

A toroidal inductor, which is a type of coil wound around a torus-shaped magnetic core. The core is dark and has a hole in the center. The wire is a thick, reddish-brown copper color and is wound in a dense, helical pattern around the core. Two thin, silver-colored metal leads extend from the top of the coil. The background is a solid yellow color.

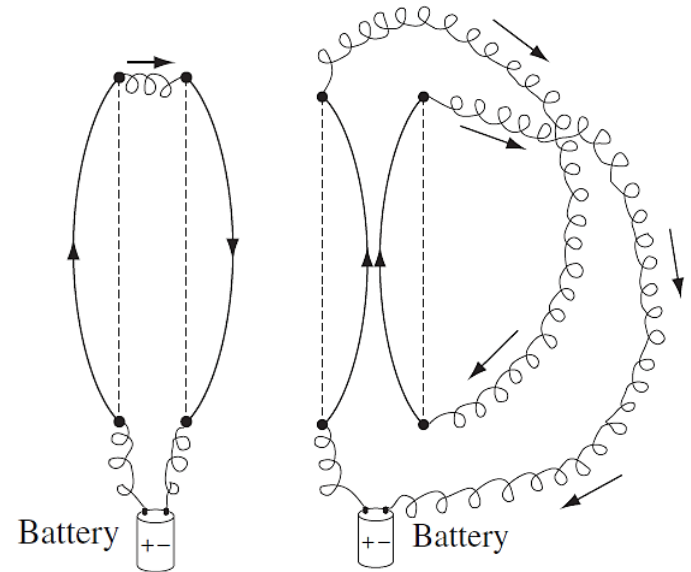
Magnetostatics

The Lorentz force law

Electrostatics: Stationary charges with constant electric fields

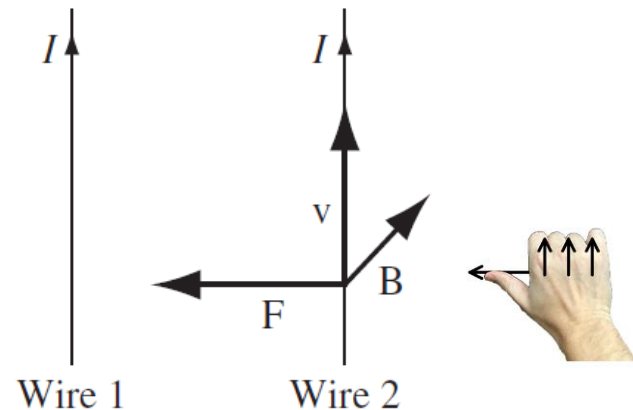
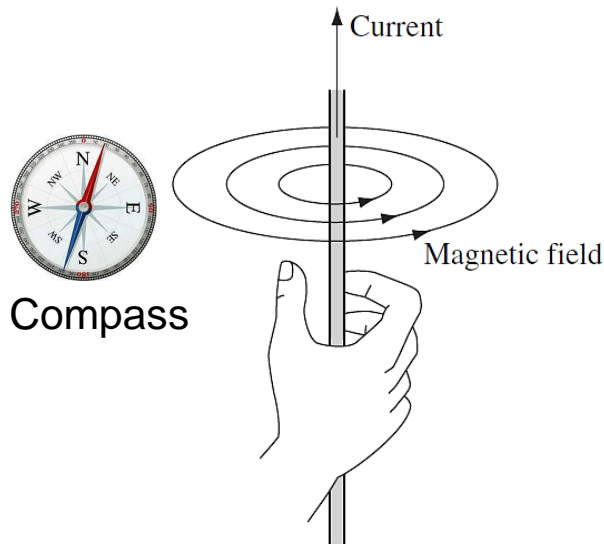
Magnetostatics: Steady currents with constant magnetic fields

- Magnetic force and magnetic field
 - Pass current in parallel wires
 - Wires repel with antiparallel current
 - Wires attract with parallel current
 - Force is not electrostatic in nature (wires are charge neutral)
 - Magnetic force



The Lorentz force law

- Magnetic force and magnetic field
 - Empirical rules for magnetic field and magnetic force
 - Straight current-carrying wire has magnetic field circling around it
 - Right-hand rule for field direction given current direction
 - Right-hand rule for force direction given current and field direction



The Lorentz force law

- The Lorentz force law
 - Magnetic force on charge Q , moving with velocity \mathbf{v} , in magnetic field \mathbf{B}

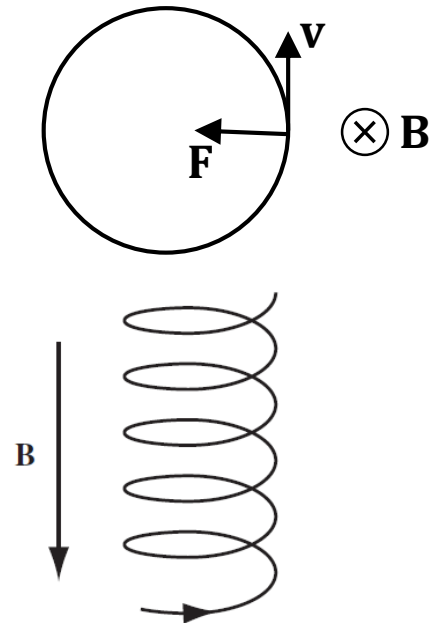
$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

- Given as an axiom without proof
 - With both \mathbf{E} and \mathbf{B} , the full Lorentz force law $\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$

- Consequences

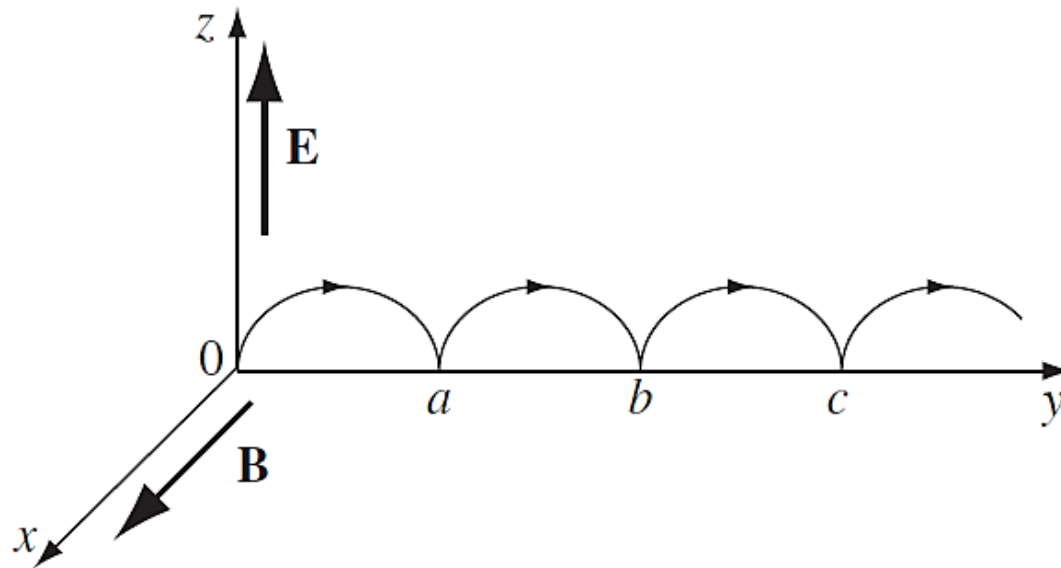
- Circular charge motion, or alike, when magnetic force acts as centripetal force
 - Magnetic forces do no work, \mathbf{B} only deflects particle direction

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \overbrace{\mathbf{v} dt}^{d\mathbf{l}} = 0$$



The Lorentz force law

Example 5.2. Cycloid Motion. A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that \mathbf{B} points in the x -direction, and \mathbf{E} in the z -direction, as shown in Fig. 5.7. A positive charge is released from the origin; what path will it follow?



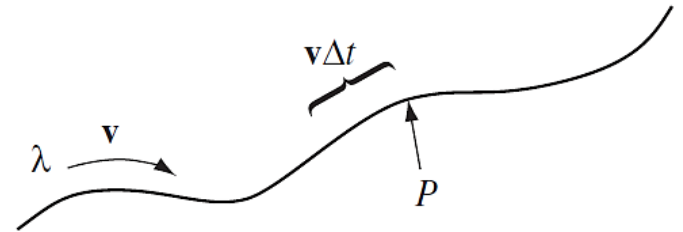
C3.exmp5.2

Currents

- Current in a wire: charge per time passing a given point

- Unit: Amperes (A) = Coulomb/second

- $\mathbf{I} = \lambda \mathbf{v}$ where λ : line charge density,
 \mathbf{v} : velocity of movement



- Positive charges moving at \mathbf{v} = negative charges moving at $-\mathbf{v}$
- Not meaningful to talk about current if it's just a single point charge moving (non-steady)

- In many problems just write the magnitude $I = \lambda v$
 - Because direction is determined by the shape of wire

- Magnetic force on a current-carrying wire $\mathbf{F}_{\text{mag}} = \int I (d\mathbf{l} \times \mathbf{B})$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \lambda dl = \int (\mathbf{I} \times \mathbf{B}) dl = \int I (d\mathbf{l} \times \mathbf{B})$$

Currents

- Surface and volume distributions of current

- Surface current density $\mathbf{K} = \sigma \mathbf{v}$ (C/m² · m/s) (σ : surface charge density)

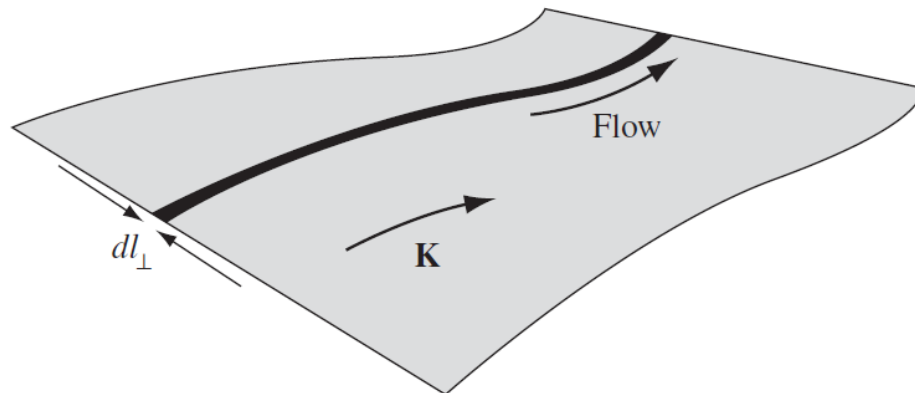
- \mathbf{K} : current per unit width (A/m)

$$d\mathbf{I} = \sigma \mathbf{v} dl_{\perp} \quad \Rightarrow \quad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}} \quad (l_{\perp}: \text{cross-sectional line segment perpendicular to } \mathbf{v})$$

- Magnetic force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \sigma da = \int (\mathbf{K} \times \mathbf{B}) da$$

* \mathbf{B} experiences discontinuity across a current-carrying surface



Currents

- Surface and volume distributions of current

- Volume current density $\mathbf{J} = \rho \mathbf{v}$ (C/m³ · m/s) (ρ : volume charge density)

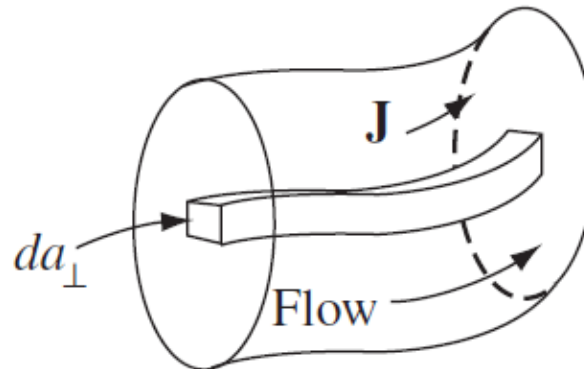
- \mathbf{J} : current per unit area (A/m²)

$$\begin{aligned} d\mathbf{I} &= \rho \mathbf{v} da_{\perp} \\ d\mathbf{I} &= \mathbf{J} \cdot d\mathbf{a} = \rho \mathbf{v} \cdot d\mathbf{a} \end{aligned} \quad \Rightarrow \quad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}} \quad (a_{\perp}: \text{cross-sectional area segment perpendicular to } \mathbf{v})$$

- Magnetic force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \rho d\tau = \int (\mathbf{J} \times \mathbf{B}) d\tau$$

- Ohm's law $\mathbf{J} = \sigma_c \mathbf{E}$ (σ_c : electrical conductivity)



Currents

- Surface and volume distributions of current

- Continuity equation for volume current density

- Current crossing a surface \mathcal{S} : $I = \int_{\mathcal{S}} J da_{\perp} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$
- Current crossing boundary of a volume \mathcal{V} :

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau$$

which must equal to the change of net charge in the volume

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau = \overbrace{-\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau}^{\text{change of net charge}} = - \int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} \right) d\tau$$

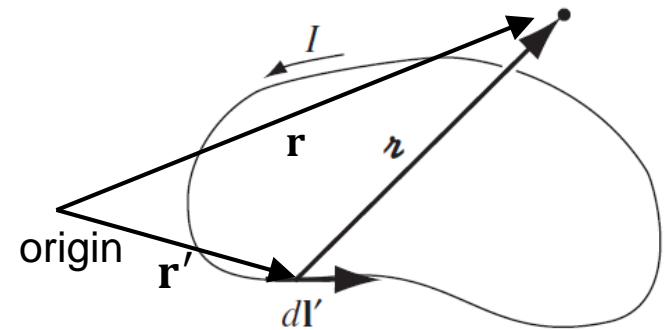
➡ $\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$ for magnetostatics $\frac{\partial \rho}{\partial t} = 0 \quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0} \quad \nabla \cdot \mathbf{J} = 0$

The Biot-Savart law

- Magnetic field generated by a steady current

- Line current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathbf{r}}}{r^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\mathbf{r}}}{r^2}$

- Unit of \mathbf{B} : Tesla (T) = N/(A·m)
- Permeability $\mu_0 = 4\pi \times 10^{-7}$ N/A²
- Separation vector $\mathbf{r} = \mathbf{r} - \mathbf{r}'$



- Surface current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} da'$

- Volume current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} d\tau'$

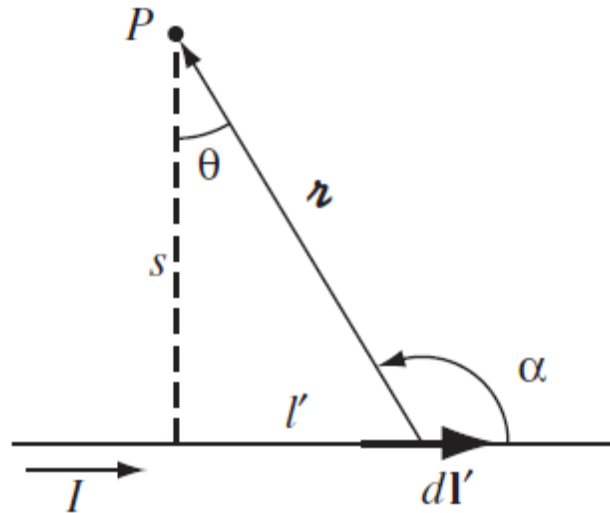
- Expressions for surface and volume currents not proved
- The law cannot be used to calculate field generated by discrete moving charges

* \mathbf{r} in textbook is typed as \mathbf{r}

The Biot-Savart law

- Application of Biot-Savart law

Example 5.5. Find the magnetic field a distance s from a long straight wire carrying a steady current I (Fig. 5.18).



$$\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$



C3.exmp5.5

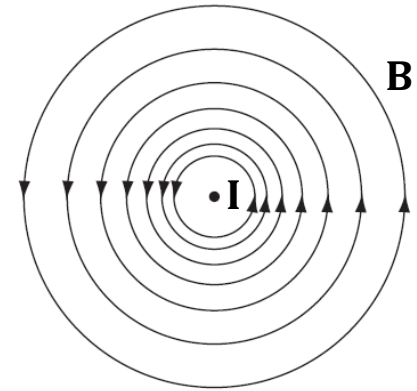
Curl of magnetic field

- Derivation of curl from Stokes theorem
 - Loop integral of \mathbf{B} around a straight-line current

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I$$

- Result still holds if path is noncircular

$$d\mathbf{l} = ds \hat{s} + \underline{s d\phi \hat{\phi}} + dz \hat{z} \quad \mathbf{B} = \underline{\frac{\mu_0 I}{2\pi s} \hat{\phi}}$$



- Loop integral of \mathbf{B} around any current distribution

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad \left(I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a} \text{ total current enclosed by the path} \right)$$

- Apply Stokes theorem

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

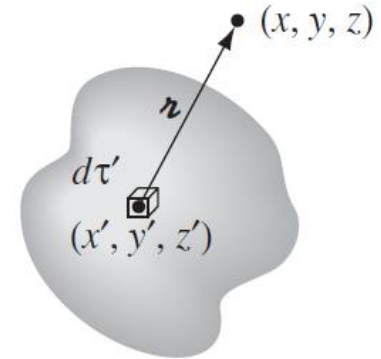
➡ $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

Curl of magnetic field

- Derivation of curl from Biot-Savart law
 - A much more formal derivation than previous slide

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau'$$

$\mathbf{B}(\mathbf{r})$ $\nabla_{\mathbf{r}}$ $\mathbf{J}(\mathbf{r}')$ $\mathbf{r} = \mathbf{r} - \mathbf{r}'$ $dx' dy' dz'$



\downarrow Product rule $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) - \underbrace{(\mathbf{J} \cdot \nabla) \frac{\hat{\mathbf{r}}}{r^2}}$$

$\downarrow \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r}),$ and second term integrates to 0 (textbook p.232)

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta^3(\mathbf{r} - \mathbf{r}') d\tau' = \mu_0 \mathbf{J}(\mathbf{r})$$

Divergence of magnetic field

- Derivation of divergence from Biot-Savart law

$$\circ \quad \nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau'$$

↓ Product rule $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) = \frac{\hat{\mathbf{r}}}{r^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\mathbf{r}}}{r^2} \right)$$

↓ $\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') = 0$, and $\nabla \times \frac{\hat{\mathbf{r}}}{r^2} = 0$ (Remember $\nabla \times \mathbf{E} = 0$)

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

Magnetic fields are divergence-free (no magnetic “free charge”)

Magnetic
monopoles?
not reproduced

First Results from a Superconductive Detector for Moving Magnetic Monopoles

Blas Cabrera
Phys. Rev. Lett. **48**, 1378 – Published 17 May 1982

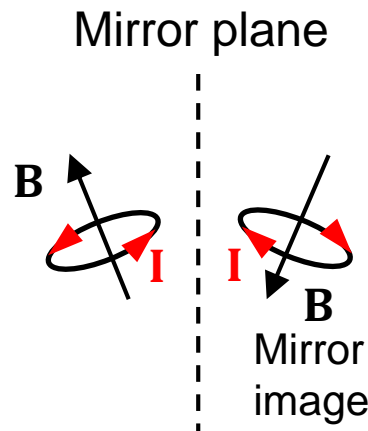
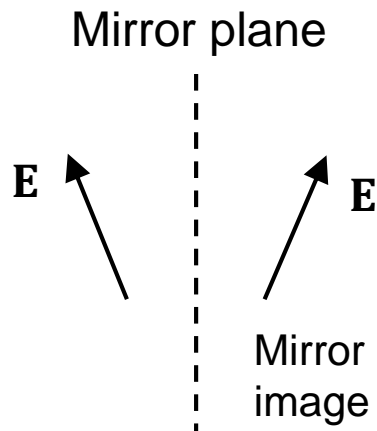
Ampère's law

Electrostatics	Magnetostatics
Coulomb's law	Biot-Savart law
Gauss's law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \quad \oint \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}}/\epsilon_0$	Ampère's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$
Gaussian surface	Amperian loop

- Use of Ampère's law to calculate magnetic field
 - Ampère's law in integral form
 - Symmetry arguments
 - Translation symmetry
 - Rotational symmetry
 - Mirror symmetry
 - Inversion symmetry
- Be careful to use when transforming \mathbf{B} :
 need to flip sign
- \mathbf{B} flips sign if \mathbf{I} flips sign (time-reversal symmetry)

Symmetry of polar vectors and axial vectors

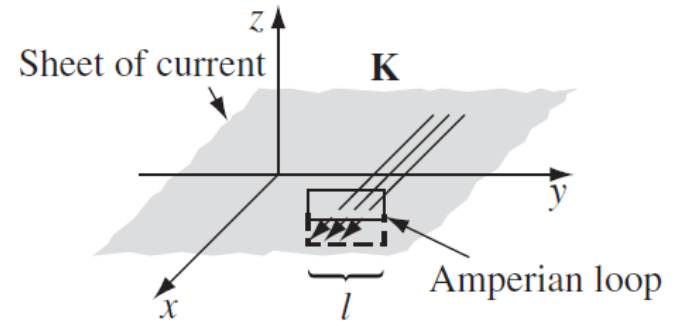
- Polar vectors and axial vectors
 - Polar vectors: \mathbf{E} , \mathbf{P} , \mathbf{D} , \mathbf{J} , \mathbf{I}
 - Usually vectors associated with electric fields
 - Axial vectors: \mathbf{B} , \mathbf{M} , \mathbf{H}
 - Usually vectors associated with magnetic fields
- Mirror operation



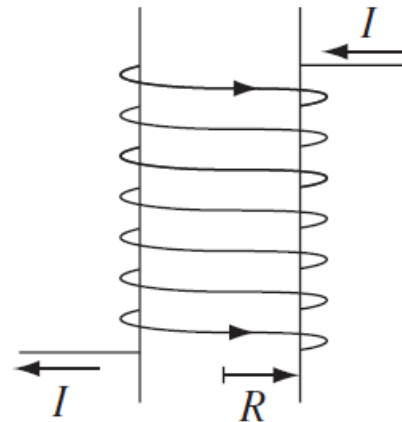
- Polar vectors: every operation as usual
- Axial vectors: **Additional sign flip for mirror and inversion operations**, other operations as usual

Ampère's law

Example 5.8. Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K \hat{\mathbf{x}}$, flowing over the xy plane (Fig. 5.33).



Example 5.9. Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R , each carrying a steady current I (Fig. 5.34).



Ampère's law

Example 5.10. A toroidal coil consists of a circular ring, or “donut,” around which a long wire is wrapped (Fig. 5.38). The winding is uniform and tight enough so that each turn can be considered a plane closed loop.

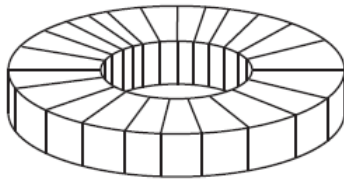
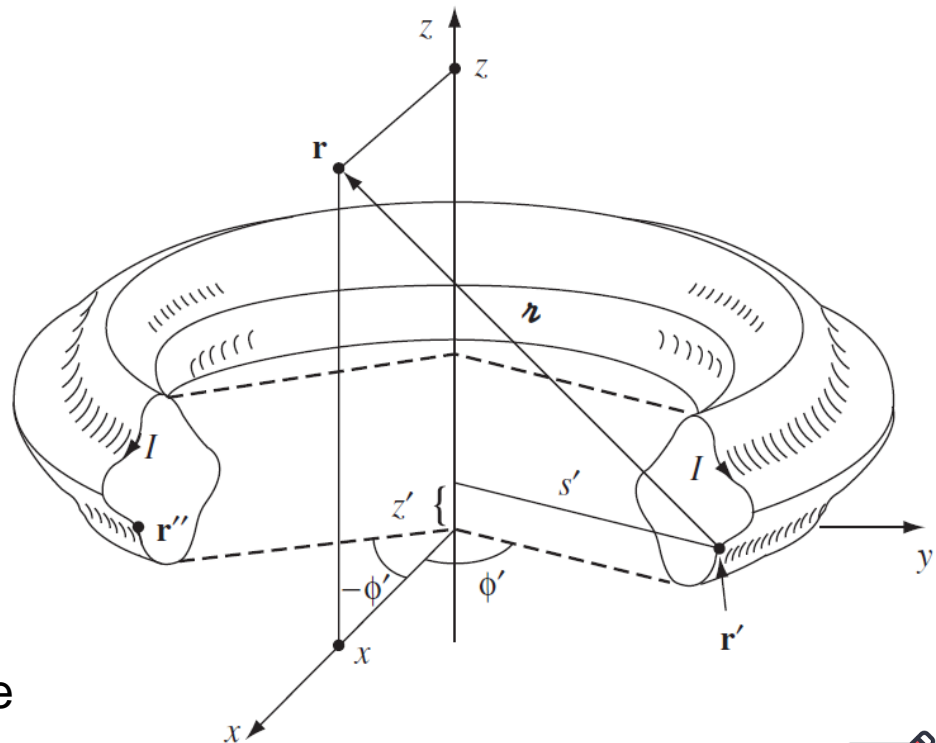


FIGURE 5.38

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \frac{\mu_0 N I}{2\pi s} \hat{\phi}, & \text{inside} \\ \mathbf{0}, & \text{outside} \end{cases}$$



Toroidal structure with any cross-sectional shape



C3.exmp5.10

Magnetic vector potential

- Vector potential \mathbf{A}

- $\mathbf{B} = \nabla \times \mathbf{A}$

- Exploiting the property $\nabla \cdot \mathbf{B} = 0$ (divergence of curl vanishes)

- $\nabla \cdot \mathbf{A} = 0$

- Chosen to be so (like choosing reference point for electric potential)

If original \mathbf{A}_0 is not, can define $\mathbf{A} = \mathbf{A}_0 + \nabla\lambda$ without varying \mathbf{B}

➡ $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2\lambda$

Can choose $\nabla^2\lambda = -\nabla \cdot \mathbf{A}_0$

- $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$

- Use the choice above, and apply Ampère's law

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\cancel{\nabla \cdot \mathbf{A}}^0) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

Magnetic vector potential

- Calculating vector potential from current

- $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ Poisson's equation, if $\mathbf{J} \rightarrow 0$ at infinity,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau' \quad (\mathbf{A} \text{ is a polar vector})$$

- Analogous to solution $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} d\tau'$ to $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

- Line current

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{r} dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{r} d\mathbf{l}'$$

- Surface current

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{r} da'$$

- Not so much simplification (vector \mathbf{A} representing vector \mathbf{B}) but equation is easier to use than the Biot-Savart law
 - No need to integrate unit vector $\hat{\mathbf{r}}$

$$* \nabla^2 \mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$$

Magnetic vector potential

- Calculating vector potential from magnetic field

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$$

where Φ is the magnetic flux

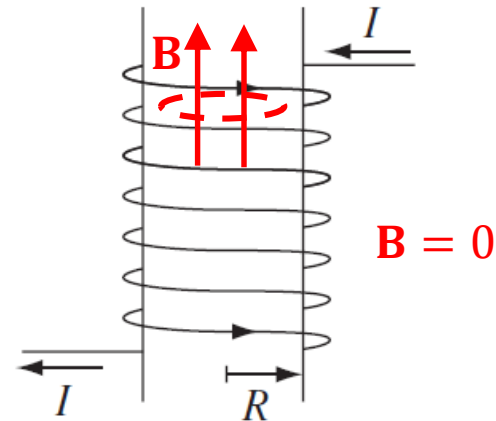
- Can calculate \mathbf{A} using equation above plus symmetry argument
- Generally \mathbf{A} mimics the direction of current

Example 5.12. Find the vector potential of an infinite solenoid with n turns per unit length, radius R , and current I .

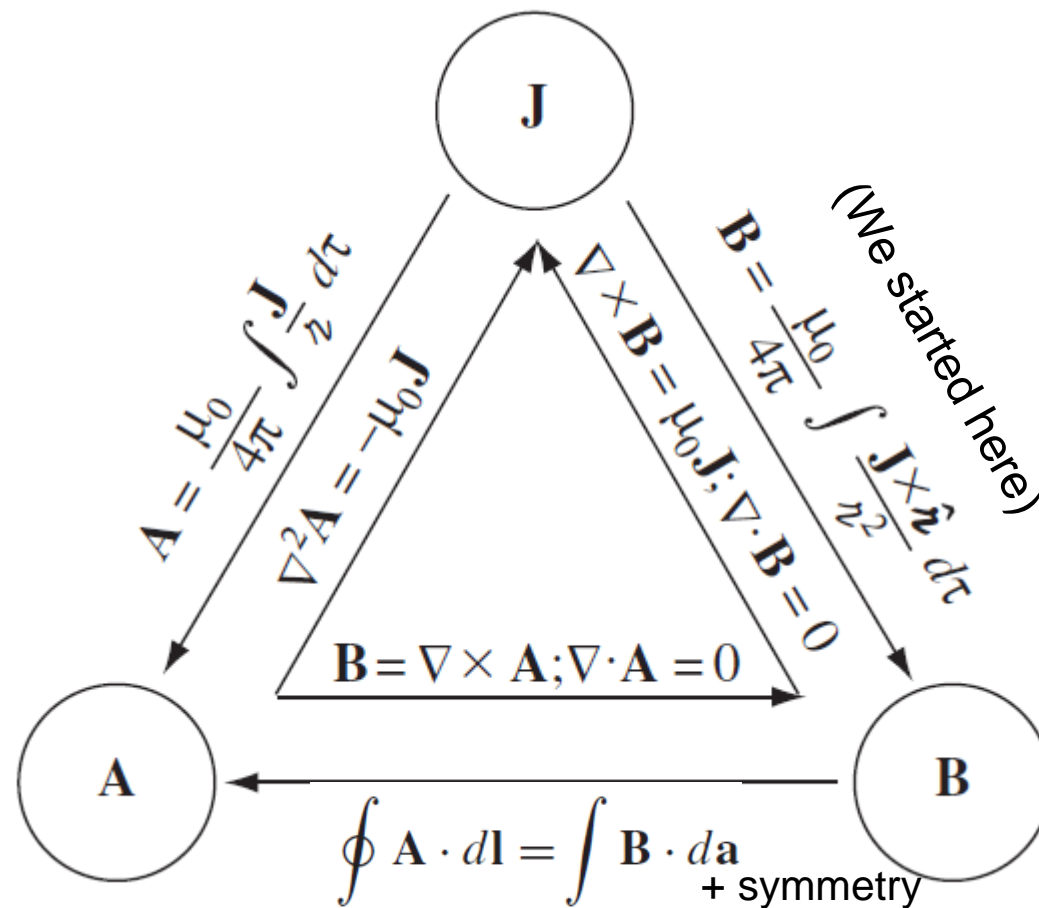
- Check $\nabla \times \mathbf{A} = \mathbf{B}$
- Check $\nabla \cdot \mathbf{A} = 0$



C3.exmp5.9&5.12



Current, magnetic field, and vector potential



Differential equations need boundary conditions to solve

Boundary conditions

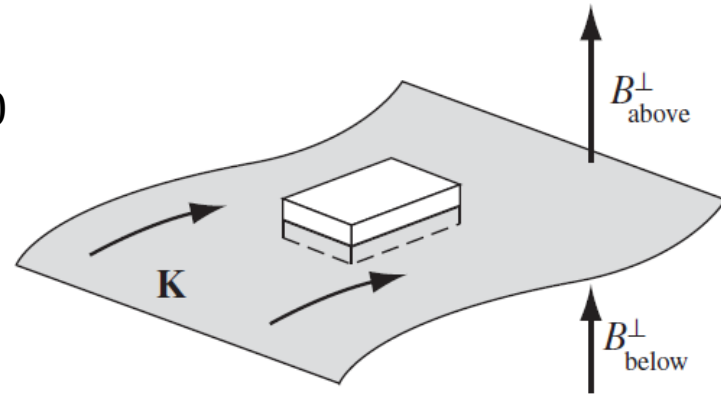
- Boundary conditions of \mathbf{B} across a 2D current surface

- Normal component of \mathbf{B}

Thin pillbox with thickness $\varepsilon \rightarrow 0$

$$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\Rightarrow B_{\text{above}}^{\perp} = B_{\text{below}}^{\perp}$$

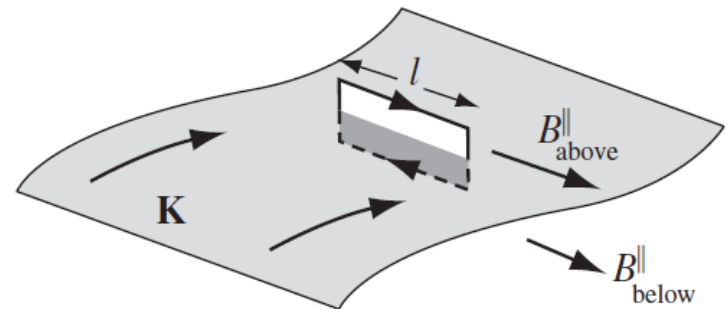


- Tangential component of \mathbf{B} that is perpendicular to current

Thin Amperian loop

$$\begin{aligned} \oint \mathbf{B} \cdot d\mathbf{l} &= (B_{\text{above}}^{\parallel} - B_{\text{below}}^{\parallel}) l \\ &= \mu_0 I_{\text{enc}} = \mu_0 K l \end{aligned}$$

$$\Rightarrow B_{\text{above}}^{\parallel} - B_{\text{below}}^{\parallel} = \mu_0 K$$



- Summarizing above $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$

Boundary conditions

- Boundary conditions of \mathbf{A} across a 2D current surface

- Vector potential \mathbf{A} is always continuous

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$$

- Normal component of \mathbf{A}

$\nabla \cdot \mathbf{A} = 0$ and select a thin pillbox

➡ $A_{\text{above}}^{\perp} = A_{\text{below}}^{\perp}$

- Tangential component of \mathbf{A}

$\nabla \times \mathbf{A} = \mathbf{B}$ and select a thin loop to integrate

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi \text{ tends to zero when loop is thin}$$

➡ $A_{\text{above}}^{\parallel} = A_{\text{below}}^{\parallel}$

- Derivative of \mathbf{A} is discontinuous

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$$



C3.Abound
(optional)

Multipole expansion of vector potential

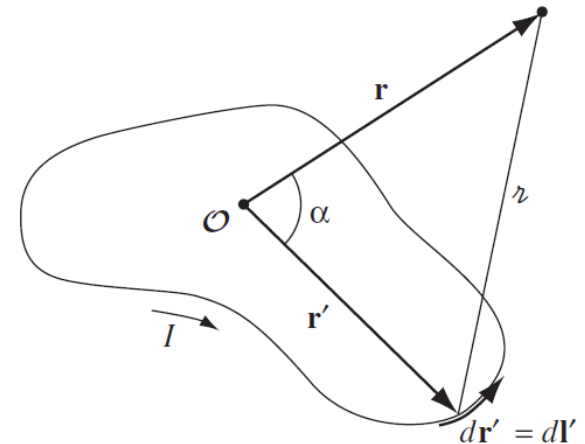
- Goal: to expand \mathbf{A} in power series of $1/r$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{r} d\mathbf{l}'$$

$$\downarrow \quad \frac{1}{r} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr' \cos \alpha}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \alpha)$$

(From electrostatic multipole expansion)

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos \alpha) d\mathbf{l}' \\ &= \frac{\mu_0 I}{4\pi} \left[\underbrace{\frac{1}{r} \oint d\mathbf{l}'}_{\text{Monopole}} + \underbrace{\frac{1}{r^2} \oint r' \cos \alpha d\mathbf{l}'}_{\text{Dipole}} \right. \\ &\quad \left. + \underbrace{\frac{1}{r^3} \oint (r')^2 \left(\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) d\mathbf{l}' + \dots}_{\text{Quadrupole}} \right] \end{aligned}$$

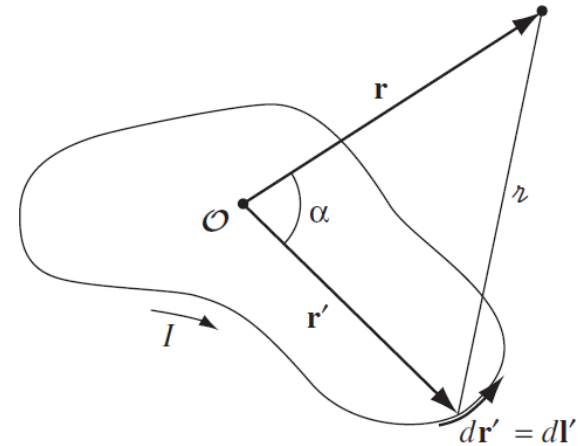


Multipole expansion of vector potential

- Magnetic monopole and magnetic dipole

- Magnetic monopole

- Always zero, because $\oint d\mathbf{l}' = \mathbf{0}$
- Also because $\nabla \cdot \mathbf{B} = 0$



- Magnetic dipole

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos \alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'$$



$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}' \quad (\text{proved by a textbook problem})$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

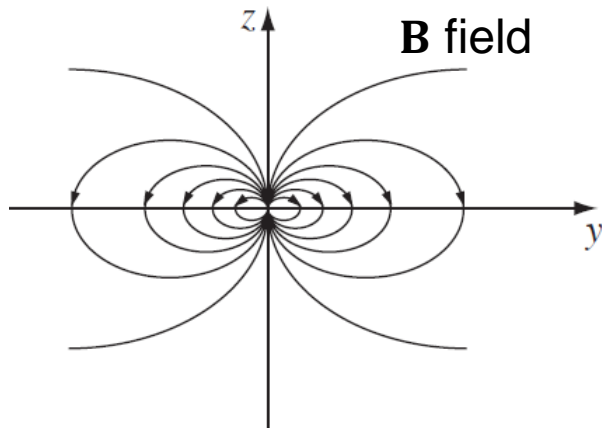
Magnetic dipole moment

$$\mathbf{m} \equiv I \int d\mathbf{a} = I \mathbf{a}$$

Multipole expansion of vector potential

- Pure dipole vs physical dipole

Pure dipole



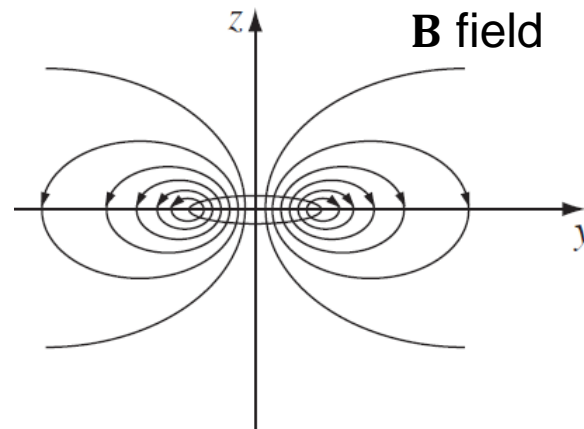
$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

Assumes $r \gg$ loop radius

$\mathbf{m} = I\mathbf{a}$ but take $I \rightarrow \infty$, $a \rightarrow 0$

Similar to pure electric dipole

Physical dipole

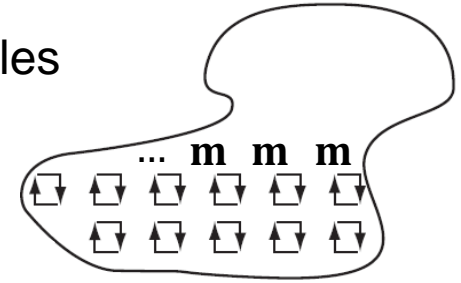


Deviations appear when closing up onto the dipole

Magnetic fields in matter

Magnetic materials

- Practically, all materials have magnetic response
 - Orbiting electrons around nuclei as magnetic dipoles
 - Electron spins as magnetic dipoles
 - Quantum object without a classical analog



- Magnetization: $\mathbf{M} = \frac{1}{V} \sum_i \mathbf{m}_i$
 - Magnetic dipole moment per unit volume
- Three types of response of matter to magnetic field
 - Paramagnets: magnetization \mathbf{M} parallel to applied \mathbf{B}
 - Diamagnets: magnetization \mathbf{M} opposite to applied \mathbf{B}
 - Ferromagnets: finite magnetization \mathbf{M} even without \mathbf{B}

Magnetic dipoles responding to field

- Paramagnetic response

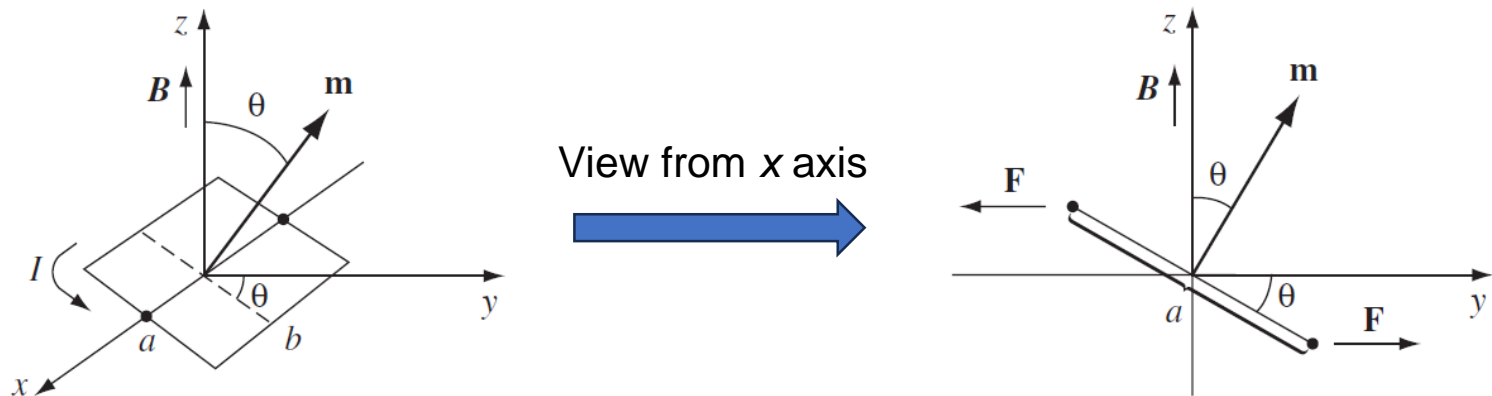
- Torque in a uniform field

- Suppose a rectangle current loop with side lengths a and b

$$\mathbf{N} = aF \sin \theta \hat{\mathbf{x}} \xrightarrow{\text{force } F = IbB} \mathbf{N} = IabB \sin \theta \hat{\mathbf{x}} = mB \sin \theta \hat{\mathbf{x}}$$

➡ $\mathbf{N} = \mathbf{m} \times \mathbf{B}$

- Torque tends to align \mathbf{m} parallel to \mathbf{B}

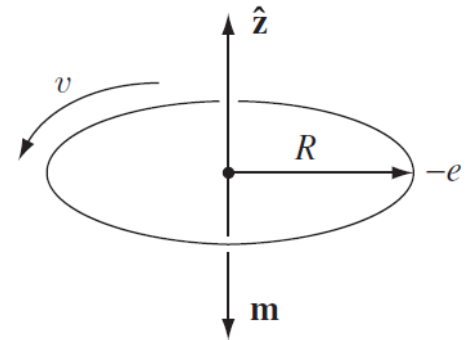


Magnetic dipoles responding to field

- Diamagnetic response

- Modification to an atomic orbit

$$I = \frac{-e}{T} = -\frac{ev}{2\pi R} \xrightarrow[\text{Orbit period}]{T = 2\pi R/v} \mathbf{m} = -\frac{1}{2}evR\hat{\mathbf{z}}$$



- Velocity without and with \mathbf{B} , balancing Coulomb and centripetal forces

without \mathbf{B} : $\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} = m_e \frac{v^2}{R}$

with \mathbf{B} : $\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} + e\bar{v}B = m_e \frac{\bar{v}^2}{R}$

v : electron velocity
 \bar{v} : electron velocity with \mathbf{B}
 m_e : electron mass

➔ $e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v} + v)(\bar{v} - v)$

➔ $\Delta v = \bar{v} - v = \frac{eRB}{2m_e}$ for small Δv

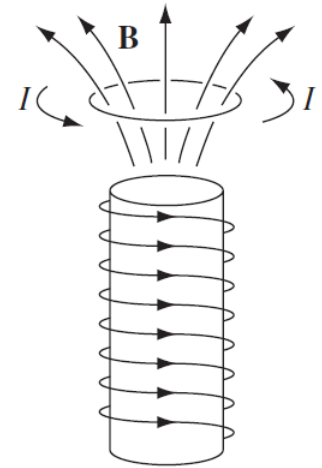
➔ $\Delta \mathbf{m} = -\frac{1}{2}e(\Delta v)R\hat{\mathbf{z}} = -\frac{e^2 R^2}{4m_e} \mathbf{B}$ Change in dipole moment works against \mathbf{B}

Forces on paramagnets and diamagnets

- General formula for small dipole in nonuniform field

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

- For paramagnets $\mathbf{m} \parallel \mathbf{B}$
 - Force directs toward more intense field regions
 - Paramagnets are attracted to magnets
- For diamagnets $\mathbf{m} \parallel -\mathbf{B}$
 - Force directs toward less intense field regions
 - Diamagnets are repelled by magnets



Levitating frog



(Ig Nobel award 2000)

Field of magnetized objects

- Bound currents
 - Vector potential of a magnetized object (neglecting the cause)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} d\tau'$$

$$= \frac{\mu_0}{4\pi} \int \left[\mathbf{M}(\mathbf{r}') \times \left(\nabla' \frac{1}{r} \right) \right] d\tau'$$

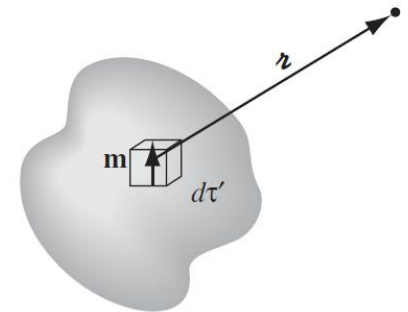
↓ Integrate by parts

$$= \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{r} [\nabla' \times \mathbf{M}(\mathbf{r}')] d\tau' - \int \nabla' \times \left[\frac{\mathbf{M}(\mathbf{r}')}{r} \right] d\tau' \right\}$$

$$= \frac{\mu_0}{4\pi} \int \frac{1}{r} [\nabla' \times \mathbf{M}(\mathbf{r}')] d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{r} [\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']$$

$$= \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_b(\mathbf{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{r} da'$$

* with $\mathbf{J}_b = \nabla \times \mathbf{M}$
 $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$



Field of magnetized objects

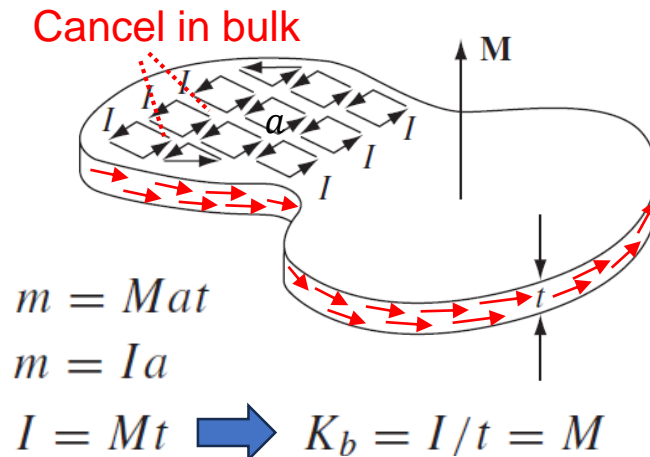
- Bound currents

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_b(\mathbf{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{r} da'$$

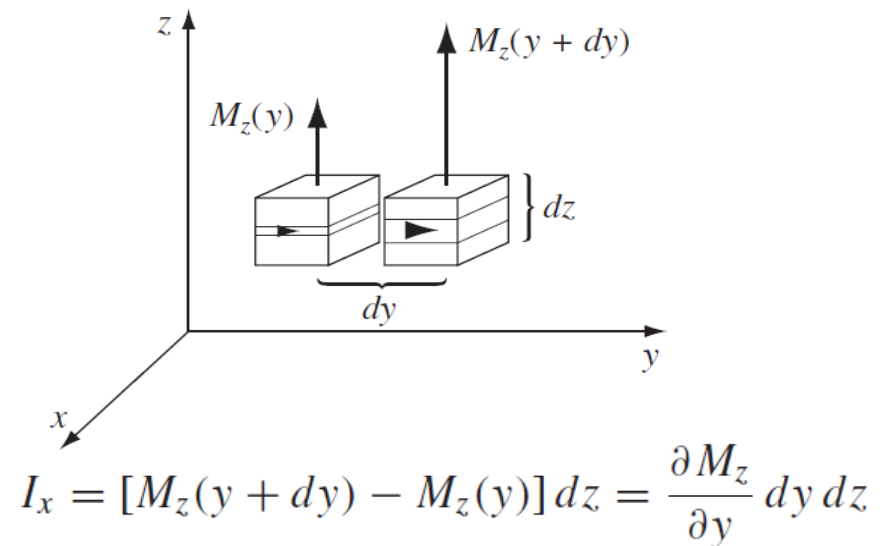
- Density of bound currents $\left\{ \begin{array}{l} \text{surface: } \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \\ \text{volume: } \mathbf{J}_b = \nabla \times \mathbf{M} \end{array} \right.$ *check $\nabla \cdot \mathbf{J}_b = 0$

- Physical picture of bound currents

Surface bound current (suppose uniform \mathbf{M})



Volume bound current



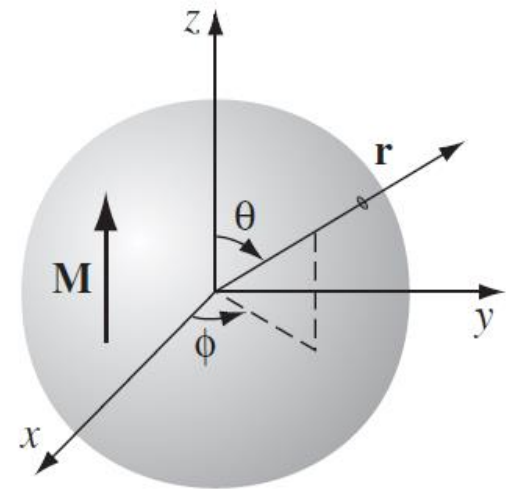
Field of magnetized objects

- Bound currents

Example 6.1. Find the magnetic field of a uniformly magnetized sphere.

- Field inside the sphere $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$
 - Uniform field
 - \mathbf{M} induces \mathbf{B} that is parallel to it, while \mathbf{P} induces \mathbf{E} that is antiparallel
- Field outside the sphere same as what would be for a dipole

$$\mathbf{m} = \frac{4}{3}\pi R^3\mathbf{M}$$



Auxiliary field

- Add the cause and the effect of magnetization

- Total magnetic field

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

- \mathbf{B} : total magnetic field
- \mathbf{J} : total current density
- \mathbf{J}_b : bound current density, due to magnetization
- \mathbf{J}_f : free current density that we control, not a result of magnetization

$$\Rightarrow \nabla \times \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f$$

- The auxiliary field

$$\mathbf{H} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

$$\Rightarrow \nabla \times \mathbf{H} = \mathbf{J}_f$$

- Ampère's law for auxiliary field

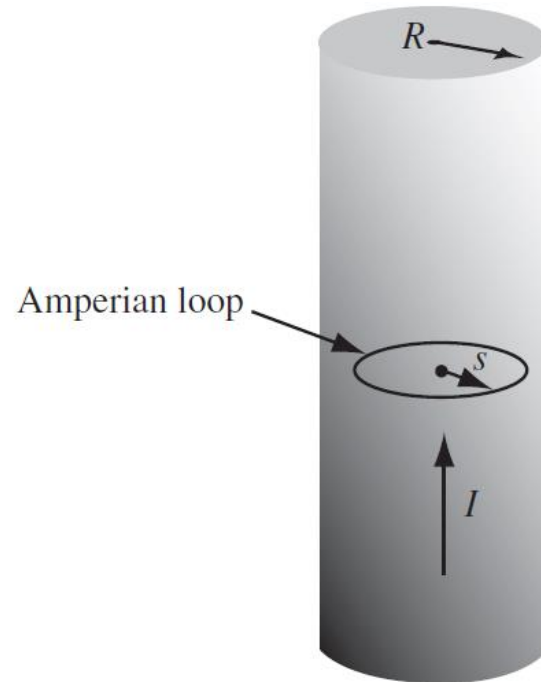
$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$$

Enclosed total
free current

Auxiliary field

- Application of auxiliary field

Example 6.2. A long copper rod of radius R carries a uniformly distributed (free) current I (Fig. 6.19). Find \mathbf{H} inside and outside the rod.

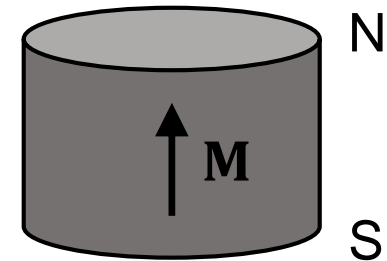


Auxiliary field

- Can \mathbf{H} be finite without applying any free current?

- Looks like so because $\oint \mathbf{H} \cdot d\mathbf{l} = I_{f\text{enc}}$

- Consider a short cylindrical magnet
- \mathbf{H} would be zero everywhere
- Then $\mathbf{B} \neq \mathbf{0}$ inside magnet, $\mathbf{B} = \mathbf{0}$ outside magnet
- Obviously wrong



- \mathbf{H} can be finite without any \mathbf{J}_F
 - Because $\nabla \cdot \mathbf{H} = \nabla \cdot \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = -\nabla \cdot \mathbf{M} \neq 0$
 - $\nabla \cdot \mathbf{M} \neq 0$ at the top and bottom surfaces of the magnet

H versus B, D versus E

- A long history of confusion
 - Whether we shall call **H** or **B** as the magnetic field
 - Some convention calls **H** the magnetic field, and **B** as the magnetic flux density
 - Our textbook takes the stance that **H** should be “auxiliary”
- Reason for such confusion
 - **H** is a lot more frequently used than **B** as **H** is given by free current, something we can control, while **B** is material dependent
 - Helmholtz coil magnetizing a specimen
 - **E** is a lot more frequently used than **D** as **D** is given by free charge, which we rarely control, while **E** is determined by voltage difference (over distance), which is what we control
 - Charging up of a parallel plate capacitor

Linear magnetic media

- Linear magnetic media

- $\mathbf{M} = \chi_m \mathbf{H}$ (χ_m : magnetic susceptibility)

➔ $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}$

- Diamagnetic susceptibility on the order of 10^{-6}

- $\mathbf{B} = \mu \mathbf{H}$ ($\mu = \mu_0(1 + \chi_m)$: permeability)

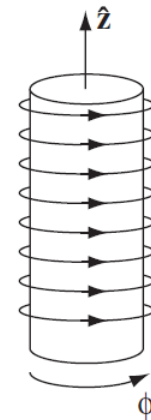
? Why this linear relation defines M-H but not M-B?

Example 6.3. An infinite solenoid (n turns per unit length, current I) is filled with linear material of susceptibility χ_m . Find the magnetic field inside the solenoid.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} \quad \text{➔} \quad \mathbf{H} = nI \hat{\mathbf{z}}$$

➔ $\mathbf{B} = \mu_0(1 + \chi_m)nI \hat{\mathbf{z}}$

Enhancement of field if paramagnetic!



Boundary conditions

- Boundary conditions reexamined
 - Earlier findings still hold, but \mathbf{J} needs to include free and bound currents

$$\left\{ \begin{array}{l} B_{\text{above}}^{\perp} = B_{\text{below}}^{\perp} \\ \mathbf{B}_{\text{above}}^{\parallel} - \mathbf{B}_{\text{below}}^{\parallel} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \end{array} \right.$$

- Easier to use the boundary conditions of \mathbf{D}

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \\ \nabla \times \mathbf{H} = \mathbf{J}_f \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \mathbf{H}_{\text{above}}^{\parallel} - \mathbf{H}_{\text{below}}^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}} \\ H_{\text{above}}^{\perp} - H_{\text{below}}^{\perp} = -(M_{\text{above}}^{\perp} - M_{\text{below}}^{\perp}) \end{array} \right.$$

Surface free
charge density
↓

Nonlinear magnetic media

- Ferromagnets: $\mathbf{M} \neq 0$ without applying any field
 - Obvious violation of the linear relation $\mathbf{M} = \chi_m \mathbf{H}$
 - Represents a quantum phenomena
 - Exchange interaction: $U = -2J \sum_{p=1}^N \mathbf{S}_p \cdot \mathbf{S}_{p+1}$ ($J > 0$)
 - Prefers spontaneous parallel alignment of spins
 - Domain formation and hysteresis loop

