PC3261: Classical Mechanics II

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National University of Singapore
Department of Physics
Faculty of Science

Lecture 0: Course Briefing

About course

- PC3261 – Classical Mechanics II
- 4 units
- Prerequisites: (PC2032 or PC2132) and PC2174A or departmental approval
- Preclusions: -

About myself

- Contact
  - Kenneth HONG Chong Ming (call me Kenneth)
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  - Email: phyhcmk@nus.edu.sg
- Education
  - <1998: primary, secondary and pre-university in Malaysia
  - 1998–2002: B. Sc. and B. Sc. (Hons.), Physics, NUS
  - 2002–2006: M. Sc. (part time), Physics, NUS
  - 2007–2013: Ph. D. (part time), Physics, NUS
- Employment
  - 2002–2006: teaching assistant, Physics, NUS
  - 2007–2014: instructor, Physics, NUS
  - 2015–2019: lecturer, Physics, NUS
  - 2020–now: senior lecturer, Physics, NUS

About syllabus

Official syllabus

This elective course assumes knowledge of and is a sequel to PC2032. A good command of calculus and linear algebra is desirable. It is intended for students who wish to acquire a deeper understanding of our Mechanical Universe. It considers the principles of relativistic, Lagrangian and Hamiltonian mechanics, and aims to establish a bridge to the principles of modern Physics. Topics covered include: dynamics with central forces, bound and unbound orbits, scattering; relativistic kinematics and dynamics of a particle, Lorentz transformations, four-dimensional notations; Lagrangian mechanics, the action principle, Euler-Lagrange equation; Hamiltonian mechanics.

About course structure

- $\sim$20 lectures, Tuesday/Friday 12–2pm S16-04-36
  - incomplete slides (before lecture) and complete slides (after lecture) will be uploaded to Canvas
- $\sim$20 in-class worksheets (LectureACT)
  - completed worksheets in PDF format are to be submitted to Canvas
- $\sim$8 assignments
  - answer scripts in PDF format are to submitted to Canvas
- 1 test: 21 March (week 9)
  - closed book with one A4-sized helpsheet
- 1 exam: 2 May 2:30–4:30pm
  - closed book with one A4-sized helpsheet

About references

- "Classical dynamics of particles and systems", 5th edition, Stephen T. Thornton and Jerry B. Marion, Cengage Learning, 2003
- "Analytical mechanics", Grant R. Fowles and George L. Cassiday, 7th edition, Cengage Learning, 2004
- "Classical mechanics", Tom W. B. Kibble and Frank H. Berkshire , 5th edition, Imperial College Press, 2004

About assessments

- Test: 15%
- LectureACT: 15%
- Assignments: 45%
- Exam: 25%

Lecture 1: Kinematics

Kronecker delta symbol

- **Kronecker delta symbol**: completely symmetric

$$\delta_{ij} = \delta_{ji}\,, \qquad \delta_{ij} \equiv \begin{cases} 0\,, & i \neq j \\ 1\,, & i = j \end{cases}\,, \qquad i,j = 1,2,3$$

- Useful identities:

$$A_i = \sum_{j=1}^{3} \delta_{ij} A_j\,, \qquad \sum_{k=1}^{3} \delta_{ik}\delta_{kj} = \delta_{ij}\,, \qquad \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} = 3$$

Levi-Civita symbol

- **Levi-Civita symbol**: completely anti-symmetric

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj}\,, \qquad \epsilon_{123} \equiv +1\,, \qquad i,j,k = 1,2,3$$

- Product of Levi-Civita symbols:

$$\epsilon_{ijk}\epsilon_{mnr} = \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{ir} \\ \delta_{jm} & \delta_{jn} & \delta_{jr} \\ \delta_{km} & \delta_{kn} & \delta_{kr} \end{vmatrix}$$

- Useful identities:

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\,, \quad \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{mjk}\epsilon_{njk} = 2\delta_{mn}\,, \quad \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{ijk} = 6$$

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

$$\begin{aligned}
\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mnk} &= \sum_{k=1}^{3} \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{ik} \\ \delta_{jm} & \delta_{jn} & \delta_{jk} \\ \delta_{km} & \delta_{kn} & \delta_{kk} \end{vmatrix} \\
&= \sum_{k=1}^{3} \delta_{im} \begin{vmatrix} \delta_{jn} & \delta_{jk} \\ \delta_{kn} & \delta_{kk} \end{vmatrix} - \sum_{k=1}^{3} \delta_{in} \begin{vmatrix} \delta_{jm} & \delta_{jk} \\ \delta_{km} & \delta_{kk} \end{vmatrix} + \sum_{k=1}^{3} \delta_{ik} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \sum_{k=1}^{3} \delta_{im} \left(\delta_{jn}\delta_{kk} - \delta_{jk}\delta_{kn}\right) - \sum_{k=1}^{3} \delta_{in} \left(\delta_{jm}\delta_{kk} - \delta_{jk}\delta_{km}\right) \\
&\qquad + \sum_{k=1}^{3} \delta_{ik} \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) \\
&= 3\delta_{im}\delta_{jn} - \delta_{im}\delta_{jn} - 3\delta_{in}\delta_{jm} + \delta_{in}\delta_{jm} + \delta_{jm}\delta_{in} - \delta_{jn}\delta_{im} \\
&= \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \qquad \blacksquare
\end{aligned}$$

Cartesian coordinate system

- Cartesian coordinates: $(x_1, x_2, x_3) \equiv (x, y, z)$

$$-\infty < x < \infty\,, \qquad -\infty < y < \infty\,, \qquad -\infty < z < \infty$$

- Cartesian unit basis vectors: $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \equiv (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \qquad \rightarrow \qquad \begin{cases} \hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_y \cdot \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z = 1 \\ \hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_y \cdot \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_x = 0 \end{cases}$$

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \sum_{k=1}^{3} \epsilon_{ijk}\, \hat{\mathbf{e}}_k \qquad \rightarrow \qquad \begin{cases} \hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_y \times \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_y \end{cases}$$

- Cartesian unit basis vectors are constant

Position vector

- **Position** of a particle in the space is specified by a vector relative to the *spatial origin* of a given *reference frame* known as **position vector**

- Position vector in the Cartesian coordinate system: (x, y, z) are the Cartesian coordinates of the particle

$$\mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z = \sum_{i=1}^{3} x_i\,\hat{\mathbf{e}}_i$$

- Motion of the particle traces a **trajectory** in the space and can be described mathematically by an one-dimensional **curve**

- Trajectory of the motion of particle can be specified by the position vector *parameterized* by **time** relative to the *temporal origin* of the reference frame

$$\mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y + z(t)\,\hat{\mathbf{e}}_z = \sum_{i=1}^{3} x_i(t)\,\hat{\mathbf{e}}_i$$

Velocity vector

- **Velocity vector**: rate of change of the position vector with respect to time

$$\mathbf{v}(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \equiv \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} \equiv \dot{\mathbf{r}}(t)$$

- Velocity vector is *tangent* to the trajectory of the particle at any given instant of time

- **Speed**: magnitude of the velocity vector

$$v(t) \equiv |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)}$$

- Cartesian coordinate system:

$$\dot{\mathbf{r}}(t) = \dot{x}(t)\,\hat{\mathbf{e}}_x + \dot{y}(t)\,\hat{\mathbf{e}}_y + \dot{z}(t)\,\hat{\mathbf{e}}_z \quad \Rightarrow \quad \dot{r}(t) \equiv |\dot{\mathbf{r}}(t)| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}$$

Acceleration vector

- **Acceleration vector**: rate of change of the velocity vector with respect to time

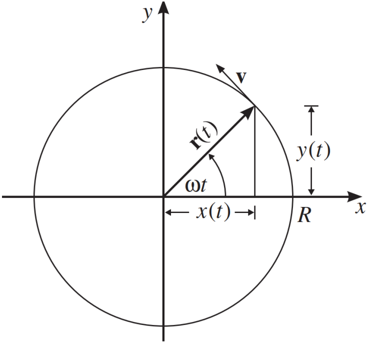
$$\mathbf{a}(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} \equiv \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} \equiv \dot{\mathbf{v}}(t) = \frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} \equiv \ddot{\mathbf{r}}(t)$$

- Cartesian coordinate system:

$$\ddot{\mathbf{r}}(t) = \ddot{x}(t)\,\hat{\mathbf{e}}_x + \ddot{y}(t)\,\hat{\mathbf{e}}_y + \ddot{z}(t)\,\hat{\mathbf{e}}_z \quad \Rightarrow \quad \ddot{r}(t) \equiv |\ddot{\mathbf{r}}(t)| = \sqrt{\ddot{x}^2(t) + \ddot{y}^2(t) + \ddot{z}^2(t)}$$

Example: Uniform circular motion

- A particle moves in a circle lying in the xy plane (centered at the origin and radius R) with constant angular speed ω counter-clockwise as viewed from $+z$ axis. The particle is on the $+x$ axis at $t = 0$



EXERCISE 1.1: Find the particle's velocity and acceleration vectors. What are the magnitude and direction of the particle's acceleration?

$$\mathbf{r}(t) = R\cos\omega t\,\hat{\mathbf{e}}_x + R\sin\omega t\,\hat{\mathbf{e}}_y \qquad \blacksquare$$

$$r(t) \equiv |\mathbf{r}(t)| = R \qquad \blacksquare$$

$$\mathbf{v}(t) \equiv \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = -R\omega\sin\omega t\,\hat{\mathbf{e}}_x + R\omega\cos\omega t\,\hat{\mathbf{e}}_y \qquad \blacksquare$$

$$\mathbf{v}(t)\cdot\mathbf{r}(t) = 0$$

$$v(t) \equiv |\mathbf{v}(t)| = R\omega \qquad \blacksquare$$

$$\mathbf{a}(t) \equiv \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = -R\omega^2\cos\omega t\,\hat{\mathbf{e}}_x - R\omega^2\sin\omega t\,\hat{\mathbf{e}}_y \qquad \blacksquare$$

$$\mathbf{a}(t)\cdot\mathbf{r}(t) = -R^2\omega^2 \qquad \blacksquare$$

$$a(t) \equiv |\mathbf{a}(t)| = R\omega^2 \qquad \blacksquare$$

Another mathematical description of trajectory

- Trajectory of the motion of particle can also be represented mathematically by the position vector parameterized by **arc length** along the trajectory

- Arc length:

$$s(t) = \int_0^t \mathrm{d}s = \int_0^t |\mathrm{d}\mathbf{r}| = \int_0^t \sqrt{\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]^2 + \left[\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right]^2 + \left[\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right]^2}\,\mathrm{d}t$$

- Speed:

$$v(t) = |\mathbf{v}(t)| = \left|\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\right| = \frac{\mathrm{d}s(t)}{\mathrm{d}t}$$

- A set of three orthogonal unit vectors, parameterized by arc length, can be constructed at each point of the trajectory