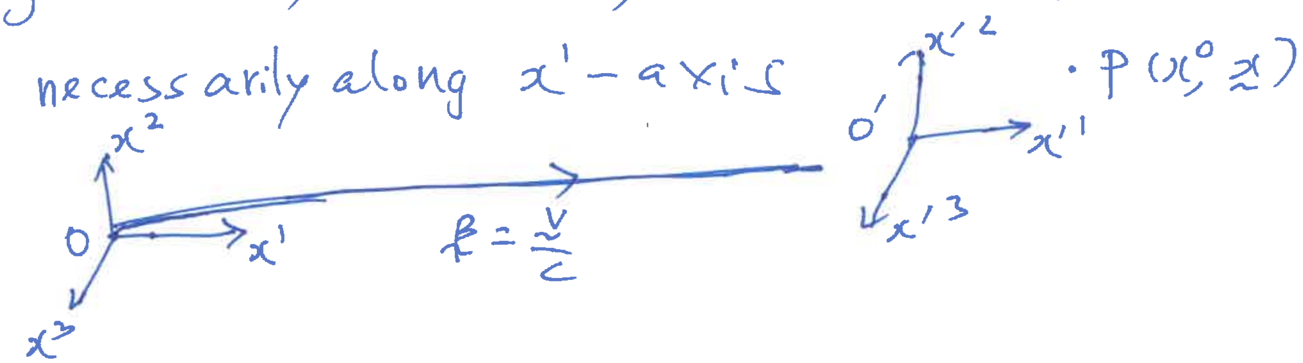
2024. 1. 30   5   1

Today

Work out most general Lorentz transformations between two inertial frames. Introduce metric tensor $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$

Already know the special Lorentz transformation along the x^1-axis, i.e. O' frame moves away from O frame along the x^1-axis of the O-frame.

Derive a more general Lorentz transformation along arbitrary direction, the velocity $\underset{\sim}{v} = \underset{\sim}{\beta} c$ not necessarily along x^1-axis



Write

$$\underset{\sim}{x} = \underset{\sim}{x}_{\parallel} + \underset{\sim}{x}_{\perp} \quad \text{by defining}$$

$$\underset{\sim}{x}_{\parallel} = \left(\underset{\sim}{x} \cdot \frac{\underset{\sim}{\beta}}{|\beta|}\right) \frac{\underset{\sim}{\beta}}{|\beta|} \qquad \left[\text{for } \underset{\sim}{\beta} = (\beta, 0, 0),\ \underset{\sim}{x}_{\parallel} = (x^1, 0, 0)\right]$$

HW

then

$$\underset{\sim}{x}_{\perp} \cdot \underset{\sim}{\beta} = 0$$

We have

$$\underset{\sim}{x}'_{\parallel} = \gamma(\underset{\sim}{x}_{\parallel} - \underset{\sim}{\beta} x^0), \quad \underset{\sim}{x}'_{\perp} = \underset{\sim}{x}_{\perp}$$

$$x'^0 = \gamma(x^0 - \underset{\sim}{\beta} \cdot \underset{\sim}{x})$$

$$\underset{\sim}{x}' = \underset{\sim}{x}'_{\parallel} + \underset{\sim}{x}'_{\perp}$$

$$\underset{\sim}{x}' = \gamma(\underset{\sim}{x}_{\parallel} - \underset{\sim}{\beta} x^0) + \underset{\sim}{x} - \underset{\sim}{x}_{\parallel}$$

$$= \underset{\sim}{x} + (\gamma - 1)\underset{\sim}{x}_{\parallel} - \gamma \underset{\sim}{\beta} x^0$$

$$= \underset{\sim}{x} + (\gamma - 1)\frac{\underset{\sim}{x}\cdot\underset{\sim}{\beta}}{|\underset{\sim}{\beta}|^2}\underset{\sim}{\beta} - \gamma \beta x^0$$

So the generalized Lorentz tran.

$$x'^0 = \gamma(x^0 - \beta \cdot \underset{\sim}{x})$$

$$\underset{\sim}{x}' = \underset{\sim}{x} + (\gamma - 1)\frac{\underset{\sim}{x}\cdot\underset{\sim}{\beta}}{|\underset{\sim}{\beta}|^2}\underset{\sim}{\beta} - \gamma \underset{\sim}{\beta} x^0$$

Next define (construct) most general Lorentz tran. between 2 inertial frames.

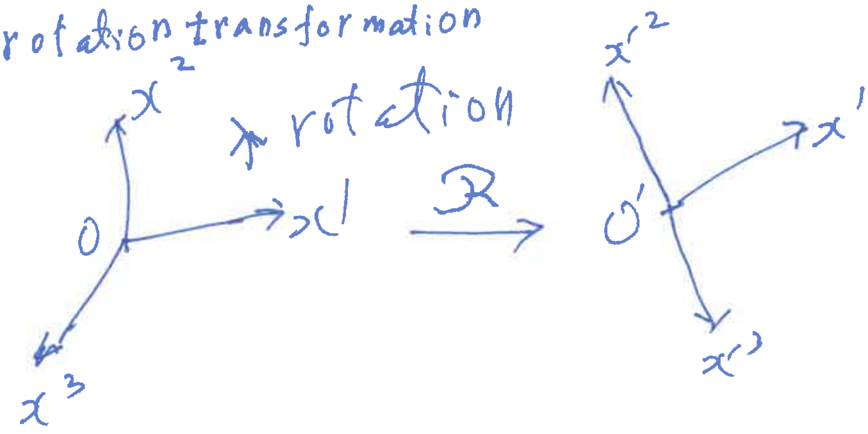
=

Write down matrix eqn for Galilean and Lorentz tran (along x^1-axis)

How to construct general tran between 2 inertial frames?

Learn from rotation in 3-dim space.

Know rotation transformation



e.g. rotation along x^3- axis by an angle θ

$$\mathcal{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

General rotation $\underset{\sim}{x} \to \underset{\sim}{x}' = \mathcal{R}\, \underset{\sim}{x}$

$$x'_i = \mathcal{R}_{ij}\, x_j \qquad (\text{sum over } j,\ j = 1, 2, 3)$$

By definition, rotation $\mathcal{R}$ preserves distance between 2 points

$$\longrightarrow \quad \mathcal{R}^t\, \mathcal{R} = 1 \to \mathcal{R}^t = \mathcal{R}^{-1}$$

orthogonal

Take this cue, define Lorentz tran as a tran that preserves distance between 2 spacetime points (events)

So for $\underset{\sim}{x}' = \mathcal{R}\, \underset{\sim}{x}$

$$x'_i = \mathcal{R}_{ij}\, x_j$$

we have

$$\underline{x} = (x^0, \underset{\sim}{x})$$

$$\underline{x}' = \Lambda\, \underline{x}$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

Distance in 3-dim space from origin is $\underset{\sim}{x}^2 = (x_1^2 + x_2^2 + x_3^2)$

Distance in 4-dim spacetime from origin is $\underline{x}^2$

what is $\underline{x}^2 = ?$

Ans $\underline{x}^2 = x^{0\,2} - x^{1\,2} - x^{2\,2} - x^{3\,2}$

(obtained by considering special Lorentz tran, e.g.

$$x'^1 = \gamma(x^1 - \beta x^0), \quad x'^0 = \gamma(x^0 - \beta x^1)$$

$$x'^2 = x^2, \quad x'^3 = x^3.$$

That is

$$\underline{x}' = \Lambda \, \underline{x}$$

$$\underline{x}'^2 = \underline{x}^2$$

$\rightarrow$ prove

$$g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

$g_{\mu\nu}$ is a metric tensor, for

Minkowski spacetime

$$g_{00} = +1, \quad g_{11} = g_{22} = g_{33} = -1$$

$$g_{\mu\nu} = 0 \quad \forall \ \mu \neq \nu$$

$g_{\mu\nu}$ tells us how to measure

distance between any two points

For Euclidean geometry,

$$g_{ij} = 0 \qquad i \neq j$$

$$g_{ij} = 1 \qquad i \neq j$$

$$\underline{x}' = \Lambda \underline{x}$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$\underline{x}'^2 = x'^{\mu} x'^{\nu} g_{\mu\nu} \quad \left(= x'^{\mu} \cdot x'_{\mu} \right)$$

$$= \left(\Lambda^{\mu}{}_{\alpha} x^{\alpha} \right) \left(\Lambda^{\nu}{}_{\beta} x^{\beta} \right) g_{\mu\nu}$$

$$= g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \cdot x^{\alpha} x^{\beta}$$

Now $\underline{x}^2 = g_{\alpha\beta} x^{\alpha} x^{\beta}$

As $\underline{x}'^2 = \underline{x}^2$,

$$\therefore g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

Assume at time $t = 0 = t'$, O' frame

and O frame coincides with respective axes

parallel to each other, also O' frame moves

along the x^1-axis of O frame.

The Lorentz transformation is

$$x'^1 = \gamma (x^1 - \beta x^0) \qquad \beta = \frac{v}{c}$$

$$x'^2 = x^2$$

$$x'^3 = x^3 \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$x'^0 = \gamma (x^0 - \beta x^1) \qquad x^0 = ct$$

$$x'^0 = ct'$$

spatial coordinates and time coordinates mix,

x'^1 contains x^1 and x^0, x'^0 contains x^0

and x^1.

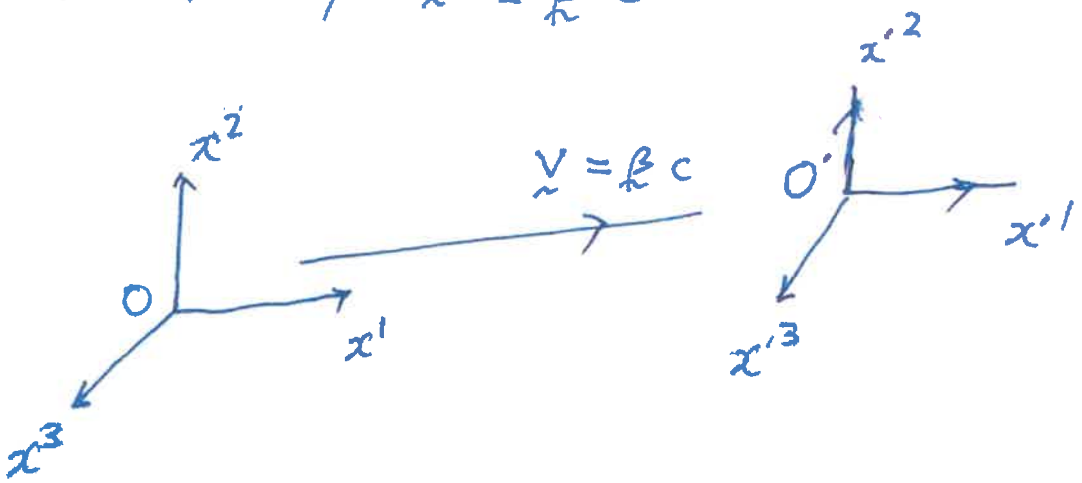
space and time both relative. →

c (speed of light) is a constant.

Write down Lorentz transformation along any

~~coordinate axis~~ direction, that is $\underset{\sim}{\beta} = \frac{\underset{\sim}{v}}{c}$, not just along x^1-axis

Lorentz transformation along any spatial direction with velocity $\underset{\sim}{V} \equiv \underset{\sim}{\beta}\, c$



Along x^1- axis

$$x'^1 = \gamma (x^1 - \beta x^0), \quad x'^2 = x^2, \quad x'^3 = x^3$$

$$x'^0 = \gamma (x^0 - \beta x^1). \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Note : spatial components perpendicular to $\underset{\sim}{V}$ unchanged (in this case, x^2, x^3)

Resolve $\underset{\sim}{x} = (x^1, x^2, x^3) = \underset{\sim}{x}_\perp + \underset{\sim}{x}_{\parallel}$

$$\underset{\sim}{x}_{\parallel} = \frac{\underset{\sim}{x} \cdot \underset{\sim}{\beta}}{|\underset{\sim}{\beta}|^2}\, \underset{\sim}{\beta}, \qquad x_\perp \cdot \underset{\sim}{\beta} = 0$$

Thus

$$\underset{\sim}{x}'_\perp = \underset{\sim}{x}_\perp$$

$$\underset{\sim}{x}'_{\parallel} = \gamma (\underset{\sim}{x}_{\parallel} - \underset{\sim}{\beta}\, x^0)$$

$$x'^0 = \gamma (x^0 - \underset{\sim}{\beta} \cdot \underset{\sim}{x})$$

$$\mathbf{x}' = \mathbf{x}'_{\perp} + \mathbf{x}'_{\parallel}$$

$$= \mathbf{x}_{\perp} + \gamma(\mathbf{x}_{\parallel} - \boldsymbol{\beta}\, x^0)$$

$$= \mathbf{x} + (\gamma - 1)\mathbf{x}_{\parallel} - \gamma \boldsymbol{\beta}\, x^0$$

$$= \mathbf{x} + (\gamma - 1)\frac{\mathbf{x}\cdot\boldsymbol{\beta}}{|\boldsymbol{\beta}|^2}\boldsymbol{\beta} - \gamma\boldsymbol{\beta}\, x^0$$

$$x'^0 = \gamma(x^0 - \boldsymbol{\beta}\cdot\mathbf{x}).$$

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}}$$

Before proceeding further, first note that Galilean transformation and Lorentz transformation can be written as matrix

put $\underset{\sim}{x} = (x^0, \underset{\sim}{x})$, $\underset{\sim}{x}$ = 4 component

$\underset{\sim}{x} = (x^1, x^2, x^3)$

3-component

For Galilean transformation along x^1-axis

$$x'^1 = x^1 - vt, \quad x'^2 = x^2, \quad x'^3 = x^3,$$

$$t' = t$$

$$\begin{pmatrix} t' \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}$$

Different values v will give different Galilean transformations

Verify all Galilean transformations form a group i.e. satisfy 4 axioms of a group (H W)

known as the Galilean group

Home work

Defn of a group

A set S of elements $\{a, b, c, \ldots d\}$

with a binary operation $\cdot$

such that (s.t)

(1) closure: If $a \in S$, $b \in S$,

then $a \cdot b \in S$

(2) $\exists$ (there exists) an identity I

$$I \cdot a = a = a \cdot I \quad \text{for any } a \in S$$

(3) Associativity:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(4) $\exists$ an inverse a^{-1} for any a

$$a^{-1} \cdot a = I \text{ (identity)}$$

$$= a \cdot a^{-1}$$

Group, usually denoted by G, is commonly used in physics; many transformations in physics form a Group. E.g., rotations form a rotation group denoted by $SO(3)$. Lorentz transformations form a group denoted by $SO(3,1)$.

similarly the Lorentz transformation along the x^1-axis can be written in a matrix form

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

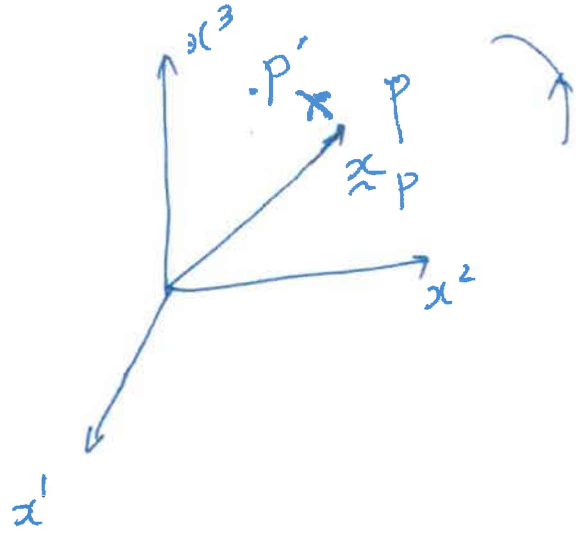
All Lorentz transformations form a group,

the Lorentz group (HW)

We now proceed to find the <u>most general</u> Lorentz transformation

we take cue from rotation transformation in 3 dimensional space

Position vector in 3 dimensional space is denoted by $\underset{\sim}{x}_p = (x_p^1, x_p^2, x_p^3)$



rotation $\mathcal{R}$, P moves to P'

$$\underset{\sim}{x}_p \xrightarrow{\mathcal{R}} \underset{\sim}{x}_{p'} = \mathcal{R}\,\underset{\sim}{x}_p$$

Distance of the point P before rotation

$$= {x_p^1}^2 + {x_p^2}^2 + {x_p^3}^2 \quad \cdots \quad (1)$$

After rotation $\mathcal{R}$, P moves to P', the distance of the point P' from the origin

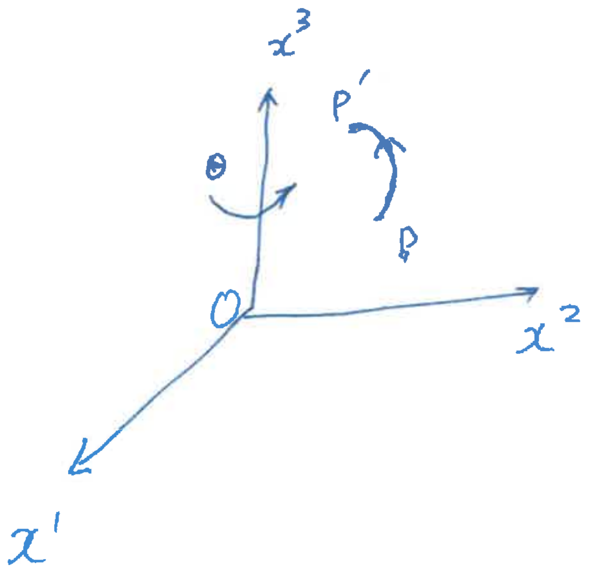
$$= {x_{p'}^1}^2 + {x_{p'}^2}^2 + {x_{p'}^3}^2. \quad \cdots\cdots \quad (2)$$

It is found : distance before rotation, eq (1)

= distance after rotation, eq (2).

We say spatial distance in 3 dimensional space is invariant under spatial rotation.

For a rotation about the x^3-axis (z-axis) by an angle θ, the rotation matrix is given by



$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It can be easily verified that for the Lorentz transformation

$$x'^0 = \gamma(x^0 - \beta x^1), \quad x'^1 = \gamma(x^1 - \beta x^0),$$
$$x'^2 = x^2, \qquad x'^3 = x^3$$

the quantity $(x^{0^2} - x^{1^2} - x^{2^2} - x^{3^2})$ is the same before and after the Lorentz transformation stated above.

In fact, one finds the interval, Δs, as defined by

$$\Delta s^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2,$$

$\Delta \underset{\sim}{x} = \underset{\sim}{x}_P - \underset{\sim}{x}_Q$, $\qquad$ P, Q two points

$\Delta x^0 = x^0_P - x^0_Q$, $\qquad$ (events) in spacetime

is unchanged (invariant) under the above Lorentz transformation $\qquad$ (HW)

We can now introduce a general Lorentz transformation as a linear transformation that preserves the interval Δs^2.

A transformation Λ is linear iff

$$\Lambda(a \underline{x}_P + b \underline{x}_Q) = a \Lambda \underline{x}_P + b \Lambda \underline{x}_Q, \quad a, b = \text{constants}$$

A Lorentz tran is a linear transformation that preserves the interval

$$\Delta s^2 = \Delta \underline{x} \cdot \Delta \underline{x} = \Delta x^{0\,2} - (\Delta \underline{x})^2$$

One denotes the Lorentz tran as $(\Lambda, \underline{a})$

$$\underline{x} \rightarrow \underline{x}' = \Lambda \underline{x}$$ (Homogeneous Lorentz tran)

or $$\underline{x}' = \Lambda \underline{x} + \underline{a}$$ (inhomogeneous Lorentz transformation = Poincaré tran.)

$\underline{a}$ = constant 4-vector

So $(\Lambda, \underline{a})$ transformation preserves the interval

$$\Delta \underline{x}' \cdot \Delta \underline{x}' = \Delta \underline{x} \cdot \Delta \underline{x}$$

For simplicity, discuss homogeneous Lorentz tran

Λ :

$$\underline{x} \rightarrow \underline{x}' = \Lambda \underline{x}$$

$\rightarrow$ $$s^2 = \underline{x} \cdot \underline{x} = \text{interval}$$

Λ preserves $\underline{x} \cdot \underline{x} = \underline{x}^2 = (x^{0\,2} - x^{1\,2} - x^{2\,2} - x^{3\,2})$

i.e. $$\underline{x}'^2 = \underline{x}^2$$

First linear : $\Lambda(a\underline{x}_1 + b\underline{x}_2) = a\Lambda\underline{x}_1 + b\Lambda\underline{x}_2$

a, b are constant

The transformation $\underline{x}' = \Lambda \underline{x}$ can be written in component form

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$\mu = 0, 1, 2, 3$
$\nu = 0, 1, 2, 3$

summation convention:

repeated indices, means summation

ν runs from 0, 1, 2, 3

Thus $x'^{\mu} = \Lambda^{\mu}{}_{0} x^{0} + \Lambda^{\mu}{}_{1} x^{1} + \Lambda^{\mu}{}_{2} x^{2} + \Lambda^{\mu}{}_{3} x^{3}$ (HW)

From $\underline{x}'^2 = \underline{x}^2$, we can derive a relation for Λ

$$\underline{x}'^2 = (\Lambda \underline{x}) \cdot (\Lambda \underline{x}) = \underline{x}^2$$

$$(\Lambda \underline{x})^{\mu} \cdot (\Lambda \underline{x})_{\mu}$$

$$\left[\rightarrow (\Lambda^{\mu}{}_{\alpha} x^{\alpha}) (\Lambda_{\mu}{}^{\beta} x_{\beta}) = \underline{x}^2 \right]$$

To proceed further, need to introduce metric tensor g

$$\underline{x}^2 = x^{0\,2} - x^{1\,2} - x^{2\,2} - x^{3\,2}$$

$$\overset{HW}{=} g_{\mu\nu} x^{\mu} x^{\nu} \quad \text{if} \quad g^{00} = +1, \quad g^{11} = g^{22} = g^{33} = -1$$

$$g^{\mu\nu} = 0 \;\; \forall \;\; \mu \neq \nu$$

$g_{\mu\nu}$ tells us how to measure 'distance'

In ordinary 3-dim space

$$\underset{\sim}{x}^2 = x^{1^2} + x^{2^2} + x^{3^2}$$
$$= g_{ij}\, x^i x^j, \qquad i\,j = 1, 2, 3$$

$g_{ij} = 0$ except $i = j$

then $g_{11} = g_{22} = g_{33}$
$= +1$

g_{ij} = metric tensor,

which defines Euclidean geometry in 3-dim space, if $g_{ij} = \delta_{ij}$

In 4-dim spacetime, the metric tensor is

$g_{\mu\nu}$, where $g_{\mu\nu} = 0 \quad \forall\ \mu \neq \nu$

and $g_{00} = +1$, $g_{11} = -1 = g_{22} = g_{33}$

which defines Minkowski geometry or the Minkowski space time

In general $g_{\mu\nu} \rightarrow$ Riemannian geometry

Now go back to $\Lambda^\mu{}_\nu$

$$\underset{\sim}{x}'^2 = g_{\mu\nu}\, x'^\mu x'^\nu \qquad O' \text{ frame}$$

$$\underset{\sim}{x}^2 = g_{\mu\nu}\, x^\mu x^\nu \qquad O \text{ frame}$$

Note: $g_{\mu\nu}$ same for both O' frame and O frame
$\therefore$ same spacetime manifold, same geometry

$$\underline{x}'^2 = g_{\mu\nu} x'^\mu x'^\nu \qquad (x'^\mu = \Lambda^\mu{}_\nu x^\nu)$$

$$= g_{\mu\nu} \Lambda^\mu{}_\alpha x^\alpha \Lambda^\nu{}_\beta x^\beta$$

$$= g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta$$

$$\underline{x}^2 = g_{\alpha\beta} x^\alpha x^\beta$$

$$\because \quad \underline{x}'^2 = \underline{x}^2$$

$$\therefore \quad \underline{g_{\mu\nu} \cdot \Lambda^\mu{}_\alpha \cdot \Lambda^\nu{}_\beta = g_{\alpha\beta}}$$

this is the relation Λ must satisfy in order for Λ to be a Lorentz transformation.

HW: what are the $\Lambda^\mu{}_\nu$ for the Lorentz transformation along x^1-axis

$$x'^0 = \gamma(x^0 - \beta x^1)$$

$$x'^1 = \gamma(x^1 - \beta x^0), \qquad x'^2 = x^2, \quad x'^3 = x^3$$

compare with $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $\therefore$

$$\Lambda^0{}_0 = \gamma, \qquad \Lambda^0{}_1 = -\gamma\beta$$

Write down the rest

$$\Lambda^\mu{}_\nu = ? \qquad \text{(HW)}$$

Some properties of Lorentz tran Λ

From definition $\quad \underline{x} \to \underline{x}' = \Lambda\, \underline{x}$

In cpt form $\quad x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$

Cf = compare

(Cf: 3-dimensional rotation

$\underset{\sim}{x} \to \underset{\sim}{x}' = \mathcal{R}\, \underset{\sim}{x}$

$\to x'_i = \mathcal{R}_{ij}\, x_j$

$\mathcal{R}_{ij}$ = 3 x 3 matrix)

So represent $\Lambda^{\mu}{}_{\nu}$ by a 4 x 4 matrix

Define a 4x4 matrix $(\Lambda)_{\mu\nu} \equiv \Lambda^{\mu}{}_{\nu}$

Thus in matrix form, for a Lorentz tran along x^1-axis

$$(\Lambda) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{c|cccc} & x^0, & x^1, & x^2, & x^3 \\ \hline x^0 & \cdot & \cdot & \cdot & \cdot \\ x^1 & \cdot & \cdot & \cdot & \cdot \\ x^2 & \cdot & \cdot & \cdot & \cdot \\ x^3 & \cdot & \cdot & \cdot & \cdot \end{array}$$

HW

$$(\Lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathcal{R} & \\ 0 & & & \end{pmatrix}$$

spatial rotation

$\mathcal{R}$ 3x3 matrix

$$(\Lambda_s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ space inversion

$$(\Lambda_t) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Time inversion

$$(\Lambda_{st}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ spacetime inversion

=

Any general Lorentz transformation Λ must satisfy

$$g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

which can be written in matrix form.

Define matrix

$$((g))_{\mu\nu} = g_{\mu\nu}$$

$$((\Lambda))_{\mu\nu} = \Lambda^{\mu}{}_{\nu}$$

Then we have

$$(g)_{\mu\nu} \ (\Lambda)_{\mu\alpha} \ (\Lambda)_{\gamma\beta} = (g)_{\alpha\beta} .$$

$$(\Lambda^t)_{\alpha\mu} \ (g)_{\mu\nu} \ (\Lambda)_{\nu\beta} = (g)_{\alpha\beta} ,$$

Λ^t = transpose of Λ

$$\longrightarrow \quad \Lambda^t \ g \ \Lambda = g$$

Taking determinant,

$$\det (\Lambda^t g \Lambda) = \det(g)$$

$$\longrightarrow \quad \det \Lambda = \pm 1 \qquad \text{(HW)}$$

cf: R = rotation in 3-dim space, $\det R = +1$

Next can show

$$\Lambda^0{}_0 > +1 \qquad \text{or} \qquad \Lambda^0{}_0 < -1$$

(HW)

General Lorentz transformation Λ

$$\underline{x} \rightarrow \underline{x}' = \Lambda\, \underline{x}$$

Λ must satisfy

$$g_{\mu\nu}\, \Lambda^{\mu}{}_{\cdot\alpha}\; \Lambda^{\nu}{}_{\cdot\beta} = g_{\alpha\beta}$$

In matrix form

$$\Lambda^t\, g\, \Lambda = g$$

Taking determinant both sides,

$$\det(\Lambda^t g \Lambda) = \det g \rightarrow \det \Lambda^t \cdot \det g \cdot \det \Lambda = \det g$$

$$\therefore \det \Lambda^t \cdot \det \Lambda = 1$$

$$(\det \Lambda)^2 = 1 \qquad \because \det \Lambda^t = \det \Lambda$$

$$\therefore \det \Lambda = \pm 1$$

cf in 3-dim space, $\det \mathcal{R} = \pm 1$, $\mathcal{R}$ = rotation matrix

Also $\quad \Lambda^0{}_0 > +1 \quad$ or $\quad \Lambda^0{}_0 < -1$

Proof:

$$g_{\mu\nu}\, \Lambda^{\mu}{}_{\alpha} \cdot \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

$\hookrightarrow$ can be written as, if $g_{\mu\nu} = g_{\nu\mu}$ symmetric,

$$g_{\mu\nu}\, \Lambda^{\nu}{}_{\alpha}\, \Lambda^{\mu}{}_{\beta} = g_{\alpha\beta}$$

setting $\alpha = 0 = \beta$

$$g_{\mu\nu}\, \Lambda^{\mu}{}_{0} \cdot \Lambda^{\nu}{}_{0} = g_{00} = +1$$

sum over μ and ν, $\quad \mu, \nu = 0, 1, 2, 3.$

say, sum over μ first. Put $\mu=0$, then $\mu=j$

$$g_{0\nu}\,\Lambda^{0}{}_{0}\,\Lambda^{\nu}{}_{0} + g_{j\nu}\,\Lambda^{j}{}_{0}\,\Lambda^{\nu}{}_{0} = +1$$

Now sum over ν

$$g_{00}\,\Lambda^{0}{}_{0}\,\Lambda^{0}{}_{0} + \overset{\nearrow 0}{g_{0i}}\,\Lambda^{0}{}_{0}\,\Lambda^{i}{}_{0}$$

$$+ \overset{\nearrow 0}{g_{j0}}\,\Lambda^{j}{}_{0}\,\Lambda^{0}{}_{0} + g_{ji}\,\Lambda^{j}{}_{0}\,\Lambda^{i}{}_{0} = +1$$

As $g_{00} = +1$, $g_{ij} = 0 \;\forall\; i \neq j$, and $g_{11} = g_{22} = g_{33} = -1$

$(g_{ij} = -\delta_{ij}$, δ_{ij} = Kronecker delta$)$

$$\therefore\; \Lambda^{0}{}_{0}\,\Lambda^{0}{}_{0} - \Lambda^{i}{}_{0}\,\Lambda^{i}{}_{0} = 1$$

$$\therefore\; (\Lambda^{0}{}_{0})^2 = 1 + \Lambda^{i}{}_{0}\,\Lambda^{i}{}_{0}$$

since $(\Lambda^{i}{}_{0}\,\Lambda^{i}{}_{0}) \geq 0$

$$\therefore\; (\Lambda^{0}{}_{0})^2 \geq 1$$

i.e. $\Lambda^{0}{}_{0} \geq +1$ or $\Lambda^{0}{}_{0} \leq -1$

So the set of Lorentz transformations can be divided into 4 subsets according to

$\det \Lambda = \pm 1$, $\Lambda^{0}{}_{0} \geq +1$, $\Lambda^{0}{}_{0} \leq -1$

e.g. $L^{\uparrow}_{+}$ $\longrightarrow$ $\Lambda^{0}{}_{0} \geq +1$ restricted Lorentz group
$\longleftarrow$ $\det \Lambda = +1$

$$L_+^\uparrow \begin{cases} \to \Lambda^0{}_0 > +1 \\ \to \det \Lambda = +1 \end{cases}$$

$L_+^\uparrow$ is a subset s. t. $\Lambda^0{}_0 > +1$

and $\det \Lambda = +1$

restricted Lorentz trans

this subset forms a group.

$L_-^\uparrow$ ($\uparrow \to \Lambda^0{}_0$, $- \to \det \Lambda$) subset contains space inversion

not a group

orthochronous transformation.

$L_+^\downarrow$ contains time-space inversion.

extended Lorentz transformations

not a gp.

$L_-^\downarrow$ contains time inversion

orthochorous trans.

not a gp

$L_+^\uparrow \cup L_-^\uparrow$ = orthochronous group

$L_+^\uparrow \cup L_+^\downarrow$ = extended Lorentz group

$L_+^\uparrow \cup L_-^\downarrow$ = orthochorous group

$L_+^\uparrow$ = restricted Lorentz group.

Introduce scalar, vector, tensor

A scalar is a one-component entity that remains unchanged under the Lorentz tran Λ

Let ϕ be a scalar, that means under Λ : $\underline{x} \to \underline{x}' = \Lambda \underline{x}$, we have

$$\to \quad \phi \xrightarrow{\Lambda} \phi' \equiv \Lambda \phi = \phi$$

If ϕ depends on space-time, then $\phi(\underline{x})$ is a scalar field which means

$$\phi(\underline{x}) \to \phi'(\underline{x}') = \phi(\underline{x})$$

$$\underline{x}' = \Lambda \underline{x}$$

Note: $\underline{x}^2$ is a scalar

$$\underline{x}'^2 = \underline{x}^2$$

$$\underline{x}^2 = \underline{x} \cdot \underline{x} = g_{\mu\nu} x^\mu x^\nu$$

A 4-component entity, say $\underline{A}$, is a vector if under Lorentz tran Λ,

$$\underline{A} \to \underline{A}' = \Lambda \underline{A} \qquad (\underline{x}' = \Lambda \underline{x})$$

If we choose basis, can write

$$A'^\mu = (\Lambda^\mu{}_\nu) A^\nu$$

(There are two types of base)