First Year Comp and Exam Solutions

March 27, 2013

Contents

Part I

Metrics

Chapter 1

Finals

1.1 Final 2010

PDF

1.1.1 Question 4 (Wolak)

1.1.1.1 Part a

$$
\begin{aligned}
Y\Gamma + XB &= E \\
\begin{bmatrix} C_t & Y_t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -\beta & 1 \end{bmatrix} + \begin{bmatrix} 1 & I_t \end{bmatrix} \begin{bmatrix} -\alpha & 0 \\ 0 & -1 \end{bmatrix} &= \begin{bmatrix} \epsilon_t & 0 \end{bmatrix} \\
\Pi &= -B\Gamma^{-1} \\
&= \frac{1}{1-\beta} \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \beta & 1 \end{bmatrix} \\
&= \frac{1}{1-\beta} \begin{bmatrix} \alpha & \alpha \\ \beta & 1 \end{bmatrix} \\
\pi_{11} &= \frac{\alpha}{1-\beta} \\
\pi_{12} &= \frac{\beta}{1-\beta} \\
\pi_{21} &= \frac{\alpha}{1-\beta} \\
\pi_{22} &= \frac{1}{1-\beta} \\
V &= E\Gamma^{-1} \\
&= \frac{1}{1-\beta} \begin{bmatrix} \epsilon_t & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \beta & 1 \end{bmatrix} \\
&= \frac{1}{1-\beta} \begin{bmatrix} \epsilon_t & \epsilon_t \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Y &= X\Pi + V \\
\begin{bmatrix} C_t & Y_t \end{bmatrix} &= \begin{bmatrix} 1 & I_t \end{bmatrix} \begin{bmatrix} \frac{\alpha}{1-\beta} & \frac{\alpha}{1-\beta} \\ \frac{\beta}{1-\beta} & \frac{1}{1-\beta} \end{bmatrix} + \frac{1}{1-\beta} \begin{bmatrix} \epsilon_t & \epsilon_t \end{bmatrix}
\end{aligned}
$$

$$
Y_t = \frac{\alpha}{1-\beta} + \frac{1}{1-\beta} I_t + \frac{1}{1-\beta} \epsilon_t
$$

1.1.1.2 Part b

$$\begin{aligned}\pi_{11} &= \frac{\alpha}{1-\beta}\\ \pi_{12} &= \frac{\beta}{1-\beta}\\ \pi_{21} &= \frac{\alpha}{1-\beta}\\ \pi_{22} &= \frac{1}{1-\beta}\end{aligned}$$

$$\beta = \frac{\pi_{12}}{\pi_{22}} = 1 - \frac{1}{\frac{1}{1-\beta}} = 1 - \frac{1}{\pi_{22}}$$

$$Y_t = \frac{\alpha}{1-\beta} + \frac{1}{1-\beta} I_t + \frac{1}{1-\beta}\epsilon_t$$

$$\begin{aligned}\hat{\pi}_{22} &= \frac{\frac{1}{T}\sum(Y_t-\bar{Y})(I_t-\bar{I})}{\frac{1}{T}\sum(I_t-\bar{I})^2}\\ Y_t - I_t &= C_t\\ \hat{\beta} &= 1 - 1/\frac{\frac{1}{T}\sum(Y_t-\bar{Y})(I_t-\bar{I})}{\frac{1}{T}\sum(I_t-\bar{I})^2}\\ &= 1 - \frac{\frac{1}{T}\sum(I_t-\bar{I})^2}{\frac{1}{T}\sum(Y_t-\bar{Y})(I_t-\bar{I})}\\ &= \frac{\frac{1}{T}\sum(C_t-\bar{C})(I_t-\bar{I})}{\frac{1}{T}\sum(Y_t-\bar{Y})(I_t-\bar{I})}\end{aligned}$$

1.2 Comp 2007 Spring

PDFOnline

1.2.1 Question 5 (Wolak)

1.2.1.1 Part a

$$\begin{aligned}y_t &= \sigma_t\epsilon_t\\ y_t^2 &= \sigma_t^2\epsilon_t^2\\ \ln y_t^2 &= \ln\sigma_t^2 + \ln\epsilon_t^2\\ \ln y_t^2 &= \lambda x_t + \eta_t + \ln\epsilon_t^2\end{aligned}$$

We want an estimator for λ of the form

$$\hat{\gamma} = \frac{\sum_{t=1}^{n} g(y_t)h(x_{t-j})}{\sum_{t=1}^{n} h(x_t)h(x_{t-j})}$$

So we can use

$$g(y_t) = \ln y_t^2$$

Checking

$$\begin{aligned} E(x_t x_{t-1}) &= E(\frac{1}{2}\xi_{t-1}^2) = \frac{1}{2} \\ E(\ln y_t^2 x_{t-1}) &= E(\lambda x_t \xi_{t-1} + \frac{1}{2}\xi_{t-2}\lambda x_t) \\ &= \frac{1}{2}\lambda \end{aligned}$$

To get

$$\begin{aligned} \hat{\gamma} &= \frac{\sum_{t=1}^{n}(\ln y_t^2)(x_{t-1})}{\sum_{t=1}^{n}(x_t)(x_{t-1})} \\ &\xrightarrow{p} \frac{E(\ln y_t^2 x_{t-1})}{E(x_t x_{t-1})} \\ &= \frac{E(\ln y_t^2 x_{t-1})}{\frac{1}{2}} \\ &= \frac{\frac{1}{2}\lambda}{\frac{1}{2}} \\ &= \lambda \end{aligned}$$

1.3 Comp 2008 Fall

PDFOnline

1.3.1 Question 4 (MaCurdy)

1.3.1.1 Part a

False

When we perform NLS then we

$$\min_{\beta} \frac{1}{T} \sum (y_t - \exp(x_t'\beta))^2$$

The FOC of this are

$$\begin{aligned}\frac{1}{2}\sum 2(y_t - \exp(x_t'\hat{\beta}))(\exp(x_t'\hat{\beta})x_t') &= 0\\ \frac{1}{2}\sum 2l_t(\hat{\beta}) &= 0\\ \sum^T l_t(\hat{\beta}) &= 0\\ l_t(\hat{\beta}) &= u_t \exp(x_t'\hat{\beta})x_t'\end{aligned}$$

So what does \hat{\beta} go to asymptotically?

$$\begin{aligned}\hat{\beta} - \beta &= -((\frac{1}{T}\sum^T \frac{\partial l_t(b)}{\partial b}|_{b=\tilde{\beta}})^{-1}\frac{1}{T}\sum l_t(b)|_{b=\beta})\\ \tilde{\beta} &\in (\beta, \hat{\beta})\end{aligned}$$

What we want to know is when is plim of this going to be 0? When is $\hat{\beta}$ consistent for true β?

The question can be rewritten as when does $\frac{1}{T}\sum l_t(b) = E(l_t(\beta)) = E(u_t \exp(x_t'\beta)x_t) = 0$

So if $E(u_t|x_t) = 0$ that is sufficient.

1.4 Comp 2008 Spring

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1.4.1 Question 1 (Wolak)

1.4.1.1 Part a

In the true DGP, we see that

$$E(y_t|x_{1t}, x_{2t}) = \beta_1 x_{1t} + \beta_2 x_{2t}$$

which shows that we are in a linear conditional mean world. Thus, we know that the plim of the unrestricted OLS regression will go the true DGP parameters:

$$\begin{aligned}b_1 &\to_p \beta_1\\ b_2 &\to_p \beta_2\end{aligned}$$

We also see that \eta is iid, so we can jump straight to saying

$$\sqrt{T}(b-\beta) \to_d N(0,\sigma^2 Q^{-1})$$

If we wanted to say a word or two more we could have written it out as

$$\sqrt{T}(b-\beta) = (\frac{X'X}{T})^{-1}(\frac{X'\eta}{\sqrt{T}})$$

We know from the setup that the first term on the right hand side goes to Q^-1 in probability.

We have to use Lindenberg-Feller for the second term because the constants are independent but different samples are not identically distributed because the scalars will be different. The second term goes to $N(0,\sigma^2 Q^{-1})$.

Do the Slusky.

1.4.1.2 Part b

We can directly compute

$$\min_\beta \sum_{t=1}^{T}(y_t - \beta x_{1t} - \beta^2 x_{2t})^2$$

FOC

$$\begin{aligned} 2\sum(y_t - \beta x_{1t} - \beta^2 x_{2t})(-x_{1t} - 2\beta x_{2t}) = 0 \\ \sum y_t x_{1t} + \beta[2\sum y_t x_{2t} - \sum x_{1t}^2] - \beta^2[3[x_{1t}x_{2t}] - \beta^3[2\sum x_{2t}^2] = 0 \end{aligned}$$

Mr. Mathematica can figure out what you are supposed to do with that mess.

1.4.1.3 Part c

Version 1:

We do our usual ABA thing

$$\begin{aligned} Q_T(\beta) &= \frac{1}{T}\sum(y_t - x_{1t}\beta - x_{2t}\beta^2)^2 \\ S_T(\beta) = \frac{\partial Q_T(\beta)}{\partial\beta} &= -\frac{2}{T}\sum(y_t - x_{1t}\beta - x_{2t}\beta^2)(x_{1t} + 2\beta x_{2t}) \\ H_T(\beta) = \frac{\partial S_T(\beta)}{\partial\beta} &= \frac{2}{T}\sum\{(x_{1t} + 2\beta x_{2t})^2 - 2x_{1t}(y_t - x_{1t}\beta - x_{2t}\beta^2)\} \end{aligned}$$

Let $b_1^\star$ be the point where $S_T(b_1^\star) = 0$ and the FOC are satisfied. By Taylor expansion:

$$\begin{aligned}
S_T(b_1^\star) = 0 &= S_T(\beta_1) + H_T(\beta_1)(b_1^\star - \beta_1) \\
0 &= S_T(\beta_1) + H_T(\beta_1)(b_1^\star - \beta_1) \\
-S_T(\beta_1) &= H_T(\beta_1)(b_1^\star - \beta_1) \\
-H_T(\beta_1)^{-1}S_T(\beta_1) &= (b_1^\star - \beta_1) \\
\beta_1 - H_T(\beta_1)^{-1}S_T(\beta_1) &= b_1^\star
\end{aligned}$$

So we can write

$$\sqrt{T}(b_1^\star - \beta) = -H_T(\tilde{\beta}_1)^{-1}\sqrt{T}S_T(\beta_1)$$

We know that

$$\begin{aligned}
-H_T(\tilde{\beta}_1)^{-1} &\to_p -H_0^{-1} \\
\sqrt{T}S_T(\beta_1) &\to_d N(0, H_0^{-1}V_0H_0^{-1})
\end{aligned}$$

Why? Because we know that $\tilde{\beta} \in (b_1^\star, \beta_1)$ and $b_1^\star \to_p \beta$ and $\beta_1 \to_p \beta$.

Let's evaluate at the true value. Also, we need to take the limit because the asymptotic variance should not have a T still in it.

$$\begin{aligned}
E(H_T(\beta)) &= \frac{2}{T}\sum E(x_{1t} + 2\beta_1 x_{2t})^2 - 2x_t\eta_t \\
&= \frac{2}{T}\sum E(x_{1t} + 2\beta_1 x_{2t})^2 \\
\lim_{t\to\infty} E(H_T(\beta)) &= \lim \frac{2}{T}\sum E(x_{1t} + 2\beta_1 x_{2t})^2 \\
&= 2\begin{bmatrix} 1 & 2\beta_1 \end{bmatrix} \begin{bmatrix} \frac{1}{T}\sum x_{1t}^2 & \frac{1}{T}\sum x_{1t}x_{2t} \\ \frac{1}{T}\sum x_{1t}x_{2t} & \frac{1}{T}\sum x_{2t}^2 \end{bmatrix} \begin{bmatrix} 1 \\ 2\beta_1 \end{bmatrix} \\
&\to_p 2u'Qu = H_0
\end{aligned}$$

$$\sqrt{T}S_t(\beta_1) = \frac{1}{\sqrt{T}}\sum 2\eta_t(x_{1t} + 2x_{2t}\beta_1)$$

$$\begin{aligned}
V_0 &= \lim Var\left(\frac{1}{\sqrt{T}}\sum 2\eta_t(x_{1t} + 2x_{2t}\beta_1)\right) \\
&= \lim \frac{4}{T}\sum (x_{1t} + 2x_{2t}\beta_1)^2 Var(\eta_t) \\
&= 4\sigma^2 u'Qu
\end{aligned}$$

So plug in

$$\sqrt{T}(b_1^\star - \beta_1) \to_d N(0, H_0^{-1}V_0H_0^{-1})$$

1.4.1.4 Part d

$$\theta = \begin{bmatrix} \beta \\ \sigma^2 \end{bmatrix}$$

$$Q_T(\theta) = -\frac{T}{2}\ln(2H) - \frac{T}{2}\ln(\sigma^2) - \frac{1}{2T}[\sum((y_t - x_{1t}\beta - x_{2t}\beta^2)/\sigma)^2]$$

1.4.1.5 Part e

Yes it reaches because it is log likelihood correctly specified.

1.4.2 Question 2 (Hansen/Hong)

Recall that in general, the LR test statistic goes in distribution to Chi-Square with 1 degree of freedom (because only one restriction in most cases).

$$\begin{aligned} H_0 &: \mu = \mu_0 \\ H_1 &: \mu \neq \mu_0 \\ X_i &\sim N(\mu, 1) \end{aligned}$$

1.4.2.1 Part a

$$\begin{aligned} \lambda_x &= 2\log L(\hat{\mu}x) - \log L(\mu_0) \\ &= \sum(x_i - \mu_0)^2 - \sum(x_i - \bar{x})^2 \\ &= \sum(x_i^2 - 2x_i + \mu_0^2 - x_i^2 + 2x_i\bar{x} - \bar{x}^2) \\ &= -2n\mu_0\bar{x} + n\mu_0^2 + \frac{2n\bar{x}^2 - n\bar{x}^2}{n\bar{x}^2} \\ &= (\frac{\bar{x} - \mu_0}{n^{-1/2}})^2 \end{aligned}$$

So we have a standard normal squared. Boom. Chi-Square.

1.4.2.2 Part b

$$
\begin{aligned}
\lambda_y &= 2\log\frac{L(\bar{x},Y)}{L(\mu_0,Y)} \\
&= \sum(y_i-\mu_0)^2 - \sum(y_i-\bar{x})^2 \\
&= -2n\mu_0\bar{y} + n\mu_0^2 + 2n\bar{x}\bar{y} - n\bar{x}^2 \\
&= n[\bar{y}^2 - 2\mu_0\bar{y} + \mu_0^2 - \bar{y}^2 + 2\bar{x}\bar{y} - \bar{x}^2] \\
&= n[(\bar{y}^2-\mu_0)^2 - (\bar{y}-\bar{x})^2] \\
&= n[2(\bar{y}-\mu_0)(\bar{x}-\mu_0) - (\bar{x}-\mu_0)^2] \\
&= 2\sqrt{n}(\bar{y}-\mu_0)\sqrt{n}(\bar{x}-\mu_0) - (\sqrt{n}(\bar{x}-\mu_0))^2 \\
z_x &= \sqrt{n}(\bar{x}-\mu_0) \\
z_y &= \sqrt{n}(\bar{y}-\mu_0) \\
E(\lambda_y) &= 2E(z_y)E(z_x) - E(z_x^2) = -1 \\
Var(2z_yz_x) &= 4(E(z_y^2)E(z_x^2) - E()^2E()^2) = 4 \\
Cov(2z_y,z_x) &= 0 \\
Var(z_x^2) &= 2 \\
Var(\lambda_y) &= Var(2z_yz_x) + Var(z_x^2) - 2Cov(2z_yz_x, z_x^2) \\
&= 4+2 \\
&= 6
\end{aligned}
$$

1.4.2.3 Part c

1.4.2.4 Part d

We know that MLE maxes the log likelihood function, so

$$
\begin{aligned}
P(\lambda_x > c) &> P(\lambda_y > c) \\
L(\bar{y};Y) &\geq L(\text{anything else};Y) \\
&\geq L(\bar{x};Y)
\end{aligned}
$$

1.4.3 Question 3 (?)

There are four cases:

y_1i	y_2i
0	0
0	1
1	0
1	1

Write out the unconditional likelihood function for a single observation using the following notation. This notation enables you to write all four cases in one summation:

$$\begin{aligned}P\left(Y_{1i}=y_{1i},Y_{2i}=y_{2i};p,\rho\right)=&(y_{1i})(y_{2i})P(Y_{1i}=1,Y_{2i}=1)\\&+(y_{1i})(1-y_{2i})P(Y_{1i}=1,Y_{2i}=0)\\&+(1-y_{1i})(y_{2i})P(Y_{1i}=0,Y_{2i}=1)\\&+(1-y_{1i})(1-y_{2i})P(Y_{1i}=0,Y_{2i}=0)\end{aligned}$$

$$\begin{aligned}P\left(Y_{1i}=y_{1i},Y_{2i}=y_{2i};p,\rho\right)=&(y_{1i})(y_{2i})P(\epsilon_{1i}\leq p,\epsilon_{2i}\leq p)\\&+(y_{1i})(1-y_{2i})P(\epsilon_{1i}\leq p,\epsilon_{2i}>p)\\&+(1-y_{1i})(y_{2i})P(\epsilon_{1i}>p,\epsilon_{2i}\leq p)\\&+(1-y_{1i})(1-y_{2i})P(\epsilon_{1i}>p,\epsilon_{2i}>p)\end{aligned}$$

Note that the probability is a joint probability. It is only separately if the \epsilon are independent. Because they are distributed normally, we can use correlation to IFF independence. Thus, they are only independent if $\rho = 0$.

We can rewrite this for convenience as

$$\begin{aligned}P\left(Y_{1i}=y_{1i},Y_{2i}=y_{2i};p,\rho\right)=&I_{1i}\Phi_{1i}\\&+I_{2i}\Phi_{2i}\\&+I_{3i}\Phi_{3i}\\&+I_{4i}\Phi_{4i}\end{aligned}$$

1.4.3.1 Part a

$$\begin{aligned}P\left(Y_{1i}=y_{1i},Y_{2i}=y_{2i};p,\rho\right)&=\sum_{j=1}^{4}I_{1i}\Phi_{1i}\\P\left(Y_{1}=y_{1},Y_{2}=y_{2};p,\rho\right)&=\prod_{i=1}^{n}\sum_{j=1}^{4}I_{1i}\Phi_{1i}\end{aligned}$$

Now write the log-likelihood using the observed data.

Useful trick: note that we can pull the indicator function out of the log because in each observation, only one of the above cases is actually realized. Thus, the log of sum of these different cases is only going to be the log of one of these cases. We therefore simplify things by pulling the indicator out of the log and moving the log inside the sum.

$$
\begin{aligned}
L\left(p,\rho;\underline{Y_1},\underline{Y_2}\right) &= \sum_{i=1}^{n}\ln(\sum_{j=1}^{4} I_{1i}\Phi_{1i}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{4} I_{1i}\ln(\Phi_{1i})
\end{aligned}
$$

The above equation is the log-likelihood whether or not they are indepedent.

Say we assumed $\rho = 0$ and they are independent. Now we only need two terms to write out the possibilities:

$$
\begin{aligned}
P\left(Y_{1i} = y_{1i}, Y_{2i} = y_{2i}; p, \rho\right) &= P\left(Y_{1i} = y_{1i}\right)P(Y_{2i} = y_{2i}) \\
&= \left[y_{1i}P(\epsilon_{1i} \leq p) + (1 - y_{1i})P(\epsilon_{1i} > p)\right] \\
&\times \left[y_{2i}P(\epsilon_{2i} \leq p) + (1 - y_{2i})P(\epsilon_{2i} > p)\right] \\
&= \left[y_{1i}\Phi(p) + (1 - y_{1i})(1 - \Phi(p)\right]\left[y_{2i}\Phi(p) + (1 - y_{2i})(1 - \Phi(p)\right]
\end{aligned}
$$

Therefore, under the assumption that \rho=0 and thus the epsilons are independent then we do have the correct likelihood function (normalized):

$$
Q_n(b) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{2} y_{1i}\ln(\Phi(b)) + (1 - y_{1i})\ln((1 - \Phi(b))
$$

1.4.3.2 Part b

$$
\hat{b} = \text{argmax } Q_n(b)
$$

Want to know if

$$
\hat{b} \xrightarrow{b} p
$$

We assumpted independence to get the likelihood function. Let's first name what this converges to. Call it $Q(b)$:

$$
\begin{aligned}
Q_n(b) &\xrightarrow{p} \lim \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{2} y_{1i}\ln(\Phi(b)) + (1 - y_{1i})\ln((1 - \Phi(b)) \\
&\xrightarrow{p} E\left[\sum_{j=1}^{2} y_{1i}\ln(\Phi(b)) + (1 - y_{1i})\ln((1 - \Phi(b))\right] \\
&= Q(b)
\end{aligned}
$$

The question is would this converge if $\rho \neq 0$ and the errors were not indepenent?

We are taking a sample average across the i observations. Important: we know that there are correlations across j, but not across i. Thus, we are able to use the regular LLN even if $\rho \neq 0$.

Now let's compute what $Q(b)$ actually is. Note that the expected value of a indicator function is the probability that it is realized.

$$\begin{aligned} E\left[\sum_{j=1}^{2} y_{1i}\ln(\Phi(b)) + (1-y_{1i})\ln((1-\Phi(b))\right] &= \sum_{j=1}^{2} E(y_{1i})\ln(\Phi(b)) + E(1-y_{1i})\ln((1-\Phi(b)) \\ &= \sum_{j=1}^{2} \Phi(p)\ln(\Phi(b)) + (1-\Phi(p))\ln((1-\Phi(b)) \end{aligned}$$

Nothing inside of the summation depends on j anymore, so re-write:

$$= 2\left[\Phi(p)\ln(\Phi(b)) + (1-\Phi(p))\ln((1-\Phi(b))\right]$$

Last, but not least, we want to actually solve for the argmax b:

$$\begin{aligned} \frac{\partial Q(b)}{\partial b} &= 2\frac{\Phi(p)}{\Phi(b)}\phi(b) - \frac{(1-\Phi(p))}{(1-\Phi(b))}\phi(b) = 0 \\ &= 2\phi(b)\left\{\frac{\Phi(p)}{\Phi(b)} - \frac{(1-\Phi(p))}{(1-\Phi(b))}\right\} = 0 \end{aligned}$$

We know that the PDF of a normal has support on the full R, so the question is when does the second term equal 0 as you vary b. The answer is that the first term is always decreasing as b increases and the second term is always increasing as b increases. Thus, they must have a single point of intersection. That point is at $b = p$.

Thus we have that

$$\hat{b} = \text{argmax } Q_n(b) \xrightarrow{p} \text{argmax } Q(b)$$

Therefore, the estimator, $\hat{b}$, is consistent for the true maximizer, p.

1.4.3.3 Part c

$$\begin{aligned} (\hat{b}_1, \hat{b}_2) &= \text{argmax } Q_n(b_1, b_2) \\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{2} y_{1i}\ln(\Phi(b_j)) + (1-y_{1i})\ln((1-\Phi(b_j)) \\ Q(b_1, b_2) &= \sum_{j=1}^{2} \Phi(p_j)\ln(\Phi(b_j)) + (1-\Phi(p_j))\ln((1-\Phi(b_j)) \end{aligned}$$

Note that this is the same as before except that the expectation of y_{1i} is now $\Phi(p_j)$.

Take first order conditions

$$\frac{\partial Q}{\partial b_j} = \phi(b_j)\left\{\frac{\Phi(p_j)}{\Phi(b_j)} - \frac{1-\Phi(p_j)}{1-\Phi(b_j)}\right\} = 0$$

So they are still good.

1.4.3.4 Part d

As we saw in part a, this likelihood function only works if they $\rho = 0$. The problem is that in general this not true. Thus, we cannot simply jump to the information matrix. We need to instead do ABA.

$$\sqrt{n}(\hat{b} - b) \to_d N(0, H_0^{-1} V_0 H_0^{-1})$$

$$\begin{aligned}
Var(\frac{1}{\sqrt{n}} S_n(p)) &= V_0 \\
E(H_n(p)) &= H_0 \\
S_n &= \frac{\partial}{\partial} Q_n(b) \\
H_n &= \frac{\partial}{\partial b \partial b} Q_n(b)
\end{aligned}$$

Computing derivatives is messy.

This is the MLE trick:

We already computed the first order condition

$$\begin{aligned}
\frac{\partial Q_n}{\partial b_j} &= \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{y_{ij}\phi(b_j)}{\Phi(b_j)} - \frac{(1-y_{1j})\phi(b_j)}{1-\Phi(b_j)}\right\} = 0 \\
&= \phi(b_j)\left\{\frac{\bar{y}}{\Phi(b_j)} - \frac{(1-\bar{y})}{1-\Phi(b_j)}\right\} = 0
\end{aligned}$$

Because nothing depends on i except y. Again, we can see that to have both of the terms in brackets equal to 1 we need to set $\Phi(b_j) = \bar{y}_j$.

Given that $\Phi(b_j) = \bar{y}_j$ is strictly increasing, we can invert to get:

$$\hat{b} = \Phi^{-1}(\bar{y}_j)$$

Now we can use the delta method to get the asymptotic distribution. The idea is to do the delta method with the function $g = \Phi^{-1}$ and use what we already know about Y being distributed as Bernoulli.

$$\sqrt{n}(\begin{array}{c}\hat{b}_1\\ \hat{b}_2\end{array} - \begin{array}{c}b_1\\ b_2\end{array}) \to_d N(0,\cdot)$$
$$\sqrt{n}(\begin{array}{c}\bar{Y}_1\\ \bar{Y}_2\end{array} - \begin{array}{c}E(y_1)\\ E(y_2)\end{array}) \to_d N(0,\Omega)$$
$$\begin{array}{c}b_1\\ b_2\end{array} = G(\begin{array}{c}\bar{Y}_1\\ \bar{Y}_2\end{array}) = (\begin{array}{c}\Phi^{-1}(\bar{Y}_1)\\ \Phi^{-1}(\bar{Y}_2)\end{array})$$

$$\frac{\partial G}{\partial y_1} = \begin{pmatrix} \frac{1}{\phi(\Phi^{-1}(\bar{y}_1))} \\ 0 \end{pmatrix}$$

$$\frac{\partial G}{\partial y_2} = \begin{pmatrix} \frac{1}{\phi(\Phi^{-1}(\bar{y}_2))} \\ 0 \end{pmatrix}$$

Evaluate at the true expected value

$$E(y_{ji}) = \Phi(p_j)$$

$$G_0 = \begin{pmatrix} \frac{1}{\phi(p_1)} & \\ & \frac{1}{\phi(p_2)} \end{pmatrix}$$

$$\begin{aligned} Var(y_1) &= \Phi(p_1)(1-\Phi(p_1) \\ Var(y_2) &= \Phi(p_2)(1-\Phi(p_2) \\ Cov(y_1,y_2) &= E(Y_1Y_2) - E(Y_1)E(Y_2) \\ &= P(Y_1 = 1, Y_2 = 1) - \Phi(p_1)\Phi(p_2) \end{aligned}$$

$$\begin{aligned} \Omega_0 = CLT &\begin{pmatrix} Var(y_1) & Cov(y_1,y_2) \\ Cov(y_1,y_2) & Var(y_2) \end{pmatrix} \\ = &\begin{pmatrix} \Phi(p_1)(1-\Phi(p_1) & P(Y_1 = 1, Y_2 = 1) - \Phi(p_1)\Phi(p_2) \\ P(Y_1 = 1, Y_2 = 1) - \Phi(p_1)\Phi(p_2) & \Phi(p_1)(1-\Phi(p_1) \end{pmatrix} \end{aligned}$$

Delta method

$$\sqrt{n}\left(G(\begin{array}{c}\bar{Y}_1\\ \bar{Y}_2\end{array}) - G(\begin{array}{c}E(\bar{Y}_1)\\ E(\bar{Y}_2)\end{array})\right) \to_d N(0, G_0\Omega_0G_0)$$

We don't really know what that last covariance thing is so no reason to plug it into the thing.

1.5 Comp 2012 Spring

PDFOnline

1.5.1 Question 5 (MaCurdy)

1.5.1.1 Part a

Start with the

$$l_t = Derivative(y_t - c - d_t a + x_t b + e_t)$$
$$L_T = \frac{1}{T}\sum^{T} l_t$$

By the First order conditions

$$L_T(\hat{a}, \hat{b}, \hat{c}) = \begin{cases} \frac{1}{T}\sum(y_t - c - d_t a + x_t b) = 0 \\ \frac{1}{T}\sum(y_t - c - d_t a + x_t b)d_t = 0 \\ \frac{1}{T}\sum(y_t - c - d_t a + x_t b)b = 0 \end{cases}$$

But let's say that we plugged in the true parameters for

$$L_T(\alpha, \beta, \tilde{c}) = \begin{cases} \frac{1}{T}\sum(y_t - \tilde{c} - d_t\alpha + x_t\beta) & = \frac{1}{T}\sum \epsilon_t - \tilde{c}_t \\ \frac{1}{T}\sum(y_t - \tilde{c} - d_t\alpha + x_t\beta)d_t & = \frac{1}{T}\sum(\epsilon_t - \tilde{c}_t)d_t \\ \frac{1}{T}\sum(y_t - \tilde{c} - d_t\alpha + x_t\beta)\beta & = \frac{1}{T}\sum(\epsilon_t - \tilde{c}_t)\beta \end{cases}$$

What is plim $\tilde{c}$?

$$\tilde{c} = y_t - d_t\alpha - x_t\beta = \epsilon_t$$
$$= \text{plim}\ \frac{1}{T}\sum \epsilon_t = \mu$$

$$\text{plim}\ L_T(\alpha, \beta, \tilde{c}) = \begin{cases} E(\epsilon_t - \tilde{c}_t) & = 0 \\ E(\epsilon_t - \tilde{c}_t)d_t & = \mu - ugliness \\ E(\epsilon_t - \tilde{c}_t)\beta & = 0 \end{cases}$$

1.5.1.2 Part b

You might be tempted to say, sure, just run

$$y_t - 2d_t = x_t\beta + \epsilon_t$$

But let's just check if that is really consistent.

$$\begin{aligned}
l_t(\beta) &= (y_t - 2d_t - x_t\beta)x_t \\
L_T(\hat{\beta}) &= \frac{1}{T}\sum(y_t - 2d_t - x_t\beta)x_t = 0 \\
\text{plim } L_T(\hat{\beta}) &= \text{plim } \frac{1}{T}\sum(y_t - 2d_t - x_t\hat{\beta})x_t \\
&= \text{plim } \frac{1}{T}\sum \epsilon_t x_t \\
&= \mu E(x) \\
&= \frac{1}{2}\mu
\end{aligned}$$

Thus, this is only consistent if you added a constant term to the regression.

1.5.1.3 Part d

Our original FOC for IV are

$$L_T(\hat{a},\hat{b},\hat{c}) = \begin{cases} \frac{1}{T}\sum(y_t - \hat{c} - d_t\hat{a} + x_t\hat{b}) = 0 \\ \frac{1}{T}\sum(y_t - \hat{c} - d_t\hat{a} + x_t\hat{b})z_t = 0 \\ \frac{1}{T}\sum(y_t - \hat{c} - d_t\hat{a} + x_t\hat{b})x_t = 0 \end{cases}$$

You would need all of these things to go to 0 to get consistent estimates. How do we get the asymptotic distribution?

$$\begin{aligned}
\theta_0 &= \begin{pmatrix} \alpha \\ \beta \\ \mu \end{pmatrix} \\
X &= \begin{pmatrix} 1 \\ d_t \\ x_t \end{pmatrix} \\
Z &= \begin{pmatrix} 1 \\ z_t \\ x_t \end{pmatrix}
\end{aligned}$$

$$\sqrt{T}(\hat{\theta} - \theta_0) \to_d N(0, S^{-1}(\theta_0)V(\theta_0)S^{-1}(\theta_0))$$

$$\begin{aligned}
S(\theta_0) &= \text{plim } \frac{1}{T}\sum\frac{\partial l_t}{\partial\theta'} = E(Z_t X_t') \\
V(\theta_0) &= \text{plim } \frac{1}{T}\sum l_t(\theta_0)l_t(\theta_0)' = E((\epsilon_t - \mu)^2 Z_t Z_t)
\end{aligned}$$

1.5.1.4 Part e

We are doing 2SLS here. We are using both exogenous variables and using a linear combination of them, just like doing a linear projection of the exogenous variables.

$$L_T(\hat{a}_1, \hat{a}_2) = \begin{cases} \frac{1}{T}\sum(d_t - \exp(x_t\hat{a}_1 + z_t\hat{a}_2))\exp(x_t\hat{a}_1 + z_t\hat{a}_2)x_t = 0 \\ \frac{1}{T}\sum(d_t - \exp(x_t\hat{a}_1 + z_t\hat{a}_2))\exp(x_t\hat{a}_1 + z_t\hat{a}_2)z_t = 0 \end{cases}$$

$$\begin{aligned} E(d_t|x_t, z_t) &= \exp(\alpha_1 x_t + \alpha_2 z_t)E(\exp(\rho\epsilon_t|x_t, z_t)) \\ &= \exp(\alpha_1 x_t + \alpha_2 z_t)E(\exp(\rho\mu + \frac{1}{2}\rho^2\sigma^2)) \\ &= \exp(\alpha_1 x_t + \alpha_2 z_t) \end{aligned}$$

$$\text{plim } L_T(\alpha_1, \alpha_2) = \begin{cases} E[E[(d_t - \exp(x_t\hat{a}_1 + z_t\hat{a}_2))\exp(x_t\hat{a}_1 + z_t\hat{a}_2)x_t|x_t, z_t]] = 0 \\ E[E[(d_t - \exp(x_t\hat{a}_1 + z_t\hat{a}_2))\exp(x_t\hat{a}_1 + z_t\hat{a}_2)z_t|x_t, z_t]] = 0 \end{cases}$$

Part II

Micro

Chapter 2

Finals

2.1 Final 2003

2.1.1 Question 1 (Bernheim)

Both Lilo and Tim have strategy sets of functions from the money in their boxes ([0,100]) to all the non-negative real value bids.

We can show that each of them naming the money in their own box is a Bayes Equailibrium.

Let m be the money in each of their boxes.

If one person plays that strategy than the best repsonse of the other person is

$$\max_{b_1} \int_0^{100} \frac{1}{100} \frac{(b_1 + m_2)}{2} 1_{\{b_1+b_2 \le m_1+m_2\}}$$

We know that given that the other player names $b_2 = m_2$then this function is increasing up to the point where $b_1 = m_1$ and after that is 0. Thus, to maximize then the person will name $b_1 = m_1$. Thus, no one wants to deviate from this equilibrium.

2.2 Final 2004

2.2.1 Question 2 (Bernheim)

	c	n
c	2a,2a	a,a+1
n	a+1,a	1,1

The setup is a little confusing. If you donate then you and everyone else get a.

N is not contribute. C is contribute.

2.2.1.1 Part a

So the NE is only N,N.

2.2.1.2 Part b

Again, this question is somewhat confusing. For SPNE the general idea is to go to the last period and figure out what the optimal thing to do is and then move backward. In the last period, it costs money to punish and there is no discernable benefit. In the first period people are going to play their best strategy which is just NE or (N,N).

2.2.1.3 Part c

i. My payoff is my monetary payoff discounted by some fraction of the amount that the other guy did better than I did (if he did better than I did) minus some fraction of how much I beat him (if I beat him).

ii. We know that in order to maximize your payoff then you should minimize the last two terms. The way get them both to be as small as possible if to have x_i=x_j. There are two ways to accomplish that. First, you can both contribute to get a payoff of 2a or you can both not contribute to get a payoff of 1. How do we know that someone won't deviate? We check the (C,C) case: $2a \geq a + 1 - \beta_i(1)$. For this to work then $a \geq 1 - \beta_i$. What about deviating for (N,N)? Need $1 \geq a - \alpha_i$, so $\alpha_i + 1 \geq a$, which must be true because α_i is already greater than 1 and a is less than 1.

iii. Say that w_1 and w_2 are the payouts from stage 1 of the game. If w_1 is greater than w_2 will w_1 ever punish? No because that player will lose a dollar for every dollar of punishment he metes out, but he will decrease his payoff by \beta_i p (1+K) which is ...???

2.3 Final 2005

2.3.1 Question 1 (Bernheim)

2.3.1.1 Part a

For Bidder #1: any strategy of bidding above v is strictly dominated by bidding 0 because going above v will be a negative payoff.

For Bidder #2: any strategy of bidding above v' because above v' is negative and bidding 0 is payoff 0.

If we iteratively delete dominated strategies then:

i. Delete Bidder #1 bidding above v

ii. Then we get that bidding above v is dominated for Bidder #2 because the payoff will be less than bidding at v.

2.3.1.2 Part b

We can run through other cases:

i. Both bid 0: no because then someone would diverge up

ii. Both bid \in (0,v): no because someone would diverge up

iii. They bid differently but within (0,v): no because someone would try to diverge over the other

iv. They both bid v: Bidder #1 would be diverge by not bidding at all because it is an all-pay auction and there is no reason for him to lose his bid. But then there is no reason for Bidder 2 to be bidding that high either.

No PSNE.

2.3.1.3 Part c

Because we deleted that strategy when doing iterated deletion of dominated strategies then we know that there will be no probability weight placed on that strategy.

2.3.1.4 Part d

$$g_2(x) = (1 - F_1(x))(-x) + F(x)(v' - x)$$

Know that any MSNE has to provide the same expected payoff for anything in the support of the strategies.

Bidder #2's CDF:

$$\begin{aligned}
(1 - F_1(x))(-x) + F_1(x)(v' - x) &= v'F_1(x) - x = C \\
F_1(v) &= 1 \\
v'F_1(v) - v &= v' - v \\
v'F_1(x) - x &= v' - v \\
F_1(x) &= \frac{v' - v + x}{v'}
\end{aligned}$$

Bidder #1's CDF:

$$\begin{aligned}
(1 - F_2(x))(-x) + F_2(x)(v - x) &= vF_2(x) - x \\
vF_2(v) - v &= 0 \\
vF_2(x) - x &= 0 \\
F_2(x) &= \frac{x}{v}
\end{aligned}$$

2.3.1.5 Part e

The expected revenue from Bidder #1

$$\int_0^v x f_1(x)\mathrm{dx} = \int_0^v x(\frac{1}{v'})\mathrm{dx} = \frac{v^2}{2v'}$$

The expected revenue from Bidder #2

$$\int_0^v x f_2(x)\mathrm{dx} = \int_0^v x(\frac{1}{v})\mathrm{dx} = \frac{v^2}{2v}$$

So the total expected revenue is

$$\frac{v^2}{2v'} + \frac{v}{2}$$

2.3.2 Question 2 (Bernheim)

2.3.2.1 Part a

2.3.3 Question 3 (Bernheim)

2.3.3.1 Part a

i.

By Rubinstein's bargaining game, we know that in the final period the proposer, i, would propose

$$(1, 0, 0)$$

Because the only alternative is for everyone to get 0 then everyone would agree.

ii.

$$(1/3, 1/3, 1/3)$$

iii.

The person doing the proposing wants to offer the exact amount such that the other two guys are indifferent between accepting the offer and their expected utility in the next round. Thus, in the first round, the proposer should offer

$$\delta * \frac{1}{3} = proposal_1$$

because in the coming round, each person has an expected utility.

iv.

If we assume that the people accept their offer when it is the same as their expected discounted utility then they have expected value of the game:

$$E = \frac{1}{3}(1 - \frac{2}{3}\delta) + \frac{2}{3}\delta\frac{1}{3}$$

2.3.3.2 Part b

For any finite game, the first round will have an offer of

$$\frac{1}{3}\delta$$

for all other players

2.3.3.3 Part c

i.

No, the proposer would still take everything and the others would still accept since this is the same to them as everyone getting zero.

ii.

The proposer would offer

$$\frac{2}{3}\delta$$

to one other person.

iii.

$$E = \frac{1}{3}(1 - \frac{1}{3}\delta) + \frac{1}{2}(\frac{2}{3}\delta\frac{1}{3})$$

iv.

2.4 Final 2006

2.4.1 Question 2 (Bernheim)

2.4.1.1 Part a

We want

$$P(p_1 \leq v) = 1 - P(v \leq p_1) = 1 - p_1$$

But we need to put some constraints to keep this a probability

$$Prob = \begin{cases} 1 - p_1 & \text{if } p_1 \leq 1 \\ 0 & \text{if } p_1 > 1 \end{cases}$$

The firm solves the problem

$$\max_{p_1}(1-p_1)p_1$$
$$p_1^* = \frac{1}{2}$$

2.4.1.2 Part b

i. If p_1 < p_2 then the buyer is only going to buy in the first period if at all. Then the probability that the buyer purchases the item is just

$$1-p_1$$

while the probability of sale in the second period is 0.

ii. Indifference would occur (given that the item is going to be bought) when

$$\tilde{v}-p_1 = \delta(\tilde{v}-p_2)$$
$$\tilde{v} = \frac{p_1-\delta p_2}{1-\delta}$$

If $\tilde{v} \leq 1$ then if you have a higher valuation then this indifference then you have that the left-side of the equation is now larger than the right side because the \delta can no longer deflate the right side to be equal. You can also see this by showing that $v - p_1 - \delta v - \delta p_2$, which is the utility from buying in the first period rather than the second is increasing in v.

iii. Say p_1 \leq p_2. The expected revenue is

$$p_1(1-p_1)$$

Say that p_1 \geq p_2 but is less than the break even point. The expected revenue is

$$p_1(1-\frac{p_1-\delta p_2}{1-\delta}) + \delta p_2(\frac{p_1-\delta p_2}{1-\delta})$$

If p_1 \geq p_2 and is greater thant he break even point then the buyer is always going to wait for the next stage

$$\delta p_2(1-p_2)$$

iv. In order to maximize the expected revnue

2.4.1.3 Part d

i. Do the FOC

$$\begin{aligned} E[\pi] &= p_2[1-F(v^*)]v^*(1-v^*) \\ p_2 &= \frac{v^*}{2} \end{aligned}$$

ii. Plug it in plug it in

$$\frac{v^*(2-\delta)}{2} = p_1$$

2.4.2 Question 3 (Bernheim)

2.4.2.1 Part a

Striclty dominated to bid over v because then even if you win then you will get negative utility. It's dominated by bidding 0 which will give you 0 utility for sure.

2.4.2.2 Part b

There are no PSNE. Consider the cases:

i. If we are both at 0 then my best response is to deviate up over you.

ii. If I am higher than your bid but below v then I should deviate down until I'm just above your bid.

iii. If we are both at v then I have an expected payoff of $\frac{1}{2}(-v)$, which is negative so I would deviate to not bidding at all.

Note also that if we both bid at $\frac{1}{2}v$ then we have expected payoff of 0 but then I would want to deviate just above, so also not PSNE.

2.4.2.3 Part c

No, becuase bidding above v is strictly dominated then we can be sure that they will never bid with positive probability in an MSNE that high.

2.4.2.4 Part d

$$\begin{aligned} g_i(b_i) &= (v-b_i)F(b_i) - b_i(1-F(b_i)) \\ &= vF(b_i) - b_iF(b_i) - b_i + b_iF(b_i) \\ &= vF(b_i) - b_i \end{aligned}$$

2.4.2.5 Part e

$$\begin{aligned} &vF(v) - v \\ &= 0 \end{aligned}$$

Thus, the payoff from a MSNE is 0.

2.4.2.6 Part f

$$\begin{aligned} 0 &= vF(b_i) - b_i \\ \frac{b_i}{v} &= F(b_i) \end{aligned}$$

2.4.2.7 Part g

$$\begin{aligned} E(payout_i) &= \int_0^v b_i \partial F(b_i) \mathrm{d}b_i \\ &= \int_0^v b_i \frac{1}{v} \mathrm{d}b_i \\ &= \frac{1}{v} \int_0^v b_i \mathrm{d}b_i \\ &= \frac{v^2}{2v} \end{aligned}$$

So the total expected payoff is v.

2.4.2.8 Part h

2.5 Final 2007

2.5.1 Question 3 (Bernheim)

2.5.1.1 Part a

We know that in the cooperative SPNE of a Bertrand with two firms, the two firms would equally share the maximized profits.

What is the stream of SPNE profits received by an individual firm from until forever:

$$\sum_t^\infty \delta^t \frac{\pi(p)}{2} = \frac{1}{1-\delta} \frac{\pi(p)}{2}$$

What would be the stream of profits if a firm deviated? If someone did then after the k_{-i} periods of detection lag, the deviating firm would get 0 profit because the other firm would play monopoly and the deviating firm would get 0 profits from then on:

$$\sum_{t=0}^{k_{-i}-1} \delta^t \pi$$

So the condition for the SPNE is

$$\frac{1}{1-\delta}\frac{\pi}{2} \geq \sum_{t=0}^{k_{-i}-1} \delta^t \pi$$
$$\frac{1}{1-\delta}\frac{1}{2} \geq \sum_{t=0}^{k_{-i}-1} \delta^t$$
$$\frac{1}{2} \geq (1-\delta) \sum_{t=0}^{k_{-i}-1} \delta^t$$
$$\frac{1}{2} \geq (\sum_{t=0}^{k_{-i}-1} \delta^t - \delta \sum_{t=0}^{k_{-i}-1} \delta^t)$$
$$\frac{1}{2} \geq 1 - \delta^{k-1}$$
$$\delta^{k-1} \geq \frac{1}{2}$$

Thus, we can maximize the left hand side using the highest sustainable price of p^m if the $\delta^k \geq 1/2$.

2.5.2 Question 4 (Bernheim)

2.5.2.1 Part a

	a	b
a	pi/2,pi/2	pi,0
b	pi,0	pi/2,pi/2

To find the min-max, we write out the possibilities. Let p be the prob that 1 plays a and q be the prob that 2 plays a.

For player 1 the min-max

$$\min_q \max_p pq\frac{\pi}{2} + p(1-q)\pi + (1-p)q\pi + (1-p)(1-q)\frac{\pi}{2}$$

$$\begin{cases} \pi - q\frac{\pi}{2} & \text{if } q < \frac{1}{2} \\ \frac{\pi}{2} + q\frac{\pi}{2} & \text{if } q > \frac{1}{2} \\ \frac{3}{4}\pi & \text{if } q = \frac{1}{2} \end{cases}$$

Thus the minmax for firm 1 is $\frac{3}{4}\pi$

For firm 2 the min-max is

$$\begin{aligned}
\min_p \max_q pq\frac{\pi}{2} + (1-p)(1-q)\frac{\pi}{2} \\
\min\max pq\frac{\pi}{2} + (1-p-q+pq)\frac{\pi}{2} \\
\min\max q(p\pi - \frac{\pi}{2}) + \frac{\pi}{2} - \frac{p\pi}{2} \\
\min\max q\frac{\pi}{2}(2p-1) + \frac{\pi}{2} - p\frac{\pi}{2}
\end{aligned}$$

$$\begin{cases} \frac{\pi}{2} - p\frac{\pi}{2} & \text{if } p < \frac{1}{2} \\ p\frac{\pi}{2} & \text{if } p > \frac{1}{2} \\ \frac{1}{4}\pi & \text{if } p = \frac{1}{2} \end{cases}$$

So the min max for firm 2 is $\frac{\pi}{4}$ when $p = 1/2$.

2.5.2.2 Part b

The folk theorem say that any NE of an infinitely repeated game must provide an average payoff greater than or equal to their minmax. But because in this game, if one firm gains then the other firm loses, so we know that the NE of an infinite game will be their minmax.

2.6 Final 2009

2.6.1 Question 2 (Bernheim)

2.6.1.1 Part a

Always remember that a PBE has two components:

i. Sequential rationality

ii. Apply Bayes where applicable.

We also know that we are looking for separating equilibria, so eqms where beliefs are 0 and 1.

Say Player 1 goes R at t_1 and L at t_2:

- Player 2 must have belief $\alpha = 1$.
- At the top right, Player 2 will play u-at-R to get his highest payoff.
- Will Player 1 diverge and instead play L then? No becuase 2>1
- At the bottom left, Player 2 will have belief $\gamma = 0$ and will play u-at-L.
- Will Player 1 diverge? No because 2>1.
- YES A SEPARATING PBE is (R,L) and (u,u)

Say Player 1 goes L at t_1 and R at t_2:

- Player 2 must have belief $\alpha = 0$.
- At the top right, Player 2 will play u-at-L to get his highest payoff.
- Will Player 1 diverge and instead play R then? Not if Player 2 plays d-at-R.
- The problem is that this is not sequentially rational since Player 2 would do better to go u-at-R.
- NOT A PBE.

2.6.1.2 Part b

Say Player 1 goes L at t_1 and L at t_2:

- Player 2 has beliefs by Bayes: $\gamma = 1/2$
- Player 2: $\frac{1}{2}3 + \frac{1}{2}4 > \frac{1}{2}0 + \frac{1}{2}1$, so she will play u-at-L.
- Will Player 1 diverge and instead play R then? At the top, to keep Player 1 from diverging to a higher payoff, Player 2 would need to play d-at-R.
- When would this be sequentially rational for Player 2? If $\alpha * 0 + (1-\alpha)2 \geq \alpha * 3 + (1-\alpha) * 0$ or $\alpha \leq \frac{2}{5}$.
- Is this a PBE? Yes, because we can choose \alpha however we want off the eqm path.
- YES a PBE: (L,L) and (u-at-Left, d-at-Right, $\alpha \leq \frac{2}{5}$)

Now to check if a sequential equilibrium:

- Put probabilities such that they create a mixed strategy for Player 1 such that he will play L-at-t_1 and L-at-t_2 and the beliefs of Player 2 will be $\alpha \leq 2/5$.
- For concretness, say that we want an SE with L-at-t_1 and L-at-t_2 and (u-at-Left, d-at-Right, $\alpha = \frac{1}{5}$)
- Say Player 1 plays L-at-t_1 with probability $1 - \epsilon_n$.
- Say Player 1 plays L-at-t_2 with probability $1 - 4\epsilon_n$.
- Say $\epsilon_n \to 0$
- By Bayes
$$\frac{(1)\epsilon_n}{\epsilon_n + 4\epsilon_n} \to \frac{1}{5}$$

2.6.1.3 Part c

There are no subgames here, so the only subgame perfect Nash will be the NEs. We can see that R at t_1 and R at t_2 and (u-at-Right, d-at-Left) will be an NE.

For the PBE, say Player 1 goes R at t_1 and R at t_2:

- By Bayes rule you can say that Player 2 has beliefs: $\alpha = 1/2$.
- Player 2: $\frac{1}{2}3 + \frac{1}{2}0 > \frac{1}{2}0 + \frac{1}{2}2$, so she will play u.
- Will Player 1 diverge and instead play L then? Not if Player 2 plays d-at-Left.
- Is that sequentially rational for Player 2
- $\gamma * 0 + (1-\gamma)1 \geq \gamma * 3 + (1-\gamma) * 4$, so we can't find beliefs to make this sequentially rational.
- NOT A PBE because not sequentially rational.

2.6.1.4 Part d

We don't really need probabilities here:

Say Player 1 mixes when at t_1 and plays L at t_2. We know that Player 2 strictly prefers playing u so Player 1 at t_1 would always go R and not mix. Not a PBE.

Say Player 1 mixes when at t_1 and plays R at t_2. Say \theta is the probability that Player 1 at t_1 goes R. Looking at Player 2's optimization problem, he compares

$$\begin{aligned}\frac{1}{2}\theta(3) + \frac{1}{2}(1-\theta)(3) + \frac{1}{2}(0) &= \frac{3}{2}\\ \frac{1}{2}\theta(0) + \frac{1}{2}(1-\theta)(0) + \frac{1}{2}(2) &= 1\end{aligned}$$

Thus, Player 2 always wants to go (u,u) no matter what the probability of the mixing strategy and hence his beliefs. Thus, in t_2 then Player 1 would want to deviate to play L-at-t_2 instead.

NOT A PBE.

2.7 Final 2011

2.7.1 Question 3 (Bernheim)

Payoff matrix

$$\begin{bmatrix} 3,3 & 1,4 \\ 4,1 & 0,0 \end{bmatrix}$$

2.7.1.1 Part a

Pure Strategy Nash Equilibria

$$(F, S)$$
$$(S, F)$$

Mixed Strategy Nash Equilibria
Let p be the prob that A plays S and q be the prob that B plays S

$$\begin{aligned} 3q + 1(1-q) &= 4q + 0(1-q) \\ 1/2 &= q \\ 3p + 1(1-p) &= 4p + 0(1-p) \\ 1/2 &= p \end{aligned}$$

Note that the mixed strategy payoff of (2,2) is greater than the Pure Nash Equilibria.

2.7.1.2 Part b

The highest symmetric payoff would be when the most probabilty is placed on S,S and the least is placed on F,F. Therefore, we can put equal probs on S,F and F,S such that

$$\begin{aligned} P(S, S) &= 1 - 2\lambda \\ P(S, F) &= \lambda \\ P(F, S) &= \lambda \\ P(F, F) &= 0 \end{aligned}$$

Now do the correlated payoffs

$$\begin{aligned} E_2(g(S, s_2)|S, \delta) &= 3\frac{1-2\lambda}{1-2\lambda+\lambda} + 1\frac{\lambda}{1-2\lambda+\lambda} = \frac{3-5\lambda}{1-\lambda} \\ E_2(g(S, s_2)|F, \delta) &= 3\lambda \\ E_2(g(F, s_2)|S, \delta) &= 4\frac{1-2\lambda}{1-2\lambda+\lambda} + 0\frac{\lambda}{1-2\lambda+\lambda} = \frac{4-8\lambda}{1-\lambda} \\ E_2(g(F, s_2)|F, \delta) &= 4 \end{aligned}$$

Plug into a correlated matrix

$$\begin{bmatrix} \frac{3-5\lambda}{1-\lambda} & \frac{4-8\lambda}{1-\lambda} \\ 3\lambda & 4 \end{bmatrix}$$

To have a correlated equilibrium then we need the highest payoffs on the diagnol

$$3 - 5\lambda \geq 4 - 8\lambda$$
$$\lambda \geq 1/3$$
$$3\lambda \geq 4$$
$$\lambda \geq 4/3$$

But to have the highest symmetric payoff we need the highest probability on S,S, so make λ as small as possible: $\lambda = 1/3$

2.7.1.3 Part c

We know that in the second period, both parties will want to play a Nash Equilibrium.

Know that the Pure Strategy Nash Equilibria make it so that one of the players will receive their minmax.

Thus, if I know that I am going to be "punished" by getting my minmax next period, then why not be "devious" and go for the max payoff I can in the first period. In other words, I have no incentive to be cooperating in the first period because you can't punish me any worse in the second period.

Therefore, if you want me people to coordinate in the first period with S,S then you need to give them incentive: make the second period be a mixed strategy such that they both receive 2 (better than they would get playing Pure Nash) if and only if they played S,S in first period.

How could we get them to adhere to this strategy? We would need them not to deviate and the only punishment that we have is the minmax. So we need

$$3 + 2\delta \geq 4 + \delta$$

That says that the payoff from playing cooperatively the first period and then playing cooperatively the mixed strategy in the second discounted period (left) is greater than going for a deviation the first period and getting punished with a minmax in the next discounted period (right).

The problem is that this says that $\delta \geq 1$. But we know that we can never have a discount rate, δ, greater than 1. Thus, there is no way to get a cooperative S,S outcome in the first that is SPE

2.7.1.4 Part d

Intution says that we can, so let's attempt to construct a strategy:

i. For the first move, play (S,S).

ii. If both players played (S,S) then play (S,F). If one person deviated then play the PSNE that gives the deviator his minmax. If both deviated then play (S,F) as if no one deviated.

iii. If both players played (S,S) in the first round and (S,F) in the second then play (F,S) in the third. If one person deviated in the last round then play the PSNE that gives the deviator his minmax. If both deviated then play (F,S) as if no one deviated.

The second and third round are just PSNE so no one wants to deviate from doing that.

The first round we need to put constraints on to make sure that they wouldn't deviate:

$$3 + \delta 1 + \delta^2 4 \geq 4 + \delta 1 + \delta^2 1$$
$$3 + \delta 4 + \delta^2 1 \geq 4 + \delta 1 + \delta^2 1$$

On the left-side these are the payoffs for someone sticking to this plan. On the right-side are the payoffs from deviating and getting punished. Thus

$$\delta \geq \frac{1}{\sqrt{3}}$$
$$\delta \geq \frac{1}{3}$$

Take the larger of the two to get the binding one.

Thus, we have found an equilibrium in which both play S in the first period.

2.7.1.5 Part e

The Folk Theorems say that we can find a discount rate such that we can sustain any feasible payoffs greater than the minmax payoffs in an infinitely repeated game.

Nash Reversion in this game will cause one player to get their minmax.

Want SS to be more appealing than a one-shot deviation followed by a whole infinity of punishment. Thus, we get that

$$\sum_{t=0}^{\infty} \delta^t * 3 \geq 4 + \sum_{t=0}^{\infty} \delta^t * 1$$
$$\frac{3}{1-\delta} \geq 4 + \frac{1}{1-\delta}$$

Set $\delta = \frac{1}{3}$ to get the mininmal discount rate.

2.8 Final 2012

2.8.1 Question 1 (Bernheim)

2.8.1.1 Part a

i. The contractor solves the problem

$$\max_{\mu} R\mu - \mu^2$$

so take the FOC and get the solution

$$\mu^*(R) = \frac{R}{2}$$

ii. Now the DoD's decision is

$$\max_{R}(B - R)\mu^*(R) = \max_{R}(B - R)\frac{R}{2}$$

Do the FOC thing

$$R^* = \frac{B}{2}$$

Now to get the probability we plug back into the optimal prob from part i

$$\mu^*(R^*) = \frac{B/2}{2} = \frac{B}{4}$$

But wait! Recall that \mu is a probability and cannot be above 1, so write

$$R^* = \begin{cases} 0 & \text{if } B \leq 0 \\ \frac{B}{2} & \text{if } 0 \leq B \leq 4 \\ 2 & \text{if } 4 \leq B \end{cases}$$

so that we get

$$\mu^* = \begin{cases} 0 & \text{if } B \leq 0 \\ \frac{B}{4} & \text{if } 0 \leq B \leq 4 \\ 1 & \text{if } 4 \leq B \end{cases}$$

2.8.1.2 Part b

i. Write out the payoffs

$$\begin{aligned} g(\mu_1, \mu_2) &= \mu_1\mu_2\frac{R}{2} + \mu_1(1 - \mu_2)R - \mu_1^2 \\ \frac{\partial}{\partial\mu_1\partial\mu_2} &= \frac{R}{2} - R \\ &= -\frac{R}{2} \end{aligned}$$

Thus, when you play aggresively then I play less aggresively. Therefore, these are strategic substitutes.

ii. Do the FOC of the payoff

$$g = \mu_1\mu_2\frac{R}{2} + \mu_1(1-\mu_2)R - \mu_1^2$$
$$FOC$$
$$\mu_1 = \mu_2\frac{R}{4} + \frac{(1-\mu_2)}{2}R$$

Set the two mu equal to each other

$$\begin{aligned}
\mu &= \mu\frac{R}{4} + \frac{(1-\mu)}{2}R \\
\mu(1-\frac{R}{4}+\frac{R}{2}) &= \frac{R}{2} \\
\mu &= \frac{R}{2(1-\frac{R}{4}+\frac{R}{2})} \\
&= \frac{R}{2-\frac{R}{2}+R} \\
&= \frac{2R}{4-R+2R} \\
&= \frac{2R}{4+R}
\end{aligned}$$

so we get that this is the optimal probability and you can take the derivative with respect to R to get that it is monotonically increasing in R.

iii. The probability is the probability that one firm makes it, that the second firm makes it, or that both make it. Do this as 1-P(both fail)

$$1-(1-\frac{2R}{4+R})^2$$

but make sure that you restrict the range of the R such that these are still probabilities.

iv. We now know the probability of success, so we can plug that into the firm's max

$$\max_R (B-R)(1-(1-\frac{2R}{4+R})^2)$$
$$R = \frac{2B}{4+B}$$

Chapter 3

Comps

3.1 Comp 2001 Spring

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3.1.1 Question 5 (Bernheim)

First, think always write out the strategy sets for player i:

i. Trade always

ii. Trade if and only if in my state, ω_i

iii. Do not trade if and only if in my state, ω_i (trade in the other states that he can't differentiate)

iv. Never trade

The payoff matrix thus look like

player 1/player2	Trade always	Trade if w_2	Dont trade if w_2	Never
Trade always	4-9p,4-9p	(-2p,p)	4-7p,4-10p	(0,0)
Trade if w_1	p,-2p	(0,0)	p,-2p	(0,0)
Not trade if w_1	4-10p,4-7p	-2p,p	4-8p,4-8p	(0,0)
Never	(0,0)	(0,0)	(0,0)	(0,0)

3.1.1.1 Part a

We can immediately see that not trading in all states will provide the same payoff regardless of what the other person does. Thus, we cannot have a strictly dominant strategy.

For player 1

i. In state 1: accepting weakly dominants rejecting

ii. In state 2/3: neither strategy domiants (rejecting weakly dominants in state 2 and accepting weakly dominants in state 3)

For player 2

i. In state 1/3: neither strategy domiants (rejecting weakly domiannts in state 1 and accepting weakly dominants in state 3)

ii. In state 2: rejecting weakly dominants.

So the weakly dominanted strategies are: {Do not trade if and only if you are ω_i, Never trade}. They are dominated by {Always trade, Trade if and only if you are in ω_i}.

3.1.1.2 Part b

Say p=2/5

player 1/player2	Trade always	Trade if w_2	~~Dont trade if w_2~~	~~Never~~
Trade always	2/5,2/5	-4/5,2/5	~~6/5,0~~	~~(0,0)~~
Trade if w_1	2/5,-4/5	0,0	~~2/5,-4/5~~	~~(0,0)~~
~~Not trade if w_1~~	~~0,6/5~~	~~-4/5,2/5~~	~~4-8p,4-8p~~	~~(0,0)~~
~~Never~~	~~0,0~~	~~0,0~~	~~0,0~~	~~(0,0)~~

We know that the weakly dominanted strategies are: {Do not trade if and only if you are ω_i, Never trade}.

We can see that the Nash equilibria are: (Trade always, Trade Always) with payoff (4-9p,4-9p) and (Trade if ω_1, Trade if ω_2) with payoff (0,0).

The expected total surplus is 8-18p in the first case and 0 in ths second.

3.1.1.3 Part c

If $p \in (2/5, 1/2)$ then

player 1/player2	Trade always	Trade if w_2	~~Dont trade if w_2~~	~~Never~~
Trade always	<-1/2,<-1/2	>-1/2,<1/2	~~4-7p,4-10p~~	~~(0,0)~~
Trade if w_1	<1/2,>-1	(0,0)	~~p,-2p~~	~~(0,0)~~
~~Not trade if w_1~~	~~4-10p,4-7p~~	~~-2p,p~~	~~4-8p,4-8p~~	~~(0,0)~~
~~Never~~	~~(0,0)~~	~~(0,0)~~	~~(0,0)~~	~~(0,0)~~

So, we have only one Nash eqm which is the lower right corner: (Trade if ω_1, Trade if ω_2) with payoff (0,0).

3.1.1.4 Part d

Go back to the big payoff matrix

player 1/player2	Trade always	Trade if w_2	Dont trade if w_2	Never
Trade always	4-9p,4-9p	(-2p,p)	4-7p,4-10p	(0,0)
Trade if w_1	p,-2p	(0,0)	p,-2p	(0,0)
Not trade if w_1	4-10p,4-7p	-2p,p	**4-8p,4-8p**	(0,0)
Never	(0,0)	(0,0)	(0,0)	(0,0)

The socially optimal way to go is to be (Don't trade if ω_1, Don't trade if ω_2), which would maximize the total surplus to 8-16p. The only state in which this occurs is when ω_3. Otherwise, someone isn't trading and they both get 0.

Unfortunately, if I know that you were going to do this dont trade unless in the recognizable state, then I would deviate to trade always because I get a +1p advantage.

3.2 Comp 2004 Spring

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3.2.1 Question 4 (Bernheim)

3.2.1.1 Part a

First, consider if neither firm quoted a price. This could not possibly be a Pure Strategy Nash Eqm because then one firm would diverge to p=v, where they could make a positive profit since v>c+k.

Second, consider if one firm quoted some price between c+k and v. In that case, the second firm would deviate to also quote a price just below the first firms price and above c+k.

Finally, consider if both firms propose a price. We know that they must be offering the same price, otherwise if one was lower than the other then the high priced firm would be making -k profit and thus shouldn't be selling in the first place. We also know that if they both are quoting a price then there is a probability of 1/2 that the customer buys from the other guy. Thus, we need price to be at least c+2k. We know that if they sell then they would get c+2k, but if they weren't picked then they would lose k for having advertised. Thus, the lowest price they would both propose would be:

$$0 = \frac{1}{2}(2k) + \frac{1}{2}(-k)$$
$$p \geq c + 2k$$

The problem is that there is still room for deviation. Each firm is going to try to go down just a little bit so that they for sure get the customer at the price

$$c + 2k > p > c + k$$

Thus, there can be not PSNE because these guys are all deviating.

3.2.1.2 Part b

F puts no weight above v because this is above the customer's reservation value, and thus the customer will not purchase the product that price.

F puts no weight below c+k because that is the cost incurred by the firm when selling the item. Selling below that level would give the firm a negative profit, and thus, the firm would be better off not selling anything.

We know that the firm has the option to not sell in the support of his strategy set. This option has expected payoff of 0. Thus, because a mixed strategy equilibrium must have all of the payoffs equal then we know that the equilibrium payoff is 0.

Next, we calculate the probability that the other firm makes an offer of p or less:

$$(1 - \pi F(p))$$

Then we calculate out expected payoff at that price given we are making an offer:

$$(1 - \pi F(p))(p - c) - k$$

Note that the k goes on the end because we pay that regardless of whetehr or not we make the sale (again, all this is conditional on us making an offer).

Using our knowledge that the expected payoff must be 0, then all of the prices in the support must be 0:

$$\begin{aligned} (1 - \pi F(p))(p - c) - k &= 0 \\ \frac{1 - \frac{k}{p-c}}{\pi} &= F(p) \\ \frac{p - c - k}{\pi(p - c)} &= F(p) \end{aligned}$$

To get \pi, we recognize that

$$\begin{aligned} F(c + k) &= 0 \\ F(v) &= 1 \end{aligned}$$

Plugging into the over equation we get

$$\pi = \frac{v - c - k}{v - k}$$

3.3 Comp 2005 Fall

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3.3.1 Question 3 (Bernheim)

3.3.1.1 Part a

i.

Need the loss in utility during the first endowed period from giving up g units to be less than the utility gained from receiving g units in the unendowed period.

$$u(W) - u(W - g) < \delta(u(g) - u(0))$$

More succintly, we can write

$$u'(W) < \delta u'(g)$$

To find the socially optimal point, just think about the points on a concave curve: expand the distance between u(0) and u(g) by increasing g until it touches u(W-g):

$$u'(W - g) = \delta u'(g)$$

ii.

Say that deviations from the optimal gift are punished by the next generation not giving anything. Under the social optimum, each generation gets

$$u(W - g) + \delta u(g)$$

If you give anything else and get punished then you receive

$$u(W - g') + \delta u(0)$$

Because the utility is strictly increasing, we know that this is less than

$$u(W) + \delta u(0)$$

But we already showed in part a of this question that this is less optimal than the sharing scheme. Thus, deviating must be less optimal than maintaining the social convention over time.

iii. The above answer shows that given any social convention that makes everyone better off, adherence, counterintuitively, does not depend on the δ. Instead, it depends on the social convention being more beneficial than doing nothing.

iv.

No this would not be possible because everyone would deviate to not following the convention. They would all stop giving the next generation, so that they would receive nothing in return. Their utility wuld revert to

$$u(W) + \delta u(0)$$

which by assumption is better than whatever sharing program they had in place before. Thus, we can also see that this does not depend on the δ.

v.

In the finite case, this all breaks down. Why would the last generation give the previous generation? Without some reciprocity, sharing with their parents is strictly dominanted. But if the last generation won't give to their parents, then their parents won't give to the previous generation either because

$$u(W) - \delta u(0) > u(W - g) - \delta u(0)$$

Thus, going up the chain, no one gives to their parents and the whole world reverts to being selfish.

3.3.1.2 Part b

NEED TO FILL IN

3.3.2 Question 4 (Bernheim)

3.3.2.1 Part a

Let v_A be the value to the consumer.

Let v_B be the value to the consumer.

Let p_A be the price quoted by firm A.

Let p_B be the price quoted by firm B.

Know that $v_A \geq v_B$ by assumption.

The comsumer wants the best deal, which we will define as

$$v_i - p_i$$

Start by rulling out $p_B > c$. We know this because the consumer will pick from the company that has the largest spread between value and price. Thus, if $p_B > c$, then firm B would lower their price slightly in an effort to get the consumer while not losing much revenue. So, $p_B = c$

Next we can rule out $v_A - p_A > v_B - p_B$. If that were the case then firm A would keep moving up p_A to increase revenue without losing the consumer.

So now we have settled that

$$v_A - p_A = v_B - p_B$$

Thus, we get that

$$p_A = v_A - v_B + p_B$$

$$p_B = c$$

and there is complete indifference between these two firms. So the consumer can flip a coin and buy from either.

3.3.2.2 Part b

i.

Then firm A is maximizing given knownledge that $p_B = c$ in the second stage and that $v_A = v_B$. Thus, the second stage is really an equilibrium such that

$$\begin{aligned} v_A + f(d) - p_A &= v_B - p_B \\ v_A + f(d) - p_A &= v_B - c \end{aligned}$$

so firm A would set its price in the second stage as

$$p_A = v_A + f(d) - v_B + c$$

Thus, firm A has profits

$$\begin{aligned} \pi_A &= p_A - c - d \\ &= v_A + f(d) - v_B + c - c - d \\ &= v_A + f(d) - v_B - d \\ &= f(d) - d \end{aligned}$$

with the last line since $v_A = v_B$ in this version.

ii.

The question is asking us to maximize the profit with respect to advertising, d:

$$\begin{aligned} \frac{\partial}{\partial d}\pi_A &= \frac{\partial}{\partial d}(f(d) - d) \\ 0 &= f'(d) - 1 \\ f'(d) &= 1 \end{aligned}$$

In the specific case of $f(d) = \sqrt{d}$:

$$\begin{aligned}\frac{1}{2}d^{-\frac{1}{2}} &= 1\\ d &= (2)^{-2}\\ &= \frac{1}{4}\end{aligned}$$

3.3.2.3 Part c

i.

$$\begin{aligned}\pi_A &= \max\{f(d_A) - f(d_B), 0\} - d_A\\ \pi_B &= \max\{f(d_B) - f(d_A), 0\} - d_B\end{aligned}$$

ii.

There couldn't be a Nash equilibrium where both firms have the same level of advertising expenditure because if there was then they would both have profits that were negative

$$\begin{aligned}\pi_A &= max\{0, 0\} - d_A\\ \pi_B &= max\{0, 0\} - d_B\end{aligned}$$

So they wouldn't even operate then.

iii.

NEED TO DO

3.4 Comp 2006 Spring

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3.4.1 Question 2 (Segall)

3.4.1.1 Part a

Definition: a normal good is one for which consumption is increasing with wealth.

We want to find increasing difference in (s_i, e_i), so that we can use Topkis to say that the optimal s_i is increasing in the strong set order.

Our maximization problem is

$$\begin{aligned}&\max u(c_i, s_i)\\ &s.t c_i + p s_i = \frac{1}{2}p + e_i\end{aligned}$$

We can rewrite without the c_i:

$$\max u(e_i + \frac{1}{2}p - ps_i, s_i)$$

So what we want to find is that when $e_i' > e_i$ then:

$$u(e_i' - \frac{p}{2}, 1) - u(e_i' + \frac{p}{2}, 0) > u(e_i - \frac{p}{2}, 1) - u(e_i + \frac{p}{2}, 0)$$

We have that u is concave with respect to its first arguement. We can use that to say

$$u(e_i' - \frac{p}{2}, 1) - u(e_i - \frac{p}{2}, 1) > u(e_i' + \frac{p}{2}, 1) - u(e_i + \frac{p}{2}, 1)$$

We also know by the fact that $u(c, 1) - u(c, 0)$ is strictly positive that

$$u(e_i' + \frac{p}{2}, 1) - u(e_i + \frac{p}{2}, 1) > u(e_i' + \frac{p}{2}, 0) - u(e_i + \frac{p}{2}, 0)$$

Thus, we have ID in (s_i, e_i) and have shown that diamonds are normal.

3.4.1.2 Part b

Player 1 gets the diamond because we just showed that there are increasing differences in endowment and the diamond. Increasing differences says that with higher endowment, the optimal consumption of diamond is weakly increasing. Thus, we would expect that the first person with his larger endowment will end up with the diamond.

3.4.1.3 Part c

We want to solve for

$$(c_1, s_1), (c_2, s_2), p$$

We have the following maximization problem

$$\begin{aligned} &\max U(c_i, s_i) \\ &\quad s.t. \\ &\qquad ps_i + c_i = \frac{1}{2}p + e_i \end{aligned}$$

We know from part b that the first consumer will get the diamond, so we are really solving for

$$(c_1, 1), (c_2, 0), p$$

We can plug into the budget constraints to get

$$\begin{aligned} c_1 + p &= \frac{1}{2}p + e_1 \\ c_2 &= \frac{1}{2}p + e_2 \end{aligned}$$

or rearranging

$$\begin{aligned} c_1 &= e_1 - \frac{1}{2}p \\ c_2 &= e_2 + \frac{1}{2}p \end{aligned}$$

To characterize the equilbrium, we can use what we know about the increasing differencees

$$\begin{aligned} u(e_1 - \frac{1}{2}p, 1) &\geq u(e_1 + \frac{1}{2}p, 0) \\ u(e_2 + \frac{1}{2}p, 0) &\geq u(e_2 - \frac{1}{2}p, 1) \end{aligned}$$

For all $x > 0$

$$u(e - x, 1) - u(e + x, 0)$$

is strcitly increasing in e.

3.4.1.4 Part d

We know that

$$e_1 \geq e_2$$

That means that we can explore both cases: $e_1 > e_2$ and $e_1 = e_2$.
We know that any equilibrium must satisfy the condition

$$\begin{aligned} u(e_1 - \frac{1}{2}p, 1) &\geq u(e_1 + \frac{1}{2}p, 0) \\ u(e_2 + \frac{1}{2}p, 0) &\geq u(e_2 - \frac{1}{2}p, 1) \end{aligned}$$

Or written differently

$$u(e_1 - \frac{1}{2}p, 1) - u(e_1 + \frac{1}{2}p, 0) \geq 0$$
$$0 \geq u(e_2 - \frac{1}{2}p, 1) - u(e_2 + \frac{1}{2}p, 0)$$

Combined to get

$$u(e_1 - \frac{1}{2}p, 1) - u(e_1 + \frac{1}{2}p, 0) \geq u(e_2 - \frac{1}{2}p, 1) - u(e_2 + \frac{1}{2}p, 0)$$

Thus, multiple equilibria

3.4.1.5 Part e

First, we figure out if the possibilities frontier is convex or concave by using the inverse function theorem

$$u_2(\bar{e} - x, 1) = (\bar{e} - g^{-1}(u_1), 1)$$
$$u_1(x, D) = g(x)$$
$$x = g^{-1}(u_1)$$

$$\frac{\partial u_2}{\partial u_1} = u_c(\bar{e} - x, 1)[-\frac{\partial g^{-1}(u_1)}{\partial u_1}]$$
$$= -\frac{u_c(\bar{e} - x, 1)}{u_c(x, 0)}$$

so we have concave.
We still need to know if

$$u(0, 1) = u_2$$
$$u(\bar{e}, 0) = u_1$$

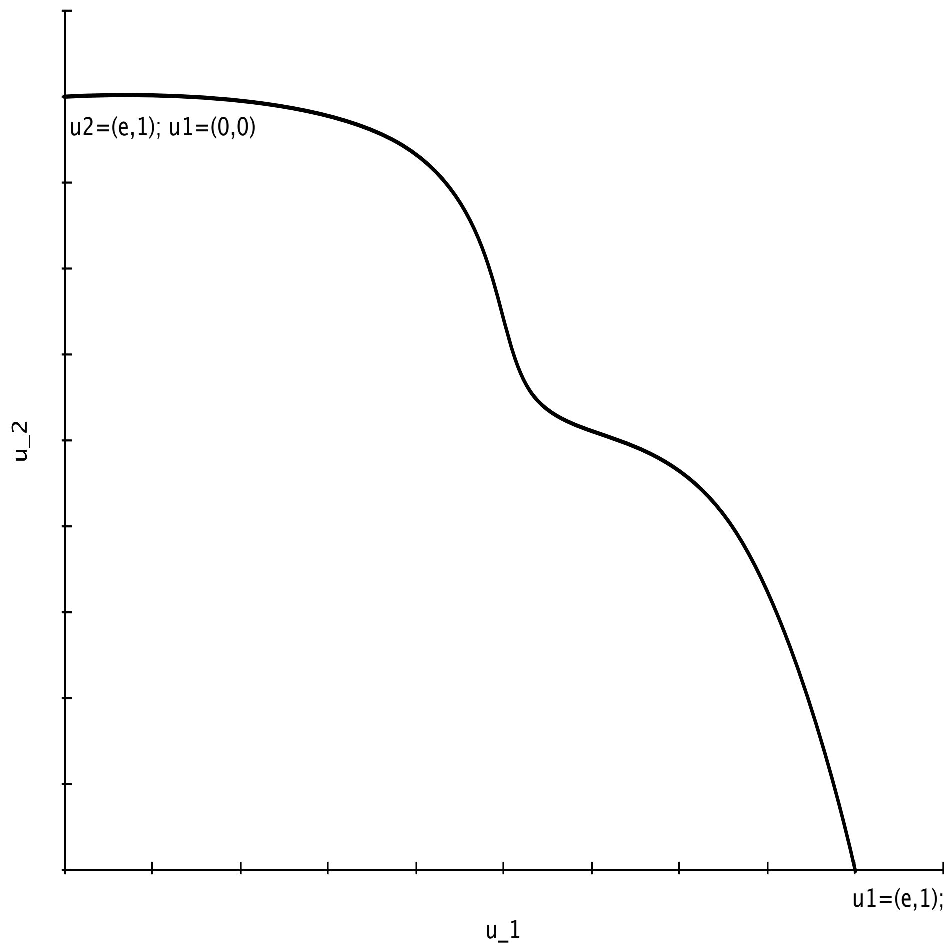
It can't be true that
$$u(0, 1) > u(\bar{e}, 0)$$
because we have an equilibrium so they are willing to trade.
Thus we get that either $u_2 = u_1$ or $u_2 < u_1$.
How do we know that there is a kink in the graph?
The marginal rate of substitution between them is

$$-\frac{u_c(\bar{e} - x, 1)}{u_c(x, 0)}$$

and

$$u_c(\bar{e}, 0) < u_c(0, 0) < u_c(0, 1)$$

and that has absolute value greater than 1.



3.4.1.6 Part f

If you look at the kind then you can see that any convex combination that has the connecting line going over the indent will provide better expected utility.

$$\begin{aligned} E(u) &= \frac{1}{2}u(c, 1) + \frac{1}{2}u(e_1 + e_2 - c, 0) \\ c &\in [0, e_1 + e_2] \end{aligned}$$

To find the max allocation just look for the slope = -1.

$$u_c(c, 1) = u_c(\bar{e} - c, 0)$$

3.5 Comp 2009 Spring

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3.5.1 Question 5 (Bernheim)

3.5.1.1 Part a

The way to start any of the spatial differentiation models is to start by finding the point of indifference for the consumers.

Say that $\tilde{\theta}$ is the preference point at which all consumers below that will go to firm A and all consumers above that will go to firm B.

$$\begin{aligned}
v - p_A - t_A\tilde{\theta} &= v - p_B - t_B|\tilde{\theta} - 1| \\
-p_A - t_A\tilde{\theta} &= -p_B - t_B|\tilde{\theta} - 1| \\
p_A + t_A\tilde{\theta} &= p_B + t_B|\tilde{\theta} - 1| \\
p_A + t_A\tilde{\theta} &= p_B + t_B 1 - \tilde{\theta} \\
\tilde{\theta} &= \frac{p_B - p_A + t_B}{t_A - t_B}
\end{aligned}$$

i. We can say that
Firm A gets

$$\frac{p_B - p_A + t_B}{t_A - t_B}$$

in sales
Firm B gets

$$1 - \frac{p_B - p_A + t_B}{t_A - t_B}$$

ii. Firm 1 gets

$$(p_A - c)\frac{p_B - p_A + t_B}{t_A - t_B}$$

in profit
Firm 2 gets

$$(p_B - c)(1 - \frac{p_B - p_A + t_B}{t_A - t_B})$$

iii.

$$\max_{p_A}(p_A - c)\frac{p_B - p_A + t_B}{t_A - t_B}$$
$$FOC$$
$$p_A = \frac{p_B + t_B + c}{2}$$

iv.

$$\begin{aligned} p_A &= \frac{\frac{1}{2}(p_A + t_A + c) + t_B + c}{2} \\ &= \frac{t_A + 2t_B + 3c}{3} \end{aligned}$$

v.

$$\pi_i = \frac{(t_i + 2t_j)^2}{9(t_i + t_j)}$$

3.5.2 Question 6 (Bernheim)

3.5.2.1 Part a

President has two information sets, one when he receives a g and one when he recieves a b: $(E, N) \times (E, N)$

Economist has three information sets, when he is ignorant, when he is knowledgable and good, when he is knowledgable and bad: $(g, b) \times (g, b) \times (g, b)$

3.5.2.2 Part b

In this setup then only sends signal when he is K and G:

$$P(g) = P(K)P(G) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

President would know that he is getting K advice so he would enact

$$P(K|g) = 1$$

3.5.2.3 Part i

$$P(b) = P(b|K) + P(b|I) = \frac{3}{4} \cdot \frac{1}{4} + \frac{3}{4} = \frac{15}{16}$$

$$P(b, G) = P(I)P(G) = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$$

$$P(G|b) = \frac{P(b, G)}{P(b)} = \frac{3/16}{15/16} = \frac{1}{5}$$

Note that he would not enact if he received a bad signal becuase

$$E(Payoff|b) = \frac{1}{5} \cdot 1 + \frac{4}{5}(-1) = -\frac{3}{5}$$

3.5.2.4 Part ii

There are two possibilities on the equilibrium path

i. He sees g and enacts

$$P(K|g) = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

ii. He sees b and doesn't enact:

$$P(K|b) = \frac{\frac{3}{4} \cdot \frac{1}{4}}{\frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4}} = \frac{1}{5}$$

Off the equilibrium path

You could have any belief between 0 and 1

3.5.2.5 Part iii

Check sequential rationality:

If the president receives a b then he will play N and the economist receives $\frac{1}{5}$.

Say instead that the ignorant economist plays g. Then the president will E. One quarter of the time the economist will get a payoff of 1. In 3/4 of the cases then the president will recieve -1 and then hit the economist with a payoff of $\mu \in [0, 1)$. For any of these mu he will prefer to play g then.

3.6 Comp 2009 Fall

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3.6.1 Question 6 (Bernheim)

3.6.1.1 Part a

A has one information set: (Undertake,Dont)

B has four information sets, one for for each of A's decisions and whether or not he is selfish: $R^+ \times R^+ \times R^+ \times R^+$

I'm using U for undertake, D for don't undertake, S for selfish, A for altruist

Writing out the payoffs is helpful for me

$$g_B(t, U|S) = 120 - t$$
$$g_B(t, D|S) = 90 - t$$
$$g_B(t, U|A) = \min 120 - t, t$$
$$g_B(t, U|A) = \min\{90 - t, 10 + t\}$$

In a subgame perfect situation then B will do whatever is best for himself in the second stage without fear of retribution. Thus, B will give 0 if S and U, 0 if S and D, 60 if A and U, 50 if A and D.

	S	A
U	120,0	60,60
D	90,10	50,50

$$g_A(U) = \pi(0) + (1 - \pi)(60)$$
$$g_A(D) = \pi(10) + (1 - \pi)(50)$$

B will give 0 if S and U, 0 if S and D, 60 if A and U, 50 if A and D.
A will only undertake if $\pi \leq \frac{1}{2}$.

3.6.1.2 Part b

i. Now B has: $g \times g \times t$ if selfish and U $\times t$ if selfish and D $\times t$ if alt and U $\times$ t if alt and D . A still has one info set: $f(g) \to U, D$

ii. If $g < g^*$ then A doesn't undertake and B and A have payoffs: if selfish (90,90) and if alt then (50,50). If $g \geq g^*$ then A undertakes so payoffs for B are $120 - g^* - t$. To keep B from deviating and not giving $g = 0$ then you would need $g^* \leq 30$ if selfish and $g^* \leq 70$ if altruistic.

iii. We know that if g^* is above 30 then no selfish B would make that gift. We know that if above 70 then not even the altruist would make that gift. Thus, the separating equilibrium must have a g in between 30 and 70 where player A will recognize it as a signal that player B is an altruist. Now that A knows what B is we need to ensure that A doesn't go off and screw things up by deviating. If a gets a gift $g \in [30, 70]$, then he knows that B is an altruist. A's payoffs when B is alt are 60 from U and 50 from Don't. Therefore, to keep A from deviating, we need the gift to him to be less than 50 or else A is not going to undertake. Therefore, we have an eqm is $g \in [30, 50]$

iv. They all have the same payoffs.

3.7 Comp 2012 Spring

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3.7.1 Question 3 (Bernheim)

3.7.1.1 Part a

Part III

Macro

Chapter 4

Finals

4.1 Final 2009

4.1.1 Question 1 (Piazzesi)

4.1.1.1 Part 1

We know that

$$q = D\psi$$

We also know that under complete markets we can invert D to get q.

Since we are only looking for the price of the Boyle contract we can just write it out

$$\begin{aligned} q_B &= \text{price of contract for Susan Boyle wins} \\ &= 100 * \psi_B \end{aligned}$$

4.1.1.2 Part 2

We know that with complete markets we have a representative agent.

We also know that

So, we can write the

$$\psi_s = p_s c_s^{-\gamma}$$

4.1.1.3 Part 3

We know that state prices are the price of a single unit of cosumption in any state is

$$\psi_s$$

Thus, the price of a riskless bond is

$$\psi_0 = \sum_s^S \psi_s = \sum_s^S p_s c_s^{-\gamma} = \frac{1}{R^f}$$

4.1.1.4 Part 4

We know that these funny risk neutral probabilities can be written

$$p_s^* = \frac{\psi_s}{\sum \psi_s} = \frac{\psi_s}{\psi_0}$$

because the price of a unit of consumption in state s is ψ_s, so ψ_s/ψ_0 will be a probability of sorts.

We can then write this using the price of a risk-free bond

$$p_s^* = \frac{\psi_s}{\sum \psi_s} = \frac{\psi_s}{\psi_0} = \frac{p_s c_s^{-\gamma}}{\sum_s^S p_s c_s^{-\gamma}}$$

Thus, we know that the price of a Susan Boyle contract is

$$p_s^* * D_{js} * \psi_0 = \psi_s$$

$$p_s^\star * 100 * \psi_0 = p_s^\star * 100 * \frac{1}{R^f}$$

4.1.1.5 Part 5

In terms of preferences, if the guy is actually risk neutral so that $\gamma = 0$, then of course the risk neutral probabilities are also the real probabilities.

Alternatively, if the aggregate endowment is the same in both states such that $e_1 = e_2 = c$

$$
\begin{aligned}
p_s^* &= \frac{p_s c^{-\gamma}}{\sum_s^S p_s c^{-\gamma}} \\
&= \frac{p_s c^{-\gamma}}{c^{-\gamma} \sum_s^S p_s} \\
&= \frac{p_s}{1} \\
&= p_s
\end{aligned}
$$

4.1.1.6 Part 6

The price of the asset is

$$\psi_s = p_s^* * 100 * \psi_0$$

If we divide by 100:

$$
\begin{aligned}
\frac{\psi_s}{100} &= p_s^* * \psi_0 \\
&= p_s^* * \frac{1}{R^f}
\end{aligned}
$$

If the time between now and the realization period is very short then there is no discounting and thus we are really looking at

$$\frac{\psi_s}{100} = p_s^*$$

Setting aggregate consumption equal in both states (which makes sense if the time period is very short)

$$\frac{\psi_s}{100} = p_s^* = p_s$$

so the price of the asset divided by 100 is equal to the probability of the realization of that state.

4.2 Final 2012

4.2.1 Question 1 (Piazzesi)

4.2.1.1 Part 1

$$U(C) = E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\alpha}}{1-\alpha}\right]$$
$$g(a) = E\left[\sum_{t=0}^{\infty} \beta^t \frac{(C_t + ax)^{1-\alpha}}{1-\alpha}\right]$$
$$g'(a) = E\left[\sum_{t=0}^{\infty} \beta^t (C_t + ax)^{-\alpha} x\right]$$

Thus, the Riesz representation is

$$\delta U(C) = E\left[\sum_{t=0}^{\infty} \beta^t (C_t)^{-\alpha} x\right]$$

and the state price deflator is

$$\pi_t = \beta^t (C_t)^{-\alpha}$$

4.2.1.2 Part 2

The pricing kernel is

$$\frac{\pi_{t+1}}{\pi_t} = \frac{\beta^{t+1}(C_{t+1})^{-\alpha}}{\beta^t (C_t)^{-\alpha}} = \beta(\frac{C_{t+1}}{C_t})^{-\alpha}$$

4.2.1.3 Part 3

The price of the risk free bond is calculated on line #43.

No change is necessary for the code because the MP model assumes complete markets so the addition of another asset (in zero net supply) will not change the price of a risk free bond. You can think about it as because we have unique state prices. The price of a risk free bond is the sum of state prices. Therefore, the price of the risk free bond doesn't change.

4.2.1.4 Part 4

$$\begin{aligned}
S_t^e &= E_t\left[\sum_{s>t}\beta^{s-t}\left(\frac{u'(c_s)}{u'(c_t)}D_s\right)\right]\\
&= E_t\left[\sum_{s>t}\beta^{s-t}\left(\frac{u'(C_s)}{u'(C_t)}C_s^b\right)\right]\\
&= E_t\left[\sum_{s>t}\beta^{s-t}\left(\frac{C_s}{C_t}\right)^{-\alpha}C_s^b\right]\\
&= E_t\left[\sum_{s>t}\beta^{s-t}\left(\frac{C_s}{C_t}\right)^{b-\alpha}C_t^b\right]
\end{aligned}$$

You can use

$$C_s = C_{s-1}x_s = ... = C_t\prod_{t+1}^{s}x_i$$

$$\begin{aligned}
S_t^e &= E_t\left[\sum_{s>t}\beta^{s-t}\left(\frac{C_t\prod_{t+1}^{s}x_i}{C_t}\right)^{b-\alpha}C_t^b\right]\\
&= E_t\left[\sum_{s>t}\beta^{s-t}\left(\prod_{t+1}^{s}x_i\right)^{b-\alpha}C_t^b\right]
\end{aligned}$$

$$\begin{aligned}
S(c,i) &= E\left[\beta\left(\frac{u'(c_{t+1})}{u'(c_t)}(S_{t+1}+D_{t+1})\right)\right]\\
&= E\left[\beta\left((\frac{c_{t+1}}{c_t})^{-\alpha}(S(c_{t+1})+D_{t+1})\right)\right]\\
&= E\left[\beta\left((\frac{\lambda c}{c})^{-\alpha}(S(\lambda c,j)+(\lambda c)^b\right)\right]\\
&= E\left[\beta\left(\lambda^{-\alpha}(S(\lambda c,j)+(\lambda c)^b\right)\right]
\end{aligned}$$

$$S(c,i) = c^b w_i$$

$$\begin{aligned}
c^b w_i &= \sum q_{ij}\beta\lambda_j^{-\alpha}(c^b\lambda_j^b w_j + (c\lambda_j)^b)\\
w_i &= \sum q_{ij}\beta\lambda_j^{-\alpha}(\lambda_j^b w_j + {\lambda_j}^b)\\
&= \sum q_{ij}\beta\lambda_j^{b-\alpha}(w_j+1)
\end{aligned}$$

Thus, we get

$$\begin{pmatrix} w_1 \\ ... \\ w_n \end{pmatrix} = \beta \begin{pmatrix} q_{11}\lambda_1^{b-a} & \\ ... & ... \\ q_{n1}\lambda_n^{b-a} & \end{pmatrix} (w+1)$$

So we would need to initialize a b value.
Replace line 54 with A(i,j) = beta * q(i,j) * lambda(j)^(b-alpha)
Replace line 66 with r(i,j) = lambda(j)^b * (1+w(j))/w(i)-1

Chapter 5

Comps

5.1 Comp 2009 Spring

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5.1.1 Question 6 (Piazzesi)

5.1.1.1 Part 1

$$
\begin{aligned}
g(\alpha) &= E_0\left[\sum \beta^t \frac{(c_t+\alpha x)^{1-\gamma}}{1-\gamma}\right] \\
&= \begin{cases} E_0\left[\sum \beta^t \frac{(c_t+\alpha x)^{1-\gamma}}{1-\gamma}\right] & \gamma \neq 1 \\ E_0\left[\sum \beta^t \log(c_t+\alpha x)\right] & \gamma = 1 \end{cases} \\
g'(\alpha) &= \begin{cases} E_0\left[\sum \beta^t (c_t+\alpha x_t)^{-\gamma} x_t\right] & \gamma \neq 1 \\ E_0\left[\sum \beta^t \frac{1}{c_t+\alpha x_t} x_t\right] & \gamma = 1 \end{cases} \\
g'(0) &= \begin{cases} E_0\left[\sum \beta^t (c_t)^{-\gamma} x_t\right] & \gamma \neq 1 \\ E_0\left[\sum \beta^t \frac{1}{c_t} x_t\right] & \gamma = 1 \end{cases} \\
&= E_0\left[\sum \beta^t (c_t)^{-\gamma} x_t\right]
\end{aligned}
$$

We know that the Riesz representation is

$$\delta U(c^\star; x) = E_0 \sum^T \pi_t x_t$$

So

$$\pi_t = \beta^t (c_t)^{-\gamma}$$