Expanding Universe Worksheet – IS 8

Part 1 is to completed individually at home **before** week 9 IS.
Part 2 is be done in class in groups **on** week 9 IS. Feel free to try it out before coming to class!
Part 3 is optional. We will do if there's time.

1 Newtonian Universe

The text below are taken from Chapter 7 of the lecture notes, slightly modified. Work your way from one equation to the next. Put a Tick beside the equation if you understand what it means and how it comes about. Draw question marks otherwise. Annotate in whatever way you like! Mentors will loo through your annotations in IS.

Expansion of the Newtonian world

Consider a spherical shell of mass m and radius R enclosing a concentric spherically symmetric mass[1] M. The shell moves radially **outwards** with rate $\dot{R} = \frac{dR}{dt}$. The kinetic energy (KE) of the shell is

$$T = \frac{1}{2}m\dot{R}^2 = \frac{1}{2}mH^2R^2. \tag{1}$$

where $H = \dfrac{\dot{R}}{R}$. Note that H is a function of time.

The potential energy (PE) due to Newtonian gravity is

$$U = -\frac{GMm}{R} = -\frac{4}{3}\pi Gm\rho R^2 \tag{2}$$

The total energy of the system is

$$\frac{1}{2}mH^2R^2 - \frac{4}{3}\pi Gm\rho R^2 = E. \tag{3}$$

Qualitatively, one can think that the KE increases the volume enclosed while the PE tries slow down or revert the expansion. The PE is dependent on the mass density ρ(of the enclosed mass).

If ρ is large enough such that PE > KE, then the mass shell's expansion will **not** continue forever.

If ρ is small such that PE < KE, then the mass shell's expansion will carry on indefinitely.

As such $E = 0$ is the critical point for which the Universe is just able to expand forever. Hence we define the critical density by

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{4}$$

Some terminologies: If $\rho > \rho_c$, the Universe is closed, if $\rho < \rho_c$, the Universe is open. Does it make sense to you?

[1] The physical size of the spherically symmetric inner mass does not matter as long the radius is less than that of the shell.

It is useful to define a quantity that represents the scale of the Universe at different times. We call this the cosmological scale factor $S(t)$. All lengths (eg. radii, wavelengths) scales with this factor.

$$\begin{aligned} R(t) &\propto S(t) \\ R(t) &= r\,S(t) \end{aligned} \tag{5}$$

where r is a constant (in time)[2]. Differentiating the above equation wrt time,

$$\dot{R}(t) = r\dot{S}(t) \quad , \quad \ddot{R}(t) = r\ddot{S}(t) \tag{6}$$

The Hubble's law can be written in terms of the scale factor:

$$H(t) = \frac{\dot{R}}{R} = \frac{\dot{S}}{S} \tag{7}$$

One may also relate the mass density to the scale factor or

$$\rho \;\propto \tfrac{1}{R^3} \propto \; S^{-3} \tag{8}$$

Substituting Eqs. (6) and (7) into Eq. (3),

$$\begin{aligned} \frac{1}{2}m\frac{\dot{S}^2}{S^2}r^2S^2 - \frac{4}{3}\pi G m\rho r^2 S^2 &= E \\ \frac{1}{2}mr^2\left(\dot{S} - \frac{8}{3}\pi G\rho S^2\right) &= E \end{aligned} \tag{9}$$

Since E, m and r are constants, we can write

$$\dot{S} - \frac{8}{3}\pi G\rho S^2 = -kc^2 \tag{10}$$

where k and c are constants. This gives

$$\frac{\dot{S}^2 + kc^2}{S^2} \;=\; \frac{8}{3}\pi G\rho \tag{11}$$

Eq. (11) is in fact one of the two Friedmann equations which are derived from the Einstein's equation for a homogeneous and isotropic Universe. The only difference is that ρ here represents mass density of matter in the Newtonian world while in the GR framework, ρ is a more general mass-energy density which includes components of matter, radiation, cosmological constant and etc.

The dynamics of the expanding shell of mass can be further probed with the Newton's law of gravitation

$$F = m\ddot{R} = -\frac{GMm}{R^2} \tag{12}$$

Rewriting in terms of the scale factor and mass density, we have

$$\ddot{S} = -\frac{4}{3}\pi G\rho S. \tag{13}$$

Comparing with Eq. (11), we have

$$\frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0 \tag{14}$$

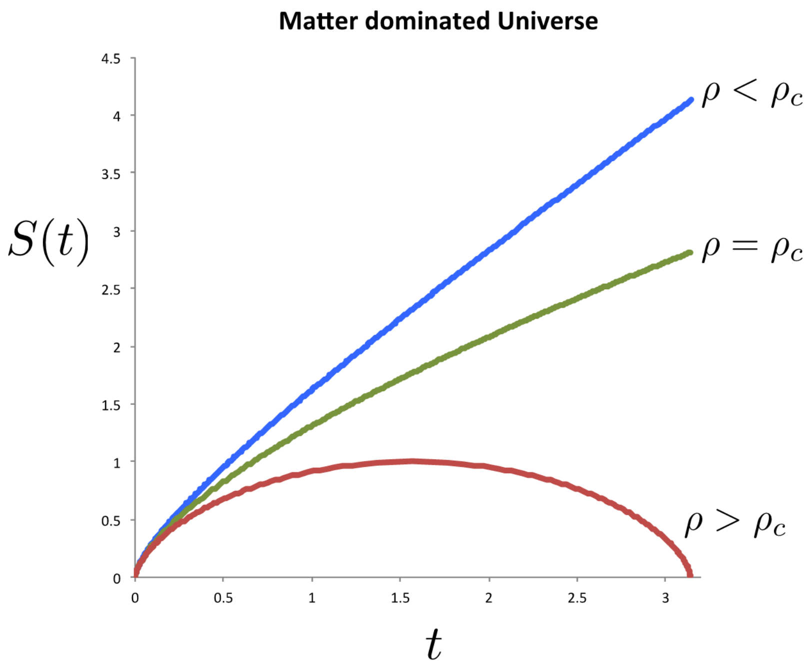
Eq. (14) turns out to be the second Friedmann equation, for the special case of a dust (pressure-less matter) filled Universe.

k can either be positive, negative or zero. What decides the value of k? Hint: look at total energy and density vs critical density.

[2] In case you wonder why I chose to use the letter r to represent a constant, the reason will be given next week.

2 Evolution of the Matter-Dominated Universe

The Friedmann equation Eq. (11) allows us to study the evolution of the (size) of the Universe. The Universe will behave differently depending on how much matter it contains. Solving the Friedmann equation, one can obtain the following scenarios summarized with the figure below.



Work in your group to solve the Friedmann equation and reproduce the above graphs.

3 The Second Friedmann Equation

Here we try to get some information from the second Friedmann equation without attempting to solve it. (We have solved the first Friedmann equation, that is good enough!)

From

$$\ddot{S} = -\frac{4}{3}\pi G\rho S$$

What can you say about $\ddot{S}(t)$?

Does it correspond with our classic idea of gravitation being an attractive force?

Does it correspond with the latest observations of our Universe?