Clebsch-Gordan Coefficents

Addition of angular momenta

$$\underset{\sim}{J}_1 \qquad\qquad \underset{\sim}{J}_2$$

$$\left|\alpha_1 j_1 m_1\right\rangle = basis \;\; for \;\; \underset{\sim}{J}_1^2 \;\; and \;\; J_{1z}$$

$$\left|\alpha_2 j_2 m_2\right\rangle = basis \;\; for \;\; \underset{\sim}{J}_2^2 \;\; and \;\; J_{2z}$$

The base vectors

$$\left|\alpha \; j_1 \; j_2 \; m_1 \; m_2\right\rangle \equiv \left|\alpha_1 \; j_1 \; m_1\right\rangle \left|\alpha_2 \; j_2 \; m_2\right\rangle$$

$$\alpha, \;\; j_1, j_2 \;\; \text{fixed},$$

$$m_1, m_2 \;\; \text{vary} \qquad\qquad -j_1 \le m_1 \le j_1$$

$$-j_2 \le m_2 \le j_2$$

span the subspace $\xi(\alpha, j_1, j_2)$.

$$\underset{\sim}{J}^2 = (\underset{\sim}{J}_1 + \underset{\sim}{J}_2)^2 \quad \text{and } J_z \text{ act on } \xi(\alpha, j_1, j_2)$$

Since $\underset{\sim}{J}_1^2$ and $\underset{\sim}{J}_2^2$ commute with $\underset{\sim}{J}^2$ *and* J_z, can also use the base

$$\left|\alpha\, j_1\, j_2\, j\, m\right\rangle, \qquad \alpha, j_1, j_2 \quad \textit{fixed}$$

$$j, m \ \text{vary}$$

$$\left|j_1 - j_2\right| \le j \le \left(j_1 + j_2\right)$$

$$-j \le m \le j$$

to generate the same subspace $\xi(\alpha, j_1, j_2)$.

The two bases are related:

$$\left|\alpha\, j_1\, j_2\, j\, m\right\rangle = \sum_{m_2=-j_2}^{j_2} \sum_{m_1=-j_1}^{j_1} \left|\alpha\, j_1 j_2 m_1 m_2\right\rangle \left\langle j_1 j_2 m_1 m_2 \middle| jm\right\rangle$$

$$\left|\alpha\, j_1\, j_2\, m_1\, m_2\right\rangle = \sum_{m=-j}^{j} \sum_{j=\left|j_1 - j_1\right|}^{(j_1+j_2)} \left|\alpha\, j_1\, j_2\, j\, m\right\rangle \left\langle j\, m \middle| j_1\, j_2\, m_1\, m_2\right\rangle$$

$$\left\langle j_1\, j_2\, m_1\, m_2 \middle| j\, m\right\rangle = \left\langle j\, m \middle| j_1\, j_2\, m_1\, m_2\right\rangle^* \quad \equiv \quad \text{Clebsch-Gordan coefficents}$$

Meaning of C.G. coeffs

(i) relating two basis vectors (just like Fourier transform)

(ii) $\langle j_1\, j_2\, m_1\, m_2 | j\, m \rangle$ = probability amplitude of finding the state $| j_1\, j_2\, m_1\, m_2 \rangle$ when the system is in state $| jm \rangle$

Properties of C.G. coeffs

(1) Selection rule:

$$\langle \alpha\, j_1 j_2 m_1 m_2 | jm \rangle = 0 \quad unless$$

$$m_1 + m_2 = m \quad and \quad |j_1 - j_2| \leq j \leq (j_1 + j_2)$$

(2) Phase convention: require

$$\langle j_1\, j_2\, j_1\, m_2 | j\, j \rangle \text{ real and } \geq 0$$

$$\mathrm{m}_2 = j - j_1$$

$$j = |j_1 - j_2|,\ |j_1 - j_2| + 1 \ldots. \qquad (j_1 + j_2)$$

Note: When $m_1 = j_1$ *and* $m = j$, it does <u>not</u> necessarily imply $m_2 = j_2$ since $j \neq (j_1 + j_2)$ in general

(3) Reality: All C.G. coeffs can be obtained from

$$\left\langle j_1\, j_2\, j_1\, m_2 \middle| j\, j \right\rangle$$

$\therefore$ all C.G. coeffs are real

(4) Orthogonality

$$\sum_{m_1\, m_2} \left\langle j_1\, j_2\, m_1\, m_2 \middle| j\, m \right\rangle \left\langle j_1\, j_2\, m_1\, m_2 \middle| j'\, m' \right\rangle = \delta_{jj'}\, \delta_{mm'}$$

$$\sum_{j\quad m} \left\langle j_1\, j_2\, m_1\, m_2 \middle| j\, m \right\rangle \left\langle j_1\, j_2\, m_1'\, m_2' \middle| j\, m \right\rangle = \delta_{m_1 m_1'}\, \delta_{m_2 m_2'}$$

Wigner-Eckart theorem

In a standard representation $\left\{ \underset{\sim}{J}^2, J_z \right\}$ whose basis vectors are denoted by $\left| \tau\, j\, m \right\rangle$,

The matrix element $\left\langle \tau\, j\, m \right| T_g^{(k)} \left| \tau^{'} j^{'} m^{'} \right\rangle$

of the q[th] standard component of a given k[th] order irreducible tensor operator, $T^{(k)}$, is equal to the product of the Clebsch-Gordan coefficient.

$$\left\langle j^{'} k\, m^{'} q \right| \left. jm \right\rangle$$

by a quantity independent of *m, m*$^{'}$ and *q* $\quad (q = -k, -k+1, \ldots. + k)$

$$\left\langle \tau\, j\, m \right| T_q^{(k)} \left| \tau^{'} j^{'} m^{'} \right\rangle = \frac{1}{\sqrt{2j+1}} \left\langle \tau\, j \,||\, T^{(k)} \,||\, \tau^{'} j^{'} \right\rangle \cdot \left\langle j^{'} k\, m^{'} q \right| \left. jm \right\rangle$$

$\left\langle \tau\, j \,||\, T^{(k)} \,||\, \tau^{'} j^{'} \right\rangle$= reduced matrix element

$\left\langle j^{'} k\, m^{'} q \right| \left. jm \right\rangle$=Clebsch-Gordan coefficient

$\neq 0$ only if $m = m^{'} + q$ and $\left| j - j^{'} \right| \leq k \leq j + j^{'}$

For a scalar operator *S* $\quad \left\langle \tau\, jm \right| s \left| \tau^{'} j^{'} m^{'} \right\rangle = \delta_{jj^{'}} \delta_{mm^{'}} S_{\tau\tau^{'}}^{(j)}$