Two Strata Case

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1 Kähler Reduction on $\mathbb{C}^n$

The simplest example of Kähler reduction with an isolated singular point is a linear action. Let $(z^1, \ldots, z^n)$ be holomorphic coordinates on $\mathbb{C}^n$. The standard Kähler form on $\mathbb{C}^n$ is given by,

$$(\omega_{std})_z = \sum_{k=1}^{n} \frac{i}{2} dz^k \wedge d\bar{z}^k \tag{1}$$

The complex structure in these coordinates is given by

$$J_{std}\left(\frac{\partial}{\partial z^j}\right) = i\frac{\partial}{\partial z^j}, \qquad J_{std}\left(\frac{\partial}{\partial \bar{z}^j}\right) = -i\frac{\partial}{\partial \bar{z}^j} \tag{2}$$

The standard hermitian metric on $\mathbb{C}^n$ is given by

$$g_{std}(z) = \omega_{std}(-, J_{std}-) = \sum_{k=1}^{n} dz^k \otimes d\bar{z}^k \tag{3}$$

Recall that $\mathbb{C}^n \backslash \{0\} \simeq (0, \infty) \times S^{2n-1}$ with coordinates on the right given by polar coordinates (r, θ). Here $r = |z|$ and θ denotes the coordinates on S^{2n-1}. The standard metric in these coordinates takes the form

$$g_{std}(r, \theta) = dr^2 + r^2 g_{S^{2n-1}}(\theta) \tag{4}$$

Now let $G \subset U(n)$ be a compact Lie group acting on $\mathbb{C}^n$. We assume that apart from the fixed point set, the action is free. Hence we have two strata given by the orbit types, $(\mathbb{C}^n)_G$ of orbit type (G) and $(\mathbb{C}^n)_e = (\mathbb{C}^n)\backslash(\mathbb{C}^n)_G$ of orbit type (e) where $e \in G$ is the identity element. Note that $(\mathbb{C}^n)_G$ is a linear symplectic subspace. Let $W := ((\mathbb{C}^n)_G)^\perp$ denote the perpendicular subspace with respect to the standard metric on $\mathbb{C}^n$. Then W is a symplectic subspace as well.

Lemma 1.1. *The subspaces $(\mathbb{C}^n)_G$ and W are symplectic and complex subspaces of $\mathbb{C}^n$*

Proof. $(\mathbb{C}^n)_G$ is a complex subspace is clear from the fact that if $v_1, v_2 \in (\mathbb{C}^n)_G$ then $av_1 + bv_2 \in (\mathbb{C}^n)_G$ for all $a, b \in \mathbb{C}$ since G action is via unitary matrices which are complex linear. W being the orthogonal complement w.r.t the □

We can write the stratum $(\mathbb{C}^n)_e$ as the product

$$(\mathbb{C}^n)_e \;=\; (\mathbb{C}^n)_G \times (W \backslash \{0\}) \tag{5}$$

Note that the standard Riemannian metric and the symlectic form restricted to $(\mathbb{C}^n)_e$ is then compatible with respect to the above decomposition. Let $(u_1, \ldots, u_m)$ be the coordinates for $(\mathbb{C}^n)_G$ and let $(w_1, \ldots, w_l)$ the coordinates for W. Then for a point $z = (u, w) \in (\mathbb{C}^n)_e$,

$$g_{std}(u, w) \;=\; \sum_{k=1}^{m} du^k \otimes d\bar{u}^k \;+\; \sum_{k=1}^{l} dw^k \otimes d\bar{w}^k \tag{6}$$

Let $A \in \mathfrak{g} \subset \mathfrak{u}(n)$. The vector field generated on $\mathbb{C}^n$ by the group action, denoted $A_{\mathbb{C}^n}$, is given by

$$A_{\mathbb{C}^n}(z) \;=\; \frac{d}{dt}\Big|_{t=0} \exp(tA) \cdot z \;=\; \begin{cases} 0 & z \in (\mathbb{C}^n)_G \\ A \cdot z & z \in (\mathbb{C}^n)_e \end{cases} \tag{7}$$

The moment map with respect to this action is given by

$$\Phi_{std}(z)(A) \;=\; (\omega_{std})_z(A_{\mathbb{C}^n}(z), z)$$

where $A \in \mathfrak{g} \subset \mathfrak{u}(n)$ is a skew-Hermitian matrix. From (7), we see that $(\mathbb{C}^n)_G \subset \Phi_{std}^{-1}(0)$.

Lemma 1.2. *The zero level set $Z_{std} := \Phi_{std}^{-1}(0)$ is a cone, i.e., $Z_{std} \simeq [0, \infty) \times L$ where $L = Z_{std} \cap S^{2n-1}$*

Proof. Let $p \in Z_{std}$. Consider the scalar multiplication of $\mathbb{R}^+$ on $\mathbb{C}^n$ denoted by $t \cdot p$. Then we have

$$\begin{aligned} \Phi_{std}(t \cdot p)(A) \;&=\; (\omega_{std})_{t \cdot p}(A_{\mathbb{C}^n}(t \cdot p), t \cdot p) \\ &=\; (\omega_{std})_p(t \cdot A_{\mathbb{C}^n}(p), t \cdot p) \\ &=\; t^2\, \Phi_{std}(p)(A) \\ &=\; 0 \end{aligned}$$

where we have used that $(\omega_{std})_p$ is independent of the point $p \in \mathbb{C}^n$ and that scalar multiplication on $\mathbb{C}^n$ is linear and commutes with the action of $\mathfrak{u}(n)$. $\square$

Note that $0 \in Z_{std} \subset \mathbb{C}^n$ is the only singular point and $Z_{std} \backslash \{0\}$ is a smooth manifold. The reduced space defined as the quotient $\pi : Z_{std} \to Z_{std}/G =: (\mathbb{C}^n)_0$ has two strata, the point $\pi(0)$ and the rest. Hence outside the point $\pi(0)$, we can talk about the symplectic form $(\omega_{std})_0$ and the Riemannian metric $(g_{std})_0$ on the manifold $(\mathbb{C}^n)_0 \backslash \{\pi(0)\}$ coming from smooth Kähler reduction.

Note that the G-action on $Z_{std} \backslash \{0\} \simeq (0, \infty) \times L$ acts only on the link L and so

$$(\mathbb{C}^n)_0 \backslash \{\pi(0)\} \simeq (0, \infty) \times (L/G) \tag{8}$$

Now combining **??**, **??**, and the above decomposition, we get that the metric on the reduced space can be written as

$$(g_{std})_0(r, \phi) = dr^2 + r^2 g_{L/G}(\phi) \tag{9}$$

where $g_{L/G}$ is the quotient metric on manifold L/G (with coordinates (ϕ)).