PC3130 W6L2 AY24

Thursday, 19 September 2024 11:51 am

$\vec{J} = \vec{J}_1 + \vec{J}_2$

acts on $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2$ "acts on $\mathcal{V}_1$" "acts on $\mathcal{V}_2$"

z-component

$\Rightarrow J_z = J_{1z} + J_{2z}$

$= \underbrace{(\vec{J}_1 \otimes \mathbb{1}_2)}_{\text{acts on } \mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2} + \underbrace{(\mathbb{1}_1 \otimes \vec{J}_2)}_{\text{acts on } \mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2}$

$\hat{\vec{J}} = \hat{J}_x \vec{e}_x + \hat{J}_y \vec{e}_y + \hat{J}_z \vec{e}_z$

$\vec{e}_x = \begin{pmatrix}1\\0\\0\end{pmatrix}$

operators acting on $\mathcal{V}$

— can be represented by matrices with dimension $n \times n$, where $\mathcal{V}$ has dimension n.

$n = 2j+1$.

J_{1x}, J_{1y}, J_{1z} matrix size $\overset{\text{generally}}{\neq}$ J_{2x}, J_{2y}, J_{2z}

$\Rightarrow$ need tensor product space.

Describe states in this tensor product space.

uncoupled representation

$|j_1, m_1\rangle \otimes |j_2, m_2\rangle$

$\equiv |j_1, m_1, j_2, m_2\rangle$

Eigenstates of

$\vec{J}_1^{\,2}, J_{1z}, \vec{J}_2^{\,2}, J_{2z}$

coupled representation

$|j, m, j_1, j_2\rangle$

Eigenstates of

$\vec{J}^{\,2}, J_z, \vec{J}_1^{\,2}, \vec{J}_2^{\,2}, \quad \vec{J} = \vec{J}_1 + \vec{J}_2$.

W6L1.

Eg 1 $\quad H = -g\mu_B S_{1z} B_z - g\mu_B S_{2z} B_z$.

Eigenstates of H are stationary states.

H commutes with $S_{1z}, S_1^2, S_{2z}, S_2^2$.

$\Rightarrow$ Eigenstates of H are $|s_1, m_1, s_2, m_2, \lambda\rangle$ (uncoupled representation)

λ: energy.

Eg 2 $\quad H = \alpha \vec{S}_1 . \vec{S}_2 . = \alpha (S_{1x} S_{2x} + S_{1y} S_{2y} + S_{1z} S_{2z})$

We showed $[H, S_{1z}] = -\alpha i\hbar S_{1y} S_{2x} + \alpha i\hbar S_{1x} S_{2y} \neq 0$ — (1)

$\vec{S}_1 . \vec{S}_2 = \frac{1}{2}(\vec{S}^2 - \vec{S}_1^{\,2} - \vec{S}_2^{\,2})$

$[\vec{S}_1 . \vec{S}_2, S^2] = 0 \quad [\vec{S}_1 . \vec{S}_2, S_1^2] = 0 \quad [\vec{S}_1 . \vec{S}_2 , S_2^2] = 0$.

$$\left\{\begin{array}{l}\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}(\vec{S}^2 - S_1^2 - S_2^2) \\ [\vec{S}_1\cdot\vec{S}_2, S^2] = 0, \quad [\vec{S}_1\cdot\vec{S}_2, S_1^2] = 0 \quad , [\vec{S}_1\cdot\vec{S}_2, S_2^2] = 0.\end{array}\right.$$

$[H, S_z] = ?$

$S_z = S_{1z} + S_{2z}$

$[H, S_{2z}] = -\alpha i\hbar S_{2y} S_{1x} + \alpha i\hbar S_{2x} S_{1y}$ — (2)

from (1) & (2), $[H, S_z] = 0$

$\Rightarrow$ Energy eigenstates of $H = \alpha\vec{S}_1\cdot\vec{S}_2$ are $|S, m, s_1, s_2, \underset{\text{energy}}{\lambda}\rangle$ (coupled representation).

Back to discussion on what operators commute with $\vec{J}^2$ and J_z?

$\vec{J}_1^2$ and $\vec{J}_2$ ✓

What about J_{1z} and J_{2z}?

$$[J^2, J_{1z}] = \underbrace{[J_1^2, J_{1z}]}_{\substack{0 \\ \text{by definition}}} + \underbrace{[J_2^2, J_{1z}]}_{\substack{0 \\ \text{different spaces}}} + 2\underbrace{[\vec{J}_1\cdot\vec{J}_2, J_{1z}]}_{\substack{\neq 0 \\ \text{from Eq. 2}}} \neq 0$$

Note: $[J_z, J_{1z}] = [J_{1z} + J_{2z}, J_{1z}] = [J_{1z}, J_{1z}] + [J_{2z}, J_{1z}] = 0.$

Poll question

If $\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3$ which of the following are a set of mutually commuting observables?

(1) $\vec{J}^2, J_z, \vec{J}_1^2, J_{1z}, \vec{J}_2^2, J_{2z}$ (X) $[\vec{J}^2, J_{1z}] \neq 0$

(2) $\vec{J}_1^2, J_{1z}, \vec{J}_2^2, J_{2z}, \vec{J}_3^2, J_{3z}$ (✓) (uncoupled representation)

(3) $\vec{J}^2, J_z, \vec{J}_1^2, \vec{J}_2^2, \vec{J}_3^2$ (✓) ("coupled represention" — missing $(\vec{J}_1+\vec{J}_2)^2$)

(4) $J_z, J_{1z}, J_{2z}, J_{3z}$ (✓) $J_z = J_{1z} + J_{2z} + J_{3z}$

$V_1 \otimes V_2 \otimes V_3 \rightarrow$ 6 quantum numbers.

$\tilde{J} = J_1 + J_2$ $\qquad J = \tilde{J} + J_3$

$j, m, \{ \tilde{j}, \tilde{m}, j_1, j_2 \}$

$J = J_1 + J_2$ $\qquad\qquad\qquad\qquad J = J + J_3$

$j_1 \; m_1$
$j_2 \; m_2$ $\Big\}$ $\tilde{j}, \tilde{m}, j_1, j_2$ (coupled rep.)
$j_3 \; m_3$

$\Big\}$ $\quad j, m \;\; \tilde{j} \;\; j_3 \;\; j_1 \;\; j_2$

$[\tilde{J}_z, J^2] \neq 0$

Eg $\quad |j=1, m=0, \underbrace{j_1 = \tfrac{1}{2}, j_2 = \tfrac{1}{2}}\rangle$

tells us that the $j=1$ system came from two $j=\frac{1}{2}$ systems.

Q) Possible values of j and m? $\quad \vec{J} = \vec{J}_1 + \vec{J}_2$.

$$\vec{J}_1^2 \, |j_1, m_1\rangle = \hbar^2 j_1 (j_1+1) \, |j_1, m_1\rangle$$

$$J_{1z} \, |j_1, m_1\rangle = \hbar m_1 \, |j_1, m_1\rangle$$

$$J_2^2 \, |j_2, m_2\rangle = \hbar^2 j_2 (j_2+1) \, |j_2, m_2\rangle$$

$$J_{2z} \, |j_2, m_2\rangle = \hbar m_2 \, |j_2, m_2\rangle$$

Let's work on m first.

$J_z = J_{1z} + J_{2z}$. $\quad$ (recall J_z, J_{1z}, J_{2z} are mutually commuting)

$$\begin{aligned}
J_z \, |j_1, m_1, j_2, m_2\rangle &= (J_{1z} + J_{2z}) \, |j_1 m_1 \, j_2 m_2\rangle \\
&= ((J_{1z} \otimes \mathbb{1}_2) + (\mathbb{1}_1 \otimes J_{2z})) \, |j_1 m_1\rangle \otimes |j_2, m_2\rangle \\
&= \hbar m_1 \, |j_1 m_1\rangle \otimes |j_2 m_2\rangle \\
&\quad + |j_1 m_1\rangle \otimes \hbar m_2 \, |j_2, m_2\rangle \\
&= \hbar (m_1 + m_2) \, |j_1, m_1, j_2, m_2\rangle
\end{aligned}$$

$\Rightarrow \quad |j_1, m_1, j_2, m_2\rangle$ is an eigenstate of J_z with eigenvalue

$\hbar (m_1 + m_2) \equiv \hbar m, \quad m = m_1 + m_2$.

Recall: $[J^2, J_z] = 0$

$\Rightarrow \exists$ a set of common eigenstates.

If there are no degeneracies, then the eigenstate must be common to J^2 and J_z.

But If there are degeneracies, this may not be the case.

$|j_1, m_1, j_2, m_2\rangle$ is not in general an eigenstate of J^2

since $[J^2, J_{1z}] \neq 0$

Reason: $\quad J_z \, |j_1, m_1, j_2, m_2\rangle = \hbar m \, |j_1, m_1, j_2, m_2\rangle \quad -(1)$

For given eigenvalue $\hbar m$, there can in general be more than one $|j_1, m_1, j_2, m_2\rangle$

that satisfies (1).

Possible m: $m = m_1 + m_2$

Possible j?

We consider an example.

Two p electrons $\quad V_1: \ l_1 = 1, \quad V_2: \ l_2 = 1$

$m_1 \in \{1, 0, -1\} \qquad m_2 \in \{1, 0, -1\}$.

$\vec{L} = \vec{L}_1 + \vec{L}_2$.

Dimension of $V = V_1 \otimes V_2 = 3 \times 3 = 9$

We know $\boxed{m = m_1 + m_2}$ $\qquad m_1 \in \{1, 0, -1\}, \quad m_2 \in \{1, 0, -1\}$

$m = 2$ $\quad |l_1=1, m_1=1, l_2=1, m_2=1\rangle \rightarrow |l=2, m=2, l_1=1, l_2=1\rangle$

no others

Can we have $l \geq 3$? No, because there is no $m = 3$ state.

$m = 1$ $\quad |l_1=1, m_1=1, l_2=1, m_2=0\rangle \quad |l_1=1, m_1=0, l_2=1, m_2=1\rangle \rightarrow |l=2, m=1, l_1=1, l_2=1\rangle$, $|l=1, m=1, l_1=1, l_2=1\rangle$

$m = 0$ $\quad |l_1=1, m_1=1, l_2=1, m_2=-1\rangle \quad |l_1=1, m_1=-1, l_2=1, m_2=1\rangle \quad |l_1=1, m_1=0, l_2=1, m_2=0\rangle$

$\hookrightarrow |l=2, m=0, l_1=1, l_2=1\rangle, \quad |l=1, m=0, l_1=1, l_2=1\rangle, \quad |l=0, m=0, l_1=1, l_2=1\rangle$

$m = -1$ $\quad |l_1=1, m_1=0, l_2=1, m_2=-1\rangle \quad |l_1=1, m_1=-1, l_2=1, m_2=0\rangle \rightarrow |l=2, m=-1, l_1=1, l_2=1\rangle$, $|l=1, m=-1, l_1=1, l_2=1\rangle$

$m = -2$ $\quad |l_1=1, m_1=-1, l_2=1, m_2=-1\rangle \rightarrow |l=2, m=-2, l_1=1, l_2=1\rangle$

no others

$$\underbrace{\underset{\text{space}}{l_1 = 1} \otimes \underset{\text{space}}{l_2 = 1}}_{\text{dimension } 3\times3 = 9} = \underbrace{\underset{\text{space}}{l = 2}}_{\substack{\text{dimension} \\ 2\times2+1 = 5}} \oplus \underbrace{\underset{\text{space}}{l = 1}}_{\substack{\text{dimension} \\ 3}} \oplus \underbrace{\underset{\text{space}}{l = 0}}_{\substack{\text{dimension} \\ 1}}$$

$l_1 = 3, \ l_2 = 3$

$9 = (2l_1+1)(2l_2+1) = (2\times2+1) + (2\times1+1) + (2\times0+1)$

"Triangularisation rule"

$\xrightarrow{j_1}\xrightarrow{\vec{j}_2}$

$\xrightarrow{\hspace{3em}} j_1 + j_2$

max possible j

$\xrightarrow{j_1}$, $\xleftarrow{j_2}$

$\longrightarrow |j_1 - j_2|$

min possible j.

Generally, if $\vec{J} = \vec{J}_1 + \vec{J}_2$,

we have $\boxed{m = m_1 + m_2}$

$m^{max} = m_1^{max} + m_2^{max} = j_1 + j_2$

$\Rightarrow j^{max} = m^{max} = j_1 + j_2$.

$$m = m_1 + m_2 \qquad - j_1, j_2$$

$$\Rightarrow \quad j^{max} = m^{max} = j_1 + j_2.$$

j^{min} ?

$$\sum_{j=j^{min}}^{j=j^{max}} (2j+1) = (2j_1+1)(2j_2+1)$$

$\therefore$ Can show that this implies $j^{min} = |j_1 - j_2|$.

$$\boxed{\text{Possible values of } j \text{ are } \quad j = |j_1-j_2|,\ (|j_1-j_2|+1),\ \dots,\ j_1+j_2.}$$

See slide on Clebsch-Gordan coeff.

$$|j, m, j_1, j_2\rangle = \underbrace{\sum_{m_1, m_2} |j_1 m_1, j_2 m_2\rangle\langle j_1 m_1, j_2 m_2|}_{\mathbb{1}} j, m, j_1, j_2\rangle$$

complete basis $\sum |n\rangle\langle n| = \mathbb{1}$

$$= \sum_{\substack{m_1, m_2 \\ m = m_1 + m_2}} |j_1 m_1, j_2 m_2\rangle \underbrace{\langle j_1 m_1, j_2 m_2 | j m j_1 j_2\rangle}_{C^{j m j_1 j_2}_{m_1, m_2}}$$

Clebsch-Gordan coeff.

Work out by hand.

Eg. Addition of two spin-$\frac{1}{2}$ particles

$s_1 = \frac{1}{2}$ $\qquad$ $s_2 = \frac{1}{2}$

$m_1 = -\frac{1}{2}, \frac{1}{2}$ $\qquad$ $m_2 = -\frac{1}{2}, \frac{1}{2}$

dimension of

$$V = V_1 \otimes V_2$$
$$= 2 \times 2$$
$$= 4$$

Possible values of s are

$$s = \underset{|s_1 - s_2|}{0}, \quad \underset{s_1 + s_2}{1}$$

check dimensions.

$\boxed{\text{singlet.}}$ $s = 0$ $\qquad$ dim $= 1$

$\boxed{\text{triplet}}$ $s = 1$ $\qquad$ dim $= 2 \times 1 + 1 = 3$

$1 + 3 = 4$ ✓

$$m = m_1 + m_2$$

dropping $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}$

$m = 1$ $\quad |s_1 = \frac{1}{2}, m_1 = \frac{1}{2}, s_2 = \frac{1}{2}, m_2 = \frac{1}{2}\rangle = |s = 1, m = 1\rangle$

$|s = 1, m = 0\rangle$

$m = 0$ $\quad |s_1 = \frac{1}{2}, m_1 = \frac{1}{2}, s_2 = \frac{1}{2}, m_2 = -\frac{1}{2}\rangle$ $\quad |s_1 = \frac{1}{2}, m_1 = -\frac{1}{2}, s_2 = \frac{1}{2}, m_2 = \frac{1}{2}\rangle$

$|s = 0, m = 0\rangle$

$m=-1 \quad |s_1=\tfrac{1}{2}, m_1=-\tfrac{1}{2}, s_2=\tfrac{1}{2}, m_2=-\tfrac{1}{2}\rangle = |s=1, m=-1\rangle$ — (*)

Apply $S_+ = S_{1+} + S_{2+}$ on the $m=-1$ state.

$$J_\pm |j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\; |j, m\pm1\rangle$$

Using RHS of (*) (coupled representation):

$$S_+ |s=1, m=-1\rangle = \hbar\sqrt{1(2)-(-1)(0)}\; |s=1, m=0\rangle$$

↑ use $s=1, m=-1$

$$= \hbar\sqrt{2}\; |s=1, m=0\rangle. \quad —(1)$$

Using LHS of (*) (uncoupled representation).

$$(S_{1+} + S_{2+})|s_1=\tfrac{1}{2}, m_1=-\tfrac{1}{2}, s_2=\tfrac{1}{2}, m_2=-\tfrac{1}{2}\rangle$$

$$= \left((S_{1+}\otimes \mathbb{1}_2) + (\mathbb{1}_1 \otimes S_{2+})\right)\left(|s_1=\tfrac{1}{2}, m_1=-\tfrac{1}{2}\rangle \otimes |s_2=\tfrac{1}{2}, m_2=-\tfrac{1}{2}\rangle\right)$$

$$= \hbar\sqrt{\tfrac{1}{2}\left(\tfrac{3}{2}\right) - \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)}\left(|s_1=\tfrac{1}{2}, m_1=\tfrac{1}{2}\rangle \otimes |s_2=\tfrac{1}{2}, m_2=-\tfrac{1}{2}\rangle\right)$$

$$+ \hbar\sqrt{\tfrac{1}{2}\left(\tfrac{3}{2}\right) - \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)}\left(|s_1=\tfrac{1}{2}, m_1=-\tfrac{1}{2}\rangle \otimes |s_2=\tfrac{1}{2}, m_2=\tfrac{1}{2}\rangle\right)$$

$$= \hbar\left(|s_1=\tfrac{1}{2}, m_1=\tfrac{1}{2}, s_2=\tfrac{1}{2}, m_2=-\tfrac{1}{2}\rangle + |s_1=\tfrac{1}{2}, m_1=-\tfrac{1}{2}, s_2=\tfrac{1}{2}, m_2=\tfrac{1}{2}\rangle\right). \quad —(2)$$

(1) & (2) $\Rightarrow$ Drop s_1, s_2

$$|s=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|m_1=\tfrac{1}{2}, m_2=-\tfrac{1}{2}\rangle + |m_1=-\tfrac{1}{2}, m_2=\tfrac{1}{2}\rangle\right)$$

($\uparrow$: $m=\frac{1}{2}$

$\downarrow$: $m=-\frac{1}{2}$)

	Coupled rep		Uncoupled rep
Triplet	$s=1$	$m=1$	$\lvert\uparrow\uparrow\rangle$
	$s=1$	$m=0$	$\frac{1}{\sqrt{2}}\left(\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle\right)$
	$s=1$	$m=-1$	$\lvert\downarrow\downarrow\rangle$

After reading week

— Quiz 3 on angular momentum.