# Magnetostatics

The Lorenz force law

Electrostatics: <u>Stationary</u> charges with constant electric fields

Magnetostatics: <u>Steady</u> currents with constant magnetic fields

- Magnetic force and magnetic field

  - Pass current in parallel wires

    - Wires repel with antiparallel current

    - Wires attract with parallel current

    - Force is not electrostatic in nature (wires are charge neutral)

    - Magnetic force

Battery

Battery
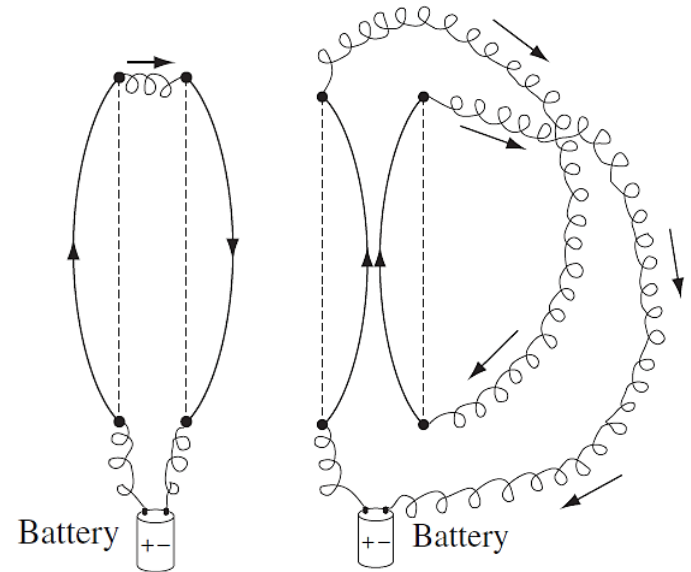
The Lorenz force law

- Magnetic force and magnetic field

  o Empirical rules for magnetic field and magnetic force

    - Straight current-carrying wire has magnetic field circling around it

    - Right-hand rule for field direction given current direction

    - Right-hand rule for force direction given current and field direction


Compass
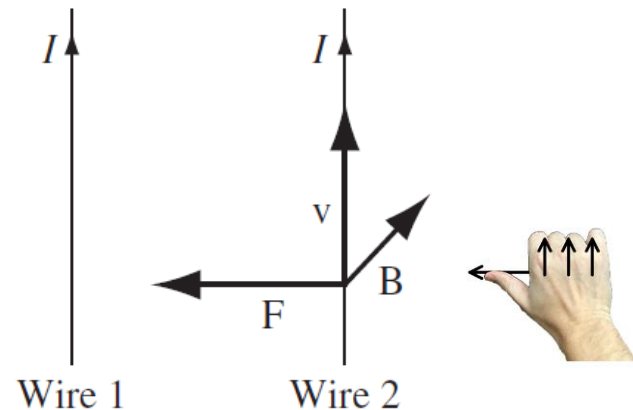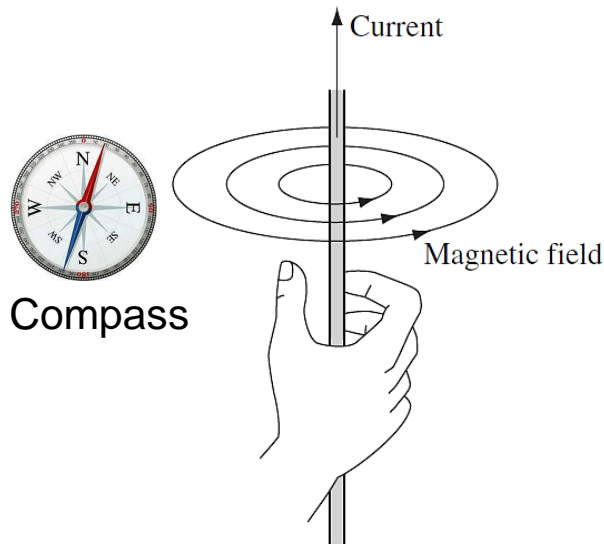
The Lorenz force law

- The Lorenz force law

  - Magnetic force on charge Q, moving with velocity $\mathbf{v}$, in magnetic field $\mathbf{B}$

  $$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

    - Given as an axiom without proof

  - With both $\mathbf{E}$ and $\mathbf{B}$, the full Lorenz force law $\boxed{\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]}$

  - Consequences

    - Circular charge motion, or alike, when magnetic force acts as centripetal force

    - Magnetic forces do <u>no</u> work, $\mathbf{B}$ only deflects particle direction

    $$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \overbrace{\mathbf{v}\, dt}^{d\mathbf{l}} = 0$$
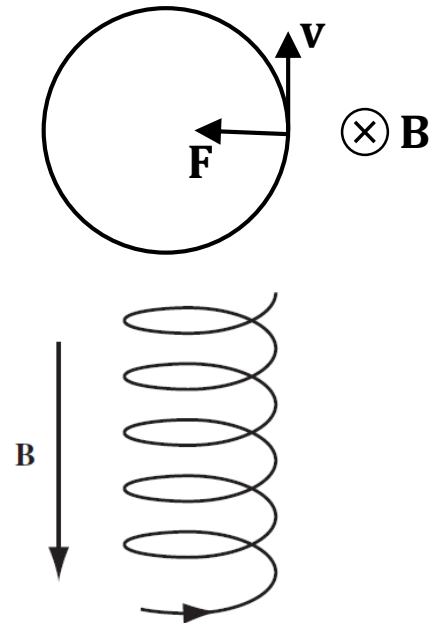
The Lorenz force law

Example 5.2. Cycloid Motion.   A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that **B** points in the x-direction, and **E** in the z-direction, as shown in Fig. 5.7. A positive charge is released from the origin; what path will it follow?
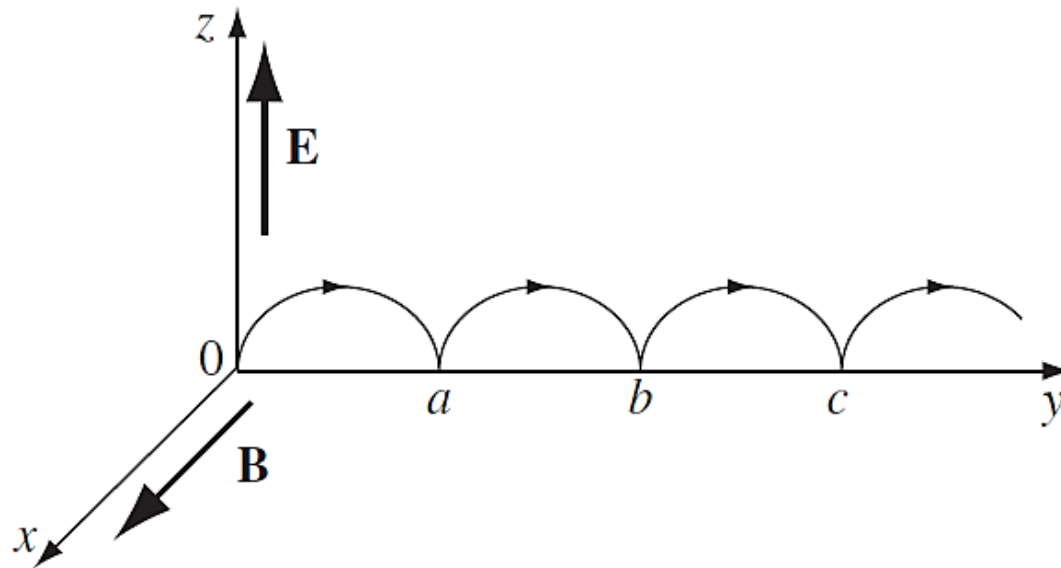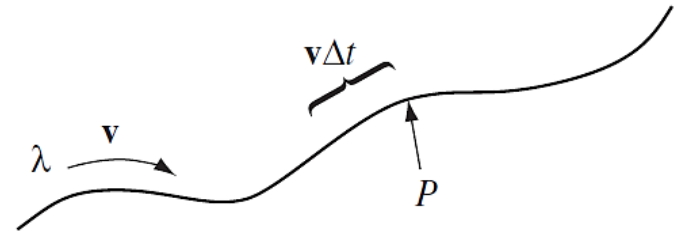
C3.exmp5.2

Currents

- Current in a wire: charge per time passing a given point

  - Unit: Amperes (A) = Coulomb/second

  - $\mathbf{I} = \lambda \mathbf{v}$ where λ: line charge density, $\mathbf{v}$: velocity of movement

  - Positive charges moving at $\mathbf{v}$ = negative charges moving at $-\mathbf{v}$

  - Not meaningful to talk about current if it's just a single point charge moving (non-steady)

  - In many problems just write the magnitude $\boxed{I = \lambda v}$

    - Because direction is determined by the shape of wire

  - Magnetic force on a current-carrying wire $\boxed{\mathbf{F}_{\text{mag}} = \int I\,(d\mathbf{l} \times \mathbf{B})}$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\,dq = \int (\mathbf{v} \times \mathbf{B})\lambda\,dl = \int (\mathbf{I} \times \mathbf{B})\,dl = \int I\,(d\mathbf{l} \times \mathbf{B})$$

Currents

- Surface and volume distributions of current

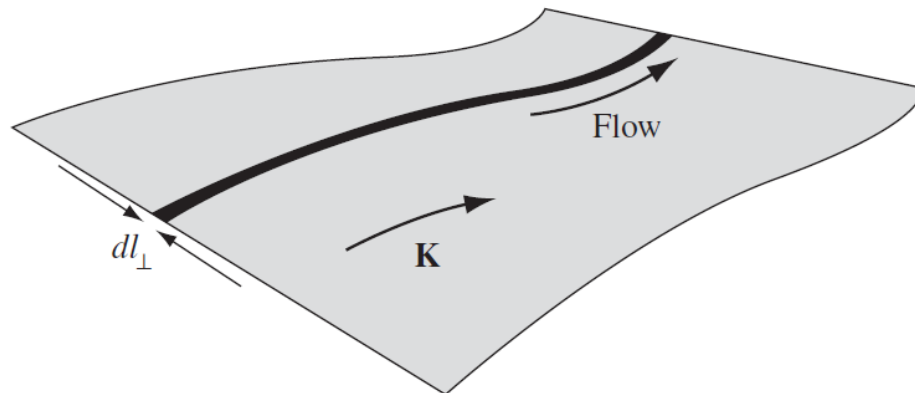 o Surface current density $\mathbf{K} = \sigma \mathbf{v}$ (C/m² · m/s)   (σ: surface charge density)

  - **K**: current per unit width (A/m)

    $$d\mathbf{I} = \sigma \mathbf{v}\, dl_\perp \qquad \Longrightarrow \qquad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp}$$

    ($l_\perp$: cross-sectional line segment perpendicular to **v**)

  - Magnetic force

    $$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B}) \sigma\, da = \int (\mathbf{K} \times \mathbf{B})\, da$$

    * **B** experiences discontinuity across a current-carrying surface

Currents

- Surface and volume distributions of current

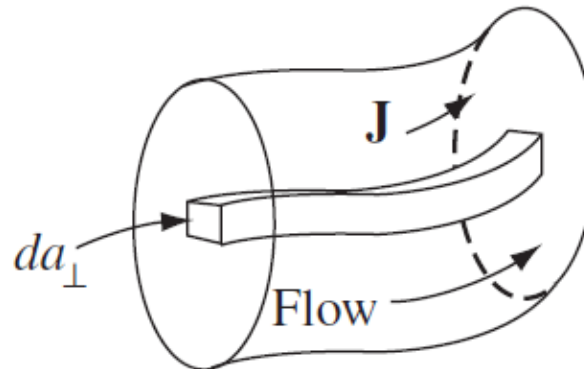 - Volume current density $\mathbf{J} = \rho \mathbf{v}$ (C/m³ · m/s) (ρ: volume charge density)

    - $\mathbf{J}$: current per unit area (A/m²)

      $$dI = \rho \mathbf{v}\, da_\perp$$
      $$dI = \mathbf{J} \cdot d\boldsymbol{a} = \rho \mathbf{v} \cdot d\boldsymbol{a}$$

      ⟹ $\mathbf{J} \equiv \dfrac{dI}{da_\perp}$ ($a_\perp$: cross-sectional area segment perpendicular to $\mathbf{v}$)

    - Magnetic force

      $$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\rho\, d\tau = \int (\mathbf{J} \times \mathbf{B})\, d\tau$$

  - Ohm's law $\mathbf{J} = \sigma_c \mathbf{E}$ (σ_c: electrical conductivity)

Currents

- Surface and volume distributions of current

    o Continuity equation for volume current density

        ▪ Current crossing a surface $\mathcal{S}$: $I = \int_{\mathcal{S}} J \, da_{\perp} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$

        ▪ Current crossing boundary of a volume $\mathcal{V}$:

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{J}) \, d\tau$$

    which must equal to the <u>change of net charge</u> in the volume

$$\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{J}) \, d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho \, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} \right) d\tau$$
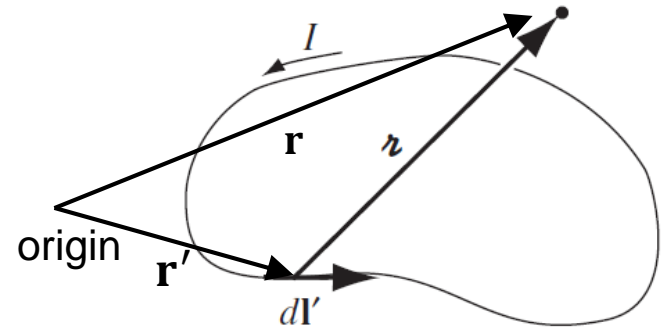
$\Longrightarrow$ $\boxed{\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$ for magnetostatics $\frac{\partial \rho}{\partial t} = 0$ $\frac{\partial \mathbf{J}}{\partial t} = \mathbf{0}$ $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$

The Biot-Savart law

- Magnetic field generated by a steady current

    o Line current $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{\mathbf{I} \times \hat{\boldsymbol{\imath}}}{\imath^2}\, dl' = \dfrac{\mu_0}{4\pi} I \displaystyle\int \dfrac{d\mathbf{l}' \times \hat{\boldsymbol{\imath}}}{\imath^2}$

    - Unit of $\mathbf{B}$: Tesla (T) = N/(A·m)

    - Permeability $\mu_0 = 4\pi \times 10^{-7}$ N/A^2

    - Separation vector $\boldsymbol{\imath} = \mathbf{r} - \mathbf{r}'$

    o Surface current $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, da'$

    o Volume current $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, d\tau'$

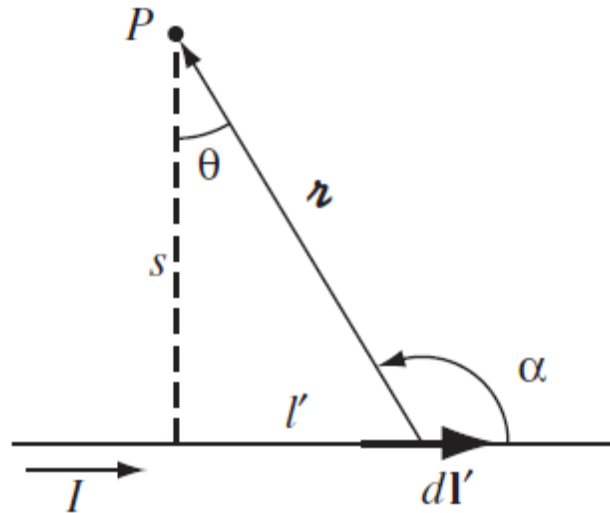   o Expressions for surface and volume currents not proved

    o The law cannot be used to calculate field generated by discrete moving charges

        * $\boldsymbol{\imath}$ in textbook is typed as $\boldsymbol{r}$

The Biot-Savart law

- Application of Biot-Savart law

Example 5.5. Find the magnetic field a distance s from a long straight wire carrying a steady current I (Fig. 5.18).

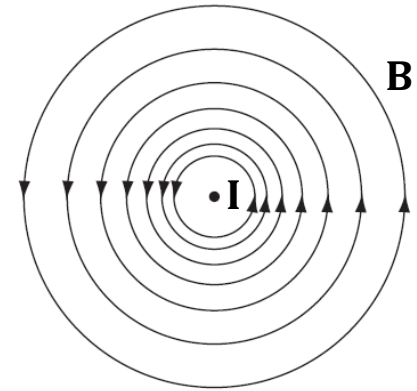$$\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}$$

C3.exmp5.5

Curl of magnetic field

- Derivation of curl from Stokes theorem

  - Loop integral of **B** around a straight-line current

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} \, dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I$$

B

•**I**

  - Result still holds if path is noncircular

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + \underline{s\,d\phi\,\hat{\boldsymbol{\phi}}} + dz\,\hat{\mathbf{z}} \qquad \mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$$

  - Loop integral of **B** around any current distribution

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad (\ I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a} \ \text{ total current enclosed by the path})$$

  - Apply Stokes theorem

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$
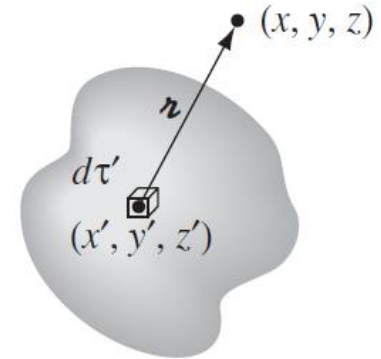
$$\Longrightarrow \boxed{\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}}$$

Curl of magnetic field

- Derivation of curl from Biot-Savart law

  o A much more formal derivation than previous slide

  o $\nabla \times \mathbf{B} = \dfrac{\mu_0}{4\pi} \displaystyle\int \nabla \times \left(\mathbf{J} \times \dfrac{\hat{\imath}}{\imath^2} \right) d\tau'$

$\mathbf{B(r)}$ $\nabla_{\mathbf{r}}$ $\mathbf{J(r')}$ $\imath = \mathbf{r} - \mathbf{r'}$ $dx' dy' dz'$

Product rule $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

$$\nabla \times \left(\mathbf{J} \times \dfrac{\hat{\imath}}{\imath^2} \right) = \mathbf{J} \left(\nabla \cdot \dfrac{\hat{\imath}}{\imath^2} \right) - (\mathbf{J} \cdot \nabla)\dfrac{\hat{\imath}}{\imath^2}$$

$\nabla \cdot \left(\dfrac{\hat{\imath}}{\imath^2} \right) = 4\pi \delta^3(\imath)$, and second term integrates to 0 (textbook p.232)

$$\nabla \times \mathbf{B} = \dfrac{\mu_0}{4\pi} \int \mathbf{J(r')} 4\pi \delta^3(\mathbf{r} - \mathbf{r'}) \, d\tau' = \mu_0 \mathbf{J(r)}$$

Divergence of magnetic field

- Derivation of divergence from Biot-Savart law

  ○ $\nabla \cdot \mathbf{B} = \dfrac{\mu_0}{4\pi} \displaystyle\int \nabla \cdot \left(\mathbf{J} \times \dfrac{\hat{\imath}}{\imath^2} \right) d\tau'$

  Product rule $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

  $\nabla \cdot \left(\mathbf{J} \times \dfrac{\hat{\imath}}{\imath^2} \right) = \dfrac{\hat{\imath}}{\imath^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \dfrac{\hat{\imath}}{\imath^2} \right)$

  $\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') = 0$, and $\nabla \times \dfrac{\hat{\imath}}{\imath^2} = 0$ (Remember $\nabla \times \mathbf{E} = \mathbf{0}$)

  $\boxed{\nabla \cdot \mathbf{B} = 0}$ Magnetic fields are divergence-free (no magnetic "free charge")

Magnetic monopoles? not reproduced

First Results from a Superconductive Detector for Moving Magnetic Monopoles

Blas Cabrera
Phys. Rev. Lett. **48**, 1378 – Published 17 May 1982

Ampère's law

Electrostatics	Magnetostatics
Coulomb's law	Biot-Savart law
Gauss's law $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \quad \oint \mathbf{E} \cdot d\mathbf{a} = Q_{\mathrm{enc}}/\varepsilon_0$	Ampère's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathrm{enc}}$
Gaussian surface	Amperian loop

- Use of Ampère's law to calculate magnetic field

  o Ampère's law in integral form

  o Symmetry arguments

    ▪ Translation symmetry

    ▪ Rotational symmetry

    ▪ Mirror symmetry

    ▪ Inversion symmetry

    Be careful to use when transforming **B**: need to flip sign

    ▪ **B** flips sign if **I** flips sign (time-reversal symmetry)

Symmetry of polar vectors and axial vectors

- Polar vectors and axial vectors

  - Polar vectors: $\mathbf{E}$, $\mathbf{P}$, $\mathbf{D}$, $\mathbf{J}$, $\mathbf{I}$

    - Usually vectors associated with electric fields

  - Axial vectors: $\mathbf{B}$, $\mathbf{M}$, $\mathbf{H}$

   - Usually vectors associated with magnetic fields

- Mirror operation

Mirror plane

E **E**

Mirror image

Mirror plane

B **I** **I** **B**

Mirror image
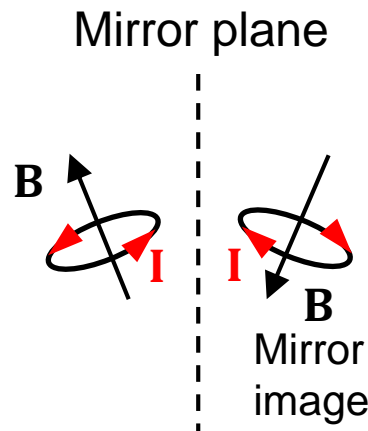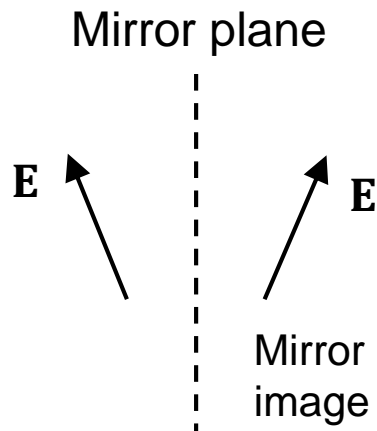
- Polar vectors: every operation as usual

- Axial vectors: Additional sign flip for mirror and inversion operations, other operations as usual
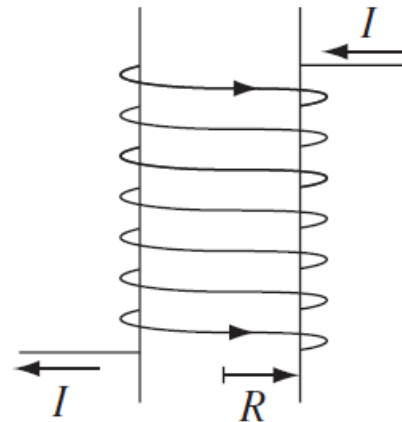
Ampère's law

Example 5.8. Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\,\hat{\mathbf{x}}$, flowing over the xy plane (Fig. 5.33).

C3.exmp5.8

Sheet of current

K

z

y

x

l

Amperian loop
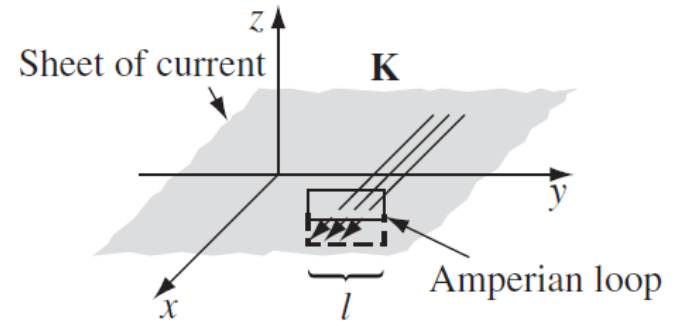
Example 5.9. Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R, each carrying a steady current I (Fig. 5.34).

C3.exmp5.9&5.12

I

I

R

Ampère's law

Example 5.10. A toroidal coil consists of a circular ring, or "donut," around which a long wire is wrapped (Fig. 5.38). The winding is uniform and tight enough so that each turn can be considered a plane closed loop.
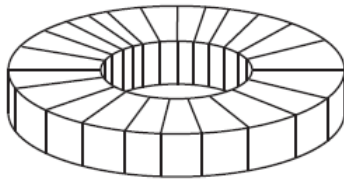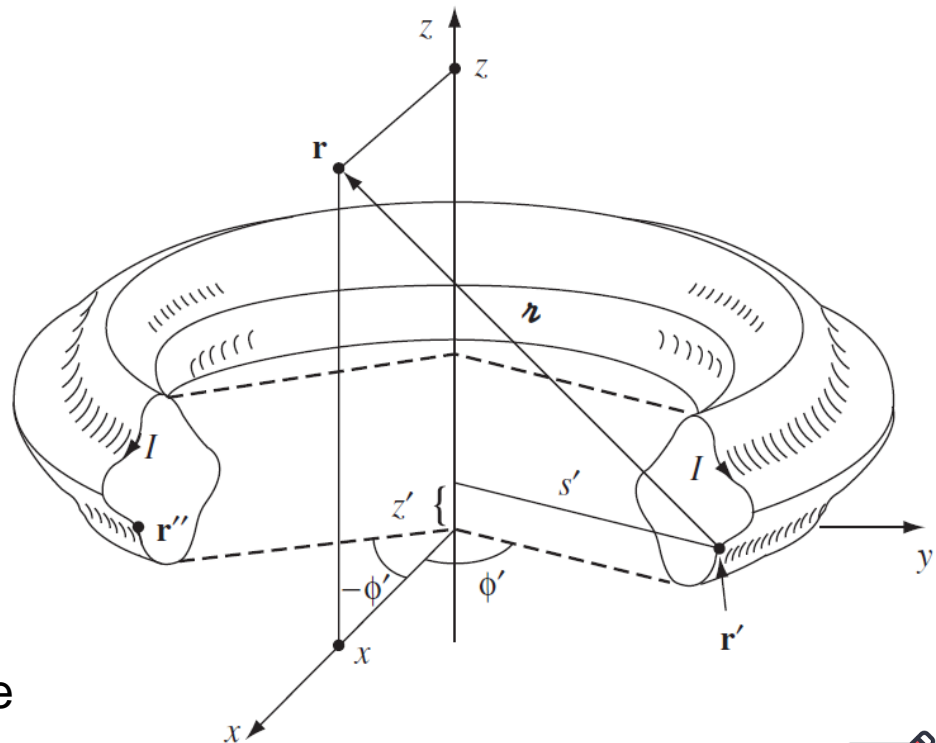
FIGURE 5.38

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\,\hat{\boldsymbol{\phi}}, & \text{inside} \\[2em] \mathbf{0}, & \text{outside} \end{cases}$$

Toroidal structure with any cross-sectional shape

C3.exmp5.10

Magnetic vector potential

- Vector potential $\mathbf{A}$

  - $\boxed{\mathbf{B} = \mathbf{\nabla} \times \mathbf{A}}$

    - Exploiting the property $\mathbf{\nabla} \cdot \mathbf{B} = 0$ (divergence of curl vanishes)

  - $\mathbf{\nabla} \cdot \mathbf{A} = 0$

    - Chosen to be so (like choosing reference point for electric potential)

      If original $\mathbf{A}_0$ is not, can define $\mathbf{A} = \mathbf{A}_0 + \mathbf{\nabla}\lambda$ without varying $\mathbf{B}$

      $\Rightarrow \quad \mathbf{\nabla} \cdot \mathbf{A} = \mathbf{\nabla} \cdot \mathbf{A}_0 + \nabla^2\lambda$

      Can choose $\nabla^2\lambda = -\mathbf{\nabla} \cdot \mathbf{A}_0$

  - $\boxed{\nabla^2\mathbf{A} = -\mu_0\mathbf{J}}$

    - Use the choice above, and apply Ampère's law

      $$\mathbf{\nabla} \times \mathbf{B} = \mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{A}) = \mathbf{\nabla}(\underset{0}{\cancel{\mathbf{\nabla} \cdot \mathbf{A}}}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

Magnetic vector potential

- Calculating vector potential from current

  - $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ Poisson's equation, if $\mathbf{J} \to 0$ at infinity,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\imath} d\tau'$$ (**A** is a polar vector)

  - Analogous to solution $V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0} \int \dfrac{\rho(\mathbf{r}')}{\imath} d\tau'$ to $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$

  - Line current $\mathbf{A} = \dfrac{\mu_0}{4\pi} \int \dfrac{\mathbf{I}}{\imath} dl' = \dfrac{\mu_0 I}{4\pi} \int \dfrac{1}{\imath} d\mathbf{l}'$

  - Surface current $\mathbf{A} = \dfrac{\mu_0}{4\pi} \int \dfrac{\mathbf{K}}{\imath} da'$

  - Not so much simplification (vector **A** representing vector **B**) but equation is easier to use than the Biot-Savart law

    - No need to integrate unit vector $\hat{\boldsymbol{\imath}}$

      * $\nabla^2 \mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$

Magnetic vector potential

- Calculating vector potential from magnetic field

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$$
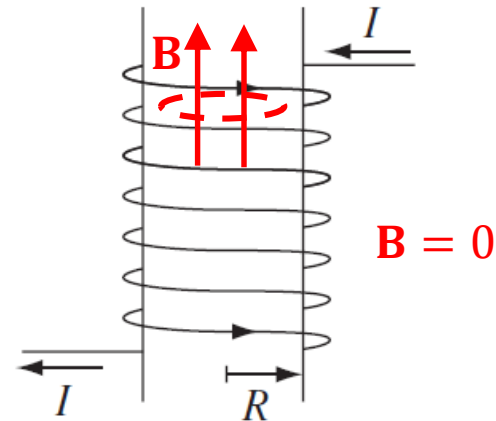
where Φ is the magnetic flux

- Can calculate **A** using equation above plus symmetry argument

- Generally **A** mimics the direction of current

Example 5.12. Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.
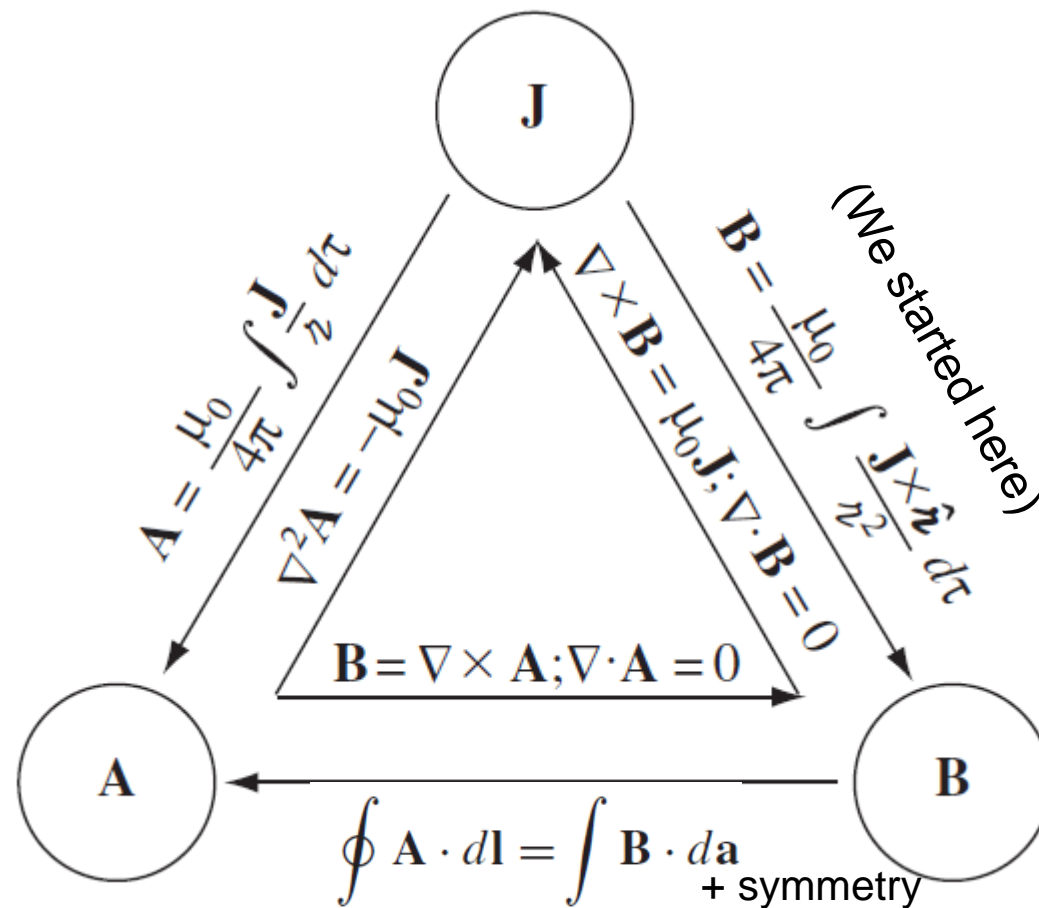
- Check $\nabla \times \mathbf{A} = \mathbf{B}$

- Check $\nabla \cdot \mathbf{A} = 0$

C3.exmp5.9&5.12

Current, magnetic field, and vector potential

Differential equations need boundary conditions to solve
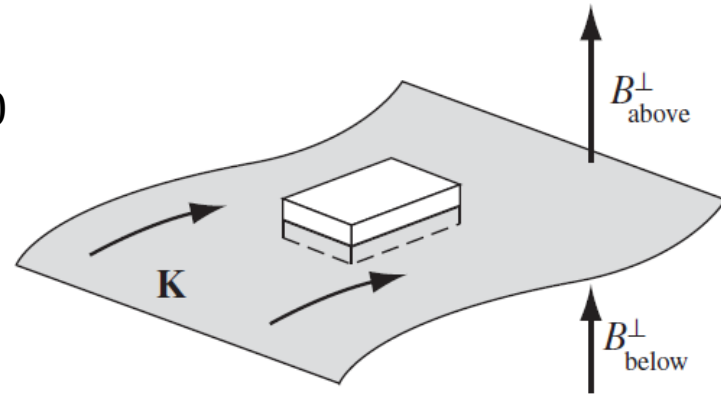
Boundary conditions

- Boundary conditions of **B** across a 2D current surface

  o Normal component of **B**

  Thin pillbox with thickness $\varepsilon \to 0$

  $$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$

  $\Rightarrow$ $\boxed{B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}}$

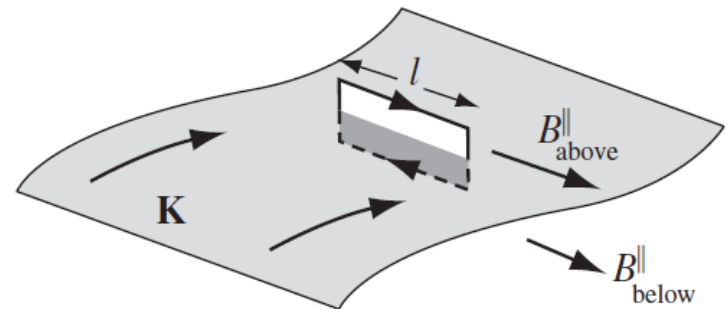 o Tangential component of **B** <u>that is perpendicular to current</u>

  Thin Amperian loop

  $$\oint \mathbf{B} \cdot d\mathbf{l} = \left(B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} \right) l$$

  $$= \mu_0 I_{\text{enc}} = \mu_0 K l$$

  $\Rightarrow$ $\boxed{B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K}$

  o Summarizing above $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$

Boundary conditions

- Boundary conditions of **A** across a 2D current surface

  - Vector potential **A** is always continuous $\boxed{\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}}$

    - Normal component of **A**

      $\nabla \cdot \mathbf{A} = 0$ and select a thin pillbox

      $\Longrightarrow \quad A^{\perp}_{\text{above}} = A^{\perp}_{\text{below}}$

    - Tangential component of **A**

      $\nabla \times \mathbf{A} = \mathbf{B}$ and select a thin loop to integrate

      $\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$ tends to zero when loop is thin

      $\Longrightarrow \quad A^{\parallel}_{\text{above}} = A^{\parallel}_{\text{below}}$

  - Derivative of **A** is discontinuous $\boxed{\dfrac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \dfrac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}}$

C3.Abound
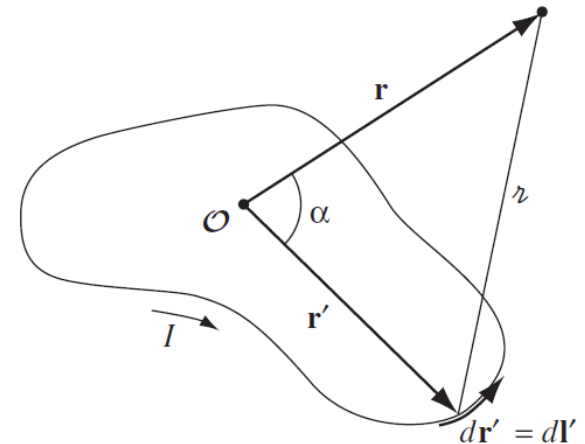(optional)

Multipole expansion of vector potential

- Goal: to expand $\mathbf{A}$ in power series of $1/r$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\imath} \, d\mathbf{l}'$$

$$\frac{1}{\imath} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

(From electrostatic multipole expansion)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\, d\mathbf{l}'$$

Monopole　　　Dipole

$$= \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}'\right.$$

$$\left. + \frac{1}{r^3}\oint (r')^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\mathbf{l}' + \cdots\right]$$
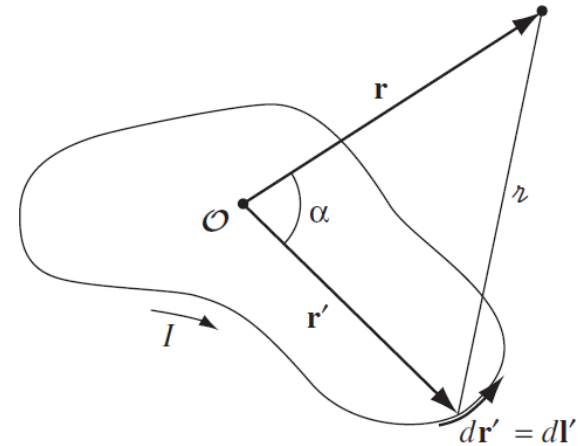
Quadrupole

Multipole expansion of vector potential

- Magnetic monopole and magnetic dipole

  - Magnetic monopole

    - Always zero, because $\oint d\mathbf{l}' = \mathbf{0}$

    - Also because $\nabla \cdot \mathbf{B} = 0$

  - Magnetic dipole

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'$$

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}' \quad \text{(proved by a textbook problem)}$$

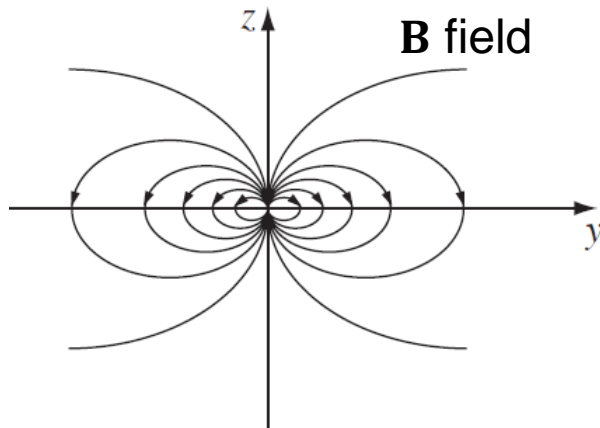$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

Magnetic dipole moment
$$\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}$$

Multipole expansion of vector potential

- Pure dipole vs physical dipole

<table>
<tr><td>Pure dipole</td><td>Physical dipole</td></tr>
</table>

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

Deviations appear when closing up onto the dipole

Assumes $r \gg$ loop radius

$\mathbf{m} = I\mathbf{a}$ but take $I \to \infty, a \to 0$
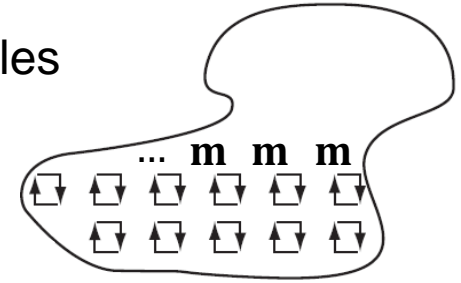
Similar to pure electric dipole

Magnetic fields in matter

Magnetic materials

- Practically, all materials have magnetic response

  - Orbiting electrons around nuclei as magnetic dipoles

  - Electron spins as magnetic dipoles

    - Quantum object without a classical analog

- Magnetization: $\mathbf{M} = \dfrac{1}{V} \displaystyle\sum_i \mathbf{m}_i$

  - Magnetic dipole moment per unit volume

- Three types of response of matter to magnetic field

  - Paramagnets: magnetization $\mathbf{M}$ parallel to applied $\mathbf{B}$

  - Diamagnets: magnetization $\mathbf{M}$ opposite to applied $\mathbf{B}$

  - Ferromagnets: finite magnetization $\mathbf{M}$ even without $\mathbf{B}$

Magnetic dipoles responding to field

- Paramagnetic response

  o Torque in a uniform field
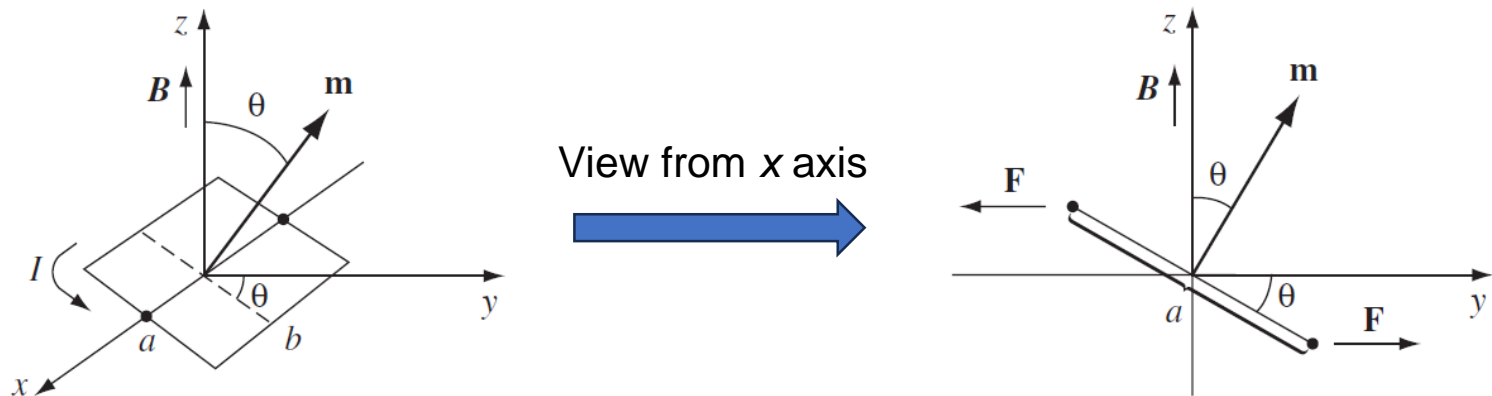
    ▪ Suppose a rectangle current loop with side lengths a and b

$$\mathbf{N} = aF \sin\theta \,\hat{\mathbf{x}} \xrightarrow{\text{force } F = IbB} \mathbf{N} = IabB \sin\theta \,\hat{\mathbf{x}} = mB \sin\theta \,\hat{\mathbf{x}}$$

➡️ $$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B}}$$

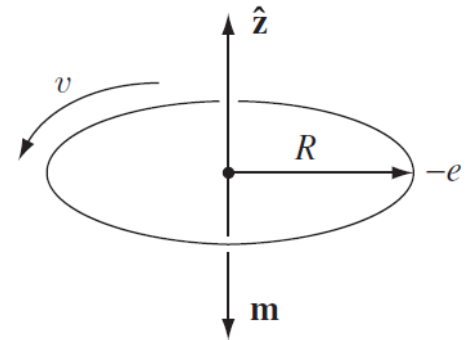 o Torque tends to align $\mathbf{m}$ parallel to $\mathbf{B}$

View from x axis

Magnetic dipoles responding to field

- Diamagnetic response

  - ○ Modification to an atomic orbit

$$I = \frac{-e}{T} = -\frac{ev}{2\pi R} \quad \xrightarrow[\text{Orbit period}]{T = 2\pi R/v} \quad \mathbf{m} = -\frac{1}{2}evR\,\hat{\mathbf{z}}$$

  - ▪ Velocity without and with **B**, balancing Coulomb and centripetal forces

    without **B**: $\dfrac{1}{4\pi\epsilon_0}\dfrac{e^2}{R^2} = m_e \dfrac{v^2}{R}$

    with **B**: $\dfrac{1}{4\pi\epsilon_0}\dfrac{e^2}{R^2} + e\bar{v}B = m_e \dfrac{\bar{v}^2}{R}$

    > v: electron velocity
    > $\bar{v}$: electron velocity with **B**
    > m_e: electron mass

➡️ $e\bar{v}B = \dfrac{m_e}{R}(\bar{v}^2 - v^2) = \dfrac{m_e}{R}(\bar{v} + v)(\bar{v} - v)$

➡️ $\Delta v = \bar{v} - v = \dfrac{eRB}{2m_e}$ for small Δv

➡️ $\Delta\mathbf{m} = -\dfrac{1}{2}e(\Delta v)R\,\hat{\mathbf{z}} = -\dfrac{e^2R^2}{4m_e}\mathbf{B}$ Change in dipole moment works against **B**

Forces on paramagnets and diamagnets

- General formula for small dipole in nonuniform field

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

  o For paramagnets $m \parallel B$

  - Force directs toward more intense field regions

  - Paramagnets are attracted to magnets

  o For diamagnets $m \parallel -B$

Levitating frog

  - Force directs toward less intense field regions

  - Diamagnets are repelled by magnets
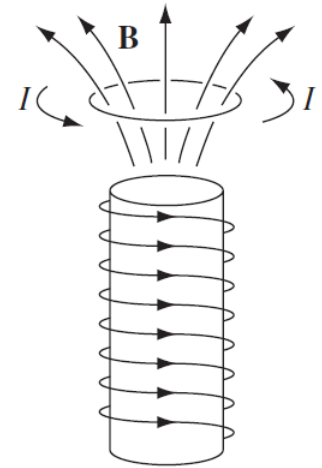
(Ig Nobel award 2000)

Field of magnetized objects

- Bound currents

  ○ Vector potential of a magnetized object (neglecting the cause)
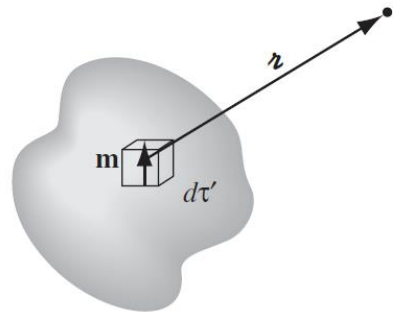
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} \, d\tau'$$

$$= \frac{\mu_0}{4\pi} \int \left[\mathbf{M}(\mathbf{r}') \times \left(\boldsymbol{\nabla}' \frac{1}{\imath} \right) \right] d\tau'$$

Integrate by parts

$$= \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{\imath} [\boldsymbol{\nabla}' \times \mathbf{M}(\mathbf{r}')] \, d\tau' - \int \boldsymbol{\nabla}' \times \left[\frac{\mathbf{M}(\mathbf{r}')}{\imath} \right] d\tau' \right\}$$

$$= \frac{\mu_0}{4\pi} \int \frac{1}{\imath} [\boldsymbol{\nabla}' \times \mathbf{M}(\mathbf{r}')] \, d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\imath} [\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']$$

C3.Curldtau

$$= \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\imath} \, d\tau' + \frac{\mu_0}{4\pi} \oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\imath} \, da'$$

> * with $\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M}$
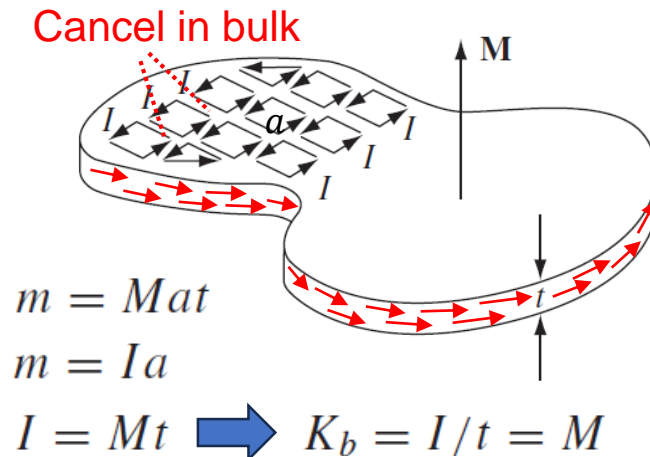> $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$

Field of magnetized objects

- Bound currents

$$\mathbf{A(r)} = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r'})}{\imath} \, d\tau' + \frac{\mu_0}{4\pi} \oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r'})}{\imath} \, da'$$

- ○ Density of bound currents $\left\{\begin{array}{l} \text{surface:} \quad \boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}} \\[2mm] \text{volume:} \quad \boxed{\mathbf{J}_b = \nabla \times \mathbf{M}} \end{array}\right.$ *check $\nabla \cdot \mathbf{J}_b = 0$

- ○ Physical picture of bound currents

**Surface bound current
(suppose uniform M)**

Cancel in bulk

$$m = Mat$$
$$m = Ia$$
$$I = Mt \implies K_b = I/t = M$$

Volume bound current

$$I_x = [M_z(y+dy) - M_z(y)]\,dz = \frac{\partial M_z}{\partial y}\,dy\,dz$$
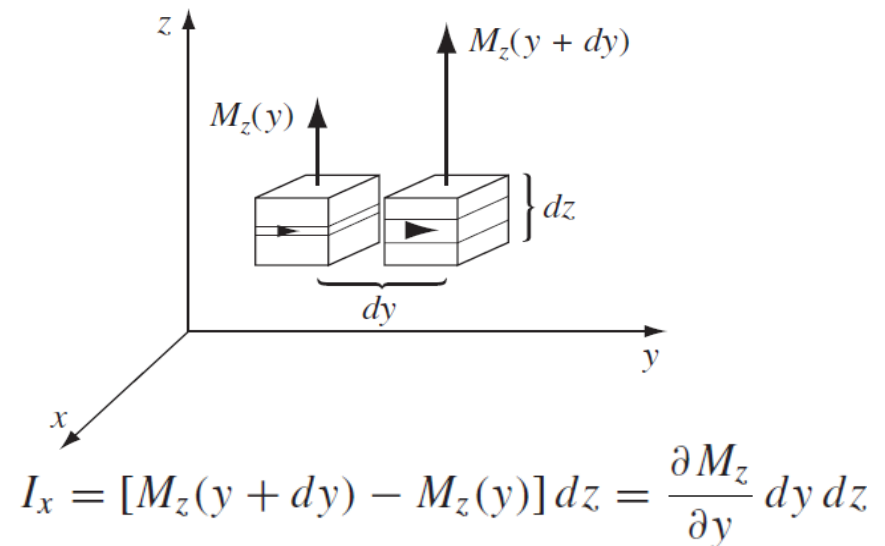
Field of magnetized objects

- Bound currents

Example 6.1. Find the magnetic field of a uniformly magnetized sphere.

- ○ Field inside the sphere $\mathbf{B} = \dfrac{2}{3}\mu_0\mathbf{M}$

  - ▪ Uniform field

  - ▪ **M** induces **B** that is parallel to it, while **P** induces **E** that is antiparallel

- ○ Field outside the sphere same as what would be for a dipole

$$\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}$$
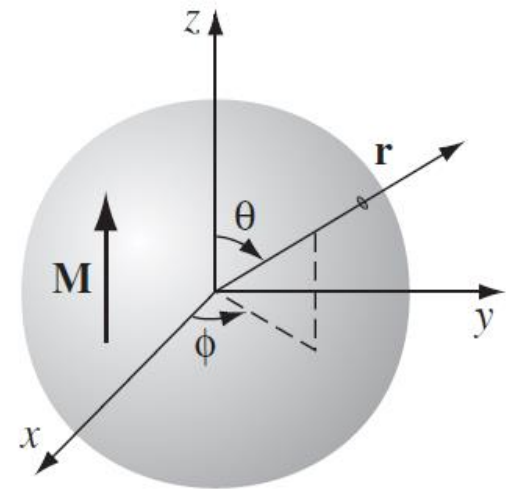
C3.exmp6.1

Auxiliary field

- Add the cause and the effect of magnetization

  - Total magnetic field

    $$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

    - $\mathbf{B}$: total magnetic field

    - $\mathbf{J}$: total current density

    - $\mathbf{J}_b$: bound current density, due to magnetization

    - $\mathbf{J}_f$: free current density that we control, not a result of magnetization

    $$\Longrightarrow \quad \nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f$$

  - The auxiliary field $\quad \boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}$

    $$\Longrightarrow \quad \boxed{\nabla \times \mathbf{H} = \mathbf{J}_f}$$

    Enclosed total free current

    - Ampère's law for auxiliary field $\quad \boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}}$

Auxiliary field

- Application of auxiliary field

Example 6.2. A long copper rod of radius R carries a uniformly distributed (free) current I (Fig. 6.19). Find **H** inside and outside the rod.
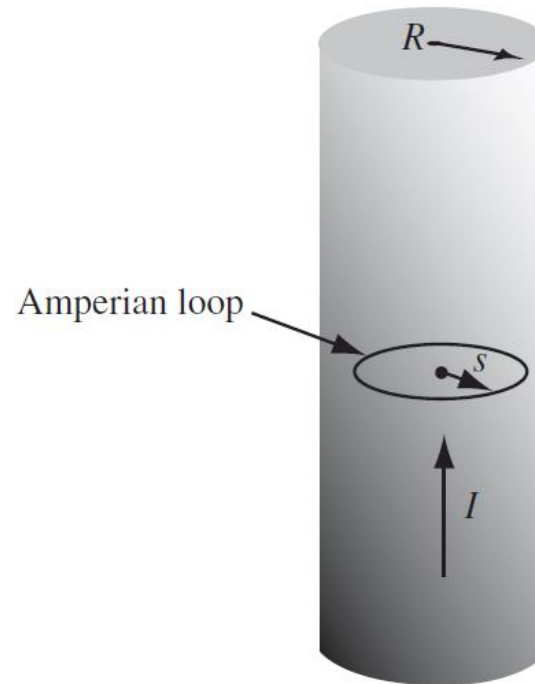
  ○ Inside

$$H(2\pi s) = I_{f_{\text{enc}}} = I\frac{\pi s^2}{\pi R^2}$$

$$\Longrightarrow \mathbf{H} = \frac{I}{2\pi R^2}s\,\hat{\boldsymbol{\phi}}$$

  ○ Outside

$$\Longrightarrow \mathbf{H} = \frac{I}{2\pi s}\,\hat{\boldsymbol{\phi}}$$

$$\Longrightarrow \mathbf{B} = \mu_0\mathbf{H} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$$
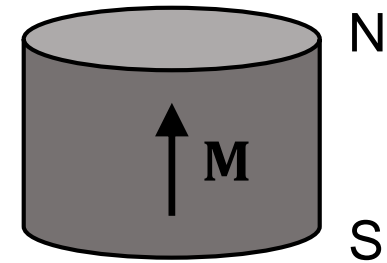

Amperian loop

Auxiliary field

- $\mathbf{H} = \mathbf{0}$ without applying any free current?

  - Looks like so because $\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$

    - Consider a short cylindrical magnet

    - $\mathbf{H}$ would be zero everywhere

    - Then $\mathbf{B} \neq \mathbf{0}$ inside magnet, $\mathbf{B} = \mathbf{0}$ outside magnet

    - Obviously wrong

  - $\mathbf{H}$ can be finite without any $\mathbf{J}_F$

    - Because $\boldsymbol{\nabla} \cdot \mathbf{H} = \boldsymbol{\nabla} \cdot \left(\dfrac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = -\boldsymbol{\nabla} \cdot \mathbf{M} \neq \mathbf{0}$

    - $\boldsymbol{\nabla} \cdot \mathbf{M} \neq \mathbf{0}$ at the top and bottom surfaces of the magnet

N

M

S

H versus B, D versus E

- A long history of confusion

  - Whether we shall call $\mathbf{H}$ or $\mathbf{B}$ as the magnetic field

    - Some convention calls $\mathbf{H}$ the magnetic field, and $\mathbf{B}$ as the magnetic flux density

    - Our textbook takes the stance that $\mathbf{H}$ should be "auxiliary"

- Reason for such confusion

  - $\mathbf{H}$ is a lot more frequently used than $\mathbf{B}$ as $\mathbf{H}$ is given by free current, something we can control, while $\mathbf{B}$ is material dependent

    - Helmholtz coil magnetizing a specimen

  - $\mathbf{E}$ is a lot more frequently used than $\mathbf{D}$ as $\mathbf{D}$ is given by free charge, which we rarely control, while $\mathbf{E}$ is determined by voltage difference (over distance), which is what we control

    - Charging up of a parallel plate capacitor

Linear magnetic media

- Linear magnetic media

  - $\boxed{\mathbf{M} = \chi_m \mathbf{H}}$ (χ_m: magnetic susceptibility)

    $\Longrightarrow$ $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}$

> **?** Why this linear relation defines M-H but not M-B?

  - Diamagnetic susceptibility on the order of 10^{-6}

  - $\boxed{\mathbf{B} = \mu\mathbf{H} \quad (\mu = \mu_0(1 + \chi_m)\text{: permeability})}$
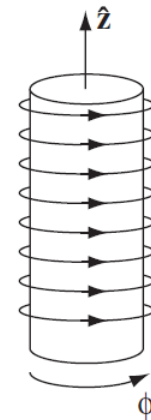
Example 6.3. An infinite solenoid (n turns per unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} \Longrightarrow \mathbf{H} = nI\,\hat{\mathbf{z}}$$

$$\Longrightarrow \mathbf{B} = \mu_0(1 + \chi_m)nI\,\hat{\mathbf{z}}$$

Enhancement of field if paramagnetic!

Boundary conditions

- Boundary conditions reexamined

  - <u>Earlier findings still hold</u>, but **J** needs to include free and bound currents

$$\begin{cases} B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \\[1em] \mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \end{cases}$$

  - Easier to use the boundary conditions of **H**

Surface free charge density

$$\begin{cases} \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \\[1em] \nabla \times \mathbf{H} = \mathbf{J}_f \end{cases} \Rightarrow \begin{cases} \mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}} \\[1em] H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}) \end{cases}$$

Nonlinear magnetic media

- Ferromagnets: $\mathbf{M} \neq 0$ without applying any field

    o Obvious violation of the linear relation $\mathbf{M} = \chi_m \mathbf{H}$

    o Represents a quantum phenomena

        ▪ Exchange interaction: $U = -2J \sum_{p=1}^{N} \boldsymbol{S}_p \cdot \boldsymbol{S}_{p+1}$ $(J > 0)$

        ▪ Prefers spontaneous parallel alignment of spins

    o Domain formation and hysteresis loop

Ferromagnetic Ni

M

+ domain saturated

B_r

$-H_c$

H

- domain saturated