Homework #4

(Due on Canvas by Sun, Nov.10)

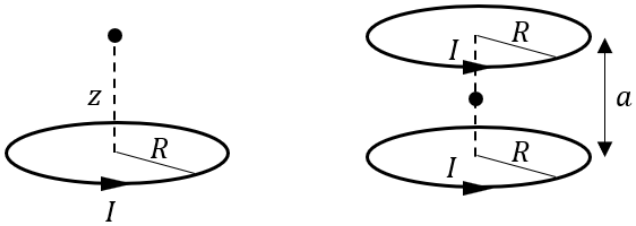
1. The Biot-Savart law

(1) See the left panel of the figure below. For a circular loop wire with radius R carrying current I, calculate the magnetic field on its center axis at a distance z above it.

(2) Two identical circular current-carrying wires share the same center axis but placed at a distance a apart, as shown in the right panel.

(i) Derive an expression of magnetic field anywhere on the center axis.

(ii) Prove that when $a = R$, the midpoint in between the coils would have the most uniform magnetic field. This is how Helmholtz coils work. (Hint: when field B is most uniform, both the first and second derivatives of B with respect to vertical displacement would vanish.)



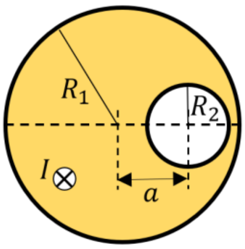
2. The Ampère's law

The Ampère's law combined with the superposition principle enables easy solutions to complex geometries. Depicted below is the cross-section of an infinitely long conducting rod (with radius R_1), through which a current I is running into the paper and distributed uniformly in the yellow region. There exists a circular hollow region with radius R_2 displaced from the center of the rod by distance a (we have $R_2 < a$).

(1) Calculate the volume current density in the yellow region.

(2) Calculate the magnetic field at the center axis of the rod.

(3) Calculate the magnetic field at the center axis of the hollow region.



3. Magnetic dipole

Consider a uniformly charged solid sphere with radius R spinning at an angular frequency of ω about the z axis. Suppose the total charge carried by the ball is q.

(1) Derive an expression for the total magnetic dipole moment of the spinning sphere.

(2) Consider this spinning sphere as a spinning electron, where $q = 1.6 \times 10^{-19}$ C and $R = 2.8 \times 10^{-15}$ m. Calculate how large the linear velocity at the electron's equator has to be in order to explain the observed value of the magnetic dipole moment carried by an electron spin $\mu_B = 9.27 \times 10^{-24}\ \mathrm{A \cdot m^2}$. Upon comparing this value to the speed of light you can get a sense of why this classical picture for electron spin is unreasonable.

4. The vector potential

The vector potential $\boldsymbol{A}$ that can describe a known magnetic field $\boldsymbol{B}$ is not unique. For a uniform magnetic field distribution $\boldsymbol{B} = B_z\hat{z}$,

(1) Try to find two possible expressions for $\boldsymbol{A}$ in the Cartesian coordinate,

(2) Try to find one possible expression for $\boldsymbol{A}$ in the cylindrical coordinate,

(3) Confirm through calculation that for all your proposed expressions, they satisfy $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\nabla \cdot \boldsymbol{A} = 0$ at the same time.

5. Uniformly magnetized sphere

A uniformly magnetized sphere (with radius R, and $\boldsymbol{M} = M\hat{z}$) is treated equal to a spherical shell carrying surface bound current density $\boldsymbol{K}_b = M \sin\theta\, \hat{\phi}$. Assuming that the point of interest is on the z axis, try to use the Biot-Savart law to prove that

(1) Magnetic field inside the sphere is uniform with $\boldsymbol{B} = \frac{2}{3}\mu_0\boldsymbol{M}$.

(2) Magnetic field outside the sphere is same as what would be produced by a dipole $\boldsymbol{m} = \frac{4}{3}\pi R^3\boldsymbol{M}$.

6. Linear magnetic media

An infinitely long solenoid (n turns per unit length, with current I) is filled with a linear magnetic medium (grey rod) with a magnetic susceptibility of χ_m.

(1) Find the magnetic field $\boldsymbol{B}$, the auxiliary field $\boldsymbol{H}$, the magnetization $\boldsymbol{M}$, and vector potential $\boldsymbol{A}$ everywhere in space.

(2) The solenoid can be approximated as a free surface current K_f wrapping around the magnetic medium. Prove that all boundary conditions that apply to $\boldsymbol{B}$, $\boldsymbol{H}$, and $\boldsymbol{A}$, holds. These boundary conditions are displayed in Chapter 3 lecture slides page 41 and page 24.