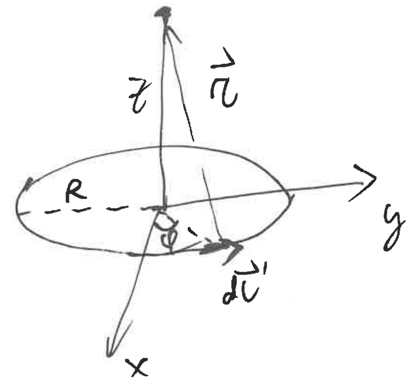
Homework 4 Solution

1.

(1) Biot-Savart law:



$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \vec{r}}{r^3}$$

For $d\vec{l}'$, we pick the one shown on the graph

where $d\vec{l}' = R\, dl\, \hat{\varphi} = R\, d\varphi\, (-\sin\varphi\, \hat{x} + \cos\varphi\, \hat{y})$

$$\vec{r} = (-R\cos\varphi, -R\sin\varphi, +z) = -R\cos\varphi\, \hat{x} - R\sin\varphi\, \hat{y} + z\hat{z}$$

$$|\vec{r}| = (R^2 + z^2)^{\frac{1}{2}}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R\, d\varphi\, (-\sin\varphi\, \hat{x} + \cos\varphi\, \hat{y}) \times (-R\cos\varphi\, \hat{x} - R\sin\varphi\, \hat{y} + z\hat{z})}{(R^2 + z^2)^{\frac{3}{2}}}$$

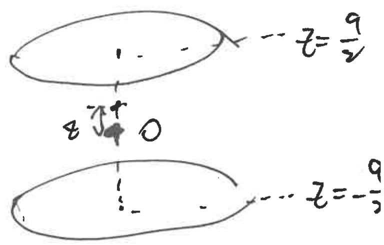
For the cross product term, we only care about the $\hat{z}$ component because $\hat{x}$ & $\hat{y}$ components would vanish by symmetry (field points center on the loop wire), so

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R\, d\varphi\, (R\sin^2\varphi\, \hat{x} \times \hat{y} - R\cos^2\varphi\, \hat{y} \times \hat{x})}{(R^2 + z^2)^{\frac{3}{2}}}$$

$$= \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R\, d\varphi\, R\hat{z}}{(R^2 + z^2)^{\frac{3}{2}}} = \frac{\mu_0 I}{4\pi} \cdot \frac{R^2 \cdot 2\pi\, \hat{z}}{(R^2 + z^2)^{\frac{3}{2}}} = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{\frac{3}{2}}}\, \hat{z}$$

(2)

(i) suppose we pick the origin at the middle point between two wires, any offset in the $\hat{z}$ direction is denoted by z.



$$B(z) = \frac{\mu_0 I R^2}{2}\left[\left(R^2 + \left(\frac{a}{2}+z\right)^2\right)^{-\frac{3}{2}} + \left(R^2 + \left(\frac{a}{2}-z\right)^2\right)^{-\frac{3}{2}} \right]$$

where $z < \frac{a}{2}$

(ii)

$$\frac{dB}{dz} = -\frac{3}{2}\,\frac{\mu_0 I R^2}{2}\Big[\left(R^2 + \left(\frac{a}{2}+z\right)^2\right)^{-\frac{5}{2}} \cdot 2\cdot\left(\frac{a}{2}+z\right)$$
$$+ \left(R^2 + \left(\frac{a}{2}-z\right)^2\right)^{-\frac{5}{2}} \cdot 2\left(\frac{a}{2}-z\right)\cdot(-1) \Big]$$

$$= \frac{-3\mu_0 I R^2}{2}\left[\left(R^2 + \left(\frac{a}{2}+z\right)^2\right)^{-\frac{5}{2}} \cdot \left(\frac{a}{2}+z\right) - \left(R^2 + \left(\frac{a}{2}-z\right)^2\right)^{-\frac{5}{2}} \left(\frac{a}{2}-z\right) \right]$$

From this, $\left.\frac{dB}{dz}\right|_{z=0,\ a=R} = 0$

$$\frac{d^2B}{dz^2} = -\frac{3\mu_0 I R^2}{2}\,\frac{d}{dz}\underbrace{\left[\left(R^2 + \left(\frac{a}{2}+z\right)^2\right)^{-\frac{5}{2}} \left(\frac{a}{2}+z\right) - \left(R^2 + \left(\frac{a}{2}-z\right)^2\right)^{-\frac{5}{2}} \left(\frac{a}{2}-z\right) \right]}_{①}$$

$$① = \left(R^2 + \left(\frac{a}{2}+z\right)^2\right)^{-\frac{5}{2}} + \left(\frac{a}{2}+z\right)^2\left(-\frac{5}{2}\right)\left(R^2 + \left(\frac{a}{2}+z\right)^2\right)^{-\frac{7}{2}} \cdot 2$$
$$+ \left(R^2 + \left(\frac{a}{2}-z\right)^2\right)^{-\frac{5}{2}} - \left(\frac{a}{2}-z\right)^2\left(-\frac{5}{2}\right)\left(R^2 + \left(\frac{a}{2}-z\right)^2\right)^{-\frac{7}{2}} \cdot 2\,(-1)$$

$$\Rightarrow ① = \left(R^2+\left(\tfrac{a}{2}+z\right)^2\right)^{-\frac{5}{2}} + \left(R^2+\left(\tfrac{a}{2}-z\right)\right)^{-\frac{5}{2}}$$

$$- 5\left(\tfrac{a}{2}+z\right)^2\left(R^2+\left(\tfrac{a}{2}+z\right)^2\right)^{-\frac{7}{2}} - 5\left(\tfrac{a}{2}-z\right)^2\left(R^2+\left(\tfrac{a}{2}-z\right)^2\right)^{-\frac{7}{2}}$$

when $a=R$, $z=0$,

$$① = \left(\tfrac{5}{4}R^2\right)^{-\frac{5}{2}} \times 2 - 5\,\tfrac{R^2}{4}\left(\tfrac{5}{4}R^2\right)^{-\frac{7}{2}} \times 2 = 0$$

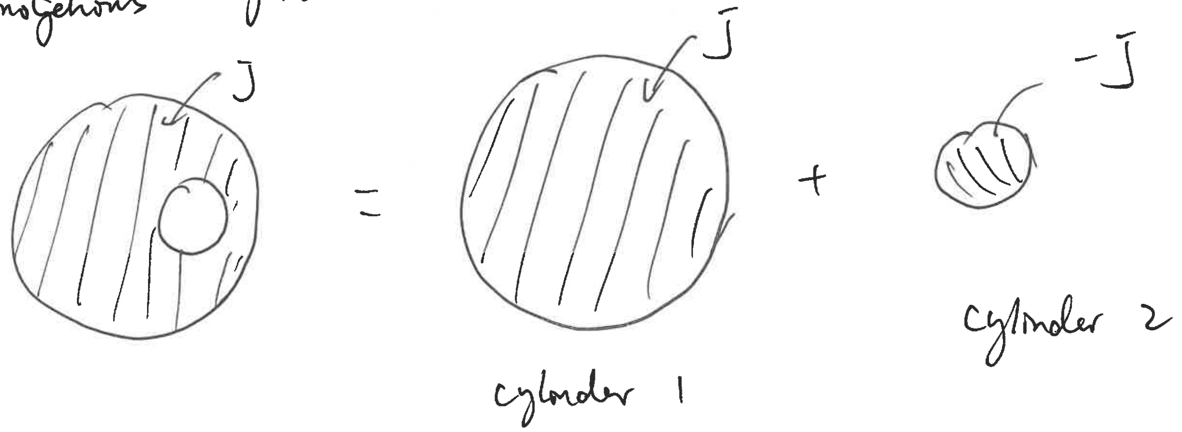
In conclusion, $\left.\frac{dB}{dz}\right|_{z=0} = \left.\frac{d^2B}{dz^2}\right|_{z=0} = 0$ when $a=R$

Therefore field is most uniform.

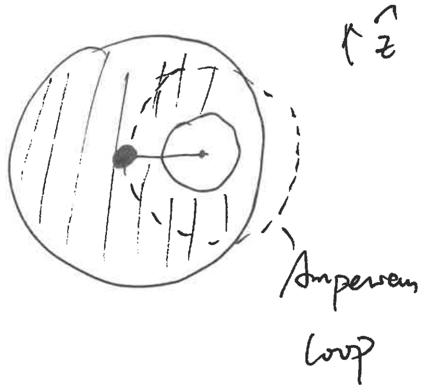
2.

(1) $J = \frac{I}{a_\perp} = \frac{I}{\pi R_1^2 - \pi R_2^2}$ since current density is uniform.

(2) We can imagine the geometry as the addition of two homogenous cylinders



At the center of the rod, cylinder 1 does not contribute any field (infinitesimal Amperian loop), so $B_1 = 0$



cylinder 2 contributes B_2 can be calculated by the Amperian loop above.

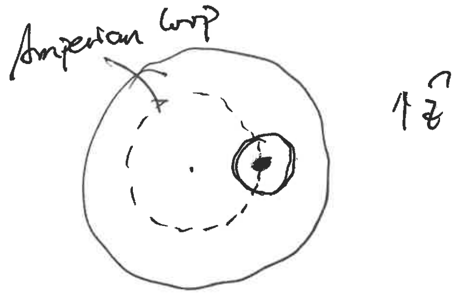
$$\oint \vec{B}_2 \cdot d\vec{l} = \mu_0 I_{enc,2}$$

$$\Rightarrow \quad B_2 \cdot 2\pi a = \mu_0 \pi R_2^2 \cdot J = \mu_0 \frac{I \pi R_2^2}{\pi R_1^2 - \pi R_2^2}$$

$$\Rightarrow \quad B_2 = \frac{\mu_0 I}{2\pi a} \frac{R_2^2}{R_1^2 - R_2^2} \Rightarrow \vec{B}_2 = \frac{\mu_0 I}{2\pi a} \frac{R_2^2}{R_1^2 - R_2^2} \cdot (-\hat{z})$$

determined by direction of current in cylinder 2

(3) At the center of the hollow region, cylinder 2 contributes zero field, to calculate cylinder 1 contribution, we choose the Amperian loop as above



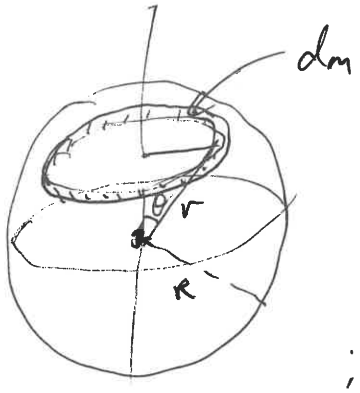
$$\oint \vec{B}_1 \cdot d\vec{l} = \mu_0 I_{enc,1}$$

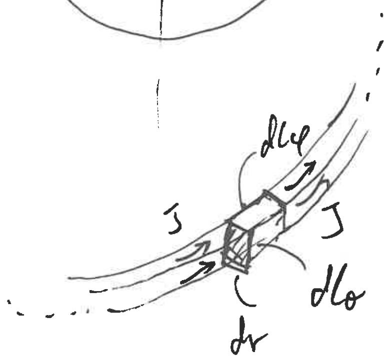
$$B_1 \cdot 2\pi a = \mu_0 \frac{\pi a^2 I}{\pi R_1^2 - \pi R_2^2}$$

$$\Rightarrow \vec{B}_1 = \frac{\mu_0 I}{2\pi} \frac{a}{R_1^2 - R_2^2} (-\hat{z})$$

3.

(1) Breakdown the sphere with rings orienting along $\hat{z}$ with radial and polar coordinates r and θ ($\varphi \in (0, 2\pi)$)





Total dipole moment $m = \int dm$

$$dm = \underset{\swarrow \text{area of loop}}{a}\ \underset{\swarrow \text{current}}{dI} = \pi (r \sin\theta)^2\, dI$$

$$dI = J \cdot da_\perp = J\, dr\, dl_\theta = J\, r\, dr\, d\theta$$

Current density $J = \rho v = \dfrac{q}{\frac{4}{3}\pi R^3}\, \omega (r \sin\theta) = \dfrac{3q}{4\pi R^3}\, \omega r \sin\theta$

$$\Rightarrow \quad dm = \pi (r\sin\theta)^2\, \frac{3q}{4\pi R^3}\, \omega r \sin\theta \cdot r\, dr\, d\theta$$

$$= \frac{3q\omega}{4R^3}\, r^4 \sin^3\theta\, dr\, d\theta$$

$$m = \int dm = \int_0^{\pi}\int_0^{R} \frac{3q\omega}{4R^3}\, r^4 \sin^3\theta\, dr\, d\theta$$

$$= \frac{3q\omega}{4R^3} \int_0^{\pi} \sin^3\theta\, d\theta \int_0^R r^4\, dr$$

$$= \frac{3q\omega}{4R^3} \int_1^{-1} -(1-u^2)\, du \;\; \frac{1}{5} r^5 \Big|_0^R$$

$$= \frac{3qw}{4R^3} \left(\frac{1}{3}u^3 - u \right) \Big|_1^{-1} \cdot \frac{R^5}{5}$$

$$= \frac{3q\omega R^2}{20} \left(-\frac{1}{3} + 1 - \frac{1}{3} + 1 \right) = \frac{q\omega R^2}{5}$$

$$\vec{\mu} = \frac{q\omega R^2}{5} \hat{z}$$

(2) for $m = \frac{q\omega R^2}{5}$

where $m = \mu_B = 9.27 \times 10^{-24}\ A \cdot m^2$

$q = 1.6 \times 10^{-19}\ C$

$R = 2.8 \times 10^{-15}\ m$

we have $\omega = 3.69 \times 10^{25}\ rad/s$

velocity at equator $v = \omega R = 1.03 \times 10^{11}\ m/s$

which is much larger than the speed of light ($3 \times 10^8\ m/s$)

4. * Note: the solution for this problem is not unique

(1) In the Cartesian coordinate $\vec{B} = \nabla \times \vec{A} = B_z \hat{z}$

converts into

$$\hat{x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)=B_z\hat{z}$$

$$\Rightarrow \quad \frac{\partial A_z}{\partial y}=\frac{\partial A_y}{\partial z}, \quad \frac{\partial A_x}{\partial z}=\frac{\partial A_z}{\partial x}, \quad \frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=B_z$$

We can choose $A_z=0$, $A_y=B_z x$, $A_x=0$

or $A_z=0$, $A_y=\frac{1}{2}B_z x$, $A_x=-\frac{1}{2}B_z y$

So two possible solutions: $\vec{A}_1=(0, B_z x, 0)$

or $\vec{A}_2=(-\frac{1}{2}B_z y, \frac{1}{2}B_z x, 0)$

(2) $\vec{A}_2=(\frac{-1}{2}B_z y, \frac{1}{2}B_z x, 0)=-\frac{1}{2}B_z y\hat{x}+\frac{1}{2}B_z x\hat{y}$

can be converted to cylindrical coordinates

Use identity
$$\begin{cases} x=s\cos\phi \\ y=s\sin\phi \\ \hat{x}=\cos\phi\,\hat{s}-\sin\phi\,\hat{\phi} \\ \hat{y}=\sin\phi\,\hat{s}+\cos\phi\,\hat{\phi} \end{cases}$$

$$\vec{A}_2=-\frac{1}{2}B_z\left(s\sin\phi(\cos\phi\,\hat{s}-\sin\phi\,\hat{\phi})-s\cos\phi(\sin\phi\,\hat{s}+\cos\phi\,\hat{\phi})\right)$$

$$=\frac{1}{2}B_z s\,\hat{\phi}$$

Alternatively one can also get an answer by examining $\nabla\times\vec{A}$ in the cylindrical coordinate

(3) $\vec{B} = \nabla \times \vec{A}$ must satisfy for $\vec{A_1}$ and $\vec{A_2}$ because we derived these expressions using this identity, so no need to prove.

Regarding $\nabla \cdot \vec{A} = 0$

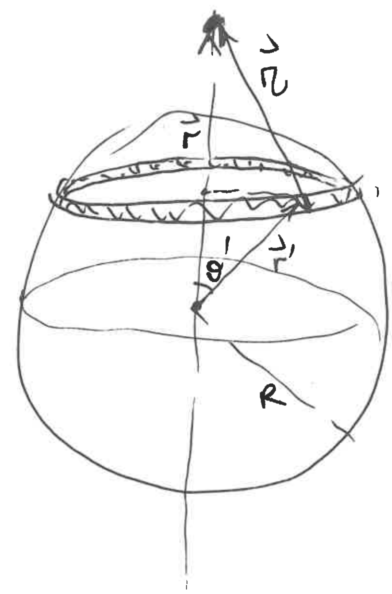
$$\nabla \cdot \vec{A_1} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(B_z x) + \frac{\partial}{\partial z}(0) = 0$$

$$\nabla \cdot \vec{A_2} = \frac{\partial}{\partial x}\left(-\tfrac{1}{2}B_z y\right) + \frac{\partial}{\partial y}\left(\tfrac{1}{2}B_z x\right) + \frac{\partial}{\partial z}(0) = 0$$

or, in cylindrical coordinates,

$$\nabla \cdot \vec{A_2} = \frac{1}{s}\frac{\partial}{\partial s}(s \cdot 0) + \frac{1}{s}\frac{\partial}{\partial \phi}\left(\tfrac{1}{2}B_z s\right) + \frac{\partial}{\partial z}(0) = 0$$

5.



Biot-Savart law that works for surface current:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'$$

where $\vec{K} = \vec{K_b} = M \sin\theta' \hat{\phi}$

From law of cosines

$$\mathscr{r}^2 = r^2 + R^2 - 2rR\cos\theta'$$

And we have $\vec{r} = (0, 0, r)$, $\vec{r}' = (R\sin\theta'\cos\varphi', R\sin\theta'\sin\varphi', R\cos\theta')$

$\Rightarrow \quad \vec{\mathscr{r}} = (R\sin\theta'\cos\varphi', -R\sin\theta'\sin\varphi', -R\cos\theta' + r)$

Further, in cylindrical coordinates $\hat{\varphi} = (-\sin\varphi, \cos\varphi', 0)$

so $(\hat{\varphi} \times \vec{\mathscr{r}}) \cdot \hat{z}$, the z-component of $\hat{\varphi} \times \vec{\mathscr{r}}$

$+\sin\varphi' \cdot R\sin\theta'\sin\varphi' + \cos\varphi' R\sin\theta'\cos\varphi' = +R\sin\theta'$

Therefore,

(where $da' = R^2\sin\theta' d\theta' d\varphi'$)

$$\hat{z} \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}_b(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} da' \cdot \hat{z}$$

$$= \frac{\mu_0}{4\pi} \int \frac{\vec{K}_b(\vec{r}') \times \vec{\mathscr{r}}}{\mathscr{r}^3} da' \cdot \hat{z}$$

$$= \frac{\mu_0}{4\pi} \int_0^{2\pi}\int_0^{\pi} \frac{M\sin\theta' (\hat{\varphi} \times \vec{\mathscr{r}}) \cdot \hat{z}}{(r^2+R^2-2rR\cos\theta)^{\frac{3}{2}}} \cdot R^2\sin\theta' d\theta' d\varphi'$$

$$= \frac{\mu_0}{4\pi} \int_0^{2\pi}\int_0^{\pi} \frac{M\sin\theta' \cdot R\sin\theta' \cdot R^2\sin\theta'}{(r^2+R^2-2rR\cos\theta)^{\frac{3}{2}}} d\theta' d\varphi'$$

$$= \frac{\mu_0 M R^3}{2} \underbrace{\int_0^{\pi} \frac{\sin^3\theta}{(r^2+R^2-2rR\cos\theta)^{\frac{3}{2}}} d\theta}_{(2)}$$

Assume substitution $u = r^2 + R^2 - 2rR\cos\theta$

$$du = 2rR\sin\theta\, d\theta$$

$$\sin^2\theta = 1-\cos^2\theta = 1-\left(\frac{r^2+R^2-u}{2rR}\right)^2$$

$$\Rightarrow \quad ② = \int_{(r-R)^2}^{(r+R)^2} \frac{1}{2rR}\left[1-\left(\frac{r^2+R^2-u}{2rR}\right)^2\right] u^{-\frac{3}{2}}\, du$$

$$= \int_{(r-R)^2}^{(r+R)^2} \frac{1}{2rR}\, \frac{4r^2R^2-(r^2+R^2-u)^2}{4r^2R^2}\, u^{-\frac{3}{2}}\, du$$

$$= \frac{1}{8r^3R^3}\int_{(r-R)^2}^{(r+R)^2} \left[(r+R)^2-u\right]\left[u-(r-R)^2\right] u^{-\frac{3}{2}}\, du$$

$$= \frac{1}{8r^3R^3}\int_{u=(r-R)^2}^{u=(r+R)^2} \left[(r+R)^2-u\right]\left[u-(r-R)^2\right](-2)\, d\left(u^{-\frac{1}{2}}\right)$$

↓ integrate by parts

$$= \frac{-2}{8r^3R^3}\Bigg\{ \underbrace{\left[(r+R)^2-u\right]\left[u-(r-R)^2\right] u^{-\frac{1}{2}}\Big|_{(r-R)^2}^{(r+R^2)}}_{\text{This term is zero}}$$

$$- \int_{(r-R)^2}^{(r+R)^2} u^{-\frac{1}{2}} \left\{\left[(r+R)^2-u\right]-\left[u-(r-R)^2\right]\right\} du \Bigg\}$$

$$= \frac{1}{4r^3R^3}\int_{(r-R)^2}^{(r+R^2)} u^{-\frac{1}{2}}\left(-2u+2r^2+2R^2\right) du$$

$$= \frac{-1}{2r^3R^3}\int_{(r-R)^2}^{(r+R^2)} \left[u^{\frac{1}{2}} - (r^2+R^2)\, u^{-\frac{1}{2}}\right] du$$

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$$= -\frac{1}{3}\frac{1}{r^3R^3}u^{\frac{3}{2}}\Big|_{(r-R)^2}^{(r+R)^2} + \frac{r^2+R^2}{r^3R^3}u^{\frac{1}{2}}\Big|_{(r-R)^2}^{(r+R)^2}$$

(1) When $r<R$

$$② = -\frac{1}{3}\frac{1}{r^3R^3}u^{\frac{3}{2}}\Big|_{(R-r)^2}^{(r+R)^2} + \frac{r^2+R^2}{r^3R^3}u^{\frac{1}{2}}\Big|_{(R-r)^2}^{(R+r)^2}$$

$$= -\frac{1}{3r^3R^3}\left[(R+r)^3-(R-r)^3\right] + \frac{r^2+R^2}{r^3R^3}(R+r-R+r)$$

$$= -\frac{1}{3r^3R^3}(2r^3+6R^2r) + \frac{2r^3+2rR^2}{r^2R^3} = \frac{4}{3}\frac{1}{R^3}$$

Therefore $\hat{z}\cdot\vec{B}(\vec{r}) = \frac{\mu_0 M R^3}{2}\cdot ② = \frac{\mu_0 M R^3}{2}\cdot\frac{4}{3}\frac{1}{R^3} = \frac{2}{3}\mu_0 M$

(2) When $r>R$

$$② = -\frac{1}{3}\frac{1}{r^3R^3}u^{\frac{3}{2}}\Big|_{(r-R)^2}^{(R+r)^2} + \frac{r^2+R^2}{r^3R^3}u^{\frac{1}{2}}\Big|_{(r-R)^2}^{(r+R)^2}$$

$$= -\frac{1}{3r^3R^3}\left[(R+r)^3-(r-R)^3\right] + \frac{r^2+R^2}{r^3R^3}\cdot 2R$$

$$= -\frac{1}{3r^3R^3}(2R^3+6r^2R) + \frac{2Rr^2+2R^3}{r^2R^3} = \frac{4}{3r^3}$$

Therefore $\hat{z}\cdot\vec{B}(\vec{r}) = \frac{\mu_0 M R^3}{2}\cdot\frac{4}{3r^3} = \frac{2}{3}\mu_0 M\frac{R^3}{r^3}$

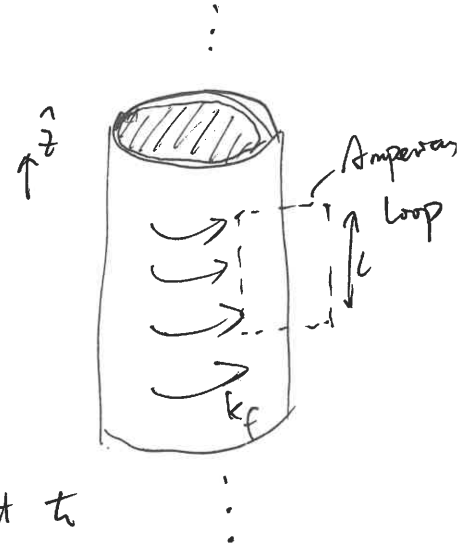
This can be rearranged as

$$\hat{z} \cdot \vec{B}(\vec{r}) = \frac{2}{3} \mu_0 M \frac{R^3}{r^3} = \frac{\mu_0}{2\pi} \left(\frac{4}{3}\pi M R^3\right) \frac{1}{r^3} = \frac{\mu_0}{2\pi} \frac{m}{r^3}$$

This is the field expected for magnetic dipole with $m = \frac{4}{3}\pi R^3 M$

according to textbook eq. (5.88)

6.



(1) Using symmetry analysis, we can argue

that $\vec{B} \parallel \vec{H} \parallel \vec{M} \parallel \hat{z}$ everywhere

and $\vec{B} = \vec{H} = \vec{M} = 0$ outside the coil. $(s > R)$

Use the rectangular Amperian Loop on the right to

calculate $\vec{H}$: $\oint \vec{H} \cdot d\vec{l} = I_{fenc}$

$$\Rightarrow H \cdot l = nIl \Rightarrow \vec{H} = nI\hat{z} \text{ inside the coil. } (s < R)$$

In the magnetic medium $\vec{B} = \mu\vec{H} = \mu_0(1+\chi_m)\vec{H} = \mu_0(1+\chi_m)nI\hat{z}$

for $(s < R)$

$$\vec{M} = \chi_m \vec{H} = \chi_m nI\hat{z} \quad (s < R)$$

For calculating vector potential $\vec{A}$, we use

$\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$ and choose the Amperian Loop below

Symmetry analysis suggests $\vec{A} \parallel \hat{\phi}$, as in textbook example 5.9

For $s < R$, $2\pi s A = \pi s^2 \cdot \mu_0 (1+\chi_m) n I$

$$\Rightarrow \vec{A} = \frac{s \mu_0 (1+\chi_m) n I}{2} \hat{\phi}$$

For $s > R$, $2\pi s A = \pi R^2 \cdot \mu_0 (1+\chi_m) n I$

$$\Rightarrow \vec{A} = \frac{R^2 \mu_0 (1+\chi_m) n I}{2s} \hat{\phi}$$

(2) $\vec{K}_f = nI\hat{\phi}$ is the effective surface current.

$$\underline{\vec{H}^{\parallel}_{above} - \vec{H}^{\parallel}_{below} = \vec{K}_f \times \hat{n}}$$

This is satisfied by $\hat{z}\left(H_z(s>R) - H_z(s<R)\right) = K_f(\hat{\phi} \times \hat{s})$

$= -nI\hat{z}$

$\Rightarrow 0 - nI\hat{z} = -nI\hat{z}$ ✓

$$\underline{H^{\perp}_{above} - H^{\perp}_{below} = -M^{\perp}_{above} - M^{\perp}_{below}}$$

Here $\perp$ components mean components along $\hat{s}$, while neither $\vec{H}$ nor $\vec{M}$ has $\hat{s}$ component, so $0 = 0$ ✓

$$\underline{B^{\perp}_{above} = B^{\perp}_{below}}$$

Because there is no $\hat{s}$ component, $0 = 0$ ✓

$$\vec{B}^{\parallel}_{above} - \vec{B}^{\parallel}_{below} = \mu_0 (\vec{K} \times \hat{n})$$

Here $\vec{K} = \vec{K}_f + \vec{K}_b$, where $\vec{K}_b$ is the bound current on surface

$$\vec{K}_b = \vec{M} \times \hat{n} = \chi_m n I \, \hat{z} \times \hat{s} = \chi_m n I \, \hat{\phi}$$

$$\therefore \vec{K} = \vec{K}_f + \vec{K}_b = (1+\chi_m) \, n I \, \hat{\phi}$$

Boundary condition satisfied by

$$\hat{z} \left(B_z(s>R) - B_z(s<R) \right) = \mu_0 K \, (\hat{\phi} \times \hat{s})$$

$$= -\mu_0 (1+\chi_m) \, n I \, \hat{z}$$

$$0 - \mu_0 (1+\chi_m) \, n I \, \hat{z} = -\mu_0 (1+\chi_m) \, n I \, \hat{z} \quad \checkmark$$

$$\vec{A}_{above} = \vec{A}_{below}$$

Match solutions for $s<R$ & $s>R$ @ $s=R$

$$\vec{A}_{below}\Big|_{s=R} = \frac{R \mu_0 (1+\chi_m) \, n I}{2} \hat{\phi}$$

$$\vec{A}_{above}\Big|_{s=R} = \frac{R^2 \mu_0 (1+\chi_m) \, n I}{2R} \hat{\phi} = \frac{R \mu_0 (1+\chi_m) \, n I}{2} = \vec{A}_{below}\Big|_{s=R}$$

$\checkmark$

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$$

Here $\vec{K} = \vec{K}_f + \vec{K}_b = (1+\chi_m)\, nI\, \hat{\phi}$

$$\frac{\partial \vec{A}_{above}}{\partial n} = \left.\frac{\partial \vec{A}\,(s>R)}{\partial s}\right|_{s=R} = \left.\frac{R^2 \mu_0 (1+\chi_m) nI}{-2s^2}\hat{\phi}\right|_{s=R} = \frac{\mu_0(1+\chi_m) nI}{-2}\hat{\phi}$$

$$\frac{\partial \vec{A}_{below}}{\partial n} = \left.\frac{\partial \vec{A}\,(s<R)}{\partial s}\right|_{s=R} = \frac{\mu_0 (1+\chi_m) nI}{2}\hat{\phi}$$

$$\Rightarrow \frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 (1+\chi_m) nI\,\hat{\phi} = -\mu_0 \vec{K} \quad ✓$$