$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

$$\sum_{n=1}^{\infty} C_n \underbrace{\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy}_{①} = \int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy \qquad n'\in\{1,2,3\ldots\}$$

$$① = \int_0^a \frac{1}{2}\left\{\cos\left[\frac{(n-n')\pi y}{a}\right] - \underbrace{\cos\left[\frac{(n+n')\pi y}{a}\right]}_{②}\right\}dy$$

$$② = \frac{a}{(n+n')\pi}\sin\left[\frac{(n+n')\pi y}{a}\right]\Bigg|_0^a = 0$$

$$① = \begin{cases} \frac{a}{2} & n=n' \\ 0 & n\neq n' \end{cases} = \frac{a}{2}\delta_{nn'}$$

$$\Rightarrow \sum_{n=1}^{\infty} C_n \frac{a}{2}\delta_{nn'} = \int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy$$

$$\Rightarrow C_{n'}\frac{a}{2} = \int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy \Rightarrow C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$