Last lecture

Symmetries — inversion symmetry — particularly helpful for us to see when some integrals are zero. (see 2nd $\frac{1}{2}$ of course)

$$[\hat{H}, \hat{\pi}] = 0$$

parity operator

$\Rightarrow$ $\exists$ a set of common eigenstates of $\hat{H}$ and $\hat{\pi}$.

$\Rightarrow$ $\frac{d}{dt}\langle \hat{\pi} \rangle = 0$ parity is conserved.

$$\left(\frac{d}{dt}\langle \hat{A} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{A}] \rangle + \frac{\partial}{\partial t}\langle \hat{A} \rangle \right)$$

Today :

Translation	Rotation
Momentum is the generator	Angular momentum is the generator

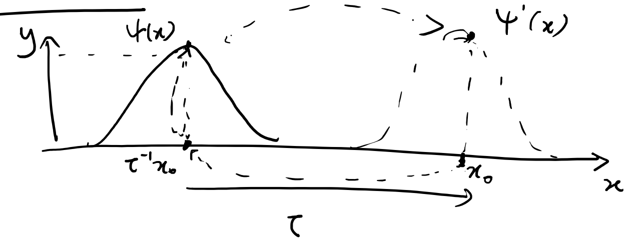
PC3130 — Angular Momentum

$\vec{L} = \vec{r} \times \vec{p}$ orbital angular momentum

Spin angular momentum

$$\boxed{[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k \text{ —— defines angular momentum}}$$

Translation



$$\psi'(x) = \psi(\tau^{-1} x)$$

It has been shown that any symmetry with a continuous parameter can be described by a unitary operator $\hat{U}$ $(\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \mathbb{1})$

$$\hat{x}_\alpha \xrightarrow{\tau} \hat{x}_\alpha + a_\alpha$$

$$\hat{U}_\tau = e^{-\frac{i a_\alpha \hat{p}_\alpha}{\hbar}} \quad \text{– not summation notation}$$

$$\psi \xrightarrow[\text{unitary transformation}]{\hat{U}_\tau} \psi' = \hat{U}_\tau \psi$$

$\equiv$ momentum is the generator of translations

(Lie – mathematician)

For $|a_\alpha| << 1$,

$$\hat{U}_\tau = e^{-\frac{i a_\alpha \hat{p}_\alpha}{\hbar}} \underset{\text{Taylor series}}{\approx} \left(\mathbb{1} - \frac{i a_\alpha}{\hbar}\hat{p}_\alpha\right)$$

$$\psi' = \left(\mathbb{1} - \frac{i a_\alpha}{\hbar}\hat{p}_\alpha\right)\psi = \psi - \frac{i a_\alpha}{\hbar}\hat{p}_\alpha \psi$$

To show: $U_\tau \psi(x) = e^{-\frac{ia}{\hbar}p}\psi(x) = \psi(x-a)$

RHS: $\psi(x-a) \underset{\text{Taylor series expansion}}{=} \psi(x) - a\psi'(x) + \frac{a^2}{2!}\psi''(x) + \frac{(-a)^3}{3!}\psi'''(x) + \dots$

$\hat{p} = -i\hbar\frac{\partial}{\partial x}$

LHS: $U_\tau\psi(x) = e^{-\frac{ia}{\hbar}\hat{p}}\psi(x)$

$$= \left(1 - \frac{ia}{\hbar}\hat{p} + \frac{1}{2!}\left(\frac{-ia}{\hbar}\right)^2\hat{p}\cdot\hat{p} + \frac{1}{3!}\left(-\frac{ia}{\hbar}\right)^3\hat{p}\cdot\hat{p}\cdot\hat{p} + \dots\right)\psi(x)$$

$$= \left(1 - a\frac{\partial}{\partial x} + \frac{a^2}{2!}\frac{\partial^2}{\partial x^2} + \frac{(-a)^3}{3!}\frac{\partial^3}{\partial x^3} + \dots\right)\psi(x)$$

$= \text{RHS}$

If a system has continuous translation symmetry, $U^\dagger_{\tau(\vec{a})} H U_{\tau(\vec{a})} = H$ for any $\vec{a}$.

(To show this implies $[\hat{H}, \hat{p}] = 0$)

$$U^\dagger_{\tau(\vec{a})} H U_{\tau(\vec{a})} = H$$

$$U_{\tau(\vec{a})} U^\dagger_{\tau(\vec{a})} H U_{\tau(\vec{a})} = U_{\tau(\vec{a})} H$$

$(UU^\dagger = \mathbb{1}) \Rightarrow \quad H U_{\tau(\vec{a})} = U_{\tau(\vec{a})} H$

$\Rightarrow \quad [H, U_{\tau(\vec{a})}] = 0$ for any $\vec{a}$.

In particular, this is true for $|\vec{a}| = \delta \ll 1$

Then $U_{\tau(\vec{a})} = \mathbb{1} - \frac{i\vec{a}\cdot\vec{p}}{\hbar}$

$\left[H, \mathbb{1} - \frac{i\vec{a}\cdot\vec{p}}{\hbar}\right] = 0$ for any $\vec{a}$ with $|\vec{a}| \ll 1$

$\rightarrow \underbrace{[H, \mathbb{1}]}_{0} - \frac{i}{\hbar}[H, \vec{a}\cdot\vec{p}] = 0$

$-\frac{i\vec{a}}{\hbar}\cdot[H, \vec{p}] = 0$

Because $\vec{a}$ is arbitrary with $|\vec{a}| \ll 1$, $\Rightarrow [H, \vec{p}] = 0$

cf. W3L1

Prove that $\Pi^\dagger = \Pi$. We used $\langle \Pi\psi | \varphi \rangle = \langle \psi | \Pi \varphi \rangle$ for any $|\psi\rangle, |\varphi\rangle$ in the Hilbert space.

$\Rightarrow \quad [\hat{H}, \hat{p}] = 0$

$\Rightarrow \quad \frac{d}{dt}\langle \hat{p} \rangle = 0$, momentum is conserved if a system has continuous translation symmetry.

Eg $\hat{H} = \frac{\hat{p}^2}{2m}$ obeys continuous translation symmetry.

Translation	Rotation
• Momentum is the generator	• Angular momentum $\hat{\vec{J}}$ is the generator
• If a system has continuous translational symmetry, $[\hat{H}, \hat{p}] = 0$	• If a system has continuous rotational symmetry, $[\hat{H}, \hat{\vec{J}}] = 0$.

c.f. for central potentials V depends only on $|\vec{r}|$.

$H = \frac{p^2}{2m} + V(|\vec{r}|)$ has continuous rotational symmetry.

Angular momentum is conserved.

$$[\hat{H}, \hat{\vec{L}}] = 0$$

Eg. in particular

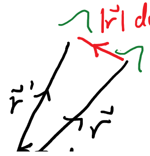
$$[\hat{H}, L_z] = 0$$

Hydrogen atom, $[\hat{H}, L_z] = 0$

(Merzbacher – optional)

$$\vec{r} \longrightarrow \vec{r}\,' = R_{\hat{n}}(d\alpha)\,\vec{r}$$

(anticlockwise)



$\underbrace{\text{arc length}}$ $|\vec{r}|\, d\alpha(\hat{n}\times\hat{r})$

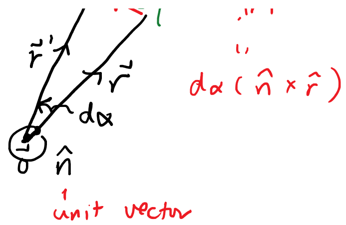
$\frac{\vec{r}}{|\vec{r}|}$

$d\alpha(\hat{n}\times\hat{r})$

$$\vec{r} \longrightarrow \vec{r}\,' = R_{\hat{n}}(d\alpha)\, \vec{r}$$

(anticlockwise)

$$= \vec{r} + (d\alpha)(\hat{n} \times \vec{r})$$



$$\boxed{\begin{array}{l} \psi \xrightarrow{U_{\hat{n}}(d\alpha)} \psi' \\ \text{where } U_{\hat{n}}(d\alpha) = e^{-\frac{i}{\hbar} d\alpha\, \hat{n}\cdot\vec{J}} \quad \text{where } \vec{J} \text{ is the angular momentum.} \end{array}}$$

Eg 1 Consider $\psi_{nlm}(r, \theta, \phi) = \langle \vec{r} | nlm \rangle = \chi(r)\, \Theta(\theta)\, e^{im\phi}$

$\Theta(\theta)$: associated Legendre polynomials

$\Theta(\theta)\, e^{im\phi}$: Spherical harmonics

Consider $\hat{n} = \hat{z}$

$$U_{\hat{z}}(d\alpha) = e^{-\frac{i}{\hbar} d\alpha\, (\hat{n}\cdot\vec{J})}$$

Choose $\vec{J} = \vec{L}$ orbital angular momentum

$$U_{\hat{z}}(d\alpha) = e^{-\frac{i}{\hbar} d\alpha\, (\hat{z}\cdot\vec{L})} = e^{-\frac{i}{\hbar} d\alpha\, \hat{L}_z}$$

$$\hat{L}_z |nlm\rangle = m\hbar |nlm\rangle$$

$$\hat{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$e^{\hat{A}} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

$$U_{\hat{z}}(d\alpha)\, \psi_{nlm}(r, \theta, \phi) = \chi(r)\, \Theta(\theta)\, e^{-\frac{i}{\hbar}(d\alpha)\hat{L}_z}\, e^{im\phi}$$

$$= \chi(r)\, \Theta(\theta)\, e^{-\frac{i}{\hbar}(d\alpha)(m\hbar)}\, e^{im\phi}$$

$$= \chi(r)\, \Theta(\theta)\, e^{im(\phi - d\alpha)}$$

$$= \psi_{nlm}(r, \theta, \phi - d\alpha)$$

Any $\Phi(r, \theta, \phi) = \sum_{nlm} c_{nlm}\, \psi_{nlm}(r, \theta, \phi)$

So $U_{\hat{z}}(d\alpha)\Phi(r, \theta, \phi) \underset{\text{linear}}{=} \sum_{nlm} c_{nlm}\, U_{\hat{z}}(d\alpha)\, \psi_{nlm}(r, \theta, \phi)$

$$= \sum_{nlm} c_{nlm}\, \psi_{nlm}(r, \theta, \phi - d\alpha) = \Phi(r, \theta, \phi - d\alpha)$$

//

Consider angle $\phi = N\, d\alpha$

Apply $U_{\hat{n}}(d\alpha)$ N times.

$$U_{\hat{n}}(\phi) = e^{-\frac{i}{\hbar} N\, d\alpha\, (\hat{n}\cdot\vec{J})} = e^{-\frac{i}{\hbar}\phi(\hat{n}\cdot\vec{J})}$$

$$U_{\hat{z}}(2\pi)\, |nlm\rangle = e^{-i\frac{(2\pi)}{\hbar} m\hbar}\, |nlm\rangle \quad \text{using } \vec{J} = \vec{L} = \vec{r} \times \vec{p}$$

$$= e^{-i(2\pi)m} |nlm\rangle$$

$$= 1\ |nlm\rangle \quad \text{since } m \text{ is an integer}$$

$$= |nlm\rangle$$

Eg 2 Two-level system described by spin-$\frac{1}{2}$ system

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_z$$

$$\left.\begin{aligned} S_x &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_x \\ S_y &= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_y \end{aligned}\right\} \text{ we can derive this later.}$$

Going to show that when we apply the rotation operator

$U_{\hat{z}}(\frac{\pi}{2})$ to $\hat{x}$, we get $\hat{y}$.

What is $\hat{x}$?

Defined as the eigenvector of σ_x with + eigenvalue.

$|x,+\rangle$

$$U_z(\tfrac{\pi}{2})\ |x,+\rangle \longrightarrow |y,+\rangle$$

What is $|x,+\rangle$.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det(\sigma_x - \lambda \mathbb{1}) = 0$$

$$\left|\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\right| = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

$$\lambda = +1: \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow a = b$$

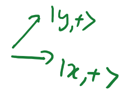
$|x,+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ —— "x-direction" in spin-$\frac{1}{2}$ system.

$|y,+\rangle$ eigenvector of σ_y with + eigenvalue

$|y,+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ —— "y-direction" in spin-$\frac{1}{2}$ system.

Now $U_{\hat z}(\theta) = e^{-\frac{i}{\hbar}\theta S_z} = e^{-\frac{i}{\hbar}\theta\frac{\hbar}{2}\sigma_z} = e^{-\frac{i}{2}\theta\begin{pmatrix}1&0\\0&-1\end{pmatrix}} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2}\end{pmatrix}$

$$U_{\hat z}\left(\tfrac{\pi}{2}\right)|x,+\rangle = \frac{1}{\sqrt2}\begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\pi/4}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$



$$= \frac{1}{\sqrt2}\begin{pmatrix} e^{-i\pi/4} \\ e^{i\pi/4}\end{pmatrix}$$

($\rightsquigarrow |y,+\rangle$?)

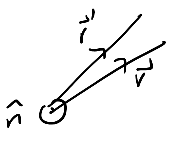
$$= \frac{1}{\sqrt2}e^{-i\pi/4}\begin{pmatrix}1\\ e^{i\pi/2}\end{pmatrix} = \frac{1}{\sqrt2}e^{-i\pi/4}\begin{pmatrix}1\\ i\end{pmatrix} = e^{-i\pi/4}|y,+\rangle.$$

Preliminaries

1) Scalar operator
- The expectation value of a scalar operator is unchanged by rotations.

2) vector operator
- The expectation value of a vector operator transforms as a vector under rotations, eg. $\vec r$, $\vec p$, $\vec L$, $\vec J$.
- The expectation value of a vector operator $\hat A$ in the rotated state $\psi'(\vec r)$ is obtained by rotating the expectation value of $\hat A$ in the original state $\psi(\vec r)$.

$\langle\psi|\hat A|\psi\rangle$ — vector under rotations.



For infinitesimal $d\alpha$, $\vec r' = \vec r + (d\alpha)(\hat n \times \vec r)$

A vector operator $\vec A$ will also transform as

$$\langle\psi'|\vec A|\psi'\rangle = \langle\psi|\vec A|\psi\rangle + (d\alpha)\left(\hat n \times \langle\psi|\vec A|\psi\rangle\right) \quad \text{— (1)}$$

— definition of vector operator.

(→ to get relation between $\vec A$ and $\vec J$)

Recall $\psi' = U\psi$

($\Rightarrow$ to get relation between $\vec{A}$ and $\vec{J}$)

Use $|\psi'\rangle = U|\psi\rangle$

Recall $\psi' = U\psi$

LHS: $\langle\psi'|\vec{A}|\psi'\rangle = \langle\psi|U^{+}\vec{A}U|\psi\rangle$

LHS = RHS in (1):

$$\langle\psi|U^{+}\vec{A}U|\psi\rangle = \langle\psi|\vec{A}|\psi\rangle + (d\alpha)(\hat{n}\times\langle\psi|\vec{A}|\psi\rangle)$$

Since this is true for all $|\psi\rangle$,

we must have $U^{+}\vec{A}U = \vec{A} + (d\alpha)(\hat{n}\times\vec{A})$ — (2)

(cf. $\overset{\text{matrix}}{A}x = 0$ if this is true for all x, then $A = 0$)

$$U = e^{-\frac{i}{\hbar}d\alpha(\hat{n}.\vec{J})}$$
$$= \left(1 - \frac{i}{\hbar}d\alpha(\hat{n}.\vec{J})\right)$$

LHS of (2): $\left(1 + \frac{i}{\hbar}d\alpha(\hat{n}.\vec{J})\right)\vec{A}\left(1 - \frac{i}{\hbar}d\alpha(\hat{n}.\vec{J})\right)$

$$= \vec{A} + \frac{i}{\hbar}d\alpha(\hat{n}.\vec{J})\vec{A} - \frac{i}{\hbar}d\alpha\vec{A}(\hat{n}.\vec{J})$$ to first order in $d\alpha$

$$= \vec{A} + \frac{i}{\hbar}d\alpha[\hat{n}.\vec{J}, \vec{A}]$$ to first order in $d\alpha$.

Comparing with RHS of (2):

We have $\frac{i}{\hbar}[\hat{n}.\vec{J}, \vec{A}] = \hat{n}\times\vec{A}$

$$\boxed{[\vec{A}, \hat{n}.\vec{J}] = i\hbar(\hat{n}\times\vec{A})}$$ — equivalent definition of a vector operator $\vec{A}$.

Let $\hat{n} = \hat{e}_j$

Then $\hat{n}.\vec{J} = J_j$

$$[\vec{A}, J_j] = i\hbar(\hat{e}_j\times\vec{A})$$

(Angular momentum $\vec{J}$ is defined by $[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k$)

Since $\vec{J}$ is a vector operator,

we can let $\vec{A}$ to be $\vec{J}$.

$$[\underset{\text{vector}}{\vec{J}}, J_j] = i\hbar\,\underset{\text{vector}}{\hat{e}_j\times\vec{J}}$$

$(\hat{e}_j\times\vec{J})_i$
$= \varepsilon_{i\tilde{j}k}(\hat{e}_j)_{\tilde{j}}J_k$
$= \varepsilon_{i\tilde{j}k}\delta_{j\tilde{j}}J_k$
$= \varepsilon_{ijk}J_k$

$(\vec{a}\times\vec{b})_i$
$\shortparallel$
$\varepsilon_{ijk}a_jb_k$
$\shortparallel$
$\varepsilon_{i\tilde{j}k}a_{\tilde{j}}b_k$

Taking the ith component of the vector:

$$\boxed{[J_i, J_j] = i\hbar\underbrace{\varepsilon_{ijk}J_k}_{\text{summing over }k}}$$ definition of angular momentum.

i & j free indices

$$\boxed{[J_i, J^2] = 0}$$ where $\vec{J}$ is angular momentum

$\boxed{[J_i, J^2] = 0}$ where $\vec{J}$ is angular momentum

(cf orbital angular momentum
$[L_z, L^2] = 0$)