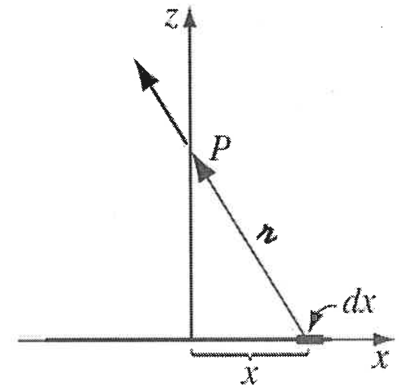
Example 2.2. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ (Fig. 2.6).



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,dl'$$

$$\vec{\mathscr{r}} = -x\hat{x} + z\hat{z} \qquad \mathscr{r} = \sqrt{x^2+z^2}$$

$$\hat{\mathscr{r}} = \vec{\mathscr{r}}/\mathscr{r} = -\frac{x\hat{x}}{\sqrt{x^2+z^2}} + \frac{z\hat{z}}{\sqrt{x^2+z^2}}$$

$$dl' = dx \qquad \lambda(\vec{r}\,') = \lambda$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{\lambda}{(x^2+z^2)^{3/2}}(-x\hat{x}+z\hat{z})\,dx$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\underbrace{\int_{-L}^{L}\frac{-x\hat{x}}{(x^2+z^2)^{3/2}}\,dx}_{=0 \;\; \int_{-L}^{L}dx \text{ of an odd function}} + \underbrace{z\hat{z}\int_{-L}^{L}\frac{1}{(x^2+z^2)^{3/2}}\,dx}_{②}\right]$$

$$② = \left.\frac{x}{z^2\sqrt{x^2+z^2}}\right|_{-L}^{L} = \frac{2L}{z^2\sqrt{L^2+z^2}}$$

$$\vec{E}(z) = \frac{2L\lambda}{4\pi\varepsilon_0 z\sqrt{L^2+z^2}}\,\hat{z}$$