PC3261: Classical Mechanics II

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Lecture 2: Newton's Laws of Motion

Newton's first law and inertia

- **Newton's first law**: a particle remains at rest or in uniform motion unless acted upon a force

- **Inertia** is the *resistance* of any particle to any change in its velocity and the quantitative measure of inertia is **mass**

- A mathematical description of the motion of a particle requires the choice of a **frame of reference –** a set of coordinates in space that can be used to specify the position, velocity and acceleration of the particle at any given instant of time

- A frame of reference at which Newton's first law is valid is called an **inertial frame of reference**

Newton's second law

- **Linear momentum** of a particle is defined as the product of its mass and velocity

$$\mathbf{p}(t) \equiv m\mathbf{v}(t)$$

- **Newton's second law**: a particle acted upon a force moves in such a manner that the time rate of change of linear momentum equals the force

$$\mathbf{F}(t) = \frac{\mathrm{d}\mathbf{p}(t)}{\mathrm{d}t}$$

- Both Newton's first and second laws remain exactly true in special relativity with a *suitably* redefinition of linear momentum

Newton's third law

- **Newton's third law**: if two particles exert forces on each other, these forces are equal in magnitude and opposite in direction

- **Central forces** are the forces acting along the line connecting two particles

- Velocity-dependent forces are non-central and Newton's third law *may* not apply

- Newton's third law is not valid in special relativity as the concept of absolute time is abandoned

Galilean relativity

- Two inertial frames, $\mathcal{O}$ and $\mathcal{O}'$, are oriented such that their spatial coordinate axes are parallel, their spatial origins are coincided when $t = t' = 0$ and $\mathcal{O}'$ moves at *uniform velocity* $\mathbf{V}$ with respect to $\mathcal{O}$

- **Galilean boost**:

$$\begin{cases} t' = t \\ \mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{V}t \end{cases}$$

- Galilean velocity transformation:

$$\mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V}$$

- Newton's laws are **Galilean invariance**

$$\begin{cases} \mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y + z(t)\,\hat{\mathbf{e}}_z \\ \mathbf{r}'(t') = x'(t')\,\hat{\mathbf{e}}_{x'} + y'(t')\,\hat{\mathbf{e}}_{y'} + z'(t')\,\hat{\mathbf{e}}_{z'} \end{cases}, \qquad \begin{cases} \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_{x'} \\ \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_{y'} \\ \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_{z'} \end{cases}$$

$$t' = t \qquad \Rightarrow \qquad \mathbf{r}'(t') = \mathbf{r}'(t) = x'(t)\,\hat{\mathbf{e}}_x + y'(t)\,\hat{\mathbf{e}}_y + z'(t)\,\hat{\mathbf{e}}_z$$

$$\mathbf{v}'(t') \equiv \frac{\mathrm{d}\mathbf{r}'(t')}{\mathrm{d}t'} = \frac{\mathrm{d}\mathbf{r}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}x'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_x + \frac{\mathrm{d}y'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_y + \frac{\mathrm{d}z'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_z \equiv \mathbf{v}'(t)$$

$$\mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{V}t \quad \Rightarrow \quad \frac{\mathrm{d}\mathbf{r}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{V}t) \quad \Rightarrow \quad \mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V} \qquad \blacksquare$$

$$\Rightarrow \quad \begin{cases} \mathbf{v}(t) \equiv \dfrac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_x + \dfrac{\mathrm{d}y(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_y + \dfrac{\mathrm{d}z(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_z \\ \mathbf{v}'(t) \equiv \dfrac{\mathrm{d}\mathbf{r}'(t)}{\mathrm{d}t} = \dfrac{\mathrm{d}x'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_{x'} + \dfrac{\mathrm{d}y'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_{y'} + \dfrac{\mathrm{d}z'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_{z'} \end{cases} \qquad \blacksquare$$

$$\begin{cases} \mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y + z(t)\,\hat{\mathbf{e}}_z \\ \mathbf{r}'(t') = x'(t')\,\hat{\mathbf{e}}_{x'} + y'(t')\,\hat{\mathbf{e}}_{y'} + z'(t')\,\hat{\mathbf{e}}_{z'} \end{cases}, \qquad \begin{cases} \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_{x'} \\ \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_{y'} \\ \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_{z'} \end{cases}$$

$$t' = t \qquad \Rightarrow \qquad \mathbf{r}'(t') = \mathbf{r}'(t) = x'(t)\,\hat{\mathbf{e}}_x + y'(t)\,\hat{\mathbf{e}}_y + z'(t)\,\hat{\mathbf{e}}_z$$

$$\mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V}$$

$$\Rightarrow \quad \frac{\mathrm{d}\mathbf{v}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} \quad \Rightarrow \quad \mathbf{a}'(t) = \mathbf{a}(t) \qquad \blacksquare$$

$$\Rightarrow \quad \begin{cases} \mathbf{a}(t) \equiv \dfrac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \dfrac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_x + \dfrac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_y + \dfrac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_z \\ \mathbf{a}'(t) \equiv \dfrac{\mathrm{d}\mathbf{v}'(t)}{\mathrm{d}t} = \dfrac{\mathrm{d}^2 x'(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_{x'} + \dfrac{\mathrm{d}^2 y'(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_{y'} + \dfrac{\mathrm{d}^2 z'(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_{z'} \end{cases} \qquad \blacksquare$$

Equation of motion

- Second order ordinary differential equation: $\mathbf{r}(0) = \mathbf{r}_0$, $\dot{\mathbf{r}}(0) = \mathbf{v}_0$

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \qquad \rightarrow \qquad \begin{cases} \mathbf{r}(t) = ? \\ \dot{\mathbf{r}}(t) = ?? \end{cases}$$

- Cartesian coordinates:

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \qquad \Rightarrow \qquad \begin{cases} m\ddot{x}(t) = F_x(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\ddot{y}(t) = F_y(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\ddot{z}(t) = F_z(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \end{cases}$$

- Polar coordinates:

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \qquad \Rightarrow \qquad \begin{cases} m\left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right] = F_\rho(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right] = F_\phi(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \end{cases}$$

First order separable ordinary differential equation

- General form:

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = f(x)\,g(y)$$

- Implicit **general solution**: existence of an *arbitrary* constant in the solution

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

First order linear ordinary differential equation

- Standard form: $a_1(x) \neq 0$

$$a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + a_0(x)\,y(x) = f(x)$$

- **Integrating factor** $\mu(x)$: integration constant is irrelevant

$$\mu(x)\,a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + \mu(x)\,a_0(x)\,y(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\,a_1(x)\,y(x)\right]$$

$$\Rightarrow \quad \mu(x) = \frac{1}{a_1(x)}\exp\left[\int^x \frac{a_0(\xi)}{a_1(\xi)}\,\mathrm{d}\xi\right]$$

- General solution: c is an arbitrary integration constant

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\,a_1(x)\,y(x)\right] = \mu(x)\,f(x) \quad \Rightarrow \quad y(x) = \frac{1}{\mu(x)\,a_1(x)}\left[\int^x \mu(\xi)\,f(\xi)\,\mathrm{d}\xi + c\right]$$

$$a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + a_0(x)\,y(x) = f(x)$$

$$\mu(x)\,a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + \mu(x)\,a_0(x)\,y(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\,a_1(x)\,y(x)\right]$$

$$\Rightarrow \quad \mu(x)\,a_0(x) = \mu(x)\,\frac{\mathrm{d}a_1(x)}{\mathrm{d}x} + \frac{\mathrm{d}\mu(x)}{\mathrm{d}x}\,a_1(x)$$

$$\Rightarrow \quad \frac{\mathrm{d}\mu}{\mu(x)} = \frac{a_0(x)}{a_1(x)}\,\mathrm{d}x - \frac{\mathrm{d}a_1}{a_1(x)}$$

$$\Rightarrow \quad \ln\mu(x) = \int^x \frac{a_0(\xi)}{a_1(\xi)}\,\mathrm{d}\xi - \ln a_1(x)$$

$$\Rightarrow \quad \mu(x) = \frac{1}{a_1(x)}\exp\left[\int^x \frac{a_0(\xi)}{a_1(\xi)}\,\mathrm{d}\xi\right] \qquad \blacksquare$$

Special case: $F_x = F_x(t)$

- Solving for $v_x(t)$: $v_x(0) = v_{x0}$

$$m\ddot{x}(t) = F_x(t) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(t) \quad \Rightarrow \quad m\int_{v_x'=v_{x0}}^{v_x} \mathrm{d}v_x' = \int_{t'=0}^{t} F_x(t')\,\mathrm{d}t'$$

$$\Rightarrow \quad v_x(t) = v_{x0} + \frac{1}{m}\int_{t'=0}^{t} F_x(t')\,\mathrm{d}t'$$

- Solving for $x(t)$: $x(0) = x_0$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v_x(t) \quad \Rightarrow \quad \int_{x'=x_0}^{x} \mathrm{d}x' = \int_{t'=0}^{t} v_x(t')\,\mathrm{d}t'$$

$$\Rightarrow \quad x(t) = x_0 + v_{x0}t + \frac{1}{m}\int_{t'=0}^{t}\left[\int_{t''=0}^{t'} F_x(t'')\,\mathrm{d}t''\right]\mathrm{d}t'$$

Special case: $F_x = F_x(x)$

- Solving for $v_x(x)$: $x = x(t) \leftrightarrow t = t(x)$

$$m\ddot{x}(t) = F_x(x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F_x(x)$$

$$\Rightarrow \quad mv_x(x)\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x} = F_x(x) \quad \Rightarrow \quad m\int_{v_x'=v_{x0}}^{v_x} v_x'\,\mathrm{d}v_x' = \int_{x'=x_0}^{x} F_x(x')\,\mathrm{d}x'$$

$$\Rightarrow \quad v_x^2(x) = v_{x0}^2 + \frac{2}{m}\int_{x'=x_0}^{x} F_x(x')\,\mathrm{d}x'$$

- Solving for $x(t)$: $x = x(t) \leftrightarrow t = t(x)$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v_x(x) \quad \Rightarrow \quad \int_{x'=x_0}^{x} \frac{\mathrm{d}x'}{v_x(x')} = \int_{t'=0}^{t} \mathrm{d}t'$$

$$\Rightarrow \quad t = \int_{x'=x_0}^{x} \frac{\mathrm{d}x'}{v_x(x')} \quad \Rightarrow \quad x(t)$$

Special case: $F_x = F_x(v_x)$

- Solving for $v_x(t)$:

$$m\ddot{x}(t) = F_x(v_x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(v_x)$$

$$\Rightarrow \quad m\int_{v'_x=v_{x0}}^{v_x} \frac{\mathrm{d}v'_x}{F_x(v'_x)} = \int_{t'=0}^{t} \mathrm{d}t' \quad \Rightarrow \quad v_x(t) \quad \Rightarrow \quad x(t)$$

- Solving for $v_x(x)$:

$$m\ddot{x}(t) = F_x(v_x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(v_x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F_x(v_x)$$

$$\Rightarrow \quad mv_x(x)\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x} = F_x(v_x) \quad \Rightarrow \quad m\int_{v'_x=v_{x0}}^{v_x} \frac{v'_x}{F_x(v'_x)}\,\mathrm{d}v'_x = \int_{x'=x_0}^{x} \mathrm{d}x'$$

$$\Rightarrow \quad v_x(x) \quad \Rightarrow \quad x(t)$$