$$J^2 |j, m\rangle = \hbar^2 j(j+1) |j,m\rangle \quad , \quad j \geq 0 \text{ and } j \text{ is an integer or half-integer}$$

$$J_z |j,m\rangle = \hbar m |j,m\rangle \quad , \quad -j \leq m \leq j$$

$$m = -j, \underbrace{-j+1}_{+1}, \ldots, j-1, j$$

Eg for $\vec{J} \equiv \vec{L} = \vec{r} \times \vec{p}$

$j \to l$

$l = 0$? $\quad l=1$? $\quad l=2$? — l has to be an integer ≥ 0.

$m_l = 0 \quad m_l = -1, 0, 1 \quad m_l = -2, -1, 0, 1, 2$

s $\quad$ p $\quad$ d

Eg hydrogen atom

$\langle \vec{r} | n\, l\, m\rangle = \psi_{nlm}(\vec{r})$

Claim (W4L1): eigenvalue for J^2 must be ≥ 0.

$$\| J |\psi\rangle \|^2 \geq 0$$

$$\| $$

$$\langle \psi | J^\dagger J |\psi\rangle$$

$$\|$$

$$\langle \psi | J^2 | \psi \rangle \quad (J^\dagger = J \ , \ J \text{ is Hermitian})$$

If $|\psi\rangle = |a,b\rangle$ using notation from W4L1,

$$J^2 |a,b\rangle = a\, |a,b\rangle$$

$$0 \leq \langle a,b | J^2 | a,b\rangle = a \langle a,b|a,b\rangle = a$$

Defined $J_\pm = J_x \pm i J_y$

Showed $J_+ |a,b\rangle = c_+ |a, b+\hbar\rangle$

$J_- |a,b\rangle = c_- |a, b-\hbar\rangle$

$$\underbrace{-\alpha\hbar}_{b_{min}} \leq b \leq \underbrace{\alpha\hbar}_{b_{max}} \quad \text{where } a = \hbar^2 \alpha(\alpha+1), \ \alpha \geq 0$$

from $b^2 - \hbar b - a \leq 0 \quad -(2)$

from $b^2 + \hbar b - a \leq 0 \quad -(1)$

J_+ returns 0

$\circlearrowleft \cdot \alpha\hbar$

$J_- \downarrow$

From,

$b^2 - \hbar b - a \leq 0 \quad -(2)$

$\| J_- |a,b\rangle \|^2 \geq 0$

'=' holds iff

$J_- |a,b\rangle = 0$

($b = -\alpha\hbar$ iff $J_- |a,b\rangle = 0$)

$\uparrow$

b_{min}

$b^2 + \hbar b - a \leq 0 \quad -(1)$

$\| J_+ |a,b\rangle \|^2 \geq 0$

'=' holds iff

$J_+ |a,b\rangle = 0$

$\uparrow$

b_{max}

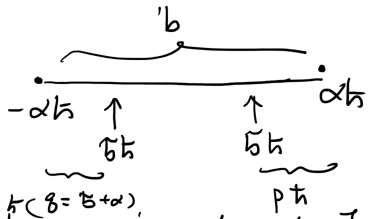
$J_+ \uparrow$ perw.. $\alpha\hbar$

$J_- \downarrow$ $(\alpha - 1)\hbar$

$J_+ \uparrow_+$ $(-\alpha+1)\hbar$

$-\alpha\hbar$

J_- returns 0.



Possible values of α?

$b \longrightarrow \tilde{b}\hbar$

If $p = \alpha - \tilde{b}$ is not an integer,

apply J_+ multiple times to $|a,b\rangle$ will lead to an eigenvector of J_+ that is not null,

and has an eigenvalue $> \alpha\hbar$.

Contradiction $\Rightarrow$ $p = \alpha - \tilde{b}$ is an integer.

Similarly, $q = \tilde{b} - (-\alpha) = \tilde{b} + \alpha$ must be an integer.

$\left.\begin{aligned} p &= \alpha - \tilde{b} \\ q &= \alpha + \tilde{b} \end{aligned}\right\}$ are integers.

$\Rightarrow$ $p + q = 2\alpha$ is an integer.

α is either integer or half-integer. ($\alpha \geq 0$ by definition)

Standard notation: $\alpha \longrightarrow j$

$\tilde{b} \longrightarrow m$

Poll questions:

a) $|j = \frac{2}{3}, m = -\frac{2}{3}\rangle$ ✗

b) $|j = 5, m = 3\rangle$ ✓

c) $|j = 2, m = \frac{1}{2}\rangle$ ✗ $m = -2, -1, 0, 1, 2$ only

d) $|j = \frac{1}{2}, m = \frac{1}{2}\rangle$ ✓

e) $|j = 1, m = 2\rangle$ ✗ $m = -1, 0, 1$ only

f) $|j = -\frac{1}{2}, m = \frac{1}{2}\rangle$ ✗ j cannot be negative

---

Now, let's obtain $J_\pm |j,m\rangle = c_\pm |j, m\pm 1\rangle$ ($J_+ |j, m_{max}\rangle = 0$

$J_- |j, m_{min}\rangle = 0$)

Note: Each $|j,m\rangle$ is normalized to 1.

We need the norms of $J_\pm |j,m\rangle$.

$$\| J_+ |j,m\rangle \|^2 = \langle j,m| J_+^\dagger J_+ |j,m\rangle$$

$$= \langle j,m| J_- J_+ |j,m\rangle$$

$$\underset{\text{(from W4L1)}}{=} \langle j,m| J^2 - J_z^2 - \hbar J_z |j,m\rangle$$

$$= \hbar^2 j(j+1) - (m\hbar)^2 - \hbar(m\hbar)$$

$$= \hbar^2 (j(j+1) - m(m+1))$$

So $J_+ |j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\ |j,m+1\rangle$

(when $m = m_{max} = j$, RHS = 0 ✓)

Similarly, $\| J_- |j,m\rangle \|^2 = \langle j,m| J_-^\dagger J_- |j,m\rangle$

$$= \langle j,m| J_+ J_- |j,m\rangle$$

$$\underset{\text{(from W4L1)}}{=} \langle j,m| J^2 - J_z^2 + \hbar J_z |j,m\rangle$$

$$= \hbar^2 j(j+1) - m^2\hbar^2 + m\hbar^2$$

$$= \hbar^2 (j(j+1) - m(m-1))$$

So $J_- |j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\ |j, m-1\rangle$

(when $m = m_{min} = -j$, RHS = 0 ✓)

$$\boxed{J_\pm |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\ |j, m\pm 1\rangle, \quad -j \le m \le j}$$

useful relation

Eg of how $J_\pm$ is useful. → to construct matrix representations of J_x and J_y, given J_z.

Start with a basis $\{|j,m\rangle\}$ - eigenvectors of J^2 and J_z.

chosen axis $\hat{z}$.

(Note: j is fixed in the description for a given physical system

eg This is a spin-$\frac{1}{2}$ system ⇒ $j = \frac{1}{2}$

" " " spin-1 system ⇒ $j = 1$)

m can take $(2j+1)$ values.

J^2 and J_z act on $\mathcal{V}$ spanned by $\{ |j, m=j\rangle, |j, m=j-1\rangle, \ldots, |j, m=-j\rangle \}$

$$\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} \quad \underbrace{\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}}_{(2j+1)}$$

Using this basis,

J^2 and J_z are diagonal.

matrix vector $\quad J_z \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \hbar j \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$

J^2 and J_z are diagonal.

$$J_z = \hbar \begin{pmatrix} j & & & \\ & (j-1) & & \\ & & (j-2) & \\ & & & \ddots \\ & & & & -j \end{pmatrix}$$

matrix vector

$A\vec{v}$

$$\begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \cdots & \vec{c}_n \end{pmatrix}$$

$\vec{c}_1 = A\vec{e}_1 \equiv A\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\vec{c}_2 = A\vec{e}_2 \equiv A\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$J_z \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \hbar j \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

How to get matrix representations of J_x and J_y?

use $J_+ = J_x + iJ_y$

$J_- = J_x - iJ_y$

Construct $J_\pm$.

Then $J_x = \frac{1}{2}(J_+ + J_-)$

$J_y = \frac{1}{2i}(J_+ - J_-)$

$$J_+ = \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{pmatrix}$$

$J_+\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\quad$ $J_+\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$|j, m=j\rangle$ $\quad$ $|j, m=-j\rangle$

Eg. $j = \frac{1}{2}$ system.

$2j+1 = 2$, $\quad m = -\frac{1}{2}, \frac{1}{2}$.

$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left(\begin{array}{l} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow |j = \frac{1}{2}, m = \frac{1}{2}\rangle \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow |j = \frac{1}{2}, m = -\frac{1}{2}\rangle \end{array} \right)$$

$$J_+ = \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix}$$

$J_+\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad$ $J_+\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$J_+\begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

$J_+|\frac{1}{2}, \frac{1}{2}\rangle = 0$ (because $m = \frac{1}{2}$ is m_{max})

$J_+\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$J_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \hbar \\ 0 \end{pmatrix}$

$J_+\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

$$J_+|\underset{j}{\frac{1}{2}}, \underset{m}{-\frac{1}{2}}\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\ |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - (-\tfrac{1}{2})(\tfrac{1}{2})}\ |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{\tfrac{3}{4} + \tfrac{1}{4}}\ |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \hbar\ |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$J_- = \begin{pmatrix} 0 & 0 \\ \hbar & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow |\frac{1}{2}, \frac{1}{2}\rangle$

$$J_- = \begin{pmatrix} 0 & \vdots & 0 \\ \hbar & \vdots & 0 \end{pmatrix}$$

$J_-\begin{pmatrix}1\\0\end{pmatrix}$ $\quad$ $J_-\begin{pmatrix}0\\1\end{pmatrix}$

$$= \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow |\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$$J_- |\tfrac{1}{2}, \underset{m}{\tfrac{1}{2}}\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{\tfrac{3}{4} + \tfrac{1}{4}}\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$= \hbar\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$J_-\begin{pmatrix}1\\0\end{pmatrix} = \hbar\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\\hbar\end{pmatrix}$$

$$J_-\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$J_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_x = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_y = \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For spin-$\frac{1}{2}$ system,

we define $\underset{\text{angular momentum}}{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$

$$\boxed{\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$ Pauli matrices. (cf W3L2)

Eigenstates of σ_z are $\{ |z,+\rangle \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix},\ |z,-\rangle \leftrightarrow \begin{pmatrix}0\\1\end{pmatrix} \}$

" " σ_x " $\{ |x,+\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ |x,-\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \}$ W3L2

" " σ_y " $\{ |y,+\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix},\ |y,-\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \}$

Ex. Work out.