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Time reversal transformation

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1. Definition of time reversal transformation U_T in Q M.

2 (i) U_T unitary and antilinear = antiunitary

(ii) $U_T^2 = 1$, $U_T^2 = -1$

3 Kramer theorem

For a physical system having a time reversal symmetry, the eigenvalue of its Hamiltonian is doubly degenerate

4. If time reversal is a perfect symmetry then electric dipole moment of a fundamental particle vanishes

$\langle \underset{\sim}{d} \rangle = 0$, $\underset{\sim}{d}$: electric dipole moment

Consider the Newton equation of motion (The second Law)

$$\underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt} \qquad \underset{\sim}{p} = m\dot{\underset{\sim}{x}} \quad \text{momentum}$$

Change the direction of the force, i.e.

$$\underset{\sim}{F} \to -\underset{\sim}{F}$$

and let $\underset{\sim}{p} \to -\underset{\sim}{p}$ (motion reversal)

we get back the same equation of motion

$$-\underset{\sim}{F} = \frac{d}{dt}(-\underset{\sim}{p}) \to \underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt}$$

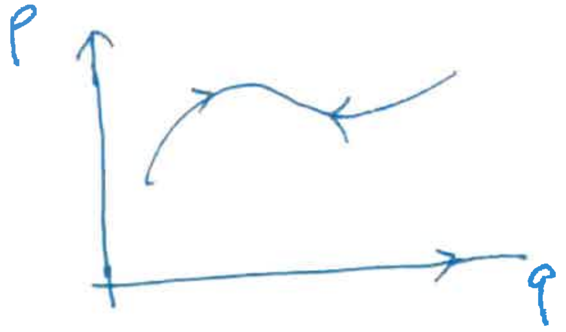
This means: if forward motion is possible in the physical world, the reversed motion is also possible in the physical world. <u>Motion reversal</u> is a symmetry.

As $\underset{\sim}{p} = m\dot{\underset{\sim}{x}} = m\frac{d\underset{\sim}{x}}{dt}$, we can realize $\underset{\sim}{p} \to -\underset{\sim}{p}$ by changing $t \to -t$.

thus <u>motion reversal</u> is realized mathematically by time reversal

The Newton equation $\underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt} = m\frac{d^2\underset{\sim}{x}}{dt^2}$ is invariant (unchanged) if $t \to -t$ (time reversal)

In classical mechanics, the trajectory of a particle can go both ways



phase space (q, p)

Motion reversal : $(q, p) \rightarrow (q, -p)$

Mathematically convenient to denote motion reversal as $t \rightarrow -t$

In quantum physics, the fundamental observables of an elementary particle are

$\underset{\sim}{x}$, $\underset{\sim}{p}$, $\underset{\sim}{J}$

So we define the time reversal operator, U_T, in terms of its action on the three fundamental observables.

In quantum mechanics (QM), define the time reversal operator U_T as follows

$$\underset{\sim}{x} \longrightarrow \underset{\sim}{x}' = U_T \, \underset{\sim}{x} \, U_T^{-1} \equiv \underset{\sim}{x}$$

$$\underset{\sim}{p} \longrightarrow \underset{\sim}{p}' = U_T \, \underset{\sim}{p} \, U_T^{-1} \equiv -\underset{\sim}{p}$$

$$\underset{\sim}{J} \longrightarrow \underset{\sim}{J}' = U_T \, \underset{\sim}{J} \, U_T^{-1} = -\underset{\sim}{J}$$

$\Rightarrow \quad [\, U_T^2 ,\ \Omega \,] = 0 \quad$ for $\Omega = \underset{\sim}{x}$, or $\underset{\sim}{p}$ or $\underset{\sim}{J}$
(HW)

From this definition, we can show that U_T has to be anti-linear

To show U_T <u>must</u> be antilinear:

By definition, an antilinear operator say A,

$$A\,\alpha|\psi\rangle = \alpha^* A|\psi\rangle$$

α = complex number

e.g $\alpha = i$, $\quad A\,i|\psi\rangle = -i\,A|\psi\rangle$

We show U_T must be antilinear in order to be consistent with the Heisenberg quantization condition

$$[x_i, x_j] = 0 = [P_i, P_j] \qquad i, j = 1, 2, 3$$

$$[x_i, P_j] = i\hbar\,\delta_{ij}$$

$\rightarrow$ Heisenberg uncertainty principle

$$\Delta x_1\, \Delta P_1 \geq \frac{\hbar}{2}$$

$$\Delta x_2\, \Delta P_2 \geq \frac{\hbar}{2}$$

$$\Delta x_3\, \Delta P_3 \geq \frac{\hbar}{2}$$

Note: No inequality between uncertainty Δx_1 and ΔP_2, for instance

Consider

$$[x_i, p_j] = i\hbar\delta_{ij}$$

$$x_i p_j - p_j x_i = i\hbar\delta_{ij}$$

Apply U_T, and U_T^{-1} to both sides

$$U_T(x_i p_j - p_j x_i)U_T^{-1} = U_T(i\hbar\delta_{ij})U_T^{-1}$$

$$U_T x_i U_T^{-1} U_T p_j U_T^{-1} - U_T p_j U_T^{-1} U_T x_i U_T^{-1}$$

$$= \hbar\delta_{ij}\, U_T(i)\, U_T^{-1}$$

$$-[x_i, p_j] = \hbar\delta_{ij}\, U_T\, i\, U_T^{-1}$$

In order to get back $[x_i, p_j] = i\hbar\delta_{ij}$

we demand $\quad U_T i = -i U_T$

so that $\quad \hbar\delta_{ij}\, U_T\, i\, U_T^{-1}$

$$= \hbar\delta_{ij}(-i)\, U_T U_T^{-1} = -i\hbar\delta_{ij}$$

$$\because U_T U_T^{-1} = 1$$

So because of $U_T i = -i U_T$

$\therefore$ U_T is antilinear.

show $\quad u_T^2 = 1 \quad$ or $\quad u_T^2 = -1$

suppose $\quad u_T : |\psi\rangle \to |\psi'\rangle = u_T|\psi\rangle$

As $\quad [u_T^2, \underset{\sim}{\Omega}] = 0, \quad \underset{\sim}{\Omega} = \underset{\sim}{x}, \underset{\sim}{p}, \underset{\sim}{J}, \quad \therefore u_T^2 = c \cdot \mathbb{1}$

identity operator

Then

$$u_T^2 |\psi\rangle = c |\psi\rangle \qquad c = \text{constant} = \text{complex number}$$

$$\langle\psi|(u_T^2)^\dagger u_T^2|\psi\rangle = c c^* \langle\psi|\psi\rangle = |c|^2$$

$$(u_T^2)^\dagger u_T^2 = u_T^\dagger u_T^\dagger u_T u_T = 1 \qquad \therefore |c|^2 = 1$$

Also $\quad u_T^2 |\phi\rangle = d |\phi\rangle, \qquad d = \text{constant} = \text{complex number}$

$$|d|^2 = 1$$

Let $\quad |\phi\rangle = |\psi\rangle + u_T|\psi\rangle$

Apply $\quad u_T^2 \quad$ on both sides

$$u_T^2|\phi\rangle = u_T^2|\psi\rangle + u_T^3|\psi\rangle$$

$$d|\phi\rangle = c|\psi\rangle + u_T c|\psi\rangle$$

$$= c|\psi\rangle + c^* u_T|\psi\rangle$$

$\rightarrow \quad d|\psi\rangle + d\, u_T|\psi\rangle = c|\psi\rangle + c^* u_T|\psi\rangle$

$\rightarrow \quad d = c, \qquad d = c^*$

$\longrightarrow \quad c = c^* \quad$ i.e. c is a real number

$$|c|^2 = 1 \Rightarrow c = +1 \quad \text{or} \quad c = -1$$

Go back $\quad u_T^2 |\psi\rangle = c\, |\psi\rangle$

i.e. $\quad u_T^2 = +1 \quad \text{or} \quad u_T^2 = -1$

HW: why $\pi^2 \neq -1$? $\quad$ HW

Interesting to note the rotation operator

$\mathcal{R}$ = rotation in 3-dim space

$$\mathcal{R}^2 = 1, \quad \mathcal{R}^2 \neq -1.$$

R = rotation in Hilbert space

$$\rightarrow \quad R^2 = 1 \quad \text{or} \quad R^2 = -1$$

For even j, $\quad R^2 = 1$

For odd j, $\quad R^2 = -1$

Kramer thm.

Given $[U_T, H] = 0$ U_T is a symmetry of physical system, then the energy E (eigenvalue of H) is doubly degenerate

'Doubly degenerate' means for the same E value, we can have two different eigenstates of H

Proof: Let $|\psi\rangle$ be eigenket of H

$$H|\psi\rangle = E|\psi\rangle$$

consider the time-reversal state $U_T|\psi\rangle$

$$H\, U_T|\psi\rangle = U_T H|\psi\rangle = U_T E|\psi\rangle = E\, U_T|\psi\rangle$$

so both $|\psi\rangle$ and $U_T|\psi\rangle$ are eigenstate of H with the same energy E.

If $U_T|\psi\rangle$ and $|\psi\rangle$ are proportional to each other, then no degeneracy

Now show $U_T|\psi\rangle$ and $|\psi\rangle$ <u>not</u> proportional. We prove by contradiction.

Assume proportionality,

$$U_T |\psi\rangle = \alpha |\psi\rangle \qquad \alpha = \text{constant}$$

Apply U_T to both sides

$$U_T^2 |\psi\rangle = U_T \alpha |\psi\rangle = \alpha^* U_T |\psi\rangle$$

$$= \alpha^* \alpha |\psi\rangle \quad \because \quad U_T|\psi\rangle = \alpha|\psi\rangle$$

Already know $U_T^2 = +1$ or $U_T^2 = -1$.

If $U_T^2 = +1$, $|\alpha|^2 = 1$, then ok

But if $U_T^2 = -1$, $|\alpha|^2 = -1$ impossible

so if $U_T^2 = -1$, then the assumption $U_T|\psi\rangle = \alpha|\psi\rangle$ is wrong

that means if $U_T^2 = -1$, the state $|\psi\rangle$ and $U_T|\psi\rangle$ are two different states

So for $U_T^2 = -1$, the energy value E is doubly degenerate (theorem is proved for states $|\psi\rangle$ s.t. $U_T^2 |\psi\rangle = -|\psi\rangle$)

Lastly we show if time reversal U_T is an exact symmetry, then electric dipole moment $\underset{\sim}{d}$ ($\underset{\sim}{d} = q\underset{\sim}{x}$ from classical electromagnetism) vanishes

i.e. $\langle \hat{d} \rangle = \langle \psi | \underset{\sim}{d} | \psi \rangle = 0$

Griffiths: suppose $\underset{\sim}{d}$ of a particle $\neq 0$ and the particle has a spin $\underset{\sim}{S}$. If U_T is a symmetry, then show $\underset{\sim}{d} = 0$

Before applying U_T, suppose $\underset{\sim}{s}$ and $\underset{\sim}{d}$ orientate in the same direction

$$\underset{\sim}{s} \uparrow \qquad \uparrow \underset{\sim}{d}$$

After applying U_T, we get the configuration

$$\underset{\sim}{s} \downarrow \qquad \uparrow \underset{\sim}{d}$$

Before U_T $\neq$ After U_T i.e.

U_T is not an exact symmetry

Two configurations different, so time reversal symmetry broken

If want time-reversal be an exact symmetry, $\rightarrow$

$$\underset{\sim}{d} = 0$$

Two configurations different, so time reversal sym. broken. Hand-waving argument $\rightarrow$ $\underset{\sim}{d} = 0$

Now use a more rigorous argument.

suppose the state of the particle is

$$|E\,j\,m\rangle$$

(cf H-atom state $|E\,l\,m\rangle$)

$H \qquad \underset{\sim}{J}^2 \qquad J_3$

assume spherical symmetry.

Consider 2 properties of the system: $\underset{\sim}{d}$, $\underset{\sim}{J}$

(cf. in CS Wu parity downfall expt, measure spin of cobolt 60 and the momentum $\underset{\sim}{p}$ of the electron emitted)

consider

$$\langle E j m | \underset{\sim}{d} | E j m \rangle, \qquad \langle E j m | \underset{\sim}{J} | E j m \rangle$$

By Wigner-Eckert theorem, we can write

$$\langle E j m | \underset{\sim}{d} | E j m \rangle = C_{Ej} \; \langle E j m | \underset{\sim}{J} | E j m \rangle \quad \cdots \quad (x)$$

Apply time-reversal transformation U_T

$$\langle E j m | U_T^\dagger U_T \, \underset{\sim}{d} \, U_T^\dagger U_T | E j m \rangle = C_{Ej} \langle E j m | U_T^\dagger U_T \underset{\sim}{J} \, U_T^\dagger U_T | E j m \rangle$$

By time-reversal transformation

$$U_T \tilde{J} U_T^\dagger = -\tilde{J}, \qquad U_T^\dagger = U_T^{-1}$$

$$U_T \tilde{x} U_T^\dagger = \tilde{x} \rightarrow U_T \tilde{d} U_T^\dagger = \tilde{d}$$

$\rightarrow$

$$\langle Ejm | U_T^\dagger \tilde{d} U_T | Ejm \rangle = -C_{Ej} \langle Ejm | U_T^\dagger \tilde{J} U_T | Ejm \rangle \quad \cdots (+)$$

Compare eq (X) and eq (+), need to know

$$U_T | Ejm \rangle = ?$$

Firstly, $[U_T, H] = 0$, $[U_T, \tilde{J}^2] = 0$ $\therefore$ $U_T|Ejm\rangle$ and $|Ejm\rangle$ same E, j

Consider $$J_3 U_T |Ejm\rangle = -U_T J_3 |Ejm\rangle$$

$$= -U_T m\hbar |Ejm\rangle = -m\hbar U_T |Ejm\rangle$$

$\rightarrow$ $U_T|Ejm\rangle$ is an eigenstate of J_3 with eigenvalue $-m\hbar$

On the hand, $$J_3 |E, j, -m\rangle = -m\hbar |E, j, -m\rangle$$

If m has no degeneracy, then

$$U_T |Ejm\rangle = \alpha |E j -m\rangle \qquad \alpha: \text{constant}$$

Eq (+) can be written as

$$\langle Ej-m | \tilde{d} | Ej-m \rangle = -C_{Ej} \langle Ej-m | \tilde{J} | Ej-m \rangle \quad \cdots (*)$$

Remember : $m = -j, -j+1, \ldots +j$

so eq (X) $\langle E j m | \underset{\sim}{d} | E j m \rangle = c_{Ej} \langle E j m | \underset{\sim}{J} | E j m \rangle$

can be written as

$$\langle E j\ -m | \underset{\sim}{d} | E j\ -m \rangle = c_{Ej} \langle E j\ -m | \underset{\sim}{J} | E j\ -m \rangle \quad \ldots \overline{(X)}$$

Adding eq (*) and eq $\overline{(X)}$, then

$$\langle E, j, -m | \underset{\sim}{d} | E, j, -m \rangle = 0$$

or $\langle E, j, m | \underset{\sim}{d} | E, j, m \rangle = 0$

If $|\psi\rangle$ is arbitrary state of the system, can write

$$|\psi\rangle = \sum_{E, j, m} k_{Ejm} |E j m\rangle, \qquad k_{Ejm} = \text{coefficients}$$

$$\rightarrow \quad \langle \psi | \underset{\sim}{d} | \psi \rangle = 0 \qquad (HW)$$

Note: In eq (X) and eq (*), $m = \{-j, -j+1, \ldots +j\}$

so eq (X) as a set for $m = j, -j+1, \ldots +j$

and eq (*) as a set for $m = j, -j+1, \ldots +j$,

the two sets are the same.