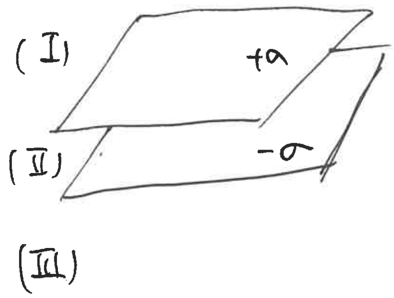
Homework 2 Solution

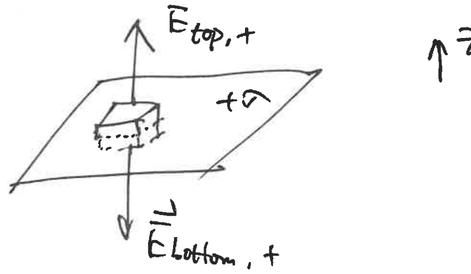
1.

(1) Let us define the 3 charge free regions by the diagram on the right. We can use



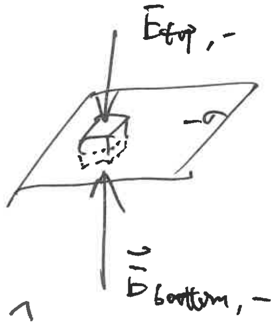
Gauss's law + superposition principle to solve the electric fields for all regions.

For $+\sigma$ plate alone, can choose Gaussian pillbox in the same way of example 2.5 (as discussed in class)



$$\Rightarrow \quad \vec{E}_{top,+} = \frac{\sigma}{2\varepsilon_0}\hat{z} \quad , \quad \vec{E}_{bottom,+} = \frac{-\sigma}{2\varepsilon_0}\hat{z}$$

For $-\sigma$ plate alone, can repeat the process



$$\Rightarrow \quad \vec{E}_{top,-} = \frac{-\sigma}{2\varepsilon_0}\hat{z} , \quad \vec{E}_{bottom,-} = \frac{\sigma}{2\varepsilon_0}\hat{z}$$

Then, according to the superposition principle in region (I), above two plates

$$\vec{E}_I = \vec{E}_{top,+} + \vec{E}_{top,-} = \frac{\sigma}{2\varepsilon_0}\hat{z} - \frac{\sigma}{2\varepsilon_0}\hat{z} = 0$$

in region (II), within the gap

$$\vec{E}_{II} = \vec{E}_{bottom,+} + \vec{E}_{top,-} = \frac{-\sigma}{2\varepsilon_0}\hat{z} - \frac{\sigma}{2\varepsilon_0}\hat{z} = \frac{-\sigma}{\varepsilon_0}\hat{z}$$

in region (III), below the two plates

$$\vec{E}_{III} = \vec{E}_{bottom,+} + \vec{E}_{bottom,-} = 0$$

(2) (i) When gap is small, and if one only cares about the field inside the gap, the two plates can be considered as infinitely charge approximately. The "edge effect" will be small.

(ii) Surface charge density $\sigma_{\pm} = \frac{\pm Q}{A}$

Electric field within the gap $\vec{E}_{gap} = \frac{-\sigma}{\varepsilon_0}\hat{z} = \frac{-Q}{\varepsilon_0 A}\hat{z}$, and is homogeneous within the gap due to the argument in (i)

Potential difference $V = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{l}$, choose path to go from $-$ plate to + plate running along $+\hat{z}$ direction

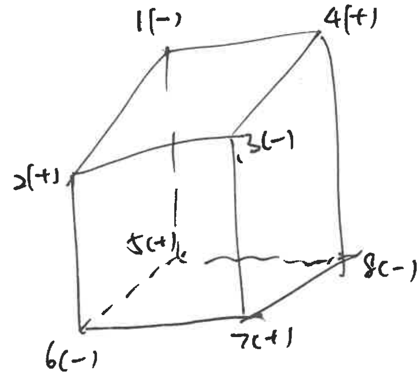
$$V = -\int_0^d \frac{-Q}{\varepsilon_0 A} \cdot dz = \frac{Qd}{\varepsilon_0 A}$$

(iii) $$C = \frac{Q}{V} = \frac{Q \varepsilon_0 A}{Q d} = \frac{\varepsilon_0 A}{d}$$

2. For discrete charge distributions, energy

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$$

potential at point $\vec{r}_i$ due to all <u>other</u> charges



Consider <u>charge 1</u>, a $-q$ charge

It has 3 nearest neighbors, 2(+), 4(+), 5(+)

with $r_{12} = r_{14} = r_{15} = a$

It has 3 ~~second~~ next nearest neighbors, 3(-), 6(-), 8(-)

with $r_{13} = r_{16} = r_{18} = \sqrt{2} a$

It has 1 farthest neighbor, 7(+) with $r_{17} = \sqrt{3} a$.

Energy associated with charge 1:

$$W_1 = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} (-q) \left[3 \cdot \frac{(+q)}{a} + 3 \cdot \frac{(-q)}{\sqrt{2} a} + \frac{(+q)}{\sqrt{3} a} \right]$$

$$= \frac{q^2}{8\pi\varepsilon_0 a} \left(-3 + \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right)$$

Notice all $(-q)$ centers, 3(-), 6(-), 8(-) all experience the same situation as 1(-), so $W_3 = W_6 = W_8 = W_1$

Further, consider charge 2, a $+q$ charge

W_2 calculation would be similar to W_1, only with the following replacement : $+q \Rightarrow -q$, and $-q \Rightarrow +q$, we end up with

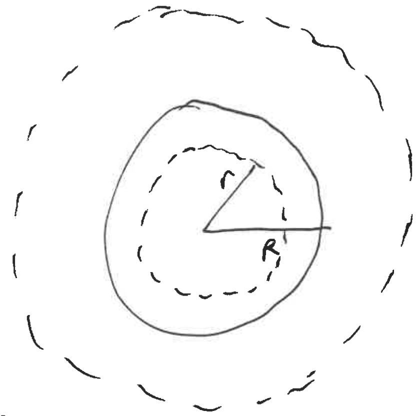
$W_2 = W_1$

And, 4(+), 5(+), 7(+) experience the same situation as 2(+)

So $W_4 = W_5 = W_7 = W_2$

Therefore, total energy $W = 8 \cdot W_1 = \dfrac{q^2}{\pi \varepsilon_0 a} \left(-3 + \dfrac{3}{\sqrt{2}} - \dfrac{1}{\sqrt{3}} \right)$

3.



(1) $V(\vec{r})$ can be calculated once we know the electric field inside & outside the sphere.

Choose Gaussian integration surface as concentric spherical shells.

For $r > R$, outside the sphere $\oint_S \vec{E} \cdot d\vec{a} = \dfrac{Q_{enc}}{\varepsilon_0}$

$\Rightarrow E \cdot 4\pi r^2 = \dfrac{1}{\varepsilon_0} \int \rho \, d\tau = \dfrac{1}{\varepsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} kr \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$

$\Rightarrow E \cdot 4\pi r^2 = \dfrac{k}{\varepsilon_0} \cdot 4\pi \int_0^R r^3 dr = \dfrac{4\pi k}{\varepsilon_0} \dfrac{R^4}{4}$

$\Rightarrow E_{out} = \dfrac{kR^4}{4\varepsilon_0 r^2}$

For $r \leq R$ inside the sphere

$$\Rightarrow \quad \bar{E} \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^{r} kr' \cdot r'^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\Rightarrow \quad \bar{E} \cdot 4\pi r^2 = \frac{k}{\varepsilon_0} \cdot 4\pi \int_0^r r'^3 dr = \frac{4\pi k}{\varepsilon_0} \frac{1}{4} r^4$$

$$\Rightarrow \quad \bar{E}_{in} = \frac{kr^2}{4\varepsilon_0}$$

Potential outside the sphere $r > R$

$V_{out} = -\int_\infty^r \vec{E} \cdot d\vec{l}$ integrate along the $-\hat{r}$ direction from ∞ to r

$$= -\int_\infty^r \frac{kR^4}{4\varepsilon_0 r'^2} dr' = -\frac{kR^4}{4\varepsilon_0} \int_\infty^r \frac{1}{r'^2} dr'$$

$$= -\frac{kR^4}{4\varepsilon_0} \left[-\frac{1}{r'} \right] \Big|_\infty^r = \frac{kR^4}{4\varepsilon_0 r}$$

Potential inside the sphere $r \leq R$, divide integral into 2 sections

$$V_{in} = -\int_\infty^r \vec{E} \cdot d\vec{l} = -\int_\infty^R \vec{E}_{out} \cdot d\vec{l} - \int_R^r \vec{E}_{in} \cdot d\vec{l}$$

$$= -\int_\infty^R \frac{kR^4}{4\varepsilon_0 r'^2} dr' - \int_R^r \frac{kr'^2}{4\varepsilon_0} dr'$$

$$= \frac{kR^4}{4\varepsilon_0 R} - \frac{k}{4\varepsilon_0} \int_R^r r'^2 dr' = \frac{kR^3}{4\varepsilon_0} - \frac{k}{4\varepsilon_0} \frac{1}{3} (r^3 - R^3)$$

$$= \frac{k}{4\varepsilon_0} \left(\frac{4}{3} R^3 - \frac{1}{3} r^3 \right)$$

(2) For continuous charge distribution

$$W = \frac{1}{2}\int \rho V_{in} d\tau \quad , \text{ notice } \rho \text{ only finite inside sphere.}$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} kr \cdot \frac{k}{4\varepsilon_0}\left(\frac{4}{3}R^3 - \frac{1}{3}r^3\right) r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{1}{2}\cdot\frac{k^2}{4\varepsilon_0}\cdot 4\pi \int_0^R \left(\frac{4}{3}R^3 r^3 - \frac{1}{3}r^6\right) dr$$

$$= \frac{\pi k^2}{2\varepsilon_0}\left[\frac{2}{6}R^3 r^4\Big|_0^R - \frac{1}{21}r^7\Big|_0^R\right]$$

$$= \frac{\pi k^2}{2\varepsilon_0}R^7\left(\frac{2}{6} - \frac{1}{21}\right) = \cancel{\quad} = \frac{\pi k^2 R^7}{7\varepsilon_0}$$

Same answer can be obtained by $W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 d\tau$

4. From example 3.2 $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\eta} + \frac{q'}{\eta'}\right)$

where image charge $q' = -\frac{R}{a}q$, and

$$\eta = (r^2 + a^2 - 2ra\cos\theta)^{\frac{1}{2}}$$

$$\eta' = (r^2 + b^2 - 2rb\cos\theta)^{\frac{1}{2}} \quad , \text{ where } b = \frac{R^2}{a}$$

(1) Simplifying expression for $V(\vec{r})$,

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r^2+a^2-2ra\cos\theta)^{\frac{1}{2}}} - \frac{1}{(R^2+(ra/R)^2-2ra\cos\theta)^{\frac{1}{2}}}\right]$$

Surface charge distribution $\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$ where $\hat{n} = \hat{r}$ is the surface normal

$$\Rightarrow \sigma = -\varepsilon_0 \frac{\partial V(\vec{r})}{\partial r}\Big|_{r=R}$$

$$= \frac{-q}{4\pi}\Big[-\tfrac{1}{2}(r^2+a^2-2ra\cos\theta)^{-\frac{3}{2}}(2r-2a\cos\theta)$$

$$+\tfrac{1}{2}(R^2+(ra/R)^2-2ra\cos\theta)^{-\frac{3}{2}}\left(\frac{2a^2}{R^2}r-2a\cos\theta\right)\Big]\Big|_{r=R}$$

$$= \frac{-q}{4\pi}(R^2+a^2-2Ra\cos\theta)^{-\frac{3}{2}}\left(-R+\cancel{a\cos\theta}+\frac{a^2}{R}-\cancel{a\cos\theta}\right)$$

$$= \frac{q}{4\pi R}(R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-\frac{3}{2}}$$

(2) $Q' = \int_S \sigma\, da$ integrate over the surface of sphere

$$= \int_0^{2\pi}\int_0^{\pi}\frac{q}{4\pi R}(R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-\frac{3}{2}}\underbrace{d\theta\, d\phi}_{R^2\sin\theta}$$

$$= \frac{q}{4\pi R}(R^2-a^2)R^2\cdot 2\pi\int_0^{\pi}(R^2+a^2-2Ra\cos\theta)^{-\frac{3}{2}}\sin\theta\, d\theta$$

$\downarrow$ use substitution $u \equiv R^2+a^2-2Ra\cos\theta$ to evaluate integral

$$= \frac{q}{2a}(a^2-R^2)\left(\frac{1}{a+R}-\frac{1}{a-R}\right) = -\frac{qR}{a} = q'$$

(3) Force acting on charge q (by q') is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(a-b)^2} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{R}{a}q^2\right)\frac{1}{(a-R^2/a)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2-R^2)}$$

← externally apply force to balance force exerted by q'

$W = \int(-\vec{F})\cdot d\vec{l}$ integrate from ∞ to a.

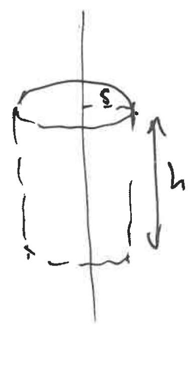
$$= \int_\infty^a +\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra'}{(a'^2-R^2)^2}\,da'$$

$$= \frac{q^2R}{4\pi\varepsilon_0}\int_\infty^a \frac{a'}{(a'^2-R^2)^2}\,da'$$

$$= \frac{q^2R}{4\pi\varepsilon_0}\left(\frac{1}{2}\right)\left(-\frac{1}{a'^2-R^2}\right)\Bigg|_\infty^a = -\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{2(a^2-R^2)}$$

5. To solve the image charge problem for a line charge we first need to find the potential $V(s)$ associated with a line charge

Consider line charge on the right, choose Gaussian surface as the cylinder with radius s, so



Gauss's law : $E\,2\pi sh = \frac{h\lambda}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 s}$

Potential $V(s) = -\int\vec{E}\cdot d\vec{l}$, notice if you choose ∞ as the reference point, $V(s) = -\int_\infty^s \frac{\lambda}{2\pi\varepsilon_0 s'}\,ds'$ will diverge

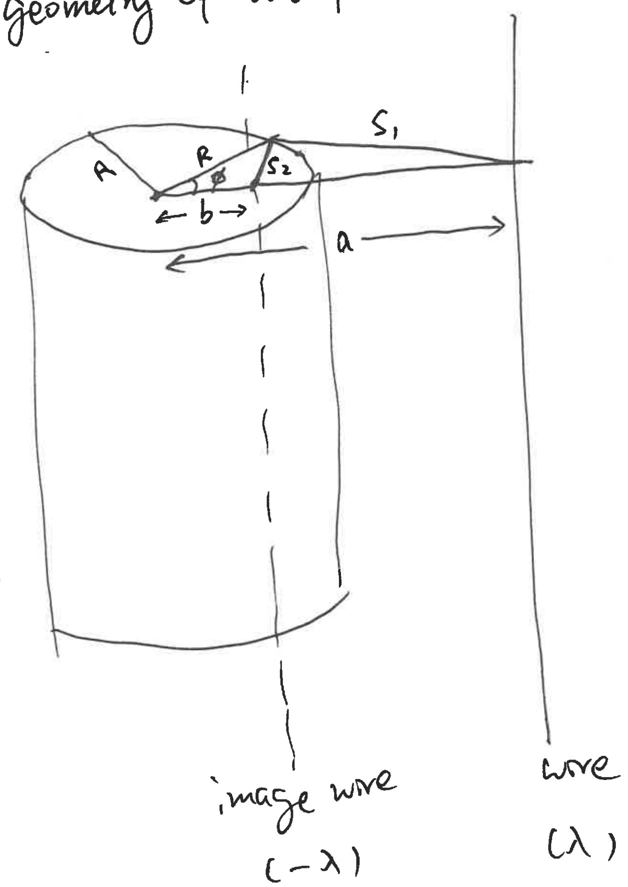
The reference point is chosen at a point with finite distance from the wire at $s = s_0$, which makes the potential become

$$V(s) = -\int_{s_0}^{s} \frac{\lambda}{2\pi\varepsilon_0 s'} ds' = \frac{\lambda}{2\pi\varepsilon_0} (-\ln s') \Big|_{s_0}^{s}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} (\ln s_0 - \ln s)$$

(1) Now we can consider the geometry of the problem

Suppose image charged is placed at $s = b$ from the center of cylinder



At the surface of cylinder, the potential contributed by charged wire is

$$V_1 = \frac{\lambda}{2\pi\varepsilon_0} (\ln s_0 - \ln s_1)$$

where $s_1^2 = a^2 + R^2 - 2aR\cos\phi$

potential contributed by image wire is

$$V_2 = \frac{-\lambda}{2\pi\varepsilon_0} (\ln s_0 - \ln s_2) \quad \text{where } s_2^2 = b^2 + R^2 - 2bR\cos\phi$$

Total potential $V = V_1 + V_2 = \frac{\lambda}{2\pi\varepsilon_0}(\ln s_2 - \ln s_1) = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{s_2}{s_1}$

This quantity must be constant @ surface of cylinder

$\Rightarrow s_2/s_1 = \sqrt{k}$, where k is a constant

$\Rightarrow s_2^2/s_1^2 = k$

$\Rightarrow b^2 + R^2 - 2bR\cos\phi = ka^2 + kR^2 - 2kaR\cos\phi$

The coefficients in front of $\cos\phi$ must be equal otherwise the equality will not hold $\Rightarrow 2bR = 2kaR \Rightarrow b = ka$

Then equation above becomes $(ka)^2 + R^2 = ka^2 + kR^2$

$\Rightarrow (k^2 - k)a^2 = (k-1)R^2$

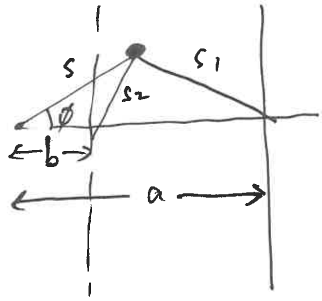
The $k=1$ solution is trivial (since then image wire will be original wire)

$\Rightarrow k = \frac{R^2}{a^2} \Rightarrow b = ka = \frac{R^2}{a}$

To conclude, image wire should be placed at $b = \frac{R^2}{a}$ from origin.

(2) Now that we know the location of the image charge, the potential outside the cylinder would be

$$V(s) = \overset{\text{by wire}}{V_1(s)} + \overset{\text{by image wire}}{V_2(s)}$$



$$= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{s_2}{s_1}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{(s^2+b^2-2sb\cos\phi)^{\frac{1}{2}}}{(s^2+a^2-2sa\cos\phi)^{\frac{1}{2}}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln \frac{s^2+b^2-2sb\cos\phi}{s^2+a^2-2sa\cos\phi}$$, and knowing $b = \frac{R^2}{a}$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln \frac{s^2+\frac{R^4}{a^2}-2s\frac{R^2}{a}\cos\phi}{s^2+a^2-2sa\cos\phi}$$

(3) $$\sigma = -\varepsilon_0 \left.\frac{\partial V(s)}{\partial s}\right|_{s=R}$$

$$\frac{\partial V(s)}{\partial s} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{2s-\frac{2R^2}{a}\cos\phi}{s^2+\frac{R^4}{a^2}-2s\frac{R^2}{a}\cos\phi} - \frac{2s-2a\cos\phi}{s^2+a^2-2sa\cos\phi}\right]$$

so $$\sigma = \frac{-\lambda}{4\pi}\left[\frac{2R-\frac{2R^2}{a}\cos\phi}{R^2+\frac{R^4}{a^2}-2\frac{R^3}{a}\cos\phi} - \frac{2R-2a\cos\phi}{R^2+a^2-2Ra\cos\phi}\right]$$

$$= \frac{-\lambda}{4\pi} \frac{2R-\frac{2R^2}{a}\cos\phi - 2R\cdot\frac{R^2}{a^2}+2a\cos\phi\frac{R^2}{a^2}}{R^2+\frac{R^4}{a^2}-2\frac{R^3}{a}\cos\phi}$$

$$= \frac{-\lambda}{4\pi} \frac{2R(1-R^2/a^2)}{R^2+\frac{R^4}{a^2}-2\frac{R^3}{a}\cos\phi}$$

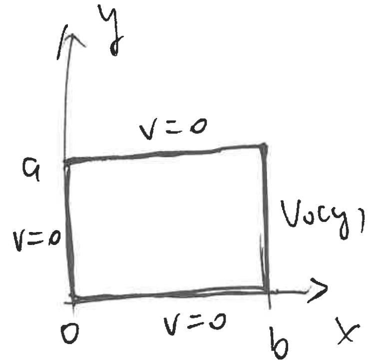
$$= \frac{-\lambda}{4\pi} \frac{2(a^2 - R^2)}{a^2 R + R^3 - 2R^2 a\cos\phi}$$

6. This problem is only different from the textbook example (that we worked in class) by closing off of the $V(0,y)$ & $V(b,y)$ boundaries

Therefore, the basic procedure of separation of variables in the Cartesian coordinates still hold, and the general solution holds as

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

(1) The goal is the determination of A, B, C, D coefficients by matching boundary conditions



$$\begin{cases} \text{(i)} & V(x,0) = 0 \\ \text{(ii)} & V(x,a) = 0 \\ \text{(iii)} & V(0,y) = 0 \\ \text{(iv)} & V(b,y) = V_0(y) \end{cases}$$

(i) $\Rightarrow D = 0$

(iii) $\Rightarrow A = -B$

(ii) $\Rightarrow \sin ka = 0 \Rightarrow k = \frac{n\pi}{a}$ where $n = 1, 2, 3 \cdots$

So $V(x,y) = A c \left(e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}}\right) \sin\left(\frac{n\pi y}{a}\right)$

To match boundary condition (iv), we need a linear combination of variable-separated solutions

$$V(x,y) = \sum_{n=1}^{\infty} C_n \left(e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}}\right) \sin\left(\frac{n\pi y}{a}\right)$$

where we use C_n instead of Ac to denote coefficient in front of each order

boundary condition

(iv) $\Rightarrow \sum_{n=1}^{\infty} C_n \left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right) \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$

Using the orthogonality conditions of the Fourier series,

$$\Rightarrow \quad C_n \left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right) = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

$$\Rightarrow \quad C_n = \frac{2}{\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right) a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

This expression for C_n together with the above expression for $V(x,y)$ gives a general expression for the solution.

(2) For $V_0(y) = V_0$, plug in to the expression for C_n

$$C_n = \frac{2}{\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)a} \int_0^a V_0 \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{2V_0}{\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)a} \cdot \frac{a}{n\pi} \left[-\cos\left(\frac{n\pi y}{a}\right)\right]\Big|_0^a$$

$$= \begin{cases} 0 & \text{(for even } n\text{)} \\ \dfrac{4V_0}{\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)n\pi} & \text{(for odd } n\text{)} \end{cases}$$