PC3261: Classical Mechanics II

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Semester II, 2024/25

Latest update: January 17, 2025 11:11am

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Lecture 2: Newton's Laws of Motion

Newton's first law and inertia

- **Newton's first law**: a particle remains at rest or in uniform motion unless acted upon a force

- **Inertia** is the *resistance* of any particle to any change in its velocity and the quantitative measure of inertia is **mass**

- A mathematical description of the motion of a particle requires the choice of a **frame of reference –** a set of coordinates in space that can be used to specify the position, velocity and acceleration of the particle at any given instant of time

- A frame of reference at which Newton's first law is valid is called an **inertial frame of reference**

Newton's second law

- **Linear momentum** of a particle is defined as the product of its mass and velocity

$$\mathbf{p}(t) \equiv m\mathbf{v}(t)$$

- **Newton's second law**: a particle acted upon a force moves in such a manner that the time rate of change of linear momentum equals the force

$$\mathbf{F}(t) = \frac{\mathrm{d}\mathbf{p}(t)}{\mathrm{d}t}$$

- Both Newton's first and second laws remain exactly true in special relativity with a *suitably* redefinition of linear momentum

Newton's third law

- **Newton's third law**: if two particles exert forces on each other, these forces are equal in magnitude and opposite in direction

- **Central forces** are the forces acting along the line connecting two particles

- Velocity-dependent forces are non-central and Newton's third law *may* not apply

- Newton's third law is not valid in special relativity as the concept of absolute time is abandoned

Galilean relativity

- Two inertial frames, $\mathcal{O}$ and $\mathcal{O}'$, are oriented such that their spatial coordinate axes are parallel, their spatial origins are coincided when $t = t' = 0$ and $\mathcal{O}'$ moves at *uniform velocity* $\mathbf{V}$ with respect to $\mathcal{O}$

- **Galilean boost**:

$$\begin{cases} t' = t \\ \mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{V}t \end{cases}$$

- Galilean velocity transformation:

$$\mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V}$$

- Newton's laws are **Galilean invariance**

Equation of motion

- Second order ordinary differential equation: $\mathbf{r}(0) = \mathbf{r}_0$, $\dot{\mathbf{r}}(0) = \mathbf{v}_0$

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \qquad \rightarrow \qquad \begin{cases} \mathbf{r}(t) = ? \\ \dot{\mathbf{r}}(t) = ?? \end{cases}$$

- Cartesian coordinates:

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \qquad \Rightarrow \qquad \begin{cases} m\ddot{x}(t) = F_x(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\ddot{y}(t) = F_y(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\ddot{z}(t) = F_z(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \end{cases}$$

- Polar coordinates:

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \quad \Rightarrow \quad \begin{cases} m\left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right] = F_\rho(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right] = F_\phi(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \end{cases}$$

First order separable ordinary differential equation

- General form:

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = f(x)\, g(y)$$

- Implicit **general solution**: existence of an *arbitrary* constant in the solution

$$\int \frac{1}{g(y)}\, \mathrm{d}y = \int f(x)\, \mathrm{d}x$$

First order linear ordinary differential equation

- Standard form: $a_1(x) \neq 0$

$$a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + a_0(x)\,y(x) = f(x)$$

- **Integrating factor** $\mu(x)$: integration constant is irrelevant

$$\mu(x)\,a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + \mu(x)\,a_0(x)\,y(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\,a_1(x)\,y(x)\right]$$

$$\Rightarrow \quad \mu(x) = \frac{1}{a_1(x)}\exp\left[\int^x \frac{a_0(\xi)}{a_1(\xi)}\,\mathrm{d}\xi\right]$$

- General solution: c is an arbitrary integration constant

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\,a_1(x)\,y(x)\right] = \mu(x)\,f(x) \quad \Rightarrow \quad y(x) = \frac{1}{\mu(x)\,a_1(x)}\left[\int^x \mu(\xi)\,f(\xi)\,\mathrm{d}\xi + c\right]$$

Special case: $F_x = F_x(t)$

- Solving for $v_x(t)$: $v_x(0) = v_{x0}$

$$m\ddot{x}(t) = F_x(t) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(t) \quad \Rightarrow \quad m\int_{v_x'=v_{x0}}^{v_x} \mathrm{d}v_x' = \int_{t'=0}^{t} F_x(t')\,\mathrm{d}t'$$

$$\Rightarrow \quad v_x(t) = v_{x0} + \frac{1}{m}\int_{t'=0}^{t} F_x(t')\,\mathrm{d}t'$$

- Solving for $x(t)$: $x(0) = x_0$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v_x(t) \quad \Rightarrow \quad \int_{x'=x_0}^{x} \mathrm{d}x' = \int_{t'=0}^{t} v_x(t')\,\mathrm{d}t'$$

$$\Rightarrow \quad x(t) = x_0 + v_{x0}t + \frac{1}{m}\int_{t'=0}^{t}\left[\int_{t''=0}^{t'} F_x(t'')\,\mathrm{d}t''\right]\mathrm{d}t'$$

Special case: $F_x = F_x(x)$

- Solving for $v_x(x)$: $x = x(t) \leftrightarrow t = t(x)$

$$m\ddot{x}(t) = F_x(x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F_x(x)$$

$$\Rightarrow \quad mv_x(x)\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x} = F_x(x) \quad \Rightarrow \quad m\int_{v_x'=v_{x0}}^{v_x} v_x'\,\mathrm{d}v_x' = \int_{x'=x_0}^{x} F_x(x')\,\mathrm{d}x'$$

$$\Rightarrow \quad v_x^2(x) = v_{x0}^2 + \frac{2}{m}\int_{x'=x_0}^{x} F_x(x')\,\mathrm{d}x'$$

- Solving for $x(t)$: $x = x(t) \leftrightarrow t = t(x)$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v_x(x) \quad \Rightarrow \quad \int_{x'=x_0}^{x} \frac{\mathrm{d}x'}{v_x(x')} = \int_{t'=0}^{t} \mathrm{d}t'$$

$$\Rightarrow \quad t = \int_{x'=x_0}^{x} \frac{\mathrm{d}x'}{v_x(x')} \quad \Rightarrow \quad x(t)$$

Special case: $F_x = F_x(v_x)$

- Solving for $v_x(t)$:

$$m\ddot{x}(t) = F_x(v_x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(v_x)$$

$$\Rightarrow \quad m\int_{v_x'=v_{x0}}^{v_x} \frac{\mathrm{d}v_x'}{F_x(v_x')} = \int_{t'=0}^{t} \mathrm{d}t' \quad \Rightarrow \quad v_x(t) \quad \Rightarrow \quad x(t)$$

- Solving for $v_x(x)$:

$$m\ddot{x}(t) = F_x(v_x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(v_x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F_x(v_x)$$

$$\Rightarrow \quad m v_x(x)\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x} = F_x(v_x) \quad \Rightarrow \quad m\int_{v_x'=v_{x0}}^{v_x} \frac{v_x'}{F_x(v_x')}\,\mathrm{d}v_x' = \int_{x'=x_0}^{x} \mathrm{d}x'$$

$$\Rightarrow \quad v_x(x) \quad \Rightarrow \quad x(t)$$

Example: Double Atwood machine

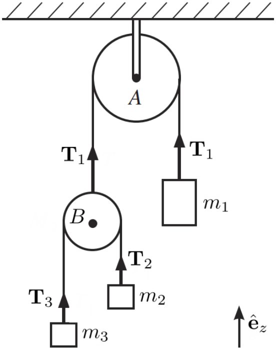
- A mass m_1 hangs at one end of a string that is led over a pulley A. The other end carries another pulley B which in tern carries a string with the masses m_2 and m_3 fixed to its ends. All pulleys and strings are assumed to be massless. Also, all strings are inextensible.

- Inextensible strings:

$$\mathbf{a}_{1A} = -\mathbf{a}_{BA}\,, \qquad \mathbf{a}_{2B} = -\mathbf{a}_{3B}$$

- Massless strings and pulleys:

$$T_2 = T_3 = T\,, \qquad T_1 = 2T_2 = 2T_3 = 2T$$



EXERCISE 2.1: Find the acceleration of all masses.

Example: Two masses on inclined plane

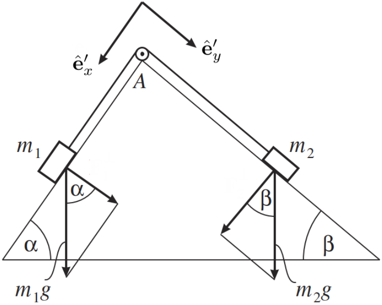
- Two masses m_1 and m_2 are lying each on one of two joined inclined planes with angles α and β with the horizontal. Both inclined planes and the horizontal make a right-angle triangle. The two masses are connected by a massless and inextensible string running over a massless and fixed pulley. The coefficients of kinetic friction of both planes are μ_k.

- Inextensible string:

$$\mathbf{a}_1 = a\,\hat{\mathbf{e}}'_x\,, \qquad \mathbf{a}_2 = -a\,\hat{\mathbf{e}}'_y$$

- Massless string and pulley:

$$\mathbf{T}_1 = -T\,\hat{\mathbf{e}}'_x\,, \qquad \mathbf{T}_2 = -T\,\hat{\mathbf{e}}'_y$$



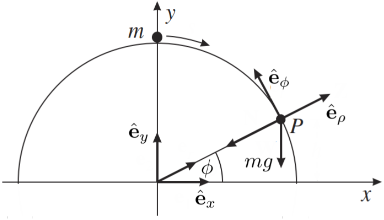
EXERCISE 2.2: Find the acceleration of the masses.

Example: Particle on a hemisphere

- A particle of mass m is located at the "North pole" of a smooth hemisphere of radius R fixed on the ground. The particle slides down the hemisphere after a small kick.

- Particle is constrained to move on the hemisphere before breaking off:

$$\rho(t) = R \quad \Rightarrow \quad \begin{cases} \dot{\rho}(t) = 0 \\ \ddot{\rho}(t) = 0 \end{cases}$$



EXERCISE 2.3: Find the angle and the speed at which the particle breaks off from the hemisphere.

Projectile with resistance

- Linear resistance: $\mathbf{F} = -mk\mathbf{v}$, $k \geq 0$

- Equation of motion:

$$\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = -g\,\hat{\mathbf{e}}_z - k\mathbf{v}(t)$$

- Initial conditions:

$$\mathbf{r}(0) = (x_0, y_0, z_0)\,, \qquad \mathbf{v}(0) = (0, v_0\cos\theta_0, v_0\sin\theta_0)$$

- Equation of motion in Cartesian coordinates:

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -kv_x(t)\,, \qquad \frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} = -kv_y(t)\,, \qquad \frac{\mathrm{d}^2z(t)}{\mathrm{d}t^2} = -g - kv_z(t)$$

Projectile with resistance: x-direction

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -k v_x(t)\,, \qquad x(0) = x_0\,, \qquad v_x(0) = 0$$

- Solving for $v_x(t)$:

$$\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = -k v_x(t) \qquad \Rightarrow \qquad v_x(t) = 0$$

- Solving for $x(t)$:

$$v_x(t) = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}x(t)}{\mathrm{d}t} = 0 \qquad \Rightarrow \qquad x(t) = x_0$$

- Motion along the x-direction is essentially stationary

Projectile with resistance: y-direction

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = -k v_y(t)\,, \qquad y(0) = y_0\,, \qquad v_y(0) = v_0 \cos\theta_0$$

- Solving:

$$v_y(t) = v_0 \cos\theta_0\, \mathrm{e}^{-kt}\,, \qquad y(t) = y_0 + \frac{v_0 \cos\theta_0}{k} \left(1 - \mathrm{e}^{-kt}\right)$$

- Zero-friction limit: $k \to 0$

$$v_y(t) \to v_0 \cos\theta_0\,, \qquad y(t) \to y_0 + v_0 \left(\cos\theta_0\right) t$$

EXERCISE 2.4: Obtain short-time and long-time behaviours for $v_y(t)$ and $y(t)$.

Projectile with resistance: z-direction

$$\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = -g - k v_z(t)\,, \qquad z(0) = z_0\,, \qquad v_z(0) = v_0 \sin\theta_0$$

- Solving:

$$v_z(t) = \left(v_0 \sin\theta_0 + \frac{g}{k}\right) \mathrm{e}^{-kt} - \frac{g}{k}\,, \qquad z(t) = z_0 + \frac{1}{k}\left(v_0 \sin\theta_0 + \frac{g}{k}\right)\left(1 - \mathrm{e}^{-kt}\right) - \frac{gt}{k}$$

- Short-time behaviour:

$$v_z(t) \to v_0 \sin\theta_0 - (g + k v_0 \sin\theta_0)\, t\,, \quad z(t) \to z_0 + v_0 (\sin\theta_0)\, t - \frac{1}{2}(g + k v_0 \sin\theta_0)\, t^2$$

- Long-time behaviour:

$$v_z(t) \to -\frac{g}{k}\,, \qquad z(t) \to z_0 + \frac{1}{k}\left(v_0 \sin\theta_0 + \frac{g}{k}\right) - \frac{gt}{k}$$

Projectile with resistance: horizontal range

- Time of the flight: $z_0 = 0$

$$z(T) = 0 \quad \Rightarrow \quad (kv_0 \sin\theta_0 + g)\left(1 - \mathrm{e}^{-kT}\right) - kgT = 0$$

- Dimensionless resistance parameter:

$$\epsilon \equiv \frac{kv_0}{g} \quad \Rightarrow \quad (\epsilon \sin\theta_0 + 1)\left(1 - \mathrm{e}^{-kT}\right) - kT = 0$$

- Perturbation calculation for *weak* friction: $\epsilon \ll 1$

$$T = \frac{2v_0 \sin\theta_0}{g}\left[1 + c_1\epsilon + c_2\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)\right]$$

- Values for c_1 and c_2 are to be determined

Projectile with resistance: horizontal range – cont'd

- Substitution, series expansion and solving:

$$T = \frac{2v_0 \sin\theta_0}{g}\left[1 - \frac{1}{3}\,\epsilon\sin\theta_0 + \frac{2}{9}\,\epsilon^2\sin^2\theta_0 + \mathcal{O}\left(\epsilon^3\right)\right]$$

- Horizontal range: $y_0 = 0$

$$R \equiv y(T) = \frac{v_0\cos\theta_0}{k}\left(1 - \mathrm{e}^{-kT}\right)$$

- Substitutions and series expansion:

$$R = \frac{2v_0^2\sin\theta_0\cos\theta_0}{g}\left[1 - \frac{4}{3}\,\epsilon\sin\theta_0 + \frac{14}{9}\,\epsilon^2\sin^2\theta_0 + \mathcal{O}\left(\epsilon^3\right)\right]$$

EXERCISE 2.5: Complete the perturbation calculations to obtain the expression for R up to ϵ^2.

Linear homogeneous ODEs

- n-order homogeneous equation with constant coefficients: $a_n \neq 0$

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1\, y^{(1)}(x) + a_0 y^{(0)}(x) = 0$$

- Characteristics equation: n-degree polynomial of λ

$$y(x) = \mathrm{e}^{\lambda x} \quad \Rightarrow \quad a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

- Characteristic roots give linearly independent solutions:
  - λ is a real root with no degeneracy: $\mathrm{e}^{\lambda x}$ is the solution
  - λ is a real root with doubly degeneracy: $\mathrm{e}^{\lambda x}$ and $x\mathrm{e}^{\lambda x}$ are solutions
  - $\lambda = \alpha \pm \mathrm{i}\beta$ are complex root with no degeneracy: $\mathrm{e}^{\alpha x}\sin\beta x$ and $\mathrm{e}^{\alpha x}\cos\beta x$ are solutions
  - $\lambda = \alpha \pm \mathrm{i}\beta$ are complex root with doubly degeneracy: $\mathrm{e}^{\alpha x}\sin\beta x$, $x\mathrm{e}^{\alpha x}\sin\beta x$, $\mathrm{e}^{\alpha x}\cos\beta x$ and $x\mathrm{e}^{\alpha x}\cos\beta x$ are solutions

Linear homogeneous ODEs – cont'd

- **Wronskian** of a set of n functions $\{f_1(x), \cdots, f_n(x)\}$:

$$W[f_1, f_2, \cdots, f_n](x) \equiv \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

- General solution: $\{y_n(x)\}$ is a set of linearly independent solutions

$$W[y_1, y_2, \cdots, y_n](x) \neq 0$$

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_{n-1} y_{n-1}(x) + C_n y_n(x)$$

- Constants C_i are to be determined from initial/boundary conditions

Charge in magnetic field

- A point charge of mass m and charge q is moving in a region of uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{e}}_y$

- Equation of motion in Cartesian coordinates: $\omega \equiv qB_0/m$

$$m\,\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = q\,\mathbf{v}(t)\times\mathbf{B} \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\omega\,\dfrac{\mathrm{d}z(t)}{\mathrm{d}t} \\[2ex] \dfrac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} = 0 \\[2ex] \dfrac{\mathrm{d}^2z(t)}{\mathrm{d}t^2} = \omega\,\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \end{cases}$$

- Initial conditions:

$$\mathbf{r}(0) = (x_0, y_0, z_0)\ , \qquad \mathbf{v}(0) = (0, v_{y0}, v_{z0})$$

Charge in magnetic field: y-direction

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = 0\,, \qquad y(0) = y_0\,, \qquad v_y(0) = v_{y0}$$

- Solving for $v_y(t)$:

$$\frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = 0 \qquad \Rightarrow \qquad v_y(t) = v_{y0}$$

- Solving for $y(t)$:

$$v_y(t) = v_{y0} \qquad \Rightarrow \qquad \frac{\mathrm{d}y(t)}{\mathrm{d}t} = v_{y0} \qquad \Rightarrow \qquad v_y(t) = v_{y0}t + y_0$$

- Motion along the y-direction is essentially uniform

Charge in magnetic field: x and z-directions

- Coupled differential equations:

$$\begin{cases} \dfrac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\omega \, \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} \\[2ex] \dfrac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = \omega \, \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \end{cases}, \qquad \begin{cases} x(0) = x_0 \,, \quad v_x(0) = 0 \\[1ex] z(0) = z_0 \,, \quad v_z(0) = v_{z0} \end{cases}$$

- Decoupling and solving:

$$\begin{cases} x(t) = C_1 \cos \omega t + C_2 \sin \omega t + C_0 \\[1ex] z(t) = D_1 \cos \omega t + D_2 \sin \omega t + D_0 \end{cases}$$

- Question: Are C_1, C_2, C_0, D_1, D_2 and D_0 all independent from each other?

EXERCISE 2.6: Obtain the general solutions for the coupled differential equations for $x(t)$ and $z(t)$.

Charge in magnetic field: x and z-directions – cont'd

- Eliminating dependencies:

$$\begin{cases} x(t) = C_1 \cos\omega t + C_2 \sin\omega t + C_0 \\ z(t) = -C_2 \cos\omega t + C_1 \sin\omega t + D_0 \end{cases}$$

- Imposing initial conditions for $x(t)$ and $z(t)$:

$$\begin{cases} x(0) = x_0 \quad \Rightarrow \quad C_0 = x_0 - C_1 \\ z(0) = z_0 \quad \Rightarrow \quad D_0 = z_0 \end{cases}$$

- Imposing initial conditions for $\dot{x}(t)$ and $\dot{z}(t)$:

$$\begin{cases} \dot{x}(0) = 0 \quad \Rightarrow \quad C_2 = 0 \\ \dot{z}(0) = v_{z0} \quad \Rightarrow \quad C_1 = \dfrac{v_{z0}}{\omega} \end{cases}$$

Charge in magnetic field: trajectory

- Position and velocity:

$$\begin{cases} x(t) = \dfrac{v_{z0}}{\omega}\cos\omega t + x_0 - \dfrac{v_{z0}}{\omega} \\ y(t) = v_{y0}t + y_0 \\ z(t) = \dfrac{v_{z0}}{\omega}\sin\omega t + z_0 \end{cases}, \qquad \begin{cases} v_x(t) = -v_{z0}\sin\omega t \\ v_y(t) = v_{y0} \\ v_z(t) = v_{z0}\cos\omega t \end{cases}$$

- Trajectory of the point charge is a circular helix of radius mv_{z0}/qB_0 centered at $(x,z) = (x_0 - mv_{z0}/qB_0, z_0)$

$$\left[x(t) - \left(x_0 - \frac{mv_{z0}}{qB_0}\right)\right]^2 + [z(t) - z_0]^2 = \left(\frac{mv_{z0}}{qB_0}\right)^2$$