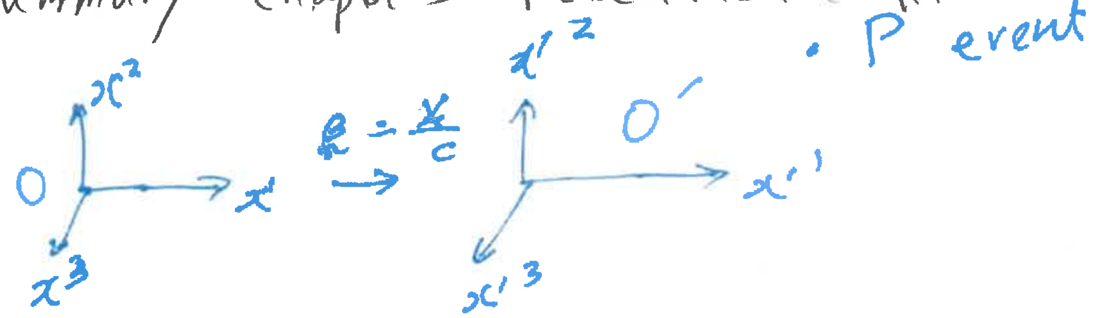
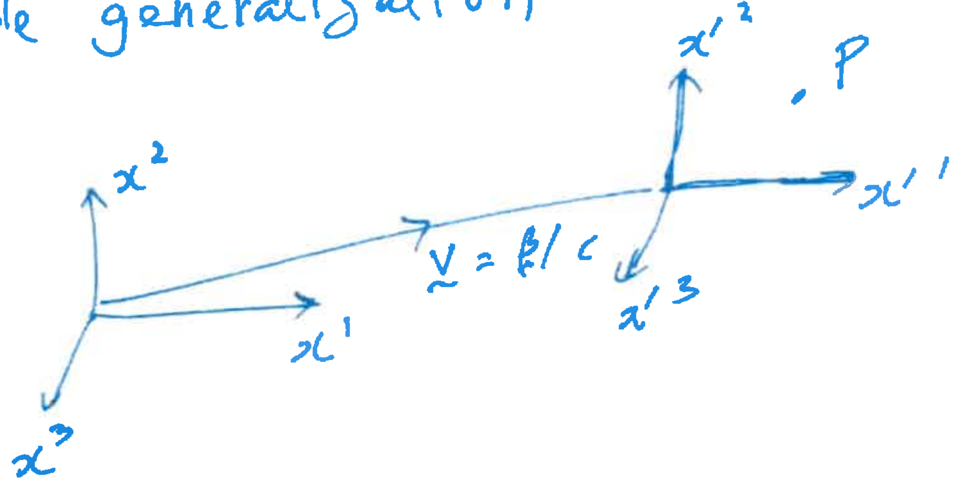
Summary chapter 3 Relativistic kinematics



$$x'^0 = \gamma(x^0 - \beta x^1)$$

$$x'^1 = \gamma(x^1 - \beta x^0), \quad x'^2 = x^2, \quad x'^3 = x^3$$

Simple generalization



$$x'^0 = \gamma(x^0 - \underset{\sim}{\beta} \cdot \underset{\sim}{x})$$

$$\underset{\sim}{x}' = \underset{\sim}{x} + (\gamma - 1)\frac{\underset{\sim}{x} \cdot \underset{\sim}{\beta}}{\beta^2}\underset{\sim}{\beta} - \gamma \underset{\sim}{\beta} x^0$$

Generalization: Any mapping that preserves the 'interval' between any two events P, Q.

Introduce metric tensor $g_{\mu\nu}$ to define distance between two points

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

A linear transformation that preserves

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu, \qquad \begin{array}{l} g_{\mu\nu} = 0 \quad \mu \neq \nu \\ g_{00} = +1 \\ g_{11} = g_{22} = g_{33} = -1 \end{array}$$

is a Lorentz transformation

Inhomongeneous Lorentz tran. $\{\Lambda, a\}$

= Poincaré transformation

Homongeneous Lorentz tran. $\{\Lambda\}$

$$\underline{x} \xrightarrow{\Lambda} \underline{x}' = \Lambda\, \underline{x}$$

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$$

Two types of basis

$\underset{\sim}{e}_i$ covariant basis
tangent basis

$\underset{\sim}{E}^i$ contra variant basis.
'normal' basis

$$\underline{A} = A_i \underset{\sim}{E}^i = A^i \underset{\sim}{e}_i$$

Define vector, scalar, tensor.

4-velocity $\underline{W} = \dfrac{d\underline{x}}{d\tau}$

$d\tau$ = proper time $= \dfrac{ds}{c} = \dfrac{dt}{\gamma}$

4 - momentum

4 - force

collision in particle physics

Define lab. frame, cm frame

Elastic, inelastic collisions

Excess energy available

Threshold energy

Examples:

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1. $p + p \rightarrow p + p + p + \bar{p}$ antiproton

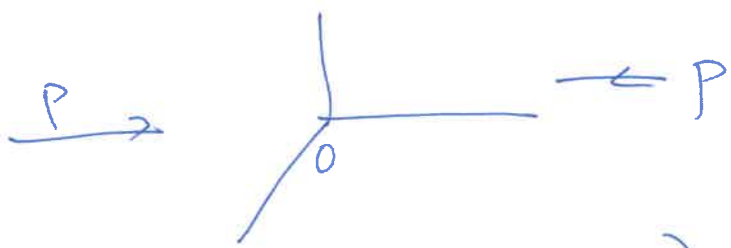
How much energy the original 2 ps should have in order to get the reaction occur i.e. to get $\bar{p}$?

Ans: Need to choose a frame. Two frames:

1. CM frame. 2. Lab frame.

Minimum excess energy ξ = threshold energy

1. CM frame



Total energy $= (P_1^0 + P_2^0) \cdot c$

$p^0 = \frac{E}{c}$

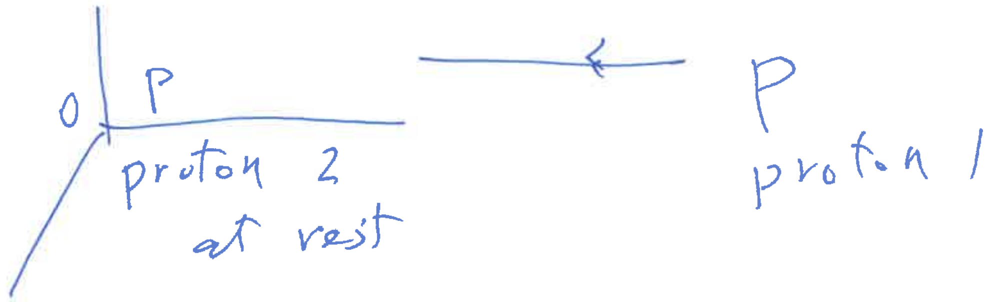
Threshold energy required

$= 4 m_p c^2$, m_p = rest mass of p

Excess energy $\xi = (P_1^0 + P_2^0)\, c - (m_1 + m_2)\, c^2$

$$= 2\, m_p c^2 \quad \text{(threshold energy)}$$

2. Lab frame



What is the total energy needed in the lab frame to get the reaction going?

$$\xi = \sqrt{(\underline{P}_1 + \underline{P}_2)^2}\; c - (m_1 + m_2)c^2$$

$$\left(\xi + (m_1 + m_2)c^2\right)^2 = (\underline{P}_1 + \underline{P}_2)^2\, c^2$$

$$= \left[(P_1^0 + P_2^0)^2 - \underset{\sim}{P}_1^{\,2}\right] c^2$$

$$= \left[m_1^2 c^2 + 2 P_1^0 P_2^0 + m_2^2 c^2\right] c^2$$

$$(\xi + 2m_p c^2)^2 = \left[2m_p^2 c^2 + 2\frac{E_1}{c} \cdot m_p c\right] c^2$$

$\because E_2 = m_2 c^2$

$$= 2m_p^2 c^4 + 2E_1 m_p c^2$$

Find E_1, $\xi = 2m_p c^2$

$$E_1 = 7 m_p c^2 \quad (\text{Hw})$$

So in the lab frame, total energy required is

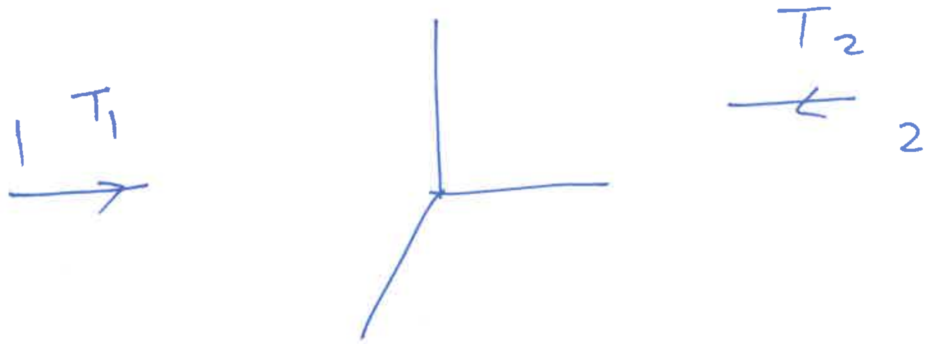
$$E_1 + E_2 = 8 m_p c^2$$

Compared with cm frame

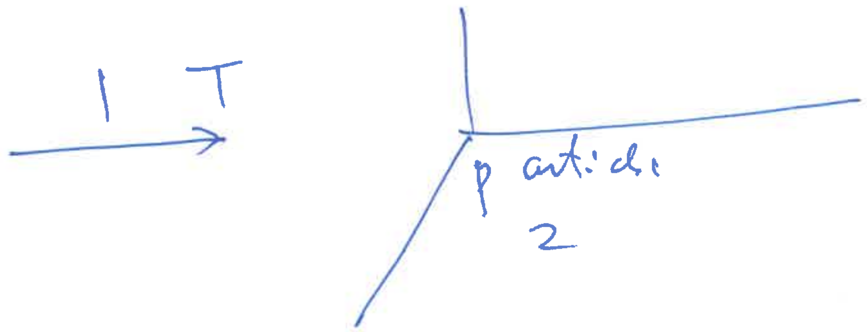
total energy $4 m_p c^2$

$\therefore$ To produce the same reaction, cm frame can save energy

2nd Example



Ask what is the KE of particle 1 wrt particle 2?



Find T in terms of T_1 and T_2 by using the idea of invariance and conservation.

$\underline{P}_1 + \underline{P}_2$ = total 4 momentum

$(\underline{P}_1 + \underline{P}_2)^2$ = dot product, invariant

= also conserved

Compute $(\underline{P}_1 + \underline{P}_2)^2$ in original frame (T_1, T_2) and in lab frame of particle 2 (T)

In the original frame (T_1, T_2), we assume is a CM frame:

$$(\underline{P}_1 + \underline{P}_2)^2\Big|_{CM} = (P_1^0 + P_2^0)^2$$

$$= \left(\frac{E_1 + E_2}{c}\right)^2 = \left(\frac{T_1 + T_2 + (m_1 + m_2)c^2}{c}\right)^2$$

$$\because E = T + mc^2$$

$$(\underline{P}_1 + \underline{P}_2)^2\Big|_{lab\ frame}$$

$$= (P_1^0 + P_2^0)^2 - \underset{\sim}{P}_1^2$$

$$= m_1^2c^2 + m_2^2c^2 + 2P_1^0 P_2^0, \qquad P_2^0 = m_2 c$$

$$P_1^0 = \frac{E_1}{c} = \frac{1}{c}(T + m_1c^2)$$

Equating

$$\left(\underline{P}_1 + \underline{P}_2\right)^2\Big|_{q\gamma} = \left(\underline{P}_1 + \underline{P}_2\right)^2\Big|_{(ab)}$$

Find T in terms of T_1 and T_2
(Hw)

Proceed to

chapter 4

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Chapter 4 Symmetries (Griffiths)

Define symmetry in physics

$$\text{Transformations} \rightarrow \text{set} \xrightarrow{\text{binary operation}} \text{group}$$

A symmetry transformation in quantum mechanics leaves transition probability invariant (unchanged)

Isospin spin symmetry SU(2)

Find ratio of scattering cross-sections for isodoublet (nucleons, n, p) and isotriplet (pions, π^+, π^0, π^-)

Discuss discrete symmetries P, C, T

Definition of a group

We define a binary operation $\bullet$ on a set S

1. $\forall \alpha, \beta \in S, \ \alpha \bullet \beta \in S$ (closure property)
2. $\exists$ an identity I such that $I \bullet \alpha = \alpha = \alpha \bullet I, \ \forall \alpha \in S$
3. Associative law: $\alpha \bullet (\beta \bullet \gamma) = (\alpha \bullet \beta) \bullet \gamma, \ \forall \alpha, \beta, \gamma \in S$

A set with the above three axioms satisfied is a semigroup.

If in addition,

4. $\forall \alpha \in S$, $\exists$ an element α^{-1} such that $\alpha^{-1} \bullet \alpha = \alpha \bullet \alpha^{-1} = I$, that is, α^{-1} is the inverse of α,

then the set S is a group with respect to the binary operator $\bullet$

If $\alpha \bullet \beta = \beta \bullet \alpha$, the group is commutative (Abelian). If $\alpha \bullet \beta \neq \beta \bullet \alpha$, the group is non-commutative (non-abelian), e.g. the $n \times n$ matrices form a group but is non-abelian. And the set of integers is an abelian group with respect to addition.

Consider a commutative group S(+). If the elements of S(+) form a semi group with respect to new binary operation, say multiplication ($\cdot$), such that the following distributive laws hold,

$$(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$$
$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma,$$

then S(+, $\cdot$) is an integral domain.
Identity element with respect to addition = zero element
Identity element with respect to multiplication = unity element.

A ring is an integral domain without a unity element with respect to multiplication.

If S(+, $\cdot$) is a commutative group with respect to addition and also a commutative group with respect to multiplication (except the zero element has no inverse with respect to multiplication), the S(+, $\cdot$) is a field F.

Let a field F act on a commutative group V(+) by scalar multiplication $\times$ such that, $\forall \underset{\sim}{\alpha} \in V(+)$ and $\forall \alpha, \beta \in F(+, \cdot)$, the following hold (omitting the $\times$)

1. $\alpha \underset{\sim}{\alpha} = \underset{\sim}{\alpha} \alpha \in V(+)$,

2. $1\underset{\sim}{a} = \underset{\sim}{a} = \underset{\sim}{a}1$

3. $0\underset{\sim}{a} = \underset{\sim}{0} = \underset{\sim}{a}0$

4. $\alpha(\underset{\sim}{a} + \underset{\sim}{b}) = \alpha\underset{\sim}{a} + \alpha\underset{\sim}{b}$

5. $(\alpha + \beta)\underset{\sim}{a} = \alpha\underset{\sim}{a} + \beta\underset{\sim}{a}$
6. $\alpha(\beta\underset{\sim}{a}) = (\alpha\beta)\underset{\sim}{a} = \alpha\beta\underset{\sim}{a} \in V(+)$,

Then the set V(+) (that is closed under addition + and scalar multiplication $\times$ by elements of the field F) is called a linear vector space over the field F, and the elements of V(+) are vectors.

Define an inner product for any two elements of V(+),

$(\underset{\sim}{a}, \underset{\sim}{b}) = \underset{\sim}{a}^* \bullet \underset{\sim}{b} \in F$, $\forall\, \underset{\sim}{a}, \underset{\sim}{b} \in V(+)$, $\underset{\sim}{a}^*$=complex conjugate of $\underset{\sim}{a}$,
then V(+) is a metric linear vector space, or a linear vector space with an inner product.

A complete linear vector space with an inner product is a Hilbert space

Definition of completeness- If the limit point of any sequence in the space belongs to the space, then the space is complete.
Consider a sequence $\{u_1, u_2, u_3, \ldots\}$, $\lim_{n\to\infty} u_n$ is known as the limit of the sequence.
Example of incompleteness:
Consider the sequence $\left\{\frac{1}{N}, N\ integer\right\}$, the limit point $\lim_{N\to\infty}\frac{1}{N}=0$
is not in the sequence, hence the sequence is incomplete.

If for any two elements of a linear vector space, we can define a commutation relation, say $\left[\underset{\sim}{a}, \underset{\sim}{b}\right]$, such that

$$\left[\underset{\sim}{a}, \underset{\sim}{b}\right] = -\left[\underset{\sim}{b}, \underset{\sim}{a}\right]$$

and

$$\left[\underset{\sim}{a}, \left[\underset{\sim}{b}, \underset{\sim}{c}\right]\right] + \left[\underset{\sim}{b}, \left[\underset{\sim}{c}, \underset{\sim}{a}\right]\right] + \left[\underset{\sim}{c}, \left[\underset{\sim}{a}, \underset{\sim}{b}\right]\right] = 0 \text{ (so called Jacobi identity)}$$

are satisfied, then we have an **algebra**.

(4)

Table 4.1 Symmetries and conservation laws.

Symmetry		**Conservation law**	
Translation in time	↔	Energy	space time
Translation in space	↔	Momentum	
Rotation	↔	Angular momentum	
Gauge transformation	↔	Charge	internal

relating symmetries and conservation laws:

Noether's Theorem: Symmetries ↔ Conservation laws

Every symmetry of nature yields a conservation law; conversely, every conservation law reflects an underlying symmetry. For example, the laws of physics are symmetrical with respect to *translations in time* (they work the same today as they did yesterday). Noether's theorem relates this invariance to conservation of energy. If a system is invariant under translations in space, then momentum is conserved; if it is symmetrical under rotations about a point, then angular momentum is conserved. Similarly, the invariance of electrodynamics under gauge transformations leads to conservation of charge (we call this an *internal* symmetry, in contrast to the space-time symmetries). I'm not going to *prove* Noether's theorem; the details are not terribly enlightening [1]. The important thing is the profound and beautiful *idea* that symmetries are associated with conservation laws (see Table 4.1).

I have been speaking rather casually about symmetries, and I cited some examples; but what precisely *is* a symmetry? It is an operation you can perform (at least conceptually) on a system that leaves it *invariant* – that carries it into a configuration indistinguishable from the original one. In the case of the function in Figure 4.1, changing the sign of the argument, $x \to -x$, and multiplying the whole thing by -1, $f(x) \to -f(-x)$, is a symmetry operation. For a meatier example, consider the equilateral triangle (Figure 4.2). It is carried into itself by a clockwise rotation through 120° (R_+), and by a counterclockwise rotation through 120° (R_-), by flipping it about the vertical axis a (R_a), or around the axis through b (R_b), or c (R_c). Is that all? Well, doing *nothing* (I) obviously leaves it invariant, so this too is a symmetry operation, albeit a pretty trivial one. And then we could combine operations – for example, rotate clockwise through 240°. But that's the same as rotating *counter* clockwise by 120° (i.e. $R_+^2 = R_-$). As it turns out, we have already identified all the distinct symmetry operations on the equilateral triangle (see Problem 4.1).

The set of all symmetry operations (on a particular system) has the following properties:

1. *Closure*: If R_i and R_j are in the set, then the *product*, R_iR_j – meaning: first perform R_j, then perform R_i* – is also in the set; that is, there exists some R_k such that $R_iR_j = R_k$.

* Note the 'backwards' ordering. Think of the symmetry operations as acting on a system to their right: $R_iR_j(\Delta) = R_i[R_j(\Delta)]$; R_j acts first, and then R_i acts on the result.

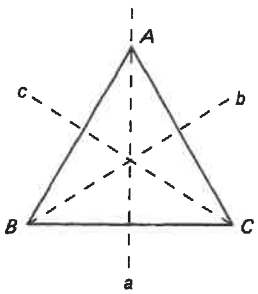

Fig. 4.2 Symmetries of the equilateral triangle.

2. *Identity*: There is an element I such that $IR_i = R_i I = R_i$ for all elements R_i.
3. *Inverse*: For every element R_i there is an *inverse*, R_i^{-1}, such that $R_i R_i^{-1} = R_i^{-1} R_i = I$.
4. *Associativity*: $R_i(R_j R_k) = (R_i R_j) R_k$.

These are the defining properties of a mathematical *group*. Indeed, group theory may be regarded as the systematic study of symmetries. Note that group elements need not *commute*: $R_i R_j \neq R_j R_i$, in general. If all the elements *do* commute, the group is called *Abelian*. Translations in space and time form Abelian groups; rotations (in three dimensions) do *not* [2]. Groups can be *finite* (like the triangle group, which has just six elements) or *infinite* (for example, the set of integers, with addition playing the role of group 'multiplication'). We shall encounter *continuous* groups (such as the group of all rotations in a plane), in which the elements depend on one or more continuous parameters* (the angle of rotation, in this case), and *discrete* groups, in which the elements may be labeled by an index that takes on only integer values (all finite groups are, of course, discrete).

As it turns out, most of the groups of interest in physics can be formulated as groups of *matrices*. The Lorentz group, for instance, consists of the set of 4×4 Λ matrices introduced in Chapter 3. In elementary particle physics, the most common groups are of the type mathematicians call $U(n)$: the collection of all unitary $n \times n$ matrices (see Table 4.2). (A *unitary matrix* is one whose inverse is equal to its transpose conjugate: $U^{-1} = \tilde{U}^*$.) If we restrict ourselves further to unitary matrices with determinant 1, the group is called $SU(n)$. (The S stands for 'special', which just means 'determinant 1'.) If we limit ourselves to *real* unitary matrices, the group is $O(n)$. (O stands for 'orthogonal'; an orthogonal matrix is one whose inverse is equal to its transpose: $O^{-1} = \tilde{O}$.) Finally, the group of real, orthogonal, $n \times n$ matrices of determinant 1 is $SO(n)$; $SO(n)$ may be thought of as the group of all *rotations* in a space of n dimensions. Thus, $SO(3)$ describes the

* If this dependence takes the form of an *analytic* function, it is called a *Lie* group. All of the continuous groups one encounters in physics are Lie groups [3].

(6)

Table 4.2 Important symmetry groups.

Group name	Dimension	Matrices in group
$U(n)$	$n \times n$	unitary ($\tilde{U}^* U = 1$)
$SU(n)$	$n \times n$	unitary, determinant 1
$O(n)$	$n \times n$	orthogonal ($\tilde{O}O = 1$)
$SO(n)$	$n \times n$	orthogonal, determinant 1

SO(3) for usual 3-dim space
SU(2) for Hilbert space

rotational symmetry of our world, a symmetry that is related by Noether's theorem to the conservation of angular momentum. Indeed, the entire quantum theory of angular momentum is really closet group theory. It so happens that $SO(3)$ is almost identical in mathematical structure to $SU(2)$, which is the most important *internal* symmetry in elementary particle physics. So the theory of angular momentum, to which we turn next, will actually serve us twice.

One final thing. Every group G can be *represented* by a group of *matrices*: for every group element a there is a corresponding matrix M_a, and the correspondence respects group multiplication, in the sense that if $ab = c$, then $M_a M_b = M_c$. A representation need not be 'faithful': there may be many distinct group elements represented by the same matrix. (Mathematically, the group of matrices is *homomorphic*, but not necessarily *isomorphic*, to G.) Indeed, there is a trivial case, in which we represent every element by the 1×1 *unit* matrix (which is to say, the number 1). If G is a group of matrices, such as $SU(6)$ or $O(18)$, then it is (obviously) a representation of *itself* – we call it the *fundamental* representation. But there will, in general, be many other representations, by matrices of various dimensions. For example, $SU(2)$ has representations of dimension 1 (the trivial one), 2 (the matrices themselves), 3, 4, 5, and in fact *every* positive integer. A major problem in group theory is the characterization of all the representations of a given group.

Adjoint representation

Of course, you can always construct a new representation by combining two old ones, thus

$$M_a = \begin{pmatrix} \boxed{M_a^{(1)}} & \text{(zeros)} \\ \text{(zeros)} & \boxed{M_a^{(2)}} \end{pmatrix}$$

But we don't count this separately; when we list the representations of a group, we are talking about the so-called *irreducible* representations, which *cannot* be decomposed into block-diagonal form. Actually, you have already encountered several examples of group representations, probably without realizing it: an ordinary scalar belongs to the one-dimensional representation of the rotation group, $SO(3)$, and a vector belongs to the three-dimensional representation; four-vectors belong to the four-dimensional representation of the Lorentz group; and the curious geometrical arrangements of Gell-Mann's Eightfold Way correspond to irreducible representations of the group $SU(3)$.

Representations.

A representation of a group G is a homomorphism of G onto a group of linear operators acting on a linear vector space,

$$D(g_i)\,D(g_j) = D(g_i g_j)$$

If a represenation is isomorphic to the group it is a faithful representation

A ray representation: $D(g_i)$ and $e^{i\alpha_i} D(g_i)$, α_i = real, are allowed

$$D(g_i)\,D(g_j) = e^{i\alpha_{ij}}\,D(g_i g_j)$$

α_{ij} = arbitary real number which can depend on the group elements g_i and g_j

If α_{ij} is restricted to take only a finite number of values, the representation is multiple-valued

Double-valued representation: $\alpha_{ij} = 0$ or $\alpha_{ij} = \pi$, i.e

$$D(g_i)\,D(g_j) = \pm\, D(g_i g_j)$$

Two representations are equivalent if one can be transformed into the other by a similarity transformation

A representation of a finite or compact Lie group can be transformed into a unitary representation by a similarity transformation

Reducible $D(g) = \begin{pmatrix} D_1(g) & X(g) \\ 0 & D_2(g) \end{pmatrix}$

Fully reducible $D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}$

A representation of a finite or compact Lie group is fully reducible

$$D = D^{(1)} \oplus D^{(2)} \oplus \cdots$$

conjugate representation

$\bar{D}$ = conjugate representation of D if we take the complex conjugte of the matrices of D

$\bar{D}(g) = (D(g))^*$ ← complex conjugate

Clebsch-Gordan Coefficents

Addition of angular momenta

$$\underset{\sim}{J}_1 \qquad \underset{\sim}{J}_2$$

$$|\alpha_1 j_1 m_1\rangle = basis\ \ for\ \ \underset{\sim}{J}_1^2\ \ and\ \ J_{1z}$$

$$|\alpha_2 j_2 m_2\rangle = basis\ \ for\ \ \underset{\sim}{J}_2^2\ \ and\ \ J_{2z}$$

$J_{13} = J_{1\,3}$ ← 3rd component

particle 1

The base vectors

$$|\alpha\, j_1\, j_2\, m_1\, m_2\rangle \equiv |\alpha_1\, j_1\, m_1\rangle |\alpha_2\, j_2\, m_2\rangle$$

$\alpha,\ j_1, j_2$ fixed,

m_1, m_2 vary $\qquad -j_1 \le m_1 \le j_1$

$\qquad\qquad\qquad -j_2 \le m_2 \le j_2$

span the subspace $\xi(\alpha, j_1, j_2)$.

$\underset{\sim}{J}^2 = (\underset{\sim}{J}_1 + \underset{\sim}{J}_2)^2 \quad and\ J_z \quad act\ on\ \ \xi(\alpha, j_1, j_2)$

Since $\underset{\sim}{J}_1^2$ and $\underset{\sim}{J}_2^2$ commute with $\underset{\sim}{J}^2\ \ and\ \ J_z$, can also use the base

$$\left|\alpha\, j_1\, j_2\, j\, m\right\rangle, \quad \alpha, j_1, j_2 \ \ fixed$$

$$j, m \ \text{ vary}$$

$$\left|j_1 - j_2\right| \le j \le \left(j_1 + j_2\right)$$

$$-j \le m \le j$$

to generate the same subspace $\xi(\alpha, j_1, j_2)$.

The two bases are related:

$$\left|\alpha\, j_1\, j_2\, j\, m\right\rangle = \sum_{m_2=-j_2}^{j_2} \sum_{m_1=-j_1}^{j_1} \left|\alpha\, j_1 j_2 m_1 m_2\right\rangle \left\langle j_1 j_2 m_1 m_2 \right| jm \rangle$$

$$\left|\alpha\, j_1\, j_2\, m_1\, m_2\right\rangle = \sum_{m=-j}^{j} \sum_{j=\left|j_1-j_1\right|}^{(j_1+j_2)} \left|\alpha\, j_1\, j_2\, j\, m\right\rangle \left\langle j\, m \right| j_1\, j_2\, m_1\, m_2 \rangle$$

$\left\langle j_1\, j_2\, m_1\, m_2 \right| j\, m \rangle = \left\langle j\, m \right| j_1\, j_2\, m_1\, m_2 \rangle^* \quad \equiv \quad$ Clebsch-Gordan coefficents

Meaning of C.G. coeffs

(i) relating two basis vectors (just like Fourier transform)

(ii) $\langle j_1\, j_2\, m_1\, m_2 | j\, m \rangle$ = probability amplitude of finding the state $| j_1\, j_2\, m_1\, m_2 \rangle$ when the system is in state $| jm \rangle$

Properties of C.G. coeffs

(1) Selection rule:

$$\langle \alpha\, j_1 j_2 m_1 m_2 | jm \rangle = 0 \quad unless$$

$$m_1 + m_2 = m \quad and \quad |j_1 - j_2| \le j \le (j_1 + j_2)$$

(2) Phase convention: require

$$\langle j_1\, j_2\, j_1\, m_2 | j\, j \rangle \ \text{real and} \ \ge 0$$

$$\mathrm{m}_2 = j - j_1$$

$$j = |j_1 - j_2|,\ |j_1 - j_2| + 1 \ldots \qquad (j_1 + j_2)$$

Note: When $m_1 = j_1$ and $m = j$, it does <u>not</u> necessarily imply $m_2 = j_2$ since $j \ne (j_1 + j_2)$ in general

(3) Reality: All C.G. coeffs can be obtained from

$$\langle j_1\, j_2\, j_1\, m_2 | j\, j \rangle$$

$\therefore$ all C.G. coeffs are real

(4) Orthogonality

$$\sum_{m_1\, m_2} \langle j_1\, j_2\, m_1\, m_2 | j\, m \rangle \langle j_1\, j_2\, m_1\, m_2 | j'\, m' \rangle = \delta_{jj'}\, \delta_{mm'}$$

$$\sum_{j\, m} \langle j_1\, j_2\, m_1\, m_2 | j\, m \rangle \langle j_1\, j_2\, m_1'\, m_2' | j\, m \rangle = \delta_{m_1 m_1'}\, \delta_{m_2 m_2'}$$

Wigner-Eckart theorem

In a standard representation $\{\underset{\sim}{J}^2, J_z\}$ whose basis vectors are denoted by $|\tau\, j m\rangle$,

The matrix element $\langle \tau\, j m | T_g^{(k)} | \tau' j' m' \rangle$

of the q[th] standard component of a given k[th] order irreducible tensor operator, $T^{(k)}$, is equal to the product of the Clebsch-Gordan coefficient.

$$\langle j' k\, m' q | j m \rangle$$

by a quantity independent of *m*, *m'* and *q* $\quad (q = -k, -k+1, \ldots. +k)$

$$\langle \tau\, j m | T_q^{(k)} | \tau' j' m' \rangle = \frac{1}{\sqrt{2j+1}} \langle \tau j \| T^{(k)} \| \tau' j' \rangle \Box \langle j' k\, m' q | j m \rangle$$

$\langle \tau j \| T^{(k)} \| \tau' j' \rangle$= reduced matrix element

$\langle j' k\, m' q | j m \rangle$=Clebsch-Gordan coefficient

$\neq 0$ only if $m = m' + q$ and $|j - j'| \leq k \leq j + j'$

For a scalar operator *S* $\quad \langle \tau\, j m | s | \tau' j' m' \rangle = \delta_{jj'} \delta_{mm'} S_{\tau\tau'}^{(j)}$

$S_{\tau\tau'}^{(j)}$ independent of m and m'

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34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2		
J		1	
M		+1	
+1/2	+1/2	1	
J		1	0
M		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
J		1	
M		−1	
−1/2	−1/2	1	

1×1/2

m_1	m_2		
J		3/2	
M		+3/2	
+1	+1/2	1	
J		3/2	1/2
M		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
J		3/2	1/2
M		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
J		3/2	
M		−3/2	
−1	−1/2	1	

2×1

m_1	m_2			
J		3		
M		+3		
+2	+1	1		
J		3	2	
M		+2	+2	
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
J		3	2	1
M		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10
J		3	2	1
M		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
J		3	2	1
M		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
J		3	2	
M		−2	−2	
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
J		3		
M		−3		
−2	−1	1		

1×1

m_1	m_2			
J		2		
M		+2		
+1	+1	1		
J		2	1	
M		+1	+1	
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
J		2	1	0
M		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
J		2	1	
M		−1	−1	
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
J		2		
M		−2		
−1	−1	1		

2×1/2

m_1	m_2		
J		5/2	
M		+5/2	
+2	+1/2	1	
J		5/2	3/2
M		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
J		5/2	3/2
M		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
J		5/2	3/2
M		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
J		5/2	3/2
M		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
J		5/2	
M		−5/2	
−2	−1/2	1	

3/2×1/2

m_1	m_2		
J		2	
M		+2	
+3/2	+1/2	1	
J		2	1
M		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
J		2	1
M		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
J		2	1
M		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
J		2	
M		−2	
−3/2	−1/2	1	

3/2×1

m_1	m_2			
J		5/2		
M		+5/2		
+3/2	+1	1		
J		5/2	3/2	
M		+3/2	+3/2	
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
J		5/2	3/2	1/2
M		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
J		5/2	3/2	1/2
M		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
J		5/2	3/2	
M		−3/2	−3/2	
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
J		5/2		
M		−5/2		
−3/2	−1	1		

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

m_1	m_2				
J		3			
M		+3			
+3/2	+3/2	1			
J		3	2		
M		+2	+2		
+3/2	+1/2	1/2	1/2		
+1/2	+3/2	1/2	−1/2		
J		3	2	1	
M		+1	+1	+1	
+3/2	−1/2	1/5	1/2	3/10	
+1/2	+1/2	3/5	0	−2/5	
−1/2	+3/2	1/5	−1/2	3/10	
J		3	2	1	0
M		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4
J		3	2	1	
M		−1	−1	−1	
+1/2	−3/2	1/5	1/2	3/10	
−1/2	−1/2	3/5	0	−2/5	
−3/2	+1/2	1/5	−1/2	3/10	
J		3	2		
M		−2	−2		
−1/2	−3/2	1/2	1/2		
−3/2	−1/2	1/2	−1/2		
J		3			
M		−3			
−3/2	−3/2	1			

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

m_1	m_2				
J		7/2			
M		+7/2			
+2	+3/2	1			
J		7/2	5/2		
M		+5/2	+5/2		
+2	+1/2	3/7	4/7		
+1	+3/2	4/7	−3/7		
J		7/2	5/2	3/2	
M		+3/2	+3/2	+3/2	
+2	−1/2	1/7	16/35	2/5	
+1	+1/2	4/7	1/35	−2/5	
0	+3/2	2/7	−18/35	1/5	
J		7/2	5/2	3/2	1/2
M		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10
J		7/2	5/2	3/2	1/2
M		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5
J		7/2	5/2	3/2	
M		−3/2	−3/2	−3/2	
0	−3/2	2/7	18/35	1/5	
−1	−1/2	4/7	−1/35	−2/5	
−2	+1/2	1/7	−16/35	2/5	
J		7/2	5/2		
M		−5/2	−5/2		
−1	−3/2	4/7	3/7		
−2	−1/2	3/7	−4/7		
J		7/2			
M		−7/2			
−2	−3/2	1			

2×2

m_1	m_2					
J		4				
M		+4				
+2	+2	1				
J		4	3			
M		+3	+3			
+2	+1	1/2	1/2			
+1	+2	1/2	−1/2			
J		4	3	2		
M		+2	+2	+2		
+2	0	3/14	1/2	2/7		
+1	+1	4/7	0	−3/7		
0	+2	3/14	−1/2	2/7		
J		4	3	2	1	
M		+1	+1	+1	+1	
+2	−1	1/14	3/10	3/7	1/5	
+1	0	3/7	1/5	−1/14	−3/10	
0	+1	3/7	−1/5	−1/14	3/10	
−1	+2	1/14	−3/10	3/7	−1/5	
J		4	3	2	1	0
M		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5
J		4	3	2	1	
M		−1	−1	−1	−1	
+1	−2	1/14	3/10	3/7	1/5	
0	−1	3/7	1/5	−1/14	−3/10	
−1	0	3/7	−1/5	−1/14	3/10	
−2	+1	1/14	−3/10	3/7	−1/5	
J		4	3	2		
M		−2	−2	−2		
0	−2	3/14	1/2	2/7		
−1	−1	4/7	0	−3/7		
−2	0	3/14	−1/2	2/7		
J		4	3			
M		−3	−3			
−1	−2	1/2	1/2			
−2	−1	1/2	−1/2			
J		4				
M		−4				
−2	−2	1				

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

proton and neutron can be regarded as two different states of a nucleon. This is isospin symmetry and the symmetry group is SU(2).

Isospin symmetry is a good symmetry for strongly interacting particles.

It can classify hadrons into (iso) multiplets.

E.g. singlet Λ, doublet (n, p), triplet (π^-, π^0, π^+)

isospin symmetry can also be used to relate scattering cross-sections of one isomultiplet to another isomultiplet, among members of the isomultiplets. We show this by an example.

Consider scattering of pions (isotriplet) with nucleons (isodoublet), we restrict to 2 incident particles to 2 outgoing particles.

There are 6 elastic processes

$$\pi^{\pm}_{0}\, p \to \pi^{\pm}_{0}\, p \;; \quad \pi^{\pm}_{0}\, n \to \pi^{\pm}_{0}\, n,$$

4 charge exchange scattering

$\pi^+ n \to \pi^0 p$, $\quad \pi^0 p \longrightarrow \pi^+ n$

$\pi^0 n \longrightarrow \pi^- p$, $\quad \pi^- p \to \pi^0 n$