Tutorial 2: Solutions

1. Rotation operator for a spin-1/2 system

(a) To prove: $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B}\,\mathbb{1} + i\sigma\cdot(\vec{A}\times\vec{B})$,

$$\vec{\sigma}\cdot\vec{A} = \sigma_i A_i = \sigma_1 A_1 + \sigma_2 A_2 + \sigma_3 A_3$$

Similar for $\vec{\sigma}\cdot\vec{B}$.

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \sigma_i A_i \sigma_j B_j = \sigma_i\sigma_j A_i B_j$$

Commutator

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\epsilon_{ijk}\sigma_k \tag{1}$$

Anti-commutator

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1} \tag{2}$$

To get $\sigma_i\sigma_j$, (1)+(2):

$$2\sigma_i\sigma_j = 2i\epsilon_{ijk}\sigma_k + 2\delta_{ij}\mathbb{1}$$
$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{1}$$

LHS gives:

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = A_i B_j(i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{1}) = i\sigma_k\epsilon_{ijk}A_iB_j + A_iB_i\mathbb{1}$$

RHS:

$$\vec{A}\cdot\vec{B}\mathbb{1} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}) = A_iB_i\mathbb{1} + i\sigma_k(\vec{A}\times\vec{B})_k = A_iB_i\mathbb{1} + i\sigma_k\epsilon_{ijk}A_iB_j = \text{LHS}$$

(b) Let $\vec{A} = \vec{B} = \hat{u}$, $\hat{u}$ is unit vector,

$$(\vec{\sigma}\cdot\hat{u})(\vec{\sigma}\cdot\hat{u}) = \hat{u}\cdot\hat{u}\mathbb{1} + i\vec{\sigma}\cdot(\hat{u}\times\hat{u}) = \mathbb{1} + 0 = \mathbb{1}$$
$$(\vec{\sigma}\cdot\hat{u})^2 = \mathbb{1}$$
$$(\vec{\sigma}\cdot\hat{u})^3 = (\vec{\sigma}\cdot\hat{u})^2(\vec{\sigma}\cdot\hat{u}) = (\vec{\sigma}\cdot\hat{u})$$
$$(\vec{\sigma}\cdot\hat{u})^n = \begin{cases} \mathbb{1} & n \text{ even} \\ \vec{\sigma}\cdot\hat{u} & n \text{ odd} \end{cases}$$

(c) To prove rotaton operators for spin-1/2, recall from class

$$U(\phi, \hat{n}) = \exp(-i\phi\hat{n} \cdot \frac{\vec{J}}{\hbar}) \quad \text{(rotation operator)}$$

Let $\phi = \alpha, \quad \vec{J} = \vec{S} = \frac{\hbar}{2}\sigma; \quad \frac{\vec{J}}{\hbar} = \frac{\vec{\sigma}}{2}$.

$$\begin{aligned}
U(\alpha, \hat{u}) &= \exp(-i\alpha\hat{u} \cdot \frac{\vec{\sigma}}{2}) \\
&= \mathbb{1} - \frac{i\alpha}{2}\vec{\sigma} \cdot \hat{u} + \frac{1}{2!}(-\frac{i\alpha}{2}\vec{\sigma} \cdot \hat{u})^2 + ... + \frac{1}{n!}(-\frac{i\alpha}{2}\vec{\sigma} \cdot \hat{u})^n + ... \\
&= [\mathbb{1} - \frac{1}{2!}(\frac{\alpha}{2})^2 \cdot \mathbb{1} + ... + \frac{(-1)^p}{(2p)!}(\frac{\alpha}{2})^{2p} \cdot \mathbb{1} + ...] \quad \text{(grouped all the even terms, p is an integer)} \\
&- i\vec{\sigma} \cdot \hat{u}[\frac{\alpha}{2} - \frac{1}{3!}(\frac{\alpha}{2})^3 + ... + \frac{(-1)^p}{(2p+1)!}(\frac{\alpha}{2})^{2p+1} + ...] \quad \text{(odd terms)} \\
&= \cos\frac{\alpha}{2}\mathbb{1} - i\vec{\sigma} \cdot \hat{u}\sin\frac{\alpha}{2}
\end{aligned}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\sigma} \cdot \hat{u} = \sigma_x u_x + \sigma_y u_y + \sigma_z u_z$$

$$\begin{aligned}
U(\alpha, \hat{u}) &= \begin{pmatrix} \cos\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2} \end{pmatrix} + \sin\frac{\alpha}{2}\begin{pmatrix} -iu_z & -iu_x - u_y \\ -iu_x + u_y & iu_z \end{pmatrix} \\
&= \begin{pmatrix} \cos\frac{\alpha}{2} - iu_z\sin\frac{\alpha}{2} & (-iu_x - u_y)\sin\frac{\alpha}{2} \\ (-iu_x + u_y)\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} + iu_z\sin\frac{\alpha}{2} \end{pmatrix}
\end{aligned}$$

(d) To rotate the eigenstate $|+z\rangle$ to $|+x\rangle$,

$$\hat{u} = \hat{y}, \quad \alpha = \frac{\pi}{2}, \quad |+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|+x\rangle = U(\frac{\pi}{2}, \hat{y})|+z\rangle = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$$

This expression for $|+x\rangle$ is the same as that obtained by diagonalizing σ_x (i.e. finding the eigenvalues and then eigenvectors) as shown in W3L2 (Page 5 of the uploaded pdf).

2. Matrix representations of angular momentum

(a) For $|\frac{3}{2}, m\rangle$, $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$.

$$J_z|\frac{3}{2}, m\rangle = m\hbar|\frac{3}{2}, m\rangle$$

$$J_z = \hbar \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

Recall $J_+|j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}|j, m+1\rangle$.

$$\begin{aligned}
J_+|\frac{3}{2}, \frac{3}{2}\rangle &= 0 \\
J_+|\frac{3}{2}, \frac{1}{2}\rangle &= \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2} - \frac{1}{2}\cdot\frac{3}{2}}|\frac{3}{2}, \frac{3}{2}\rangle = \sqrt{3}\hbar|\frac{3}{2}, \frac{3}{2}\rangle \\
J_+|\frac{3}{2}, -\frac{1}{2}\rangle &= \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2} - (-\frac{1}{2})\cdot\frac{1}{2}}|\frac{3}{2}, \frac{1}{2}\rangle = 2\hbar|\frac{3}{2}, \frac{1}{2}\rangle \\
J_+|\frac{3}{2}, -\frac{3}{2}\rangle &= \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2} - (-\frac{3}{2})\cdot(-\frac{1}{2})}|\frac{3}{2}, -\frac{1}{2}\rangle = \sqrt{3}\hbar|\frac{3}{2}, -\frac{1}{2}\rangle
\end{aligned}$$

Recall matrix elements of operator A are determined by $\langle\phi_m|A|\phi_n\rangle$ for m$^{\text{th}}$ row and n$^{\text{th}}$ column.

From $J_+|\frac{3}{2}, \frac{3}{2}\rangle = 0$, the 1$^{\text{st}}$ column of J_+ are all zeros.

$\langle\frac{3}{2}, \frac{3}{2}|J_+|\frac{3}{2}, \frac{1}{2}\rangle = \langle\frac{3}{2}, \frac{3}{2}|\sqrt{3}\hbar|\frac{3}{2}, \frac{3}{2}\rangle = \sqrt{3}\hbar$, other entries in the 2$^{\text{nd}}$ column of J_+ are zeros.

...

$$J_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Similarly,

$$\begin{aligned}
J_-|\frac{3}{2}, \frac{3}{2}\rangle &= \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2} - \frac{3}{2}\cdot\frac{1}{2}}|\frac{3}{2}, \frac{1}{2}\rangle = \sqrt{3}\hbar|\frac{3}{2}, \frac{1}{2}\rangle \\
J_-|\frac{3}{2}, \frac{1}{2}\rangle &= \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2} - \frac{1}{2}\cdot(-\frac{1}{2})}|\frac{3}{2}, -\frac{1}{2}\rangle = 2\hbar|\frac{3}{2}, -\frac{1}{2}\rangle \\
J_-|\frac{3}{2}, -\frac{1}{2}\rangle &= \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2} - (-\frac{1}{2})\cdot(-\frac{3}{2})}|\frac{3}{2}, -\frac{3}{2}\rangle = \sqrt{3}\hbar|\frac{3}{2}, -\frac{3}{2}\rangle \\
J_-|\frac{3}{2}, -\frac{3}{2}\rangle &= 0
\end{aligned}$$

$$J_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$\left.\begin{aligned} J_+ &= J_x + iJ_y \\ J_- &= J_x - iJ_y \end{aligned}\right\} \quad J_x = \frac{1}{2}(J_+ + J_-); \quad J_y = \frac{1}{2i}(J_+ - J_-)$$

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad J_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

(b) Verify $[J_x, J_y] = i\hbar J_z$.

$$\begin{aligned} [J_x, J_y] &= J_x J_y - J_y J_x \\ &= \frac{\hbar^2}{4i}\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} - \frac{\hbar^2}{4i}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \\ &= i\hbar^2\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \\ &= i\hbar J_z \end{aligned}$$