1 Vector Analysis

1.1 Vector Algebra

1.1.1 Vector Operations

Vectors - have direction and magnitude
i.e. displacment, velocity, acceleration, force & momentum

- denoted as $\boldsymbol{A}$
- represented by arrows with length proportional to the magnitude of the vector and arrowhead indicating direction

  e.g. Minus $\boldsymbol{A}$ ($-\boldsymbol{A}$) is a vector with the same magnitude as $\boldsymbol{A}$ but of opposite direction `(Figure 1.1-1)`

Scalars - have direction but no magnitude
e.g. mass, charge, density & temperature

- denoted as $|\boldsymbol{A}|$ or A

Vector operations:

1. Addition of two vectors
   Placing the tail of $\boldsymbol{B}$ at the head of $\boldsymbol{A}$ `(Figure 1.1-2)`, the resultant vector from the tail of $\boldsymbol{A}$ to the head of $\boldsymbol{B}$ is the sum -

   $$\boldsymbol{A}+\boldsymbol{B} \tag{1}$$

   For the subtraction of two vectors, add its opposite `(Figure 1.1-3)`

   $$\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-\boldsymbol{B}) \tag{2}$$

   - Commutative

     $$\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A} \tag{3}$$

   - Associative

     $$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C}) \tag{4}$$

2. Multiplication by a scalar
   Multiplies the magnitude but leaves the direction unchanged `(Figure 1.1-4)`

   - if a is -ve, the direction is reversed
   - Distributive

     $$a(\boldsymbol{A}+\boldsymbol{B})=a\boldsymbol{A}+a\boldsymbol{B} \tag{5}$$

3. Dot product of two vectors

   $$\boldsymbol{A}\cdot\boldsymbol{B}\equiv AB\cos\theta \tag{6}$$

   where θ is the angle $\boldsymbol{A}$ & $\boldsymbol{B}$ form when placed tail-to-tail `(Figure 1.1-5)` and it results in a scalar.

   - Commutative

     $$\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{A} \tag{7}$$

- Distributive

$$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C} \tag{8}$$

- Geometrically, $\boldsymbol{A} \cdot \boldsymbol{B}$ is the product of A times the projection of $\boldsymbol{B}$ along $\boldsymbol{A}$

$$\left.\begin{array}{ll} \Rightarrow & \boldsymbol{A} \cdot \boldsymbol{B} = AB, \quad \text{if } \boldsymbol{A} \text{ \& } \boldsymbol{B} \text{ are parallel} \\ & \boldsymbol{A} \cdot \boldsymbol{B} = 0, \quad \text{if } \boldsymbol{A} \text{ \& } \boldsymbol{B} \text{ are perpendicular} \end{array}\right\} \tag{9}$$

4. Cross product of two vectors

$$\boldsymbol{A} \times \boldsymbol{B} \equiv AB \sin\theta \; \hat{\boldsymbol{n}} \tag{10}$$

where $\hat{\boldsymbol{n}}$ is a unit vector pointing perpendicular to the plane of $\boldsymbol{A}$ & $\boldsymbol{B}$.

- with direction `(Figure 1.1-6)` resolved by the right-hand rule
  i.e. the direction that of the thumb at when the fingers point towards the direction of the first vector while curling towards the second
- Geometrically, $|\boldsymbol{A} \times \boldsymbol{B}|$ is the area of the parallelogram generated by $\boldsymbol{A}$ & $\boldsymbol{B}$ `(Figure 1.1-6)`
- Distributive

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) + (\boldsymbol{A} \times \boldsymbol{C}) \tag{11}$$

- Not Commutative

$$(\boldsymbol{B} \times \boldsymbol{A}) = -(\boldsymbol{A} \times \boldsymbol{B}) \tag{12}$$

1.1.2 Vector Algebra: Component Form

Consider a 3-D coordinate system in Cartesian coordinates.
Let $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ & $\hat{\boldsymbol{z}}$ be unit vectors parallel to the x, y and z axes respectively `(Figure 1.2-1)`.
A vector $\boldsymbol{A}$ `(Figure 1.2-2)` can be expanded in terms of the aforementioned basis vectors as -

$$\boldsymbol{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}} \tag{13}$$

where A_x, A_y & A_z are the components of $\boldsymbol{A}$ or geometrically as projections of $\boldsymbol{A}$ along the three coordinate axes -

- $A_x = \boldsymbol{A} \cdot \hat{\boldsymbol{x}}$
- $A_y = \boldsymbol{A} \cdot \hat{\boldsymbol{y}}$
- $A_z = \boldsymbol{A} \cdot \hat{\boldsymbol{z}}$

This allows for the 4 vector operations to be expressed in terms of its components -

1. Vector addition - add like components

$$\begin{aligned} \boldsymbol{A} + \boldsymbol{B} &= (A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}) + (B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}) \\ &= (A_x + B_x)\hat{\boldsymbol{x}} + (A_y + B_y)\hat{\boldsymbol{y}} + (A_z + B_z)\hat{\boldsymbol{z}} \end{aligned} \tag{14}$$

2. Multiplication by a scalar - multiply each component

$$a\boldsymbol{A} = (aA_x)\hat{\boldsymbol{x}} + (aA_y)\hat{\boldsymbol{y}} + (aA_z)\hat{\boldsymbol{z}} \tag{15}$$

3. Dot product of two vectors - multiply like components, then add
   Since $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are mutually perpendicular -

$$\left.\begin{aligned} \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{z}} &= 1 \\ \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}} = \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{z}} = \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{z}} &= 0 \end{aligned}\right\} \tag{16}$$

$$\begin{aligned} \Rightarrow \boldsymbol{A}\cdot\boldsymbol{B} &= (A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}})\cdot(B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}) \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned} \tag{17}$$

i.e. For the case of $\boldsymbol{A}\cdot\boldsymbol{A} = A_x^2 + A_y^2 + A_z^2$

$$|\boldsymbol{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{18}$$

4. Cross product of two vectors - form the determinant whose first row are the unit vectors parallel to the coordinate axes, whose second row are the components of $\boldsymbol{A}$, and whose third row are the components of $\boldsymbol{B}$
   Since $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are mutually perpendicular -

$$\left.\begin{aligned} \hat{\boldsymbol{x}}\times\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}\times\hat{\boldsymbol{y}} &= \hat{\boldsymbol{z}}\times\hat{\boldsymbol{z}} = 0 \\ \hat{\boldsymbol{x}}\times\hat{\boldsymbol{y}} = -\hat{\boldsymbol{y}}\times\hat{\boldsymbol{x}} &= \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{y}}\times\hat{\boldsymbol{z}} = -\hat{\boldsymbol{z}}\times\hat{\boldsymbol{y}} &= \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}} = -\hat{\boldsymbol{x}}\times\hat{\boldsymbol{z}} &= \hat{\boldsymbol{y}} \end{aligned}\right\} \tag{19}$$

$$\begin{aligned} \Rightarrow \boldsymbol{A}\times\boldsymbol{B} &= (A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}})\times(B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}) \\ &= (A_yB_z - A_zB_y)\hat{\boldsymbol{x}} + (A_zB_x - A_xB_z)\hat{\boldsymbol{y}} + (A_xB_y - A_yB_x)\hat{\boldsymbol{z}} \\ &\equiv \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \end{aligned} \tag{20}$$

1.1.3 Triple Products