Tutorial 3: Solutions

1. State transformations

$$\begin{aligned} e^{-iH't'/\hbar} &= e^{-i(H_0\otimes\mathbf{1}+\mathbf{1}\otimes H_0)t'/\hbar} \\ &= e^{-i(H_0t'/\hbar\otimes\mathbf{1}+\mathbf{1}\otimes H_0t'/\hbar)} \\ &= e^{i(K\otimes\mathbf{1}+\mathbf{1}\otimes K)} \end{aligned}$$

Using the given identity, we have

$$\begin{aligned} e^{i(K\otimes\mathbf{1}+\mathbf{1}\otimes K)}|\psi\rangle &= \big(e^{iK}\otimes e^{iK}\big)\big(\alpha|x;+\rangle\otimes|x;+\rangle+\beta|x;-\rangle\otimes|x;-\rangle\big) \\ &= \alpha e^{iK}|x;+\rangle\otimes e^{iK}|x;+\rangle+\beta e^{iK}|x;-\rangle\otimes e^{iK}|x;-\rangle \end{aligned}$$

To satisfy $e^{-iH't'/\hbar}|\psi\rangle = |\psi'\rangle$, we need $e^{iK}|x;+\rangle = |+\rangle$ and $e^{iK}|x;-\rangle = |-\rangle$.

The transformation from $|x;+\rangle$ to $|+\rangle$ is a rotation by $\pi/2$ in the anticlockwise direction, about the rotation axis $-\hat{\vec{y}}$. Similarly, the transformation from $|x;-\rangle$ to $|-\rangle$ is also a rotation by $\pi/2$ in the anticlockwise direction, about the rotation axis $-\hat{\vec{y}}$. So from what we learnt about angular momentum as a generator of rotations, the rotation operator can be written as $\exp\big(-\frac{i(\pi/2)(-S_y)}{\hbar}\big)$, where $-S_y = -\vec{S}\cdot\hat{\vec{y}}$.

(Comment: Here, we use the angular momentum $\vec{J} = \vec{S}$, because we have a spin 1/2 particle, and we commonly use the notation S for spin 1/2 particles. But you could have used J in the notation above as well.)

Therefore

$$K = \frac{\pi}{2\hbar}S_y = \frac{\pi}{4}\sigma_y$$

2. Conservation of total angular momentum in the hydrogen atom

$$H = H_0 + U \ , \quad H_0 = -\frac{h^2\nabla^2}{2m_e} - \frac{e^2}{r}$$

$$[H_0, \vec{L}^2] = [H_0, L_z] = 0$$

(a)

$$
\begin{aligned}
[H, L_z] &= [H_0 + U, L_z] = [U, L_z] \\
[\vec{S}\cdot\vec{L}, L_z] &= \vec{S}\cdot[\vec{L}, L_z] + \underbrace{\cancel{[\vec{S}, L_z]}}_{0}\cdot\vec{L} \\
&= S_x[L_x, L_z] + S_y[L_y, L_z] \\
&= -S_x(i\hbar L_y) + S_y(i\hbar L_x) \\
&\neq 0
\end{aligned}
$$

So $[U, L_z] \neq 0 \Rightarrow [H, L_z] \neq 0$. Note that $[\vec{S}, L_z] = 0$ because $\vec{S}$ and L_z act in different vector spaces.

(b)

$$
\begin{aligned}
[H, \vec{L}^2] = [H_0 + U, \vec{L}^2] &= [H_0, \vec{L}^2] + [U, \vec{L}^2] \\
&= [U, \vec{L}^2] \\
[\vec{S}\cdot\vec{L}, \vec{L}^2] = \vec{S}\cdot[\vec{L}, \vec{L}^2] &+ [\vec{S}, \vec{L}^2]\cdot\vec{L} \\
= 0 &
\end{aligned}
$$

because $[\vec{L}, \vec{L}^2] = 0$ by property of angular momentum, and $[\vec{S}, \vec{L}^2] = 0$ since $\vec{S}$ and $\vec{L}^2$ are operators in different vector spaces.

$$
\begin{aligned}
&\Rightarrow [U, \vec{L}^2] = 0 \\
&\Rightarrow [H, \vec{L}^2] = 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
\vec{J} &= \vec{S} + \vec{L} \\
\vec{J}^2 &= (\vec{S} + \vec{L})^2 = \vec{S}^2 + \vec{L}^2 + 2\vec{S}\cdot\vec{L} \quad (\text{since } [\vec{S}, \vec{L}] = 0)
\end{aligned}
$$

$$
[H, \vec{J}^2] = [H_0 + U, \vec{J}^2] = [H_0, \vec{J}^2] + [U, \vec{J}^2]
$$

$$
\begin{aligned}
[H_0, \vec{L}^2] &= [H_0, \vec{S}^2] = 0 \\
[H_0, \vec{S}\cdot\vec{L}] &= [H_0, \vec{S}]\cdot\vec{L} + \vec{S}\cdot[H_0, \vec{L}] \\
&= 0 \text{ since } [H_0, \vec{S}] = [H_0, \vec{L}] = 0
\end{aligned}
$$

So $[H_0, \vec{J}^2] = 0$

$$
\begin{aligned}
[\vec{S}\cdot\vec{L},\vec{J}^2] &= [\vec{S}\cdot\vec{L},\vec{S}^2] + [\vec{S}\cdot\vec{L},\vec{L}^2] + \cancelto{0}{2[\vec{S}\cdot\vec{L},\vec{S}\cdot\vec{L}]} \\
&= \vec{S}\cdot[\vec{L},\vec{S}^2] + [\vec{S},\vec{S}^2]\cdot\vec{L} \\
&\quad + \vec{S}\cdot[\vec{L},\vec{L}^2] + [\vec{S},\vec{L}^2]\cdot\vec{L} \\
&= 0
\end{aligned}
$$

since $\{\vec{L},\vec{L}^2\}$ and $\{\vec{S},\vec{S}^2\}$ act in different vector spaces and $[\vec{L},\vec{L}^2] = [\vec{S},\vec{S}^2] = 0$ by property of angular momentum.

$$
\begin{aligned}
&\Rightarrow [U,\vec{J}^2] = 0 \\
&\Rightarrow [H,\vec{J}^2] = 0
\end{aligned}
$$

(d)

$$
\begin{aligned}
[J_z,\vec{S}\cdot\vec{L}] &= [L_z + S_z, S_xL_x + S_yL_y + S_zL_z] \\
&= [L_z + S_z, S_xL_x + S_yL_y] \text{ since } [L_z,S_z] = 0
\end{aligned}
$$

$$
\begin{aligned}
[L_z, S_xL_x] &= [L_z,S_x]L_x + S_x[L_z,L_x] \\
&= 0 + S_x(i\hbar\epsilon_{zxy}L_y) \\
&= i\hbar S_xL_y && (1) \\
\text{Similarly, } [L_z, S_yL_y] &= S_y[L_z,L_y] \\
&= -i\hbar S_yL_x && (2) \\
[S_z, S_xL_x] &= [S_z,S_x]L_x \\
&= i\hbar S_yL_x && (3) \\
[S_z, S_yL_y] &= [S_z,S_y]L_y \\
&= -i\hbar S_xL_y && (4)
\end{aligned}
$$

(1) + (2) + (3) + (4) gives

$$
\begin{aligned}
[J_z,\vec{S}\cdot\vec{L}] &= i\hbar S_xL_y - i\hbar S_yL_x + i\hbar S_yL_x - i\hbar S_xL_y \\
&= 0
\end{aligned}
$$

3. Addition of angular momenta in the hydrogen atom

(a)

$$\vec{J} = \vec{L} + \vec{S}$$
$$l = 1\ ,\ s = \frac{1}{2}$$
$$\text{Then } j^{max} = l + s = \frac{3}{2}$$
$$j^{min} = |l - s| = \frac{1}{2}$$

and possible values of j are $\frac{1}{2}$ and $\frac{3}{2}$.
[Check: $l = 1$ has $2l + 1 = 3$ possible $|l, m_z\rangle$ states
$s = \frac{1}{2}$ has $2s + 1 = 2$ possible $|s, m_s\rangle$ states.
Total number of states $= 3 \times 2 = 6$.
$(2 \cdot \frac{1}{2} + 1) + (2 \cdot \frac{3}{2} + 1) = 2 + 4 = 6$]

(b)

$$l = 1,\ m_l = -1, 0, 1$$
$$s = \frac{1}{2},\ m_s = -\frac{1}{2}, \frac{1}{2}$$

$$|j = \frac{3}{2}, m = \frac{3}{2}\rangle = |l = 1, m_l = 1\rangle \otimes |s = \frac{1}{2}, m_s = \frac{1}{2}\rangle$$
$$\equiv |1, 1\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle \tag{5}$$
$$\text{Also } |j = \frac{3}{2}, m = -\frac{3}{2}\rangle = |1, -1\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle \tag{6}$$

Apply J_- to the LHS and RHS of (5):
Apply J_- to LHS of (5):

$$J_-|j = \frac{3}{2}, m = \frac{3}{2}\rangle = \hbar\sqrt{j(j+1) - m(m-1)}|j = \frac{3}{2}, m = \frac{1}{2}\rangle$$
$$= \hbar\sqrt{\frac{3}{2}(\frac{5}{2}) - \frac{3}{2}(\frac{1}{2})}|j = \frac{3}{2}, m = \frac{1}{2}\rangle$$
$$= \hbar\sqrt{3}|j = \frac{3}{2}, m = \frac{1}{2}\rangle \tag{7}$$

Apply J_- to RHS of (5):

$$
\begin{aligned}
J_-|1,1\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle &= (L_- + S_-)(|1,1\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle) \\
&= (L_-|1,1\rangle) \otimes |\frac{1}{2},\frac{1}{2}\rangle + |1,1\rangle \otimes (S_-|\frac{1}{2},\frac{1}{2}\rangle) \\
&= (\hbar\sqrt{1(2)-1(0)}|1,0\rangle) \otimes |\frac{1}{2},\frac{1}{2}\rangle + |1,1\rangle \otimes \hbar\sqrt{\frac{1}{2}\big(\frac{3}{2}\big) - \frac{1}{2}\big(-\frac{1}{2}\big)}|\frac{1}{2},-\frac{1}{2}\rangle \\
&= \hbar\sqrt{2}|1,0\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle + \hbar|1,1\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle \qquad (8)
\end{aligned}
$$

Comparing (7) and (8),

$$
|j=\frac{3}{2},m=\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1,0\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle + \frac{1}{\sqrt{3}}|1,1\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle \qquad (9)
$$

Apply J_+ to the LHS and RHS of (6):

Apply J_+ to LHS of (6):

$$
\begin{aligned}
J_+|j=\frac{3}{2},m=-\frac{3}{2}\rangle &= \hbar\sqrt{\frac{3}{2}\big(\frac{5}{2}\big) - \big(-\frac{3}{2}\big)\big(-\frac{1}{2}\big)}|j=\frac{3}{2},m=-\frac{1}{2}\rangle \\
&= \hbar\sqrt{3}|j=\frac{3}{2},m=-\frac{1}{2}\rangle \qquad (10)
\end{aligned}
$$

Apply J_+ to RHS of (6):

$$
\begin{aligned}
J_+(|1,-1\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle) &= \hbar\sqrt{1(2)-(-1)(0)}|1,0\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle \\
&\quad + |1,-1\rangle \otimes \hbar\sqrt{\frac{1}{2}\big(\frac{3}{2}\big) - \big(-\frac{1}{2}\big)\big(\frac{1}{2}\big)}|\frac{1}{2},\frac{1}{2}\rangle \\
&= \hbar\sqrt{2}|1,0\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle + \hbar|1,-1\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \qquad (11)
\end{aligned}
$$

Comparing (10) and (11):

$$
|j=\frac{3}{2},m=-\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1,0\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle + \frac{1}{\sqrt{3}}|1,-1\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \qquad (12)
$$

$|j=\frac{1}{2},m=\frac{1}{2}\rangle$ is orthonormal to $|j=\frac{3}{2},m=\frac{1}{2}\rangle$.
So from (9),

$$
|j=\frac{1}{2},m=\frac{1}{2}\rangle = -\frac{1}{\sqrt{3}}|1,0\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|1,1\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle
$$

Similarly, $|j=\frac{1}{2},m=-\frac{1}{2}\rangle$ is orthonormal to $|j=\frac{3}{2},m=-\frac{1}{2}\rangle$.
So from (12),

$$
|j=\frac{1}{2},m=-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|1,0\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|1,-1\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle
$$