Tutorial 7: Solutions

1. Time-dependent perturbation theory: Gaussian pulse

(a)

$$
\begin{aligned}
P_{n\leftarrow 0} &= \frac{1}{\hbar^2}\Big|\int_{t_0}^{\infty} V_{n0}\frac{e^{-\frac{t_1^2}{2\tau^2}}}{\sqrt{2\pi\tau^2}} e^{\frac{i(E_n-E_0)(t_1-t_0)}{\hbar}}\, dt_1\Big|^2 \\
&= \frac{|V_{n0}|^2}{2\pi\hbar^2\tau^2}\Big|\int_{t_0}^{\infty} e^{-\frac{1}{2\tau^2}(t_1-\beta)^2} e^{\frac{\beta^2}{2\tau^2}}\, dt_1\Big|^2 \\
\beta &= \frac{i(E_n-E_0)}{2\hbar(\frac{1}{2\tau^2})} = \frac{i(E_n-E_0)\tau^2}{\hbar} \\
&\Big(at^2+bt = a\big(t+\frac{b}{2a}\big)^2 - \frac{b^2}{4a}\Big) \\
P_{n\leftarrow 0} &= \frac{|V_{n0}|^2}{2\pi\hbar^2\tau^2} e^{\frac{\beta^2}{\tau^2}}\Big|\sqrt{\frac{\pi}{(1/(2\tau^2))}}\Big|^2 \\
&= \frac{|V_{n0}|^2}{\hbar^2} e^{-\frac{(E_n-E_0)^2}{\hbar^2}\tau^2}
\end{aligned}
$$

(b) (i) As $\tau \to 0$, $P_{n\leftarrow 0} \to \frac{|V_{n0}|^2}{\hbar^2}$.
(ii) As $\tau \to \infty$, $P_{n\leftarrow 0} \to 0$. As $\tau \to \infty$, the change is very slow. The adiabatic approximation is valid and the system remains in the eigenstate with quantum number 0. $H_{t\to\infty} \approx H_{t\to-\infty}$. So $|0\rangle_{t\to\infty} \approx |0\rangle_{t\to-\infty}$ and $P_{n\leftarrow 0} \to 0$.

2. Interaction of a hydrogen atom with an electromagnetic wave

(a) $\lambda \gg a_0$

(b) (1) $L_z = (\vec{r}\times\vec{p})_z = xp_y - yp_x$ does not involve z or p_z. Hence $[L_z, z] = 0$.

$$
\begin{aligned}
&\Rightarrow \langle n'\ell'm'|L_z z|n\ell m\rangle = \langle n'\ell'm'|zL_z|n\ell m\rangle \\
&\Rightarrow \hbar m'\langle n'\ell'm'|z|n\ell m\rangle = \hbar m\langle n'\ell'm'|z|n\ell m\rangle \\
&\Rightarrow \hbar(m'-m)\langle n'\ell'm'|z|n\ell m\rangle = 0 \\
&\Rightarrow \text{ if } m'\neq m, \langle n'\ell'm'|z|n\ell m\rangle = 0.
\end{aligned}
$$

(2) $L_z = xp_y - yp_x$

$$[L_z, x] = -y[p_x, x] = -y(-i\hbar) = i\hbar y$$
$$[L_z, y] = x[p_y, y] = x(-i\hbar) = -i\hbar x$$

where we have used the fact that

$$[x, y] = 0$$
$$[x, p_y] = [y, p_x] = 0.$$

$$\langle n'\ell'm'|L_z x - xL_z|n\ell m\rangle = i\hbar\langle n'\ell'm'|y|n\ell m\rangle$$
$$\Rightarrow \hbar(m'-m)\langle n'\ell'm'|x|n\ell m\rangle = i\hbar\langle n'\ell'm'|y|n\ell m\rangle \tag{1}$$
$$\langle n'\ell'm'|L_z y - yL_z|n\ell m\rangle = -i\hbar\langle n'\ell'm'|x|n\ell m\rangle$$
$$\Rightarrow \hbar(m'-m)\langle n'\ell'm'|y|n\ell m\rangle = -i\hbar\langle n'\ell'm'|x|n\ell m\rangle \tag{2}$$

(1) and (2):

$$\Rightarrow i(m'-m)^2\langle n'\ell'm'|y|n\ell m\rangle = i\langle n'\ell'm'|y|n\ell m\rangle$$
$$\Rightarrow [(m'-m)^2 - 1]\langle n'\ell'm'|y|n\ell m\rangle = 0$$
$$\Rightarrow \text{ if } (m'-m) \neq \pm 1, \langle n'\ell'm'|y|n\ell m\rangle = 0$$

Similarly,

$$-i(m'-m)^2\langle n'\ell'm'|x|n\ell m\rangle = -i\langle n'\ell'm'|x|n\ell m\rangle$$
$$\Rightarrow \text{ if } (m'-m) \neq \pm 1, \langle n'\ell'm'|x|n\ell m\rangle = 0$$

3. Born approximation in scattering theory

(a) The states of the free particle form a continuum, with a continuum of possible directions of $\vec{k}$ and also a continuum of energies close to $E_k = \frac{\hbar^2 k^2}{2m}$. Thus, the transition occurs to a continuum of final states. Also, the perturbing potential $V(r)$ is constant in time. Thus, Fermi's Golden Rule can be applied.

(b) By Fermi's Golden Rule for a constant $V(r)$, the rate of transition $P_{kk'}$ from state $|\vec{k'}\rangle$ to state $|\vec{k}\rangle$ with energy $E_k = E_{k'}$ is

$$P_{kk'} = \frac{2\pi}{\hbar} g(E_k) |\langle\vec{k}|V(r)|\vec{k'}\rangle|^2.$$

So

$$
\begin{aligned}
P_{kk'} &= \frac{2\pi}{\hbar}\frac{mL^3k}{2\pi^2\hbar^2}\frac{1}{L^6}\left(\left|\int V(r)e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}d^3\vec{r}\right|^2\right) \\
&= \frac{mk}{L^3\hbar^3\pi}\left|\int V(r)e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}d^3\vec{r}\right|^2
\end{aligned}
$$

$$
\begin{aligned}
\bar{w}_{kk'} &= \frac{d\Omega}{4\pi}P_{kk'} \\
&= \frac{d\Omega}{4\pi}\left(\frac{mk}{L^3\hbar^3\pi}\left|\int V(r)e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}d^3\vec{r}\right|^2\right)
\end{aligned}
$$

where the integral over all solid angles is 4π. ($\int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} \, d\phi = 4\pi$)

(c)

$$
\begin{aligned}
\vec{J}_{inc} &= \frac{i\hbar}{2m}\big(\psi\nabla\psi^* - \psi^*\nabla\psi\big) \\
&= \frac{i\hbar}{2m}\frac{1}{L^3}(-i\vec{k}' - i\vec{k}') \\
&= \frac{\hbar\vec{k}'}{mL^3}
\end{aligned}
$$

$$
\begin{aligned}
|\vec{J}_{inc}|d\sigma &= \bar{w}_{kk'} \text{ by definition.} \\
\frac{\hbar|\vec{k}'|}{mL^3}d\sigma &= \frac{d\Omega}{4\pi}\frac{mk}{L^3\hbar^3\pi}\left|\int V(r)e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}d^3\vec{r}\right|^2 \\
&\text{also } E_k = E_{k'} \Rightarrow |k'| = |k| \\
\Rightarrow \frac{d\sigma}{d\Omega} &= \left(\frac{m}{2\pi\hbar^2}\right)^2\left|\int V(r)e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}d^3\vec{r}\right|^2
\end{aligned}
$$