PC3246
Astrophysics I

Assignment 5

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Question 1. In general, an *integral average* of some continuous function $f(t)$ over an interval τ is given by

$$\langle f(t)\rangle = \frac{1}{\tau}\int_0^{\tau} f(t)dt. \tag{1}$$

Beginning with an expression for the integral average, prove that

$$\langle U\rangle = -G\frac{M\mu}{a}, \tag{2}$$

a binary system's gravitational potential energy, averaged over one period, equals the value of the instantaneous potential energy of the system when the two masses are separated by the distance a, the semimajor axis of the orbit of the reduced mass about the center of mass.
Hints:

- *You may find the following definite integral useful:*

$$\int_0^{2\pi} \frac{d\theta}{1+e\cos\theta} = \frac{2\pi}{\sqrt{1-e^2}} \tag{3}$$

- *Express U as a function of θ, use $dt = \frac{dt}{d\theta}d\theta$ and $L = \mu r^2 \frac{d\theta}{dt}$.*

Solution: Starting from the given integral average, we first perform a change of variables,

$$\langle U\rangle = \frac{1}{\tau}\int_0^{\tau} U(t)dt = \frac{1}{\tau}\int_0^{2\pi} U(\theta)\frac{dt}{d\theta}d\theta. \tag{4}$$

From the angular momenta L of a reduced mass μ and central body M subjected to a central force,

$$L = \mu r^2\frac{d\theta}{dt} \qquad \Longrightarrow \qquad \frac{dt}{d\theta} = \frac{\mu r^2}{L}, \tag{5}$$

and the reduced mass system will have a potential energy U,

$$U(r) = -\frac{GM\mu}{r}. \tag{6}$$

Hence,

$$\langle U\rangle = \frac{1}{\tau}\int_0^{2\pi}\left(-\frac{GM\mu}{r}\right)\frac{\mu r^2}{L}d\theta = -\frac{GM\mu^2}{L\tau}\int_0^{2\pi} r d\theta. \tag{7}$$

For a Keplerian orbit, the reduced mass μ draws out an ellipse about the central body M as described by Kepler's 1st Law,

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}. \tag{8}$$

Therefore,

$$\langle U \rangle = -\frac{GM\mu^2}{L\tau}\int_0^{2\pi}\frac{a(1-e^2)}{1+e\cos\theta}d\theta = -\frac{GM\mu^2 a(1-e^2)}{L\tau}\frac{2\pi}{\sqrt{1-e^2}} = -\frac{2\pi GM\mu^2 a\sqrt{1-e^2}}{L\tau}, \tag{9}$$

where by Kepler's 2nd Law, an elliptical orbit has angular momenta L given by,

$$L = \mu\sqrt{GMa(1-e^2)}. \tag{10}$$

Together,

$$\langle U \rangle = -\frac{2\pi GM\mu^2 a\sqrt{1-e^2}}{\mu\sqrt{GMa(1-e^2)}\tau} = -\frac{2\pi\mu\sqrt{GMa}}{\tau}, \tag{11}$$

and by Kepler's 3rd Law, an elliptical orbit has period τ given by,

$$\tau = 2\pi\sqrt{\frac{a^3}{GM}}. \tag{12}$$

Such that,

$$\langle U \rangle = -\frac{2\pi\mu\sqrt{GMa}}{2\pi\sqrt{\frac{a^3}{GM}}} = -\frac{GM\mu}{a}. \tag{13}$$

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Question 2. Verify that Kepler's third law in the form of Eq. 14 applies to the four moons that Galileo discovered orbiting Jupiter (the Galilean moons: Io, Europa, Ganymede, and Callisto).

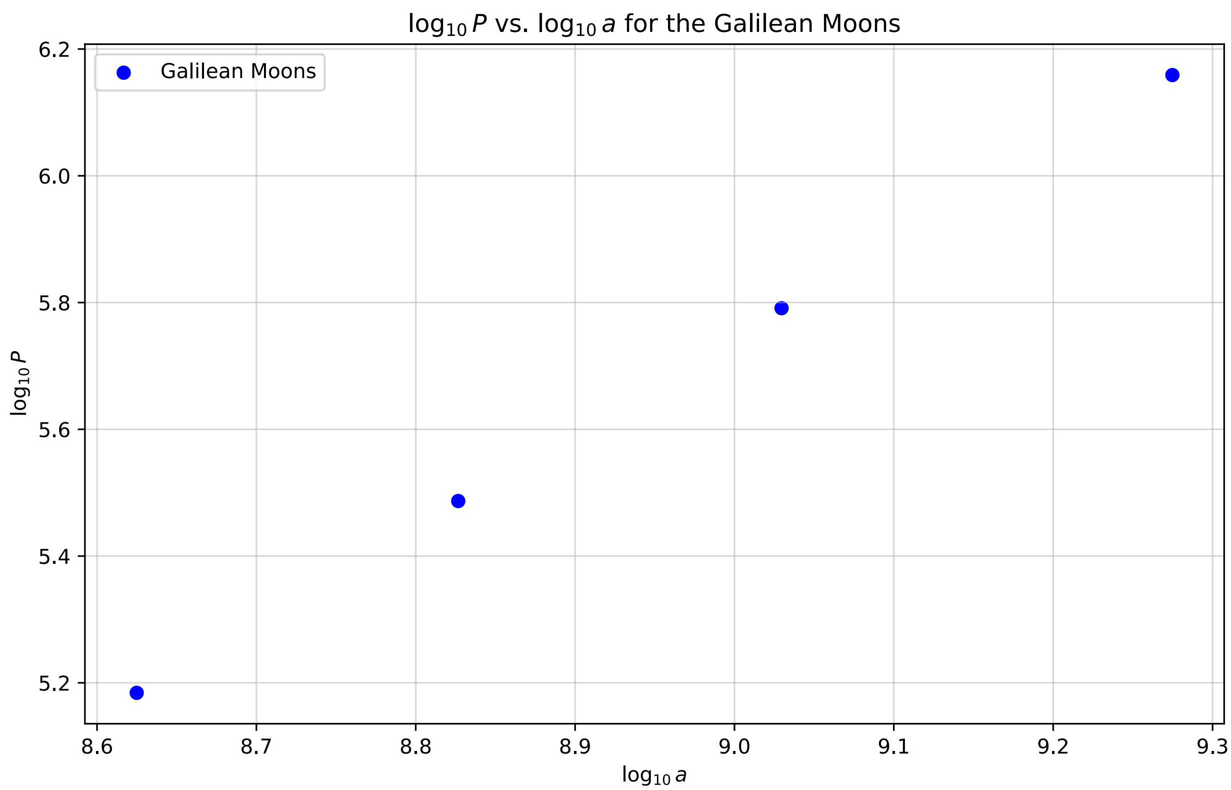
$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3. \tag{14}$$

(a) Using the data available in Appendix: Solar-System Data, create a graph of $\log_{10} P$ vs. $\log_{10} a$

(b) From the graph, show that the slope of the best-fit straight line through the data is 3/2.

(c) Calculate the mass of Jupiter from the value of the y-intercept.

Solution: (a)



(b)

```python
# Linear regression
slope, intercept, r_value, p_value, std_err = linregress(galilean["log_10a"], galilean["log_10P"])

print("Linear regression results:")
print(f"\tSlope = {slope:.3f}")
print(f"\tIntercept = {intercept:.3f}")
```

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Linear regression results:
    Slope = 1.49980
    Intercept = -7.75139
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$$m \approx 1.500 \quad (4 \text{ s.f.}) \tag{15}$$

(c) Since $m_\text{J} \gg m_\text{moon}$, we can simplify Kepler's third law as

$$P^2 \approx \frac{4\pi^2}{Gm_\text{J}} a^3. \tag{16}$$

Taking the $\log_{10}$ on both sides,

$$\log_{10} P^2 = \log_{10}\left(\frac{4\pi^2}{Gm_\text{J}} a^3\right) \qquad \implies \qquad 2\log_{10} P = \log_{10}\left(\frac{4\pi^2}{Gm_\text{J}}\right) + 3\log_{10} a. \tag{17}$$

At the y-intercept of the $\log_{10} P$ vs. $\log_{10} a$ graph, $\log_{10} P = -7.7514$ when $\log_{10} a = 3$. Hence,

$$\log_{10}\left(\frac{4\pi^2}{GM_\text{J}}\right) = 2(-7.7514) - 3(0) \tag{18}$$

$$\begin{aligned} M_\text{J} &= \frac{4\pi^2}{G\left[10^{2(-7.7514)}\right]} \approx 1.88249 \times 10^{27}\text{kg} \\ &\approx 1.882 \times 10^{27}\text{kg} \quad (4 \text{ s.f.}) \end{aligned} \tag{19}$$

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Question 3. Consider a planet with an axial tilt of 90°. Does the Sun's gravitational force on the equatorial bulge create a torque on the planet? Explain your answer.

Solution: For a planet with an axial tilt of 90°, the Sun's gravitational force on the equatorial bulge does not create a torque on the planet.

If a planet has an axial tilt of 90° such that the Sun remains directly overhead at one of its pole throughout its entire orbit, as the rotation axis of the planet's rotation is parallel to the plane of the Solar System due to its axial tilt of 90°, the planet will observe equatorial bulges along a plane perpendicular to the plane of the Solar System. Since the distance between any point on the equatorial bulge and the Sun is the same, the gravitational force experienced by the planet at any point of the equatorial bulge is symmetrically the same as well. As such, there is no difference in gravitational force experienced by any two opposite points on the equatorial bulge and hence the Sun's gravitational force on the equatorial bulge does not create a torque on the planet. ■

Question 4. Consider a planet with a retrograde rotation (i.e. clockwise rotation as seen from above Earth's north pole). Does the Sun's gravitational torque cause the planet's rotation axis to precess clockwise or counterclockwise as seen from above Earth's north pole? Explain your answer.

Solution: For a planet with retrograde rotation, the Sun's gravitational torque causes the planet's rotation axis to precess counterclockwise as seen from above Earth's north pole.

For a planet with prograde rotation, the near side of the planet's equatorial bulge from the Sun experiences a stronger gravitational force than the far side. This results in a gravitational torque exerted onto the planet about an axis that is perpendicular to the planet's axis of rotation and on the plane of the Solar System. Since this torque does not change the rotation period of the planet, the direction of the planet's axis of rotation changes (precesses), resulting in a clockwise precession. Hence, if we consider a planet with retrograde rotation, the angular momentum vector $\mathbf{L}$ describing the rotation of the planet will now point in the opposite direction and thus the planet will precess counterclockwise.

Additionally as established in Q3, if the planet has an axial tilt of 0° or 90°, there will be no precession. ■