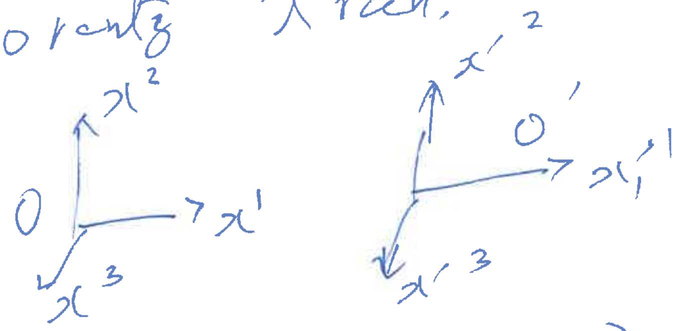
2025. 2. 4. L6 ①

Relativistic kinematics

Frames of reference

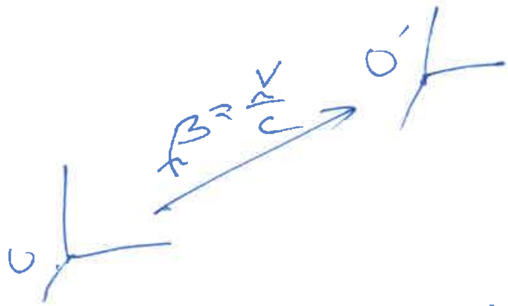
Transformation. $\begin{cases} \text{Galilean tran.} \\ \text{Lorentz tran.} \end{cases}$

special Lorentz tran.



$$x'^0 = \gamma(x^0 - \beta x^1)$$
$$x'^1 = \gamma(x^1 - \beta x^0)$$
$$x'^2 = x^2, \qquad x'^3 = x^3.$$

Generalize



Most general Lorentz tran.

In 3-dim space, distance

$$(\Delta \underset{\sim}{x} \cdot \Delta \underset{\sim}{x}) = \Delta x^i \Delta x^i$$
$$= \Delta x^1 \Delta x^1 + (\Delta x^2)^2 + (\Delta x^3)^2$$

Metric tensor $g_{\mu\nu}$

4-dim spacetime

$$(\Delta \underline{x})^2 = \Delta x_\mu \cdot \Delta x^\mu$$

$$= g_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

Minkowski geometry

$$g_{00} = +1, \quad g_{11} = g_{22} = g_{33} = -1$$

Distance between 2 points (events)

$$(\Delta s)^2 = \Delta \underline{x} \cdot \Delta \underline{x}$$

$$= g_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

$$= (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$$

$$= (\Delta x^0)^2 - \Delta x^i \Delta x^i$$

$i = 1, 2, 3.$

Most general Lorentz trans $(\Lambda, \underline{a})$ that keeps $(\Delta s)^2$ unchanged.

$\underline{a} = (a^0, a^1, a^2, a^3)$

= translation

Λ = homogeneous Lorentz tran.

$(\Lambda, \underline{a})$ = inhomogeneous

= Poincaré tran.

Properties of Λ.

$$g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

matrix rep of Λ

e.g. space inversion

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Meaning of scalar, vector, tensor.

scalar, vector, tensor must be defined

wrt transformations

In physics trans form a

group.

Define scalar, vector etc wrt

Lorentz gp. $\{ \Lambda_1, \Lambda_2, \Lambda_3, \ldots \}$

so ϕ is a scalar, if

under Λ

$$\phi \xrightarrow{\Lambda} \phi' = \phi$$

$\phi(\underline{x})$ is a scalar field if

$$\phi(x) \xrightarrow{\Lambda} \phi'(\underline{x}') = \phi(\underline{x})$$

$$\underline{x}' = \Lambda \underline{x}$$

Vector $\underset{\sim}{A}$

$$\underset{\sim}{A} \rightarrow \underset{\sim}{A}' = \Lambda \underset{\sim}{A}$$

similarly

$$\underset{\sim}{A}(\underset{\sim}{x}) \rightarrow \underset{\sim}{A}'(\underset{\sim}{x}') = \Lambda \underset{\sim}{A}(\underset{\sim}{x})$$

Tensor $\underset{\approx}{T}$

$$\underset{\approx}{T} \rightarrow \underset{\approx}{T}' = \Lambda \Lambda \underset{\approx}{T}$$

e.g. m_0 = rest mass scalar

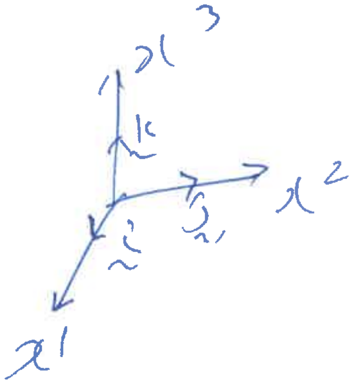
$\underset{\sim}{x} = (x^0, \underset{\sim}{x})$ 4-vector

$\underset{\sim}{P} = (P^0, \underset{\sim}{P})$ $\quad P^0 = \frac{E}{c}$

Tensor

$F_{\mu\nu}$ (electromagnetic field)

Above in abstract, in practice use frames of reference



$\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k} \to (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$

$\underset{\sim}{A}$ in a fram has component. How to write down the component (cpt) that depends on the way we construct basis

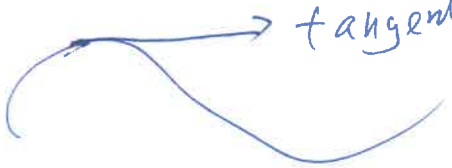
$\underset{\sim}{A}$ in terms of ~~$\{\underset{\sim}{e}_i\}$~~ $\{\underset{\sim}{e}_\mu\}$

$\mu = 0, 1, 2, 3.$

$$\underset{\sim}{A} = A^\mu \underset{\sim}{e}_\mu$$

But there are two types of bases, $\longrightarrow$ geometry

Curve



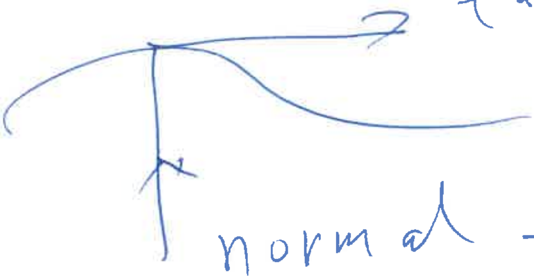
tangent (vector

Multidimensional, tangent space

Bases in tangent space → $\underset{\sim}{e}_i$; $i = 1, \ldots n$.

covariant basis

tangent



normal → $\{\underset{\sim}{E}^j\}$, $j = 1, 2 \ldots$

contravariant basis

$\underset{\sim}{e}_i \cdot \underset{\sim}{E}^j = \delta_i^j$

$$\underset{\sim}{A} = A^\mu \underset{\sim}{e}_\mu = A_\mu \underset{\sim}{E}^\mu$$

A^μ = contravariant cpt

A_μ = covariant cpt.

We have $\underset{\sim}{e}_i \cdot \underset{\sim}{E}^j = \delta_i^j$

$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = g_{ij}$ (metric tensor)

$\underset{\sim}{E}^i \cdot \underset{\sim}{E}^j = g^{ij}$

Examples.

4 - position $\underline{x} = (x^0, x^i)$

4 - ~~vecto~~ velocity

proper time $:= \frac{dt}{\gamma} \equiv d\tau$

$$ds^2 = dx^{0\,2} - dx^i dx^i$$

$$= dx^{0\,2}(1 - \beta \cdot \beta), \quad \beta^i = \frac{dx^i}{dx^0}$$

$$= \frac{1}{c}\frac{dx^i}{dt}$$

$$= \frac{1}{c} v^i$$

$$= \frac{dx^{0\,2}}{\gamma^2}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$= \frac{dt^2}{c^2\gamma^2}$$

$$\frac{dt^2}{\gamma^2} = \frac{ds^2}{c^2} \qquad \text{scalar (Lorentz)}$$

4 - velocity $\underline{w} = \frac{d\underline{x}}{d\tau}$

$$w^\mu = \frac{dx^\mu}{d\tau}.$$

$$\underline{w}^2 = \underline{w} \cdot \underline{w} = c^2 \qquad (\text{Hw})$$

4-mom. $\underset{\sim}{P} = m_0 \underset{\sim}{W}$

m_0 = rest mass = scalar.

$\underset{\sim}{P}^2 = m_0^2 c^2$ (HW)

Newton law $\underset{\sim}{F} = \frac{d\underset{\sim}{P}}{dt}$

4-force $\underset{\sim}{f} = \frac{d\underset{\sim}{P}}{d\tau}$

Today

(1) Understand the concept of scalar, vector, tensor

(2) Two types of basis. Contravariant, covariant

(3) Examples in mechanics, electrodynamics

(4) collision of particles
Lab frames, CM frames
Elastic collision, inelastic collision
Excess energy available for inelastic process

Introduce scalar, vector, tensor

A scalar is a one-component entity that remains unchanged under the Lorentz tran Λ

Let ϕ be a scalar, that means under Λ : $\underline{x} \to \underline{x}' = \Lambda \underline{x}$, we have

$$\to \quad \phi \xrightarrow{\Lambda} \phi' \equiv \Lambda \phi = \phi$$

If ϕ depends on space-time, then $\phi(\underline{x})$ is a scalar field which means

$$\phi(\underline{x}) \to \phi'(\underline{x}') = \phi(\underline{x})$$
$$\underline{x}' = \Lambda \underline{x}$$

Note: $\underline{x}^2$ is a scalar

$$\underline{x}'^2 = \underline{x}^2$$
$$\underline{x}^2 = \underline{x} \cdot \underline{x} = g_{\mu\nu} x^\mu x^\nu$$

A 4-component entity, say $\underline{A}$, is a vector if under Lorentz tran Λ,

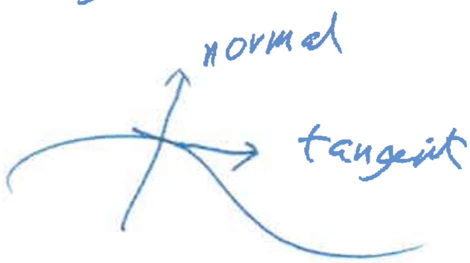
$$\underline{A} \to \underline{A}' = \Lambda \underline{A} \qquad (\underline{x}' = \Lambda \underline{x})$$

If we choose basis, can write

$$A'^\mu = (\Lambda^\mu{}_\nu) A^\nu$$

(There are two types of base)

: Define a vector by tangent to a curve



n dim

At any point of a curve, can draw tangent or normal

$\rightarrow$ 2 types of basis

In the (vector) tangent space, basis $\underset{\sim}{e}_i$

In the 'normal' space, (covector) basis $\underset{\sim}{E}^i$

$\underset{\sim}{e}_i \cdot \underset{\sim}{E}^j = \delta_i^j$

Define

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = g_{ij} \qquad i,j = 1, 2, \cdots n$$

$$\underset{\sim}{E}^i \cdot \underset{\sim}{E}^j = g^{ij}$$

Given an abstract vector $\underset{\sim}{A}$, we can use $\underset{\sim}{e}_i$ as a basis or $\underset{\sim}{E}^i$ as a basis,

$$\underset{\sim}{A} = A^i \underset{\sim}{e}_i \quad \text{or} \quad \underset{\sim}{A} = A_i \underset{\sim}{E}^i$$

To relate A^i with A_i:

$$A^i \underset{\sim}{e}_i = A_j \underset{\sim}{E}^j \qquad i,j = 1, \cdots n$$

$$A^i \underset{\sim}{e}_i \cdot \underset{\sim}{e}_l = A_j \underset{\sim}{E}^j \cdot \underset{\sim}{e}_l = A_j \delta^j_l = A_l$$

(by construction)

$$LHS = A^i g_{il}$$

$$\longrightarrow \quad A^i g_{il} = A_l$$

A^i = contravariant

A_i = covariant

symmetric

$\rightarrow \quad A_i = g_{il} A^l \qquad (g_{il} = g_{li})$

$\rightarrow \quad A_\mu = g_{\mu\nu} A^\nu, \qquad A^\mu = g^{\mu\nu} A_\nu$

Examples:

$\underline{x} = (x^0, \underline{x})$ 4-vector

Define 4-vector velocity or 4-velocity

$$\underline{W} = \frac{d\underline{x}}{d\tau} \qquad \tau = \text{proper time}$$

$$ds^2 = dx_\mu dx^\mu = g_{\mu\nu} dx^\mu dx^\nu$$

$$ds^2 = dx^{0\,2} - dx^i dx^i$$

$$= dx^{0\,2} \left(1 - \frac{dx^i}{dx^0}\frac{dx^i}{dx^0}\right) \qquad x^0 = ct$$

$$= dx^{0\,2} \left(1 - \frac{1}{c^2} v^i v^i\right) \qquad v^i \equiv \frac{dx^i}{dt}$$

$$= dx^{0\,2} (1 - \beta^2) \qquad \beta = \frac{v}{c}$$

$$\gamma^2 = \frac{1}{1-\beta^2}$$

$$= \frac{dx^{0\,2}}{\gamma^2} = \frac{c^2 dt^2}{\gamma^2}$$

$d\tau$ = proper time

$$\equiv \frac{ds}{c} = \frac{1}{\gamma} dt$$

As ds is a scalar and c is a scalar wrt Lorentz tran, so $d\tau$ is a scalar. proper time is a scalar

The 4-velocity $\underline{W} = \frac{d\underline{x}}{d\tau} = \frac{4\text{-vector}}{\text{scalar}}$

$\longrightarrow$ $\underline{W}$ is a 4-vector

$\longrightarrow$ $\underline{W}^2 \overset{\underline{W}\cdot\underline{W}}{=} W_\mu W^\mu = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}$

$= \frac{ds^2}{d\tau \cdot d\tau}$ $\quad \because ds^2 = dx_\mu dx^\mu$

$d\tau = \frac{ds}{c}$

$= c^2$

HW: $w^0 = ?$

$w^i = ?$ $\quad i=1,2,3$

So, for the 4-velocity $\underline{W}$, its magnitude squared is a constant, c^2

Define 4-momentum

$\underline{P} = m_0 \underline{W}$ $\qquad m_0$ = rest mass

m_0 is a scalar or invariant under Lorentz tran $\therefore$

$\underline{P}^2 = \underline{P}\cdot\underline{P} = P_\mu P^\mu$

$= g_{\mu\nu} P^\nu \cdot P^\mu$

$\underline{P}^2 = m_0^2 \underline{W}^2$

$\left(P^{0 2} - \underset{\sim}{P}^2 = m_0^2 c^2 \right) \longleftarrow \underline{P}^2 = m_0^2 c^2$ HW

rest mass

Define 4-force,

$\underline{f} = \frac{d\underline{P}}{d\tau} = m_0 \frac{d\underline{W}}{d\tau}$. $\quad \underset{\sim}{f} = \frac{d\underset{\sim}{P}}{d\tau} = \gamma \frac{d\underset{\sim}{P}}{dt}$

As $\underline{W}^2 = c^2$, $\therefore \frac{d\underline{W}}{d\tau}\cdot\underline{W} = 0$ i.e. $\underline{f}\cdot\underline{W} = f_\mu W^\mu = 0$

4-momentum $\underline{P} = m_0 \underline{W}$. $\quad p^0 = m_0 \frac{dx^0}{d\tau} = m_0 \gamma c = mc = \frac{E}{c}$

$$\underline{P} = (p^0, \underset{\sim}{p}) \qquad \underset{\sim}{p} = m_0 \frac{d\underset{\sim}{x}}{d\tau} = m_0 \gamma \frac{d\underset{\sim}{x}}{dt}$$

$$= \left(\frac{E}{c}, \underset{\sim}{p}\right) \qquad p^0 = \frac{E}{c} = \frac{1}{c}(m_0 \gamma c^2)$$

4-momentum of a photon $\qquad = mc$, m = relativistic mass $= \gamma \cdot m_0$

4 current $\quad \underline{j} = (j^0, \underset{\sim}{j})$

$$= (\rho c, \underset{\sim}{\hat{j}}) \qquad \rho = \text{charge density}$$

$\underset{\sim}{\hat{j}}$ = usual current density

4 - vector potential in electrodynamics

$$\underline{A} = \left(\frac{\phi}{c}, \underset{\sim}{A}\right), \qquad A^0 = \frac{\phi}{c}$$

ϕ = Electric potential

$\underset{\sim}{A}$ = magnetic vector potential

$$\underset{\sim}{E} \text{ (electric field)} = -\nabla\phi - \frac{\partial \underset{\sim}{A}}{\partial t}$$

$$\underset{\sim}{B} \text{ (magnetic field)} = \nabla \wedge \underset{\sim}{A}$$

An entity $\underline{\underline{T}}$ is a tensor if under the Lorentz tran Λ,

rank 2

$$\underline{\underline{T}} \rightarrow \underline{\underline{T}}' = \Lambda \Lambda \underline{\underline{T}}$$

In component form

Contravariant $$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} T^{\alpha\beta}$$

Covariant $$T'_{\mu\nu} = \Lambda_{\mu}{}^{\alpha} \Lambda_{\nu}{}^{\beta} T_{\alpha\beta}$$

mixed $$T'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda_{\nu}{}^{\beta} T^{\alpha}{}_{\beta}$$

Example

Electromagnetic field tensor

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

$$= \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}}$$

$\frac{\partial}{\partial x^{\mu}}$ is covariant vector

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$\frac{\partial}{\partial x_{\mu}}$ is contravariant vector

$\underset{\sim}{A}$ = 4-vector potential $= \left(\frac{\phi}{c}, \underset{\sim}{A}\right)$

e.g

$$F^{i0} = \partial^i A^0 - \partial^0 A^i$$

$$= \frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x_0} \qquad x^i = -x_i$$

$$= -\frac{\partial A^0}{\partial x^i} - \frac{1}{c}\frac{\partial A^i}{\partial t} \qquad x^0 = x_0$$

$$= \frac{1}{c}\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A^i}{\partial t}\right) \qquad \because A^0 = \frac{\phi}{c}$$

But $\underset{\sim}{E}$ (electric field) $= -\nabla\phi - \frac{\partial \underset{\sim}{A}}{\partial t}$

i.e. $E^i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A^i}{\partial t}$, $\boxed{(\nabla\phi)^i \equiv \frac{\partial \phi}{\partial x^i}}$

definition

i.e. $\underline{F^{i0} = \frac{E^i}{c}}$

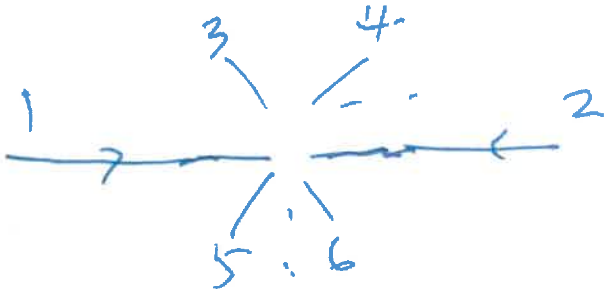
Can show $B^i = -\frac{1}{2}\varepsilon^{ijk} F_{jk}$

(HW)

$\underset{\sim}{B} = \nabla \wedge \underset{\sim}{A}$ $\qquad \wedge$ = cross product

$$\varepsilon_{ilm}\,\varepsilon_{ipq} = \delta_{lp}\,\delta_{mq} - \delta_{lq}\,\delta_{mp}$$

Consider collision of 2 particles



Frames of reference

Lab frame: A lab frame of particle 1 is the inertial frame at which particle 1 is at rest
particle 1 = target, particle 2 = projectile.

CM frame:
centre of mass frame:
Define centre of mass $\underset{\sim}{X}_G$

$$\underset{\sim}{X}_G = \frac{\sum_{i=1}^{n} m_i \underset{\sim}{x}_i}{\sum_{j=1}^{n} m_j} \qquad \sum_{i=1}^{n} m_i = M$$

Velocity of centre of mass

$$\dot{\underset{\sim}{X}}_G = \frac{\sum_{i=1}^{n} m_i \dot{\underset{\sim}{x}}_i}{\sum_j m_j}$$

A centre of mass frame is a frame at which the centre of mass is at rest i.e.

$$\dot{\underset{\sim}{X}}_G = 0$$

In relativistic collisions, centre of mass frame not useful ∵ (1) The total rest mass needs not be conserved. (2) photon has no rest mass

In relativistic collisions, one uses centre of momentum frame. A CM (centre of momentum) is a frame of reference in which the sum total of spatial momenta is zero i.e.

$$\sum_{i=1}^{n} \underset{\sim}{p}_i = 0 \quad \text{particle } i$$

(assume total n particles involved)

Consider

$$x'^0 = \gamma(x^0 - \beta x^1)$$

$$x'^1 = \gamma(x^1 - \beta x^0),$$

$$x'^2 = x^2, \qquad x'^3 = x^3$$

So for the 4 momentum

$$p_i'^0 = \gamma(p_i^0 - \beta p_i^1) \qquad i = 1, 2, \cdots n$$

$$p_i'^1 = \gamma(p_i^1 - \beta p_i^0) \qquad n \text{ particles}$$

$$p_i'^2 = p_i^2, \qquad p_i'^3 = p_i^3$$

To get CM frame:

$$\sum_i p'^1_i = \gamma\left(\sum_i p^1_i - \sum_i \beta p^0_i\right)$$

In CM frame $\sum_i p'^1_i = 0$

$$\rightarrow \beta = \frac{\sum_i p^1_i}{\sum_i p^0_i}$$

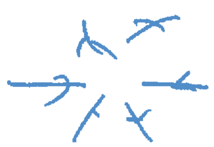
so if O' has a speed β wrt O, then O' is a CM frame because in O' frame, total spatial momentum $= 0$

Elastic and inelastic collisions

In any collision if the initial total KE (kinetic energy $T = E - m_0 c^2$) is same find total KE, then collision is elastic

Inelastic if initial total KE $\neq$ final total KE

Inelastic collison: Explosive collision
sticky collision



Final KE > initial KE
Explosive

Final KE < initial KE
sticky

Consider 2 examples.

1. What is the excess energy available for inelastic process?

Consider two incident particles. How much energy of these 2 particles can be used to produce other particles

To answer this, use CM frame.

The excess energy ξ

rest mass rest mass

$$= E_1 + E_2 - m_1 c^2 - m_2 c^2 = T_1 + T_2$$

E_1 = energy of particle 1 T_i = KE of particle i

In this expression, ξ is not invariant apparently.

To make ξ invariant, we rewrite it as

$$\xi = (p_1^0 + p_2^0)c - m_1 c^2 - m_2 c^2, \quad p^0 = \frac{E}{c}$$

$$\underset{\text{frame}}{\overset{\text{CM}}{=}} \sqrt{(\underline{p}_1 + \underline{p}_2)^2 c^2} - (m_1 + m_2) c^2$$

so $\xi = c\sqrt{(\underline{p}_1 + \underline{p}_2)^2} - (m_1 + m_2)c^2$ is an invariant definition of excess energy

- Example: what is the threshold energy (minimum excess energy) for the following process

$$p + p \to p + p + p + \bar{p}$$

i.e. threshold energy to produce an antiproton?

Ans this in CM frame and lab frame

In CM frame, answer is obvious

$$\xi = 2 m_p c^2$$

(HW)

rest mass

$m_p = m_{\bar{p}}$
= mass of proton
= mass of antiproton

Now do in the lab frame of a proton:

$$\xi = c\sqrt{(\underset{\sim}{P}_1 + \underset{\sim}{P}_2)^2} - 2 m_p c^2$$

∴ rest frame of proton 2

$$\xi = c\sqrt{(P_1^0 + P_2^0)^2 - (\underset{\sim}{P}_1 + \underset{\sim}{P}_2)^2} - 2 m_p c^2$$

$$= c\sqrt{(P_1^0 + P_2^0)^2 - \underset{\sim}{P}_1^2} - 2 m_p c^2 \qquad \because \underset{\sim}{P}_2 = 0$$

$$(\xi + 2 m_p c^2)^2 = c^2\left[(P_1^0 + P_2^0)^2 - \underset{\sim}{P}_1^2\right]$$

$$= c^2\left[P_1^{0\,2} + P_2^{0\,2} + 2P_1^0 P_2^0 - \underset{\sim}{P}_1^2\right] = c^2\left[m_p^2 c^2 + P_2^{0\,2} + 2P_1^0 P_2^0\right.$$