2024. 3. 21 18 (D)

chapter 6 Griffith

Transition prob. per unit time

$$= \frac{2\pi}{\hbar} \cdot |\mathcal{M}|^2 \cdot \text{phase space factor}$$

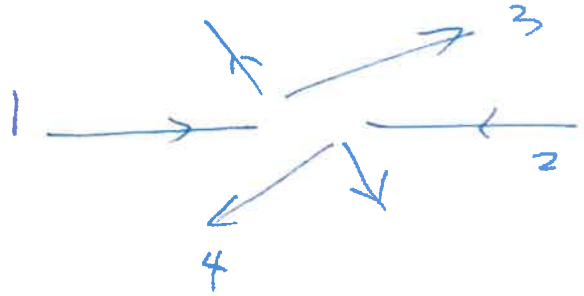
Decay rate Γ $\quad 1 \to 2 + 3 \cdots + N$

Differential decay rate

$$d\Gamma = \frac{S}{2\hbar \cdot m_1} |\mathcal{M}|^2 \cdot (2\pi)^4 \delta^{(4)}(p_1 - p_2 \cdots - p_N)$$

$$\prod_{j=2}^{N} \frac{d^4 p_j}{(2\pi)^4} (2\pi)\, \delta(p_j^2 - m^2 c^2)\, \theta(p_j^0)$$

Differential cross section $\quad 1 + 2 \to 3 + 4 + \cdots + N$



$$d\sigma = \frac{S \hbar^2}{4 \cdot \sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 \cdots p_N)$$

$$\prod_{j=3}^{N} \frac{d^4 p_j}{(2\pi)^4} \theta(p_j^0) (2\pi)\, \delta(p_j^2 - m_j^2 c^2)$$

$\theta(p_j^0)$ = step function

Consider 2-particle decays

$$1 \to 2 + 3$$

Assume particle 1 at rest and decays

The decay rate is given by (page (6a))

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(\underset{\sim}{P}_1 - \underset{\sim}{P}_2 - \underset{\sim}{P}_3) \cdot$$

$$\cdot \prod_{j=2}^{3} \left(\frac{1}{2p_j^0} \frac{d^3 \underset{\sim}{P}_j}{(2\pi)^3} \right)$$

scattering amplitude $\mathcal{M}$
$= \mathcal{M}(\underset{\sim}{P}_1, \underset{\sim}{P}_2, \underset{\sim}{P}_3 \ldots)$

$$= \frac{S}{8\pi^2 \hbar m_1} \int |\mathcal{M}|^2 \delta^{(4)}(\underset{\sim}{P}_1 - \underset{\sim}{P}_2 - \underset{\sim}{P}_3) \frac{d^3 \underset{\sim}{P}_2}{2p_2^0} \cdot \frac{d^3 \underset{\sim}{P}_3}{2p_3^0}$$

$$= \frac{S}{8\pi^2 \hbar m_1} \int |\mathcal{M}|^2 \delta(p_1^0 - p_2^0 - p_3^0) \delta^{(3)}(\underset{\sim}{P}_1 - \underset{\sim}{P}_2 - \underset{\sim}{P}_3) \cdot$$

$$\cdot \frac{d^3 \underset{\sim}{P}_2}{2p_2^0} \frac{d^3 \underset{\sim}{P}_3}{2p_3^0}$$

As the decaying particle 1 is at rest,

$\underset{\sim}{p}_1 = 0.$

$\delta^{(3)}(\underset{\sim}{p}_2 + \underset{\sim}{p}_3)$ $\because \delta(-x) = \delta(x)$

$\therefore$

$$\Gamma = \frac{S}{8\pi^2 \hbar m_1} \int |M|^2 \, \delta(p_1^0 - p_2^0 - p_3^0) \, \delta^{(3)}(\underset{\sim}{p}_2 - \underset{\sim}{p}_3) \cdot \frac{d^3 \underset{\sim}{p}_2}{2 p_2^0} \frac{d^3 \underset{\sim}{p}_3}{2 p_3^0}$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

$$\int f(\underset{\sim}{x})\, \delta^{(3)}(\underset{\sim}{x} - \underset{\sim}{a})\, d^3\underset{\sim}{x} = f(\underset{\sim}{a})$$

$$= \frac{S}{8\pi^2 \hbar m_1} \int |M|^2 \, \delta(p_1^0 - p_2^0 - p_3^0) \frac{d^3 \underset{\sim}{p}_2}{2 p_2^0 \cdot 2 p_3^0}$$

where $\underset{\sim}{p}_3 = -\underset{\sim}{p}_2$

$p_3^0 = \sqrt{\underset{\sim}{p}_3^2 + m_3^2 c^2}$

The volume differential $d^3 \underset{\sim}{p}_2$ can be written as

$$d^3 \underset{\sim}{p}_2 = |\underset{\sim}{p}_2|^2 \cdot d|\underset{\sim}{p}_2| \cdot d\Omega_{\underset{\sim}{p}_2} \qquad \left(d^3 \underset{\sim}{x} = r^2\, dr\, d\Omega \right)$$

Integrating $d\Omega_{\underset{\sim}{p}_2}$ and assuming $|M|^2$ does not depend on $\Omega_{\underset{\sim}{p}_2}$, we have

$$\Gamma = \frac{S}{8\pi\hbar m_1} \int |M|^2 \, \delta(p_1^0 - p_2^0 - p_3^0) \cdot \frac{|\underset{\sim}{p}_2|^2 \cdot d|\underset{\sim}{p}_2|}{p_2^0 \cdot p_3^0}$$

where $\underset{\sim}{p}_3 = -\underset{\sim}{p}_2$

Changing the integration variable by defining

$$p^0 = p_2^0 + p_3^0$$

$$\therefore \; dp^0 = dp_2^0 + dp_3^0$$

$p_i^{02} = \underset{\sim}{p}_i^2 + m_i^2 c^2$

$i = 2, 3$

$$= \frac{|\underset{\sim}{p}_2| \cdot d|\underset{\sim}{p}_2|}{p_2^0} + \frac{|\underset{\sim}{p}_3| \cdot d|\underset{\sim}{p}_3|}{p_3^0}$$

$2p^0 \, dp^0$

$= 2 \, |\underset{\sim}{p}| \cdot d|p|$

$$= \frac{p_3^0 + p_3^0}{p_2^0 \cdot p_3^0} \cdot |\underset{\sim}{p}_2| \cdot d|\underset{\sim}{p}_2|$$

$\because \; \underset{\sim}{p}_3 = -\underset{\sim}{p}_2$

$\therefore \; |\underset{\sim}{p}_3| = |\underset{\sim}{p}_2|$

$$\therefore \; \frac{dp^0}{p^0} = \frac{|\underset{\sim}{p}_2| \cdot d|\underset{\sim}{p}_2|}{p_2^0 \cdot p_3^0}$$

Note: we have changed integration variable $|\underset{\sim}{p}|$ to p^0

Thus

$$P = \frac{S}{8\pi\hbar\cdot m_1} \int |\mathcal{M}|^2 \, \delta(p_1^0 - p^0) \cdot \frac{|\underset{\sim}{p}_2| \cdot dp^0}{p^0}$$

$$= \frac{S}{8\pi\hbar\cdot m_1} |\mathcal{M}|^2 \frac{|\underset{\sim}{p}_2|}{p_1^0}$$

where $\underset{\sim}{p}_3 = -\underset{\sim}{p}_2$

$p^0 \equiv p_2^0 + p_3^0 = p_1^0$

As particle 1 is at rest, $p_1^0 = m_1 c$

$\therefore$

$$\Gamma = \frac{S}{8\pi\hbar\cdot m_1^2 c} |\mathcal{M}|^2 \cdot |\underset{\sim}{p}_2|$$

where $\underset{\sim}{p}_3 = -\underset{\sim}{p}_2$

$p_2^0 + p_3^0 = p_1^0 = m_1 c$

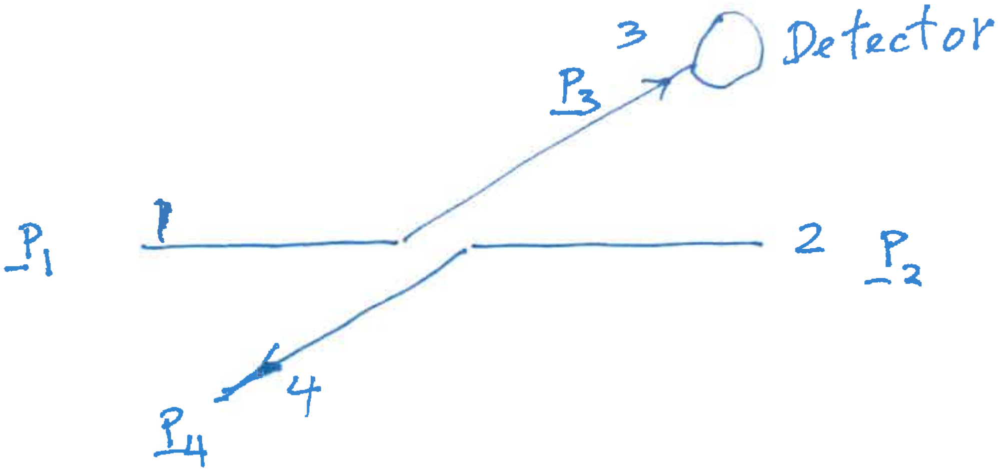
To find $|\underset{\sim}{p}_2|$:

As $p_1^0 = p_2^0 + p_3^0$, $\therefore$

$$m_1 c = \sqrt{\underset{\sim}{p}_2^2 + m_2^2 c^2} + \sqrt{\underset{\sim}{p}_3^2 + m_3^2 c^2} \quad \because \underset{\sim}{p}_3 = -\underset{\sim}{p}_2$$

$$\therefore \underset{\sim}{p}_2^2 = \frac{c^2}{4m_1^2} \cdot \left[(m_1^4 + m_2^4 + m_3^4) - 2(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2) \right] \quad \text{(HW)}$$

Consider 2 particles to 2 particles scattering



Using the Fermi golden rule, the differential cross section can be written as (page 7) → previous lecture

$\mathcal{M}$ = scattering amplitude

$$d\sigma = \frac{S\,\hbar^2}{4\cdot\sqrt{(\underline{p}_1\cdot\underline{p}_2)^2-(m_1 m_2 c^2)^2}}\cdot|\mathcal{M}|^2\cdot$$

$$\cdot(2\pi)^4\,\delta^{(4)}(\underline{p}_1+\underline{p}_2-\underline{p}_3-\underline{p}_4)\cdot$$

$$\prod_{j=3}^{4}\frac{d^4 p_j}{(2\pi)^4}(2\pi)\,\delta(\underline{p}_j^2-m_j^2c^2)\cdot\Theta(p_j^0)$$

Integrating away the energy p_j^0

We get

$$d\sigma = \frac{s\hbar^2}{4\sqrt{(\underset{\sim}{p}_1\cdot \underset{\sim}{p}_2)^2 - (m_1 m_2 c^2)^2}} \cdot |\mathcal{M}|^2 \cdot$$

$$(2\pi)^4\, \delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4) \cdot \prod_{j=3}^{4} \frac{d^3 \underset{\sim}{p}_j}{(2\pi)^3}\, \frac{1}{2 p_j^0}$$

$$= \frac{s\hbar^2}{4\cdot\sqrt{(\underline{p}_1\cdot \underline{p}_2)^2 - (m_1 m_2 c^2)^2}} \; |\mathcal{M}|^2 \cdot (2\pi)^4\, \delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4)$$

$$\frac{d^3 \underset{\sim}{p}_3}{(2\pi)^3}\; \frac{d^3 \underset{\sim}{p}_4}{(2\pi)^3} \cdot \frac{1}{2 p_3^0} \cdot \frac{1}{2 p_4^0}$$

Integrating away $\int d^3 \underset{\sim}{p}_4$ by using the Dirac delta function $\delta^{(3)}(\underset{\sim}{p}_1 + \underset{\sim}{p}_2 - \underset{\sim}{p}_3 - \underset{\sim}{p}_4)$,

$$d\sigma = \frac{s\hbar^2}{4\cdot\sqrt{(\underline{p}_1\cdot \underline{p}_2)^2 - (m_1 m_2 c^2)^2}} \cdot |\mathcal{M}|^2 \cdot$$

$$\delta(p_1^0 + p_2^0 - p_3^0 - p_4^0) \cdot \frac{d^3 \underset{\sim}{p}_3}{(2\pi)^2} \cdot \frac{1}{4\, p_3^0 \cdot p_4^0}$$

where $$\underset{\sim}{p}_4 = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 - \underset{\sim}{p}_3$$

We assume the detector is detecting particle 3

so $$d^3 \underset{\sim}{p}_3 = |\underset{\sim}{p}_3|^2 \cdot d|\underset{\sim}{p}_3| \cdot d\Omega_{\underset{\sim}{p}_3}$$

and compute $\dfrac{d\sigma}{d\Omega_{\underset{\sim}{p}_3}}$.

We write

$$\frac{d\sigma}{d\Omega_{\underset{\sim}{p}_3}} = \frac{S\hbar^2}{4\sqrt{(\underset{\sim}{p}_1 \cdot \underset{\sim}{p}_2)^2 - (m_1 m_2 c^2)^2}} \cdot$$

$$\cdot \int \frac{|\underset{\sim}{p}_3|^2 \cdot d|\underset{\sim}{p}_3|}{(4\pi)^2 \, p_3^0 \, p_4^0} \cdot |\mathcal{M}|^2 \cdot \delta(p_1^0 + p_2^0 - p_3^0 - p_4^0)$$

$$\underset{\sim}{p}_4 = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 - \underset{\sim}{p}_3$$
$$= -\underset{\sim}{p}_3 \quad (\text{CM frame})$$

Changing the integrating variable $|\underset{\sim}{p}_3|$ by defining

$$p^0 = p_3^0 + p_4^0 ,$$

$$dp^0 = dp_3^0 + dp_4^0$$

$$= \frac{|\underset{\sim}{p}_3| \cdot d|\underset{\sim}{p}_3|}{p_3^0} + \frac{|\underset{\sim}{p}_3| \cdot d|\underset{\sim}{p}_3|}{p_4^0}$$

$$\because \; p_3^0 = \sqrt{\underset{\sim}{p}_3^2 + m_3^2 c^2}$$

$$\underset{\sim}{p}_4 = -\underset{\sim}{p}_3$$

$$= \frac{p^0}{p_3^0 \cdot p_4^0} \cdot |\underset{\sim}{p}_3| \cdot d|\underset{\sim}{p}_3|$$

$$\therefore \; \frac{dp^0}{p^0} = \frac{|\underset{\sim}{p}_3| \cdot d|\underset{\sim}{p}_3|}{p_3^0 \cdot p_4^0}$$

We get

$$\frac{d\sigma}{d\Omega} = \frac{s\hbar^2}{4 \cdot \sqrt{(\underset{\sim}{p}_1 \cdot \underset{\sim}{p}_2)^2 - (m_1 m_2 c^2)^2}} \cdot \frac{1}{(4\pi)^2} \cdot$$

$$\int |\underset{\sim}{p}_3| \cdot \frac{dp^0}{p^0} \cdot |\mathcal{M}|^2 \cdot \delta(p_1^0 + p_2^0 - p^0)$$

Integrating $\int dp^0$,

$$\frac{d\sigma}{d\Omega} = \frac{s\hbar^2}{(8\pi)^2 \sqrt{(\underset{\sim}{p_1}\cdot \underset{\sim}{p_2})^2 - (m_1 m_2 c^2)^2}} \cdot \frac{|M|^2 \cdot |\underset{\sim}{p_3}|}{(p_1^0 + p_2^0)}$$

$$\underset{\sim}{p_3} = -\underset{\sim}{p_4} \quad (\text{CM frame})$$

$$p^0 = p_3^0 + p_4^0 = p_1^0 + p_2^0$$

Only unknown is $|\underset{\sim}{p_3}|$

We can find $|\underset{\sim}{p_3}|$ by using

$$p_1^0 + p_2^0 = p_3^0 + p_4^0$$

$$= \sqrt{\underset{\sim}{p_3}^2 + m_3^2 c^2} + \sqrt{\underset{\sim}{p_3}^2 + m_4^2 c^2}$$

As $(p_1^0 + p_2^0)$ is fixed and known, so can get $|\underset{\sim}{p_3}|$ from the above relation

$$\underset{\sim}{p_3}^2 = \frac{\left(K^2 + (m_4^2 - m_3^2) c^2\right)^2}{4 K^2} - m_4^2 c^2$$

$$K \equiv p_1^0 + p_2^0$$