Tuesday, 1 October 2024 9:50 am

Tutorial 3 : Due on Thurs 3 Oct.

Addition of two spin-$\frac{1}{2}$ particles

$S_1 = \frac{1}{2}$ $\quad\quad$ $S_2 = \frac{1}{2}$

$m_1 = -\frac{1}{2}, \frac{1}{2}$ $\quad\quad$ $m_2 = -\frac{1}{2}, \frac{1}{2}$

Possible values of s where $\vec{S} = \vec{S}_1 + \vec{S}_2$.

$s = 0, 1$

$S_{min} = |\frac{1}{2} - \frac{1}{2}| = 0$ $\quad\quad$ $S_{max} = \frac{1}{2} + \frac{1}{2}$

Recall coupled vs uncoupled representations.

Find the coupled representation in terms of the uncoupled rep.

— Refer to Clebsch-Gordon Table.

By hand

$m = m_1 + m_2$

$m = 1$ $\quad |s_1 = \frac{1}{2}, m_1 = \frac{1}{2}, s_2 = \frac{1}{2}, m_2 = \frac{1}{2}\rangle = |s=1, m=1\rangle$

2D subspace.

$m = 0$ $\quad \underbrace{|s_1 = \frac{1}{2}, m_1 = \frac{1}{2}, s_2 = \frac{1}{2}, m_2 = -\frac{1}{2}\rangle}_{|\uparrow\downarrow\rangle}, \underbrace{|s_1 = \frac{1}{2}, m_1 = -\frac{1}{2}, s_2 = \frac{1}{2}, m_2 = \frac{1}{2}\rangle}_{|\downarrow\uparrow\rangle}$ $\quad$ $|S=1, m=0\rangle$, $|S=0, m=0\rangle$

$m = -1$ $\quad |s_1 = \frac{1}{2}, m_1 = -\frac{1}{2}, s_2 = \frac{1}{2}, m_2 = -\frac{1}{2}\rangle = |s=1, m=-1\rangle$

Last time:

Obtained $|s=1, m=0\rangle$

using $S_+ = S_{1+} + S_{2+}$ on the $m=-1$ state

(Alternative: use $S_- = S_{1-} + S_{2-}$ on the $m=1$ state)

<u>Result</u>. (using $\overset{m=+\frac{1}{2}}{|\uparrow\rangle}$ $\overset{m=-\frac{1}{2}}{|\downarrow\rangle}$ for $m = \pm\frac{1}{2}$)

Addition of two spin $\frac{1}{2}$ systems

	Coupled rep s	m	Uncoupled rep
triplet	1	1	$\vert\uparrow\uparrow\rangle$
	1	0	$\frac{1}{\sqrt{2}}\left(\vert\uparrow\downarrow\rangle + \vert\downarrow\uparrow\rangle\right)$
	1	-1	$\vert\downarrow\downarrow\rangle$
singlet	0	0	$\frac{1}{\sqrt{2}}\left(\vert\uparrow\downarrow\rangle - \vert\downarrow\uparrow\rangle\right)$

To obtain the singlet,
use orthogonality.

$$|s=0,\ m=0\rangle = \alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle$$

$$\langle s=0,\ m=0|s=1,\ m=0\rangle = 0$$

$$\alpha^*\langle\uparrow\downarrow|\uparrow\downarrow\rangle + \beta^*\langle\downarrow\uparrow|\downarrow\uparrow\rangle = 0$$

$$\alpha^* + \beta^* = 0$$

$$\alpha = -\beta$$

$\frac{1}{\sqrt{2}}$ from normalization.

So $|s=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

What if $s_1 = 1$, $s_2 = 1$?

$$S = 0,\quad 1,\quad 2$$

m

2 (dim 1) ($S = 2$) ✓

$\downarrow S_-$

1 (dim 2) ($S = 2$ ✓ or $S = 1$) ✓ orthonormality

(dim 3) ($S = 2$ ✓ or $S = 1$ ✓ or $S = 0$) ✓ orthonormality

1 (dim 2) (S = 2 or S = 1)

0 (dim 3) (S = 2 ✓ or S = 1 or S = 0) ✓ orthonormality

$\uparrow S_+$ $\uparrow S_+$ ③

−1 (dim 2) (S = 2 ✓ or S = 1) ✓ from. orthonormality

$\uparrow S_+$ ②

−2 (dim 1) (S = 2) ✓

①

Looking ahead:

- Identical & indistinguishable particles (Griffiths Chapter 5)
  - constraints on the many-body state.
- Approximate methods to solve for the eigenstates of the Hamiltonian, or for time-propagation of the quantum state.

Identical & indistinguishable particles

Classically,

⇒ can track the paths & velocities.

2 identical cars

In QM, we cannot measure the position & momentum at the same time. ($\Delta x \Delta p \geq \frac{\hbar}{2}$)

So identical particles are indistinguishable.

If we write:

$$\text{many-body state } |\Psi\rangle = |g\ e\rangle = \underset{\substack{\uparrow \\ v_1}}{|g\rangle_1} \otimes \underset{\substack{\uparrow \\ v_2}}{|e\rangle_2}$$

where g: ground state
e: excited state

this is wrong if the two particles are identical & indistinguishable,

because $|g\rangle_1 \otimes |e\rangle_2$ means that we know the particle in because $|g\rangle_1 \otimes |e\rangle_2$ means that we know the particle in v_1 is in the ground state & the particle in v_2 —— " — excited state.

We could perhaps fix this by having

$$|\Psi\rangle = \alpha\, |g\rangle_1 \otimes |e\rangle_2 + \beta\, |e\rangle_1 \otimes |g\rangle_2$$

for some carefully chosen α and β.

To choose α and β:

Think about what happens when we exchange two identical & indistinguishable particles.

$\Rightarrow$ Your observables should not change.

$$\langle\psi|\hat{A}|\psi\rangle \, , \quad |\langle\phi|\psi\rangle|^2$$

Define an operator denoting exchange of particles 1 & 2 to be P_{12}.

It can be shown that P_{12} has eigenvalues ± 1.

$P_{12}\,|\psi_+\rangle = |\psi_+\rangle$; $|\psi_+\rangle$ symmetric w.r.t. exchange - bosons

$P_{12}\,|\psi_-\rangle = -|\psi_-\rangle$; $|\psi_-\rangle$ antisymmetric wrt exchange.

fermions

Bosons vs Fermions

Bosons: Wavefunction / State stays the same under exchange of two identical & indistinguishable particles

symmetric many-body wavefunctions

Fermions: wavefunction/state acquires a negative sign when we exchange two identical & indistinguishable particles

(iid)

Antisymmetric many-body

Antisymmetric many-body wavefunctions sign when we exchange two identical & (iid) indistinguishable particles.

Going back to α and β:

$$|\psi\rangle = \alpha|g\,e\rangle + \beta|e\,g\rangle$$

Suppose we have two iid bosons.

Then $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|g\,e\rangle + |e\,g\rangle\right)$

Suppose we have two iid fermions:

Then $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|g\,e\rangle - |e\,g\rangle\right)$

Examples of bosons and fermions

Bosons

Spin: $\mathbb{Z}$ integer

Typically, elementary bosons mediate interactions

eg. phonons, photons

Composite bosons:

Eg. Even number of fermions

Fermions

Spin: $\frac{\mathbb{Z}}{2}$ - odd $\frac{1}{2}$-integer

Typically, elementary fermions are constituents of matter

eg. electron, proton,

Composite fermions:

Eg. Odd number of fermions

Pauli's exclusion principle:

No two fermions can occupy the same quantum state.

Example

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|g\,e\rangle - |e\,g\rangle\right)$$ for two fermions

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\alpha\beta\rangle - |\beta\alpha\rangle \right)$$

If $|\alpha\rangle \equiv |\beta\rangle$, $|\psi\rangle \equiv 0$

This is an example of Pauli's exclusion principle.

Now we consider more than two iid particles.

Eg. four bosons.

Each boson can be in state $|g\rangle$ or $|e\rangle$.

$$|\psi\rangle = A \left(|ggge\rangle + |ggeg\rangle + |gegg\rangle + |eggg\rangle \right)$$

$$\langle\psi|\psi\rangle = 1 \Rightarrow 4A^2 = 1$$

$$A = \frac{1}{2}$$

Eg Fermions.

Suppose we have 3 fermions, A, B, C.

3 single-particle states $|\phi\rangle$, $|\chi\rangle$, $|\omega\rangle$.

$$|\psi\rangle \propto \begin{vmatrix} |\phi\rangle_A & |\phi\rangle_B & |\phi\rangle_C \\ |\chi\rangle_A & |\chi\rangle_B & |\chi\rangle_C \\ |\omega\rangle_A & |\omega\rangle_B & |\omega\rangle_C \end{vmatrix}$$

Slater determinant.

$$\begin{aligned} &= |\phi\rangle_A \left(|\chi\rangle_B |\omega\rangle_C - |\omega\rangle_B |\chi\rangle_C \right) \\ &\quad - |\phi\rangle_B \left(|\chi\rangle_A |\omega\rangle_C - |\omega\rangle_A |\chi\rangle_C \right) \\ &\quad + |\phi\rangle_C \left(|\chi\rangle_A |\omega\rangle_B - |\omega\rangle_A |\chi\rangle_B \right) \end{aligned}$$

Antisymmetric wrt exchange.