PC3130 W4L1 AY24

Tuesday, 3 September 2024 9:42 am

W3L2

Angular momentum $\vec{J}$ is defined as

$$\boxed{[J_i, J_j] = i\hbar\,\varepsilon_{ijk} J_k} \quad ☆$$

- generator of rotations.

$$[A_i, J_\ell] = i\hbar\,\varepsilon_{ijk}\delta_{j\ell} A_k = i\hbar\,\varepsilon_{i\ell k} A_k$$

$$[J_\ell, A_i] = -[A_i, J_\ell] = -i\hbar\,\varepsilon_{i\ell k} A_k = i\hbar\,\varepsilon_{\ell i k} A_k$$

For any vector operator $\vec{A}$, we have

$$[\vec{A}, \hat{n}\cdot\vec{J}] = i\hbar\,\hat{n}\times\vec{A}$$

Another important commutator relation for $\vec{J}$:

$$\boxed{[J_i, J^2] = [J^2, J_i] = 0} \quad ☆$$

Recall

For central potentials (i.e. V depends only on $|\vec{r}|$),

$H = T + V$ commutes with $\vec{L} = \vec{r}\times\vec{p}$.

KE

Claim: $[\vec{L}, |\vec{r}|] = 0$

$[\vec{L}, |\vec{r}|^2] = 0$

$[L_i, \vec{r}\cdot\vec{r}] = 0$ $\quad r_1r_1 + r_2r_2 + r_3r_3$.

$$\left(\sum a_i b_i c_i\right) \quad a_i b_i + c_i d_i$$

$$[L_i, \vec{r}\cdot\vec{r}] = [L_i, r_j r_j]$$

j appears twice in each term

$$= r_j[L_i, r_j] + [L_i, r_j]\,r_j$$

$\vec{r}$ is a vector operator.

$$= r_j(i\hbar\,\varepsilon_{ijk} r_k) + (i\hbar\,\varepsilon_{ijk} r_k)\,r_j$$

$$= 2i\hbar\,(\vec{r}\times\vec{r})_i \qquad (\varepsilon_{ijk} r_j r_k = (\vec{r}\times\vec{r})_i)$$

$$= 0$$

Likewise, $[L_i, \vec{p}\cdot\vec{p}] = 0$

Similarly, we can show $[J_i, \vec{J}\cdot\vec{J}] = 0$ or $[J_i, J^2] = 0$

Tut 1 Q3

To show $[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$ using $[r_i, p_j] = i\hbar\delta_{ij}$

and $\vec{L} = \vec{r} \times \vec{p}$; $L_i = \varepsilon_{ijk} r_j p_k$

$$[L_i, L_j] = [\varepsilon_{ilk} r_l p_k, \ \varepsilon_{jmn} r_m p_n]$$

anything except i & j.

any letter except i, j, l, k

$$= \underbrace{\varepsilon_{ilk}\varepsilon_{jmn}}\ \underbrace{[r_l p_k, r_m p_n]}$$

$\delta_{lm}\delta_{kn} - \delta_{ln}\delta_{km}$

$[ab, c]$
$= a[b, c] + [a, c]b$.

We will now work toward:

$[J^2, J_i] = 0$

$\Rightarrow \exists$ common eigenstates for J^2 & J_i

label them $|j, m\rangle$ (cf. $|n l m\rangle$

$$\boxed{\begin{array}{l} J^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle, \\ \qquad j \geq 0 \text{ and } j \text{ is integer or half-integer} \\ J_z |j, m\rangle = \hbar m |j, m\rangle, \\ \qquad -j \leq m \leq j \\ \qquad m = -j, -j+1, -j+2, \dots, j-1, j \end{array}}$$

(Recall) 1D Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$a = \sqrt{\frac{m\omega}{\ }}\, x + \frac{i}{\ } p$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + \frac{i}{\sqrt{2m\hbar\omega}}\, p$$

$$a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{i}{\sqrt{2m\hbar\omega}}\, p$$

Can show $[a, a^{\dagger}] = \mathbb{1}$

Can show $a^{\dagger}a = \frac{m\omega}{2\hbar} x^2 + \frac{1}{2m\hbar\omega} p^2 + \underbrace{\frac{i}{2\hbar} [x, p]}_{\text{non-classical}}$

$$\Rightarrow \quad \hbar\omega (a^{\dagger} a) = \underbrace{\frac{1}{2} m\omega^2 x^2 + \frac{p^2}{2m}}_{H} + \frac{i\omega}{2} (i\hbar \mathbb{1})$$

$$\Rightarrow \quad H = \hbar\omega \left((a^{\dagger}a) + \frac{1}{2}\mathbb{1} \right)$$

$$\equiv \hbar\omega \left(N + \frac{1}{2}\mathbb{1} \right), \quad N = a^{\dagger}a$$

(final result: eigenvalues of H are $E_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$)

$$\boxed{\text{What are possible values of } n, \text{ where } N|n\rangle = n|n\rangle \text{ ?}}$$

Commutator relations: $[a, a^{\dagger}] = \mathbb{1}$

$$[N, a] = -a$$

$$[N, a^{\dagger}] = a^{\dagger}.$$

Is $a^{\dagger}|n\rangle$ also an eigenstate of N?

$a^{\dagger}$ ↑ $(n+1)$, $|n+1\rangle$

• n, $|n\rangle$

a ↓ $(n-1)$, $|n-1\rangle$

$$\underbrace{N\,(a^{\dagger}|n\rangle)}_{= ?\, (a^{\dagger}|n\rangle)\, ??} = a^{\dagger} \underbrace{N|n\rangle}_{\text{we know this}} + [N, a^{\dagger}]\,|n\rangle$$

$$= n\, a^{\dagger}|n\rangle + a^{\dagger}|n\rangle$$

$$= (n+1)\, a^{\dagger}|n\rangle$$

$\Rightarrow \quad a^{\dagger}|n\rangle$ is an eigenvector of N, with eigenvalue $(n+1)$.

i.e. $a^\dagger|n\rangle = c_+|n+1\rangle$

$$N(a|n\rangle) = aN|n\rangle + [N, a]|n\rangle$$
$$= n\,a|n\rangle - a|n\rangle$$
$$= (n-1)\,a|n\rangle$$

$\Rightarrow$ $a|n\rangle$ is an eigenvector of N, with eigenvalue $(n-1)$.

i.e. $a|n\rangle = c_-|n-1\rangle$

Some bounds on n?

$$n = \langle n|N|n\rangle$$
$$= \langle n|a^\dagger a|n\rangle \qquad (\ (a|n\rangle)^\dagger = \langle n|a^\dagger\ )$$
$$= \|a|n\rangle\|^2$$
$$\geq 0$$

$n=0$ iff $\|a|n\rangle\| = 0$

i.e. $a|n\rangle = 0$.

$\Big\}$ $a|0\rangle = 0$.

$\uparrow$ $n=0$.

Claim: n can only be integer-valued.

Proof by contradiction

Suppose $\exists\ n' = m - \delta$, $0<\delta<1$, $m = \lceil n' \rceil$

$a^{m-1}|n'\rangle \propto |1-\delta\rangle$

$(n' - (m-1)) = 1-\delta)$

i.e., $a^m|n'\rangle$ has eigenvalue for N to be $(1-\delta)-1 = -\delta < 0$

Contradiction

$\because$ all eigenvalues of $N \geq 0$.

because

$\rightarrow$ n is an integer.

$a\ \phi$ — $|0\rangle$ ☺

No $|-1\rangle$

Suppose

a — $|\tfrac{1}{2}\rangle$

— $|-\tfrac{1}{2}\rangle$

problem

because $n \geq 0$.

$$H = \hbar\omega\left(N + \tfrac{1}{2}\mathbb{1}\right)$$

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right), \qquad n = 0, 1, 2, \ldots$$

integers ≥ 0.

Find c_+, c_- :

$$a^\dagger |n\rangle = c_+ \underbrace{|n+1\rangle}$$

normalized to 1.

$$\| a^\dagger |n\rangle \|^2 = \langle n | a a^\dagger | n \rangle$$
$$= \langle n | (a^\dagger a + 1) | n \rangle$$
$$= (n+1)$$

So $a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$.

$$\| a |n\rangle \|^2 = \langle n | a^\dagger a | n \rangle = n$$

So $a|n\rangle = \sqrt{n}\, |n-1\rangle$.

Angular momentum

$$\boxed{[J_i, J_j] = i\hbar\, \varepsilon_{ijk} J_k}$$

$$\boxed{[J^2, J_i] = 0}$$

$\exists\ |a, b\rangle$ s.t.

$$J^2 |a, b\rangle = a\, |a, b\rangle$$
$$J_z |a, b\rangle = b\, |a, b\rangle$$

eigenvalues

labels.

What are the possible values of a and b?

Use ladder operators

$$J_+ = J_x + i J_y$$
$$J_- = J_x - i J_y = J_+^\dagger$$

$\Big\}$ ladder operators bring us from one eigenstate of

$$J_- = J_x - i J_y = J_+^\dagger$$

bring us from one eigenstate of J_z to the next.

$\Downarrow$

$[J_z, J_+]$, ?

$[J_z, J_-]$?

(a, $a^\dagger$ bringing us from one eigenstate of N to the next. $[N, a^\dagger]$, $[N, a]$.)

Find $[J_z, J_+]$, $[J_z, J_-]$.

$$\begin{aligned}
[J_z, J_+] &= [J_z, J_x + iJ_y] \\
&= [J_z, J_x] + i[J_z, J_y] \\
&= i\hbar J_y + i(-i\hbar J_x) \\
&= \hbar (J_x + iJ_y) \\
&= \hbar J_+
\end{aligned}$$

(recall $[N, a^\dagger] = a^\dagger$)

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k.$$

$$[J_z, J_x] = i\hbar \varepsilon_{zxy} J_y$$

$i=z$, $j=x$

$$= i\hbar J_y$$

$$[J_z, J_y] = i\hbar \varepsilon_{zyx} J_x$$

$i=z$, $j=y$

$$= i\hbar(-1) J_x = -i\hbar J_x$$

$$\begin{aligned}
[J_z, J_-] &= [J_z, J_x - iJ_y] \\
&= [J_z, J_x] - i[J_z, J_y] \\
&= i\hbar J_y - i(-i\hbar J_x) \\
&= -\hbar(J_x - iJ_y) \\
&= -\hbar J_-
\end{aligned}$$

(recall $[N, a] = -a$)

$[J^2, J_+]$? , $[J^2, J_-]$?

$$\begin{aligned}
[J^2, J_\pm] &= [J^2, J_x \pm iJ_y] \\
&= [J^2, J_x] \pm i[J^2, J_y] \\
&= 0 \pm 0 \\
&= 0
\end{aligned}$$

$$[J_z, J_+] = \hbar J_+, \qquad [J_z, J_-] = -\hbar J_-$$

Starting with $|a, b\rangle$,

Is $J_+|a,b\rangle$ also an eigenstate of J_z and J^2?

If so, what are the eigenvalues?

Consider:

know this

Consider:

$$J_z\left(J_+|a,b\rangle\right) = J_+ \overbrace{J_z|a,b\rangle}^{\text{know this}} + [J_z, J_+]|a,b\rangle$$

$$= J_+\, b\,|a,b\rangle + \hbar J_+|a,b\rangle$$

$$= \underbrace{(b+\hbar)}_{\text{eigenvalue of } J_z}\left(J_+|a,b\rangle\right)$$

eigenstates of J_z & J^2.

$$J_z\left(J_-|a,b\rangle\right) = J_- J_z|a,b\rangle + [J_z, J_-]|a,b\rangle$$

$$= J_-\, b|a,b\rangle - \hbar J_-|a,b\rangle$$

$$= \underbrace{(b-\hbar)}_{\text{eigenvalue of } J_z}\left(J_-|a,b\rangle\right)$$

J_+ $+\hbar$

J_- $-\hbar$

Consider also

$$J^2 J_+|a,b\rangle = J_+ J^2|a,b\rangle = \underbrace{a}_{\text{eigenvalue of } J^2}\, J_+|a,b\rangle$$

$$J^2 J_-|a,b\rangle = J_- J^2|a,b\rangle = a\, J_-|a,b\rangle$$

So
$$J_+|a,b\rangle = c_+|a, b+\hbar\rangle$$
$$J_-|a,b\rangle = c_-|a, b-\hbar\rangle$$

<u>What are possible values for a and b?</u>

We use the idea of norms as before: (recall $\||\alpha n\rangle\|^2 \geq 0$ was used earlier)

$$\|J_+|a,b\rangle\|^2 \geq 0$$

$$\shortparallel$$

$$\langle a,b|J_+^\dagger J_+|a,b\rangle$$

$$\shortparallel$$

$$\langle a,b|J_- J_+|a,b\rangle \quad (J_x^2+J_y^2+J_z^2)$$

eigenvector for J^2 and J_z.

$$\shortparallel$$

$$\langle a,b|J^2 - J_z^2 - \hbar J_z|a,b\rangle$$

$$\shortparallel$$

$$a - b^2 - \hbar b$$

$$\boxed{\begin{aligned} &J_- J_+ \\ &= (J_x - iJ_y)(J_x + iJ_y) \\ &= J_x^2 + J_y^2 - i[J_y, J_x] \\ &= J^2 - J_z^2 - i(-i\hbar J_z) \\ &= J^2 - J_z^2 - \hbar J_z \end{aligned}}$$

$(J_x^2|a,b\rangle = ?)$

So $\quad a - b^2 - \hbar b \geq 0$

$$b^2 + \hbar b - a \leq 0 \quad \text{—} \quad (1)$$

$$\boxed{\begin{aligned} J_+ J_- &= (J_x + iJ_y)(J_x - iJ_y) \\ &= J_x^2 + J_y^2 + i[J_y, J_x] \\ &= J^2 - J_z^2 + \hbar J_z \end{aligned}}$$

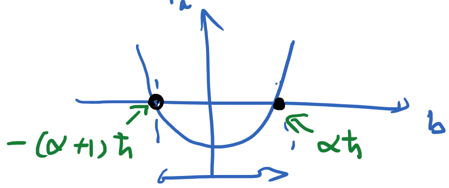
Likewise $\quad \| J_- |a,b\rangle \|^2 \geq 0$

$$\begin{aligned} &\| J_- |a,b\rangle \|^2 \\ &= \langle a,b| J_-^{\dagger} J_- |a,b\rangle \\ &= \langle a,b| J_+ J_- |a,b\rangle \\ &= \langle a,b| J^2 - J_z^2 + \hbar J_z |a,b\rangle \\ &= a - b^2 + \hbar b \end{aligned}$$

So $\quad a - b^2 + \hbar b \geq 0$

$$b^2 - \hbar b - a \leq 0 \quad \text{—} \quad (2)$$

(1): $\quad f_a(b) = b^2 + \hbar b - a$



For (1) to hold,

$$-(\alpha+1)\hbar \leq b \leq \alpha\hbar$$

(1): $\quad b^2 + \hbar b - a \leq 0$

When $b^2 + \hbar b - a = 0$,

$$\begin{aligned} b &= \frac{-\hbar \pm \sqrt{\hbar^2 - 4(-a)}}{2} \\ &= \frac{\hbar}{2}\left(-1 \pm \sqrt{1 + 4\gamma}\right), \quad a = \hbar^2 \gamma \\ &= \frac{\hbar}{2}\left(-1 \pm \sqrt{(2\alpha+1)^2}\right) \end{aligned}$$

where $\gamma = \alpha(\alpha+1)$, $\alpha \geq 0$

i) a is the eigenvalue for J^2

$a \geq 0$

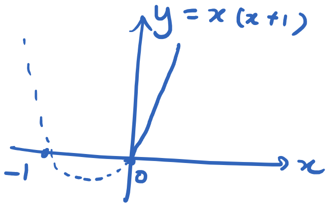
$\gamma \geq 0$

ii) $\gamma \geq 0 \Rightarrow \gamma = \alpha(\alpha+1)$, $\alpha \geq 0$

We can write any real, non-negative number as $\alpha(\alpha+1)$, $\alpha \geq 0$

$$\begin{aligned} &= \frac{\hbar}{2}\left(-1 \pm (2\alpha+1)\right) \\ &= \begin{cases} \frac{\hbar}{2}(2\alpha) = \alpha\hbar \\ \frac{\hbar}{2}\left(-1 - (2\alpha+1)\right) = -(\alpha+1)\hbar \end{cases} \end{aligned}$$

$$\left(1 + 4\alpha(\alpha+1) = 4\alpha^2 + 4\alpha + 1 = (2\alpha+1)^2 \right)$$



For (1) to hold,

$$-(\alpha+1)\hbar \le b \le \alpha\hbar \quad \text{—} \quad (1)'$$

$$a = \hbar^2\alpha(\alpha+1)$$

For (2) to hold,

...

$$-\alpha\hbar \le b \le (\alpha+1)\hbar \quad \text{—} \quad (2)'$$

$\alpha \geq 0$

For **BOTH** (1)' & (2)' to hold,

$$-\alpha\hbar \le b \le \alpha\hbar$$

from (2) from (1)

$J_-|a, b_{min}\rangle = 0$ $J_+|a, b_{max}\rangle = 0$

(recall $a|0\rangle = 0$)