CP violation in the weak decays of kaon mesons.

The neutral kaon mesons are produced by strong interaction, K^0, $\overline{K^0}$ $K^0 = d\bar{s}$, $\overline{K^0} = \bar{d}s$

But they decay as K_L, and K_S

Three features

(i) Oscillation $K^0 \rightleftharpoons \overline{K^0}$

(ii) kaon decay, two different lifetimes
τ_L, τ_S, decay as 2 pions, or 3 pions

(iii) CP is minutely violated.

K_S : $(|K^0\rangle - |\overline{K^0}\rangle)$, $CP = +1$ $\because$ $CP|K^0\rangle = -|\overline{K^0}\rangle$
$CP|\overline{K^0}\rangle = -|K^0\rangle$

K_L : $(|K^0\rangle + |\overline{K^0}\rangle)$, $CP = -1$

$CP\,|K^0\rangle = -|\overline{K^0}\rangle$, $CP\,|\overline{K^0}\rangle = -|K^0\rangle$

$CP\,|\pi^0\rangle = -|\pi^0\rangle$, 2 pions, $CP = +1$
3 pions $CP = -1$

If CP is conserved

$K_S \to 2$ pions, $K_L \to 3$ pions

1964 experiment Cronin - Fitch

source $\cdot$ K^0 K_S $\bar{K}^0$ K_L produced $\longrightarrow$ $\cdot$ 57 ft $\approx$ 17.4 m away Detectors.

Only K_L remains.

expect to see only 3-pion decays

But found 45 2-pion decays among 22700 decays

If CP is conserved, K_L can only decay into 3 pions

This experiment indicates CP violation only violated minutely

$$\frac{45}{22700} \sim 2 \times 10^{-3} \approx 0.2\%$$

$K_2 \to 2\pi$ (rare decay)

Work out the mathematics of kaons

(i) Oscillation $K^0 \rightleftharpoons \overline{K^0}$

(ii) CP violation

(iii) Decay of kaons. K_L, K_S can decay with different life times τ_L, τ_S

The idea:

(i) start with K^0, $\overline{K^0}$ produced by strong interaction.

Treating as 2-state physical system, using QM to explain occuring of oscillation.

(similar to coupled pendulum problem in classical mechanics)

(ii) K^0, $\overline{K^0}$ are not eigenstates of CP.

construct CP eigenstates K_1, K_2

$$K_1 = \frac{1}{\sqrt{2}}(K^0 - \overline{K^0}), \qquad K_2 = \frac{1}{\sqrt{2}}(K^0 + \overline{K^0})$$

Note: K_1, K_2 just mathematical construct,

actual physical particles are K^0, $\overline{K^0}$; K_L; K_S

Modify the states K_1, K_2 to get slightly CP broken states K_I, K_{II}

(iii) Introduce 'effective' Hamiltonian to explain decay. Change the actual Hamiltonian H to a non-Hermitian $\bar{H}$, $\bar{H} \neq \bar{H}^\dagger$

Use quantum mechanics to account for the oscillation. The kaon is a 2-state system

$\{K^0, \bar{K}^0\} \longrightarrow \{K_S, K_L\}$

produced by strong interaction

Decay by wk interaction

Assume CP is conserved, we have shown

short lifetime $|K_S\rangle \sim |K^0\rangle - |\bar{K}^0\rangle \rightarrow CP = +1$

Long life time $|K_L\rangle \sim |K^0\rangle + |\bar{K}^0\rangle \longrightarrow CP = -1$

$CP|K^0\rangle = C(-)|K^0\rangle$
$= -C|K^0\rangle$
$= -|\bar{K}^0\rangle$

Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

$$H = H_{st} + H_{em} + H_{wk} = H_0 + H_{wk}$$

$H_0 \equiv H_{st} + H_{em}$

Discrete basis for n-state system

$$|\psi\rangle = \sum_{i=1}^{n} c_i |i\rangle$$

$i = 1, 2, \ldots n$

General expression

$$i\hbar \frac{\partial}{\partial t} c_i(t) = \sum_{j=1}^{n} H_{ij} c_j(t)$$

$i = 1, 2, \ldots n$

$$H_{ij} = \langle i|H|j\rangle$$

For 2-state system

$$i\hbar\, \boxed{\partial_t c_1(t)} = H_{11} c_1(t) + H_{12} c_2(t)$$

$$i\hbar\, \boxed{\partial_t c_2(t)} = H_{21} c_1(t) + H_{22} c_2(t)$$

$\xrightarrow{\text{notation}}$ $\partial_t c_1 \equiv \frac{\partial c_1}{\partial t}$

$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

c_1, c_2 unknown

To solve this, choose a basis.

A basis can be $(|K^0\rangle, |\bar{K}^0\rangle)$ or (K_S, K_L)

We choose

$$|K^0\rangle = |1\rangle, \quad |\bar{K}^0\rangle = |2\rangle.$$

Hence

$$H_{11} = \langle 1 | H | 1\rangle = \langle K^0 | H | K^0\rangle$$

$$= \langle K^0 | H_0 + H_{wk} | K^0\rangle$$

$$= \langle K^0 | H_0 | K^0\rangle + \langle K^0 | H_{wk} | K^0\rangle$$

Assume $|K^0\rangle, |\bar{K}^0\rangle$ are eigenstates of H_0

$= H_{st} + H_{em}$

$$H_0 |K^0\rangle = E_0 |K^0\rangle$$

$$H_0 |\bar{K}^0\rangle = E_0 |\bar{K}^0\rangle$$

$$\therefore \quad H_{11} = E_0 \langle K^0 | K^0\rangle + \langle K^0 | H_{wk} | K^0\rangle$$

$$= E_0 + \underbrace{\langle K^0 | H_{wk} | K^0\rangle}_{\rightarrow \text{small}}$$

$$\simeq E_0$$

$$\rightarrow \quad H_{22} = E_0$$

$$H_{12} = \langle k^0 | H | \bar{k}^0 \rangle$$

$$= \langle k^0 | H_0 + H_{wk} | \bar{k}^0 \rangle$$

$$= \langle k^0 | H_0 | \bar{k}^0 \rangle + \langle k^0 | H_{wk} | \bar{k}^0 \rangle$$

$$= E_0 \langle k^0 | \bar{k}^0 \rangle + \langle k^0 | H_{wk} | \bar{k}^0 \rangle$$

$$= \langle k^0 | H_{wk} | \bar{k}^0 \rangle = -A$$

$$\rightarrow \quad i\hbar \frac{\partial}{\partial t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\rightarrow \quad i\hbar \frac{\partial}{\partial t} c_1 = E_0 c_1 - A c_2$$

$$i\hbar \frac{\partial}{\partial t} c_2 = -A c_1 + E_0 c_2$$

unknown c_1, c_2 $\qquad \partial_t \equiv \frac{\partial}{\partial t}$

Adding, $i\hbar \partial_t (c_1 + c_2) = (E_0 - A) c_1 + (E_0 - A) c_2$

$$= (E_0 - A)(c_1 + c_2)$$

subtracting $i\hbar \partial_t (c_1 - c_2) = (E_0 + A)(c_1 - c_2)$

solution obvious

differentiate LHS yielded same expression but multiplied by a constant so ansatz (c1+c2) yields an exponential expressn

$$c_1 + c_2 = a\, e^{-\frac{i}{\hbar}(E_0 - A)t}$$

a = arbitrary constant

$$c_1 - c_2 = b\, e^{-\frac{i}{\hbar}(E_0 + A)t}$$

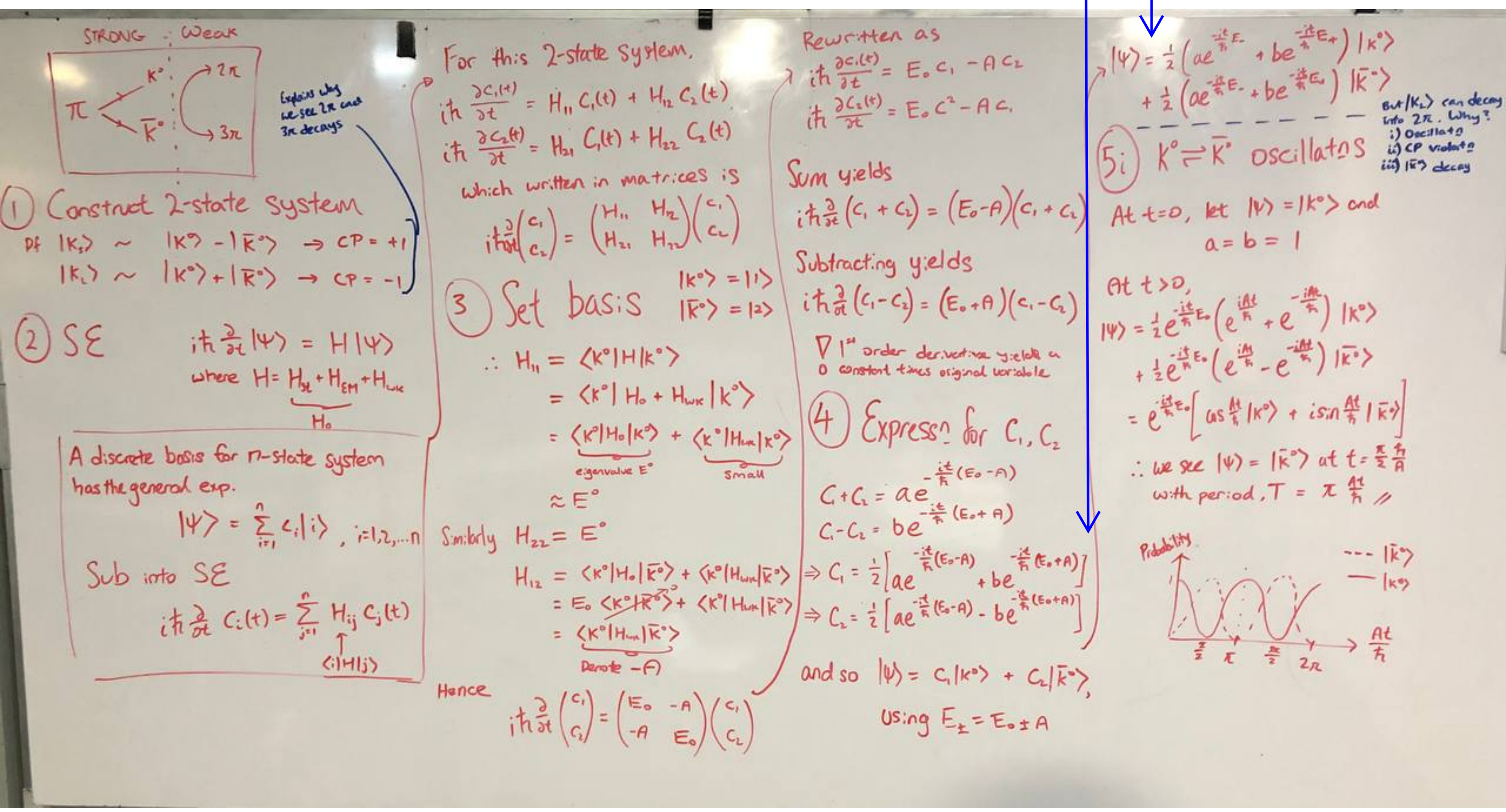
$\rightarrow \quad c_1 =$

HW

$c_2 =$

$\therefore \quad |\psi\rangle = c_1 |K^0\rangle + c_2 |\bar{K}^0\rangle$

$=$



$$|\psi\rangle = c_1 |K^0\rangle + c_2 |\bar{K}^0\rangle$$

$$= \frac{1}{2}\left(a e^{-iE_2 t/\hbar} + b e^{-iE_1 t/\hbar}\right) |K^0\rangle$$

$E_2 = E_0 - A$

$E_1 = E_0 + A$

$$+ \frac{1}{2}\left(a e^{-iE_2 t/\hbar} - b e^{-iE_1 t/\hbar}\right) |\bar{K}^0\rangle \qquad \text{(HW)}$$

3 different situations

(i) At time $t = 0$, let $|\psi\rangle = |K^0\rangle$

$$a = b = 1$$

So at time $t \geq 0$

$$|\psi\rangle = \frac{1}{2} e^{-iE_0 t/\hbar}\left(e^{iAt/\hbar} + e^{-iAt/\hbar}\right) |K^0\rangle$$

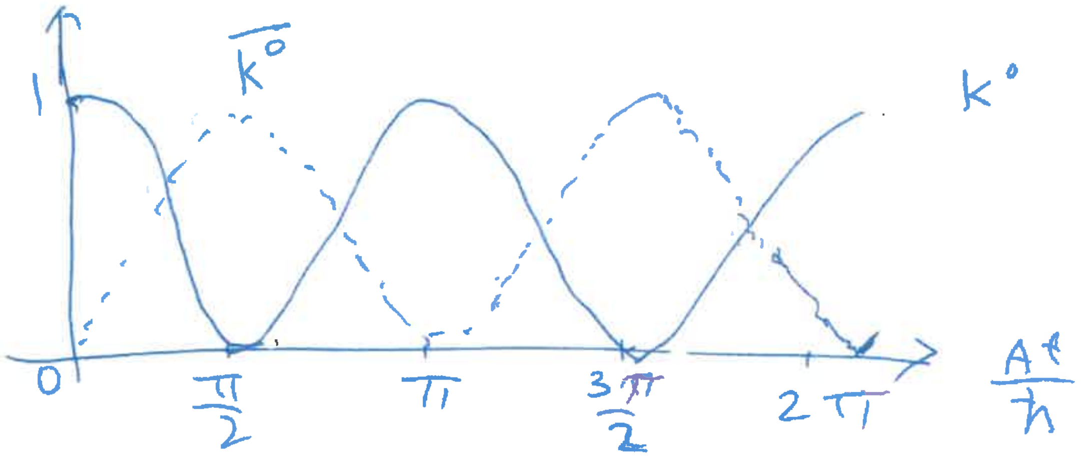
$$+ \frac{1}{2} e^{-iE_0 t/\hbar}\left(e^{iAt/\hbar} - e^{-iAt/\hbar}\right) |\bar{K}^0\rangle$$

$$= e^{-iE_0 t/\hbar}\left[\cos\frac{At}{\hbar} |K^0\rangle + i \sin\frac{At}{\hbar} |\bar{K}^0\rangle\right]$$

This explains oscillation $\because$ at $t = 0$,

$|\psi\rangle = |K^0\rangle$, at $t = \frac{\pi}{2}\frac{\hbar}{A}$, $|\psi\rangle = |\bar{K}^0\rangle$

period of oscillation = HW



So far, oscillation $K^0 \rightleftharpoons \overline{K^0}$ is explained and is due to off-diagonal elements $H_{12} = H_{21} = -A \neq 0$

Next consider

(ii) Assume $b = 0$ (arbitrary constant), the state

$$|\psi\rangle = \frac{a}{2} e^{-iE_2 t/\hbar} |K^0\rangle + \frac{a}{2} e^{-iE_2 t/\hbar} |\overline{K^0}\rangle$$

$$= \frac{a}{2} e^{-iE_2 t/\hbar} \left(|K^0\rangle + |\overline{K^0}\rangle \right)$$

$\langle \psi | \psi \rangle = 1$, $\quad \frac{|a|^2}{4}(1 + 1) = \frac{|a|^2}{2}$

$$\rightarrow \quad a = \sqrt{2}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} e^{-iE_2 t/\hbar} \left(|K^0\rangle + |\overline{K^0}\rangle \right)$$

$$= e^{-iE_2 t/\hbar} |K_2\rangle, \quad |K_2\rangle \equiv \frac{1}{\sqrt{2}} \left(|K^0\rangle + |\overline{K^0}\rangle \right)$$

So the $|K_2\rangle$ is an eigenstate of the Hamiltonian H with eigenvalue $E_2 = E_0 - A$

i.e. $H|k_2\rangle = E_2|k_2\rangle$

proof:
substitute $|\psi\rangle = e^{-iE_2t/\hbar}|k_2\rangle$ into the S.E.

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

$$LHS = e^{-\frac{iE_2t}{\hbar}} \cdot E_2 |k_2\rangle$$

$$RHS = H|\psi\rangle = H e^{-iE_2t/\hbar}|k_2\rangle$$

$$= e^{-iE_2t/\hbar} H|k_2\rangle$$

$\xrightarrow{LHS = RHS}$ $H|k_2\rangle = E_2|k_2\rangle$

Note $|k_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle)$ is also an eigenstate of CP

$$CP|k_2\rangle = \frac{1}{\sqrt{2}}(CP|K^0\rangle + CP|\bar{K}^0\rangle)$$

$$= -|k_2\rangle \qquad (\text{H w})$$

i.e. $|k_2\rangle$ is a common eigenstate of H and CP.

can we identify $|k_2\rangle$ as $|k_L\rangle$?

Assume $a = 0$. Find $|\psi\rangle =$ H w

$$\frac{1}{2} b e^{-iE_1t/\hbar}(|K^0\rangle - |\bar{K}^0\rangle)$$

$$= e^{-iE_1t/\hbar}|K_1\rangle \quad |k_1\rangle \equiv \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle)$$

$$\rightarrow \; |K_1\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right)$$

$$H\,|K_1\rangle = E_1\,|K_1\rangle\,, \qquad E_1 = E_0 + A$$

so $|K_1\rangle$ is a common eigenstate of CP and the Hamiltonian, i.e.

$$[CP, H] = 0$$

for the state $|K_1\rangle$ and $|K_2\rangle$

At this stage, we cannot identify $|K_1\rangle \sim |K_S\rangle$, $|K_2\rangle \sim |K_L\rangle$

because K_1 conserves CP but not K_S; similarly K_2 cannot be identified as K_L.

Next to construct a model that accounts for CP violation; one way is to write

$$|K_S\rangle \approx \frac{\left(|K_1\rangle + \varepsilon\,|K_2\rangle\right)}{\sqrt{1+|\varepsilon|^2}}$$

$|\varepsilon|$ small parameter

$$|K_L\rangle = \frac{|K_2\rangle + \varepsilon' |K_1\rangle}{\sqrt{1+|\varepsilon'|^2}} \qquad (11)$$

$|\varepsilon'|$ small ~ 0

Another way to incorporate CP violation is to write

$$|K_I\rangle = \frac{1}{\sqrt{|p|^2+|q|^2}}\left(p|K^0\rangle - q|\bar{K}^0\rangle\right)$$

$$|K_{II}\rangle = \frac{1}{\sqrt{|p|^2+|q|^2}}\left(p|K^0\rangle + q|\bar{K}^0\rangle\right)$$

$|p| \sim 1$, $|q| \sim 1$

Note if $|p|=1$, $|q|=1$, then $|K_I\rangle = |K_1\rangle$

$|K_{II}\rangle = |K_2\rangle$.

solve the S.E. all over again. Note CP is violated minutely at the beginning

by assuming $|p| \sim 1$ but $|p| \neq 1$

$|q| \sim 1$ but $|q| \neq 1$

This treatment does not explain the decay of K_S, K_L, just accounts for CP violation.

c(ii)

To account for CP violation, and decay of K_s, K_L particles, we introduce the basis and an 'effective' Hamiltonian $\bar{H}$.

$$|K_{I,II}\rangle = \frac{1}{\sqrt{|p|^2 + |q|^2}}\left(p|K^0\rangle \mp q|\bar{K}^0\rangle\right)$$

$\bar{H}|K_I\rangle = E_I|K_I\rangle$, $\bar{H}|K_{II}\rangle = E_{II}|K_{II}\rangle$

The state of the system at time t is

$$|\psi\rangle = a_I(t)|K_I\rangle + a_{II}(t)|K_{II}\rangle$$

and can be found by solving the S.E.

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \bar{H}|\psi\rangle, \quad \bar{H} = H - \frac{i\Gamma}{2}$$

$\bar{H}$ = effective Hamiltonian

$H^\dagger = H$, $\Gamma^\dagger = \Gamma$

$\bar{H}^\dagger \neq \bar{H}$

H is hermitian and so is Γ but Hbar is not hermitian

Instead of solving the S.E. we use evolution operator

$$|\psi(t)\rangle = e^{-i\bar{H}t/\hbar}|\psi(0)\rangle$$

$$e^{-i\bar{H}t/\hbar} = \text{evolution operator}$$

Let the state at time $t=0$ be

$$|\psi(0)\rangle = a_I(0)|K_I\rangle + a_{II}(0)|K_{II}\rangle$$

$$\equiv a_I|K_I\rangle + a_{II}|K_{II}\rangle$$

$a_I = a_I(0)$, $a_{II} = a_{II}(0)$ constants

13

$$|\psi(t)\rangle = e^{-i\bar{H}t/\hbar}|\psi(0)\rangle$$

$$= a_I e^{-i\bar{H}t/\hbar}|K_I\rangle + a_{II} e^{-i\bar{H}t/\hbar}|K_{II}\rangle$$

$$= a_I e^{-iE_I t/\hbar}|K_I\rangle + a_{II} e^{-iE_{II}t/\hbar}|K_{II}\rangle$$

They are not quite 'energies' that are observable. Rather, they are complex eigenvalues

$$E_I = E_1 - i\frac{\Gamma_1}{2} = E_0 + A - \frac{i\Gamma_1}{2}$$

think

$$E_{II} = E_2 - i\frac{\Gamma_2}{2} = E_0 - A - \frac{i\Gamma_2}{2}$$

They are 'constructed' in the same way we constructed the effective Hamiltonian

$$\Gamma \underset{\uparrow}{=} \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}$$

basis $|K_I\rangle, |K_{II}\rangle$

$$= e^{-iE_0 t/\hbar}\Big(a_I e^{-i(A-\frac{i\Gamma_1}{2})t/\hbar}|K_I\rangle$$

HW

$$+ a_{II} e^{+i(A+i\frac{\Gamma_2}{2})t/\hbar}|K_{II}\rangle$$

Now express in terms of K^0, $\overline{K^0}$

$$|\psi(t)\rangle = e^{-iE_0 t/\hbar} N\Big[a_I e^{-i(A-\frac{i\Gamma_1}{2})t/\hbar}\,(p|K^0\rangle - q|\overline{K^0}\rangle)$$

$$+ a_{II} e^{i(A+\frac{i\Gamma_2}{2})t/\hbar}(p|K^0\rangle + q|\overline{K^0}\rangle)\Big]$$

$$N = \frac{1}{\sqrt{|p|^2+|q|^2}}$$

Assume $|\psi(0)\rangle = |K^0\rangle$, $a_I = a_{II} = \frac{1}{2Np}$

HW

$$|\psi(t)\rangle = \frac{1}{2} e^{-iE_0 t/\hbar} \left[\left(e^{-i\frac{At}{\hbar}} e^{-\frac{\Gamma_1 t}{2\hbar}} + e^{i\frac{At}{\hbar}} e^{-\frac{\Gamma_2 t}{2\hbar}} \right) |K^0\rangle - \frac{q}{p} \left(e^{-i\frac{At}{\hbar}} e^{-\frac{\Gamma_1 t}{2\hbar}} - e^{i\frac{At}{\hbar}} e^{-\frac{\Gamma_2 t}{2\hbar}} \right) |\bar{K}^0\rangle \right]$$

prob. of finding K^0 at time t

$$= \frac{1}{4} \left[e^{-\frac{\Gamma_1 t}{\hbar}} + e^{-\frac{\Gamma_2 t}{\hbar}} + 2 \cos\frac{2At}{\hbar} e^{-\bar{\Gamma} t/\hbar} \right]$$

$$\bar{\Gamma} \equiv \frac{\Gamma_1 + \Gamma_2}{2} \qquad \text{(HW)}$$

prob of getting $\bar{K}^0$ at time t

$$= \frac{1}{4} \left|\frac{q}{p}\right|^2 \left[e^{-\Gamma_1 t/\hbar} + e^{-\frac{\Gamma_2 t}{\hbar}} - 2\cos\frac{2At}{\hbar} e^{-\bar{\Gamma} t/\hbar} \right]$$

$2A = 3.5 \times 10^{-6}\, eV/c^2$, $\quad \Gamma_1 = \frac{\hbar}{\tau_S}$, $\quad \Gamma_2 = \frac{\hbar}{\tau_L}$

$\tau_S = 0.89 \times 10^{-10}\, s$, $\quad \tau_L = 5.2 \times 10^{-8}\, s$

$\hbar = 6.582 \times 10^{-22}\, MeV \cdot s$

Plot probs. vs time (HW) for different values of A, Γ_1, Γ_2

15

Page 11-15

Now remember that K^0 and $\bar{K}^0$ are each linear combinations of K_1 and K_2. In Eqs. (11.54) the amplitudes have been chosen so that at $t = 0$ the $\bar{K}^0$ parts cancel each other out by interference, leaving only a K^0 state. But the $|K_1\rangle$ state *changes with time*, and the $|K_2\rangle$ state *does not*. After $t = 0$ the interference of C_1 and C_2 will give finite amplitudes for both K^0 and $\bar{K}^0$.

What does all this mean? Let's go back and think of the experiment we sketched in Fig. 11-5. A π^- meson has produced a Λ^0 particle and a K^0 meson which is tooting along through the hydrogen in the chamber. As it goes along, there is some small but uniform chance that it will collide with a hydrogen nucleus. At first, we thought that strangeness conservation would prevent the K-particle from making a Λ^0 in such an interaction. Now, however, we see that that is not right. For although our K-particle *starts out* as a K^0—which cannot make a Λ^0—it does not *stay* this way. After a while, there *is some amplitude* that it will have flipped to the $\bar{K}^0$ state. We can, therefore, sometimes expect to see a Λ^0 produced along the K-particle track. The chance of this happening is given by the amplitude C_-, which we can [by using Eq. (11.50)) backwards] relate to C_1 and C_2. The relation is

$$C_- = \frac{1}{\sqrt{2}}(C_1 - C_2) = \frac{1}{2}(e^{-\beta t}e^{-i\alpha t} - 1).$$

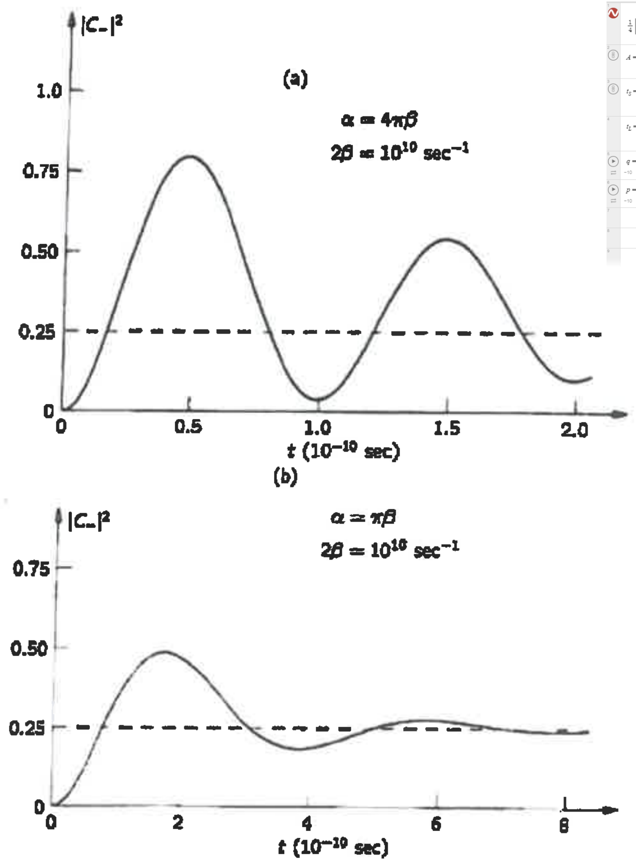
As our K-particle goes along, the probability that it will "act like" a $\bar{K}^0$ is equal to $|C_-|^2$, which is

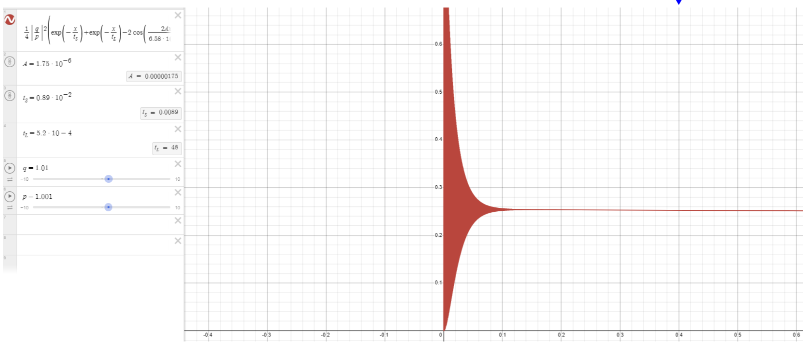
$$|C_-|^2 = \frac{1}{4}(1 + e^{-2\beta t} - 2e^{-\beta t}\cos\alpha t).$$

A complicated and strange result!

This, then, is the remarkable prediction of Gell-Mann and Pais: when a K^0 is produced, the chance that it will turn into a $\bar{K}^0$—as it can demonstrate by being able to produce a Λ^0—varies with time according to Eq. (11.56). This prediction came from using only sheer logic and the basic principles of the quantum mechanics—with no knowledge at all of the inner workings of the K-particle. Since nobody knows anything about the inner machinery, that is as far as Gell-Mann and Pais could go. They could not give any theoretical values for α and β. And nobody has been able to do so to this date. They were able to give a value of β obtained from the experimentally observed rate of decay into two π's ($2\beta = 10^{10}$ sec^{-1}), but they could say nothing about α.

We have plotted the function of Eq. (11.56) for two values of α in Fig. 11-6. You can see that the form depends very much on the ratio of α to β. There is no $\bar{K}^0$ probability at first; then it builds up. If α is large, the probability would have large oscillations. If α is small, there will be little or no oscillation—the probability will just rise smoothly to 1/4





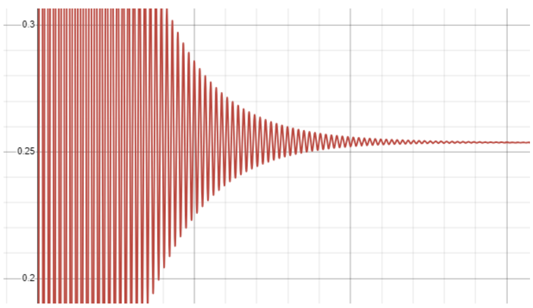


Fig. 11-6. The function of Eq. (11.56): (a) for $\alpha = 4\pi\beta$, (b) for $\alpha = \pi\beta$ (with $2\beta = 10^{10}$ sec^{-1}).

Now, typically, the K-particle will be travelling at a constant speed near the speed of light. The curves of Fig. 11-6 then also represent the probability along the track of observing a $\bar{K}^0$—with typical distances of several centimeters. You can see why this prediction is so remarkably peculiar. You produce a single particle and instead of just disintegrating, it does something else. Sometimes it disintegrates, and other times it turns into a different kind of a particle. Its characteristic

Recap

Kaon decays and CP violation

1964 Experiment. Cronin - Fitch

strong interaction

weak decay

Mostly 3-pion decay.
seen 45 2-pion decay
among 22700 decays

Theoretical understanding.

(1) Hamiltonian $H = H_{st} + H_{em} + H_{wk} \equiv H_0 + H_{wk}$

Kaons $K^0, \overline{K^0}$ eigenstates of H_0 but not H;

H_{wk} causes the oscillation $K^0 \rightleftharpoons \overline{K^0}$

Construct common eigenstates of H and CP

(2) $|K_{1,2}\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle \mp |\overline{K^0}\rangle\right)$ K^0 and Kbar^0 are the 2-state basis used here

K_1 same CP as 2 pions, K_2 as 3 pions

CP is conserved. $K_1 \sim K_S$, $K_2 \sim K_L$

Althougk K_1 has same CP (= +1) quantum number as 2 pions and K_2 same CP (= −1) as 3 pions, K_1 and K_2 are eigenstates of H and hence cannot decay.

Now modify the states $|K_1\rangle$, $|K_2\rangle$ and the Hamiltonian H

To account for CP violation, introduce

$$|K_{\substack{I\\II}}\rangle = \frac{1}{\sqrt{|p|^2+|q|^2}}\left(p\,|K^0\rangle \mp q\,|\bar{K}^0\rangle\right)$$

$$|p| \simeq 1, \quad |p| \neq 1; \quad |q| \simeq 1, \quad |q| \neq 1$$

two parameters p, q are used.

To account for decays, introduce an "effective" Hamiltonian

$$\bar{H} = H - \frac{i}{2}\Gamma, \qquad H^\dagger = H, \qquad \Gamma^\dagger = \Gamma$$

$$\bar{H}^\dagger \neq \bar{H}$$

K_I may be identified with the particle K_S

K_{II} may be identified with the particle K_L

Instead of solving the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \bar{H} |\Psi\rangle$$

we make use of the evolution operator $e^{-i\bar{H}t/\hbar}$

to find the general state $|\Psi(t)\rangle$,

$$|\Psi(t)\rangle = e^{-i\bar{H}t/\hbar} |\Psi(0)\rangle$$

Using $|K_I\rangle$ and $|K_{II}\rangle$ as a basis, can write

$$|\Psi(0)\rangle = a_I |K_I\rangle + a_{II} |K_{II}\rangle,$$

the coefficients a_I, a_{II} are constants

thus

$$|\Psi(t)\rangle = e^{-iE_0 t/\hbar} \left[a_I e^{-i(A - \frac{i\Gamma_1}{2})t/\hbar} |K_I\rangle + a_{II} e^{+i(A + i\frac{\Gamma_2}{2})t/\hbar} |K_{II}\rangle \right]$$

$$= N e^{-iE_0 t/\hbar} \left[\left(a_I e^{-i(A - \frac{i\Gamma_1}{2})t/\hbar} + a_{II} e^{i(A + i\frac{\Gamma_2}{2})t/\hbar} \right) p |K^0\rangle - \left(a_I e^{-i(A - \frac{i\Gamma_1}{2})t/\hbar} - a_{II} e^{i(A + \frac{i\Gamma_2}{2})t/\hbar} \right) q |\bar{K}^0\rangle \right]$$

$$N = \frac{1}{\sqrt{(|p|^2 + |q|^2)}}$$

CP violation

Hilbert space

$K^0, \overline{K^0}$

$H_0 = H_{st} + H_{em}$

$K^0, \overline{K^0}$ eigenstates of H_0

$K^0 \rightleftharpoons \overline{K^0}$ oscillation

$\xrightarrow[\text{on}]{H_{wk}}$

$K^0, \overline{K^0}$

K_1, K_2

$H = H_0 + H_{wk}$

$|K_{1,2}\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle \mp |\overline{K^0}\rangle)$

$\rightarrow$

$|K_1\rangle, |K_2\rangle$ are eigenstates of CP, also eigenstates of H with eigenvalues $E_1 = E_0 + A$, $E_2 = E_0 - A$ respectively. (analogous to coupled pendulum)

Decay $\xrightarrow[\text{on}]{\frac{\Gamma}{2}}$

$K^0, \overline{K^0}$

K_1, K_2

K_I, K_{II}

$\overline{H} = H - \frac{i}{2}\Gamma$

K_I, K_{II} eigenstates of $\overline{H}$

$$|K_{I,II}\rangle = \frac{1}{\sqrt{|p|^2 + |q|^2}} \cdot (p|K^0\rangle \mp q|\overline{K^0}\rangle)$$

Today:

Time reversal symmetry U_T antilinear

Kramer's theorem

Electric dipole moment.

L. Ballentine, Quantum Mechanics

Time reversal transformation

1. Definition of time reversal transformation U_T in Q M.

2 (i) U_T unitary and antilinear = antiunitary

(ii) $U_T^2 = 1$, $U_T^2 = -1$

3 Kramer theorem

For a physical system having a time reversal symmetry, the eigenvalue of its Hamiltonian is doubly degenerate

"degenerate" -> there is more than one eigenstate with the same eigenvalue

"doubly degen" -> specifically TWO eigenstates with the same eigenvalue

4. If time reversal is a perfect symmetry then electric dipole moment of a fundamental particle vanishes

$\langle \underset{\sim}{d} \rangle = 0$, $\underset{\sim}{d}$: electric dipole moment

Consider the Newton equation of motion (The second Law)

$$\underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt} \qquad \underset{\sim}{p} = m\dot{\underset{\sim}{x}} \quad \text{momentum}$$

Change the direction of the force, i.e.

$$\underset{\sim}{F} \to -\underset{\sim}{F}$$

and let $\underset{\sim}{p} \to -\underset{\sim}{p}$ (motion reversal)

we get back the same equation of motion

$$-\underset{\sim}{F} = \frac{d}{dt}(-\underset{\sim}{p}) \longrightarrow \underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt}$$

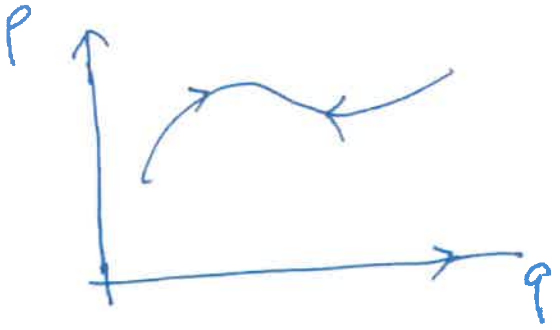
This means: if forward motion is possible in the physical world, the reversed motion is also possible in the physical world. Motion reversal is a symmetry.

As $\underset{\sim}{p} = m\dot{\underset{\sim}{x}} = m\dfrac{d\underset{\sim}{x}}{dt}$, we can realize $\underset{\sim}{p} \to -\underset{\sim}{p}$ by changing $t \to -t$.

Thus <u>motion reversal</u> is realized mathematically by <u>time reversal</u>

The Newton equation $\underset{\sim}{F} = \dfrac{d\underset{\sim}{p}}{dt} = m\dfrac{d^2\underset{\sim}{x}}{dt^2}$ is invariant (unchanged) if $t \to -t$ (time reversal)

In classical mechanics, the trajectory of a particle can go both ways



phase space (q, p)

Motion reversal : $(q, p) \rightarrow (q, -p)$

Mathematically convenient to denote motion reversal as $t \rightarrow -t$

In quantum physics, the fundamental observables of an elementary particle are

$\underset{\sim}{x}, \quad \underset{\sim}{p}, \quad \underset{\sim}{J}$

So we define the time reversal operator, $\underset{\sim}{U}_T$, in terms of its action on the three fundamental observables.

In quantum mechanics (QM), define the time reversal operator U_T as follows

$$\underset{\sim}{x} \longrightarrow \underset{\sim}{x}' = U_T \underset{\sim}{x} U_T^{-1} \equiv \underset{\sim}{x}$$

$$\underset{\sim}{p} \longrightarrow \underset{\sim}{p}' = U_T \underset{\sim}{p} U_T^{-1} \equiv -\underset{\sim}{p}$$

$$\underset{\sim}{J} \longrightarrow \underset{\sim}{J}' = U_T \underset{\sim}{J} U_T^{-1} = -\underset{\sim}{J}$$

$\Rightarrow$ $[U_T^2, \Omega] = 0$ for $\Omega = \underset{\sim}{x}$, or $\underset{\sim}{p}$ or $\underset{\sim}{J}$ (HW)

From this definition, we can show that

U_T has to be anti-linear

To show U_T must be antilinear:

By definition, an antilinear operator say A,

$$A\,\alpha|\psi\rangle = \alpha^* A|\psi\rangle$$

α = complex number

e.g. $\alpha = i$, $\quad A\,i|\psi\rangle = -i\,A|\psi\rangle$

We show U_T must be antilinear in order to be consistent with the Heisenberg quantization condition

$$[x_i, x_j] = 0 = [p_i, p_j] \qquad i, j = 1, 2, 3$$

$$[x_i, p_j] = i\hbar\,\delta_{ij}$$

$\rightarrow$ Heisenberg uncertainty principle

$$\Delta x_1\,\Delta p_1 \geq \frac{\hbar}{2}$$

$$\Delta x_2\,\Delta p_2 \geq \frac{\hbar}{2}$$

$$\Delta x_3\,\Delta p_3 \geq \frac{\hbar}{2}$$

Note: No inequality between uncertainty Δx_1 and Δp_2, for instance consider

$$[x_i, p_j] = i\hbar\delta_{ij}$$

$$x_i p_j - p_j x_i = i\hbar\delta_{ij}$$

Apply u_T, and u_T^{-1} to both sides

$$u_T(x_i p_j - p_j x_i)u_T^{-1} = u_T(i\hbar\delta_{ij})u_T^{-1}$$

$$u_T x_i u_T^{-1} u_T p_j u_T^{-1} - u_T p_j u_T^{-1} u_T x_i u_T^{-1}$$

$$= \hbar\delta_{ij}\ u_T(i)u_T^{-1}$$

$$-[x_i, p_j] = \hbar\delta_{ij}\ u_T\ i\ u_T^{-1}$$

In order to get back $[x_i, p_j] = i\hbar\delta_{ij}$

we demand $\quad u_T i = -i u_T$

so that $\quad \hbar\delta_{ij}\ u_T\ i\ u_T^{-1}$

$$= \hbar\delta_{ij}(-i)u_T u_T^{-1} = -i\hbar\delta_{ij}$$

$$\because u_T u_T^{-1} = 1$$

so because of $\quad u_T i = -i u_T$

$\therefore \quad u_T$ is antilinear.

identity operator

show $u_T^2 = 1$ or $u_T^2 = -1$

suppose $u_T : |\psi\rangle \to |\psi'\rangle = u_T|\psi\rangle$

Then

$$u_T^2 |\psi\rangle = c|\psi\rangle \qquad c = \text{constant}$$

$$|c|^2 = 1.$$

can check the norm of $u_T^2|\psi\rangle = 1$

Also $u_T^2 |\phi\rangle = d|\phi\rangle$, $d = \text{constant}$

$$|d|^2 = 1$$

Let $|\phi\rangle = |\psi\rangle + u_T|\psi\rangle$

Apply u_T^2 on both sides

$$u_T^2|\phi\rangle = u_T^2|\psi\rangle + u_T^3|\psi\rangle$$

$$d|\phi\rangle = c|\psi\rangle + u_T c|\psi\rangle$$

$$= c|\psi\rangle + c^* u_T|\psi\rangle$$

$\rightarrow d|\psi\rangle + d\,u_T|\psi\rangle = c|\psi\rangle + c^* u_T|\psi\rangle$

$\rightarrow$ $d = c$, $d = c^*$

$\longrightarrow$ $c = c^*$ i.e. c is a real number

$$|c|^2 = 1 \Rightarrow c = +1 \quad \text{or} \quad c = -1$$

Go back $\quad u_T^2 |\psi\rangle = c\,|\psi\rangle$

i.e. $\quad u_T^2 = +1 \quad \text{or} \quad u_T^2 = -1$

HW: why $\pi^2 \neq -1$? $\quad$ HW

Interesting to note about the rotation operator

Not examinable

$\mathcal{R}$ = rotation in 3-dim space

$$\mathcal{R}^2 = 1, \quad \mathcal{R}^2 \neq -1.$$

R = rotation in Hilbert space

$$\longrightarrow \quad R^2 = 1 \quad \text{or} \quad R^2 = -1$$

① Definitn U_T

$\underset{\sim}{x} \rightarrow \underset{\sim}{x}' = U_T \underset{\sim}{x} U_T^{-1} = \underset{\sim}{x}$

$\underset{\sim}{p} \rightarrow \underset{\sim}{p}' = U_T \underset{\sim}{p} U_T^{-1} = -\underset{\sim}{p}$

$\underset{\sim}{J} \rightarrow \underset{\sim}{J}' = U_T \underset{\sim}{J} U_T^{-1} = -\underset{\sim}{J}$

② Show $[U_T^2, \Omega] = 0$

LHS $\quad U_T^2 \underset{\sim}{x} - \underset{\sim}{x} U_T^2$

from df $\quad U_T^2 (U_T \underset{\sim}{x} U_T^{-1}) - (U_T \underset{\sim}{x} U_T^{-1}) U_T^2$

$U^3 \underset{\sim}{x} U_T^{-3} - U_T \underset{\sim}{x} U_T^{-1}$

$\underset{\sim}{x} - \underset{\sim}{x} = 0$ //

LHS $\quad U_T^2 \underset{\sim}{p} - \underset{\sim}{p} U_T^2$

from df $\quad U_T^2 (-U_T \underset{\sim}{p} U_T^{-1}) + (U_T \underset{\sim}{p} U_T^{-1}) U_T^2$

$-U_T^3 \underset{\sim}{p} U^{-3} + U_T \underset{\sim}{p} U_T^{-1}$

$-(-\underset{\sim}{p}) + -\underset{\sim}{p} = 0$ //

③ Show anti-linear

(df) $\hat{A}\alpha|\psi\rangle = \alpha^* \hat{A}|\psi\rangle$

From $[x, p] = i\hbar$

LHS $\quad U_T (xp - px) U_T^{-1} = i\hbar$

$U_T x U_T^{-1} U_T p U_T^{-1} - U_T p U_T^{-1} U_T x U_T^{-1}$

$x(-p) - (-p)(x)$

$-[x, p]$

$\therefore -[x, p] = U_T i\hbar$

If linear, $-(i\hbar) = i\hbar$ ✗

If antilinear, $-(i\hbar) = -i\hbar$ ✓

④ Show $U_T^2 = -\mathbb{1}$ or $\mathbb{1}$

Given $U_T^2 |\psi\rangle = c|\psi\rangle$, $c \in \mathbb{C}$ and $|c|^2 = 1$

For some arbitrary state $|\phi\rangle$,

$U_T^2 |\phi\rangle = d|\phi\rangle$, $d \in \mathbb{C}$, $|d|^2 = 1$

Let $|\phi\rangle = |\psi\rangle + U_T|\psi\rangle$ and apply U_T to both sides,

$U_T^2 |\phi\rangle = U_T^2 |\psi\rangle + U_T^3 |\psi\rangle$

$d\,|\phi\rangle = c|\psi\rangle + U_T(c|\psi\rangle)$

$d(|\psi\rangle + U_T|\psi\rangle) = c|\psi\rangle + c^* U_T|\psi\rangle$

$\therefore d = c$ AND $d = c^*$

so $c \in \mathbb{R}$

Since $|c|^2 = 1$ then

$c = \pm 1$

Hence $U_T^2 |\psi\rangle = c|\psi\rangle$

$U_T^2 = \mathbb{1}$ or $-\mathbb{1}$ //

Kramer thm.

Given $[U_T, H] = 0$ U_T is a symmetry of physical system, then the energy E (eigenvalue of H) is doubly degenerate

'Doubly degenerate' means for the same E value, we can have two different eigenstates of H

Proof: Let $|\psi\rangle$ be eigenket of H

$$H|\psi\rangle = E|\psi\rangle$$

consider the time-reversal state $U_T|\psi\rangle$

$$H\, U_T|\psi\rangle = U_T\, H|\psi\rangle = U_T\, E|\psi\rangle = E\, U_T|\psi\rangle$$

so both $|\psi\rangle$ and $U_T|\psi\rangle$ are eigenstates of H with the same energy E.

If $U_T|\psi\rangle$ and $|\psi\rangle$ are proportional to each other, then no degeneracy.

Now show $U_T|\psi\rangle$ and $|\psi\rangle$ not proportional. We prove by contradiction.

Assume proportionality,

$$u_T |\psi\rangle = \alpha |\psi\rangle \qquad \alpha = \text{constant}$$

Apply u_T to both sides

$$u_T^2 |\psi\rangle = u_T \alpha |\psi\rangle = \alpha^* u_T |\psi\rangle$$

$$= \alpha^* \alpha |\psi\rangle \quad \because \ u_T|\psi\rangle = \alpha|\psi\rangle$$

Already know $u_T^2 = +1$ or $u_T^2 = -1$.

If $u_T^2 = +1$, $|\alpha|^2 = 1$, then ok

But if $u_T^2 = -1$, $|\alpha|^2 = -1$ impossible

so if $u_T^2 = -1$, then the assumption

$u_T|\psi\rangle = \alpha|\psi\rangle$ is wrong

that means if $u_T^2 = -1$, the state $|\psi\rangle$

and $u_T|\psi\rangle$ are two different states

So for $u_T^2 = -1$, the energy value E

is doubly degenerate (theorem is proved

for states $|\psi\rangle$ s.t. $u_T^2 |\psi\rangle = -|\psi\rangle$)

Lastly we show if time reversal U_T is an exact symmetry, then electric dipole moment $\underset{\sim}{d}$ ($\underset{\sim}{d} = q\underset{\sim}{x}$ from classical electromagnetism) vanishes

i.e. $\langle \underset{\sim}{d} \rangle = \langle \psi | \underset{\sim}{d} | \psi \rangle = 0$

Griffiths: suppose $\underset{\sim}{d}$ of a particle $\neq 0$ and the particle has a spin $\underset{\sim}{S}$. If U_T is a symmetry, then show $\underset{\sim}{d} = 0$

Before applying U_T, suppose $\underset{\sim}{s}$ and $\underset{\sim}{d}$ orientate in the same direction

$$\underset{\sim}{s}\uparrow \qquad \uparrow\underset{\sim}{d}$$

After applying U_T, we get the configuration

$$\underset{\sim}{s}\downarrow\uparrow\underset{\sim}{d}$$

Before U_T $\neq$ After U_T i.e.
U_T is not an exact symmetry

Two configurations different, so time reversal symmetry broken

If want time-reversal be an exact symmetry, $\rightarrow$

$$\underset{\sim}{d} = 0$$

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(1)

Given $[U_T, H] = 0$, show E is doubly degenerate.

$\rightarrow$ Suppose E is not degen. $H|\psi\rangle = E|\psi\rangle$

$$U_T H|\psi\rangle = U_T E|\psi\rangle$$
$$U_T H|\psi\rangle = E U_T|\psi\rangle$$

$\rightarrow$ If we assume $U_T|\psi\rangle = \alpha|\psi\rangle$, $\alpha \in \mathbb{R}$,

$$U_T H|\psi\rangle = E\alpha|\psi\rangle$$
$$U_T^2 H|\psi\rangle = U_T(-E\alpha)|\psi\rangle = E\alpha^* U_T|\psi\rangle = E\alpha^*\alpha|\psi\rangle$$

Since $U_T^2 = +1$ or -1, (+1): $H|\psi\rangle = E|\alpha|^2|\psi\rangle$, $\because |\alpha|^2 = 1$

(-1): $-H|\psi\rangle = E|\alpha|^2|\psi\rangle$, $\because |\alpha|^2 = -1$ NOT allowed!

Hence $U_T|\psi\rangle = \alpha|\psi\rangle$ is wrong and so if $U_T^2 = -1$ then $|\psi\rangle$ and $U_T|\psi\rangle$ are diff states with energy E //

(2)

If U_T is an exact symmetry, show dipole moment $\underset{\sim}{d} = q\underset{\sim}{x}$ is 0 for a particle

$\rightarrow$ Suppose the particle has spin $\underset{\sim}{s}$ with $\underset{\sim}{d} \neq 0$

$\rightarrow$ Suppose $\underset{\sim}{s}$ and $\underset{\sim}{d}$ orientate in same direction before apply U_T

$\boxed{\underset{\sim}{s}\uparrow \ \uparrow\underset{\sim}{d} \xrightarrow{U_T} \underset{\sim}{s}\downarrow \ \uparrow\underset{\sim}{d}}$

$\rightarrow$ Apply U_T: $U_T \underset{\sim}{d} U_T^\dagger \Leftrightarrow U_T \underset{\sim}{x} U_T^\dagger$

$$= \underset{\sim}{d}$$

$$U_T \underset{\sim}{s} U_T^\dagger = -\underset{\sim}{s}$$

diff

$U_T \stackrel{!}{=}$ exact symmetry $\therefore \underset{\sim}{d} = 0$ //

(1) Show E is doubly degenerate
(2) If U_T is an exact symmetry, show dipole moment =0
(3) Show dipole moment = 0 using WET

(3)

$\rightarrow$ Consider $\underset{\sim}{J}$ and $\underset{\sim}{d}$ for H-atom. By Wigner-Eckert Theorem,

$$\langle E j m|\underset{\sim}{d}|E j m\rangle = C_{Ej} \langle E j m|\underset{\sim}{J}|E j m\rangle \quad (\times)$$

$\rightarrow$ Apply U_T

$$\langle Ejm| U_T^\dagger U_T \underset{\sim}{d} U_T^\dagger U_T |Ejm\rangle = C_{Ej} \langle Ejm| U_T^\dagger U_T \underset{\sim}{J} U_T^\dagger U_T |Ejm\rangle$$
$$\langle Ejm| U_T^\dagger \underset{\sim}{d} U_T|Ejm\rangle = -C_{Ej} \langle Ejm| U_T^\dagger \underset{\sim}{J} \underline{U_T|Ejm\rangle} \quad (+)$$

$\rightarrow$ To find $U_T|Ejm\rangle$, consider $J_3 U_T|Ejm\rangle = -U_T J_3|Ejm\rangle$

$$= -U_T(m\hbar)|Ejm\rangle$$
$$= -m\hbar\, U_T|Ejm\rangle$$

We also know $J_3|Ej{-m}\rangle = -m\hbar\,|Ej{-m}\rangle$

$\rightarrow$ If m has NO degeneracies, $U_T|Ejm\rangle = \alpha|Ej{-m}\rangle$, $\alpha \in \mathbb{R}$

hence (+) becomes

$$|\alpha|^2\langle Ej{-m}|\underset{\sim}{d}|Ej{-m}\rangle = -C_{Ej}|\alpha|^2\langle Ej{-m}|\underset{\sim}{J}|Ej{-m}\rangle, \text{ cancel } |\alpha|^2$$
$$\langle Ej{-m}|\underset{\sim}{d}|Ej{-m}\rangle = -C_{Ej}\langle Ej{-m}|\underset{\sim}{J}|Ej{-m}\rangle$$

$\rightarrow$ Rewrite $(\times)$ for $|Ej{-m}\rangle$

$$\langle Ej{-m}|\underset{\sim}{d}|Ej{-m}\rangle = C_{Ej}\langle Ej{-m}|\underset{\sim}{J}|Ej{-m}\rangle$$

$2\langle Ej{-m}|\underset{\sim}{d}|Ej{-m}\rangle = 0$

$\therefore \langle\psi|\underset{\sim}{d}|\psi\rangle = 0$ // SHOWN

Two configurations different, so time reversal sym. broken. Hand-waving argument $\to$ $\underset{\sim}{d} = 0$

Now use a more rigorous argument.

suppose the state of the particle is

$$|E j m\rangle$$

(cf H-atom state $|E \ell m\rangle$)

$$H \qquad \underset{\sim}{J}^2 \qquad J_3$$

$|Ejm\rangle = |nlm\rangle$ from H-atom

assume spherical symmetry.

Consider 2 properties of the system: $\underset{\sim}{d}$, $\underset{\sim}{J}$

(cf in CS Wu parity downfall expt, measure spin of cobalt 60 and the momentum $\underset{\sim}{p}$ of the electron emitted)

consider

$$\langle E j m | \underset{\sim}{d} | E j m\rangle, \qquad \langle E j m | \underset{\sim}{J} | E j m\rangle$$

By Wigner-Eckert theorem, we can write **Given**

$$\boxed{\langle E j m | \underset{\sim}{d} | E j m\rangle = C_{Ej} \langle E j m | \underset{\sim}{J} | E j m\rangle \quad \cdots (*)}$$

Apply time-reversal transformation u_T

$$\langle E j m | u_T^\dagger u_T \underset{\sim}{d} u_T^\dagger u_T | E j m\rangle = C_{Ej} \langle E j m | u_T^\dagger u_T \underset{\sim}{J} u_T^\dagger u_T | E j m\rangle$$

By time-reversal transformation

$$u_T \, \underset{\sim}{J} \, u_T^\dagger = -\underset{\sim}{J}, \qquad u_T^\dagger = u_T^{-1}$$

$$u_T \, \underset{\sim}{x} \, u_T^\dagger = \underset{\sim}{x} \rightarrow u_T \, \underset{\sim}{d} \, u_T^\dagger = \underset{\sim}{d}$$

$\rightarrow$

$$\langle Ej m | u_T^\dagger \, \underset{\sim}{d} \, u_T | Ej m \rangle = -C_{Ej} \langle Ej m | u_T^\dagger \underset{\sim}{J} u_T | Ej m \rangle \quad \text{-- } (+)$$

Compare eq (x) and eq (+), need to know

$$u_T | E j m \rangle = ?$$

Consider $\quad J_3 \, u_T | Ej m \rangle = - u_T \, J_3 | E j m \rangle$

$$= - u_T \, m\hbar \, | E j m \rangle = - m\hbar \, u_T | E j m \rangle$$

$\rightarrow \quad u_T | E j m \rangle$ is an eigenstate of J_3 with eigenvalue $-m\hbar$

On the hand, $\quad J_3 | E j -m \rangle = -m\hbar | E j -m \rangle$

If m has no degeneracy, then

$$u_T | E j m \rangle = \alpha \, | E j -m \rangle \qquad \alpha: \text{constant}$$

Eq (+) can be written as

$$\langle E j -m | \underset{\sim}{d} | E j -m \rangle = -C_{Ej} \langle E j -m | \underset{\sim}{J} | E j -m \rangle \quad \text{-- } (*)$$

Remember : $m = -j, -j+1, \ldots +j$

so eq (x) $\langle E j m | \underset{\sim}{d} | E j m \rangle = c_{Ej} \langle E j m | \underset{\sim}{J} | E j m \rangle$

can be written as

$$\langle E j \; -m | \underset{\sim}{d} \; | E \; j \; -m \rangle = c_{Ej} \langle E j \; -m | \underset{\sim}{J} | E j -m \rangle \quad \ldots \overline{(x)}$$

Adding eq (*) and eq $\overline{(x)}$, then

$$\langle E, j, -m | \underset{\sim}{d} | E, j, -m \rangle = 0$$

or $\langle E, j, m | \underset{\sim}{d} | E, j, m \rangle = 0$

If $|\psi\rangle$ is arbitrary state of the system, can write

$$|\psi\rangle = \sum_{E,j,m} k_{Ejm} | E j m \rangle, \qquad k_{Ejm} = \text{coefficients}$$

$$\rightarrow \quad \langle \psi | \underset{\sim}{d} | \psi \rangle = 0 \qquad \text{(HW)}$$