PC3130 W5L2 AY24

Thursday, 12 September 2024 11:48 am

Tut 2 due next Tues 17 Sep.

Today - continue — spin angular momentum.

Recap
- — Pauli matrices
- — Spin magnetic moment
- — Spin-orbit coupling
- — Dirac equation (not examinable)

- Tensor products $\Rightarrow$ Addition of angular momentum in W6.

Pauli matrices - Spin-$\frac{1}{2}$ systems.

$$\{\sigma_k, \sigma_j\} = 2\delta_{kj}\mathbb{1}$$ — anticommutator relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$ — commutator relations.

For all k, $\sigma_k^2 = \mathbb{1}$.

Spin magnetic moment.

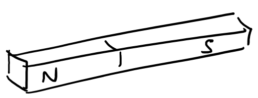
For an electron, $\vec{\mu} = -g_e\mu_B\frac{\vec{S}}{\hbar}$, $g_e \simeq 2$

$$\mu_B = \frac{e\hbar}{2m_e c} > 0 \text{ (CGS)}$$

Spin-orbit coupling (SOC)

- spin couples to lattice only through the spin-orbit coupling.
  (spatial degrees of freedom)

- magnetic anisotropy — preferred spatial orientation of spin, and along with that, spin magnetic moment.
  (from SOC)

-  $\vec{\mu}$ points along the axis of the magnet.

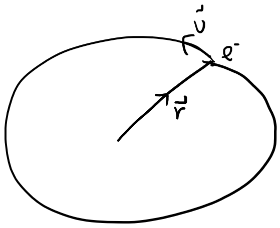
- skyrmions (slides).

Spin-orbit coupling $\boxed{\underset{\text{energy}}{U} \propto \vec{S}\cdot\vec{L}}$

idea: $\vec{S} \Rightarrow \vec{\mu}$ magnetic moment

$\vec{L} \rightarrow \vec{B}$ internal magnetic field felt by the electron.

$U \propto -\vec{\mu}\cdot\vec{B}$.



$\vec{v}$ & $\vec{E}$ are given in the frame of the nucleus.

$\vec{E} = f(r)\,\vec{r}$

$\vec{E} = -\vec{\nabla}\phi$

$= -\frac{\partial\phi}{\partial r}\hat{r} + 0$

$= -\frac{\partial\phi}{\partial r}\frac{1}{r}\vec{r}$

$f(r) = -\frac{\partial\phi}{\partial r}\frac{1}{r}$

In an inertial frame (non-accelerating) moving at velocity $\vec{v}$, the particle feels a $\vec{B}$ field

$$\vec{B} = -\frac{\vec{v}}{c}\times\vec{E} \quad \text{(from special relativity)}$$

(CGS)

Here, in the frame of the electron, which is precessing around the nucleus, we need an additional factor of $\frac{1}{2}$, known as the Thomas precession factor; the electron feels a B field

$$\vec{B} = -\frac{1}{2}\frac{\vec{v}}{c}\times\vec{E} \quad \text{(not tested)}$$

(CGS)

Energy
$$U = -\vec{\mu}\cdot\vec{B}$$
$$= \frac{1}{2}\vec{\mu}\cdot\left(\frac{\vec{v}}{c}\times f(r)\,\vec{r}\right)$$
$$= -\frac{1}{2}\frac{f(r)}{c}\vec{\mu}\cdot(\vec{r}\times\vec{v})$$
$$= -\frac{1}{2}\frac{f(r)}{cm_e}\vec{\mu}\cdot(\vec{r}\times\vec{p})$$
$$= -\frac{1}{2}\frac{f(r)}{cm_e}\vec{\mu}\cdot\vec{L}$$

Spin μ
$$= -\frac{1}{2}\frac{f(r)}{cm_e}\left(-g_e\frac{e\hbar}{2m_ec}\frac{1}{\hbar}\vec{S}\right)\cdot\vec{L}$$
$$= \frac{e\,f(r)\,g_e}{4(m_ec)^2}\vec{S}\cdot\vec{L}$$

$f(r) = -\frac{d\phi}{dr}\frac{1}{r}$
$$= -\frac{g_e\,e}{4(m_ec)^2}\frac{1}{r}\frac{d\phi}{dr}\vec{S}\cdot\vec{L}$$

Spin-orbit coupling.

Tensor products

- Addition of angular momentum.

Eg Addition of angular momentum.

$\vec{L} + \vec{S}$?

$l = 1$ $\quad$ $s = \frac{1}{2}$

dimension $2l+1 = 3$ $\quad$ dimension $2s+1 = 2$

Spaces that $\vec{L}$ & $\vec{S}$ operate on have different dimensionalities.

eventually see: $\underset{(3\times3)}{\vec{L}} \otimes \mathbb{1}_{2\times2} + \mathbb{1}_{3\times3} \otimes \underset{(2\times2)}{\vec{S}}$

Tensor product space $\mathcal{E}$

Given two vector spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, we can define another vector space $\mathcal{E}$ which is called the tensor product space of $\mathcal{E}_1$ and $\mathcal{E}_2$.

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$$

Suppose $\mathcal{E}_1$ is spanned by $\{|u_i(1)\rangle\}$ (o.n. basis) and $\mathcal{E}_2$ is spanned by $\{|v_j(2)\rangle\}$ (o.n. basis).

Then $\mathcal{E}$ is spanned by $\{|u_i(1)\rangle \otimes |v_j(2)\rangle\}$ (o.n. basis)

Any vector in $\mathcal{E}$ can be written as $|\psi\rangle = \sum_{i,j} c_{ij} |u_i(1)\rangle \otimes |v_j(2)\rangle$.

(Definition of $\mathcal{E}$)

Properties of vectors in $\mathcal{E}$: For $|\varphi(1)\rangle \in \mathcal{E}_1$, $|\chi(2)\rangle \in \mathcal{E}_2$,

Linearity:
$$(\lambda|\varphi(1)\rangle) \otimes |\chi(2)\rangle = \lambda(|\varphi(1)\rangle \otimes |\chi(2)\rangle)$$
$$|\varphi(1)\rangle \otimes (\mu|\chi(2)\rangle) = \mu(|\varphi(1)\rangle \otimes |\chi(2)\rangle)$$
$$|\varphi(1)\rangle \otimes (|\chi_1(2)\rangle + |\chi_2(2)\rangle) = |\varphi(1)\rangle \otimes |\chi_1(2)\rangle + |\varphi(1)\rangle \otimes |\chi_2(2)\rangle$$
$$(|\varphi_1(1)\rangle + |\varphi_2(1)\rangle) \otimes |\chi(2)\rangle = |\varphi_1(1)\rangle \otimes |\chi(2)\rangle + |\varphi_2(1)\rangle \otimes |\chi(2)\rangle$$

For every pair of vectors $|\varphi(1)\rangle \in \mathcal{E}_1$ and $|\chi(2)\rangle \in \mathcal{E}_2$,

$$\exists \text{ a vector } |\varphi(1)\rangle \otimes |\chi(2)\rangle \in \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2. \quad \text{— (1)}$$

But not all vectors in $\mathcal{E}$ can be written as $|\varphi(1)\rangle \otimes |\chi(2)\rangle$.

Those that cannot be written this way are called entangled.

Eg. $\mathcal{E}_1$ is spanned by $\{ |0\rangle_1, |1\rangle_1 \}$.

$\mathcal{E}_2$ —— " —— $\{ |0\rangle_2, |1\rangle_2 \}$.

<u>Claim</u>: $|\phi\rangle = \frac{1}{\sqrt{2}} \left(|1\rangle_1 \otimes |0\rangle_2 - |0\rangle_1 \otimes |1\rangle_2 \right)$ cannot be written as $|\psi(1)\rangle \otimes |\chi(2)\rangle$

$\equiv \frac{1}{\sqrt{2}} \left(|1,0\rangle - |0,1\rangle \right)$ abbreviation.

$|1,0\rangle = |1\rangle_1 \otimes |0\rangle_2$

$|0,1\rangle = |0\rangle_1 \otimes |1\rangle_2$

But $|\phi\rangle \in \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$

where $|\psi(1)\rangle = a_1 |0\rangle_1 + b_1 |1\rangle_1$,

and $|\chi(2)\rangle = a_2 |0\rangle_2 + b_2 |1\rangle_2$. —(2).

<u>Let's verify the claim</u>.

Suppose $|\phi\rangle = |\psi(1)\rangle \otimes |\chi(2)\rangle$ where $|\psi(1)\rangle$, $|\chi(2)\rangle$ are given in (2)

Then $|\phi\rangle = (a_1 |0\rangle_1 + b_1 |1\rangle_1) \otimes (a_2 |0\rangle_2 + b_2 |1\rangle_2)$

$= \underline{a_1 a_2 |0,0\rangle} + a_1 b_2 |0,1\rangle + b_1 a_2 |1,0\rangle + \underline{b_1 b_2 |1,1\rangle}$

LHS $= \frac{1}{\sqrt{2}} \left(|1,0\rangle - |0,1\rangle \right)$

$\Rightarrow$ We need $a_1 a_2 = 0$ <u>and</u> $b_1 b_2 = 0$

$\Rightarrow$ ($a_1 = 0$ <u>or</u> $a_2 = 0$), <u>and</u> ($b_1 = 0$ <u>or</u> $b_2 = 0$).

Either $a_1 b_2 = 0$ or $a_2 b_1 = 0$ or both.

contradiction.

So $|\phi\rangle = \frac{1}{\sqrt{2}} \left(|1,0\rangle - |0,1\rangle \right)$ is an entangled state.

— basis of quantum teleportation, cryptography, etc.

(Ballentine).

textbook.

Eg. $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\overset{1}{0}\,\overset{2}{0}\,\overset{3}{0}\rangle + |\overset{1}{0}\,\overset{2}{1}\,\overset{3}{0}\rangle \right)$ $\equiv$ $\underset{\mathcal{E}_1}{| \;\rangle} \otimes \underset{\mathcal{E}_2}{| \;\rangle} \otimes \underset{\mathcal{E}_3}{| \;\rangle}$

$= \frac{1}{\sqrt{2}} \left(|0\rangle_1 \otimes \left(|0\rangle_2 + |1\rangle_2 \right) \otimes |0\rangle_3 \right)$

— not entangled.

$|\varphi(1)\rangle \otimes \left(|\chi_1(2)\rangle + |\chi_2(2)\rangle \right)$

$= |\varphi_1(1)\rangle \otimes |\chi_1(2)\rangle + |\varphi_1(1)\rangle \otimes |\chi_2(2)\rangle$

Eg. $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|1\,0\,0\rangle + |1\,0\,1\rangle \right)$

$= \frac{1}{\sqrt{2}} \left(|1\rangle_1 \otimes \left(|0\rangle_2 + |1\rangle_2 \right) \right)$ — not entangled.

Another operation for vectors:

Scalar products in $\mathcal{E}$.

$$\langle \varphi_1' \chi_2' | \varphi_1 \chi_2 \rangle = \langle \varphi_1' | \varphi_1 \rangle \langle \chi_2' | \chi_2 \rangle$$

i.e. $\left(\langle \varphi_1' | \otimes \langle \chi_2' | \right) \left(|\varphi_1\rangle \otimes |\chi_2\rangle \right) = \langle \varphi_1' | \varphi_1 \rangle \langle \chi_2' | \chi_2 \rangle$

Now we cover operators on tensor product space.

$\hat{A}_1$ operator on $\mathcal{E}_1$, $\hat{B}_2$ operator on $\mathcal{E}_2$

Define tensor product of operators:

$(\hat{A}_1 \otimes \hat{B}_2)$ acting on $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$.

$$(\hat{A}_1 \otimes \hat{B}_2) \underbrace{(|\varphi_1\rangle \otimes |\chi_2\rangle)}_{\in \, \mathcal{E}_1 \otimes \mathcal{E}_2} = \underbrace{(\hat{A}_1 |\varphi_1\rangle)}_{\in \, \mathcal{E}_1} \otimes \underbrace{(\hat{B}_2 |\chi_2\rangle)}_{\in \, \mathcal{E}_2} \in \mathcal{E}_1 \otimes \mathcal{E}_2$$

Consider $\mathcal{E}_1$ & $\mathcal{E}_2$ are each of dimension 2.

If $\hat{A}_1 \longmapsto A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\hat{B}_2 \longrightarrow B_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

Basis for $\mathcal{E}_1 = \{ |e_1\rangle, |e_2\rangle \}$ Basis for $\mathcal{E}_2 = \{ |f_1\rangle, |f_2\rangle \}$.

Eg. $a_{11} = \langle e_1 | \hat{A}_1 | e_1 \rangle$

$a_{12} = \langle e_1 | \hat{A}_1 | e_2 \rangle$

Convention in linear algebra.

$$\underbrace{A_1 \otimes B_2}_{\text{matrices}} = \begin{array}{c|c} & \begin{array}{cccc} |g_1\rangle & |g_2\rangle & |g_3\rangle & |g_4\rangle \end{array} \\ \begin{array}{c} \langle g_1| \\ \langle g_2| \\ \langle g_3| \\ \langle g_4| \end{array} & \left(\begin{array}{c|c} a_{11} B_2 & a_{12} B_2 \\ \hline a_{21} B_2 & a_{22} B_2 \end{array} \right) \end{array}$$

$$a_{12} B_2 = \begin{pmatrix} a_{12} b_{11} & a_{12} b_{12} \\ a_{12} b_{21} & a_{12} b_{22} \end{pmatrix}$$

4x4 matrix.

Basis for $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is $\{ |g_1\rangle = |e_1\rangle \otimes |f_1\rangle,$

$|g_2\rangle = |e_1\rangle \otimes |f_2\rangle,$

$|g_3\rangle = |e_2\rangle \otimes |f_1\rangle$

$|g_4\rangle = |e_2\rangle \otimes |f_2\rangle \}$

Properties of tensor products of operators.

- linear: $(\alpha_1 \hat{A}_1 + \alpha_2 \hat{A}_2) \otimes \hat{B} = \alpha_1 \hat{A}_1 \otimes \hat{B} + \alpha_2 \hat{A}_2 \otimes \hat{B}$.

- linear: $\underbrace{(\alpha_1 \hat{A}_1 + \alpha_2 \hat{A}_2)}_{\text{acts on } \mathcal{E}_1} \otimes \underset{\substack{\uparrow \\ \text{acts on } \mathcal{E}_2}}{\hat{B}} = \alpha_1 \hat{A}_1 \otimes \hat{B} + \alpha_2 \hat{A}_2 \otimes \hat{B}$.

- $(\hat{A}' \otimes \hat{B}')(\hat{A} \otimes \hat{B}) \underset{\substack{\cap \\ \mathcal{E}_1 \otimes \mathcal{E}_2}}{|\Psi\rangle} = (\hat{A}'\hat{A}) \otimes (\hat{B}'\hat{B}) |\Psi\rangle$

- Trace. $Tr_{\mathcal{E}}(\hat{A} \otimes \hat{B}) = Tr_{\mathcal{E}_1}(\hat{A})\, Tr_{\mathcal{E}_2}(\hat{B})$

- Adjoint $(\underset{\substack{\downarrow \\ \text{acts on} \\ \mathcal{E}_1}}{\hat{A}} \otimes \underset{\substack{\downarrow \\ \text{acts on} \\ \mathcal{E}_2}}{\hat{B}})^{+} |\Psi\rangle = \underset{\substack{\downarrow \\ \text{acts on} \\ \mathcal{E}_1}}{\hat{A}^{+}} \otimes \underset{\substack{\downarrow \\ \text{acts on } \mathcal{E}_2.}}{\hat{B}^{+}} |\Psi\rangle$

Contrast with:
Single-particle Hilbert space $\mathcal{E}$
where $\hat{Q}$ acts on $\mathcal{E}$. and $\hat{R}$ also acts on $\mathcal{E}$.

$$(\hat{Q}\hat{R})^{+} |\varphi\rangle = (\hat{R}^{+} \hat{Q}^{+}) |\varphi\rangle$$

$\mathcal{E}_1$ dimension n_1

$\mathcal{E}_2$ " n_2

$\mathcal{E}_1 \otimes \mathcal{E}_2$ " $n_1 n_2$.

<u>Example</u> Spin-orbit coupling.

$$U = \lambda \underset{\substack{\nearrow \\ \text{acts on } \mathcal{E}_1}}{\vec{S}} \cdot \underset{\substack{\nwarrow \\ \text{acts on } \mathcal{E}_2.}}{\vec{L}}$$

$\mathcal{E}_1$ spanned by $\{ \underset{\substack{\shortparallel \\ |e_1\rangle}}{|+\rangle_z}, \underset{\substack{\shortparallel \\ |e_2\rangle}}{|-\rangle_z} \}$. Spin-$\frac{1}{2}$ system.

Let's say $l = 1$.

$\mathcal{E}_2$ spanned by angular momentum eigenstates for l:

$$\{ \underset{\substack{\shortparallel \\ |f_1\rangle}}{|l=1, m=1\rangle}, \underset{\substack{\shortparallel \\ |f_2\rangle}}{|l=1, m=0\rangle}, \underset{\substack{\shortparallel \\ |f_3\rangle}}{|l=1, m=-1\rangle} \}$$

$$\vec{S} \cdot \vec{L} = \frac{\hbar}{2} (\sigma_x L_x + \sigma_y L_y + \sigma_z L_z)$$

$\vec{S} = \frac{\hbar}{2}(\sigma_x \hat{e}_x + \sigma_y \hat{e}_y + \sigma_z \hat{e}_z)$

$S_x = \frac{\hbar}{2} \sigma_x$

What does this mean?

Eg. $\underbrace{\sigma_x}_{\text{acts on } \mathcal{E}_1} \underbrace{L_x}_{\text{acts on } \mathcal{E}_2} = \sigma_x \otimes L_x$

$$(\sigma_x \otimes L_x)(|+\rangle_z \otimes |f_1\rangle) = (\sigma_x |+\rangle_z) \otimes (L_x |f_1\rangle)$$

Dimension of $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is $2 \times 3 = 6$