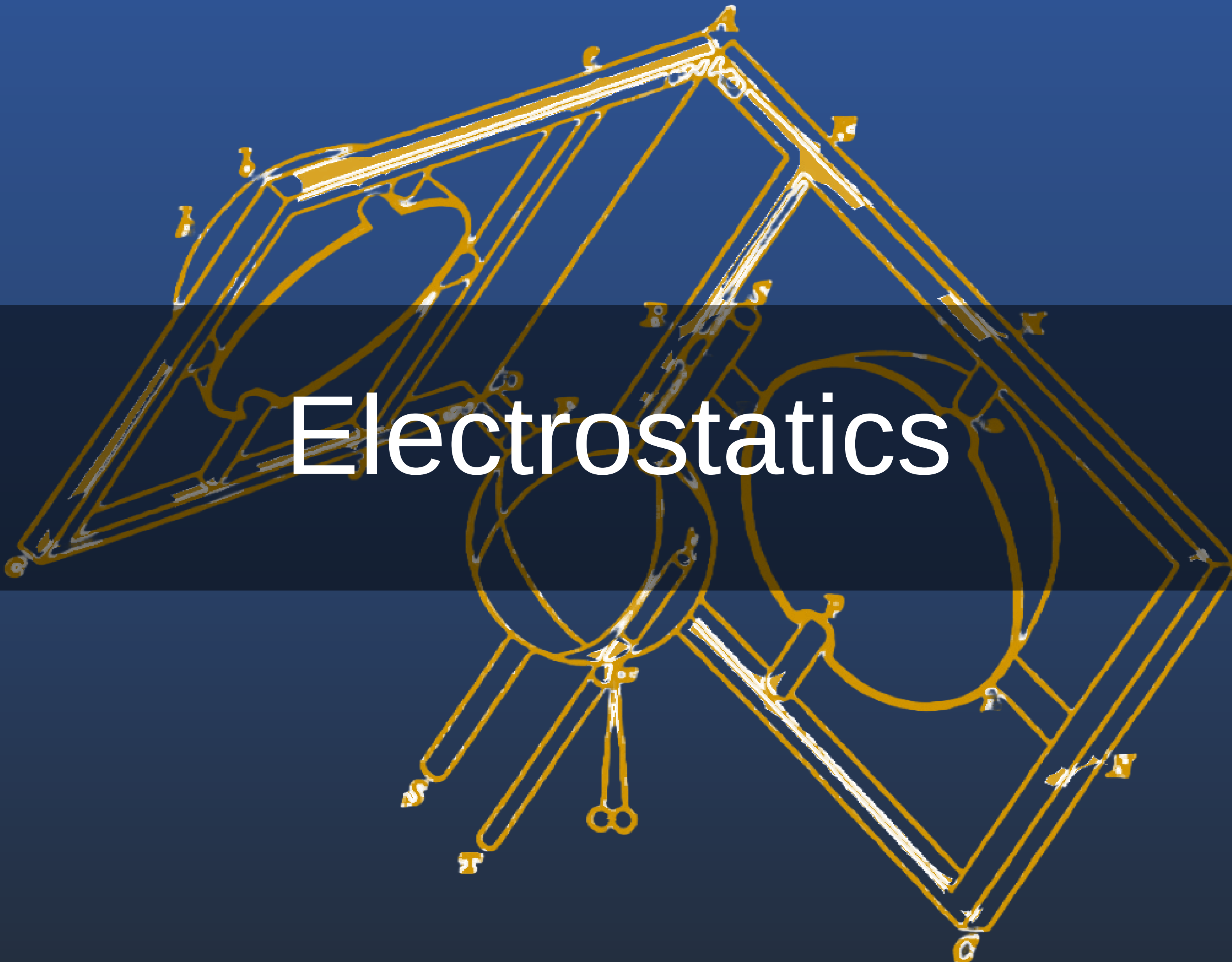
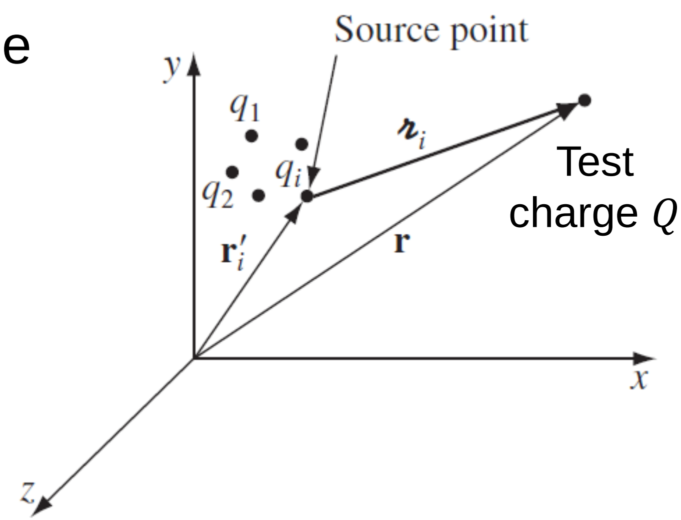
Electrostatics



Charge, electric field, and potential

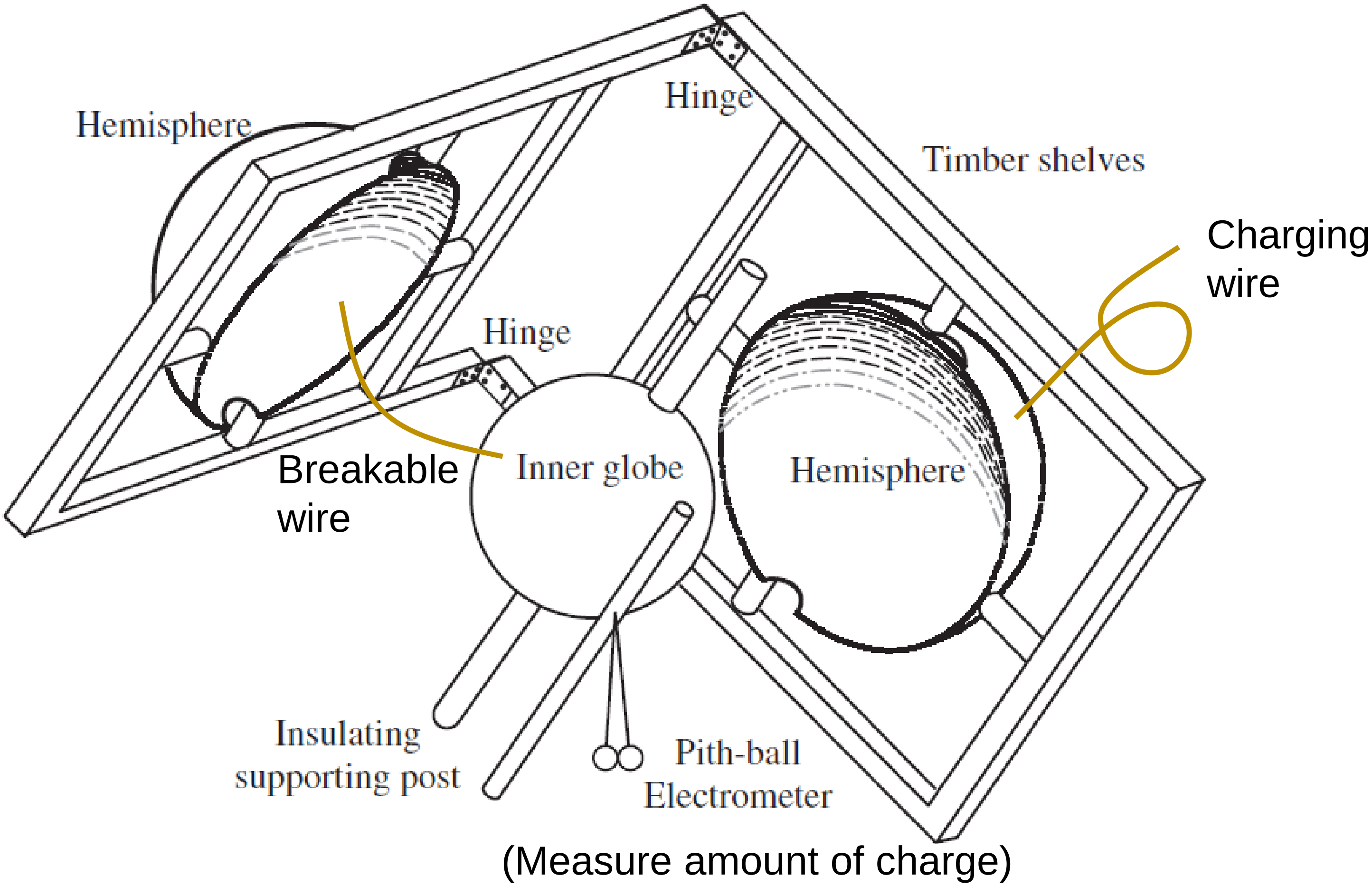
Coulomb's law

- Force of n source charges on a test charge
  - Force from source charge q_i acting on test charge Q
    - Coulomb's law $\boxed{F_i = \frac{1}{4\pi\varepsilon_0}\frac{q_i Q}{\boldsymbol{r}_i^2}\widehat{\boldsymbol{r}_i}}$
    - Permittivity $\varepsilon_0 = 8.85 \times 10^{12}$ C²/(N m²)
    - Separation vector $\boldsymbol{r}_i = \boldsymbol{r} - \boldsymbol{r}_i'$
    - Location of Q: $\boldsymbol{r}$, location of q_i: $\boldsymbol{r}_i'$
  - Principle of superposition
    - Total force acting on test charge $F = \sum_{i=1}^{n} F_i$
    - Not a necessity, but an experimental fact



* $\boldsymbol{\mathscr{r}}$ in textbook is typed as $\boldsymbol{r}$ in our slides (Cursive "r")

Coulomb's law



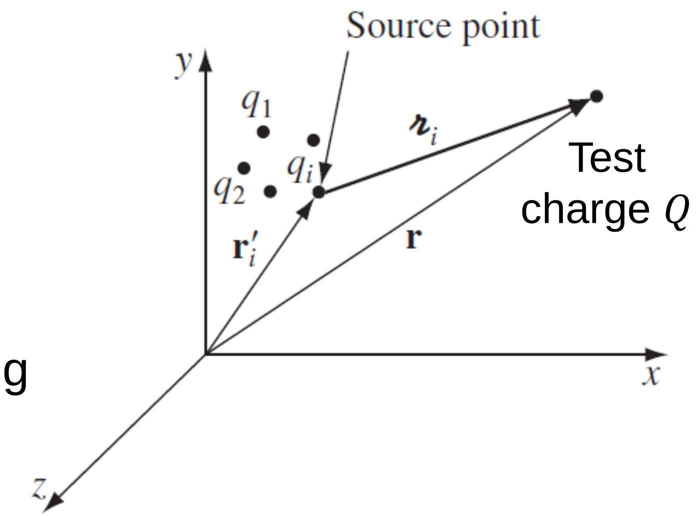
Cavendish's apparatus for determining $F \propto r^{-2}$ in Coulomb's law

Electric field induced by charge

- Relation of force and electric field

$$\boxed{\mathbf{F} = Q\mathbf{E}}$$

  - Electric field: force per unit charge
  - Real physical entity, as a vector field filling the space around charges
  - Negated theory of "ether"



- Electric field induced by discrete charges

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i$$

  - Separation vector $\boldsymbol{\mathscr{r}}_i = \boldsymbol{r} - \boldsymbol{r}'_i$, contains $\boldsymbol{r}$
  - Principle of superposition also holds

Electric field induced by charge

- Electric field induced by continuous charge distribution

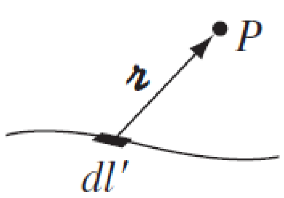
$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, dq}$$

  - Add up contributions from infinitesimal charge elements dq
  - Three ways dq can be distributed

Line charge

$dq \to \lambda\, dl'$

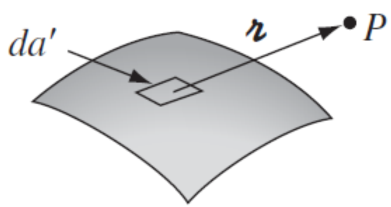
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, dl'$$



Surface charge

$dq \to \sigma\, da'$

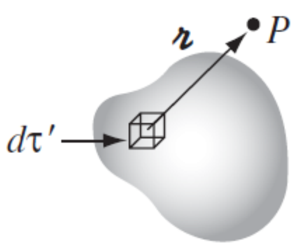
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, da'$$



Volume charge

$dq \to \rho\, d\tau'$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, d\tau'$$

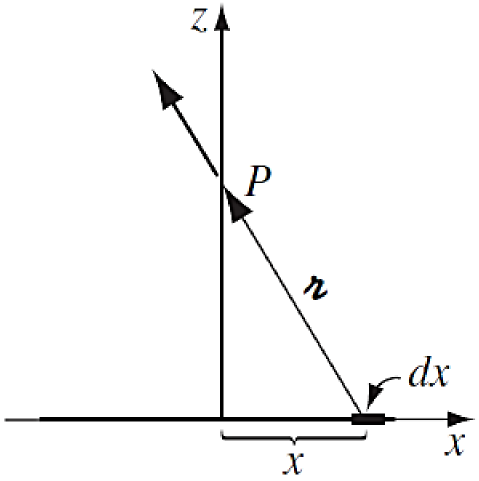


*λ, σ, ρ: charge per unit length, area, volume

Electric field induced by charge

- Electric field induced by continuous charge distribution

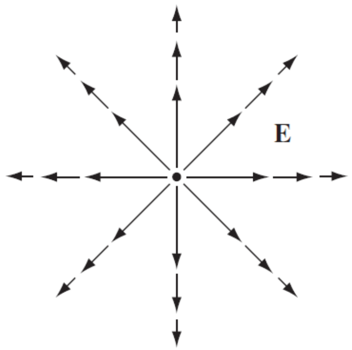
Example 2.2. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ (Fig. 2.6).



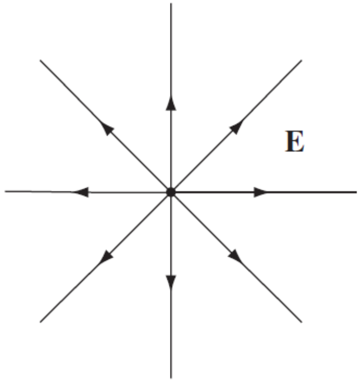
- Integration sometimes can get formidable, need to device new tools to simplify problems.

Gauss's law

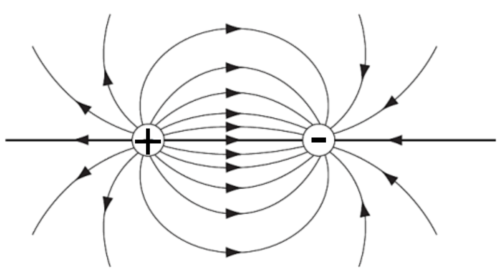
- Electric field lines
  - Source charge q at the origin $$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}}$$

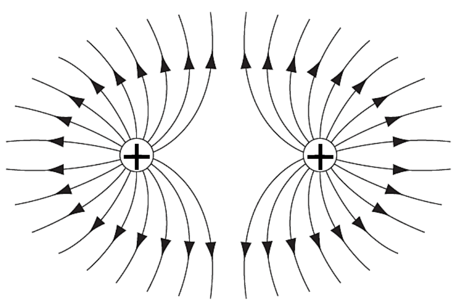


  - Draw vector field – field falls off like $1/r^2$
  - Connect up the arrows – electric field lines
    - Direction of line indicates field direction
    - Density of line indicates field magnitude



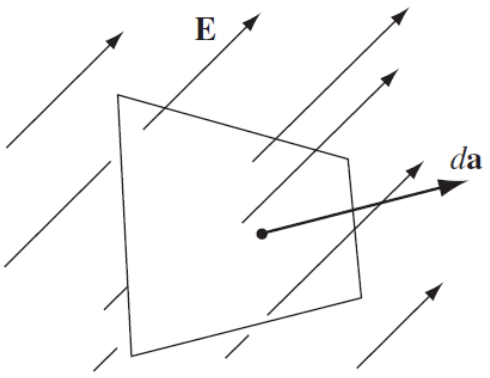
  - Field lines begin from positive charges and end on negative ones





Gauss's law

- Electric field flux $\Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$
  - A measure of the number of field lines passing through an area



- Gauss's law
  - The flux through any closed surface is a measure of the total charge inside

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot \underbrace{(r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}})}_{d\boldsymbol{a}} = \frac{1}{\epsilon_0} q$$

Spherical surface of radius r

  - The surface integral can be any shape, not necessarily spherical
  - Multiple charges $\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n}\left(\oint \mathbf{E}_i \cdot d\mathbf{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\epsilon_0} q_i\right)$

$$\Rightarrow \boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}}$$

(Q_{enc}: total charge enclosed in the integrated surface)

Gauss's law

- Gauss's law
  - Gauss's law in the differential form

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$ (integral form)

Divergence theorem

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau$$

Consider volume distribution

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho\, d\tau$$

$$\Rightarrow \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau$$

$$\Rightarrow \boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho}$$ (differential form)

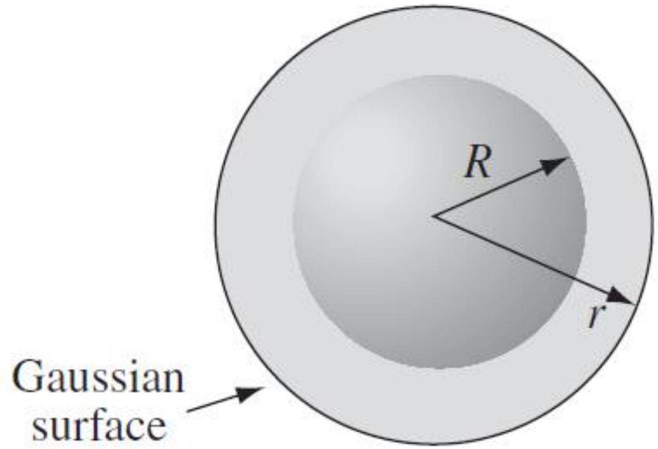
    - Differential form more compact, but integral form easier to use

  - Use of Gauss's law to calculate electric field
    - Need (1) Gauss's law in integral form and (2) symmetry arguments

Gauss's law

- Application of Gauss's law

Example 2.3. Find the field outside a uniformly charged solid sphere of radius R and total charge q.



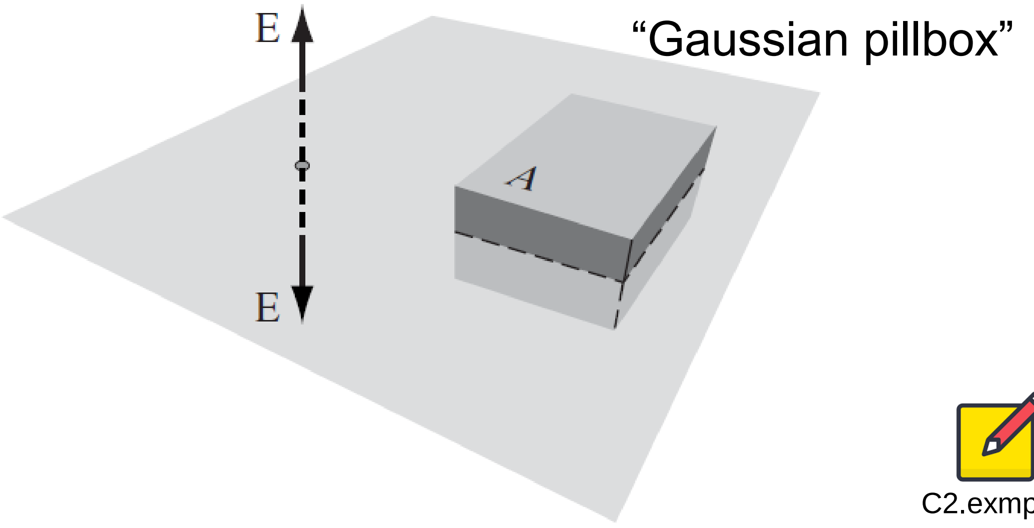
C2.exmp2.3

- The field outside the sphere is exactly the same as it would have been if all the charge had been concentrated at the center

Electric potential

- Application of electric potential

Example 2.5. An infinite plane carries a uniform surface charge σ. Find its electric field.



C2.exmp2.5

Divergence of electric field

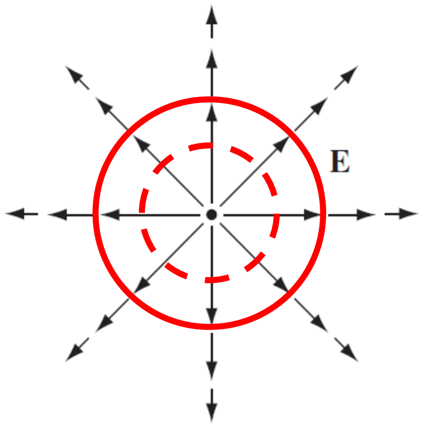
- Directly calculate divergence
  - According to Coulomb's law

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')\,d\tau'$$

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \boxed{\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)}\rho(\mathbf{r}')\,d\tau'$$

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \nabla\cdot\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0$$

? This seems to contradict the Gauss's law, what went wrong



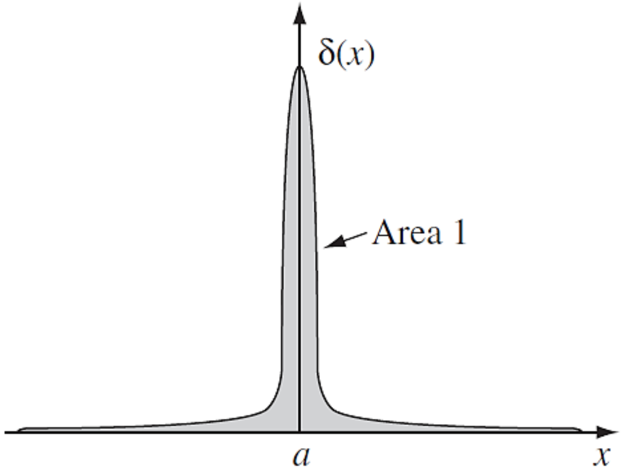
  - The derivation above is correct anywhere but the origin ($r = 0$), where the divergence should go to infinity
    - Consider special case of point charge and Gauss's law with varying volume to integrate

Divergence of electric field

- Delta function
  - Infinitely high, infinitesimally narrow
  - 1D Delta function

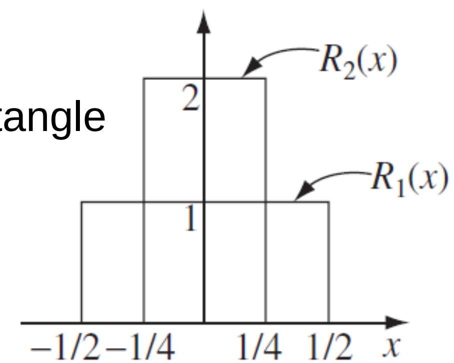
$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

$$\text{with} \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

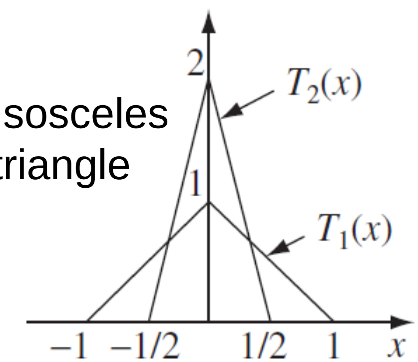


- Can be understood as the limit of a sequence of functions

Rectangle



Isosceles triangle



Divergence of electric field

- Delta function
  - 1D Delta function
    - When in an integral, "picks out" the value of a function

      Since $\delta(x)$ anywhere 0 but at $x = 0$

      [$f(x)$ being an ordinary function not going to infinity]

      $$f(x)\delta(x) = f(0)\delta(x)$$

      $$\Rightarrow \int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0)$$

      And, one can shift $\delta(x)$ to $\delta(x-a)$ to pick out another one

      $$f(x)\delta(x-a) = f(a)\delta(x-a)$$

      $$\Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)}$$

    - A frequently used expression $\delta(kx) = \dfrac{1}{|k|}\delta(x)$

Divergence of electric field

- Delta function
  - 3D Delta function $\boxed{\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z)}$

    with $\displaystyle\int_{\text{all space}} \delta^3(\mathbf{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$

    - Picks out a function value $\displaystyle\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\,d\tau = f(\mathbf{a})$

- Back to calculating divergence of electric field

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\rho(\mathbf{r}')\,d\tau'$$

$$\Big\downarrow\ \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

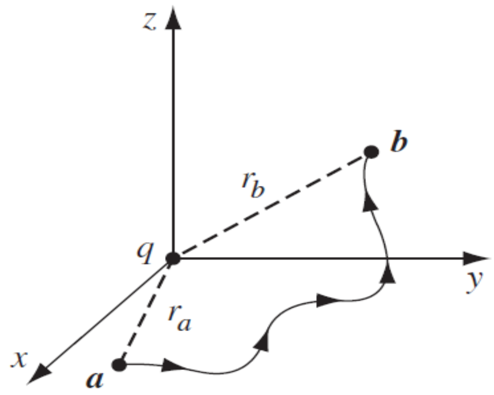
C2.delta

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int 4\pi\delta^3(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\,d\tau' = \frac{1}{\epsilon_0}\rho(\mathbf{r})$$

Gauss's law recovered

Curl of electric field

- Calculate curl for point charge at origin



$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}}\frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

- For any closed loop ($r_a = r_b$) $\oint \mathbf{E}\cdot d\mathbf{l} = 0$

$\boxed{\nabla\times\mathbf{E} = \mathbf{0}}$ due to Stoke's theorem

Stoke's theorem

$$\int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

- Any static charge distribution

$$\nabla\times\mathbf{E} = \nabla\times(\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla\times\mathbf{E}_1) + (\nabla\times\mathbf{E}_2) + \ldots = \mathbf{0}$$

Electric potential

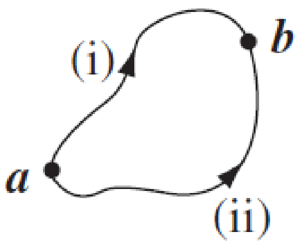
- Vector field **E** cannot take arbitrary form
  - Crucial constraint: $\nabla \times \mathbf{E} = \mathbf{0}$

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \qquad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} \qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

? Any chance the vector field can be described more easily?

- Electric potential: $\boxed{V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}}$
  - Unit: joules per coulomb
  - $\mathcal{O}$: a reference point (usually taken as infinity)
  - Integral does not depend on path
    - $\nabla \times \mathbf{E} = \mathbf{0}$
    - $\oint \mathbf{E} \cdot d\mathbf{l} = 0$
    - $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$ is path independent

Equivalent statements



Electric potential

- Electric potential: $V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$
  - Potential difference between two points is more meaningful

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$
$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

On the other hand, the theorem for gradient gives

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} \quad \Rightarrow \quad \boxed{\mathbf{E} = -\nabla V}$$

  - Scalar field V gives full information of vector field **E**
  - Can be off by a constant if choosing a different reference point

$$V'(\mathbf{r}) = -\int_{\mathcal{O}'}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{O}'}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = K + V(\mathbf{r})$$

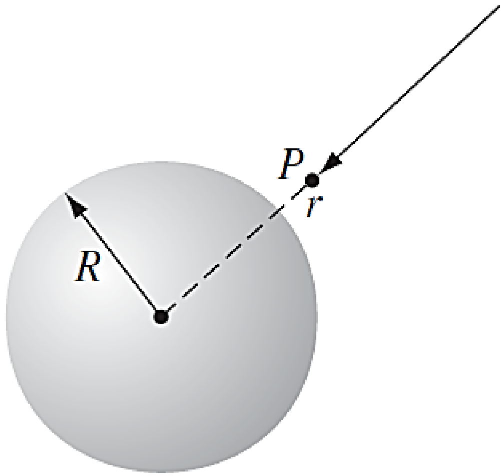
new

Electric potential

- Application of electric potential

Example. Find the potential of a point charge q at origin

Example 2.7. Find the potential inside and outside a spherical shell of radius R (Fig. 2.31) that carries a uniform surface charge. Set the reference point at infinity.



Electric potential

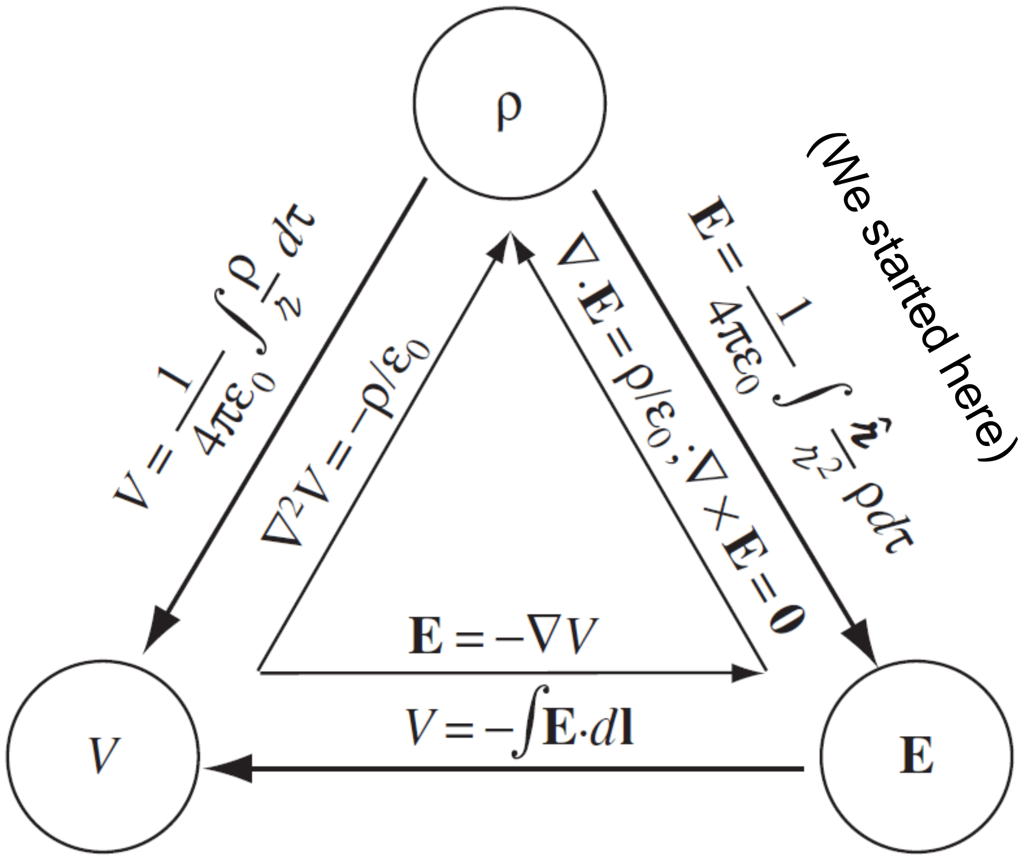
- Poisson's equation of potential
  - Poisson's equation $\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$

    $$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$
  - In regions with no charge, Laplace's equation $\boxed{\nabla^2 V = 0}$
  - Curl of a gradient always zero $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = \mathbf{0}$

- Potential of a localized charge distribution
  - Pick infinity as the reference point $\mathcal{O} = \infty$
  - Principle of superposition holds $V = V_1 + V_2 + \ldots$
  - Discrete charges $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i}$

    $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau'$$
  - Continuous charge $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$ Can check

Charge, electric field, and potential

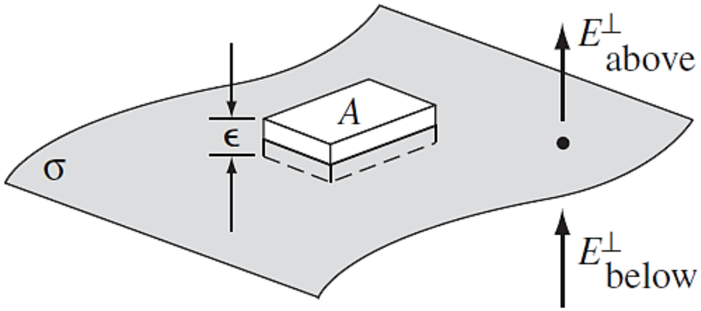


Differential equations need boundary conditions to solve

Boundary conditions

- Boundary conditions of **E** across a 2D charged surface
  - Normal component of **E**

    "Gaussian pillbox" with $\varepsilon \to 0$

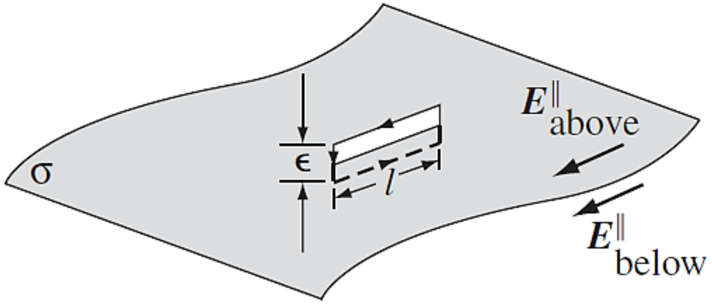


$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \sigma A$$

$$\Rightarrow \boxed{E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma}$$

  - Tangential component of **E**

    Thin loop with $\varepsilon \to 0$



$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\Rightarrow \boxed{\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}}}$$

  - Summarizing above $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$

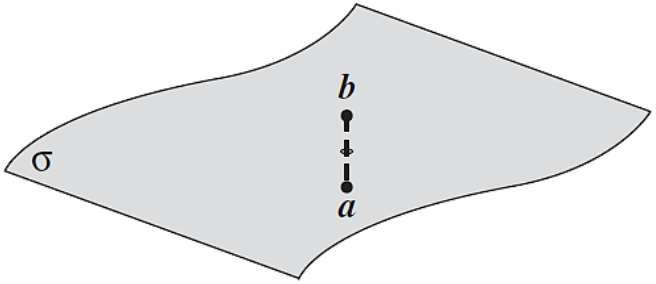
Boundary conditions

- Boundary conditions of V across a 2D charged surface
  - Potential is continuous (across any boundary)

$$V_{\text{above}} - V_{\text{below}} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

Path length → 0

$$V_{\text{above}} = V_{\text{below}}$$



  - Gradient of potential is discontinuous

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

$$\mathbf{E} = -\nabla V$$

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0}\sigma\hat{\mathbf{n}}$$

$$\Rightarrow \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0}\sigma$$

where we define normal derivative of V

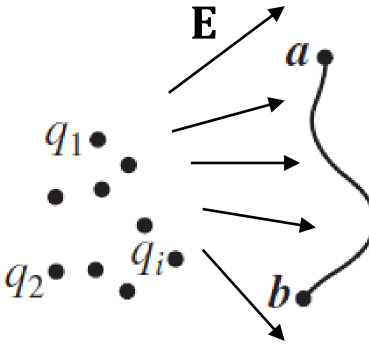
$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

Energy in electrostatics

- Work done to move a charge
  - Integrate force over distance

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

$$= Q[V(\mathbf{b}) - V(\mathbf{a})]$$



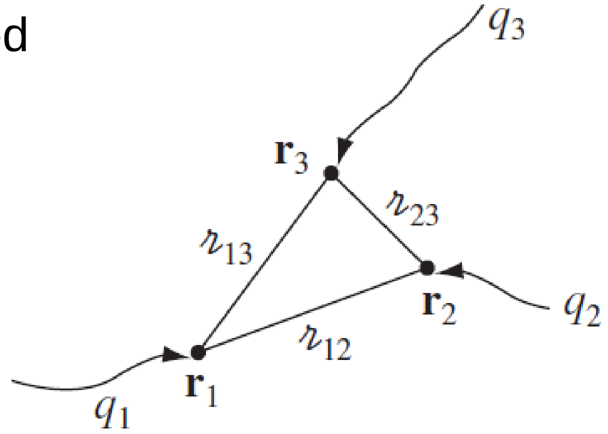
  - Electrostatic force is conservative (path independent)
  - Can confirm the unit of electric potential
  - Work for bringing from infinitely far to $\boldsymbol{r}$

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

$$W = QV(\mathbf{r})$$ with the potential reference point set to infinity

Energy in electrostatics

- Energy of a point charge configuration
  - Equals to the work required to bring charges together from infinity
    - First charge q_1 to $\boldsymbol{r_1}$, no work required
    - q_2 to $\mathbf{r_2}$ $\quad W_2 = \dfrac{1}{4\pi\epsilon_0} q_2 \left(\dfrac{q_1}{\mathscr{r}_{12}}\right)$
    - q_3 to $\mathbf{r_3}$ $\quad W_3 = \dfrac{1}{4\pi\epsilon_0} q_3 \left(\dfrac{q_1}{\mathscr{r}_{13}} + \dfrac{q_2}{\mathscr{r}_{23}}\right)$
    - $W = W_1 + W_2 + W_3$



  - Total work (energy) for n charges

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right)}_{V(\mathbf{r}_i)}$$

Count once for each pair

$$\Rightarrow \boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)}$$

Potential q_i feels due to all **<u>other</u>** charges

Energy in electrostatics

- Energy of a continuous charge distribution
  - Generalize point charge equation to

$$W = \frac{1}{2}\int \rho V \, d\tau$$

with V: actual potential, without excluding the charge of interest

$$\Big\downarrow \; \rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

$$W = \frac{\epsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V \, d\tau$$

$\Big\downarrow$ Integrate by parts $\int_{\mathcal{V}} f(\nabla \cdot \mathbf{A})\, d\tau = -\int_{\mathcal{V}} \mathbf{A} \cdot (\nabla f)\, d\tau + \oint_{\mathcal{S}} f\mathbf{A} \cdot d\mathbf{a}$

$$W = \frac{\epsilon_0}{2}\left[-\int \mathbf{E} \cdot (\nabla V)\, d\tau + \oint V\mathbf{E} \cdot d\mathbf{a}\right] = \frac{\epsilon_0}{2}\left(\int_{\mathcal{V}} E^2\, d\tau + \underbrace{\oint_{\mathcal{S}} V\mathbf{E} \cdot d\mathbf{a}}_{\text{Vanishes when } \mathcal{V} \to \infty}\right)$$

$$\Rightarrow \boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2\, d\tau}$$

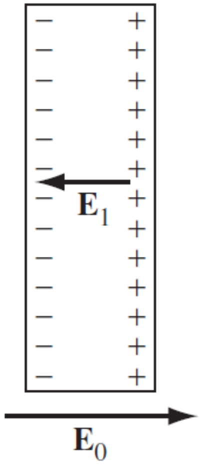
  - <u>Cannot</u> be directly compared to equation of point charge, see textbook

Conductor electrostatics

- Conductors
  - Free electrons – solid-state metals and doped semiconductors
  - Free ions – Electrolyte, salt water, lithium ion battery
  - Unlimited supply of free charges, which are free to move
- Electrostatics of perfect conductors
  - $\mathbf{E} = 0$ inside a conductor
    - If not, charge will flow to induce a new surface charge distribution that exactly cancels the internal field
  - $\rho = 0$ (net charge volume density) inside a conductor
    - Because $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
  - A conductor is an equipotential
    - For any two points, $V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$

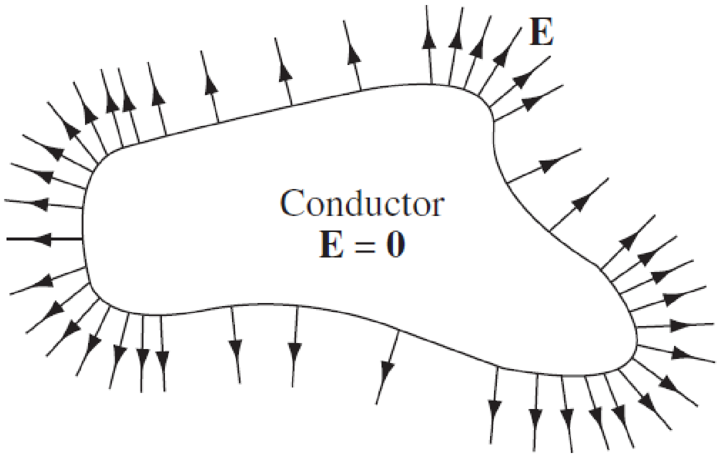
Conductor electrostatics

- Electrostatics of perfect conductors
  - Any net charge only resides on the surface (minimizes energy)
  - Surface net charges serves to cancel the internal field
  - $\mathbf{E}$ is always perpendicular to the surface, just outside the conductor
    - Recall boundary conditions



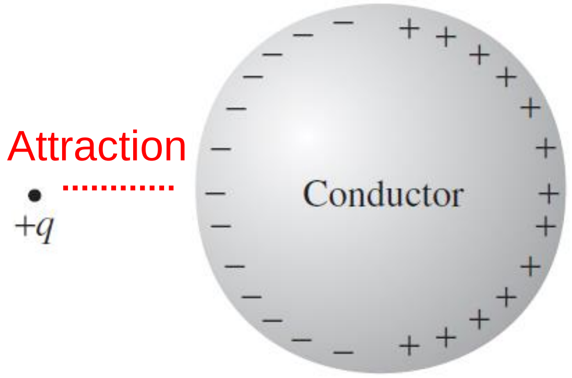
$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma$$

$$\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} = 0$$

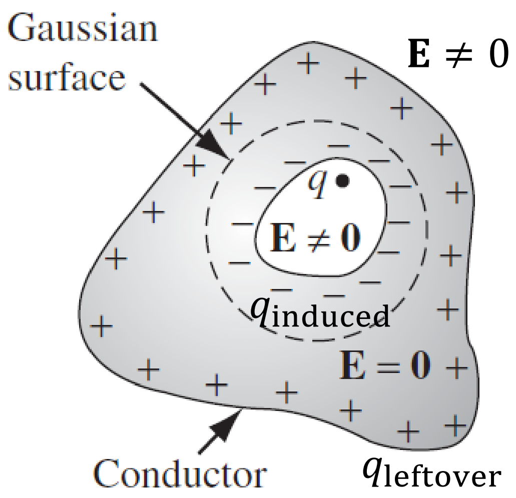


Conductor electrostatics

- Induced charges
  - Charge placed outside a metal
    - Induced charge serves to cancel field inside conductor
    - Net force of attraction



  - Charge in the cavity of a hollow metal
    - Inside the cavity: $\mathbf{E} \neq 0$
    - Induced charge $q_{\text{induced}} = -q$ at inner wall
    - Inside the conductor: $\mathbf{E} = 0$
    - Leftover charge $q_{\text{induced}} = q$ at outer wall
    - Outside the conductor: $\mathbf{E} \neq 0$



Conductor electrostatics

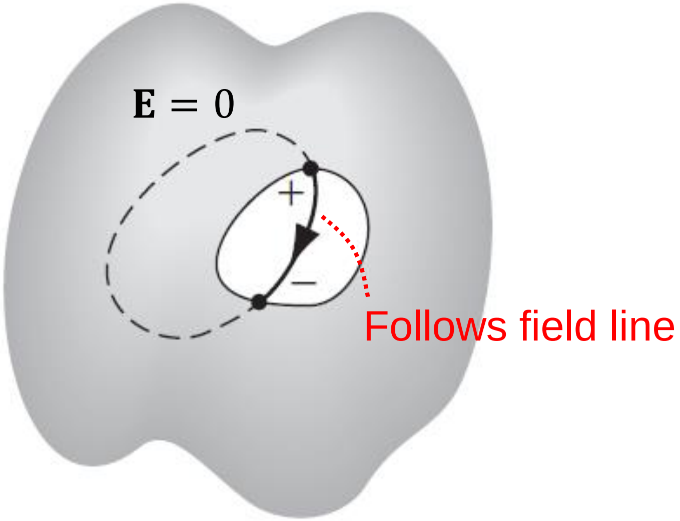
- Induced charges
  - Faraday cage
    - If no charge is placed in the cavity of a hollow conductor, $\mathbf{E} = 0$ in the cavity regardless of the outside conditions

      If not, can construct a loop of integration, whose trajectory in the cavity follows the field line

      ⇨ $\oint \mathbf{E} \cdot d\mathrm{l} \neq 0$

      ⇨ Contradicts $\nabla \times \mathbf{E} = 0$

      ⇨ $\mathbf{E} = 0$

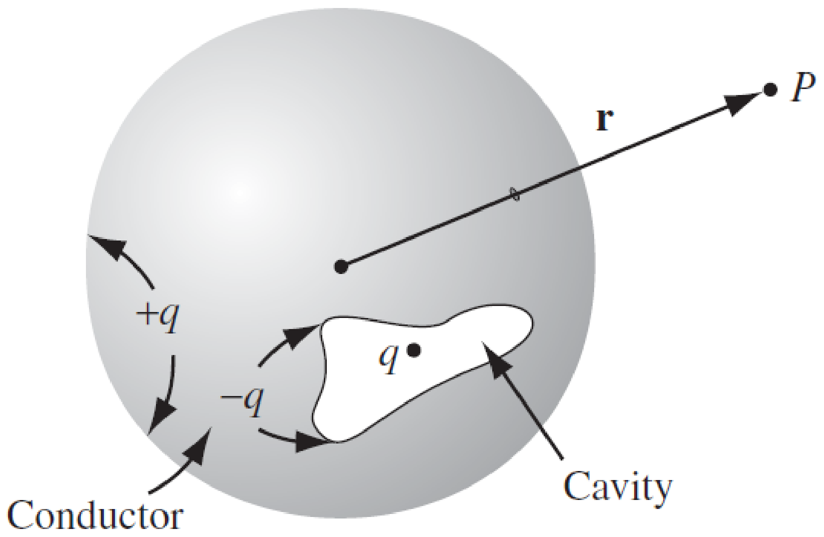


    - Protects sensitive apparatus inside the cavity by shielding out external electric fields

Conductor electrostatics

- Induced charges

Example 2.10. An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it (Fig. 2.46). Somewhere within the cavity is a charge q. *Question:* What is the field outside the sphere?



Conductor electrostatics

- Surface charge and force on a conductor
  - Boundary conditions

$$\begin{cases} \mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \dfrac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \\ \dfrac{\partial V_{\text{above}}}{\partial n} - \dfrac{\partial V_{\text{below}}}{\partial n} = -\dfrac{1}{\epsilon_0}\sigma \end{cases} \xrightarrow{\text{On the surface of a perfect conductor}} \begin{cases} \mathbf{E} = \dfrac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \\ \sigma = -\epsilon_0 \dfrac{\partial V}{\partial n} \end{cases}$$

  - Force (per unit area) exerted on the conductor
    - Can prove (textbook p.104) : for any surface across which is discontinuous, force needs to be calculated by

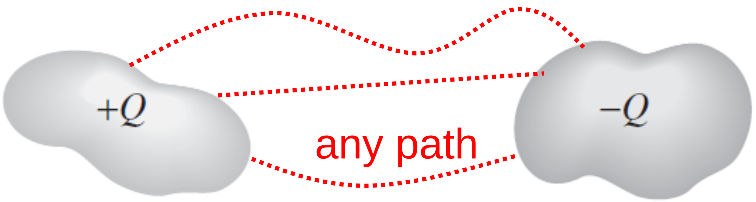
$$\mathbf{f} = \sigma \mathbf{E}_{\text{average}} = \frac{1}{2}\sigma(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}})$$

    - For conductors $\mathbf{f} = \dfrac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}}$

Conductor electrostatics

- Capacitors
  - We can define a potential difference between two conductors, without specifying locations of the integral

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}$$



  - Although $\mathbf{E}$ is geometry dependent, we know $\mathbf{E} \propto Q$, and $V \propto Q$
  - Can define ratio as capacitance $C \equiv \dfrac{Q}{V}$
    - A purely geometrical quantity, determined by shapes, sizes, and separation of the two conductors
    - Unit: farads (F), or Coulomb per volt
    - Always positive