

# Magnetostatics

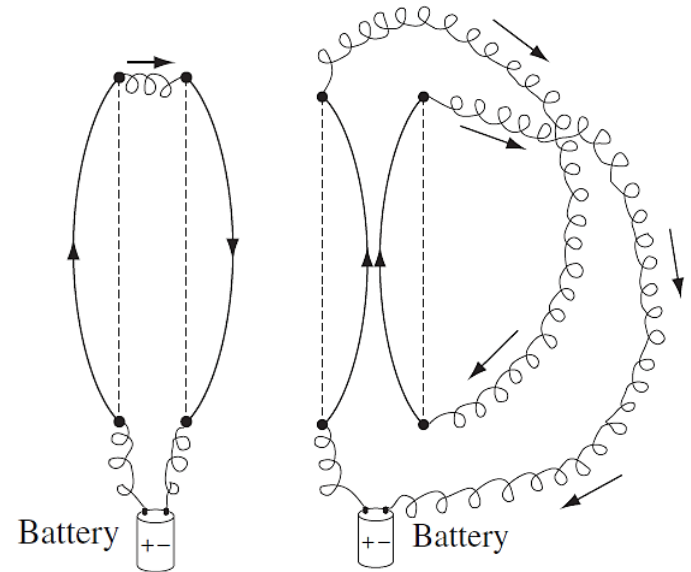
A photograph of a toroidal coil, which is a donut-shaped inductor made of many turns of copper wire. The coil is mounted on a dark, rectangular core. Two thin, vertical metal rods are positioned symmetrically on either side of the coil's center, and a horizontal metal rod passes through the middle of the toroid. The entire setup is set against a dark background. A semi-transparent yellow rectangle is superimposed over the center of the image, containing the word 'Magnetostatics' in a white, sans-serif font.

# The Lorenz force law

**Electrostatics:** Stationary charges with constant electric fields

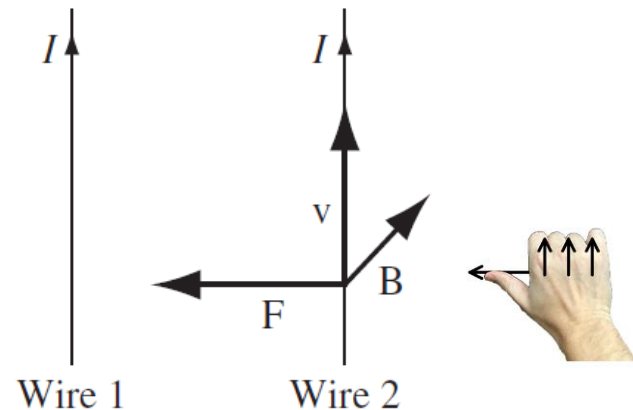
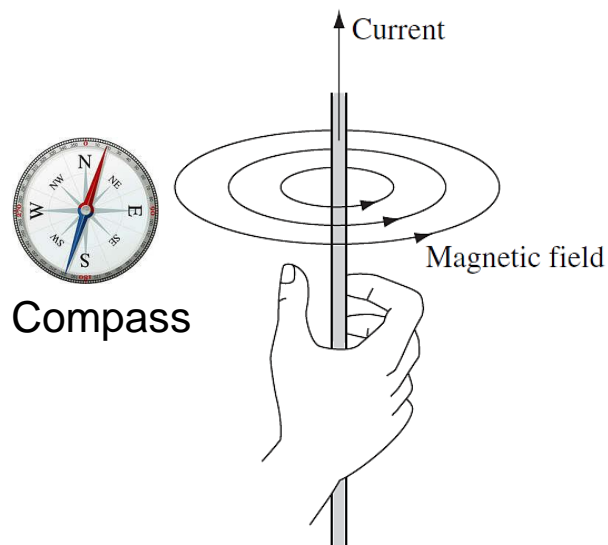
**Magnetostatics:** Steady currents with constant magnetic fields

- Magnetic force and magnetic field
  - Pass current in parallel wires
    - Wires repel with antiparallel current
    - Wires attract with parallel current
    - Force is not electrostatic in nature (wires are charge neutral)
    - Magnetic force



# The Lorentz force law

- Magnetic force and magnetic field
  - Empirical rules for magnetic field and magnetic force
    - Straight current-carrying wire has magnetic field circling around it
    - Right-hand rule for field direction given current direction
    - Right-hand rule for force direction given current and field direction



# The Lorentz force law

- The Lorentz force law
  - Magnetic force on charge  $Q$ , moving with velocity  $\mathbf{v}$ , in magnetic field  $\mathbf{B}$

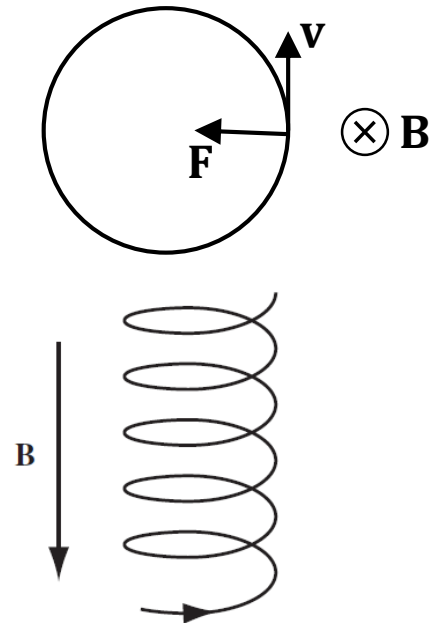
$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

- Given as an axiom without proof
  - With both  $\mathbf{E}$  and  $\mathbf{B}$ , the full Lorentz force law  $\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$

- Consequences

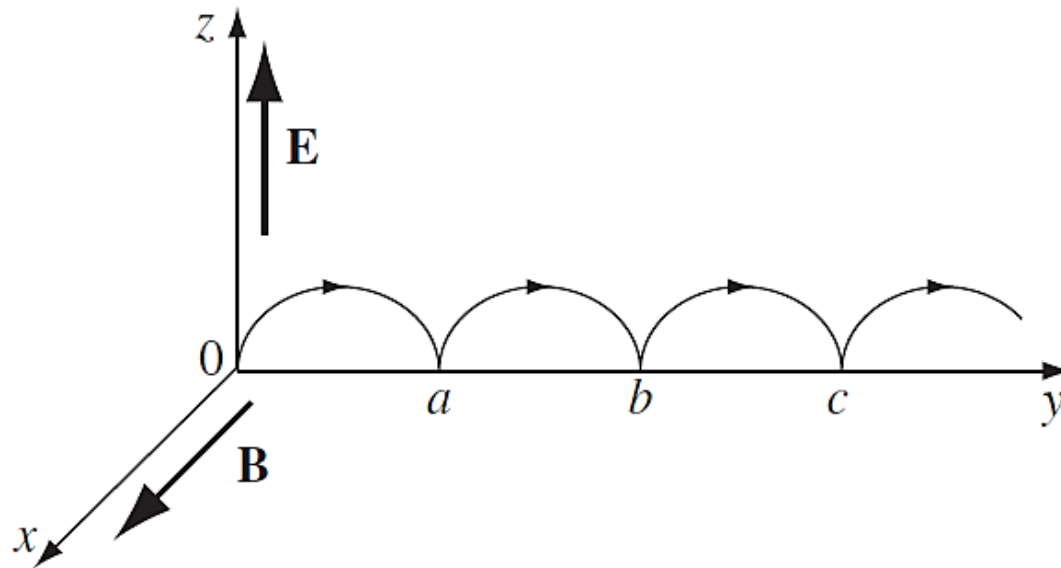
- Circular charge motion, or alike, when magnetic force acts as centripetal force
  - Magnetic forces do no work,  $\mathbf{B}$  only deflects particle direction

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \overbrace{\mathbf{v} dt}^{d\mathbf{l}} = 0$$



# The Lorentz force law

**Example 5.2. Cycloid Motion.** A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that  $\mathbf{B}$  points in the  $x$ -direction, and  $\mathbf{E}$  in the  $z$ -direction, as shown in Fig. 5.7. A positive charge is released from the origin; what path will it follow?



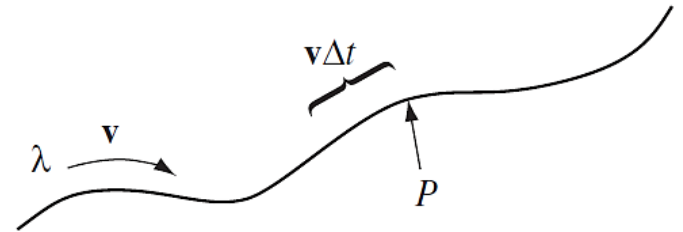
C3.exmp5.2

# Currents

- Current in a wire: charge per time passing a given point

- Unit: Amperes (A) = Coulomb/second

- $\mathbf{I} = \lambda \mathbf{v}$  where  $\lambda$ : line charge density,  
 $\mathbf{v}$ : velocity of movement



- Positive charges moving at  $\mathbf{v}$  = negative charges moving at  $-\mathbf{v}$
- Not meaningful to talk about current if it's just a single point charge moving (non-steady)

- In many problems just write the magnitude  $I = \lambda v$ 
  - Because direction is determined by the shape of wire

- Magnetic force on a current-carrying wire  $\mathbf{F}_{\text{mag}} = \int I (d\mathbf{l} \times \mathbf{B})$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \lambda dl = \int (\mathbf{I} \times \mathbf{B}) dl = \int I (d\mathbf{l} \times \mathbf{B})$$

# Currents

- Surface and volume distributions of current

- Surface current density  $\mathbf{K} = \sigma \mathbf{v}$  (C/m<sup>2</sup> · m/s) ( $\sigma$ : surface charge density)

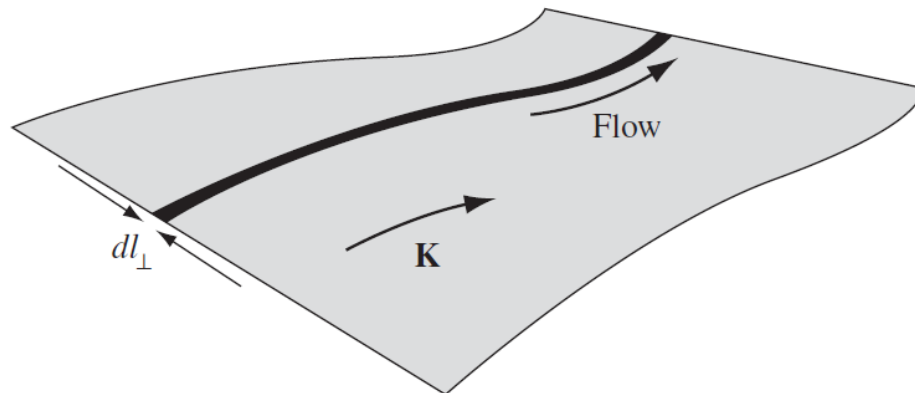
- $\mathbf{K}$ : current per unit width (A/m)

$$d\mathbf{I} = \sigma \mathbf{v} dl_{\perp} \quad \Rightarrow \quad \mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}} \quad (l_{\perp}: \text{cross-sectional line segment perpendicular to } \mathbf{v})$$

- Magnetic force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \sigma da = \int (\mathbf{K} \times \mathbf{B}) da$$

\*  $\mathbf{B}$  experiences discontinuity across a current-carrying surface



# Currents

- Surface and volume distributions of current

- Volume current density  $\mathbf{J} = \rho \mathbf{v}$  (C/m<sup>3</sup> · m/s) ( $\rho$ : volume charge density)

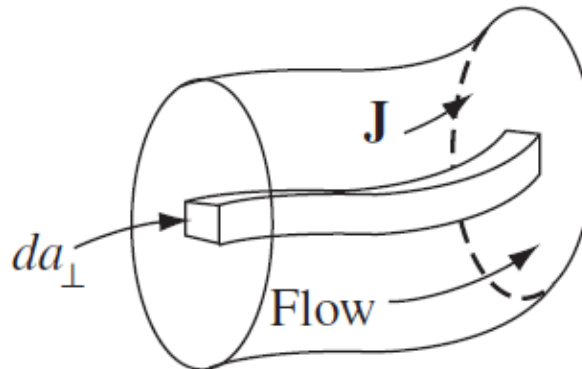
- $\mathbf{J}$ : current per unit area (A/m<sup>2</sup>)

$$\begin{aligned} d\mathbf{I} &= \rho \mathbf{v} da_{\perp} \\ d\mathbf{I} &= \mathbf{J} \cdot d\mathbf{a} = \rho \mathbf{v} \cdot d\mathbf{a} \end{aligned} \quad \Rightarrow \quad \mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}} \quad (a_{\perp}: \text{cross-sectional area segment perpendicular to } \mathbf{v})$$

- Magnetic force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \rho d\tau = \int (\mathbf{J} \times \mathbf{B}) d\tau$$

- Ohm's law  $\mathbf{J} = \sigma_c \mathbf{E}$  ( $\sigma_c$ : electrical conductivity)





# Currents

- Surface and volume distributions of current

- Continuity equation for volume current density

- Current crossing a surface  $\mathcal{S}$ :  $I = \int_{\mathcal{S}} J da_{\perp} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$
- Current crossing boundary of a volume  $\mathcal{V}$ :

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau$$

which must equal to the change of net charge in the volume

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau = \overbrace{-\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau}^{\text{change of net charge}} = - \int_{\mathcal{V}} \left( \frac{\partial \rho}{\partial t} \right) d\tau$$

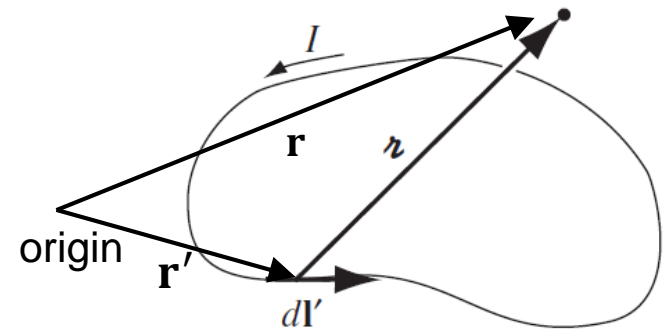
➡  $\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$  for magnetostatics  $\frac{\partial \rho}{\partial t} = 0 \quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0} \quad \nabla \cdot \mathbf{J} = 0$

# The Biot-Savart law

- Magnetic field generated by a steady current

- Line current  $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathbf{r}}}{r^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\mathbf{r}}}{r^2}$

- Unit of  $\mathbf{B}$ : Tesla (T) = N/(A·m)
- Permeability  $\mu_0 = 4\pi \times 10^{-7}$  N/A<sup>2</sup>
- Separation vector  $\mathbf{r} = \mathbf{r} - \mathbf{r}'$



- Surface current  $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} da'$

- Volume current  $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} d\tau'$

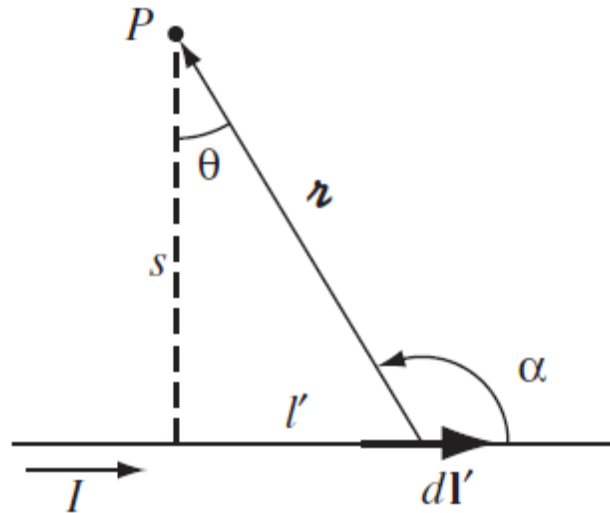
- Expressions for surface and volume currents not proved
- The law cannot be used to calculate field generated by discrete moving charges

\*  $\mathbf{r}$  in textbook is typed as  $\mathbf{r}$

# The Biot-Savart law

- Application of Biot-Savart law

**Example 5.5.** Find the magnetic field a distance  $s$  from a long straight wire carrying a steady current  $I$  (Fig. 5.18).



$$\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$



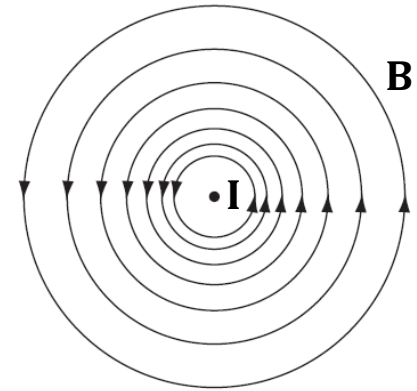
# Curl of magnetic field

- Derivation of curl from Stokes theorem
  - Loop integral of  $\mathbf{B}$  around a straight-line current

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I$$

- Result still holds if path is noncircular

$$d\mathbf{l} = ds \hat{s} + \underline{s d\phi \hat{\phi}} + dz \hat{z} \quad \mathbf{B} = \underline{\frac{\mu_0 I}{2\pi s} \hat{\phi}}$$



- Loop integral of  $\mathbf{B}$  around any current distribution

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad (I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a} \text{ total current enclosed by the path})$$

- Apply Stokes theorem

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

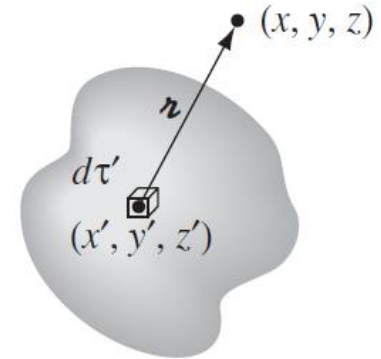
➡  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

# Curl of magnetic field

- Derivation of curl from Biot-Savart law
  - A much more formal derivation than previous slide

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left( \mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau'$$

$\mathbf{B}(\mathbf{r})$        $\nabla_{\mathbf{r}}$        $\mathbf{J}(\mathbf{r}')$        $\mathbf{r} = \mathbf{r} - \mathbf{r}'$        $dx' dy' dz'$



$\downarrow$  Product rule  $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

$$\nabla \times \left( \mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) = \mathbf{J} \left( \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) - \underbrace{(\mathbf{J} \cdot \nabla) \frac{\hat{\mathbf{r}}}{r^2}}$$

$\downarrow \nabla \cdot \left( \frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r}),$  and second term integrates to 0 (textbook p.232)

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta^3(\mathbf{r} - \mathbf{r}') d\tau' = \mu_0 \mathbf{J}(\mathbf{r})$$

# Divergence of magnetic field

- Derivation of divergence from Biot-Savart law

$$\circ \quad \nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left( \mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau'$$

↓

Product rule  $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$$\nabla \cdot \left( \mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) = \frac{\hat{\mathbf{r}}}{r^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left( \nabla \times \frac{\hat{\mathbf{r}}}{r^2} \right)$$

↓

$\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') = 0$ , and  $\nabla \times \frac{\hat{\mathbf{r}}}{r^2}$  (Remember  $\nabla \times \mathbf{E} = 0$ )

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

Magnetic fields are divergence-free (no magnetic “free charge”)

Magnetic  
monopoles?  
not reproduced

First Results from a Superconductive Detector for Moving Magnetic Monopoles

Blas Cabrera  
Phys. Rev. Lett. **48**, 1378 – Published 17 May 1982

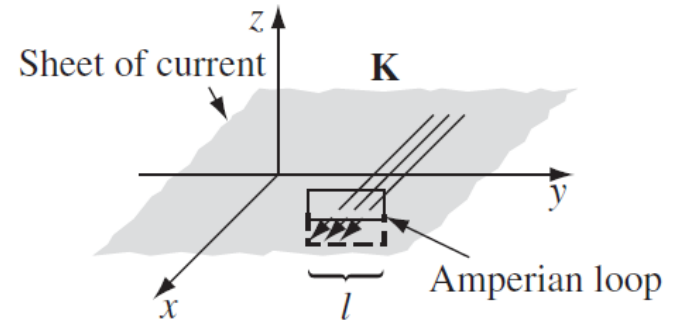
# Ampère's law

Electrostatics	Magnetostatics
Coulomb's law	Biot-Savart law
Gauss's law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \quad \oint \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}}/\epsilon_0$	Ampère's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$
Gaussian surface	Amperian loop

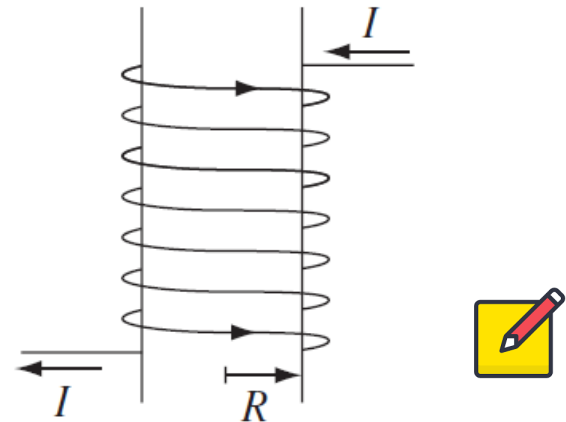
- Use of Ampère's law to calculate magnetic field
    - Ampère's law in integral form
    - Symmetry arguments
      - Translation symmetry
      - Rotational symmetry
      - Mirror symmetry
      - Inversion symmetry
- Be careful to use when transforming  $\mathbf{B}$ :  
 need to flip sign
- $\mathbf{B}$  flips sign if  $\mathbf{I}$  flips sign (time-reversal symmetry)

# Ampère's law

**Example 5.8.** Find the magnetic field of an infinite uniform surface current  $\mathbf{K} = K \hat{\mathbf{x}}$ , flowing over the  $xy$  plane (Fig. 5.33).



**Example 5.9.** Find the magnetic field of a very long solenoid, consisting of  $n$  closely wound turns per unit length on a cylinder of radius  $R$ , each carrying a steady current  $I$  (Fig. 5.34).





# Ampère's law

**Example 5.10.** A toroidal coil consists of a circular ring, or “donut,” around which a long wire is wrapped (Fig. 5.38). The winding is uniform and tight enough so that each turn can be considered a plane closed loop.

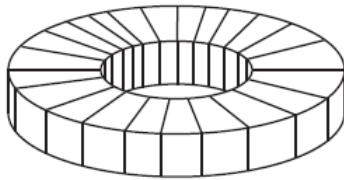
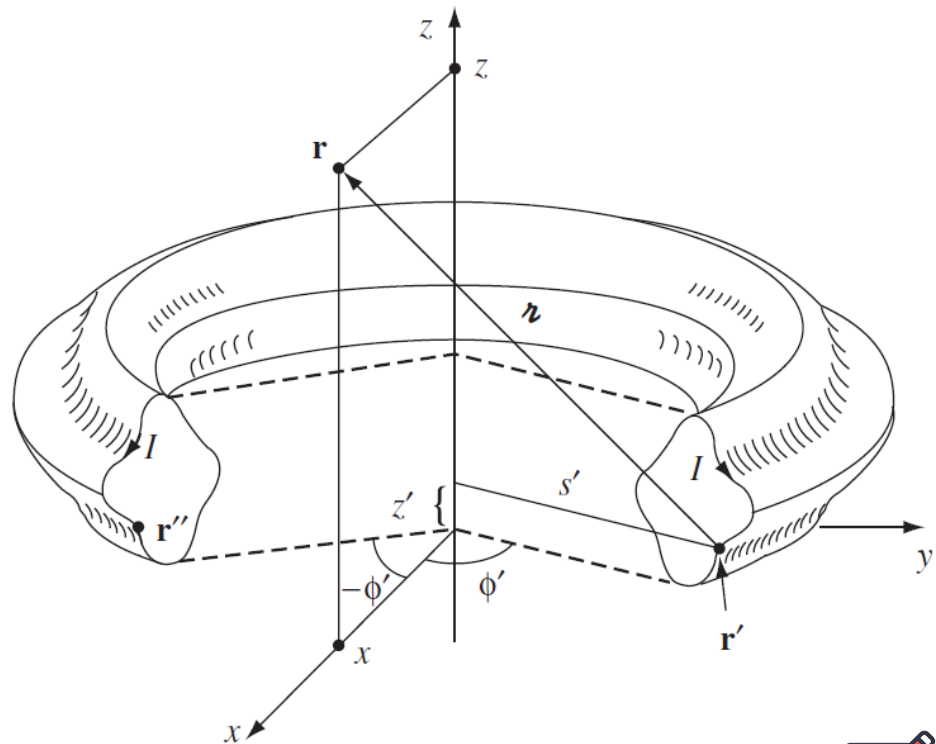


FIGURE 5.38



Toroidal structure with any cross-sectional shape



# Magnetic vector potential

- Vector potential  $\mathbf{A}$

- $\mathbf{B} = \nabla \times \mathbf{A}$

- Exploiting the property  $\nabla \cdot \mathbf{B} = 0$  (divergence of curl vanishes)

- $\nabla \cdot \mathbf{A} = 0$

- Chosen to be so (like choosing reference point for electric potential)

If original  $\mathbf{A}_0$  is not, can define  $\mathbf{A} = \mathbf{A}_0 + \nabla\lambda$  without varying  $\mathbf{B}$

➡  $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2\lambda$

Can choose  $\nabla^2\lambda = -\nabla \cdot \mathbf{A}_0$

- $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$

- Use the choice above, and apply Ampère's law

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\cancel{\nabla \cdot \mathbf{A}}^0) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

# Magnetic vector potential

- Calculating vector potential from current

- $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$  Poisson's equation, if  $\mathbf{J} \rightarrow 0$  at infinity,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau'$$

- Analogous to solution  $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} d\tau'$  to  $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

- Line current

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{r} dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{r} d\mathbf{l}'$$

- Surface current

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{r} da'$$

- Not so much simplification (vector  $\mathbf{A}$  representing vector  $\mathbf{B}$ ) but equation is easier to use than the Biot-Savart law
    - No need to integrate unit vector  $\hat{\mathbf{r}}$

$$* \nabla^2 \mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$$

# Magnetic vector potential

- Calculating vector potential from magnetic field

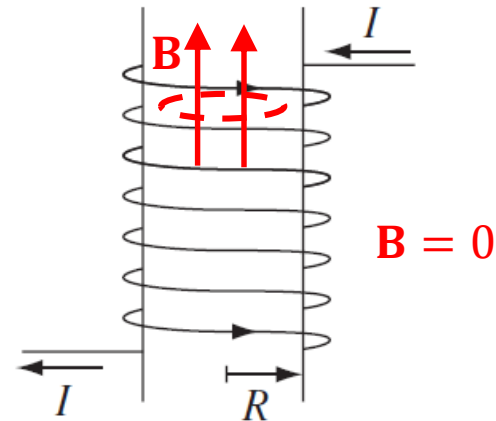
$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$$

where  $\Phi$  is the magnetic flux

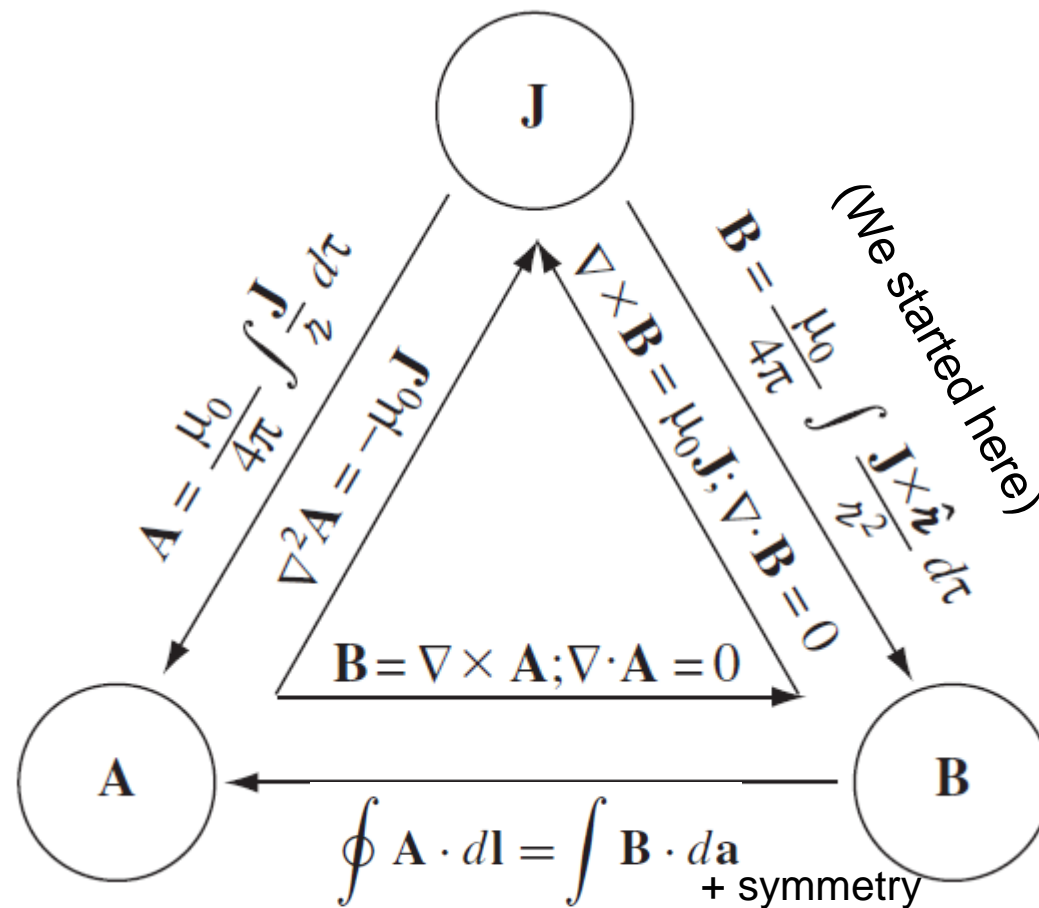
- Can calculate  $\mathbf{A}$  using equation above plus symmetry argument
- Generally  $\mathbf{A}$  mimics the direction of current

**Example 5.12.** Find the vector potential of an infinite solenoid with  $n$  turns per unit length, radius  $R$ , and current  $I$ .

- Check  $\nabla \times \mathbf{A} = \mathbf{B}$
- Check  $\nabla \cdot \mathbf{A} = 0$



# Current, magnetic field, and vector potential



Differential equations need boundary conditions to solve

# Boundary conditions

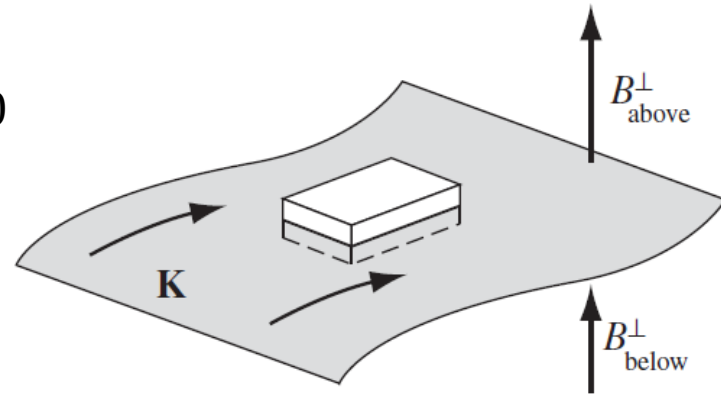
- Boundary conditions of  $\mathbf{B}$  across a 2D current surface

- Normal component of  $\mathbf{B}$

Thin pillbox with thickness  $\varepsilon \rightarrow 0$

$$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$

➡  $B_{\text{above}}^{\perp} = B_{\text{below}}^{\perp}$

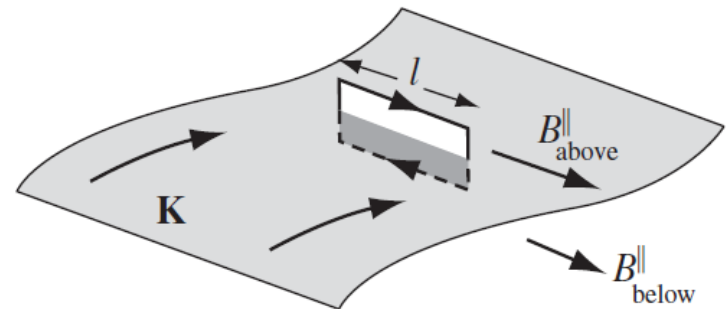


- Tangential component of  $\mathbf{B}$  that is perpendicular to current

Thin Amperian loop

$$\begin{aligned} \oint \mathbf{B} \cdot d\mathbf{l} &= (B_{\text{above}}^{\parallel} - B_{\text{below}}^{\parallel}) l \\ &= \mu_0 I_{\text{enc}} = \mu_0 K l \end{aligned}$$

➡  $B_{\text{above}}^{\parallel} - B_{\text{below}}^{\parallel} = \mu_0 K$



- Summarizing above  $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$

# Boundary conditions

- Boundary conditions of  $\mathbf{A}$  across a 2D current surface

- Vector potential  $\mathbf{A}$  is always continuous  $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$

- Normal component of  $\mathbf{A}$

$\nabla \cdot \mathbf{A} = 0$  and select a thin pillbox

➡  $A_{\text{above}}^{\perp} = A_{\text{below}}^{\perp}$

- Tangential component of  $\mathbf{A}$

$\nabla \times \mathbf{A} = \mathbf{B}$  and select a thin loop to integrate

$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$  tends to zero when loop is thin

➡  $A_{\text{above}}^{\parallel} = A_{\text{below}}^{\parallel}$

- Derivative of  $\mathbf{A}$  is discontinuous  $\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$



# Multipole expansion of vector potential

- Goal: to expand  $\mathbf{A}$  in power series of  $1/r$

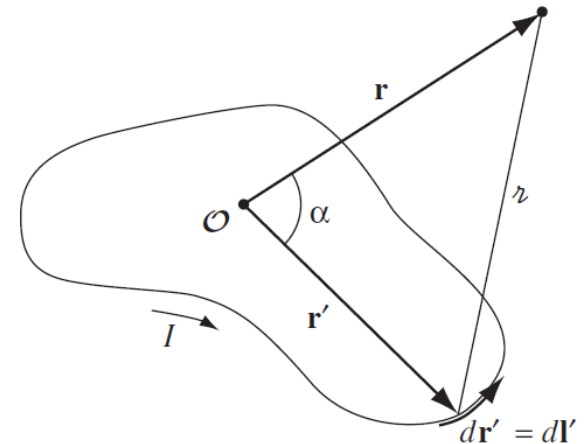
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{r} d\mathbf{l}'$$

$$\downarrow \quad \frac{1}{r} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr' \cos \alpha}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \alpha)$$

(From electrostatic multipole expansion)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos \alpha) d\mathbf{l}'$$

$$= \frac{\mu_0 I}{4\pi} \left[ \underbrace{\frac{1}{r} \oint d\mathbf{l}'}_{\text{Monopole}} + \underbrace{\frac{1}{r^2} \oint r' \cos \alpha d\mathbf{l}'}_{\text{Dipole}} + \underbrace{\frac{1}{r^3} \oint (r')^2 \left( \frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) d\mathbf{l}'}_{\text{Quadrupole}} + \dots \right]$$



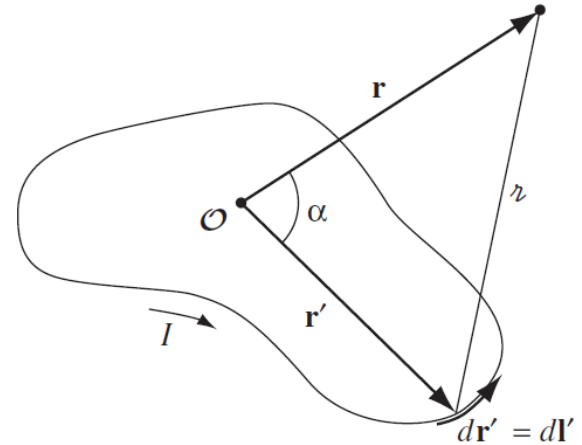


# Multipole expansion of vector potential

- Magnetic monopole and magnetic dipole

- Magnetic monopole

- Always zero, because  $\oint d\mathbf{l}' = \mathbf{0}$
- Also because  $\nabla \cdot \mathbf{B} = 0$



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos \alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'$$



$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'$$



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

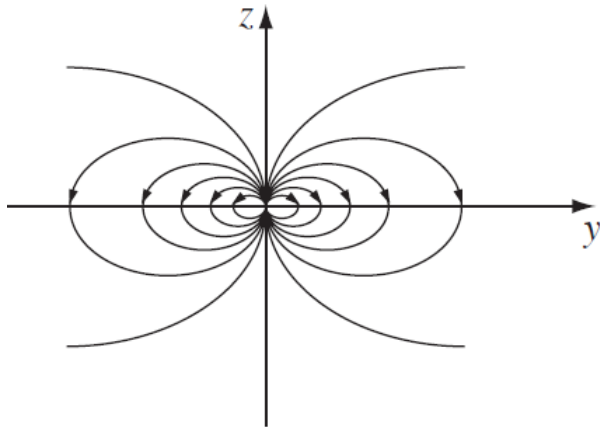
Magnetic dipole moment

$$\mathbf{m} \equiv I \int d\mathbf{a} = I \mathbf{a}$$

# Multipole expansion of vector potential

- Pure dipole vs physical dipole

Pure dipole



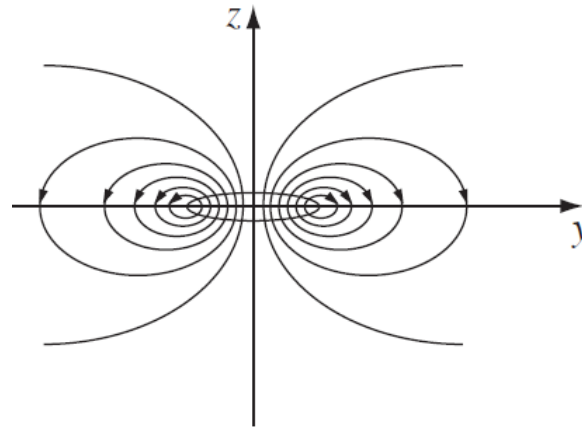
$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

Assumes  $r \gg$  loop radius

$\mathbf{m} = I\mathbf{a}$  but take  $I \rightarrow \infty$ ,  $a \rightarrow 0$

Similar to pure electric dipole

Physical dipole

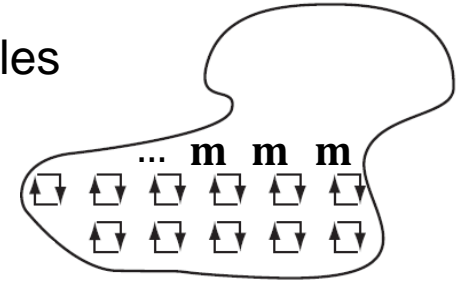


Deviations appear when closing up onto the dipole

# Magnetic fields in matter

# Magnetic materials

- Practically, all materials have magnetic response
  - Orbiting electrons around nuclei as magnetic dipoles
  - Electron spins as magnetic dipoles
    - Quantum object without a classical analog



- Magnetization:  $\mathbf{M} = \frac{1}{V} \sum_i \mathbf{m}_i$ 
  - Magnetic dipole moment per unit volume
- Three types of response of matter to magnetic field
  - Paramagnets: magnetization  $\mathbf{M}$  parallel to applied  $\mathbf{B}$
  - Diamagnets: magnetization  $\mathbf{M}$  opposite to applied  $\mathbf{B}$
  - Ferromagnets: finite magnetization  $\mathbf{M}$  even without  $\mathbf{B}$

# Magnetic dipoles responding to field

- Paramagnetic response

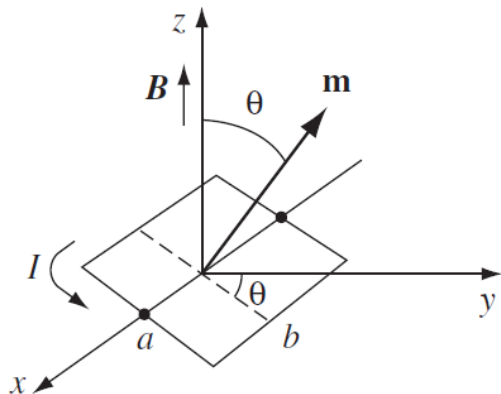
- Torque in a uniform field

- Suppose a rectangle current loop with side lengths  $a$  and  $b$

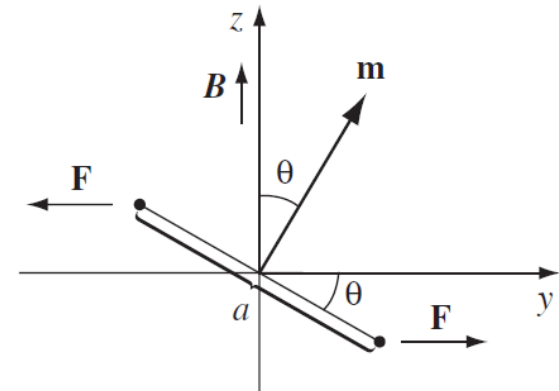
$$\mathbf{N} = aF \sin \theta \hat{\mathbf{x}} \xrightarrow{\text{force } F = IbB} \mathbf{N} = IabB \sin \theta \hat{\mathbf{x}} = mB \sin \theta \hat{\mathbf{x}}$$

➡  $\mathbf{N} = \mathbf{m} \times \mathbf{B}$

- Torque tends to align  $\mathbf{m}$  parallel to  $\mathbf{B}$



View from x axis

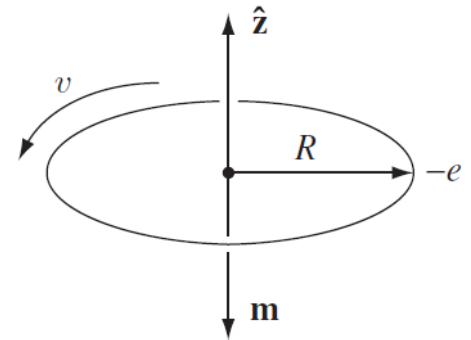


# Magnetic dipoles responding to field

- Diamagnetic response

- Modification to an atomic orbit

$$I = \frac{-e}{T} = -\frac{ev}{2\pi R} \xrightarrow[\text{Orbit period}]{T = 2\pi R/v} \mathbf{m} = -\frac{1}{2}evR\hat{\mathbf{z}}$$



- Velocity without and with  $\mathbf{B}$ , balancing Coulomb and centripetal forces

without  $\mathbf{B}$ :  $\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} = m_e \frac{v^2}{R}$

with  $\mathbf{B}$ :  $\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} + e\bar{v}B = m_e \frac{\bar{v}^2}{R}$

$v$ : electron velocity  
 $\bar{v}$ : electron velocity with  $\mathbf{B}$   
 $m_e$ : electron mass

➔  $e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v} + v)(\bar{v} - v)$

➔  $\Delta v = \bar{v} - v = \frac{eRB}{2m_e}$  for small  $\Delta v$

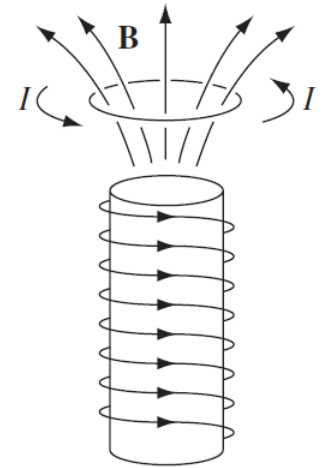
➔  $\Delta \mathbf{m} = -\frac{1}{2}e(\Delta v)R\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}$  Change in dipole moment works against  $\mathbf{B}$

# Forces on paramagnets and diamagnets

- General formula for small dipole in nonuniform field

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

- For paramagnets  $\mathbf{m} \parallel \mathbf{B}$ 
  - Force directs toward more intense field regions
  - Paramagnets are attracted to magnets
- For diamagnets  $\mathbf{m} \parallel -\mathbf{B}$ 
  - Force directs toward less intense field regions
  - Diamagnets are repelled by magnets



Levitating frog



(Ig Nobel award 2000)

# Field of magnetized objects

- Bound currents
  - Vector potential of a magnetized object (neglecting the cause)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} d\tau'$$

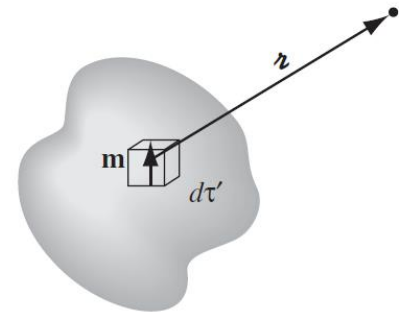
$$= \frac{\mu_0}{4\pi} \int \left[ \mathbf{M}(\mathbf{r}') \times \left( \nabla' \frac{1}{r} \right) \right] d\tau'$$

↓ Integrate by parts

$$= \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{r} [\nabla' \times \mathbf{M}(\mathbf{r}')] d\tau' - \int \nabla' \times \left[ \frac{\mathbf{M}(\mathbf{r}')}{r} \right] d\tau' \right\}$$

$$= \frac{\mu_0}{4\pi} \int \frac{1}{r} [\nabla' \times \mathbf{M}(\mathbf{r}')] d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{r} [\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']$$

$$= \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_b(\mathbf{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{r} da'$$



\* with  $\mathbf{J}_b = \nabla \times \mathbf{M}$   
 $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$



# Field of magnetized objects

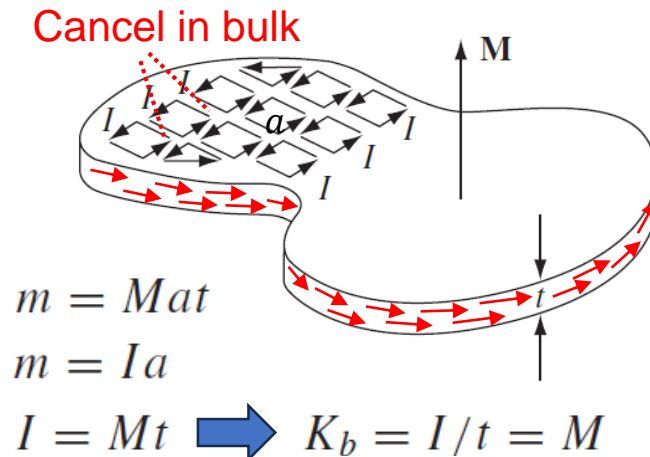
- Bound currents

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_b(\mathbf{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{r} da'$$

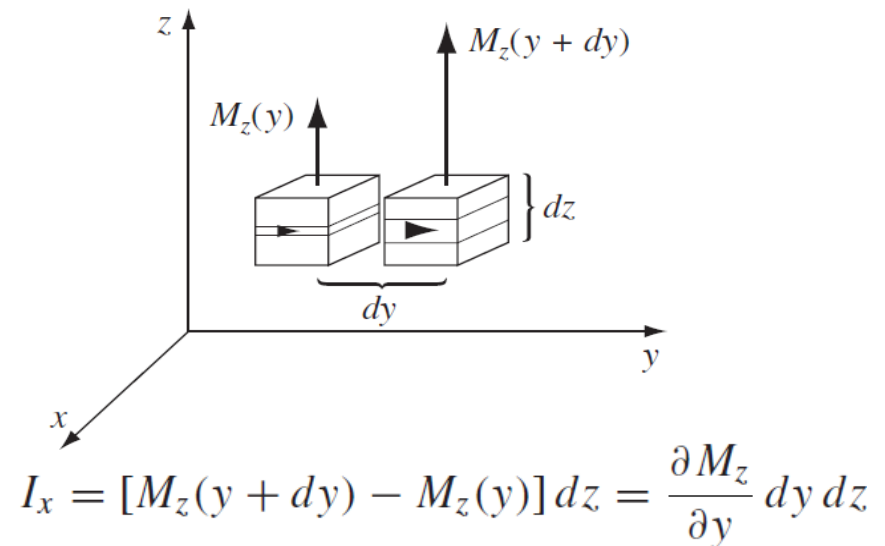
- Density of bound currents  $\left\{ \begin{array}{l} \text{surface: } \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \\ \text{volume: } \mathbf{J}_b = \nabla \times \mathbf{M} \end{array} \right.$  \*check  $\nabla \cdot \mathbf{J}_b = 0$

- Physical picture of bound currents

## Surface bound current (suppose uniform $\mathbf{M}$ )



## Volume bound current



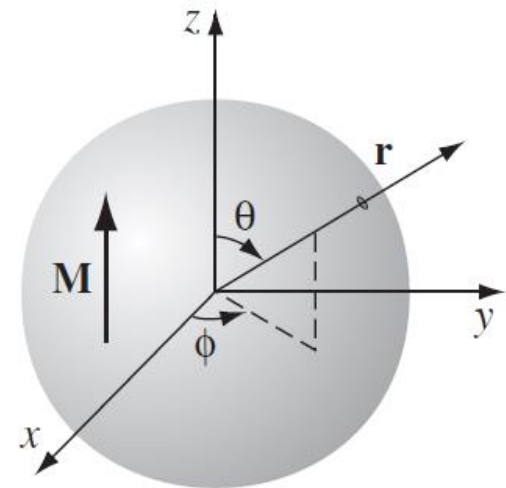
# Field of magnetized objects

- Bound currents

**Example 6.1.** Find the magnetic field of a uniformly magnetized sphere.

- Field inside the sphere  $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$ 
  - Uniform field
  - $\mathbf{M}$  induces  $\mathbf{B}$  that is parallel to it, while  $\mathbf{P}$  induces  $\mathbf{E}$  that is antiparallel
- Field outside the sphere same as what would be for a dipole

$$\mathbf{m} = \frac{4}{3}\pi R^3\mathbf{M}$$



# Auxiliary field

- Add the cause and the effect of magnetization

- Total magnetic field

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

- $\mathbf{B}$ : total magnetic field
- $\mathbf{J}$ : total current density
- $\mathbf{J}_b$ : bound current density, due to magnetization
- $\mathbf{J}_f$ : free current density that we control, not a result of magnetization

$$\Rightarrow \nabla \times \left( \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f$$

- The auxiliary field

$$\mathbf{H} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

$$\Rightarrow \nabla \times \mathbf{H} = \mathbf{J}_f$$

- Ampère's law for auxiliary field

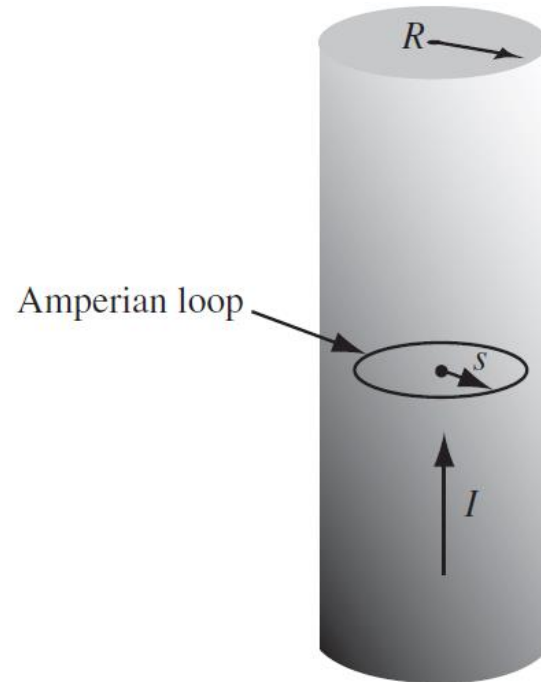
$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$$

Enclosed total  
free current

# Auxiliary field

- Application of auxiliary field

**Example 6.2.** A long copper rod of radius  $R$  carries a uniformly distributed (free) current  $I$  (Fig. 6.19). Find  $\mathbf{H}$  inside and outside the rod.

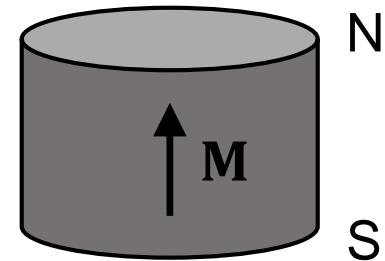


# Auxiliary field

- Can  $\mathbf{H}$  be finite without applying any free current?

- Looks like so because  $\oint \mathbf{H} \cdot d\mathbf{l} = I_{f\text{enc}}$

- Consider a short cylindrical magnet
- $\mathbf{H}$  would be zero everywhere
- Then  $\mathbf{B} \neq \mathbf{0}$  inside magnet,  $\mathbf{B} = \mathbf{0}$  outside magnet
- Obviously wrong



- $\mathbf{H}$  can be finite without any  $\mathbf{J}_F$ 
  - Because  $\nabla \cdot \mathbf{H} = \nabla \cdot \left( \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = -\nabla \cdot \mathbf{M} \neq 0$
  - $\nabla \cdot \mathbf{M} \neq 0$  at the top and bottom surfaces of the magnet

# H versus B, D versus E

- A long history of confusion
  - Whether we shall call **H** or **B** as the magnetic field
    - Some convention calls **H** the magnetic field, and **B** as the magnetic flux density
    - Our textbook takes the stance that **H** should be “auxiliary”
- Reason for such confusion
  - **H** is a lot more frequently used than **B** as **H** is given by free current, something we can control, while **B** is material dependent
    - Helmholtz coil magnetizing a specimen
  - **E** is a lot more frequently used than **D** as **D** is given by free charge, which we rarely control, while **E** is determined by voltage difference (over distance), which is what we control
    - Charging up of a parallel plate capacitor

# Linear magnetic media

- Linear magnetic media

- $\mathbf{M} = \chi_m \mathbf{H}$  ( $\chi_m$ : magnetic susceptibility)

➔  $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}$

- Diamagnetic susceptibility on the order of  $10^{-6}$

- $\mathbf{B} = \mu \mathbf{H}$  ( $\mu = \mu_0(1 + \chi_m)$ : permeability)

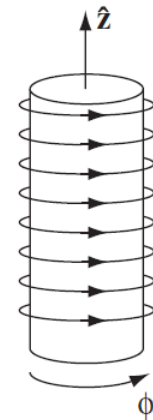
? Why this linear relation defines M-H but not M-B?

**Example 6.3.** An infinite solenoid ( $n$  turns per unit length, current  $I$ ) is filled with linear material of susceptibility  $\chi_m$ . Find the magnetic field inside the solenoid.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} \quad \text{➔} \quad \mathbf{H} = nI \hat{\mathbf{z}}$$

➔  $\mathbf{B} = \mu_0(1 + \chi_m)nI \hat{\mathbf{z}}$

Enhancement of field if paramagnetic!



# Boundary conditions

- Boundary conditions reexamined
  - Earlier findings still hold, but  $\mathbf{J}$  needs to include free and bound currents

$$\left\{ \begin{array}{l} B_{\text{above}}^{\perp} = B_{\text{below}}^{\perp} \\ \mathbf{B}_{\text{above}}^{\parallel} - \mathbf{B}_{\text{below}}^{\parallel} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \end{array} \right.$$

- Easier to use the boundary conditions of  $\mathbf{D}$

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \\ \nabla \times \mathbf{H} = \mathbf{J}_f \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \mathbf{H}_{\text{above}}^{\parallel} - \mathbf{H}_{\text{below}}^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}} \\ H_{\text{above}}^{\perp} - H_{\text{below}}^{\perp} = -(M_{\text{above}}^{\perp} - M_{\text{below}}^{\perp}) \end{array} \right.$$

Surface free  
charge density  
↓



# Nonlinear magnetic media

- Ferromagnets:  $\mathbf{M} \neq 0$  without applying any field
  - Obvious violation of the linear relation  $\mathbf{M} = \chi_m \mathbf{H}$
  - Represents a quantum phenomena
    - Exchange interaction:  $U = -2J \sum_{p=1}^N \mathbf{S}_p \cdot \mathbf{S}_{p+1}$  ( $J > 0$ )
    - Prefers spontaneous parallel alignment of spins
  - Domain formation and hysteresis loop

