General Lorentz transformation Λ

$$\underline{x} \rightarrow \underline{x}' = \Lambda \underline{x}$$

Λ must satisfy

$$g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

In matrix form

$$\Lambda^t g \Lambda = g$$

Taking determinant both sides,

$$\det(\Lambda^t g \Lambda) = \det g \rightarrow \det \Lambda^t \cdot \det g \cdot \det \Lambda = \det g$$

$\therefore$ $\det \Lambda^t \cdot \det \Lambda = 1$

$(\det \Lambda)^2 = 1$ $\qquad \because \det \Lambda^t = \det \Lambda$

$\therefore$ $\det \Lambda = \pm 1$

cf in 3-dim space, $\det \mathcal{R} = \pm 1$, $\mathcal{R}$ = rotation matrix

Also $\Lambda^0{}_0 > +1$ or $\Lambda^0{}_0 < -1$

Proof:

$$g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \cdot \Lambda^{\mu}{}_{\beta} = g_{\alpha\beta}$$

$\hookrightarrow$ can be written as, if $g_{\mu\nu} = g_{\nu\mu}$ symmetric,

setting $\alpha = 0 = \beta$ $\qquad g_{\mu\nu} \Lambda^{\nu}{}_{\alpha} \Lambda^{\mu}{}_{\beta} = g_{\alpha\beta}$

$$g_{\mu\nu} \Lambda^{\mu}{}_{0} \cdot \Lambda^{\nu}{}_{0} = g_{00} = +1$$

sum over μ and ν, $\quad \mu, \nu = 0, 1, 2, 3.$

say, sum over μ first. Put $\mu=0$, then $\mu=j$

$$g_{0\nu}\Lambda^0{}_0\Lambda^\nu{}_0 + g_{j\nu}\Lambda^j{}_0\Lambda^\nu{}_0 = +1$$

Now sum over ν

$$g_{00}\Lambda^0{}_0\Lambda^0{}_0 + \underset{\to 0}{g_{0i}}\Lambda^0{}_0\Lambda^i{}_0$$

$$+ \underset{\to 0}{g_{j0}}\Lambda^j{}_0\Lambda^0{}_0 + g_{ji}\Lambda^j{}_0\Lambda^i{}_0 = +1$$

As $g_{00}=+1$, $g_{ij}=0\ \forall\, i\neq j$, and $g_{11}=g_{22}=g_{33}=-1$

$(g_{ij} = -\delta_{ij}$, δ_{ij} = Kronecker delta$)$

$$\therefore \Lambda^0{}_0\Lambda^0{}_0 - \Lambda^i{}_0\Lambda^i{}_0 = 1$$

$$\therefore (\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0\Lambda^i{}_0$$

since $(\Lambda^i{}_0\Lambda^i{}_0) \geqslant 0$

$$\therefore (\Lambda^0{}_0)^2 \geqslant 1$$

i.e. $\Lambda^0{}_0 \geqslant +1$ or $\Lambda^0{}_0 \leqslant -1$

So the set of Lorentz transformations can be divided into 4 subsets according to

$\det\Lambda = \pm 1$, $\Lambda^0{}_0 \geqslant +1$, $\Lambda^0{}_0 \leqslant -1$

e.g. $L^{\uparrow}_{+}$ $\longrightarrow$ $\Lambda^0{}_0 \geqslant +1$ restricted Lorentz group

$\longleftarrow$ $\det\Lambda = +1$

say, sum over μ first. Put $\mu=0$, then $\mu=j$

$$g_{0\nu}\Lambda^0{}_0\Lambda^\nu{}_0 + g_{j\nu}\Lambda^j{}_0\Lambda^\nu{}_0 = +1$$

Now sum over ν

$$g_{00}\Lambda^0{}_0\Lambda^0{}_0 + \overset{0}{g_{0i}}\Lambda^0{}_0\Lambda^i{}_0$$

$$+ \overset{0}{g_{j0}}\Lambda^j{}_0\Lambda^0{}_0 + g_{ji}\Lambda^j{}_0\Lambda^i{}_0 = +1$$

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since $(\Lambda^i{}_0\Lambda^i{}_0) \geq 0$

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i.e. $\Lambda^0{}_0 \geq +1$ or $\Lambda^0{}_0 \leq -1$

So the set of Lorentz transformations can be divided into 4 subsets according to

$\det\Lambda = \pm 1$, $\Lambda^0{}_0 \geq +1$, $\Lambda^0{}_0 \leq -1$

e.g. $L_+^\uparrow$ $\longrightarrow$ $\Lambda^0{}_0 \geq +1$ restricted Lorentz group
$\longleftarrow$ $\det\Lambda = +1$

$$L_+^\uparrow \begin{cases} \to \Lambda^0{}_0 > +1 \\ \to \det \Lambda = +1 \end{cases}$$

$L_+^\uparrow$ is a subset s.t. $\Lambda^0{}_0 > +1$

and $\det \Lambda = +1$

restricted Lorentz trans

this subset forms a group.

$L_-^\uparrow$ ($\uparrow \to \Lambda^0{}_0$, $- \to \det \Lambda$) subset contains space inversion

not a group

orthochronous transformation.

$L_+^\downarrow$ contains time-space inversion.

extended Lorentz transformations

not a gp.

$L_-^\downarrow$ contains time inversion

orthochorous trans.

not a gp

$L_+^\uparrow \cup L_-^\uparrow$ = orthochronous group

$L_+^\uparrow \cup L_+^\downarrow$ = extended Lorentz group

$L_+^\uparrow \cup L_-^\downarrow$ = orthochorous group

$L_+^\uparrow$ = restricted Lorentz group.

Introduce scalar, vector, tensor

A scalar is a one-component entity that remains unchanged under the Lorentz tran Λ

Let ϕ be a scalar, that means under Λ : $\underline{x} \to \underline{x}' = \Lambda \underline{x}$, we have

$$\to \qquad \phi \xrightarrow{\Lambda} \phi' \equiv \Lambda \phi = \phi$$

If ϕ depends on space time, then $\phi(\underline{x})$ is a scalar field which means

$$\phi(\underline{x}) \to \phi'(\underline{x}') = \phi(\underline{x})$$
$$\underline{x}' = \Lambda \underline{x}$$

Note: $\underline{x}^2$ is a scalar

$$\underline{x}'^2 = \underline{x}^2$$
$$\underline{x}^2 = \underline{x} \cdot \underline{x} = g_{\mu\nu} x^\mu x^\nu$$

A 4-component entity, say $\underline{A}$, is a vector if under Lorentz tran Λ,

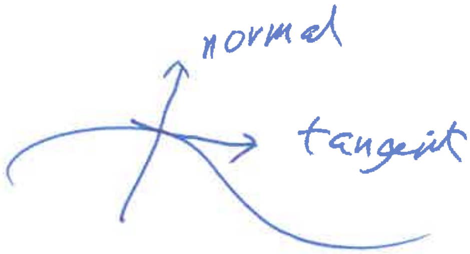
$$\underline{A} \to \underline{A}' = \Lambda \underline{A} \qquad (\underline{x}' = \Lambda \underline{x})$$

If we choose basis, can write

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu$$

(There are two types of base)

Define a vector by tangent to a curve



n dim

At *any* point of a curve, can draw tangent or normal

$\longrightarrow$ 2 types of basis

In the tangent (vector) space, basis $\underset{\sim}{e}_i$

In the 'normal' space, basis $\underset{\sim}{E}^i$ (covector)

$\underset{\sim}{e}_i \cdot \underset{\sim}{E}^j = \delta_i^j$

Define $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = g_{ij}$ $\quad i, j = 1, 2, \cdots n$

$\underset{\sim}{E}^i \cdot \underset{\sim}{E}^j = g^{ij}$

Given an abstract vector $\underset{\sim}{A}$, we can use $\underset{\sim}{e}_i$ as a basis or $\underset{\sim}{E}^i$ as a basis,

$$\underset{\sim}{A} = A^i \underset{\sim}{e}_i \quad \text{or} \quad \underset{\sim}{A} = A_i \underset{\sim}{E}^i$$

To relate A^i with A_i:

$$A^i \underset{\sim}{e}_i = A_j \underset{\sim}{E}^j \qquad i, j = 1, \cdots n$$

$$A^i \underset{\sim}{e}_i \cdot \underset{\sim}{e}_l = A_j \underset{\sim}{E}^j \cdot \underset{\sim}{e}_l = A_j \delta^j_l = A_l$$

(by construction)

$$LHS = A^i g_{il}$$

$$\longrightarrow \quad A^i g_{il} = A_l$$

A^i = contravariant

A_i = covariant

symmetric

$\rightarrow \quad A_i = g_{il} A^l \qquad (g_{il} = g_{li})$

$\rightarrow \quad A_\mu = g_{\mu\nu} A^\nu, \qquad A^\mu = g^{\mu\nu} A_\nu$

Examples:

$\underset{\sim}{x} = (x^0, \underset{\sim}{x}) \qquad$ 4-vector

Define 4-vector velocity or 4-velocity

$$\underset{\sim}{W} = \frac{d\underset{\sim}{x}}{d\tau} \qquad \tau = \text{proper time}$$

$$ds^2 = dx_\mu dx^\mu = g_{\mu\nu} dx^\mu dx^\nu$$

$$ds^2 = dx^{0\,2} - dx^i dx^i$$

$$= dx^{0\,2} \left(1 - \frac{dx^i}{dx^0}\frac{dx^i}{dx^0}\right) \qquad x^0 = ct$$

$$= dx^{0\,2} \left(1 - \frac{1}{c^2} v^i v^i\right) \qquad v^i \equiv \frac{dx^i}{dt}$$

$$= dx^{0\,2} (1 - \beta^2) \qquad \beta = \frac{\underset{\sim}{v}}{c}$$

$$\gamma^2 = \frac{1}{1-\beta^2}$$

$$= \frac{dx^{0\,2}}{\gamma^2} = \frac{c^2}{\gamma^2} dt^2$$

$d\tau$ = proper time

$$\equiv \frac{ds}{c} = \frac{1}{\gamma} dt$$

As ds is a scalar and c is a scalar wrt Lorentz tran, so $d\tau$ is a scalar. proper time is a scalar

The 4-velocity $\underline{W} = \dfrac{d\underline{x}}{d\tau} = \dfrac{\text{4-vector}}{\text{scalar}}$

$\longrightarrow$ $\underline{W}$ is a 4-vector

$$\rightarrow \quad \underline{W}^2 \overset{\underline{W}\cdot\underline{W}}{=} W_\mu W^\mu = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}$$

$$= \frac{ds^2}{d\tau \cdot d\tau} \qquad \because ds^2 = dx_\mu dx^\mu$$

$$d\tau = \frac{ds}{c}$$

$$= c^2$$

HW: $W^0 = ?$

$W^i = ?$ $i = 1, 2, 3$

So, for the 4-velocity $\underline{W}$, its magnitude squared is a constant, c^2

Define 4-momentum

$$\underline{P} = m_0 \underline{W} \qquad m_0 = \text{rest mass}$$

m_0 is a scalar or invariant under Lorentz tran $\therefore$

$$\underline{P}^2 = \underline{P}\cdot\underline{P} = P_\mu P^\mu$$

$$= g_{\mu\nu} P^\nu \cdot P^\mu$$

$$P^2 = m_0^2 W^2$$

$$(p^{0\,2} - \underline{p}^2 = m_0^2 c^2) \leftarrow \underline{P}^2 = m_0^2 c^2 \qquad \text{HW}$$

rest mass

Define 4-force,

$$\underline{f} = \frac{d\underline{P}}{d\tau} = m_0 \frac{d\underline{W}}{d\tau} \quad . \quad \underline{f} = \frac{d\underline{P}}{d\tau} = \gamma \frac{d\underline{p}}{dt}$$

As $\underline{W}^2 = c^2$, $\therefore \dfrac{d\underline{W}}{d\tau}\cdot \underline{W} = 0$ i.e. $\underline{f}\cdot\underline{W} = f_\mu W^\mu = 0$

4- momentum $\underline{P} = m_0 \underline{W}$. $p^0 = m_0 \frac{dx^0}{d\tau} = m_0 \gamma c = mc = \frac{E}{c}$ (7)

$$\underline{P} = (p^0, \underset{\sim}{p})$$

$$\underset{\sim}{p} = m_0 \frac{d\underset{\sim}{x}}{d\tau} = m_0 \gamma \frac{d\underset{\sim}{x}}{dt}$$

$$= (\tfrac{E}{c}, \underset{\sim}{p})$$

$$p^0 = \frac{E}{c} = \frac{1}{c}(m_0 \gamma c^2)$$

4-momentum of a photon

$= mc$, m = relativistic mass $= \gamma \cdot m_0$

4 current $\underline{j} = (j^0, \underset{\sim}{j})$

$$= (\rho c, \underset{\sim}{j})$$

ρ = charge density

$\underset{\sim}{j}$ = usual current density

4-vector potential in electrodynamics

$$\underline{A} = (\tfrac{\phi}{c}, \underset{\sim}{A}), \qquad A^0 = \frac{\phi}{c}$$

ϕ = Electric potential

$\underset{\sim}{A}$ = magnetic vector potential

$\underset{\sim}{E}$ (electric field) $= -\nabla\phi - \frac{\partial \underset{\sim}{A}}{\partial t}$

$\underset{\sim}{B}$ (magnetic field) $= \nabla \wedge \underset{\sim}{A}$

An entity $\underline{\underline{T}}$ is a tensor if under the Lorentz tran Λ, rank 2

$$\underline{\underline{T}} \rightarrow \underline{\underline{T}}' = \Lambda \Lambda \underline{\underline{T}}$$

In component form

Contravariant $\quad T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} T^{\alpha\beta}$

Covariant $\quad T'_{\mu\nu} = \Lambda_{\mu}{}^{\alpha} \Lambda_{\nu}{}^{\beta} T_{\alpha\beta}$

mixed $\quad T'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda_{\nu}{}^{\beta} T^{\alpha}{}_{\beta}$

Example

Electromagnetic field tensor

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

$$= \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}}$$

$\frac{\partial}{\partial x^{\mu}}$ is covariant vector

$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

$\frac{\partial}{\partial x_{\mu}}$ is contravariant vector

$\underset{\sim}{A} = $ 4-vector potential $= \left(\frac{\phi}{c}, \underset{\sim}{A}\right)$ (9)

e.g

$$F^{i0} = \partial^i A^0 - \partial^0 A^i$$

$$= \frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x_0} \qquad x^i = -x_i$$

$$= -\frac{\partial A^0}{\partial x^i} - \frac{1}{c}\frac{\partial A^i}{\partial t} \qquad x^0 = x_0$$

$$= \frac{1}{c}\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A^i}{\partial t}\right) \qquad \because\ A^0 = \frac{\phi}{c}$$

But $\underset{\sim}{E}$ (electric field) $= -\nabla\phi - \frac{\partial \underset{\sim}{A}}{\partial t}$

i.e. $E^i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A^i}{\partial t}$, $\quad (\nabla\phi)^i \equiv \frac{\partial \phi}{\partial x^i}$

i.e. $F^{i0} = \frac{E^i}{c}$

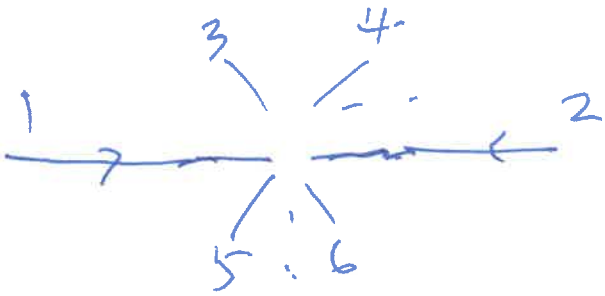
Can show $B^i = -\frac{1}{2}\varepsilon^{ijk} F_{jk}$

(HW)

$$\underset{\sim}{B} = \nabla \wedge \underset{\sim}{A} \qquad \wedge = \text{cross product}$$

$$\varepsilon_{ilm}\,\varepsilon_{ipq} = \delta_{lp}\,\delta_{mq} - \delta_{lq}\,\delta_{mp}$$

Consider collision of 2 particles



Frames of reference

Lab frame: A lab frame of particle 1 is the inertial frame at which particle 1 is at rest particle 1 = target, particle 2 = projectile.

CM frame:

centre of mass frame:

Define centre of mass $\underset{\sim}{X}_G$

$$\underset{\sim}{X}_G = \frac{\sum_{i=1}^{n} m_i \underset{\sim}{x}_i}{\sum_{j=1}^{n} m_j} \qquad \sum_{i=1}^{n} m_i = M$$

Velocity of centre of mass

$$\dot{\underset{\sim}{X}}_G = \frac{\sum_{i=1}^{n} m_i \dot{\underset{\sim}{x}}_i}{\sum_j m_j}$$

A centre of mass frame is a frame at which the centre of mass is at rest i.e.

$$\dot{\underset{\sim}{X}}_G = 0$$

In relativistic collisions, centre of mass frame not useful ∵ (1) The total rest mass needs not be conserved. (2) photon has no rest mass

In relativistic collisions, one use centre of momentum frame. A CM (centre of momentum) is a frame of reference in which the sum total of spatial momenta is zero i.e.

$$\sum_{i=1}^{n} \underset{\sim}{p}_i = 0$$

particle i

(assume total n particles involved)

Consider

$$x'^0 = \gamma(x^0 - \beta x^1)$$

$$x'^1 = \gamma(x^1 - \beta x^0),$$

$$x'^2 = x^2, \qquad x'^3 = x^3$$

So for the 4-momentum

$$p_i'^0 = \gamma(p_i^0 - \beta p_i^1)$$

$$p_i'^1 = \gamma(p_i^1 - \beta p_i^0)$$

$$p_i'^2 = p_i^2, \qquad p_i'^3 = p_i^3$$

$i = 1, 2, \cdots n$

n particles

To get CM frame:

$$\sum_i p'^1_i = \gamma \left(\sum_i p^1_i - \sum_i \beta p^0_i \right)$$

In CM frame $\sum_i p'^1_i = 0$

$$\rightarrow \quad \beta = \frac{\sum_i p^1_i}{\sum_i p^0_i}$$

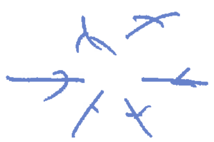
so if O' has a speed β wrt O, then O' is a CM frame. because in O' frame, total spatial momentum $= 0$

Elastic and inelastic collisions

In any collision if the initial total KE (kinetic energy $T = E - m_0 c^2$) is same final total KE, then collision is elastic

Inelastic if initial total KE $\neq$ final total KE

Inelastic collison: Explosive collision
sticky collision



Final KE $>$ initial KE
Explosive

Final KE $<$ initial KE
sticky

Consider 2 examples.

1. What is the excess energy available for inelastic process?

Consider two incident particles. How much energy of these 2 particles can be used to produce other particles

To answer this, use CM frame.

The excess energy ξ

$$= E_1 + E_2 - \underbrace{m_1 c^2}_{\text{rest mass}} - \underbrace{m_2 c^2}_{\text{rest mass}} = T_1 + T_2$$

E_1 = energy of particle 1 $\qquad T_i$ = KE of particle i

In this expression, ξ is not invariant apparently.

To make ξ invariant, we rewrite it as

$$\xi = (p_1^0 + p_2^0)c - m_1 c^2 - m_2 c^2, \qquad p^0 = \frac{E}{c}$$

$$\overset{\text{CM}}{\underset{\text{frame}}{=}} \sqrt{(\underline{p}_1 + \underline{p}_2)^2 c^2} - (m_1 + m_2) c^2$$

So $\xi = c\sqrt{(\underline{p}_1 + \underline{p}_2)^2} - (m_1 + m_2)c^2$ is an invariant definition of excess energy

Example: what is the threshold energy (minimum excess energy) for the following process

$$p + p \rightarrow p + p + p + \bar{p}$$

i.e. threshold energy to produce an antiproton?

Ans this in CM frame and lab frame

In CM frame, answer is obvious

$$\xi = 2 m_p c^2 \quad \text{(HW)}$$

rest mass $m_p = m_{\bar{p}}$ = mass of proton = mass of antiproton

Now do in the lab frame of a proton:

$$\xi = c\sqrt{(\underset{\sim}{p}_1 + \underset{\sim}{p}_2)^2} - 2 m_p c^2$$

∴ rest frame of proton 2

$$\xi = c\sqrt{(p_1^0 + p_2^0)^2 - (\underset{\sim}{p}_1 + \underset{\sim}{p}_2)^2} - 2 m_p c^2$$

$$= c\sqrt{(p_1^0 + p_2^0)^2 - \underset{\sim}{p}_1^2} - 2 m_p c^2 \quad \because \underset{\sim}{p}_2 = 0$$

$$(\xi + 2 m_p c^2)^2 = c^2 [(p_1^0 + p_2^0)^2 - \underset{\sim}{p}_1^2]$$

$$= c^2 [p_1^{02} + p_2^{02} + 2 p_1^0 p_2^0 - \underset{\sim}{p}_1^2] = c^2 [m_p^2 c^2 + p_2^{02} + 2 p_1^0 p_2^0]$$

$$p_2^0 = \frac{E_2}{c} = \frac{m_p c^2}{c} = m_p c$$ (2nd proton at rest, $E_2 = m_p c^2$)

$$(\xi + 2 m_p c^2)^2 = c^2 (2 m_p^2 c^2 + 2 p_1^0 \cdot m_p c)$$

$$p_1^0 = \frac{(\xi + 2 m_p c^2)^2 - 2 m_p^2 c^4}{2 m_p c^3}$$

$$= \frac{\xi^2 + 4 \xi m_p c^2 + 2 m_p^2 c^4}{2 m_p c^3}$$

$$\rightarrow \quad E_1 = \frac{\xi^2 + 4 \xi m_p c^2 + 2 m_p^2 c^4}{2 m_p c^2}$$

$$\xi = 2 m_p c^2$$

$$= 7 \, m_p c^2$$

That means to produce the same excess energy, in the CM frame the total energy involved $= 2 m_p c^2 + 2 m_p c^2 = 4 m_p c^2$

in the lab frame, the total energy required

$$= E_1 + E_2 = 7 m_p c^2 + m_p c^2$$

$$= 8 \, m_p c^2$$

So to produce antiproton (or extra number of proton and antiproton) it is more economical to use CM frame than a lab frame

Bevatron was used at Berkeley (USA) to produce antiproton

2nd example

What is the KE of the particle 2 wrt the particle 1, given that in the O frame of reference, particle 1 has KE T_1 and particle 2 has KE T_2 ?

In dealing collision problems of particles, good to make use of conservation of 4-momentum and invariants.

Conservation refers to a same frame of reference:

$$\text{quantity}\Big|_{\substack{\text{before}\\ \text{collision}}} = \text{quantity}\Big|_{\text{after collision}}$$

e.g. Total energy before = total energy after

or

$$p^\mu\Big|_{\text{before}} = p^\mu\Big|_{\text{after}} \qquad \mu = 0, 1, 2, 3$$

Invariants refer to different frames of reference. That is, an invariant quantity is always the same no matter what frame of reference is used.

e.g. scalar product is an invariant under Lorentz transformations.

Thus $\underline{P}^2 = P_\mu p^\mu = g_{\mu\nu} p^\mu p^\nu$ an invariant

The excess energy $\mathcal{E} = c\sqrt{(\underline{P}_1 + \underline{P}_2)^2} - (m_1 + m_2)c^2$ also an invariant. Rest mass an invariant, $\underline{P}^2 = m^2 c^2$

An invariant needs not be conserved,

e.g. rest mass is invariant but not conserved.

A conserved quantity needs not be invariant,

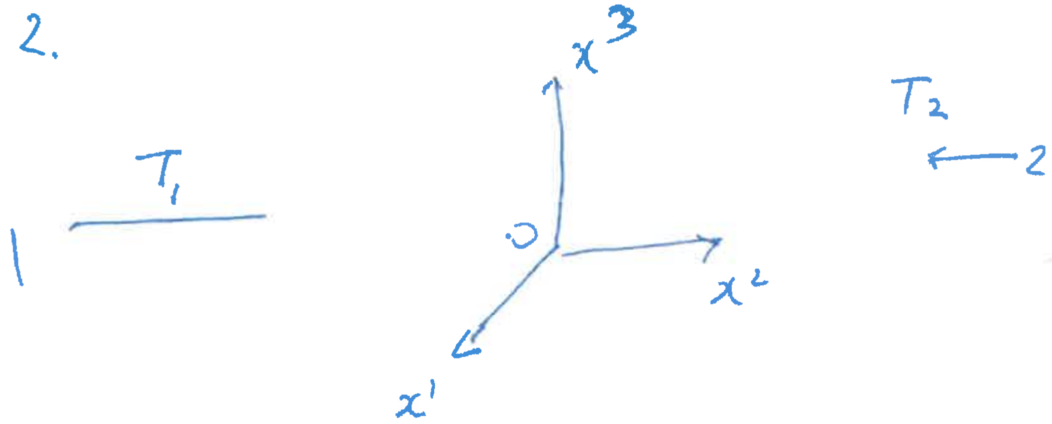
e.g. energy is conserved but not invariant

The total 4-momentum square $\underline{P}^2 = \underline{P} \cdot \underline{P}$

$= P_\mu P^\mu = g_{\mu\nu} P^\mu P^\nu = g^{\mu\nu} P_\mu P_\nu$

is an invariant and also conserved

Ex 2.



T_1 = KE of particle 1.

Ask: what is the KE, T, of particle 1 wrt particle 2? $1 \rightarrow \quad \bullet 2$

i.e. Given T_1 and T_2, find T?

We use invariant to solve this problem

$(\underline{P}_1 + \underline{P}_2)^2$ is an invariant

$\underline{P}_1$ = 4-mom. of particle 1

We compute $(\underline{P}_1 + \underline{P}_2)^2$ in the present frame O and also the lab frame of particle 2

O frame: $(\underline{P}_1 + \underline{P}_2)^2 = (P_1^0 + P_2^0)^2$ assume O is a c m frame

$$= \left(\frac{T_1 + m_1c^2 + T_2 + m_2c^2}{c}\right)^2$$

$$= \frac{(\xi + (m_1 + m_2)c^2)^2}{c^2}$$

$\because E = T + mc^2$

$\xi \equiv T_1 + T_2$

Lab frame of particle 2:

$$(\underline{P}_1 + \underline{P}_2)^2 = \left(P_1^0 + \frac{m_2 c^2}{c}\right)^2 - (\underset{\sim}{P}_1 + 0)^2$$

$$= \underline{(P_1^{0\,2} - \underline{P}_1^2)} + 2\, P_1^0 m_2 c + m_2^2 c^2$$

$$(\underline{P}_1 + \underline{P}_2)^2 = (P_1^0 + P_2^0)^2 - (\underset{\sim}{P}_1 + \underset{\sim}{P}_2)^2$$

$$= \underline{m_1^2} c^2 + m_2^2 c^2 + 2 P_1^0 m_2 c \qquad \because \underline{P}^2 = m^2 c^2$$

$$P_1^0 = \frac{E_1}{c} = \frac{T + m_1 c^2}{c}$$

Equating

$$\frac{\xi^2 + 2\xi (m_1 + m_2) c^2 + (m_1 + m_2)^2 c^4}{c^2} = (m_1^2 + m_2^2) c^2 + 2 P_1^0 m_2 c$$

$$\frac{\xi^2 + 2\xi(m_1 + m_2) c^2}{c^2} + 2\, m_1 m_2\, c^2 = 2\left(\frac{T + m_1 c^2}{c}\right) m_2 c$$

i.e. $$T = \frac{\xi^2 + 2\xi (m_1 + m_2) c^2}{2\, m_2 c^2}$$

$$\xi = T_1 + T_2$$

Compare with the previous example: $p + p \to p + p + p + \bar{p}$

$$E_1 = \frac{\xi^2 + 4\xi m_p c^2 + 2 m_p^2 c^4}{2 m_p c^2} = \frac{\xi^2 + 4\xi m_p c^2}{2 m_p c^2} + m_p c^2$$

$$T_1 = E_1 - m_p c^2 = \frac{\xi^2 + 4\xi m_p c^2}{2 m_p c^2} \quad \text{(rest frame particle 2)}$$