

PC3261: Classical Mechanics II

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Lecture 2: Newton's Laws of Motion

Newton's first law and inertia

Notes

- **Newton's first law:** a particle remains at rest or in uniform motion unless acted upon a force
- **Inertia** is the *resistance* of any particle to any change in its velocity and the quantitative measure of inertia is **mass**
- A mathematical description of the motion of a particle requires the choice of a **frame of reference** – a set of coordinates in space that can be used to specify the position, velocity and acceleration of the particle at any given instant of time
- A frame of reference at which Newton's first law is valid is called an **inertial frame of reference**

Newton's second law

Notes

- **Linear momentum** of a particle is defined as the product of its mass and velocity

$$\mathbf{p}(t) \equiv m\mathbf{v}(t)$$

- **Newton's second law:** a particle acted upon a force moves in such a manner that the time rate of change of linear momentum equals the force

$$\mathbf{F}(t) = \frac{d\mathbf{p}(t)}{dt}$$

- Both Newton's first and second laws remain exactly true in special relativity with a *suitably* redefinition of linear momentum

Newton's third law

Notes

- **Newton's third law:** if two particles exert forces on each other, these forces are equal in magnitude and opposite in direction
- **Central forces** are the forces acting along the line connecting two particles
- Velocity-dependent forces are non-central and Newton's third law *may* not apply
- Newton's third law is not valid in special relativity as the concept of absolute time is abandoned

Galilean relativity

Notes

- Two inertial frames, \mathcal{O} and \mathcal{O}' , are oriented such that their spatial coordinate axes are parallel, their spatial origins are coincided when $t = t' = 0$ and \mathcal{O}' moves at *uniform velocity* \mathbf{V} with respect to \mathcal{O}

- **Galilean boost:**

$$\begin{cases} t' = t \\ \mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{V}t \end{cases}$$

- Galilean velocity transformation:

$$\mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V}$$

- Newton's laws are **Galilean invariance**

Equation of motion

Notes

- Second order ordinary differential equation: $\mathbf{r}(0) = \mathbf{r}_0, \dot{\mathbf{r}}(0) = \mathbf{v}_0$

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \quad \rightarrow \quad \begin{cases} \mathbf{r}(t) = ? \\ \dot{\mathbf{r}}(t) = ?? \end{cases}$$

- Cartesian coordinates:

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \quad \Rightarrow \quad \begin{cases} m\ddot{x}(t) = F_x(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\ddot{y}(t) = F_y(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\ddot{z}(t) = F_z(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \end{cases}$$

- Polar coordinates:

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \quad \Rightarrow \quad \begin{cases} m[\ddot{\rho}(t) - \rho(t)\dot{\phi}^2(t)] = F_\rho(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m[\rho(t)\ddot{\phi}(t) + 2\dot{\rho}(t)\dot{\phi}(t)] = F_\phi(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \end{cases}$$

First order separable ordinary differential equation

Notes

- General form:

$$\frac{dy(x)}{dx} = f(x)g(y)$$

- Implicit **general solution**: existence of an *arbitrary* constant in the solution

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

First order linear ordinary differential equation

Notes

- Standard form: $a_1(x) \neq 0$

$$a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = f(x)$$

- **Integrating factor** $\mu(x)$: integration constant is irrelevant

$$\mu(x) a_1(x) \frac{dy(x)}{dx} + \mu(x) a_0(x) y(x) \equiv \frac{d}{dx} [\mu(x) a_1(x) y(x)]$$

$$\Rightarrow \mu(x) = \frac{1}{a_1(x)} \exp \left[\int^x \frac{a_0(\xi)}{a_1(\xi)} d\xi \right]$$

- General solution: c is an arbitrary integration constant

$$\frac{d}{dx} [\mu(x) a_1(x) y(x)] = \mu(x) f(x) \quad \Rightarrow \quad y(x) = \frac{1}{\mu(x) a_1(x)} \left[\int^x \mu(\xi) f(\xi) d\xi + c \right]$$

Special case: $F_x = F_x(t)$

Notes

- Solving for $v_x(t)$: $v_x(0) = v_{x0}$

$$m\ddot{x}(t) = F_x(t) \quad \Rightarrow \quad m \frac{dv_x(t)}{dt} = F_x(t) \quad \Rightarrow \quad m \int_{v'_x=v_{x0}}^{v_x} dv'_x = \int_{t'=0}^t F_x(t') dt'$$

$$\Rightarrow v_x(t) = v_{x0} + \frac{1}{m} \int_{t'=0}^t F_x(t') dt'$$

- Solving for $x(t)$: $x(0) = x_0$

$$\frac{dx(t)}{dt} = v_x(t) \quad \Rightarrow \quad \int_{x'=x_0}^x dx' = \int_{t'=0}^t v_x(t') dt'$$

$$\Rightarrow x(t) = x_0 + v_{x0}t + \frac{1}{m} \int_{t'=0}^t \left[\int_{t''=0}^{t'} F_x(t'') dt'' \right] dt'$$

Special case: $F_x = F_x(x)$

Notes

- Solving for $v_x(x)$: $x = x(t) \leftrightarrow t = t(x)$

$$\begin{aligned}
 m\ddot{x}(t) = F_x(x) &\Rightarrow m \frac{dv_x(t)}{dt} = F_x(x) \Rightarrow m \frac{dv_x(x)}{dx} \frac{dx(t)}{dt} = F_x(x) \\
 \Rightarrow mv_x(x) \frac{dv_x(x)}{dx} = F_x(x) &\Rightarrow m \int_{v'_x=v_{x0}}^{v_x} v'_x dv'_x = \int_{x'=x_0}^x F_x(x') dx' \\
 \Rightarrow v_x^2(x) = v_{x0}^2 + \frac{2}{m} \int_{x'=x_0}^x F_x(x') dx'
 \end{aligned}$$

- Solving for $x(t)$: $x = x(t) \leftrightarrow t = t(x)$

$$\begin{aligned}
 \frac{dx(t)}{dt} = v_x(x) &\Rightarrow \int_{x'=x_0}^x \frac{dx'}{v_x(x')} = \int_{t'=0}^t dt' \\
 \Rightarrow t = \int_{x'=x_0}^x \frac{dx'}{v_x(x')} &\Rightarrow x(t)
 \end{aligned}$$

Special case: $F_x = F_x(v_x)$

Notes

- Solving for $v_x(t)$:

$$\begin{aligned}
 m\ddot{x}(t) = F_x(v_x) &\Rightarrow m \frac{dv_x(t)}{dt} = F_x(v_x) \\
 \Rightarrow m \int_{v'_x=v_{x0}}^{v_x} \frac{dv'_x}{F_x(v'_x)} = \int_{t'=0}^t dt' &\Rightarrow v_x(t) \Rightarrow x(t)
 \end{aligned}$$

- Solving for $v_x(x)$:

$$\begin{aligned}
 m\ddot{x}(t) = F_x(v_x) &\Rightarrow m \frac{dv_x(t)}{dt} = F_x(v_x) \Rightarrow m \frac{dv_x(x)}{dx} \frac{dx(t)}{dt} = F_x(v_x) \\
 \Rightarrow mv_x(x) \frac{dv_x(x)}{dx} = F_x(v_x) &\Rightarrow m \int_{v'_x=v_{x0}}^{v_x} \frac{v'_x}{F_x(v'_x)} dv'_x = \int_{x'=x_0}^x dx' \\
 \Rightarrow v_x(x) &\Rightarrow x(t)
 \end{aligned}$$

Example: Double Atwood machine

Notes

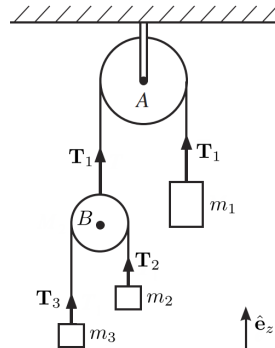
- A mass m_1 hangs at one end of a string that is led over a pulley A . The other end carries another pulley B which in turn carries a string with the masses m_2 and m_3 fixed to its ends. All pulleys and strings are assumed to be massless. Also, all strings are inextensible.

- Inextensible strings:

$$\mathbf{a}_{1A} = -\mathbf{a}_{BA}, \quad \mathbf{a}_{2B} = -\mathbf{a}_{3B}$$

- Massless strings and pulleys:

$$T_2 = T_3 = T, \quad T_1 = 2T_2 = 2T_3 = 2T$$



EXERCISE 2.1: Find the acceleration of all masses.

Example: Two masses on inclined plane

Notes

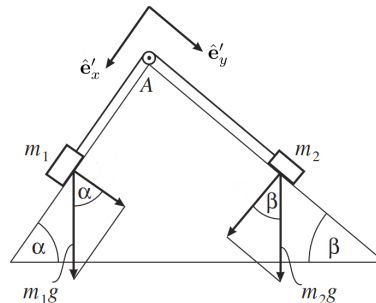
- Two masses m_1 and m_2 are lying each on one of two joined inclined planes with angles α and β with the horizontal. Both inclined planes and the horizontal make a right-angle triangle. The two masses are connected by a massless and inextensible string running over a massless and fixed pulley. The coefficients of kinetic friction of both planes are μ_k .

- Inextensible string:

$$\mathbf{a}_1 = a \hat{\mathbf{e}}'_x, \quad \mathbf{a}_2 = -a \hat{\mathbf{e}}'_y$$

- Massless string and pulley:

$$\mathbf{T}_1 = -T \hat{\mathbf{e}}'_x, \quad \mathbf{T}_2 = -T \hat{\mathbf{e}}'_y$$



EXERCISE 2.2: Find the acceleration of the masses.

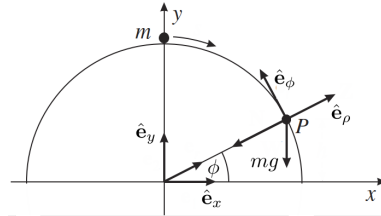
Example: Particle on a hemisphere

Notes

- A particle of mass m is located at the “North pole” of a smooth hemisphere of radius R fixed on the ground. The particle slides down the hemisphere after a small kick.

- Particle is constrained to move on the hemisphere before breaking off:

$$\rho(t) = R \Rightarrow \begin{cases} \dot{\rho}(t) = 0 \\ \ddot{\rho}(t) = 0 \end{cases}$$



EXERCISE 2.3: Find the angle and the speed at which the particle breaks off from the hemisphere.

Projectile with resistance

Notes

- Linear resistance: $\mathbf{F} = -mk\mathbf{v}$, $k \geq 0$

- Equation of motion:

$$\frac{d^2\mathbf{r}(t)}{dt^2} = -g\hat{\mathbf{e}}_z - k\mathbf{v}(t)$$

- Initial conditions:

$$\mathbf{r}(0) = (x_0, y_0, z_0), \quad \mathbf{v}(0) = (0, v_0 \cos \theta_0, v_0 \sin \theta_0)$$

- Equation of motion in Cartesian coordinates:

$$\frac{d^2x(t)}{dt^2} = -kv_x(t), \quad \frac{d^2y(t)}{dt^2} = -kv_y(t), \quad \frac{d^2z(t)}{dt^2} = -g - kv_z(t)$$

Projectile with resistance: x -direction

Notes

$$\frac{d^2x(t)}{dt^2} = -kv_x(t), \quad x(0) = x_0, \quad v_x(0) = 0$$

- Solving for $v_x(t)$:

$$\frac{dv_x(t)}{dt} = -kv_x(t) \quad \Rightarrow \quad v_x(t) = 0$$

- Solving for $x(t)$:

$$v_x(t) = 0 \quad \Rightarrow \quad \frac{dx(t)}{dt} = 0 \quad \Rightarrow \quad x(t) = x_0$$

- Motion along the x -direction is essentially stationary

Projectile with resistance: y -direction

Notes

$$\frac{d^2y(t)}{dt^2} = -kv_y(t), \quad y(0) = y_0, \quad v_y(0) = v_0 \cos \theta_0$$

- Solving:

$$v_y(t) = v_0 \cos \theta_0 e^{-kt}, \quad y(t) = y_0 + \frac{v_0 \cos \theta_0}{k} (1 - e^{-kt})$$

- Zero-friction limit: $k \rightarrow 0$

$$v_y(t) \rightarrow v_0 \cos \theta_0, \quad y(t) \rightarrow y_0 + v_0 (\cos \theta_0) t$$

EXERCISE 2.4: Obtain short-time and long-time behaviours for $v_y(t)$ and $y(t)$.

Projectile with resistance: z -direction

Notes

$$\frac{d^2 z(t)}{dt^2} = -g - k v_z(t), \quad z(0) = z_0, \quad v_z(0) = v_0 \sin \theta_0$$

- Solving:

$$v_z(t) = \left(v_0 \sin \theta_0 + \frac{g}{k} \right) e^{-kt} - \frac{g}{k}, \quad z(t) = z_0 + \frac{1}{k} \left(v_0 \sin \theta_0 + \frac{g}{k} \right) (1 - e^{-kt}) - \frac{gt}{k}$$

- Short-time behaviour:

$$v_z(t) \rightarrow v_0 \sin \theta_0 - (g + k v_0 \sin \theta_0) t, \quad z(t) \rightarrow z_0 + v_0 (\sin \theta_0) t - \frac{1}{2} (g + k v_0 \sin \theta_0) t^2$$

- Long-time behaviour:

$$v_z(t) \rightarrow -\frac{g}{k}, \quad z(t) \rightarrow z_0 + \frac{1}{k} \left(v_0 \sin \theta_0 + \frac{g}{k} \right) - \frac{gt}{k}$$

Projectile with resistance: horizontal range

Notes

- Time of the flight: $z_0 = 0$

$$z(T) = 0 \quad \Rightarrow \quad (k v_0 \sin \theta_0 + g) (1 - e^{-kT}) - k g T = 0$$

- Dimensionless resistance parameter:

$$\epsilon \equiv \frac{k v_0}{g} \quad \Rightarrow \quad (\epsilon \sin \theta_0 + 1) (1 - e^{-kT}) - kT = 0$$

- Perturbation calculation for *weak* friction: $\epsilon \ll 1$

$$T = \frac{2 v_0 \sin \theta_0}{g} [1 + c_1 \epsilon + c_2 \epsilon^2 + \mathcal{O}(\epsilon^3)]$$

- Values for c_1 and c_2 are to be determined

Projectile with resistance: horizontal range – cont'd

Notes

- Substitution, series expansion and solving:

$$T = \frac{2v_0 \sin \theta_0}{g} \left[1 - \frac{1}{3} \epsilon \sin \theta_0 + \frac{2}{9} \epsilon^2 \sin^2 \theta_0 + \mathcal{O}(\epsilon^3) \right]$$

- Horizontal range: $y_0 = 0$

$$R \equiv y(T) = \frac{v_0 \cos \theta_0}{k} (1 - e^{-kT})$$

- Substitutions and series expansion:

$$R = \frac{2v_0^2 \sin \theta_0 \cos \theta_0}{g} \left[1 - \frac{4}{3} \epsilon \sin \theta_0 + \frac{14}{9} \epsilon^2 \sin^2 \theta_0 + \mathcal{O}(\epsilon^3) \right]$$

EXERCISE 2.5: Complete the perturbation calculations to obtain the expression for R up to ϵ^2 .

Linear homogeneous ODEs

Notes

- n -order homogeneous equation with constant coefficients: $a_n \neq 0$

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y^{(1)}(x) + a_0 y^{(0)}(x) = 0$$

- Characteristics equation: n -degree polynomial of λ

$$y(x) = e^{\lambda x} \quad \Rightarrow \quad a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

- Characteristic roots give linearly independent solutions:
 - λ is a real root with no degeneracy: $e^{\lambda x}$ is the solution
 - λ is a real root with doubly degeneracy: $e^{\lambda x}$ and $x e^{\lambda x}$ are solutions
 - $\lambda = \alpha \pm i\beta$ are complex root with no degeneracy: $e^{\alpha x} \sin \beta x$ and $e^{\alpha x} \cos \beta x$ are solutions
 - $\lambda = \alpha \pm i\beta$ are complex root with doubly degeneracy: $e^{\alpha x} \sin \beta x$, $x e^{\alpha x} \sin \beta x$, $e^{\alpha x} \cos \beta x$ and $x e^{\alpha x} \cos \beta x$ are solutions

Linear homogeneous ODEs – cont'd

Notes

- **Wronskian** of a set of n functions $\{f_1(x), \dots, f_n(x)\}$:

$$W[f_1, f_2, \dots, f_n](x) \equiv \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

- General solution: $\{y_n(x)\}$ is a set of linearly independent solutions

$$W[y_1, y_2, \dots, y_n](x) \neq 0$$

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_{n-1} y_{n-1}(x) + C_n y_n(x)$$

- Constants C_i are to be determined from initial/boundary conditions

Charge in magnetic field

Notes

- A point charge of mass m and charge q is moving in a region of uniform magnetic field $\mathbf{B} = B_0 \hat{\mathbf{e}}_y$

- Equation of motion in Cartesian coordinates: $\omega \equiv qB_0/m$

$$m \frac{d^2 \mathbf{r}(t)}{dt^2} = q \mathbf{v}(t) \times \mathbf{B} \quad \Rightarrow \quad \begin{cases} \frac{d^2 x(t)}{dt^2} = -\omega \frac{dz(t)}{dt} \\ \frac{d^2 y(t)}{dt^2} = 0 \\ \frac{d^2 z(t)}{dt^2} = \omega \frac{dx(t)}{dt} \end{cases}$$

- Initial conditions:

$$\mathbf{r}(0) = (x_0, y_0, z_0), \quad \mathbf{v}(0) = (0, v_{y0}, v_{z0})$$

Charge in magnetic field: y -direction

Notes

$$\frac{d^2 y(t)}{dt^2} = 0, \quad y(0) = y_0, \quad v_y(0) = v_{y0}$$

- Solving for $v_y(t)$:

$$\frac{dv_y(t)}{dt} = 0 \quad \Rightarrow \quad v_y(t) = v_{y0}$$

- Solving for $y(t)$:

$$v_y(t) = v_{y0} \quad \Rightarrow \quad \frac{dy(t)}{dt} = v_{y0} \quad \Rightarrow \quad v_y(t) = v_{y0}t + y_0$$

- Motion along the y -direction is essentially uniform

Charge in magnetic field: x and z -directions

Notes

- Coupled differential equations:

$$\begin{cases} \frac{d^2 x(t)}{dt^2} = -\omega \frac{dz(t)}{dt} \\ \frac{d^2 z(t)}{dt^2} = \omega \frac{dx(t)}{dt} \end{cases}, \quad \begin{cases} x(0) = x_0, & v_x(0) = 0 \\ z(0) = z_0, & v_z(0) = v_{z0} \end{cases}$$

- Decoupling and solving:

$$\begin{cases} x(t) = C_1 \cos \omega t + C_2 \sin \omega t + C_0 \\ z(t) = D_1 \cos \omega t + D_2 \sin \omega t + D_0 \end{cases}$$

- Question: Are C_1 , C_2 , C_0 , D_1 , D_2 and D_0 all independent from each other?

EXERCISE 2.6: Obtain the general solutions for the coupled differential equations for $x(t)$ and $z(t)$.

Charge in magnetic field: x and z -directions – cont'd

Notes

- Eliminating dependencies:

$$\begin{cases} x(t) = C_1 \cos \omega t + C_2 \sin \omega t + C_0 \\ z(t) = -C_2 \cos \omega t + C_1 \sin \omega t + D_0 \end{cases}$$

- Imposing initial conditions for $x(t)$ and $z(t)$:

$$\begin{cases} x(0) = x_0 & \Rightarrow & C_0 = x_0 - C_1 \\ z(0) = z_0 & \Rightarrow & D_0 = z_0 \end{cases}$$

- Imposing initial conditions for $\dot{x}(t)$ and $\dot{z}(t)$:

$$\begin{cases} \dot{x}(0) = 0 & \Rightarrow & C_2 = 0 \\ \dot{z}(0) = v_{z0} & \Rightarrow & C_1 = \frac{v_{z0}}{\omega} \end{cases}$$

Charge in magnetic field: trajectory

Notes

- Position and velocity:

$$\begin{cases} x(t) = \frac{v_{z0}}{\omega} \cos \omega t + x_0 - \frac{v_{z0}}{\omega} \\ y(t) = v_{y0}t + y_0 \\ z(t) = \frac{v_{z0}}{\omega} \sin \omega t + z_0 \end{cases}, \quad \begin{cases} v_x(t) = -v_{z0} \sin \omega t \\ v_y(t) = v_{y0} \\ v_z(t) = v_{z0} \cos \omega t \end{cases}$$

- Trajectory of the point charge is a circular helix of radius mv_{z0}/qB_0 centered at $(x, z) = (x_0 - mv_{z0}/qB_0, z_0)$

$$\left[x(t) - \left(x_0 - \frac{mv_{z0}}{qB_0} \right) \right]^2 + [z(t) - z_0]^2 = \left(\frac{mv_{z0}}{qB_0} \right)^2$$