Derive first derivatives in spherical coordinates

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial r}$$

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix}}_{T_1(r,\theta,\phi)} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

We know
$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = T_1^{-1}(r,\theta,\phi) \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix}$$

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \underbrace{\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}}_{T_2(\theta,\phi)} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = T_2^{-1}(\theta,\phi) \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

① & ②

$$\Rightarrow \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\Rightarrow \nabla T = \hat{r}\frac{\partial T}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial T}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

Divergence

$$\nabla \cdot \vec{v} = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \cdot \left(v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}\right)$$

multiply by "like" components

$$= \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \quad \times$$

$$= \hat{r} \cdot \frac{\partial}{\partial r}\left(v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}\right)$$

$$+ \hat{\theta} \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\left(v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}\right)$$

$$+ \hat{\phi} \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}\right)$$

e.g. $\hat{\theta} \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(v_r\hat{r})$

$$= \cancel{\hat{\theta} \cdot \frac{1}{r}\hat{r}\frac{\partial v_r}{\partial \theta}} + \hat{\theta} \cdot \frac{v_r}{r}\boxed{\frac{\partial \hat{r}}{\partial \theta}} = \frac{v_r}{r}$$

$\hat{\theta}$

$$\frac{\partial \hat{r}}{\partial r} = \frac{\partial \hat{\theta}}{\partial r} = \frac{\partial \hat{\phi}}{\partial r} = 0$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \qquad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0$$

$$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta\,\hat{\phi} \qquad \frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\,\hat{\phi}$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -(\hat{r}\sin\theta + \hat{\theta}\cos\theta)$$