Chapter 7

The Expanding Unviverse

The history of astronomy is a history of receding horizons. – Edwin Hubble

Learning Objectives

In this chapter, you will first examine Hubble's 1929 data, and with the cosmological principle in mind develop an understanding of the expanding Universe. You will then apply some of the concepts you have learned in earlier chapters to construct models of the expanding Universe. By solving the cosmological models, you will be able to determine the evolution of our Universe in different energy conditions.

Learning flow

We will spend 2 weeks for this chapter.

In week 8 lecture, we begin with Hubble's law and describe the expanding Universe using a Newtonian model (Section 7.2 and 7.3). We will also start on the spacetime metric for an expanding Universe (Section 7.4) for flat and sppherical space.

In week 9 IS, we will solve the differential equations (Friedmann equations) for the Newtonian expanding Universe numerically. We will also understand some of the consequences of the equations via the discussion questions (Q1 and Q2).

In week 9 lecture, we will continue the section (7.4) on the spacetime metric for expanding Universe. We will finally discuss the cosmological models that are derived from Einstein's theory of General Relativity.

In week 10 lecture, we wil review what we have learnt so far and if time permits, discuss some of the questions at the end of the chapter.

Contents

7.1 Pre-Lesson Homework

7.1.1 Hubble's data

In the 1920-1930s, Edwin Hubble and Milton Humason collated and measured the distances and velocities of galaxies. Below is a table comprising some of their early data. Analyse the data and deduce a relationship between velocities and distance (if you have not already done so in previous IS).

Object	r (distance in units of 10^6 parsecs)	v (measured velocities in km/s)
S. Mag.	0.032	170
L. Mag.	0.034	290
N.G.C.6822	0.214	-130
N.G.C.598	0.263	-70
N.G.C.221	0.275	-185
N.G.C.224	0.275	-220
N.G.C.5457	0.45	200
N.G.C.4736	0.5	290
N.G.C.5194	0.5	270
N.G.C.4449	0.63	200
N.G.C.4214	0.8	300
N.G.C.3031	0.9	-30
N.G.C.3627	0.9	650
N.G.C.4826	0.9	150
N.G.C.5236	0.9	500
N.G.C.1068	1	920
N.G.C.5055	1.1	450
N.G.C.7331	1.1	500
N.G.C.4258	1.4	500
N.G.C.4151	1.7	960
N.G.C.4382	2	500
N.G.C.4472	2	850
N.G.C.4486	2	800
N.G.C.4649	2	1090

7.1.2 The Cosmological Principle

The cosmological principle states that the Universe is **homogeneous** and **isotropic**.

What do the words in bold mean?

Reason that the cosmological principle implies that any observer looking at the motion of galaxies will find the radial velocities proportional to the distance between her and the galaxy.

7.1.3 Pythagoras theorem in spherical coordinates

Some intense mathy stuffs below. Just read, no need to work out.

Spherical coordinates are used to specify points on the sphere and are related to the Cartesian coordinates by

$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned} \tag{7.1}$$

The corresponding differentials are

$$\begin{aligned} dx &= \sin\theta\cos\phi dR + R\cos\theta\cos\phi d\theta - R\sin\theta\sin\phi d\phi \\ dy &= \sin\theta\sin\phi dR + R\cos\theta\sin\phi d\theta + R\sin\theta\cos\phi d\phi \\ dz &= \cos\theta dR - R\sin\theta d\theta \end{aligned} \tag{7.2}$$

The infinitesimal length between two spatial points is given by

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

To write this in spherical coordinates, we'll use Eq.(7.2), and some elbow grease.

$$\begin{aligned} (dl)^2 = &(\sin\theta\cos\phi dR + R\cos\theta\cos\phi d\theta - R\sin\theta\sin\phi d\phi)^2 \\ &+ (\sin\theta\sin\phi dR + R\cos\theta\sin\phi d\theta + R\sin\theta\cos\phi d\phi)^2 \\ &+ (\cos\theta dR - R\sin\theta d\theta)^2 \\ = &(dR)^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta) \\ &+ (d\theta)^2(R^2\cos^2\theta\cos^2\phi + R^2\cos^2\theta\sin^2\phi + R^2\sin^2\theta) \\ &+ (d\phi)^2(R^2\sin^2\theta\sin^2\phi + R^2\sin^2\theta\cos^2\phi) \\ &+ 2dRd\theta(R\sin\theta\cos^2\phi\cos\theta + R\sin\theta\sin^2\phi\cos\theta - R\sin\theta\cos\theta) \\ &+ 2dRd\phi(-R\sin^2\theta\cos\phi\sin\phi + R\sin^2\theta\sin\phi\cos\phi) \\ &+ 2d\theta d\phi(-R^2\cos\theta\cos\phi\sin\theta\sin\phi + R^2\cos\theta\sin\phi\sin\theta\cos\phi) \\ = &(dR)^2 + R^2(d\theta)^2 + R^2\sin^2\theta(d\phi)^2 \end{aligned}$$

Yeah finally done! This equation will be used later.

Oh one note, it is really troublesome to write the parenthesis for the squares of the differentials. From this point onward, we will not write them. Hence we have

$$\begin{aligned} dl^2 &= dx^2 + dy^2 + dz^2 \\ &= dR^2 + R^2 d\theta^2 + R^2\sin^2\theta d\phi^2 \end{aligned} \tag{7.3}$$

7.2 Hubble's Law

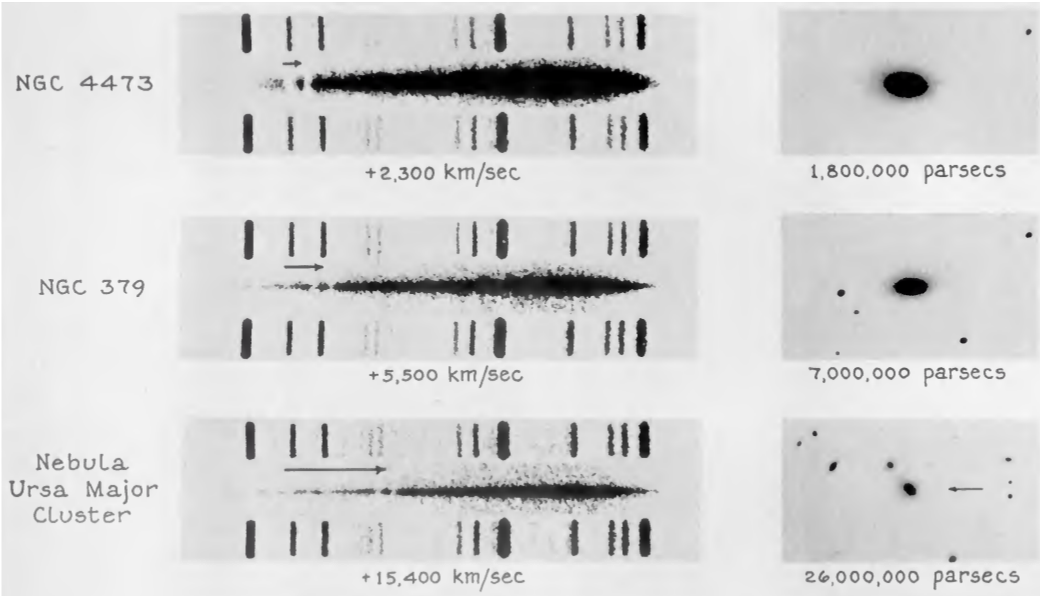

Figure 7.1: Photographs of some galaxies and their spectra captured by Milton Humason using the 100 inch Mount Wilson Telescope. As shown the fainter the galaxy (thus further), the more red-shifted its spectrum is.

In 1920s, Edwin Hubble and Milton Humason began to measure spectra of galaxies. Hubble found that the amount of redshift of a galaxy is not a random value, but directly proportional to the distance away from us. Using the observational data (see Sec. 7.1.1), he published in 1929 a relationship between distance and recession velocity of a galaxy, now known as the Hubble's law

$$v = H_0 d. \tag{7.4}$$

The Hubble parameter H_0 is an empirically fitted value. The subscript 0 is a convention that astronomers use to indicate that the value is as measured at current time t_0. (Theoretically the Hubble parameter can be a function of time, so it could be very different say 10 billion years ago.) Since Hubble's 1929 report, various groups have obtained different values of the Hubble's constant. Recent measurements narrows the value to be around 70 $\mathrm{kms^{-1}Mpc^{-1}}$.[1]

The redshift of galaxies that Hubble found follow a rule that is applicable to the entire Universe. For this reason, we call this redshift the **cosmoogical redshift**.

Ex. Cosmology is the study of the Universe as a whole. How did Hubble's work impact the field of cosmology?

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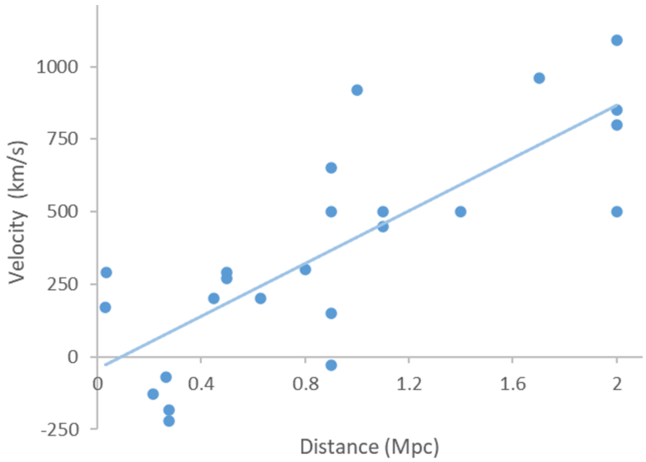
[1] See this article for an overview of recent techniques and results on the the measurement of Hubble's constant.

Ex. See Pre-Lecture homework on Hubble's data and the Cosmological principle. Why do you think that Hubble proposed a linear relationship based on his seemingly scattered data?



Ex. What is causing the galaxies to "move away" from us (and from each other)?

Ex. What is the difference between Doppler redshift and Cosmological redshift?

7.3 Expansion of the Newtonian world

7.3.1 Derivation of the Friedmann equations for a matter dominated Universe

A Newtonian world is a matter-dominated Universe, governed by Newton's laws of mechanics and gravity. In this section, we shall derive the set of differential equations that describes the expansion of the Universe.

Consider a spherical shell of mass m and radius R enclosing a spherically symmetric mass[2] M located at the center. The shell moves radially **outwards** with the rate $\dot{R} = \frac{dR}{dt}$. The space encompassed by the shell expands as a result. The kinetic energy (KE) of the shell is

$$T = \frac{1}{2}m\dot{R}^2 = \frac{1}{2}mH^2R^2. \tag{7.5}$$

where $H = \dfrac{\dot{R}}{R}$. Note that H is a function of time.

The potential energy (PE) due to Newtonian gravity is

$$U = -\frac{GMm}{R} = -\frac{4}{3}\pi Gm\rho R^2 \tag{7.6}$$

The total energy of the system is

$$\frac{1}{2}mH^2R^2 - \frac{4}{3}\pi Gm\rho R^2 = E. \tag{7.7}$$

Qualitatively, one can think that the KE increases the volume enclosed while the PE tries slow down or revert the expansion. The PE is dependent on the mass density ρ(of the enclosed mass).

If ρ is large enough such that PE $>$ KE, then the mass shell's expansion will **not** continue forever.

If ρ is small such that PE $<$ KE, then the mass shell's expansion will carry on indefinitely.

As such $E = 0$ is the critical point for which the Universe is just able to expand forever. Hence we define the critical density by

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{7.8}$$

If $\rho > \rho_c$, the Universe is "closed", if $\rho < \rho_c$, the Universe is "open".

It is useful to define a quantity that represents the scale of the Universe at different times. We call this the cosmological scale factor $S(t)$. All lengths (eg. radii, wavelengths) scales with this factor.

$$\begin{aligned} R(t) &\propto S(t) \\ R(t) &= r\,S(t) \end{aligned} \tag{7.9}$$

where r is a constant (in time)[3]. Differentiating the above equation wrt time,

$$v(t) = \dot{R}(t) = r\dot{S}(t) \tag{7.10}$$

$$\ddot{R}(t) = r\ddot{S}(t) \tag{7.11}$$

The parameter $H(t) = v/R$ can hence be written as,

$$H(t) = \frac{\dot{S}(t)}{S(t)} \tag{7.12}$$

One may also relate the mass density to the scale factor or

$$\rho \propto \tfrac{1}{R^3} \propto S^{-3} \tag{7.13}$$

Substituting Eqs. (7.10) and (7.12) into Eq. (7.7),

[2] The physical size of the spherically symmetric inner mass does not matter as long the radius is less than the radius of the shell.

[3] In case you wonder why I chose to use the letter r to represent a constant, the reason will be given next week.

$$\frac{1}{2}m\frac{\dot{S}^2}{S^2}r^2S^2 - \frac{4}{3}\pi Gm\rho r^2S^2 = E$$
$$\frac{1}{2}mr^2\left(\dot{S}^2 - \frac{8}{3}\pi G\rho S^2\right) = E \tag{7.14}$$

Since E, m and r are constants, we can write

$$\dot{S}^2 - \frac{8}{3}\pi G\rho S^2 = -kc^2 \tag{7.15}$$

This gives

$$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8}{3}\pi G\rho \tag{7.16}$$

The dynamics of the expanding shell of mass can be further probed with the Newton's law of gravitation

$$F = m\ddot{R} = -\frac{GMm}{R^2} \tag{7.17}$$

Rewriting in terms of the scale factor and mass density, we have

$$\ddot{S} = -\frac{4}{3}\pi G\rho S. \tag{7.18}$$

Comparing with Eq. (7.16), we have

$$\frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0 \tag{7.19}$$

Eqs. (7.16) and (7.19) are the differential equations for the scale factor $S(t)$. Solving them to find $S(t)$ allows us to track how the size of the Universe varies with time.

Ex. k can either be positive, negative or zero. What decides the value of k?

Hint: look at total energy and density vs critical density.

7.3.2 Evolution of the Matter-Dominated Universe

The Universe will behave differently depending on how much matter it contains. Putting Eq.(7.13) into the differential equation Eq. (7.16) and solving it numerically, one can obtain the following scenarios summarized with the figure below.

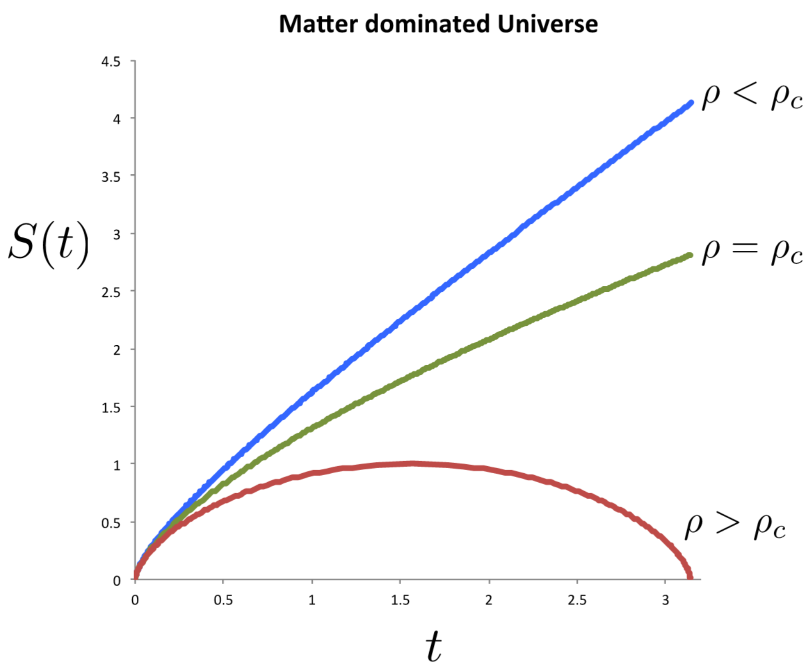


Figure 7.2: Plot of scale factor over time in a matter-dominated expanding universe. When the density of the Universe is above a certain critical value ρ_c, the expansion halts and starts to contract at some point in time. Otherwise, the expansion continues forever. In all cases the second derivative of the graphs are negative, meaning that the rates of expansion slow down over time.

Ex. Discuss in groups how to solve the differential equation Eq. (7.16) and reproduce the graphs of Figure 7.2. (We will do this slowly in IS!)

7.4 Expanding Universe in the Spacetime framework

Go to Activity 1.

7.4.1 Flat space (no curvature)

Suppose the space is flat or Euclidean, the metric of space is given by

$$dl^2 = dx^2 + dy^2 + dz^2 \tag{7.20}$$

In a homogeneous, isotropic, size-changing Universe, one will see the galaxies moving radially away/towards us equally in all directions. It is thus apt to express the metric in spherical coordinates (R, θ, ϕ):

$$dl^2 = dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \tag{7.21}$$

Looks familiar?

We are now going to do something that requires your brain to stretch a bit..

First, we allow R to vary with time, i.e. $R = R(t)$. This will mean that if we were to measure the distance between 2 points (ΔR), the value will depend on *when* you measure.

Since the space is homogeneous and isotropic, the change in distance is the same anywhere and in any directions. This allows us to think of the change of ΔR to be due to the scale (size) of the entire system that changes. We then define a time-dependent scale factor $S(t)$ and a time-independent "comoving" coordinate r where

$$R(t) = r\,S(t) \tag{7.22}$$

Note that r, θ and ϕ are **not** functions of time. This means that if we use (r, θ, ϕ) to label positions of objects, then these coordinate points will not change in time even if the space expands or contract. Using Eq.(7.22), we allow the time dependency of space to be encoded by the scale factor $S(t)$. For example, $S(t_2) > S(t_1)$ will mean that distance measured between two objects at t_2 will be larger than that in t_1.

Ex. In the space below, illustrate the above idea by draw 2 diagrams, one at time t_1 and the other at t_2. Each diagram should contain 2 points P and Q. P has the same (r, θ, ϕ) coordinates in t_1 and t_2. Similarly for Q.

Putting Eq.(7.22) into Eq.(7.21), we obtain

$$dl^2 = S(t)^2 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{7.23}$$

Eq.(7.23) is the space metric for flat space, written in spherical coordinates.

Space need not be flat.

If one allows space to be curved, yet maintaining universal homogeneity, one may have (i) flat (zero curvature), (ii) spherical (constant positive curvature) or (ii) hyperbolic (constant negative curvature) space.

7.4.2 Spherical space (positive curvature)

A 2-sphere is the 2D surface of a 3D ball. It is defined by the equation

$$x^2 + y^2 + z^2 = R^2$$

where R is the radius of the ball/sphere. The metric for the 2 sphere is

$$ds^2 = dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2$$

Note that $dR = 0$ since R is a constant here.

Now extend your imagination to think of the surface of a 4-dimensional ball. This surface is a curved 3D space which we call the **3-sphere**. Being one dimension higher than the 2-sphere, the equation for the 3-sphere is

$$w^2 + x^2 + y^2 + z^2 = S^2 \tag{7.24}$$

where S is a constant in space (but not a constant in time). Let $R^2 = x^2 + y^2 + z^2$. In this case R is not a constant. To find the space metric, first we work out the differentials

$$\begin{aligned} w^2 &= S^2 - R^2 \\ 2wdw &= 0 - 2RdR \\ w^2 dw^2 &= R^2 dR^2 \\ dw^2 &= \frac{R^2}{S^2 - R^2} dR^2 \end{aligned}$$

The metric of the 3-sphere is

$$\begin{aligned} dl^2 &= dw^2 + dx^2 + dy^2 + dz^2 \\ &= \frac{R^2 dR^2}{S^2 - R^2} + dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \qquad (7.25) \\ &= S^2 \left(\frac{dr^2}{1 - r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \qquad (7.26) \end{aligned}$$

where in the last step we had applied Eq.(7.22) on Eq.(7.25) to obtain Eq.(7.26).

7.4.3 Hyperbolic space (negative curvature)

The space of an isotropic and homogeneous negatively curved space takes on a surface of a hyperboloid, whose equation is given by

$$w^2 - x^2 - y^2 - z^2 = S^2$$

The metric of the 3-hyperboloid is

$$dl^2 = -dw^2 + dx^2 + dy^2 + dz^2$$

Using a similar treatment (as we did for the spherical case), the metric for a hyperbolic space is

$$dl^2 = S^2 \left(\frac{dr^2}{1 + r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \tag{7.27}$$

Ex. Derive Eq.(7.27).

7.4.4 Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

Putting the above altogether, allowing S to change with time, and adding the time component, the spacetime metric for an expanding Universe is

$$ds^2 = -c^2 dt^2 + S(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \tag{7.28}$$

where

$$k = \begin{cases} +1 & \text{, positively curved space} \\ 0 & \text{, flat space} \\ -1 & \text{, negatively curved space} \end{cases}$$

Eq. (7.28) is known as the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric. It is the spacetime metric of a homogeneous and isotropic (expanding) Universe.

7.4.5 The Cosmological Redshift

Consider light from a galaxy situated at a r_1. Let us narrow down to see just two photons, emitted one after the other at times t_1 and $t_1 + 1/f_1$, with f_1 being the frequency of the photon emitted. Assume the photons travel freely across the intergalactic distance to us, reaching our detectors at times t_0 and $t_0 + 1/f_0$. Photons propagating in an isotropic and homogeneous Universe are described by

$$ds^2 = c^2 dt^2 - S(t)^2 \frac{dr^2}{1 - kr^2} = 0 \tag{7.29}$$

where $S(t)$ is the cosmological scale factor characterizing the size of the Universe as it expands. For the first and the second photon, we have respectively

$$c \int_{t_1}^{t_0} \frac{dt}{S(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} \tag{7.30}$$

and

$$c \int_{t_1 + 1/f_1}^{t_0 + 1/f_0} \frac{dt}{S(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}}. \tag{7.31}$$

Since the RHS of the equations are the same, we can equate the LHS of both equations,

$$\begin{aligned} \int_{t_1}^{t_0} \frac{dt}{S(t)} &= \int_{t_1+1/f_1}^{t_0+1/f_0} \frac{dt}{S(t)} \\ &= \int_{t_1+1/f_1}^{t_0} \frac{dt}{S(t)} + \int_{t_0}^{t_0+1/f_0} \frac{dt}{S(t)} \\ &= \int_{t_1}^{t_0} \frac{dt}{S(t)} - \int_{t_1}^{t_1+1/f_1} \frac{dt}{S(t)} + \int_{t_0}^{t_0+1/f_0} \frac{dt}{S(t)} \\ \int_{t_1}^{t_1+1/f_1} \frac{dt}{S(t)} &= \int_{t_0}^{t_0+1/f_0} \frac{dt}{S(t)} \end{aligned} \tag{7.32}$$

The time interval(s) between the first and second photons emitted (and received) is very short and it will be reasonable to assume that the Universe do not expand much in that short time. We can therefore assume S to be constant within the limits of integration and take it out of the integration sign(s). This gives

$$\frac{1}{S(t_1)} \frac{1}{f_1} = \frac{1}{S(t_0)} \frac{1}{f_0} \tag{7.33}$$

Finally we define the cosmological redshift z by

$$1 + z = \frac{\lambda_0}{\lambda_1} = \frac{f_1}{f_0} = \frac{S(t_0)}{S(t_1)} \tag{7.34}$$

7.5 Cosmology from Einstein's General Relativity

Einstein's theory of general relativity (GR). GR is a theory that relates the energy content of a system and its spacetime geometry. The theory hinges on the idea that spacetime can be curved, and the curvature dependent on its mass-energy content. Particles move in (free-falling) motion by following paths of least action governed by the geometry of curved spacetime.

The dynamics is encoded in a set of differential equations known as the Einstein field equation:

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

The LHS of the Einstein field equation consists of geometrical terms (R_{ab}, R and g_{ab}) that can be found with the metric. The RHS of the field equation (T_{ab}) contain information about the mass-energy content in the system. In essense, the equation relates mass-energy with spacetime. As aptly summarised by John Wheeler with 12 words,

> "Spacetime tells matter how to move; matter tells spacetime how to curve."

The theory of general relativity is applicable for isolated systems such as stars and black holes, as well as for the Universe as a whole. While a full discussion of the theory is out of the scope of this course, we will examine the results of the theory in the context expanding Universe.

7.5.1 Friedmann Equations from General Relativity

The FLRW metric

$$ds^2 = -c^2 dt^2 + S(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right)$$

describes the spacetime geometry of an expanding homogeneous and isotropic Universe. Fitting it into the Einstein field equations, one obtain the following equations[4]:

$$\frac{\dot{S}(t)^2 + kc^2}{S(t)^2} = \frac{8}{3}\pi G \rho(t) \tag{7.35}$$

and

$$-2\frac{\ddot{S}(t)}{S(t)} - \frac{\dot{S}(t)^2 + kc^2}{S(t)^2} = \frac{8\pi G}{c^2} p(t) \tag{7.36}$$

Eqs (7.35) and (7.36) are known as the first and second Friedmann equations respectively. They are differential equations of the scale factor $S(t)$ that models the evolution of the expanding Universe. The equations are first derived by Alexander Friedmann in 1922, six years after Einstein published the theory of general relativity.

We find that the GR-derived Friedmann equations, are similar to Eqs (7.16) and (7.19) which were derived using Newtonian gravity. The difference, as we shall see below, is in the energy content.

Recall that in the Newtonian derivation of the Friedmann equations, the Universe contain matter where

$$\rho_{\text{matter}} \propto S^{-3} \tag{7.37}$$

This is rather intuitive as energy dilutes as the universe expands, and in our 3D world, volume $\propto S^{-3}$.

In the context of general relativity, the energy content of the Universe may not come only from matter. Other forms of energy such as radiation and vacuum energy can be considered in the GR-derived Friedmann equations. In the following section, we will examine how the different energy content affects the expansion of the Universe.

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[4] I have for some years taken GR out of the course in order to maintain sanity for most (including myself). If you are interest to learn a tiny bit of GR to see how the following equatons are derived, See Appendix A.

Ex. p stands for "pressure". What is the pressure for gas particles with 0 Kelvins?
What about pressure for photons. Google for radiation pressure.

7.5.2 Matter, Radiation, Vacuum and Dark Energy

In the Friedmann equations, there are three time dependent functions $S(t)$,$\rho(t)$ and $p(t)$. The ultimate goal is to solve the Friedmann equations to obtain the scale factor $S(t)$. However we cannot solve for $S(t)$ without knowing the energy density ρ and pressure p. To do so, some assumptions and simplifications needs to be made.

We begin with the first Friedmann equation Eq.(7.35)

$$\dot{S}^2 + kc^2 = \frac{8}{3}\pi G\rho S^2$$

Differentiating this with respect to time, we have

$$2\dot{S}\ddot{S} = \frac{8\pi G}{3}\left(\dot{\rho}S^2 + \rho 2S\dot{S}\right)$$

Putting this into the second Friedmann equation Eq.(7.36), we have

$$\begin{aligned}
-2\frac{\ddot{S}(t)}{S(t)} - \frac{\dot{S}(t)^2 + kc^2}{S(t)^2} &= \frac{8\pi G}{c^2}p(t) \\
-\frac{1}{S}\frac{8\pi G}{3\dot{S}}\left(\dot{\rho}S^2 + \rho 2S\dot{S}\right) - \frac{8}{3}\pi G\rho &= \frac{8\pi G}{c^2}p \\
-\frac{\dot{\rho}S}{3\dot{S}} - \frac{2\rho}{3} - \frac{\rho}{3} &= \frac{p}{c^2} \\
-\frac{\dot{\rho}S}{3\dot{S}} &= \rho + \frac{p}{c^2} \\
\dot{\rho} &= -\frac{3\dot{S}}{S}\left(\rho + \frac{p}{c^2}\right) \\
\frac{\dot{\rho}}{\rho} &= -\frac{3\dot{S}}{S}\left(1 + \frac{p}{\rho c^2}\right) \\
\frac{\dot{\rho}}{\rho} &= -\frac{3\dot{S}}{S}\left(1 + w\right)
\end{aligned}$$

where

$$w = \frac{p}{\rho c^2} \tag{7.38}$$

w is known as the "equation of state". We now assume for our cosmological models that density and pressure are proportional, meaning that while p and ρ are functions of time, their ratio w is a constant. Continuing from the above working,

$$\begin{aligned}
\frac{1}{\rho}\frac{d\rho}{dt} &= -3(1+w)\frac{1}{S}\frac{dS}{dt} \\
\frac{d}{dt}\ln\rho &= -3(1+w)\frac{d}{dt}\ln S \\
\ln\rho &= \ln S^{-3(1+w)} + \text{constant} \\
\rho &\propto S^{-3(1+w)}
\end{aligned} \tag{7.39}$$

Eq. (7.39) relates the energy content (characterized by w) to the scale factor. Putting this into Eq. (7.35), we have

$$\dot{S}^2 + kc^2 \propto \frac{8}{3}\pi G S^{-3(1+w)} S$$

It is now (almost) possible to solve this differential equation!

We still need to know the value for w. Different types of energy content have different values of w. Common energy contents that are discussed in modern cosmology are

- Matter $w_{\mathrm{m}} = 0$
- Radiation $w_{\mathrm{r}} = \frac{1}{3}$
- Vacuum energy (aka cosmological constant) $w_\Lambda = -1$
- Dark Energy $w_{de} < -\frac{1}{3}$

7.5.2.1 Matter-Dominated Universe

In a matter-dominated Universe, $w = 0$. From Eq.(7.39),

$$\rho \propto S^{-3}$$

The Friedmann equations become

$$\dot{S}^2 + kc^2 \propto S^{-1}$$
$$\ddot{S} \propto S^{-2}$$

Ex. Does this looks familiar? What will the solutions for $S(t)$ look like for $k = -1, 0$ and 1?

7.5.2.2 Radiation-Dominated Universe

The early Universe is often modeled as a radiation dominated Universe. Matter in the early Universe will be highly energetic, moving so fast at near light speed that they can be approximated to be radiation-like.

Putting $w_{\mathrm{r}} = \frac{1}{3}$ into Eq.(7.39), we find that the energy density of radiation relates to the scale factor as:

$$\rho_{\mathrm{r}} \propto S^{-4} \tag{7.40}$$

Ex. How do we understand Eq.(7.40) intuitively?

Hint: Compare with Eq.(7.37) and think about cosmological redshift.

7.5.2.3 Dark Energy-Dominated Universe

"Dark-energy" sounds too evil...

In the 1990s, two competing groups of astronomers were hunting down supernovae. Their goal was to extend Hubble's plot to very far-away objects. The research question is "How does the Hubble parameter change over time?" Based on cosmological models known at the time, the researchers were pretty sure that the expanding universe is decelerating (see Fig.(7.2)).

In 1998 and 1999, both groups published their results: The expansion of the Universe is not slowing down. It is in fact accelerating! This shook the scientific community because no one was expecting that, yet both (competing) groups got the same results.

On one hand, if we think about it, the expansion of the Universe is in itself an astonishing "fact" (backed up by observational evidences from Hubble and after). But we can always attribute the expansion to an *initial condition*, that the Universe started off expanding. We could religiously accept this initial condition, or apply the anthropic principle and reason that if it is otherwise, the Universe would not have evolved to this stage for us to discuss about this.

Acceleration, on the other hand is a different game. The cosmological models that we know of, the energy contents that we are familiar with, all point towards a Universe whose rate of expansion decreases over time. We cannot accept a deccelerating Universe by putting on an initial condition. It needs us to either tweak in the (cosmological) model, or accept a new kind of energy content. While it seems neccessary for to believe in the existence of a new energy content in the Universe to account for the new observational evidence for an accelerating Universe, no one has a concrete idea of what this energy is. It has never been directly or even indirectly observed. All we could do is to "reason" that it must be there, in dominating amounts! Astronomers call this mysterious energy "**dark energy**"

To create an accelerating Universe (on paper), one can choose some value for w and put it into the Friedmann equations to obtain $\ddot{S}(t) > 0$. The choice for w is not unique (we shall see that later), but a popular choice that astronomers like to use is $w = -1$, corresponding to what we call the vacuum energy.

Historically the vacuum energy was first introduced by Einstein for another purpose. He called it the cosmological constant Λ. Nowadays these terms are used interchangably. Let us put $w_\Lambda = -1$ into Eq.(7.39), (7.38), (7.35) and (7.36):

$$\begin{aligned} w_\Lambda &= -1 \\ \rho_\Lambda &\propto S^{-3(1-1)} = 1 \\ \Rightarrow \rho_\Lambda &= \text{constant} \\ \text{Let } \rho_\Lambda &= \frac{\Lambda}{8\pi G} \end{aligned}$$

where Λ is a positive constant. The energy density associated with vacuum energy is a constant.

Ex. Normally constants are simple good stuffs, maybe even a bit boring. In the context of an expanding Universe however, a constant (vacuum) energy density is a strange strange thing. Why so?

The pressure associated with Λ can be found using the equation of state Eq.(7.38):

$$\begin{aligned} -1 &= \frac{p_\Lambda}{\rho_\Lambda c^2} \\ \Rightarrow p_\Lambda &= -\frac{\Lambda}{8\pi G} c^2 \end{aligned}$$

The first Friedmann equation Eq.(7.35) becomes

$$\frac{\dot{S}(t)^2 + kc^2}{S(t)^2} = \frac{8}{3}\pi G \frac{\Lambda}{8\pi G} = \frac{\Lambda}{3}$$

Solving this equation gives the evolution of a expanding vacuum energy dominated Universe.

The second Friedmann equation Eq.(7.36) becomes

$$
\begin{aligned}
-2\frac{\ddot{S}}{S} - \frac{\Lambda}{3} &= \frac{8\pi G}{c^2}\frac{-\Lambda}{8\pi G}c^2 \\
-2\frac{\ddot{S}}{S} &= \frac{\Lambda}{3} - \Lambda \\
\ddot{S} &= \frac{\Lambda}{3}S \ > 0
\end{aligned}
$$

Hence a vacuum energy dominated Universe follows an accelerated expansion.

Ex. The vacuum energy falls in the catergory known as dark energy. We have used $w_\Lambda = -1$ to obtain an accelerating Universe ($\ddot{S} > 0$). Show that an accelerating Univese can be made more generally with

$$
w < -\frac{1}{3}
$$

Ex. Assume that the Universe was initally dominated by radiation, and $\rho_{\mathrm{r},0} > \rho_{\mathrm{m},0} > \rho_\Lambda$. Plot energy densities of radiation, matter and vacuum energy against cosmological scale factor. Use log-log scale.

The 2011 Physcis Nobel Prize which was awarded to astronomers who discovered that our Universe is accelerating. For more information on the accelerating Universe and dark energy, read the following:

https://www.nobelprize.org/prizes/physics/2011/advanced-information/

7.6 In-class Activities

Activity 1: The Curved Space of a 2-Sphere

Triangles on a 2-sphere

Blow a balloon into an approximately nice spherical shape. Define the north and south poles on your balloon. Draw a set of longitudes using a fine marker. Draw the equator as well.

Form a triangle with 2 longitudinal lines and the equator, draw it out with a thick marker.

Measure the interior angles of the triangle and find the sum of the three angles.

Geodesics on a 2-sphere

A geodesic is a generalised notion of a straight line. It is the shortest distance between two points in space. It is the line whereby a non-accelerated body will move on.

Blow another balloon into a sphere. Mark out two points on the balloon and call them NiuLang and ZhiNü. How would you draw the shortest distance, aka the geodesic, between them?

7.7 Discussion Questions

1. The Friedmann equations derived using Newton's laws are:

$$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8}{3}\pi G\rho \tag{7.41}$$

$$\ddot{S} = -\frac{4}{3}\pi G\rho S \tag{7.42}$$

From the second Friedmann equation Eq.(7.42) , what can you say about $\ddot{S}(t)$?

Does it correspond with our classical idea of gravitation being an attractive force.

Does it correspond with current observations of our Universe?

2. The reciprocal of the Hubble's constant is often used as an estimate for the age of the Universe. It can be shown that

$$t_0 < \frac{1}{H_0} \tag{7.43}$$

where H_0 is the current value of the Hubble's constant and t_0 is the current age of the Universe.
Discuss why Hubble's original value of $H_0 = 550\ \mathrm{kms^{-1}Mpc^{-1}}$ is inappropriate.

3. It is possible to obtain Eq.(7.43) by first showing that

$$H_0 t_0 = \int_0^1 \left(1 + \frac{\rho_0}{\rho_{c,0}}\left(\frac{1}{x} - 1\right)\right)^{-1/2} dx \tag{7.44}$$

Show that the intergration on the RHS is less than 1.
If you are interested in deriving Eq.(7.44), please see Appendix B.

4. **Einstein's Static Universe.** Einstein's theory of general relativity (GR) was formulated in 1916. It was based on the premise that the metric of spacetime can be dependent on space and time, and such dependency is related to the energy content in the system. In 1917, Einstein and his contemporaries began to apply GR to cosmology. Since this more than a decade before Hubble's 1929 observations on receeding galaxies, Einstein believed that our universe has to be **static**. However, the theory as-is could not result give rise to a static universe. Perhaps then the theory needs to be modified when applied to cosmology? Einstein thought.

GR can be modified in a number of ways . The simplest modification was to add a constant term. With this added term in the equation, the Friedmann equations become

$$\frac{\dot{S}^2 + kc^2}{S^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho \tag{7.45}$$

$$-2\frac{\ddot{S}}{S} - \frac{\dot{S}^2 + kc^2}{S^2} + \Lambda = \frac{8\pi G}{c^2}p \tag{7.46}$$

where Λ is a constant, also known as the cosmological constant.

(a) Show that if one set $\dot{S} = 0, \ddot{S} = 0$ and $p = 0$, then

$$\rho = \frac{\Lambda}{4\pi G} \tag{7.47}$$

(b) S is a constant as well. Why?

(c) Since S is a constant, let us set $S = 1$. Show that

$$\Lambda = kc^2 \tag{7.48}$$

Einstein's static matter-dominated Universe constructed! We have:

$$S = 1, \dot{S} = 0, \ddot{S} = 0, p = 0, \rho = \frac{\Lambda}{4\pi G}, \Lambda = kc^2$$

which means that for a static matter-dominated Universe, we need a positive cosmological constant Λ and a positive curvature k.

5. **Einstein's Blunder.** Einstein found out from Hubble in 1929 that the Universe is expanding. The addition of the cosmological constant into the theory is pointless after all! Einstein thought that this was "the greatest blunder of his life".

Perhaps Einstein was a little harsh on himself. He was just too ahead of his time! Or perhaps he knew that the static Universe model is flawed purely from a theoretical point of view too.

(a) Suppose there is a small perturbation to the scale factor

$$S = 1 + S'$$

where S' is a function of time and $|S'| \ll 1$. Show that

$$\rho = \rho_0 S^{-3} \approx \rho_0(1 - 3S')$$

(b) Using $\rho = \rho_0(1 - 3S')$ and $S = 1 + S'$, show that

$$\frac{d^2 S'}{dt^2} = \Lambda S'$$

(c) Solve $\dfrac{d^2 S'}{dt^2} = \Lambda S'$ with the initial conditions $S'(0) = S'_0, \dot{S}'(0) = 0$. Discuss how S' evolves.

Hence Einstein's static universe is an unstable one. A positive perturbation will cause it to expand while a negative pertubation will cause it to contract.

6. **Cosmological constant revived.** We saw that the idea of the cosmological constant to create a static Universe was flawed. But rather than banishing the cosmological constant, what if it can be used for another purpose instead? Recent astronomical observations suggests that the Universe is accelerating ($\ddot{S} > 0$). As we saw in Section 3.7.3, this can be acheived with some form of energy with equation of state $w < -\frac{1}{3}$. Can the cosmological constant be seen as a form of energy?

Let us look at Eq.(7.45) and (7.46), but now write the cosmological constant term on the energy side (RHS) of the equations:

$$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{7.49}$$

$$-2\frac{\ddot{S}}{S} - \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{c^2}p - \Lambda \tag{7.50}$$

These can be rewritten as

$$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3}\left(\rho + \rho_\Lambda\right) \tag{7.51}$$

$$-2\frac{\ddot{S}}{S} - \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{c^2}\left(p + p_\Lambda\right) \tag{7.52}$$

where

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}$$

and

$$p_\Lambda = -\frac{\Lambda c^2}{8\pi G}$$

are intepreted as the energy density and pressure of the cosmological constant respectively. The equation of state (Eq.(7.38)) is

$$w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda c^2} = -1$$

Since $w_\Lambda < -\frac{1}{3}$, it can indeed be used to model an accelerating Universe! I wonder if Einstein will be happy to know this....

The "deceleration parameter" q is define by

$$q = -\frac{\ddot{S}S}{\dot{S}^2} = -\frac{\ddot{S}}{SH^2}$$

a. Show that

$$q = \frac{1}{2}\left[\frac{\rho}{\rho_c}(1 + 3w) + \frac{\rho_\Lambda}{\rho_c}(1 + 3w_\Lambda)\right]$$

where $\rho_c = \frac{3H^2}{8\pi G}$

b. Using the above expression for the deceleration parameter, discuss the sign (positive/negative) of it for the cases below.

(i) Matter dominated Universe ($\rho \gg \rho_\Lambda$)

(ii) Cosmological constant dominated Universe ($\rho_\Lambda \gg \rho$).

(iii) The dominant energies of the Universe are matter and cosmological constant (only), with comparable energy densities.

NB. "Comparable" means quantitatively similar in order of magnitude. Of course one may be larger than the other, and permutations of this should be considered in your answer.

Appendix A: Einstein's Field equation to Friedmann Equations

For those who are interested...

Introducing particles with mass into space will result in a gravitational field. Einstein wanted to describe this gravitational field using geometry of spacetime. In other words, the presence of the massive particles (with energy and momentum) should influence the curvature of the spacetime.

With that Einstein went in search for an equation that links between energy-momentum and geometry of spacetime. But how do we link these two totally different things together? The hint comes when we consider that there is a law that governs the conservation of energy-momentum. The conservation law for energy momentum is:

$$\nabla_c T_{ab} = 0 \tag{7.53}$$

where ∇_c is some form of differentiation, and T_{ab} is a quantity that describe energy-momentum content. Can we find a similar one for geometry, like

$$\nabla_c G_{ab} = 0 \tag{7.54}$$

where G_{ab} is some geometrical term? Indeed after many years of searching, Einstein found a "simple" geometrical term that satisfy the above equation.

Without doing the derivation, we state the Einstein's field equation as follows:

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab} \tag{7.55}$$

The terms on the left hand side, $R_{ab} - \frac{1}{2} R g_{ab}$ is Einstein's choice for G_{ab} (Yes it satisfies Eq.(7.54). Each of the terms is only dependent on the metric.

$$\text{Ricci tensor:}\quad R_{ab} = \sum_c \partial_c \Gamma^c_{ab} - \sum_c \partial_b \Gamma^c_{ac} + \sum_{c,d} \Gamma^c_{ab} \Gamma^d_{cd} - \sum_{c,d} \Gamma^c_{ad} \Gamma^d_{cb} \tag{7.56}$$

$$\Gamma^a_{bc} = \Gamma^a_{cb} = \sum_d \frac{1}{2} g^{da} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) \tag{7.57}$$

$$\text{Ricci scalar:}\quad R = \sum_{a,b} g^{ab} R_{ab} \tag{7.58}$$

$$\sum_b g^{ab} g_{bc} = \delta^a{}_c \tag{7.59}$$

The right hand side of Einstein's field equation is one that describe the matter/energy content in the system.

$$\text{Energy momentum tensor } T_{ab} = \sum_d g_{ad} T^d{}_b \tag{7.60}$$

In the case of a homogeneous and isotropic Universe,

$$\begin{aligned} T^0{}_0 &= -\rho(t) c^2 \\ T^1{}_1 &= T^2{}_2 = T^3{}_3 = p(t) \end{aligned} \tag{7.61}$$

where $\rho(t)$ is the energy density and $p(t)$ is the pressure of the matter/energy content in the system (the Universe in this case).

From the FLRW metric

$$ds^2 = -c^2 dt^2 + S(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right),$$

$$g_{00} = -c^2,\ g_{11} = S(t)^2 \frac{1}{1 - kr^2},\ g_{22} = S(t)^2 r^2,\ g_{33} = S(t)^2 r^2 \sin^2\theta \tag{7.62}$$

All other g_{ab} are zero.

$$g^{00} = \frac{-1}{c^2},\, g^{11} = \frac{1}{S^2}(1-kr^2),\, g^{22} = \frac{1}{S^2r^2},\, g^{33} = \frac{1}{S^2r^2\sin^2\theta} \tag{7.63}$$

All other g^{ab} are zero.

We are not going to torture ourselves with calculating R_{ab} and R here. For now the result will be stated.
The non-zero components of the Ricci tensor are

$$\begin{aligned}
R_{00} &= -\frac{3\ddot{S}}{S} \\
R_{11} &= \frac{1}{1-kr^2}\frac{1}{c^2}(S\ddot{S} + 2\dot{S}^2 + 2kc^2) \\
R_{22} &= r^2\frac{1}{c^2}(S\ddot{S} + 2\dot{S}^2 + 2kc^2) \\
R_{33} &= r^2\sin^2\theta\frac{1}{c^2}(S\ddot{S} + 2\dot{S}^2 + 2kc^2)
\end{aligned} \tag{7.64}$$

The Ricci scalar is

$$R = \frac{6}{c^2}(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}) \tag{7.65}$$

With the above information, we can derive the first and second Friedmann equations using Einstein's field equation by
(i) setting $a = 0, b = 0$,

$$\frac{\dot{S}(t)^2 + kc^2}{S(t)^2} = \frac{8}{3}\pi G\rho(t)$$

.
(ii) set $a = 1, b = 1$ (or both to 2 or both to 3)

$$-2\frac{\ddot{S}(t)}{S(t)} - \frac{\dot{S}(t)^2 + kc^2}{S(t)^2} = \frac{8\pi G}{c^2}p(t)$$

Appendix B

To obtain Eq.(7.44), we need to first get the following:

$$H = \frac{\dot{S}}{S}\ ,\ H_0 = \frac{\dot{S}_0}{S_0}\ ,\ \rho_c = \frac{3H^2}{8\pi G}\ ,\ \rho_{c,0} = \frac{3H_0^2}{8\pi G}\ ,\ \rho S^3 = \rho_0 S_0^3$$

From the first Friedmann equation,

$$\begin{aligned}
\dot{S}^2 - \frac{8}{3}\pi G\rho S^2 = kc^2 = \dot{S}_0^2 - \frac{8}{3}\pi G\rho_0 S_0^2 \\
\dot{S}^2 &= \dot{S}_0^2\left(1 - \frac{8}{3}\pi G\rho_0\frac{S_0^2}{\dot{S}_0^2} + \frac{8}{3}\pi G\rho\frac{S^2}{\dot{S}_0^2}\right) \\
&= \dot{S}_0^2\left(1 - \frac{H_0^2}{\rho_{c,0}}\rho_0\frac{S_0^2}{\dot{S}_0^2} + \frac{H_0^2}{\rho_{c,0}}\rho_0\frac{S_0^3}{S^3}\frac{S^2}{\dot{S}_0^2}\right) \\
&= \dot{S}_0^2\left(1 - \frac{1}{\rho_{c,0}}\rho_0 + \frac{1}{\rho_{c,0}}\rho_0\frac{S_0}{S}\right) \\
&= \dot{S}_0^2\left(1 + \frac{\rho_0}{\rho_{c,0}}\left(\frac{1}{x} - 1\right)\right)
\end{aligned}$$

where $x = \dfrac{S}{S_0}$

$$\begin{aligned}
H_0 t_0 &= H_0\int_0^{t_0} dt \\
&= H_0\int_0^{S_0}\frac{dt}{dS}dS \\
&= H_0\int_0^1(\dot{S})^{-1}S_0 dx \\
&= H_0\int_0^1\dot{S}_0^{-1}\left(1 + \frac{\rho_0}{\rho_{c,0}}\left(\frac{1}{x} - 1\right)\right)^{-1/2} S_0 dx \\
&= \int_0^1\left(1 + \frac{\rho_0}{\rho_{c,0}}\left(\frac{1}{x} - 1\right)\right)^{-1/2} dx
\end{aligned}$$