3})(n^2)(n-1)(n-2)x^{n-3}$

$$\sum_{n=3}^{\infty}(n-\frac{1}{3})(n^2)(n-1)(n-2)x^{n-3}$$

$$\sum_{n=3}^{\infty} n^3(n-1)(n-2)x^{n-3} - \frac{1}{3}\sum_{n=3}^{\infty} n^2(n-1)(n-2)x^{n-3}$$

We can take the infinite sequence of $1+x+x^2+x^3+....$ and do some differentiation on the sequence to reach the desired sequence mentioned above. Finally we'll come to a solution which is

$$\frac{6x^2+60x+54}{(1-x)^6} - \frac{1}{3}\frac{18+6x}{(1-x)^5}$$

$$\frac{6x^2+60x+54}{(1-x)^6} - \frac{6+2x}{(1-x)^5}$$

$$\frac{6x^2+60x+54-6-2x+6x+2x^2}{(1-x)^6}$$

$$\frac{8x^2+64x+48}{(1-x)^6}$$

$$\frac{8(x^2+8x+6)}{(1-x)^6}$$

Now substituting everything back to the equation results us in

$$xT^{(4)}(x) + 3T^{(3)}(x) = \frac{8(x^2+8x+6)}{(1-x)^6} + \frac{12}{(1-x)^2}(xT^{(2)}(x) + T^{(1)}(x))$$

$$\boxed{x(1-x)^2T^{(4)}(x) + 3(1-x)^2T^{(3)}(x) - 12xT^{(2)}(x) - 12T^{(1)}(x) = \frac{8(x^2+8x+6)}{(1-x)^4}}$$

Q1) d

From 1c we have

$$x(1-x)^2T^{(4)}(x)+3(1-x)^2T^{(3)}(x)-12xT^{(2)}(x)-12T^{(1)}(x)=\frac{8(x^2+8x+6)}{(1-x)^4}$$

Let's try to solve the LHS if it's a form of a derivative of a function. We can write the above function in this format too.

$$x(1-x)^2T^{(4)}(x)\ +(1-x)(1-3x)T^{(3)}(x)\ +2(1-x)T^{(3)}(x)\ -2T^{(2)}(x)\ +(2-12x)T^{(2)}(x)\ -12T^{(1)}(x)$$

This above equation is nothing but

$$\frac{d}{dz}(x(1-x)^2T^{(3)}(x)\ \ +2(1-x)T^{(2)}(x)\ \ +2(1-6x)T^{(1)}(x)+c_1)$$

Now let's try to solve the RHS. We can write in the form as

$$\frac{8(x^2-2x+1-10+10x+15)}{(1-x)^4}$$

$$8(\frac{1}{(1-x)^2}-\frac{10}{(1-x)^3}+\frac{15}{(1-x)^4})$$

We can write this in this form too

$$\frac{d}{dz}(8(\frac{1}{1-x}-\frac{5}{(1-x)^2}+\frac{5}{(1-x)^3})+c_2)$$

$$\frac{d}{dz}(\frac{8(x^2+3x+1)}{(1-x)^3}+c_2)$$

Now equating both LHS and RHS provides us

$$x(1-x)^2T^{(3)}(x)\ \ +2(1-x)T^{(2)}(x)\ \ +2(1-6x)T^{(1)}(x)=\frac{8(x^2+3x+1)}{(1-x)^3}+c$$

Now when we substitute x=0 and we know the values of $T^{(1)}(0)=t_1=0$, $T^{(2)}(0)=2t_2=2$ and $T^{(3)}(0)=6t_3=18$ Substituting these values and solving we get $c=-4$. So finally

$$\boxed{x(1-x)^2T^{(3)}(x)\ \ +2(1-x)T^{(2)}(x)\ \ +2(1-6x)T^{(1)}(x)=\frac{8(x^2+3x+1)}{(1-x)^3}-4}$$

Q1) e

From 1d we have

$$x(1-x)^2T^{(3)}(x)\ +2(1-x)T^{(2)}(x)\ +2(1-6x)T^{(1)}(x) = \frac{8(x^2+3x+1)}{(1-x)^3} - 4$$

We know that

$$T(x) = \sum_{n=0}^{\infty} t_n x^n$$

Let's take a derivative of this, we get

$$T^{(1)}(x) = \sum_{n=1}^{\infty} n t_n x^{n-1}$$

In the Similar manner, we get

$$T^{(2)}(x) = \sum_{n=2}^{\infty} n(n-1)t_n x^{n-2},\ T^{(3)}(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)t_n x^{n-3}$$

Lets put these equations in our first equation that we got from 1d, we get

$$x(1-x)^2(\sum_{n=3}^{\infty} n(n-1)(n-2)t_n x^{n-3})+2(1-x)(\sum_{n=2}^{\infty} n(n-1)t_n x^{n-2})+2(1-6x)(\sum_{n=1}^{\infty} n t_n x^{n-1}) = \frac{8(x^2+3x+1)}{(1-x)^3} - 4$$

Now let's find the coefficient of x^{n-1} in LHS and RHS and equate them.
Coefficient of x^{n-1} in LHS =

$$(n-1)(n-2)(n-3)t_{n-1} - 2n(n-1)(n-2)t_n + (n+1)(n)(n-1)t_{n+1} + 2(n+1)(n)t_{n+1}$$

$$-2n(n-1)t_n + 2nt_n - 12(n-1)t_{n-1}$$

$$= t_n[-2n(n-1)(n-2) - 2n(n-1) + 2n]\ +\ t_{n+1}[(n+1)n(n-1) + 2(n+1)n]$$

$$+\ t_{n-1}[(n-1)(n-2)(n-3) - 12(n-1)]$$

$$= t_n[-2n(n-1)(n-2+1)+2n]\ +\ t_{n+1}[(n+1)n(n-1+2)]\ +\ t_{n-1}[(n-1)((n-2)(n-3)-12)]$$

$$= t_n[-2n(n-1)^2 + 2n]\ +\ t_{n+1}[n(n+1)^2]\ +\ t_{n-1}[(n-1)(n^2-5n-6)$$

$$= t_n[2n(1-(n-1)^2)]\ +\ t_{n+1}[n(n+1)^2]\ +\ t_{n-1}[(n-1)(n^2-5n-6)$$

$$= t_n[2n(2n-n^2)]\ +\ t_{n+1}[n(n+1)^2]\ +\ t_{n-1}[(n-1)(n-6)(n+1)$$

$$= t_n[2n^2(2-n)]\ +\ t_{n+1}[n(n+1)^2]\ +\ t_{n-1}[(n-1)(n-6)(n+1)]$$

Before finding the Coefficient of x^{n-1} in RHS, let's solve it a further we know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ... = \sum_{n=0}^{\infty} x^n$$

Taking derivative on both sides we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + ... = \sum_{n=1}^{\infty} nx^{n-1}$$

Taking a derivative again we get

$$\frac{2}{(1-x)^3} = 2 + 3.2x + 4.3x^2 + 5.4x^3 + ... = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}(2 + 3.2x + 4.3x^2 + 5.4x^3 + ...) = \frac{1}{2}(\sum_{n=2}^{\infty} n(n-1)x^{n-2})$$

Let's replace this with the denominator of RHS, we get

$$\frac{8(x^2+3x+1)}{(1-x)^3} - 4 = (8x^2 + 24x + 8)(\frac{1}{2}(\sum_{n=2}^{\infty} n(n-1)x^{n-2})) - 4$$

$$= (4x^2 + 12x + 4)(\sum_{n=2}^{\infty} n(n-1)x^{n-2}) - 4$$

Now Let's find the coefficient of x^{n-1} in this term.
Coefficient of x^{n-1} in RHS =

$$= \ 4(n-1)(n-2)+12(n)(n-1)+4(n+1)(n) \ = \ 4n^2-12n+8+12n^2-12n+4n^2+4n \ = \ 20n^2-20n+8$$

Equating both coefficients from LHS and RHS we get

$$t_n[2n^2(2-n)] \ + \ t_{n+1}[n(n+1)^2] \ + \ t_{n-1}[(n-1)(n-6)(n+1)] \ = \ 20n^2 - 20n + 8$$

$$t_{n+1}[n(n+1)^2] \ + \ t_{n-1}[(n-1)(n-6)(n+1)] - (20n^2 - 20n + 8) \ = \ -(t_n[2n^2(2-n)])$$

$$-(t_n[2n^2(2-n)]) = t_{n+1}[n(n+1)^2] \ + \ t_{n-1}[(n-1)(n-6)(n+1)] - (20n^2 - 20n + 8)$$

$$t_n[2n^2(n-2)] = (n+1)(t_{n+1}[n(n+1)] \ + \ t_{n-1}[(n-1)(n-6)]) - 20(n^2 - n) - 8$$

$$t_n[n^2(n-2)] = \frac{1}{2}(n+1)[]t_{n+1}[n(n+1)] \ + \ t_{n-1}(n-1)(n-6)] - 10n(n-1) - 4$$

$$\boxed{n^2(n-2)t_n = \frac{1}{2}(n+1)[(n-1)(n-6)t_{n-1} \ + \ n(n+1)t_{n+1}] - 10n(n-1) - 4}$$

Q1) f

From 1e we have

$$n^2(n-2)t_n = \frac{1}{2}(n+1)[(n-1)(n-6)t_{n-1} + n(n+1)t_{n+1}] - 10n(n-1) - 4$$

We know asymptotically, we can assume that $t_n \approx t_{n-1} \approx t_{n+1}$. So by substituting this back to the equation we get

$$n^2(n-2)t_n = \frac{1}{2}(n+1)[(n-1)(n-6)t_n + n(n+1)t_n] - 10n(n-1) - 4$$

$$t_n((n^2 - 3n + 3)(n+1) - n^2(n-2)) = 10n(n-1) + 4$$

$$t_n(n^3 + n^2 - 3n^2 - 3n + 3n + 3 - n^3 + 2n^2) = 10n(n-1) + 4$$

$$t_n = \frac{10n(n-1)+4}{3}$$

To give a bound we'll do this.

$$t_n = \Theta(\frac{10n(n-1)+4}{3})$$

$$t_n = \Theta(\frac{10n^2}{3} - \frac{10n}{3} + \frac{4}{3})$$

$$t_n = \Theta(\frac{10n^2}{3})$$

$$\boxed{t_n = \Theta(n^2)}$$

Q2) a

According to the question the two recurrence function is given by

$$s_n = s_{n-1} + 2t_{n-1} \text{ ....(1)}$$

$$t_n = 3s_n + 2t_{n-1} \text{ ....(2)}$$

Given base conditions for Comp(s) and Comp(T) as:

$$s_0 = 1$$

$$t_0 = 0$$

So, we can write recurrence relation of s_i as:

$$S(x) = s_0 + s_1x + s_2x........s_kx + ...\infty$$

$$S(x) = \sum_{k=0}^{\infty} s_k x^k$$

$$S(x) = s_0 + \sum_{k=1}^{\infty} s_k x^k$$

On substituting recurrence relation of s_n from equation (1) we get:

$$S(x) = s_0 + \sum_{k=1}^{\infty} (s_{k-1} + 2t_{k-1})x^k$$

$$S(x) = s_0 + \sum_{k=1}^{\infty} s_{k-1}x^k + 2\sum_{k=1}^{\infty} t_{k-1}x^k$$

$$S(x) = s_0 + x\sum_{k=1}^{\infty} s_{k-1}x^{k-1} + 2x\sum_{k=1}^{\infty} t_{k-1}x^{k-1}$$

$$S(x) = s_0 + xS(x) + 2xT(x) \quad s_0 = 1$$

$$S(x) = 1 + xS(x) + 2xT(x) \text{ ....(3)}$$

similarly, we can write recurrence relation of t_i as:

$$T(x) = t_0 + t_1x + t_2x........t_kx + ...\infty$$

$$T(x) = \sum_{k=0}^{\infty} t_i x^i$$

$$T(x) = t_0 + \sum_{k=1}^{\infty} t_k x^k$$

On substituting recurrence relation of t_n from equation (2) we get:

$$T(x) = t_0 + \sum_{k=1}^{\infty}(3s_k + t_{k-1})x^k$$

$$T(x) = t_0 + 3\sum_{k=1}^{\infty} s_k x^k + \sum_{k=1}^{\infty} t_{k-1}x^k$$

$$T(x) = t_0 + 3\sum_{k=1}^{\infty} s_k x^{k-1} + x\sum_{k=1}^{\infty} t_{k-1}x^{k-1}$$

$$T(x) = t_0 + 3x(S(x) - 1) + xT(x) \quad t_0 = 0$$

$$T(x) = 3x(S(x) - 1) + xT(x) \text{ ....(4)}$$

Solving equation 3 & 4 we get:

$$S(x) = \frac{1 - 7x}{x_2 - 8x + 1}$$

$$T(x) = \frac{3x}{x^2 - 8x + 1}$$

Evaluating S(x) :
we can write Denominator of S(x) as:

$$x^2 - 8x + 1 = (1 - (4 + \sqrt{15})x)(1 - (4 - \sqrt{15})x)$$

so,

$$S(x) = \frac{1 - 7x}{(1 - (4 + \sqrt{15})x)(1 - (4 - \sqrt{15})x)}$$

$$S(x) = \frac{1 - 7x}{2\sqrt{15}x}\left(\frac{1}{(1 - (4 + \sqrt{15})x)} - \frac{1}{(1 - (4 - \sqrt{15})x)}\right)$$

$$S(x) = \frac{1}{2\sqrt{15}}\left(\frac{1}{x} - 7\right)\left(\frac{1}{(1 - (4 + \sqrt{15})x)} - \frac{1}{(1 - (4 - \sqrt{15})x)}\right)$$

To solve of Comp-S(n) we need to solve for coefficient of n in S(x) i.e, we need coefficient of n+1 for 1/x and coefficient of n for -7

So coefficient of 1/x is :

$$\frac{1}{2\sqrt{15}}((4 + \sqrt{15})^{n+1} - (4 - \sqrt{15})^{n+1})$$

coefficient of -7 is:

$$\frac{-7}{2\sqrt{15}}((4 + \sqrt{15})^{n} - (4 - \sqrt{15})^{n+1})$$

Therefore coefficient of n is:

$$\frac{1}{2\sqrt{15}}((4 + \sqrt{15})^{n+1} - (4 - \sqrt{15})^{n+1}) - \frac{7}{2\sqrt{15}}((4 + \sqrt{15})^{n} - (4 - \sqrt{15})^{n+1})$$

On solving we obtain coefficient of n as:

$$\frac{1}{2\sqrt{15}}((\sqrt{15}-3)(4+\sqrt{15})^n+(\sqrt{15}+3)(4-\sqrt{15})^n)$$

Therefore Comp-S(n) is coefficient of nth power in S(n):

$$\boxed{COMP-S(n)=\frac{1}{2\sqrt{15}}((\sqrt{15}-3)(4+\sqrt{15})^n+(\sqrt{15}+3)(4-\sqrt{15})^n)}$$

Evaluating T(x) :

$$T(x)=\frac{3x}{x^2-8x+1}$$

$$x^2-8x+1=(1-(4+\sqrt{15})x)(1-(4-\sqrt{15})x)$$

We can write T(x) as:

$$T(x)=\frac{3x}{(1-(4+\sqrt{15})x)(1-(4-\sqrt{15})x)}$$

$$T(x)=\frac{3}{2\sqrt{15}}(\frac{1}{(1-(4+\sqrt{15})x)}-\frac{1}{(1-(4-\sqrt{15})x)})$$

COMP-T(n) is the nth Coefficient of T(x) which is:

$$\boxed{COMP-T(n)=\frac{3}{(2\sqrt{15})}((4+\sqrt{15})^n-(4-\sqrt{15})^n)}$$

Q2) b

Time complexity recurrence relation for Comp-S(n) is:

$$s_n = s_{n-1} + t_{n-1} + O(1) \text{ ....(1)}$$

s_n is the time complexity of Comp-S(n)

Time complexity recurrence relation for Comp-T(n) is:

$$t_n = s_n + t_{n-1} + O(1) \text{ ....(2)}$$

t_n is the time complexity of Comp-T(n)
Hence from these recurrence relations we can write generating functions S(n) and T(n) where $s_i's$ are coefficients of S(n) and $t_i's$ are coefficients of T(n)

Given base conditions are both are O(1) time:

$$s_0 = O(1)$$

$$t_0 = O(1)$$

Generating function S(n) is :

$$S(x) = s_0 + s_1x + s_2x^2 + \text{..............} + s_nx^n + \text{..........}\infty$$

$$S(x) = \sum_{k=0}^{\infty} s_k x^k$$

$$S(x) = s_0 + \sum_{k=1}^{\infty} s_k x^k$$

$$S(x) = s_0 + \sum_{k=1}^{\infty} (s_{k-1} + t_{k-1} + O(1))x^k$$

$$S(x) = s_0 + \sum_{k=1}^{\infty} s_{k-1}x^k + \sum_{k=1}^{\infty} t_{k-1}x^k + O(1)\sum_{k=1}^{\infty} x^k$$

$$S(x) = s_0 + x\sum_{k=1}^{\infty} s_{k-1}x^{k-1} + x\sum_{k=1}^{\infty} t_{k-1}x^{k-1} + O(1)\sum_{k=1}^{\infty} x^k$$

Since $s_0 = O(1)$

$$S(x) = O(1) + xS(x) + xT(x) + O(1)\sum_{k=1}^{\infty} x^k$$

$$S(x) = xS(x) + xT(x) + O(1)\sum_{k=0}^{\infty} x^k$$

$$S(x) = xS(x) + xT(x) + O(1)\frac{1}{1-x}$$

using base condition $s_0 = 1$

$$S(x) = xS(x) + xT(x) + O(1)\frac{1}{1-x} \text{ ....(3)}$$

Generating function T(n) is :

$$T(n) = t_0 + t_1 x + t_2 x^2 + \text{...............} + t_n x^n + \text{..........}\infty$$

$$T(x) = \sum_{k=0}^{\infty} t_k x^k$$

$$T(x) = t_0 + \sum_{k=1}^{\infty} t_k x^k$$

$$T(x) = t_0 + \sum_{k=1}^{\infty} (s_k + t_{k-1} + O(1))x^k$$

$$T(x) = t_0 + \sum_{k=1}^{\infty} s_k x^k + \sum_{k=1}^{\infty} t_{k-1} x^k + O(1)\sum_{k=1}^{\infty} x^k$$

$$T(x) = t_0 - s_0 + \sum_{k=0}^{\infty} s_k x^k + x\sum_{k=1}^{\infty} t_{k-1} x^{k-1} + O(1)\sum_{k=1}^{\infty} x^k$$

Since $t_0 = s_0$

$$T(x) = \sum_{k=0}^{\infty} s_k x^k + x\sum_{k=1}^{\infty} t_{k-1} x^{k-1} + O(1)\sum_{k=0}^{\infty} x^k - O(1)$$

$$T(x) = S(x) + xT(x) + O(1)(\frac{1}{1-x} - 1)$$

$$T(x) = S(x) + xT(x) + O(1)\frac{x}{1-x}$$

Solving equations (3) and (4) we get:

$$S(x) = O(1)(\frac{1 - x + x^2}{(1-x)(x^2 - 3x + 1)})$$

$$T(x) = O(1)(\frac{1 + x - x^2}{(1-x)(1 + x^2 - 3x)})$$

Lets consider the equation which is denominator for both S(x) and T(x):

$$F(x) = \frac{1}{(1-x)(1+x^2-3x)}$$

we can write F(x) as :

$$F(x) = \frac{1}{\sqrt{5}(x)(1-x)}\Big(\frac{1}{1-\phi x} - \frac{1}{1-\hat{\phi}x}\Big)$$

$$\phi = \frac{3+\sqrt{5}}{2}$$

$$\hat{\phi} = \frac{3-\sqrt{5}}{2}$$

The above expression can be written in terms of generating sequence as:

$$F(x) = \frac{1}{\sqrt{5}x}(1+x+x^2+x^3+.........\infty)[(1+(\phi x)^1+(\phi x)^2+.....\infty)-(1+(\hat{\phi}x)^1+(\hat{\phi}x)^2+.....\infty)]$$

$$F(x) = \frac{1}{\sqrt{5}x}(1+x+x^2+x^3+.........\infty)[((\phi x)^1+(\phi x)^2+.....\infty)-((\hat{\phi}x)^1+(\hat{\phi}x)^2+.....\infty)]$$

$$F(x) = \frac{1}{\sqrt{5}x}(1+x+x^2+x^3+.........\infty)[(\phi-\hat{\phi})x+(\phi^2-\hat{\phi}^2)x^2+(\phi^3-\hat{\phi}^3)x^2........\infty]$$

$$F(x) = \frac{1}{\sqrt{5}}(1+x+x^2+x^3+.........\infty)[(\phi-\hat{\phi})+(\phi^2-\hat{\phi}^2)x^1+(\phi^3-\hat{\phi}^3)x^2........\infty]$$

So coefficient of x^n is given by:

$$= \frac{1}{\sqrt{5}}[(\phi-\hat{\phi})+(\phi^2-\hat{\phi}^2)+(\phi^3-\hat{\phi}^3)+..........+(\phi^n-\hat{\phi}^n)]$$

$$= \frac{1}{\sqrt{5}}\Big[\frac{(\phi^{n+1}-\phi)}{\phi-1} - \frac{(\hat{\phi}^{n+1}-\hat{\phi})}{\hat{\phi}-1}\Big]$$

$$S(x) = O(1)(1-x+x^2)*F(x)$$

Hence the coefficients of S(x) are the above sequence of coefficients of x^n , coefficient of x^{n-1} , coefficient of x^{n-2} in F(x)

$$= O(1)\frac{1}{\sqrt{5}}\Big[\frac{(\phi^{n+1}-\phi)}{\phi-1} - \frac{(\hat{\phi}^{n+1}-\hat{\phi})}{\hat{\phi}-1}\Big] - O(1)\frac{1}{\sqrt{5}}\Big[\frac{(\phi^{n}-\phi)}{\phi-1} - \frac{(\hat{\phi}^{n}-\hat{\phi})}{\hat{\phi}-1}\Big] + O(1)\frac{1}{\sqrt{5}}\Big[\frac{(\phi^{n-1}-\phi)}{\phi-1} - \frac{(\hat{\phi}^{n-1}-\hat{\phi})}{\hat{\phi}-1}\Big]$$

$$T(x) = O(1)(1+x-x^2)*F(x)$$

Hence the coefficients of T(x) are the above sequence of coefficients of x^n , coefficient of x^{n-1} , coefficient of x^{n-2} in F(x)

$$= O(1)\frac{1}{\sqrt{5}}\Big[\frac{(\phi^{n+1}-\phi)}{\phi-1}-\frac{(\hat{\phi}^{n+1}-\hat{\phi})}{\hat{\phi}-1}\Big]+O(1)\frac{1}{\sqrt{5}}\Big[\frac{(\phi^{n}-\phi)}{\phi-1}+\frac{(\hat{\phi}^{n}-\hat{\phi})}{\hat{\phi}-1}\Big]-O(1)\frac{1}{\sqrt{5}}\Big[\frac{(\phi^{n-1}-\phi)}{\phi-1}-\frac{(\hat{\phi}^{n-1}-\hat{\phi})}{\hat{\phi}-1}\Big]$$

Since we know that $\hat{\phi} < 1$, for large n values $\hat{\phi}^n$ is negligible compared to ϕ^n. So our equations on both algorithms follow exponential order of ϕ^n

$$O(1)\phi^n + O(1)\phi^n + O(1)\phi^n = O(\phi^n) = O(((3+\sqrt{5})/2)^n) = O((2.618)^n)$$

Running times of both the algorithms are exponential

$$\boxed{COMP - S(n) = O((2.618)^n)}$$

$$\boxed{COMP - T(n) = O((2.618)^n)}$$

Q3)

Given a recursive algorithm for random bit flips. The only anomaly in the problem is t←Random(1,10). Here our algorithm doesn't follow a single recurrence and it depends on the value of t.But since we are asked to calculate the Expected running time of the algorithm, lets use the probability of t being a number.

$$P(t\ divisible\ by\ 2) = 5/10$$
$$P(t\ divisible\ by\ 3) = 2/10$$
$$P(t\ divisible\ by\ 5) = 1/10$$
$$P(t\ divisible\ by\ 7) = 1/10$$

With which we get the following recurrence relation.

$$T(n) = \frac{5}{10}T(n) + \frac{2}{10}T(n/4) + \frac{1}{10}T(n/4) + \frac{1}{10}T(n/2) + \Theta(n^{\log_2 \frac{(\sqrt{61}-1)}{6}})$$

$$\frac{T(n)}{2} = \frac{3}{10}T(n/4) + \frac{1}{10}T(n/2) + \Theta(n^{\log_2 \frac{(\sqrt{61}-1)}{6}})$$

$$T(n) = \frac{3}{5}T(n/4) + \frac{1}{5}T(n/2) + 2\Theta(n^{\log_2 \frac{(\sqrt{61}-1)}{6}})$$

$$T(n) = \begin{cases} \Theta(1)\ if\ n \leq 4 \\ \frac{3}{5}T(n/4) + \frac{1}{5}T(n/2) + 2\Theta(n^{\log_2 \frac{(\sqrt{61}-1)}{6}}), otherwise \end{cases}$$

Lets solve this equation using Akra-Bazzi Recurrences, since we can't solve this using master's theorem.

First lets check if it satisfies all the conditions.

1. k≥1, here k=2 and is an integer constant

2. $a_i > 0$, and here 3/5 is $>$ 0 and 1/5 is $>$ 0

3. $b_i \in$ (0,1), and here 1/4 and 1/2 $\in$ (0,1) and they are constants.

4. n $\geq$ 1 is a real number.

5. $n_0 \geq \max\left\{\frac{1}{b_i}, \frac{1}{1-b_i}\right\}$ is a constant and here $4 \geq \max\left\{4, \frac{4}{3}\right\}$ and $4 \geq \max\left\{2, 2\right\}$

6. $n^{\log_2 \frac{(\sqrt{61}-1)}{6}}$ is a non-negative function

Now let's find a p such that $a_1 b_1^p + a_2 b_2^p = 1$

$$\frac{3}{5}\left(\frac{1}{4}\right)^p + \frac{1}{5}\left(\frac{1}{2}\right)^p = 1$$

$$\frac{3}{5}\Big(\frac{1}{4}\Big)^p + \frac{1}{5}\Big(\frac{1}{4}\Big)^{p/2} = 1$$

Let's assume $(\frac{1}{4})^{p/2}$ as x, then our equation becomes

$$\frac{3}{5}x^2 + \frac{1}{5}x = 1$$
$$3x^2 + x - 5 = 0$$

from this our x values are $\frac{-1+\sqrt{61}}{6}$ and $\frac{-1-\sqrt{61}}{6}$, since x is cannot be negative, since it is of the form a^b

$$x = \frac{-1+\sqrt{61}}{6}$$
$$(\frac{1}{4})^{p/2} = \frac{-1+\sqrt{61}}{6}$$

applying log on both sides, we get

$$\frac{p}{2}(\log_2(1/4)) = \log_2(\frac{-1+\sqrt{61}}{6})$$

$$\frac{p}{2}(\log_2(1) - \log_2(4)) = \log_2(\frac{-1+\sqrt{61}}{6})$$

$$\frac{p}{2}(0-2)) = \log_2(\frac{-1+\sqrt{61}}{6})$$

$$p = -\log_2(\frac{-1+\sqrt{61}}{6})$$

Since p is unique real number for which $a_1 b_1^p + a_2 b_2^p = 1$. Then

$$T(n) = \Theta\Bigg(n^p\Bigg(1 + \int_1^n \frac{g(u)}{u^{p+1}}du\Bigg)\Bigg)$$

We know that g(u) = $u^{\log_2 \frac{(\sqrt{61}-1)}{6}}$ and for the time being let's assume $\log_2 \frac{(\sqrt{61}-1)}{6}$ = k, then g(u) = u^k and p becomes -k

$$T(n) = \Theta\Bigg(n^{-k}\Bigg(1 + \int_1^n \frac{u^k}{u^{-k+1}}du\Bigg)\Bigg)$$

$$T(n) = \Theta\Bigg(n^{-k}\Bigg(1 + \int_1^n u^{2k-1}du\Bigg)\Bigg)$$

$$T(n) = \Theta\left(n^{-k}\left(1 + \left[\frac{u^{2k}}{2k}\right]_1^n\right)\right)$$

$$T(n) = \Theta\left(n^{-k}\left(1 + \frac{n^{2k}}{2k} + \frac{1}{2k}\right)\right)$$

$$T(n) = \Theta\left(\frac{n^k}{2k} + n^{-k}(1 + \frac{1}{2k})\right)$$

and for large n we can ignore the n^{-k} term, Therefore

$$T(n) = \Theta\left(\frac{n^k}{2k}\right)$$

after substituting k = $\log_2^{\frac{(\sqrt{61}-1)}{6}}$ and rewriting $\Theta(\frac{n^k}{c})$ as $\Theta(n^k)$ we get

$$\boxed{T(n) = \Theta\left(n^{\log_2^{\frac{(\sqrt{61}-1)}{6}}}\right)}$$

$$\boxed{T(n) = \Theta\left(n^{0.182745190}\right)}$$