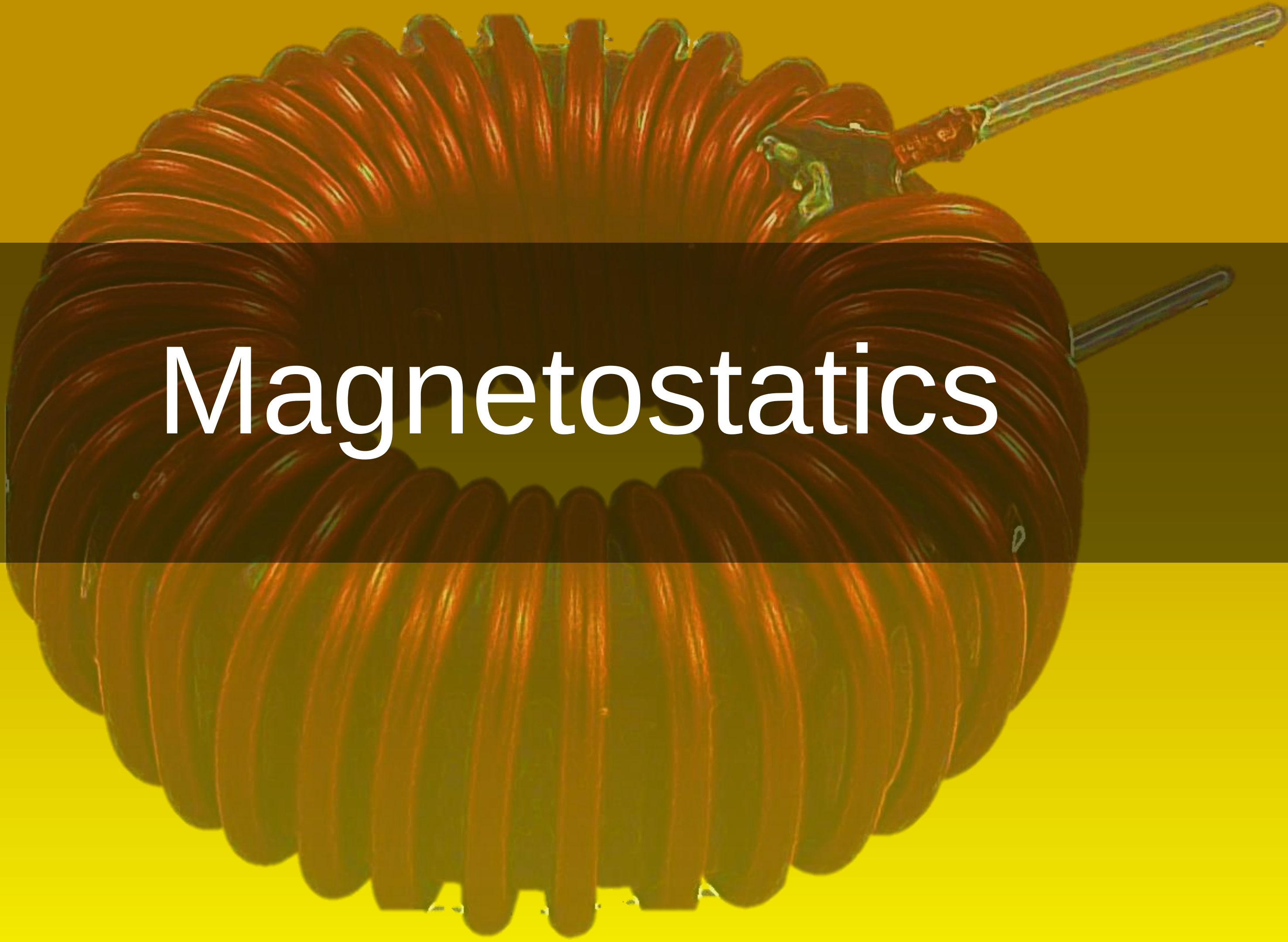
Magnetostatics

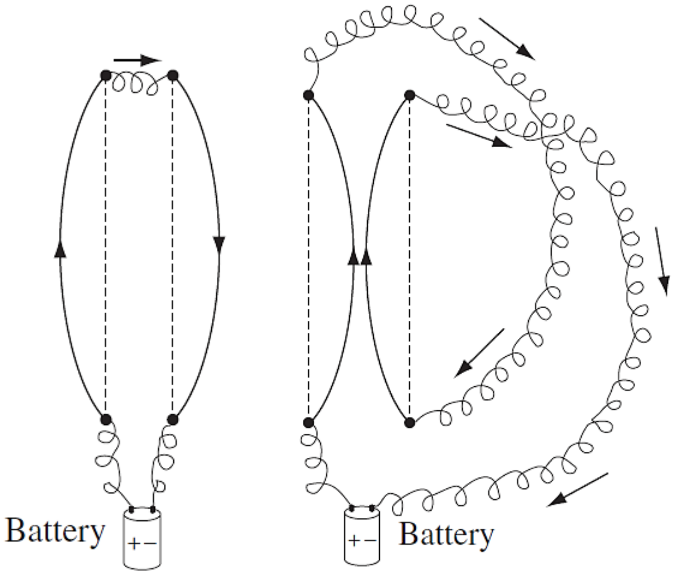


The Lorenz force law

Electrostatics: Stationary charges with constant electric fields

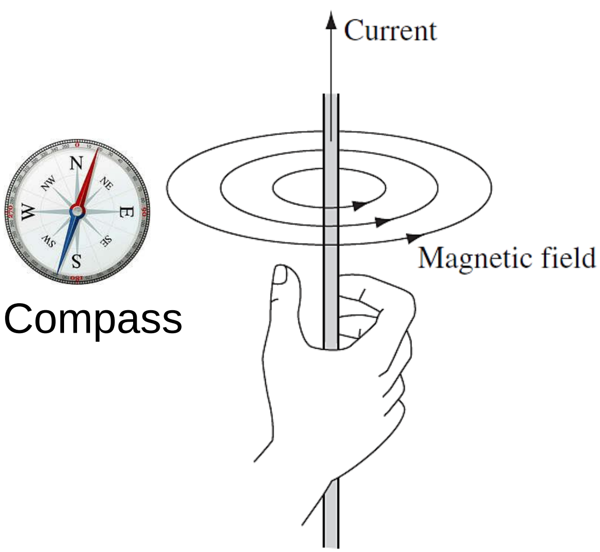
Magnetostatics: Steady currents with constant magnetic fields

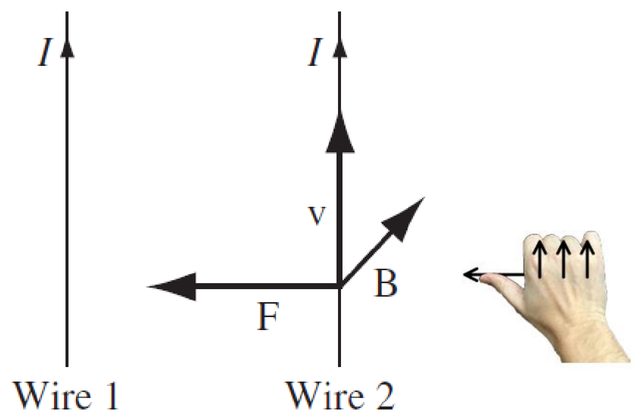
- Magnetic force and magnetic field
  - Pass current in parallel wires
    - Wires repel with antiparallel current
    - Wires attract with parallel current
    - Force is not electrostatic in nature (wires are charge neutral)
    - Magnetic force



The Lorenz force law

- Magnetic force and magnetic field
  - Empirical rules for magnetic field and magnetic force
    - Straight current-carrying wire has magnetic field circling around it
    - Right-hand rule for field direction given current direction
    - Right-hand rule for force direction given current and field direction



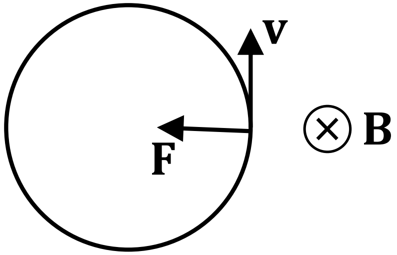


The Lorenz force law

- The Lorenz force law
  - Magnetic force on charge Q, moving with velocity $\mathbf{v}$, in magnetic field $\mathbf{B}$

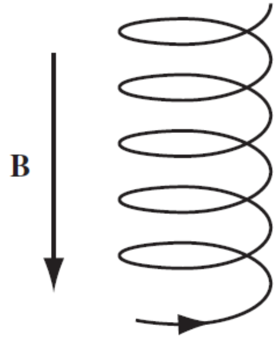
$$\boxed{\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})}$$

    - Given as an axiom without proof
  - With both $\mathbf{E}$ and $\mathbf{B}$, the full Lorenz force law $\boxed{\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]}$
  - Consequences
    - Circular charge motion, or alike, when magnetic force acts as centripetal force



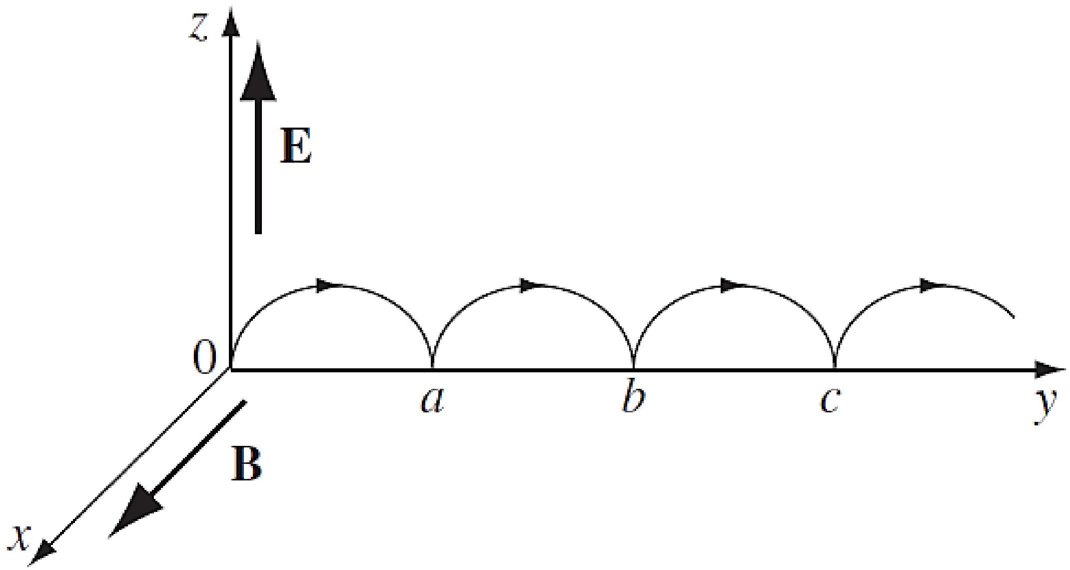
    - Magnetic forces do <u>no</u> work, $\mathbf{B}$ only deflects particle direction

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \overbrace{\mathbf{v}\,dt}^{d\mathbf{l}} = 0$$



The Lorenz force law

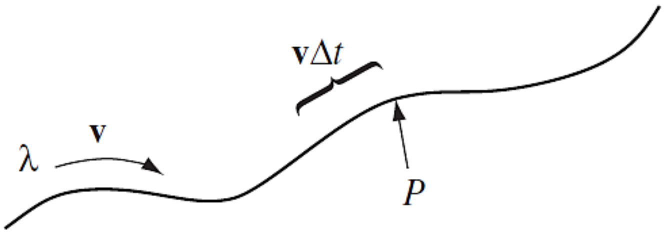
Example 5.2. Cycloid Motion. A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that $\mathbf{B}$ points in the x-direction, and $\mathbf{E}$ in the z-direction, as shown in Fig. 5.7. A positive charge is released from the origin; what path will it follow?



C3.exmp5.2

Currents

- Current in a wire: charge per time passing a given point
  - Unit: Amperes (A) = Coulomb/second
  - $\boxed{\mathbf{I} = \lambda \mathbf{v}}$ where λ: line charge density, $\mathbf{v}$: velocity of movement



  - Positive charges moving at $\mathbf{v}$ = negative charges moving at $-\mathbf{v}$
  - Not meaningful to talk about current if it's just a single point charge moving (non-steady)
  - In many problems just write the magnitude $\boxed{I = \lambda v}$
    - Because direction is determined by the shape of wire
  - Magnetic force on a current-carrying wire $\boxed{\mathbf{F}_{\text{mag}} = \int I\,(d\mathbf{l} \times \mathbf{B})}$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\lambda\, dl = \int (\mathbf{I} \times \mathbf{B})\, dl = \int I\,(d\mathbf{l} \times \mathbf{B})$$

Currents

- Surface and volume distributions of current
  - Surface current density $\boxed{\mathbf{K} = \sigma \mathbf{v}}$ (C/m^2 · m/s) (σ: surface charge density)
    - **K**: current per unit width (A/m)

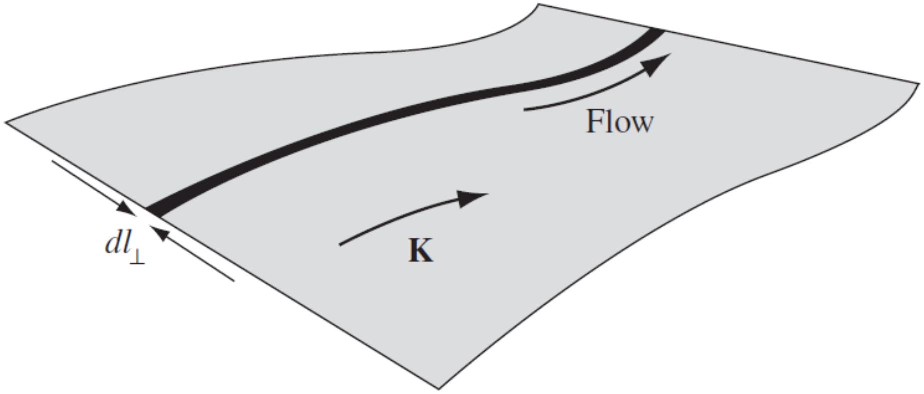
$$d\mathbf{I} = \sigma \mathbf{v}\, dl_\perp \Rightarrow \boxed{\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp}}$$

($l_\perp$: cross-sectional line segment perpendicular to $\mathbf{v}$)

    - Magnetic force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\sigma\, da = \int (\mathbf{K} \times \mathbf{B})\, da$$

* **B** experiences discontinuity across a current-carrying surface



Currents

- Surface and volume distributions of current
  - Volume current density $\boxed{\mathbf{J} = \rho \mathbf{v}}$ (C/m³ · m/s) (ρ: volume charge density)
    - **J**: current per unit area (A/m²)

$$d\mathbf{I} = \rho \mathbf{v}\, da_{\perp}$$
$$d\mathrm{I} = \mathbf{J} \cdot d\boldsymbol{a} = \rho \mathbf{v} \cdot d\boldsymbol{a}$$

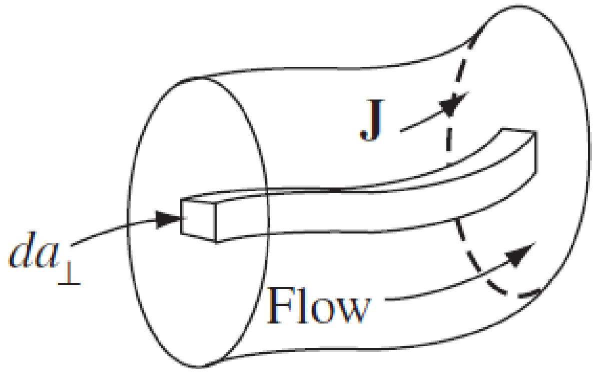
$$\Rightarrow \boxed{\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}}$$

($a_{\perp}$: cross-sectional area segment perpendicular to $\mathbf{v}$)

    - Magnetic force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B}) \rho \, d\tau = \int (\mathbf{J} \times \mathbf{B})\, d\tau$$

  - Ohm's law $\mathbf{J} = \sigma_c \mathbf{E}$ (σ_c: electrical conductivity)



Currents

- Surface and volume distributions of current
  - Continuity equation for volume current density
    - Current crossing a surface $\mathcal{S}$: $I = \int_{\mathcal{S}} J \, da_{\perp} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$
    - Current crossing boundary of a volume $\mathcal{V}$:

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau$$

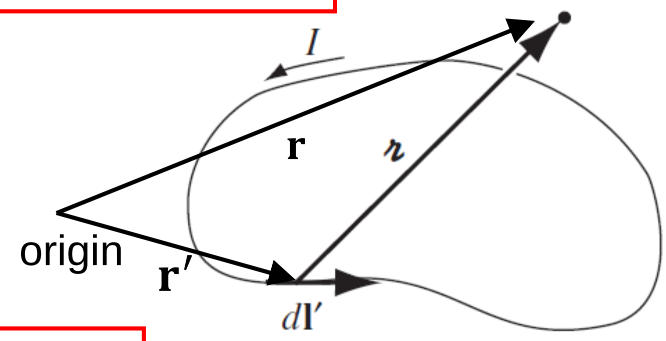
which must equal to the <u>change of net charge</u> in the volume

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau = \overbrace{-\frac{d}{dt} \int_{\mathcal{V}} \rho \, d\tau} = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} \right) d\tau$$

$\Rightarrow$ $\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$ for magnetostatics $\frac{\partial \rho}{\partial t} = 0$ $\quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0}$ $\quad \nabla \cdot \mathbf{J} = 0$

The Biot-Savart law

- Magnetic field generated by a steady current
  - Line current $\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}}$
    - Unit of $\mathbf{B}$: Tesla (T) = N/(A·m)
    - Permeability $\mu_0 = 4\pi \times 10^{-7}$ N/A^2
    - Separation vector $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$



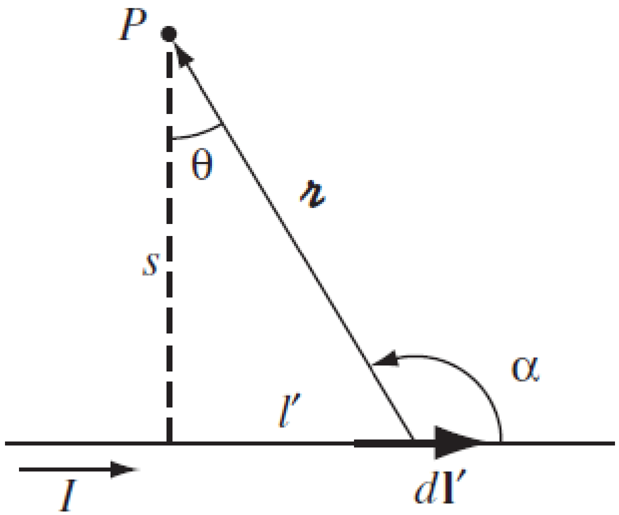
  - Surface current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} da'$
  - Volume current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau'$
  - Expressions for surface and volume currents not proved
  - The law cannot be used to calculate field generated by discrete moving charges

* $\mathscr{r}$ in textbook is typed as $\boldsymbol{\mathscr{r}}$

The Biot-Savart law

- Application of Biot-Savart law

Example 5.5. Find the magnetic field a distance s from a long straight wire carrying a steady current I (Fig. 5.18).



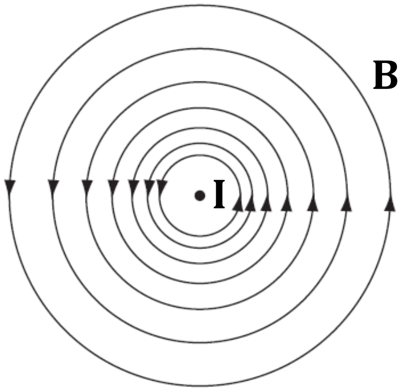
$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$$

C3.exmp5.5

Curl of magnetic field

- Derivation of curl from Stokes theorem
  - Loop integral of $\mathbf{B}$ around a straight-line current

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I$$



    - Result still holds if path is noncircular

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + \underline{s\,d\phi\,\hat{\boldsymbol{\phi}}} + dz\,\hat{\mathbf{z}} \qquad \underline{\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}}$$

  - Loop integral of $\mathbf{B}$ around any current distribution

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \qquad \left(I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a} \text{ total current enclosed by the path}\right)$$

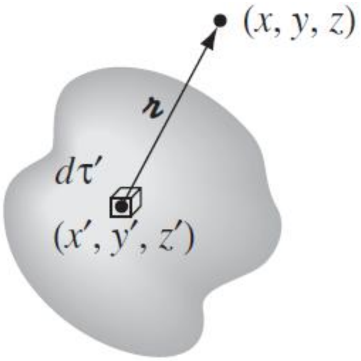
  - Apply Stokes theorem

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\Rightarrow \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

Curl of magnetic field

- Derivation of curl from Biot-Savart law
  - A much more formal derivation than previous slide
  - $\nabla \times \underbrace{\mathbf{B}}_{\mathbf{B}(\mathbf{r})} = \dfrac{\mu_0}{4\pi} \displaystyle\int \underbrace{\nabla}_{\nabla_{\mathbf{r}}} \times \left(\underbrace{\mathbf{J}}_{\mathbf{J}(\mathbf{r}')} \times \dfrac{\hat{\boldsymbol{\mathscr{r}}}}{\underbrace{\mathscr{r}}_{\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'}^2} \right) \underbrace{d\tau'}_{dx'dy'dz'}$



$\big\downarrow$ Product rule $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) - \underline{(\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}}$$

$\big\downarrow$ $\nabla \cdot \left(\dfrac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\boldsymbol{\mathscr{r}})$, and second term integrates to 0 (textbook p.232)

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta^3(\mathbf{r} - \mathbf{r}')\, d\tau' = \mu_0 \mathbf{J}(\mathbf{r})$$

Divergence of magnetic field

- Derivation of divergence from Biot-Savart law
  - $\nabla \cdot \mathbf{B} = \dfrac{\mu_0}{4\pi} \displaystyle\int \nabla \cdot \left(\mathbf{J} \times \dfrac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'$

Product rule $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right)$$

$\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') = 0$, and $\nabla \times \dfrac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 0$ (Remember $\nabla \times \mathbf{E} = \mathbf{0}$)

$\boxed{\nabla \cdot \mathbf{B} = 0}$

Magnetic fields are divergence-free (no magnetic "free charge")

Magnetic monopoles? not reproduced

First Results from a Superconductive Detector for Moving Magnetic Monopoles

Blas Cabrera

Phys. Rev. Lett. **48**, 1378 – Published 17 May 1982

Ampère's law

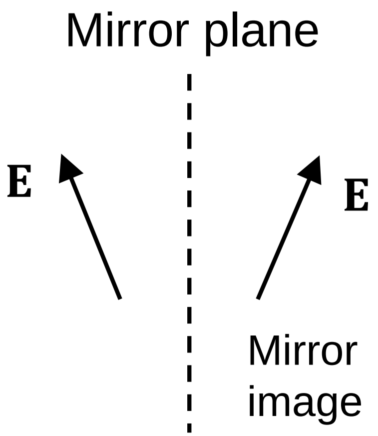
Electrostatics	Magnetostatics
Coulomb's law	Biot-Savart law
Gauss's law $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \quad \oint \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}}/\varepsilon_0$	Ampère's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$
Gaussian surface	Amperian loop

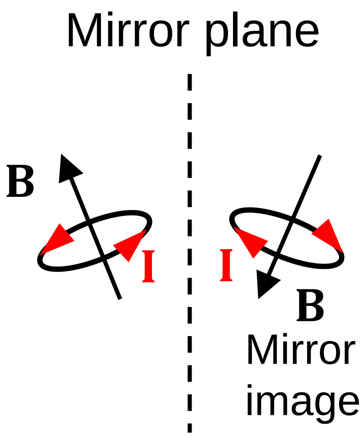
- Use of Ampère's law to calculate magnetic field
  - Ampère's law in integral form
  - Symmetry arguments
    - Translation symmetry
    - Rotational symmetry
    - Mirror symmetry
    - Inversion symmetry

      Be careful to use when transforming $\mathbf{B}$: need to flip sign
    - $\mathbf{B}$ flips sign if $\mathbf{I}$ flips sign (time-reversal symmetry)

Symmetry of polar vectors and axial vectors

- Polar vectors and axial vectors
  - Polar vectors: **E**, **P**, **D**, **J**, **I**
    - Usually vectors associated with electric fields
  - Axial vectors: **B**, **M**, **H**
    - Usually vectors associated with magnetic fields
- Mirror operation

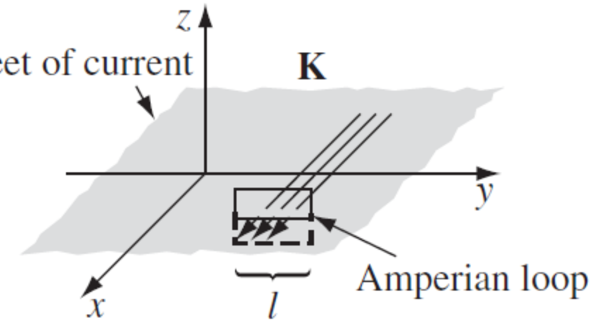




- Polar vectors: every operation as usual
- Axial vectors: Additional sign flip for mirror and inversion operations, other operations as usual

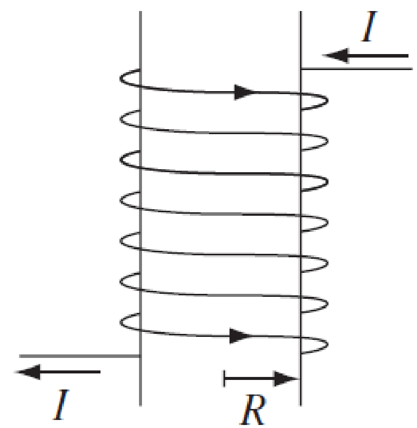
Ampère's law

Example 5.8. Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\,\hat{\mathbf{x}}$, flowing over the xy plane (Fig. 5.33).



C3.exmp5.8

Example 5.9. Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R, each carrying a steady current I (Fig. 5.34).



C3.exmp5.9&5.12

Ampère's law

Example 5.10. A toroidal coil consists of a circular ring, or "donut," around which a long wire is wrapped (Fig. 5.38). The winding is uniform and tight enough so that each turn can be considered a plane closed loop.

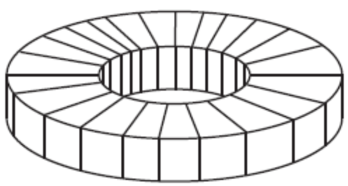
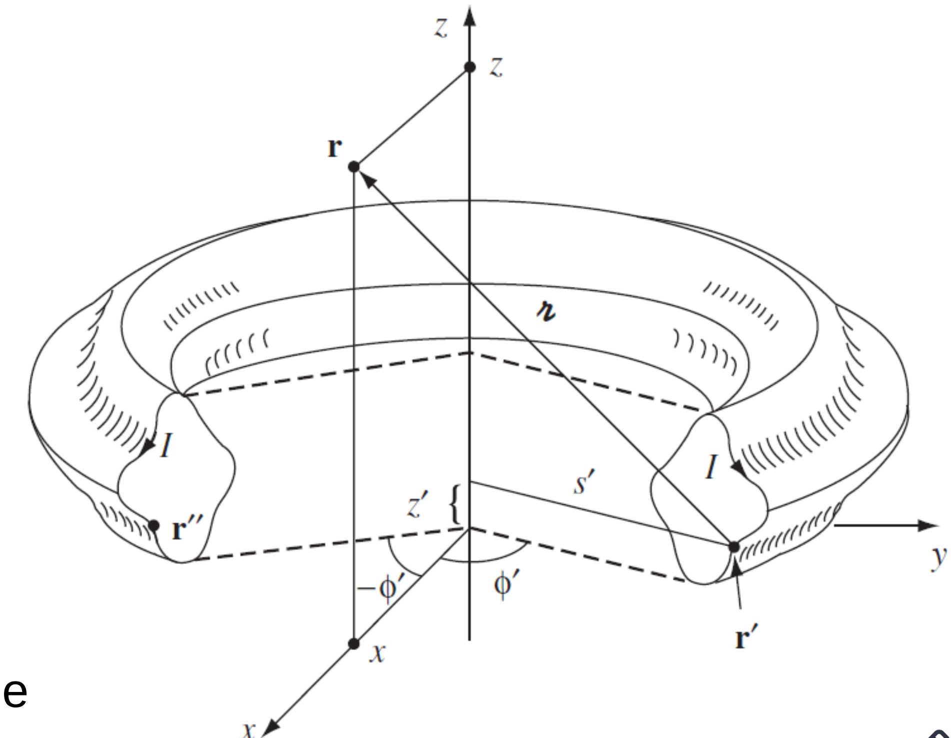


FIGURE 5.38



$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{inside} \\ \mathbf{0}, & \text{outside} \end{cases}$$

Toroidal structure with any cross-sectional shape

C3.exmp5.10

Magnetic vector potential

- Vector potential $\mathbf{A}$
  - $\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$
    - Exploiting the property $\nabla \cdot \mathbf{B} = 0$ (divergence of curl vanishes)
  - $\boxed{\nabla \cdot \mathbf{A} = 0}$
    - Chosen to be so (like choosing reference point for electric potential)

      If original $\mathbf{A}_0$ is not, can define $\mathbf{A} = \mathbf{A}_0 + \nabla\lambda$ without varying $\mathbf{B}$

      $$\Rightarrow \nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2\lambda$$

      Can choose $\nabla^2\lambda = -\nabla \cdot \mathbf{A}_0$
  - $\boxed{\nabla^2\mathbf{A} = -\mu_0\mathbf{J}}$
    - Use the choice above, and apply Ampère's law

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\cancelto{0}{\nabla \cdot \mathbf{A}}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

Magnetic vector potential

- Calculating vector potential from current
  - $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ Poisson's equation, if $\mathbf{J} \to 0$ at infinity,

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, d\tau'}$$

($\mathbf{A}$ is a polar vector)

    - Analogous to solution $V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau'$ to $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$
  - Line current $\boxed{\mathbf{A} = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\mathbf{I}}{\mathscr{r}} \, dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathscr{r}} \, d\mathbf{l}'}$
  - Surface current $\boxed{\mathbf{A} = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\mathbf{K}}{\mathscr{r}} \, da'}$
  - Not so much simplification (vector $\mathbf{A}$ representing vector $\mathbf{B}$) but equation is easier to use than the Biot-Savart law
    - No need to integrate unit vector $\hat{\mathscr{r}}$

* $\nabla^2 \mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$

Magnetic vector potential

- Calculating vector potential from magnetic field

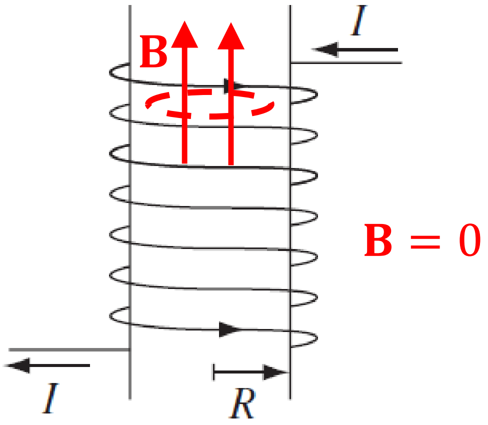
$$\boxed{\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi}$$

where Φ is the magnetic flux

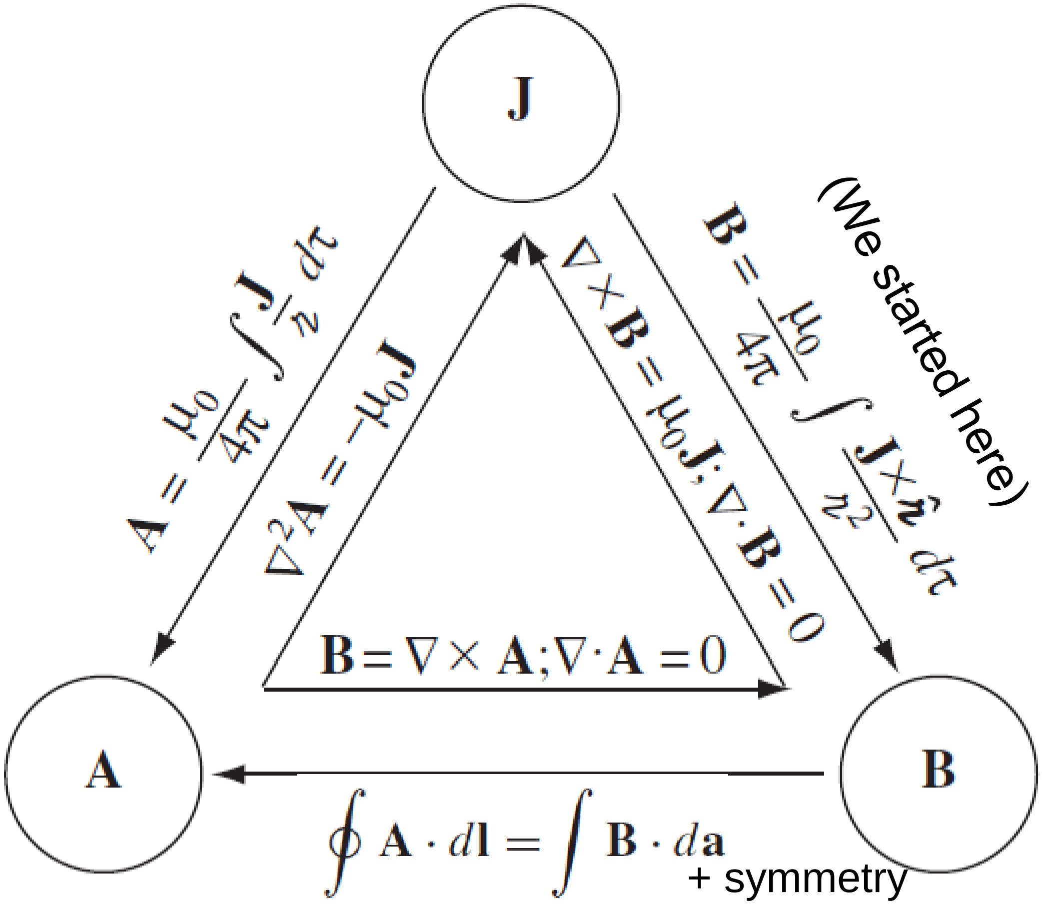
  - Can calculate $\mathbf{A}$ using equation above plus symmetry argument
  - Generally $\mathbf{A}$ mimics the direction of current

Example 5.12. Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

  - Check $\nabla \times \mathbf{A} = \mathbf{B}$
  - Check $\nabla \cdot \mathbf{A} = 0$



Current, magnetic field, and vector potential



Differential equations need boundary conditions to solve

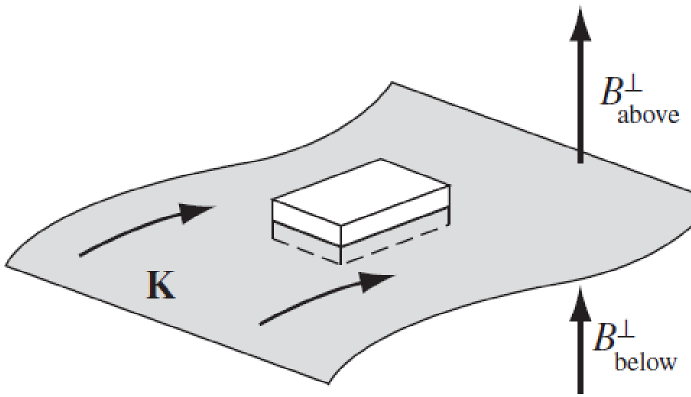
Boundary conditions

- Boundary conditions of **B** across a 2D current surface
  - Normal component of **B**

    Thin pillbox with thickness $\varepsilon \to 0$

    $$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$

    $$\Rightarrow \boxed{B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}}$$



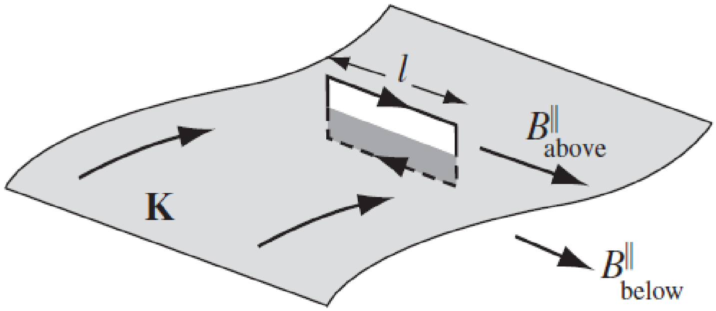
  - Tangential component of **B** <u>that is perpendicular to current</u>

    Thin Amperian loop

    $$\oint \mathbf{B} \cdot d\mathbf{l} = \left(B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}}\right) l$$

    $$= \mu_0 I_{\text{enc}} = \mu_0 K l$$

    $$\Rightarrow \boxed{B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K}$$



  - Summarizing above $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}})$

Boundary conditions

- Boundary conditions of $\mathbf{A}$ across a 2D current surface
  - Vector potential $\mathbf{A}$ is always continuous $\boxed{\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}}$
    - Normal component of $\mathbf{A}$

      $\nabla \cdot \mathbf{A} = 0$ and select a thin pillbox

      $\Rightarrow$ $A^{\perp}_{\text{above}} = A^{\perp}_{\text{below}}$
    - Tangential component of $\mathbf{A}$

      $\nabla \times \mathbf{A} = \mathbf{B}$ and select a thin loop to integrate

      $\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$ tends to zero when loop is thin

      $\Rightarrow$ $A^{\parallel}_{\text{above}} = A^{\parallel}_{\text{below}}$
  - Derivative of $\mathbf{A}$ is discontinuous $\boxed{\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}}$

C3.Abound (optional)

Multipole expansion of vector potential

- Goal: to expand $\mathbf{A}$ in power series of $1/r$

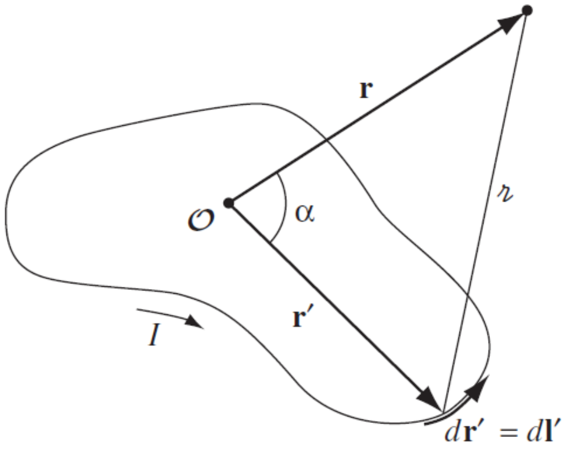
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\mathscr{r}}\, d\mathbf{l}'$$

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

(From electrostatic multipole expansion)

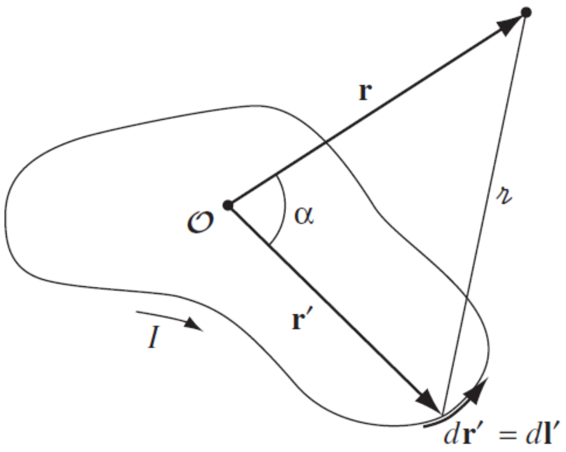
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\, d\mathbf{l}'$$

$$= \frac{\mu_0 I}{4\pi}\left[\overbrace{\underline{\frac{1}{r}\oint d\mathbf{l}'}}^{\text{Monopole}} + \overbrace{\underline{\frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}'}}^{\text{Dipole}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\mathbf{l}'}_{\text{Quadrupole}} + \cdots\right]$$



Multipole expansion of vector potential

- Magnetic monopole and magnetic dipole
  - Magnetic monopole
    - Always zero, because $\oint d\mathbf{l}' = \mathbf{0}$
    - Also because $\nabla \cdot \mathbf{B} = 0$



  - Magnetic dipole

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha \, d\mathbf{l}'} = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'$$

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'$$

(proved by a textbook problem)

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}}$$

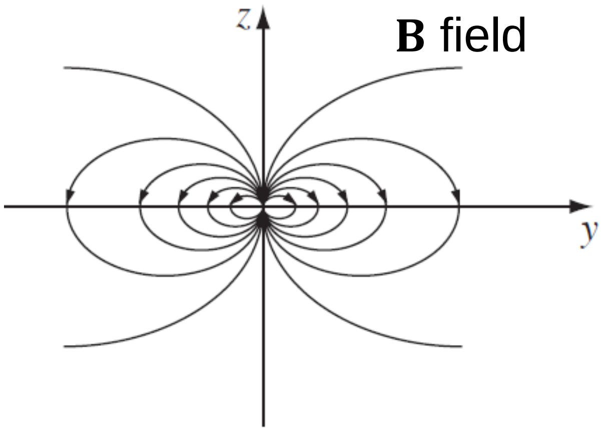
Magnetic dipole moment

$$\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}$$

Multipole expansion of vector potential

- Pure dipole vs physical dipole

Pure dipole



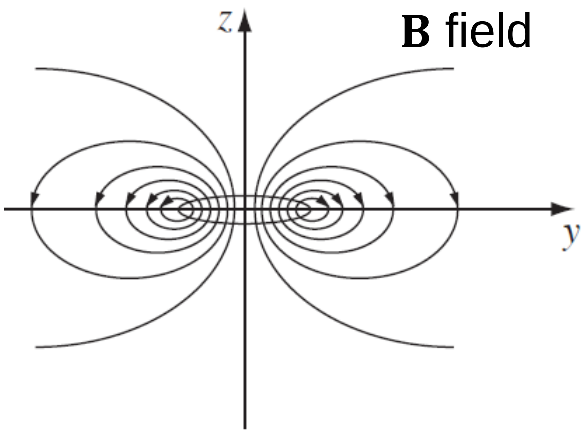
$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

Assumes $r \gg$ loop radius

$\mathbf{m} = I\mathbf{a}$ but take $I \to \infty$, $a \to 0$

Similar to pure electric dipole

Physical dipole

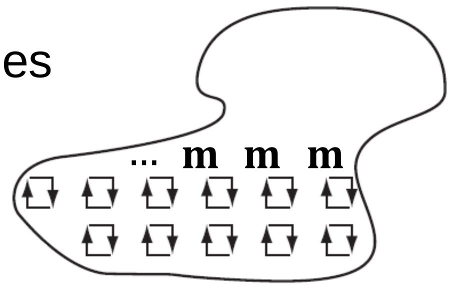


Deviations appear when closing up onto the dipole

Magnetic fields in matter

Magnetic materials

- Practically, all materials have magnetic response
  - Orbiting electrons around nuclei as magnetic dipoles
  - Electron spins as magnetic dipoles
    - Quantum object without a classical analog



- Magnetization: $\mathbf{M} = \frac{1}{V}\sum_i \mathbf{m}_i$
  - Magnetic dipole moment per unit volume

- Three types of response of matter to magnetic field
  - Paramagnets: magnetization $\mathbf{M}$ parallel to applied $\mathbf{B}$
  - Diamagnets: magnetization $\mathbf{M}$ opposite to applied $\mathbf{B}$
  - Ferromagnets: finite magnetization $\mathbf{M}$ even without $\mathbf{B}$

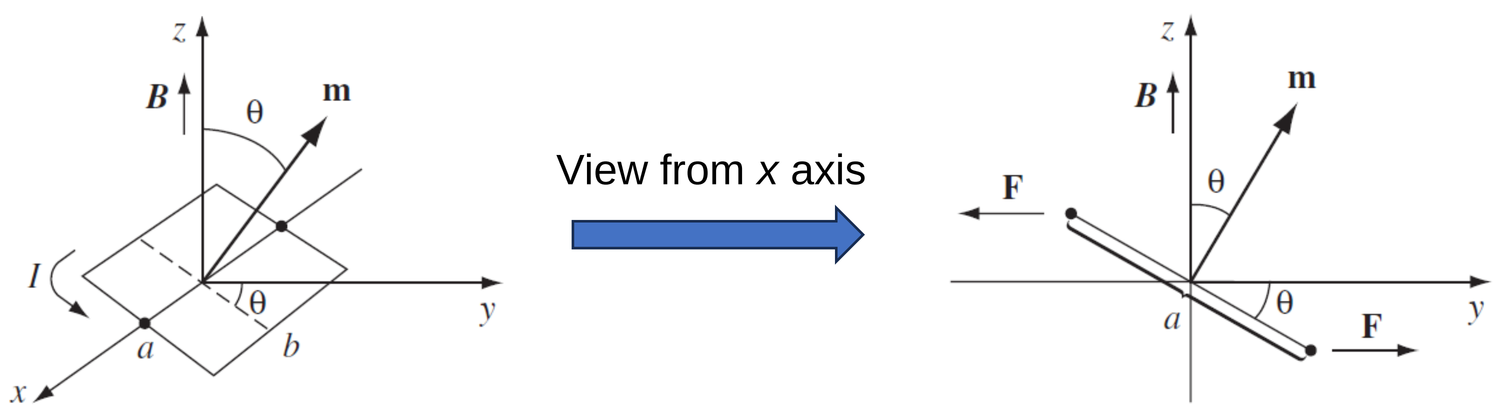
Magnetic dipoles responding to field

- Paramagnetic response
  - Torque in a uniform field
    - Suppose a rectangle current loop with side lengths a and b

$$\mathbf{N} = aF\sin\theta\,\hat{\mathbf{x}} \xrightarrow{\text{force } F = IbB} \mathbf{N} = IabB\sin\theta\,\hat{\mathbf{x}} = mB\sin\theta\,\hat{\mathbf{x}}$$

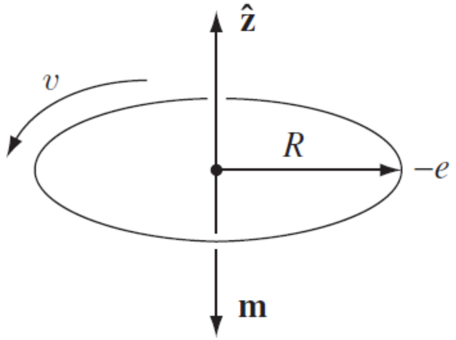
$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B}}$$

  - Torque tends to align $\mathbf{m}$ parallel to $\mathbf{B}$



Magnetic dipoles responding to field

- Diamagnetic response
  - Modification to an atomic orbit



$$I = \frac{-e}{T} = -\frac{ev}{2\pi R} \xrightarrow[\text{Orbit period}]{T = 2\pi R/v} \mathbf{m} = -\frac{1}{2} evR\,\hat{\mathbf{z}}$$

- Velocity without and with $\mathbf{B}$, balancing Coulomb and centripetal forces

without $\mathbf{B}$: $$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e \frac{v^2}{R}$$

with $\mathbf{B}$: $$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e \frac{\bar{v}^2}{R}$$

v: electron velocity
$\bar{v}$: electron velocity with $\mathbf{B}$
m_e: electron mass

$$\Rightarrow \quad e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v} + v)(\bar{v} - v)$$

$$\Rightarrow \quad \Delta v = \bar{v} - v = \frac{eRB}{2m_e}$$ for small Δv

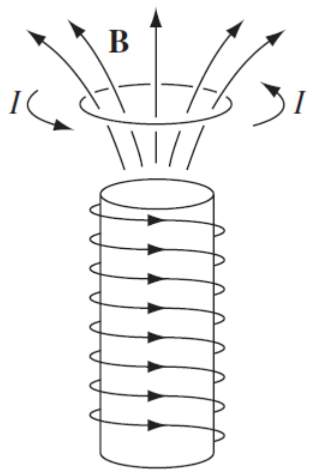
$$\Rightarrow \quad \boxed{\Delta\mathbf{m} = -\frac{1}{2} e(\Delta v)R\,\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}}$$

Change in dipole moment works against $\mathbf{B}$

Forces on paramagnets and diamagnets

- General formula for small dipole in nonuniform field

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$



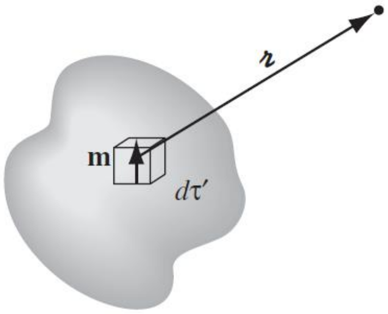
  - For paramagnets $m \parallel B$
    - Force directs toward more intense field regions
    - Paramagnets are attracted to magnets
  - For diamagnets $m \parallel -B$
    - Force directs toward less intense field regions
    - Diamagnets are repelled by magnets

Levitating frog

(Ig Nobel award 2000)

Field of magnetized objects

- Bound currents
  - Vector potential of a magnetized object (neglecting the cause)



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \left[\mathbf{M}(\mathbf{r}')\times\left(\nabla'\frac{1}{\mathscr{r}}\right)\right] d\tau'$$

Integrate by parts

$$= \frac{\mu_0}{4\pi}\left\{\int \frac{1}{\mathscr{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]\,d\tau' - \int \nabla'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right] d\tau'\right\}$$

$$= \frac{\mu_0}{4\pi}\int \frac{1}{\mathscr{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]\,d\tau' + \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}')\times d\mathbf{a}']$$

C3.Curldtau

$$= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da'$$

* with $\mathbf{J}_b = \nabla\times\mathbf{M}$

$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$

Field of magnetized objects

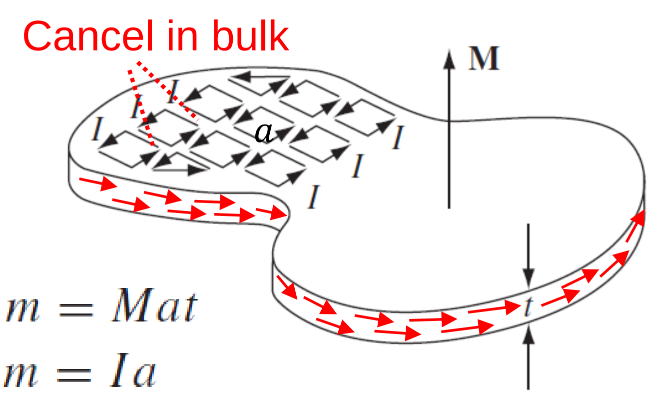
- Bound currents

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\, da'$$

  - Density of bound currents $\begin{cases} \text{surface:} & \boxed{\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}} \\ \text{volume:} & \boxed{\mathbf{J}_b = \nabla\times\mathbf{M}} \end{cases}$ *check $\nabla\cdot\mathbf{J}_b = 0$

  - Physical picture of bound currents

Surface bound current (suppose uniform $\mathbf{M}$)

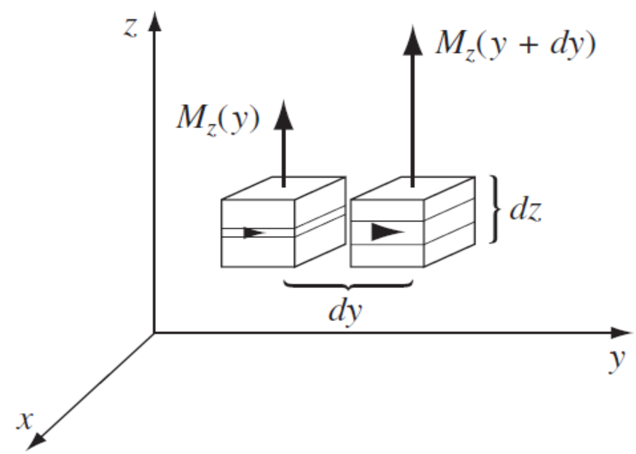


$m = Mat$

$m = Ia$

$I = Mt \;\Rightarrow\; K_b = I/t = M$

Volume bound current

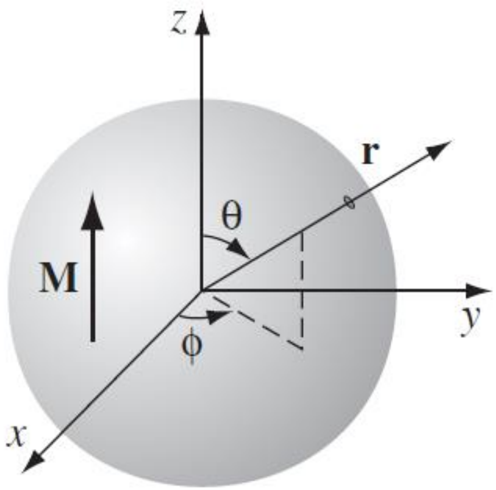


$$I_x = [M_z(y+dy) - M_z(y)]\,dz = \frac{\partial M_z}{\partial y}\,dy\,dz$$

Field of magnetized objects

- Bound currents

Example 6.1. Find the magnetic field of a uniformly magnetized sphere.



- Field inside the sphere $\mathbf{B} = \frac{2}{3}\mu_0 \mathbf{M}$
  - Uniform field
  - **M** induces **B** that is parallel to it, while **P** induces **E** that is antiparallel
- Field outside the sphere same as what would be for a dipole

$$\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}$$

C3.exmp6.1

Auxiliary field

- Add the cause and the effect of magnetization
  - Total magnetic field

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

    - $\mathbf{B}$: total magnetic field
    - $\mathbf{J}$: total current density
    - $\mathbf{J}_b$: bound current density, due to magnetization
    - $\mathbf{J}_f$: free current density that we control, not a result of magnetization

$$\Rightarrow \quad \nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_f$$

  - The auxiliary field $\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}$

$$\Rightarrow \quad \boxed{\nabla \times \mathbf{H} = \mathbf{J}_f}$$

    - Ampère's law for auxiliary field $\boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}}$

Enclosed total free current

Auxiliary field

- Application of auxiliary field

Example 6.2. A long copper rod of radius R carries a uniformly distributed (free) current I (Fig. 6.19). Find $\mathbf{H}$ inside and outside the rod.

- Inside

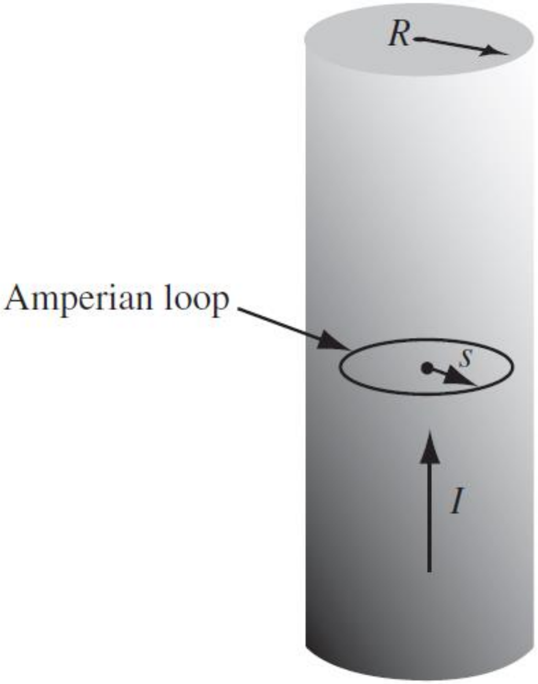
$$H(2\pi s) = I_{f_{\text{enc}}} = I\frac{\pi s^2}{\pi R^2}$$

$$\Rightarrow \mathbf{H} = \frac{I}{2\pi R^2}s\,\hat{\boldsymbol{\phi}}$$

- Outside

$$\Rightarrow \mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}$$

$$\Rightarrow \mathbf{B} = \mu_0\mathbf{H} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$$



Linear magnetic media

- Linear magnetic media
  - $\boxed{\mathbf{M} = \chi_m \mathbf{H}}$ (χ_m: magnetic susceptibility)

    $\Rightarrow$ $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}$

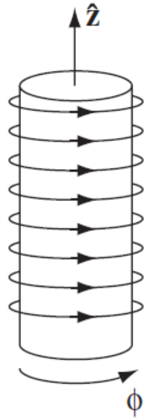
    - Diamagnetic susceptibility on the order of 10^{-6}
  - $\boxed{\mathbf{B} = \mu \mathbf{H} \quad (\mu = \mu_0(1 + \chi_m)\text{: permeability})}$

? Why this linear relation defines M-H but not M-B?

Example 6.3. An infinite solenoid (n turns per unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} \Rightarrow \mathbf{H} = nI\,\hat{\mathbf{z}}$$

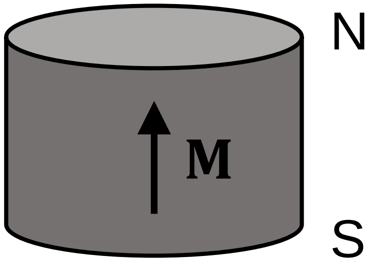
$$\Rightarrow \mathbf{B} = \mu_0(1 + \chi_m)nI\,\hat{\mathbf{z}}$$



Enhancement of field if paramagnetic!

Auxiliary field

- $\mathbf{H} = \mathbf{0}$ without applying any free current?
  - Looks like so because $\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$
    - Consider a short cylindrical magnet
    - $\mathbf{H}$ would be zero everywhere
    - Then $\mathbf{B} \neq \mathbf{0}$ inside magnet, $\mathbf{B} = \mathbf{0}$ outside magnet
    - Obviously wrong



  - $\mathbf{H}$ can be finite without any $\mathbf{J}_F$
    - Because $\nabla \cdot \mathbf{H} = \nabla \cdot \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = -\nabla \cdot \mathbf{M} \neq \mathbf{0}$
    - $\nabla \cdot \mathbf{M} \neq \mathbf{0}$ at the top and bottom surfaces of the magnet

H versus B, D versus E

- A long history of confusion
  - Whether we shall call $\mathbf{H}$ or $\mathbf{B}$ as the magnetic field
    - Some convention calls $\mathbf{H}$ the magnetic field, and $\mathbf{B}$ as the magnetic flux density
    - Our textbook takes the stance that $\mathbf{H}$ should be “auxiliary”
- Reason for such confusion
  - $\mathbf{H}$ is a lot more frequently used than $\mathbf{B}$ as $\mathbf{H}$ is given by free current, something we can control, while $\mathbf{B}$ is material dependent
    - Helmholtz coil magnetizing a specimen
  - $\mathbf{E}$ is a lot more frequently used than $\mathbf{D}$ as $\mathbf{D}$ is given by free charge, which we rarely control, while $\mathbf{E}$ is determined by voltage difference (over distance), which is what we control
    - Charging up of a parallel plate capacitor

Boundary conditions

- Boundary conditions reexamined
  - Earlier findings still hold, but $\mathbf{J}$ needs to include free and bound currents

$$\begin{cases} B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}} \\ \mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \end{cases}$$

  - Easier to use the boundary conditions of $\mathbf{H}$

Surface free charge density

$$\begin{cases} \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \\ \nabla \times \mathbf{H} = \mathbf{J}_f \end{cases} \Rightarrow \boxed{\begin{cases} \mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}} \\ H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}) \end{cases}}$$

Nonlinear magnetic media

- Ferromagnets: $\mathbf{M} \neq 0$ without applying any field
  - Obvious violation of the linear relation $\mathbf{M} = \chi_m \mathbf{H}$
  - Represents a quantum phenomena
    - Exchange interaction: $U = -2J \sum_{p=1}^{N} \boldsymbol{S}_p \cdot \boldsymbol{S}_{p+1}$ $(J > 0)$
    - Prefers spontaneous parallel alignment of spins
  - Domain formation and hysteresis loop