Example: Double Atwood machine

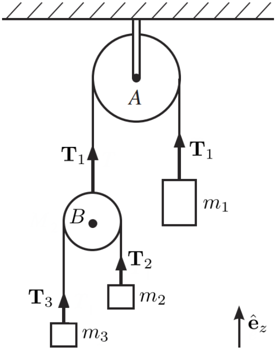
- A mass m_1 hangs at one end of a string that is led over a pulley A. The other end carries another pulley B which in tern carries a string with the masses m_2 and m_3 fixed to its ends. All pulleys and strings are assumed to be massless. Also, all strings are inextensible.

- Inextensible strings:

$$\mathbf{a}_{1A} = -\mathbf{a}_{BA}\,, \qquad \mathbf{a}_{2B} = -\mathbf{a}_{3B}$$

- Massless strings and pulleys:

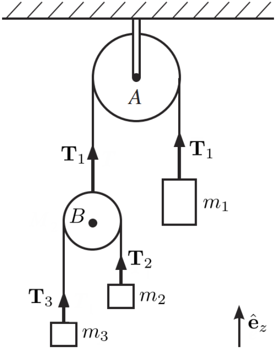
$$T_2 = T_3 = T\,, \qquad T_1 = 2T_2 = 2T_3 = 2T$$



EXERCISE 2.1: Find the acceleration of all masses.

$$\begin{cases} \mathbf{a}_{1A} = -\mathbf{a}_{BA} \\ \mathbf{a}_{2B} = -\mathbf{a}_{3B} \end{cases}$$

$$\mathbf{a}_{2B} = -\mathbf{a}_{3B} \quad \Rightarrow \quad a_2 + a_1 = -\left(a_3 + a_1\right) \quad \Rightarrow \quad a_1 = -\frac{1}{2}\left(a_2 + a_3\right) \qquad \blacksquare$$



$$\begin{cases} T_1 - m_1 g = m_1 a_1 \\ T_2 - m_2 g = m_2 a_2 \\ T_3 - m_3 g = m_3 a_3 \end{cases} \Rightarrow \begin{cases} 2T - m_1 g = -\dfrac{m_1}{2}\left(a_2 + a_3\right) \\ T - m_2 g = m_2 a_2 \\ T - m_3 g = m_3 a_3 \end{cases}$$

$$\Rightarrow \quad \begin{cases} a_2 = -\dfrac{4m_2 m_3 + m_1\left(m_2 - 3m_3\right)}{m_1\left(m_2 + m_3\right) + 4m_2 m_3}\, g \\ a_3 = -\dfrac{4m_2 m_3 + m_1\left(m_3 - 3m_2\right)}{m_1\left(m_2 + m_3\right) + 4m_2 m_3}\, g \\ T = \dfrac{4m_1 m_2 m_3}{m_1\left(m_2 + m_3\right) + 4m_2 m_3}\, g \end{cases} \qquad \blacksquare$$

$$a_1 = -\frac{1}{2}\left(a_2 + a_3\right) = \frac{4m_2 m_3 - m_1\left(m_2 + m_3\right)}{m_1\left(m_2 + m_3\right) + 4m_2 m_3}\, g \qquad \blacksquare$$

Example: Two masses on inclined plane

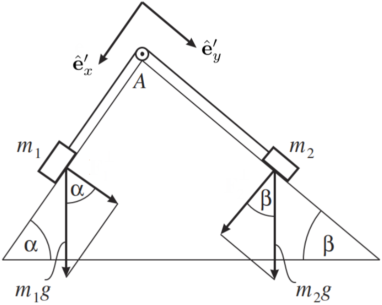
- Two masses m_1 and m_2 are lying each on one of two joined inclined planes with angles α and β with the horizontal. Both inclined planes and the horizontal make a right-angle triangle. The two masses are connected by a massless and inextensible string running over a massless and fixed pulley. The coefficients of kinetic friction of both planes are μ_k.

- Inextensible string:

$$\mathbf{a}_1 = a\,\hat{\mathbf{e}}'_x\,, \qquad \mathbf{a}_2 = -a\,\hat{\mathbf{e}}'_y$$

- Massless string and pulley:

$$\mathbf{T}_1 = -T\,\hat{\mathbf{e}}'_x\,, \qquad \mathbf{T}_2 = -T\,\hat{\mathbf{e}}'_y$$

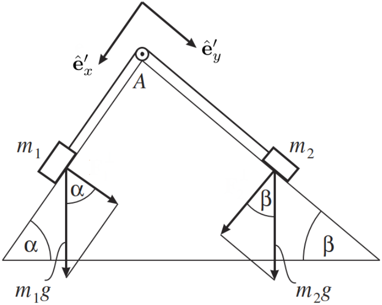


EXERCISE 2.2: Find the acceleration of the masses.

$$\mathbf{F}_1 = (m_1 g \sin\alpha - T - \mu_k N_1)\ \hat{\mathbf{e}}'_x + (m_1 g \cos\alpha - N_1)\ \hat{\mathbf{e}}'_y$$

$$\mathbf{F}_1 = m_1 \mathbf{a}_1 \quad \Rightarrow \quad \begin{cases} m_1 g \sin\alpha - T - \mu_k N_1 = m_1 a \\ m_1 g \cos\alpha - N_1 = 0 \end{cases}$$

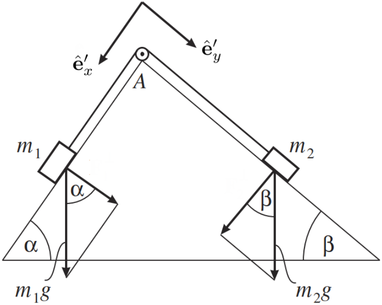
$$\Rightarrow \quad m_1 g \sin\alpha - T - \mu_k m_1 g \cos\alpha = m_1 a \qquad \blacksquare$$



$$\mathbf{F}_2 = (m_2 g\cos\beta - N_2)\ \hat{\mathbf{e}}'_x + (m_2 g\sin\beta - T + \mu_k N_2)\ \hat{\mathbf{e}}'_y$$

$$\mathbf{F}_2 = m_2\mathbf{a}_2 \quad \Rightarrow \quad \begin{cases} m_2 g\cos\beta - N_2 = 0 \\ m_2 g\sin\beta - T + \mu_k N_2 = -m_2 a \end{cases}$$

$$\Rightarrow \quad m_2 g\sin\beta - T + \mu_k m_2 g\cos\beta = -m_2 a \qquad \blacksquare$$



$$\begin{cases} m_1 g \sin\alpha - T - \mu_k m_1 g \cos\alpha = m_1 a \\ m_2 g \sin\beta - T + \mu_k m_2 g \cos\beta = -m_2 a \end{cases}$$

$$\Rightarrow \quad \begin{cases} a = \dfrac{(m_1 \sin\alpha - m_2 \sin\beta) - \mu_k\,(m_1 \cos\alpha + m_2 \cos\beta)}{m_1 + m_2}\, g \\ T = \dfrac{m_1 m_2 g}{m_1 + m_2}\left[(\sin\alpha + \sin\beta) - \mu_k\,(\cos\alpha - \cos\beta)\right] \end{cases} \qquad \blacksquare$$

$$\mu_k \to 0 \qquad \Rightarrow \qquad a \to \frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2}\, g \qquad \blacksquare$$

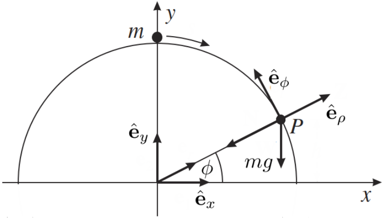
$$\alpha = \beta = \frac{\pi}{2} \qquad \Rightarrow \qquad a \to \frac{m_1 - m_2}{m_1 + m_2}\, g \qquad \blacksquare$$

Example: Particle on a hemisphere

- A particle of mass m is located at the "North pole" of a smooth hemisphere of radius R fixed on the ground. The particle slides down the hemisphere after a small kick.

- Particle is constrained to move on the hemisphere before breaking off:

$$\rho(t) = R \quad \Rightarrow \quad \begin{cases} \dot{\rho}(t) = 0 \\ \ddot{\rho}(t) = 0 \end{cases}$$



EXERCISE 2.3: Find the angle and the speed at which the particle breaks off from the hemisphere.

$$\mathbf{F}(t) = [N(t) - mg\sin\phi(t)]\;\hat{\mathbf{e}}_\rho(t) - mg\cos\phi(t)\,\hat{\mathbf{e}}_\phi(t)$$

$$\begin{cases} N(t) - mg\sin\phi(t) = -mR\,\dot{\phi}^2(t) \\ -mg\cos\phi(t) = mR\,\ddot{\phi}(t) \end{cases}$$

$$-\,mg\sin\phi(t_0) = -mR\,\dot{\phi}^2(t_0) \quad \Rightarrow \quad \dot{\phi}^2(t_0) = \frac{g}{R}\sin\phi(t_0) \qquad \blacksquare$$

$$\ddot{\phi}(t) = -\frac{g}{R}\cos\phi(t) \quad \Rightarrow \quad \frac{\mathrm{d}\dot{\phi}(\phi)}{\mathrm{d}\phi}\,\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = -\frac{g}{R}\cos\phi(t)$$

$$\Rightarrow \quad \int_{\dot{\phi}'=0}^{\dot{\phi}(t_0)} \dot{\phi}'\,\mathrm{d}\dot{\phi}' = -\frac{g}{R}\int_{\phi'=\pi/2}^{\phi(t_0)} \cos\phi'\,\mathrm{d}\phi'$$

$$\Rightarrow \quad \frac{1}{2}\,\dot{\phi}^2(t_0) = -\frac{g}{R}\,[\sin\phi(t_0) - 1] \qquad \blacksquare$$

$$\begin{cases} \dot{\phi}^2(t_0) = \dfrac{g}{R}\sin\phi(t_0) \\[2ex] \dfrac{1}{2}\,\dot{\phi}^2(t_0) = -\dfrac{g}{R}\,\left[\sin\phi(t_0) - 1\right] \end{cases}$$

$$\Rightarrow \quad \begin{cases} \sin\phi(t_0) = \dfrac{2}{3} \\[2ex] \dot{\phi}^2(t_0) = \dfrac{2g}{3R} \end{cases} \qquad \blacksquare$$

$$\phi(t_0) = \sin^{-1}\frac{2}{3} \approx 42^\circ \qquad \blacksquare$$

$$\mathbf{v}(t_0) = \dot{\rho}(t_0)\,\hat{\mathbf{e}}_\rho + \rho(t_0)\,\dot{\phi}(t_0)\,\hat{\mathbf{e}}_\phi = -\sqrt{\frac{2Rg}{3}}\,\hat{\mathbf{e}}_\phi \quad \Rightarrow \quad v(t_0) = \sqrt{\frac{2Rg}{3}} \qquad \blacksquare$$

Projectile with resistance

- Linear resistance: $\mathbf{F} = -mk\mathbf{v}$, $k \geq 0$

- Equation of motion:

$$\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = -g\,\hat{\mathbf{e}}_z - k\mathbf{v}(t)$$

- Initial conditions:

$$\mathbf{r}(0) = (x_0, y_0, z_0)\,, \qquad \mathbf{v}(0) = (0, v_0\cos\theta_0, v_0\sin\theta_0)$$

- Equation of motion in Cartesian coordinates:

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -kv_x(t)\,, \qquad \frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} = -kv_y(t)\,, \qquad \frac{\mathrm{d}^2z(t)}{\mathrm{d}t^2} = -g - kv_z(t)$$

Projectile with resistance: x-direction

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -k v_x(t)\,, \qquad x(0) = x_0\,, \qquad v_x(0) = 0$$

- Solving for $v_x(t)$:

$$\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = -k v_x(t) \qquad \Rightarrow \qquad v_x(t) = 0$$

- Solving for $x(t)$:

$$v_x(t) = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}x(t)}{\mathrm{d}t} = 0 \qquad \Rightarrow \qquad x(t) = x_0$$

- Motion along the x-direction is essentially stationary

Projectile with resistance: y-direction

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = -k v_y(t)\,, \qquad y(0) = y_0\,, \qquad v_y(0) = v_0 \cos\theta_0$$

- Solving:

$$v_y(t) = v_0 \cos\theta_0\, \mathrm{e}^{-kt}\,, \qquad y(t) = y_0 + \frac{v_0 \cos\theta_0}{k}\left(1 - \mathrm{e}^{-kt}\right)$$

- Zero-friction limit: $k \to 0$

$$v_y(t) \to v_0 \cos\theta_0\,, \qquad y(t) \to y_0 + v_0\,(\cos\theta_0)\,t$$

EXERCISE 2.4: Obtain short-time and long-time behaviours for $v_y(t)$ and $y(t)$.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = -k v_y(t)\,, \qquad y(0) = y_0\,, \qquad v_y(0) = v_0 \cos\theta_0$$

$$\frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = -k v_y(t) \quad \Rightarrow \quad \int_{v'_y = v_0 \cos\theta_0}^{v_y} \frac{\mathrm{d}v'_y}{v'_y} = -k \int_{t'=0}^{t} \mathrm{d}t$$

$$\Rightarrow \quad v_y(t) = v_0 \cos\theta_0\, \mathrm{e}^{-kt} \qquad \blacksquare$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = v_0 \cos\theta_0\, \mathrm{e}^{-kt} \quad \Rightarrow \quad \int_{y'=y_0}^{y} \mathrm{d}y' = \int_{t'=0}^{t} v_0 \cos\theta_0\, \mathrm{e}^{-kt'}\, \mathrm{d}t'$$

$$\Rightarrow \quad y(t) = y_0 + \frac{v_0 \cos\theta_0}{k} \left(1 - \mathrm{e}^{-kt}\right) \qquad \blacksquare$$

$$v_y(t) = v_0 \cos\theta_0 \, \mathrm{e}^{-kt}\,, \qquad y(t) = y_0 + \frac{v_0 \cos\theta_0}{k}\left(1 - \mathrm{e}^{-kt}\right)$$

$$t \ll \frac{1}{k} \qquad \to \qquad \begin{cases} v_y(t) \to v_0 \cos\theta_0 \,(1 - kt) \\ y(t) \to y_0 + v_0 \,(\cos\theta_0)\, t - \dfrac{1}{2} k v_0 \,(\cos\theta_0)\, t^2 \end{cases} \qquad \blacksquare$$

$$t \gg \frac{1}{k} \qquad \to \qquad \begin{cases} v_y(t) \to 0 \\ y(t) \to y_0 + \dfrac{v_0 \cos\theta_0}{k} \end{cases} \qquad \blacksquare$$

$$\frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = -k v_y(t) \quad \Rightarrow \quad \frac{\mathrm{d}v_y(y)}{\mathrm{d}y}\,\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -k v_y(t)$$

$$\Rightarrow \quad \int_{v_y' = v_0 \cos\theta_0}^{v_y} \mathrm{d}v_y' = -\int_{y'=y_0}^{y} k \,\mathrm{d}y' \quad \Rightarrow \quad v_y(y) = v_0 \cos\theta_0 - k\,(y - y_0) \qquad \blacksquare$$

Projectile with resistance: z-direction

$$\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = -g - k v_z(t)\,, \qquad z(0) = z_0\,, \qquad v_z(0) = v_0 \sin\theta_0$$

- Solving:

$$v_z(t) = \left(v_0 \sin\theta_0 + \frac{g}{k}\right) \mathrm{e}^{-kt} - \frac{g}{k}\,, \qquad z(t) = z_0 + \frac{1}{k}\left(v_0 \sin\theta_0 + \frac{g}{k}\right)\left(1 - \mathrm{e}^{-kt}\right) - \frac{gt}{k}$$

- Short-time behaviour:

$$v_z(t) \to v_0 \sin\theta_0 - (g + k v_0 \sin\theta_0)\, t\,, \quad z(t) \to z_0 + v_0 (\sin\theta_0)\, t - \frac{1}{2}(g + k v_0 \sin\theta_0)\, t^2$$

- Long-time behaviour:

$$v_z(t) \to -\frac{g}{k}\,, \qquad z(t) \to z_0 + \frac{1}{k}\left(v_0 \sin\theta_0 + \frac{g}{k}\right) - \frac{gt}{k}$$

$$\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = -g - k v_z(t)\,, \qquad z(0) = z_0\,, \qquad v_z(0) = v_0 \sin\theta_0$$

$$\frac{\mathrm{d}v_z(t)}{\mathrm{d}t} = -g - k v_z(t) \quad \Rightarrow \quad \int_{v_z' = v_0 \sin\theta_0}^{v_z} \frac{\mathrm{d}v_z'}{g + k v_z'} = -\int_{t'=0}^{t} \mathrm{d}t'$$

$$\Rightarrow \quad \frac{1}{k} \ln \frac{g + k v_z(t)}{g + k v_0 \sin\theta_0} = -t \quad \Rightarrow \quad v_z(t) = \left(v_0 \sin\theta_0 + \frac{g}{k}\right) \mathrm{e}^{-kt} - \frac{g}{k} \qquad \blacksquare$$

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \left(v_0 \sin\theta_0 + \frac{g}{k}\right) \mathrm{e}^{-kt} - \frac{g}{k}$$

$$\Rightarrow \quad \int_{z'=z_0}^{z} \mathrm{d}z' = \int_{t'=0}^{t} \left[\left(v_0 \sin\theta_0 + \frac{g}{k}\right) \mathrm{e}^{-kt'} - \frac{g}{k}\right] \mathrm{d}t'$$

$$\Rightarrow \quad z(t) = z_0 + \frac{1}{k}\left(v_0 \sin\theta_0 + \frac{g}{k}\right)\left(1 - \mathrm{e}^{-kt}\right) - \frac{gt}{k} \qquad \blacksquare$$