PC3261: Classical Mechanics II

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Lecture 1: Kinematics

Kronecker delta symbol

- **Kronecker delta symbol**: completely symmetric

$$\delta_{ij} = \delta_{ji}\,, \qquad \delta_{ij} \equiv \begin{cases} 0\,, & i \neq j \\ 1\,, & i = j \end{cases}\,, \qquad i, j = 1, 2, 3$$

- Useful identities:

$$A_i = \sum_{j=1}^{3} \delta_{ij} A_j\,, \qquad \sum_{k=1}^{3} \delta_{ik}\delta_{kj} = \delta_{ij}\,, \qquad \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} = 3$$

Levi-Civita symbol

- **Levi-Civita symbol**: completely anti-symmetric

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj}\,, \qquad \epsilon_{123} \equiv +1\,, \qquad i,j,k = 1,2,3$$

- Product of Levi-Civita symbols:

$$\epsilon_{ijk}\epsilon_{mnr} = \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{ir} \\ \delta_{jm} & \delta_{jn} & \delta_{jr} \\ \delta_{km} & \delta_{kn} & \delta_{kr} \end{vmatrix}$$

- Useful identities:

$$\sum_{k=1}^{3}\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\,, \quad \sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{mjk}\epsilon_{njk} = 2\delta_{mn}\,, \quad \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\epsilon_{ijk} = 6$$

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

$$\begin{aligned}
\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mnk} &= \sum_{k=1}^{3} \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{ik} \\ \delta_{jm} & \delta_{jn} & \delta_{jk} \\ \delta_{km} & \delta_{kn} & \delta_{kk} \end{vmatrix} \\
&= \sum_{k=1}^{3} \delta_{im} \begin{vmatrix} \delta_{jn} & \delta_{jk} \\ \delta_{kn} & \delta_{kk} \end{vmatrix} - \sum_{k=1}^{3} \delta_{in} \begin{vmatrix} \delta_{jm} & \delta_{jk} \\ \delta_{km} & \delta_{kk} \end{vmatrix} + \sum_{k=1}^{3} \delta_{ik} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \sum_{k=1}^{3} \delta_{im} \left(\delta_{jn}\delta_{kk} - \delta_{jk}\delta_{kn}\right) - \sum_{k=1}^{3} \delta_{in} \left(\delta_{jm}\delta_{kk} - \delta_{jk}\delta_{km}\right) \\
&\qquad + \sum_{k=1}^{3} \delta_{ik} \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) \\
&= 3\delta_{im}\delta_{jn} - \delta_{im}\delta_{jn} - 3\delta_{in}\delta_{jm} + \delta_{in}\delta_{jm} + \delta_{jm}\delta_{in} - \delta_{jn}\delta_{im} \\
&= \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \qquad \blacksquare
\end{aligned}$$

Cartesian coordinate system

- Cartesian coordinates: $(x_1, x_2, x_3) \equiv (x, y, z)$

$$-\infty < x < \infty\,, \qquad -\infty < y < \infty\,, \qquad -\infty < z < \infty$$

- Cartesian unit basis vectors: $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \equiv (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \qquad \rightarrow \qquad \begin{cases} \hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_y \cdot \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z = 1 \\ \hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_y \cdot \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_x = 0 \end{cases}$$

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \sum_{k=1}^{3} \epsilon_{ijk}\, \hat{\mathbf{e}}_k \qquad \rightarrow \qquad \begin{cases} \hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_y \times \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_y \end{cases}$$

- Cartesian unit basis vectors are constant

Position vector

- **Position** of a particle in the space is specified by a vector relative to the *spatial origin* of a given *reference frame* known as **position vector**

- Position vector in the Cartesian coordinate system: (x, y, z) are the Cartesian coordinates of the particle

$$\mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z = \sum_{i=1}^{3} x_i\,\hat{\mathbf{e}}_i$$

- Motion of the particle traces a **trajectory** in the space and can be described mathematically by an one-dimensional **curve**

- Trajectory of the motion of particle can be specified by the position vector *parameterized* by **time** relative to the *temporal origin* of the reference frame

$$\mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y + z(t)\,\hat{\mathbf{e}}_z = \sum_{i=1}^{3} x_i(t)\,\hat{\mathbf{e}}_i$$

Velocity vector

- **Velocity vector**: rate of change of the position vector with respect to time

$$\mathbf{v}(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \equiv \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} \equiv \dot{\mathbf{r}}(t)$$

- Velocity vector is *tangent* to the trajectory of the particle at any given instant of time

- **Speed**: magnitude of the velocity vector

$$v(t) \equiv |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)}$$

- Cartesian coordinate system:

$$\dot{\mathbf{r}}(t) = \dot{x}(t)\,\hat{\mathbf{e}}_x + \dot{y}(t)\,\hat{\mathbf{e}}_y + \dot{z}(t)\,\hat{\mathbf{e}}_z \quad \Rightarrow \quad \dot{r}(t) \equiv |\dot{\mathbf{r}}(t)| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}$$

Acceleration vector

- **Acceleration vector**: rate of change of the velocity vector with respect to time

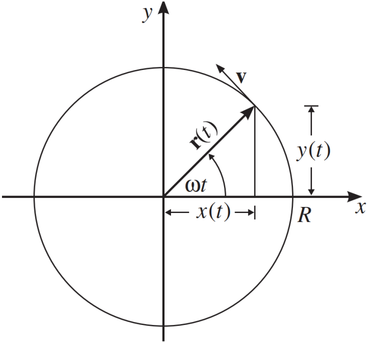
$$\mathbf{a}(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} \equiv \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} \equiv \dot{\mathbf{v}}(t) = \frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} \equiv \ddot{\mathbf{r}}(t)$$

- Cartesian coordinate system:

$$\ddot{\mathbf{r}}(t) = \ddot{x}(t)\,\hat{\mathbf{e}}_x + \ddot{y}(t)\,\hat{\mathbf{e}}_y + \ddot{z}(t)\,\hat{\mathbf{e}}_z \quad \Rightarrow \quad \ddot{r}(t) \equiv |\ddot{\mathbf{r}}(t)| = \sqrt{\ddot{x}^2(t) + \ddot{y}^2(t) + \ddot{z}^2(t)}$$

Example: Uniform circular motion

- A particle moves in a circle lying in the xy plane (centered at the origin and radius R) with constant angular speed ω counter-clockwise as viewed from $+z$ axis. The particle is on the $+x$ axis at $t = 0$



EXERCISE 1.1: Find the particle's velocity and acceleration vectors. What are the magnitude and direction of the particle's acceleration?

$$\mathbf{r}(t) = R\cos\omega t\,\hat{\mathbf{e}}_x + R\sin\omega t\,\hat{\mathbf{e}}_y \qquad \blacksquare$$

$$r(t) \equiv |\mathbf{r}(t)| = R \qquad \blacksquare$$

$$\mathbf{v}(t) \equiv \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = -R\omega\sin\omega t\,\hat{\mathbf{e}}_x + R\omega\cos\omega t\,\hat{\mathbf{e}}_y \qquad \blacksquare$$

$$\mathbf{v}(t)\cdot\mathbf{r}(t) = 0$$

$$v(t) \equiv |\mathbf{v}(t)| = R\omega \qquad \blacksquare$$

$$\mathbf{a}(t) \equiv \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = -R\omega^2\cos\omega t\,\hat{\mathbf{e}}_x - R\omega^2\sin\omega t\,\hat{\mathbf{e}}_y \qquad \blacksquare$$

$$\mathbf{a}(t)\cdot\mathbf{r}(t) = -R^2\omega^2 \qquad \blacksquare$$

$$a(t) \equiv |\mathbf{a}(t)| = R\omega^2 \qquad \blacksquare$$

Another mathematical description of trajectory

- Trajectory of the motion of particle can also be represented mathematically by the position vector parameterized by **arc length** along the trajectory

- Arc length:

$$s(t) = \int_0^t \mathrm{d}s = \int_0^t |\mathrm{d}\mathbf{r}| = \int_0^t \sqrt{\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]^2 + \left[\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right]^2 + \left[\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right]^2}\,\mathrm{d}t$$

- Speed:

$$v(t) = |\mathbf{v}(t)| = \left|\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\right| = \frac{\mathrm{d}s(t)}{\mathrm{d}t}$$

- A set of three orthogonal unit vectors, parameterized by arc length, can be constructed at each point of the trajectory

Moving trihedral

- Tangent and normal vectors: κ is called the **curvature**

$$\hat{\mathbf{e}}_T(s) \equiv \frac{\mathrm{d}\mathbf{r}(s)}{\mathrm{d}s} \quad \Rightarrow \quad \mathbf{v}(s) = v(s)\,\hat{\mathbf{e}}_T(s)$$

$$\hat{\mathbf{e}}_N(s) \equiv \frac{1}{\kappa(s)}\,\frac{\mathrm{d}\hat{\mathbf{e}}_T(s)}{\mathrm{d}s}$$

- Binormal vector: τ is called the **torsion**

$$\hat{\mathbf{e}}_B(s) \equiv \hat{\mathbf{e}}_T(s) \times \hat{\mathbf{e}}_N(s)\,, \qquad \frac{\mathrm{d}\hat{\mathbf{e}}_B(s)}{\mathrm{d}s} \equiv -\tau(s)\,\hat{\mathbf{e}}_N(s)$$

EXERCISE 1.2: Show that the acceleration of a particle moving along a trajectory $\mathbf{r}(t)$ is give by

$$\mathbf{a}(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + \frac{v^2(t)}{\rho}\,\hat{\mathbf{e}}_N\,,$$

where $\rho \equiv 1/\kappa$ is its radius of curvature.

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}s(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathbf{r}(s)}{\mathrm{d}s} = v(t)\,\hat{\mathbf{e}}_T \qquad \blacksquare$$

$$\begin{aligned}
\mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} &= \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + v(t)\,\frac{\mathrm{d}\hat{\mathbf{e}}_T}{\mathrm{d}t} \\
&= \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + v(t)\,\frac{\mathrm{d}s(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\hat{\mathbf{e}}_T}{\mathrm{d}s} \\
&= \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + v^2(t)\,\kappa\,\hat{\mathbf{e}}_N \\
&= \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + \frac{v^2(t)}{\rho}\,\hat{\mathbf{e}}_N \qquad \blacksquare
\end{aligned}$$

Example: Circular helix

- Position vector: a, b and ω are constants

$$\mathbf{r}(t) = a\cos\omega t\,\hat{\mathbf{e}}_x + a\sin\omega t\,\hat{\mathbf{e}}_y + b\omega t\,\hat{\mathbf{e}}_z$$

- Curvature and torsion: circular helix is the unique curve with non-zero constant curvature and torsion

$$\kappa(t) = \frac{a}{a^2+b^2}\,, \qquad \tau(t) = \frac{b}{a^2+b^2}$$

EXERCISE 1.3: Find the tangent, normal and binormal vectors, as well as, curvature and torsion for the circular helix.

$$\mathbf{r}(t) = a\cos\omega t\,\hat{\mathbf{e}}_x + a\sin\omega t\,\hat{\mathbf{e}}_y + b\omega t\,\hat{\mathbf{e}}_z$$

$$\dot{\mathbf{r}}(t) = -a\omega\sin\omega t\,\hat{\mathbf{e}}_x + a\omega\cos\omega t\,\hat{\mathbf{e}}_y + b\omega\,\hat{\mathbf{e}}_z$$

$$s(t) = \int_0^t |\dot{\mathbf{r}}(t)|\,\mathrm{d}t = \omega\sqrt{a^2+b^2}\,t \quad \Rightarrow \quad \frac{\mathrm{d}s(t)}{\mathrm{d}t} = \omega\sqrt{a^2+b^2}$$

$$\hat{\mathbf{e}}_T(t) = \frac{\mathrm{d}\mathbf{r}(s)}{\mathrm{d}s} = \frac{\dfrac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}}{\dfrac{\mathrm{d}s(t)}{\mathrm{d}t}} = \frac{\dot{\mathbf{r}}(t)}{\dot{s}(t)} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\omega t\,\hat{\mathbf{e}}_x + a\cos\omega t\,\hat{\mathbf{e}}_y + b\,\hat{\mathbf{e}}_z\right) \qquad \blacksquare$$

$$\hat{\mathbf{e}}_T(t) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\omega t\,\hat{\mathbf{e}}_x + a\cos\omega t\,\hat{\mathbf{e}}_y + b\,\hat{\mathbf{e}}_z\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}t} = \frac{a\omega}{\sqrt{a^2+b^2}}\left(-\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\frac{\mathrm{d}\mathbf{e}_T(t)}{\mathrm{d}s} = \frac{\dfrac{\mathrm{d}\mathbf{e}_T(t)}{\mathrm{d}t}}{\dfrac{\mathrm{d}s(t)}{\mathrm{d}t}} = \frac{a}{a^2+b^2}\left(-\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y\right) \quad \Rightarrow \quad \left|\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}s}\right| = \frac{a}{a^2+b^2}$$

$$\hat{\mathbf{e}}_N(t) = \frac{1}{\kappa(t)}\,\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}s} \quad \Rightarrow \quad \kappa(t) = \left|\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}s}\right| = \frac{a}{a^2+b^2} \qquad \blacksquare$$

$$\hat{\mathbf{e}}_N(t) = \frac{1}{\kappa(t)}\,\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}s} = -\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y \qquad \blacksquare$$

$$\hat{\mathbf{e}}_T(t) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\omega t\,\hat{\mathbf{e}}_x + a\cos\omega t\,\hat{\mathbf{e}}_y + b\,\hat{\mathbf{e}}_z\right)\;, \quad \hat{\mathbf{e}}_N(t) = -\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y$$

$$\hat{\mathbf{e}}_B(t) = \hat{\mathbf{e}}_T(t)\times\hat{\mathbf{e}}_N(t) = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\omega t\,\hat{\mathbf{e}}_x - b\cos\omega t\,\hat{\mathbf{e}}_y + a\,\hat{\mathbf{e}}_z\right) \qquad \blacksquare$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}t} = \frac{b\omega}{\sqrt{a^2+b^2}}\left(\cos\omega t\,\hat{\mathbf{e}}_x + \sin\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}s} = \frac{\dfrac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}t}}{\dfrac{\mathrm{d}s(t)}{\mathrm{d}t}} = \frac{b}{a^2+b^2}\left(\cos\omega t\,\hat{\mathbf{e}}_x + \sin\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}s} = -\tau(t)\,\hat{\mathbf{e}}_N(t) \quad \Rightarrow \quad \tau(t) = -\hat{\mathbf{e}}_N(t)\cdot\frac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}s} = \frac{b}{a^2+b^2} \qquad \blacksquare$$

$$\hat{\mathbf{e}}_N(t) = -\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y\,, \qquad \hat{\mathbf{e}}_B(t) = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\omega t\,\hat{\mathbf{e}}_x - b\cos\omega t\,\hat{\mathbf{e}}_y + a\,\hat{\mathbf{e}}_z\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_N(t)}{\mathrm{d}t} = \omega\left(\sin\omega t\,\hat{\mathbf{e}}_x - \cos\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_N(t)}{\mathrm{d}s} = \frac{\dfrac{\mathrm{d}\hat{\mathbf{e}}_N(t)}{\mathrm{d}t}}{\dfrac{\mathrm{d}s(t)}{\mathrm{d}t}} = \frac{1}{\sqrt{a^2+b^2}}\left(\sin\omega t\,\hat{\mathbf{e}}_x - \cos\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\hat{\mathbf{e}}_N(s)\cdot\hat{\mathbf{e}}_B(s) = 0 \quad\Rightarrow\quad \hat{\mathbf{e}}_N(s)\cdot\frac{\mathrm{d}\hat{\mathbf{e}}_B(s)}{\mathrm{d}s} + \frac{\mathrm{d}\hat{\mathbf{e}}_N(s)}{\mathrm{d}s}\cdot\hat{\mathbf{e}}_B(s) = 0$$

$$\Rightarrow\quad -\tau(s)\,\hat{\mathbf{e}}_N(s)\cdot\hat{\mathbf{e}}_N(s) + \frac{\mathrm{d}\hat{\mathbf{e}}_N(s)}{\mathrm{d}s}\cdot\hat{\mathbf{e}}_B(s) = 0 \quad\Rightarrow\quad \tau(s) = \hat{\mathbf{e}}_B(s)\cdot\frac{\mathrm{d}\hat{\mathbf{e}}_N(s)}{\mathrm{d}s}$$

$$\tau(t) = \hat{\mathbf{e}}_B(t)\cdot\frac{\mathrm{d}\hat{\mathbf{e}}_N(t)}{\mathrm{d}s} = \frac{b}{a^2+b^2} \qquad \blacksquare$$

2D polar coordinate system

- Polar coordinates: $(u_1, u_2) = (\rho, \phi)$

ρ: distance from the origin, $0 \leq \rho < \infty$

ϕ: azimuthal angle from $+x$-axis, $0 \leq \phi < 2\pi$

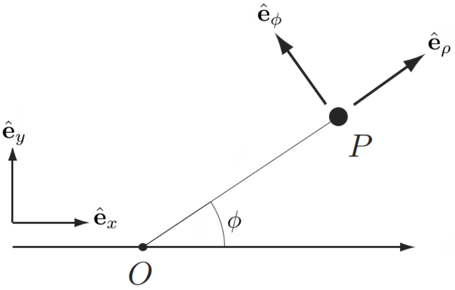
- Coordinate transformation between polar and Cartesian coordinates:

$$\begin{cases} x = \rho \cos\phi \\ y = \rho \sin\phi \end{cases} \quad \Leftrightarrow \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

- Unit basis vectors $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi)$ are *not* constant!

EXERCISE 1.4: Establish the relationship between unit basis vectors $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi)$ of the polar coordinate system and the unit basis vectors $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y)$ of the Cartesian coordinate system.

$$
\begin{cases}
\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{e}}_x + \sin\phi\,\hat{\mathbf{e}}_y \\[2ex]
\hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y
\end{cases}
\qquad \blacksquare
$$



$$\begin{pmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}^{-1} \begin{pmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\phi \end{pmatrix}$$

$$\Rightarrow \quad \begin{cases} \hat{\mathbf{e}}_x = \cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_y = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \end{cases} \qquad \blacksquare$$

Kinematics in 2D polar coordinates

- Position vector:

$$\mathbf{r}(t) = \rho(t)\,\hat{\mathbf{e}}_\rho$$

- Velocity:

$$\mathbf{v}(t) = \dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi$$

- Acceleration:

$$\mathbf{a}(t) = \left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right]\hat{\mathbf{e}}_\rho + \left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right]\,\hat{\mathbf{e}}_\phi$$

EXERCISE 1.5: Express the velocity and acceleration vectors in 2D polar coordinates.

$$
\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \end{cases}, \qquad \begin{cases} \hat{\mathbf{e}}_\rho = \cos\phi(t)\,\hat{\mathbf{e}}_x + \sin\phi(t)\,\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_\phi = -\sin\phi(t)\,\hat{\mathbf{e}}_x + \cos\phi(t)\,\hat{\mathbf{e}}_y \end{cases}
$$

$$
\mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y = r_\rho\,\hat{\mathbf{e}}_\rho + r_\phi\,\hat{\mathbf{e}}_\phi
$$

$$
\begin{cases} r_\rho = \hat{\mathbf{e}}_\rho \cdot \mathbf{r}(t) = x(t)\cos\phi(t) + y(t)\sin\phi(t) = \rho(t) \\ r_\phi = \hat{\mathbf{e}}_\phi \cdot \mathbf{r}(t) = -x(t)\sin\phi(t) + y(t)\cos\phi(t) = 0 \end{cases}
$$

$$
\Rightarrow \quad \mathbf{r}(t) = \rho(t)\,\hat{\mathbf{e}}_\rho \qquad \blacksquare
$$

$$\begin{cases} \hat{\mathbf{e}}_\rho = \cos\phi(t)\,\hat{\mathbf{e}}_x + \sin\phi(t)\,\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_\phi = -\sin\phi(t)\,\hat{\mathbf{e}}_x + \cos\phi(t)\,\hat{\mathbf{e}}_y \end{cases}$$

$$\Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}\hat{\mathbf{e}}_\rho}{\mathrm{d}t} = -\dot{\phi}(t)\sin\phi(t)\,\hat{\mathbf{e}}_x + \dot{\phi}(t)\cos\phi(t)\,\hat{\mathbf{e}}_y = \dot{\phi}(t)\,\hat{\mathbf{e}}_\phi \\ \dfrac{\mathrm{d}\hat{\mathbf{e}}_\phi}{\mathrm{d}t} = -\dot{\phi}(t)\cos\phi(t)\,\hat{\mathbf{e}}_x - \dot{\phi}(t)\sin\phi(t)\,\hat{\mathbf{e}}_y = -\dot{\phi}(t)\,\hat{\mathbf{e}}_\rho \end{cases}$$

$$\begin{aligned} \mathbf{v}(t) &= \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(t)\,\hat{\mathbf{e}}_\rho\right] \\ &= \dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi \qquad \blacksquare \end{aligned}$$

$$\mathbf{v}(t) = \dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi$$

$$\begin{cases} \dfrac{\mathrm{d}\hat{\mathbf{e}}_\rho}{\mathrm{d}t} = \dot{\phi}(t)\,\hat{\mathbf{e}}_\phi \\[2ex] \dfrac{\mathrm{d}\hat{\mathbf{e}}_\phi}{\mathrm{d}t} = -\dot{\phi}(t)\,\hat{\mathbf{e}}_\rho \end{cases}$$

$$\begin{aligned} \mathbf{a}(t) &= \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi\right] \\ &= \left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right]\hat{\mathbf{e}}_\rho + \left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right]\,\hat{\mathbf{e}}_\phi \qquad \blacksquare \end{aligned}$$

Cylindrical coordinate system

- Cylindrical coordinates: $(u_1, u_2, u_3) = (\rho, \phi, z)$

ρ: polar distance from the z axis, $0 \le \rho < \infty$

ϕ: azimuthal angle from the x axis on the xy-plane, $0 \le \phi < 2\pi$

z: coordinate along the z axis, $-\infty < z < \infty$

- Coordinate transformation between cylindrical and Cartesian coordinates:

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

- Velocity and acceleration:

$$\begin{cases} \mathbf{v}(t) = \dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi + \dot{z}(t)\,\hat{\mathbf{e}}_z \\ \mathbf{a}(t) = \left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right]\hat{\mathbf{e}}_\rho + \left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right]\,\hat{\mathbf{e}}_\phi + \ddot{z}(t)\,\hat{\mathbf{e}}_z \end{cases}$$

Spherical coordinate system

- Spherical coordinates: $(u_1, u_2, u_3) = (r, \theta, \phi)$

r: radial distance from the origin, $0 \leq r < \infty$

θ: polar angle from the z axis, $0 \leq \theta \leq \pi$

ϕ: azimuthal angle from the x axis on the xy-plane, $0 \leq \phi < 2\pi$

- Coordinate transformation between spherical and Cartesian coordinates:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \quad \Leftrightarrow \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

EXERCISE 1.6: Express the spherical unit basis vectors $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi)$ in terms of Cartesian unit basis vectors $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$.

$$\mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z = r\sin\theta\cos\phi\,\hat{\mathbf{e}}_x + r\sin\theta\sin\phi\,\hat{\mathbf{e}}_y + r\cos\theta\,\hat{\mathbf{e}}_z$$

$$\begin{cases} \dfrac{\partial \mathbf{r}}{\partial r} = \sin\theta\;\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z \\[2ex] \dfrac{\partial \mathbf{r}}{\partial \theta} = r\cos\theta\cos\phi\,\hat{\mathbf{e}}_x + r\cos\theta\sin\phi\,\hat{\mathbf{e}}_y - r\cos\theta\,\hat{\mathbf{e}}_z \\[2ex] \dfrac{\partial \mathbf{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{\mathbf{e}}_x + r\sin\theta\cos\phi\,\hat{\mathbf{e}}_y \end{cases}$$

$$\Rightarrow \quad \begin{cases} \hat{\mathbf{e}}_r \equiv \dfrac{\dfrac{\partial \mathbf{r}}{\partial r}}{\left|\dfrac{\partial \mathbf{r}}{\partial r}\right|} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z \\[4ex] \hat{\mathbf{e}}_\theta \equiv \dfrac{\dfrac{\partial \mathbf{r}}{\partial \theta}}{\left|\dfrac{\partial \mathbf{r}}{\partial \theta}\right|} = \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z \\[4ex] \hat{\mathbf{e}}_\phi \equiv \dfrac{\dfrac{\partial \mathbf{r}}{\partial \phi}}{\left|\dfrac{\partial \mathbf{r}}{\partial \phi}\right|} = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_z \end{cases} \qquad \blacksquare$$

$$
\begin{cases}
\hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z \\
\hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z \\
\hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_z
\end{cases}
$$

$$
\begin{aligned}
\hat{\mathbf{e}}_r \cdot (\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\phi) &= \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \\
&= -\sin\phi \begin{vmatrix} \sin\theta\sin\phi & \cos\theta \\ \cos\theta\sin\phi & -\sin\theta \end{vmatrix} - \cos\phi \begin{vmatrix} \sin\theta\cos\phi & \cos\theta \\ \cos\theta\cos\phi & -\sin\theta \end{vmatrix} \\
&= -\sin\phi\left(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi\right) - \cos\phi\left(-\sin^2\theta\cos\phi - \cos^2\theta\cos\phi\right) \\
&= 1 \qquad \blacksquare
\end{aligned}
$$

Kinematics in spherical coordinates

- Position vector:

$$\mathbf{r}(t) = r(t)\,\hat{\mathbf{e}}_r$$

- Velocity vector:

$$\mathbf{v}(t) = \dot{r}(t)\,\hat{\mathbf{e}}_r + r(t)\,\dot{\theta}(t)\,\hat{\mathbf{e}}_\theta + r(t)\,\dot{\phi}(t)\sin\theta(t)\,\hat{\mathbf{e}}_\phi$$

- Acceleration vector:

$$\begin{aligned}\mathbf{a}(t) = &\left[\ddot{r}(t) - r(t)\,\dot{\phi}^2(t)\sin^2\theta(t) - r(t)\,\dot{\theta}^2(t)\right]\hat{\mathbf{e}}_r \\ &+ \left[r(t)\,\ddot{\theta}(t) + 2\dot{r}(t)\,\dot{\theta}(t) - r(t)\,\dot{\phi}^2(t)\sin\theta(t)\cos\theta(t)\right]\hat{\mathbf{e}}_\theta \\ &+ \left[r(t)\,\ddot{\phi}(t)\sin\theta(t) + 2\dot{r}(t)\,\dot{\phi}(t)\sin\theta(t) + 2r(t)\,\dot{\theta}(t)\,\dot{\phi}(t)\cos\theta(t)\right]\hat{\mathbf{e}}_\phi\end{aligned}$$

$$\hat{\mathbf{e}}_r = \sin\theta(t)\cos\phi(t)\,\hat{\mathbf{e}}_x + \sin\theta(t)\sin\phi(t)\,\hat{\mathbf{e}}_y + \cos\theta(t)\,\hat{\mathbf{e}}_z$$

$$\begin{aligned}
\frac{\mathrm{d}\hat{\mathbf{e}}_r}{\mathrm{d}t} &= \frac{\partial\hat{\mathbf{e}}_r}{\partial\theta}\,\dot{\theta} + \frac{\partial\hat{\mathbf{e}}_r}{\partial\phi}\,\dot{\phi} \\
&= (\cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z)\,\dot{\theta} + (-\sin\theta\sin\phi\,\hat{\mathbf{e}}_x + \sin\theta\cos\phi\,\hat{\mathbf{e}}_y)\,\dot{\phi} \\
&= \dot{\theta}\,\hat{\mathbf{e}}_\theta + \sin\theta\,\dot{\phi}\,\hat{\mathbf{e}}_\phi \qquad \blacksquare
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}(t) \equiv \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\,\hat{\mathbf{e}}_r\right] \\
&= \dot{r}(t)\,\hat{\mathbf{e}}_r + r(t)\,\frac{\mathrm{d}\hat{\mathbf{e}}_r}{\mathrm{d}t} \\
&= \dot{r}(t)\,\hat{\mathbf{e}}_r + r(t)\,\dot{\theta}(t)\,\hat{\mathbf{e}}_\theta + r(t)\,\dot{\phi}(t)\,\sin\theta(t)\,\hat{\mathbf{e}}_\phi \qquad \blacksquare
\end{aligned}$$