# SP3176 The Universe Assignment 0
# MURDEROUS MATHEMATICS

1. **Move things around in an equation**
   Let
   $$F = \frac{mv^2}{r} = k\frac{Ze^2}{r^2}$$
   and
   $$L = mvr = \frac{nh}{2\pi}.$$
   Express $r$ in terms of $n, h, m, k, Z$ and $e$.

2. **A first look at differential equations**
   The following equation is variable separable differential equation
   $$\frac{df}{dt} = g(f)\, h(t)$$
   Briefly describe how do we obtain $f(t)$ from the above? (If needed, consult the internet for "variable seperable differenential equation")

3. **Discover the most important thing about exponential function**
   Consider the exponential function
   $$f(t) = ke^{\lambda t} \tag{1}$$
   (a) Substitute Eq.(1) into the differential equation:
   $$a\frac{d^2f}{dt^2} + b\frac{df}{dt} + cf = 0$$
   and work through it until there is no more differentiation operations.
   (b) There is something nice about the above result. What is it that is nice?

4. **Learn how to use Wolfram Alpha**
   On Wolfram Alpha, plot the functions
   $$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$$
   and
   $$g(x) = \sin x$$
   What do you observe from your plots?

5. **Transformation**
The polar coordinates $(r, \theta)$ are related to the cartesian coordinates $(x, y)$ by the following transformation equations:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

(a) Invert the above equations to find $r(x, y)$ and $\theta(x, y)$.
(b) Find the polar coordinates of the points whose cartesian coodinates are (i) $(0, 1)$, (ii) $(1/\sqrt{2}, -1/\sqrt{2})$ and (iii) $(-1/\sqrt{2}, -1/\sqrt{2})$.

6. **Learn how to use (and play) with Desmos**
Visualise a curve plotted with polar coordinates in Desmos. Enter the following lines

$$a = 2$$

$$E = 0.1$$

$$r = \frac{a(1 - E^2)}{1 + E\cos\theta}$$

Scroll the slider bars for $a$ and $E$. Describe how $a$ and $E$ affects the shape of the plot. (Restrict the range of $E$ and reduce the step size if neccessary).

7. **Why rabbits are dangerous creatures**
1859, in the land down under, Thomas Austin released 24 gray rabbits that he imported from England, as game for shooting parties. Being prolific creatures, the population of rabbits exploded to millions within a few years. Population dynamics may be modelled by the following ordinary differential equation:

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{r}\right)$$

where $N$ is the number of rabbits as a function of time $t$, $k$ and $r$ are constants that represents the growth rate and carrying capacity respectively.
(a) A steady state refers to the lack of change in a system. In this scenario, what is the value of N at steady state?
(b) Solve the differential equation, i.e. obtain $N(t)$. You may choose to do it analytically or numerically.
(c) Show that when $r$ is large, $N$ is effectively a simple exponential function of time.

# Administrative details

This assignment is to be done individually, and holds 5% weightage of the total assessment for The Universe.

You may type or hand-write your solutions. If you choose the latter, scan the hand-written document with a mobile phone and combine into a single pdf file for submission. File name for the assignment submission: A0_A#######X.pdf where A#######X is your matric number.

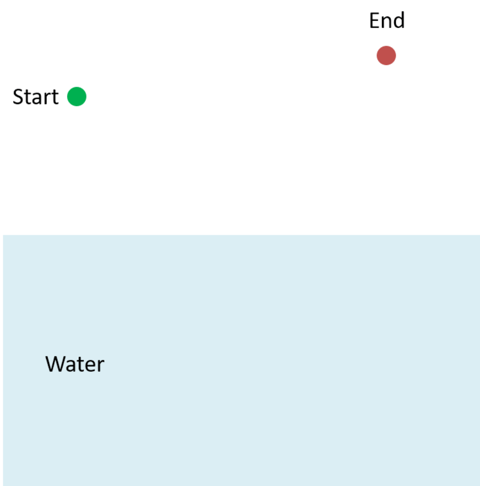
Submission: Week 3 Saturday

## Optional questions

I find them fun but you may not. If you are not doing them, try to at least appreciate the results.

1. **Fermat's principle of least time**
   Pierre de Fermat gave a cool idea to account for reflection and refraction of light. The velocity of light is different in different media. As light travels from one medium to another, the velocity will change. The light path is such that it should reach its destination as quickly as possible. To illustrate this principle of least time, let us imagine a soldier kenna tekan.

   **(a) Reflection**
   A soldier is told to run from the "Start" point, kiss the water, turn back and run to the "End" point. What path he should take to reach the destination soonest?




   **(b) Refraction**
   The "End" point is now in the water. The soldier has now to run from the "Start", and then swim to reach the "End". He runs faster than he swims. $v_{\text{land}} > v_{\text{water}}$. What path he should take to reach the destination soonest? (How do you prove that this is the fastest path?)

2. **Focus light with a parabolic mirror**
Let $P(x_1, y_1)$ be a point on the curve $y^2 = 4px$. This shape is called a parabola. The parabola and its tangent at point $P$ are drawn on the diagram below. Let $F(p, 0)$ be a point on the $x-$axis. Let $\beta$ be the (acute) angle between $y = y_1$ and the tangent of the parabola. Let $\alpha$ be the (acute) angle between $FP$ and the tangent of the parabola.
Show that $\alpha = \beta$.