

Magnetostatics

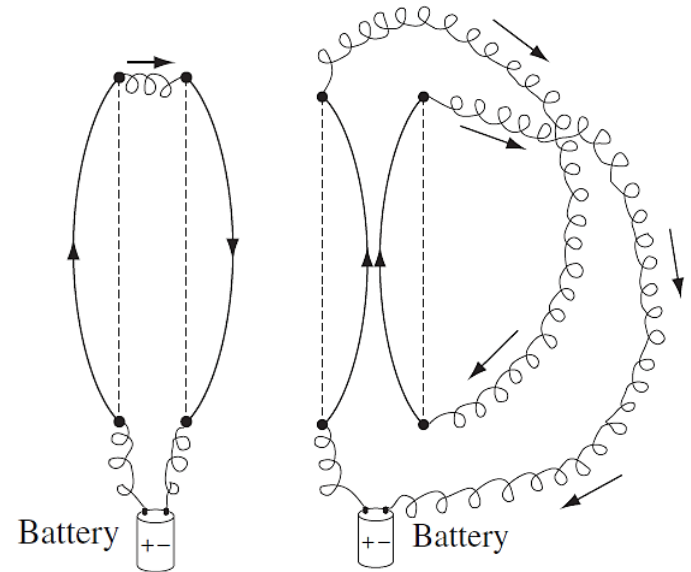
A toroidal coil of copper wire, which is a donut-shaped inductor. The wire is wound in a dense, uniform helix around a central core. Two thin, straight metal leads extend from the top of the coil, one pointing towards the upper right and the other towards the lower right. The coil is set against a background that is a gradient of yellow and orange, with a dark horizontal band across the middle where the text is located.

The Lorenz force law

Electrostatics: Stationary charges with constant electric fields

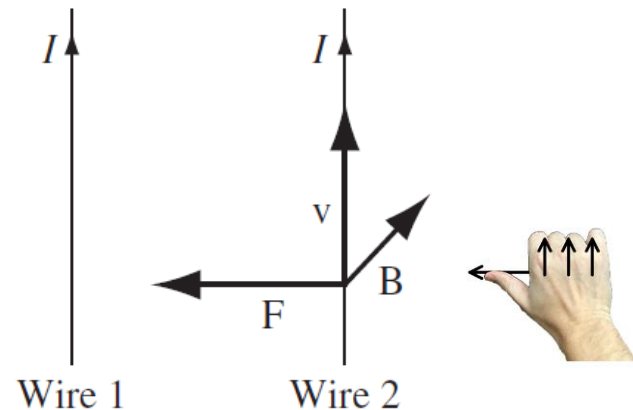
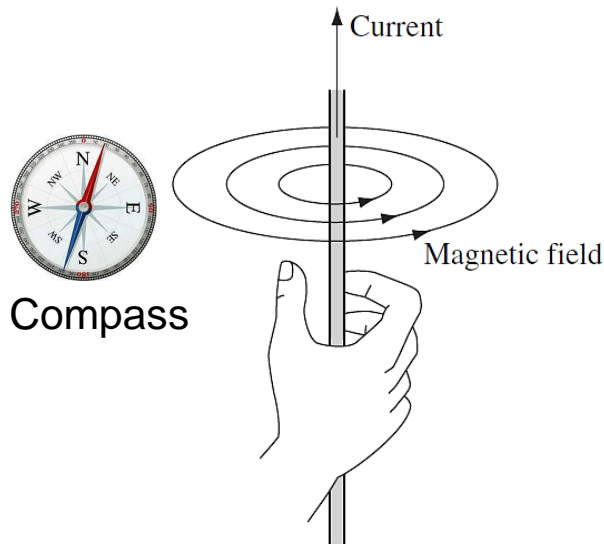
Magnetostatics: Steady currents with constant magnetic fields

- Magnetic force and magnetic field
 - Pass current in parallel wires
 - Wires repel with antiparallel current
 - Wires attract with parallel current
 - Force is not electrostatic in nature (wires are charge neutral)
 - Magnetic force



The Lorentz force law

- Magnetic force and magnetic field
 - Empirical rules for magnetic field and magnetic force
 - Straight current-carrying wire has magnetic field circling around it
 - Right-hand rule for field direction given current direction
 - Right-hand rule for force direction given current and field direction



The Lorentz force law

- The Lorentz force law
 - Magnetic force on charge Q , moving with velocity \mathbf{v} , in magnetic field \mathbf{B}

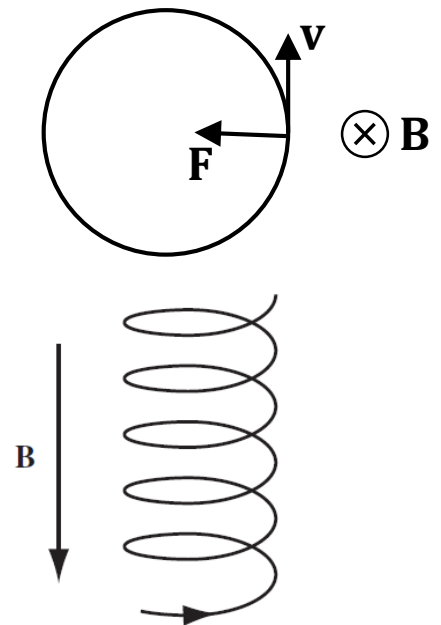
$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

- Given as an axiom without proof
 - With both \mathbf{E} and \mathbf{B} , the full Lorentz force law $\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$

- Consequences

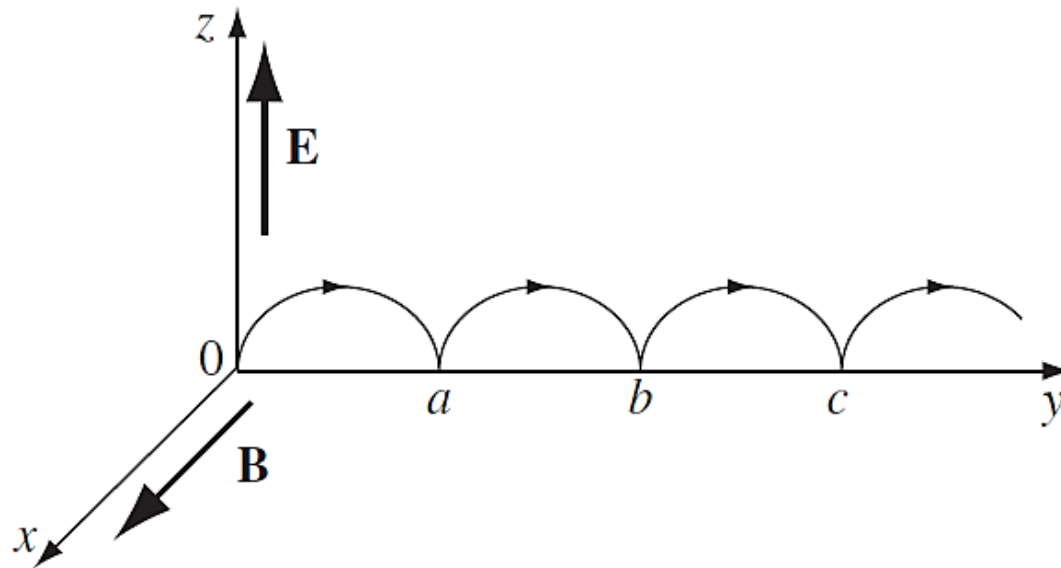
- Circular charge motion, or alike, when magnetic force acts as centripetal force
 - Magnetic forces do no work, \mathbf{B} only deflects particle direction

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \overbrace{\mathbf{v} dt}^{d\mathbf{l}} = 0$$



The Lorentz force law

Example 5.2. Cycloid Motion. A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that \mathbf{B} points in the x -direction, and \mathbf{E} in the z -direction, as shown in Fig. 5.7. A positive charge is released from the origin; what path will it follow?



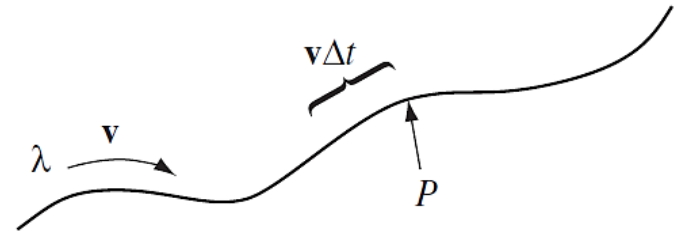
C3.exmp5.2

Currents

- Current in a wire: charge per time passing a given point

- Unit: Amperes (A) = Coulomb/second

- $\mathbf{I} = \lambda \mathbf{v}$ where λ : line charge density,
 \mathbf{v} : velocity of movement



- Positive charges moving at \mathbf{v} = negative charges moving at $-\mathbf{v}$
- Not meaningful to talk about current if it's just a single point charge moving (non-steady)

- In many problems just write the magnitude $I = \lambda v$
 - Because direction is determined by the shape of wire

- Magnetic force on a current-carrying wire $\mathbf{F}_{\text{mag}} = \int I (d\mathbf{l} \times \mathbf{B})$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \lambda dl = \int (\mathbf{I} \times \mathbf{B}) dl = \int I (d\mathbf{l} \times \mathbf{B})$$

Currents

- Surface and volume distributions of current

- Surface current density $\mathbf{K} = \sigma \mathbf{v}$ (C/m² · m/s) (σ : surface charge density)

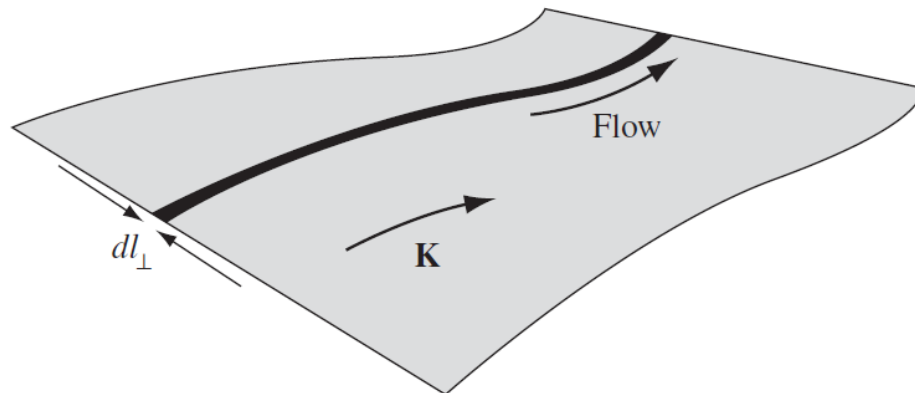
- \mathbf{K} : current per unit width (A/m)

$$d\mathbf{I} = \sigma \mathbf{v} dl_{\perp} \quad \Rightarrow \quad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}} \quad (l_{\perp}: \text{cross-sectional line segment perpendicular to } \mathbf{v})$$

- Magnetic force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \sigma da = \int (\mathbf{K} \times \mathbf{B}) da$$

* \mathbf{B} experiences discontinuity across a current-carrying surface



Currents

- Surface and volume distributions of current

- Volume current density $\mathbf{J} = \rho \mathbf{v}$ (C/m³ · m/s) (ρ : volume charge density)

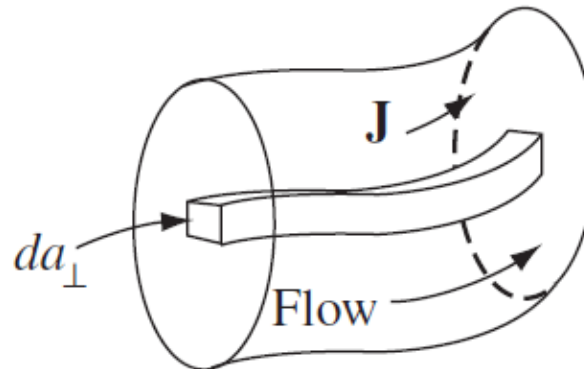
- \mathbf{J} : current per unit area (A/m²)

$$\begin{aligned} d\mathbf{I} &= \rho \mathbf{v} da_{\perp} \\ d\mathbf{I} &= \mathbf{J} \cdot d\mathbf{a} = \rho \mathbf{v} \cdot d\mathbf{a} \end{aligned} \quad \Rightarrow \quad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}} \quad (a_{\perp}: \text{cross-sectional area segment perpendicular to } \mathbf{v})$$

- Magnetic force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \rho d\tau = \int (\mathbf{J} \times \mathbf{B}) d\tau$$

- Ohm's law $\mathbf{J} = \sigma_c \mathbf{E}$ (σ_c : electrical conductivity)



Currents

- Surface and volume distributions of current

- Continuity equation for volume current density

- Current crossing a surface \mathcal{S} : $I = \int_{\mathcal{S}} J da_{\perp} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$
- Current crossing boundary of a volume \mathcal{V} :

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau$$

which must equal to the change of net charge in the volume

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau = \overbrace{-\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau}^{\text{change of net charge}} = - \int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} \right) d\tau$$

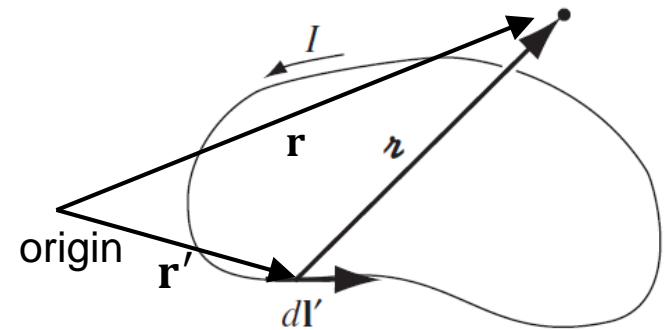
➡ $\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$ for magnetostatics $\frac{\partial \rho}{\partial t} = 0 \quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0} \quad \nabla \cdot \mathbf{J} = 0$

The Biot-Savart law

- Magnetic field generated by a steady current

- Line current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathbf{r}}}{r^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\mathbf{r}}}{r^2}$

- Unit of \mathbf{B} : Tesla (T) = N/(A·m)
- Permeability $\mu_0 = 4\pi \times 10^{-7}$ N/A²
- Separation vector $\mathbf{r} = \mathbf{r} - \mathbf{r}'$



- Surface current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} da'$

- Volume current $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} d\tau'$

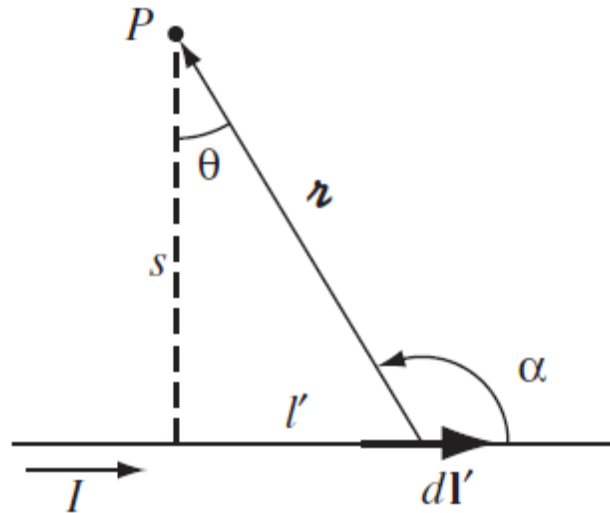
- Expressions for surface and volume currents not proved
- The law cannot be used to calculate field generated by discrete moving charges

* \mathbf{r} in textbook is typed as \mathbf{r}

The Biot-Savart law

- Application of Biot-Savart law

Example 5.5. Find the magnetic field a distance s from a long straight wire carrying a steady current I (Fig. 5.18).



$$\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$



C3.exmp5.5

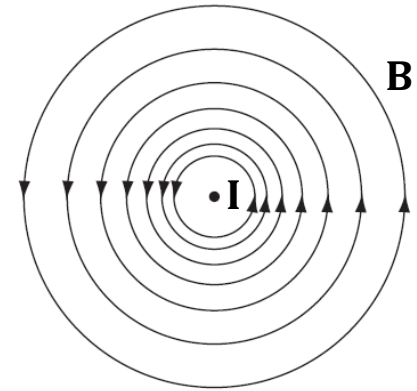
Curl of magnetic field

- Derivation of curl from Stokes theorem
 - Loop integral of \mathbf{B} around a straight-line current

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I$$

- Result still holds if path is noncircular

$$d\mathbf{l} = ds \hat{s} + \underline{s d\phi \hat{\phi}} + dz \hat{z} \quad \mathbf{B} = \underline{\frac{\mu_0 I}{2\pi s} \hat{\phi}}$$



- Loop integral of \mathbf{B} around any current distribution

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad \left(I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a} \text{ total current enclosed by the path} \right)$$

- Apply Stokes theorem

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

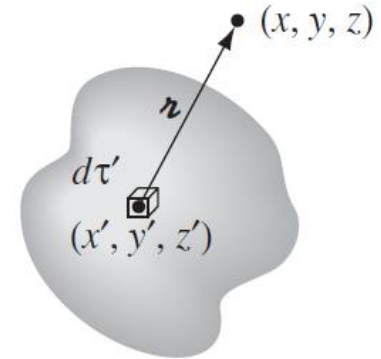
$$\Rightarrow \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

Curl of magnetic field

- Derivation of curl from Biot-Savart law
 - A much more formal derivation than previous slide

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau'$$

$\mathbf{B}(\mathbf{r})$ $\nabla_{\mathbf{r}}$ $\mathbf{J}(\mathbf{r}')$ $\mathbf{r} = \mathbf{r} - \mathbf{r}'$ $dx' dy' dz'$



\downarrow Product rule $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) - \underbrace{(\mathbf{J} \cdot \nabla) \frac{\hat{\mathbf{r}}}{r^2}}$$

$\downarrow \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r}),$ and second term integrates to 0 (textbook p.232)

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta^3(\mathbf{r} - \mathbf{r}') d\tau' = \mu_0 \mathbf{J}(\mathbf{r})$$

Divergence of magnetic field

- Derivation of divergence from Biot-Savart law

$$\circ \quad \nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau'$$

↓ Product rule $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) = \frac{\hat{\mathbf{r}}}{r^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\mathbf{r}}}{r^2} \right)$$

↓ $\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') = 0$, and $\nabla \times \frac{\hat{\mathbf{r}}}{r^2} = 0$ (Remember $\nabla \times \mathbf{E} = 0$)

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

Magnetic fields are divergence-free (no magnetic “free charge”)

Magnetic
monopoles?
not reproduced

First Results from a Superconductive Detector for Moving Magnetic Monopoles

Blas Cabrera
Phys. Rev. Lett. **48**, 1378 – Published 17 May 1982

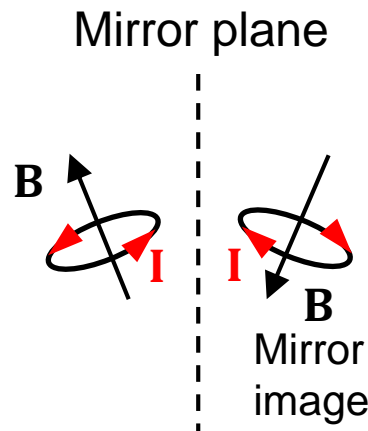
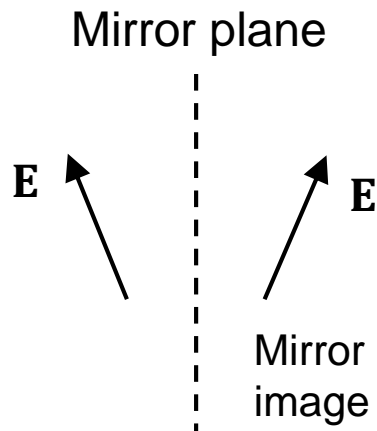
Ampère's law

Electrostatics	Magnetostatics
Coulomb's law	Biot-Savart law
Gauss's law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \quad \oint \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}}/\epsilon_0$	Ampère's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$
Gaussian surface	Amperian loop

- Use of Ampère's law to calculate magnetic field
 - Ampère's law in integral form
 - Symmetry arguments
 - Translation symmetry
 - Rotational symmetry
 - Mirror symmetry
 - Inversion symmetry
- Be careful to use when transforming \mathbf{B} :
 need to flip sign
- \mathbf{B} flips sign if \mathbf{I} flips sign (time-reversal symmetry)

Symmetry of polar vectors and axial vectors

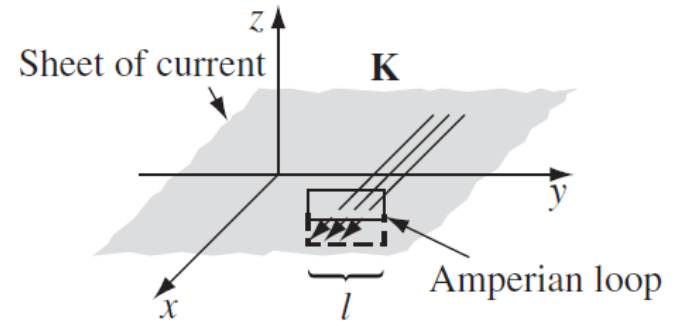
- Polar vectors and axial vectors
 - Polar vectors: \mathbf{E} , \mathbf{P} , \mathbf{D} , \mathbf{J} , \mathbf{I}
 - Usually vectors associated with electric fields
 - Axial vectors: \mathbf{B} , \mathbf{M} , \mathbf{H}
 - Usually vectors associated with magnetic fields
- Mirror operation



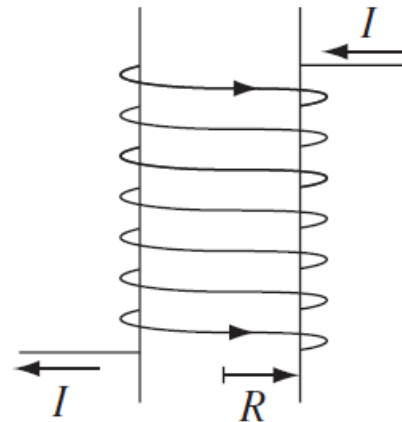
- Polar vectors: every operation as usual
- Axial vectors: **Additional sign flip for mirror and inversion operations**, other operations as usual

Ampère's law

Example 5.8. Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K \hat{\mathbf{x}}$, flowing over the xy plane (Fig. 5.33).



Example 5.9. Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R , each carrying a steady current I (Fig. 5.34).



Ampère's law

Example 5.10. A toroidal coil consists of a circular ring, or “donut,” around which a long wire is wrapped (Fig. 5.38). The winding is uniform and tight enough so that each turn can be considered a plane closed loop.

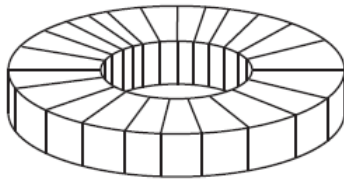
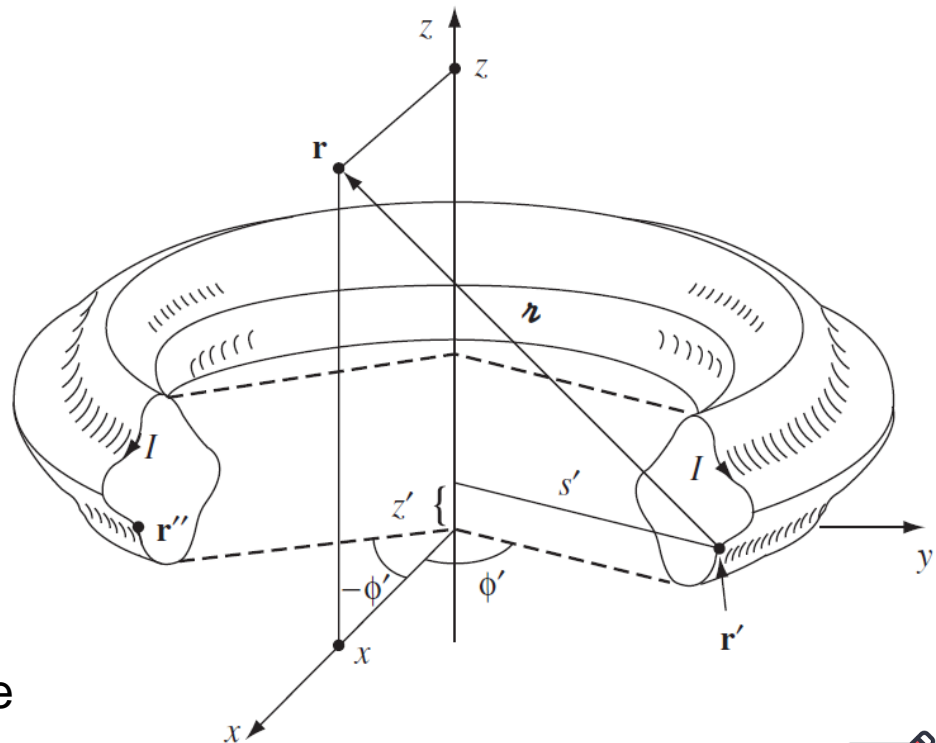


FIGURE 5.38

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \frac{\mu_0 N I}{2\pi s} \hat{\phi}, & \text{inside} \\ \mathbf{0}, & \text{outside} \end{cases}$$



Toroidal structure with any cross-sectional shape



C3.exmp5.10

Magnetic vector potential

- Vector potential \mathbf{A}

- $\mathbf{B} = \nabla \times \mathbf{A}$

- Exploiting the property $\nabla \cdot \mathbf{B} = 0$ (divergence of curl vanishes)

- $\nabla \cdot \mathbf{A} = 0$

- Chosen to be so (like choosing reference point for electric potential)

If original \mathbf{A}_0 is not, can define $\mathbf{A} = \mathbf{A}_0 + \nabla\lambda$ without varying \mathbf{B}

➡ $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2\lambda$

Can choose $\nabla^2\lambda = -\nabla \cdot \mathbf{A}_0$

- $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$

- Use the choice above, and apply Ampère's law

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\cancel{\nabla \cdot \mathbf{A}}^0) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

Magnetic vector potential

- Calculating vector potential from current

- $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ Poisson's equation, if $\mathbf{J} \rightarrow 0$ at infinity,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau' \quad (\mathbf{A} \text{ is a polar vector})$$

- Analogous to solution $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} d\tau'$ to $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

- Line current

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{r} dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{r} d\mathbf{l}'$$

- Surface current

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{r} da'$$

- Not so much simplification (vector \mathbf{A} representing vector \mathbf{B}) but equation is easier to use than the Biot-Savart law
 - No need to integrate unit vector $\hat{\mathbf{r}}$

$$* \nabla^2 \mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$$

Magnetic vector potential

- Calculating vector potential from magnetic field

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$$

where Φ is the magnetic flux

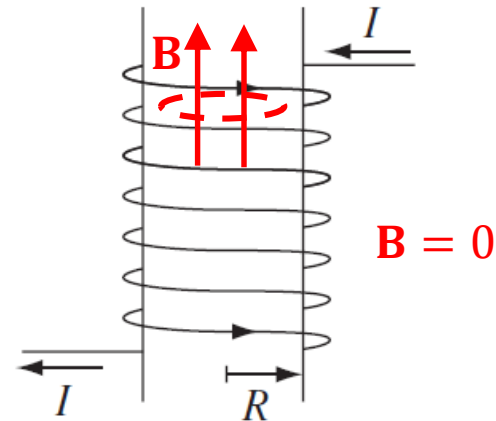
- Can calculate \mathbf{A} using equation above plus symmetry argument
- Generally \mathbf{A} mimics the direction of current

Example 5.12. Find the vector potential of an infinite solenoid with n turns per unit length, radius R , and current I .

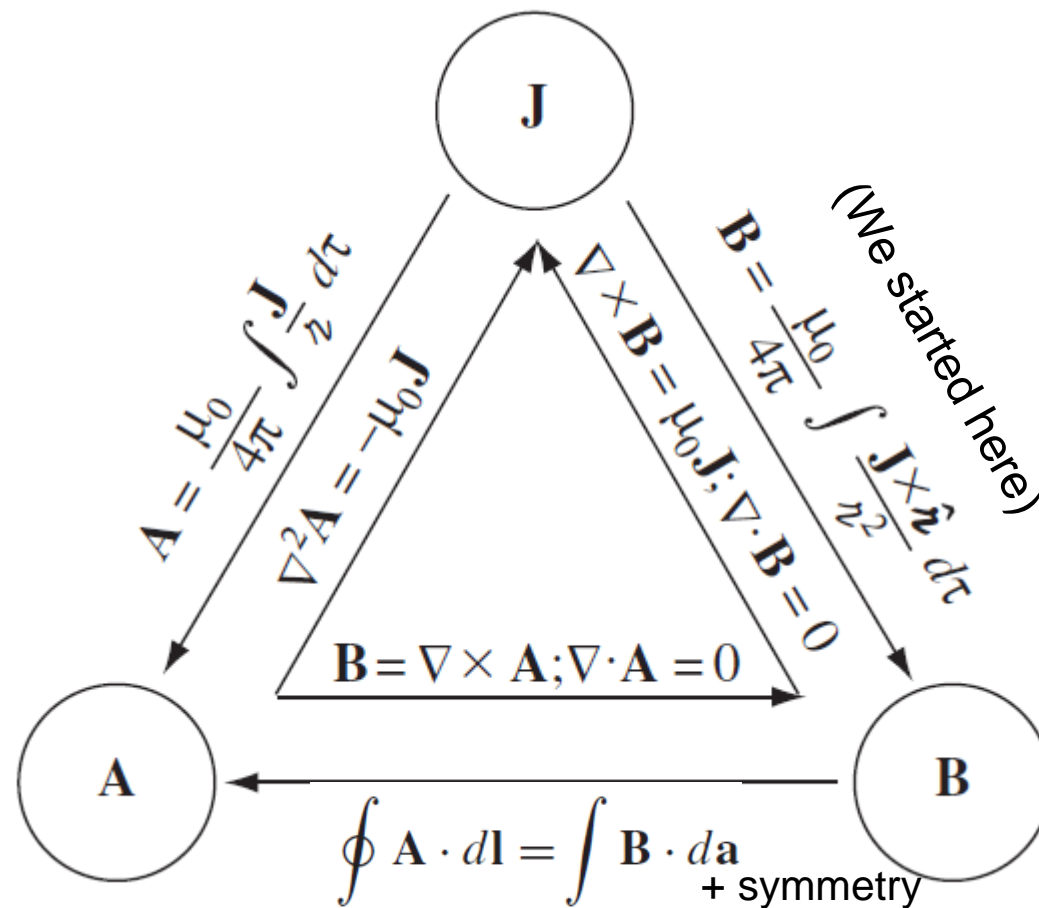
- Check $\nabla \times \mathbf{A} = \mathbf{B}$
- Check $\nabla \cdot \mathbf{A} = 0$



C3.exmp5.9&5.12



Current, magnetic field, and vector potential



Differential equations need boundary conditions to solve

Boundary conditions

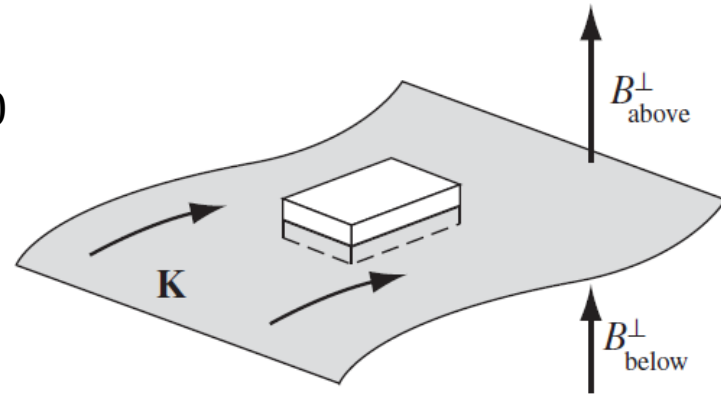
- Boundary conditions of \mathbf{B} across a 2D current surface

- Normal component of \mathbf{B}

Thin pillbox with thickness $\varepsilon \rightarrow 0$

$$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\Rightarrow B_{\text{above}}^{\perp} = B_{\text{below}}^{\perp}$$

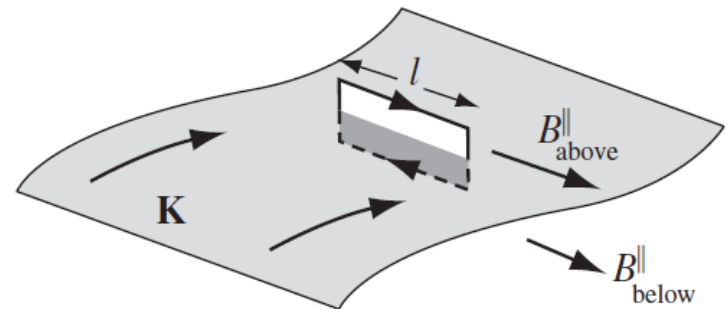


- Tangential component of \mathbf{B} that is perpendicular to current

Thin Amperian loop

$$\begin{aligned} \oint \mathbf{B} \cdot d\mathbf{l} &= (B_{\text{above}}^{\parallel} - B_{\text{below}}^{\parallel}) l \\ &= \mu_0 I_{\text{enc}} = \mu_0 K l \end{aligned}$$

$$\Rightarrow B_{\text{above}}^{\parallel} - B_{\text{below}}^{\parallel} = \mu_0 K$$



- Summarizing above $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$

Boundary conditions

- Boundary conditions of \mathbf{A} across a 2D current surface

- Vector potential \mathbf{A} is always continuous $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$

- Normal component of \mathbf{A}

$\nabla \cdot \mathbf{A} = 0$ and select a thin pillbox

➡ $A_{\text{above}}^{\perp} = A_{\text{below}}^{\perp}$

- Tangential component of \mathbf{A}

$\nabla \times \mathbf{A} = \mathbf{B}$ and select a thin loop to integrate

$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$ tends to zero when loop is thin

➡ $A_{\text{above}}^{\parallel} = A_{\text{below}}^{\parallel}$

- Derivative of \mathbf{A} is discontinuous $\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$



Multipole expansion of vector potential

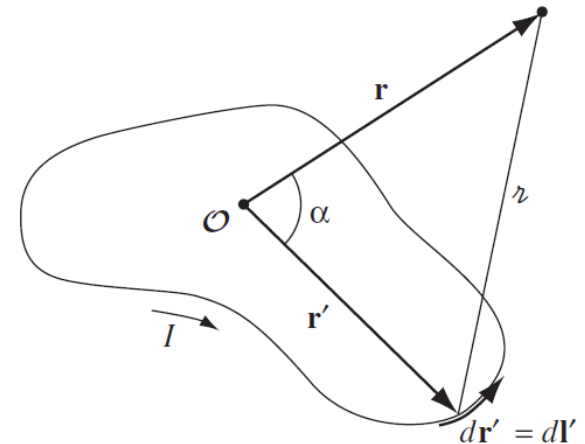
- Goal: to expand \mathbf{A} in power series of $1/r$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{r} d\mathbf{l}'$$

$$\downarrow \quad \frac{1}{r} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr' \cos \alpha}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \alpha)$$

(From electrostatic multipole expansion)

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos \alpha) d\mathbf{l}' \\ &= \frac{\mu_0 I}{4\pi} \left[\underbrace{\frac{1}{r} \oint d\mathbf{l}'}_{\text{Monopole}} + \underbrace{\frac{1}{r^2} \oint r' \cos \alpha d\mathbf{l}'}_{\text{Dipole}} \right. \\ &\quad \left. + \underbrace{\frac{1}{r^3} \oint (r')^2 \left(\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) d\mathbf{l}' + \dots}_{\text{Quadrupole}} \right] \end{aligned}$$

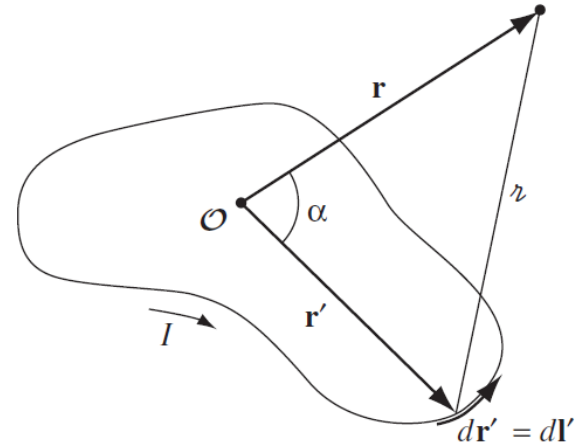


Multipole expansion of vector potential

- Magnetic monopole and magnetic dipole

- Magnetic monopole

- Always zero, because $\oint d\mathbf{l}' = \mathbf{0}$
- Also because $\nabla \cdot \mathbf{B} = 0$



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos \alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'$$



$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'$$



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

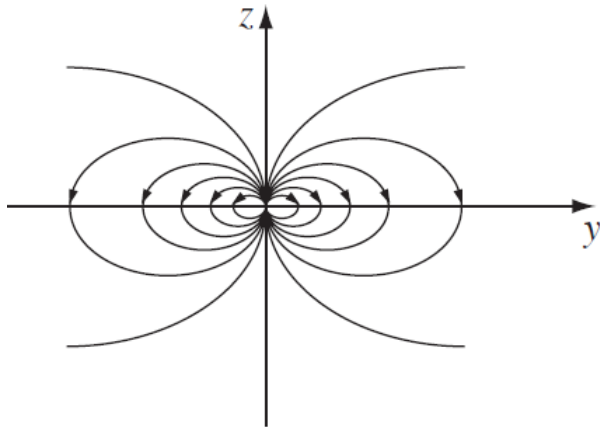
Magnetic dipole moment

$$\mathbf{m} \equiv I \int d\mathbf{a} = I \mathbf{a}$$

Multipole expansion of vector potential

- Pure dipole vs physical dipole

Pure dipole



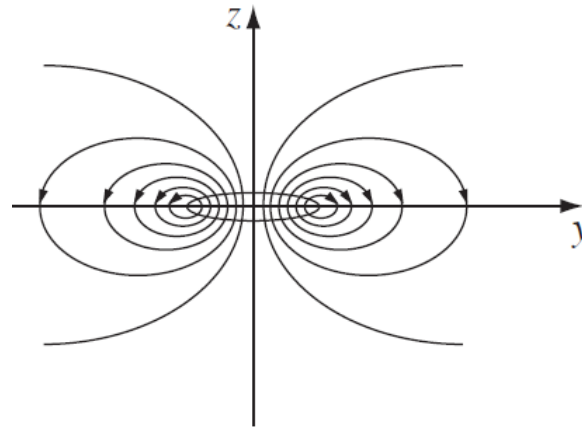
$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

Assumes $r \gg$ loop radius

$\mathbf{m} = I\mathbf{a}$ but take $I \rightarrow \infty$, $a \rightarrow 0$

Similar to pure electric dipole

Physical dipole

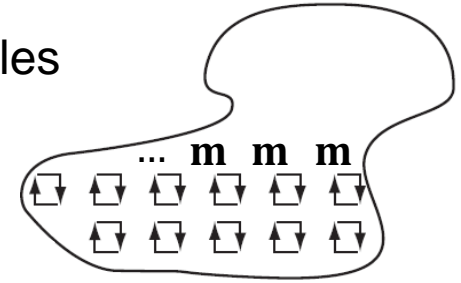


Deviations appear when closing up onto the dipole

Magnetic fields in matter

Magnetic materials

- Practically, all materials have magnetic response
 - Orbiting electrons around nuclei as magnetic dipoles
 - Electron spins as magnetic dipoles
 - Quantum object without a classical analog



- Magnetization: $\mathbf{M} = \frac{1}{V} \sum_i \mathbf{m}_i$
 - Magnetic dipole moment per unit volume
- Three types of response of matter to magnetic field
 - Paramagnets: magnetization \mathbf{M} parallel to applied \mathbf{B}
 - Diamagnets: magnetization \mathbf{M} opposite to applied \mathbf{B}
 - Ferromagnets: finite magnetization \mathbf{M} even without \mathbf{B}

Magnetic dipoles responding to field

- Paramagnetic response

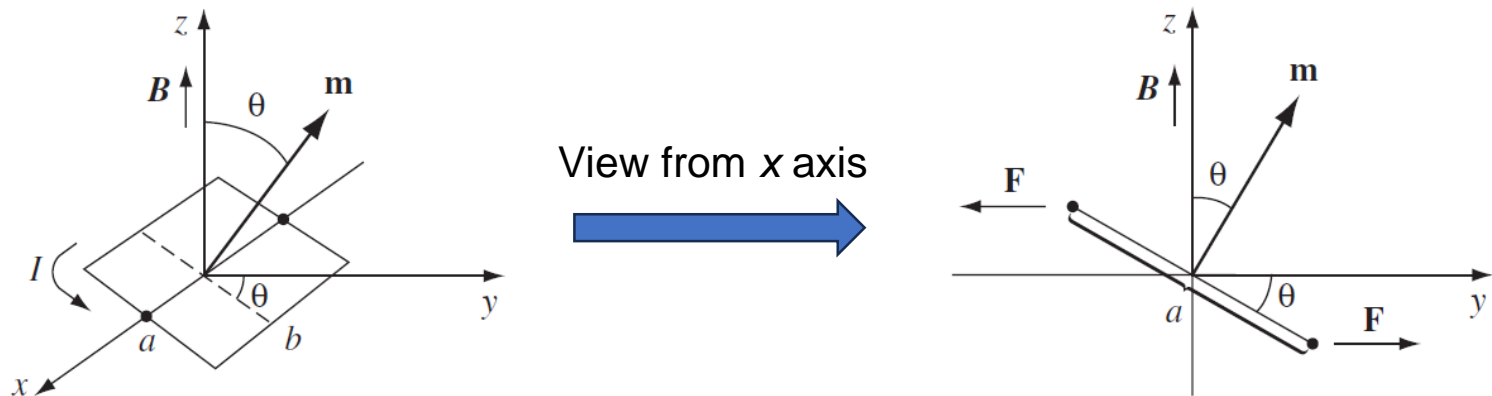
- Torque in a uniform field

- Suppose a rectangle current loop with side lengths a and b

$$\mathbf{N} = aF \sin \theta \hat{\mathbf{x}} \xrightarrow{\text{force } F = IbB} \mathbf{N} = IabB \sin \theta \hat{\mathbf{x}} = mB \sin \theta \hat{\mathbf{x}}$$

➡ $\mathbf{N} = \mathbf{m} \times \mathbf{B}$

- Torque tends to align \mathbf{m} parallel to \mathbf{B}

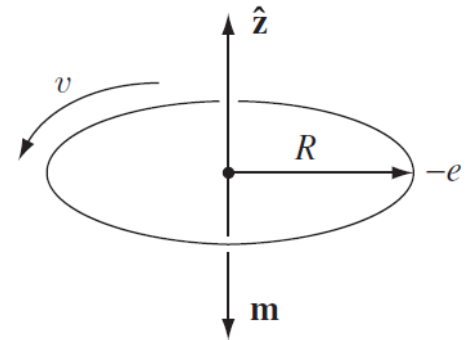


Magnetic dipoles responding to field

- Diamagnetic response

- Modification to an atomic orbit

$$I = \frac{-e}{T} = -\frac{ev}{2\pi R} \xrightarrow[\text{Orbit period}]{T = 2\pi R/v} \mathbf{m} = -\frac{1}{2}evR\hat{\mathbf{z}}$$



- Velocity without and with \mathbf{B} , balancing Coulomb and centripetal forces

without \mathbf{B} : $\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} = m_e \frac{v^2}{R}$

with \mathbf{B} : $\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} + e\bar{v}B = m_e \frac{\bar{v}^2}{R}$

v : electron velocity
 \bar{v} : electron velocity with \mathbf{B}
 m_e : electron mass

➡ $e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v} + v)(\bar{v} - v)$

➡ $\Delta v = \bar{v} - v = \frac{eRB}{2m_e}$ for small Δv

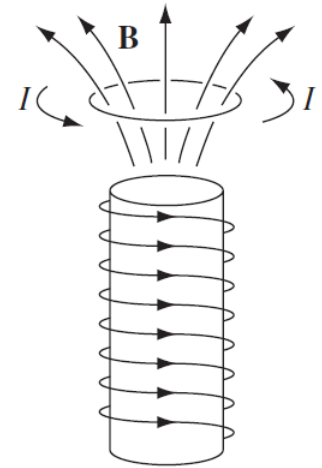
➡ $\Delta \mathbf{m} = -\frac{1}{2}e(\Delta v)R\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}$ Change in dipole moment works against \mathbf{B}

Forces on paramagnets and diamagnets

- General formula for small dipole in nonuniform field

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

- For paramagnets $\mathbf{m} \parallel \mathbf{B}$
 - Force directs toward more intense field regions
 - Paramagnets are attracted to magnets
- For diamagnets $\mathbf{m} \parallel -\mathbf{B}$
 - Force directs toward less intense field regions
 - Diamagnets are repelled by magnets



Levitating frog



(Ig Nobel award 2000)

Field of magnetized objects

- Bound currents
 - Vector potential of a magnetized object (neglecting the cause)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} d\tau'$$

$$= \frac{\mu_0}{4\pi} \int \left[\mathbf{M}(\mathbf{r}') \times \left(\nabla' \frac{1}{r} \right) \right] d\tau'$$

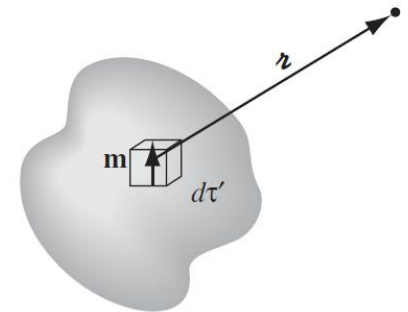
↓ Integrate by parts

$$= \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{r} [\nabla' \times \mathbf{M}(\mathbf{r}')] d\tau' - \int \nabla' \times \left[\frac{\mathbf{M}(\mathbf{r}')}{r} \right] d\tau' \right\}$$

$$= \frac{\mu_0}{4\pi} \int \frac{1}{r} [\nabla' \times \mathbf{M}(\mathbf{r}')] d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{r} [\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']$$

$$= \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_b(\mathbf{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{r} da'$$

* with $\mathbf{J}_b = \nabla \times \mathbf{M}$
 $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$



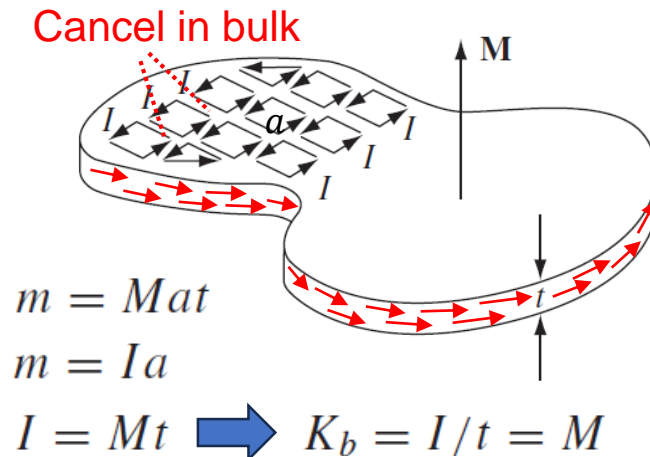
Field of magnetized objects

- Bound currents

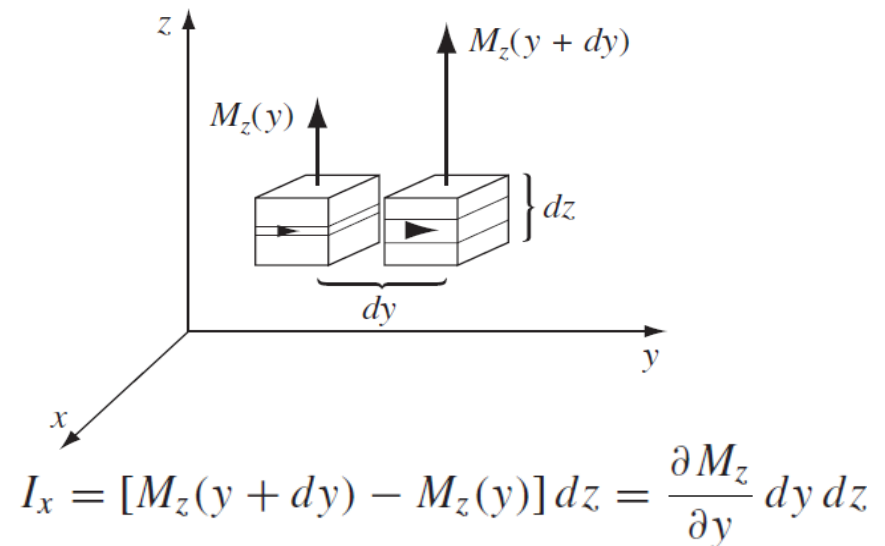
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_b(\mathbf{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{r} da'$$

- Density of bound currents $\left\{ \begin{array}{l} \text{surface: } \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \\ \text{volume: } \mathbf{J}_b = \nabla \times \mathbf{M} \end{array} \right.$ *check $\nabla \cdot \mathbf{J}_b = 0$
- Physical picture of bound currents

Surface bound current (suppose uniform \mathbf{M})



Volume bound current



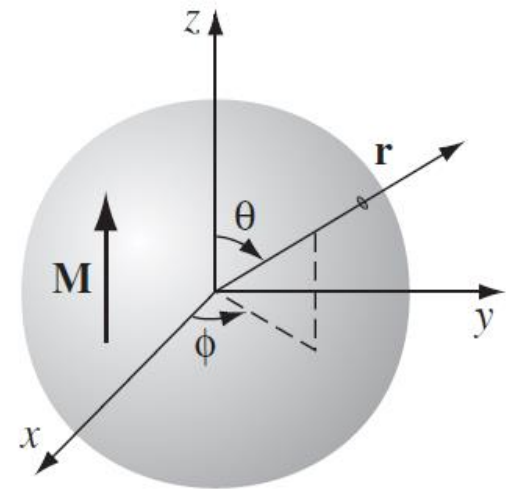
Field of magnetized objects

- Bound currents

Example 6.1. Find the magnetic field of a uniformly magnetized sphere.

- Field inside the sphere $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$
 - Uniform field
 - \mathbf{M} induces \mathbf{B} that is parallel to it, while \mathbf{P} induces \mathbf{E} that is antiparallel
- Field outside the sphere same as what would be for a dipole

$$\mathbf{m} = \frac{4}{3}\pi R^3\mathbf{M}$$



Auxiliary field

- Add the cause and the effect of magnetization

- Total magnetic field

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

- \mathbf{B} : total magnetic field
- \mathbf{J} : total current density
- \mathbf{J}_b : bound current density, due to magnetization
- \mathbf{J}_f : free current density that we control, not a result of magnetization

$$\Rightarrow \nabla \times \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f$$

- The auxiliary field

$$\mathbf{H} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

$$\Rightarrow \nabla \times \mathbf{H} = \mathbf{J}_f$$

- Ampère's law for auxiliary field

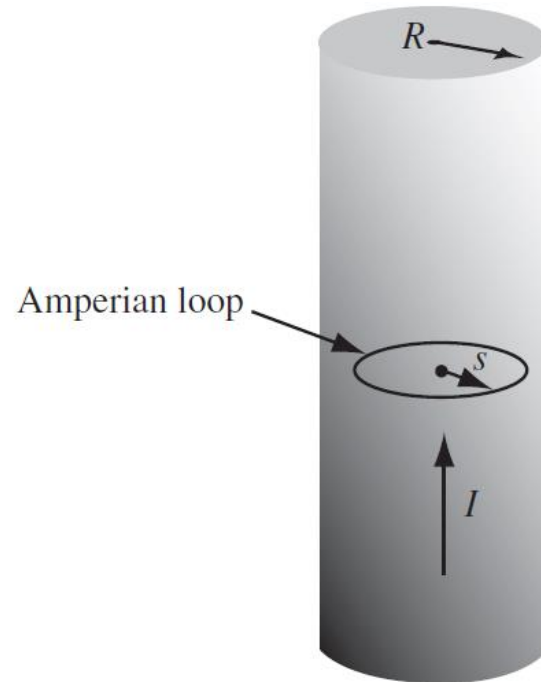
$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$$

Enclosed total
free current

Auxiliary field

- Application of auxiliary field

Example 6.2. A long copper rod of radius R carries a uniformly distributed (free) current I (Fig. 6.19). Find \mathbf{H} inside and outside the rod.

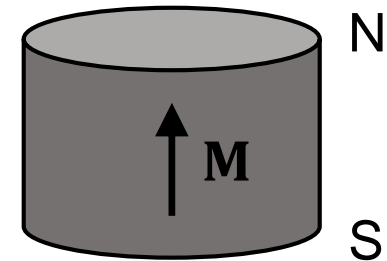


Auxiliary field

- Can \mathbf{H} be finite without applying any free current?

- Looks like so because $\oint \mathbf{H} \cdot d\mathbf{l} = I_{f\text{enc}}$

- Consider a short cylindrical magnet
- \mathbf{H} would be zero everywhere
- Then $\mathbf{B} \neq \mathbf{0}$ inside magnet, $\mathbf{B} = \mathbf{0}$ outside magnet
- Obviously wrong



- \mathbf{H} can be finite without any \mathbf{J}_F
 - Because $\nabla \cdot \mathbf{H} = \nabla \cdot \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = -\nabla \cdot \mathbf{M} \neq 0$
 - $\nabla \cdot \mathbf{M} \neq 0$ at the top and bottom surfaces of the magnet

H versus B, D versus E

- A long history of confusion
 - Whether we shall call **H** or **B** as the magnetic field
 - Some convention calls **H** the magnetic field, and **B** as the magnetic flux density
 - Our textbook takes the stance that **H** should be “auxiliary”
- Reason for such confusion
 - **H** is a lot more frequently used than **B** as **H** is given by free current, something we can control, while **B** is material dependent
 - Helmholtz coil magnetizing a specimen
 - **E** is a lot more frequently used than **D** as **D** is given by free charge, which we rarely control, while **E** is determined by voltage difference (over distance), which is what we control
 - Charging up of a parallel plate capacitor

Linear magnetic media

- Linear magnetic media

- $\mathbf{M} = \chi_m \mathbf{H}$ (χ_m : magnetic susceptibility)

➔ $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}$

- Diamagnetic susceptibility on the order of 10^{-6}

- $\mathbf{B} = \mu \mathbf{H}$ ($\mu = \mu_0(1 + \chi_m)$: permeability)

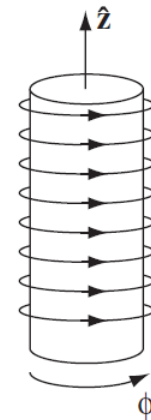
? Why this linear relation defines M-H but not M-B?

Example 6.3. An infinite solenoid (n turns per unit length, current I) is filled with linear material of susceptibility χ_m . Find the magnetic field inside the solenoid.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} \quad \text{➔} \quad \mathbf{H} = nI \hat{\mathbf{z}}$$

➔ $\mathbf{B} = \mu_0(1 + \chi_m)nI \hat{\mathbf{z}}$

Enhancement of field if paramagnetic!



Boundary conditions

- Boundary conditions reexamined
 - Earlier findings still hold, but \mathbf{J} needs to include free and bound currents

$$\left\{ \begin{array}{l} B_{\text{above}}^{\perp} = B_{\text{below}}^{\perp} \\ \mathbf{B}_{\text{above}}^{\parallel} - \mathbf{B}_{\text{below}}^{\parallel} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \end{array} \right.$$

- Easier to use the boundary conditions of \mathbf{D}

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \\ \nabla \times \mathbf{H} = \mathbf{J}_f \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \mathbf{H}_{\text{above}}^{\parallel} - \mathbf{H}_{\text{below}}^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}} \\ H_{\text{above}}^{\perp} - H_{\text{below}}^{\perp} = -(M_{\text{above}}^{\perp} - M_{\text{below}}^{\perp}) \end{array} \right.$$

Surface free
charge density
↓

Nonlinear magnetic media

- Ferromagnets: $\mathbf{M} \neq 0$ without applying any field
 - Obvious violation of the linear relation $\mathbf{M} = \chi_m \mathbf{H}$
 - Represents a quantum phenomena
 - Exchange interaction: $U = -2J \sum_{p=1}^N \mathbf{S}_p \cdot \mathbf{S}_{p+1}$ ($J > 0$)
 - Prefers spontaneous parallel alignment of spins
 - Domain formation and hysteresis loop

