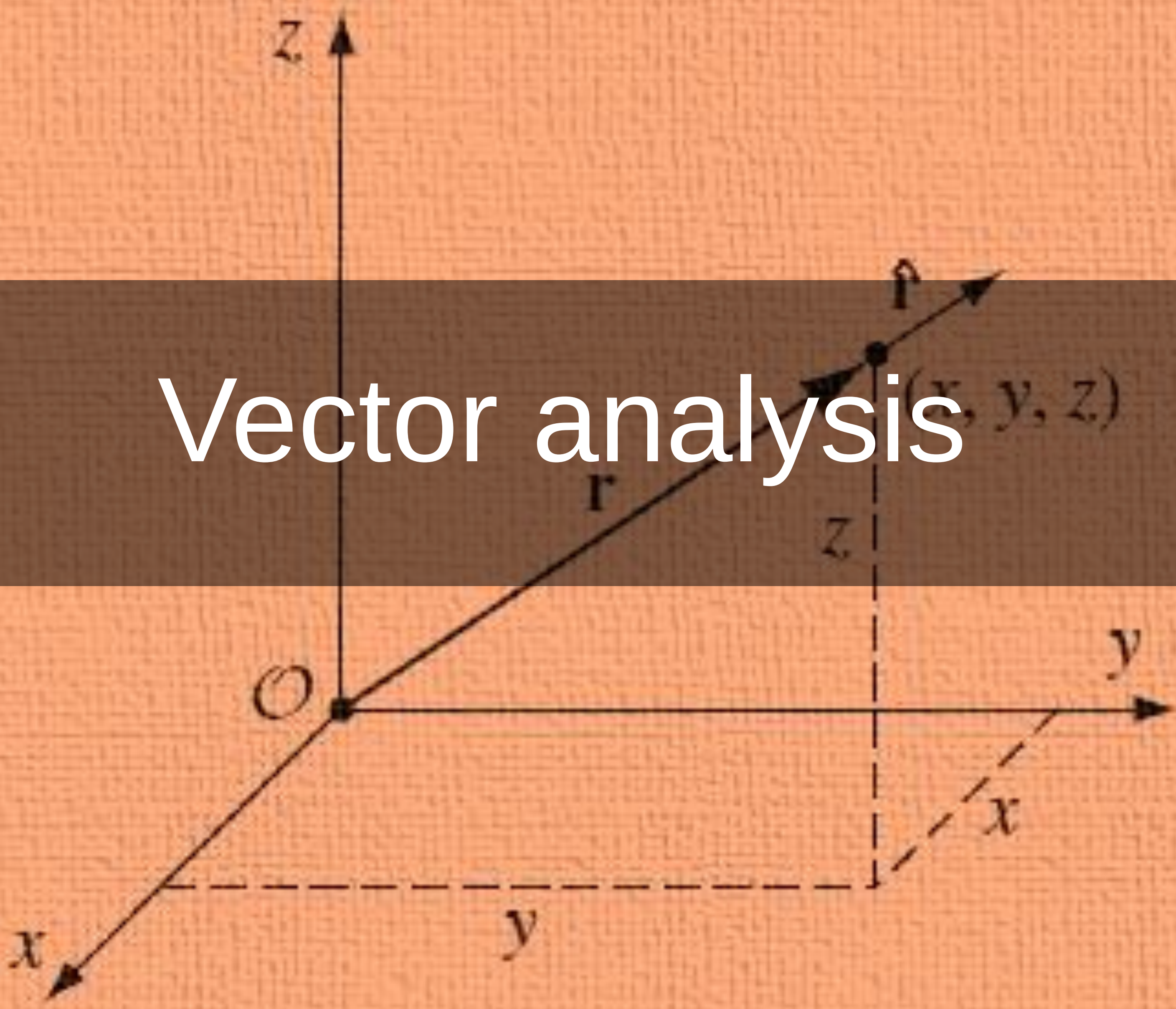
Vector analysis

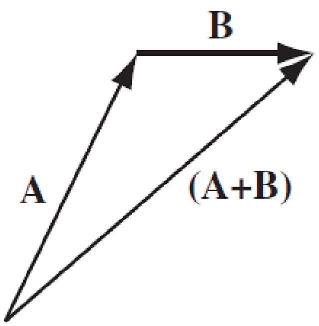


Vector algebra

Vector operations

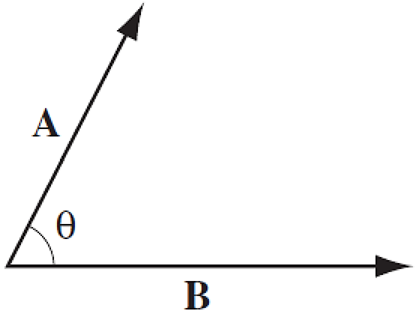
- A vector has direction and magnitude, <u>but no location</u>

- Addition of vectors
  - Commutative $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - Associative $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$



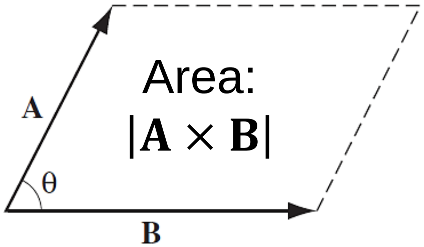
- Multiplication of vectors by a scalar
  - Distributive $a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$

- Dot product of vectors $\mathbf{A} \cdot \mathbf{B} \equiv AB\cos\theta$
  - Commutative $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
  - Distributive $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$



Vector operations

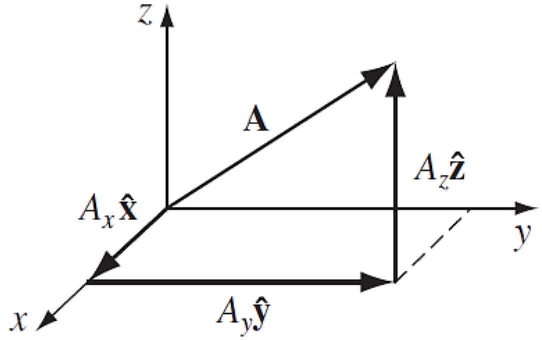
- Cross product of vectors $\mathbf{A} \times \mathbf{B} \equiv AB \sin\theta\, \hat{\mathbf{n}}$
  - Unit vector $\hat{\mathbf{n}} \perp \mathbf{A}$, $\hat{\mathbf{n}} \perp \mathbf{B}$
  - Orientation of $\hat{\mathbf{n}}$: right-hand rule
  - Distributive $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$
  - Anticommutative $(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B})$



- Component form (as opposed to abstract form) of vectors

$$\mathbf{A} = \underline{A_x}\hat{\mathbf{x}} + \underline{A_y}\hat{\mathbf{y}} + \underline{A_z}\hat{\mathbf{z}}$$

Components



  - Components: projections of the vector on three coordinate axes
  - $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are mutually perpendicular unit vectors

Vertical operations

- Addition (in component form)

$$\mathbf{A} + \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) + (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$= (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}.$$

- Multiplication (in component form)

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}$$

- Dot product (in component form)

$$\mathbf{A} \cdot \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$= A_xB_x + A_yB_y + A_zB_z.$$

  - Modulus (magnitude) of a vector $\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2$

$$\Rightarrow \quad A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Vector operations

- Cross product (in component form)

$$\mathbf{A} \times \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}$$

  - A more neat form $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

C1.cross

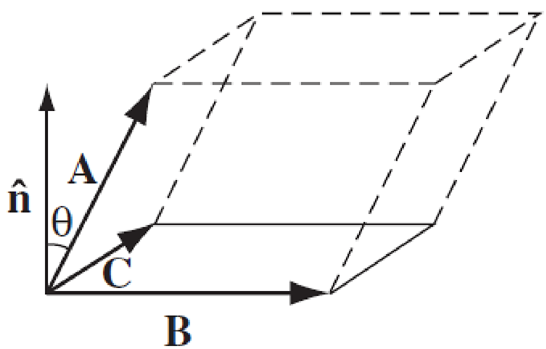
- Triple product
  - Scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$
    - Can "rotate"

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

  - Vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
    - "BAC-CAB" rule

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Volume $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$



Vector operations

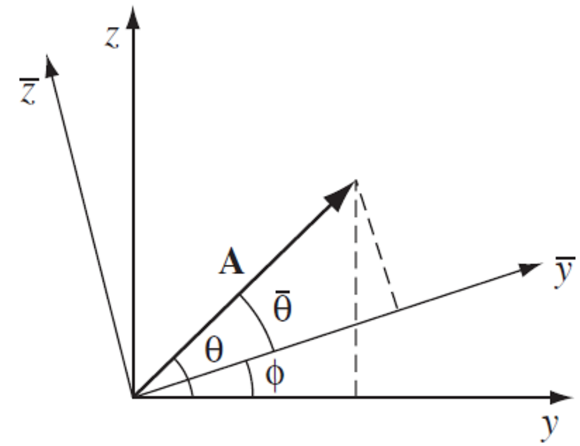
- Vector transformation
  - Consider axes rotation about x
    - In original coordinates

$$A_y = A\cos\theta, \qquad A_z = A\sin\theta,$$

    - In new coordinates

$$\bar{A}_y = A\cos\bar{\theta} = A\cos(\theta-\phi) = A(\cos\theta\cos\phi + \sin\theta\sin\phi)$$
$$= \cos\phi\, A_y + \sin\phi\, A_z,$$
$$\bar{A}_z = A\sin\bar{\theta} = A\sin(\theta-\phi) = A(\sin\theta\cos\phi - \cos\theta\sin\phi)$$
$$= -\sin\phi\, A_y + \cos\phi\, A_z.$$

$$\Rightarrow \begin{pmatrix} \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$



  - Generally, for 3D rotation

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Vector calculus

Differential calculus

- Ordinary derivative for function $f(x)$

$$\underbrace{df}_{\text{Effect}} = \underbrace{\left(\frac{df}{dx}\right)}_{\text{Slope}} \underbrace{dx}_{\text{Cause}}$$

- For a scalar function of three variables $T(x, y, z)$
  - How T varies with a movement depends on direction, and

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

- Gradient $\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$

$$dT = \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}})$$

$$= (\nabla T) \cdot (d\mathbf{l}),$$

*dT is largest when $d\mathbf{l}$ is directed along ∇T

Differential calculus

- The Del operator $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$
  - Can “act upon” functions
  - Not a vector by itself
  - But follows arithmetic rules that apply to ordinary vectors
  - Major vector operator responsible for <u>all three types of derivatives</u> in vector calculus

- The three types of derivatives
  - Gradient ∇T (Directly act on a scalar function)
  - Divergence $\nabla \cdot \mathbf{v}$ (Dot product with a vector function)
  - Curl $\nabla \times \mathbf{v}$ (Cross product with a vector function)

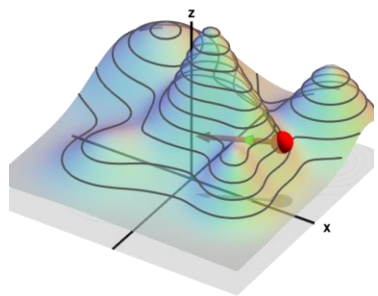
Differential calculus

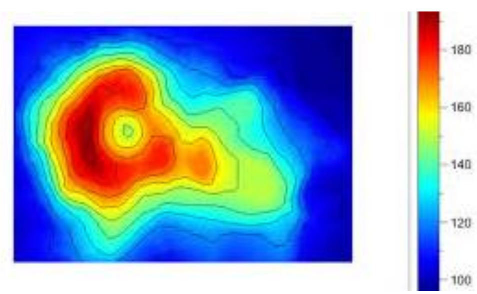
- Scalar field versus vector field (2D example)

Scalar field

$$T(x, y)$$

- A scalar number associated with each x, y location

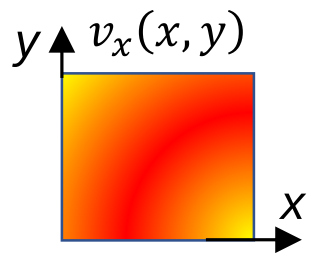


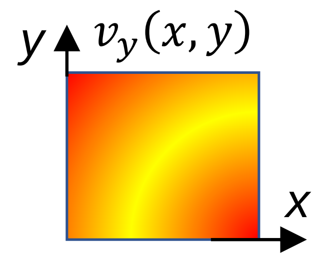


Vector field

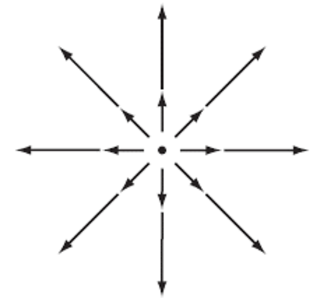
$$\mathbf{v}(x, y) = v_x(x, y)\hat{\mathbf{x}} + v_y(x, y)\hat{\mathbf{y}}$$

- A vector (with 2 components) associated with each x, y location
  - 1^{st} way to draw





  - 2^{nd} way to draw



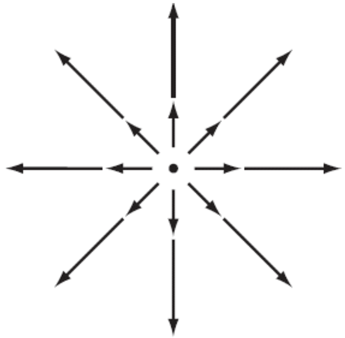
Differential calculus

- Divergence

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$$

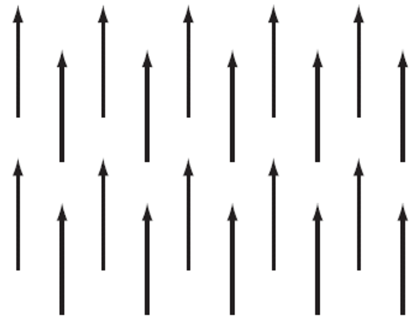
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

  - Is a measure of how much the vector spreads out (diverges) for a point in question



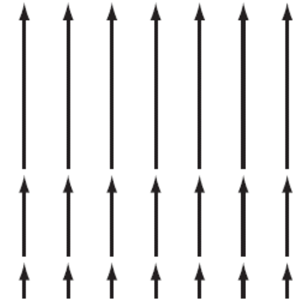
$\mathbf{v} = \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

$\nabla \cdot \mathbf{v} = 3$



$\mathbf{v} = \hat{\mathbf{z}}$

$\nabla \cdot \mathbf{v} = 0$



$\mathbf{v} = z\hat{\mathbf{z}}$

$\nabla \cdot \mathbf{v} = 1$

  - $\nabla \cdot \mathbf{v}$ can also be a scalar field

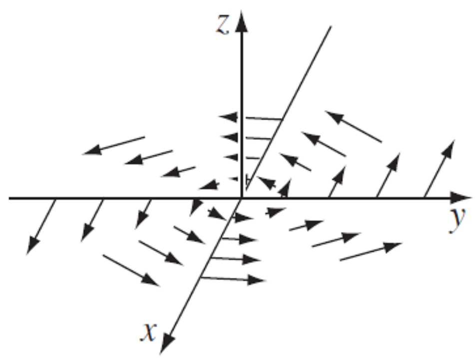
Differential calculus

- Curl

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

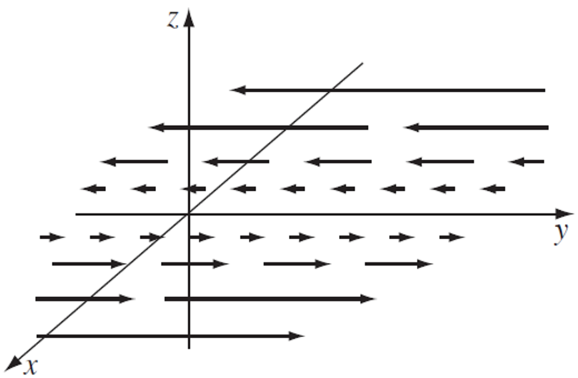
$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

  - Is a measure of how much the vector swirls around a point in question



$$\mathbf{v} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$$

$$\nabla \times \mathbf{v} = 2\hat{\mathbf{z}}$$



$$\mathbf{v} = x\hat{\mathbf{y}}$$

$$\nabla \times \mathbf{v} = \hat{\mathbf{z}}$$

  - $\nabla \times \mathbf{v}$ can also be a vector field

Differential calculus

- Ordinary derivatives versus derivatives in vector calculus
  - Trivial ones

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \quad \Rightarrow \quad \nabla(f+g) = \nabla f + \nabla g$$

$$\frac{d}{dx}(kf) = k\frac{df}{dx} \quad \Rightarrow \quad \begin{cases} \nabla(kf) = k\nabla f \\ \nabla \cdot (k\mathbf{A}) = k(\nabla \cdot \mathbf{A}) \\ \nabla \times (k\mathbf{A}) = k(\nabla \times \mathbf{A}) \end{cases}$$

  - Product rules $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$\Rightarrow \begin{cases} \nabla(fg) = f\nabla g + g\nabla f \\ \nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \\ \nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f) \\ \nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \\ \nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f) \\ \nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) \end{cases}$$

Differential calculus

- Ordinary derivatives versus derivatives in vector calculus

  - Quotient rules (easily derived from product rules)

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} \Rightarrow \begin{cases} \nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \\ \nabla\cdot\left(\frac{\mathbf{A}}{g}\right) = \frac{g(\nabla\cdot\mathbf{A}) - \mathbf{A}\cdot(\nabla g)}{g^2} \\ \nabla\times\left(\frac{\mathbf{A}}{g}\right) = \frac{g(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla g)}{g^2} \end{cases}$$

  - How? (Proof of a few expressions)

C1.derivatives

  - Why?

    - To make full use of the known quantities: derivatives of individual scalar & vector field

Differential calculus

- Second derivatives
  - Divergence of gradient (Laplace) $\nabla \cdot (\nabla T) = \nabla^2 T$

$$\nabla \cdot (\nabla T) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right)$$
$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}.$$

  - Curl of gradient (always zero) $\nabla \times (\nabla T) = \mathbf{0}$
  - Gradient of divergence (nothing special) $\nabla(\nabla \cdot \mathbf{v})$
  - Divergence of curl (always zero) $\nabla \cdot (\nabla \times \mathbf{v}) = \mathbf{0}$
  - Curl of curl $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$

    where $\nabla^2 \mathbf{v} \equiv (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}$

Integral calculus

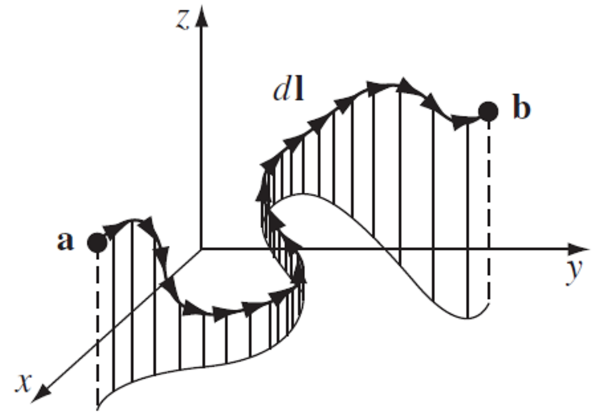
- Line integral

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l}$$

From point a to point b

Vector field

Infinitesimal displacement vector



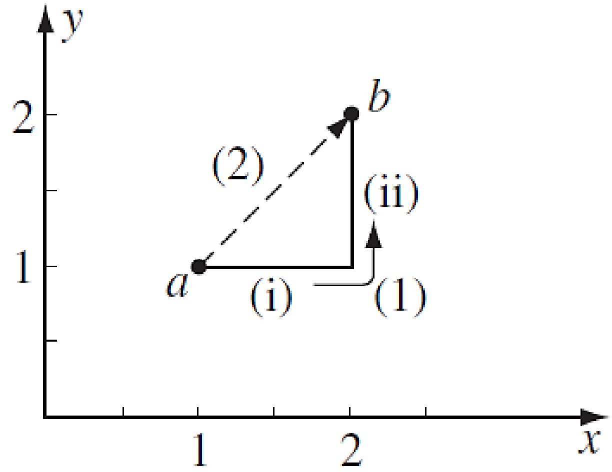
  - If path forms a closed loop, denote as $\oint \mathbf{v} \cdot d\mathbf{l}$
  - Generally line integrals depend on the path taken from a to b
  - But sometimes it can be independent from the path chosen as long as points a and b are fixed, for certain $\mathbf{v}$
  - 2D example: $\mathbf{v}(x, y) = v_x(x, y)\hat{\mathbf{x}} + v_y(x, y)\hat{\mathbf{y}}$

$$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}}$$

Integral calculus

- Line integral

Example 1.6. Calculate the line integral of the function $\mathbf{v} = y^2\,\hat{\mathbf{x}} + 2x(y+1)\,\hat{\mathbf{y}}$ from the point $\mathbf{a} = (1, 1, 0)$ to the point $\mathbf{b} = (2, 2, 0)$, along the paths (1) and (2) in Fig. 1.21. What is $\oint \mathbf{v} \cdot d\mathbf{l}$ for the loop that goes from $\mathbf{a}$ to $\mathbf{b}$ along (1) and returns to $\mathbf{a}$ along (2)?

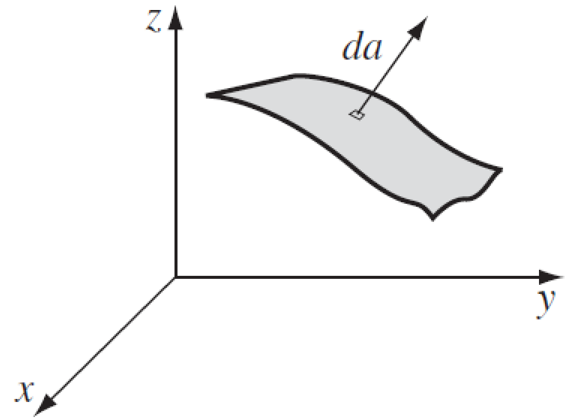


C1.exmp1.6

Integral calculus

- Surface integral

$$\underbrace{\int_S}_{\text{The specified surface}} \underbrace{\mathbf{v}}_{\text{Vector field}} \cdot \underbrace{d\mathbf{a}}_{\text{Infinitesimal patch of area}}$$

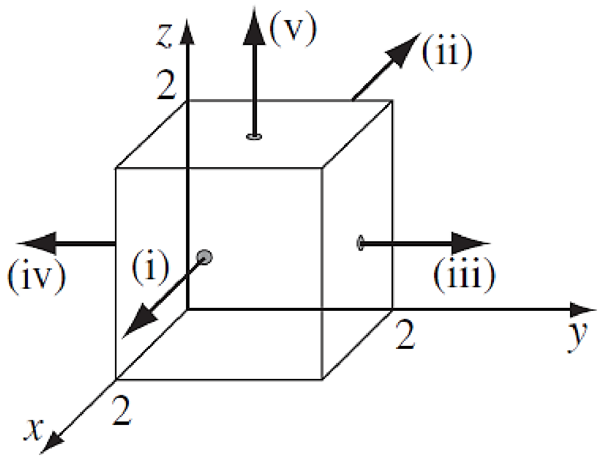


  - If surface is closed, denote as $\oint \mathbf{v} \cdot d\mathbf{a}$
  - Orientation of surface normal is intrinsically ambiguous
  - But when surface is closed, take “outward” direction as positive
  - Sometimes the integral is independent of surface chosen and is determined by the boundary line, for certain $\mathbf{v}$

Integral calculus

- Surface integral

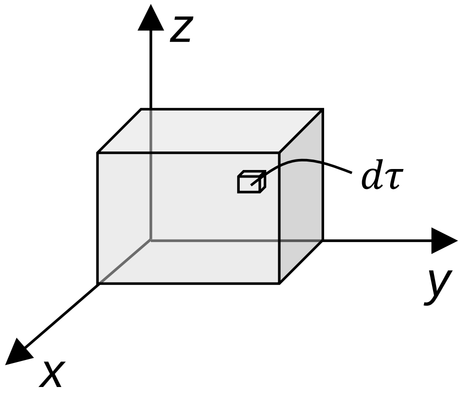
Example 1.7. Calculate the surface integral of $\mathbf{v} = 2xz\,\hat{\mathbf{x}} + (x+2)\,\hat{\mathbf{y}} + y(z^2-3)\,\hat{\mathbf{z}}$ over five sides (excluding the bottom) of the cubical box (side 2) in Fig. 1.23. Let "upward and outward" be the positive direction, as indicated by the arrows.



Integral calculus

- Volume integral

$$\underbrace{\int_{\mathcal{V}}}_{\text{The specified volume}} \underbrace{T}_{\text{Scalar field}} \, \underbrace{d\tau}_{\text{Infinitesimal volume element}}$$



  - In Cartesian coordinates $d\tau = dx\, dy\, dz$

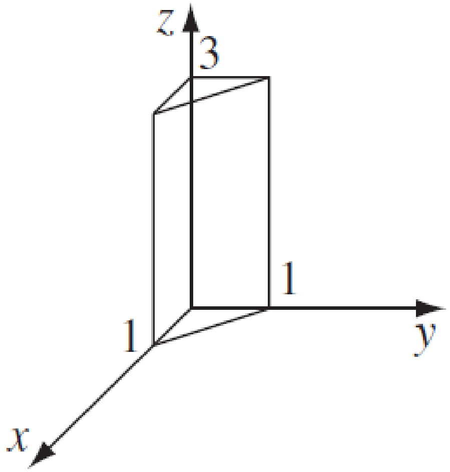
  - Integrand is usually scalar fields, but can also be vector fields (trivial extension)

$$\int \mathbf{v}\, d\tau = \int (v_x\, \hat{\mathbf{x}} + v_y\, \hat{\mathbf{y}} + v_z\, \hat{\mathbf{z}}) d\tau = \hat{\mathbf{x}} \int v_x d\tau + \hat{\mathbf{y}} \int v_y d\tau + \hat{\mathbf{z}} \int v_z d\tau$$

Integral calculus

- Volume integral

Example 1.8. Calculate the volume integral of $T = xyz^2$ over the prism in Fig. 1.24.



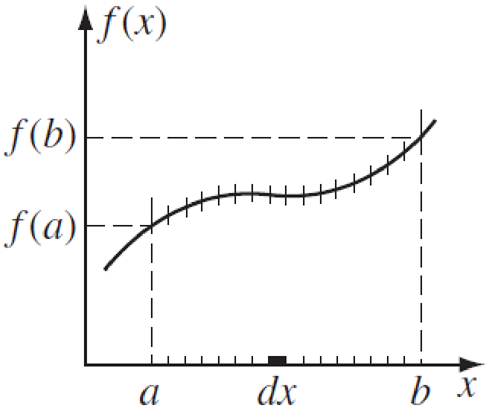

C1.exmp1.8

Integral calculus

- Fundamental theorem of calculus

$$\int_a^b \underline{F(x)}\, dx = f(b) - f(a)$$

where $df = \underline{(df/dx)}dx$



  - $F(x)dx$: infinitesimal change to function f when x changes to $x + dx$

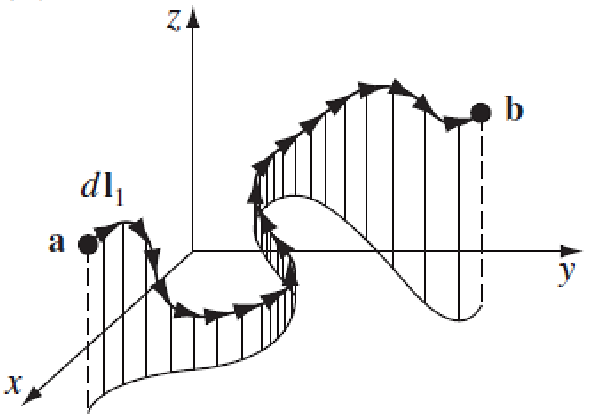
  - **Integral of a derivative over some region is given by the value of the function at the boundaries (end points)**

  - The format above appears for **all three types of derivatives** when attempting to integrate them

Integral calculus

- Fundamental theorem for gradients

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$



  - Segment by segment

$$dT = (\nabla T) \cdot d\mathbf{l}_1$$
$$dT = (\nabla T) \cdot d\mathbf{l}_2$$
$$\vdots$$

  - As long as end points are fixed, integration of a gradient is independent of the path
    - $$\underbrace{\int_{\mathbf{a}}^{\mathbf{b}}}_{\text{any path}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$
    - $$\oint (\nabla T) \cdot d\mathbf{l} = 0$$

Integral calculus

- Fundamental theorem for divergences

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$$

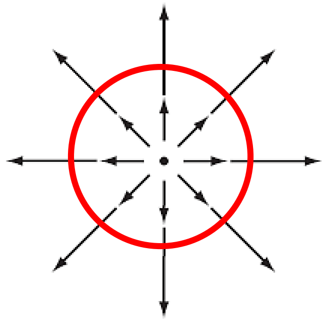
  - Volume integral of divergence equals a integral of the field with respect to the surface (that encloses the volume)

  - Named Gauss's theorem, Green's theorem, or divergence theorem

  - Two ways of knowing how many sources are there

    - Count the "faucets" $\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau$

    - Measure how much "water" flows out through the boundary $\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$

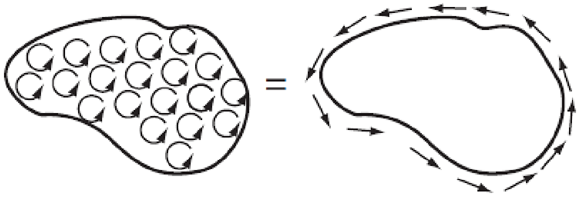


Integral calculus

- Fundamental theorem for curls

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

  - Surface integral of curl equals a integral of the field with respect to the boundary line
  - As long as the boundary line remains the same, integration of curl is independent of the choice of surface
  - Named Stokes' theorem
  - Two ways of knowing how many swirls are there
    - Count the swirls $\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$
    - Measure how much flow is following the boundary $\oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$



Integral calculus

- Integration by parts

$$\int_a^b f\left(\frac{dg}{dx}\right)dx = -\int_a^b g\left(\frac{df}{dx}\right)dx + fg\Big|_a^b$$

  - Can transfer the derivative from g to f, at the cost of a minus sign and a boundary term

- Integration by parts for gradient and curl

$$\int_{\mathcal{V}} f(\boldsymbol{\nabla}\cdot\mathbf{A})\,d\tau = -\int_{\mathcal{V}} \mathbf{A}\cdot(\boldsymbol{\nabla} f)\,d\tau + \oint_{\mathcal{S}} f\mathbf{A}\cdot d\mathbf{a}$$

$$\boldsymbol{\nabla}\cdot(f\mathbf{A}) = f(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla} f)$$

$$\int_{\mathcal{S}} f(\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{a} = \int_{\mathcal{S}} [\mathbf{A}\times(\boldsymbol{\nabla} f)]\cdot d\mathbf{a} + \oint_{\mathcal{P}} f\mathbf{A}\cdot d\mathbf{l}$$

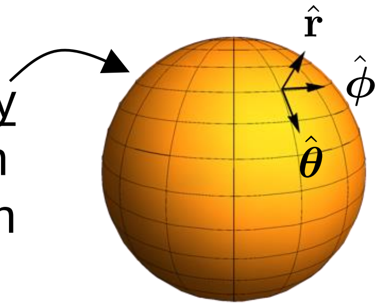
$$\boldsymbol{\nabla}\times(f\mathbf{A}) = f(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla} f)$$

$$\int_{\mathcal{V}} \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})\,d\tau = \int_{\mathcal{V}} \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,d\tau + \oint_{\mathcal{S}} (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$$

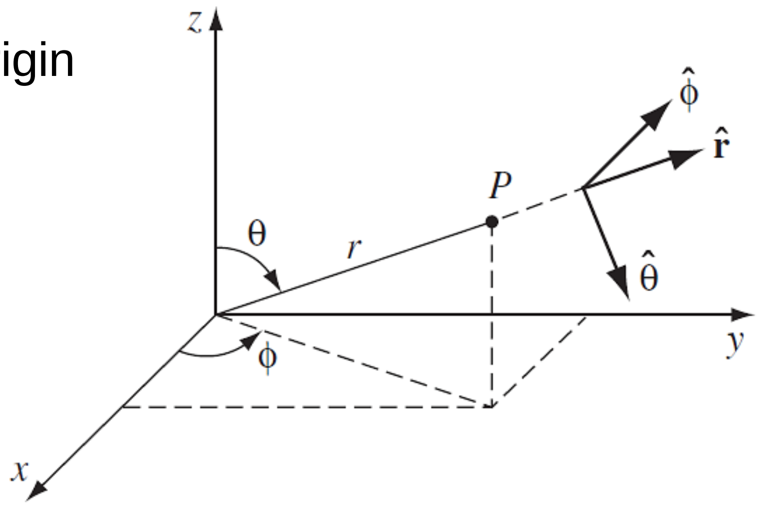
Curvilinear coordinates

- Curvilinear coordinates
  - Coordinate systems where the coordinate lines may be curved, and can be more convenient to use than Cartesian coordinates for problems adopting certain geometries



- Spherical coordinates
  - Represent a point by (r, θ, ϕ) instead of (x, y, z)
    - r: distance of the point from the origin
    - θ: angle down from the $+z$ axis
    - ϕ: azimuthal angle

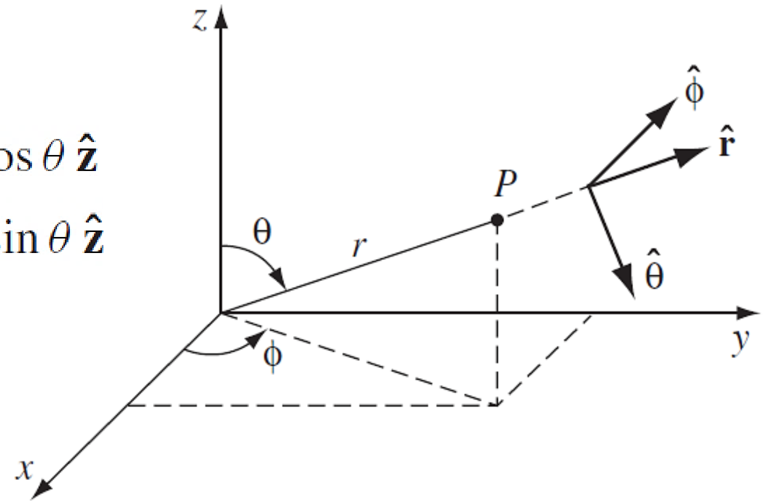
$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$



Curvilinear coordinates

- Spherical coordinates
  - Unit vectors

$$\begin{cases} \hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$



- Mutually orthogonal
- a vector can be expressed as

$$\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}}$$

- Unit vectors depend on position of the point of interest

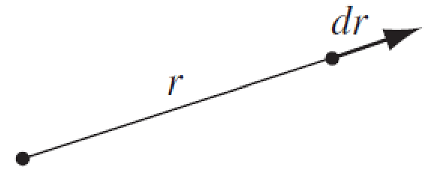
$$\hat{\boldsymbol{r}}(\theta,\phi) \qquad \hat{\boldsymbol{\theta}}(\theta,\phi) \qquad \hat{\boldsymbol{\phi}}(\theta,\phi)$$

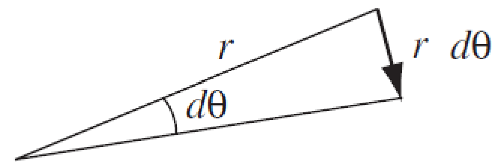
Example: charges on a sphere interacting with an electric field

Curvilinear coordinates

- Spherical coordinates
  - Infinitesimal element of length

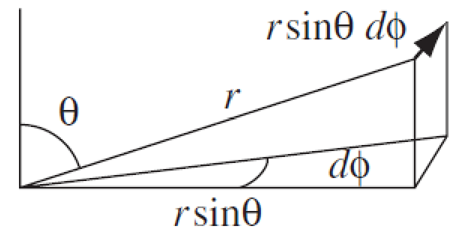
$$\begin{cases} dl_r = dr \\ dl_\theta = r\, d\theta \\ dl_\phi = r \sin\theta\, d\phi \end{cases}$$





  - Infinitesimal displacement vector

$$d\mathbf{l} = dr\, \hat{\mathbf{r}} + r\, d\theta\, \hat{\boldsymbol{\theta}} + r \sin\theta\, d\phi\, \hat{\boldsymbol{\phi}}$$



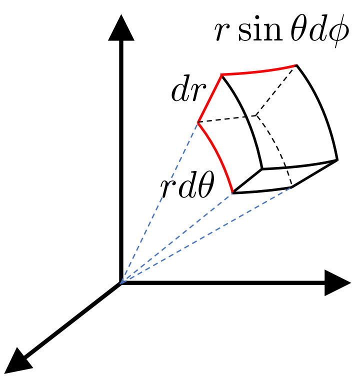
  - Infinitesimal element of volume

$$d\tau = dl_r\, dl_\theta\, dl_\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

  - Infinitesimal element of surface
    - Geometry dependent

      e.g. on a sphere

$$d\mathbf{a}_1 = dl_\theta\, dl_\phi\, \hat{\mathbf{r}} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$$



Curvilinear coordinates

- Spherical coordinates
  - Derivatives
    - Gradient $$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$
    - Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$
    - Curl

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.$$

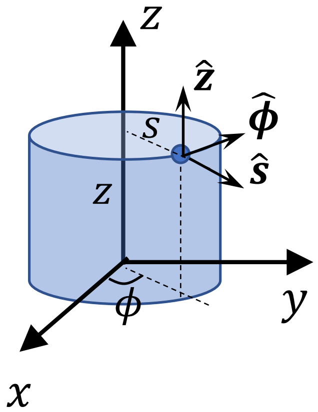
C1.spherical

- Laplacian

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

Curvilinear coordinates

- Cylindrical coordinates
  - Coordinates
    - s: distance of the point to the z axis
    - ϕ: azimuthal angle
    - z: same as the z in Cartesian coordinates



  - Unit vectors

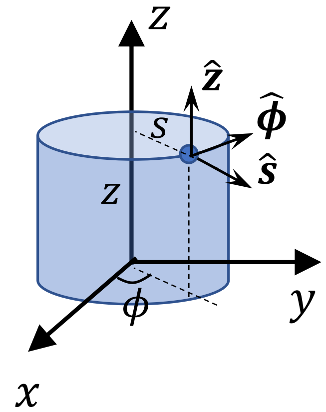
$$\begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

    - Mutually orthogonal, just like all other coordinates

Curvilinear coordinates

- Cylindrical coordinates
  - Infinitesimal element of length

$$\begin{cases} dl_s = ds \\ dl_\phi = s\, d\phi \\ dl_z = dz \end{cases}$$



  - Infinitesimal displacement vector

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

  - Infinitesimal element of volume

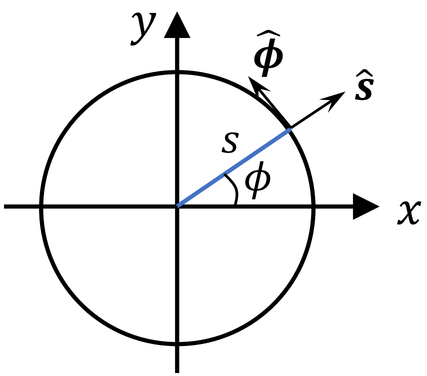
$$d\tau = s\,ds\,d\phi\,dz$$

  - Derivatives
    - Gradient $$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$
    - Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curvilinear coordinates

- Polar coordinates (in 2D)
  - Spherical coordinates without the θ, or cylindrical coordinates without the z
  - Unit vectors

$$\begin{aligned}\hat{\mathbf{s}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}\end{aligned}$$



  - Infinitesimal element of length

$$\begin{cases} dl_s = ds \\ dl_\phi = s\,d\phi \end{cases}$$

  - Infinitesimal displacement vector $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}}$
  - Infinitesimal element of area $da = s\,ds\,d\phi$