Homework 3 Solution

1.

(1) Monopole contribution

$$V_{mono} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \rho(\vec{r}\,')\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_0^{2\pi}\int_0^{\pi}\int_{R-d}^{R}\frac{\sigma_0\cos\theta'}{d}\,r'^2\sin\theta'\,dr'\,d\theta'\,d\varphi'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\cdot 2\pi\int_{R-d}^{R}\frac{\sigma_0}{d}\,r'^2\,dr'\int_0^{\pi}\cos\theta'\sin\theta'\,d\theta'$$

$$= \frac{1}{2\varepsilon_0 r}\frac{\sigma_0}{d}\,\frac{1}{3}\,r'^3\Big|_{R-d}^{R}\cdot\underbrace{\int_{\theta=0}^{\theta=\pi}(-\cos\theta')\,d[\cos\theta]}_{=0}$$

$$= 0$$

Dipole contribution

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\cos\theta'\,\rho(\vec{r}\,')\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\int_0^{2\pi}\int_0^{\pi}\int_{R-d}^{R} r'\cos\theta'\cdot\frac{\sigma_0\cos\theta'}{d}\cdot r'^2\sin\theta'\,dr'\,d\theta'\,d\varphi'$$

$$= \frac{2\pi\,\sigma_0}{4\pi\varepsilon_0 r^2 d}\int_{R-d}^{R} r'^3\,dr'\int_0^{\pi}\cos^2\theta'\sin\theta'\,d\theta'$$

$$= \frac{\sigma_0}{2\varepsilon_0 r^2 d} \; \frac{1}{4} r'^4 \Big|_{R-d}^{R} \int_{\theta=0}^{\theta=\pi} [-\cos^2\theta'] \, d(\cos\theta')$$

$(u \equiv \cos\theta')$

$$= \frac{\sigma_0}{8\varepsilon_0 r^2 d} \left[R^4 - (R-d)^4 \right] \left(-\frac{1}{3}\right) u^3 \Big|_{1}^{-1}$$

$$= \frac{\sigma_0}{12\varepsilon_0 r^2 d} \left[R^4 - (R-d)^4 \right]$$

Quadrupole contribution

$$V_q = \frac{1}{4\pi\varepsilon_0 r^3} \int r'^2 \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2} \right) \rho(\vec{r}') \, d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0 r^3} \int_0^{2\pi} \int_0^{\pi} \int_{R-d}^{R} r'^2 \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2} \right) \frac{\sigma_0 \cos\theta'}{d} \cdot r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi'$$

$$= \frac{\sigma_0}{2\varepsilon_0 r^3 d} \int_{R-d}^{R} r'^4 \, dr' \int_0^{\pi} \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2} \right) \cos\theta' \sin\theta' \, d\theta'$$

$$= \frac{\sigma_0}{2\varepsilon_0 r^3 d} \; \frac{1}{5} r'^5 \Big|_{R-d}^{R} \cdot \int_{\theta=0}^{\theta=\pi} \left(-\frac{3}{2}\cos^2\theta' + \frac{1}{2} \right) \cos\theta' \, d(\cos\theta')$$

$$= \frac{\sigma_0}{2\varepsilon_0 r^3 d} \; \frac{1}{5} \left[(R)^5 - (R-d)^5 \right] \int_1^{-1} \left(-\frac{3}{2}u^2 + \frac{1}{2} \right) u \, du$$

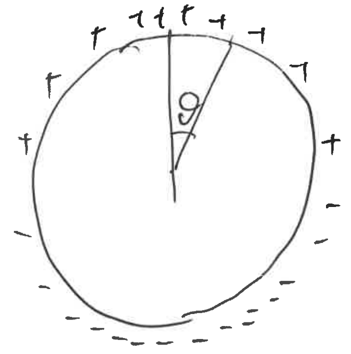
$$= \left(-\frac{3}{8}u^4 + \frac{1}{4}u^2 \right) \Big|_1^{-1} = 0$$

$$= 0$$

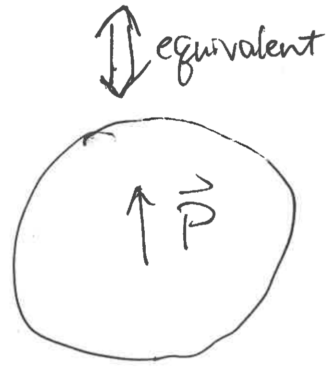
(2) Monopole & Quadrupole contributions vanish while the Dipole term remains. This is because of the following

In the $d \to 0$ limit, $\rho(r,\theta,d) \to \infty$, volume charge density diverges, but the surface charge density

$$\sigma(R,\theta,d) = \rho(r=R,\theta,d)\cdot d = \sigma_0 \cos\theta$$



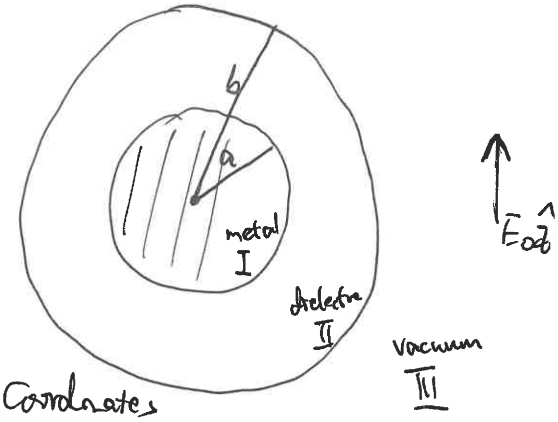
This charge distribution, depicted on the right, resembles the surface bound charge of a uniformly polarized sphere.



This is textbook example 4.2. A uniformly ~~charged~~ polarized sphere consists of many tiny electric dipoles aligned along the same direction. That is why the potential it produces would only contain the dipole term. Quadrupole contribution is zero because we didn't put quadrupoles in the polarized sphere in the first place.

2.

(1) Suppose a geometry shown on the right



We can suppose a general solution from

~~Spets~~ separation of variables in spherical coordinates

as

$$
\begin{cases}
V_{I}\ (0 \le r \le a) = 0 \\
V_{II}\ (a \le r \le b) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l \cos\theta) \\
V_{III}\ (b \le r) = \sum_{l=0}^{\infty} \left(A'_l r^l + \frac{B'_l}{r^{l+1}}\right) P_l (\cos\theta)
\end{cases}
$$

With applicable boundary conditions

$$
\begin{cases}
V_{II}\ (r=a) = V_{I}\ (r=a) = 0 & \text{(i)} \\
V_{II}\ (r=b) = V_{III}\ (r=b) & \text{(ii)} \\
\left(\varepsilon_0 \frac{\partial V_{III}}{\partial r} - \varepsilon_r \varepsilon_0 \frac{\partial V_{II}}{\partial r}\right)\Big|_{r=b} = \sigma_f = 0 & \text{(iii)} \\
V_{III} \to -E_0 r\cos\theta \text{ when } r \to \infty & \text{(iv)}
\end{cases}
$$

free charge $= 0$

From (iv) $\Rightarrow$ $A'_1 = -E_0\cos\theta$, while all other $A'_l = 0$ $(l \ne 1)$

$$\Rightarrow V_{III} = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B'_l}{r^{l+1}} P_l(\cos\theta)$$

From (i) $\Rightarrow \sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}}\right) P_l(\cos\theta) = 0$

$$\Rightarrow \quad A_l a^l + \frac{B_l}{a^{l+1}} = 0 \quad \Rightarrow \quad B_l = -a^{2l+1} A_l$$

From (ii) $\Rightarrow \sum_{l=0}^{\infty} A_l \left(b^l - \frac{a^{2l+1}}{b^{l+1}}\right) P_l(\cos\theta) = -E_0 b\cos\theta + \sum_{l=0}^{\infty} \frac{B_l'}{b^{l+1}} P_l(\cos\theta)$ (2)

From (iii) $\Rightarrow -E_0\cos\theta + \sum_{l=0}^{\infty} (-l-1)\frac{B_l'}{b^{l+2}} P_l(\cos\theta)$

$$= \varepsilon_r \left[A_l\left(\sum_{l=0}^{\infty} l b^{l-1} - \frac{a^{2l+1}}{b^{l+2}}(-l-1)\right) P_l(\cos\theta) \right] \quad (3)$$

Now let's examon the coefficients for different orders.

$\boxed{\text{For } l \neq 1}$

Eq. (2) $\Rightarrow \left(b^l - \frac{a^{2l+1}}{b^{l+1}}\right) A_l = \frac{B_l'}{b^{l+1}}$

Eq. (3) $\Rightarrow (-l-1)\frac{B_l'}{b^{l+2}} = A_l\left[l b^{l-1} - \frac{a^{2l+1}}{b^{l+2}}(-l-1)\right]$

This gives $A_l = 0$, $B_l' = 0$ for all $l \neq 1$

For $\boxed{l = 1}$

Eq. (2) $\Rightarrow \left(b - \frac{a^3}{b^2}\right) A_1 = \frac{B_1'}{b^2} - E_0 b$ (4)

Eq. (3) $\Rightarrow -E_0 - 2\frac{B_1'}{b^3} = \varepsilon_r A_1\left(1 + \frac{2a^3}{b^3}\right)$ (5)

Eq.(4) $\Rightarrow \quad B_1' = (b^3-a^3)A_1 + E_0 b^3$

Eq.(5) $\Rightarrow \quad B_1' = \frac{\varepsilon_r}{2}(-b^3-2a^3)A_1 - \frac{E_0}{2}b^3$

$$\Rightarrow \quad (b^3-a^3)A_1 + E_0 b^3 = \frac{\varepsilon_r}{2}(-b^3-2a^3)A_1 - \frac{E_0}{2}b^3$$

$$\Rightarrow \quad A_1 = \frac{-3b^3 E_0}{2(b^3-a^3)+\varepsilon_r(b^3+2a^3)} \qquad (6)$$

$$B_1' = \frac{-3b^3(b^3-a^3)E_0}{2(b^3-a^3)+\varepsilon_r(b^3+2a^3)} + E_0 b^3$$

$$= \frac{2b^6-2a^3b^3+\varepsilon_r b^6+2\varepsilon_r a^3b^3-3b^6+3a^3b^3}{2(b^3-a^3)+\varepsilon_r(b^3+2a^3)} E_0$$

$$= \frac{(\varepsilon_r-1)b^6+(2\varepsilon_r+1)a^3b^3}{2(b^3-a^3)+\varepsilon_r(b^3+2a^3)} E_0 \qquad (7)$$

These coefficients, plugged in V_{II}, V_{IV} will generate our solution

$$V_{II} = \frac{-3b^3 r + 3a^3b^3/r^2}{2(b^3-a^3)+\varepsilon_r(b^3+2a^3)} E_0\cos\theta = \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

$$= \left(A_1 r - \frac{a^3}{r^2}A_1\right)\cos\theta$$

$$V_{IV} = \frac{(\varepsilon_r-1)b^6+(2\varepsilon_r+1)a^3b^3}{2(b^3-a^3)+\varepsilon_r(b^3+2a^3)} \frac{E_0}{r^2}\cos\theta - E_0 r\cos\theta = \frac{B_1'}{r^2}\cos\theta - E_0 r\cos\theta$$

(2)

From the solution

$$\begin{cases} V_{I} = 0 \\ V_{II} = \left(r - \frac{a^3}{r^2}\right) A_1 \cos\theta \\ V_{III} = -E_0 r\cos\theta + \frac{B_1'}{r^2}\cos\theta \end{cases}$$

<u>At the metal - dielectric interface</u>

$$\sigma_{I} = -\varepsilon_0 \left.\frac{\partial V_{II}}{\partial r}\right|_{r=a}$$ which consists of two parts:

= "free" charge due to polarized metal $\sigma_{fI} = -\varepsilon \left.\frac{\partial V_{II}}{\partial r}\right|_{r=a}$

bound charge due to polarized dielectric $\sigma_{bI} = (\varepsilon - \varepsilon_0) \left.\frac{\partial V_{II}}{\partial r}\right|_{r=a}$

$$\Rightarrow \sigma_{I} = -\varepsilon_0 \left(1 + \frac{2a^3}{a^3}\right) A_1 \cos\theta = -3\varepsilon_0 A_1 \cos\theta$$

<u>At the insulator vacuum interface</u>

$$\sigma_{II} = -\varepsilon_0 \left[\left.\frac{\partial V_{III}}{\partial r}\right|_{r=b} - \left.\frac{\partial V_{II}}{\partial r}\right|_{r=b} \right]$$ which consists of two parts:

= "free" charge $\sigma_{fII} = 0$ (consistent with boundary condition (iii))

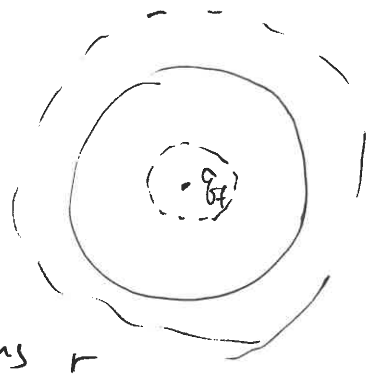
= "bound" charge $\sigma_{bII} = \sigma_{II}$

$$\sigma_{II} = -\varepsilon_0 \left[-E_0\cos\theta - \frac{2B_1'}{b^3}\cos\theta - \left(1 + \frac{2a^3}{b^3}\right) A_1 \cos\theta \right]$$

Where we can find expressions for A_l & B_l in Eq. (6) & (7).

3.

(1) Apply the Gauss's law for electric displacement with a Gaussian surface of a sphere having radius r



For $r<R$ inside the sphere

$$\oint \vec{D}_{in} \cdot d\vec{a} = Q_{fenc} = q_f \Rightarrow \vec{D}_{in} = \frac{q_f}{4\pi r^2}\hat{r}$$

For $r>R$ outside the sphere it's the same

$$\oint \vec{D}_{out} \cdot d\vec{a} = q_f \Rightarrow \vec{D}_{out} = \frac{q_f}{4\pi r^2}\hat{r}$$

$$\Rightarrow \vec{E}_{in} = \frac{1}{\varepsilon}\vec{D}_{in} = \frac{1}{\varepsilon_0\varepsilon_r}\vec{D}_{in} = \frac{1}{\varepsilon_0}\frac{1}{1+\chi_e}\vec{D}_{in} = \frac{q_f}{4\pi\varepsilon_0(1+\chi_e)r^2}\hat{r}$$

$$\vec{E}_{out} = \frac{1}{\varepsilon_0}\vec{D}_{out} = \frac{q_f}{4\pi\varepsilon_0 r^2}\hat{r}$$

(2) $$\vec{P} = \varepsilon_0\chi_e\vec{E}_{in} = \frac{\chi_e q_f}{4\pi(1+\chi_e)r^2}\hat{r}$$

(3) surface bound charge density

$$\sigma_b = \vec{P}\cdot\hat{r}\Big|_{r=R} = \frac{\chi_e q_f}{4\pi(1+\chi_e)r^2}\Big|_{r=R} = \frac{\chi_e q_f}{4\pi(1+\chi_e)R^2}$$

Volume bound charge density

$$\rho_b = -\nabla \cdot \vec{P}$$

$$= -\frac{\chi_e q_f}{4\pi(1+\chi_e)} \underbrace{\nabla \cdot \left(\frac{\hat{r}}{r^2}\right)}_{4\pi \delta^3(\vec{r})} = -\frac{\chi_e q_f}{1+\chi_e} \delta^3(\vec{r})$$

$\downarrow$

$4\pi \delta^3(\vec{r})$ according to lecture note

4.

(1) Judging from the right hand rule, q is positive

(2) Charged particle in a magnetic field engages in a cyclotron motion, whose trajectory is circular, The radius of that circle is R and one has

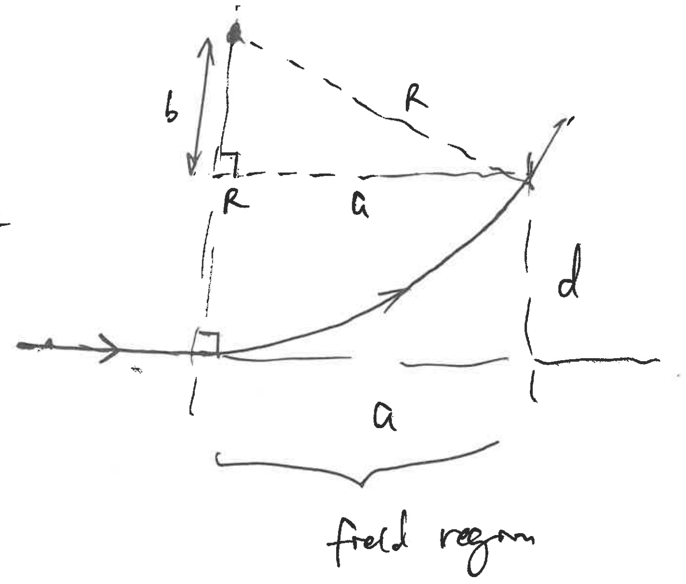
$$\underline{Q v B} = \underline{m \frac{v^2}{R}}$$

$\uparrow$ $\uparrow$

balance of Lorentz force & centripetal force

$$\Rightarrow \quad QB = P/R \quad \Rightarrow \quad R = \frac{P}{QB}$$ where P is momentum

Now we know in the field region with width a, the trajectory does not complete a full circle. Rather, it is an arc with radius R.



Then we have $b = \sqrt{R^2 - a^2}$

$$d = R - b = R - \sqrt{R^2 - a^2}$$

Then applying $R = \dfrac{P}{QB}$

$$\Rightarrow \quad d = \frac{P}{QB} - \sqrt{\frac{P^2}{Q^2B^2} - a^2}$$

$$\Rightarrow \quad \cancel{\frac{P^2}{Q^2B^2}} - a^2 = \cancel{\frac{P^2}{Q^2B^2}} + d^2 - 2\frac{Pd}{QB}$$

$$\Rightarrow \quad P = (a^2 + d^2)\frac{QB}{2d}$$

(3) In fact, the force experienced by the proton is irrelavant to where the proton is in the field region, because B field does no work and v is not changed in magnitude.

$E = 5\ MeV$

$\Rightarrow \frac{mv^2}{2} = 5 \times 10^6 \times 1.6 \times 10^{-19}$

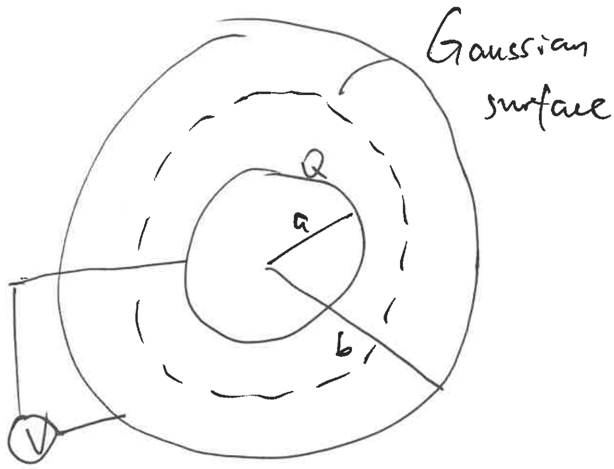
with proton mass $m = 1.67 \times 10^{-27}$ kg

$v = 3.1 \times 10^7\ m/s$

$F = QVB = 1.6 \times 10^{-19} \times 3.1 \times 10^7 \times 1.5 = 7.4 \times 10^{-12}\ N$

(* No relativistic effect was considered)

5.



Suppose inner shell carries charge Q.

Note that both shells need to be charged to create the potential difference

From Gauss's law $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \cdot \hat{r}$ in between the shells

From Ohm's law $\vec{J} = \sigma\vec{E} = \frac{\sigma Q}{4\pi\varepsilon_0 r^2} \cdot \hat{r}$

So total current $I = \oint \vec{J} \cdot d\vec{a} = 4\pi r^2 \frac{\sigma Q}{4\pi\varepsilon_0 r^2} = \frac{\sigma Q}{\varepsilon_0}$

On the other hand, the votage difference between the shells $\Delta V = -\int_{out}^{in} \vec{E} \cdot d\vec{r}$

$$= -\int_b^a \frac{Q}{4\pi \varepsilon_0 r^2} dr = \frac{Q}{4\pi \varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

The effective resistance $R = \frac{\Delta V}{I} = \frac{Q}{4\pi \varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \cdot \frac{\varepsilon_0}{\sigma Q}$

$$= \frac{1}{4\pi \sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$