Chapter 6

Space and Time

> Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. - Hermann Minkowski (1864 - 1909)

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Learning Objectives

By the end of this chapter, you will learn about concepts of space and time in the mathematical framework of special relativity. Through the eyes of two observers who are in constant motion relative to each other, you will uncover bizarre consequences such as a moving rod measuring a shorter length and a moving clock ticking slower.

Contents

6.1 Pre-Lesson Homework

6.1.1 An interplanetary murder?

Suppose there are Martians. On a beautiful afternoon, Dr U emerged from the shore of a tropical (Earth) island in his wetsuit and dive gear. Something streaked across the sky and Dr U fell. It was 4:15 pm. In the investigation of the murder, it was found that at exactly 3 minutes before Dr U fell, a weapon capable of interplanetary murder was fired in Mars. It was known that Mars was 60 million km away from Earth at the time of murder. Did the Martian kill Dr U?

6.1.2 Different perspectives

Get yourself a tennis ball. Also get a friend who is happy to do geeky stuffs with you. Take turns to do the following actions. While one do, the other stay still and observe.

Action 1: Stand still, throw the tennis ball vertically up into the air and catch it when it goes back down.

Action 2: Start walking at a constant pace. While walking, throw the ball up vertically and catch it when it goes back down.

How does the motion of the ball in Action 1 and 2 differ? How does the observation of Action 2 differ for the person doing the action (the actor) and for the stationary observer?

Draw the paths of the balls of Action 2 as seen by both the actor and the observer?

Compare the distance traveled by both the actor and observer.

Also compare the time taken for the ball to go up and down.

Compare the average speed of the ball as seen by you and your friend.

6.1.3 A thought experiment

Physicists like to fantasize about experiments that are simple enough to think of but impossible to carry out due to physical constraints. Einstein's "happiest thought" was one such *gedanken* (thought) experiment!

Let us begin our fantasy with something rather down to Earth - a clock. Think of a clock as a device which exhibit periodic observable phenomena. The turning of a clock needle is an example. A pendulum is another.

Now imagine a special "clock" consisting of two perfectly reflective plane mirrors facing each other, one above the other. When a photon is released in between the mirrors, in a direction parallel to the line joining the 2 planes, the photon bounces up and down, forever. The motion of the photon is periodic and thus serves as a good clock.

Take the "light clock" and do what you did in the previous experiment (tennis ball $\rightarrow$ photon). Will your previous conclusions stay the same?

6.2 Galilean relativity

When we consider the term "relativity", it should not take too long to convince ourselves that most physical measurements are made relative to a chosen reference system. The position vector of a particle, $\mathbf{r} = (x, y, z)$, means that its position vector has components (x, y, z) relative to some chosen origin and a chosen set of axes. An event occurring at time $t = 10$ s means that t is 10 seconds relative to a chosen origin of time, $t = 0$ s. When we measure the kinetic energy of a car, it makes a difference whether the kinetic energy is measured relative to a reference frame fixed to the road or to one fixed on the car. Every one of us carries along our individual set of reference frame, relative to which physical measurements are made.

Many of the ideas on relativity are already present in classical physics. Newton's laws hold in **inertial reference frames**, any one of which moves at constant velocity to any other.

> An **inertial reference frame** is defined to be a frame in which a free object is observed to move at a constant velocity (can be at rest) everywhere in the frame.

The above definition implies that that the frame is at rest or moving with constant velocity. In other words, the frame is non-accelerating.

Imagine, for example, playing a game of billiards in a train going at constant speed down a smooth straight track. The game will proceed exactly the same as it would if the train is at rest; you do not have to "correct" your shots for the fact that the train is moving. If you look out from the windows, from your perspective, it seems as if the environment outside is the one *moving*! Likewise, an observer outside will justifiably claim that your train is the one that is moving from his or her perspective. Furthermore, if you pulled down all the curtains, you would have no way of knowing whether the train is moving or not.

Notice, however, that you know *immediately* if the train speeds up, or slows down, or rounds a corner, or goes over a bump. The tell tale signs are the billiard balls moving in curved trajectories and yourself feeling a lurch. The laws of mechanics, therefore, behave quite differently when the frame is accelerating.

> The laws of mechanics remain the same for observers in inertial frames that are in uniform motion (can be at rest) with respect to one another.

The above is known as the **Galilean principle of relativity**. In its application to classical physics, the principle of relativity is nothing new and makes intuitive sense, as first stated by Galileo Galilei (1564-1642).

6.2.1 Galilean transformation

Consider two inertial frames, S and S', that are aligned in the same way: x' axis is parallel to x axis, y' axis parallel to y and z' axis parallel to z. Suppose that S' moves at a constant velocity of $\mathbf{v}$ along the x-axis with respect to S. It was a fundamental assumption of classical physics that there is a single universal time t. Therefore, if observers in S and S' agree to synchronise their clocks (and use the same unit of time), then $t = t'$. Finally, we can choose origins O and O' such that they coincide at $t = t' = 0$. This is illustrated in Figure 6.1.

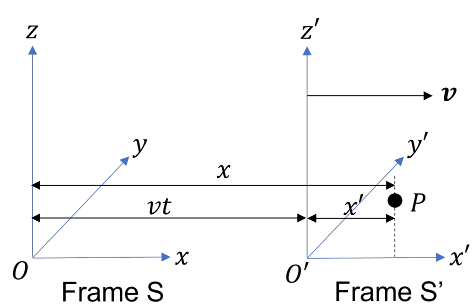


Figure 6.1: Two inertial frames, S and S' whose spatial coordinate axes are aligned and spatial origins coincide at $t = t' = 0$. S' moves in the $+x$ direction relative to S with a speed v.

Consider some event P. As measured by observers in S, this event occurs at $\mathbf{r} = (x, y, z)$ and at time t. As measured by observers in S', the same event occurs at $\mathbf{r'} = (x', y', z')$ and at time t'.

We can establish a mathematical relationship between (t, x, y, z) and (t', x', y', z') just by inspecting Figure 6.1.

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{6.1}$$

These four equations are known collectively as the **Galilean transformation**. They are the mathematical expression of the classical ideas of space and time.

The Galilean transformation relates the coordinates measured in two inertial frames arranged with corresponding axes parallel and with relative velocity along the x-axis. This is not the most general configuration. For example, if the relative velocity $\mathbf{v}$ is in an arbitrary direction, we can rewrite (6.1) compactly in the form of vectors as

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} - \mathbf{v}t \\ t' &= t \end{aligned} \tag{6.2}$$

With (6.5), we can relate the velocities of an object, as measured in the two inertial frames. If $\mathbf{u} = \frac{d\mathbf{r}}{dt}$ is the velocity of an object measured in S and $\mathbf{u}' = \frac{d\mathbf{r}'}{dt'}$ is the velocity of the object measured in S', then by differentiating (6.5), we have

$$\begin{aligned} \frac{d\mathbf{r}'}{dt'} &= \frac{d\mathbf{r}}{dt'} - \mathbf{v}\frac{dt}{dt'} \\ \frac{d\mathbf{r}'}{dt'} &= \frac{dt}{dt'}\frac{d\mathbf{r}}{dt} - \mathbf{v} \\ \frac{d\mathbf{r}'}{dt'} &= \frac{d\mathbf{r}}{dt} - \mathbf{v} \\ \mathbf{u}' &= \mathbf{u} - \mathbf{v} \end{aligned} \tag{6.3}$$

This is the non-relativistic velocity addition formula. According to the ideas of non-relativistic physics, relative velocities add (and subtract) according to the rules of vector arithmetic.

6.3 The 'Light' problem

As stated by the Galilean principle of relativity, Newton's laws hold in all inertia frames and these inertia frames can be related by Galilean transformation. However, Physics is not all about mechanics. What about the laws of electromagnetism? As we shall soon see, unlike the laws of mechanics, the laws of electromagnetism change under Galilean transformation and are different between inertia frames.

A fundamental law of electromagnetism, as revealed from Maxwell's equations, is that light (and, more generally, any electromagnetic wave) propagates through vacuum in any direction with speed, $c = 2.998 \times 10^8$ m/s.

Suppose a light beam travels at speed $u = c$ in the x direction, as measured in frame S. Now, consider a second frame S' travelling at a constant speed v along the x-axis of S, and imagine the same beam of light travelling in the same direction. Using the classical velocity addition formula (6.3), the speed of light as measured by S' will be

$$u' = c - v \tag{6.4}$$

in the x direction. Similarly, if the light beam is travelling in the opposite direction, then according to (6.3), S' will measure the speed of light to be

$$u' = c + v. \tag{6.5}$$

Depending on the direction of the light beam, S' will measure a speed of u' that varies anywhere between $c - v$ and $c + v$. Therefore, while the speed of light is constant in any direction in frame S, it is not so in a different inertia frame S' that moves at constant velocity relative to S. It seems that there is a *unique* frame in which light will travel at the same speed in all directions. This supposed frame is sometimes called the ether frame.

6.4 The search for ether

It was once thought that there exists an invisible substance that permeates all space, known as **ether**. Since the Earth travels at a considerable speed in a continually changing direction around the Sun, it seemed that Earth must spend most of its time moving relative to this ether frame and hence, the speed of light measured on Earth should be different in different directions. The effect, however, was expected to be very small.

Nevertheless, in 1880, Albert Michelson (1852-1931) and Edward Morley (1838-1923) cleverly devised an interferometer that should have detected the expected discrepancies in the speed of light. To their surprise and disappointment, they found absolutely no variation in their measurement of the speed of light in different directions. Although it yielded null results, this experiment would later be famously known as the **Michelson-Morley experiment**. For more details involving the experiment, look at: Michelson-Morley experiment

The Michelson-Morley experiment, together with many different experiments with the same objective, have been repeated and have never found any reproducible evidence of discrepancies in the speed of light relative to Earth. Contrary to expectations, the speed of light is the same in all directions relative to Earth's frame. In other words, it is not true that there is only a unique frame in which light has the same speed in all directions.

This conclusion was so surprising that it was not taken seriously for many years. Instead, several other ingenious theories were put forth to explain the null results of Michelson-Morley experiment, while preserving the idea of a unique ether frame. However, none of those theories were able to explain all of the observed facts, at least not in a reasonable and economical way. Today, almost all physicists have accepted that there is no ether frame and the speed of light is a universal constant in all directions in all inertia frames. The first person who accepted this was Albert Einstein (1879-1955), while first proposing his special theory of relativity.

6.5 Postulates of special relativity

The special theory of relativity is based on the acceptance of the universality of the speed of light. In developing his special theory of relativity, Einstein proposed two postulates (assumptions that are taken to be true). The first postulates asserts the existence of many inertia frames, travelling at constant velocity relative to one another.

> **First postulate of special relativity**: The laws of physics remain the same for observers in inertia frames that are in uniform motion (even at rest) with respect to one another.

Notice how the above is similar to the Galilean principle of relativity, with "laws of mechanics" replaced by "laws of physics". Einstein claimed that the principle of relativity applies to *all* laws of physics, the laws of electromagnetism inclusive! In other words, among all inertia frames, there is no *preferred* frame; the laws of physics single out no one inertia frame as being in any way special than any other inertia frame.

The second postulate specifies the fundamental law of electromagnetism that is universal in all inertia frames.

> **Second postulate of special relativity**: The speed of light (in vacuum) has the same value c in every direction in all inertia frames.

This is, of course, the result from the Michelson-Morley experiment. To put it in another way, the speed of light is **invariant** in all inertia frames. Let's take this postulate to the extreme and consider light as a single photon (Figure 6.2).

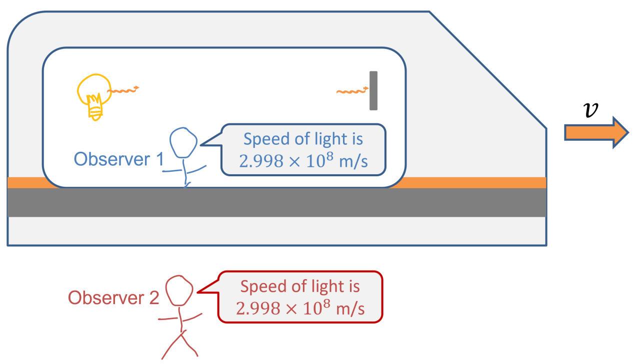


Figure 6.2: Observer 1 is in a train moving at constant speed v with respect to Observer 2. A photon was emitted in the train and hits a screen. Both observer made measurements of the speed of the photon and reported the same numerical value.

If the photon is replaced with any other object travelling at a speed much less than the speed of light, Observer 2 will have seen the object move faster than Observer 1 according to the classical velocity addition formula (6.3). This is however, not the case for light, even as a photon! Although the second postulate flies in the face of our everyday experiences, it is by now a firmly established experimental fact.

In the next few sections, we shall explore the bizarre and surprising consequences of Einstein's postulates, all of which seem to contradict our everyday experiences.

6.6 Time dilation: An informal derivation

The second postulate of special relativity forces us to abandon the classical notion of a single universal time. Instead, we shall see that the time of any one event, as measured in two different inertia frames travelling relative to each other, is generally different.

Let us return to the thought experiment in Section 6.1.3. Suppose you bring the 'light-clock' with you onto a train carriage and the train moves at constant speed, while your friend watches the train (with you and the 'light-clock') moves. Going back to our familiar two frames, we have S (your friend) anchored to the ground and S' (you) travelling at speed v relative to S. For simplicity, let us assume the train moves in the $+x$-direction.

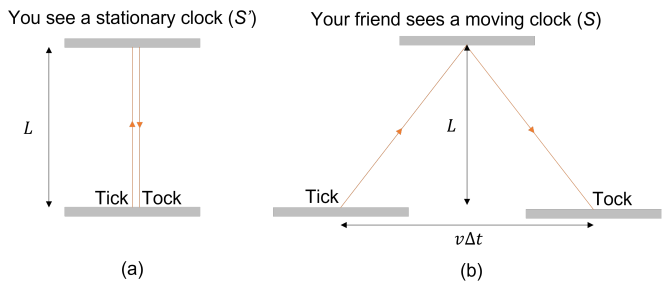


Figure 6.3: (a): The experiment as seen in S' (by you). The photon travels straight up and down, and the "tick" and "tock" occur at the same place. (b): As seen in S (by your friend), the "tick" and "tock" are separated by the distance $v\Delta t$. Note that in S, your friend will require another person to help time the two events, since "tick" and "tock" occur at two different locations.

Let $\Delta t'$ be the time you record for a "tick-tock" to happen in S'.

$$\Delta t' = \frac{2L}{c} \tag{6.6}$$

Let Δt be the time your friend records for a "tick-tock" to happen in S. Geometrically, from Figure 6.3

$$\Delta t = \frac{\sqrt{(2L)^2 + (v\Delta t)^2}}{c} \tag{6.7}$$

Notice that this is where we use the second postulate, that c is the invariant speed of light as observed in S and S'. We now make c the subject in both Equations (6.6) and (6.7) before equating them together.

$$\begin{aligned}
c = \frac{2L}{\Delta t'} &= \frac{\sqrt{(2L)^2 + (v\Delta t)^2}}{\Delta t} \\
\left(\frac{2L}{\Delta t'}\Delta t\right)^2 &= 4L^2 + v^2(\Delta t)^2 \\
(\Delta t)^2\left(\frac{4L^2}{(\Delta t')^2} - v^2\right) &= 4L^2 \\
(\Delta t)^2 &= \frac{4L^2(\Delta t')^2}{4L^2 - v^2(\Delta t')^2} \\
&= \frac{(\Delta t')^2}{1 - v^2/c^2} \\
\Delta t &= \gamma \Delta t' \qquad (6.8)
\end{aligned}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

For $v < c$, $\gamma > 1$ and $\Delta t' < \Delta t$. In other words, your friend that measures Δt sees a slower clock than you who measures $\Delta t'$!

6.7 Length contraction: An informal derivation

The postulates of relativity have led us to conclude that time is relative - the time between two events is different when measured in different inertia frames. This, in turn, implies that the length of an object is likewise dependent on the frame in which it is measured.

To see this, imagine now you (still on the train) set up the "light-clock" such that the photon source is at one end of the train carriage while the mirror is at the other end, such that a photon can be sent down and back. (Figure 6.4a)

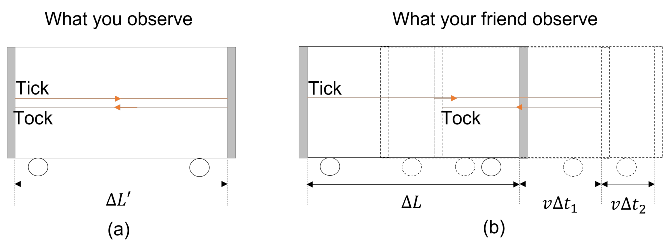


Figure 6.4: (a): The experiment as seen in S' (by you). The photon travels the length of the train, and the "tick" and "tock" occur at the same place. (b): The photon as seen in S (by your friend). The "tick" and "tock" occur at different locations. Δt_1 is the time taken for the photon to reach the front end while Δt_2 is the return time.

Question: How long does the photon takes to complete one round trip? To you, the answer is

$$\Delta t' = \frac{2\Delta L'}{c}, \tag{6.9}$$

where $\Delta L'$ is the length of the train as measured by you in S'. However, to your friend, the situation is a little more complicated since the train is moving (Figure 6.4b). Let Δt_1 be the time taken for the photon to reach the front end while Δt_2 be the return time. Then geometrically, from Figure 6.4b, we have

$$\begin{aligned} \Delta t_1 &= \frac{\Delta L + v\Delta t_1}{c} \\ \Delta t_2 &= \frac{\Delta L - v\Delta t_2}{c}, \end{aligned} \tag{6.10}$$

where ΔL is the length of the train as measured by your friend in S. Note that once again, we have used the second postulate. Solving for Δt_1 and Δt_2, we have

$$\begin{aligned} \Delta t_1 &= \frac{\Delta L}{c - v} \\ \Delta t_2 &= \frac{\Delta L}{c + v}. \end{aligned} \tag{6.11}$$

So, your friend will measure the round-trip time as

$$\Delta t = \Delta t_1 + \Delta t_2 = 2\frac{\Delta L}{c}\frac{1}{1 - v^2/c^2}. \tag{6.12}$$

Finally, recoginising that Δt and $\Delta t'$ are related through time dilation (6.8) and using Equation (6.9), we arrive at

$$\Delta L' = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta L = \gamma \Delta L. \tag{6.13}$$

For $v < c$, $\gamma > 1$ and $\Delta L < \Delta L'$. In other words, your friend measures a shorter train![1]

[1]Dimensions perpendicular to the velocity are not length contracted.

6.8 Lorentz Transformation

In Sections 6.6 and 6.7, we showed the relativity of time and space between different observers in uniform relative motion. According to the classical notions of time and space, the Galilean transformation equations (6.1) between coordinates in two inertia frames S and S' that we saw in Section 6.2.1 cannot be the actual relations.

Instead, the general mathematical relations between coordinates in two inertia frames S and S' moving with constant relative velocity $\mathbf{v}$ along the x-axis are given by the **Lorentz transformation** equations

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned} \tag{6.14}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{v}{c}$, v is the relative speed between the two observers, and c is the speed of light. Notice that there is now a transformation for the time coordinate, unlike Galilean transformation. Also, $y' = y$ and $z' = z$ since the relative motion between S and S' is along the x-axis. The equations can be inverted to obtain the coordinates (t, x, y, z) in terms of (t', x', y', z'), giving the **inverse Lorentz transformation**.

$$\begin{aligned} ct &= \gamma(ct' + \beta x) \\ x &= \gamma(x' + \beta ct) \\ y &= y' \\ z &= z' \end{aligned} \tag{6.15}$$

The Lorentz transformation equations expresses all the properties of space and time that follow from the postulates of special relativity. Using it, one can calculate all of the kinematic relations between measurements made in different inertia frames. Go to **Activity 1**.

6.8.1 Minkowski Space-Time Diagram

Up to now, we have been representing space-time events relative to the spatial axes of inertia observers in uniform relative motion, while representing time by drawing the axes at different positions, with the set of axes remaining fixed being the coordinate system of the observer from whose perspective we are observing the event.

In this section, we shall represent space-time events in a more efficient and useful (though perhaps more abstract) manner through the use of **Minkowski space-time diagrams**, proposed by the German mathematician Hermann Minkowski (1864-1909). Go to **Activity 2**.

A space-time diagram easily allows us to visualise a sequence of events. Suppose we track a flashing beacon as it moves along the x-axis. Each flash records a space-time event with the position it is located and the time of the flash, so we get a string of several points on the space-time diagram. If the flashing frequency is increased to infinity, then the space-time points form a continuous curve that tracks the space-time history of the moving object. Such a curve is known as a **worldline**.

6.8.2 Invariance of Space-time Interval

A set of coordinate transformations - such as the set of rotations in two or three dimensional space - can often be characterised by quantities that remain unchanged, or remain invariant after the transformation.

Consider two points in a two-dimensional space. Let one of the points be located at (x, y) while the other an infinitesimal distance from it, at $(x+dx, y+dy)$. This infinitesimal distance, ds, is therefore given by the Pythagoras Theorem in two dimensions,

$$(ds)^2 = (dx)^2 + (dy)^2. \tag{6.16}$$

What happens if we perform a rotation of the coordinate axes in this two dimensional space while keeping the two points fixed? Well, the pair of coordinates of each point changes with respect to the transformed axes. However, the quantity ds remains unchanged. That is, the infinitesimal distance between the two points remains unchanged even after the rotation!

We can see this idea easily applies to the three dimensional space. Consider two points located infinitesimally apart in three dimensional space with coordinates (x, y, z) and $(x + dx, y + dy, z + dz)$. Using Pythagoras Theorem in three dimensions, the infinitesimal distance between them is

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2. \tag{6.17}$$

Like its two dimensional counterpart, the infinitesimal distance ds between the two points in three dimension remains unchanged even after we arbitrarily rotate the three coordinate axes. Now, here's the catch. Lorentz transformations equations ((6.14) and (6.15)) describe a four dimensional "rotation" between two inertia frames by transforming the space-time coordinates of an event in one frame (t, x, y, z) to that of another frame (t', x', y', z').

Motivated from its two dimensional and three dimensional counterparts, we can therefore define the four dimensional infinitesimal "distance" between two events at (t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz)$ as observed in an inertia frame S,

$$(ds)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2, \tag{6.18}$$

such that this "distance" remains unchanged even after we apply Lorentz transformation. This quantity is known as the **space-time interval**. Note that it is the simply the extension of Pythagoras Theorem from three dimensions to four dimensions, with the exception of the negative sign in the first term. We shall see its importance in the **Activity 3** and the subsequent section. It is because of this negative sign that Lorentz transformations of space-time is not exactly the same as rotations of coordinate axes in ordinary space.

In another inertia frame S' in relative motion with respect to S, the space-time interval between the same two events is given by

$$(ds')^2 = -(cdt')^2 + (dx')^2 + (dy')^2 + (dz')^2. \tag{6.19}$$

Go to **Activity 3**.

6.8.3 Invariance of the speed of light from space-time interval

The space-time interval is essentially a mathematical construct. We are interested to know how the invariance of spacetime interval translates to something physical, such as speed.

Consider an object in an inertia frame S moving in the x-direction. An observer in this frame will measure that the speed of the object to be $\dfrac{dx}{dt}$. Since the object only moves in the x-direction, $dy' = 0$ and $dz' = 0$. Therefore, in an inertia frame S, the space-time interval of the object is given by

$$\begin{aligned}(ds)^2 &= -(cdt)^2 + (dx)^2 \\ &= (dt)^2\left(-c^2 + \left(\frac{dx}{dt}\right)^2\right).\end{aligned} \tag{6.20}$$

Suppose an observer in S a photon travelling in x-direction with speed $\dfrac{dx}{dt} = c$. Then

$$(ds)^2 = (dt)^2\left(-c^2 + c^2\right) = 0. \tag{6.21}$$

Consider an observer in another inertia frame S' that is in uniform relative motion to S. By the invariance of space-time interval

$$\begin{aligned}(ds')^2 = (ds)^2 &= 0 \\ (ds')^2 = -(cdt')^2 + (dx')^2 &= 0 \\ \left(\frac{dx'}{dt'}\right)^2 &= c^2 \\ \frac{dx'}{dt'} &= c\end{aligned} \tag{6.22}$$

Therefore, the speed of light is invariant between two inertia frames that are in uniform relative motion. This provides the mathematical framework for the second postulate of special relativity.

We shall see an alternative method of mathematically showing the second postulate under Discussion Questions.

6.8.4 The relativity of simultaneity

Let us go back to the thought experiment with you being on the train that is moving at constant speed and your friend being the ground observer. Imagine now you set up your photon source (i.e. light clock) in the very center of the train. Your "light clock" now emits two photons at the same time, one in a direction towards the front end of the train while the other in a direction towards the back end of the train. Because your "light clock" is equidistant from the two ends, you will observe that the photon reaches the front end at the same instant as the other photon reaches the back end. The two events: (a) photon reaches the front end (and maybe a buzzer goes off) and (b) the other photon reaches the back end (another buzzer sounds) occur *simultaneously*.

However, to your friend on the ground, these same two events are *not* simultaneous. For as the two photons travel out from the "light clock" (going at speed c according to the second postulate), the train itself is moving forward, so the photon going to the back end need only travel a shorter distance compared to the other photon going forward. As far as your friend is concerned, event (b) happens *before* event (a).

> Two events that are simultaneous in one inertia frame are not, in general, simultaneous in another inertia frame.

Now that we have seen the relativity of time and length between observers in two inertia frames (Sections 6.6 and 6.7), this should not be too surprising of a result. Naturally, your train has to be going awfully fast before the discrepancy becomes detectable.

Go to **Activity 4**.

6.8.5 Time dilation (Re-visited)

We derived time dilation equation (6.8) using a hypothetical clock and geometrical methods in Section 6.6. We shall now re-derive the same results using Lorentz transformation ((6.14) and (6.15)).

Consider two inertia frames S and S', with S' moving with constant velocity of $\mathbf{v}$ along the x-axis with respect to S. Imagine a light bulb on a table in front of an observer in frame S'. The light bulb is stationary with respect to frame S'. Let Event 1 be the event where the bulb is switched on and Event 2 be the later event where it is switched off.

$$\text{Time between Events 1 and 2 as measured by an observer in } S' \; = t_2' - t_1' = \Delta t'$$

Since the light bulb is stationary with respect to S', we fixed the position of it, say at $x' = a$, for both t_1' and t_2'. Then, from the inverse Lorentz transformation equations (6.15),

$$\begin{aligned} ct_1 &= \gamma(ct_1' + \beta a) \\ ct_2 &= \gamma(ct_2' + \beta a). \end{aligned} \tag{6.23}$$

Subtracting the first equation from the second,

$$\begin{aligned} c(t_2 - t_1) &= \gamma(ct_2' - ct_1') \\ t_2 - t_1 &= \gamma \Delta t'. \end{aligned} \tag{6.24}$$

Letting the time between the events measured by an observer in S be $\Delta t = t_2 - t_1$, we arrived at $\Delta t = \gamma \Delta t'$.

6.8.6 Length contraction (Re-visited)

Similarly, we can re-derive the length contraction equation (6.13) in Section 6.7 using Lorentz transformation ((6.14) and (6.15)). Consider now a rod that is stationary with respect to an observer in frame S. Let the left and right ends of the rod be located at coordinates x_1' and x_2' respectively, according to S'.

$$\text{Length of rod measured by an observer in } S' = x_2' - x_1' = \Delta L'$$

Now let's put ourselves in the shoes of an observer in S. Since the rod is stationary with respect to S', the rod will be moving at speed v in the x-direction according to an observer in S. How should he measure the length of the moving rod? Most logically, he would subtract the positions of the two ends at *one instant of his time*! Say, at the instant $t = a$, an observer in S observes the left and right ends to be at x_1 and x_2 respectively. Then, from Lorentz transformation equations (6.14),

$$\begin{aligned} x_1' &= \gamma(x_1 - \beta ca) \\ x_2' &= \gamma(x_2 - \beta ca). \end{aligned} \tag{6.25}$$

Subtracting the first equation from the second,

$$\begin{aligned} x_2' - x_1' &= \gamma(x_2 - x_1) \\ \Delta L' &= \gamma(x_2 - x_1). \end{aligned} \tag{6.26}$$

Letting the length of the rod as measured by an observer in S to be $\Delta L = x_2 - x_1$, we finally arrived at $\Delta L' = \gamma \Delta L$.

6.9 Causality and the light cone

Without any constraint on how fast information can travel, the causal structure of space and time can be illustrated in Figure 6.5.

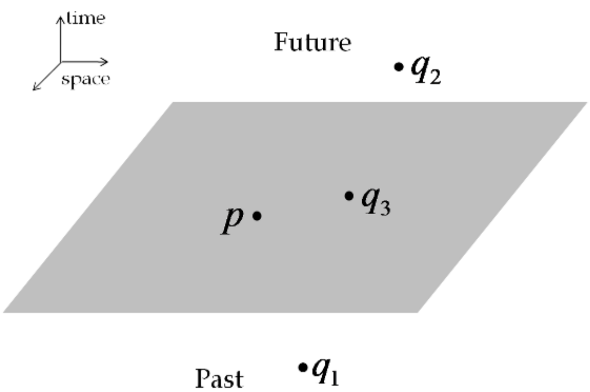


Figure 6.5: The causal structure of space and time before relativity.

Two events p and q will fall in either of the following mutually exclusive cases: (1) q is to the past of p, thus possible for q to influence p ; (2) q is to the future of p , thus possible for p to influence q; (3) p and q are simultaneous events, thus impossible for them to influence each other.

However, the above description of causality is incorrect! Information does take time to travel, and without information transfer, one event cannot influence another. As we have learnt from special relativity, the speed limit at which information can travel is the speed of light $c = 2.998 \times 10^8$ m/s. By taking into account this speed limit, the new causal structure of space and time is as illustrated below. (Figure 6.6)

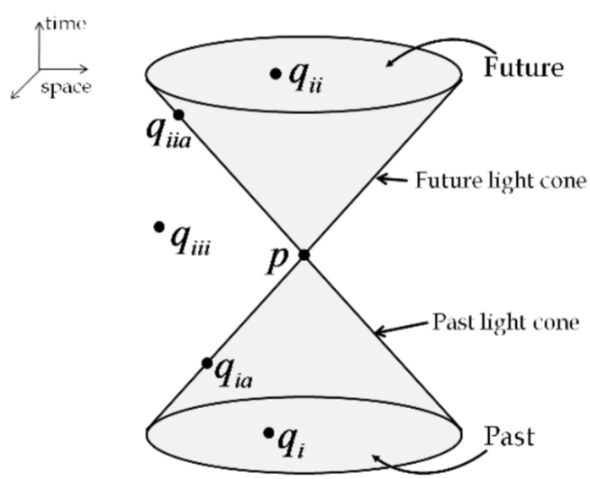


Figure 6.6: The more accurate causal structure of space and time. The events in the past and future of p that have possibilities of cause and effect are constrained within the past and future *light cones*.

Note that the light cones in Figure 6.6 are the three-dimensional generalisation of the world lines of a light signal in the Minkowski space-time diagram (Section 6.8.1). Here the events p and q will fall in one of the following categories. (i) q is in the definite past of p, thus possible for its information to reach p ; (ia) q is on the past light cone of p. Nothing other than light can reach p from q. (ii) q is in the definite future of p, thus possible for its information to reach p; (iia) q is on the future light cone of p. Nothing other than light can reach q from p. (iii) Events p and q are separated in such a way that no material body nor light can go from one event to another. Some observers may see p then q, some may see q then p and some may even see p and q simultaneously.

6.10 In-class Activities

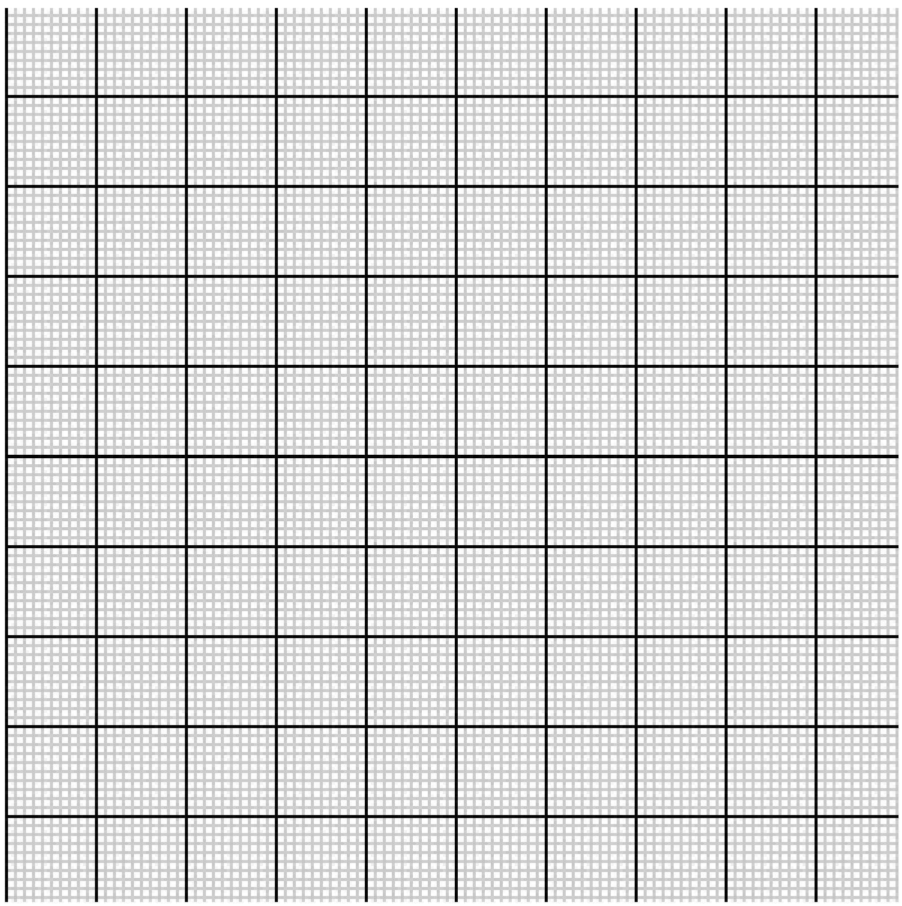
Activity 1: Lorentz transformation in the low speed limit

In the low speed limit where $v \ll c$, $\beta \approx 0$ and $\gamma \approx 1$. What do the time dilation (6.8), length contraction (6.13) and Lorentz transformation equations (6.14) become? Does this make intuitive sense?

Activity 2: Minkowski space-time diagram

Let us visualise how the space-time coordinates of two inertia observers in uniform relative motion are related to each other in a space-time diagram. For simplicity, we consider two inertia frames (S and S') where the relative motion between them occurs only along the x-axis. Specifically, suppose S' moves with a constant speed v in the $+x$-direction relative to S.

On the grid below, draw a vertical axis and label it ct. Draw a horizontal axis and label it x. This set of axes represent the inertia frame S. With this construction, a space-time event from the perspective of an observer in S is simply a point located in the plane of the two axes. The coordinate position and coordinate time (ct, x) can be read off the axes directly. Notice that position is represented on the horizontal axis while time is represented on the vertical axis. In relativity, it is conventional to represent time on the vertical axis. Moreover, the space-time diagram gives both axes the same units by scaling the vertical axis by the speed of light, c.



Next, we represent the space-time diagram, (ct', x'), for frame S' on the same grid. To draw the ct' axis, we examine various coordinate points where $x' = 0$. First, using Lorentz transformation (6.14), show that $x = \gamma\beta$ and $ct = \gamma$ when $\{x' = 0, ct' = 1\}$.

Repeat the previous step to find the corresponding x and ct coordinates for $\{x' = 0, ct' = 2\}$ and so on. Plot the points on the (ct, x) graph for $\beta = 0.6$. Draw a line through these points and label it as ct'.

To draw the x' axis, find the corresponding x and ct coordinates for $\{x' = 1, ct' = 0\}$, $\{x' = 2, ct' = 0\}$ and so on. Plot the points on the (ct, x) graph for $\beta = 0.6$. Draw a line through these points and label it as x'.

Suppose an observer in S observes an event occurring at $\{x = 3\gamma, ct = 4\gamma\}$. With the help of your space-time diagrams, what are the corresponding ct' and x' coordinates for the same event from the perspective of an observer in S' when $\beta = 0.6$?

How will the ct' and x' axes change for higher or lower values of β ?

What happens if $\beta = 1$?

Activity 3: Invariance of the space-time interval

Use the Lorentz transformation equations (6.14) to show the invariance of the space-time interval between two events at (t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz)$ when transforming from one inertia frame S to another inertia frame S', i.e. $ds = ds'$.

From Lorentz transformation (6.14), we have

$$
\begin{aligned}
cdt' &= \gamma(cdt - \beta dx) \\
dx' &= \gamma\left(dx - \beta cdt\right),
\end{aligned}
$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$.

Activity 4: Relativity of simultaneity

Consider inertia frame S' which moves at a constant speed (comparable to the speed of light) relative to inertia frame S. On the same Minkowski space-time diagram, sketch a pair of (ct', x') axes and (ct, x) axes for frames S' and S respectively. Mark out 2 events on (ct', x') axes that are simultaneous in frame S'.

Show that two events that are simultaneous in one inertia frame is not simultaneous in another inertia frame.

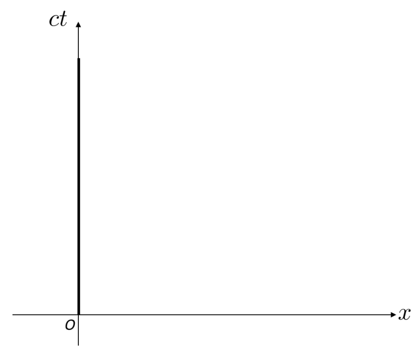
6.11 Discussion Questions

1. (Challenging problem) **Relativistic velocity addition.** Consider two inertial observers, S and S', whose frames of reference are related by Lorentz transformation ((6.14) and (6.15)). Observer S' sees an object moving in the x-direction with speed $u' = \dfrac{dx'}{dt'} = \alpha c$ where $\alpha < 1$. Find $u = \dfrac{dx}{dt}$, the speed at which Observer S sees the object is moving at, in terms of u'.

Suppose Observer S' sees a photon travelling with speed $u' = c$, use the relation you have derived to determine the speed of the photon as seen by Observer S. Verify that the second postulate of special relativity is satisfied.

Note: This relation is the relativistic version of the classical velocity addition formula (6.3).

2. **Worldline trajectories.** Consider a (1+1 D) Minkowski space-time diagram where the vertical axis is c multiply by time, and the horizontal axis is spatial position. A person located at the origin that is not moving and standing perfectly still will have his worldline indicated by the vertical thick black line on the space-time diagram as shown below.



Plot the worldline of the someone on the above spacetime graph if he/she is

- Moving with a constant velocity v such that $\dfrac{v}{c} = \dfrac{dx}{d(ct)} < 1$
- Move with a constant velocity $v = c$.
- Move with a constant velocity $v > c$.
- Begins motion at origin with $v = 0$, and reaches a velocity of $v = 0.999\,c$ after some time.

3. **The barn and pole paradox.** Once upon a time, there was a farmer who had a pole too long to store in his barn. After learning about relativity, he instructed his son to run with the pole as fast as he could, such that the moving pole would Lorentz-contract to a size the barn could accomodate. However, his son, who also knew about relativity, argued that the *barn* should be the one that Lorentz-contract, not the pole. So the fit would be even worse. Who was right? Would the pole fit inside the barn, or would it not?

4. **Life-time of a muon.** A muon is an elementary particle similar to an electron, with negative one electronic charge and a rest mass of 105.7 MeV/c^2. Muons are unstable and decay at an exponential rate into electrons (and neutrinos) with a half-life of 1.5 μs. Muons are produced at the upper atmosphere of Earth by collisions of cosmic rays (energetic protons from outer space) with the atmospheric molecules. Assume that muons are produced at 15 km above Earth surface at a rate of 10^6 per minute and travel radially down to Earth at a speed of $0.99c$.

(a) Without relativistic consideration, determine the rate of detection of muons on the Earth surface.

Now with relativistic consideration. The muons see the surface of Earth moving at $0.99c$. From the muon point of view, the length between the upper atmosphere and the surface of the Earth is not 15 km but rather a “contracted” length.

(b) Find the length that the muons move through (from the muons’ perspective).

(c) Using this length, calculate the rate of detection of muons on the Earth surface.

5. Re-look at the muon problem but from the perspective of a observer on Earth seeing the muon move. The half-life of the muon can be taken as the “biological clock” of the muon. What is the half-life of the muon as measured by the Earthling?