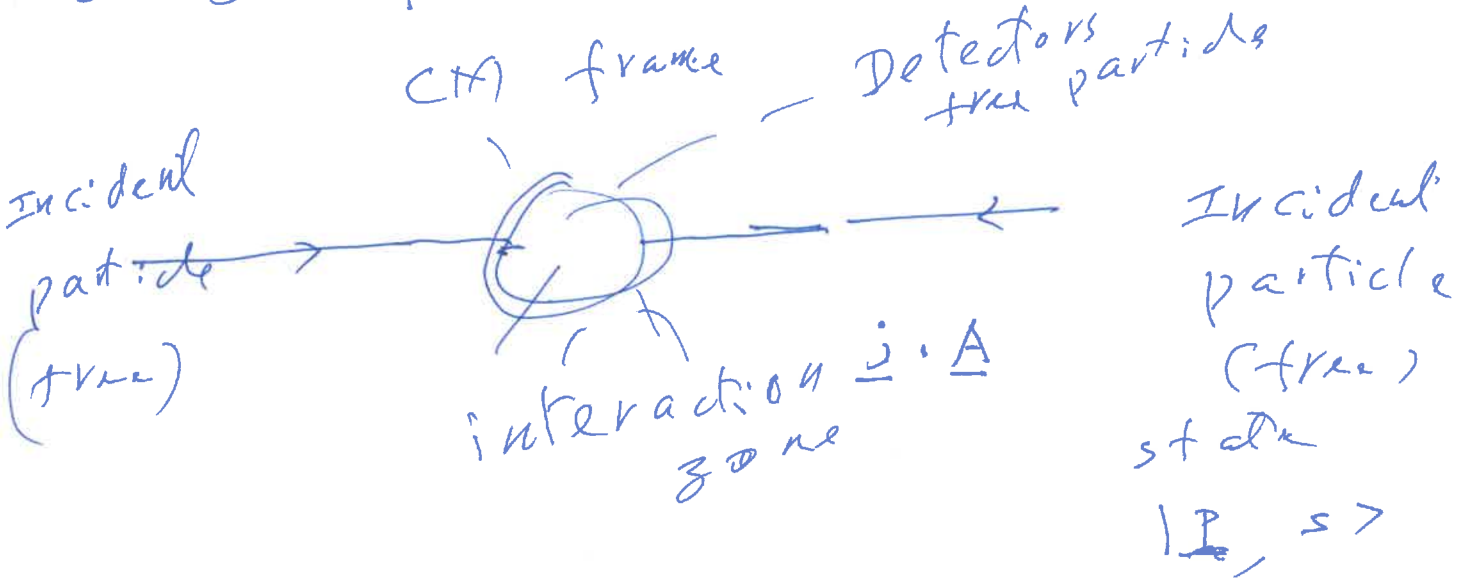
2024. 4. 9 ①

Chapter 7 QED

interaction of a photon with a charged particle (electron)



1st : Describe free particles.

Next : Discuss interactions ← Topic today

Re cap: free particles

photon $\underline{A}$, $\quad \Box^2 \underline{A} = 0$

solution $\quad \underline{A} = \text{const}\ e^{-i p\cdot x/\hbar}\ \underline{\varepsilon}(p)$

$$p^2 = 0, \quad p \cdot \underline{\varepsilon} = 0$$

Coulomb gauge $\quad \varepsilon^0(p) = 0, \quad \underset{\sim}{p} \cdot \underset{\sim}{\varepsilon} = 0$

$\underline{\varepsilon}(p)$ = polarization vector

How to find $\underset{\sim}{\varepsilon}(\underset{\sim}{p})$? Let $\underset{\sim}{p} = (0, 0, p) \Rightarrow$

$$\underset{\sim}{\varepsilon}^{(1)} = (1, 0, 0), \quad \underset{\sim}{\varepsilon}^{(2)} = (0, 1, 0)$$

Charged particle : Relativistic quantum equation

Klein-Gordon equ

$$\underline{p}^2 \phi(\underline{x}) = m^2 c^2 \phi(\underline{x})$$

soln $\quad \phi(\underline{x}) = \text{const} \; e^{-i\underline{p}\cdot\underline{x}/\hbar}$

$$\underline{p}^2 = m^2 c^2$$

K. G. describes scalar (pseudo scalar) particle. No spin.

For particle with spin, use Dirac equ

$\not{p} = p_\mu \gamma^\mu$

$$\not{p}\, \psi(\underline{x}) = mc\, \psi(\underline{x})$$

$$\psi(\underline{x}) = \text{Const} \; e^{-i\underline{p}\cdot\underline{x}/\hbar} \; u(\underline{p})$$

$$u(\underline{p}) = \begin{pmatrix} u_1(\underline{p}) \\ u_2(\underline{p}) \\ u_3(\underline{p}) \\ u_4(\underline{p}) \end{pmatrix}$$

Need to find $u(\underline{p})$

$$p^0 = \pm \sqrt{\underline{p}^2 + m^2c^2}$$

For $p^0 = \sqrt{\underline{p}^2 + m^2c^2}$

$$u_+^{(s)}(\underline{p}) = \sqrt{p^0 + mc} \begin{pmatrix} w^{(s)}(\underline{p}) \\ \frac{\underline{\sigma}\cdot\underline{p}}{p^0+mc} w^{(s)}(\underline{p}) \end{pmatrix}$$

$s = 1, 2$

$$p^0 = -\sqrt{\underline{p}^2 + m^2c^2}$$

$$u_-^{(s)}(\underline{p}) = \sqrt{mc - p^0} \begin{pmatrix} \frac{\underline{\sigma}\cdot\underline{p}}{mc - p^0} w^{(s)} \\ w^{(s)} \end{pmatrix}$$

$\rightarrow$ $u_-^{(s)}(-\underline{p})$ (positive energy soln)

Ansatz

$$\psi(\underline{x}) = \text{const}\; e^{i\underline{p}\cdot\underline{x}/\hbar}\, V(\underline{p})$$

$$p^0 = \sqrt{\underline{p}^2 + m^2c^2}, \quad V(\underline{p}) = \sqrt{p^0 + mc} \begin{pmatrix} \frac{\underline{\sigma}\cdot\underline{p}}{p^0+mc} w^{(s)} \\ w^{(s)} \end{pmatrix}$$

$\longrightarrow$ antiparticle.

As we have already learned how to describe a free photon $A_\mu(x)$ and a free electron $\psi(x)$ as plane wave solutions of the Maxwell equations and the Dirac equation respectively, we can proceed to study their interaction.

The interaction is dictated by the gauge symmetry or the principle of gauge invariance.

Instead of using Hamiltonian

$$H = H_{photon} + H_{electron} + H_I$$

Lagrangian density is used

$$\mathcal{L} = \mathcal{L}_{photon} + \mathcal{L}_{electron} + \mathcal{L}_I$$

$$\mathcal{L}_I = q\, j_\mu A^\mu = q\, j^\mu A_\mu$$

$$j^\mu = c\, \bar{\psi}(x) \cdot \gamma^\mu \psi(x)$$

We shall proceed the study, using Feynman rules and Feynman diagrams.

Instead of using quantum field theoretic method to derive the transition amplitude (scattering amplitude) and hence the differential cross section, we use a diagramatic method, the Feynman diagram

For any physical process, we first sketch the Feynman diagram for the process (we learned in chapter 2). Then using a dictionary (Feynman rules), each piece of the diagram can be translated to mathematical expression (symbol)

These mathematical expressions are joined up together to give the scattering amplitude.

We now list out the Feynman rules

Using examples, we illustrate how scattering amplitudes can be derived from a Feynman diagram using Feynman rules

Summary

e^- $\qquad\qquad$ e^+

Wave functions

$$\psi(\underset{\sim}{x}) = e^{-i\underset{\sim}{p}\cdot\frac{\underset{\sim}{x}}{\hbar}} u^{(s)}(\underset{\sim}{p}) \qquad \psi(\underset{\sim}{x}) = e^{i\underset{\sim}{p}\cdot\frac{\underset{\sim}{x}}{\hbar}} v^{(s)}(\underset{\sim}{p})$$

s=1 spin up
s= 2 spin down

s=1 spin down
s=2 spin up

and

$$(\not{p} - mc)u = 0 \qquad (\not{p} + mc)v = 0$$

$$\bar{u}(\not{p} - mc) = 0 \qquad \bar{v}(\not{p} + mc) = 0$$

Orthonormality

$$\bar{u}^{(s_1)}\, u^{(s_2)} = 2mc\, \delta_{s_1 s_2} \qquad \bar{v}^{(s_1)}\, v^{(s_2)} = -2mc\, \delta_{s_1 s_2}$$

$s_1, s_2 = 1, 2$

Completeness

$$\sum_{s=1}^{2} u^{(s)}\, \bar{u}^{(s)} = (\not{p} + mc) \qquad \sum_{s=1}^{2} v^{(s)}\, \bar{v}^{(s)} = (\not{p} - mc)$$

Photon

Plane Wave

$$A^\mu(\underset{\sim}{x}) = e^{-i\underset{\sim}{p}\cdot\frac{\underset{\sim}{x}}{\hbar}}\ \varepsilon^\mu_{(s)}$$

s=1, 2 for the two polarization states

Polarization vector $\varepsilon^\mu_{(s)}$ statistics $\quad p_\mu\, \varepsilon^\mu_{(s)} = 0$.

Orthonormality

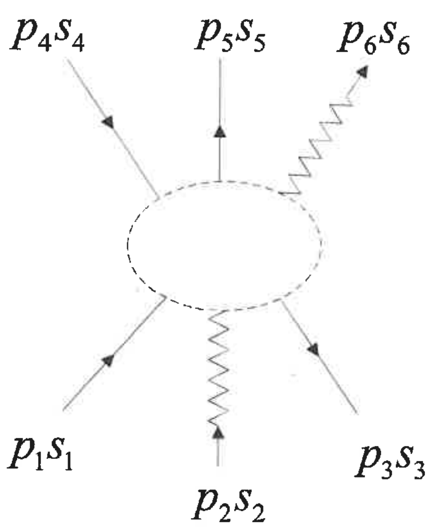
$$\varepsilon^{\mu *}_{(s_1)} \;\; \varepsilon_{\mu(s_2)} = \delta_{s_1 s_2}$$

Coulomb gauge $\quad \varepsilon^{\circ} = 0, \quad \underset{\sim}{\varepsilon} \cdot \underset{\sim}{p} = 0$

Completeness

$$\sum_{s=1}^{2} (\varepsilon_{(s)})_i \, (\varepsilon^*_{(s)})_j = \delta_{ij} - \hat{p}_i \hat{p}_j \qquad\qquad \hat{p}_i = p_i / |\underset{\sim}{p}|$$

Feynman rules QED



Notations

Label external lines by momentum p_i and spin s_i,

Label internal lines by momenta q_i

Arrows on external fermion lines indicate

e^- (forward in time) $\qquad\qquad$ e^+ (backward in time)

Arrows on internal fermion lines are assigned so that direction of the flow of 4-momenta through the diagram is kept.

Arrows on external photon lines point forward; for internal photon lines, the choice is arbitrary.

(i) External lines

e^-	incoming		$: u$
	outgoing		$: \bar{u}$
e^+	incoming		$: \bar{v}$
	outgoing		$: v$
γ	incoming		$: \varepsilon^{\mu}$
	outgoing		$: \varepsilon^{\mu^*}$

(ii)Vertex

Each vertex contributes a factor $ig\gamma^{\mu}$

g = dimensionless coupling constant = $\sqrt{4\pi\alpha}$

(iii) Propagators (internal lines)

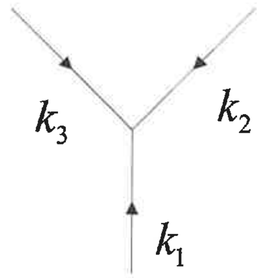
HW ?

$$e^- \; or \; e^+ : \frac{i}{\not q - mc} = \frac{i(\not q + mc)}{q^2 - m^2c^2}$$

$\not q = q_\mu \gamma^\mu$

$$\gamma: \quad \frac{-ig_{\mu\nu}}{q^2}$$

(iv) Conservation of 4 - momentum P_μ:



For each vertex, write $(2\pi)^4\delta^{(4)}(\underline{k}_1 + \underline{k}_2 + \underline{k}_3)$

(v) Integrate over internal momenta

$$\int \frac{d^4q}{(2\pi)^4}$$

(vi) Cancel the overall delta function

$$(2\pi)^4\delta^{(4)}(\underline{p}_1 + \underline{p}_2 \cdots \underline{p}_n)$$

what remains is the $-i\mathcal{M}$, $\mathcal{M}$= scattering amplitude

(vii) Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) $e^{-'}s(ore^{+'}s)$

or of an incoming e^- with an outgoing e^+ (or vice versa)

(viii) Charge is conserved at each vertex.

Lepton number etc must also be conserved.

(ix) For a closed fermion loop, include a factor -1 and take the trace.

(viii) Charge is conserved at each vertex.
Lepton number etc must also be conserved.

(ix) For a closed fermion loop, include a factor -1 and take the trace.

Examples

(i) $e^- \mu^- \to e^- \mu^-$ electron muon scattering

(ii) $e^- e^- \to e^- e^-$ Møller scattering

(iii) $e^- e^+ \to e^- e^+$ Bhabha scattering

(iv) $e^- \gamma \to e^- \gamma$ Compton scattering

(v) $e^- e^+ \to \gamma \gamma$ annihilation

$\underline{u}$ (for e^-), $\underline{v}$ (for e^+), $\underline{\varepsilon}$ (for γ)

Calculating scattering amplitudes at the tree level for the following processes:

(1) $e^- \mu^- \to e^- \mu^-$

(2) $e^- e^- \to e^- e^-$ Møller scatt

(3) $e^+ e^- \to e^+ e^-$ Bhabha scatt.

(4) $e^- \gamma \to e^- \gamma$ Compton scatt.

(5) $e^+ e^- \to \gamma\gamma$ photo production

The scattering amplitude is a function of the initial and final states, meaning that

$$\mathcal{M} = \mathcal{M}(p_1, s_1, p_2 s_2 ; p_3, s_3, p_4, s_4)$$

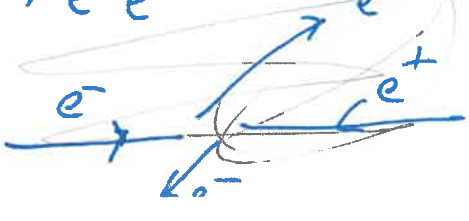
differential cross section $\frac{d\sigma}{d\Omega}$ = kinematic part $\cdot$ $|\mathcal{M}|^2$

dynamic part

Casimir trick to sum $|\mathcal{M}|^2$

Proceed to compute $\mathcal{M}$

(3) $e^+ e^- \to e^+ e^-$



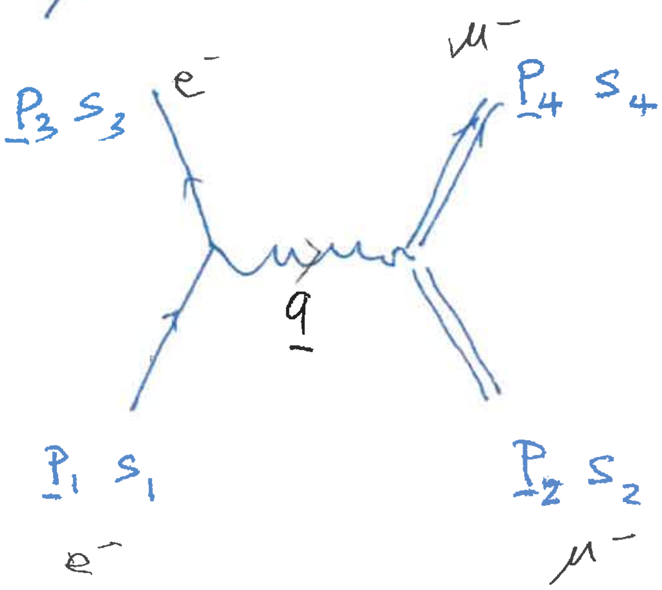
We show, by using examples, how the scattering amplitude $\mathcal{M}$ can be computed from the Feynman diagram using the Feynman rules.

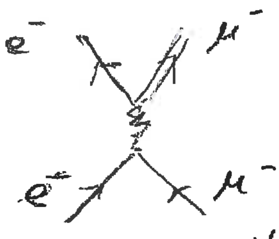
To begin with, we consider a simple process electron - muon scattering

$$e^- \mu^- \rightarrow e^- \mu^-$$

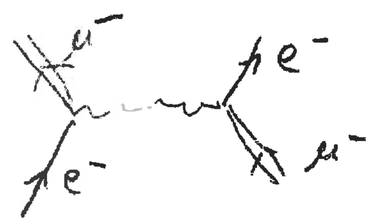
Interaction governed by the vertex

At the tree-level, only one possibility of joining up the lines with the only allowed vertex





not allowed
(charge not conserved)



not allowed ?
(lepton number conservation)

Use Feynman's rules to translate the diagram into mathmatical expression.

Read the diagram from left to Right and below to top. Write the expressions from Right to left.

Thus

start here

$$(2\pi)^4 \delta^{(4)}(\underline{p}_1 - \underline{p}_3 - \underline{q}) \cdot \frac{-i g_{\mu\nu}}{q^2} \bar{u}(\underline{p}_3, s_3) \cdot i g \gamma^{\mu} \cdot u(\underline{p}_1, s_1)$$

$$\int \frac{d^4 q}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(\underline{q} + \underline{p}_2 - \underline{p}_4)\, \bar{u}(\underline{p}_4, s_4)\, i g \gamma^{\nu} u(\underline{p}_2, s_2)$$

$$= i g^2 (2\pi)^4 \int d^4 q\, \underbrace{\bar{u}(\underline{p}_4, s_4) \gamma^{\nu} u(\underline{p}_2, s_2)}_{\text{complex number}} \cdot \delta^{(4)}(\underline{p}_1 - \underline{p}_3 - \underline{q}) \cdot$$

$$\cdot \frac{g_{\mu\nu}}{q^2} \cdot \underline{\bar{u}(\underline{p}_3, s_3) \gamma^{\mu} u(\underline{p}_1, s_1)}\, \delta^{(4)}(\underline{q} + \underline{p}_2 - \underline{p}_4)$$

$$= i g^2 (2\pi)^4\, \bar{u}(\underline{p}_4, s_4) \gamma^{\nu} u(\underline{p}_2, s_2) \frac{g_{\mu\nu}}{(\underline{p}_1 - \underline{p}_3)^2} \cdot$$

$$\bar{u}(\underline{p}_3, s_3) \gamma^{\mu} u(\underline{p}_1, s_1) \cdot \delta^{(4)}(\underline{p}_1 - \underline{p}_3 + \underline{p}_2 - \underline{p}_4)$$

δ

Throw away the overall delta function for 4-momentum conservation, we get

$$-i\mathcal{M} = ig^2\ \bar{u}(\underline{P}_4\ s_4)\ \gamma^\nu\ u(\underline{P}_2\ s_2)\ \frac{g_{\mu\nu}}{(\underline{P}_1 - \underline{P}_3)^2}\cdot$$
$$\bar{u}(\underline{P}_3,\ s_3)\ \gamma^\mu\ u(\underline{P}_1\ s_1)$$

Multiplying by i to get $\mathcal{M}$, the scattering amplitude

$$\mathcal{M} = -g^2\ \bar{u}(\underline{P}_4,\ s_4)\ \gamma^\nu\ u(\underline{P}_2,\ s_2)\cdot\frac{g_{\mu\nu}}{(\underline{P}_1 - \underline{P}_3)^2}$$
$$\bar{u}(\underline{P}_3,\ s_3)\ \gamma^\mu\ u(\underline{P}_1,\ s_1)$$

If u and $\bar{u}$ are known explicitly, then $(\bar{u}\ \ \gamma^\nu\ \ u)$ is just a complex number

i.e. if all the u, $\bar{u}$ are known explicitly, then the scattering amplitude $\mathcal{M}$ is just a complex number.

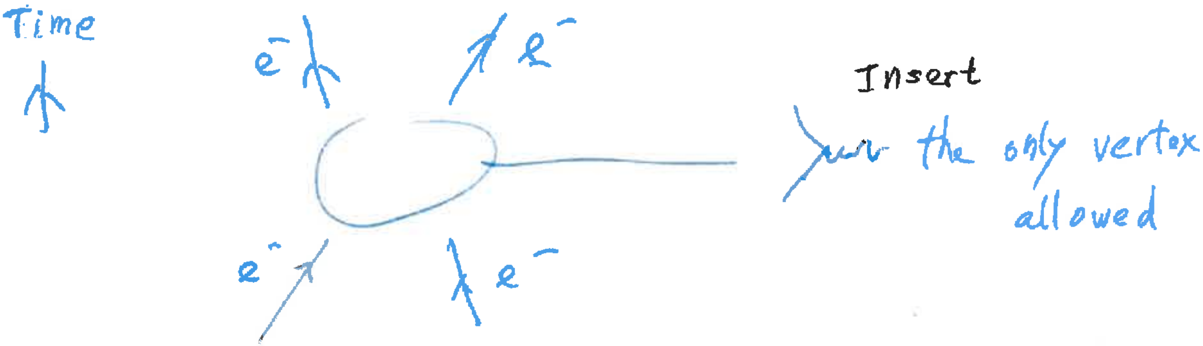
simplify the notations: $(\underline{P}_1, s_1) \to (1)$, $(\underline{P}_i, s_i) \to (i)$

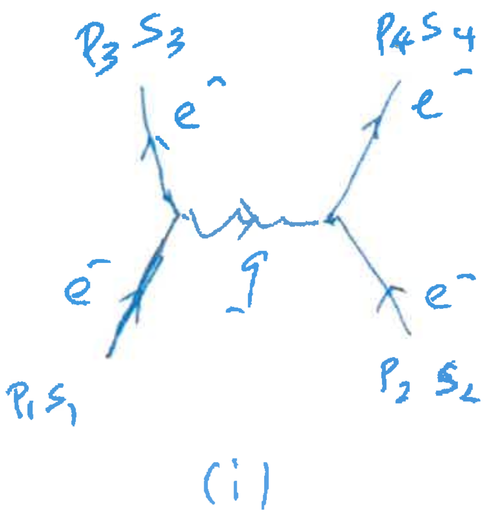
$$\mathcal{M} = -g^2\ \bar{u}(4)\ \gamma^\nu\ \bar{u}(2)\ \frac{g_{\mu\nu}}{(\underline{P}_1 - \underline{P}_3)^2}\cdot\bar{u}(3)\ \gamma^\mu\ u(1).$$

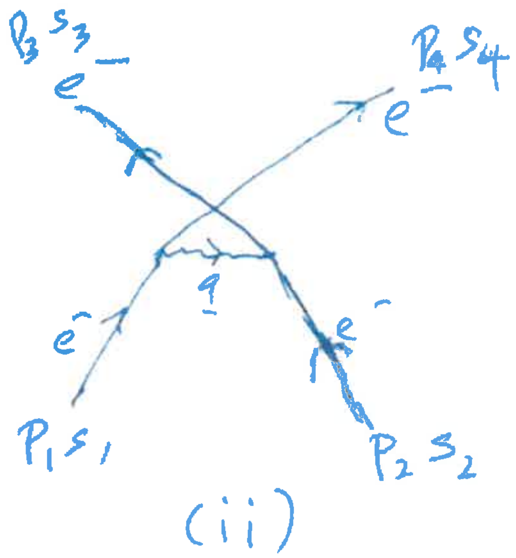
continue to get scattering amplitude of a physical process by using Feynman diagrams.

(ii) $e^- e^- \to e^- e^-$ Möller scattering

Time







2 diagrams $\longrightarrow$ 2 amplitudes $\mathcal{M}_{(i)}$, $\mathcal{M}_{(ii)}$

For diagram (i), it is like the $e^- \mu^- \to e^- \mu^-$, so we copy the result from previous example