Tutorial 2

Due: Tue 17 Sep, 2024 at 23:59 on Canvas

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indexnumber_name_Tutorial2.pdf

1. Rotation operator for a spin-1/2 system

In this tutorial, we will explicitly derive the rotation operator for a spin-1/2 system and apply the operator to derive some familiar results.

(a) Prove the following relations, also given in lecture:

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B}\,\mathbb{1} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}) \tag{1}$$

where the components of $\vec{\sigma}$ are the three Pauli matrices, and $\vec{A}$, $\vec{B}$ are two arbitrary vectors.

[Hint: Use the commutator and anti-commutator relations between the Pauli matrices to get an expression for $\sigma_i\sigma_j$. Also note that the i-th component of $\vec{A}\times\vec{B}$ is given by $(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$ where ϵ_{ijk} is the Levi-Cevita symbol and Einstein's summation convention has been used.]

(b) Set $\vec{A}=\vec{B}=\hat{u}$, where $\hat{u}$ is a unit vector. What is $(\vec{\sigma}\cdot\hat{u})^n$ when n is even? How about when n is odd?

(c) Use your result from (b) to prove that the operator for rotating a spin-1/2 system by angle α about the axis $\hat{u}$ can be written as

$$U(\alpha,\hat{u}) = \cos\frac{\alpha}{2}\mathbb{1} - i\sigma\cdot\hat{u}\sin\frac{\alpha}{2}. \tag{2}$$

Write out the rotation operator $U(\alpha,\hat{u})$ as a 2×2 matrix.

(d) Use your result from (c) to obtain the eigenstate $|{+x}\rangle$ of σ_x by rotating the eigenstate $|{+z}\rangle$ of σ_z through a suitable angle α about the axis $\hat{y}$. Compare this to the eigenstate $|{+x}\rangle$ of σ_x obtained in class from the diagonalization of the matrix for σ_x in the basis $\{|{+z}\rangle, |{-z}\rangle\}$.

2. Matrix representations of angular momentum

A quantum particle is known to have total angular momentum $j=\frac{3}{2}$.

(a) Use the eigenstates of J_z (denote them as $|j,m\rangle = |3/2,m\rangle$) as a basis and find the matrix representation of the three operators, J_x, J_y and J_z in this four-dimensional

subspace.
[Hint: You know the action of the operators J_z and $J_\pm$ on the states $|3/2, m\rangle$, and you know the relation between $J_\pm$ and J_x and J_y].
(b) Verify that the matrices you found in (a) satisfy the commutation relation $[J_x, J_y] = i\hbar J_z$.