First Year Summaries

March 27, 2013

Contents

Part I

Macro

Chapter 1

Dynamic Programming

1.1 Definition of a Competitive Equilibrium (AD)

Given a sequence of prices {p_t}_{t=0}^\infty

$$\max_c \sum \beta^t \ln(c_t^i)$$

such that

$$\sum p_t c_t^i \leq \sum p_t e_t^i$$

and markets clear

$$c_t^1 + c_t^2 = e_t^1 + e_t^2$$

1.2 Solving with FOC

$$\frac{\beta^t}{c_t^i} = \lambda_i p_t$$
$$\frac{\beta^{t+1}}{c_{t+1}^i} = \lambda_i p_{t+1}$$

so

$$p_{t+1} c_{t+1}^i = \beta p_t c_t^i$$

1.3 Pareto Optimality and FWT

i. Feasible

i) $c_t^i \geq 0$

ii) $c_t^1 + c_t^2 = e_t^1 + e_t^2$

ii. There are no other feasible allocations such that everyone is at least as well off and someone is a little bit better off

1.4 Negishi's Method

The social planner's problem doesn't involve prices.

Social Planner's problem

$$\max_c \alpha \mathrm{u}(c^1) + (1-\alpha)u(c^2) = \max_{c^1,c^2} \sum \beta^t[\alpha \ln(c_t^1) + (1-\alpha)\ln(c_t^2)]$$
$$s.t.$$
$$c_t^i \geq 0$$
$$c_t^1 + c_t^2 = e_t^1 + e_t^2$$

An allocation is going to be Pareto efficient if and only if it solves the social planners problem for some $\alpha \in [0, 1]$.

In Kruerger he attached Lagrange multipliers of $\mu/2$, so you get

$$\frac{\alpha\beta^t}{c_t^1} = \frac{\mu_t}{2}$$
$$\frac{(1-\alpha)\beta^t}{c_t^2} = \frac{\mu_t}{2}$$

so

$$\frac{c_t^1}{c_t^2} = \frac{\alpha}{1-\alpha}$$

Plug back into the resource constraint to get

$$c_t^1 = 2\alpha$$

Then the Lagrange multiplier gives

$$\mu_t = \frac{2\alpha\beta}{c_t^1} = \beta^t$$

We can compare these FOC to those of the competiitve equilibrium to see that if we choose $\lambda_1 = \frac{1}{2\alpha}$ and $\mu_t = p_t$ then we get identical FOC.

How do we now get from all the potential equilibria found by doing the social planner's problem to the particular competitive equilibrium?

Define a transfer function

$$t^i(\alpha) = \sum \mu_t[c_t^i(\alpha) - e_t^i]$$

this is the amount of good that would need to be transfered in order for agent i to be able to afford the Pareto optimal allocation indexed by \alpha.

To find the competitive equilibrium then we need to solve for t^1 = t^2 = 0.

Remark 1. Note that the Lagrange multiplier on a budget constraint is a single multiplier that says that the budget constraint holds across all periods. The multiplier on the resource constraint is for each time period and says that the resource constraint holds in every time period.

1.5 Sequential Equilibrium

Let r_{t+1} be the interest rate on one period bonds from t to t+1. This is a promise to pay one unit of consumption in period t+1 in exchange for \frac{1}{1+r_{t+1}} units of consumption good at t.

Interpret the q_t = \frac{1}{1+r_{t+1}} as the relative price of one unit of consumption good in t+1 in terms of period t consumption good.

Let a^i_{t+1} be the amount of bonds purchased by agent i in period t and carried over to period t+1. so a household budget constraint is

$$c_t^i + \frac{a_{t+1}^i}{(1 + r_{t+1})} \leq e_t^i + a_t^i$$

Thus, we can write the SE as:

i. For given interest rates

ii. {c_t} and {a_t+1} are sequences

iii. $\max_{c_t, a_{t+1}} \sum \beta^t \ln(c_t^i)$

iv. s.t.

v. $c_t^i + \frac{a_{t+1}}{1+r_{t+1}} \leq e_t^i + a_t^i$

vi. c_t^i \geq 0

vii. $a_{t+1}^i \geq -\bar{A}$

viii. and for all people

ix. $\sum^i c_t^i \leq \sum^i e_t^i$

x. $\sum^i a_{t+1} = 0$

Arrow Debreu and Sequential Markets will give the same final results.

1.6 Neoclassical Growth Model

1.6.1 Enviornment

- Labor, Capital, and final output
- $y_t = F(k_t, n_t)$
- Time separable utility function \sum_t \beta^t U(c_t)
- Endowments are being born with \bar{k}(0) and a unit of productive labor per time period
- No uncertainity

1.6.2 Pareto Optimality

Consider a social planner attempting to maximize the utility for the rep agent.
Feasible if

$$F(k_t, n_t) = c_t + k_{t+1} - (1-\delta)k_t$$

The problem of the social planner is

$$\begin{aligned} w(\bar{k}_0) &= \max_{c_t, k_t, n_t} \sum \beta^t U(c_t) \\ & s.t. \\ F(k_t, n_t) &= c_t + k_{t+1} - (1-\delta)k_t \\ & ... \end{aligned}$$

Assumptions

i. U is continuously differentiable, strictly increasing, strictly concave, and bounded. It also satisfies the Inada conditions.

ii. F is continuously differentiable and homogenous of degree one, strcitly increasing, strictly quasiconcave. F(0,n)=F(k,0)=0.

Consequences

i. People don't value their leisure so they should always give $n_t = 1$

ii. $f(k) = F(k, 1) + (1-\delta)k$, which the total amount of final good available for either investment or consuming.

A rewrite of the social planner's problem would be

$$\max_{k_{t+1}} \sum \beta^t U(f(k_t) - k_{t+1})$$

$$s.t.$$

$$0 \leq k_{t+1} \leq f(k_t)$$

The reason for solving for this social planner problem is that by the welfare theorems we will have solved for the competitive equilibria.

1.6.3 Exploiting Sationarity

We can write the social planner's problem from above as a dynamic programming problem

$$\begin{aligned}
w(k_0) &= \max_{\{k_{t+1}\},0\leq k_{t+1}\leq f(k_t)} \sum_{t=0} \beta^t U(f(k_t) - k_{t+1}) \\
&= \max_{\{k_{t+1}\},0\leq k_{t+1}\leq f(k_t)} U(f(k_0) - k_1) + \sum_{t=1} \beta^t U(f(k_t) - k_{t+1}) \\
&= \max_{\{k_{t+1}\},0\leq k_{t+1}\leq f(k_t)} U(f(k_0) - k_1) + \beta \sum_{t=1} \beta^{t-1} U(f(k_t) - k_{t+1}) \\
&= \max_{k_1,0\leq k_1\leq f(k_0)} U(f(k_0) - k_1) + \beta[\max_{k_{t+1}} \sum_{t=1} \beta^{t-1} U(f(k_t) - k_{t+1})] \\
&= \max_{k_1,0\leq k_1\leq f(k_0)} U(f(k_0) - k_1) + \beta[\max_{\{k_{t+2}\}} \sum_{t=0} \beta^{t} U(f(k_{t+1}) - k_{t+2})] \\
&= \max_{0\leq k_1\leq f(k_0)} \{U(f(k_0) - k_1) + \beta w(k_1)\} \\
v(k) &= \max_{0\leq k'\leq f(k)} \{U(f(k) - k') + \beta v(k')\}
\end{aligned}$$

When are we allowed to do this and when is it progress?

Call v the value function of the recursive formulation and the w the function for the sequential formulation.

The capital stock of the current period summarizes all of the decisions leading up to this point and thus is called the state variable.

The capital tomorrow is the control variable because it is controlled by the social planner today.

The recursive formulation is known as the Bellman equation, which has a function as a solution.

This is an improvement because we are maximizing the question of the utility of the agent today versus the discounted utility of all future periods. This should be easier than trying to finding the perfect sequence of capital stocks. That said, it also means that we have to maximize for every possible k stock today value.

We do this by solving a functional equation: the value v and an optimal function k^\prime = g(k) known as the policy function that descibes the best k^\prime as a function of the current k.

When does a functional equation exist and is it unique? Could we get there if we guessed any v? When are v and w equivalent?

1.6.4 Guess and Verify

Guess the form. Plug in the Bellman.

Step

i. Guess the form

ii. Solve the RHS maximization problem using FOC to get the k^\prime.

iii. Evaluate the RHS at the optimal k^\prime.

iv. Check if the LHS now equls the RHS and solve for the A and B. Note that one side is going to be constant and that the other side will need to always be constant so will probably be 0.

v. Plug that A and B into the k^\prime you found before to get the policy function

1.6.5 Value Function Iteration: Analytical Approach

Steps

i. Guess a value function v_0

ii. Plug in and perform maximization to get v_1

iii. Solve again using the new v_1 to get the v_2

iv. Rinse and repeat

1.6.6 Value Function Iteration: Numerical Approach

We can only evaluate for k at a finite number of points

Steps

i. Create grid

ii. Make a guess for v_0

iii. Solve the maximization problem using that v_0 on the RHS

iv. Get a k^\prime from that optimization.

v. Plug this value back into the grid points to get a vector of v_1 values for all those grid points.

1.6.7 Euler Equation Finite Case

Form the Lagrange multiplier

$$L = U(f(k_0) - k_1) + ... + \beta^T U(f(k_T) - k_{T+1})$$
$$\frac{\partial L}{\partial k_{t+1}} = -\beta^t U'(f(k_t) - k_{t+1}) + \beta^{t+1} U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}$$
$$U'(f(k_t) - k_{t+1}) = \beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1})$$

Cost in utiluty for saving 1 more unit for capital t+1 = Discounted additional utility from one more unit of c

Costs need to equal benefits at the optimum. This is saying that the COST associated with saving one more unit of capital for t+1 should be equal to the BENEFIT as calculated as the discounted unit of consumption times the additional production of the consumption units that you get from producing with the additional unit of capital saved.

This equation is known as the **Euler Equation**

$$U'(f(k_t) - k_{t+1}) = \beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1})$$

While we have T+1 unknowns and T equations, we also have that $k_{T+1} = 0$

1.6.8 Euler Equation Infinite Case

We no longer have a terminal thing, but we do have the TVC.

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) k_t = 0$$

(the value in discounted terms of capital)*(total capital stock) = 0

Note that this does not say that the capital stock itself is going to zero. It merely says that the shadow value of the capital stock goes to zero.

Theorem 2. *If we satisfy the above assumptions (U is strictly increasing, differnetiable, concave, Inada) and (F is homogenous degree 1 and etc) then an allocation {k_t+1}_{t=0}^\infty that satisfies the Euler equations and the TVC solves the sequential planner problem for a given k_0.*

Note that this theorem does not apply to unbounded utility functions, such as log (although log can be fixed by Stokey Lucas).

1.6.9 Steady State

A steady state is one in which the consumption and the capital are constant.

Use the Euler equations to characterize the steady state

$$\beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) = U'(f(k_t) - k_{t+1})$$
$$\beta U'(c_{t+1}) f'(k_{t+1}) = U'(c_t)$$

Because in steady state the c_t = c_t+1 = c^*

$$\begin{aligned}
\beta U'(c^\star) f'(k_{t+1}) &= U'(c^\star) \\
\beta f'(k_{t+1}) &= 1 \\
f'(k_{t+1}) &= 1+\rho \\
f'(k_{t+1}) &= F_k(k,1) + (1-\delta) \\
F_k(k^*,1) - \delta &= \rho
\end{aligned}$$

This is the Modified Golden Rule: equate the real interest rate and the time discount rate.

1.6.10 Balanced Growth

The NGM does not generate long-run growth.

1.6.11 Competitive Equilibrium

Given a sequences of p_t, w_t, r_t firms face

$$\begin{aligned}
\pi &= \max_{y_t,k_t,n_t} \sum_t p_t(y_t - r_t k_t - w_t n_t) \\
&s.t. \\
y_t &= F(k_t, n_t) \\
y_t, k_t, n_t &\geq 0
\end{aligned}$$

The firm problem is totally static.
Give p_t, w_t, r_t, households face

$$\begin{gathered}
\max_{c_t,i_t,x_{t+1},k_t,n_t} \sum \beta^t U(c_t) \\
s.t. \\
\sum p_t(c_t + i_t) \leq \sum p_t(r_t k_t + w_t n_t) + \pi \\
x_{t+1} = (1-\delta)x_t + i_t
\end{gathered}$$

Remark 3. Note that there is only one AD budget constraint because the market is only open once.

A competitive equilibrium is therefore defined by

i. prices p_t, w_t, r_t

ii. alllocations for firms

iii. allocations for households

iv. such that given prices

v. the firms solve

vi. the households solve

vii. y_t = c_t + i_t

viii. n_t^d = n_t^s

ix. k_t^d = k_t^s

1.7 Contraction Mapping

Theorem 4. *Let (S,d) be a metric space and T:S \rightarrow S be a function mapping S into itself. The function T is a contraction mapping if there exists a number \beta \in (0,1) satisfying* $d(Tx, Ty) \leq \beta d(x, y)$

Theorem 5. *If (S,d) is complete and T is a contraction mapping then the operator T has exactly one fixed point. Also,* $d(T^n v_0, v^\star) \leq \beta^n (v_0, v^\star)$

so starting from any guess, you can zero-in on the right answers

Theorem 6. *Blackwell's Theorem. If T operates on bounded spaces and satisfies monotonicity and discounting then the operator is a contraction mapping.*

Monotinicty is that $f(x) \leq g(x)$ *then* $(Tf)(x) \leq (Tg)(x)$.

Discounting says that if we define (f+a)(x)=f(x)+a then $[T(f + a)](x) \leq [Tf](x) + \beta a$

1.8 Optimality

Want to go from solution to

$$v(x) = \sup\{F(x, y) + \beta v(y)\}$$

to the solution of

$$w(x_0) = \sup \sum_t \beta^t F(x_t, x_{t+1})$$
$$s.t.$$
$$x_{t+1} \in \Gamma(x_t)$$

Assumptions

i. \Gamma is nonempty

ii. for all feasible plans, \lim_{n\rightarrow \infty} \sum \beta^t F(x_t,x_{t+1}) exists although it can be infinity.

Theorem 7. *Principle of Optimality*
If we satisfy the two above assumptions then we get

i. the function w that satisfies the FE

ii. if for all x_0 and all $\bar{x}$ a solution v to the functional equation FE satisfies $\lim \beta^n v(x_0) = 0$

Then $v = w$.

The second condition here is something like a TVC for the functional equation.

What aobut going in the opposite direction from SP to FE?

Theorem 8. *Support satisfy the first two assumptions and*

i. \bar{x} is a feasible plan that attains a supremum in the SP. Then

$$w(\bar{x}_t) = F(\bar{x}_t, \bar{x}_{t+1}) + \beta w(\bar{x}_{t+1})$$

ii. Let \hat{x} be a feasible plan satisfiying

$$w(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta w(\hat{x}_{t+1})$$

and

$$\limsup \beta^t w(\hat{x}_t) \leq 0$$

then \hat{x} attains the superumum. for the SP for the initial condition x_0.

Any optimal plan in the SP then using the w as the value function in the FE we will satisfy the FE.

The second part says that for the right fixed point of the function equation w the corresponding g that generates a plan \hat{x} solves the SP if the additional limit holds. If w is a fixed point of FE and has a correpsonding policy function g then it solves the SP if the limit condition holds.

Chapter 2

Growth

We know that countries are growing at remarkably different rates.

Barrow and Sala-i-Martin argue that we should focus not on unconditional growth differences, but on conditional growth differences: do countries with similar characterisitics have similar or dissimilar income gaps?

He employs the Barrow growth regression

$$g_{i,t,t-1} = \alpha \log y_{i,t-1} + X_{i,t-1}^T \beta + \epsilon_{i,t}$$

There is no evidence of unconditonal convergence int he world income distribution over the postwar era. There is some evidence ofr conditional convegeence meaning that the income gap between countries with similar characterisitcs are getting smaller.

2.1 Solow Growth Model (Discrete)

2.1.1 Enviornment

- All households are identical, so it trivially admits a representative household.
- Individuals save a constant portion fot heir incomes.
- All firms have the same production function, so a representative firm with aggregate production

$$Y(t) = F(K(t), L(t), A(t))$$

- Capital is the same as the final good. Capital is the stuff that isn't eaten and is instead put into the production of more goods.
- Technology is free.

2.1.2 Assumptions

i. $F_K(K,L,A) = \frac{\partial F(K,L,A)}{\partial K} > 0$

ii. $F_{KK}(K,L,A) = \frac{\partial^2 F(K,L,A)}{\partial K^2} < 0$

iii. $F_L(K,L,A) = \frac{\partial F(K,L,A)}{\partial L} > 0$

iv. $F_L(K,L,A) = \frac{\partial F(K,L,A)}{\partial L^2} > 0$

v. F has constant returns to scale in K and L

   i) Because of constant returns to scale: linearly homogenous of degree 1 in K
   ii) Because of constant returns to scale: linearly homogenous of degree 1 in L
   iii) By Euler's law: $F(K,L,A) = F_K(K,L,A) + F_L(K,L,A)$
   iv) And F_K and F_L are homogenous of degree 0 in K and L

vi. Households own all the labor, which they supply inelastically: $L(t) = \bar{L}(t)$

vii. Households own the capital stock and rent tit ot eh firms

viii. Market clearing conditon says that $K(t) = \bar{K}(t)$

ix. Initial capital $K(0) \geq 0$

x. Capital depreciates at an exponential rate of $\delta \in (0,1)$. Out of every 1 unit of capital this period, only $1-\delta$ is left for next period.

xi. Interest rate is $r(t) = R(t) - \delta$. Why does this make sense? Because a unit of good can either be consumed today or you can rent it to a firm. If you rent it then the capital then you lose \delta from it. Thus, if the household rents it then it gives up a unit of good at $t-1$ and receives $1 + r(t) = R(t) + 1 - \delta$ back at time t.

2.1.3 Firm Optimization

$$\max_{K,L} F(K,L,A(t)) - R(t)K - w(t)L$$

Because F has constant returns to scale this does not have a well defined solution (either infinity or zero)

Because labor and capital supply must equal labor and capital demand then the representative firm must make zero profits since otherwise it would want to hire more labor and more capital forever.

By FOC

- $w(t) = F_L(K(t), L(t), A(t))$

- $R(t) = F_K(K(t), L(t), A(t))$

By Euler's Theorem we get that

$$Y(t) = w(t)L(t) + R(t)K(t)$$

2.1.4 Inada Conditions

Page 33-34 in Acemoglu.

2.1.5 Laws of Motion

- $K(t+1) = (1-\delta)K(t) + I(t)$
- Because a closed economy: $Y(t) = C(t) + I(t)$
- $S(t) = sY(t)$ and $C(t) = (1-s)Y(t)$
- Fundamental Law of Motion: $K(t+1) = sF(K(t), L(t), A(t)) + (1-\delta)K(t)$

2.1.6 Equilibrium without Population or Technological Progress

- $K(t), Y(t), C(t), w(t), R(t)_{t=0}^{\infty}$
- s.t.
- $K(t+1) = sF(K(t), L(t), A(t)) + (1-\delta)K(t)$
- $Y(t) = F(K, L, A)$
- $C(t) = (1-s)Y(t)$
- $w(t) = F_L(K(t), L(t), A(t))$
- $R(t) = F_K(K(t), L(t), A(t))$
- $L(t) = L$
- $A(t) = A$
- Define $k(t) = \frac{K(t)}{L}$
- Assume constant returns to scale per capita

$$\begin{aligned} y(t) &= F(\frac{K(t)}{L}, 1, A) \\ &= f(k(t)) \end{aligned}$$

- Thus by FOC

$$R(t) = f'(k(t)) > 0$$
$$w(t) = f(k(t)) - k(t)f'(k(t)) > 0$$

- The law of motion in per capita: $k(t+1) = sf(k(t)) + (1-\delta)k(t)$

2.1.7 Steady State

- Steady State equilibrium without technological progress and population growtih is an equilibrium path in which $k(t) = k^\star$ for all t.
- Capital-labor ratio remains constant
- Since there is no population growth, the level of the capital stock
- The point of intersection between $sf(k(t)) + (1-\delta)k(t)$ and the 45-degree line is when $\frac{f(k^\star)}{k^\star} = \frac{\delta}{s}$.
- Let us assume now that production is Hicks-neutral

$$f(k) = A\tilde{f}(k)$$

- Then the steady state level of capital-labor ratio is $k^\star(A, s, \delta)$ and th steady state level of ouput y^\star:
  - $\frac{\partial k^\star(A,s,\delta)}{\partial A} > 0$
  - $\frac{\partial y^\star(A,s,\delta)}{\partial A} > 0$
  - $\frac{\partial k^\star(A,s,\delta)}{\partial s} > 0$
  - $\frac{\partial y^\star(A,s,\delta)}{\partial s} > 0$
  - $\frac{\partial k^\star(A,s,\delta)}{\partial \delta} < 0$
  - $\frac{\partial y^\star(A,s,\delta)}{\partial \delta} < 0$
  - Economies with higher savings rates and better technologies will have higher capital-labor ratios and will be richer.
  - Those with higher technological deprciation will tend to have lower capital-labor ratios and will be poorer.
- The steady state consumption per capital is not monotone in the saings rate.
  - There is an s_{gold} that maximizes the steady-state level of consumption.
  - $c^\star(s) = (1-s)f(k^\star(s)) = f(k^\star(s)) - \delta k^\star(s)$
  - because $sf(k) = \delta k$
  - $\frac{\partial c^\star(s)}{\partial s} = [f'(k^\star(s)) - \delta]\frac{\partial k^\star}{\partial s}$
  - The golden rule saving rate is when $\frac{\partial c^\star(s_{gold})}{\partial s} = 0$
  - In the Solow Model the capital-gold is $f'(k^\star_{gold}) = \delta$.

2.1.8 Transitional Dynamics

The question is will an economy tend toward the steady state and what will it do along the equilibrium path?

- If the assumptions regarding increasing and concave hold as well as the Inada conditions then the steady state equilibrium of the Solow Growth model described by $k(t+1) = sf(k(t)) + (1-\delta)k(t)$ will be globally asymptotically stable and starting from any k(0)>0, k(t) monotonically onverges to k^*.
- If k(0)<k^* then $w(t)_{t=0}^{\infty}$ is an increasing sequence and $R(t)_{t=0}^{\infty}$ is a decreasing sequence.
  - When the economy starts with too little capital relative to the labor supply
  - Capital-labor ratio increases
  - The marginal product of capital will fall due to diminishing returns to capital, so R is decreasing.
  - The wage rate will increase.

2.2 Solow Growth Model (Continuous)

2.2.1 Laws of Motion

- S(t)=sY(t)
- Population growth $L(t) = \exp(nt)L(0)$
- $\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{\dot{K}(t)}{K(t)} - n$
- Law of Motion of the Capital Stock:

$$\dot{K}(t) = sF(K(t), L(t), A(t)) - \delta K(t)$$
$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (n+\delta)$$

2.2.2 Equilibrium

- Steady state of k

$$\frac{f(k^\star)}{k^\star} = \frac{n+\delta}{s}$$

- Steady state k needs to both replenish the depreciating capital and ensure that the capital labor ratio remains constant.
- $\frac{\partial k^\star(A,s,\delta)}{\partial A} > 0$

- $\frac{\partial y^\star(A,s,\delta)}{\partial A} > 0$
- $\frac{\partial k^\star(A,s,\delta)}{\partial s} > 0$
- $\frac{\partial y^\star(A,s,\delta)}{\partial s} > 0$
- $\frac{\partial k^\star(A,s,\delta)}{\partial \delta} < 0$
- $\frac{\partial y^\star(A,s,\delta)}{\partial \delta} < 0$
- $\frac{\partial k^\star(A,s,\delta)}{\partial n} < 0$
- $\frac{\partial y^\star(A,s,\delta)}{\partial n} < 0$
  - When higher population growht rate then reduce the capita-labor ratio and output per capita and thus there are lower incomes per person.

2.2.3 Sustained Growth

Sometimes the transition back to steady state is very slow. If we have Cobb-Douglas with \alpha=1 then we get the AK model

$$F(K(t), L(t), A(t)) = AK(t)$$

Combined with the law of motion of capital

$$\frac{\dot{k}(t)}{k(t)} = sA - \delta - n$$

so when sA-\delta-n>0 then there will be sustained growth in the capital labor ratio and thus in output per capita.

2.2.4 Balanced Growth

Kaldor Facts

- while output per capita increases the capital-output ratio, the interest rate, and the distribution of income between capital and labor remain constant.
- BGP refes to: output grows at a constant rate, capital-output ratio, the interest rate, and factor shares remain constant.

Uzawa's Theorem

- BGP means constancy of factor shares and constancy of the capital-output ratio K(t)/Y(t)
- $\alpha_K(t) = \frac{R(t)K(t)}{Y(t)}$
- $\alpha_L(t) = \frac{w(t)L(t)}{Y(t)}$

2.2.5 Comparative Dynamics

$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (n + \delta + g)$$

Say there is a one time permanent increase from s to s'.

Chapter 3

Finance Quarter

The basic idea of the quarter is to figure out how to price assets.

3.1 Static Model

The setup: one good, two period (one in which you trade assets and one in which you consume), and S states of the world. The states of the world are characterists of a given time period (e.g. the time period and there is a drought). There are m agents with e^i endowments and U^i utility functions.

Finally, there are N assets. These can be things like bonds, equities, or derivatives. We represent them by their payoffs in various states. The payoff matrix D is N \times S with D_js being the payoff of asset j in state s. An Arrow security, for example, would be an asset providing 1 unit of consumption in some state of the world.

\theta is a portfolio vector of assets that you are either purchasing or shorting. Thus, **portfolio payoff** is $D'\theta$

The price of purchasing one of these assets at time 0 is q_j. It is a scalar and the primary goal of this whole course is how to find q.

The **market value** of of a portfolio is $q'\theta$ at time 0. This is the total market value of the protfolio using prices that are given in the market. We are going to want to understand where these prices come from.

Complete markets means that the rank of the payoff matrix D is S. We can span the entire space of payoffs by creating portfolios using these assets.

3.1.1 Individual Agent's Problem

$$\sup_c U^i(c)$$

such that
budget set

$$B(q, e^i) = \{c \in R^S : \exists \theta \in R^N : q'\theta \le 0, c = e^i + D'\theta\}$$

If utility is increasing then appeal to Walras's Law and say that $q'\theta = 0$
In complete markets

$$\sup U^i(x) \qquad s.t. \quad \psi' x \leq \psi' e^i$$

Then the FOC

$$\partial U^i(c) = \alpha_i \psi$$

so with complete markets then everyone has colinear marginal utilities and state price vectors.

3.1.2 Equilibrium

$$((c^i, \theta^i)_{i=1}^m, q)$$

such that

i. $c^i = e^i + D'\theta^i$solves the agents problem

ii. goods markets clear $\sum_i^m c^i = \sum_i^m e^i$

iii. asset markets clear $\sum_i^m \theta^i = 0$

3.1.3 Arbitrage

A portfolio such that either

$$q'\theta \leq 0$$
$$D'\theta > 0$$

and

$$q'\theta < 0$$
$$D'\theta \geq 0$$

When do you know that there are no arbitrage opportunities?

i. There are no arbitrage opportunities if and only if there are state price vectors

ii. If there is a solution to the agent's problem then there can't be an arbitrage because then you would always be able to get more utility by explotiing it

iii. If U is continuous and there is no arbitrage then we can solve the individual's problem

3.1.4 State Price Vectors

The basis for figuring out how to price assets is to think about the value of consumption in a particular state of the world.

A **state price** ψ_s is the value of a unit of consumption in state s.

State prices help convert payoffs into prices

$$q = D\psi$$

The way to see that this equation makes some sense is to think of an asset which pays 1 unit in one period (an Arrow secruity)

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

The payoff matrix times the state price is going to give you value of consumption in that one period.

Now what would the price of a riskless bond paying 1 unit in all periods

$$D = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

it would be the sum of the state prices.

Using a linear combination of the state prices, we can price any asset.

i. If there are state prices if and only if there is no arbitrage.

ii. If markets are complete then the state prices are unique

iii. If markets are complete then the welfare theorems apply

3.1.5 Price Kernels

$$M_s = \frac{\psi_s}{p_s}$$

$$q = D\psi = \sum^{S} p_s M_s D_{js} = E(MD)$$

Basically the state price vector is what enables you to convert the pricing function from just a scalar operation to something that looks like an expectation involving the true probabilities.

$$\begin{aligned} q_j &= E[MD_j] = E[M]E[D] + cov(M, D_j) \\ q_{bond} &= 1/R^f \\ q_j &= \frac{1}{R^f} E[D_j] + cov(M, D_j) \end{aligned}$$

An asset if valueable if it has a high covariance. Why? Because you want when M is up (so hunger in the state is up) then the payoffs are up too.

3.1.6 Budget Sets

General Budget Sets

$$\begin{aligned} B(q,e) &= \{c \in R^S : \exists \theta \in R^N : q'\theta \leq 0, c = e + D'\theta\} \\ &= \{c \in R^S : \exists \theta \in R^N : \psi' D'\theta \leq 0, c = e + D'\theta\} \\ &= \{c \in R^S : \psi'(c-e) \leq 0; c - e \in span(D')\} \end{aligned}$$

Complete markets budget set

$$\begin{aligned} B &= \{x \in R^S : \psi'(x-e) \leq 0; x - e \in span(D')\} \\ &= \{x \in R^S : \psi'(x-e) \leq 0; x - e \in span(D')\} \end{aligned}$$

3.1.7 Risk Neutral Probabilities

$$\begin{aligned} p_s^\star &= \frac{\psi_s}{\sum^S \psi_s} \\ E^\star[D_j] &= \sum^S p^* D_{js} \end{aligned}$$

Note that

$$R^f = \frac{1}{E[M]}$$

Part II

Metrics

Chapter 4

Estimation

4.1 Method of Moments

Our goal is to look at data generate from some distribution based on a parameter $\theta_0 \in \Theta$ and infer what the origonal \theta_0 was.

The idea of the method of moments is that we define a function that has expectation=0 and use that as a system of equations to get the estimate we are interested in.

In this notation, our objective function is $l(t)$, so

$$
\begin{aligned}
l_t(\theta) &= l(y(t), z(t), \theta) \\
E[l_t(\theta_0)] &= 0 \\
L_t(\theta) &= \frac{1}{T}\sum_t^{\infty} l(t) \\
L_t(\hat{\theta}) &= 0 \\
dim(l_t) &= dim(\theta) = k \\
S_t(\theta) &= \frac{\partial L_t}{\partial \theta} = \frac{1}{T}\sum_t^{\infty} \frac{\partial l_t}{\partial \theta'} \\
V_T(\theta) &= \frac{1}{T}\sum_t^{T} l_t(\theta) l_t'(\theta)
\end{aligned}
$$

These are assumptions and definitions. Along with the strong mixing conditions and regularity conditions, we get the obvious consistentcy results that would help us know that estimation using these methods is going to be useful. Namely

i. $L_t(\theta) \to_p \lim E(L_t(\theta))$

ii. etc

iii. $\sqrt{T} L_T(\theta_0) \rightarrow_d N(0, V(\theta_0))$ or in other words $L_T(\theta_0)\ (0, \frac{\bar{\sigma}_T^2}{T})$

How does this get us an estimator? How dowe know that the method of moments is going to give us consistent estimates?

We know that $L_T(\hat{\theta}) = 0$. We can use an exact first order taylor approximation then to say

$$
\begin{aligned}
L_T(\hat{\theta}) &= 0 \\
L_T(\hat{\theta}) &= L_T(\theta_0) + S_T(\tilde{\theta})(\hat{\theta} - \theta_0) = 0 \\
\tilde{\theta} &\in (\theta_0, \hat{\theta}) \\
L_T(\theta_0) + S_T(\tilde{\theta})(\hat{\theta} - \theta_0) &= 0 \\
-S_T(\tilde{\theta})(\hat{\theta} - \theta_0) &= L_T(\theta_0) \\
(\hat{\theta} - \theta_0) &= -S_T^{-1}(\tilde{\theta}) L_T(\theta_0) \\
\text{plim } (\hat{\theta} - \theta_0) &= -\text{plim } S_T^{-1}(\tilde{\theta}) \text{ plim } L_T(\theta_0) \\
&= -Q \cdot 0 \\
&= 0
\end{aligned}
$$

What does this tell us? It tell us that using this estimation approach, we can get a consistent estimate of the parameter θ_0 using $\hat{\theta}$.

What is the distribution of the method of moments estimator?

$$
\begin{aligned}
\sqrt{T}(\hat{\theta} - \theta_0) &= -S_T^{-1}(\tilde{\theta}) \sqrt{T} L_T(\theta_0) \\
\text{plim } \sqrt{T}(\hat{\theta} - \theta_0) &= \text{plim} - S_T^{-1}(\tilde{\theta}) \text{ plim } \sqrt{T} L_T(\theta_0) \\
&\rightarrow_d N(0, S_T(\theta_0) V(\theta_0) S_T'(\theta_0))
\end{aligned}
$$

Corollary 9. *If $S_T \stackrel{A}{=} \alpha V_T$ then we have that $\hat{\theta} \stackrel{A}{\sim} N(\theta_0, \frac{1}{T} \frac{1}{\alpha^2} V_T(\hat{\theta})^{-1})$*

Example 10. Least Squares