$\underset{\sim}{p} = \frac{\hbar}{i}\nabla$ $\quad \partial_\mu A^\mu = 0 \longrightarrow \partial_\mu \partial^\mu \lambda(\underset{\sim}{x}) = 0$ (4)

$p_\mu = i\hbar\partial_\mu \longrightarrow p \cdot A = 0$

still not sufficient to specify $A^\mu(\underset{\sim}{x})$ uniquely.

Next use coulomb gauge condition to make

$$A^0 = 0$$

$\underset{\sim}{p} \cdot \underset{\sim}{A} = 0 \quad \nabla \cdot \underset{\sim}{A} = 0$ coulomb gauge

With the Lorentz condition and coulomb gauge, $\underset{\sim}{A}$ has 2 independent components

The <u>free</u> photon equation

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0$$

Look at $\partial_\mu F^{\mu\nu} = 0$

$\partial_\mu = \frac{\partial}{\partial x^\mu}$

$\partial^\mu = \frac{\partial}{\partial x_\mu}$

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$$

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

HW

$$\partial^\nu \partial_\mu A^\mu = 0$$

$\because \partial_\mu A^\mu = 0$ (Lorentz condition)

$$\rightarrow \partial_\mu \partial^\mu A^\nu = 0$$

$$\rightarrow \Box^2 A^\nu = 0, \quad \dots (2)$$

$\Box^2 =$ D'Alembertian

$\equiv \partial_\mu \partial^\mu$

$= \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2$

solution is Ansatz

$$A_\mu(x) = \text{Const} \cdot e^{-ip\cdot x/\hbar} \, \varepsilon_\mu(\underset{\sim}{p}) \quad --(3)$$

$\underset{\sim}{\varepsilon}(\underset{\sim}{p}) =$ polarization vector

(5)

$\underline{p} = \hbar \underline{k} \qquad \underset{\sim}{k} = (k^0, \underset{\sim}{k}) \qquad k^0 = \frac{\omega}{c}$

plane wave in S.E. $\psi(\underline{x}) = e^{i(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{x})}$ (schrodinger eqn.)

Expression (3) is a solution of eq (2) iff

$$\underline{p}^2 = 0 \quad \text{i.e.} \quad p^0 = |\underset{\sim}{p}| \qquad (HW)$$

and also (3) must satisfy the Lorentz condition

$$\partial_\mu A^\mu = 0$$

$$\underline{p} \cdot \underline{\varepsilon} = p_\mu \varepsilon^\mu = 0 \qquad (HW)$$

For coulomb gauge

$$\varepsilon^0 = 0 \quad \because \quad A^0 = 0$$

$$\longrightarrow \qquad \underset{\sim}{p} \cdot \underset{\sim}{\varepsilon} = 0 \qquad (\nabla \cdot \underset{\sim}{A} = 0)$$

So the free photon is

$$\longrightarrow \qquad \underline{A}(\underline{x}) = \text{const}\ e^{-i \underline{p} \cdot \underline{x}/\hbar}\ \underline{\varepsilon}$$

and $\quad \underline{p}^2 = 0, \quad \varepsilon^0 = 0, \quad \underset{\sim}{p} \cdot \underset{\sim}{\varepsilon} = 0$

If photon propagates along x^3- direction,

$$\underset{\sim}{p} = (0, 0, p)$$

Then solutions for $\underset{\sim}{p} \cdot \underset{\sim}{\varepsilon} = 0$ are given by

$$\underset{\sim}{\varepsilon}_{(1)} = (1, 0, 0), \qquad \varepsilon_{(2)} = (0, 1, 0)$$

The polarization $\underset{\sim}{\varepsilon}$ is perpendicular to the photon propagation direction. The free photon is transversely polarized.

The two solutions $\underset{\sim}{\varepsilon}_{(1)} = (1, 0, 0)$, $\underset{\sim}{\varepsilon}_{(2)} = (0, 1, 0)$ describe linearly polarized em field (Transverse polarization)

For circular polarization, the polarization vector $\underset{\sim}{\varepsilon}(\underset{\sim}{p})$ can be written as

$$\underset{\sim}{\varepsilon}_{(\pm)} = \mp \frac{\varepsilon_{(1)} \pm i\varepsilon_{(2)}}{\sqrt{2}}$$

$\underset{\sim}{\varepsilon}_{+}$ = R H circularly polarized $= \frac{-1}{\sqrt{2}}(1, i, 0)$

$\underset{\sim}{\varepsilon}_{-}$ = L H circularly polarized $= \frac{1}{\sqrt{2}}(1, -i, 0)$

Thus we have obtained free photon solution

$$A_\mu(\underline{x}) = (\text{Constant}) \cdot e^{-i\underline{p}\cdot\underline{x}/\hbar} \varepsilon_\mu(\underline{p}), \quad \underline{p}^2 = 0$$

In the Coulomb gauge, $\varepsilon_0(\underline{p}) = 0 = A_0(\underline{x})$, $\nabla \cdot \underset{\sim}{A}(\underline{x}) = 0$, the $\underset{\sim}{\varepsilon}(\underline{p})$ is as given above.

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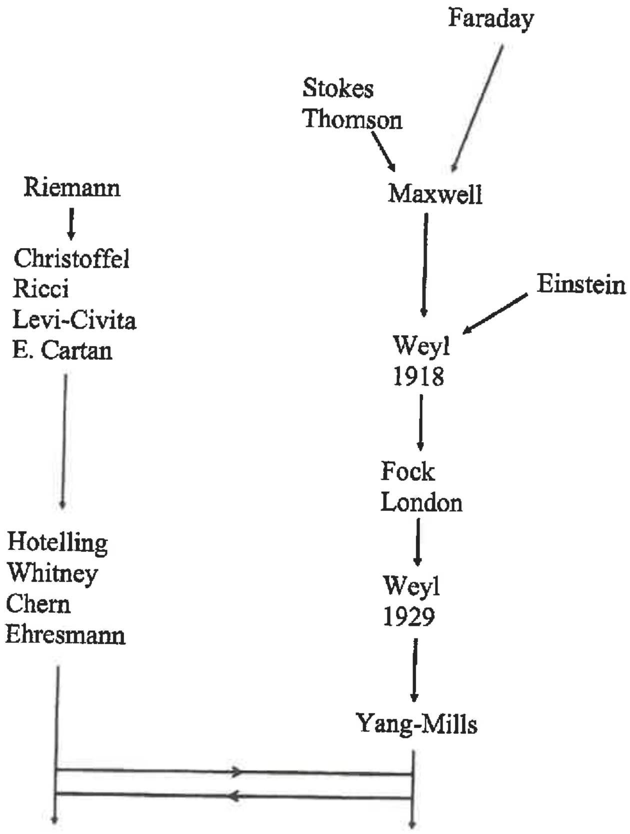

Fig. 1. Flow of ideas in the evolution of the concept of the vector potential.

describable beautifully and precisely by field theories, and that all these theories have mathematical structures required by the concept of symmetry. Hence the principle: *symmetry dictates interaction*. The conceptual history of this remarkable development is the subject of the present paper.

Playing an important part in this history is the vector potential **A**, which first made its appearance in the 19th century. There was certain freedom, now called gauge freedom in its definition, which was early recognized as a simple but somewhat annoying mathematical property. It is this freedom which has now metamorphosed into the key symmetry principle that dictates the exact equations describing the fundamental forces of nature.

Very remarkably, the mathematics of this symmetry principle was in the meantime developed by geometers in the theory of *fiber bundles*, entirely independently of the developments in physics. When this became known, a renewed cross-fertilization of basic ideas between the disciplines of physics and mathematics happily resulted.

Throughout this paper our emphasis is on the early motivation and evolution of the key ideas. There is a vast literature about various aspects of the history we

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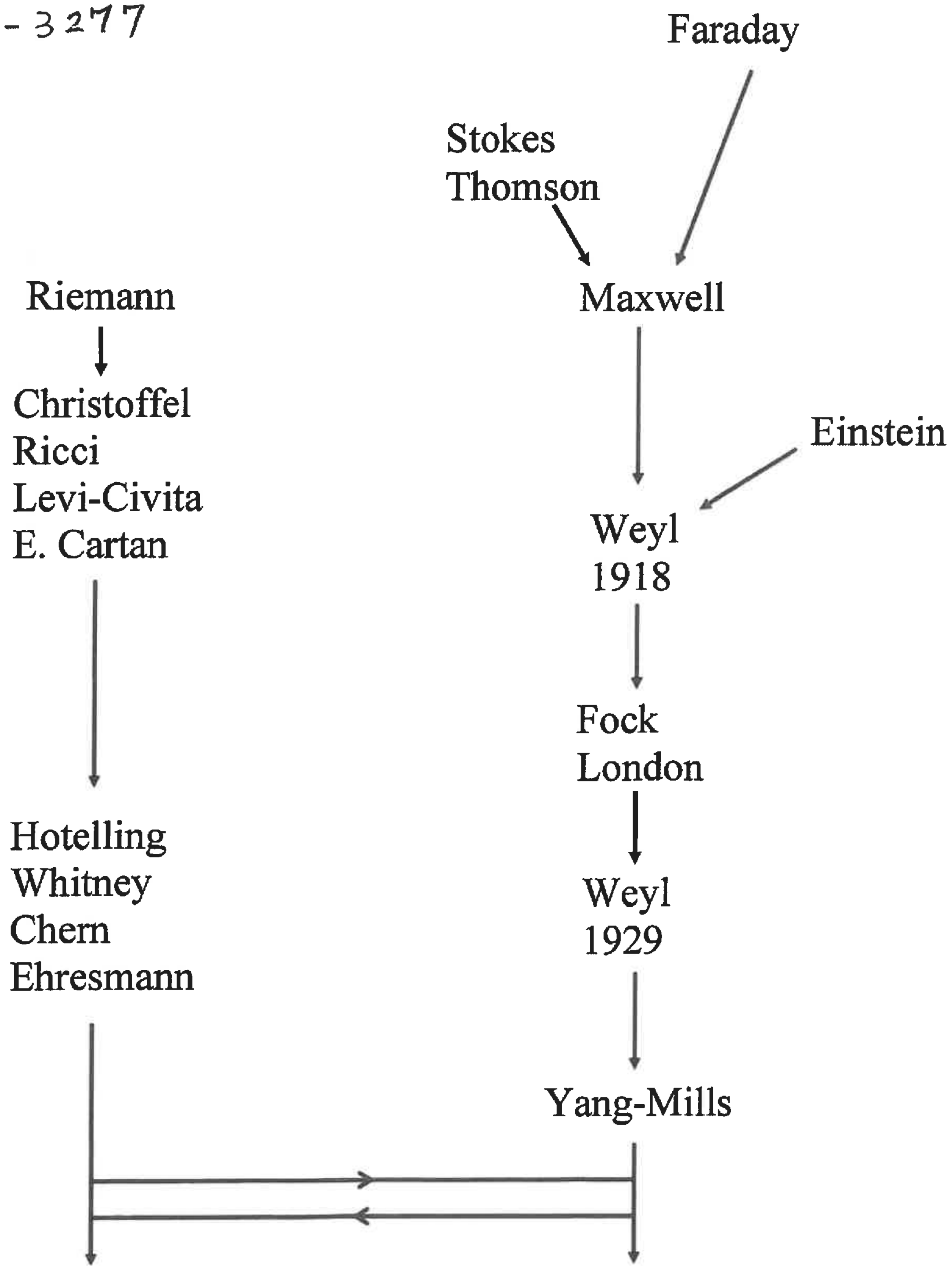

Figure 1

Relativistic equation for electron

The familiar Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(\underset{\sim}{x}) = H\psi(\underset{\sim}{x}) \qquad E = \frac{\underset{\sim}{p}^2}{2m}$$

$$H = \frac{\underset{\sim}{p}^2}{2m} \qquad \text{(free electron kinetic energy)}$$

$$\underset{\sim}{p} = \frac{\hbar}{i}\nabla$$

is relativistically not correct because time is first order derivative whilst space second order derivative, so time and space not treated equally

Two ways to 'derive' relativistically 'correct' equations:

(i) $\frac{\partial^2}{\partial t^2}$, $\frac{\partial^2}{\partial x^{i2}}$ both second order

(ii) $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x^i}$ both first order

First way, change $\frac{\partial}{\partial t}$ in Schrödinger equation to $\frac{\partial^2}{\partial t^2}$.

In Special Relativity, for a free particle,

$$P^2 = m^2 c^2 \qquad \text{i.e.} \quad p^{0\,2} - \underset{\sim}{p}^2 = m^2 c^2$$

The 'correct' equation should be

$$P^2\,\psi(\underset{\sim}{x}) = m^2 c^2\,\psi(\underset{\sim}{x}) \qquad (4)$$

$$P_\mu = i\hbar\partial_\mu = i\hbar\frac{\partial}{\partial x^\mu} \to P_0 = i\hbar\frac{\partial}{\partial x^0} = i\hbar\frac{1}{c}\frac{\partial}{\partial t}, \quad P_i = i\hbar\frac{\partial}{\partial x^i}$$

$$p^i = -P_i = \frac{\hbar}{i}\frac{\partial}{\partial x^i}$$

Then the 'correct' equation eq (4) becomes

$$\left(\Box^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(\underset{\sim}{x}) = 0 \qquad (5) \qquad \text{HW}$$

known as Klein-gordon equation

plane wave solution

$$\psi(\underline{x}) = \text{const}\; e^{-i\, \underline{p}\cdot \underline{x}/\hbar}$$

$$\underline{p}^2 = m^2c^2, \quad p^0 = \pm\sqrt{\underline{p}^2 + m^2c^2}$$

this allows $-$ve energy $p^0 = -\sqrt{\underline{p}^2 + m^2c^2}$

in the plane wave solution. This is the first difficulty about the K. G. eqn

Next is $\psi(\underline{x})$ a wave function (probability amplitude) ?

In S.E., prob. density $= |\psi|^2 = \psi^* \psi = \rho$

probability current density

$$\underline{j} = \frac{1}{2m}\left(\psi^* \underline{p}\, \psi + (\underline{p}\psi)^* \psi\right)$$

$$\underline{p} = \frac{\hbar}{i}\nabla$$

should we do the same for the K. G.- $\psi(\underline{x})$?

If do the same, then $\partial_\mu j^\mu \neq 0$

i.e. prob. is not conserved.

In order to ensure $\partial_\mu j^\mu = 0$ for the K.G.. case,
(conservation of probability)

one puts

$$j^\mu = \frac{1}{2m}\left(\phi^* p^\mu \phi + (p^\mu \phi)^* \phi\right) \quad \text{--(6)}$$

change K.G $\psi(\underline{x})$
to $\phi(\underline{x})$

This definition does lead to

$$\partial_\mu j^\mu = 0 \qquad \text{(HW)}$$

But problem remains because

$$\rho = \frac{j^0}{c} = \frac{1}{2mc}\left(\phi^* p^0 \phi + (p^0\phi)^* \phi\right)$$

$p^0 = i\hbar \frac{\partial}{c\partial t}$

as obtained from j^0 (defined by eq (6))

can be −ve. That means probability density ρ can be −ve, not allowed!

So K. G. equation is wrong if $\phi(\underline{x})$ is a prob. amp. However nowadays we regard K. G equation is relativistically correct for spin 0 particle such as pion, but then here $\phi(x)$ is interpreted as a field operator.

Next comes the Dirac equation.

change S. E. (Schrödinger Equation) $\left(\frac{\partial}{\partial t}, \frac{\partial^2}{\partial x^2}\right)$ to

$\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$ 1st order derivatives.

S. E. $\quad i\hbar\partial_t \psi(\underset{\sim}{x}) = \frac{\underset{\sim}{p}^2}{2m}\psi(\underset{\sim}{x})$

K. G $\quad \underline{p}^2 \phi(\underline{x}) = m^2c^2\phi(\underline{x})$

How to change 2nd order derivative in space $\frac{\partial^2}{\partial x^2}$ to 1st order $\frac{\partial}{\partial x}$?

Take square root of the operator $\underline{p}^2$

$$\underline{p}^2 = -\hbar^2\Box^2 = -\hbar^2\partial_\mu\partial^\mu$$

Let $\psi(\underline{x})$ be multicomponent $\psi_i(\underline{x}),\ i=1,2,\cdots N,$

$$\psi(\underline{x}) \rightarrow \psi(\underline{x}) = \begin{pmatrix}\psi_1(\underline{x})\\ \vdots \\ \psi_N(\underline{x})\end{pmatrix}$$

i.e. $\sqrt{\underline{p}^2}$ must be a matrix

Dirac introduced $\not{p}$

$$\not{p} = p_\mu\gamma^\mu, \qquad \mu=0,1,2,3.$$

and obtained the Dirac equation

$$\not{p}\,\psi(\underline{x}) = mc\,\psi(\underline{x}), \qquad \psi(x) = \begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\end{pmatrix}$$

Then look for plane wave solution and (11)

also construct prob. current density $j_\mu(x)$

s. t. $\partial_\mu j^\mu = 0$

As $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}$, so γ^μ $(\mu = 0, 1, 2, 3)$

must be $N \times N$ matrices, $\mu = 0, 1, 2, 3$

Note: It turns out γ^μ is not a 4-vector

so $\not{p} = p_\mu \gamma^\mu$ is not a scalar although

p_μ is a 4-vector.

$\not{p} = p_\mu \gamma^\mu - \gamma^\mu p_\mu$ is not a scalar wrt

Lorentz transformations

Dirac equation

$$\not{p}\,\psi(\underline{x}) = mc\,\psi(\underline{x})$$

literally taking the square root of the K. G. eqn $\underline{p}^2\phi(\underline{x}) = m^2c^2\phi(\underline{x})$

$$\not{p} = p_\mu \gamma^\mu \qquad \mu = 0, 1, 2, 3$$

$$\psi(\underline{x}) = \begin{pmatrix} \psi_1(\underline{x}) \\ \psi_2(\underline{x}) \\ \vdots \\ \psi_N(\underline{x}) \end{pmatrix}$$

$$\gamma^\mu = N \times N \quad \text{matrix}$$

Today study properties of γ^μ and find plane wave solution of the Dirac eqn.

Properties of γ^μ:

1st the Dirac equation must yield

$$\underline{p}^2 = m^2c^2$$

in order to be consistent with sp. Relativity for a free particle.

For this, we 'square' the Dirac eqn

Apply $\not{p}$ to $\not{p}\,\psi = mc\,\psi$

$$\not{p}^2\psi = \not{p}\,mc\,\psi = mc\,\not{p}\,\psi = m^2c^2\psi$$