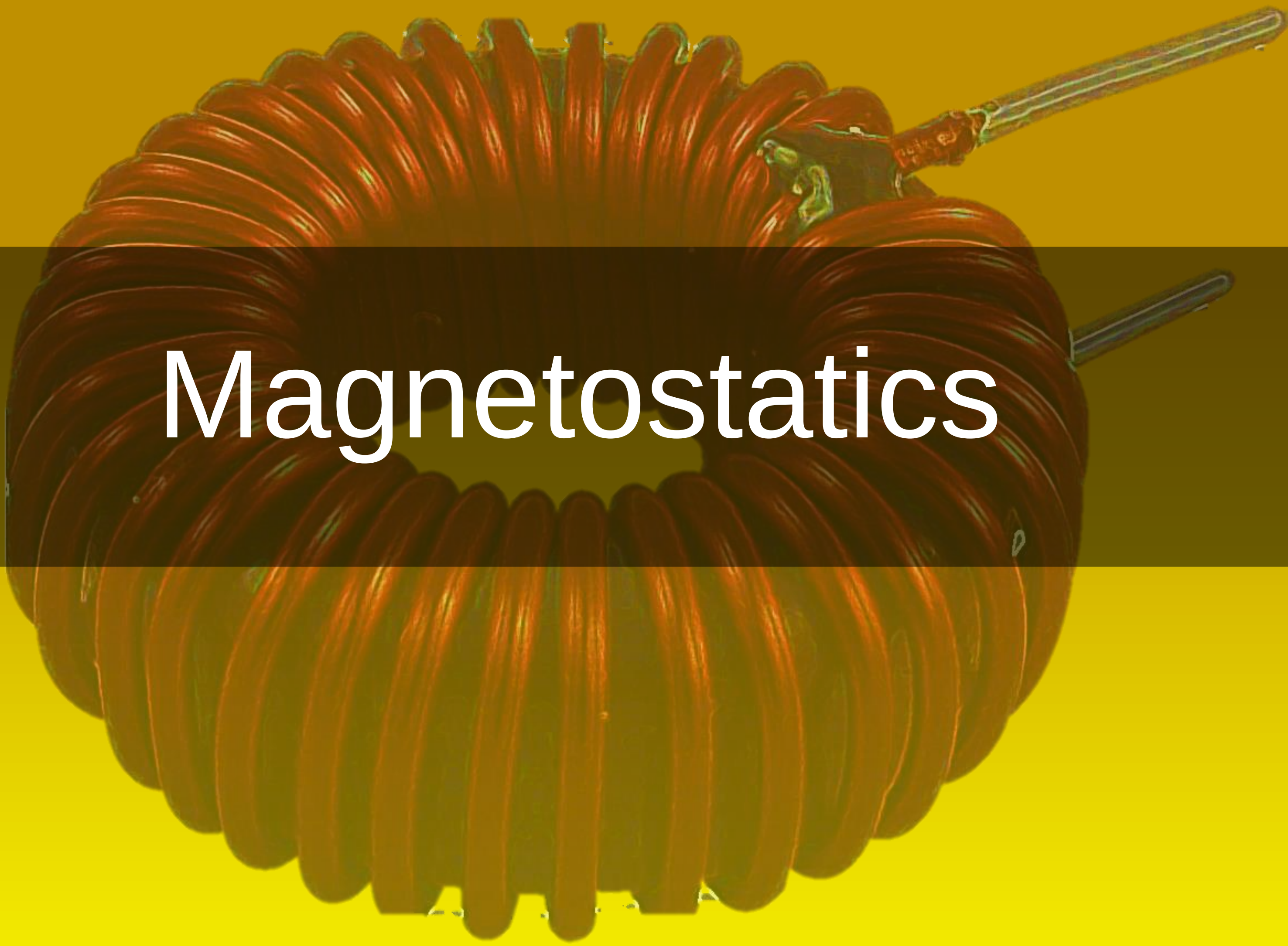
# Magnetostatics

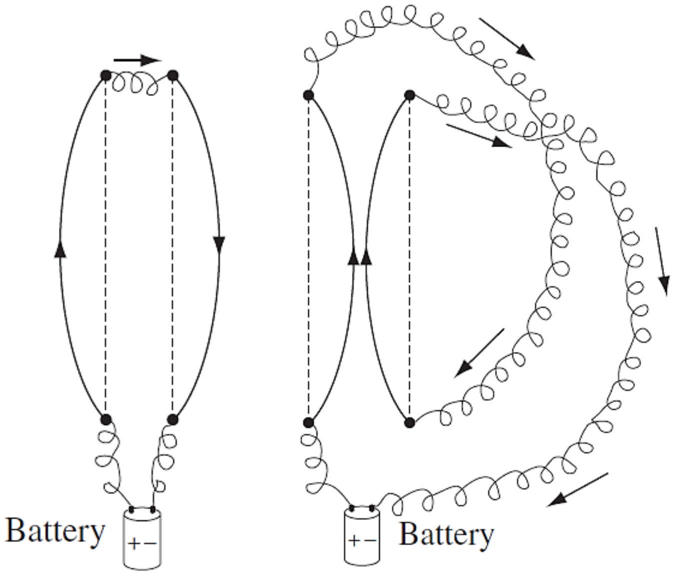


# The Lorenz force law

**Electrostatics**: Stationary charges with constant electric fields

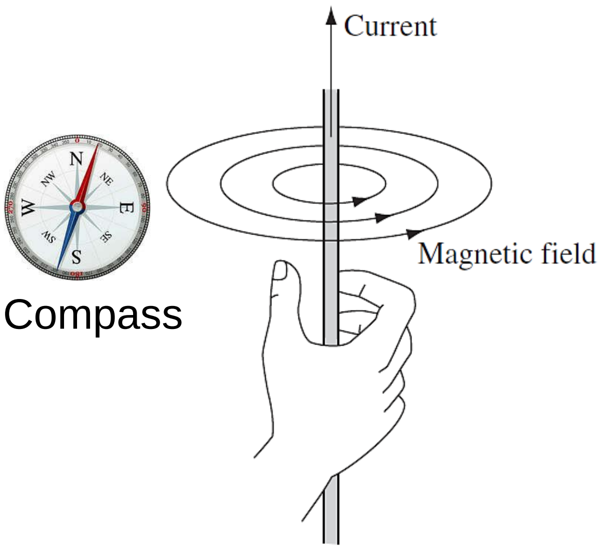
**Magnetostatics**: Steady currents with constant magnetic fields

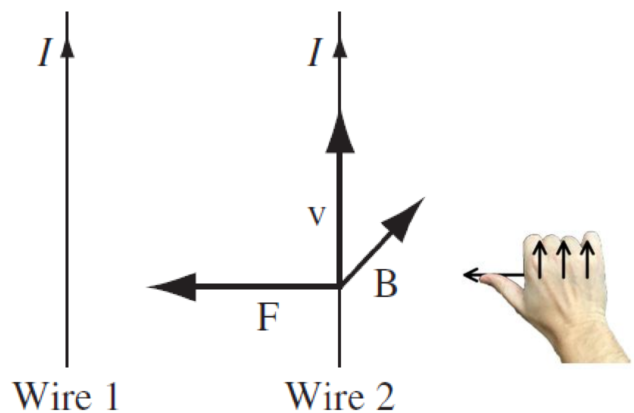
- Magnetic force and magnetic field
  - Pass current in parallel wires
    - Wires repel with antiparallel current
    - Wires attract with parallel current
    - Force is not electrostatic in nature (wires are charge neutral)
    - Magnetic force



# The Lorenz force law

- Magnetic force and magnetic field
  - Empirical rules for magnetic field and magnetic force
    - Straight current-carrying wire has magnetic field circling around it
    - Right-hand rule for field direction given current direction
    - Right-hand rule for force direction given current and field direction



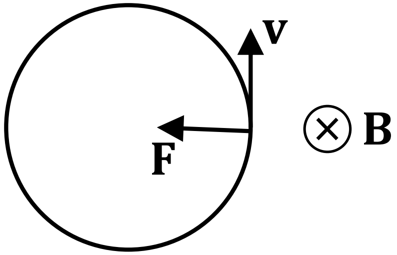



# The Lorenz force law

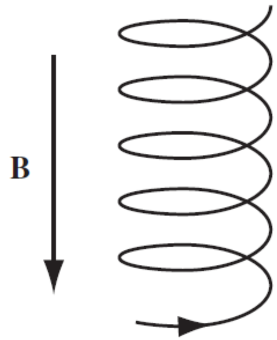
- The Lorenz force law
  - Magnetic force on charge $Q$, moving with velocity $\mathbf{v}$, in magnetic field $\mathbf{B}$

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

    - Given as an axiom without proof
  - With both $\mathbf{E}$ and $\mathbf{B}$, the full Lorenz force law $\boxed{\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]}$
  - Consequences
    - Circular charge motion, or alike, when magnetic force acts as centripetal force



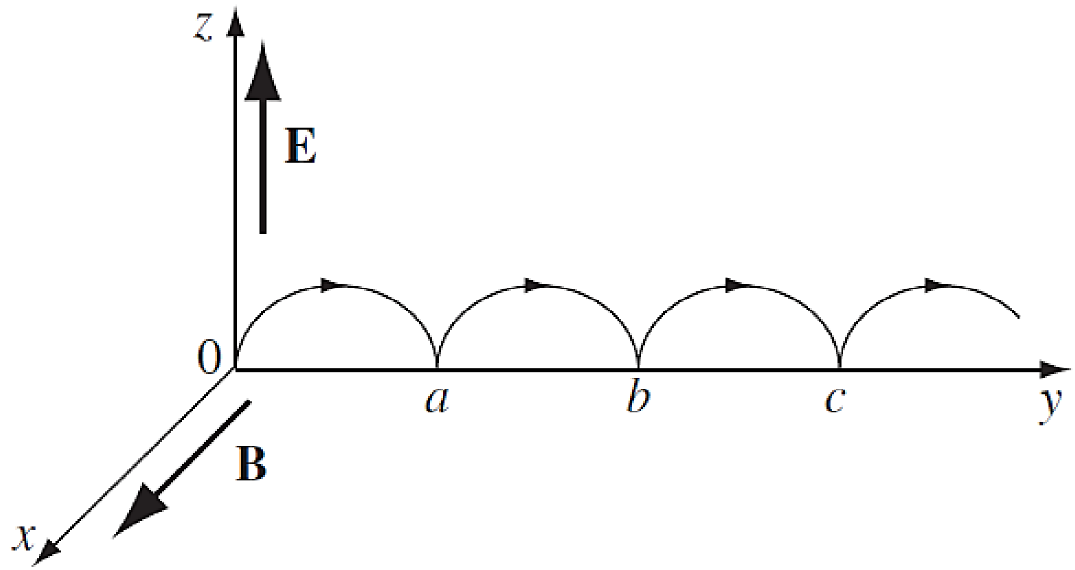

    - Magnetic forces do <u>no</u> work, $\mathbf{B}$ only deflects particle direction




$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \overbrace{\mathbf{v}\,dt}^{d\mathbf{l}} = 0$$

# The Lorenz force law

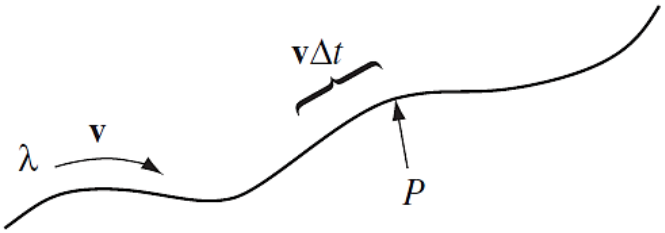
**Example 5.2. Cycloid Motion.** A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that $\mathbf{B}$ points in the $x$-direction, and $\mathbf{E}$ in the $z$-direction, as shown in Fig. 5.7. A positive charge is released from the origin; what path will it follow?




C3.exmp5.2

# Currents

- Current in a wire: charge per time passing a given point
  - Unit: Amperes (A) = Coulomb/second
  - $\mathbf{I} = \lambda \mathbf{v}$ where $\lambda$: line charge density, $\mathbf{v}$: velocity of movement




  - Positive charges moving at $\mathbf{v}$ = negative charges moving at $-\mathbf{v}$
  - Not meaningful to talk about current if it's just a single point charge moving (non-steady)
  - In many problems just write the magnitude $\boxed{I = \lambda v}$
    - Because direction is determined by the shape of wire
  - Magnetic force on a current-carrying wire $\boxed{\mathbf{F}_{\text{mag}} = \int I\,(d\mathbf{l} \times \mathbf{B})}$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\,dq = \int (\mathbf{v} \times \mathbf{B})\lambda\,dl = \int (\mathbf{I} \times \mathbf{B})\,dl = \int I\,(d\mathbf{l} \times \mathbf{B})$$

# Currents

- Surface and volume distributions of current
  - Surface current density $\mathbf{K} = \sigma \mathbf{v}$ (C/m$^2$ · m/s) ($\sigma$: surface charge density)
    - **K**: current per unit width (A/m)

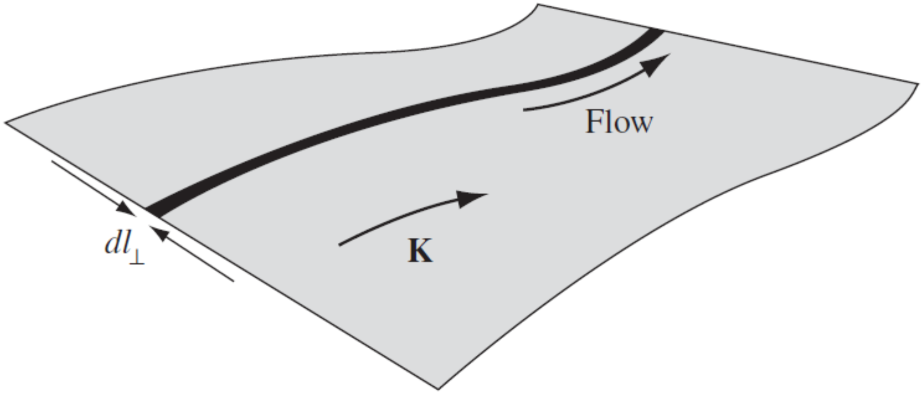
$$d\mathbf{I} = \sigma \mathbf{v}\, dl_\perp \Rightarrow \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp}$$

($l_\perp$: cross-sectional line segment perpendicular to $\mathbf{v}$)

    - Magnetic force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B}) \sigma\, da = \int (\mathbf{K} \times \mathbf{B})\, da$$

* **B** experiences discontinuity across a current-carrying surface



# Currents

- Surface and volume distributions of current
  - Volume current density $\mathbf{J} = \rho\mathbf{v}$ (C/m³ · m/s) ($\rho$: volume charge density)
    - **J**: current per unit area (A/m²)

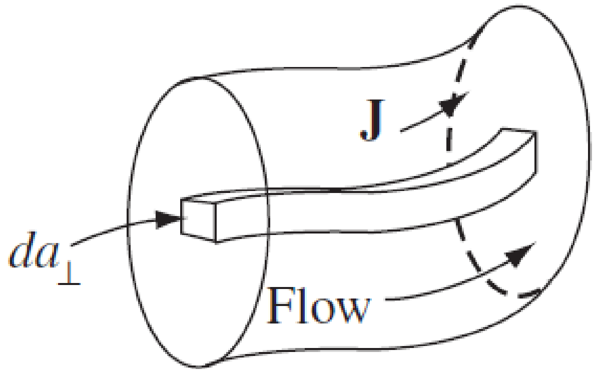
$$dI = \rho\mathbf{v}\, da_{\perp}$$
$$dI = \mathbf{J}\cdot d\boldsymbol{a} = \rho\mathbf{v}\cdot d\boldsymbol{a} \quad\Rightarrow\quad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}$$

($a_{\perp}$: cross-sectional area segment perpendicular to $\mathbf{v}$)

    - Magnetic force

$$\mathbf{F}_{\mathrm{mag}} = \int (\mathbf{v}\times\mathbf{B})\, dq = \int (\mathbf{v}\times\mathbf{B})\rho\, d\tau = \int (\mathbf{J}\times\mathbf{B})\, d\tau$$

  - Ohm's law $\mathbf{J} = \sigma_c\, \mathbf{E}$ ($\sigma_c$: electrical conductivity)



# Currents

- Surface and volume distributions of current
  - Continuity equation for volume current density
    - Current crossing a surface $\mathcal{S}$: $I = \int_{\mathcal{S}} J\, da_{\perp} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$
    - Current crossing boundary of a volume $\mathcal{V}$:

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau$$

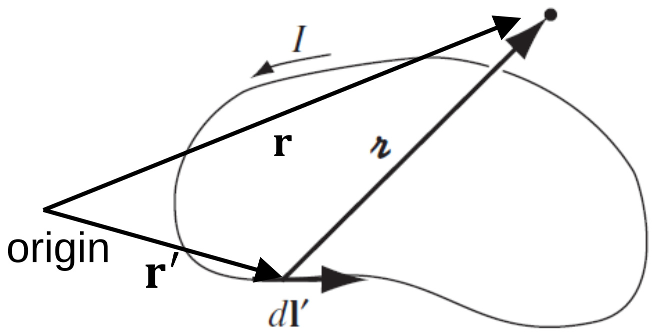
which must equal to the change of net charge in the volume

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = \overbrace{-\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau} = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

$\Rightarrow$ $\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$ for magnetostatics $\frac{\partial \rho}{\partial t} = 0$ $\frac{\partial \mathbf{J}}{\partial t} = \mathbf{0}$ $\nabla \cdot \mathbf{J} = 0$

# The Biot-Savart law

- Magnetic field generated by a steady current
  - Line current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$
    - Unit of $\mathbf{B}$: Tesla (T) = N/(A·m)
    - Permeability $\mu_0 = 4\pi \times 10^{-7}$ N/A$^2$
    - Separation vector $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$



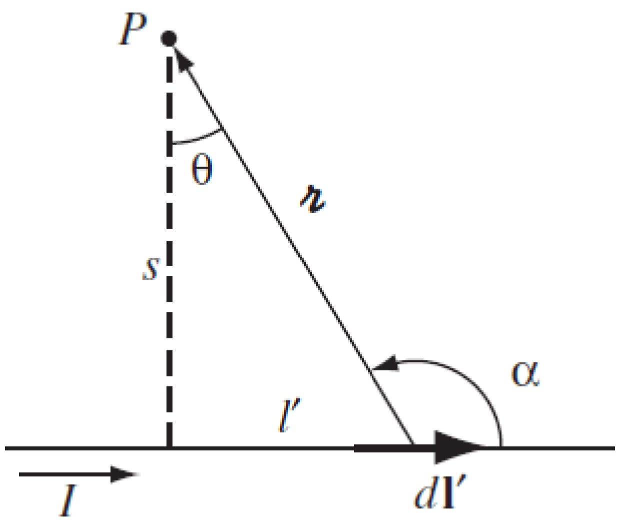

  - Surface current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,da'$
  - Volume current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$
  - Expressions for surface and volume currents not proved
  - The law cannot be used to calculate field generated by discrete moving charges

* $\mathscr{r}$ in textbook is typed as $\boldsymbol{r}$

# The Biot-Savart law

- Application of Biot-Savart law

**Example 5.5.** Find the magnetic field a distance $s$ from a long straight wire carrying a steady current $I$ (Fig. 5.18).



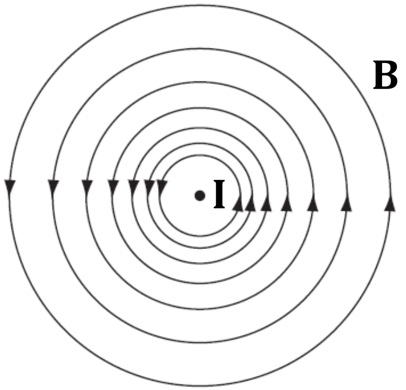

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$$


C3.exmp5.5

# Curl of magnetic field

- Derivation of curl from Stokes theorem
  - Loop integral of **B** around a straight-line current

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I$$




    - Result still holds if path is noncircular

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + \underline{s\,d\phi\,\hat{\boldsymbol{\phi}}} + dz\,\hat{\mathbf{z}} \qquad \underline{\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}}$$

  - Loop integral of **B** around any current distribution

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \left( I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a} \text{ total current enclosed by the path} \right)$$

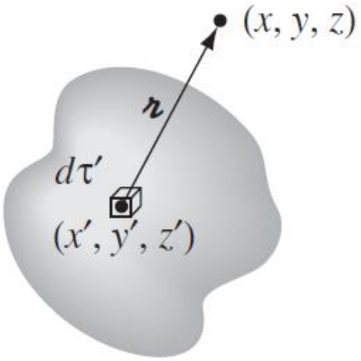
  - Apply Stokes theorem

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\Rightarrow \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

# Curl of magnetic field

- Derivation of curl from Biot-Savart law
  - A much more formal derivation than previous slide
  - $$\nabla \times \underbrace{\mathbf{B}}_{\mathbf{B}(\mathbf{r})} = \frac{\mu_0}{4\pi}\int \underbrace{\nabla}_{\nabla_{\mathbf{r}}} \times \left(\underbrace{\mathbf{J}}_{\mathbf{J}(\mathbf{r}')} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\underbrace{\mathscr{r}^2}_{\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'}}\right) \underbrace{d\tau'}_{dx'dy'dz'}$$




Product rule $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \mathbf{J}\left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) - \underline{(\mathbf{J}\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}}$$

$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$, and second term integrates to 0 (textbook p.232)

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')4\pi\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau' = \mu_0\mathbf{J}(\mathbf{r})$$

# Divergence of magnetic field

- Derivation of divergence from Biot-Savart law
  - $$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left( \mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'$$

Product rule $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$$\nabla \cdot \left( \mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left( \nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right)$$

$\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') = 0$, and $\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 0$ (Remember $\nabla \times \mathbf{E} = \mathbf{0}$)

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

Magnetic fields are divergence-free (no magnetic "free charge")

Magnetic monopoles? not reproduced

First Results from a Superconductive Detector for Moving Magnetic Monopoles

Blas Cabrera

Phys. Rev. Lett. **48**, 1378 – Published 17 May 1982

# Ampère's law

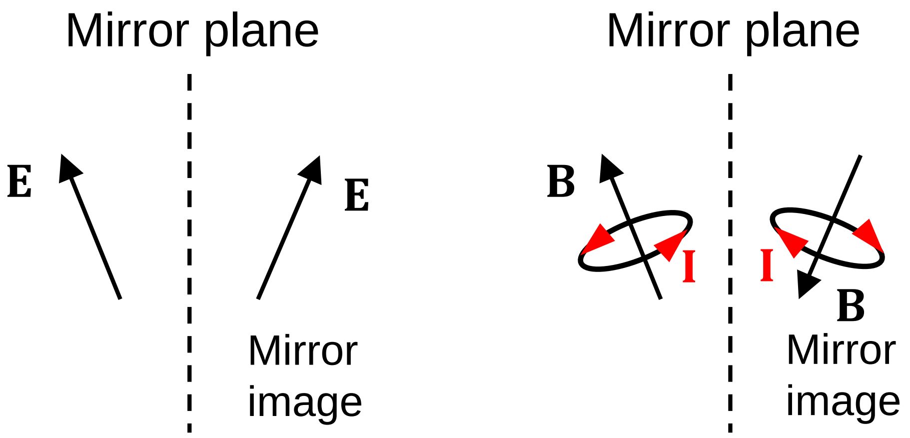
| Electrostatics | Magnetostatics |
|---|---|
| Coulomb's law | Biot-Savart law |
| Gauss's law<br>$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \quad \oint \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}}/\varepsilon_0$ | Ampère's law<br>$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$ |
| Gaussian surface | Amperian loop |

- Use of Ampère's law to calculate magnetic field
  - Ampère's law in integral form
  - Symmetry arguments
    - Translation symmetry
    - Rotational symmetry
    - Mirror symmetry
    - Inversion symmetry

      Be careful to use when transforming $\mathbf{B}$: need to flip sign
    - $\mathbf{B}$ flips sign if $\mathbf{I}$ flips sign (time-reversal symmetry)

# Symmetry of vectors and pseudovectors

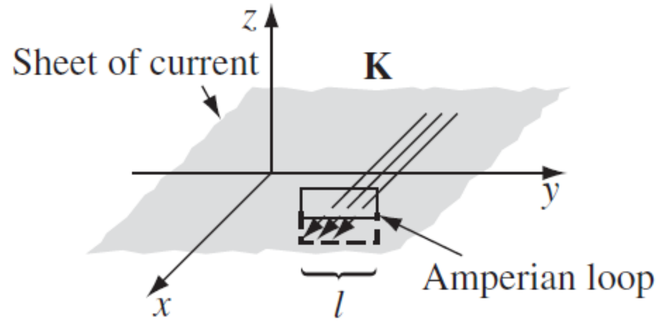
- Polar vectors and axial vectors
  - Polar vectors: **E**, **P**, **D**, **J**, **I**
    - Usually vectors associated with electric fields
  - Axial vectors: **B**, **M**, **H**
    - Usually vectors associated with magnetic fields
- Mirror operation



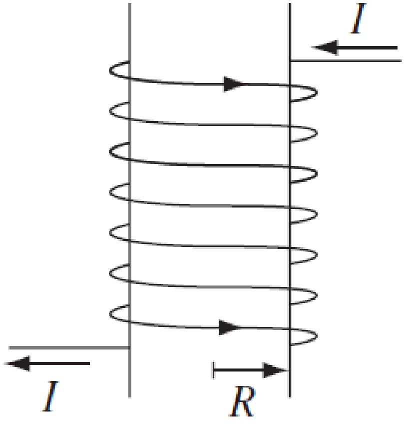

- Polar vectors: every operation as usual
- Axial vectors: Additional sign flip for mirror and inversion operations, other operations as usual

# Ampère's law

**Example 5.8.** Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\,\hat{\mathbf{x}}$, flowing over the $xy$ plane (Fig. 5.33).




**Example 5.9.** Find the magnetic field of a very long solenoid, consisting of $n$ closely wound turns per unit length on a cylinder of radius $R$, each carrying a steady current $I$ (Fig. 5.34).



# Ampère’s law

**Example 5.10.** A toroidal coil consists of a circular ring, or “donut,” around which a long wire is wrapped (Fig. 5.38). The winding is uniform and tight enough so that each turn can be considered a plane closed loop.

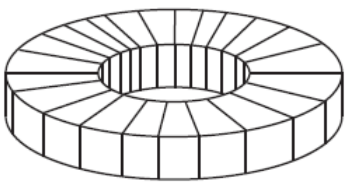
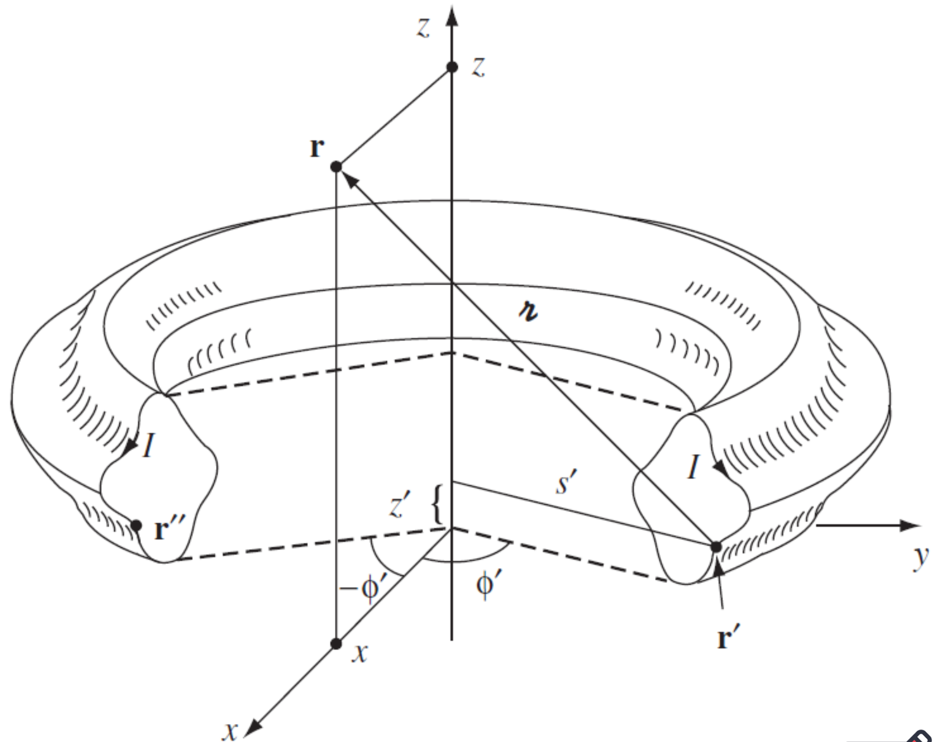


FIGURE 5.38




Toroidal structure with any cross-sectional shape

# Magnetic vector potential

- Vector potential $\mathbf{A}$
  - $\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$
    - Exploiting the property $\nabla \cdot \mathbf{B} = 0$ (divergence of curl vanishes)
  - $\nabla \cdot \mathbf{A} = 0$
    - Chosen to be so (like choosing reference point for electric potential)

      If original $\mathbf{A}_0$ is not, can define $\mathbf{A} = \mathbf{A}_0 + \nabla\lambda$ without varying $\mathbf{B}$

      $$\Rightarrow \nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2\lambda$$

      Can choose $\nabla^2\lambda = -\nabla \cdot \mathbf{A}_0$
  - $\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$
    - Use the choice above, and apply Ampère's law

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\cancelto{0}{\nabla \cdot \mathbf{A}}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

# Magnetic vector potential

- Calculating vector potential from current
  - $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ Poisson's equation, if $\mathbf{J} \to 0$ at infinity,

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'}$$

    - Analogous to solution $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$ to $\nabla^2 V = -\frac{\rho}{\epsilon_0}$
  - Line current $\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}} dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathscr{r}} d\mathbf{l}'}$
  - Surface current $\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\mathscr{r}} da'}$
  - Not so much simplification (vector $\mathbf{A}$ representing vector $\mathbf{B}$) but equation is easier to use than the Biot-Savart law
    - No need to integrate unit vector $\hat{\mathscr{r}}$

* $\nabla^2 \mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$

# Magnetic vector potential

- Calculating vector potential from magnetic field

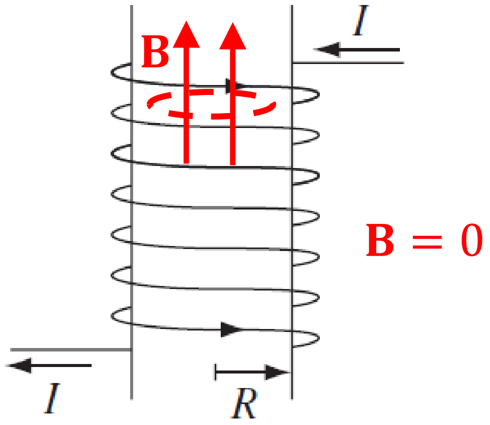
$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$$

where $\Phi$ is the magnetic flux

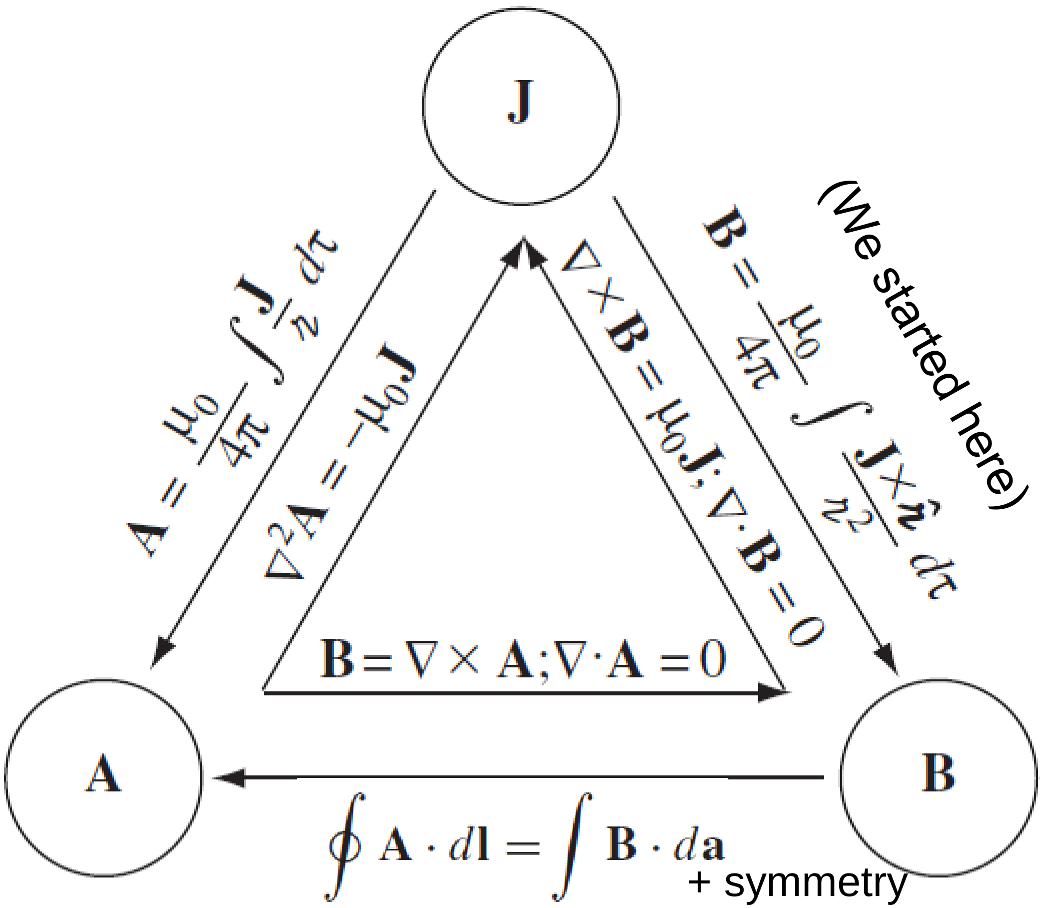
  - Can calculate $\mathbf{A}$ using equation above plus symmetry argument
  - Generally $\mathbf{A}$ mimics the direction of current

**Example 5.12.** Find the vector potential of an infinite solenoid with $n$ turns per unit length, radius $R$, and current $I$.

  - Check $\nabla \times \mathbf{A} = \mathbf{B}$
  - Check $\nabla \cdot \mathbf{A} = 0$



# Current, magnetic field, and vector potential




Differential equations need boundary conditions to solve

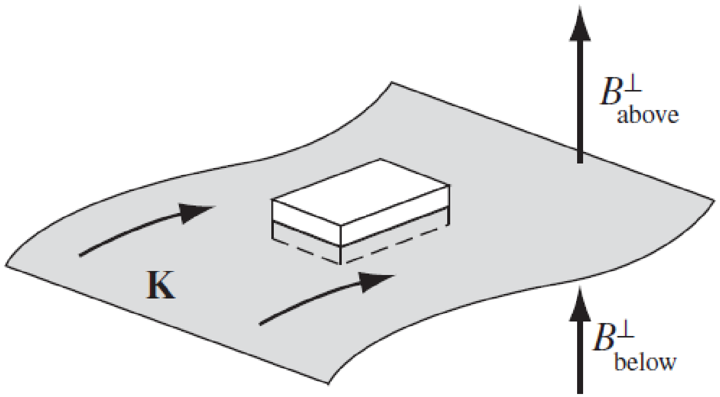
# Boundary conditions

- Boundary conditions of **B** across a 2D current surface
  - Normal component of **B**

    Thin pillbox with thickness $\varepsilon \to 0$

    $$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$

    $$\Rightarrow \boxed{B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}}$$



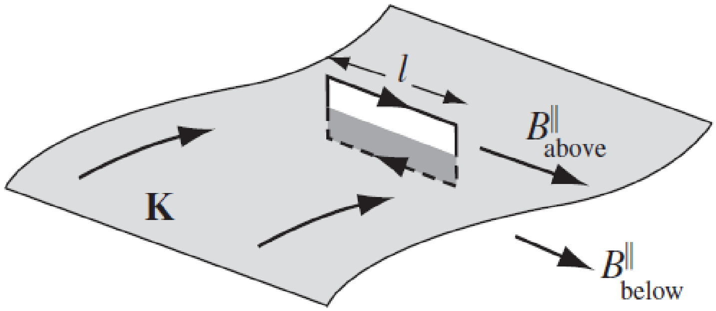

  - Tangential component of **B** <u>that is perpendicular to current</u>

    Thin Amperian loop

    $$\oint \mathbf{B} \cdot d\mathbf{l} = \left(B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}}\right) l$$

    $$= \mu_0 I_{\text{enc}} = \mu_0 K l$$

    $$\Rightarrow \boxed{B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K}$$




  - Summarizing above $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}})$

# Boundary conditions

- Boundary conditions of $\mathbf{A}$ across a 2D current surface
  - Vector potential $\mathbf{A}$ is always continuous $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$
    - Normal component of $\mathbf{A}$

      $\nabla \cdot \mathbf{A} = 0$ and select a thin pillbox

      $\Rightarrow A^{\perp}_{\text{above}} = A^{\perp}_{\text{below}}$

    - Tangential component of $\mathbf{A}$

      $\nabla \times \mathbf{A} = \mathbf{B}$ and select a thin loop to integrate

      $\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$ tends to zero when loop is thin

      $\Rightarrow A^{\parallel}_{\text{above}} = A^{\parallel}_{\text{below}}$

  - Derivative of $\mathbf{A}$ is discontinuous $\dfrac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \dfrac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$

# Multipole expansion of vector potential

- Goal: to expand $\mathbf{A}$ in power series of $1/r$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\mathscr{r}} \, d\mathbf{l}'$$

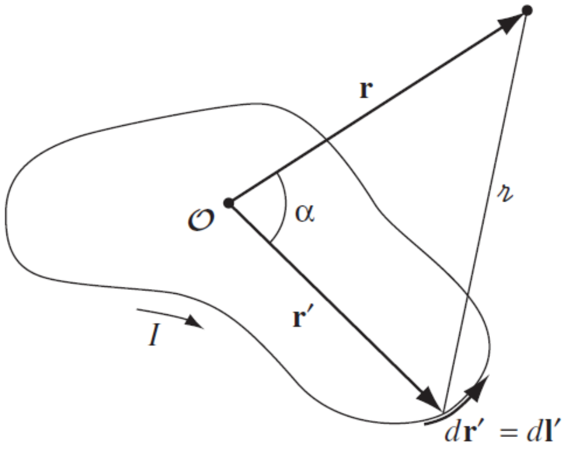
$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\alpha}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

(From electrostatic multipole expansion)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\alpha) \, d\mathbf{l}'$$

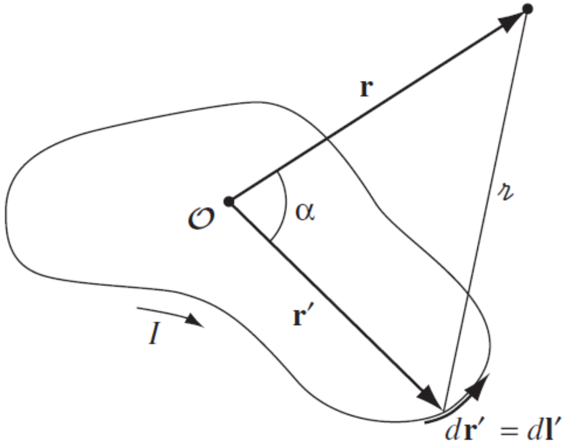
$$= \frac{\mu_0 I}{4\pi} \left[ \underbrace{\frac{1}{r} \oint d\mathbf{l}'}_{\text{Monopole}} + \underbrace{\frac{1}{r^2} \oint r' \cos\alpha \, d\mathbf{l}'}_{\text{Dipole}} \right.$$

$$\left. + \underbrace{\frac{1}{r^3} \oint (r')^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\mathbf{l}'}_{\text{Quadrupole}} + \cdots \right]$$



# Multipole expansion of vector potential

- Magnetic monopole and magnetic dipole
  - Magnetic monopole
    - Always zero, because $\oint d\mathbf{l}' = \mathbf{0}$
    - Also because $\nabla \cdot \mathbf{B} = 0$
  - Magnetic dipole




$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'$$

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'$$

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}}$$

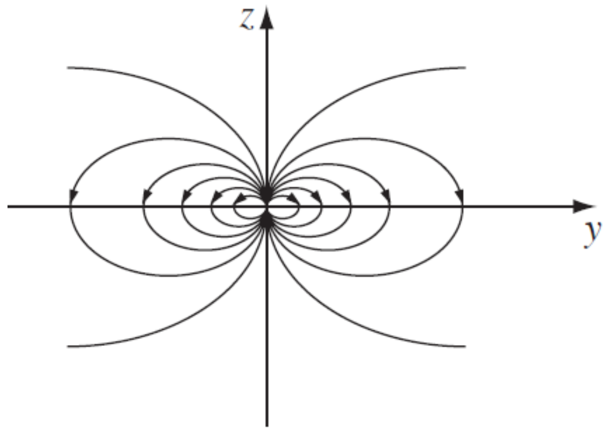
Magnetic dipole moment

$$\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}$$

# Multipole expansion of vector potential

- Pure dipole vs physical dipole

Pure dipole



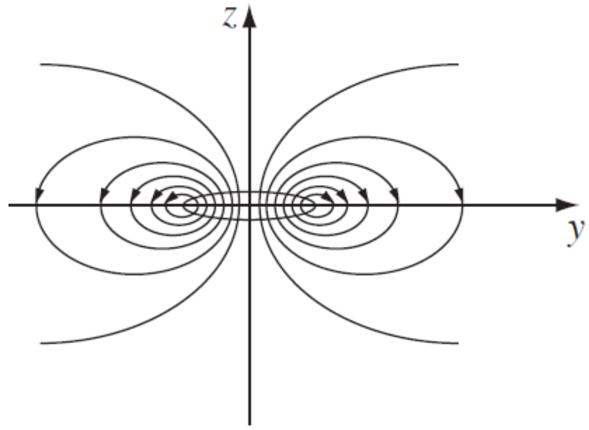

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

Assumes $r \gg$ loop radius

$\mathbf{m} = I\mathbf{a}$ but take $I \to \infty$, $a \to 0$

Similar to pure electric dipole

Physical dipole

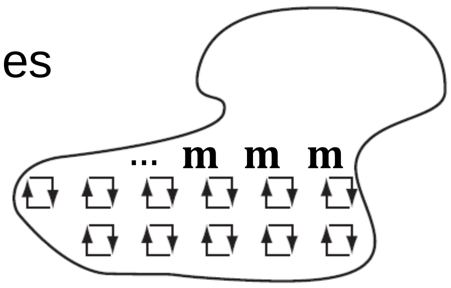



Deviations appear when closing up onto the dipole

# Magnetic fields in matter

# Magnetic materials

- Practically, all materials have magnetic response
  - Orbiting electrons around nuclei as magnetic dipoles
  - Electron spins as magnetic dipoles
    - Quantum object without a classical analog




- Magnetization: $\mathbf{M} = \frac{1}{V}\sum_i \mathbf{m}_i$
  - Magnetic dipole moment per unit volume

- Three types of response of matter to magnetic field
  - Paramagnets: magnetization $\mathbf{M}$ parallel to applied $\mathbf{B}$
  - Diamagnets: magnetization $\mathbf{M}$ opposite to applied $\mathbf{B}$
  - Ferromagnets: finite magnetization $\mathbf{M}$ even without $\mathbf{B}$

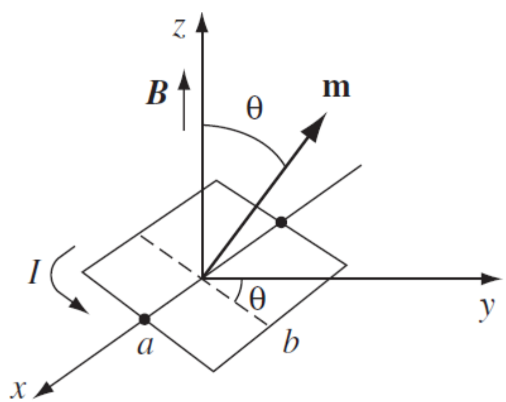
# Magnetic dipoles responding to field

- Paramagnetic response
  - Torque in a uniform field
    - Suppose a rectangle current loop with side lengths $a$ and $b$

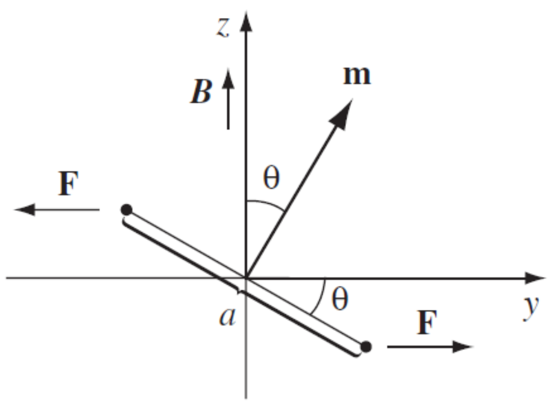
$$\mathbf{N} = aF\sin\theta\,\hat{\mathbf{x}} \xrightarrow{\text{force } F = IbB} \mathbf{N} = IabB\sin\theta\,\hat{\mathbf{x}} = mB\sin\theta\,\hat{\mathbf{x}}$$

$$\Rightarrow \boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B}}$$

  - Torque tends to align $\mathbf{m}$ parallel to $\mathbf{B}$

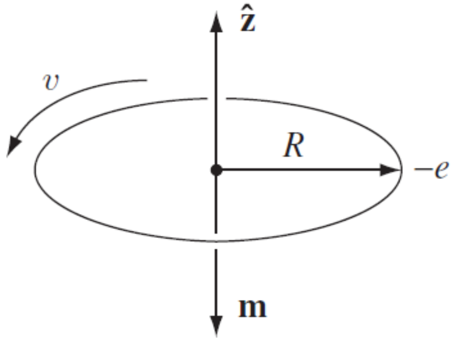



View from $x$ axis



# Magnetic dipoles responding to field

- Diamagnetic response
  - Modification to an atomic orbit




$$I = \frac{-e}{T} = -\frac{ev}{2\pi R} \xrightarrow[\text{Orbit period}]{T = 2\pi R/v} \mathbf{m} = -\frac{1}{2} e v R \, \hat{\mathbf{z}}$$

- Velocity without and with $\mathbf{B}$, balancing Coulomb and centripetal forces

without $\mathbf{B}$: $$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e \frac{v^2}{R}$$

with $\mathbf{B}$: $$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e \frac{\bar{v}^2}{R}$$

$v$: electron velocity
$\bar{v}$: electron velocity with $\mathbf{B}$
$m_e$: electron mass

$$\Rightarrow \quad e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v)$$

$$\Rightarrow \quad \Delta v = \bar{v} - v = \frac{eRB}{2m_e}$$ for small $\Delta v$

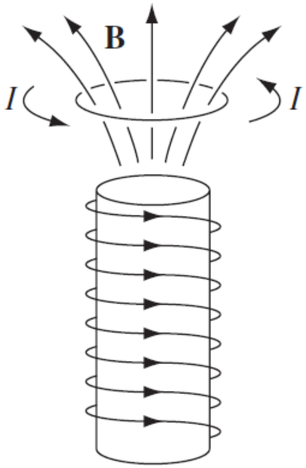
$$\Rightarrow \quad \Delta\mathbf{m} = -\frac{1}{2}e(\Delta v)R\,\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}$$

Change in dipole moment works against $\mathbf{B}$

# Forces on paramagnets and diamagnets

- General formula for small dipole in nonuniform field

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$



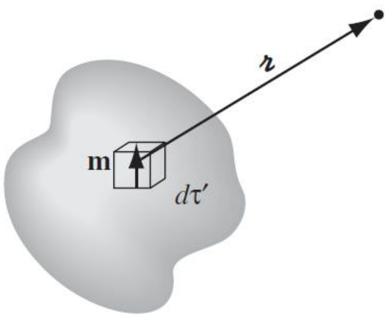

  - For paramagnets $m \parallel B$
    - Force directs toward more intense field regions
    - Paramagnets are attracted to magnets
  - For diamagnets $m \parallel -B$
    - Force directs toward less intense field regions
    - Diamagnets are repelled by magnets

Levitating frog

(Ig Nobel award 2000)

# Field of magnetized objects

- Bound currents
  - Vector potential of a magnetized object (neglecting the cause)




$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \left[\mathbf{M}(\mathbf{r}') \times \left(\boldsymbol{\nabla}'\frac{1}{\mathscr{r}}\right)\right] d\tau'$$

$\downarrow$ Integrate by parts

$$= \frac{\mu_0}{4\pi}\left\{\int \frac{1}{\mathscr{r}}[\boldsymbol{\nabla}' \times \mathbf{M}(\mathbf{r}')]\, d\tau' - \int \boldsymbol{\nabla}' \times \left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right] d\tau'\right\}$$

$$= \frac{\mu_0}{4\pi}\int \frac{1}{\mathscr{r}}[\boldsymbol{\nabla}' \times \mathbf{M}(\mathbf{r}')]\, d\tau' + \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']$$

$$= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\, da'$$

* with $\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M}$

$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$

# Field of magnetized objects

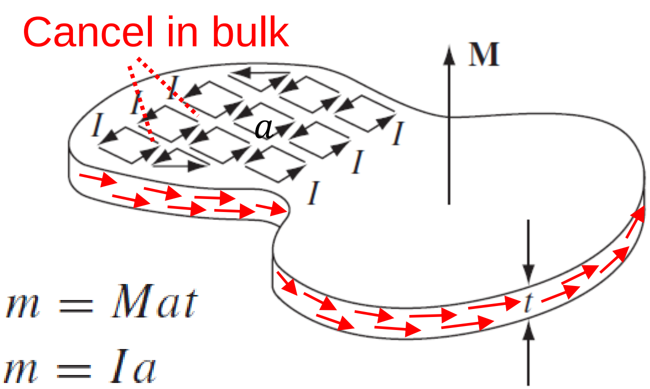
- Bound currents

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\, da'$$

  - Density of bound currents $\begin{cases} \text{surface: } \boxed{\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}} \\ \text{volume: } \boxed{\mathbf{J}_b = \nabla\times\mathbf{M}} \end{cases}$ *check $\nabla\cdot\mathbf{J}_b = 0$

  - Physical picture of bound currents

**Surface bound current (suppose uniform M)**

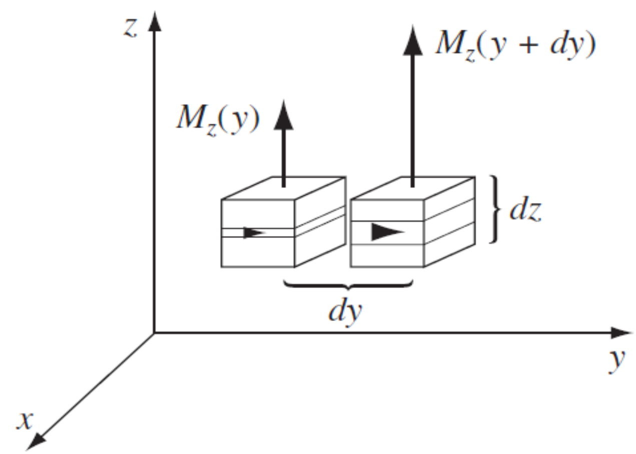



$m = Mat$

$m = Ia$

$I = Mt \Rightarrow K_b = I/t = M$

**Volume bound current**

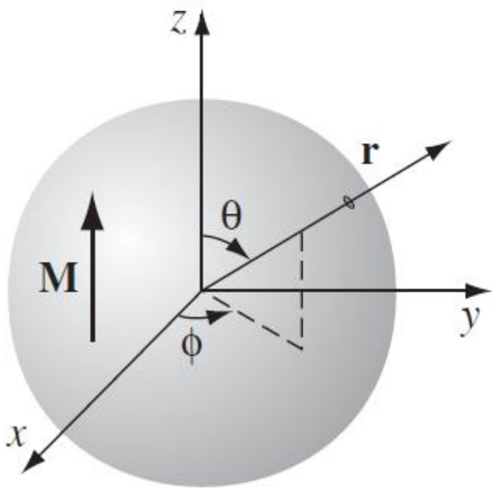



$$I_x = [M_z(y+dy) - M_z(y)]\,dz = \frac{\partial M_z}{\partial y}\,dy\,dz$$

# Field of magnetized objects

- Bound currents

**Example 6.1.** Find the magnetic field of a uniformly magnetized sphere.




- Field inside the sphere $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$
  - Uniform field
  - **M** induces **B** that is parallel to it, while **P** induces **E** that is antiparallel
- Field outside the sphere same as what would be for a dipole

$$\mathbf{m} = \frac{4}{3}\pi R^3\mathbf{M}$$

# Auxiliary field

- Add the cause and the effect of magnetization
  - Total magnetic field

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

    - $\mathbf{B}$: total magnetic field
    - $\mathbf{J}$: total current density
    - $\mathbf{J}_b$: bound current density, due to magnetization
    - $\mathbf{J}_f$: free current density that we control, not a result of magnetization

$$\Rightarrow \quad \nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_f$$

  - The auxiliary field $\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}$

$$\Rightarrow \quad \boxed{\nabla \times \mathbf{H} = \mathbf{J}_f}$$

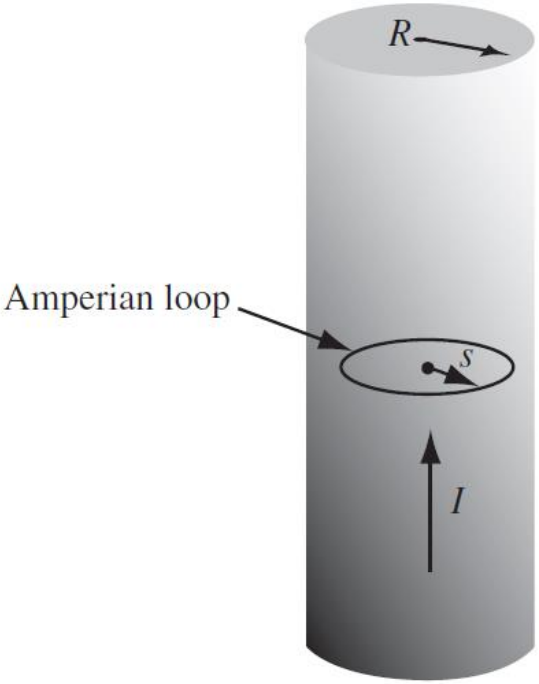
    - Ampère's law for auxiliary field $\boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}}$

Enclosed total free current

# Auxiliary field

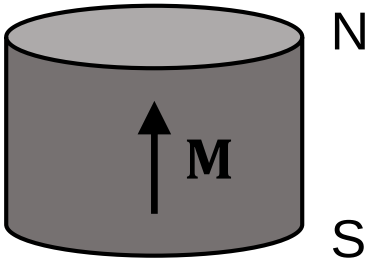
- Application of auxiliary field

**Example 6.2.** A long copper rod of radius $R$ carries a uniformly distributed (free) current $I$ (Fig. 6.19). Find $\mathbf{H}$ inside and outside the rod.



# Auxiliary field

- Can $\mathbf{H}$ be finite without applying any free current?
  - Looks like so because $\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$
    - Consider a short cylindrical magnet
    - $\mathbf{H}$ would be zero everywhere
    - Then $\mathbf{B} \neq \mathbf{0}$ inside magnet, $\mathbf{B} = \mathbf{0}$ outside magnet
    - Obviously wrong




  - $\mathbf{H}$ can be finite without any $\mathbf{J}_F$
    - Because $\boldsymbol{\nabla} \cdot \mathbf{H} = \boldsymbol{\nabla} \cdot \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = -\boldsymbol{\nabla} \cdot \mathbf{M} \neq \mathbf{0}$
    - $\boldsymbol{\nabla} \cdot \mathbf{M} \neq \mathbf{0}$ at the top and bottom surfaces of the magnet

# H versus B, D versus E

- A long history of confusion
  - Whether we shall call $\mathbf{H}$ or $\mathbf{B}$ as the magnetic field
    - Some convention calls $\mathbf{H}$ the magnetic field, and $\mathbf{B}$ as the magnetic flux density
    - Our textbook takes the stance that $\mathbf{H}$ should be "auxiliary"
- Reason for such confusion
  - $\mathbf{H}$ is a lot more frequently used than $\mathbf{B}$ as $\mathbf{H}$ is given by free current, something we can control, while $\mathbf{B}$ is material dependent
    - Helmholtz coil magnetizing a specimen
  - $\mathbf{E}$ is a lot more frequently used than $\mathbf{D}$ as $\mathbf{D}$ is given by free charge, which we rarely control, while $\mathbf{E}$ is determined by voltage difference (over distance), which is what we control
    - Charging up of a parallel plate capacitor

# Linear magnetic media

- Linear magnetic media
  - $\boxed{\mathbf{M} = \chi_m \mathbf{H}}$ ($\chi_m$: magnetic susceptibility)

    $\Rightarrow$ $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}$

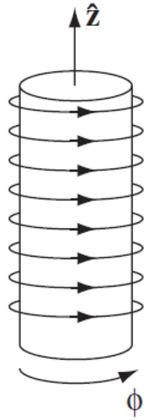
    - Diagnetic susceptibility on the order of $10^{-6}$
  - $\boxed{\mathbf{B} = \mu \mathbf{H} \quad (\mu = \mu_0(1 + \chi_m)\text{: permeability})}$

? Why this linear relation defines M-H but not M-B?

**Example 6.3.** An infinite solenoid ($n$ turns per unit length, current $I$) is filled with linear material of susceptibility $\chi_m$. Find the magnetic field inside the solenoid.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} \Rightarrow \mathbf{H} = nI\,\hat{\mathbf{z}}$$

$$\Rightarrow \mathbf{B} = \mu_0(1 + \chi_m)nI\,\hat{\mathbf{z}}$$




Enhancement of field if paramagnetic!

# Boundary conditions

- Boundary conditions reexamined
  - Earlier findings still hold, but $\mathbf{J}$ needs to include free and bound currents

$$\begin{cases} B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \\ \mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \end{cases}$$

  - Easier to use the boundary conditions of $\mathbf{D}$

Surface free charge density

$$\begin{cases} \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \\ \nabla \times \mathbf{H} = \mathbf{J}_f \end{cases} \Rightarrow \boxed{\begin{cases} \mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}} \\ H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}) \end{cases}}$$

# Nonlinear magnetic media

- Ferromagnets: $\mathbf{M} \neq 0$ without applying any field
  - Obvious violation of the linear relation $\mathbf{M} = \chi_m \mathbf{H}$
  - Represents a quantum phenomena
    - Exchange interaction: $U = -2J \sum_{p=1}^{N} \boldsymbol{S}_p \cdot \boldsymbol{S}_{p+1}$ $(J > 0)$
    - Prefers spontaneous parallel alignment of spins
  - Domain formation and hysteresis loop