2024. 2. 8

8 ①

pion + nucleon $\xrightarrow{\text{scattering}}$ pion + nucleon

2 particles $\longrightarrow$ 2 particles

use the underlying isospin symmetry SU(2) to relate the different crosssections of $\pi N \to \pi N$

N → nucleon

$\pi = \pi^+, \pi^0, \pi^-$, $N = (n, p)$

↓ isotriplet

$I = 1$, $I_3 = +1, 0, -1$

isodoublet

$I = \frac{1}{2}$, $I_3 = +\frac{1}{2}, -\frac{1}{2}$

List of reactions

$\pi^{\pm 0} p \to \pi^{\pm 0} p$, $\pi^{\pm 0} n \to \pi^{\pm 0} n$

charge-exchange reactions

$\pi^+ n \to \pi^0 p$, $\pi^0 p \to \pi^+ p$

$\pi^- p^+ \to \pi^0 n$, $\pi^0 n \to \pi^- p^+$

calculate 3 reactions

(i) $\pi^- p \rightarrow \pi^- p$

(ii) $\pi^+ p \rightarrow \pi^+ p$

(iii) $\pi^- p \rightarrow \pi^0 n$

scattering crosssection ← crosssection is area

scatt. crosssection = $|$ scattering amp $|^2$

$$= \left| {}_{out}\langle \; | \; \rangle_{in} \right|^2$$

(i) $\pi^- p \rightarrow \pi^- p$

in-state $\pi^- p$, out-state $\pi^- p$

$| \; \rangle_{in}$ $| \; \rangle_{out}$

Focuss on isospin, neglect all other quantum numbers

$|I_1 \, I_2 \, m_1 \, m_2\rangle$

(cf $|j_1 j_2 m_1 m_2\rangle$)

Instate $|1 \, \tfrac{1}{2} \, -1 \, \tfrac{1}{2}\rangle_{in}$

Outstate $|1 \, \tfrac{1}{2} \, -1 \, \tfrac{1}{2}\rangle_{out}$

total

Isospin is conserved in st. intn.

So better label the in-state and out-state in terms of total isospin

i.e. Don't use $|I_1\ I_2\ m_1\ m_2\rangle$

use $|I_1\ I_2\ I\ m\rangle$

We know $I = I_1 + I_2,\ I_1 + I_2 - 1,\ \ldots\ |I_1 - I_2|$

i.e. convert $|I_1\ I_2\ m_1\ m_2\rangle$ to

$|I_1\ I_2\ I\ m\rangle$ by using

C. G. coeffs.

$$\begin{array}{ll} & \quad\ I_1\ I_2\ \ m_1\ \ m_2 \\ \text{Consider}\quad \pi^- p = & |1\ \tfrac{1}{2}\ -1\ \tfrac{1}{2}\rangle \end{array}$$

$$\text{to set}\quad |\underset{I_1}{1}\ \underset{I_2}{\tfrac{1}{2}}\ I\ m\rangle$$

$$|1\ \tfrac{1}{2}\ -1\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1\ \tfrac{1}{2}\ \tfrac{3}{2}\ -\tfrac{1}{2}\rangle$$

$$-\sqrt{\tfrac{2}{3}}\,|1\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

so

$$|\pi p^-\rangle_{in} = |1\ \tfrac{1}{2}\ -1\ \tfrac{1}{2}\rangle_{in}$$

$$= \sqrt{\tfrac{1}{3}}\ |1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in} - \sqrt{\tfrac{2}{3}}\ |1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}$$

$$|\pi p^-\rangle_{out} = \sqrt{\tfrac{1}{3}}\ |1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{out} - \sqrt{\tfrac{2}{3}}\ |1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{out}$$

scatt. amp for $\pi^- p \to \pi^- p$ is given by

$$\mathcal{M}_{\pi^- p \to \pi^- p}$$

$$= \tfrac{1}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2} | 1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in}$$

$$+ \tfrac{2}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2} | 1\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}$$

$$\left(\because\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2} | 1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in} = 0 \quad \text{et et}\right)$$

Do the same calculation for othe reactions

(ii) $\pi^+ p \to \pi^+ p$, (iii) $\pi^- p \to \pi^0 n$

see lecture notes

We show that all these 10 cross sections are related, due to underlying SU(2) isospin symmetry.

For example, consider

$$\pi^+ p \to \pi^+ p \qquad (i)$$
$$\pi^- p \to \pi^- p \qquad (ii)$$
$$\pi^- p \to \pi^0 n \qquad (iii)$$

To compute scattering cross section, we need scatt. amp.,

scatt amplitude $= {}_{out}\langle \quad | \quad \rangle_{in}$ $\qquad | \ \rangle_{in}$ = in-state

For our example, specify the <u>in-state</u> and <u>out-state</u> in terms of isospins, or better still, <u>total isospins</u>.

Consider process (i), the individual isospin of the particle involved is known

$$p = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \qquad \pi^+ = |1, +1\rangle$$

$\uparrow \ \uparrow$
$I^2 \quad I_3$

{ like ang. mom $|j\,m\rangle$, we have $|I\,m\rangle$ }

So the in-state in process (i) is given by

$$\pi^+ p \to |1 \ {+1}\rangle |\tfrac{1}{2} \ \tfrac{1}{2}\rangle$$
$$= |\underset{I_1}{1} \ \underset{I_2}{\tfrac{1}{2}}, \ \underset{m_{I_1}}{1} \ \underset{m_{I_2}}{\tfrac{1}{2}}\rangle$$

Express in terms of total isospin quantum numbers.

Recall $\underset{(j_1 m_1)}{\underset{\sim}{J_1}}, \ \underset{(j_2 m_2)}{\underset{\sim}{J_2}} \longrightarrow \underset{(j\ m)}{\underset{\sim}{J}} = \underset{\sim}{J_1} + J_2$

Addition of two angular momenta, $\underset{\sim}{J_1}$, $\underset{\sim}{J_2}$

$$j = (j_1+j_2),\ j_1+j_2-1,\ \cdots\ |j_1-j_2|$$

$$m = m_1 + m_2$$

Clebsch-Gordan expansion,

$$|j_1 j_2\ m_1\ m_2\rangle = \sum_{j,m} |j_1 j_2\, j\, m\rangle\langle j\, m|j_1 j_2\, m_1\, m_2\rangle$$

$$|j_1 j_2\ j\ m\rangle = \sum_{m_1 m_2} |j_1 j_2\, m_1, m_2\rangle\langle m_1, m_2|j_1 j_2\, j\, m\rangle$$

So express $\pi^+ p = |\underset{I_1}{1}\ \underset{I_2}{\tfrac{1}{2}}\ \underset{m_{I_1}}{1}\ \underset{m_{I_2}}{\tfrac{1}{2}}\rangle$

in terms of $|I_1\ I_2\ I\ m\rangle$

since $I_1 = 1$, $I_2 = \frac{1}{2}$ $\therefore$ $I = \frac{3}{2}, \frac{1}{2}$,

Using Clebsch-Gordan table

$$\pi^+ p = |\underset{I_1}{1}\ \underset{I_2}{\tfrac{1}{2}}\ \underset{m_{I_1}}{1}\ \underset{m_{I_2}}{\tfrac{1}{2}}\rangle$$

$$= |\underset{I_1}{1}\ \underset{I_2}{\tfrac{1}{2}}\ \underset{I}{\tfrac{3}{2}}\ \underset{m}{\tfrac{3}{2}}\rangle_{in}$$

Similarly, for process (i), the out-state is

$$\pi^+ p = |1\ \tfrac{1}{2}\ 1\ \tfrac{1}{2}\rangle = |1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}\rangle_{out}$$

So scatt. amp for (i) is

$$\mathcal{M}_{(i)} = {}_{out}\langle 1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}|1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}\rangle_{in}$$

For (ii) $\pi^- p \to \pi^- p$

$$\pi^- p = |1\ {-1}\rangle\,|\tfrac{1}{2}\ {+\tfrac{1}{2}}\rangle$$

with $I_1 = 1$, $m_{I_1} = -1$, $I_2 = \tfrac{1}{2}$, $m_{I_2} = +\tfrac{1}{2}$:

$$\equiv |\underset{I_1}{1}\ \underset{I_2}{\tfrac{1}{2}}\ \underset{m_{I_1}}{-1}\ \underset{m_{I_2}}{+\tfrac{1}{2}}\rangle$$

e.g.

$$\overset{\text{Table}}{=} \sqrt{\tfrac{1}{3}}\ |1\tfrac{1}{2}\ \tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{2}{3}}\ |1\tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$

scatt. amp $\mathcal{M}_{(ii)} = \frac{1}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\,|\,1\tfrac{1}{2}\ \tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle_{in}$

$$+\ \frac{2}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\,|\,1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}$$ HW

For (iii) $\pi^- p \to \pi^0 n$

$$\pi^0 n = |1\ 0\rangle\,|\tfrac{1}{2}\ \tfrac{-1}{2}\rangle = |1\tfrac{1}{2}\ 0\ \tfrac{-1}{2}\rangle$$

$$= \sqrt{\tfrac{2}{3}}\ |1\tfrac{1}{2}\ \tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{1}{3}}\ |1\tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$ (To check, HW)

scatt. amp $= {}_{out}\langle \pi^0 n\,|\,\pi^- p\rangle_{in} = \mathcal{M}_{(iii)}$

$$= \frac{\sqrt{2}}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\,|\,1\tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in} - \frac{\sqrt{2}}{3}\ {}_{out}\langle 1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\,|\,1\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}$$

The 3 scatt. amps $\mathcal{M}_{(i)}$, $\mathcal{M}_{(ii)}$ and $\mathcal{M}_{(iii)}$ are related as will be shown below.

$$\mathcal{M}_{(i)} : \mathcal{M}_{(ii)} : \mathcal{M}_{(iii)} = {}_{out}\langle \overset{I}{\tfrac{3}{2}}\ \overset{m}{\tfrac{3}{2}} | \overset{I}{\tfrac{3}{2}}\ \overset{m}{\tfrac{3}{2}} \rangle_{in} :$$

$$\left(\tfrac{1}{3}\ {}_{out}\langle \tfrac{3}{2}\ \tfrac{-1}{2} | \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in} + \tfrac{2}{3}\ {}_{out}\langle \tfrac{1}{2}\ \tfrac{-1}{2} | \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}\right) :$$

$$\left(\tfrac{\sqrt{2}}{3}\ {}_{out}\langle \tfrac{3}{2}\ \tfrac{-1}{2} | \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in} - \tfrac{\sqrt{2}}{3}\ \langle \tfrac{1}{2}\ \tfrac{-1}{2} | \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{in}\right)$$

Put

$${}_{out}\langle \tfrac{3}{2}\ \tfrac{3}{2} | \tfrac{3}{2}\ \tfrac{3}{2}\rangle_{in} = \mathcal{M}_{\frac{3}{2}}$$

different $m$

$$\overset{?}{=} {}_{out}\langle \tfrac{3}{2}\ \tfrac{-1}{2} | \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{in}$$

Why? = Wigner-Eckart theorem

i.e. $\mathcal{M}_{3/2}$ depends on $I = 3/2$ but not on $m$

$$\langle \tfrac{1}{2}\ -\tfrac{1}{2} | \tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \mathcal{M}_{\frac{1}{2}}$$

$$\mathcal{M}_{(i)} : \mathcal{M}_{(ii)} : \mathcal{M}_{(iii)} = \mathcal{M}_{\frac{3}{2}} : \left(\tfrac{1}{3}\mathcal{M}_{\frac{3}{2}} + \tfrac{2}{3}\mathcal{M}_{\frac{1}{2}}\right) :$$

$$\left(\tfrac{\sqrt{2}}{3}\mathcal{M}_{\frac{3}{2}} - \tfrac{\sqrt{2}}{3}\mathcal{M}_{\frac{1}{2}}\right)$$

scattering amplitude depends on energy of incident particles,

At CM energy $1232\ \mathrm{MeV}^2$, $\mathcal{M}_{\frac{3}{2}} \gg \mathcal{M}_{\frac{1}{2}}$,

Then

$$\mathcal{M}_{(i)} : \mathcal{M}_{(ii)} : \mathcal{M}_{(iii)} = \left(1 : \tfrac{1}{3} : \tfrac{\sqrt{2}}{3}\right)$$

$$= (3 : 1 : \sqrt{2})$$

Cross section $= |\mathcal{M}|^2$

$$\sigma_{(i)} : \sigma_{(ii)} : \sigma_{(iii)} = |\mathcal{M}_{3/2}|^2 : \left|\frac{1}{3}\mathcal{M}_{3/2} + \frac{2}{3}\mathcal{M}_{\frac{1}{2}}\right|^2$$

$$: \left|\frac{\sqrt{2}}{3}\mathcal{M}_{3/2} - \frac{\sqrt{2}}{3}\mathcal{M}_{\frac{1}{2}}\right|^2$$

$$\simeq |\mathcal{M}_{\frac{3}{2}}|^2 : \left|\frac{1}{3}\mathcal{M}_{3/2}\right|^2 : \left|\frac{\sqrt{2}}{3}\mathcal{M}_{\frac{3}{2}}\right|^2$$

$$= 1 : \frac{1}{9} : \frac{2}{9} = 9 : 1 : 2$$

If we are interested in the cross section ratio of $\sigma_{\pi^+ p}$ and $\sigma_{\pi^- p}$ for the 3 processes

$$\frac{\sigma_{\pi^+ p}}{\sigma_{\pi^- p}} \simeq \frac{9}{1+2} = 3$$

where $\sigma_{\pi^- p} = \sigma_{(ii)} + \sigma_{(iii)}$

The calculated ratio agrees with the experimental result. see Fig in page 21

We now extend isospin SU(2) to higher flavour symmetries, SU(3), SU(4), ... SU(6)

2



$$\sigma_a : \sigma_c : \sigma_j = 9|\mathcal{M}_3|^2 : |\mathcal{M}_3 + 2\mathcal{M}_1|^2 : 2|\mathcal{M}_3 - \mathcal{M}_1|^2 \tag{4.49}$$

At a CM energy of 1232 MeV there occurs a famous and dramatic bump in pion-nucleon scattering, first discovered by Fermi in 1951;[7] here the pion and nucleon join to form a short-lived "resonance" state—the $\Delta$. We know the $\Delta$ carries $I = \frac{3}{2}$, so we expect that at this energy $\mathcal{M}_3 \gg \mathcal{M}_1$, and hence

$0.6 \times 10^{-23}$ s

$$\sigma_a : \sigma_c : \sigma_j = 9 : 1 : 2 \tag{4.50}$$

Experimentally, it is easier to measure the total cross sections, so (c) and (j) are combined:

$$\frac{\sigma_{\text{tot}}(\pi^+ + p)}{\sigma_{\text{tot}}(\pi^- + p)} = 3 \tag{4.51}$$

As you can see in Figure 4.6, this prediction is well satisfied by the data.

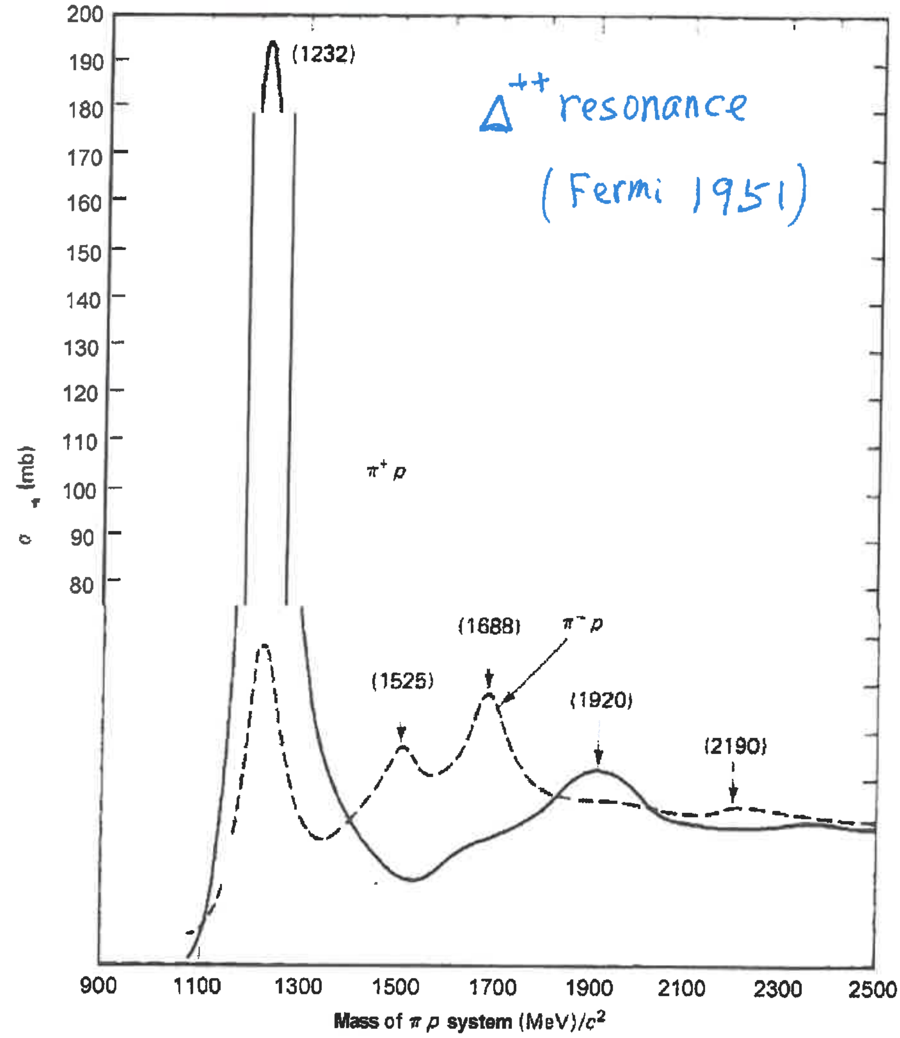



**Figure 4.6** Total cross sections for $\pi^+p$ (solid line) and $\pi^-p$ (dashed line) scattering. (Source: S. Gasiorowicz, Elementary Particle Physics (New York: Wiley, copyright © 1966, page 294. Reprinted by permission of John Wiley and Sons, Inc.)

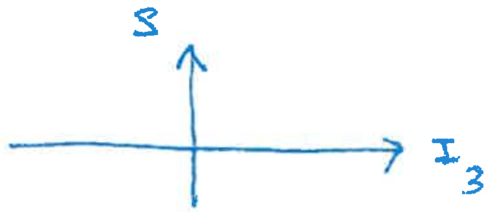
# Quark flavour

$J^2$ = Casimir operator

Originally (Heisenberg, 1932) isospin was introduced to classify elementary particles into doublet $(p, n)$, or triplet $(\pi^+, \pi^0, \pi^-)$ etc. The isospin group is $SU(2)$

$|jm\rangle$, $|I\, m_I\rangle$

In early 1960, many more elementary particles were found, $SU(2)$ isospin as a classification scheme is not adequate. A new quantum number, strangeness $S$, was introduced.




Many particles can then be accommodated into representations of a bigger symmetry group $SU(3)$

Mesons form singlet or octet representations of $SU(3)$

Baryons form singlet, octet (eight fold way) decuplet representations of $SU(3)$

Questions were then raised why only these three types of representation of $SU(3)$ are realized by elementary particles at that time?

The quark model (3 quarks) explains this. Mesons are made of quark and antiquark. Baryon are made of 3 quarks