1 Matrices, Vectors, and Vector Calculus

1.1 Introduction

Results from applying physical "laws" to particular situations
- must be independent of the choice of coordinate system
- must be independent of the exact choice of origin for the coordinates

$\Longrightarrow$ Physical phenomena can be discussed through vector methods

A vector – quantity that can be represented as a directed line segment
- related to coordinate transformations through matrices & matrix notation

1.2 Concept of a Scalar

A scalar – quanitty that is invariant under coordinate transformations
e.g.

$$M(x,\ y) = M(x',\ y') \tag{1}$$

where the x-y axis has been transformed (rotated & displaced) to x'-y' `(Figure 1.2-1)`

1.3 Coordinate Transformations

Looking at coordinate transformations on a 2-D coordinate system rotated about its origin
From `(Figure 1.3-1)`,

$$\begin{aligned} x'_1 &= (\text{projection of } x_1 \text{ onto the } x'_1\text{-axis}) + (\text{projection of } x_2 \text{ onto the } x'_1\text{-axis}) \\ &= \overline{Oa} + \left(\overline{ab} + \overline{bc}\right), \quad \text{where } \overline{ab} + \overline{bc} = \overline{dx_2} \\ &= \overline{Oa} + \overline{dx_2} \\ &= x_1 \cos\theta + x_2 \sin\theta \\ &= x_1 \cos\theta + x_2 \cos\left(\frac{\pi}{2} - \theta\right) \end{aligned} \tag{2}$$

$$\begin{aligned} x'_2 &= (\text{projection of } x_1 \text{ onto the } x'_2\text{-axis}) + (\text{projection of } x_2 \text{ onto the } x'_2\text{-axis}) \\ &= -\overline{de} + \overline{Od}, \quad \text{where } \overline{de} = \overline{Of} \\ &= -\overline{Of} + \overline{Od} \\ &= -x_1 \sin\theta + x_2 \cos\theta \\ &= x_1 \cos\left(\frac{\pi}{2} + \theta\right) + x_2 \cos\theta \end{aligned} \tag{3}$$

Next, let $(x'_i,\ x_j)$ be the angle between the x'_i-axis & the x_j-axis
and a set of numbers λ_{ij}

$$\lambda_{ij} \equiv \cos(x'_i,\ x_j) \tag{4}$$

Thus, in the case of 2-D coordinate transformations,

$$\left.\begin{aligned} \lambda_{11} &= \cos(x'_1,\ x_1) = \cos\theta \\ \lambda_{12} &= \cos(x'_1,\ x_2) = \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta \\ \lambda_{21} &= \cos(x'_2,\ x_1) = \cos\left(\tfrac{\pi}{2} + \theta\right) = -\sin\theta \\ \lambda_{22} &= \cos(x'_2,\ x_2) = \cos\theta \end{aligned}\right\} \tag{5}$$

which allows us to express the equations of transformation x'_1 & x'_2 as

$$\left.\begin{aligned} x'_1 &= x_1 \cos(x'_1,\ x_1) + x_2 \cos(x'_1,\ x_2) = \lambda_{11} x_1 + \lambda_{12} x_2 \\ x'_2 &= x_1 \cos(x'_2,\ x_1) + x_2 \cos(x'_2,\ x_2) = \lambda_{21} x_1 + \lambda_{22} x_2 \end{aligned}\right\} \tag{6}$$

or for a 3-D coordinate transformation,

$$\left.\begin{aligned}
x_1' &= x_1\cos(x_1',\ x_1) + x_2\cos(x_1',\ x_2) + x_3\cos(x_1',\ x_3) = \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3 \\
x_2' &= x_1\cos(x_2',\ x_1) + x_2\cos(x_2',\ x_2) + x_3\cos(x_2',\ x_3) = \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3 \\
x_3' &= x_1\cos(x_3',\ x_1) + x_2\cos(x_3',\ x_2) + x_3\cos(x_3',\ x_3) = \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3
\end{aligned}\right\} \tag{7}$$

which in summation notation surmises to

$$x_i' = \sum_{j_1}^{3} \lambda_{ij}x_j, \quad i = 1,\ 2,\ 3 \tag{8}$$

and inversely for equation of transformation x_1, x_2 & x_3,

$$\left.\begin{aligned}
x_1 &= x_1'\cos(x_1',\ x_1) + x_2'\cos(x_2',\ x_1) + x_3'\cos(x_3',\ x_1) = \lambda_{11}x_1' + \lambda_{21}x_2' + \lambda_{31}x_3' \\
x_2 &= x_1'\cos(x_1',\ x_2) + x_2'\cos(x_2',\ x_2) + x_3'\cos(x_3',\ x_2) = \lambda_{12}x_1' + \lambda_{22}x_2' + \lambda_{32}x_3' \\
x_3 &= x_1'\cos(x_1',\ x_3) + x_2'\cos(x_2',\ x_3) + x_3'\cos(x_3',\ x_3) = \lambda_{13}x_1' + \lambda_{23}x_2' + \lambda_{33}x_3'
\end{aligned}\right\} \tag{9}$$

which in summation notation surmises to

$$x_i = \sum_{j=1}^{3} \lambda_{ji}x_j', \quad i = 1,\ 2,\ 3 \tag{10}$$

where the quantity λ_{ij} is the direction cosine of the x_i'-axis relative to the x_j-axis

The set of numbers λ_{ij} for all i & j combinations can be arranged into a square matrix $\boldsymbol{\lambda}$ denoting the totality of the individual elements λ_{ij}

i.e.

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \tag{11}$$

in which $\boldsymbol{\lambda}$ is the transformation matrix (rotation matrix)

$\hookrightarrow$ specifies the transformation properties of the coordinates

1.4 Properties of Rotation Matrices

Consider a line segment extending in a certain direction in space with the origin of the coordinate system lying at some point on the line `(Figure 1.4-1)`

The direction cosines of the line – cosines of definite angles the line makes with each coordinate axes

$\hookrightarrow$ relates to unity as

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{12}$$

Another similar line segment is added such that it intersects the original line segment at the origin `(Figure 1.4-2)`

The cosine of the angle θ between the two lines relates to the direction cosines of both lines as

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' \tag{13}$$

Looking at coordinate transformations on a 3-D coordinate system rotated about some axis through the origin.

We specify the coordinate transformations through the 9 individual elements λ_{ij} of the transformation matrix $\boldsymbol{\lambda}$

– 6 quantities are related

Suppose the x_1'-, x_2'- & x_3'-axes are lines in the $(x_1,\ x_2,\ x_3)$ coordinate system such that lines x_1', x_2' & x_3' are defined with direction cosines $(\lambda_{11},\ \lambda_{12},\ \lambda_{13})$, $(\lambda_{21},\ \lambda_{22},\ \lambda_{23})$ & $(\lambda_{31},\ \lambda_{32},\ \lambda_{33})$ respectively

For each line,

$$\left.\begin{aligned}\lambda_{11}^2+\lambda_{12}^2+\lambda_{13}^2=1\\ \lambda_{21}^2+\lambda_{22}^2+\lambda_{23}^2=1\\ \lambda_{31}^2+\lambda_{32}^2+\lambda_{33}^2=1\end{aligned}\right\} \tag{14}$$

which in summation notation surmises to

$$\sum_j \lambda_{ij}\lambda_{kj}=1, \quad i=k \tag{15}$$

As each line is perpendicular to each other,

$$\left.\begin{aligned}\lambda_{11}\lambda_{21}+\lambda_{12}\lambda_{22}+\lambda_{13}\lambda_{23}=\cos\left(\tfrac{\pi}{2}\right)=0\\ \lambda_{21}\lambda_{31}+\lambda_{22}\lambda_{32}+\lambda_{23}\lambda_{33}=\cos\left(\tfrac{\pi}{2}\right)=0\\ \lambda_{11}\lambda_{31}+\lambda_{12}\lambda_{32}+\lambda_{13}\lambda_{33}=\cos\left(\tfrac{\pi}{2}\right)=0\end{aligned}\right\} \tag{16}$$

which in summation notation surmises to

$$\sum_j \lambda_{ij}\lambda_{kj}=0, \quad i\neq k \tag{17}$$

– 3 independent quantities

Of the 6 non-independent quantities, we can combine the summation results into

$$\sum_j \lambda_{ij}\lambda_{kj}=\delta_{ik} \tag{18}$$

where δ_{ik} is the Kronecker delta symbol

$$\hookrightarrow \delta_{ik}=\begin{cases}0, & i\neq k\\ 1, & i=k\end{cases} \tag{19}$$

The above result is the orthogonality condition[1] – true when coordinate axes in each of the systems specified in the rotation are mutually perpendicular (orthogonal)

The transformation of coordinates and properties of transformation matrices are mathematically similar for differing constructions

- the transformation acts on the point P, giving a new state of the point (point P') expressed with respect to a fixed coordinate system `(Figure 1.4-3)`
- the transformation acts on the frame of reference `(Figure 1.4-4)`

$\hookrightarrow$ The coordinates of point P' in the former construction is equivalent to the new coordinates $(x_1', \ x_2')$ of point P in the latter construction

1.5 Matrix Operations

Square Matrix – equal no. of rows & columns

e.g.

transformation matrix $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda}=\begin{pmatrix}\lambda_{11} & \lambda_{12} & \lambda_{13}\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\\ \lambda_{31} & \lambda_{32} & \lambda_{33}\end{pmatrix} \tag{20}$$

Column Matrix

e.g.

$$\boldsymbol{x}=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \tag{21}$$

Row Matrix

e.g.

$$\boldsymbol{x}=\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix} \tag{22}$$

[1] If instead the x_1-, x_2- & x_3-axes were taken as lines in the $(x_1', \ x_2', \ x_3')$ coordinate system, it will yield the relation $\sum_i \lambda_{ij}\lambda_{ik}=\delta_{ik}$, which is mathematically similar to the result from the former construction

Matrix multiplication

– the product $\boldsymbol{AB}$ is given by

$$\begin{aligned}\boldsymbol{C} &= \boldsymbol{AB}\\ C_{ij} &= [\boldsymbol{AB}]_{ij} = \sum_k A_{ik}B_{kj}\end{aligned} \tag{23}$$

e.g.

$$\begin{aligned}\boldsymbol{x}' &= \boldsymbol{\lambda x}\\ x'_i &= \sum_j \lambda_{ij}x_j\\ \begin{pmatrix}x'_1\\x'_2\\x'_3\end{pmatrix} &= \begin{pmatrix}\lambda_{11} & \lambda_{12} & \lambda_{13}\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\\ \lambda_{31} & \lambda_{32} & \lambda_{33}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\\ &= \begin{pmatrix}\lambda_{11}x_1+\lambda_{12}x_2+\lambda_{13}x_3\\ \lambda_{21}x_1+\lambda_{22}x_2+\lambda_{23}x_3\\ \lambda_{31}x_1+\lambda_{32}x_2+\lambda_{33}x_3\end{pmatrix}\end{aligned} \tag{24}$$

– is defined if

(no. of columns in $\boldsymbol{A}$) = (no. of rows in $\boldsymbol{B}$)

$\hookrightarrow$ multiplying matrix $\boldsymbol{A}$ with m rows & n columns with matrix $\boldsymbol{B}$ with n rows & p columns result in product matrix $\boldsymbol{C}$ with m rows & p columns

– is NOT commutative

i.e.

If $\boldsymbol{A}$ & $\boldsymbol{B}$ are both square matrices, then in general

$$\begin{aligned}\boldsymbol{AB} &\neq \boldsymbol{BA}\\ \sum_k A_{ik}B_{kj} &\neq \sum_k B_{ik}A_{kj}\end{aligned} \tag{25}$$

1.6 Further Definitions

Transposed Matrix – matrix derived from an original matrix by interchange of rows & columns

– denoted by a superscript 't' on the original matrix

i.e.

the transpose of $\boldsymbol{A}$ is $\boldsymbol{A}^t$

e.g.

$$\lambda^t_{ij} = \lambda_{ji} \quad \text{and that} \quad \left(\boldsymbol{\lambda}^t\right)^t = \boldsymbol{\lambda} \tag{26}$$

Identity Matrix – matrix which when multiplied by another matrix, leaves the latter unaffected

e.g.

$$\begin{aligned}\mathbf{1}\boldsymbol{A} = \boldsymbol{A} &\implies \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}A_1\\A_2\end{pmatrix} = \begin{pmatrix}1A_1+0A_2\\0A_1+1A_2\end{pmatrix} = \begin{pmatrix}A_1\\A_2\end{pmatrix}\\ \boldsymbol{B}\mathbf{1} = \boldsymbol{B} &\implies \begin{pmatrix}B_1 & B_2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}B_1(1)+B_2(0) & B_1(0)+B_2(1)\end{pmatrix} = \begin{pmatrix}B_1 & B_2\end{pmatrix}\end{aligned} \tag{27}$$

Inverse of a matrix – matrix which, when multiplied by the original matrix, produces the identity matrix

– denoted by a superscript '-1' on the original matrix

i.e.

the inverse of $\boldsymbol{A}$ is $\boldsymbol{A}^{-1}$

e.g.

$$\boldsymbol{\lambda\lambda}^{-1} = \mathbf{1} \tag{28}$$

Rule of Matrix Algebra

1. Matrix multiplication is not communicative in general
   i.e.
   $$\boldsymbol{AB} \neq \boldsymbol{BA} \tag{29}$$
   except when
   - multiplication of a matrix & its inverse
     i.e.
     $$\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = 1 \tag{30}$$
   - multiplication of a matrix & an identity matrix
     i.e.
     $$\boldsymbol{1A} = \boldsymbol{A1} = \boldsymbol{A} \tag{31}$$
2. Matrix multiplication is associative
   i.e.
   $$(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC}) \tag{32}$$
3. Matrix addition
   - the sum $\boldsymbol{A} + \boldsymbol{B}$ is given by
     $$\begin{aligned} \boldsymbol{C} &= \boldsymbol{A} + \boldsymbol{B} \\ C_{ij} &= A_{ij} + B_{ij} \end{aligned} \tag{33}$$
   - is defined if $\boldsymbol{A}$ & $\boldsymbol{B}$ have the same dimensions

For orthogonal rotation matrices, their transpose & inverse matrices are identical

$$\boldsymbol{\lambda}^t = \boldsymbol{\lambda}^{-1} \tag{34}$$

this results from the product between the orthogonal rotation matrix & its transpose matrix, where

$$\begin{aligned} \boldsymbol{\lambda\lambda}^t &= \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_{11}\lambda_{11} + \lambda_{22}\lambda_{22} & \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} \\ \lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12} & \lambda_{21}\lambda_{21} + \lambda_{22}\lambda_{22} \end{pmatrix}, \quad \text{where } \sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{1} = \boldsymbol{\lambda\lambda}^{-1} \end{aligned} \tag{35}$$

1.7 Geometric Significance of Transformation Matrices

Looking at coordinate transformations on a 3-D coordinate system
If the coordinate system undergo a series of rotations, the entire coordinate transformation can be described by the matrix product of each respective rotation's transformation matrix
Let $\boldsymbol{\lambda}_3$ be the trasnformation matrix describing a series of rotations `(Figure 1.7-1)`

$$\begin{aligned} x' &= \boldsymbol{\lambda}_1 x \quad \& \quad x'' = \boldsymbol{\lambda}_2 x' \\ x'' &= \boldsymbol{\lambda}_2(\boldsymbol{\lambda}_1 x) = \boldsymbol{\lambda}_2\boldsymbol{\lambda}_1 x \equiv \boldsymbol{\lambda}_3 x \\ \Longrightarrow \boldsymbol{\lambda}_3 &= \boldsymbol{\lambda}_2\boldsymbol{\lambda}_1 \end{aligned} \tag{36}$$

If the order of rotations change while each respective rotation remains the same, the matrix product will change as matrix multiplication is non-commutative
Let $\boldsymbol{\lambda}_4$ be the trasnformation matrix describing a series of rotations `(Figure 1.7-2)`

$$\begin{aligned} x' &= \boldsymbol{\lambda}_2 x \quad \& \quad x'' = \boldsymbol{\lambda}_1 x' \\ x'' &= \boldsymbol{\lambda}_1(\boldsymbol{\lambda}_2 x) = \boldsymbol{\lambda}_1\boldsymbol{\lambda}_2 x \equiv \boldsymbol{\lambda}_4 x \\ \Longrightarrow \boldsymbol{\lambda}_4 &= \boldsymbol{\lambda}_1\boldsymbol{\lambda}_2 \end{aligned} \tag{37}$$

$$\therefore \boldsymbol{\lambda}_3 \neq \boldsymbol{\lambda}_4 \tag{38}$$

Generalising $\boldsymbol{\lambda}_1$ & $\boldsymbol{\lambda}_2$, consider the $(x_1,\ x_2,\ x_3)$ coordinate system rotated θ counterclockwise about the x_2-axis. `(Figure 1.7-3)`

The elements λ_{ij} of the transformation matrix $\boldsymbol{\lambda}_5$ describing such a coordinate transformation are

$$\begin{aligned}
\cos(x_1',\ x_1) &= \cos\theta & \cos(x_2',\ x_1) &= \cos\left(\frac{\pi}{2}+\theta\right) & \cos(x_3',\ x_1) &= \cos\left(\frac{\pi}{2}\right)\\
&= \lambda_{11} & &= -\sin\theta = \lambda_{21} & &= 0 = \lambda_{31}\\
\cos(x_1',\ x_2) &= \cos\left(\frac{\pi}{2}-\theta\right) & \cos(x_2',\ x_2) &= \cos\theta & \cos(x_3',\ x_2) &= \cos\left(\frac{\pi}{2}\right)\\
&= \sin\theta = \lambda_{12} & &= \lambda_{22} & &= 0 = \lambda_{32}\\
\cos(x_1',\ x_3) &= \cos\left(\frac{\pi}{2}\right) & \cos(x_2',\ x_3) &= \cos\left(\frac{\pi}{2}\right) & \cos(x_3',\ x_3) &= \cos(0)\\
&= 0 = \lambda_{13} & &= 0 = \lambda_{23} & &= 1 = \lambda_{33}
\end{aligned} \tag{39}$$

which surmises to

$$\boldsymbol{\lambda}_5 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{40}$$

Inversion – transformation that results in the reflection through the origin for all the axes

i.e.

$$x_i' = -x_i \tag{41}$$

e.g.

Let $\boldsymbol{\lambda}_6$ be the trasnformation matrix describing an inversion `(Figure 1.7-4)`

$$\boldsymbol{x}' = \boldsymbol{\lambda}_6\boldsymbol{x}, \quad \text{where } \boldsymbol{\lambda}_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{42}$$

Looking at coordinate transformations on a 3-D coordinate system

Consider successive applications of orthogonal transformation described by transformation matrices $\boldsymbol{\lambda}$ & $\boldsymbol{\mu}$ respectively

$$\begin{aligned}
x_i' &= \sum_j \lambda_{ij}x_j \quad \& \quad x_k'' = \sum_i \mu_{ki}x_i' \\
x_k'' &= \sum_j \left(\sum_i \mu_{ki}\lambda_{ij}\right)x_j = \sum_j [\boldsymbol{\mu\lambda}]_{kj}x_k
\end{aligned} \tag{43}$$

As $(\boldsymbol{\mu\lambda}) = \boldsymbol{\lambda}^t\boldsymbol{\mu}^t$, and that $\boldsymbol{\lambda}^t = \boldsymbol{\lambda}^{-1}$ & $\boldsymbol{\mu}^t = \boldsymbol{\mu}^{-1}$,

$$\begin{aligned}
(\boldsymbol{\mu\lambda})^t\boldsymbol{\mu\lambda} = \boldsymbol{\lambda}^t\boldsymbol{\mu}^t\boldsymbol{\mu\lambda} = \boldsymbol{\lambda}^t\mathbf{1}\boldsymbol{\lambda} &= \boldsymbol{\lambda}^t\boldsymbol{\lambda} \\
&= \mathbf{1} \\
&= (\boldsymbol{\mu\lambda})^{-1}\boldsymbol{\mu\lambda} \\
\implies (\boldsymbol{\mu\lambda})^t &\equiv (\boldsymbol{\mu\lambda})^{-1}
\end{aligned} \tag{44}$$

↪ successive orthogonal transformations always results in an orthogonal transformation

Orthogonal transformations can be characterised by the determinant of its transformation matrix

- Proper rotations – rotations starting from the original set of axes
  In general, such as in the coordinate transformation described by transformation matrix $\boldsymbol{\lambda}_5$. `(Figure 1.7-3)`

$$
\begin{aligned}
|\boldsymbol{\lambda}_5| &= \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \cos\theta \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} \\
&= \cos\theta\ [\cos\theta\ (1) - 0(0)] - \sin\theta\ [-\sin\theta\ (1) - 0(0)] + 0[-\sin\theta\ (0) - \cos\theta\ (0)] \\
&= \cos^2\theta - \left(-\sin^2\theta\right) - 0 \\
&= 1
\end{aligned} \tag{45}
$$

$\hookrightarrow$ determinants equal to '+1'

- Improper rotations – rotation that cannot be generated by any series of proper rotations from the original set of axes Such as in the coordinate transformation described by transformation matrix $\boldsymbol{\lambda}_6$. `(Figure 1.7-4)`

$$
\begin{aligned}
|\boldsymbol{\lambda}_6| &= \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \\
&= -1 \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} + 0 \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \\
&= -1[-1(-1) - 0(0)] - 0[0(-1) - 0(0)] + 0[-1(-1) - 0(0)] \\
&= -1 - 0 + 0 \\
&= -1
\end{aligned} \tag{46}
$$

$\hookrightarrow$ determinants equal to '-1'

1.8 Definitions of a Scalar and a Vector in Terms of Transformation Properties

Looking at coordinate transformations from the x_i-system to the x'_i-system defined by

$$x'_i = \sum_j \lambda_{ij} x_j \quad \& \quad \sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik} \tag{47}$$

Scalar (scalar invariant) – a quantity ϕ that remains unaffected under such a transformation
Vector – a quantity $\boldsymbol{A}$ representing a set of quantities A_i that transforms as the coordinates of a point to a set of quantities A'_i (represented by the quantity $\boldsymbol{A}'$)
i.e.

$$
\begin{aligned}
&\boldsymbol{A}' = \boldsymbol{\lambda}\boldsymbol{A} \\
&A'_i = \sum_j \lambda_{ij} A_j
\end{aligned} \tag{48}
$$

1.9 Elementary Scalar and Vector Operations

For vectors $\boldsymbol{A}$, $\boldsymbol{B}$ & $\boldsymbol{C}$ (with components A_i, B_i & C_i respectively) and scalars ψ, ϕ & η
Addition

- Vectors

$$A_i + B_i = B_i + A_i \quad \text{Commutative law} \tag{49}$$
$$A_i + (B_i + C_i) = (A_i + B_i) + C_i \quad \text{Associative law} \tag{50}$$

- Scalars

$$\phi + \psi = \psi + \phi \quad \text{Commutative law} \tag{51}$$
$$\phi + (\psi + \eta) = (\phi + \psi) + \eta \quad \text{Associative law} \tag{52}$$

Multiplication by a scalar η
- vector

$$\eta \boldsymbol{A} = \boldsymbol{B} \tag{53}$$

results in a vector
- scalar

$$\eta \phi = \psi \tag{54}$$

results in a scalar

1.10 Scalar Product of Two Vectors

Scalar product (dot product)

$$\boldsymbol{A} \cdot \boldsymbol{B} = \sum_i A_i B_i \tag{55}$$

- the scalar product can be expressed in terms of the distances from the origin to points $(A_1,\ A_2,\ A_3)$ & $(B_1,\ B_2,\ B_3)$ and the angle between vectors $\boldsymbol{A}$ & $\boldsymbol{B}$

i.e.

The direction cosines Λ_i of position vectors $\boldsymbol{A}$ or $\boldsymbol{B}$ are given by

$$\Lambda_i^A = \frac{A_i}{|\boldsymbol{A}|} \quad \& \quad \Lambda_i^B = \frac{B_i}{|\boldsymbol{B}|} \tag{56}$$

and from equation (13), the expression $\sum_i \Lambda_i^A \Lambda_i^B$ equates to the cosine of the angle between $\boldsymbol{A}$ & $\boldsymbol{B}$

$$\begin{aligned}\cos(\boldsymbol{A},\ \boldsymbol{B}) &= \sum_i \Lambda_i^A \Lambda_i^B \\ &= \sum_i \frac{A_i}{|\boldsymbol{A}|}\frac{B_i}{|\boldsymbol{B}|} = \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|} \\ \implies \boldsymbol{A} \cdot \boldsymbol{B} &= |\boldsymbol{A}||\boldsymbol{B}| \cos(\boldsymbol{A},\ \boldsymbol{B})\end{aligned} \tag{57}$$

Consider position vectors $\boldsymbol{A}$ & $\boldsymbol{B}$ on a 3-D coordinate system `(Figure 1.10-1)`

Position vector – vector extending from the origin to a point

e.g.

The vector $\boldsymbol{A}$ extending from the origin to the point $(A_1,\ A_2,\ A_3)$ on a 3-D coordinate system
- has magnitude

$$|\boldsymbol{A}| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}} = \sqrt{\sum_i A_i A_i} = \sqrt{\sum_i A_i^2} \tag{58}$$

$\hookrightarrow$ which gives the distance from the origin to the point

Difference vector – vector connecting a point and any other point

e.g.

The vector connecting points $(A_1,\ A_2,\ A_3)$ & $(B_1,\ B_2,\ B_3)$ is $\boldsymbol{A} - \boldsymbol{B}$
- has magnitude

$$\sqrt{\sum_i (A_i - B_i)^2} = \sqrt{(\boldsymbol{A} - \boldsymbol{B}) \cdot (\boldsymbol{A} - \boldsymbol{B})} = |\boldsymbol{A} - \boldsymbol{B}| \tag{59}$$

$\hookrightarrow$ which gives the distance between the two points

Looking at coordinate transformations of vectors $\boldsymbol{A}$ & $\boldsymbol{B}$ on a 3-D coordinate system

Position vectors $\boldsymbol{A}$ & $\boldsymbol{B}$ transforms to $\boldsymbol{A'}$ & $\boldsymbol{B'}$ as

$$A'_i = \sum \lambda_{ij} A_j \quad \& \quad B'_i = \sum_k \lambda_{ik} B_k \tag{60}$$

$$
\begin{aligned}
\boldsymbol{A}'\cdot\boldsymbol{B}' = \sum_i A_i'B_i' &= \sum_i \left(\sum_j \lambda_{ij}A_j\right)\left(\sum_k \lambda_{ik}B_k\right) \\
&= \sum_{j,\ k}\left(\sum_i \lambda_{ij}\lambda_{ik}\right)A_jB_k, \quad \sum_i \lambda_{ij}\lambda_{ik} = \delta_{jk} \\
&= \sum_j \left(\sum_k \delta_{jk}A_jB_k\right), \quad \delta_{jk}A_jB_k = \begin{cases} 0, & j \neq k \\ A_jB_j, & j = k \end{cases} \\
&= \sum_j A_jB_j \\
&= \boldsymbol{A}\cdot\boldsymbol{B} \\
\implies \boldsymbol{A}\cdot\boldsymbol{B} &= \boldsymbol{A}'\cdot\boldsymbol{B}'
\end{aligned}
\tag{61}
$$

$\hookrightarrow$ the value of the scalar product remains unaffected under a coordinate transformation

$\hookrightarrow$ the distance between a point and any other point remains unchanged

i.e.

$$
|\boldsymbol{A}-\boldsymbol{B}| = |\boldsymbol{A}'-\boldsymbol{B}'| \tag{62}
$$

$\hookrightarrow$ orthogonal transformations are distance preserving transformations

Scalar products obey the commutative and distributive laws

– Commutative

$$
\begin{aligned}
\boldsymbol{A}\cdot\boldsymbol{B} &= \sum_i A_iB_i \\
&= \sum_i B_iA_i = \boldsymbol{B}\cdot\boldsymbol{A}
\end{aligned}
\tag{63}
$$

– Distributive

$$
\begin{aligned}
\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C}) &= \sum_i A_i(B+C)_i \\
&= \sum_i A_i(B_i+C_i) = \sum_i (A_iB_i + A_iC_i) = \boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}
\end{aligned}
\tag{64}
$$

1.11 Unit Vectors

Unit vetors – a vector with length equal to the unit of length used along the particular coordinate axes

i.e.

unit vector along the radial direction

$$
\boldsymbol{e}_R = \frac{\boldsymbol{R}}{|\boldsymbol{R}|} \tag{65}
$$

e.g.

variants of the symbols for unit vectors

$$
\left.
\begin{array}{l}
(\boldsymbol{i},\ \boldsymbol{j},\ \boldsymbol{k}) \\
(\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3) \\
(\boldsymbol{e}_r,\ \boldsymbol{e}_\theta,\ \boldsymbol{e}_\phi) \\
\left(\hat{\boldsymbol{r}},\ \hat{\boldsymbol{\theta}},\ \hat{\boldsymbol{\phi}}\right)
\end{array}
\right\}
\tag{66}
$$

vector $\boldsymbol{A} = (A_1,\ A_2,\ A_3)$ to be expressed in terms of unit vectors

$$\left.\begin{aligned}\boldsymbol{A} &= A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}\\ \boldsymbol{A} &= \boldsymbol{e}_1A_1 + \boldsymbol{e}_2A_2 + \boldsymbol{e}_3A_3 = \textstyle\sum_i \boldsymbol{e}_iA_i\end{aligned}\right\} \tag{67}$$

The components of a vector can be obtained by projection onto its axes

$$A_i = \boldsymbol{e}_i \cdot \boldsymbol{A} \tag{68}$$

For any two orthogonal unit vectors,

$$\begin{aligned}\boldsymbol{e}_i \cdot \boldsymbol{e}_j = |\boldsymbol{e}_i||\boldsymbol{e}_j|\cos(i,\ j) &= \begin{cases} 0, & i \neq j\\ 1, & i = j\end{cases}\\ &= \delta_{ij}\end{aligned} \tag{69}$$

1.12 Vector Products of Two Vectors

Consider vectors $\boldsymbol{A}$ & $\boldsymbol{B}$ on a 3-D coordinate system
Vector product (cross product)

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} \tag{70}$$

where $\boldsymbol{C}$ is a vector resulting from this operation with components

$$C_i \equiv \sum_{j,k} \epsilon_{ijk}A_jB_k \tag{71}$$

in which ϵ_{ijk} is the permutation symbol (Levi-Civita density)

$$\hookrightarrow\ \epsilon_{ijk} = \begin{cases} 0 & , \text{ if any index is equal to any other index}\\ +1 & , \text{ if } i,\ j,\ k \text{ form an } \textit{even} \text{ permutation of } 1,\ 2,\ 3\\ -1 & , \text{ if } i,\ j,\ k \text{ form an } \textit{odd} \text{ permutation of } 1,\ 2,\ 3\end{cases}$$

- An even permutation has an even number of exchanges of position of symbols
  e.g.
  Cyclic permutations $123 \to 312 \to 231$
- An odd permutation has an odd number of exchanges of position of symbols
  e.g.
  Cyclic permutations $132 \to 213 \to 321$

- has magnitude

$$\begin{aligned}|\boldsymbol{C}| &= \sqrt{C_1^2 + C_2^2 + C_3^2} = \sqrt{\sum_i\left(\sum_{j,k}\epsilon_{ijk}A_jB_k\right)^2}\\ &= \sqrt{(A_2B_3 - A_3B_2)^2 + (A_3B_1 - A_1B_3)^2 + (A_1B_2 - A_2B_1)^2}\\ &= \Big[\big(A_2^2B_3^2 - 2A_2B_3A_3B_2 + A_3^2B_2^2\big) + \big(A_3^2B_1^2 - 2A_3B_1A_1B_3 + A_1^2B_3^2\big)\\ &\qquad + \big(A_1^2B_2^2 - 2A_1B_2A_2B_1 + A_2^2B_1^2\big)\Big]^{\frac{1}{2}}\\ &= \Big[\big(A_2^2B_3^2 + A_3^2B_2^2 + A_3^2B_1^2 + A_1^2B_3^2 + A_1^2B_2^2 + A_2^2B_1^2 + A_1^2B_1^2 + A_2^2B_2^2 + A_3^2B_3^2\big)\\ &\qquad - \big(2A_2B_3A_3B_2 + 2A_3B_1A_1B_3 + 2A_1B_2A_2B_1 + A_1^2B_1^2 + A_2^2B_2^2 + A_3^2B_3^2\big)\Big]^{\frac{1}{2}}\\ &= \sqrt{\left(\sum_i A_i^2\right)\left(\sum_i B_i^2\right) - (A_1B_1 + A_2B_2 + A_3B_3)^2} = \sqrt{|\boldsymbol{A}|^2|\boldsymbol{B}|^2 - \left(\sum_i A_iB_i\right)^2}\\ &= \sqrt{|\boldsymbol{A}|^2|\boldsymbol{B}|^2 - |\boldsymbol{A}|^2|\boldsymbol{B}|^2\cos^2(\boldsymbol{A},\ \boldsymbol{B})} = \sqrt{|\boldsymbol{A}|^2|\boldsymbol{B}|^2\sin^2(\boldsymbol{A},\ \boldsymbol{B})}\\ &= |\boldsymbol{A}||\boldsymbol{B}|\sin(\boldsymbol{A},\ \boldsymbol{B})\end{aligned} \tag{72}$$

Geometrically, the expression $|\boldsymbol{A}||\boldsymbol{B}|\sin(\boldsymbol{A},\ \boldsymbol{B})$ is equivalent to the area of the parallelogram defined by vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ `(Figure 1.12-1)`.
$\hookrightarrow$ The vector $\boldsymbol{C}$ represents a vector that describes such a plane area

As $\boldsymbol{A}\cdot\boldsymbol{C}=0$ & $\boldsymbol{B}\cdot\boldsymbol{C}=0$,
$\hookrightarrow$ $\boldsymbol{C}$ is perpendicular to the plane defined by $\boldsymbol{A}$ and $\boldsymbol{B}$
- The +ve direction of $\boldsymbol{C}$ is chosen to be the direction of advance of a right-hand screw when rotated from $\boldsymbol{A}$ to $\boldsymbol{B}$

which gives the results

$$\boldsymbol{A}\times\boldsymbol{B}=-\boldsymbol{B}\times\boldsymbol{A} \tag{73}$$

and in general,

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})\neq(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} \tag{74}$$

Consider unit vectors $\boldsymbol{e}_i$, $\boldsymbol{e}_j$ & $\boldsymbol{e}_k$ of a 3-D coordinate system.
The orthogonality of the unit vectors requires the vector product of the unit vectors to be

$$\begin{aligned}\boldsymbol{e}_i\times\boldsymbol{e}_j&=\boldsymbol{e}_k,\quad i,\ j,\ k \text{ in cyclic order}\\ &=\sum_k \boldsymbol{e}_k\epsilon_{ijk}\end{aligned} \tag{75}$$

Therefore we can express the cross product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in such a 3-D coordinate system as

$$\begin{aligned}\boldsymbol{C}=\boldsymbol{A}\times\boldsymbol{B}&=\sum_{i,j,k}\epsilon_{ijk}\boldsymbol{e}_iA_jB_k\\ &\equiv\begin{vmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}\end{aligned} \tag{76}$$

More identities involving Vector Products:
1. $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A})=\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})\equiv\boldsymbol{ABC}$
2. $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$
3. $\left.\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=\boldsymbol{A}\cdot[\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{D})]\quad&=\boldsymbol{A}\cdot[(\boldsymbol{B}\cdot\boldsymbol{D})\boldsymbol{C}-(\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{D}]\\ &=(\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})\end{aligned}\right\}$
4. $\left.\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D})\quad&=[(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{D}]\boldsymbol{C}-[(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}]\boldsymbol{D}\\ &=(\boldsymbol{ABD})\boldsymbol{C}-(\boldsymbol{ABC})\boldsymbol{D}=(\boldsymbol{ACD})\boldsymbol{B}-(\boldsymbol{BCD})\boldsymbol{A}\end{aligned}\right\}$

1.13 Differentiation of a Vector with Respect to a Scalar

Looking at coordinate transformations
A derivative of a scalar function $\phi=\phi(s)$ differentiated with respect to the scalar variable s is a scalar.

$$\begin{aligned}\phi=\phi'\quad&\Longrightarrow\quad d\phi=d\phi'\\ \&\quad s=s'\quad&\Longrightarrow\quad ds=ds'\end{aligned} \tag{77}$$

$$\Longrightarrow\ \frac{d\phi}{ds}=\frac{d\phi'}{ds'}=\left(\frac{d\phi}{ds}\right)' \tag{78}$$

Consider a vector $\boldsymbol{A}$
The components of $\boldsymbol{A}$ transform as

$$A_i'=\sum_j\lambda_{ij}A_j \tag{79}$$

On differentiation, we obtain,

$$\frac{dA_i'}{ds'} = \frac{d}{ds'}\sum_j \lambda_{ij}A_j = \sum_j \lambda_{ij}\frac{dA_j}{ds'} \quad \& \quad \frac{dA_i'}{ds'} = \left(\frac{dA_i}{ds}\right)' = \sum_j \lambda_{ij}\left(\frac{dA_j}{ds}\right) \tag{80}$$

$$\implies \frac{dA_i'}{ds'} = \sum_j \lambda_{ij}\frac{dA_j}{ds'} \equiv \sum_j \lambda_{ij}\left(\frac{dA_j}{ds}\right) \tag{81}$$

$\hookrightarrow$ the quantities $\frac{dA_j}{ds}$ transform as do the components of a vector and hence are the components of a vector $\frac{d\boldsymbol{A}}{ds}$ (vector $\boldsymbol{A}$ differentiated with respect to the scalar variable s)

Geometrically, for the vector $\frac{dA}{ds}$ to exist, $\boldsymbol{A}$ must be a continuous function of the variable s: $\boldsymbol{A} = \boldsymbol{A}(s)$. Suppose the function $\boldsymbol{A}(s)$ to be represented by the continuous curve Γ `(Figure 1.13-1)`
At point P, $x_1 = s$;
at point Q, $x_2 = s + \Delta s$
From the limit definition of a derivative,

$$\frac{d\boldsymbol{A}}{ds} = \lim_{\Delta s \to 0}\frac{\Delta \boldsymbol{A}}{\Delta s} = \lim_{\Delta s \to 0}\frac{\boldsymbol{A}(s+\Delta s) - \boldsymbol{A}(s)}{\Delta s} \tag{82}$$

Derivatives of vector sums and products obey the rules of ordinary vector calculus as follows

$$\left.\begin{array}{l} \frac{d}{ds}(\boldsymbol{A}+\boldsymbol{B}) = \frac{d\boldsymbol{A}}{ds} + \frac{d\boldsymbol{B}}{ds} \\ \frac{d}{ds}(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\frac{d\boldsymbol{B}}{ds} + \frac{d\boldsymbol{A}}{ds}\cdot\boldsymbol{B} \\ \frac{d}{ds}(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}\times\frac{d\boldsymbol{B}}{ds} + \frac{d\boldsymbol{A}}{ds}\times\boldsymbol{B} \\ \frac{d}{ds}(\phi\boldsymbol{A}) = \phi\frac{d\boldsymbol{A}}{s} + \frac{d\phi}{ds}\boldsymbol{A} \end{array}\right\} \tag{83}$$

1.14 Examples of Derivatives- Velocity and Acceleration

Motion of particles can be represented by vectors.
- *position* of a particle with respect to a certain reference frame ($\boldsymbol{r}$)

$$\boldsymbol{r} = \boldsymbol{r}(t) \tag{84}$$

- *velocity* vector ($\boldsymbol{v}$)

$$\boldsymbol{v} \equiv \frac{d\boldsymbol{r}}{t} = \dot{\boldsymbol{r}} \tag{85}$$

- *acceleration* vector ($\boldsymbol{a}$)

$$\boldsymbol{a} \equiv \frac{d\boldsymbol{v}}{t} = \frac{d^2\boldsymbol{r}}{dt^2} = \ddot{\boldsymbol{r}} \tag{86}$$

Consider a point particle in a rectangular coordinate system
The vectors $\boldsymbol{r}$, $\boldsymbol{v}$, $\boldsymbol{a}$ describing a point particle can be expressed as

$$\left.\begin{array}{ll} \boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = \sum_i x_i\boldsymbol{e}_i & \text{Position} \\ \boldsymbol{v} = \dot{\boldsymbol{r}} = \sum_i \dot{x}_i\boldsymbol{e}_i = \sum_i \frac{dx_i}{dt}\boldsymbol{e}_i & \text{Velocity} \\ \boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \sum_i \ddot{x}_i\boldsymbol{e}_i = \sum_i \frac{d^2x_i}{dt^2}\boldsymbol{e}_i & \text{Acceleration} \end{array}\right\} \tag{87}$$

In non-rectangular coordinate systems, the unit vectors at the position of the point particle might not necessarily remain constant with time
Consider a moving object tracing out the curve in `(Figure 1.14-1)` in a plane polar coordinate system
As the object moves along the curve $s(t)$ from point $P^{(1)}$ to point $P^{(2)}$ in time interval $t_2 - t_1 = dt$, the object experiences a change in $\boldsymbol{e}_r$ and change in $\boldsymbol{e}_\theta$

$$
\left.\begin{aligned}
\boldsymbol{e}_r^{(1)} - \boldsymbol{e}_r^{(2)} &= d\boldsymbol{e}_r \\
\boldsymbol{e}_\theta^{(1)} - \boldsymbol{e}_\theta^{(2)} &= d\boldsymbol{e}_\theta
\end{aligned}\right\} \tag{88}
$$

where $d\boldsymbol{e}_r$ is a vector normal to $\boldsymbol{e}_r$ (in the direction of $\boldsymbol{e}_\theta$) and $d\boldsymbol{e}_\theta$ is a vector normal to $\boldsymbol{e}_\theta$ (in the direction opposite of $\boldsymbol{e}_r$). Therefore,

$$
\left.\begin{aligned}
d\boldsymbol{e}_r &= d\theta\boldsymbol{e}_\theta & \implies \quad \dot{\boldsymbol{e}}_r &= \dot{\theta}\boldsymbol{e}_\theta \\
d\boldsymbol{e}_\theta &= -d\theta\boldsymbol{e}_r & \implies \quad \dot{\boldsymbol{e}}_\theta &= -\dot{\theta}\boldsymbol{e}_r
\end{aligned}\right\} \tag{89}
$$

With expressions for $\dot{\boldsymbol{e}}_r$ and $\dot{\boldsymbol{e}}_\theta$, we can resolve $\boldsymbol{v}$ as

$$
\begin{aligned}
\boldsymbol{v} = \dot{\boldsymbol{r}} &= \frac{d}{dt}(r\boldsymbol{e}_r) \\
&= \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r \\
&= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta
\end{aligned} \tag{90}
$$

and $\boldsymbol{a}$ as

$$
\begin{aligned}
\boldsymbol{a} = \dot{\boldsymbol{v}} &= \frac{d}{dt}\left(\dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta\right) \\
&= \ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\boldsymbol{e}}_r + \dot{r}\dot{\theta}\boldsymbol{e}_\theta + r\ddot{\theta}\boldsymbol{e}_\theta + r\dot{\theta}\dot{\boldsymbol{e}}_\theta \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\boldsymbol{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\boldsymbol{e}_\theta
\end{aligned} \tag{91}
$$

into distinct *radial* and *angular* (*traverse*) components.

In summary, we have the following expressions for $d\boldsymbol{s}$, ds^2, v^2, and $\boldsymbol{v}$ described in various 3-D coordinate systems:

– Rectangular coordinates $(x,\ y,\ z)$

$$
\left.\begin{aligned}
d\boldsymbol{s} &= dx_1\boldsymbol{e}_1 + dx_2\boldsymbol{e}_2 + dx_3\boldsymbol{e}_3 \\
ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 \\
v^2 &= \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \\
\boldsymbol{v} &= \dot{x}_1\boldsymbol{e}_1 + \dot{x}_2\boldsymbol{e}_2 + \dot{x}_3\boldsymbol{e}_3
\end{aligned}\right\} \tag{92}
$$

– Spherical coordinates $(r,\ \theta,\ \phi)$

$$
\left.\begin{aligned}
d\boldsymbol{s} &= dr\boldsymbol{e}_r + rd\theta\boldsymbol{e}_\theta + r\sin\theta\ d\phi\boldsymbol{e}_\phi \\
ds^2 &= dr^2 + r^2d\theta^2 + r^2\sin^2\theta\ d\phi^2 \\
v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\ \dot{\phi}^2 \\
\boldsymbol{v} &= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r\sin\theta\ \dot{\phi}\boldsymbol{e}_\phi
\end{aligned}\right\} \tag{93}
$$

– Cylinrical coordinates $(r,\ \phi,\ z)$

$$
\left.\begin{aligned}
d\boldsymbol{s} &= dr\boldsymbol{e}_r + rd\phi\boldsymbol{e}_\phi + dz\boldsymbol{e}_z \\
ds^2 &= dr^2 + r^2d\phi^2 + dz^2 \\
v^2 &= \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 \\
\boldsymbol{v} &= \dot{r}\boldsymbol{e}_r + r\dot{\phi}\boldsymbol{e}_\phi + \dot{z}\boldsymbol{e}_z
\end{aligned}\right\} \tag{94}
$$

1.15 Angular Velocity

Consider a particle moving in `(Figure 1.15-1)` in a cylindrical coordinate system with the origin lying at an arbitrary point O on the axis of rotation

The path the particle describes in an infinitesimal time δt may be represented as an infinitesimal arc of a circle with the instantaneous axis of rotation being the line passing through the center of the circle perpendicular to the instantaneous direciton of motion

As the position vector of a point changes from $\boldsymbol{r}$ to $\boldsymbol{r} + \delta\boldsymbol{r}$,

$$\delta \boldsymbol{r} = \delta \boldsymbol{\theta} \times \boldsymbol{r} \tag{95}$$

where the quantity $\delta\boldsymbol{\theta}$ has magnitude equal to the infinitesimal rotation angle and direction along the instantaneous axis of rotation
Only such infinitesimal rotations can be represented by vectors

- Finite rotations – do not commute; gives different results depending on order of rotations
  i.e.

$$\begin{aligned} \boldsymbol{\lambda}_3 = \boldsymbol{\lambda}_1\boldsymbol{\lambda}_2 \quad &\Longrightarrow \quad \boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} \\ \boldsymbol{\lambda}_4 = \boldsymbol{\lambda}_2\boldsymbol{\lambda}_1 \quad &\Longrightarrow \quad \boldsymbol{D} = \boldsymbol{B} + \boldsymbol{A} \end{aligned} \tag{96}$$

  In general $\boldsymbol{\lambda}_3 \not\equiv \boldsymbol{\lambda}_4$ even though $\boldsymbol{C} \equiv \boldsymbol{D}$
  $\hookrightarrow$ successive applications of finite rotations do not commute
- Infinitesimal rotations – do commute
  i.e.
  Suppose a point moves as in rotations $\delta\boldsymbol{\theta}_1$ and $\delta\boldsymbol{\theta}_2$ `(Figure 1.15-2)`
  As the rotation $\delta\boldsymbol{\theta}_1$ takes $\boldsymbol{r}$ into $\boldsymbol{r} + \delta\boldsymbol{r}_1$,

$$\delta\boldsymbol{r}_1 = \delta\boldsymbol{\theta}_1 \times \boldsymbol{r} \tag{97}$$

  a successive rotaton $\delta\boldsymbol{\theta}_2$ from $\boldsymbol{r}+\delta\boldsymbol{r}_1$ will result in

$$\delta\boldsymbol{r}_2 = \delta\boldsymbol{\theta}_2 \times (\boldsymbol{r} + \delta\boldsymbol{r}_1) \tag{98}$$

  which gives

$$\begin{aligned} \boldsymbol{r} + \delta\boldsymbol{r}_{12} &= \boldsymbol{r} + [\delta\boldsymbol{\theta}_1 \times \boldsymbol{r} + \delta\boldsymbol{\theta}_2 \times (\boldsymbol{r} + \delta\boldsymbol{r}_1)] \\ \delta\boldsymbol{r}_{12} &\approx \delta\boldsymbol{\theta}_1 \times \boldsymbol{r} + \delta\boldsymbol{\theta}_2 \times \boldsymbol{r} \end{aligned} \tag{99}$$

  Or instead if the rotation $\delta\boldsymbol{\theta}_2$ suceeds $\delta\boldsymbol{\theta}_1$,

$$\begin{aligned} \delta\boldsymbol{r}_2 &= \delta\boldsymbol{\theta}_2 \times \boldsymbol{r} \\ \delta\boldsymbol{r}_1 &= \delta\boldsymbol{\theta}_1 \times (\boldsymbol{r} + \delta\boldsymbol{r}_2) \end{aligned} \tag{100}$$

  which gives

$$\begin{aligned} \boldsymbol{r} + \delta\boldsymbol{r}_{21} &= \boldsymbol{r} + [\delta\boldsymbol{\theta}_2 \times \boldsymbol{r} + \delta\boldsymbol{\theta}_1 \times (\boldsymbol{r} + \delta\boldsymbol{r}_2)] \\ \delta\boldsymbol{r}_{21} &\approx \delta\boldsymbol{\theta}_2 \times \boldsymbol{r} + \delta\boldsymbol{\theta}_1 \times \boldsymbol{r} \end{aligned} \tag{101}$$

  Comparing both sets of successive rotations, $\delta\boldsymbol{r}_{12} \equiv \delta\boldsymbol{r}_{21}$
  $\hookrightarrow$ successive applications of infinitesimal rotations do commute

The time rate of change of the position vector is the linear velocity vector of the particle
i.e.

$$\dot{\boldsymbol{r}} = \boldsymbol{v} \tag{102}$$

- magnitude

$$\begin{aligned} v = R\frac{d\theta}{dt} &= R\omega, \quad \text{where } R = r\sin\alpha \\ &= r\omega\sin\alpha \end{aligned} \tag{103}$$

- direction perpendicular to $\boldsymbol{r}$ on the plane of the circle

Angular velocity vector $\boldsymbol{\omega}$ – rate of change of the angular position vector

- defined along the normal of the plane with the positive direction corresponding to the direction of advance of a right-hand screw when turned in the same sense as the rotation of the particle
- described as the ratio of an infinitesimal rotation angle to an infinitesimal time
  i.e.

$$\boldsymbol{\omega} = \frac{\delta\boldsymbol{\theta}}{\delta t} \tag{104}$$

From equation (95),

$$\begin{aligned} &\frac{\delta\boldsymbol{r}}{\delta t} = \frac{\delta\boldsymbol{\theta}}{\delta t} \times \boldsymbol{r} \\ \Longrightarrow\ &\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}, \quad \text{as } \delta t \to 0 \end{aligned} \tag{105}$$

1.16 Gradient Operator

Looking at coordinate transformations on a 3-D coordinate system
Under a coordinate transformation that transforms x_i into x'_i, a scalar ϕ remains invariant

i.e.

$$\phi(x_1,\ x_2,\ x_3) = \phi'(x'_1,\ x'_2,\ x'_3) \tag{106}$$

with partial derivatives ϕ' expanded using chain rule as

$$\frac{\partial \phi'}{\partial x'_1} = \sum_j \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial x'_1}, \quad \frac{\partial \phi'}{\partial x'_2} = \sum_j \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial x'_2} \quad \& \quad \frac{\partial \phi'}{\partial x'_3} = \sum_j \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial x'_3} \tag{107}$$

$$\implies \frac{\partial \phi'}{\partial x'_i} = \sum_j \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial x'_i} \tag{108}$$

and to transform x' to x,

$$x_j = \sum_k \lambda_{kj} x'_k \tag{109}$$

with partial derivatives

$$\begin{aligned}\frac{\partial x_j}{\partial x'_i} = \frac{\partial}{\partial x'_i}\left(\sum_k \lambda_{kj} x'_k\right) = \sum_k \lambda_{kj}\left(\frac{\partial x'_k}{\partial x'_i}\right) &= \begin{cases} 0, & i \neq k \\ \sum_k \lambda_{kj}, & i = k \end{cases}, \quad \text{where } \frac{\partial x'_k}{\partial x'_i} \equiv \delta_{ik} \\ &= \sum_k \lambda_{kj}\delta_{ik} \\ &= \lambda_{ij}\end{aligned} \tag{110}$$

which gives the partial derivatives of ϕ' as

$$\frac{\partial \phi'}{\partial x'_i} = \sum_j \lambda_{ij}\frac{\partial \phi}{\partial x_j} \tag{111}$$

$\hookrightarrow$ $\frac{\partial \phi}{\partial x_j}$ transforms as components of a vector

Vector gradient operator ($\boldsymbol{\nabla}$)

$$\mathbf{grad} = \boldsymbol{\nabla} = \sum_i \boldsymbol{e}_i \frac{\partial}{\partial x_i} \tag{112}$$

which describes the differential operator in equation (111)
– can operate directly on a scalar function

$$\mathbf{grad}\ \phi = \boldsymbol{\nabla}\phi = \sum_i \boldsymbol{e}_i \frac{\partial \phi}{\partial x_i} \tag{113}$$

Consider 3-D topographical maps as in `(Figure 1.16-1)`
Let the scalar ϕ denote the height at any point $\phi = \phi(x_1,\ x_2,\ x_3)$.
An infinitesimal change in ϕ ($d\phi$) is

$$\begin{aligned} d\phi = \sum_i \frac{\partial \phi}{\partial x_i} dx_i &= \sum_i (\boldsymbol{\nabla}\phi)_i dx_i \\ &= (\boldsymbol{\nabla}\phi)\cdot d\boldsymbol{s}, \quad \text{where } d\boldsymbol{s} = (dx_1,\ dx_2,\ dx_3)\end{aligned} \tag{114}$$

1. $\boldsymbol{\nabla}\phi$ is normal to lines or surfaces for which ϕ is constant
2. $\boldsymbol{\nabla}\phi$ is in the direction of the maximum change in ϕ
3. the rate of change of ϕ in the direction of a unit vector $\boldsymbol{n}$ (directional derivative of ϕ) is

$$\boldsymbol{n}\cdot\boldsymbol{\nabla}\phi \equiv \frac{\partial \phi}{\partial n} \tag{115}$$

divergence (div) of $\boldsymbol{A}$ – scalar product with a vector function

$$\mathbf{div}\ \boldsymbol{A} = \boldsymbol{\nabla}\cdot\boldsymbol{A} = \sum_i \frac{\partial A_i}{\partial x_i} \tag{116}$$

curl of $\boldsymbol{A}$ – vector product with a vector function

$$\textbf{curl } \boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{A} = \sum_{i,j,k} \boldsymbol{e}_{ijk} \frac{\partial A_k}{\partial x_j} \boldsymbol{e}_i \tag{117}$$

Laplacian – successive operation of the gradient operator

$$\boldsymbol{\nabla}^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \sum_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} = \sum_i \frac{\partial^2}{\partial x_i^2} \tag{118}$$

e.g.

$$\boldsymbol{\nabla}^2 \psi = \sum_i \frac{\partial^2 \phi}{\partial x_i^2} \tag{119}$$

1.17 Integration of Vectors

Volume integration of a vector function $\boldsymbol{A} = \boldsymbol{A}(x_i)$ through a volume V

$$\int_V \boldsymbol{A} dv = \left(\int_V A_1 dv \int_V A_2 dv \int_V A_3 dv \right) \tag{120}$$

Consider a surface S in 3-D coordinate system as in `(Figure 1.17-1)`
An integral over S of the projection of a vector function $\boldsymbol{A} = \boldsymbol{A}(x_i)$ onto the normal of the surface

$$\int_S \boldsymbol{A} \cdot d\boldsymbol{a} \tag{121}$$

where $d\boldsymbol{a}$ is an element of area of S
- with magnitude da
- with direction normal to the surface
  e.g. for the unit normal vector $\boldsymbol{n}$

$$d\boldsymbol{a} = \boldsymbol{n} da \tag{122}$$

for a closed surface, the *outward* normal is taken as the positive direction following conventions
Similarly, the components of $d\boldsymbol{a}$ are projections of the element of area on the three mutually perpentually planes defined by the rectangular axes
e.g. $da_1 = dx_2 dx_3, \quad da_2 = dx_1 dx_3 \quad \& \quad da_3 = dx_1 dx_2$
Therefore,

$$\int_S \boldsymbol{A} \cdot d\boldsymbol{a} = \int_S \boldsymbol{A} \cdot \boldsymbol{n} da \tag{123}$$

which shows that the integral of $\boldsymbol{A}$ over S is the integral of the normal component of $\boldsymbol{A}$ over S or

$$\int_S \boldsymbol{A} \cdot d\boldsymbol{a} = \int_S \sum_i A_i da_i \tag{124}$$

Consider a path extending from point B to C `(Figure 1.17-2)`
The line integral along the path from point B to C of a vector function $\boldsymbol{A} = \boldsymbol{A}(x_i)$ is the integral of the component of $\boldsymbol{A}$ along the path

$$\int_{BC} \boldsymbol{A} \cdot d\boldsymbol{s} = \int_{BC} \sum_i A_i dx_i \tag{125}$$

where $d\boldsymbol{s}$ is an element of the length with magnitude ds and direction along the direction of the path traversed.
Consider a closed volume V enclosed by the surface S `(Figure 1.17-3)`
Let the vector $\boldsymbol{A}$ and its first derivatives be continuous throughout the volume.
Gauss's theorem (divergence theorem) – the surface integral of $\boldsymbol{A}$ over the closed surface S is equal to the volume integral of the divergence of $\boldsymbol{A}$ ($\boldsymbol{\nabla} \cdot \boldsymbol{A}$) throughout the volume V enclosed by the surface S

$$\int_S \boldsymbol{A} \cdot d\boldsymbol{a} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{A} dv \tag{126}$$

Consider an open surface S and the contour path C that defines the surface `(Figure 1.17-4)`
Stoke's theorem – the line integral of the vector $\boldsymbol{A}$ around the contour path C is equal to the surface integral of the curl of $\boldsymbol{A}$ over the surface defined by C

$$\int_C \boldsymbol{A} \cdot d\boldsymbol{s} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot d\boldsymbol{a} \tag{127}$$

where the line integral is around the closed contour path C and $\boldsymbol{\nabla} \times \boldsymbol{A}$ must exist and be integrable over the entire surface S.

2 Newtonian Mechanics- Single Particle

2.1 Introduction

2.2 Newton's Laws

2.3 Frames of Reference