

PC3261: Classical Mechanics II

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Lecture 3: Linear Momentum

Linear momentum of two-particle system

Notes

- Forces are *assumed* to obey principle of superposition of forces: \mathbf{f}_{12} is the force acting on m_1 due to m_2

$$\begin{cases} \mathbf{F}_1(t) = \mathbf{F}_1^{\text{ext}}(t) + \mathbf{f}_{12}(t) \\ \mathbf{F}_2(t) = \mathbf{F}_2^{\text{ext}}(t) + \mathbf{f}_{21}(t) \end{cases}$$

- Total linear momentum of the system: forces between particles are assumed to obey Newton's third law of motion

$$\mathbf{P}(t) \equiv \mathbf{p}_1(t) + \mathbf{p}_2(t) \quad \Rightarrow \quad \dot{\mathbf{P}}(t) = \mathbf{F}_1^{\text{ext}}(t) + \mathbf{F}_2^{\text{ext}}(t)$$

- Newton's second law: the time rate of change of total linear momentum of the two-particle system equals to the total *external* force acting upon it

$$\dot{\mathbf{P}}(t) = \mathbf{F}^{\text{ext}}(t), \quad \mathbf{F}^{\text{ext}}(t) \equiv \mathbf{F}_1^{\text{ext}}(t) + \mathbf{F}_2^{\text{ext}}(t)$$

Linear momentum of multi-particle system

Notes

- Total force acting on the α -particle: $\mathbf{f}_{\alpha\beta}$ is the force acting on m_α due to m_β

$$\mathbf{F}_\alpha(t) = \mathbf{F}_\alpha^{\text{ext}}(t) + \sum_{\beta=1, \beta \neq \alpha}^N \mathbf{f}_{\alpha\beta}(t), \quad \alpha = 1, 2, 3, \dots, N$$

- Total linear momentum of multi-particle system:

$$\mathbf{P}(t) \equiv \sum_{\alpha=1}^N \mathbf{p}_\alpha(t)$$

- Newton's second law: the time rate of change of total linear momentum of multi-particle system equals to the total *external* force acting upon it

$$\dot{\mathbf{P}}(t) = \mathbf{F}^{\text{ext}}(t), \quad \mathbf{F}^{\text{ext}}(t) \equiv \sum_{\alpha=1}^N \mathbf{F}_\alpha^{\text{ext}}(t)$$

Impulse-Momentum theorem

Notes

- **Impulse-Momentum theorem** (integral form of the Newton's second law): change of total linear momentum equals to the time integral of the total external force

$$\mathbf{F}^{\text{ext}}(t) = \frac{d\mathbf{P}(t)}{dt} \quad \rightarrow \quad \int_{t_1}^{t_2} \mathbf{F}^{\text{ext}}(t) dt = \mathbf{P}(t_2) - \mathbf{P}(t_1)$$

- **Conservation law of linear momentum:** if the total external force on a multi-particle system is zero, then the total linear momentum of the multi-particle is a constant

$$\dot{\mathbf{P}}(t) = \mathbf{F}^{\text{ext}}(t) = \mathbf{0} \quad \Rightarrow \quad \mathbf{P}(t) = \text{constant} \quad \Rightarrow \quad \mathbf{P}(t_1) = \mathbf{P}(t_2) \quad \forall \quad t_1, t_2$$

- The validity of the conservation law of linear momentum depends crucially on the *experimental* basis of the Newton's third law!

A violation of Newton's third law???

Notes

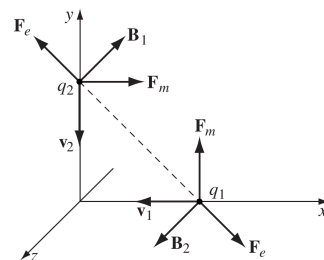
- Two point charges, q_1 and q_2 , are moving at uniform velocities \mathbf{v}_1 and \mathbf{v}_2 respectively

- Electric fields and forces:

$$\begin{cases} \mathbf{E}_1(\mathbf{r}_2) = \frac{q_1}{4\pi\epsilon_0} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ \mathbf{E}_2(\mathbf{r}_1) = \frac{q_2}{4\pi\epsilon_0} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \end{cases} \quad \Rightarrow \quad \begin{cases} \mathbf{F}_{e,12} = q_1 \mathbf{E}_2(\mathbf{r}_1) \\ \mathbf{F}_{e,21} = q_2 \mathbf{E}_1(\mathbf{r}_2) \end{cases}$$

- Electric forces obey Newton's third law

$$\mathbf{F}_{e,12} = -\mathbf{F}_{e,21}$$



A violation of Newton's third law??? – cont'd

Notes

- Magnetic fields and forces:

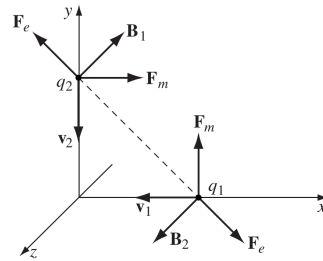
$$\begin{cases} \mathbf{B}_1(\mathbf{r}_2) = \frac{\mu_0 q_1}{4\pi} \frac{\mathbf{v}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ \mathbf{B}_2(\mathbf{r}_1) = \frac{\mu_0 q_2}{4\pi} \frac{\mathbf{v}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \end{cases} \Rightarrow \begin{cases} \mathbf{F}_{m,12} = q_1 \mathbf{v}_1 \times \mathbf{B}_2(\mathbf{r}_1) \\ \mathbf{F}_{m,21} = q_2 \mathbf{v}_2 \times \mathbf{B}_1(\mathbf{r}_2) \end{cases}$$

- Magnetic forces do not obey Newton's third law!

$$\mathbf{F}_{m,12} \neq -\mathbf{F}_{m,21}$$

- Electromagnetic linear momentum density: fields also possess linear momentum!

$$\mathbf{g}_{\text{EM}}(\mathbf{r}) = \epsilon_0 \mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})$$



System with variable mass

Notes

- Newton's second law with variable mass:

$$\mathbf{F}^{\text{ext}}(t) = \frac{d}{dt} [m(t) \mathbf{v}(t)] \xrightarrow{???} \mathbf{F}^{\text{ext}}(t) = \frac{dm(t)}{dt} \mathbf{v}(t) + m \frac{d\mathbf{v}(t)}{dt}$$

- Galilean velocity transformation: $\mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V}$

$$\frac{d}{dt} [m(t) \mathbf{v}'(t)] = \mathbf{F}^{\text{ext}}(t) \not\leftrightarrow \frac{d}{dt} [m(t) \mathbf{v}(t)] = \mathbf{F}^{\text{ext}}(t)$$

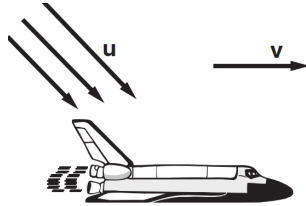
- There is *no* fundamental difficulty in handling any system with variable mass provided the same set of particles is included *throughout* the time interval t_1 to t_2

$$\int_{t_1}^{t_2} \mathbf{F}^{\text{ext}}(t) dt = \mathbf{P}(t_2) - \mathbf{P}(t_1)$$

Example: Spacecraft and dust particles

Notes

- A spacecraft with mass M moves through space with constant velocity \mathbf{v} . The spacecraft encounters a stream of dust particles that embed themselves in the hull at rate dm/dt . The dust has velocity \mathbf{u} just before it hits.



EXERCISE 3.1: Find the external force necessary to keep the spacecraft moving uniformly.

Newton's second law with variable mass

Notes

- A system with mass $m(t)$ moves at velocity $\mathbf{v}(t)$. Particles are added to the system at a rate $dm(t)/dt$. These particles have velocity $\mathbf{u}(t)$ just before entering the system.

- Newton's second law:

$$\mathbf{F}^{\text{ext}}(t) = \frac{d}{dt} [m(t) \mathbf{v}(t)] - \frac{dm(t)}{dt} \mathbf{u}(t)$$
$$\Rightarrow m(t) \dot{\mathbf{v}}(t) = \mathbf{F}^{\text{ext}}(t) + \dot{m}(t) [\mathbf{u}(t) - \mathbf{v}(t)]$$

- Galilean invariance is preserved:

$$\mathbf{F}^{\text{ext}}(t) = \frac{d}{dt} [m(t) \mathbf{v}(t)] - \dot{m}(t) \mathbf{u}(t) \quad \leftrightarrow \quad \mathbf{F}^{\text{ext}}(t) = \frac{d}{dt} [m(t) \mathbf{v}'(t)] - \dot{m}(t) \mathbf{u}'(t)$$

Example: Rocket in a constant gravitational field

Notes

- A rocket is taking off from rest in a uniform gravitation field $\mathbf{g} = -g \hat{\mathbf{e}}_z$. The fuel is ejected at a constant rate $\dot{m}(t) = -k$ at a constant exhaust speed u relative to the rocket.

- Linear momentum of the system:

$$\begin{cases} \mathbf{P}(t) = m(t)\mathbf{v}(t) \\ \mathbf{P}(t + \Delta t) = [m(t) + \Delta m][\mathbf{v}(t) + \Delta \mathbf{v}] + (-\Delta m)[\mathbf{v}(t) + \Delta \mathbf{v} + \mathbf{u}(t + \Delta t)] \end{cases}$$

- Newton's second law:

$$m(t) \dot{\mathbf{v}}(t) - \mathbf{u}(t) \dot{m}(t) = \mathbf{F}^{\text{ext}}(t)$$

EXERCISE 3.2: Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the rocket in its subsequent motion given that the initial mass of the rocket is m_0 .

Center of mass

Notes

- Position vector of the **center of mass** of a multi-particle system:

$$\mathbf{R}_{\text{CM}}(t) \equiv \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} \mathbf{r}_{\alpha}(t), \quad M \equiv \sum_{\alpha=1}^N m_{\alpha}$$

- Velocity of the center of mass: total linear momentum of the system is equal to the linear momentum of the center of mass *as if* it were a particle of mass M with velocity $\mathbf{V}_{\text{CM}}(t)$

$$\mathbf{V}_{\text{CM}}(t) \equiv \dot{\mathbf{R}}_{\text{CM}}(t) = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} \dot{\mathbf{r}}_{\alpha}(t) \quad \Rightarrow \quad \mathbf{P}(t) = M \mathbf{V}_{\text{CM}}(t)$$

- Acceleration of the center of mass: center of mass moves exactly *as if* it were a single particle of mass M subjected to the total external force on the system

$$\mathbf{A}_{\text{CM}}(t) \equiv \ddot{\mathbf{R}}_{\text{CM}}(t) = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} \ddot{\mathbf{r}}_{\alpha}(t) \quad \Rightarrow \quad \dot{\mathbf{P}}(t) = M \mathbf{A}_{\text{CM}}(t)$$

Example: Projectile motion

Notes

- A rigid object consists of two masses m_1 and m_2 separated by a light rod of length L . It is thrown into the air.

- Center of mass:

$$\mathbf{R}_{\text{CM}}(t) = \frac{m_1 \mathbf{r}_1(t) + m_2 \mathbf{r}_2(t)}{m_1 + m_2}$$

- Equation of motion of the center of mass:

$$\mathbf{F}^{\text{ext}}(t) = (m_1 + m_2) \ddot{\mathbf{R}}_{\text{CM}}(t) \Rightarrow \ddot{\mathbf{R}}_{\text{CM}}(t) = \mathbf{g}$$

- The center of mass follows the parabolic trajectory of a single mass, $m_1 + m_2$, in a uniform gravitational field (motions of m_1 and m_2 about the center of mass are to be analyzed separately)

Center of mass of extended body

Notes

- Visualize mass element dm of volume dV located at position $\mathbf{r}(t)$ with mass density $\rho(\mathbf{r})$:

$$\mathbf{R}_{\text{CM}}(t) = \frac{1}{M} \iiint_V \mathbf{r}(t) \rho(\mathbf{r}) dV$$

- Center of mass of a uniform solid (upper) hemisphere: mass M and radius R

$$\mathbf{R}_{\text{CM}} = \frac{3}{8} R \hat{\mathbf{e}}_z$$

EXERCISE 3.3: A thin non-uniform plate lies on the xy -plane with corners $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b) . Its surface mass density is $\sigma(x, y) = \sigma_0 xy/ab$ where σ_0 is a constant. Find its center of mass.

Center-of-mass frame

Notes

- **Center-of-mass frame** is a reference frame at which the center of mass remains at the origin:

$$\mathbf{r}'_{\alpha}(t) \equiv \mathbf{r}_{\alpha}(t) - \mathbf{R}_{\text{CM}}(t) \Rightarrow \mathbf{R}'_{\text{CM}}(t) = \mathbf{0}$$

- Velocity of the center of mass in the center-of-mass frame: center of mass is stationary in the center-of-mass frame

$$\mathbf{V}'_{\text{CM}}(t) = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} \dot{\mathbf{r}}'_{\alpha}(t) = \mathbf{0}$$

- Acceleration of the center of mass in the center-of-mass frame:

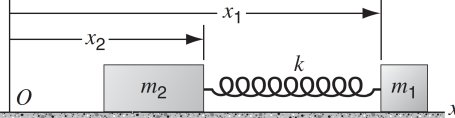
$$\mathbf{A}'_{\text{CM}}(t) = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} \ddot{\mathbf{r}}'_{\alpha}(t) = \mathbf{0}$$

Example: Two-body oscillations

Notes

- Two identical blocks 1 and 2 each of mass m slide without friction on a straight track. They are connected by a massless spring with unstretched length L_0 and spring constant k . Initially, the system is at rest. At $t = 0$, block 1 is hit sharply giving it an instantaneous velocity v_0 to the right.

- Equations of motion in the center-of-mass frame:

$$\begin{cases} m\ddot{x}'_1(t) = -k[x'_1(t) - x'_2(t) - L_0] \\ m\ddot{x}'_2(t) = +k[x'_1(t) - x'_2(t) - L_0] \end{cases}$$


EXERCISE 3.4: Find the velocities of each block at later times with respect to the track.