PC3261
Classical Mechanics II

Worksheet 2

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Exercise 1.2. Show that the acceleration of a particle moving along a trajectory $\mathbf{r}(t)$ is given by

$$\mathbf{a}(t) = \frac{dv(t)}{dt}\hat{\mathbf{e}}_{\mathrm{T}} + \frac{v^2(t)}{\rho}\hat{\mathbf{e}}_{\mathrm{N}}, \tag{1}$$

where $\rho \equiv \frac{1}{\kappa}$ is its radius of curvature.

Solution: *Proof.* Given that the velocity vector $\mathbf{v}$ of the particle can be expressed in TNB basis as,

$$\mathbf{v}(s) = v(s)\hat{\mathbf{e}}_{\mathrm{T}}, \tag{2}$$

where given that $s : t \mapsto s(t)$,

$$\begin{aligned}\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2} = \frac{d\mathbf{v}(t)}{dt} &= \frac{dv(t)}{dt}\hat{\mathbf{e}}_{\mathrm{T}} + v(t)\frac{d\hat{\mathbf{e}}_{\mathrm{T}}}{dt} = \frac{dv(t)}{dt}\hat{\mathbf{e}}_{\mathrm{T}} + v(t)\left[\frac{d\hat{\mathbf{e}}_{\mathrm{T}}}{ds}\frac{ds(t)}{dt}\right] \\ &= \frac{dv(t)}{dt}\hat{\mathbf{e}}_{\mathrm{T}} + v(t)\kappa(s)\hat{\mathbf{e}}_{\mathrm{N}}v(t) \\ &= \frac{dv(t)}{dt}\hat{\mathbf{e}}_{\mathrm{T}} + \frac{v^2(t)}{\rho}\hat{\mathbf{e}}_{\mathrm{N}}.\end{aligned} \tag{3}$$

□

Exercise 1.3. Find the tangent, normal and binormal vectors, as well as, curvature and torsion for the circular helix.

Solution: Starting with the position vector of a moving particle with a trajectory of a circular helix,

$$\mathbf{r} = a\cos\omega t\hat{\mathbf{e}}_x + a\sin\omega t\hat{\mathbf{e}}_y + b\omega t\hat{\mathbf{e}}_z. \tag{4}$$

From the definition of the velocity vector as the rate of change of the position vector w.r.t. time,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a\omega\sin\omega t\hat{\mathbf{e}}_x + a\omega\cos\omega t\hat{\mathbf{e}}_y + b\omega\hat{\mathbf{e}}_z. \tag{5}$$

From which, we can obtain the trajectory arc length w.r.t. $t = 0$,

$$s = \int_0^t |\mathbf{v}|dt' = \int_0^t \left|-a\omega\sin\omega t' + a\omega\cos\omega t' + b\omega\right|dt' = \omega t\sqrt{a^2+b^2}, \tag{6}$$

and,

$$\frac{ds}{dt} = \omega\sqrt{a^2+b^2}. \tag{7}$$

Given the definition of the tangent vector,

$$\begin{aligned}\hat{\mathbf{e}}_{\mathrm{T}} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} &= (-a\omega\sin\omega t\hat{\mathbf{e}}_x + a\omega\cos\omega t\hat{\mathbf{e}}_y + b\omega\hat{\mathbf{e}}_z)\left(\frac{1}{\omega\sqrt{a^2+b^2}}\right) \\ &= \frac{1}{\sqrt{a^2+b^2}}(-a\sin\omega t\hat{\mathbf{e}}_x + a\cos\omega t\hat{\mathbf{e}}_y + b\hat{\mathbf{e}}_z),\end{aligned} \tag{8}$$

and,

$$\frac{d\hat{\mathbf{e}}_{\mathrm{T}}}{dt} = \frac{1}{\sqrt{a^2+b^2}}(-a\omega\cos\omega t\hat{\mathbf{e}}_x - a\omega\sin\omega t\hat{\mathbf{e}}_y) = -\frac{a\omega}{\sqrt{a^2+b^2}}(\cos\omega t\hat{\mathbf{e}}_x + \sin\omega t\hat{\mathbf{e}}_y) \tag{9}$$

$$\begin{aligned}\implies \quad \frac{d\hat{\mathbf{e}}_{\mathrm{T}}}{ds} = \frac{d\hat{\mathbf{e}}_{\mathrm{T}}}{dt}\frac{dt}{ds} &= -\frac{a\omega}{\sqrt{a^2+b^2}}(\cos\omega t\hat{\mathbf{e}}_x + \sin\omega t\hat{\mathbf{e}}_y)\left(\frac{1}{\omega\sqrt{a^2+b^2}}\right) \\ &= -\frac{a}{a^2+b^2}(\cos\omega t\hat{\mathbf{e}}_x + \sin\omega t\hat{\mathbf{e}}_y).\end{aligned} \tag{10}$$

Given the definition of the normal vector,

$$\hat{\mathbf{e}}_{\mathrm{N}} \equiv \underbrace{\left|\frac{1}{\frac{d\hat{\mathbf{e}}_{\mathrm{T}}}{ds}}\right|}_{1/\kappa}\frac{d\hat{\mathbf{e}}_{\mathrm{T}}}{ds}. \tag{11}$$

$$\begin{aligned}\therefore \quad \kappa = \left|\frac{d\hat{\mathbf{e}}_{\mathrm{T}}}{ds}\right| &= \frac{a}{a^2+b^2}\left(\cos^2\omega t + \sin^2\omega t\right) \\ &= \frac{a}{a^2+b^2}.\end{aligned} \tag{12}$$

Hence,

$$\begin{aligned}\hat{\mathbf{e}}_{\mathrm{N}} = \frac{1}{\kappa}\frac{d\hat{\mathbf{e}}_{\mathrm{T}}}{ds} &= \frac{a^2+b^2}{a}\left[-\frac{a}{a^2+b^2}(\cos\omega t\hat{\mathbf{e}}_x + \sin\omega t\hat{\mathbf{e}}_y)\right] \\ &= -(\cos\omega t\hat{\mathbf{e}}_x + \sin\omega t\hat{\mathbf{e}}_y),\end{aligned} \tag{13}$$

and,

$$\begin{aligned}\frac{d\hat{\mathbf{e}}_{\mathrm{N}}}{ds} = \frac{d\hat{\mathbf{e}}_{\mathrm{N}}}{dt}\frac{dt}{ds} &= -(-\omega\sin\omega t\hat{\mathbf{e}}_x + \omega\cos\omega t\hat{\mathbf{e}}_y)\left(\frac{1}{\omega\sqrt{a^2+b^2}}\right) \\ &= \frac{1}{\sqrt{a^2+b^2}}(\sin\omega t\hat{\mathbf{e}}_x - \cos\omega t\hat{\mathbf{e}}_y).\end{aligned} \tag{14}$$

Given the definition of the binormal vector as orthonormal to the tangent and normal vectors,

$$
\begin{aligned}
\hat{\mathbf{e}}_{\mathrm{B}} \equiv \hat{\mathbf{e}}_{\mathrm{T}} \times \hat{\mathbf{e}}_{\mathrm{N}} &= \left[\frac{1}{\sqrt{a^2+b^2}}(-a\sin\omega t\hat{\mathbf{e}}_x + a\cos\omega t\hat{\mathbf{e}}_y) + b\hat{\mathbf{e}}_z\right] \times \left[-(\cos\omega t\hat{\mathbf{e}}_x + \sin\omega t\hat{\mathbf{e}}_y)\right] \\
&= -\frac{1}{\sqrt{a^2+b^2}}\begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ -a\sin\omega t & a\cos\omega t & b \\ \cos\omega t & \sin\omega t & 0 \end{vmatrix} \\
&= -\frac{1}{\sqrt{a^2+b^2}}\left[-b\sin\omega t\hat{\mathbf{e}}_x - (-b\cos\omega t\hat{\mathbf{e}}_y) + \left(-a\sin^2\omega t - a\cos^2\omega t\right)\hat{\mathbf{e}}_z\right] \\
&= \frac{1}{\sqrt{a^2+b^2}}(b\sin\omega t\hat{\mathbf{e}}_x - b\cos\omega t\hat{\mathbf{e}}_y + a\hat{\mathbf{e}}_z),
\end{aligned} \tag{15}
$$

and,

$$
\frac{d\hat{\mathbf{e}}_{\mathrm{B}}}{ds} \equiv -\tau\hat{\mathbf{e}}_{\mathrm{N}}. \tag{16}
$$

Since the set of basis vectors $\{\hat{\mathbf{e}}_{\mathrm{T}},\ \hat{\mathbf{e}}_{\mathrm{N}},\ \hat{\mathbf{e}}_{\mathrm{B}}\}$ are mutually orthonormal,

$$
\hat{\mathbf{e}}_{\mathrm{N}} \cdot \hat{\mathbf{e}}_{\mathrm{B}} = 0, \tag{17}
$$

and thus,

$$
\hat{\mathbf{e}}_{\mathrm{N}} \cdot \underbrace{\frac{d\hat{\mathbf{e}}_{\mathrm{B}}}{ds}}_{-\tau\hat{\mathbf{e}}_{\mathrm{N}}} + \frac{d\hat{\mathbf{e}}_{\mathrm{N}}}{ds} \cdot \hat{\mathbf{e}}_{\mathrm{B}} = 0 \tag{18}
$$

$$
\begin{aligned}
\implies \qquad \tau &= -\frac{d\hat{\mathbf{e}}_{\mathrm{N}}}{ds} \cdot \hat{\mathbf{e}}_{\mathrm{B}} \\
&= -\left[-\frac{1}{\sqrt{a^2+b^2}}(\sin\omega t\hat{\mathbf{e}}_x - \cos\omega t\hat{\mathbf{e}}_y)\right] \cdot \left[\frac{1}{\sqrt{a^2+b^2}}(b\sin\omega t\hat{\mathbf{e}}_x - b\cos\omega t\hat{\mathbf{e}}_y + a\hat{\mathbf{e}}_z)\right] \\
&= \frac{1}{a^2+b^2}\left(b\sin^2\omega t + b\cos^2\omega t\right) \\
&= \frac{b}{a^2+b^2}.
\end{aligned} \tag{19}
$$

Exercise 1.4. Establish the relationship between unit basis vectors $(\hat{\mathbf{e}}_\rho,\ \hat{\mathbf{e}}_\phi)$ of the polar coordinate system and the unit basis vectors $(\hat{\mathbf{e}}_x,\ \hat{\mathbf{e}}_y)$ of the Cartesian coordinate system.

Solution: Geometrically,

$$
\begin{cases} \hat{\mathbf{e}}_\rho = \cos\phi\hat{\mathbf{e}}_x + \sin\phi\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_\phi = -\sin\phi\hat{\mathbf{e}}_x + \cos\phi\hat{\mathbf{e}}_y \end{cases}. \tag{20}
$$

This transformation can be cast into a transformation matrix $\boldsymbol{R}$ as,

$$
\boldsymbol{R} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{21}
$$

Since this transformation matrix is a rotation matrix,

$$\boldsymbol{R}^{-1} = \boldsymbol{R}^{\intercal} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{22}$$

$$\implies \quad \begin{pmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \end{pmatrix} = \boldsymbol{R}^{-1} \begin{pmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi\hat{\mathbf{e}}_\rho - \sin\phi\hat{\mathbf{e}}_\phi \\ \sin\phi\hat{\mathbf{e}}_\rho + \cos\phi\hat{\mathbf{e}}_\phi \end{pmatrix}, \tag{23}$$

and,

$$\begin{cases} \hat{\mathbf{e}}_x = \cos\phi\hat{\mathbf{e}}_\rho - \sin\phi\hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_y = \sin\phi\hat{\mathbf{e}}_\rho + \cos\phi\hat{\mathbf{e}}_\phi \end{cases}. \tag{24}$$

Exercise 1.5. Express the velocity and acceleration vectors in 2D polar coordinates.

Solution:

Exercise 1.6. Express the spherical unit basis vectors $(\hat{\mathbf{e}}_r,\ \hat{\mathbf{e}}_\rho,\ \hat{\mathbf{e}}_\phi)$ in terms of Cartesian unit basis vectors $(\hat{\mathbf{e}}_x,\ \hat{\mathbf{e}}_y,\ \hat{\mathbf{e}}_z)$.

Solution: