Homework 1 solution

1. $\vec{r} = r\hat{r}$ in spherical coordinates

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ in Cartesian coordinates

Can switch freely to calculate the expressions.

(1) $\nabla \times \left(\frac{\vec{r}}{r}\right) = \nabla \times (1 \cdot \hat{r})$

check the formula for $\nabla \times v$ in spherical coordinates

$$= \frac{1}{r\sin\theta}\left[\frac{\partial(0)}{\partial\theta} - \frac{\partial(0)}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial(1)}{\partial\phi} - \frac{\partial}{\partial r}(0)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial(0)}{\partial r} - \frac{\partial(1)}{\partial\theta}\right]\hat{\phi}$$

$$= 0$$

(2) $\nabla \cdot \left(\frac{\vec{r}}{r}\right) = \nabla \cdot (1 \cdot \hat{r})$

check formula for $\nabla \cdot v$ in spherical coordinates

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot 1) + \frac{1}{r\sin\theta}\frac{\partial(0)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(0)}{\partial\phi} = \frac{2}{r}$$

(3) $\nabla(\vec{a} \cdot \vec{r})$

product rule

$$= \underbrace{\vec{a} \times (\nabla \times \vec{r})}_{1} + \underbrace{\vec{r} \times (\nabla \times \vec{a})}_{2} + \underbrace{(\vec{a} \cdot \nabla)\vec{r}}_{3} + \underbrace{(\vec{r} \cdot \nabla)\vec{a}}_{4}$$

2 & 4 both zero, because $\vec{a}$ is a constant vector

For 1, $\nabla \times \vec{r} = 0$ (verify with expression of $\nabla \times$ in spherical coordinates)

$$= (\vec{a} \cdot \nabla)\vec{r}$$

$$= \left(a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$$

$= a_x \hat{x} + a_y \hat{y} + a_z \hat{z} = \vec{a}$

Another way to show $(\vec{a} \cdot \nabla) \vec{r} = \vec{a}$ is to use spherical coordinates

$(\vec{a} \cdot \nabla) \vec{r}$

$= \left(a_r \frac{\partial}{\partial r} + \frac{a_\theta}{r} \frac{\partial}{\partial \theta} + \frac{a_\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \right) (r \hat{r})$

$= \underbrace{\hat{r}\, a_r \frac{\partial r}{\partial r}}_{a_r \hat{r}} + \frac{a_\theta}{r} \cdot r \underbrace{\frac{\partial \hat{r}}{\partial \theta}}_{= \hat{\theta}} + \frac{a_\phi}{r \sin\theta} \cdot r \underbrace{\frac{\partial \hat{r}}{\partial \phi}}_{= \sin\theta \hat{\phi}}$

$= a_r \hat{r} + a_\theta \hat{\theta} + a_\phi \hat{\phi} = \vec{a}$

(4) $\nabla \cdot [(\vec{a} \cdot \vec{r}) \vec{b}]$

$\downarrow$ product rule, treat $(\vec{a} \cdot \vec{r})$ as scalar field f

$= (\vec{a} \cdot \vec{r})(\nabla \cdot \vec{b}) + \vec{b} \cdot [\nabla (\vec{a} \cdot \vec{r})]$

$\downarrow$ $\nabla \cdot \vec{b} = 0$, & $\nabla (\vec{a} \cdot \vec{r}) = \vec{a}$

$= \vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}$

(5) $\nabla \times [(\vec{a} \cdot \vec{r}) \vec{r}]$

$= (\vec{a} \cdot \vec{r})(\nabla \times \vec{r}) - \vec{r} \times [\nabla (\vec{a} \cdot \vec{r})]$

$\downarrow$ $\nabla \times \vec{r} = 0$, $\nabla (\vec{a} \cdot \vec{r}) = \vec{a}$

$= -\vec{r} \times \vec{a} = \vec{a} \times \vec{r}$

2. The divergence theorem reads as

$$\int_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

For $\vec{v} = x^2 \hat{x} + y^2 \hat{y} + z \hat{z}$

$$\nabla \cdot \vec{v} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z) = 2x + 2y + 1$$

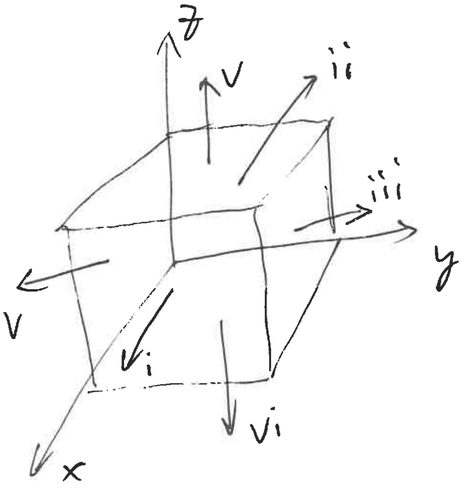
In the defined volume

$$\int_V (\nabla \cdot \vec{v})\, d\tau = \int_0^a \int_0^a \int_0^a (2x + 2y + 1)\, dx\, dy\, dz$$

$$= \iiint 2x\, dx\, dy\, dz + \iiint 2y\, dx\, dy\, dz + \iiint dx\, dy\, dz$$

$$= a \cdot a \cdot x^2 \Big|_0^a + a \cdot a \cdot y^2 \Big|_0^a + a \cdot a \cdot a$$

$$= 2a^4 + a^3$$



For the surface integral

$$\oint_S \vec{v} \cdot d\vec{a} = \int_i \vec{v} \cdot d\vec{a} + \int_{ii} \vec{v} \cdot d\vec{a} + \dots + \int_{vi} \vec{v} \cdot d\vec{a}$$

6 surfaces intotal defined in the graph

$$\int_i \vec{v} \cdot d\vec{a} = \int_0^a \int_0^a (x^2 \hat{x} + y^2 \hat{y} + z \hat{z}) \cdot (\hat{x}\, dy\, dz) = \int_0^a \int_0^a x^2\, dy\, dz$$

$x = a$ on surface i $\Rightarrow \int_i \vec{v} \cdot d\vec{a} = a^4$

$$\int_{ii} \vec{v}\cdot d\vec{a} = \int_0^a\int_0^a (x^2\hat{x}+y^2\hat{y}+z\hat{z})\cdot(-\hat{x}\,dy\,dz) = -\int_0^a\int_0^a x^2\,dy\,dz$$

$x=0$ on surface ii $\Rightarrow \int_{ii} \vec{v}\cdot d\vec{a} = 0$

$$\int_{iii} \vec{v}\cdot d\vec{a} = \int_0^a\int_0^a (x^2\hat{x}+y^2\hat{y}+z\hat{z})\cdot(\hat{y}\,dx\,dz) = \int_0^a\int_0^a y^2\,dx\,dz$$

$y=a$ on surface iii $\Rightarrow \int_{iii} \vec{v}\cdot d\vec{a} = a^4$

$\int_{iv} \vec{v}\cdot d\vec{a} = \int_0^a\int_0^a -y^2\,dx\,dz$ while $y=0 \Rightarrow \int_{iv} \vec{v}\cdot d\vec{a} = 0$

$$\int_{v} \vec{v}\cdot d\vec{a} = \int_0^a\int_0^a (x^2\hat{x}+y^2\hat{y}+z\hat{z})\cdot(\hat{z}\,dx\,dy) = \int_0^a\int_0^a z\,dx\,dy$$

$z=a$ on surface v $\Rightarrow \int_{v} \vec{v}\cdot d\vec{a} = a^3$

$\int_{vi} \vec{v}\cdot d\vec{a} = \int_0^a\int_0^a (-z)\,dx\,dy$ while $z=0 \Rightarrow \int_{vi} \vec{v}\cdot d\vec{a} = 0$

In total, $\oint_S \vec{v}\cdot d\vec{a} = a^4+0+a^4+0+a^3+0 = 2a^4+a^3$

Noticing ~~...~~ $\int_V (\nabla\cdot\vec{v})\,d\tau = 2a^4+a^3$, proved ✓.

3. The Stoke's theorem reads as

$$\int_S (\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$$

For $\vec{v} = s^2\hat{\phi} + \phi\hat{z}$

$$\nabla \times \vec{v} = \left[\frac{1}{s}\frac{\partial(\phi)}{\partial \phi} - \frac{\partial(s^2)}{\partial z}\right]\hat{s} + \left(-\frac{\partial \phi}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial(s \cdot s^2)}{\partial s}\right]\hat{z}$$

$$= \frac{1}{s}\hat{s} + 3s\hat{z}$$

In the defined surface $d\vec{a} = \hat{z}\, s \cdot ds\, d\phi$

$$\Rightarrow \int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \int_0^{2\pi}\int_0^a \left(\frac{1}{s}\hat{s} + 3s\hat{z}\right) \cdot \hat{z}\, s\, ds\, d\phi$$

$$= \int_0^{2\pi}\int_0^a 3s^2\, ds\, d\phi = 2\pi\, s^3\Big|_0^a = 2\pi a^3$$

Next we examine $\oint_P \vec{v} \cdot d\vec{l}$ along the boundary of the surface

$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$ in cylindrical coordinates

On the circular boundary line, $s = a$, $z = 0$, $\Rightarrow ds = dz = 0$

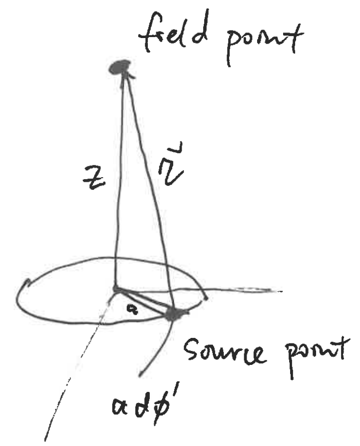
$\Rightarrow d\vec{l} = s\, d\phi\, \hat{\phi}$

$$\oint_P \vec{v} \cdot d\vec{l} = \int_0^{2\pi} (s^2\hat{\phi} + \phi\hat{z}) \cdot \hat{\phi}\, s\, d\phi$$

$$= \int_0^{2\pi} s^3\, d\phi = a^3 \int_0^{2\pi} d\phi = 2\pi a^3$$

Noticing $\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = 2\pi a^3$ above, proved ✓.

4. $\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2} \hat{\mathscr{r}} \, dl'$



In cylindrical coordinates

source point $\vec{r}\,' = a\hat{s}$

field point $\vec{r} = z\hat{z}$

separation $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = -a\hat{s} + z\hat{z}$

$\mathscr{r} = \sqrt{a^2 + z^2}$

On the other hand $dl' = a\,d\phi'$, $\lambda(\vec{r}\,') = \lambda$, constant

z - component of $\vec{E}$

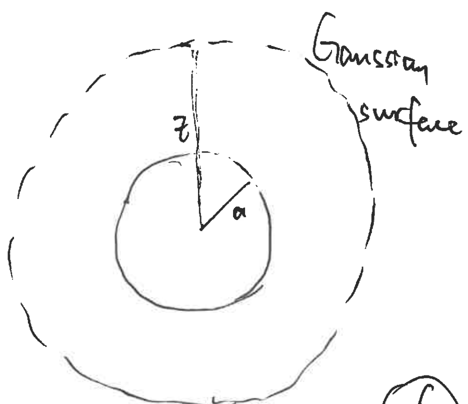
$$\vec{E}(\vec{r}) \cdot \hat{z} = \frac{a\lambda}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{z}{(a^2+z^2)^{\frac{3}{2}}} d\phi'$$

$$= \frac{a\lambda}{4\pi\varepsilon_0} \frac{z}{(a^2+z^2)^{\frac{3}{2}}} \cdot 2\pi = \frac{a\lambda}{2\varepsilon_0} \frac{z}{(a^2+z^2)^{\frac{3}{2}}}$$

5. For use ~~of~~ of the Gauss's law

Construct a spherical shell with radius z

as the Gaussian surface for integration



$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

spherical symmetry $\Rightarrow \vec{E} = E\hat{r}$

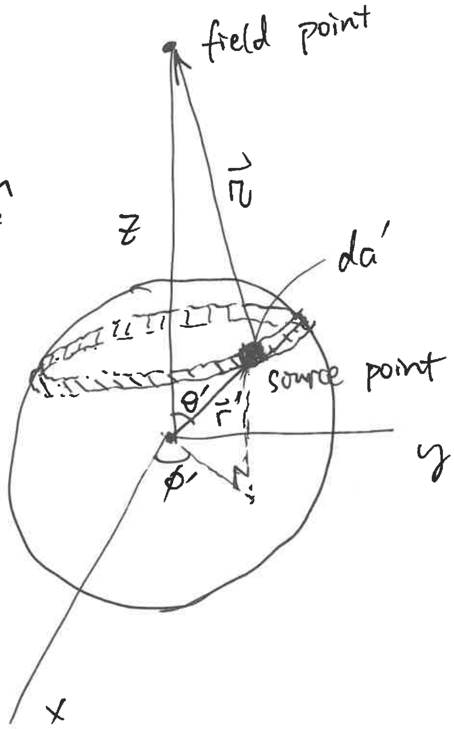
$$\oint \vec{E} \cdot d\vec{a} = E \oint da = E \cdot 4\pi z^2$$

On the other hand, $Q_{enc} = \sigma \cdot \underbrace{A}_{\text{surface area of charge-carrying ball}} = \sigma \cdot 4\pi a^2$

$$\Rightarrow E \cdot 4\pi z^2 = \frac{\sigma}{\varepsilon_0} 4\pi a^2 \Rightarrow E = \frac{\sigma a^2}{\varepsilon_0 z^2}$$

Now we use Coulomb's law $\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(r')}{\mathscr{r}^2} \hat{\mathscr{r}} \, da'$

Using the law of cosines:



$$\mathscr{r}^2 = z^2 + a^2 - 2za\cos\theta'$$

$$\hat{\mathscr{r}} = a\sin\theta'\cos\phi'\hat{x} + a\sin\theta'\sin\phi'\hat{y} + (z - a\cos\theta')\hat{z}$$

$\sigma(r') = \sigma$, uniform thoughout the shell

$$da' = a^2 \sin\theta \, d\theta \, d\phi$$

Therefore

$$\vec{E} \cdot \hat{z} = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\pi} \frac{a^2 (z - a\cos\theta')\sin\theta'}{(z^2 + a^2 - 2za\cos\theta')^{\frac{3}{2}}} d\theta' d\phi'$$

$$= \frac{\sigma \cdot 2\pi}{4\pi\varepsilon_0} \int_0^{\pi} \frac{a^2 (z - a\cos\theta')\sin\theta'}{(z^2 + a^2 - 2za\cos\theta')^{\frac{3}{2}}} d\theta'$$

Define $u \equiv z^2 + a^2 - 2za\cos\theta'$

$\Rightarrow \quad du = 2za \sin\theta' d\theta'$

$z - a\cos\theta' = \frac{1}{2z}(u + z^2 - a^2)$ appears in the integral

$$\vec{E}\cdot\hat{z} = \frac{\sigma\cdot 2\pi}{4\pi\varepsilon_0}\int_{(z-a)^2}^{(z+a)^2} a^2 \cdot \frac{1}{2z}(u+z^2-a^2)\cdot\frac{1}{2za}\cdot u^{-\frac{3}{2}}\, du$$

$$= \frac{\sigma}{2\varepsilon_0}\cdot\frac{a}{4z^2}\int_{(z-a)^2}^{(z+a)^2}(u+z^2-a^2)\,u^{-\frac{3}{2}}\,du$$

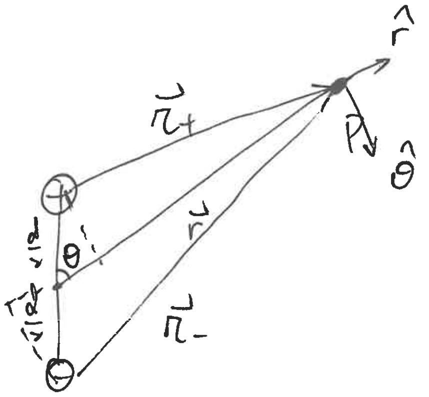
$$= \frac{\sigma a}{8\varepsilon_0 z^2}\cdot\left[\int_{(z-a)^2}^{(z+a)^2} u^{-\frac{1}{2}}\,du + (z^2-a^2)\int_{(z-a)^2}^{(z+a)^2} u^{-\frac{3}{2}}\,du\right]$$

$$= \frac{\sigma a}{8\varepsilon_0 z^2}\cdot\left[2u^{\frac{1}{2}}\Big|_{(z-a)^2}^{(z+a)^2} + (z^2-a^2)(-2)\,u^{-\frac{1}{2}}\Big|_{(z-a)^2}^{(z+a)^2}\right]$$

$$= \frac{\sigma a}{8\varepsilon_0 z^2}\cdot\left[4a + 4a\right] = \frac{\sigma a^2}{\varepsilon_0 z^2}$$

This result agrees with that using the Gauss's law.

6. In spherical coordinates



$$\vec{r}_+ = (r - \frac{d}{2}\cos\theta)\,\hat{r} + \frac{d}{2}\sin\theta\,\hat{\theta}$$

$$\vec{r}_- = (r + \frac{d}{2}\cos\theta)\,\hat{r} - \frac{d}{2}\sin\theta\,\hat{\theta}$$

$$r_+^2 = r^2 + (\frac{d}{2})^2 - dr\cos\theta \approx r^2 - dr\cos\theta$$

$$r_-^2 = r^2 + (\frac{d}{2})^2 + dr\cos\theta \approx r^2 + dr\cos\theta$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left(\frac{\hat{r}_+}{r_+^2} - \frac{\hat{r}_-}{r_-^2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}_+}{r_+^3} - \frac{\vec{r}_-}{r_-^3}\right)$$

$$\vec{E}\cdot\hat{r} = \frac{q}{4\pi\varepsilon_0}\left[\frac{r - \frac{d}{2}\cos\theta}{(r^2 - dr\cos\theta)^{\frac{3}{2}}} - \frac{r + \frac{d}{2}\cos\theta}{(r^2 + dr\cos\theta)^{\frac{3}{2}}}\right]$$

$$\approx \frac{q}{4\pi\varepsilon_0}\left[\frac{r - \frac{d}{2}\cos\theta}{(r - \frac{d}{2}\cos\theta)^3} - \frac{r + \frac{d}{2}\cos\theta}{(r + \frac{d}{2}\cos\theta)^3}\right]$$

$$\approx \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r - \frac{d}{2}\cos\theta)^2} - \frac{1}{(r + \frac{d}{2}\cos\theta)^2}\right]$$

Note that $(r-\frac{d}{2}\cos\theta)^2 = r^2(1-\frac{d}{2r}\cos\theta)^2 \approx r^2(1-\frac{d}{r}\cos\theta)$

$$\vec{E}\cdot\hat{r} \approx \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\left[\frac{1}{1-\frac{d}{r}\cos\theta} - \frac{1}{1+\frac{d}{r}\cos\theta}\right]$$

$$\approx \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{2d}{r}\cos\theta = \frac{2qd}{4\pi\varepsilon_0 r^3}\cos\theta$$

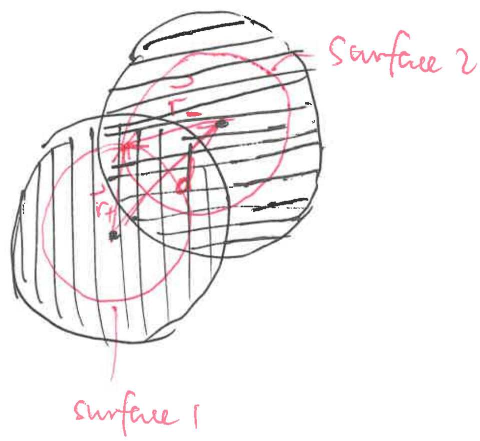
$$\vec{E}\cdot\hat{\theta} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\frac{d}{2}\sin\theta}{(r^2-dr\cos\theta)^{\frac{3}{2}}} - \frac{-\frac{d}{2}\sin\theta}{(r^2+dr\cos\theta)^{\frac{3}{2}}}\right]$$

$$\approx \frac{q}{4\pi\varepsilon_0}\frac{\frac{d}{2}\sin\theta - (-\frac{d}{2}\sin\theta)}{r^3}$$

$$= \frac{qd\sin\theta}{4\pi\varepsilon_0 r^3}$$

Summarizing above, $\vec{E} = \frac{2qd}{4\pi\varepsilon_0 r^3}\cos\theta\,\hat{r} + \frac{qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{\theta}$

7. First apply Gauss's law to "+" charge cloud on surface 1 (which intersects that point in the overlapping region.



$\vec{r}_\pm$ denotes vector going from the "±" charge center to the point of interest

Gauss's law for surface 1 on "+" charge

$$\oint_1 \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\oint \vec{E} \cdot d\vec{a} = E \cdot 4\pi r_+^2 \quad , \quad \frac{Q_{enc}}{\varepsilon_0} = \frac{\frac{4}{3}\pi r_+^3 \cdot \rho}{\varepsilon_0}$$

$$\Rightarrow \quad E_+ = \frac{1}{4\pi r_+^2} \frac{\frac{4}{3}\pi r_+^3 \cdot \rho}{\varepsilon_0} = \frac{\rho r_+}{3\varepsilon_0}$$

$$\vec{E}_+ = \frac{\rho r_+}{3\varepsilon_0} \hat{r}_+ = \frac{\rho \vec{r}_+}{3\varepsilon_0}$$

Gauss's law for surface 2 on "−" charge

$$\vec{E}_- = \frac{-\rho \vec{r}_-}{3\varepsilon_0}$$

According to superposition principle,

$$\vec{E}_{tot} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{3\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{3\varepsilon_0}\vec{d}$$

Which is a constant