We show that all these 10 cross sections are related, due to underlying SU(2) isospin symmetry.

For example, consider

$$\pi^+ p \to \pi^+ p \quad \text{(i)}$$
$$\pi^- p \to \pi^- p \quad \text{(ii)}$$
$$\pi^- p \to \pi^0 n \quad \text{(iii)}$$

To compute scattering cross section, we need scatt. amp.,

scatt amplitude $= {}_{out}\langle \quad | \quad \rangle_{in}$ $\qquad | \; \rangle_{in}$ = in-state

For our example, specify the in-state and out-state in terms of isospins, or better still, total isospins.

Consider process (i), the individual isospin of the particle involved is known

$$p = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \qquad \pi^+ = |1, +1\rangle$$

$\uparrow \; I^2 \quad \uparrow \; I_3$

{ like ang. mom $|j\, m\rangle$, we have $|I\, m\rangle$ }

So the in-state in process (i) is given by

$$\pi^+ p \to |1\, +1\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$
$$= |\underset{I_1}{1}\, \underset{I_2}{\tfrac{1}{2}}, \underset{m_{I_1}}{1}\, \underset{m_{I_2}}{\tfrac{1}{2}}\rangle$$

Express in terms of total isospin quantum numbers.

Recall $\underset{(j_1\, m_1)}{\underset{\sim}{J}_1}, \underset{(j_2\, m_2)}{\underset{\sim}{J}_2} \longrightarrow \underset{(j\, m)}{\underset{\sim}{J}} = \underset{\sim}{J}_1 + J_2$

Addition of two angular momenta, $\underset{\sim}{J}_1$, $\underset{\sim}{J}_2$

$$j = (j_1+j_2),\quad j_1+j_2-1,\quad \cdots\quad |j_1-j_2|$$

$$m = m_1 + m_2$$

Clebsch-Gordan expansion,

$$|j_1 j_2\, m_1 m_2\rangle = \sum_{j,m} |j_1 j_2\, j m\rangle \langle j m | j_1 j_2\, m_1 m_2\rangle$$

$$|j_1 j_2\, j m\rangle = \sum_{m_1 m_2} |j_1 j_2\, m_1 m_2\rangle \langle m_1 m_2 | j_1 j_2\, j m\rangle$$

So express $\pi^+ p = |\underset{I_1}{\overset{\uparrow}{1}}\ \underset{I_2}{\overset{\uparrow}{\tfrac{1}{2}}}\ \underset{m_{I_1}}{\overset{\uparrow}{1}}\ \underset{m_{I_2}}{\overset{\uparrow}{\tfrac{1}{2}}}\rangle$

in terms of $|I_1\, I_2\, I\, m\rangle$

since $I_1 = 1$, $I_2 = \frac{1}{2}$ $\therefore$ $I = \frac{3}{2}, \frac{1}{2}$,

Using Clebsch-Gordan table

$$\pi^+ p = |\underset{I_1}{\overset{\uparrow}{1}}\ \underset{I_2}{\overset{\uparrow}{\tfrac{1}{2}}}\ \underset{m_{I_1}}{\overset{\uparrow}{1}}\ \underset{m_{I_2}}{\overset{\uparrow}{\tfrac{1}{2}}}\rangle$$

$$= |\underset{I_1}{\overset{\uparrow}{1}}\ \underset{I_2}{\overset{\uparrow}{\tfrac{1}{2}}}\ \underset{I}{\overset{\uparrow}{\tfrac{3}{2}}}\ \underset{m}{\overset{\uparrow}{\tfrac{3}{2}}}\rangle_{in}$$

Similarly, for process (i), the out-state is

$$\pi^+ p = |1\ \tfrac{1}{2}\ 1\ \tfrac{1}{2}\rangle = |1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}\rangle_{out}$$

So scatt. amp for (i) is

$$\mathcal{M}_{(i)} = {}_{out}\langle 1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{3}{2} | 1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}\rangle_{in}$$

For (ii) $\pi^- p \to \pi^- p$

$$\pi^- p = |1\ {-1}\rangle\, |\tfrac{1}{2}\ {+\tfrac{1}{2}}\rangle$$
$$\quad\ \ \uparrow\ \ \ \uparrow\qquad \uparrow\ \ \ \uparrow$$
$$\quad\ \ I_1\ \ m_{I_1}\quad I_2\ \ m_{I_2}$$

$$\equiv |1\ \tfrac{1}{2}\ \ {-1}\ \ {+\tfrac{1}{2}}\rangle$$
$$\ \ \uparrow\ \ \uparrow\quad \uparrow\quad\ \uparrow$$
$$\ \ I_1\ I_2\ \ m_{I_1}\ \ m_{I_2}\rangle$$

e.g.

$$\underset{\text{Table}}{=} \sqrt{\tfrac{1}{3}}\ |1\tfrac{1}{2}\ \tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{2}{3}}\ |1\tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$

scatt. amp $\mathcal{M}_{(ii)} = \frac{1}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\,|\,1\tfrac{1}{2}\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle_{in}$

$$+ \tfrac{2}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\,|\,1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}$$ HW

For (iii) $\pi^- p \to \pi^0 n$

$$\pi^0 n = |1\,0\rangle\, |\tfrac{1}{2}\ \tfrac{-1}{2}\rangle = |1\tfrac{1}{2}\ \ 0\ \tfrac{-1}{2}\rangle$$

$$= \sqrt{\tfrac{2}{3}}\ |1\tfrac{1}{2}\ \tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{1}{3}}\ |1\tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$ (To check, HW)

scatt. amp $= {}_{out}\langle \pi^0 n\,|\,\pi^- p\rangle_{in} = \mathcal{M}_{(iii)}$

$$= \frac{\sqrt{2}}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\,|\,1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in} - \frac{\sqrt{2}}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\,|\,1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}$$

The 3 scatt. amps $\mathcal{M}_{(i)}$, $\mathcal{M}_{(ii)}$ and $\mathcal{M}_{(iii)}$ are related as will be shown below.

$$\mathcal{M}_{(i)} : \mathcal{M}_{(ii)} : \mathcal{M}_{(iii)} = {}_{out}\langle \overset{I}{\tfrac{3}{2}}\ \overset{m}{\tfrac{3}{2}} | \overset{I}{\tfrac{3}{2}}\ \overset{m}{\tfrac{3}{2}} \rangle_{in} :$$

$$\left(\tfrac{1}{3}\ {}_{out}\langle \tfrac{3}{2}\ \tfrac{-1}{2} | \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in} + \tfrac{2}{3}\ {}_{out}\langle \tfrac{1}{2}\ \tfrac{-1}{2} | \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}\right) :$$

$$\left(\tfrac{\sqrt{2}}{3}\ {}_{out}\langle \tfrac{3}{2}\ \tfrac{-1}{2} | \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in} - \tfrac{\sqrt{2}}{3}\ \langle \tfrac{1}{2}\ \tfrac{-1}{2} | \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}\right)$$

Put

$${}_{out}\langle \tfrac{3}{2}\ \tfrac{3}{2} | \tfrac{3}{2}\ \tfrac{3}{2}\rangle_{in} = \mathcal{M}_{\frac{3}{2}}$$

different m

$$\overset{?}{=}\ {}_{out}\langle \tfrac{3}{2}\ \tfrac{-1}{2} | \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in}$$

Why? = Wigner-Eckart theorem

i.e. $\mathcal{M}_{3/2}$ depends on $I = 3/2$ but not on m

$$\langle \tfrac{1}{2}\ -\tfrac{1}{2} | \tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \mathcal{M}_{\frac{1}{2}}$$

$$\mathcal{M}_{(i)} : \mathcal{M}_{(ii)} : \mathcal{M}_{(iii)} = \mathcal{M}_{\frac{3}{2}} : \left(\tfrac{1}{3}\mathcal{M}_{\frac{3}{2}} + \tfrac{2}{3}\mathcal{M}_{\frac{1}{2}}\right) :$$

$$\left(\tfrac{\sqrt{2}}{3}\mathcal{M}_{\frac{3}{2}} - \tfrac{\sqrt{2}}{3}\mathcal{M}_{\frac{1}{2}}\right)$$

scattering amplitude depends on energy of incident particles,

At CM energy 1232 MeV, $\mathcal{M}_{\frac{3}{2}} \gg \mathcal{M}_{\frac{1}{2}}$.

Then

$$\mathcal{M}_{(i)} : \mathcal{M}_{(ii)} : \mathcal{M}_{(iii)} = \left(1 : \tfrac{1}{3} : \tfrac{\sqrt{2}}{3}\right)$$

$$= (3 : 1 : \sqrt{2})$$

Cross section $= |\mathcal{M}|^2$

$$\sigma_{(i)} : \sigma_{(ii)} : \sigma_{(iii)} = |\mathcal{M}_{3/2}|^2 : \left|\frac{1}{3}\mathcal{M}_{3/2} + \frac{2}{3}\mathcal{M}_{\frac{1}{2}}\right|^2$$

$$: \left|\frac{\sqrt{2}}{3}\mathcal{M}_{3/2} - \frac{\sqrt{2}}{3}\mathcal{M}_{\frac{1}{2}}\right|^2$$

$$\simeq |\mathcal{M}_{\frac{3}{2}}|^2 : \left|\frac{1}{3}\mathcal{M}_{3/2}\right|^2 : \left|\frac{\sqrt{2}}{3}\mathcal{M}_{\frac{3}{2}}\right|^2$$

$$= 1 : \frac{1}{9} : \frac{2}{9} = 9 : 1 : 2$$

If we are interested in the cross section ratio of $\sigma_{\pi^+ p}$ and $\sigma_{\pi^- p}$ for the 3 processes

$$\frac{\sigma_{\pi^+ p}}{\sigma_{\pi^- p}} = \frac{9}{1+2} = 3$$

where $\sigma_{\pi^- p} = \sigma_{(ii)} + \sigma_{(iii)}$

The calculated ratio agrees with the experimental result. see Fig in page 21

We now extend isospin SU(2) to higher flavour symmetries, SU(3), SU(4), ... SU(6)

2

$$\sigma_a : \sigma_c : \sigma_j = 9|\mathcal{M}_3|^2 : |\mathcal{M}_3 + 2\mathcal{M}_1|^2 : 2|\mathcal{M}_3 - \mathcal{M}_1|^2 \tag{4.49}$$

At a *CM* energy of 1232 MeV there occurs a famous and dramatic bump in pion-nucleon scattering, first discovered by Fermi in 1951;[7] here the pion and nucleon join to form a short-lived "resonance" state—the **Δ**. We know the **Δ** carries I $= \frac{3}{2}$, so we expect that at this energy $\mathcal{M}_3 \gg \mathcal{M}_1$, and hence

0.6 × 10^-23 s

$$\sigma_a : \sigma_c : \sigma_j = 9 : 1 : 2 \tag{4.50}$$

Experimentally, it is easier to measure the total cross sections, so (c) and (j) are combined:

$$\frac{\sigma_{\text{tot}}(\pi^+ + p)}{\sigma_{\text{tot}}(\pi^- + p)} = 3 \tag{4.51}$$

As you can see in Figure 4.6, this prediction is well satisfied by the data.

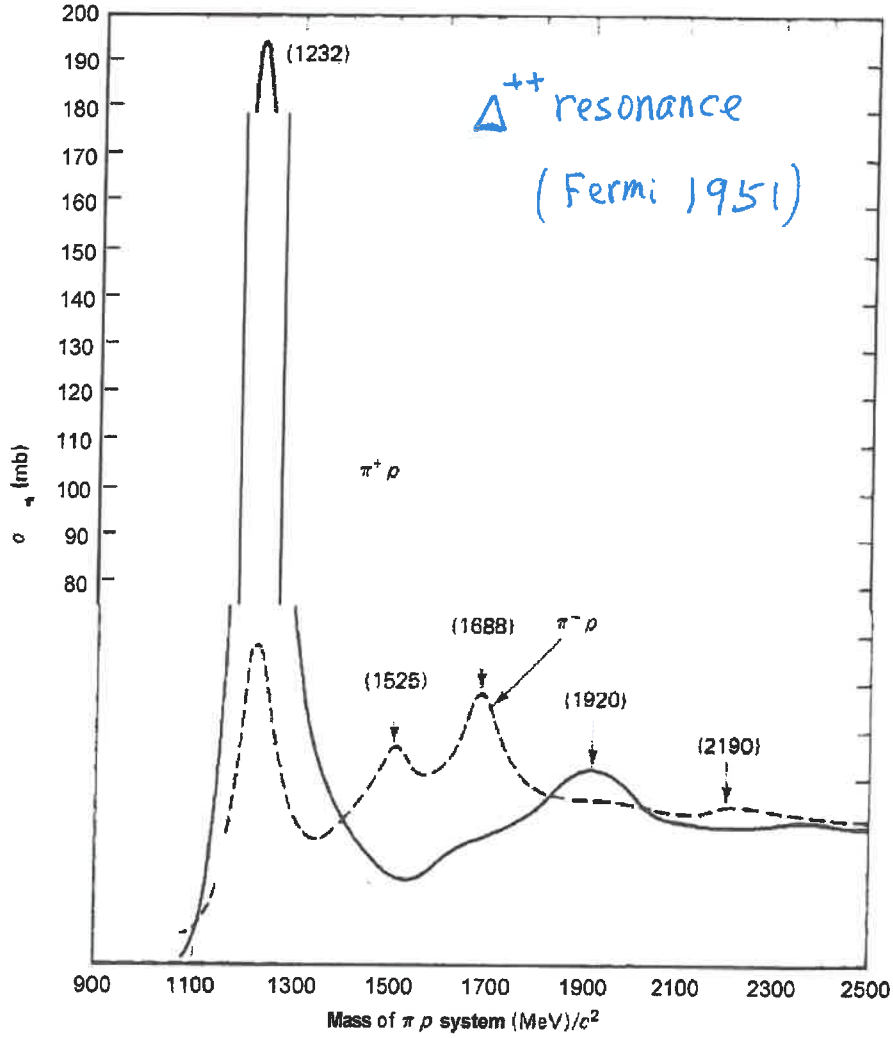

Figure 4.6 Total cross sections for π^+p (solid line) and π^-p (dashed line) scattering. (Source: S. Gasiorowicz, Elementary Particle Physics (New York: Wiley, copyright © 1966, page 294. Reprinted by permission of John Wiley and Sons, Inc.)

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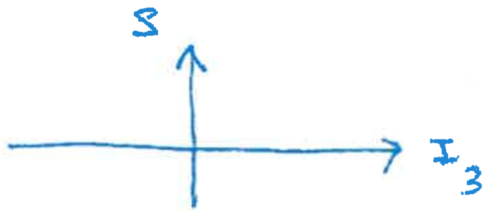
Quark flavour

J^2 = Casimir operator

Originally (Heisenberg, 1932) isospin was introduced to classify elementary particles into doublet (p, n), or triplet (π^+, π^0, π^-) etc. The isospin group is $SU(2)$

$|jm\rangle$, $|I\, m_I\rangle$

In early 1960, many more elementary particles were found, $SU(2)$ isospin as a classification scheme is not adequate. A new quantum number, strangeness S, was introduced.



Many particles can then be accommodated into representations of a bigger symmetry group $SU(3)$

Mesons form singlet or octet representations of $SU(3)$

Baryons form singlet, octet (eight fold way) decuplet representations of $SU(3)$

Questions were then raised why only these three types of representation of $SU(3)$ are realized by elementary particles at that time?

The quark model (3 quarks) explains this. Mesons are made of quark and antiquark. Baryon are made of 3 quarks

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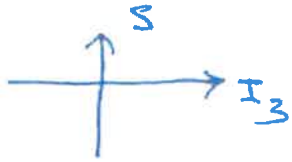
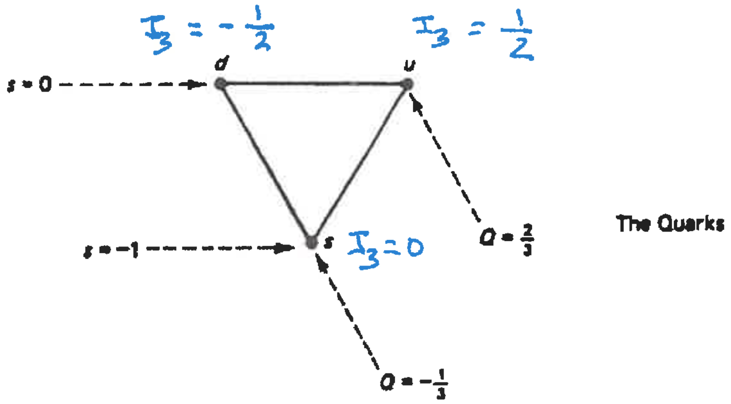
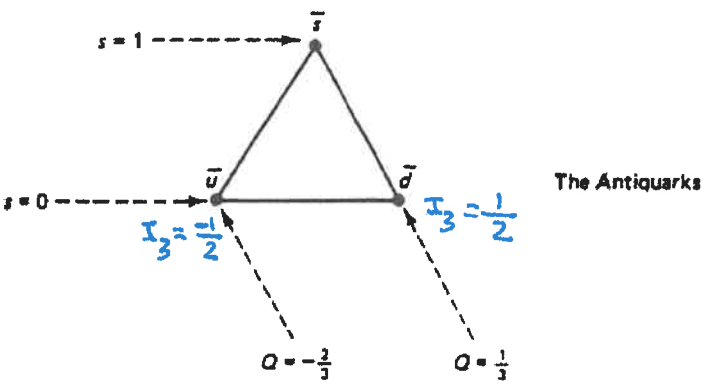
Assume the 3 quarks form the fundamental respresentation of SU(3), triplet 3





Antiquarks form the conjugate representation, denoted by $\bar{3}$



From the fundamental representation, one can construct higher dimensional representation:

$$3 \otimes \bar{3} = 1 \oplus 8$$

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$

1 = singlet
8 = Octet
10 = Decuplet
wrt SU(3) transformations

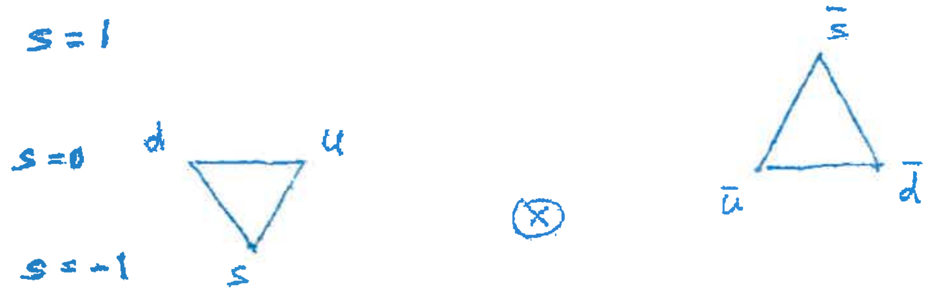
Outer multiplication of two matrices

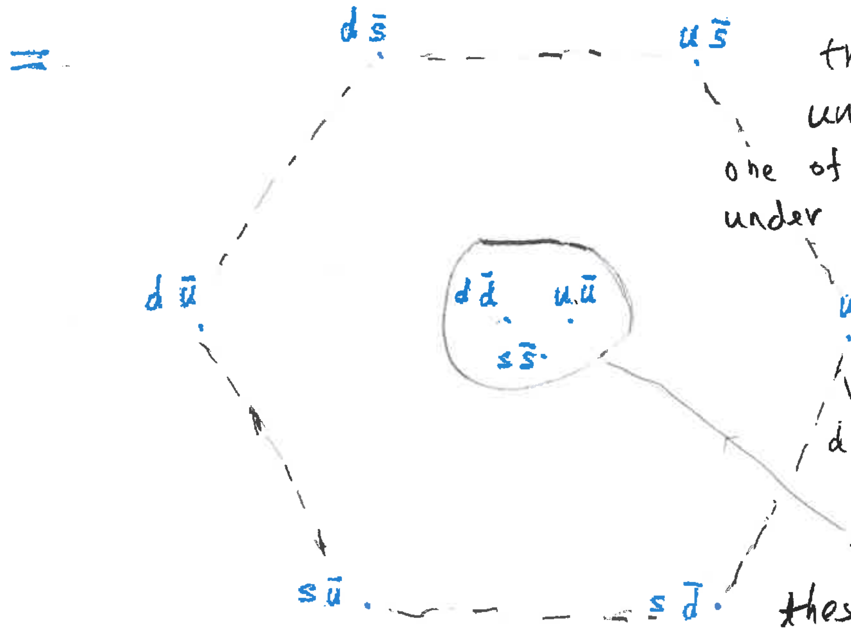
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

Mesons are made of quark and antiquark

$$3 \otimes \bar{3} = 8 \oplus 1$$





Among the nine states, nonet, 8 of them transform into each other under SU(3) transformation, one of them is invariant under SU(3) transformations.

So the nonet can be decomposed into an octet plus a singlet.

We say the nonet is a direct sum $\oplus$ of $\underline{8}$ and $\underline{1}$

$$3 \otimes \bar{3} = 1 \oplus 8$$

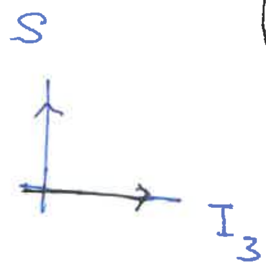
one of combinations of these three is unchanged under SU(3) transformations, that is singlet wrt SU(3)

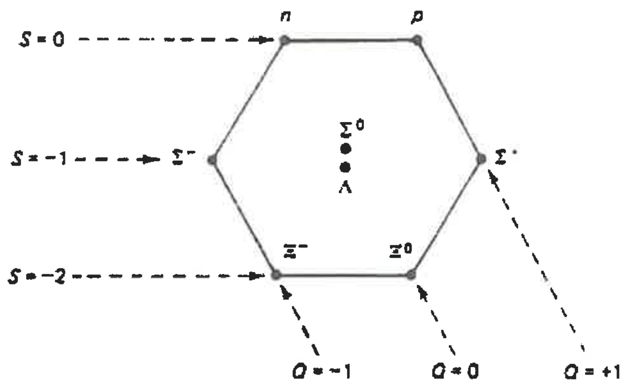
For baryons:

$$3 \otimes 3 \otimes 3 = \underbrace{10}_{\text{Decuplet}} \oplus 8 \oplus 8 \oplus 1$$

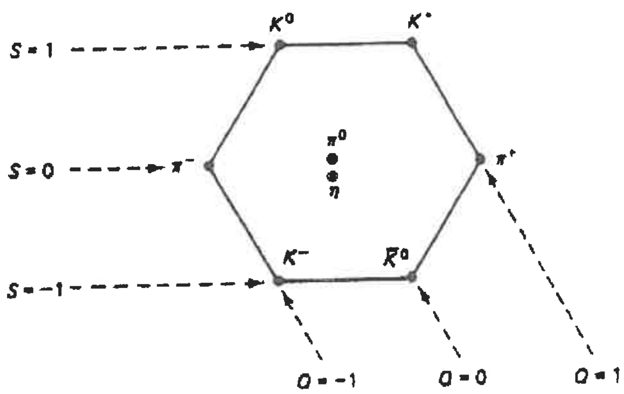
Note: The above nonet is a direct sum of octet and a singlet, Octet, singlet wrt SU(3)
The octet consists of 2 isodoublets, 1 isotriplet and 1 isosinglet wrt SU(2)

25

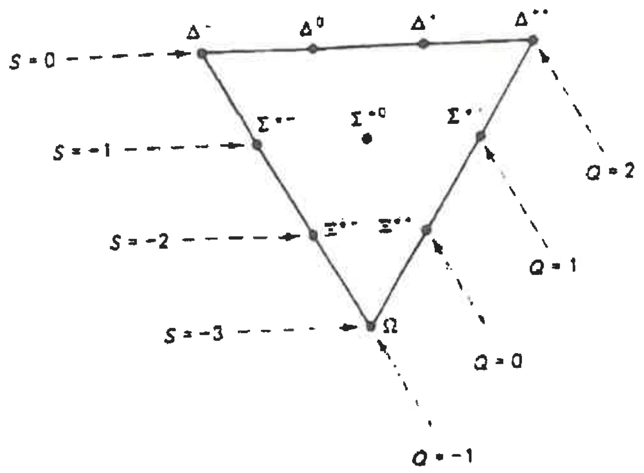




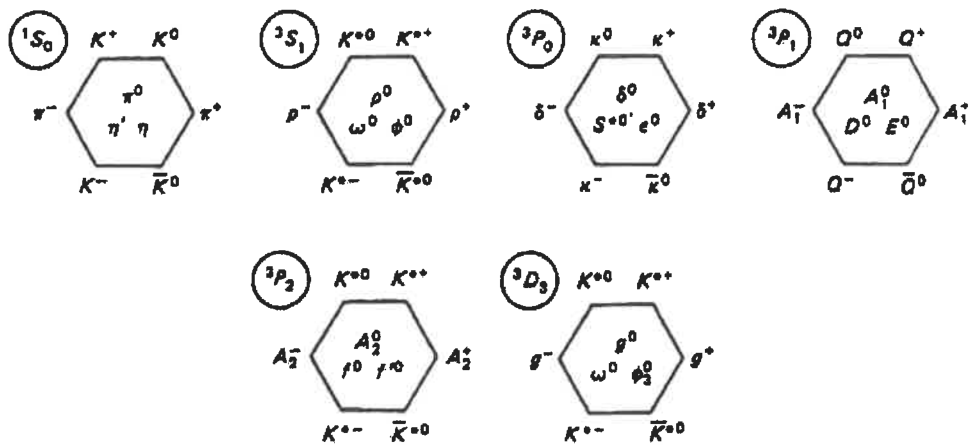
The Baryon Octet



The Meson Octet



The Baryon Decuplet



$^{2S+1}L_j$

Established meson nonets. Obviously, we are running out of letters. It is customary to distinguish different particles represented by the same letter by indicating the mass parenthetically (in MeV/c^2), thus $K^*(892)$, $K^*(1430)$, $K^*(1650)$, and so on. In this figure the supermultiplets are labeled in spectroscopic notation At present, there are no complete baryon supermultiplets beyond the octet and decuplet, although there are many partially filled diagrams.

So SU(3) scheme with 3 quarks (u, d, s) as the fundamental constituents of matter was a very good scheme for classifying mesons and baryons into SU(3) singlets, octets, decuplets

Charm quark was discovered in Nov. 1974

Then SU(3) was extended to SU(4) scheme with four constituent quarks: u, d, c, s.

And when bottom quark b and top quark t were discovered, SU(6) is used to include b, t quarks

Unfortunately all these higher groups are badly broken, due to the large mass differences among the 6 quarks. Members of the multiplet have very different masses.

In the SU(2) scheme, proton and neutron almost same mass, so are the pions (π^+, $\underline{\underline{\pi^0}}$, π^-, triplet)

Depending on the circumstances, one assigns
effective (constituent) mass or
current (bare) mass to quarks

(28)

current

Table 4.4 Quark masses (MeV/c^2)

constituent

Quark flavor	Bare mass	Effective mass
u	2	336
d	5	340
s	95	486
c	1300	1550
b	4200	4730
t	174 000	177 000

Warning: These numbers are somewhat speculative and model dependent [12].

However, there is an important *caveat* in this neat hierarchy: isospin, $SU(2)$, is a very 'good' symmetry; the members of an isospin multiplet differ in mass by at most 2 or 3%, which is about the level at which electromagnetic corrections would be expected.* But the Eightfold Way, $SU(3)$, is a badly 'broken' symmetry; mass splittings within the baryon octet are around 40%. The symmetry breaking is even worse when we include charm; the $\Lambda_c^+(udc)$ weighs more than twice the $\Lambda(uds)$, although they are in the same $SU(4)$ supermultiplet. It is worse still with bottom, and absolutely terrible with top, which doesn't form bound states at all.

Why is isospin such a good symmetry, the Eightfold Way fair, and flavor $SU(6)$ so poor? The Standard Model blames it all on the quark masses. Now, the theory of quark masses is a slippery business, given the fact that they are not accessible to direct experimental measurement. Various arguments [9] suggest that the u and d quarks are *intrinsically* very light, about 10 times the mass of the electron. However, within the confines of a hadron, their *effective* mass is much greater. The precise value, in fact, depends on the context; it tends to be a little higher in baryons than in mesons (more on this in Chapter 5). In somewhat the same way, the effective inertia of a spoon is greater when you're stirring honey than when you're stirring tea, and in either case it exceeds the true mass of the spoon. Generally speaking, the effective mass of a quark in a hadron is about 350 MeV/c^2 greater than its bare mass [10] (see Table 4.4). Compared to this, the quite different *bare* masses of up and down quarks are practically irrelevant; they *function* as though they had identical masses. But the s quark is distinctly heavier, and the c, b, and t quarks are widely separated. Apart from the differences in quark masses, the strong interactions treat all flavors equally. Thus isospin is a good symmetry because the effective u and d masses are so nearly equal (which is to say, on a more fundamental level, because their *bare* masses are so small); the Eightfold Way is a *fair* symmetry because the effective mass of the strange quark is not *too* far from that of the u and d. But strong interaction

* Indeed, it used to be thought that isospin was an *exact* symmetry of the strong interactions, and *all* of the symmetry breaking was attributable to electromagnetic contamination. The fact that the $n-p$ mass splitting is in the wrong direction to be purely electromagnetic was troubling, however, and we now believe that $SU(2)$ is only an *approximate* symmetry of the strong interactions.

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chapter 4

Discrete symmetries

P = parity, space inversion (mirror reflection)

C = charge conjugation (positive charge $\rightleftharpoons$ negative charge)

T = time reversal (motion reversal, $\underset{\sim}{p} \to -\underset{\sim}{p}$)

First discuss space inversion, P

Introduce space inversion in 3-dimensional physical space, then as an operator in quantum mechanics

Parity operator π, π unitary, Hermitian (observable)

Downfall of parity conservation in weak interaction

C. S. Wu experiment 1956

parity broken in weak decay.

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9

Symmetry Transformation in Quantum Mechanics

In QM, state $|\psi\rangle$ and operators are the key elements in analyzing a physical problem. Clearly, a symmetry transformation is associated with an operator in Hilbert space.

Define symmetry transformation in QM:
A symmetry transformation operator U is a 1-1 mapping that maps a dynamically possible state, say $|\psi\rangle$, to another dynamically possible state $|\psi'\rangle$, namely $U: |\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$, such that the transition probability is preserved, i.e. no change in the transition probability.

$$\text{Transition probability from state } |\psi\rangle \text{ to state } |\varphi\rangle = |\langle\varphi|\psi\rangle|^2$$
$$\text{Transition probability from state } |\psi'\rangle \text{ to state } |\varphi'\rangle = |\langle\varphi'|\psi'\rangle|^2$$
$$\text{Transition probability is preserved means: } |\langle\varphi|\psi\rangle|^2 = |\langle\varphi'|\psi'\rangle|^2$$

In other words, the transition probability before applying the symmetry transformation U is the same as the transition probability after the applying the same symmetry transformation.

From the definition, we can show that
Theorem
(i) U is unitary.
(ii) U is linear or anti-linear.
(iii) If U does not depend on time explicitly, then $[U, H] = 0$

Proof:
(i) An operator A is unitary if $A^+ = A^{-1}$.
We want to show that symmetry transformation U is unitary.
Given that $|\langle\varphi|\psi\rangle|^2 = |\langle\varphi'|\psi'\rangle|^2$, $|\psi'\rangle = U|\psi\rangle$, $|\varphi'\rangle = U|\varphi\rangle \Rightarrow \langle\varphi'| = \langle\varphi|U^+$, then

$$|\langle\varphi'|\psi'\rangle|^2 = |\langle\varphi|U^+U|\psi\rangle|^2 = |\langle\varphi|\psi\rangle|^2$$

This is true for any arbitrary state $|\varphi\rangle$ and $|\psi\rangle$, hence $U^+U = 1$.

By associativity rule $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, we can show that $UU^+ = 1$.
From the definition of an inverse operator, $U^{-1}U = 1 = UU^{-1}$, so we have $U^{-1} = U^+$, i.e. U is unitary.

(ii) To show U is linear or anti-linear, we consider a state $|\psi\rangle$ and a state $\alpha|\psi\rangle$, where α is a complex number.

U is linear if $U(\alpha|\psi\rangle) = \alpha(U|\psi\rangle)$;
U is anti-linear if $U(\alpha|\psi\rangle) = \alpha^*(U|\psi\rangle)$, where α^* = complex conjugate of α.

Given $|\langle\varphi|\psi\rangle|^2 = |\langle\varphi'|\psi'\rangle|^2$, one can have either $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$ or $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^*$. Then U is linear if $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$; U is anti-linear if $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^*$.

Consider first the case $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$,
LHS: $\langle\varphi'|\psi'\rangle = \langle\varphi'|U|\psi\rangle$, let $|\psi\rangle = \lambda|\Omega\rangle$, where λ = constant, then

$$\langle\varphi'|\psi'\rangle = \langle\varphi'|U\lambda|\Omega\rangle$$

RHS: $\langle\varphi|\psi\rangle = \langle\varphi|\lambda|\Omega\rangle = \lambda\langle\varphi|\Omega\rangle = \lambda\langle\varphi'|\Omega'\rangle = \lambda\langle\varphi'|U|\Omega\rangle = \langle\varphi'|\lambda U|\Omega\rangle$

Since $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$, then $\langle\varphi'|U\lambda|\Omega\rangle = \langle\varphi'|\lambda U|\Omega\rangle$.

As $\langle\varphi'|$ and $|\Omega\rangle$ are arbitrary, so $U\lambda = \lambda U$, i.e. U is linear.

If we start from $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^*$ instead of $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$, then we can show $U\lambda = \lambda^* U$, i.e. U is anti-linear.
LHS=$\langle\varphi'|\psi'\rangle$=$\langle\varphi'|U|\psi\rangle = \langle\varphi'|U\lambda|\omega\rangle$
RHS= $\langle\varphi|\psi\rangle^* = (\langle\varphi|\lambda|\omega\rangle)^* = (\lambda\langle\varphi|\omega\rangle)^* = \lambda^*\langle\varphi|\omega\rangle^* = \lambda^*\langle\varphi'|\omega'\rangle = \lambda^*\langle\varphi'|U|\omega\rangle = \langle\varphi'|\lambda^* U|\omega\rangle$

LHS=RHS gives $\langle\varphi'|U\lambda|\omega\rangle$=$\langle\varphi'|\lambda^* U|\omega\rangle$, that is, $U\lambda|\omega\rangle$=$\lambda^* U|\omega\rangle$since $\langle\varphi'|$ is arbitrary. Or, $U\lambda = \lambda^* U$.
Note: Put $|\omega\rangle = a|\omega_1\rangle + b|\omega_2\rangle$, then $U\lambda|\omega\rangle = U\lambda(a|\omega_1\rangle + b|\omega_2\rangle) = U(\lambda a|\omega_1\rangle + \lambda b|\omega_2\rangle) = U\lambda a|\omega_1\rangle + U\lambda b|\omega_2\rangle$

$$= \lambda^* a^* U|\omega_1\rangle + \lambda^* b^* U|\omega_2\rangle$$

That is equivalent to $U(a|\varphi\rangle + b|\psi\rangle) = a^* U|\varphi\rangle + b^* U|\psi\rangle$.

So we have shown that U is either linear or anti-linear.

A linear unitary operator is usually called a unitary operator. An anti-linear unitary operator is called anti-unitary operator. In nature, most of the symmetry transformations are associated with unitary operators. Time reversal and charge conjugation are associated with anti-unitary operators.

(iii) To show $[U,H] = 0$ if $\frac{\partial U}{\partial t} = 0$, we consider 2 dynamically possible states $|\psi\rangle$ and $|\psi'\rangle = U|\psi\rangle$.

By definition, a dynamically possible state is a state that satisfies the TDSE:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

$$i\hbar \frac{\partial}{\partial t}|\psi'\rangle = H|\psi'\rangle$$

From the 2nd equation above, we have LHS:

$$i\hbar \frac{\partial}{\partial t} U|\psi\rangle = i\hbar \left(\frac{\partial U}{\partial t}\right)|\psi\rangle + Ui\hbar \frac{\partial}{\partial t}|\psi\rangle = i\hbar \left(\frac{\partial U}{\partial t}\right)|\psi\rangle + UH|\psi\rangle$$

For RHS: $H|\psi'\rangle = HU|\psi\rangle$

Since LHS = RHS, we have

$$i\hbar \left(\frac{\partial U}{\partial t}\right)|\psi\rangle + UH|\psi\rangle = HU|\psi\rangle$$

If U does not depend on time explicitly, i.e. $\frac{\partial U}{\partial t} = 0$, then we have

$$(UH - HU)|\psi\rangle = 0$$

As $|\psi\rangle$ is any dynamically possible state, so we have $[U, H] = 0$.

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Definition

Linear: $U(a|\phi\rangle + b|\psi\rangle)$

$= a\,U|\phi\rangle + b\,U|\psi\rangle$

a, b = complex numbers

Anti linear:

$U(a|\phi\rangle + b|\psi\rangle)$

$= a^* U|\phi\rangle + b^* |\psi\rangle$

Restrict (sufficient)

$U(\lambda|\phi\rangle) = \lambda\,U|\phi\rangle$

$U\lambda|\phi\rangle) = \lambda^* U|\phi\rangle$

λ = complex number

From defn of sym: $|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2$

we get $\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle$

or

$\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle^*$

show linear or antilinear using the above 2 equalities.

Proof:

LHS: $\langle\phi'|\psi'\rangle = \langle\phi'|U|\psi\rangle$

Trick $|\psi\rangle = \lambda|\omega\rangle$

$\langle\phi'|\psi'\rangle = \langle\phi'|U\lambda|\omega\rangle$

RHS: $\langle\phi|\psi\rangle = \langle\phi|\lambda|\omega\rangle = \lambda\langle\phi|\omega\rangle$

RHS:

$$\langle \phi | \psi \rangle = \langle \phi | \lambda | \omega \rangle = \lambda \langle \phi | \omega \rangle$$

$$= \lambda \langle \phi' | \omega' \rangle$$

$$= \lambda \langle \phi' | U | \omega \rangle$$

We have shown

$$\text{LHS} \left(= \langle \phi' | \psi' \rangle\right) = \langle \phi' | U \lambda | \omega \rangle$$

$$\text{RHS} \left(= \langle \phi | \psi \rangle\right) = \lambda \langle \phi' | U | \omega \rangle$$
$$= \langle \phi' | \lambda U | \omega \rangle$$

$$\Rightarrow \quad U \lambda = \lambda U$$

i.e. U is a linear operator.

Using the second equality

$$\langle \phi' | \psi' \rangle = \langle \phi | \psi \rangle^*$$

and using the same way, can show

$$U \lambda = \lambda^* U$$

λ^* = complex conjugate of λ

i.e. U antilinear