SP3176 The Universe Assignment 4
Cosmological Models

The Friedmann equations

$$\dot{S}^2 + kc^2 = \frac{8\pi G}{3}\rho S^2 \tag{1}$$

and

$$-2\frac{\ddot{S}(t)}{S(t)} - \frac{\dot{S}(t)^2 + kc^2}{S(t)^2} = \frac{8\pi G}{c^2}p(t) \tag{2}$$

model the dynamics of the expanding Universe. By solving the differential equation(s) to obtain the scale factor S as a function of time, one can plot and visualise how the size of the Universe evolves. As Eq.(1) is a first order differential equation while Eq.(2) is second order, one will usually try to obtain the solution for $S(t)$ using Eq.(1). While ρ and p are functions of time, we will assume that their ratio $\frac{p}{\rho c^2} = w$ is constant. There are different types of energy content that one can have. Each type has its characteristic value of w. See Section 3.7 of the lecture notes for details.

1. Before we plunge in and solve the differential equation Eq.(1) numerically, let us obtain the analytical solutions for the simpler cases of flat space. Putting $k = 0$ and

$$\rho \propto S^{-3(1+w)} \tag{3}$$

into Eq.(1), we have

$$\begin{aligned}
\dot{S}^2 &= \text{constant } \cdot S^{-3-3w+2} \\
\frac{dS}{dt} &= c_1 S^{(-1-3w)/2}
\end{aligned}$$

For the case of matter, where $w = 0$,

$$\begin{aligned}
\frac{dS}{dt} &= c_1 S^{-1/2} \\
\int S^{1/2} dS &= \int c_1 dt \\
\frac{S^{3/2}}{3/2} &= c_1 t + c_2
\end{aligned}$$

Suppose at $t = 0$, the scale factor is S_0. This helps to determine c_2:

$$\begin{aligned}
\frac{S_0^{3/2}}{3/2} &= c_2 \\
c_2 &= \frac{2}{3}S_0^{3/2}
\end{aligned}$$

Hence

$$S = \left(\frac{3}{2}c_1 t + S_0^{3/2}\right)^{2/3}$$

If S_0 is very very small (Universe is very very small to begin with), then $S \propto t^{2/3}$.

Similarly, obtain analytical solutions of $S(t)$ in the cases of

(a) $k = 0, w = \frac{1}{3}$ (flat, radiation-dominated Universe), and

(b) $k = 0, w = -1$ (flat, Λ-dominated Universe).

2. Solve the first Friedmann equation Eq.(1) numerically for Universes that are

(a) matter-dominated $w = 0$

(i) flat $k = 0$

(ii) negatively curved $k = -1$

(iii) positively curved $k = 1$

(b) radiation-dominated $w = \frac{1}{3}$

(i) flat $k = 0$

(ii) negatively curved $k = -1$

(iii) positively curved $k = 1$

(c) Λ-dominated $w = -1$

(i) flat $k = 0$

(ii) negatively curved $k = -1$

(iii) positively curved $k = 1$

Set S_0 (value of S at $t = 0$) to be the same in all cases.

Present your solutions graphically, in ways where the solutions can be easily compared by the reader.[1]

Briefly discuss your findings.

Administrative details

This assignment is to be submitted individually, and carries 5% weightage of the total assessment for The Universe. Q1 can be either handwritten or typed. The graphical solutions for Q2 should be be clearly formatted and presented. All solutions should be contained within a single pdf ducument. Your numerical codes can either be a attached to your solution document, or shared via a hyperlink. File name for the assignment submission: A4_A#######X.pdf where A#######X is your matric number.

Submission: Week 12 Saturday.

[1] The stupidity of your reader (zhihan) is not within your control, but your clarity is!