Tutorial 1: Solutions

1. States: Vectors and projectors

(a)

$$P_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$P_0 + P_1 = \mathbb{1}$, illustrating the completeness relation $\sum_n |n\rangle\langle n| = \mathbb{1}$ for the two-dimensional Hilbert space.

(b)

$$P_+ = |+\rangle\langle +| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_- = |-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Again, $P_+ + P_- = \mathbb{1}$, illustrating the completeness relation.

(c)

$$P_{e^{i\theta}} = e^{i\theta}|0\rangle e^{-i\theta}\langle 0| = |0\rangle\langle 0| = P_0$$

($e^{i\theta}$ is global phase; $P_{e^{i\theta}}$ is independent of θ and the global phase does not change the projector.)

$$P_\theta = |\theta\rangle\langle\theta| = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)\frac{1}{\sqrt{2}}(\langle 0| + e^{-i\theta}\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix}, \text{ depends on } \theta$$

(d) Make a measurement of spin along x. Then $|\pm\rangle$ are distinguishable and correspond to the two eigenstates of σ_x.

2. Position and momentum operators

(a)

$$\langle x|\hat{x}|\psi\rangle = x\langle x|\psi\rangle = x\psi(x)$$

We also know that, in the position representation, $\hat{p}|\psi\rangle = \int dx(-i\hbar\frac{\partial}{\partial x}\psi(x))|x\rangle$. So $\langle x|\hat{p}|\psi\rangle = \langle x|\hat{p}\psi\rangle = -i\hbar\frac{\partial}{\partial x}\psi(x) = -i\hbar\frac{\partial}{\partial x}\langle x|\psi\rangle$. We thus have $\langle x|\hat{p} = -i\hbar\frac{\partial}{\partial x}\langle x|$.

$$\langle x|\hat{p}^2|\psi\rangle = \langle x|\hat{p}\cdot\hat{p}|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\hat{p}|\psi\rangle = -i\hbar\frac{\partial}{\partial x}(-i\hbar\frac{\partial}{\partial x}\psi(x)) = -\hbar^2\frac{\partial^2}{\partial x^2}\psi(x)$$

$$\langle x|\hat{x}\hat{p}|\psi\rangle = x\langle x|\hat{p}|\psi\rangle = -i\hbar x\frac{\partial}{\partial x}\psi(x)$$

$$\langle x|\hat{p}\hat{x}|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\hat{x}|\psi\rangle = -i\hbar\frac{\partial}{\partial x}(x\psi(x)) = -i\hbar\psi(x) - i\hbar x\frac{\partial}{\partial x}\psi(x)$$

(b)

$$\langle p|\hat{x}|\psi\rangle = \langle p|\hat{x}\int dx\,|x\rangle\langle x||\psi\rangle = \int dx\,\langle p|\hat{x}|x\rangle\langle x|\psi\rangle \text{ (Here, } \mathbb{1} = \int dx\,|x\rangle\langle x|)$$

$$\langle p|\hat{x}|x\rangle = x\langle p|x\rangle = x\langle x|p\rangle^* = x(\frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar}) = \frac{d}{dp}(e^{-ipx/\hbar})\frac{ih}{\sqrt{2\pi\hbar}} = i\hbar\frac{d}{dp}\langle p|x\rangle$$

$$\langle p|\hat{x}|\psi\rangle = \int dx\,i\hbar\frac{d}{dp}\langle p|x\rangle\langle x|\psi\rangle = i\hbar\frac{d}{dp}\langle p|\int dx\,|x\rangle\langle x|\psi\rangle = i\hbar\frac{d}{dp}\langle p|\psi\rangle$$

This tells us that $\langle p|\hat{x} = i\hbar\frac{\partial}{\partial p}\langle p|$.

(c)

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$$

$$\begin{aligned}
\langle p|\hat{x}\hat{p} - \hat{p}\hat{x}|\psi\rangle &= i\hbar\frac{d}{dp}\langle p|\hat{p}|\psi\rangle - p\langle p|\hat{x}|\psi\rangle \\
&= i\hbar\frac{d}{dp}(p\psi(p)) - pi\hbar\frac{d}{dp}\psi(p) \\
&= i\hbar\psi(p) + i\hbar p\frac{d}{dp}\psi(p) - pi\hbar\frac{d}{dp}\psi(p) \\
&= i\hbar\psi(p)
\end{aligned}$$

3. Commutator relations for orbital angular momentum

$$[r_i, p_j] = i\hbar\delta_{ij}$$

To show: $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$.

$$\begin{aligned}
&\vec{L} = \vec{r}\times\vec{p} \\
&L_i = \epsilon_{ilk}r_l p_k \\
&L_j = \epsilon_{jmn}r_m p_n \quad \text{from definition of cross product.}
\end{aligned}$$

$$\begin{aligned}
[L_i, L_j] &= [\epsilon_{ilk} r_l p_k, \epsilon_{jmn} r_m p_n] \\
&= \epsilon_{ilk}\epsilon_{jmn}[r_l p_k, r_m p_n] \\
&= \epsilon_{ilk}\epsilon_{jmn}(r_l[p_k, r_m p_n] + [r_l, r_m p_n]p_k) \\
&= \epsilon_{ilk}\epsilon_{jmn}(r_l[p_k, r_m]p_n + r_m[r_l, p_n]p_k) \\
&= \epsilon_{ilk}\epsilon_{jmn}(i\hbar)(-r_l\delta_{km}p_n + r_m\delta_{ln}p_k) \\
&= (i\hbar)(\epsilon_{ilk}\epsilon_{jkn}(-r_l p_n) + \epsilon_{ilk}\epsilon_{jml} r_m p_k) \\
&= (i\hbar)(\epsilon_{kil}\epsilon_{knj}(-r_l p_n) + \epsilon_{lki}\epsilon_{ljm} r_m p_k) \\
&= (i\hbar)((\delta_{in}\delta_{lj} - \delta_{ij}\delta_{ln})(-r_l p_n) + (\delta_{kj}\delta_{im} - \delta_{km}\delta_{ij}) r_m p_k) \\
&= (i\hbar)(-r_j p_i + \delta_{ij} r_l p_l + r_i p_j - \delta_{ij} r_k p_k) \\
&= (i\hbar)(r_i p_j - r_j p_i) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
(i\hbar)\epsilon_{ijk} L_k &= (i\hbar)\epsilon_{ijk}\epsilon_{klm} r_l p_m \\
&= (i\hbar)\epsilon_{kij}\epsilon_{klm} r_l p_m \\
&= (i\hbar)(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(r_l p_m) \\
&= (i\hbar)(r_i p_j - r_j p_i) \qquad (2)
\end{aligned}$$

Comparing (1) and (2):

$$[L_i, L_j] = i\hbar\epsilon_{ijk} L_k \text{ as required.}$$