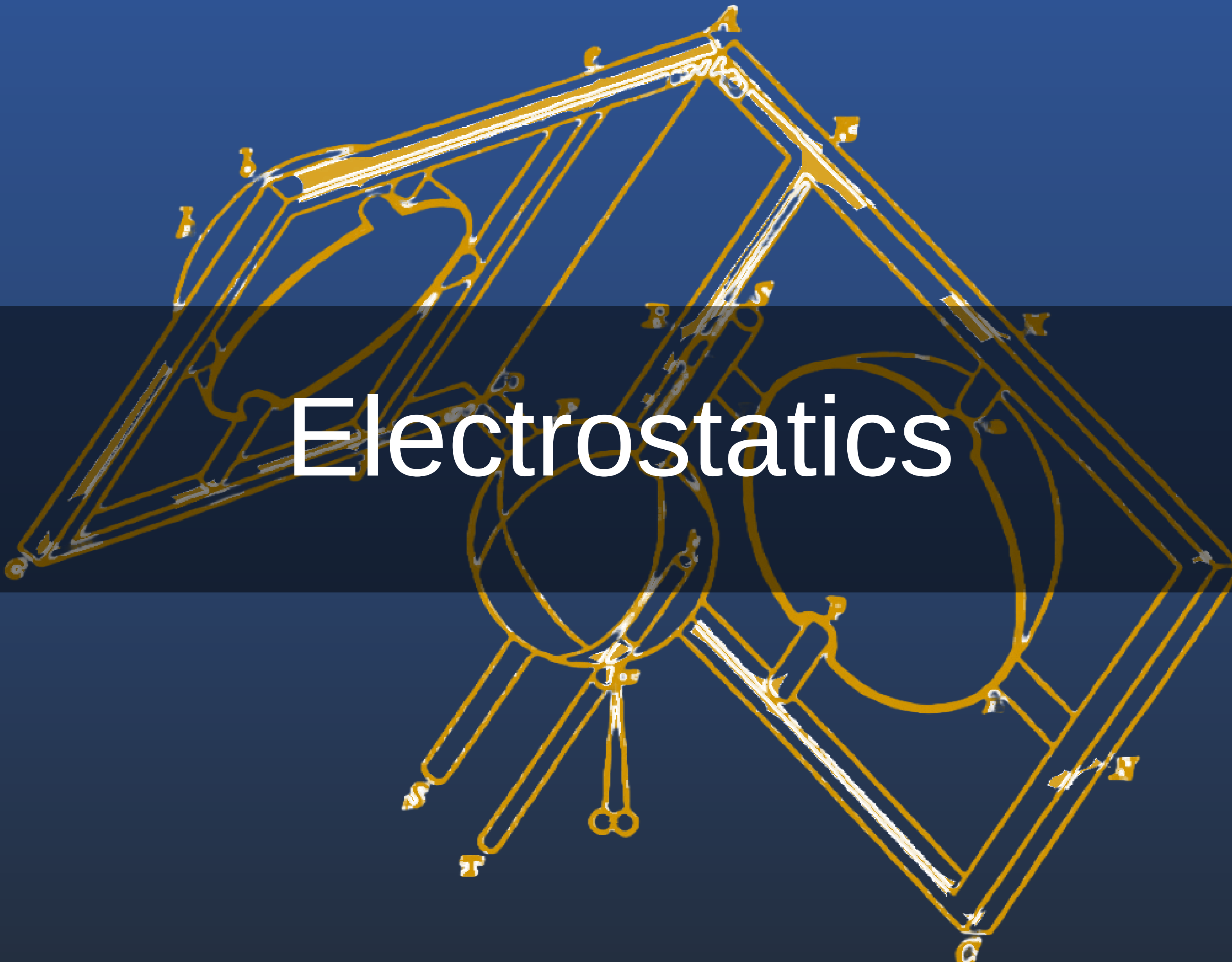
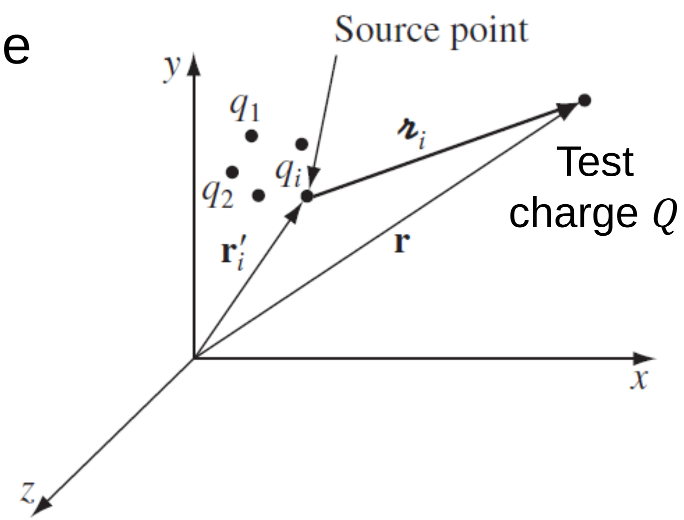
Electrostatics



Charge, electric field, and potential

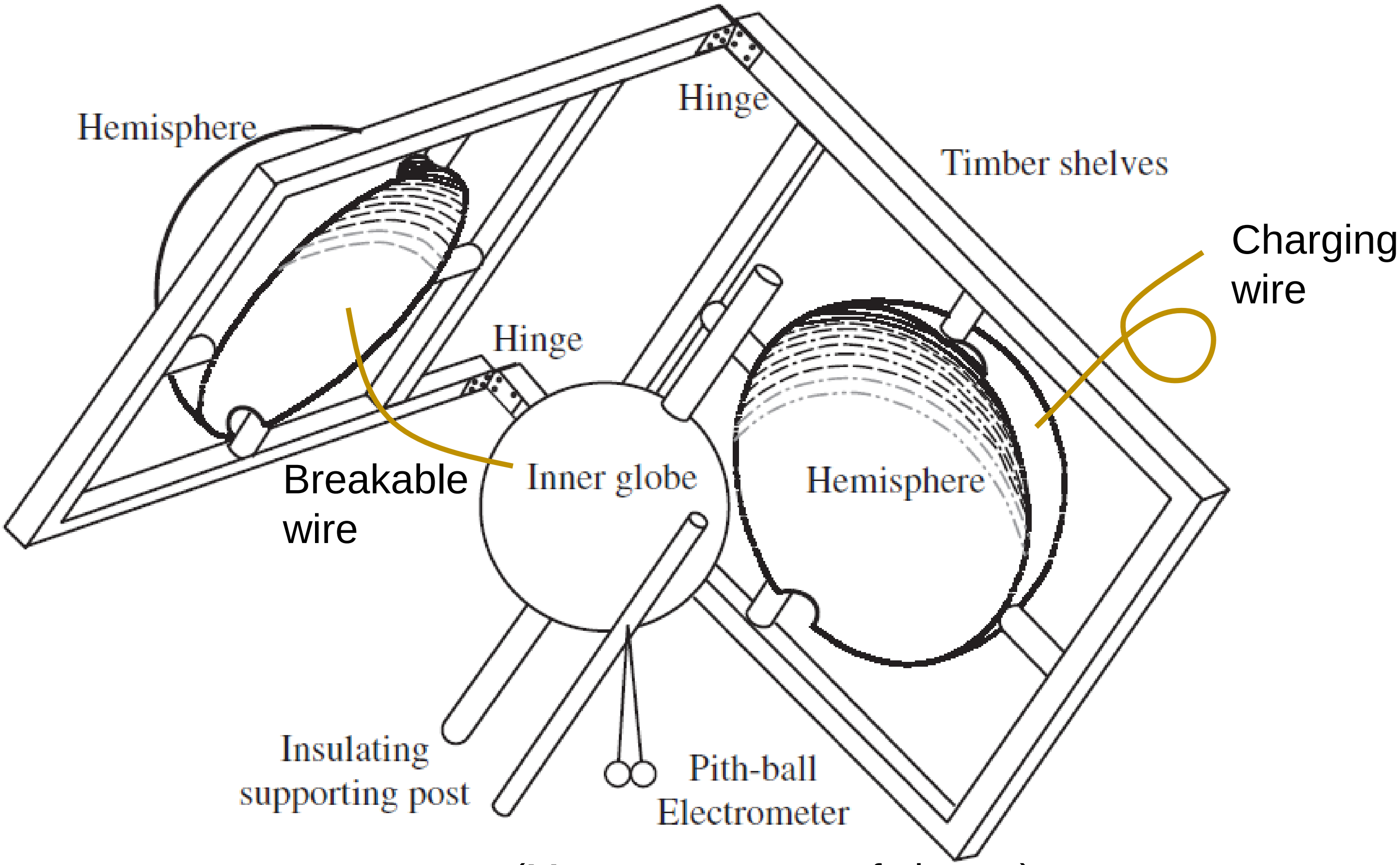
Coulomb's law

- Force of n source charges on a test charge
  - Force from source charge q_i acting on test charge Q
    - Coulomb's law $\boxed{F_i = \frac{1}{4\pi\varepsilon_0}\frac{q_i Q}{\boldsymbol{r}_i^2}\widehat{\boldsymbol{r}_i}}$
    - Permittivity $\varepsilon_0 = 8.85 \times 10^{12}$ C^2/(N m^2)
    - Separation vector $\boldsymbol{r}_i = \boldsymbol{r} - \boldsymbol{r}'_i$
    - Location of Q: $\boldsymbol{r}$, location of q_i: $\boldsymbol{r}'_i$
  - Principle of superposition
    - Total force acting on test charge $F = \sum_{i=1}^{n} F_i$
    - Not a necessity, but an experimental fact



* $\boldsymbol{r}$ in textbook is typed as $\boldsymbol{r}$ in our slides (Cursive "r")

Coulomb's law



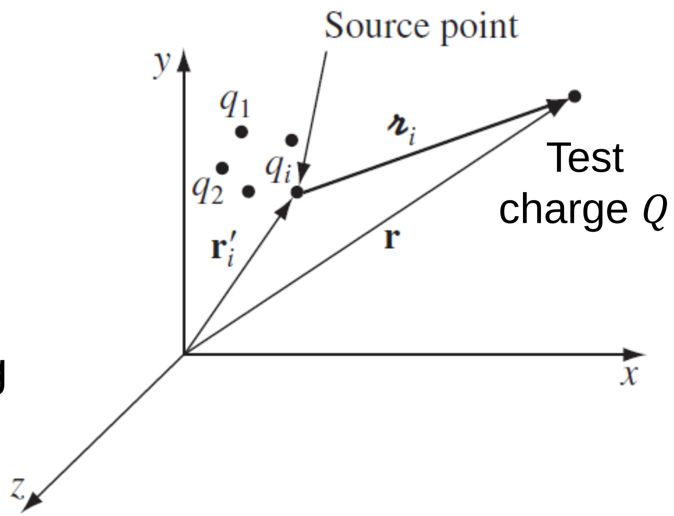
Cavendish's apparatus for determining $F \propto r^{-2}$ in Coulomb's law

Electric field induced by charge

- Relation of force and electric field

$$\boxed{\mathbf{F} = Q\mathbf{E}}$$

  - Electric field: force per unit charge
  - Real physical entity, as a vector field filling the space around charges
  - Negated theory of "ether"



- Electric field induced by discrete charges

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i$$

  - Separation vector $\boldsymbol{\mathscr{r}}_i = \boldsymbol{r} - \boldsymbol{r}'_i$, contains $\boldsymbol{r}$
  - Principle of superposition also holds

Electric field induced by charge

- Electric field induced by continuous charge distribution

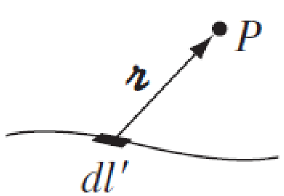
$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, dq}$$

  - Add up contributions from infinitesimal charge elements dq
  - Three ways dq can be distributed

Line charge

$dq \to \lambda\, dl'$

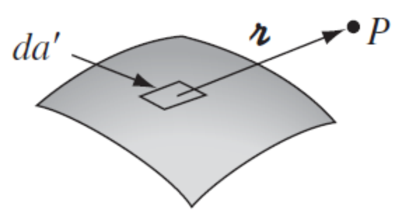
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, dl'$$



Surface charge

$dq \to \sigma\, da'$

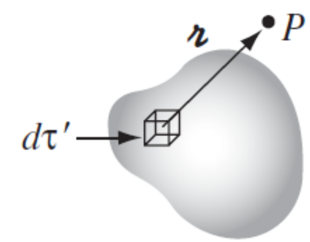
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, da'$$



Volume charge

$dq \to \rho\, d\tau'$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, d\tau'$$

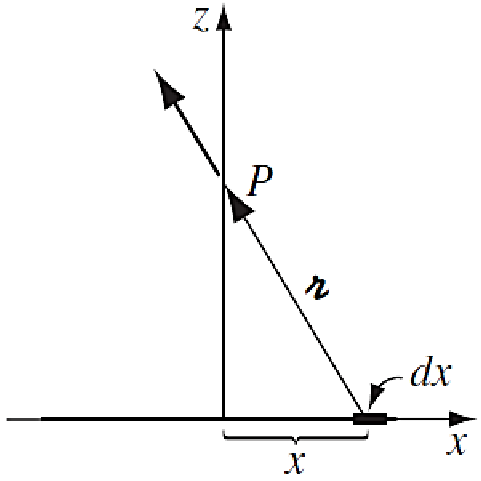


*λ, σ, ρ: charge per unit length, area, volume

Electric field induced by charge

- Electric field induced by continuous charge distribution

Example 2.2. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ (Fig. 2.6).

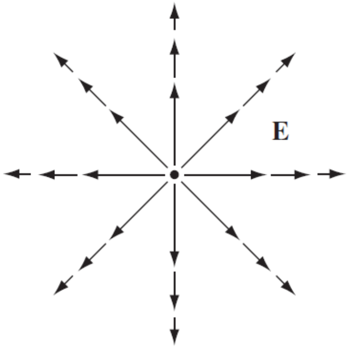


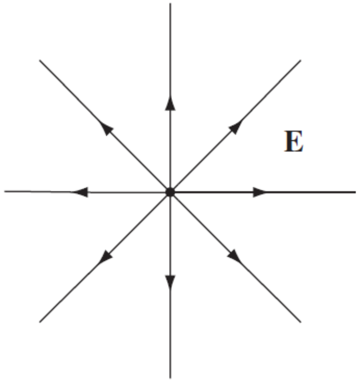
C2.exmp2.2

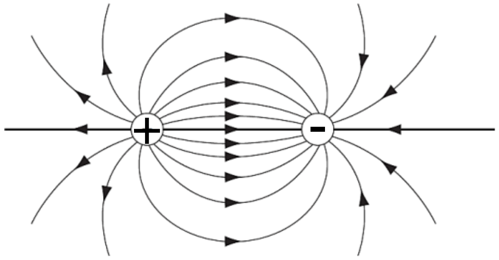
  - Integration sometimes can get formidable, need to device new tools to simplify problems.

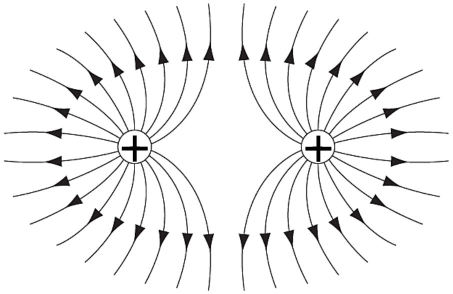
Gauss's law

- Electric field lines
  - Source charge q at the origin $$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}}$$
  - Draw vector field – field falls off like $1/r^2$
  - Connect up the arrows – electric field lines
    - Direction of line indicates field direction
    - Density of line indicates field magnitude
  - Field lines begin from positive charges and end on negative ones



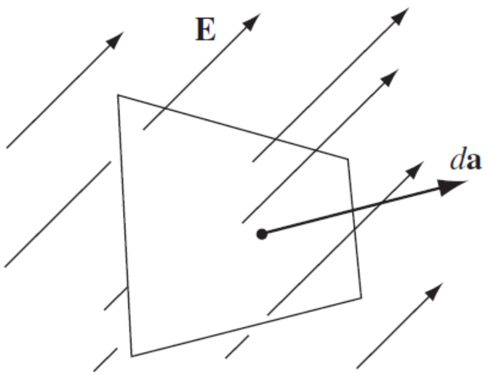






Gauss's law

- Electric field flux $\Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$



  - A measure of the number of field lines passing through an area

- Gauss's law

  - The flux through any closed surface is a measure of the total charge inside

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot \underbrace{(r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}})}_{d\boldsymbol{a}} = \frac{1}{\epsilon_0} q$$

Spherical surface of radius r

  - The surface integral can be any shape, not necessarily spherical

  - Multiple charges $\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n}\left(\oint \mathbf{E}_i \cdot d\mathbf{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\epsilon_0} q_i\right)$

$$\Rightarrow \boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\mathrm{enc}}}$$

(Q_{enc}: total charge enclosed in the integrated surface)

Gauss's law

- Gauss's law
  - Gauss's law in the differential form

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$ (integral form)

Divergence theorem

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau$$

Consider volume distribution

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho \, d\tau$$

$$\Rightarrow \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau$$

$$\Rightarrow \boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho}$$ (differential form)

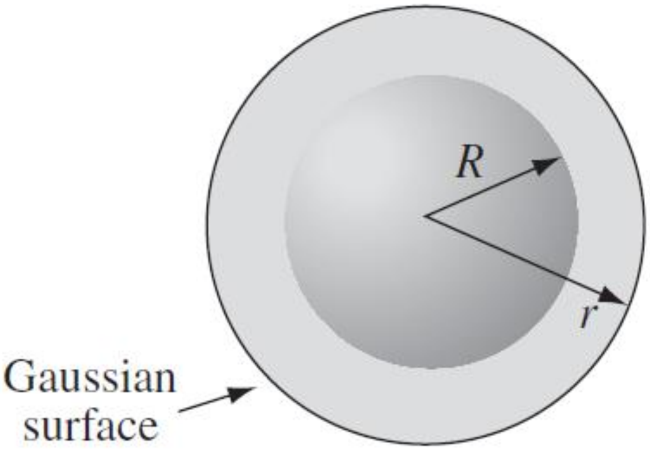
    - Differential form more compact, but integral form easier to use

  - Use of Gauss's law to calculate electric field
    - Need (1) Gauss's law in integral form and (2) symmetry arguments

Gauss's law

- Application of Gauss's law

Example 2.3. Find the field outside a uniformly charged solid sphere of radius R and total charge q.

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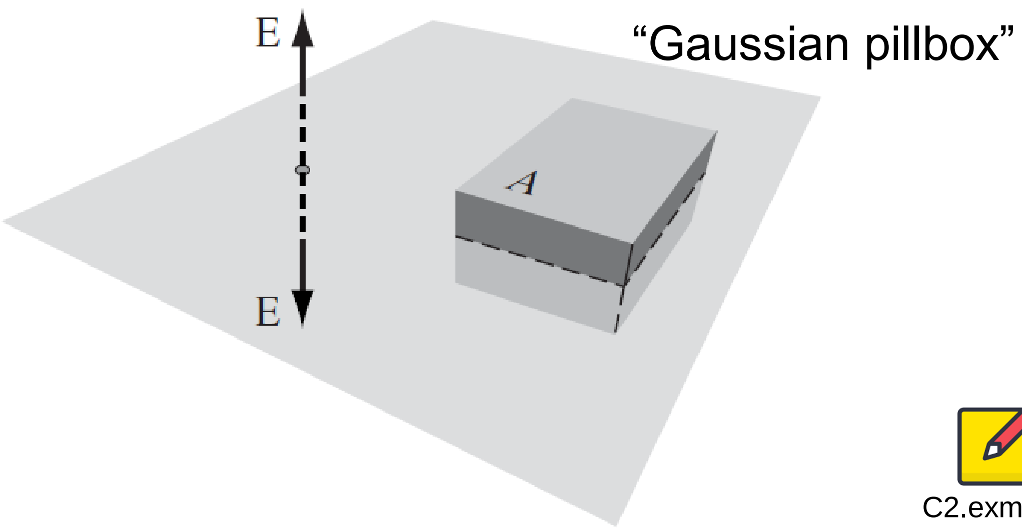
- ○ The field outside the sphere is exactly the same as it would have been if all the charge had been concentrated at the center

Gauss's law

- Application of Gauss's law

Example 2.5. An infinite plane carries a uniform surface charge σ. Find its electric field.



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Divergence of electric field

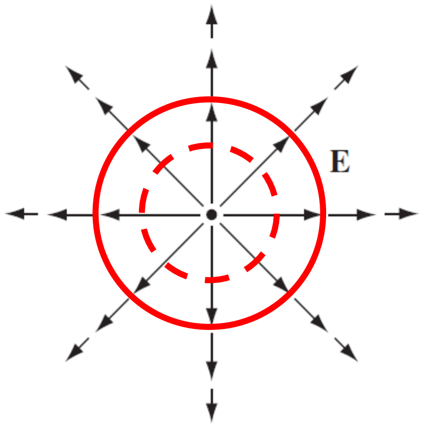
- Directly calculate divergence
  - According to Coulomb's law

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \rho(\mathbf{r}')\, d\tau'$$

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \boxed{\nabla \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right)} \rho(\mathbf{r}')\, d\tau'$$

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) = \nabla \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$$

This seems to contradict the Gauss's law, what went wrong



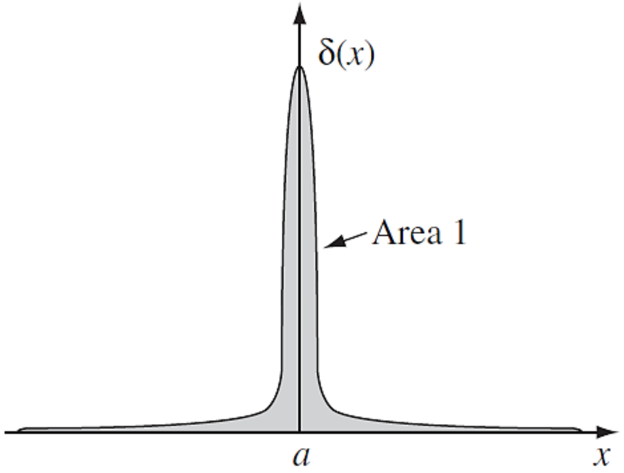
  - The derivation above is correct anywhere but the origin ($r = 0$), where the divergence should go to infinity
    - Consider special case of point charge and Gauss's law with varying volume to integrate

Divergence of electric field

- Delta function
  - Infinitely high, infinitesimally narrow
  - 1D Delta function

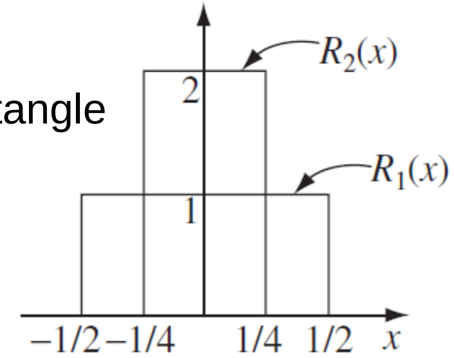
$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

with $$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

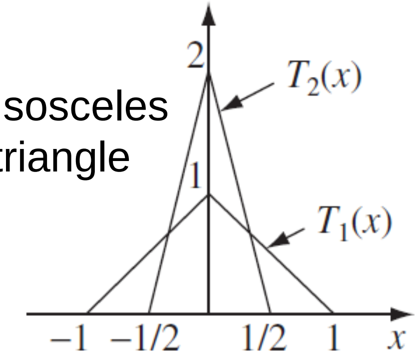


- Can be understood as the limit of a sequence of functions

Rectangle



Isosceles triangle



Divergence of electric field

- Delta function
  - 1D Delta function
    - When in an integral, "picks out" the value of a function

Since $\delta(x)$ anywhere 0 but at $x = 0$

[$f(x)$ being an ordinary function not going to infinity]

$$f(x)\delta(x) = f(0)\delta(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0)$$

And, one can shift $\delta(x)$ to $\delta(x-a)$ to pick out another one

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)}$$

- A frequently used expression $\delta(kx) = \frac{1}{|k|}\delta(x)$

Divergence of electric field

- Delta function
  - 3D Delta function $\boxed{\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z)}$

    with $$\int_{\text{all space}} \delta^3(\mathbf{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$$

    - Picks out a function value $\displaystyle\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\,d\tau = f(\mathbf{a})$

- Back to calculating divergence of electric field

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\rho(\mathbf{r}')\,d\tau'$$

$$\Big\downarrow\ \nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

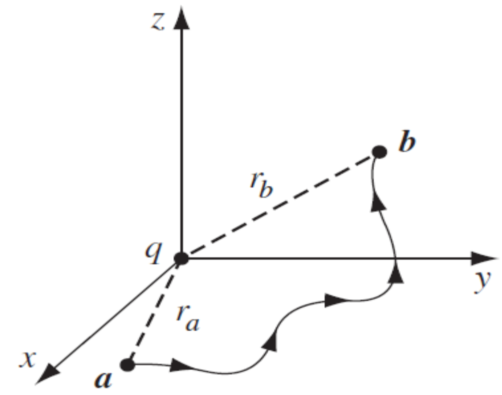
C2.delta

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int 4\pi\delta^3(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\,d\tau' = \frac{1}{\epsilon_0}\rho(\mathbf{r})$$

Gauss's law recovered

Curl of electric field

- Calculate curl for point charge at origin



$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

- For any closed loop ($r_a = r_b$) $\oint \mathbf{E}\cdot d\mathbf{l} = 0$

$\Rightarrow$ $\boxed{\nabla\times\mathbf{E} = \mathbf{0}}$ due to Stoke's theorem

Stoke's theorem

$$\int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

- Any static charge distribution

$$\nabla\times\mathbf{E} = \nabla\times(\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla\times\mathbf{E}_1) + (\nabla\times\mathbf{E}_2) + \ldots = \mathbf{0}$$

Electric potential

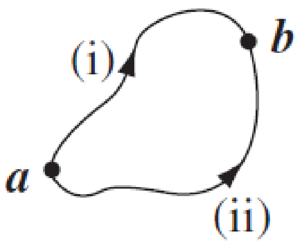
- Vector field **E** cannot take arbitrary form
  - Crucial constraint: $\nabla \times \mathbf{E} = \mathbf{0}$

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \qquad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} \qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

? Any chance the vector field can be described more easily?

- Electric potential: $\boxed{V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}}$
  - Unit: joules per coulomb
  - $\mathcal{O}$: a reference point (usually taken as infinity)
  - Integral does not depend on path
    - $\nabla \times \mathbf{E} = \mathbf{0}$
    - $\oint \mathbf{E} \cdot d\mathbf{l} = 0$
    - $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$ is path independent

Equivalent statements



Electric potential

- Electric potential: $V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$
  - Potential difference between two points is more meaningful

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

On the other hand, the theorem for gradient gives

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} \quad \Rightarrow \quad \boxed{\mathbf{E} = -\nabla V}$$

  - Scalar field V gives full information of vector field **E**
  - Can be off by a constant if choosing a different reference point

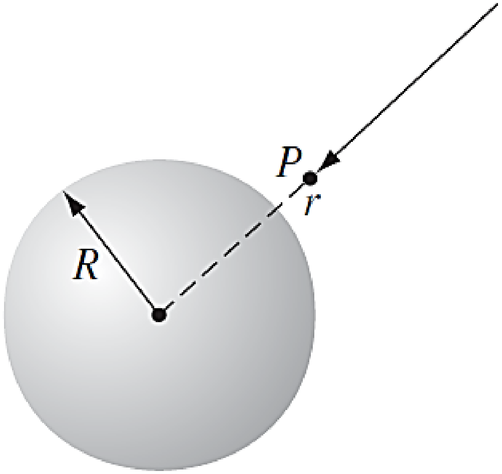
$$V'(\mathbf{r}) = -\int_{\underbrace{\mathcal{O}'}_{\text{new}}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{O}'}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = K + V(\mathbf{r})$$

Electric potential

- Application of electric potential

Example. Find the potential of a point charge q at origin

Example 2.7. Find the potential inside and outside a spherical shell of radius R (Fig. 2.31) that carries a uniform surface charge. Set the reference point at infinity.

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Electric potential

- Poisson's equation of potential
  - Poisson's equation $\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

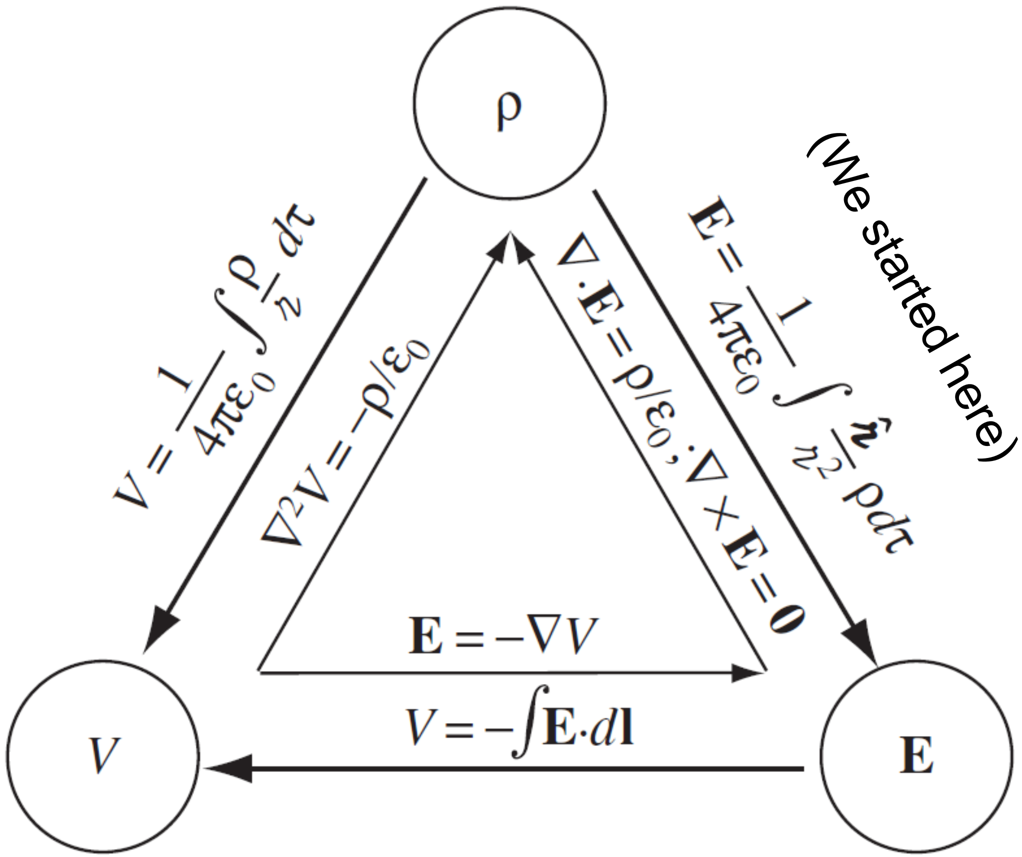
  - In regions with no charge, Laplace's equation $\boxed{\nabla^2 V = 0}$
  - Curl of a gradient always zero $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = \mathbf{0}$

- Potential of a localized charge distribution
  - Pick infinity as the reference point $\mathcal{O} = \infty$
  - Principle of superposition holds $V = V_1 + V_2 + \ldots$
  - Discrete charges $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i}$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2} \hat{\mathscr{r}} \, d\tau'$$

  - Continuous charge $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau'$ Can check

Charge, electric field, and potential



Differential equations need boundary conditions to solve

Boundary conditions

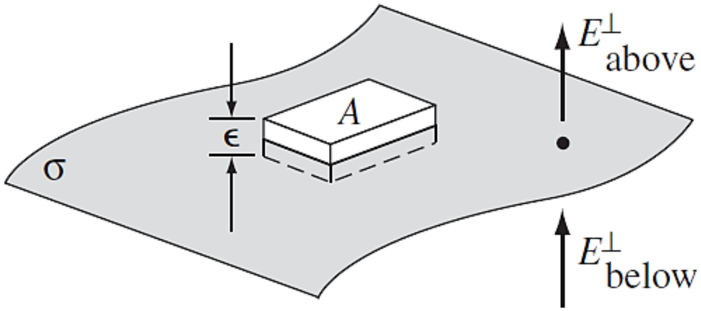
- Boundary conditions of **E** across a 2D charged surface

  - Normal component of **E**

    "Gaussian pillbox" with $\varepsilon \to 0$

    $$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \sigma A$$

    $$\Rightarrow \boxed{E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma}$$

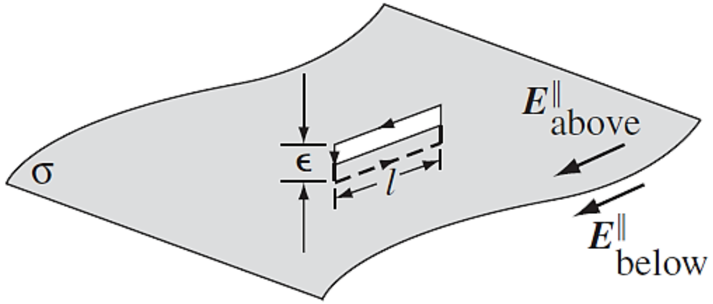


  - Tangential component of **E**

    Thin loop with $\varepsilon \to 0$

    $$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

    $$\Rightarrow \boxed{\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}}}$$



  - Summarizing above $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$

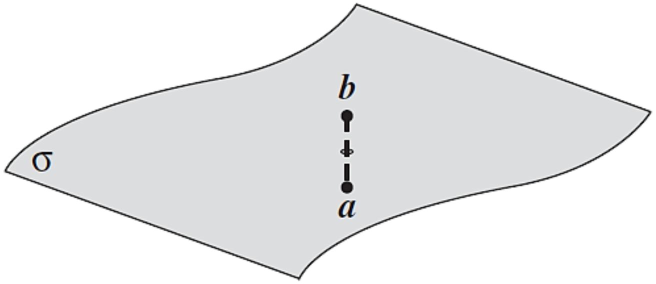
Boundary conditions

- Boundary conditions of V across a 2D charged surface
  - Potential is continuous (across any boundary)

$$V_{\text{above}} - V_{\text{below}} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

Path length → 0

$$V_{\text{above}} = V_{\text{below}}$$



  - Gradient of potential is discontinuous

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

$$\mathbf{E} = -\nabla V$$

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0}\sigma\hat{\mathbf{n}}$$

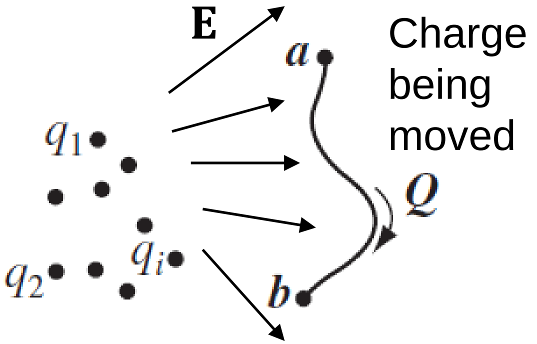
$$\Rightarrow \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0}\sigma$$

where we define normal derivative of V

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

Energy in electrostatics

- Work done to move a charge
  - Integrate force over distance



$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

$$= Q[V(\mathbf{b}) - V(\mathbf{a})]$$

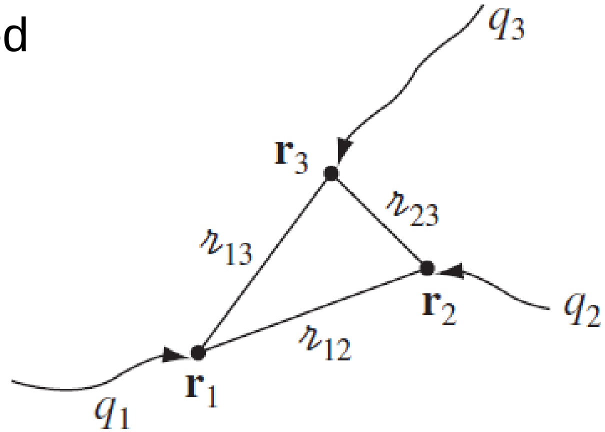
  - Electrostatic force is conservative (path independent)
  - Can confirm the unit of electric potential
  - Work for bringing from infinitely far to $\boldsymbol{r}$

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

$W = QV(\mathbf{r})$ with the potential reference point set to infinity

Energy in electrostatics

- Energy of a point charge configuration
  - Equals to the work required to bring charges together from infinity
    - First charge q_1 to $\boldsymbol{r_1}$, no work required
    - q_2 to $\mathbf{r_2}$ $\quad W_2 = \dfrac{1}{4\pi\epsilon_0} q_2 \left(\dfrac{q_1}{\mathscr{r}_{12}} \right)$
    - q_3 to $\mathbf{r_3}$ $\quad W_3 = \dfrac{1}{4\pi\epsilon_0} q_3 \left(\dfrac{q_1}{\mathscr{r}_{13}} + \dfrac{q_2}{\mathscr{r}_{23}} \right)$
    - $W = W_1 + W_2 + W_3$



  - Total work (energy) for n charges

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{\mathscr{r}_{ij}} \right)}_{V(\mathbf{r}_i)}$$

Count once for each pair

$$\boxed{W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i)}$$

Potential q_i feels due to all **<u>other</u>** charges

Energy in electrostatics

- Energy of a continuous charge distribution
  - Generalize point charge equation to

$$W = \frac{1}{2}\int \rho V \, d\tau$$

with V: actual potential, without excluding the charge of interest

$$\Big\downarrow \; \rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

$$W = \frac{\epsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V \, d\tau$$

$\Big\downarrow$ Integrate by parts $\int_{\mathcal{V}} f(\nabla \cdot \mathbf{A})\, d\tau = -\int_{\mathcal{V}} \mathbf{A} \cdot (\nabla f)\, d\tau + \oint_{\mathcal{S}} f\mathbf{A} \cdot d\mathbf{a}$

$$W = \frac{\epsilon_0}{2}\left[-\int \mathbf{E} \cdot (\nabla V)\, d\tau + \oint V\mathbf{E} \cdot d\mathbf{a}\right] = \frac{\epsilon_0}{2}\left(\int_{\mathcal{V}} E^2 \, d\tau + \underbrace{\oint_{\mathcal{S}} V\mathbf{E} \cdot d\mathbf{a}}_{\text{Vanishes when } \mathcal{V} \to \infty}\right)$$

$$\Rightarrow \boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau}$$

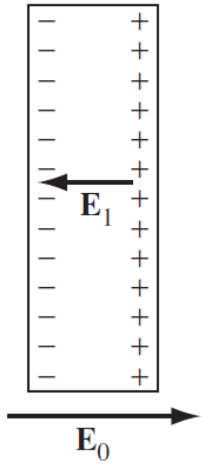
  - Cannot be directly compared to equation of point charge, see textbook

Conductor electrostatics

- Conductors
  - Free electrons – solid-state metals and doped semiconductors
  - Free ions – Electrolyte, salt water, lithium ion battery
  - Unlimited supply of free charges, which are free to move
- Electrostatics of perfect conductors
  - $\mathbf{E} = 0$ inside a conductor
    - If not, charge will flow to induce a new surface charge distribution that exactly cancels the internal field
  - $\rho = 0$ (net charge volume density) inside a conductor
    - Because $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
  - A conductor is an equipotential
    - For any two points, $V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$

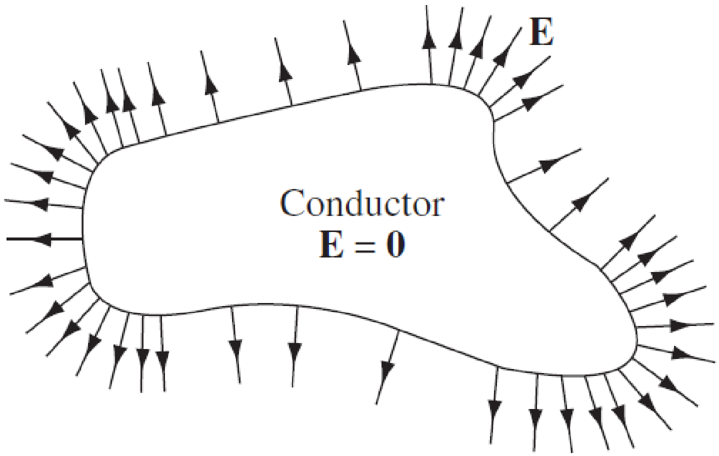
Conductor electrostatics

- Electrostatics of perfect conductors
  - Any net charge only resides on the surface (minimizes energy)
  - Surface net charges serves to cancel the internal field
  - $\mathbf{E}$ is always perpendicular to the surface, just outside the conductor
    - Recall boundary conditions



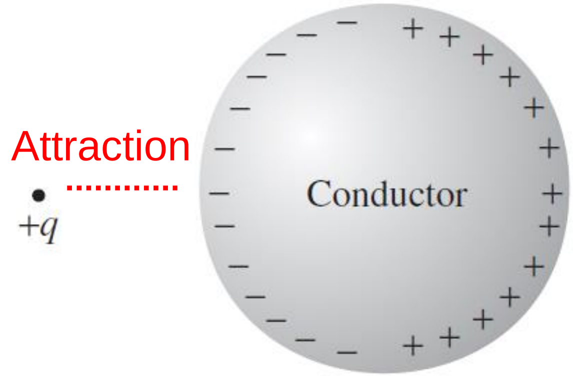
$$E^{\perp}_{\text{above}} - \cancelto{0}{E^{\perp}_{\text{below}}} = \frac{1}{\epsilon_0}\sigma$$

$$\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} = 0$$

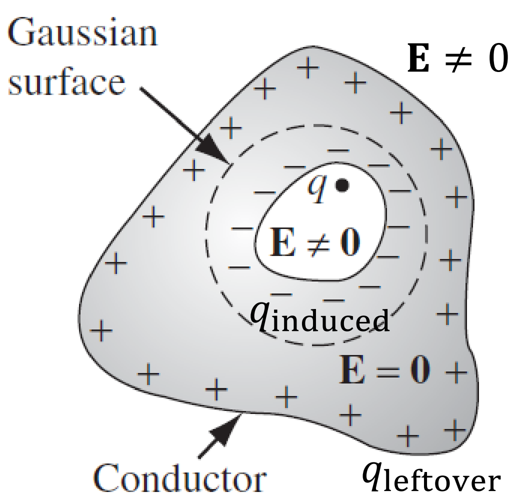


Conductor electrostatics

- Induced charges
  - Charge placed outside a metal
    - Induced charge serves to cancel field inside conductor
    - Net force of attraction



  - Charge in the cavity of a hollow metal
    - Inside the cavity: $\mathbf{E} \neq 0$
    - Induced charge $q_{\text{induced}} = -q$ at inner wall
    - Inside the conductor: $\mathbf{E} = 0$
    - Leftover charge $q_{\text{leftover}} = q$ at outer wall
    - Outside the conductor: $\mathbf{E} \neq 0$



Conductor electrostatics

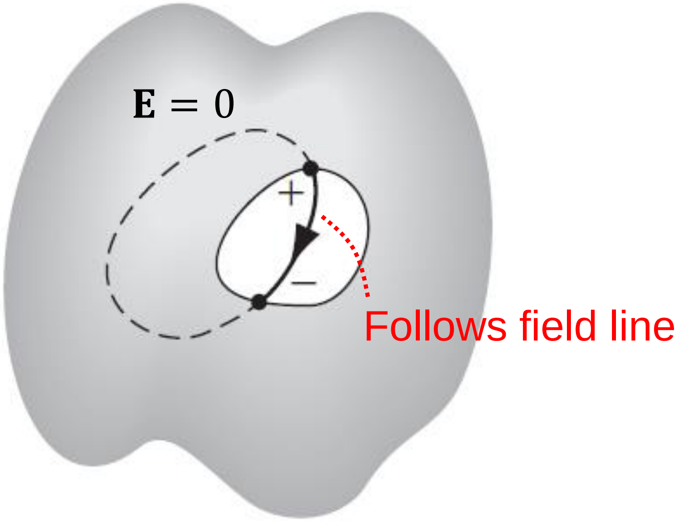
- Induced charges
  - Faraday cage
    - If no charge is placed in the cavity of a hollow conductor, $\mathbf{E} = 0$ in the cavity regardless of the outside conditions

      If not, can construct a loop of integration, whose trajectory in the cavity follows the field line

      ➡ $\oint \mathbf{E} \cdot d\mathrm{l} \neq 0$

      ➡ Contradicts $\nabla \times \mathbf{E} = 0$

      ➡ $\mathbf{E} = 0$

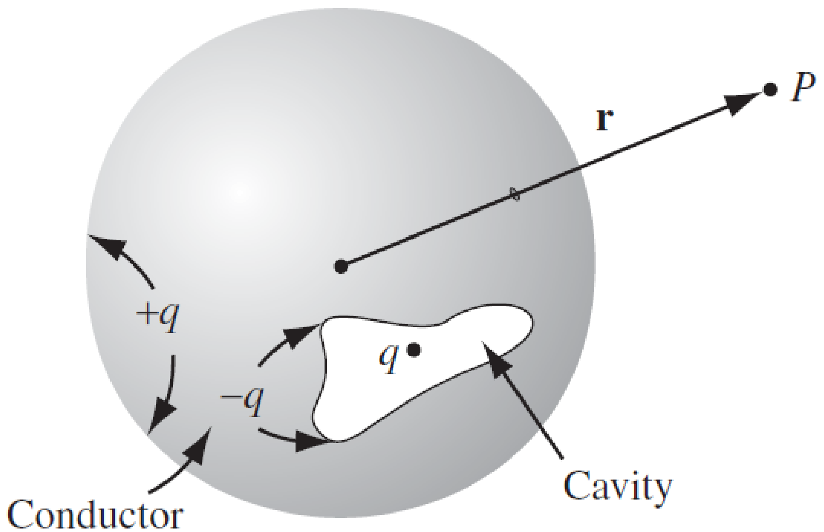


    - Protects sensitive apparatus inside the cavity by shielding out external electric fields

Conductor electrostatics

- Induced charges

Example 2.10. An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it (Fig. 2.46). Somewhere within the cavity is a charge q. *Question:* What is the field outside the sphere?



Conductor electrostatics

- Surface charge and force on a conductor
  - Boundary conditions

$$\begin{cases} \mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \dfrac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \\ \dfrac{\partial V_{\text{above}}}{\partial n} - \dfrac{\partial V_{\text{below}}}{\partial n} = -\dfrac{1}{\epsilon_0}\sigma \end{cases} \xrightarrow{\text{On the surface of a perfect conductor}} \begin{cases} \mathbf{E} = \dfrac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \\ \sigma = -\epsilon_0 \dfrac{\partial V}{\partial n} \end{cases}$$

  - Force (per unit area) exerted on the conductor
    - Can prove (textbook p.104) : for any surface across which is discontinuous, force needs to be calculated by

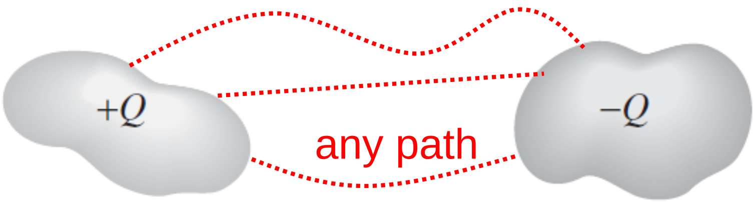
$$\mathbf{f} = \sigma \mathbf{E}_{\text{average}} = \frac{1}{2}\sigma(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}})$$

    - For conductors $\mathbf{f} = \dfrac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}}$

Conductor electrostatics

- Capacitors
  - We can define a potential difference between two conductors, without specifying locations of the integral

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}$$



  - Although $\mathbf{E}$ is geometry dependent, we know $\mathbf{E} \propto Q$, and $V \propto Q$
  - Can define ratio as capacitance $C \equiv \dfrac{Q}{V}$
    - A purely geometrical quantity, determined by shapes, sizes, and separation of the two conductors
    - Unit: farads (F), or Coulomb per volt
    - Always positive

Potentials

Laplace equation

- Why Laplace equation is of interest
  - Three ways to solve electrostatic problems
    - $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$ Involves lengthy integrations
    - $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$
    - $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ + boundary conditions

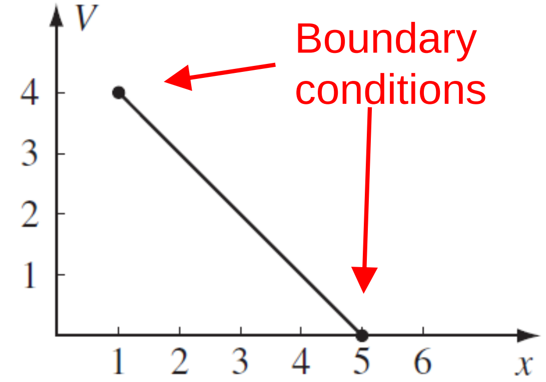
    $\Big\}$ Similar
  - Many problems are only concerned with charge-free regions
    - Laplace equation $\nabla^2 V = 0$ plus boundary conditions
    - Charges can exist elsewhere
    - Laplace equation has ubiquitous usage: electrostatics, theory of gravity, magnetism, theory of heat…

Laplace equation

- Laplace equation in 1D

$$\frac{d^2V}{dx^2} = 0$$

  - Solution: $V(x) = mx + b$ where m and b are constants
  - Trivial solution, but two notable features (generalizable to higher-dimension equations)
    - $V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$ <u>for any a</u>
    - Solution has no local maximum or minimum, extrema must exist at boundaries

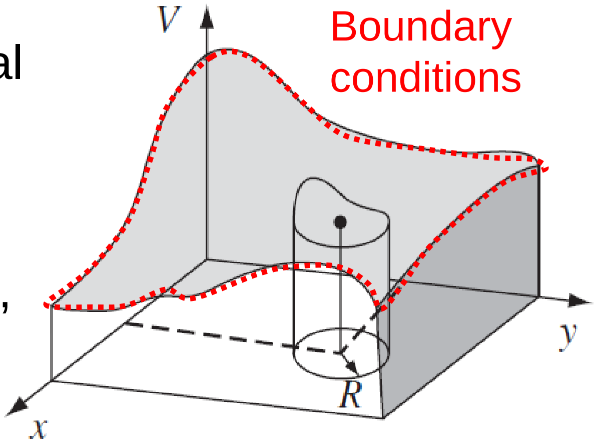


Laplace equation

- Laplace equation in 2D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

  - Partial differential equation, where general solutions of the closed form is not possible
  - $V(x, y)$ is the average of those around the point
    - $V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V\, dl$ where path is a circle centered at (x, y)
    - Method of relaxation to reach a numerical solution by iteratively using the equation above
  - Solution has no local maximum or minimum, extrema must exist at boundaries



Laplace equation

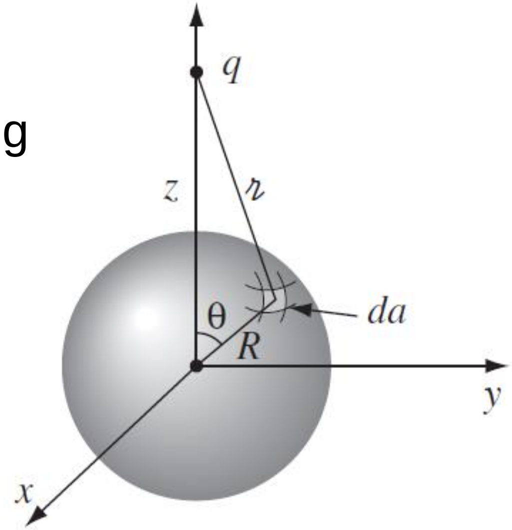
- Laplace equation in 3D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

  - $V(\mathbf{r})$ is the average of V over a spherical surface centered at $\mathbf{r}$
    - $V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da$ where area is a sphere centered at (x, y)

Example. Verify the above property by calculating the potential induced by a single charge q

C2.3DLaplace



  - Solution has no local maximum or minimum

Uniqueness theorems

- The first uniqueness theorem

> The solution to Laplace's equation in some volume is uniquely determined if V is specified on the boundary surface.

- Relevant to apparatus whose parts are connected to battery or ground
- Proof by contradiction

Suppose there are two solutions

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0$$

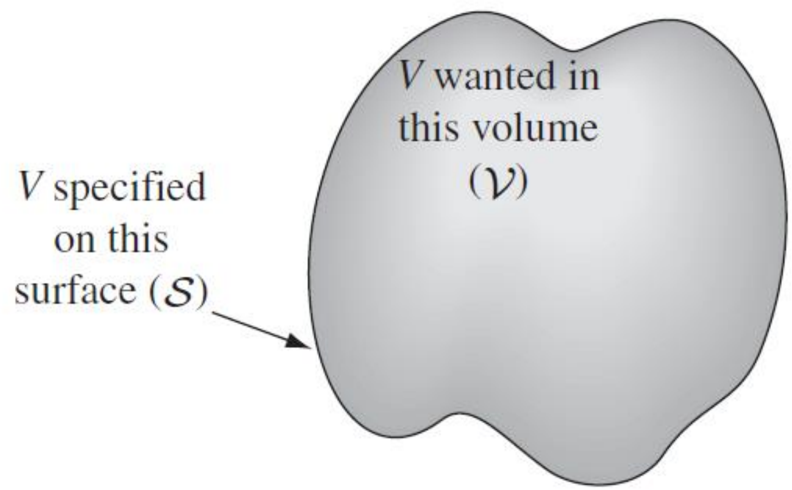
Define $V_3 \equiv V_1 - V_2$

Then $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$

with $V_3 = 0$ on all boundaries

⇒ $V_3 = 0$ everywhere as there is no extrema expect on boundaries

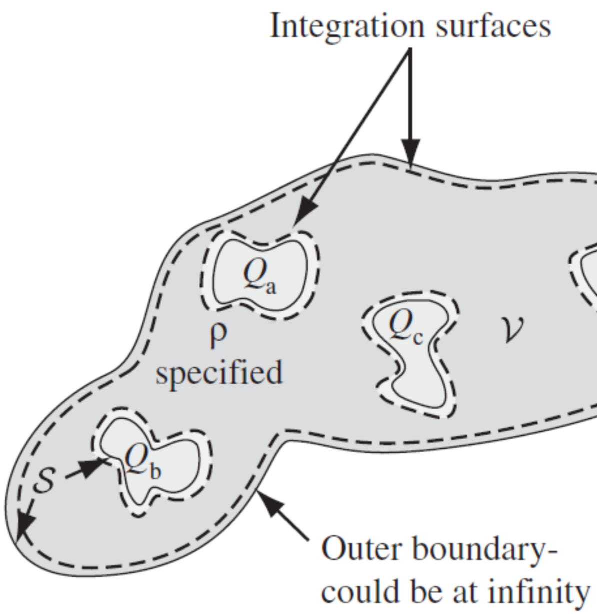
⇒ $V_1 = V_2$



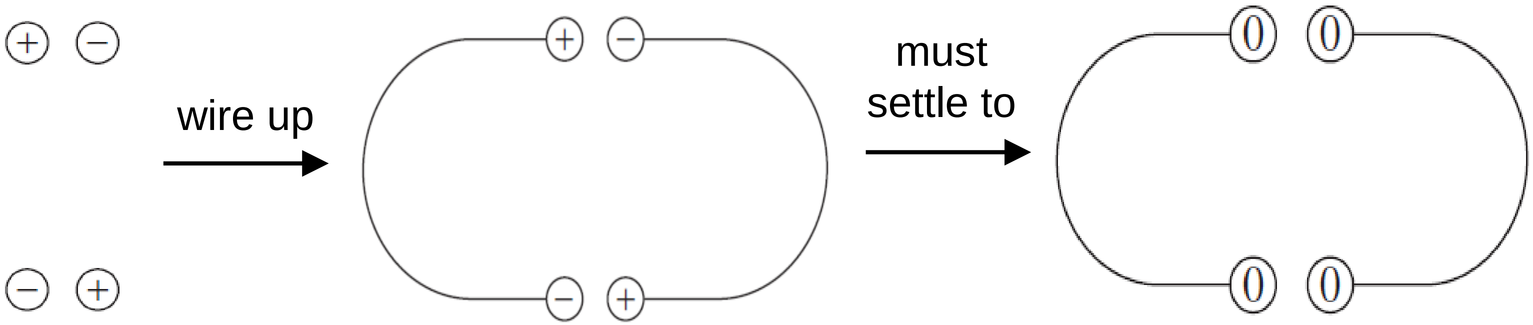
Uniqueness theorems

- The second uniqueness theorem

  In a volume surrounded by conductors with a specified charge density, the electric field is uniquely determined if the total charge (not the charge distribution) on each conductor is given.

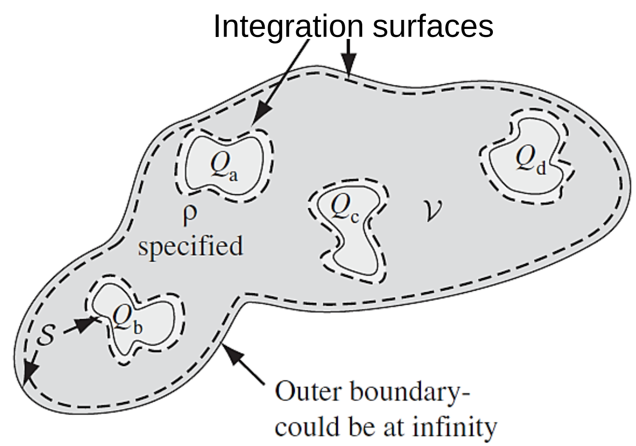


  - Useful for conductor electrostatics
  - Purcell's example



Uniqueness theorems

- The second uniqueness theorem
  - Proof by contradiction



Suppose there are two solutions

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0}\rho, \qquad \nabla \cdot \mathbf{E}_2 = \frac{1}{\epsilon_0}\rho$$

$$\Rightarrow \oint_{i\text{th conducting surface}} \mathbf{E}_{1,2} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_i \qquad \oint_{\text{outer boundary}} \mathbf{E}_{1,2} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}$$

Define $\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2$ then $\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0$

Consider the following expression (with product rule and intuition)

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3(\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2$$

$$\Rightarrow \int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{E}_3)\, d\tau = \oint_{\mathcal{S}} V_3 \mathbf{E}_3 \cdot d\mathbf{a} = -\int_{\mathcal{V}} (E_3)^2\, d\tau$$

also

$$= V_3 \oint \mathbf{E}_3 \cdot d\mathbf{a} = 0$$

$\Rightarrow$ $E_3 = 0$ everywhere

Three ways to solve the Laplace equation in the charge-free region

1. The method of images

2. Separation of variables

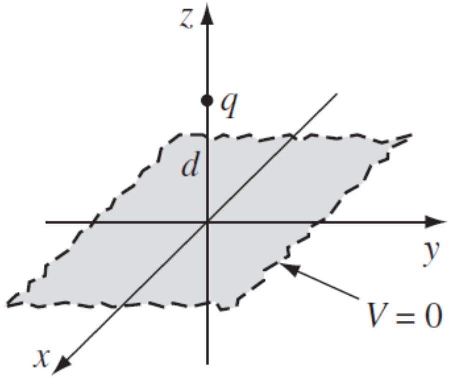
3. Multipole expansion

The method of images

- Usually works for problems involving charge(s) and conductor(s)

- The classical image problem

  - Point charge q held a distance d from a grounded ($V = 0$) conducting plate



  - Ask: the potential in the region above the plane

  - The 1st uniqueness theorem applicable - V at all boundaries known

    - $V = 0$ at $z = 0$

    - $V = 0$ at $\sqrt{x^2 + y^2 + z^2} \to \infty$

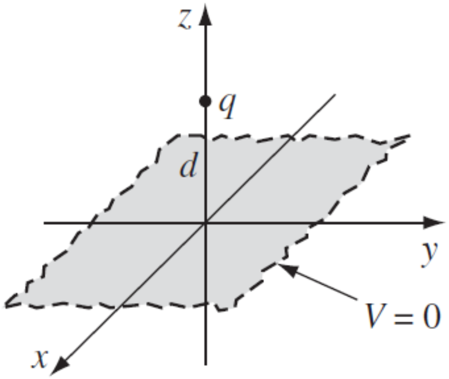
    - Can guess a $V(x, y, z)$ that is consistent with the Poisson's equation (in the region of interest) and these requirements

The method of images

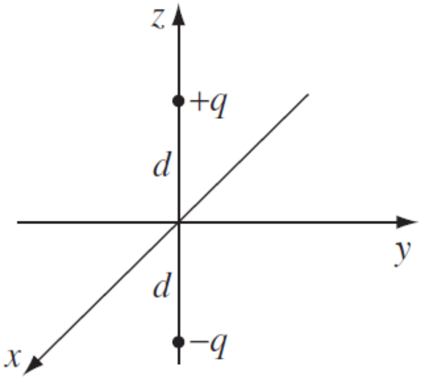
- The classical image problem
  - Guess: potential due to two point charges at $(0,0,\pm d)$ in free space

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

    - Charge configuration same as original problem at $z > 0$ ✓
    - $V = 0$ at $z = 0$ ✓
    - $V = 0$ at $\sqrt{x^2+y^2+z^2} \to \infty$ ✓



Guess

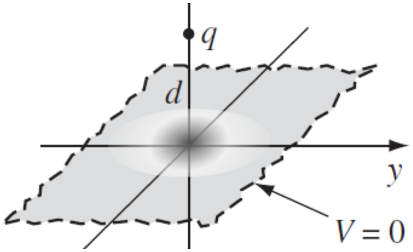


The method of images

- The classical image problem

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

  - Induced surface charge on the conducting plate



$$\sigma = -\epsilon_0\frac{\partial V}{\partial n} = -\epsilon_0\frac{\partial V}{\partial z}\bigg|_{z=0}$$

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}$$

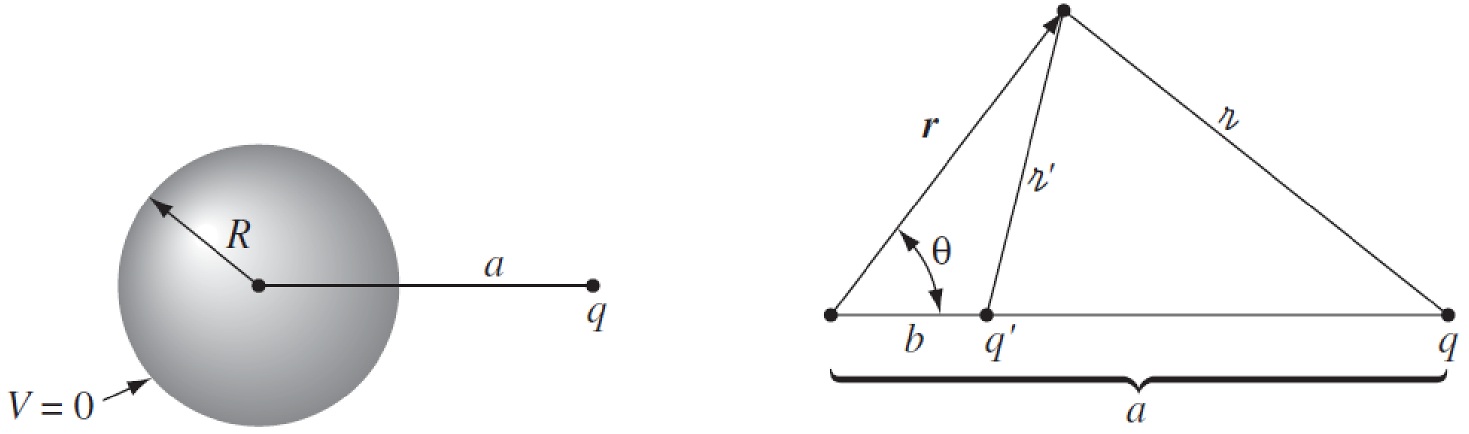
$$\sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

    - Total surface charge (calculate with polar coordinate)

$$Q = \int_0^{2\pi}\int_0^{\infty}\frac{-qd}{2\pi(r^2+d^2)^{3/2}}r\,dr\,d\phi = \frac{qd}{\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q$$

The method of images

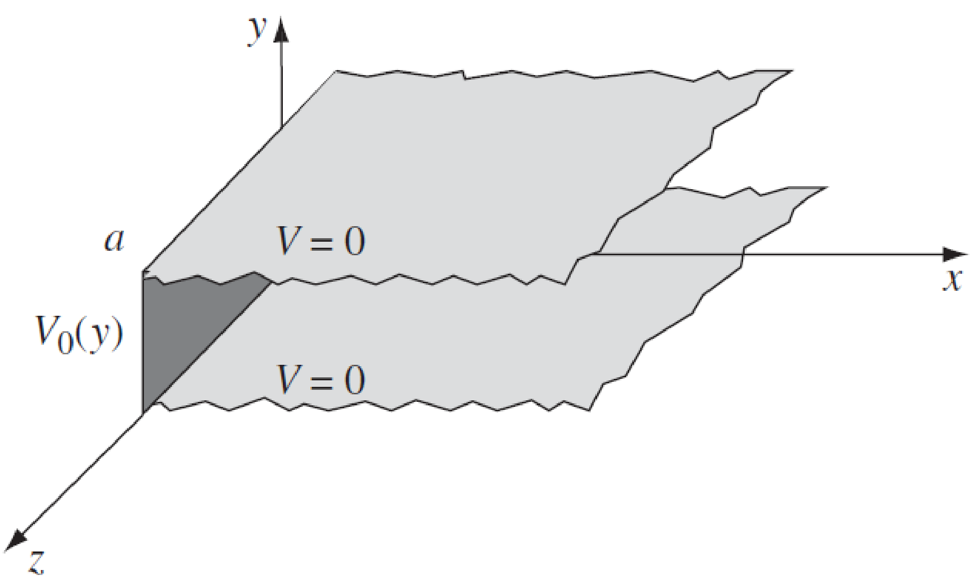
Example 3.2. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R (Fig. 3.12). Find the potential outside the sphere.



Separation of variables

- Assume general solutions are products of single-variable functions
  - Applicability strongly depends on problem
- Separation of variables in Cartesian coordinates

Example 3.3. Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$ (Fig. 3.17). The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."



Separation of variables

- Example 3.3

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$

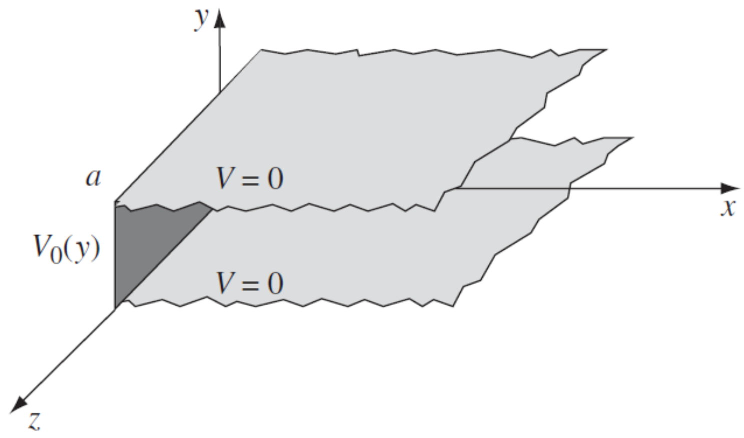
  - Boundary conditions

    (1) $V(x,y) = 0$ when $y = 0$

    (2) $V(x,y) = 0$ when $y = a$

    (3) $V(x,y) = V_0(y)$ when $x = 0$

    (4) $V(x,y) \to 0$ when $x \to \infty$



  - Trial solution $V(x,y) = X(x)Y(y)$

$$\Rightarrow \quad \underline{\frac{1}{X}\frac{d^2X}{dx^2}} + \underline{\frac{1}{Y}\frac{d^2Y}{dy^2}} = 0$$

Two terms only depend on x or y

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \text{with} \quad C_1 + C_2 = 0$$

Separation of variables

- Example 3.3

  ? What would happen if you assume $C_1 < 0$, $C_2 > 0$

  - Trial solution $V(x,y) = X(x)Y(y)$
    - Suppose $C_1 = k^2 > 0$, $C_2 = -k^2 < 0$ (k being a constant)

$$\Rightarrow \begin{cases} \dfrac{d^2X}{dx^2} = k^2 X \\ \dfrac{d^2Y}{dy^2} = -k^2 Y \end{cases} \Rightarrow \begin{cases} X(x) = Ae^{kx} + Be^{-kx} \\ Y(y) = C \sin ky + D \cos ky \end{cases}$$

$$\Rightarrow V(x,y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)$$

  - Determine A, B, C, D by matching boundary conditions

(4) $V(x,y) \to 0$ when $x \to \infty$ $\Rightarrow$ $A = 0$

(1) $V(x,y) = 0$ when $y = 0$ $\Rightarrow$ $D = 0$

(2) $V(x,y) = 0$ when $y = a$ $\Rightarrow$ $k = \dfrac{n\pi}{a}$ $\quad (n = 1,2,3 \ldots)$

Separation of variables

- Example 3.3
  - Determine A, B, C, D by matching boundary conditions

$$V(x,y) = Ce^{-kx}\sin ky \qquad (k = \frac{n\pi}{a})$$

(3) $V(x,y) = V_0(y)$ when $x = 0$, how to match this condition?

- Create a linear combination of solutions

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a)$$

(C_n: coefficients to be determined)

$$\Rightarrow \quad V(0,y) = \underline{\sum_{n=1}^{\infty} C_n \sin(n\pi y/a)} = V_0(y)$$

Fourier series expansion of $V_0(y)$

$$\Rightarrow \quad C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy$$

Separation of variables

- Example 3.3
  - Examine the solution

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

with $C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy$

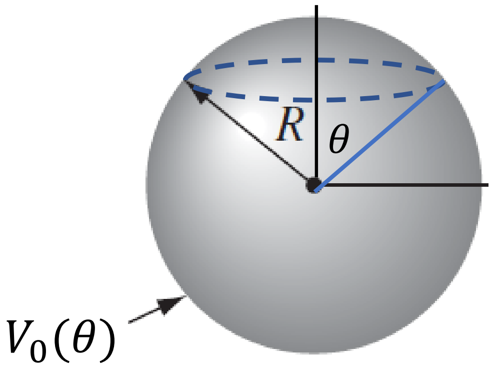
- The assumption of $V(x, y) = X(x)Y(y)$ was bodacious
- Validity of this method highly depend on the system's geometry
- The final solution written as a linear combination of trial solutions is not separated in variables

Separation of variables

- Separation of variables in spherical coordinates
  - Applicable to round objects

Example 3.6. The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside the sphere.

Example 3.7. The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R, but this time we are asked to find the potential *outside*, assuming there is no charge there.



Separation of variables

- Examples 3.6 and 3.7

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

$$\Big\downarrow \text{ Problem \& solution must be } \phi\text{-independent}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

  - Boundary conditions

    (1) $V(R,\theta)=V_0(\theta)$ at the surface of the sphere

    (2) $V(0,\theta)$ finite

    (3) $V(R,\theta)\to 0$ when $R\to\infty$

  - Trial solution $V(r,\theta)=R(r)\Theta(\theta)$

Two terms only depend on r or θ

$$\Rightarrow \underline{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}+\underline{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}=0$$

Separation of variables

- Examples 3.6 and 3.7
  - Trial solution $V(r,\theta) = R(r)\Theta(\theta)$

$$\Rightarrow \begin{cases} \dfrac{1}{R}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right) = l(l+1) \\ \dfrac{1}{\Theta\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d\Theta}{d\theta}\right) = -l(l+1) \end{cases} \Rightarrow \begin{cases} R(r) = Ar^l + \dfrac{B}{r^{l+1}} \\ \Theta(\theta) = P_l(\cos\theta) \end{cases}$$

$$\Rightarrow V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta)$$

$\Rightarrow$ General solution $V(r,\theta) = \displaystyle\sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

  - Legendre polynomial $P_l(x) \equiv \dfrac{1}{2^l l!}\left(\dfrac{d}{dx}\right)^l (x^2-1)^l$

$$\begin{aligned} P_0(x) &= 1 & P_3(x) &= (5x^3-3x)/2 \\ P_1(x) &= x & P_4(x) &= (35x^4-30x^2+3)/8 \\ P_2(x) &= (3x^2-1)/2 & P_5(x) &= (63x^5-70x^3+15x)/8 \end{aligned}$$

Separation of variables

- Examples 3.6 and 3.7
  - Determine A_l, B_l by matching boundary conditions

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

    - For $V(r,\theta)$ inside the sphere

(2) $V(0,\theta)$ finite $\Rightarrow$ $B_l = 0$

(1) $V(R,\theta) = V_0(\theta)$ $\Rightarrow$ $V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta)$

Orthogonality of Legendre polynomial

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta = \frac{2\delta_{ll'}}{2l+1}$$

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta$$

Separation of variables

- Examples 3.6 and 3.7
  - Determine A_l, B_l by matching boundary conditions

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

- For $V(r,\theta)$ outside the sphere

(3) $V(\infty,\theta) \to 0 \Rightarrow A_l = 0$

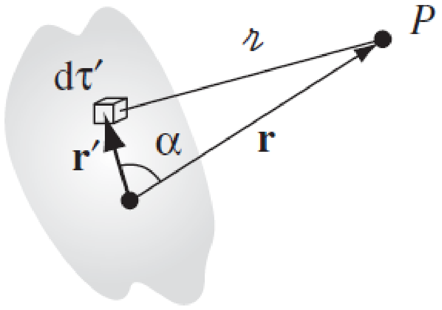
(1) $V(R,\theta) = V_0(\theta) \Rightarrow V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta)$

$\downarrow$ Orthogonality of Legendre polynomial

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

Multipole expansion

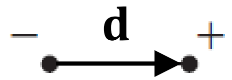
- Why not exact solution but use an expansion?
  - Exact solution $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$
    - For many problems hard to integrate, because of $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$
    - For many problem not necessary, if one only cares about solution for large $\boldsymbol{r}$
  - Multipole expansion
    - An expansion that examines from low to high order multipole contributions progressively
  - Types of multipoles
    - Monopole (1 charge), dipole (2 charges), quadrupole (4 charges), octupole (8 charges)



Multipole expansion

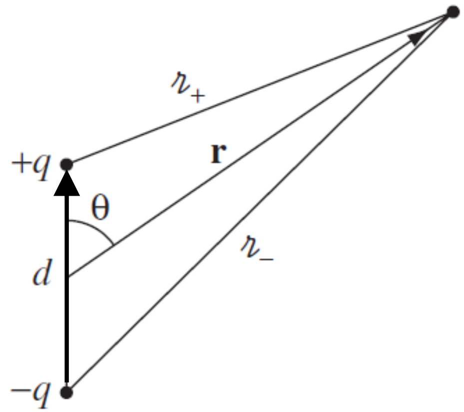
- Types of multipoles
  - Monopole = total charge, $Q = \sum_i q_i$, or $Q = \int \rho(\boldsymbol{r}')d\tau'$
  - Dipole between 2 charges

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i \quad \text{or} \quad \mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau'$$



$\mathbf{p} = q\mathbf{d}$
(points from $-q$ to $+q$)

Example 3.10. A (physical) **electric dipole** consists of two equal and opposite charges ($\pm q$) separated by a distance d. Find the approximate potential at points far from the dipole.

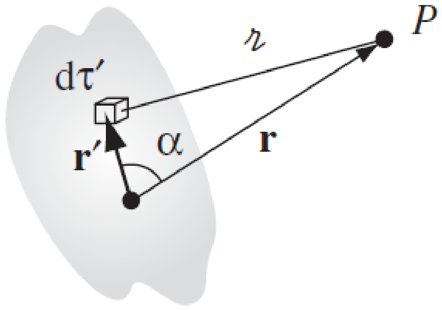


- Dipole field at large $\boldsymbol{r}$: $V_{\text{dip}}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0} \dfrac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$

Multipole expansion

- Formal multipole expansion

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$



  - Focus on expanding $1/\mathscr{r}$

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$\mathscr{r} = r\sqrt{1+\epsilon}$ where $\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

Taylor expansion knowing $\epsilon \ll 1$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

Multipole expansion

- Formal multipole expansion
  - Focus on expanding $1/\mathscr{r}$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \dots\right]$$

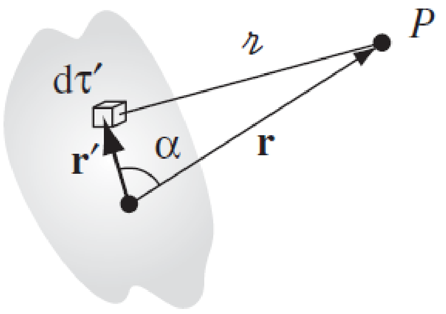
$\downarrow$ Sort by different powers of r'/r

$$= \frac{1}{r}\left[\underline{1} + \left(\frac{r'}{r}\right)\underline{(\cos\alpha)} + \left(\frac{r'}{r}\right)^2\underline{\left(\frac{3\cos^2\alpha - 1}{2}\right)} + \left(\frac{r'}{r}\right)^3\underline{\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right)} + \dots\right] = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

In the form of Legendre polynomials

Multipole expansion

- Formal multipole expansion



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau'$$

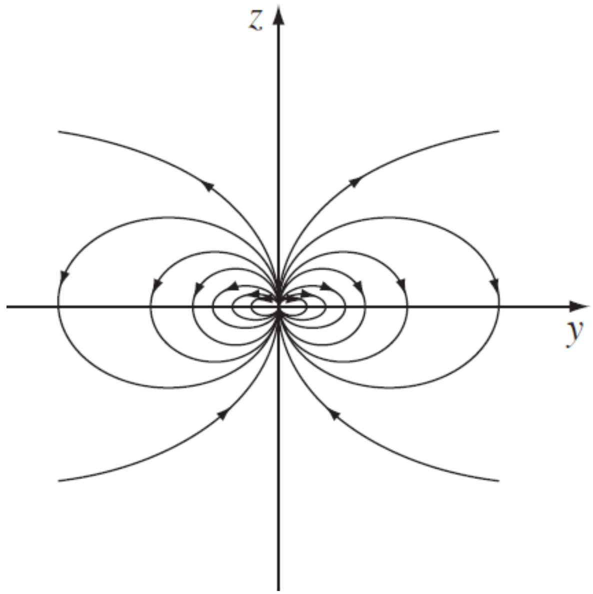
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\Bigg[\overbrace{\underline{\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau'}}^{\text{Monopole}} + \overbrace{\underline{\frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau'}}^{\text{Dipole}}$$
$$+ \underbrace{\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau'}_{\text{Quadrupole}} + \ldots\Bigg]$$

  - Merits of multipole expansion
    - r moved out of the integral ($\alpha = \theta$ polar angle if we set $\boldsymbol{r}$ along $\hat{z}$)
    - Can truncate to finite terms
  - Caveat: terms can depend on choice of origin (such as dipole term)

Multipole expansion

- Pure dipole vs physical dipole

Pure dipole

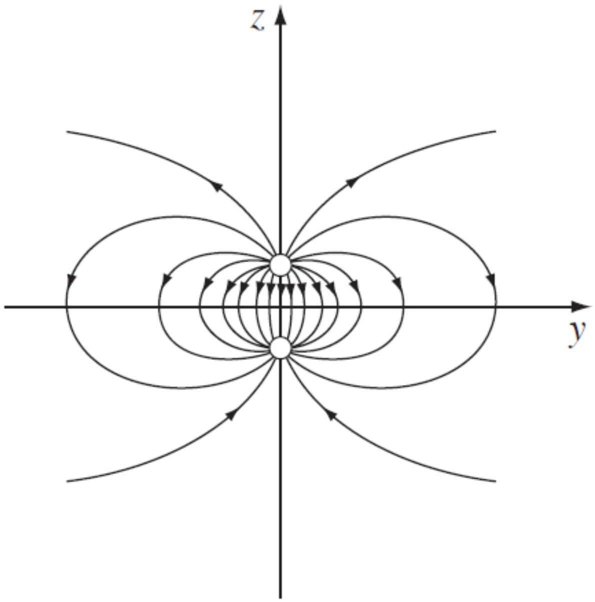


$$V_{\mathrm{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

Assumes $r \gg d$ always holds

$\mathbf{p} = q\mathbf{d}$ but take $q \to \infty$, $d \to 0$

Physical dipole



Deviations appear when closing up onto the dipole

$$\mathbf{p} = q\mathbf{d}$$