Addition of angular momentum Today!

W5L2 – tensor products & tensor product spaces.

$\hat{O}_1$ acting on V_1 ; $\hat{O}_2$ acting on V_2

We consider $\hat{O}_1 \hat{O}_2$ or $\hat{O}_1 + \hat{O}_2$

— Don't make sense rigorously, especially when $\dim(V_1) \neq \dim(V_2)$

→ use tensor products.

$$\hat{O}_1 \hat{O}_2 \equiv \underbrace{\hat{O}_1}_{\text{acts on } V_1} \otimes \underbrace{\hat{O}_2}_{\text{acts on } V_2.} \quad ; \quad \hat{O}_1 + \hat{O}_2 = \underbrace{\hat{O}_1}_{\text{acts on } V_1} \otimes \underbrace{\mathbb{1}_2}_{\text{acts on } V_2} + \underbrace{\mathbb{1}_1}_{\text{acts on } V_1} \otimes \underbrace{\hat{O}_2}_{\text{acts on } V_2.}$$

Another example (continuing W5L2):

$$\vec{J} = \vec{S} + \vec{L}$$

$J_x = S_x + L_x$
$J_y = S_y + L_y$
$J_z = S_z + L_z$

$(s = \frac{1}{2})$
$\vec{S}$ acts on $\mathcal{E}_1$, spanned by $\{ |e_1\rangle \equiv |+\rangle_z, |e_2\rangle \equiv |-\rangle_z \}$

$\vec{L}$ " " $\mathcal{E}_2$, " " $\{ |f_1\rangle \equiv |l=1, m=1\rangle,$
$(l=1)$ $\quad |f_2\rangle \equiv |l=1, m=0\rangle,$
$\quad |f_3\rangle \equiv |l=1, m=-1\rangle \}$

$$\vec{S}\,(\overbrace{|e_i\rangle \otimes |f_j\rangle}^{\text{basis vector in } \mathcal{E}_1 \otimes \mathcal{E}_2.}) = (\vec{S} \otimes \mathbb{1})(|e_i\rangle \otimes |f_j\rangle)$$
$$= (\vec{S}\,|e_i\rangle) \otimes (\mathbb{1}\,|f_j\rangle)$$
$$= \vec{S}\,|e_i\rangle \otimes |f_j\rangle$$

$$\vec{L}\,(|e_i\rangle \otimes |f_j\rangle) = (\mathbb{1} \otimes \vec{L})(|e_i\rangle \otimes |f_j\rangle)$$
$$= (\mathbb{1}\,|e_i\rangle) \otimes (\vec{L}\,|f_j\rangle)$$
$$= |e_i\rangle \otimes \vec{L}\,|f_j\rangle$$

$$\vec{J}\,(|e_i\rangle \otimes |f_j\rangle) = ((\vec{S} \otimes \mathbb{1}) + (\mathbb{1} \otimes \vec{L}))(|e_i\rangle \otimes |f_j\rangle)$$
$$= (\vec{S}\,|e_i\rangle \otimes |f_j\rangle) + (|e_i\rangle \otimes \vec{L}\,|f_j\rangle).$$

If $\hat{O}^{(1)}$ acts on V_1, $\hat{O}^{(2)}$ acts on V_2,

then $[\hat{O}^{(1)}, \hat{O}^{(2)}] = 0$ //

Show this: Take any $v \in V_1$, $w \in V_2$.

$$\hat{O}^{(1)}\hat{O}^{(2)}(|v\rangle \otimes |w\rangle) = (\hat{O}^{(1)} \otimes \hat{O}^{(2)})(|v\rangle \otimes |w\rangle)$$
$$= \hat{O}^{(1)}|v\rangle \otimes \hat{O}^{(2)}|w\rangle$$

$$\hat{O}^{(2)}\hat{O}^{(1)}(\underset{(1)}{|v\rangle} \otimes \underset{(2)}{|w\rangle}) = \hat{O}^{(2)}(\hat{O}^{(1)} \otimes \mathbb{1}^{(2)})(|v\rangle \otimes |w\rangle)$$
$$= \hat{O}^{(2)}(\hat{O}^{(1)}|v\rangle \otimes |w\rangle)$$
$$= (\mathbb{1}^{(1)} \otimes \hat{O}^{(2)})(\hat{O}^{(1)}|v\rangle \otimes |w\rangle)$$
$$= \hat{O}^{(1)}|v\rangle \otimes \hat{O}^{(2)}|w\rangle$$
$$= \hat{O}^{(1)}\hat{O}^{(2)}(|v\rangle \otimes |w\rangle) \quad \text{// Shown.}$$

Addition of angular momentum

Fundamental result:

Suppose $\vec{J}^{(1)}$ is an angular momentum operator acting on V_1,
and $\vec{J}^{(2)}$ ——— " ——— V_2.

Then there exists a new angular momentum

$$\vec{J} = (\vec{J}^{(1)} \otimes \mathbb{1}^{(2)}) + (\mathbb{1}^{(1)} \otimes \vec{J}^{(2)}) \text{ acting on } V_1 \otimes V_2.$$

$$(\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)})$$

Show this:

Need to show that $[J_i, J_j] = i\hbar\,\varepsilon_{ijk} J_k$ in $V_1 \otimes V_2$.

$$[J_i, J_j] = [J_i^{(1)} + J_i^{(2)}, J_j^{(1)} + J_j^{(2)}]$$
$$= [J_i^{(1)}, J_j^{(1)}] + [J_i^{(2)}, J_j^{(2)}]$$
$$+ \underbrace{\cancel{[J_i^{(1)}, J_j^{(2)}]}}_{0} + \underbrace{\cancel{[J_i^{(2)}, J_j^{(1)}]}}_{0}$$

since $J^{(1)}$ and $J^{(2)}$ operate on different vector spaces.

on different vector spaces.

$$= i\hbar\,\varepsilon_{ijk} J_k^{(1)} + i\hbar\,\varepsilon_{ijk} J_k^{(2)}$$

$[J_i^{(1)}, J_j^{(1)}] = i\hbar\varepsilon_{ijk} J_k^{(1)}$ since $\vec{J}^{(1)}$ is an angular momentum operator

$$= i\hbar\,\varepsilon_{ijk}\left(J_k^{(1)} + J_k^{(2)}\right)$$

$$= i\hbar\,\varepsilon_{ijk} J_k \quad //$$

$|j^{(1)}, m^{(1)}\rangle$ — $j^{(1)}$ fixed, $m^{(1)}$ vary — $\vec{J}^{(1)}$

$|j^{(2)}, m^{(2)}\rangle$ — $j^{(2)}$ fixed, $m^{(2)}$ vary — $\vec{J}^{(2)}$

$\rightarrow$ uncoupled representation

$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$$

$\vec{J}^{(1)} \otimes 1^{(2)}$ $\quad$ $1^{(1)} \otimes \vec{J}^{(2)}$

$|j^{(1)}, m^{(1)}\rangle \otimes |j^{(2)}, m^{(2)}\rangle$

$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$ new angular momentum.

$\downarrow$

$|j, m\rangle$ $\rightarrow$ coupled representation.

Q1) What are the possible values of j and m for $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$?

Q2) How many, and which, quantum numbers specify the eigenstates of $\vec{J}^2$ acting on $V_1 \otimes V_2$?

Recall :

Angular momentum :

$$\vec{J}^2\,|j, m\rangle = \hbar^2 j(j+1)\,|j, m\rangle\,, \quad j \geq 0, \text{ integer or half-integer}$$

$$J_z\,|j, m\rangle = \hbar m\,|j, m\rangle\,, \quad m = -j, -j+1, \ldots, j$$

$+1$ $\quad$ $+1$

$[\vec{J}^2, J_z] = 0$

$$[\vec{J}^2, J_x] = 0\,, \quad [\vec{J}^2, J_y] = 0$$

We answer Q2 first.

Today, we will be discussing which quantum numbers can be simultaneously specified.

We will do so by showing that the operators associated with the quantum numbers commute.

Show that j_1, m_1, j_2, m_2 can be simultaneously specified.

Switch notation.

$\vec{J}_1$ acts on V_1 , $\vec{J}_2$ acts on V_2.

$$\vec{J}_1^{\,2} |j_1, m_1\rangle = \hbar^2 j_1(j_1+1) |j_1, m_1\rangle \qquad \vec{J}_2^{\,2} |j_2, m_2\rangle = \hbar^2 j_2(j_2+1) |j_2, m_2\rangle$$

$$J_{1z} |j_1, m_1\rangle = \hbar m_1 |j_1, m_1\rangle \qquad J_{2z} |j_2, m_2\rangle = \hbar m_2 |j_2, m_2\rangle$$

$$[\vec{J}_1^{\,2}, J_{1z}] = 0 \qquad [\vec{J}_2^{\,2}, J_{2z}] = 0$$

$$[\vec{J}_1^{\,2}, J_{1z}] = [\vec{J}_2^{\,2}, J_{2z}] = 0 \quad \checkmark$$

$$[\vec{J}_1^{\,2}, J_2^2] = [\vec{J}_1^{\,2}, J_{2z}] = [\vec{J}_2^{\,2}, J_{1z}] = [J_{1z}, J_{2z}] = 0$$

(operate on different spaces) ✓

So j_1, m_1, j_2, m_2 are good quantum numbers. — 4 of them.

Basis state $|j_1, m_1, j_2, m_2\rangle$ (uncoupled representation)

Coupled representation.

$$\left.\begin{aligned} \vec{J}^{\,2} |j, m\rangle &= \hbar^2 j(j+1) |j, m\rangle \\ J_z |j, m\rangle &= \hbar m |j, m\rangle \end{aligned}\right\} \text{ only 2 quantum numbers so far.}$$

Need to find 2 more quantum numbers. $j \neq j_1 + j_2$ not so simple.

What do $\vec{J}^{\,2}$, J_z commute with?

Possible options: $\vec{J}_1^{\,2}$, $\vec{J}_2^{\,2}$, J_{1z}, J_{2z}.

$$\vec{J}^2 = (\vec{J}_1 + \vec{J}_2).(\vec{J}_1 + \vec{J}_2)$$
$$= \vec{J}_1^2 + \vec{J}_2^2 + \vec{J}_1.\vec{J}_2 + \vec{J}_2.\vec{J}_1$$
$$= \vec{J}_1^2 + \vec{J}_2^2 + 2\vec{J}_1.\vec{J}_2 \quad \text{since } [\vec{J}_1, \vec{J}_2] = 0.$$

$$[\vec{J}^2, \vec{J}_1^2] = \underbrace{[\vec{J}_1^2, \vec{J}_1^2]}_{0} + \underbrace{[\vec{J}_2^2, \vec{J}_1^2]}_{0} + 2[\vec{J}_1.\vec{J}_2, \vec{J}_1^2]$$
$$= 2\vec{J}_1.\underbrace{[\vec{J}_2, \vec{J}_1^2]}_{0} + 2\underbrace{[\vec{J}_1, \vec{J}_1^2]}_{0}.\vec{J}_2$$
$$= 0$$

since $\vec{J}_1^2$ commutes with all components of $\vec{J}_1$.

$[\vec{J}_1^2, J_{1x}] = [\vec{J}_1^2, J_{1y}] = [\vec{J}_1^2, J_{1z}] = 0.$

Similarly, $[\vec{J}^2, \vec{J}_2^2] = 0.$

$\vec{J} = \vec{J}_1 + \vec{J}_2$

$J_z = J_{1z} + J_{2z}$
$= J_{1z} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes J_{2z}$

$$[J_z, \vec{J}_1^2] = [J_{1z} + J_{2z}, \vec{J}_1^2]$$
$$= \underbrace{[J_{1z}, \vec{J}_1^2]}_{0} + \underbrace{[J_{2z}, \vec{J}_1^2]}_{0}$$

$\vec{J}_1$ is angular momentum

different spaces

$$= 0$$

Similarly, $[J_z, \vec{J}_2^2] = 0$

$\vec{J}_1^2 \otimes \mathbb{1}_2 \quad \mathbb{1}_1 \otimes \vec{J}_2^2$

So $\{\vec{J}^2, J_z, \vec{J}_1^2, \vec{J}_2^2\}$ are a set of commuting observables in $V_1 \otimes V_2$.

Their common eigenstates are specified as $\{|j, m, j_1, j_2\rangle\}$.

$\vec{J}^2 \quad J_z \quad \vec{J}_1^2 \quad \vec{J}_2^2$

$\|j_1, m_1, j_2, m_2\rangle$	$\|j, m, j_1, j_2\rangle$
$\in V_1 \otimes V_2.$	$\in V_1 \otimes V_2$
uncoupled representation.	Coupled representation

uncoupled representation.

eigenstate of
$\vec{J}_1^2,\ J_{1z},\ \vec{J}_2^2,\ J_{2z}.$

Dimension of $V_1 \otimes V_2$
$= (\text{Dimension of } V_1)(\text{Dimension of } V_2)$
$= (2j_1+1)(2j_2+1)$

Eg. if $j_1 = 1,\ j_2 = 2,$
Dimension of $V_1 \otimes V_2$
$= 3 \times 5$
$= 15.$

coupled representation

eigenstate of
$\vec{J}^2,\ J_z,\ \vec{J}_1^2,\ \vec{J}_2^2.$
where $\vec{J} = \vec{J}_1 + \vec{J}_2.$

# of possible m values $= 2j+1.$

For a given j,
of $\{|j, m, j_1, j_2\rangle\} = 2j+1.$

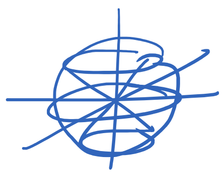
If $j = j_1 + j_2 = 3,$
$2j+1 = 7 \neq 15.$

If $j = j_2 - j_1 = 1$
$2j+1 = 3$

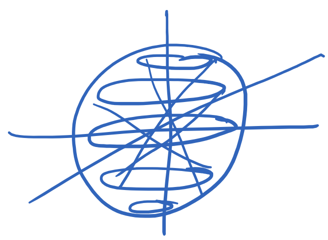
If $j = 2,$
$2j+1 = 5$

$7+3+5 = 15.$ — maybe!

$j_1 = 1$



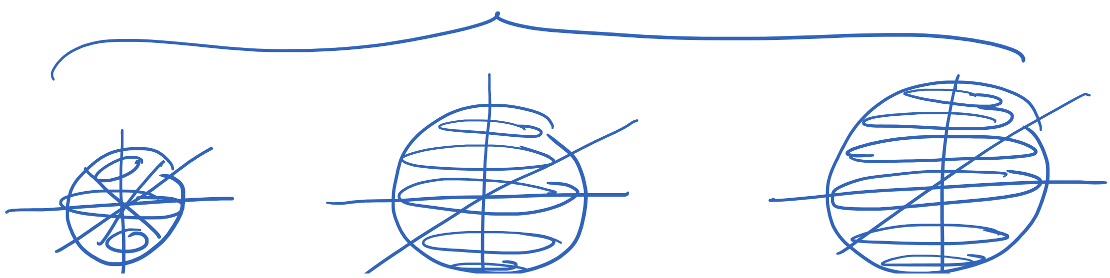
$j_2 = 2$

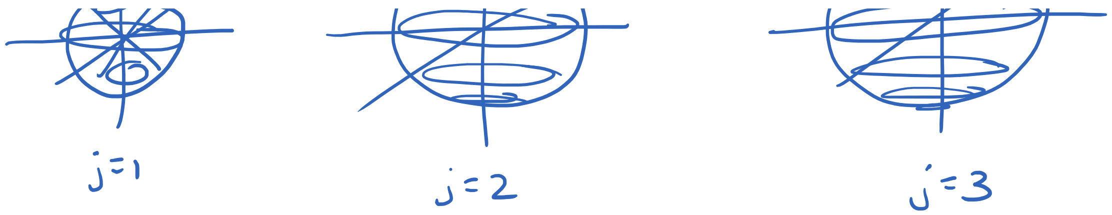


Add $\vec{J} = \vec{J}_1 + \vec{J}_2$ together

↓

more than one "new" angular momentum.





We know $\{|j_1, m_1, j_2, m_2\rangle\}$ forms an o.n. basis for $V_1 \otimes V_2$.

So $$|j, m, j_1, j_2\rangle = \sum_{m_1, m_2} C^{(j,m,j_1,j_2)}_{m_1,m_2} |j_1, m_1, j_2, m_2\rangle$$

$m_1 : -j_1$ to j_1
$m_2 : -j_2$ to j_2
$m = m_1 + m_2$

$\nwarrow$ Clebsch-Gordan coefficients.

$m_1 + m_2 = ?$

$J_{1z} |j_1, m_1\rangle = \hbar m_1 |j_1, m_1\rangle$

$J_{2z} |j_2, m_2\rangle = \hbar m_2 |j_2, m_2\rangle$

$J_z = J_{1z} + J_{2z}$

$$(J_{1z} + J_{2z})(|j_1, m_1, j_2, m_2\rangle = \hbar m_1 (|j_1, m_1\rangle) \otimes |j_2, m_2\rangle + (|j_1, m_1\rangle) \otimes \hbar m_2 |j_2, m_2\rangle = \hbar(m_1 + m_2)(|j_1, m_1, j_2, m_2\rangle)$$

$J_z |j, m, j_1, j_2\rangle = \hbar m |j, m, j_1, j_2\rangle$

$m = m_1 + m_2$.

When do we use the uncoupled representation or the coupled representation?

Eg 1 Consider two spin-$\frac{1}{2}$ systems, interacting with an external magnetic field $\vec{B}$, // z-axis.

$$H = H_1 + H_2 = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2$$

$$\vec{S} \cdot \vec{B} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix} = S_z B_z.$$

$$= -g\mu_B S_{1z}B_z - g\mu_B S_{2z}B_z$$

$\{H, S_{1z}, S_1^2, S_{2z}, S_2^2\}$ are a set of mutually commuting observables.

Energy eigenstates can be labelled as $|s_1, m_1, s_2, m_2, \lambda\rangle$ (λ ↑ energy).

[uncoupled representation].

Eg 2 Consider two spin-$\frac{1}{2}$ systems.

Here, the two spins interact with one another.

$$H = \alpha \vec{S}_1 \cdot \vec{S}_2$$

$$= \alpha(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z})$$

Since eg $[S_{1x}, S_{1z}] \neq 0$

$[H, S_{1z}] \neq 0$

$[H, S_{2z}] \neq 0$.

$$[H, S_{1z}] = [\alpha(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}), S_{1z}]$$

$$= \alpha[S_{1x}, S_{1z}]S_{2x} + \alpha[S_{1y}, S_{1z}]S_{2y} + \alpha\underbrace{[S_{1z}, S_{1z}]}_{0}S_{2z}$$

$$= -\alpha i\hbar S_{1y}S_{2x} + \alpha i\hbar S_{1x}S_{2y}$$

$$\neq 0$$

$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k$

$i: x$
$j: z$
$\varepsilon_{xzy} = -1$
$\varepsilon_{xzx} = \varepsilon_{xzz} = 0$

$i: y$
$j: z$
$\varepsilon_{yzx} = +1$
$\varepsilon_{yzz} = \varepsilon_{yzy} = 0$

Similarly for $[H, S_{2z}]$.

So m_1 and m_2 are "bad" quantum numbers.

Let $\vec{S} = \vec{S}_1 + \vec{S}_2$.

$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\cdot\vec{S}_2.$$

$$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2)$$

$$[\vec{S}_1\cdot\vec{S}_2, \vec{S}^2] = \frac{1}{2}([\vec{S}^2, \vec{S}^2] - [\vec{S}_1^2, \vec{S}^2] - [\vec{S}_2^2, \vec{S}^2])$$

$$[\vec{S}_1 \cdot \vec{S}_2, \vec{S}^2] = \frac{1}{2}\Big(\underbrace{[\vec{S}^2, \vec{S}^2]}_{0} - \underbrace{[\vec{S}_1^2, \vec{S}^2]}_{\substack{0 \\ \text{from this} \\ \text{lecture}}} - \underbrace{[\vec{S}_2^2, \vec{S}^2]}_{\substack{\text{from this} \\ \text{lecture.}}} \Big)$$

$$= 0$$

$$[\vec{S}_1 \cdot \vec{S}_2, \vec{S}_1^2] = \frac{1}{2}\Big(\underbrace{[\vec{S}^2, \vec{S}_1^2]}_{0} - \underbrace{[\vec{S}_1^2, \vec{S}_1^2]}_{0} - \underbrace{[\vec{S}_2^2, \vec{S}_1^2]}_{0} \Big)$$

$$= 0$$

Similarly, $[\vec{S}_1 \cdot \vec{S}_2, \vec{S}_2^2] = 0$.

So far: $\underset{\propto \vec{S}_1 \cdot \vec{S}_2}{H}$ commutes with $\vec{S}^2$, $\vec{S}_1^2$ and $\vec{S}_2^2$.

How about $[H, S_z]$?