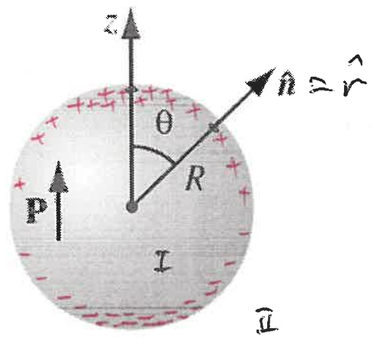
Example 4.2. Find the electric field produced by a uniformly polarized sphere of radius R.



uniform $\vec{P}$ $\qquad \rho_b = -\nabla\cdot\vec{P} = 0$

$$\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$$

from separation of variables

General solution: $V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$

$$V_I(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$V_{II}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Boundary conditions $\begin{cases} (1)\ V_I(R,\theta) = V_{II}(R,\theta) \\ (2)\ \left[\frac{\partial V_{II}(r,\theta)}{\partial r} - \frac{\partial V_I(r,\theta)}{\partial r}\right]\Big|_{r=R} = -\frac{\sigma_b}{\epsilon_0} \end{cases}$

(1) $\Rightarrow \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$

$\Rightarrow A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow A_l = \frac{B_l}{R^{2l+1}}, \quad B_l = R^{2l+1} A_l$

(2) $\Rightarrow$

$$\left[\sum_{l=0}^{\infty} -(l+1)\frac{B_l}{r^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} (A_l r^{l-1} P_l(\cos\theta))\right]\Big|_{r=R} = -\frac{\sigma_b}{\epsilon_0}$$

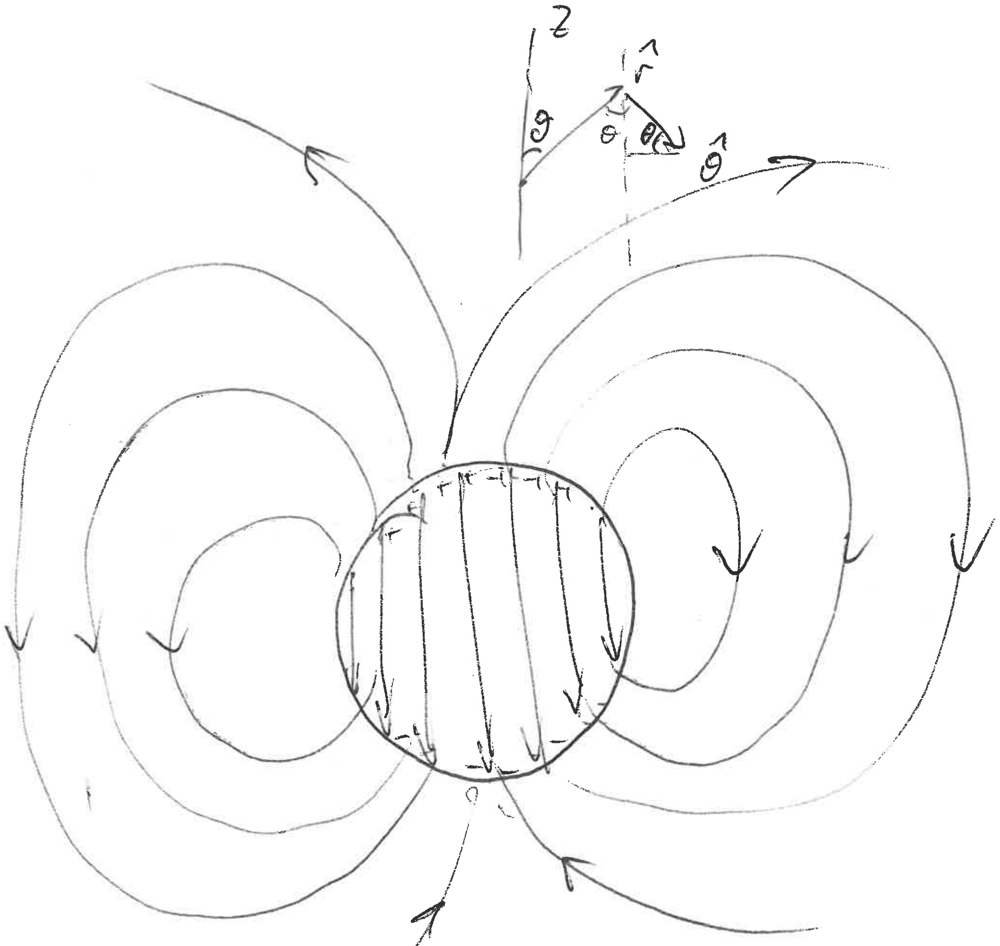
$$\Rightarrow \sum_{l=0}^{\infty} (2l+1) A_l r^{l-1} P_l(\cos\theta) = \frac{P\cos\theta}{\epsilon_0}$$

$$A_1 = \frac{P}{3\epsilon_0}, \quad B_1 = R^3 A_1$$

$$\Rightarrow \begin{cases} V_I(r,\theta) = \frac{P r\cos\theta}{3\epsilon_0} & (r \le R) \\ V_{II}(r,\theta) = \frac{P R^3}{3\epsilon_0 r^2}\cos\theta & (r \ge R) \end{cases}$$

$$V_{I}(r,\theta) = \frac{P r \cos\theta}{3\varepsilon_0} \quad (r \le R)$$

$$V_{II}(r,\theta) = \frac{P R^3}{3\varepsilon_0 r^2} \cos\theta \quad (r \geq R)$$



$$\vec{E}_{I} = -\nabla V_{I} = -\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\right)\frac{P r \cos\theta}{3\varepsilon_0}$$

$$= \frac{-P}{3\varepsilon_0}\cos\theta\,\hat{r} + \frac{P}{3\varepsilon_0}\sin\theta\,\hat{\theta}$$

$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$= \frac{-P\hat{z}}{3\varepsilon_0} = \frac{-\vec{P}}{3\varepsilon_0}$$

$$V_{II} = \frac{P R^3}{3\varepsilon_0 r^2}\cos\theta = \frac{P\frac{4}{3}\pi R^3}{4\pi\varepsilon_0 r^2}\cos\theta = \frac{\vec{P}_d \cdot \hat{r}}{4\pi\varepsilon_0 r^2} \quad \text{where } |\vec{P}_d| = P\frac{4}{3}\pi R^3$$