Tuesday, 10 September 2024 9:46 am

Tut 2 Eg (1)

$$\underset{2\times 2}{(\vec{\sigma}\cdot\vec{A})}\underset{2\times 2}{(\vec{\sigma}\cdot\vec{B})} = \overbrace{\vec{A}\cdot\vec{B}}^{1\times 1}\, \underset{2\times 2}{\mathbb{1}_{2\times 2}} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}), \quad \vec{\sigma} = \sigma_x\hat{e}_x + \sigma_y\hat{e}_y + \sigma_z\hat{e}_z$$

2×2 matrices

$$\vec{A} = A_1\hat{e}_x + A_2\hat{e}_y + A_3\hat{e}_z$$

Quiz 2

Work out the matrices for L_+ and L_- for $l=1$.

using as your basis $|e_1\rangle = |l=1, m=1\rangle$

$|e_2\rangle = |l=1, m=0\rangle$

$|e_3\rangle = |l=1, m=-1\rangle$

Hence check if $\frac{1}{\sqrt{2}}(|e_2\rangle + |e_3\rangle)$ is an eigenstate of L_x.

$L_+ |l=1, m=1\rangle = 0$ $\quad$ m=1 is m_{max}. $\quad$ $L_+|e_1\rangle = 0$

$(= 0\, |l=1, m=2\rangle)$

$$L_+|l=1, m=0\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\,|l=1, m=1\rangle$$

$$= \hbar\sqrt{2}\,|l=1, m=1\rangle$$

$L_+|e_2\rangle = \hbar\sqrt{2}\,|e_1\rangle$

$$L_+|l=1, m=-1\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\,|l=1, m=0\rangle$$

$$= \hbar\sqrt{2 - (-1)(-1+1)}\,|l=1, m=0\rangle$$

$$= \hbar\sqrt{2}\,|l=1, m=0\rangle$$

$L_+|e_3\rangle = \hbar\sqrt{2}\,|e_2\rangle$

$$L_+ = \left(L_+\vec{e}_1 \,\vdots\, L_+\vec{e}_2 \,\vdots\, L_+\vec{e}_3 \right)$$

$$= \begin{pmatrix} 0 & \hbar\sqrt{2} & 0 \\ 0 & 0 & \hbar\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \hbar\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_- = L_+^\dagger = \hbar\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$L_- = L_+^\dagger = \hbar\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_x = \frac{1}{2}(L_+ + L_-) = \frac{\hbar\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

To check if $\underbrace{\frac{1}{\sqrt{2}}(|e_2\rangle + |e_3\rangle)}_{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$ is an eigenstate of L_x,

$$L_x \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ not } \parallel \text{ to } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow \frac{1}{\sqrt{2}}(|e_2\rangle + |e_3\rangle)$ is not an eigenstate of L_x. //

Angular momentum — general (last few lectures) ← j = integer or $\frac{1}{2}$-integer ≥ 0

- orbital angular momentum ← l = integer ≥ 0. ($\vec{L} = \vec{r} \times \vec{p}$)

↳ Spin angular momentum

Orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$

Eigenstates of L_z for given l are $|l, m\rangle$

Spherical harmonics:

$$Y_{l,m}(\theta, \phi) = \langle \theta, \phi | l, m \rangle$$
$$= f_{l,m}(\theta)\, e^{im\phi}$$

We have seen: l can take non-negative integer values.

For fixed l, $m = -l, -l+1, \ldots, l-1, l$ (also integers)

(W3L2) Rotate Y_{lm} by $2\pi \longrightarrow$ same $Y_{l,m}$ because m is an integer.

Now, we will show that for $\vec{L} = \vec{r} \times \vec{p}$, l cannot be half-integer.

To do this, we will explicitly use $\vec{L} = \vec{r} \times \vec{p}$

We will just show that l cannot be $\frac{1}{2}$.

We will just show that l cannot be $\frac{1}{2}$.

Prove by contradiction.

Assume $l=\frac{1}{2}$ is allowed, $m=-\frac{1}{2},\frac{1}{2}$.

Define $Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi)$ and $Y_{\frac{1}{2},\frac{1}{2}}(\theta,\phi)$

$$Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) = f_{\frac{1}{2},-\frac{1}{2}}(\theta)\, e^{-\frac{i}{2}\phi} \qquad Y_{\frac{1}{2},\frac{1}{2}}(\theta,\phi) = f_{\frac{1}{2},\frac{1}{2}}(\theta)\, e^{\frac{i}{2}\phi}$$

Show that $L_- Y_{\frac{1}{2},\frac{1}{2}}$ does not give anything proportional to $Y_{\frac{1}{2},-\frac{1}{2}}$. //

Prove by contradiction

(from Griffiths 4.3.2)

$$\vec{L} = \vec{r}\times\vec{p} = -i\hbar(\vec{r}\times\vec{\nabla})$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

gives $\vec{L} = -i\hbar\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$.

$$\left.\begin{aligned} L_z &= -i\hbar\frac{\partial}{\partial\phi} \\ L_\pm &= \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)\end{aligned}\right\} (*)$$

Assume $l=\frac{1}{2}$ is allowed.

Let's find the corresponding spherical harmonics

$Y_{\frac{1}{2},\frac{1}{2}}(\theta,\phi)$, $Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi)$ using $(*)$.

(We will show that the resulting forms do not satisfy the requirements for angular momentum eigenstates.)

$$L_z Y_{\frac{1}{2},\frac{1}{2}} = \frac{\hbar}{2} Y_{\frac{1}{2},\frac{1}{2}}$$

$$-i\hbar \frac{\partial}{\partial\phi} Y_{\frac{1}{2},\frac{1}{2}} = \frac{\hbar}{2} Y_{\frac{1}{2},\frac{1}{2}}$$

$$Y_{\frac{1}{2},\frac{1}{2}}(\theta,\phi) = f_{\frac{1}{2},\frac{1}{2}}(\theta)\, e^{i\phi/2}$$

$$L_+ Y_{\frac{1}{2},\frac{1}{2}} = 0 \qquad (m = m_{max})$$

$$\underbrace{\hbar e^{i\phi}}_{\text{to the left of } \frac{\partial}{\partial\phi}} \left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi} \right) f_{\frac{1}{2},\frac{1}{2}}(\theta)\, e^{i\phi/2} = 0$$

- factor out.

$$\frac{\partial f}{\partial\theta} e^{i\phi/2} + i\cot\theta\, f(\theta)\, \frac{i}{2} e^{i\phi/2} = 0$$

$$\frac{\partial f}{\partial\theta} - \frac{f}{2}\cot\theta = 0$$

$$\frac{\partial f}{\partial\theta} = \frac{\cot\theta}{2} f$$

$$\frac{1}{f} df = \frac{1}{2} \frac{\cos\theta}{\sin\theta} d\theta$$

$$\ln f = \ln\left(k(\sin\theta)^{1/2}\right)$$

$$f = k\sqrt{\sin\theta}$$

$$\text{So } Y_{\frac{1}{2},\frac{1}{2}}(\theta,\phi) = \underset{\substack{\uparrow \\ k \text{ determined by normalization.}}}{k} \sqrt{\sin\theta}\; e^{i\phi/2}$$

Likewise $L_- Y_{\frac{1}{2},-\frac{1}{2}} = 0 \qquad (m = m_{min})$

$$\underbrace{\hbar e^{-i\phi}}_{\text{factor out}} \left(-\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi} \right) f e^{-i\phi/2} = 0$$

$$-\frac{\partial f}{\partial\theta} e^{-i\phi/2} + i f \cot\theta \left(-\frac{i}{2}\right) e^{-i\phi/2} = 0$$

$$\frac{\partial f}{\quad} = \frac{\cot\theta}{\quad} f$$

$$\frac{\partial f}{\partial \theta} = \frac{\cot\theta}{2} f$$

$$\Rightarrow \quad Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) = \underset{\substack{\uparrow \\ \text{same } k \text{ as for } Y_{\frac{1}{2},\frac{1}{2}}}}{k} \sqrt{\sin\theta}\; e^{-i\phi/2}$$

Now we will show that $L_- Y_{\frac{1}{2},\frac{1}{2}}$ does not give anything proportional to $Y_{\frac{1}{2},-\frac{1}{2}}$.

$$L_- Y_{\frac{1}{2},\frac{1}{2}} = k\hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right)\sqrt{\sin\theta}\; e^{i\phi/2}$$

$$= k\hbar e^{-i\phi}\left(e^{i\phi/2}\left(-\tfrac{1}{2}(\sin\theta)^{-1/2}\cos\theta\right) + i\cot\theta\sqrt{\sin\theta}\left(\tfrac{i}{2}\right)e^{i\phi/2}\right)$$

$$= -k\hbar e^{-i\phi/2}\left(\frac{\cot\theta}{2}\sqrt{\sin\theta} + \frac{1}{2}\cot\theta\sqrt{\sin\theta}\right)$$

$$= -k\hbar e^{-i\phi/2}\cot\theta\sqrt{\sin\theta}$$

not a constant times $Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi)$.

Contradiction.

So orbital angular momentum cannot have $\ell = \frac{1}{2}$. //

Spin angular momentum

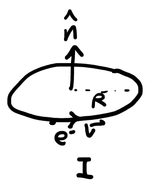
Most direct evidence — Stern-Gerlach experiment 1922. (S-G)

Background and motivation for S-G experiment

- Stern and Gerlach were suspicious about the quantization of orbital angular momentum, predicted by Bohr in his model of the atom.
- They wanted to test Bohr's model.

How to test Bohr's model?

Angular momentum in charged particles, like electrons, is associated with a magnetic moment μ.



$\vec{A} = \hat{n}$ (Area enclosed

$$\vec{\mu} = \frac{1}{c}|I|\vec{A} \quad (CGS), \qquad \vec{\mu} = |I|\vec{A} \quad (SI)$$

$\vec{I} = \lambda\vec{v}$, λ linear charge density

$\vec{A} = \hat{n}$ (Area enclosed by loop)

$$\vec{I} = \lambda \vec{v} \quad , \lambda \text{ linear charge density}$$

$$= \frac{Q}{2\pi R} \vec{v}$$

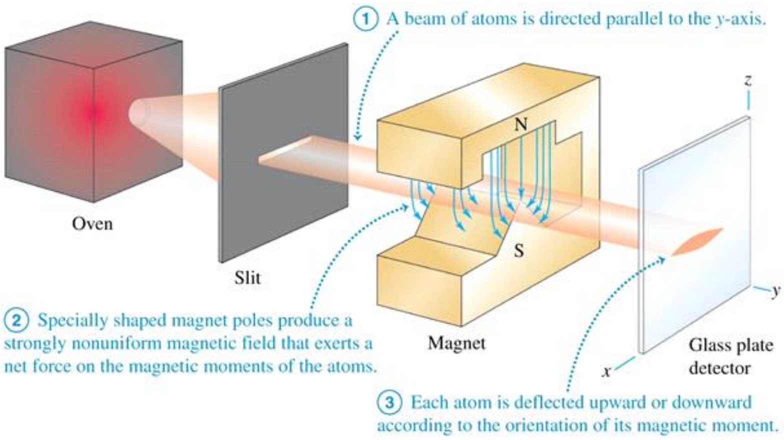
$$\vec{A} = \hat{n} \pi R^2$$

$$\vec{\mu} = \frac{1}{c} \hat{n} \frac{Q}{2\pi R} |\vec{v}| \pi R^2$$

$$= \frac{1}{c} \hat{n} \frac{Q}{2} |\vec{v}| R$$

$$\vec{L} = \vec{r} \times \vec{p} = \hat{n} m v R \quad , v = |\vec{v}|$$

So $\boxed{\vec{\mu} = \frac{Q}{2mc} \vec{L} \quad \text{(CGS)}}$



magnetic field with a gradient in the z-direction.

S-G experiment $\rightarrow$ measure $\vec{\mu}$.

How? Use B field. $\quad U = -\vec{\mu} \cdot \vec{B}$

Need a force $\vec{F} = -\vec{\nabla} U$

$$= \vec{\nabla}(\vec{\mu} \cdot \vec{B})$$

$$\underset{\text{see fig}}{=} \mu_z \frac{dB}{dz} \hat{z}$$

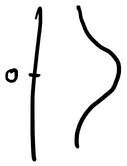
S & G used Ag atoms.

- charge neutral. So Lorentz force $\vec{F} = \frac{q}{c}(\vec{v} \times \vec{B})$ is zero.

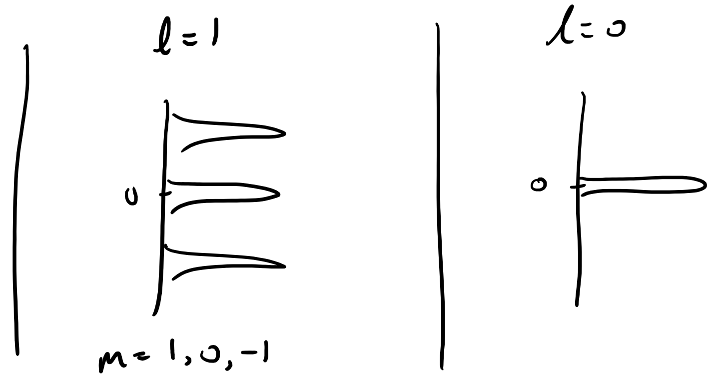
only force $\vec{F} = \mu_z \frac{dB}{dz} \hat{z}$.

Possible observations?

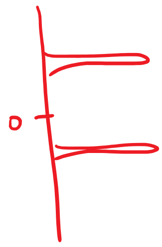
Classical (no quantization)



μ_z takes a continuous range of values



Actually observed



surprising

$\Rightarrow$ exactly two distinct values of μ_z.

$\Rightarrow$ ______ a ______ J_z

$$J_z |j, m\rangle = m\hbar |j, m\rangle$$

$\uparrow$ $(2j+1)$ possible values of m for given j.

$$2j+1 = 2 \Rightarrow j = \tfrac{1}{2}$$

$\Rightarrow$ (earlier in lecture) This is not orbital angular momentum.

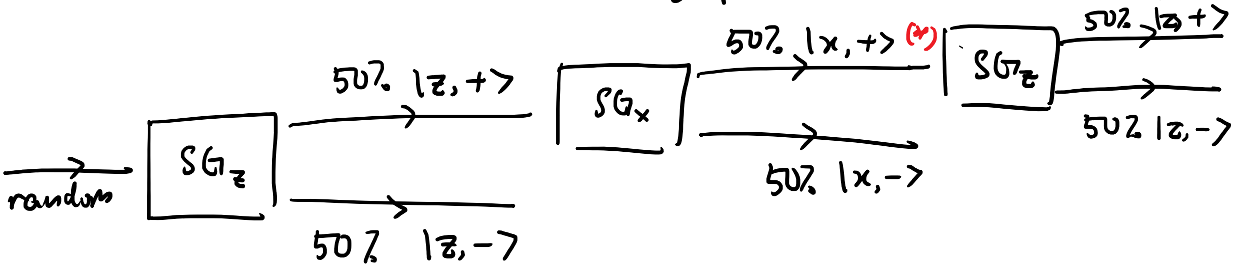
Note: Ag atoms have a single valence electron — s orbital.

s electron: $l = 0$

(Next time: addition of angular momentum. $\vec{J} = \vec{L} + \vec{S}$.

$l = 0, \quad s = \tfrac{1}{2}$

Only possible value of $j = \tfrac{1}{2}$.)



(*)

Given you are in state $|z,+\rangle$,

Probability of measuring $|x,+\rangle$

$$= |\langle x,+|z,+\rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1) \right|^2$$

$$= \frac{1}{2} \quad //$$

$|z,+\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|z,-\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|x,+\rangle \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\langle x,+| \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix}$

It took a while to understand spin.

<u>What we know now</u> :

- Spin is an <u>intrinsic</u> property of the electron
  (<u>not</u> the electron spinning about its axis)
- Spin is a form of angular momentum.
- In general, spin angular momentum can take integer or half-integer values
  eg. multiple spin-$\frac{1}{2}$ particles.
- Other quantum particles can also have spin.
- A single electron is a spin-$\frac{1}{2}$ particle ($s \equiv j = \frac{1}{2}$)
  and has two possible spin states (called spin up and spin down)
  $m = \pm\frac{1}{2}$.
- Spin eigenstates are abstract. — cannot be visualized in real space.

(for info) $\psi(r) = \begin{pmatrix} \psi_{\uparrow}(r) \\ \psi_{\downarrow}(r) \end{pmatrix}$ spinor

<u>Spin magnetic moment of electrons</u>

$\vec{\mu}$ is anti-parallel to $\vec{S}$ for an electron.

$\uparrow \langle \vec{S} \rangle$

$\downarrow \langle \vec{\mu} \rangle$

Before: $\vec{\mu} = \frac{Q}{2m_e} \vec{L}$ (CGS)

For an e^-, $Q = -e$, $e > 0$.

orbital angular momentum magnetic moment
$$\vec{\mu} = -\underbrace{\left(\frac{e\hbar}{2m_e c}\right)}_{\text{Bohr magneton}} \frac{\vec{L}}{\hbar} = -\mu_B \frac{\vec{L}}{\hbar}$$

$$\mu_B = \frac{e\hbar}{2m_e c} \ \text{(CGS)}, \qquad \mu_B = \frac{e\hbar}{2m_e} \ \text{(SI)}$$

$$|\mu_B| \sim 6 \times 10^{-5} \ \text{eV/Tesla}.$$

For spin angular momentum in an electron, we need a factor g_e (Landé g-factor)

$$\boxed{\vec{\mu} = -g_e \mu_B \frac{\vec{S}}{\hbar}}, \quad g_e \approx 2 \ \text{for an electron.}$$

$g_e > 0$, $\mu_B > 0$

$\Rightarrow$ $\vec{\mu}$ is antiparallel to $\vec{S}$ for an electron.

Take the z-component

$$\mu_z = -g_e \mu_B \frac{S_z}{\hbar}$$

$$\frac{S_z}{\hbar} \begin{array}{l} \nearrow \ \frac{1}{2} \ \text{spin up} \\ \searrow \ -\frac{1}{2} \ \text{spin down.} \end{array}$$

$$\mu_z = \begin{cases} -\mu_B & \text{spin up} \\ \mu_B & \text{spin down} \end{cases}$$

W5L2.

Spin angular momentum (continued)

- Spin magnetic moment (S-G expt)

$j = \frac{1}{2}$

$\vec{\mu}$ anti-// to $\vec{S}$ for an e^-.

- Spin-orbit coupling
- Dirac equation (not examinable) – relativistic description of a free particle.
- Pauli matrices

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Properties of Pauli matrices

For all k, $\sigma_k^2 = \sigma_k \sigma_k = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ — (1)

$\sigma_1 \sigma_2 = -\sigma_2 \sigma_1$, $\sigma_1 \sigma_3 = -\sigma_3 \sigma_1$, $\sigma_2 \sigma_3 = -\sigma_3 \sigma_2$

$\{ \sigma_k, \sigma_j \} = \sigma_k \sigma_j + \sigma_j \sigma_k = 2 \delta_{kj} \mathbb{1}$ — (2)

anti-commutator

(1) & (2) are properties of matrices that show up in Dirac's equation

- also a see a term $-\vec{\mu} \cdot \vec{B}$ where $\vec{\mu} \propto \vec{\sigma}$ and see a term – spin-orbit term.