AY 2022-2023 Final exam paper

1) $l = 3$

$s = \frac{1}{2}$

$\vec{J} = \vec{L} + \vec{S}$

$\vec{J}^2 |j, m_j\rangle = \hbar^2 j(j+1) |j, m_j\rangle$

$J_z |j, m_j\rangle = \hbar m_j |j, m_j\rangle$

(a) $\max j = l + s = 3 + \frac{1}{2} = \frac{7}{2}$

$\min j = |l - s| = |3 - \frac{1}{2}| = \frac{5}{2}$

Allowed values of $j$: $\frac{5}{2}, \frac{7}{2}$

$+1$

(b) Possible outcomes when measuring $\vec{J}^2$ are:

$\hbar^2 j(j+1)$ where $j = \frac{5}{2}, \frac{7}{2}$.

$j = \frac{5}{2}$, $\hbar^2 j(j+1) = \hbar^2 \frac{5}{2} \cdot \frac{7}{2} = \frac{35}{4} \hbar^2$

$j = \frac{7}{2}$, $\hbar^2 j(j+1) = \hbar^2 \frac{7}{2} \cdot \frac{9}{2} = \frac{63}{4} \hbar^2$

(c) $\vec{L} \cdot \vec{S}$?

$$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{J}\cdot\vec{J} = (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S})$$

$$= \vec{L}^2 + \vec{S}^2 + \vec{S}\cdot\vec{L} + \vec{L}\cdot\vec{S}$$

$$= \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S}$$

( $\vec{L}$ & $\vec{S}$ commute since they operate on different spaces)

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

$$\langle j, m_j, l, s | \vec{L}\cdot\vec{S} | j, m_j, l, s\rangle \qquad \left(l = 3,\ s = \tfrac{1}{2}\right)$$

$$= \frac{1}{2}\langle j, m_j, l, s | \vec{J}^2 - \vec{L}^2 - \vec{S}^2 | j, m_j, l, s\rangle$$

$$= \frac{\hbar^2}{2}\left(j(j+1) - l(l+1) - s(s+1)\right)$$

$$= \frac{1}{2}\hbar^2 j(j+1) - \frac{\hbar^2}{2}3.4 - \frac{\hbar^2}{2}\frac{1}{2}\cdot\frac{3}{2}$$

$$= \frac{1}{2}\hbar^2 j(j+1) - \hbar^2\frac{51}{8}$$

From (b):

$j = \frac{5}{2}$ : $\hbar^2 j(j+1) = \frac{35}{4}\hbar^2$

Outcome : $\frac{1}{2}\cdot\frac{35}{4}\hbar^2 - \frac{51}{8}\hbar^2 = -2\hbar^2$

$j = \frac{7}{2}$ : $\quad \hbar^2 j(j+1) = \frac{63}{4}\hbar^2$

outcome: $\quad \frac{1}{2}\cdot\frac{63}{4}\hbar^2 - \frac{51}{8}\hbar^2 = \frac{3}{2}\hbar^2$

(d) $\quad j = \frac{7}{2}$

$m_j = j = \frac{7}{2} \qquad\qquad (\hat{L}_z \rightarrow \hbar m_\ell)$

Recall W6L2, $\quad m_j = m_\ell + m_s$

$$m_j^{max} = m_\ell^{max} + m_s^{max}$$

$$|j = \tfrac{7}{2}, \underbrace{m_j = \tfrac{7}{2}}_{\substack{\text{max } m_j \\ (\ell=3,\ s=\frac{1}{2})}}\rangle = |m_\ell = m_\ell^{max} = 3\rangle \otimes |m_s = m_s^{max} = \tfrac{1}{2}\rangle$$

Possible outcome for $L_z$ : $m_\ell\hbar = 3\hbar$

with probability one.

2.

$$H_{xxz} = \frac{J_x}{2}\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y\right) + J_z\sigma_1^z\sigma_2^z$$

where $J_x$ and $J_z$ are constants

From W5L2:

Basis for $V_1 = \{|e_1\rangle, |e_2\rangle\}$

Basis for $V_2 = \{|f_1\rangle, |f_2\rangle\}$

Basis for $V_1 \otimes V_2 = \{ |e_1 f_1\rangle, |e_1 f_2\rangle, |e_2 f_1\rangle, |e_2 f_2\rangle \}$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} & a_2 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ a_3 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} & a_4 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \end{pmatrix}$$

(a) $\sigma_1^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Basis for tensor product space.

$\{ |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \}$

So

$$\sigma_1^x \sigma_2^x = \begin{pmatrix} 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(b) $H_{xxz} = \frac{J_x}{2} \left( \sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y \right) + J_z \sigma_1^z \sigma_2^z$

$$\sigma_1^y \otimes \sigma_2^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_1^z \otimes \sigma_2^z = \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & -1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So $H_{xxz}$

$$= \frac{J_x}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{J_x}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{J_x}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{J_x}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$+ J_z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} J_z & 0 & 0 & 0 \\ 0 & -J_z & J_x & 0 \\ 0 & J_x & -J_z & 0 \\ 0 & 0 & 0 & J_z \end{pmatrix}$$

(c) Basis $\{ |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \}$

$$|\chi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right)$$

$$\langle \chi | H_{xxz} | \chi \rangle = ?$$

$$|\chi\rangle \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\langle \chi | \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\langle \chi | H_{xxz} | \chi \rangle$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} J_z & 0 & 0 & 0 \\ 0 & -J_z & J_x & 0 \\ 0 & J_x & -J_z & 0 \\ 0 & 0 & 0 & J_z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -J_z + J_x \\ J_x - J_z \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\left(-J_z + J_x + J_x - J_z\right)$$

$$= J_x - J_z$$

(d) spatial states $\{ |\nu=0\rangle, |\nu=1\rangle \}$

$$|\chi\rangle = \frac{1}{\sqrt{2}}\left( |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right)$$

spin - $\frac{1}{2}$ atoms — fermions

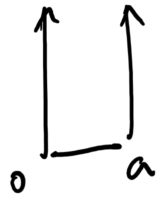
$\Rightarrow$ overall antisymmetric states wrt exchange of particles.

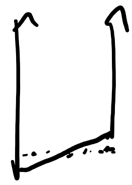
Since the spin part is symmetric, the spatial part must be antisymmetric.

Allowed $|\Psi\rangle = \frac{1}{\sqrt{2}}\left( |\nu=0, \nu=1\rangle - |\nu=1, \nu=0\rangle \right)$

3) similar to Tutorial 5 Q1.

$V_1(x)$ : infinite square well



$V_2(x)$ 

$\Rightarrow V_2(x) = V_1(x) + \tilde{V}_2(x)$

where $\tilde{V}_2(x) = \frac{b}{a}\left(x - \frac{a}{2}\right)$

$V_2(x) = \frac{b}{a}\left(x - \frac{a}{2}\right)$

for $0 \leq x \leq a$

& $\infty$ otherwise.

(a) $H = T + V_2$

Define $H_0 = T + V_1$

$H = H_0 + \tilde{V}_2$

No degeneracies.

$E_n^{(1)} = \langle \psi_n^{(0)} | \tilde{V}_2 | \psi_n^{(0)} \rangle$

where $\psi_n^{(0)}(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$

By symmetry,

$\tilde{V}_2(x)$ is odd about $x = \frac{a}{2}$,

$\psi_n^{(0)}(x)$ is either even or odd about $x = \frac{a}{2}$.

Therefore $E_n^{(1)} = 0$ by symmetry.

To 1st order in $\frac{b}{a}$,

$E_n = E_n^{(0)} + E_n^{(1)} = \frac{n^2 \hbar^2 \pi^2}{2ma^2}$

(b) $H = T + V_3$

$$V_3(x) = V_2(x) + \frac{b}{2}$$

So to 1st order in $\frac{b}{a}$, $E_n = \frac{n^2 \hbar^2 \pi^2}{2ma^2} + \frac{b}{2}$,

using the result in (a).

(c)

$V_4$ : $b \approx a$

$V_2, V_3$ : $b << a$, $\frac{b}{a} << 1$

For $V_2$ and $V_3$, we can use perturbation theory because $\frac{b}{a} << 1$

But for $V_4$, $\frac{b}{a} \sim 1$ and we cannot use perturbation theory.

(d) $E_1^*$ — lowest energy eigenvalue for $H = T + V_4$.

— unknown.

$$E_1 = \frac{\hbar^2 \pi^2}{2ma^2}$$

Correct answer is $E_1^* < E_1$ (3).

$$E_1 = \langle \psi_1^{(0)} | T + V_1 | \psi_1^{(0)} \rangle.$$

$$V_4 = V_1 + \tilde{V}_4 ,$$

$$\tilde{V}_4 = \frac{b}{a}\left(x - \frac{a}{2}\right)$$

$$\therefore \quad \langle \psi_1^{(0)} | T + V_4 | \psi_1^{(0)} \rangle$$

$$= \langle \psi_1^{(0)} | T + V_1 | \psi_1^{(0)} \rangle + \underbrace{\langle \psi_1^{(0)} | \tilde{V}_4 | \psi_1^{(0)} \rangle}_{0}$$

$$= E_1$$

by symmetry (see (a))

By the variational principle,

$$E_1^* \leq E_1$$

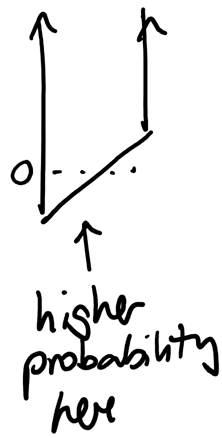
But $E_1^* \neq E_1$ because

$$\psi_1^{(0)}(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$$

cannot be the ground state wavefunction.

(e)

In the ground state, the particle should have a higher probability of being in the region of -ve V.



Compared to the square well potential, which is zero for $0 \le x \le a$, we expect the ground state energy to be smaller.

4) (a) (from W11 L1 & W12 L1)

In the interaction picture,

$$i\hbar \frac{\partial U_I}{\partial t}(t, t_0) = V_I(t) U_I(t, t_0)$$

so that

$$U_I(t, t_0) = \mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t') U_I(t', t_0)\, dt' \qquad -(1)$$

By substituting expressions for $U_I$ to $n$th order in the RHS of (1), we can obtain expressions for $U_I$ to $(n+1)$th order in $V$.

In the Schrödinger representation,

$$U(t, t_0) = \mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t} H(t') U(t', t_0)\, dt'$$

In this case, the same procedure gives expressions for $U$ in orders of $H$ but this expansion does not converge because $H = H_0 + V$ is not small in general.

(b) zeroth order $U_I(t, t_0) = \mathbb{1}$.

(b) zeroth order $U_I(t,t_0) = \mathbb{1}$.

$$U_I(t,t_0) = \mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t')\,\mathbb{1}\,dt'$$

( 1st order in $V_I$ )

Apply $U_I(t,t_0)$ to $\psi_I(t_0)$ :

$$\psi_I(t) = \psi_I(t_0) + \underbrace{\frac{1}{i\hbar}\int_{t_0}^{t} V_I(t')\,dt'\,\psi_I(t_0)}_{\text{1st order correction}}$$

$$\psi_I^{(1)}(t) = \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t')\,dt'\,\psi_I(t_0).$$

(c) Expectation value of A is

$$\langle \psi_I(t) \,|\, A \,|\, \psi_I(t)\rangle$$

interaction picture.

( zeroth order :

$$\langle \psi_I(t_0) \,|A|\, \psi_I(t_0)\rangle$$

1st order correction :

If $|\psi_I(t)\rangle = |\psi_I(t_0)\rangle + |\psi_I^{(1)}(t)\rangle$

we have

$$\langle \psi_I(t_0) + \psi_I^{(1)}(t) \,|\, A \,|\, \psi_I(t_0) + \psi_I^{(1)}(t)\rangle$$

$$= \langle \psi_I(t_0) | A | \psi_I(t_0) \rangle$$

$$\left.\begin{aligned} &+ \langle \psi_I^{(1)}(t) | A | \psi_I(t_0) \rangle \\ &+ \langle \psi_I(t_0) | A | \psi_I^{(1)}(t) \rangle \end{aligned}\right\} \text{1st order corrections (one power of } V\text{)}$$

$$+ \langle \psi_I^{(1)}(t) | A | \psi_I^{(1)}(t) \rangle$$

The 1st order correction for $\langle A \rangle$ is

$$\langle \psi_I^{(1)}(t) | A | \psi_I(t_0) \rangle + \langle \psi_I(t_0) | A | \psi_I^{(1)}(t) \rangle.$$

(d) Need to consider

$$|\psi_I(t)\rangle = |\psi_I(t_0)\rangle + |\psi_I^{(1)}(t)\rangle + \underbrace{|\psi_I^{(2)}(t)\rangle}_{\text{2nd order correction to } |\psi^I\rangle}$$

$$\langle \psi_I(t) | A | \psi_I(t) \rangle$$

$$= \langle \psi_I(t_0) + \psi_I^{(1)}(t) + \psi_I^{(2)}(t) | A | \psi_I(t_0) + \psi_I^{(1)}(t) + \psi_I^{(2)}(t) \rangle$$

2nd order correction to $\langle A \rangle$: $\langle \psi |$ ↓ $|\psi\rangle$ ↓

$\langle \psi_I^{(1)}(t) | A | \psi_I^{(1)}(t) \rangle$ (1st order 1st order)

$+ \langle \psi_I(t_0) | A | \psi_I^{(2)}(t) \rangle$ (0th order 2nd order)

$+ \langle \psi_I^{(2)}(t) | A | \psi_I(t_0) \rangle$ (2nd order 0th order)