$$\not{p}^2 = \not{p}\not{p} = p_\mu \gamma^\mu \cdot p_\nu \gamma^\nu$$

$$= p_\mu p_\nu \gamma^\mu \gamma^\nu$$

$$= p_\nu p_\mu \gamma^\nu \gamma^\mu$$

(Recall: want $\not{p}^2 \to \underline{p}^2$

$$= \frac{1}{2}(p_\mu p_\nu \gamma^\mu \gamma^\nu + p_\nu p_\mu \gamma^\nu \gamma^\mu)$$

$p_\mu p_\nu = p_\nu p_\mu$

$$= p_\mu p_\nu \frac{1}{2}[\gamma^\mu, \gamma^\nu]_+$$

anticommutator $[A, B]_+ = AB + BA$

cf $\underline{p}^2 = p_\mu p^\mu = g_{\mu\nu} p^\mu p^\nu$

In order for $\not{p}^2 = \underline{p}^2$ then demand

$$\frac{1}{2}[\gamma^\mu, \gamma^\nu]_+ = g^{\mu\nu}$$

i.e. $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$

which defines the Dirac matrix γ^μ

What are the properties of γ^μ, $\mu = 0, 1, 2, 3$?

(1) $\gamma^{0\,2} = 1$, $\gamma^{i\,2} = -1$, $i = 1, 2, 3$

proof: $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$

put $\mu = 0 = \nu \to 2\gamma^{0\,2} = 2g^{00} = 2 \quad \because g^{00} = +1$

i.e. $\gamma^{0\,2} = 1$

The correct equation of motion for a relativistic particle with spin $\frac{1}{2}$ is the Dirac equation

$$\not{p}\, \psi(\underline{x}) = mc\, \psi(\underline{x})$$

$$\not{p} = p_\mu \gamma^\mu = \gamma^\mu p_\mu, \qquad p_\mu = i\hbar \partial_\mu = i\hbar \frac{\partial}{\partial x^\mu}$$

$$[\gamma^\mu, \gamma^\nu]_+ = 2 g^{\mu\nu}$$

The anticommutator for the Dirac matrix γ^μ defines γ^μ. One can show

(i) $\gamma^{0\,2} = 1, \quad \gamma^{i\,2} = -1, \quad i = 1, 2, 3$

(ii) $\mathrm{Tr}\ \gamma^\mu = 0, \quad \mu = 0, 1, 2, 3$

(iii) γ^μ is $N \times N$ matrix, N = even integer

Dirac put $N = 4$

(iv) Hermiticity of γ^μ,

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

can define $\beta = \gamma^0$, $\quad \underset{\sim}{\alpha} = \gamma^0 \underset{\sim}{\gamma}$, or $\alpha^i = \gamma^0 \gamma^i$
$i = 1, 2, 3$

Then $\quad \beta^\dagger = \beta, \qquad \alpha^{i\dagger} = \alpha^i$

The Hamiltonian of a free Dirac particle is

$$H = c\, \underset{\sim}{\alpha} \cdot \underset{\sim}{p} + \beta m c^2$$

(v) Representations of γ^μ, $\quad \mu = 0, 1, 2, 3$

The Dirac representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli matrix

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note added:

Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underset{\sim}{\gamma} = \begin{pmatrix} 0 & \underset{\sim}{\sigma} \\ -\underset{\sim}{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Majorana representation

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}. \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note in the Majorana representation, all elements of the gamma (γ) matrices are imaginary number

Derive properties of the Dirac matrix γ^μ

from the definiting equation

$$[\gamma^\mu, \gamma^\nu]_+ = 2 g^{\mu\nu} , \qquad \mu, \nu = 0, 1, 2, 3$$

(i) $\mu = 0 = \nu$

$$[\gamma^0, \gamma^0]_+ = 2 g^{00} = 2 \qquad \because g^{00} = +1$$

$$\therefore \quad {\gamma^0}^2 = \gamma^0 \gamma^0 = 1 \quad \text{(identity matrix)}$$

Similarly, $\quad {\gamma^1}^2 = \gamma^1 \gamma^1 = -1 \qquad \because g^{11} = -1$

$$\rightarrow \quad {\gamma^i}^2 = -1 , \qquad i = 1, 2, 3$$

(HW)

(ii) $\mu = 0, \quad \nu = i , \qquad i = 1, 2, 3$

$$[\gamma^0, \gamma^i]_+ = 2 g^{0i} = 0$$

$$\gamma^0 \gamma^i = - \gamma^i \gamma^0$$

Multiply γ^0 on both sides

$$\gamma^0 \gamma^0 \gamma^i = - \gamma^0 \gamma^i \gamma^0$$

$$\therefore \quad \gamma^i = - \gamma^0 \gamma^i \gamma^0 \qquad \because {\gamma^0}^2 = 1$$

Taking trace of both sides,

$$\mathrm{Tr}\,\gamma^i = \mathrm{Tr}\,(-\gamma^0\gamma^i\gamma^0) = -\mathrm{Tr}(\gamma^0\gamma^i\gamma^0)$$

$$= -\mathrm{Tr}(\gamma^0\gamma^0\gamma^i) \qquad \because \mathrm{Tr}\,AB = \mathrm{Tr}\,BA$$

$$= -\mathrm{Tr}\,\gamma^i \qquad \because {\gamma^0}^2 = 1$$

$$\therefore \quad \mathrm{Tr}\,\gamma^i = 0 \qquad i = 1, 2, 3$$

similarly $\quad \mathrm{Tr}\,\gamma^0 = 0 \qquad$ (HW)

Thus $\quad \mathrm{Tr}\,\gamma^\mu = 0, \quad \mu = 0, 1, 2, 3.$

(iii) Let γ^μ be a $N \times N$ matrix

show $N =$ even integer

Use $\quad \gamma^i = -\gamma^0\gamma^i\gamma^0$

Taking determinant both sides

$$\det\gamma^i = \det(-\gamma^0\gamma^i\gamma^0) = (-1)^N \det(\gamma^0\gamma^i\gamma^0)$$

As $\det AB = \det BA$,

$$\det(\gamma^0\gamma^i\gamma^0) = \det(\gamma^i\gamma^0\gamma^0) = \det\gamma^i$$

$$\because {\gamma^0}^2 = 1$$

Thus $\quad \det\gamma^i = (-1)^N \det\gamma^i$

$\therefore$ N = even integer,

$N = 2, 4, 6, 8 \ldots$

Dirac chose $N=4$

From now onwards, put $N=4$, i.e.

$\gamma^\mu = 4 \times 4$ matrix

and the wavefunction $\psi(\underset{\sim}{x})$

$$\psi(\underset{\sim}{x}) = \begin{pmatrix} \psi_1(\underset{\sim}{x}) \\ \psi_2(\underset{\sim}{x}) \\ \psi_3(\underset{\sim}{x}) \\ \psi_4(\underset{\sim}{x}) \end{pmatrix}$$

(iv) Is γ^μ Hermitian?

To answer this, find the Dirac Hamiltonian first from the Dira equation

Dirac equation is

$$\not{p}\,\psi(\underset{\sim}{x}) = mc\,\psi(\underset{\sim}{x}) \qquad \not{p} = p_\mu \gamma^\mu$$

$$\therefore (\gamma^0 p^0 - \underset{\sim}{\gamma} \cdot \underset{\sim}{p})\,\psi(\underset{\sim}{x}) = mc\,\psi(\underset{\sim}{x})$$

$$\gamma^0 p^0 \psi(\underset{\sim}{x}) = (\underset{\sim}{\gamma}\cdot\underset{\sim}{p} + mc)\psi(\underset{\sim}{x})$$

$$p^0 \psi(\underset{\sim}{x}) = (\gamma^0 \underset{\sim}{\gamma}\cdot\underset{\sim}{p} + mc\gamma^0)\psi(\underset{\sim}{x})$$

cf Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\underset{\sim}{x}) = H\psi(\underset{\sim}{x})$$

As $p_\mu = i\hbar\frac{\partial}{\partial x^\mu}$

$$\therefore\quad i\hbar\frac{\partial}{\partial x^0}\psi(\underset{\sim}{x}) = (\gamma^0 \underset{\sim}{\gamma}\cdot\underset{\sim}{p} + mc\gamma^0)\psi(\underset{\sim}{x})$$

$$i\hbar\frac{\partial}{\partial t}\psi(\underset{\sim}{x}) = (c\gamma^0\underset{\sim}{\gamma}\cdot\underset{\sim}{p} + mc^2\gamma^0)\psi(\underset{\sim}{x})$$

$x^0 = ct$

$$\therefore\quad H = c\gamma^0\underset{\sim}{\gamma}\cdot\underset{\sim}{p} + mc^2\gamma^0$$
$$= c\,\underset{\sim}{\alpha}\cdot\underset{\sim}{p} + \beta mc^2,$$

$\underset{\sim}{\alpha} \equiv \gamma^0\underset{\sim}{\gamma}$

$\beta \equiv \gamma^0$

Now $H^\dagger = H$

$\therefore\quad \underset{\sim}{\alpha}^\dagger = \underset{\sim}{\alpha}, \quad \beta^\dagger = \beta$

can show $\gamma^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0$ (HW)

$\gamma^{0\dagger} = \gamma, \quad \gamma^i = \gamma^0\gamma^{i\dagger}\gamma^0$

$$H^\dagger = H$$

$$H = c\,\underset{\sim}{\alpha}\cdot\underset{\sim}{p} + \beta m c^2 , \qquad \underset{\sim}{p}^\dagger = \underset{\sim}{p}$$

$$\underset{\sim}{\alpha}^\dagger = \underset{\sim}{\alpha} , \qquad \beta^\dagger = \beta$$

$$\beta = \gamma^0 \qquad \therefore \gamma^{0\dagger} = \gamma^0$$

Now $\quad \underset{\sim}{\alpha} = \gamma^0 \underset{\sim}{\gamma} , \qquad \underset{\sim}{\alpha}^\dagger = \underset{\sim}{\gamma}^\dagger \gamma^0$

$$\underset{\sim}{\alpha} = \underset{\sim}{\alpha}^\dagger \rightarrow \gamma^0 \underset{\sim}{\gamma} = \underset{\sim}{\gamma}^\dagger \gamma^0$$

$$\gamma^0 \gamma^\dagger \gamma^0 = \gamma^{0\,2} \underset{\sim}{\gamma} = \underset{\sim}{\gamma}$$

can write

$$\gamma^\mu = \gamma^0 \gamma^{\mu\dagger} \gamma^0 \qquad (HW)$$

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

We have already obtained a correct relativistic equation for a spin $\frac{1}{2}$ particle, the Dirac equation

$$\not{p}\,\psi(x) = mc\,\psi(x) \qquad \not{p} = p_\mu \gamma^\mu$$

We now want to construct a free particle solution of the Dirac equation.

Recall:

In non-relativistic quantum mechanics, the equation of motion is the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(\underline{x}) = H\psi(\underline{x}), \qquad H = \frac{\underline{p}^2}{2m} + V(\underline{x})$$

For a free particle $H = \frac{\underline{p}^2}{2m}$, no potential force field, $V(\underline{x}) = 0$, $\rightarrow$ $i\hbar\frac{\partial}{\partial t}\psi(\underline{x}) = -\frac{\hbar^2}{2m}\nabla^2\psi(\underline{x})$

The free particle is a plane wave

$$\psi(\underline{x}, t) = \text{const.}\ e^{i(\underline{k}\cdot\underline{x} - \omega t)} \text{ or } e^{-i(\underline{k}\cdot\underline{x} + \omega t)}$$

$$\underline{p} = \hbar \underline{k}, \qquad E = \hbar\omega, \qquad E = \frac{\underline{p}^2}{2m}$$

Note: $e^{-i(\underline{k}\cdot\underline{x} - \omega t)}$, $e^{i(\underline{k}\cdot\underline{x} + \omega t)}$ not allowed photon is described by the Maxwell equation

$$\partial_\mu \partial^\mu \underline{A}(\underline{x}) = 0 \qquad \text{or} \qquad \Box^2 \underline{A}(\underline{x}) = 0$$

$\Box^2$ = D'lembertian

$$\partial_\mu A^\mu(\underline{x}) = 0 \qquad \text{Lorentz condition}$$

Free photon is a plane wave

$$A_\mu(\underline{x}) = \text{const}\ e^{-i\underline{p}\cdot\underline{x}/\hbar}\ \varepsilon_\mu(\underline{p}),\quad \underline{p}^2 = 0$$

or $\underline{A}(\underline{x}) = \text{const}\ e^{-i\underline{p}\cdot\underline{x}/\hbar}\ \underline{\varepsilon}(\underline{p})$

and $\partial_\mu A^\mu(\underline{x}) = 0 \longrightarrow \underline{p}\cdot\underline{\varepsilon}(\underline{p}) = 0$

$\underline{\varepsilon}(\underline{p})$ = polarization

The relativistic spin-0 particle is described by the Klein-Gordon equation

$$\underline{p}^2\,\phi(\underline{x}) = m^2c^2\,\phi(\underline{x}),\quad \underline{p}^2 = -\hbar^2\Box^2$$

The free particle is a plane-wave

$$\phi(\underline{x}) = \text{const}\ e^{-i(\underline{p}\cdot\underline{x})/\hbar},\qquad \underline{p}^2 = m^2c^2$$

spin 0 particle or scalar particle or pseudo-scalar particle, e.g. π^0, π^+, π^- mesons

Construct the free particle solution of the Dirac equation.

$$\not{p}\,\psi(\underline{x}) = mc\,\psi(\underline{x})$$

The plane wave solution can be written as

$$\psi(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar}\,u(\underline{p})$$

or

$$\psi_\alpha(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar}\,u_\alpha(\underline{p}),$$

$$\alpha = 1, 2, 3, 4$$

$$u(\underline{p}) = \begin{pmatrix} u_1(\underline{p}) \\ u_2(\underline{p}) \\ u_3(\underline{p}) \\ u_4(\underline{p}) \end{pmatrix}$$

The unknowns are $\underline{p}$ and $u(\underline{p})$

4-momentum of the particle

bispinor

Substituting

$$\psi(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar}\, u(\underline{p})$$

into the Dirac equation

$$\gamma^\mu p_\mu \psi(\underline{x}) = mc\,\psi(\underline{x}),$$

we get

$$\gamma^\mu p_\mu u(\underline{p}) = mc\, u(\underline{p}),$$

p_μ are four numbers, not a differential operator.

Using the Dirac representation for the matrix γ^μ,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

$$\gamma^\mu i\hbar\partial_\mu \psi(\underline{x}) = mc\,\psi(\underline{x}) \rightarrow \gamma^\mu p_\mu u(\underline{p}) = mc\, u(\underline{p})$$

we have

$$(\gamma^0 p_0 + \gamma^i p_i)\, u(\underline{p}) = mc\, u(\underline{p})$$

$$(\gamma^0 p^0 - \gamma^i p^i)\, u(\underline{p}) = mc\, u(\underline{p})$$