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(22)

Quark flavour

$$J^2 = \text{Casimir operator}$$

Originally (Heisenberg, 1932) isospin was introduced to classify elementary particles into doublet (p, n), or triplet (π^+, π^0, π^-) etc. The isospin group is $SU(2)$ $|j, m\rangle, |I, m_I\rangle$

In early 1960, many more elementary particles were found, $SU(2)$ isospin as a classification scheme is not adequate. A new quantum number, strangeness S , was introduced.



Many particles can then be accommodated into representations of a bigger symmetry group $SU(3)$

Mesons form singlet or octet representations of $SU(3)$

Baryons form singlet, octet (eight fold way) decuplet representations of $SU(3)$

Questions were then raised why only these three types of representation of $SU(3)$ are realized by elementary particles at that time?

The quark model (3 quarks) explains this. Mesons are made of quark and antiquark. Baryons are made of 3 quarks

Outer multiplication of two matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

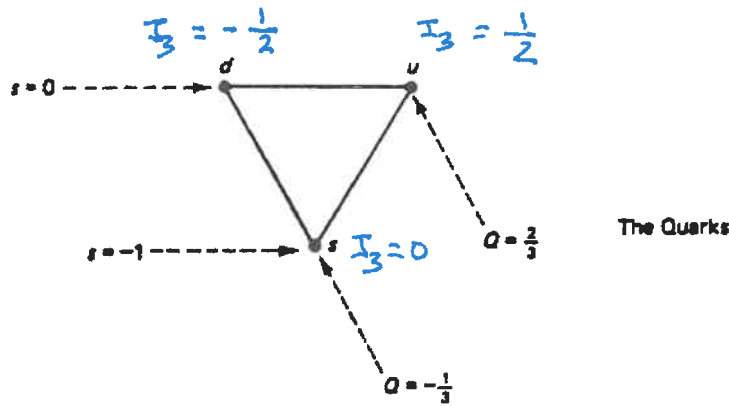
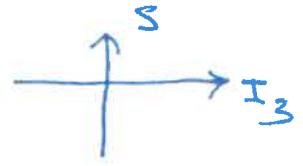
$$= \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

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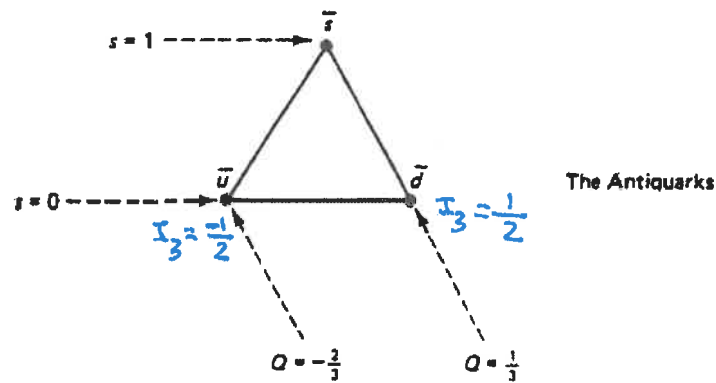
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Assume the 3 quarks form the fundamental representation of $SU(3)$,
triplet 3



Antiquarks form the conjugate representation, denoted by $\bar{3}$



From the fundamental representation, one can construct higher dimensional representation:

$$3 \otimes \bar{3} = 1 \oplus 8$$

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$

1 = singlet

8 = Octet

10 = Decuplet

Wrt $SU(3)$ transformations

Mesons are made of quark and antiquark

24

3

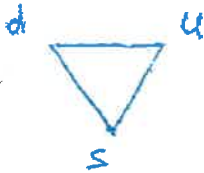
⊗

$$\bar{3} = 8 \oplus 1$$

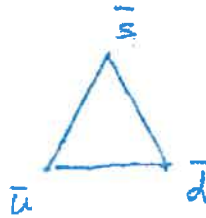
$s=1$

$s=0$

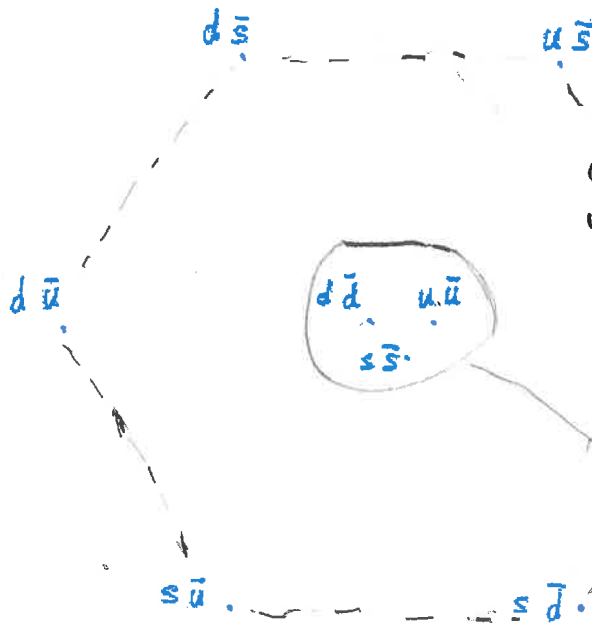
$s=-1$



⊗



=



Among the nine states, nonet, 8 of them transform into each other under $SU(3)$ transformation one of them is invariant under $SU(3)$ transformations.

So the nonet can be decomposed into an octet plus a singlet.

We say the nonet is a direct sum \oplus of 8 and 1
 $3 \otimes \bar{3} = 1 \oplus 8$

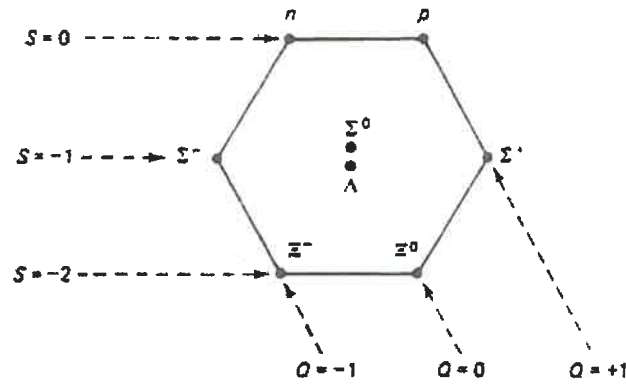
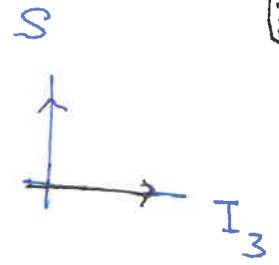
one of combinations of these three is unchanged under $SU(3)$ transformations, that is singlet wrt $SU(3)$

For baryons:

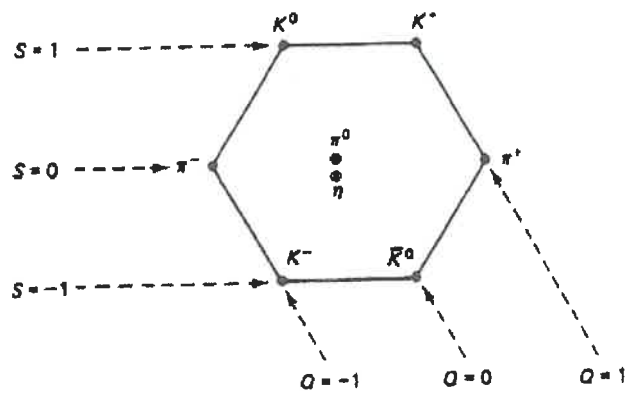
$$3 \otimes 3 \otimes 3 = \underline{10} \oplus 8 \oplus 8 \oplus 1$$

Decuplet

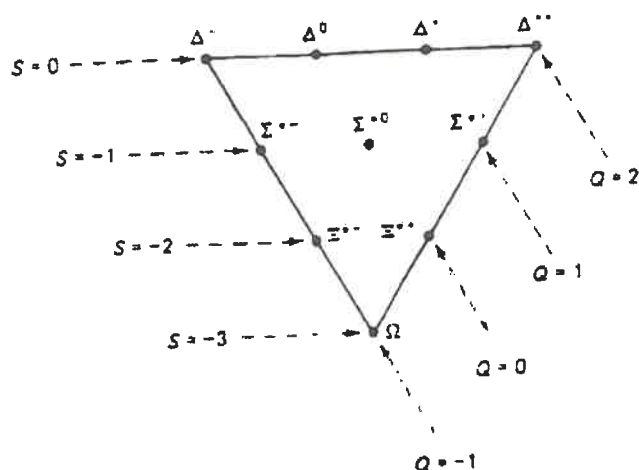
Note: The above nonet is a direct sum of octet and a singlet, octet, singlet wrt $SU(3)$
 The octet consists of 2 isodoublets, 1 isotriplet and 1 isosinglet wrt $SU(2)$



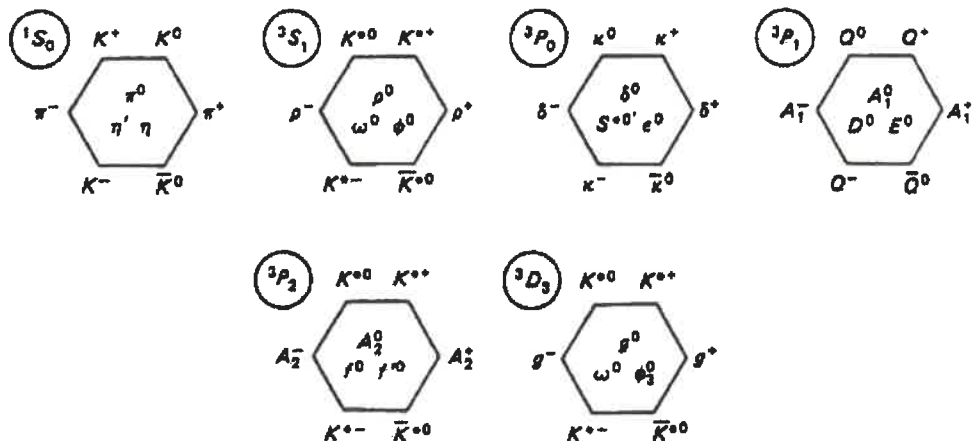
The Baryon Octet



The Meson Octet



The Baryon Decuplet



$2S+1$ L_j

Established meson nonets. Obviously, we are running out of letters. It is customary to distinguish different particles represented by the same letter by indicating the mass parenthetically (in MeV/c^2), thus $K^*(892)$, $K^*(1430)$, $K^*(1650)$, and so on. In this figure the supermultiplets are labeled in spectroscopic notation. At present, there are no complete baryon supermultiplets beyond the octet and decuplet, although there are many partially filled diagrams.

So $SU(3)$ scheme with 3 quarks (u, d, s) as the fundamental constituents of matter was a very good scheme for classifying mesons and baryons into $SU(3)$ singlets, octets, decuplets.

Charm quark was discovered in Nov. 1974

Then $SU(3)$ was extended to $SU(4)$ scheme with four constituent quarks: u, d, c, s .

And when bottom quark b and top quark t were discovered, $SU(6)$ is used to include b, t quarks.

Unfortunately all these higher groups are badly broken, due to the large mass differences among the 6 quarks. Members of the multiplet have very different masses.

In the $SU(2)$ scheme, proton and neutron almost same mass, so are the pions (π^+, π^0, π^- , triplet).

Depending on the circumstances, one assigns effective (constituent) mass or current (bare) mass to quarks.

Table 4.4 Quark masses (MeV/c^2)

Quark flavor	Bare mass	Effective mass
u	2	336
d	5	340
s	95	486
c	1300	1550
b	4200	4730
t	174 000	177 000

Warning: These numbers are somewhat speculative and model dependent [12].

However, there is an important *caveat* in this neat hierarchy: isospin, $SU(2)$, is a very 'good' symmetry; the members of an isospin multiplet differ in mass by at most 2 or 3%, which is about the level at which electromagnetic corrections would be expected.* But the Eightfold Way, $SU(3)$, is a badly 'broken' symmetry; mass splittings within the baryon octet are around 40%. The symmetry breaking is even worse when we include charm; the $\Lambda_c^+(udc)$ weighs more than twice the $\Lambda(uds)$, although they are in the same $SU(4)$ supermultiplet. It is worse still with bottom, and absolutely terrible with top, which doesn't form bound states at all.

Why is isospin such a good symmetry, the Eightfold Way fair, and flavor $SU(6)$ so poor? The Standard Model blames it all on the quark masses. Now, the theory of quark masses is a slippery business, given the fact that they are not accessible to direct experimental measurement. Various arguments [9] suggest that the u and d quarks are *intrinsically* very light, about 10 times the mass of the electron. However, within the confines of a hadron, their *effective mass* is much greater. The precise value, in fact, depends on the context; it tends to be a little higher in baryons than in mesons (more on this in Chapter 5). In somewhat the same way, the effective inertia of a spoon is greater when you're stirring honey than when you're stirring tea, and in either case it exceeds the true mass of the spoon. Generally speaking, the effective mass of a quark in a hadron is about $350 \text{ MeV}/c^2$ greater than its bare mass [10] (see Table 4.4). Compared to this, the quite different *bare* masses of up and down quarks are practically irrelevant; they *function* as though they had identical masses. But the s quark is distinctly heavier, and the c , b , and t quarks are widely separated. Apart from the differences in quark masses, the strong interactions treat all flavors equally. Thus isospin is a good symmetry because the effective u and d masses are so nearly equal (which is to say, on a more fundamental level, because their *bare* masses are so small); the Eightfold Way is a fair symmetry because the effective mass of the strange quark is not too far from that of the u and d . But

* Indeed, it used to be thought that isospin was an *exact* symmetry of the strong interactions, and all of the symmetry breaking was attributable to electromagnetic contamination. The fact that the n - p mass splitting is in the

wrong direction to be purely electromagnetic was troubling, however, and we now believe that $SU(2)$ is only an *approximate* symmetry of the strong interactions.

current

→ constituent

strong interaction

Symmetry Transformation in Quantum Mechanics

In QM, state $|\psi\rangle$ and operators are the key elements in analyzing a physical problem. Clearly, a symmetry transformation is associated with an operator in Hilbert space.

Define symmetry transformation in QM:

A symmetry transformation operator U is a 1-1 mapping that maps a dynamically possible state, say $|\psi\rangle$, to another dynamically possible state $|\psi'\rangle$, namely $U:|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$, such that the transition probability is preserved, i.e. no change in the transition probability.

$$\text{Transition probability from state } |\psi\rangle \text{ to state } |\varphi\rangle = |\langle\varphi|\psi\rangle|^2$$

$$\text{Transition probability from state } |\psi'\rangle \text{ to state } |\varphi'\rangle = |\langle\varphi'|\psi'\rangle|^2$$

$$\text{Transition probability is preserved means: } |\langle\varphi|\psi\rangle|^2 = |\langle\varphi'|\psi'\rangle|^2$$

In other words, the transition probability before applying the symmetry transformation U is the same as the transition probability after the applying the same symmetry transformation.

From the definition, we can show that

Theorem

- (i) U is unitary.
- (ii) U is linear or anti-linear.
- (iii) If U does not depend on time explicitly, then $[U, H] = 0$

Proof:

- (i) An operator A is unitary if $A^\dagger = A^{-1}$.

We want to show that symmetry transformation U is unitary.

Given that $|\langle\varphi|\psi\rangle|^2 = |\langle\varphi'|\psi'\rangle|^2$, $|\psi'\rangle = U|\psi\rangle$, $|\varphi'\rangle = U|\varphi\rangle \Rightarrow \langle\varphi'| = \langle\varphi|U^\dagger$, then

$$|\langle\varphi'|\psi'\rangle|^2 = |\langle\varphi|U^\dagger U|\psi\rangle|^2 = |\langle\varphi|\psi\rangle|^2$$

This is true for any arbitrary state $|\varphi\rangle$ and $|\psi\rangle$, hence $U^\dagger U = 1$.

By associativity rule $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, we can show that $UU^\dagger = 1$.

From the definition of an inverse operator, $U^{-1}U = 1 = UU^{-1}$, so we have $U^{-1} = U^\dagger$, i.e. U is unitary.

- (ii) To show U is linear or anti-linear, we consider a state $|\psi\rangle$ and a state $\alpha|\psi\rangle$, where α is a complex number.

U is linear if $U(\alpha|\psi\rangle) = \alpha(U|\psi\rangle)$;

U is anti-linear if $U(\alpha|\psi\rangle) = \alpha^*(U|\psi\rangle)$, where α^* = complex conjugate of α .

Given $|\langle\varphi|\psi\rangle|^2 = |\langle\varphi'|\psi'\rangle|^2$, one can have either $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$ or $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^*$. Then U is linear if $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$; U is anti-linear if $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^*$.

Consider first the case $\langle \varphi' | \psi' \rangle = \langle \varphi | \psi \rangle$,

LHS: $\langle \varphi' | \psi' \rangle = \langle \varphi' | U | \psi \rangle$, let $|\psi\rangle = \lambda |\Omega\rangle$, where $\lambda = \text{constant}$, then

$$\langle \varphi' | \psi' \rangle = \langle \varphi' | U \lambda |\Omega\rangle$$

$$\text{RHS: } \langle \varphi | \psi \rangle = \langle \varphi | \lambda |\Omega\rangle = \lambda \langle \varphi | \Omega \rangle = \lambda \langle \varphi' | \Omega' \rangle = \lambda \langle \varphi' | U |\Omega\rangle = \langle \varphi' | \lambda U |\Omega\rangle$$

Since $\langle \varphi' | \psi' \rangle = \langle \varphi | \psi \rangle$, then $\langle \varphi' | U \lambda |\Omega\rangle = \langle \varphi' | \lambda U |\Omega\rangle$.

As $\langle \varphi' |$ and $|\Omega\rangle$ are arbitrary, so $U\lambda = \lambda U$, i.e. U is linear.

If we start from $\langle \varphi' | \psi' \rangle = \langle \varphi | \psi \rangle^*$ instead of $\langle \varphi' | \psi' \rangle = \langle \varphi | \psi \rangle$, then we can show $U\lambda = \lambda^* U$, i.e. U is anti-linear.

$$\text{LHS} = \langle \varphi' | \psi' \rangle = \langle \varphi' | U | \psi \rangle = \langle \varphi' | U \lambda |\omega\rangle$$

$$\text{RHS} = \langle \varphi | \psi \rangle^* = (\langle \varphi | \lambda |\omega\rangle)^* = (\lambda \langle \varphi | \omega \rangle)^* = \lambda^* \langle \varphi | \omega \rangle^* = \lambda^* \langle \varphi' | \omega' \rangle = \lambda^* \langle \varphi' | U |\omega\rangle = \langle \varphi' | \lambda^* U |\omega\rangle$$

LHS=RHS gives $\langle \varphi' | U \lambda |\omega\rangle = \langle \varphi' | \lambda^* U |\omega\rangle$, that is, $U \lambda |\omega\rangle = \lambda^* U |\omega\rangle$ since $\langle \varphi' |$ is arbitrary. Or, $U \lambda = \lambda^* U$.

$$\begin{aligned} \text{Note: Put } |\omega\rangle &= a |\omega_1\rangle + b |\omega_2\rangle, \text{ then } U \lambda |\omega\rangle = U \lambda (a |\omega_1\rangle + b |\omega_2\rangle) = \\ U(\lambda a |\omega_1\rangle + \lambda b |\omega_2\rangle) &= U \lambda a |\omega_1\rangle + U \lambda b |\omega_2\rangle \\ &= \lambda^* a^* U |\omega_1\rangle + \lambda^* b^* U |\omega_2\rangle \end{aligned}$$

$$\text{That is equivalent to } U(a |\varphi\rangle + b |\psi\rangle) = a^* U |\varphi\rangle + b^* U |\psi\rangle.$$

So we have shown that U is either linear or anti-linear.

A linear unitary operator is usually called a unitary operator. An anti-linear unitary operator is called anti-unitary operator. In nature, most of the symmetry transformations are associated with unitary operators. Time reversal and charge conjugation are associated with anti-unitary operators.

- (iii) To show $[U, H] = 0$ if $\frac{\partial U}{\partial t} = 0$, we consider 2 dynamically possible states $|\psi\rangle$ and $|\psi'\rangle = U|\psi\rangle$.

By definition, a dynamically possible state is a state that satisfies the TDSE:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle$$

$$i\hbar \frac{\partial}{\partial t} |\psi'\rangle = H |\psi'\rangle$$

From the 2nd equation above, we have LHS:

$$i\hbar \frac{\partial}{\partial t} U |\psi\rangle = i\hbar \left(\frac{\partial U}{\partial t} \right) |\psi\rangle + U i\hbar \frac{\partial}{\partial t} |\psi\rangle = i\hbar \left(\frac{\partial U}{\partial t} \right) |\psi\rangle + UH |\psi\rangle$$

For RHS: $H |\psi'\rangle = HU |\psi\rangle$

Since LHS = RHS, we have

$$i\hbar \left(\frac{\partial U}{\partial t} \right) |\psi\rangle + UH |\psi\rangle = HU |\psi\rangle$$

If U does not depend on time explicitly, i.e. $\frac{\partial U}{\partial t} = 0$, then we have

$$(UH - HU) |\psi\rangle = 0$$

As $|\psi\rangle$ is any dynamically possible state, so we have $[U, H] = 0$.

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chapter 4

Discrete symmetries

P = parity, space inversion (mirror reflection)

C = charge conjugation (positive charge \Rightarrow negative charge)

T = time reversal (motion reversal, $\underline{p} \rightarrow -\underline{p}$)

First discuss space inversion, P

Introduce space inversion in 3-dimensional physical space, then as an operator in quantum mechanics
Parity operator π , π unitary, Hermitian (observable)

Downfall of parity conservation in weak interaction

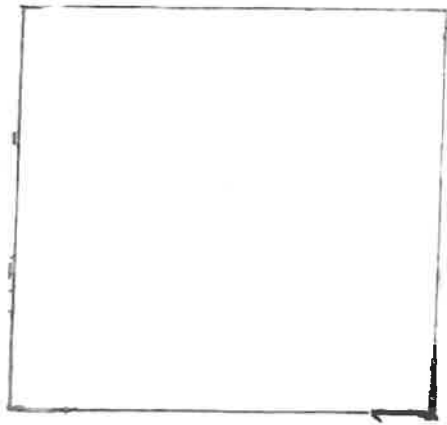
C. S. Wu experiment 1956

parity broken in weak decay.

Feynman question

0

A square mirror puzzle



L - R symmetrical

U - d symmetrical

However,

image is L R reversed
but not U d reversed

①

Mirror reflection : Why L-R inverted
and not Up-down inverted ?

Answer :

It is the axis normal to the mirror that
is inverted



L - R or Up-down depends on how
the observer views the object and
its image

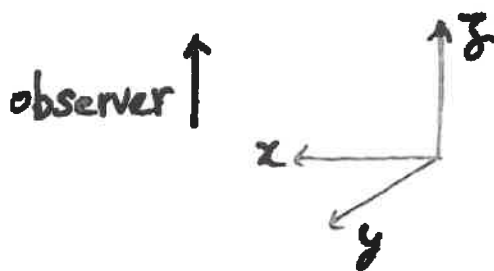
(2)

Observer facing \leftarrow x axis sees the
 \swarrow y -axis on his RHS

But when the observer facing the mirror
 image (facing the \dashrightarrow x axis) the
 \swarrow y -axis is on his LHS

In the above the observer turns 180° about
 the z -axis

Consider the observer facing \leftarrow x -axis again
 He sees the z -axis above his head.



Now the observer turn 180° about the
 y -axis



③

to face the mirror.

He faces $\rightarrow x$ axis

He then sees the z axis

below his feet

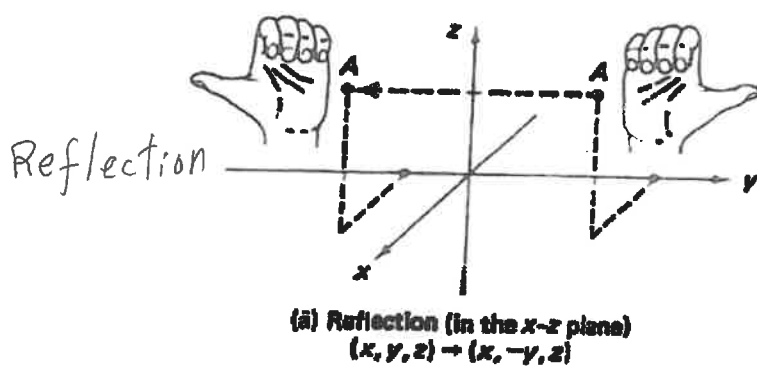
The image is up side down !

Because we move on 2-dimensional plane, so usually we turn about z -axis and mirror image appears to be L-R inverted

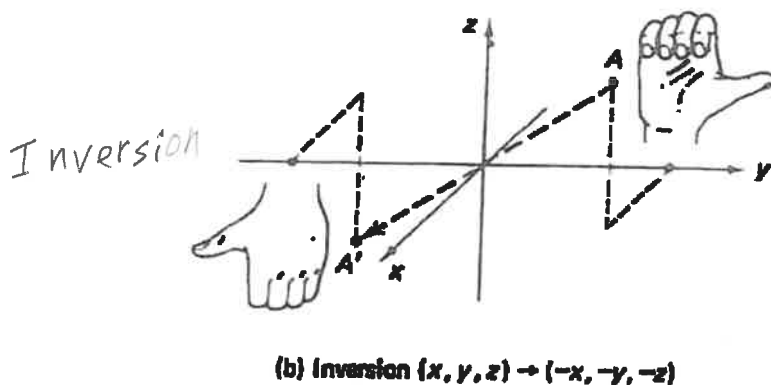
If we are free to move in 3-dimensional space, then we can easily turn about the y -axis and the mirror image would appear up-down inverted.

Recap: Mirror reflection is just one of coordinate axes inverted.

whether Left-Right or Up-down depends on how the observer viewing the image



This requires
 where to put the
 mirror



No mirror needed

Figure 4.11 Reflections and inversions.

So more convenient to discuss space
 inversion instead of mirror reflection

(5)

Space inversion ;

$$\underline{x} \longrightarrow \underline{x}' = -\underline{x}$$

Same as mirror reflection plus a

rotation of 180° about an axis

Consider a point P in front of the x^3 - x^1 plane.

Its coordinates = $(0, x^2, x^3) = \underline{x}_P$

Mirror reflection on the x^3 - x^1 plane,

$$\underline{x}_P \rightarrow \underline{x}_P^M = (0, -x^2, x^3).$$

Rotation about the x^2 -axis by 180° ,

$$\underline{x}_P^M \rightarrow \underline{x}_P^{MR} = (0, -x^2, -x^3) = -\underline{x}_P$$

