Chapter 2

Falling Apples and Orbiting Planets

> If I have seen further it is by standing on the shoulders of Giants. – Isaac Newton

To study the motion of objects under the influence of forces, Newton developed a new branch of mathematics, now known as Calculus. Calculus is the study of infinitesimal changes of variables, how one change impact another. The technique turns out to be not just useful for studying motion of particles. It can be applied to study all things that change (smoothly).

There used to be a time when all Science students take a course on Calculus. This too have changed.

Learning Objectives

At the end of this lesson, you should be able to understand how dynamical situations can be modelled with differential equations (DEs). You will be able to solve some simple DEs analytically and numerically. The ultimate task is to solve the three-body (gravitational system) problem using Newton's laws.

Learning Flow

Before the lecture, you will read section 2.1 to learn about how initial conditions affects motion of objects under gravitation. Most of the lecture (in week 2) will be spent on differential equations where we will first solve a few cases by hand, (analytically) and then introduce numerical methods (which you may recall learning them in SP2273). In the tutorial (in week 3), we will move on to study the case of a 2-body system and visualise the solution with animations that you make. If we have time left, we will do the discussion questions at the end.

Contents

2.1 The apple, the cannonball and the Moon

An apple falls straight down. The moon orbits Earth in a circle (almost). We know for a "fact" now that both motions are due to (the force of) gravity between Earth and the apple/moon. But this was not an obvious fact before Newton, simply because the motions looks so different.

In the modern day, if you ask a mathematician/physicist why the apple and the moon follow different paths despite both being acted by the same type of force, you will probably get a reply like:

> "Oh yea gravity is Universal! It acts in the same way for both the apple and the Moon. It is just that the **initial conditions** were different, hence different paths exhibited."

The reply probably would not enlighten many people. To visualise and make sense of the above, I'll borrow the story used by Leonhard Euler (1707–1783) in one of his letters to a German princess.

There is a cannon on a cliff, positioned such that it is parallel to the ground. If the cannonball is released without any firepower, the ball will fall straight down the cliff, much like how an apple falls down a tree. If however, the cannon shoots the ball with some firepower, the path will be a curve (projectile), and the ball will fall some distance away from the cliff. The more firepower one uses to shoot the ball, the greater the initial horizontal/tangential[1] speed of the ball, and the further away the ball will land.

Now we know that while our Earth looks flat locally, it is actually round. Suppose we have so much firepower at our expense, we can shoot the ball with so much initial tangential speed such that the ball goes almost round the Earth and fall right *behind* us! And if we were to be so silly to shoot with just a little more tangential speed, the ball will not land at all, but go round the Earth and hit us from behind!!

Perhaps we did shoot the cannonball with such an initial tangential speed, but we pack the cannon and move away while the cannonball circumnavigated the Earth. The ball will go round and round the Earth forever (assuming no air resistance), much like how the Moon orbits the Earth.

Ex. Provide an illustration to the above story.

Ex. What happens if one uses even greater firepower than the amount neccessary for the ball to circumnavigate the Earth?

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[1] Euler explained the concept of tangents in an earlier letter.

2.2 Science is a differential equation. Religion is a boundary condition.

The title of this section is a quote by Alan Turing. Initial conditions discussed earlier is a type of boundary condition. We have seen how important these conditions are in nfluencing outcomes. To fully appreciate Turing's quote, we need to know what are differential equations.

2.2.1 Differential Equations Model Things that Change

Differential equations (DEs) are equations that have terms involving differentiations. DEs are widely used as mathematical statements to describe how quantities change. Such statements are often logical.

Beer foam dynamics

For example, beer foam are air bubbles , and bubbles pop. When there are more bubbles, there will be more popping. When bubbles pop, there will be less bubbles. The rate of change of the number of bubbles in a column of beer foam is proportional to the number of bubbles there are. I hope this sounds logical. If not, get a drink.. We model this scenario wth

$$\frac{dN}{dt} = -kN$$

where N is the number of bubbles at time t, and k is a positive constant.

Particle dynamics

A more relevant example to this chapter is Newton's law of motion. We know that a force is capable of changing an object's velocity v. More force, more change. Newton says that the resultant force applied to a particle is proportional to the rate of change in velocity

$$F = m\frac{dv}{dt}$$

where m is a constant known as (inertial) mass.

Love dynamics

Love is a more complicated affair. Romeo loves Juliet. The more attention he gets from Juliet, the more feelings he have for her. Juliet is different (and Romeo don't get it). The more affection Romeo displays, the more repulsive she finds him. But when Romeo do not give her as much attention, she starts to develop more feelings for him. We can model the lovers' feelings as follows:

$$\begin{aligned}\frac{dR}{dt} &= k_1 J \\ \frac{dJ}{dt} &= -k_2 R\end{aligned}$$

Quantum mechanics

The modern view of how molecules, atoms and subatomic particles behave is that they follow rules of quantum mechanics. The Schrödinger equation (presented in the 1-dimensional, time independent form) below is a differential equation that describe how the wavefunction $\psi(x)$ of a particle changes with position x in different physical conditions $V(x)$. The wavefunction itself relates to the probability of finding the particle. The bigger the wavefunction at a position, the higher the chance of finding the particle around that position.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V(x) - E)\,\psi(x) = 0$$

2.2.2 Solving DEs Analytically

2.2.2.1 Case 1: Beer foam

Let $N(t)$ be the number of bubbles at time t.
Let $N(0) = N_0$.
As introduced above,

$$\frac{dN}{dt} = -kN$$

where k is a positive constant.
Bringing N to the LHS and then integrating both sides with respect to t,

$$\begin{aligned}\int \frac{1}{N} dN &= \int -k dt \\ \ln N &= -kt + c \\ N &= e^{-kt+c} \\ &= e^c e^{-kt}\end{aligned}$$

Using $N(0) = N_0$ to find c,

$$\begin{aligned}N_0 &= e^c e^0 = e^c \\ \Rightarrow N &= N_0 e^{-kt}\end{aligned}$$

Go to Activity 1.

2.2.2.2 Case 2: Constant force on particle

Assume a constant force F acting on a particle with initial velocity $v(0) = v_0$.
We apply Newton's law of mechanics:

$$\begin{aligned}F &= m\frac{dv}{dt} \\ \frac{dv}{dt} &= \frac{F}{m} \\ \int dv &= \int \frac{F}{m} dt \\ v &= \frac{F}{m}t + c\end{aligned}$$

Throw in the initial condition $v(0) = v_0$,

$$\begin{aligned}v_0 &= 0 + c \\ \Rightarrow v &= \frac{F}{m}t + v_0\end{aligned}$$

Let us define acceleration of the particle a as

$$a = \frac{dv}{dt}$$

It follows that $a = \dfrac{dv}{dt} = \dfrac{F}{m}$ is a constant in this case. We can now write

$$v = v_0 + at$$

Let us also define the position of the particle $x(t)$ with

$$v = \frac{dx}{dt}$$

This allows us to solve for x:

$$\frac{dx}{dt} = v_0 + at$$
$$\int dx = \int (v_0 + at)dt$$
$$x = v_0 t + \frac{1}{2}at^2 + c$$

Let $x(0) = 0$. (We can do this cos we can set the origin at anywhere!)

$$0 = 0 + c$$
$$\Rightarrow x = v_0 t + \frac{1}{2}at^2$$

Derived above are the so-called kinematic equations which you may be familiar with.

2.2.2.3 Case 3: Restoring force / Harmonic Oscillator

Think of a spring with one end attached on a wall and the other end to a particle. The longer you pull the , the greater the force in the spring that wants to restore it back to the original state. The scenario is described with a vector equation:

$$F = -kx$$

where F is the spring force and x the displacement from the equilibrium position of the spring. Applying Newton's law,

$$F = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = -kx$$
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

An educated guess of the solution of x is

$$x = ce^{\lambda t}$$

The reason behind this guess is that differentiating an exponential function gives back an exponential function. With this guess, we carry on to differentiate it and fit it back to the DE:

$$\frac{dx}{dt} = c\lambda e^{\lambda t}$$
$$\frac{d^2x}{dt^2} = c\lambda^2 e^{\lambda t}$$
$$c\lambda^2 e^{\lambda t} = -\frac{k}{m}ce^{\lambda t}$$
$$\lambda^2 = -\frac{k}{m}$$
$$\lambda = \pm\sqrt{-\frac{k}{m}}$$
$$= \pm i\sqrt{\frac{k}{m}}$$

Hence the solution is

$$x = c_1 e^{i\sqrt{k/m}t} + c_2 e^{-i\sqrt{k/m}t}$$

Assuming the following **initial conditions**:

$$x(0) = x_0$$
$$v(0) = 0$$

This is the scenario where you pull the spring by a certain extension (x_0), and then let go.

$$\begin{aligned}
v(t) &= ic_1\sqrt{\frac{k}{m}}e^{i\sqrt{k/m}t} - ic_2\sqrt{\frac{k}{m}}e^{-i\sqrt{k/m}t}\\
v(0) = 0 &= ic_1 - ic_2\\
c_1 &= c_2
\end{aligned}$$

$$\begin{aligned}
x(0) = c_1 + c_2 &= x_0\\
c_1 &= \frac{x_0}{2}\\
x &= \frac{x_0}{2}\left(e^{i\sqrt{k/m}t} + e^{-i\sqrt{k/m}t}\right)\\
&= x_0 \cos\left(\sqrt{\frac{k}{m}}t\right)
\end{aligned}$$

2.2.2.4 Case 4: Love (is not that complicated)

Recall Romeo and Juliet

$$\begin{aligned}
\frac{dR}{dt} &= k_1 J\\
\frac{dJ}{dt} &= -k_2 R
\end{aligned}$$

where k_1 and k_2 are (positive) constants.

Since both R and J are functions of time, the above is a set of coupled DEs. One cannot solve for R without knowing J, and vice versa. One way to decouple them is to take the time derivative on one of them, say R:

$$\begin{aligned}
\frac{d}{dt}\frac{dR}{dt} &= \frac{d}{dt}k_1 J\\
\frac{d^2R}{dt^2} &= k_1\frac{dJ}{dt}\\
\frac{d^2R}{dt^2} &= -k_1k_2R
\end{aligned}$$

If we look at the above equation carefully we will notice that it is the DE for a restoring force / harmonic oscillator ($x \to R, m \to 1, k \to k_1k_2$)!

Assume the initial conditions

$$\begin{aligned}
R(0) &= R_0\\
J(0) &= 0,
\end{aligned}$$

(This is equivalent to $x(0) = x_0, v(0) = 0$ in the harmonic oscillator case.)

The solutions are

$$\begin{aligned}
R &= R_0\cos(\sqrt{k_1k_2}t)\\
J &= \frac{1}{k_1}\frac{dR}{dt} = -\frac{1}{k_1}R_0\sqrt{k_1k_2}\sin(\sqrt{k_1k_2}t) = -\sqrt{\frac{k_2}{k_1}}R_0\sin(\sqrt{k_1k_2}t)
\end{aligned}$$

Go to Activity 2.

2.2.3 Solving DEs Numerically

We will use the simple Euler's method to numerically solve DEs. Simply approximate $\frac{df}{dt}$ with $\frac{\Delta f}{\Delta t}$ and choose an appropriately small Δt.

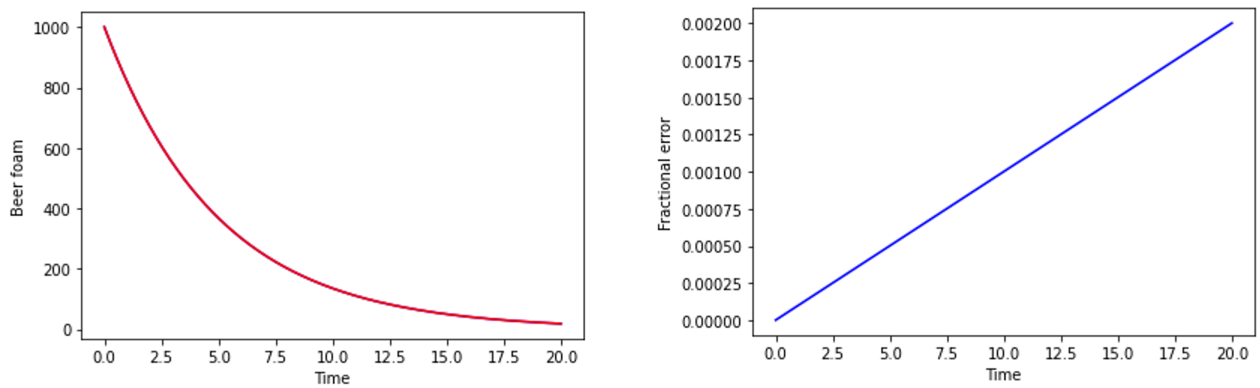


Figure 2.1: Left: Numerical solution of $N(t)$ using Euler's method. Right: Fractional error against time. We can see that the fractional error accumulates with time.

2.2.3.1 Case 1: Beer foam

Let

$$\begin{aligned}\frac{\Delta N}{\Delta t} &= \frac{dN}{dt} = -kN \\ N(0) &= 1000 = N_0 \\ k &= 0.2 \\ \Delta t &= 0.01\end{aligned}$$

We know N_0 at first. The numerical schemes finds N_1 using N_0, then finds N_2 using N_1 and so on.N

$$\begin{aligned}N_{i+1} &= N_i + \Delta N \\ &= N_i + \frac{\Delta N}{\Delta t}\Delta t \\ &= N_i - kN_i\Delta t\end{aligned}$$

We can implement this on in a computer (Python) and plot the solution.

Ex. Do the above!

Any numerical solution will have an error. As the analytical solution is known, we can track the error with

$$\text{fractional error} = \frac{\text{numerical solution} - \text{analytical solution}}{\text{analytical solution}}$$

2.2.3.2 Case 2: Love (is in the numbers)

$$\begin{aligned}\frac{\Delta R}{\Delta t} &\approx \frac{dR}{dt} = k_1 J \\ \frac{\Delta J}{\Delta t} &\approx \frac{dJ}{dt} = -k_2 R \\ R(0) &= R_0 = 1 \\ J(0) &= J_0 = 0\end{aligned}$$

Euler method:

$$R_{i+1} = R_i + \Delta R = R_i + k_1 J_i \Delta t$$
$$J_{i+1} = J_i + \Delta J = J_i - k_2 R_i \Delta t$$

We can implement this on a computer as was done previously for the case of beer foam.

Ex. Do the above for $k_1 = 1$, $k_2 = 0.8$, $\Delta t = 0.001$.

Plot $R(t)$ and $J(t)$ on the same graph.

Plot J vs R.

Introducing a modified Euler method known as the Euler-Cromer method[2]:

$$R_{i+1} = R_i + \Delta R = R_i + k_1 J_i \Delta t$$
$$J_{i+1} = J_i + \Delta J = J_i - k_2 R_{i+1} \Delta t$$

Note the difference with the original Euler method?

Go to Activity 3.

2.3 Two-Body Gravitating System

Let us now apply what we have learnt in an application of differential equation that is perhaps the most famous work of Newton: How does a planet orbit he Sun. We will use a familiar 2-body system, the Sun and the Earth to illustrate.

First fill in the following:

Radius of Sun $R =$

Average distance from Sun to Earth $R_e =$

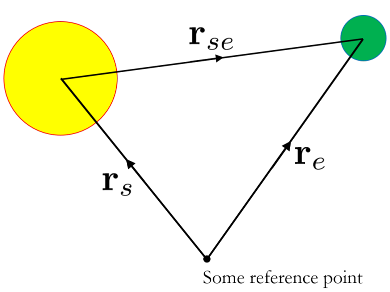
Mass of Sun $M_s =$

Mass of Earth $M_e =$

Gravitational constant $G =$

Linear momentum of Earth $p_e =$

(We can get p_e by knowing that the Earth circles around the Sun in about 365 days. How?)



Let $\mathbf{r}_s$ and $\mathbf{r}_e$ be the position vectors of the Sun and Earth respectively. Express $\mathbf{r}_{se}$ in terms of the two postition vectors.

$$\mathbf{r}_{se} =$$

Newton's Law of Gravitation:

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

[2]A. Cromer, Stable solutions using the Euler Approximation, American Journal of Physics, 49, 455 (1981)

Force that the Sun exerts on the Earth:

$$\mathbf{F}_{se} =$$

Force that the Earth exerts on the Sun:

$$\mathbf{F}_{es} =$$

Newton's second law (express in terms of rate of change of momentum)

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Applying Euler's scheme for solving differential equation numerically

$$\begin{aligned} d\mathbf{p} &= \mathbf{F}dt \\ \mathbf{p}_{i+1} &= \mathbf{p}_i + d\mathbf{p} \\ &= \mathbf{p}_i + \mathbf{F}_i dt \end{aligned}$$

Relationship between momentum and position vector is given by $\mathbf{p} = m\frac{d\mathbf{r}}{dt}$
Applying Euler's scheme for solving differential equation numerically

$$\begin{aligned} d\mathbf{r} &= \\ \mathbf{r}_{i+1} &= \mathbf{r}_i + d\mathbf{r} \\ &= \end{aligned}$$

Go to Activity 4.

2.4 In-class Activities

2.4.1 Activity 1: How to win an Ig Nobel Prize

Observe the decay of beer foam and think of science..

2.4.2 Activity 2: Love plot

For some chosen values of R_0, k_1 and k_2, plot $R(t)$ and $J(t)$ on the same graph on Desmos.

$$\begin{aligned} R &= R_0 \cos(\sqrt{k_1 k_2} t) \\ J &= -\sqrt{\frac{k_2}{k_1}} R_0 \sin(\sqrt{k_1 k_2} t) \end{aligned}$$

Open a new Desmos window and plot R vs J.
Can you form a conserved quantity using R and J? (A quantity that do not change with time t.)

2.4.3 Activity 3: Numerical solution for love

Solve the case of love numerically with

1. Euler Method
2. Euler-Cromer Method

Compare the numerical solutions with the analytical solution.
Also use the conserved quantity to compare the two numerical methods.

2.4.4 Activity 4: VPython code for the Earth-Sun system

Take a close look at the code below. In you group, discuss and make sense of the code below.

```
GlowScript 2.2 VPython
#constants
R=15e9
Re=150e9
Ms=2e30
Me=6e24
G=6.67e-11

#creating the Sun, set the initial position
sun=sphere(pos=vector(0,0,0), radius=R, color=color.yellow)
sun.m=Ms
#sun.p=vector(0,0,0)*sun.m

#creating Earth, set the initial position and momentum
earth=sphere(pos=vector(Re,0,0), radius=0.4*R, color=color.green)
earth.m=Me
earth.p=vector(0,30e3,0)*earth.m

#Some physics
#here I set the momentum of sun so that the total momentum is zero
sun.p=-(earth.p)

#aesthetics
attach_trail(sun)
attach_trail(earth)

#initial time and time step
t=0
dt=50

#now the serious coding
while t<15000000000:
    rate(10**5)

    #vector from sun to earth
    rse=earth.pos-sun.pos

    #Newton’s law of gravitation
    #Fse is the force the Sun exerts on the Earth
    Fse=-G*sun.m*earth.m*norm(rse)/mag(rse)**2

    #Fes is the force the Earth exerts on the Sun. By Newton’s third law,
    Fes=-Fse

    #update momentum (with total vector force)
    #Newton’s second law
    earth.p=earth.p+(Fse)*dt
    sun.p=sun.p+(Fes)*dt

    #update position
    #relation between momentum and position vectors
    sun.pos=sun.pos+sun.p*dt/sun.m
    earth.pos=earth.pos+earth.p*dt/earth.m
    t=t+dt
```

Run the code in https://trinket.io/glowscript/7bdf9fcfab

In your program, change parameters one at a time and observe the difference in results. For example change the initial Earth momentum to

```
earth.p=vector(0,38e3,0)*earth.m
```

Play around and record any notable “discoveries”.

Get an account in https://trinket.io Sign in, create a new trinket program (select Glowscript) and copy and paste the the Sun-Earth code. Now the program is yours!

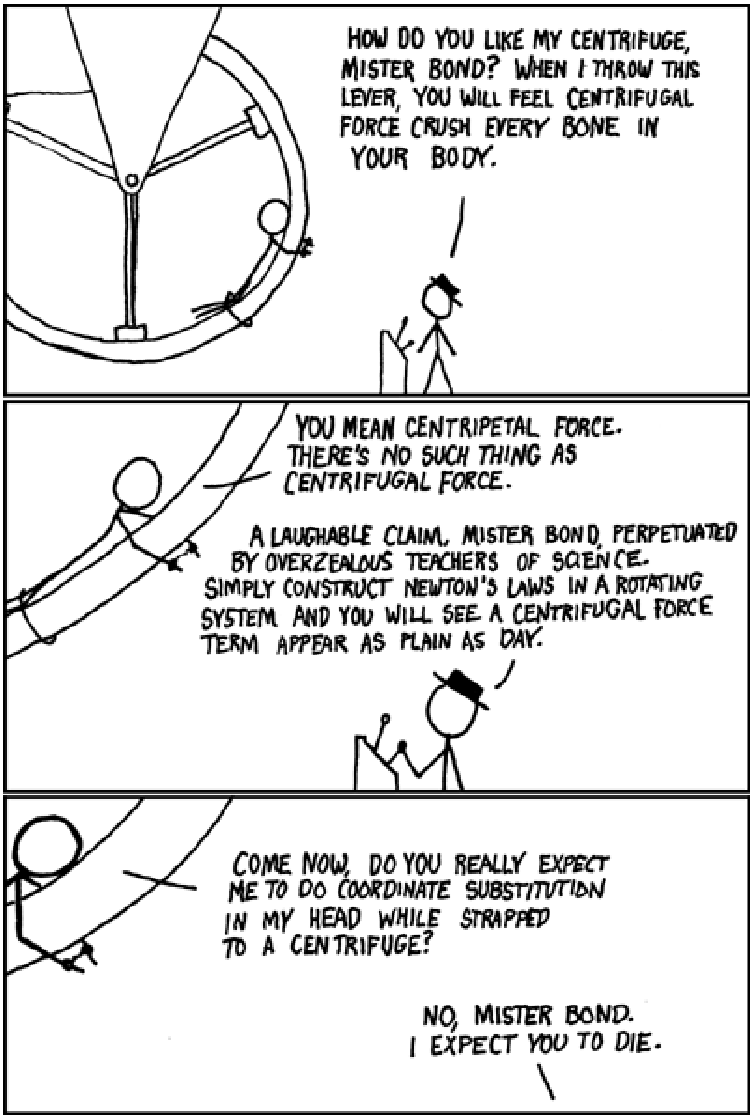
2.5 Discussion Questions

1. Give some more examples of how differential equations are used to model stuffs in science.

2. What is the benefit of solving the planetary motion problem numerically? What is the benefit of solving the planetary motion problem analytically (see for example https://brilliant.org/wiki/deriving-keplers-laws/) ?

3. We saw a conserved quantity for the case of love. For planetary orbits, we saw that Kepler's second law is equivalent to the conservation of angular momentum. Other than angular momentum, is there another conserved quantity for planetary orbit?

4. Study the comic below. Do you understand it?



5. The above is rather sadistic. A nicer application is the centrifuge machine. Explain how the centrifuge is able to separate particles of different sizes and masses.