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Transition prob. per unit time

$$= \frac{2\pi}{\hbar} \cdot |\mathcal{M}|^2 \cdot \text{phase space factor}$$

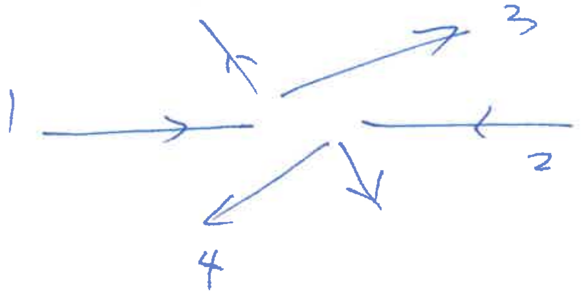
Decay rate Γ $\quad 1 \rightarrow 2 + 3 \cdots + N$

Differential decay rate

$$d\Gamma = \frac{S}{2\hbar \cdot m_1} |\mathcal{M}|^2 \cdot (2\pi)^4 \delta^{(4)}(p_1 - p_2 \cdots - p_N)$$

$$\prod_{j=2}^{N} \frac{d^4 p_j}{(2\pi)^4} (2\pi)\, \delta(p_j^2 - m^2 c^2)\, \theta(p_j^0)$$

Differential cross section $\quad 1 + 2 \rightarrow 3 + 4 + \cdots N$



$$d\sigma = \frac{S\, \hbar^2}{4 \cdot \sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 \cdots p_N)$$

$$\prod_{j=3}^{N} \frac{d^4 p_j}{(2\pi)^4} \theta(p_j^0)\, (2\pi)\, \delta(p_j^2 - m_j^2 c^2)$$

$\theta(p_j^0)$
$=$ step function

Chapter 6 Griffiths

Decays and cross section (scattering)

Experimentally, the spectroscopic investigation (spectral lines) provides information about bound states of the particles (e. g. hydrogen atom H as bound state of e^- and p). Another approach is scattering of the particles, observe decay of these particles.

scattering can reveal the nature of particle interactio

→ Formulate decay process and scattering mathematically.

A typical decay process : $1 \rightarrow 2 + 3 + 4 \cdots + N$

Define decay rate = probability a particle decay per unit time $= \Gamma$

The probability a particle will decay in time $\delta t = \Gamma \cdot \delta t$

If there are N particles at time t, then the number of particles will decay in time δt $= N \cdot \Gamma \cdot \delta t$

→ $\delta N = - N \Gamma \delta t$

$$\frac{dN}{dt} = - \Gamma N$$

solving

→ $$N = N_0 e^{-\Gamma t}$$

N_0 = # of particles at time $t=0$.

Mean life time of a particle = τ

$$= \frac{\text{Sum of the lifetimes of all the decayed particles}}{\text{sum of all the decayed particles}}$$

$$= \frac{1}{\Gamma}$$ (as shown below)

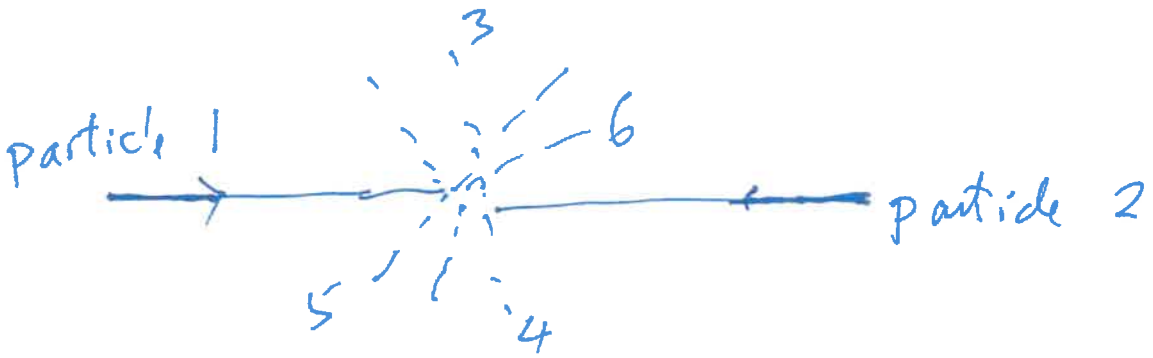
Suppose at time t, we have N particles, and at time $t+\delta t$, δN particles decay away, that means life time of all δN particles

$= t \cdot \delta N$ (each of the δN particles has a lifetime t).

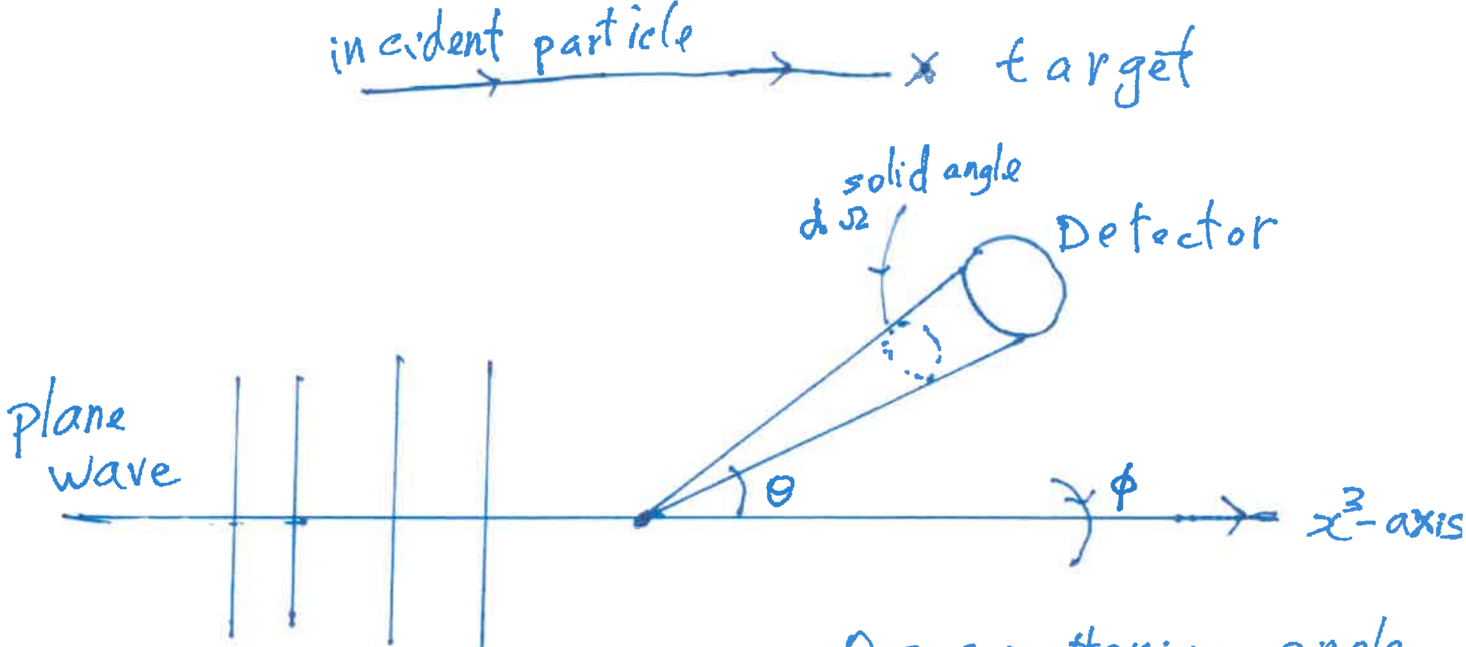
$$\tau = \frac{\int_{N_0}^{0} t\, dN}{\int_{N_0}^{0} dN} = \frac{1}{\Gamma}$$ (HW)

(using $N = N_0 e^{-\Gamma t}$, HW)

Scattering $1+2 \to 3+4+5+\cdots$



Lab frame



θ = scattering angle

Scattering amplitude

Differential cross section $= \dfrac{\text{scattered flux}}{\text{Incident flux}}$

$= \dfrac{d\sigma}{d\Omega}$

Total cross section σ

Incident flux = $\overset{\text{number}}{\#}$ of particles incident per unit area per unit time

= I_0 (probability current density $\vec{j}$)

Scattered flux = # of particles scattered into the solid angle direction (θ, ϕ) per solid angle per unit time [the detector is at the angular position (θ, ϕ)]

= $I(\theta, \phi)$

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{I(\theta, \phi)}{I_0} \ ,$$

Total cross section $\sigma = \int_{\text{all angles}} \frac{d\sigma}{d\Omega} \, d\Omega$

Today discuss how to compute Γ and $\frac{d\sigma}{d\Omega}$

Γ = decay rate

To do that we need a formula from quantum mechanics, the Fermi golden rule. We quote:

→ Transition probability per unit time = $\frac{2\pi}{\hbar} \cdot |M|^2 \cdot$ phase space factor

Compute $|M|^2$ from dynamics, M = scattering amplitude

phase space factor from kinematics.

$\mathcal{M}$ = scattering amplitude (matrix element)

can be obtained by solving equation of motion

or using Feynman diagrams with Feynman rules.

phase spce factor denotes the states available for

the finally produced particles to occupy

The larger the phase spce factor, the more likely

the process will be.

Using this transition probability formula, one

can derive, the differential decay rate $d\Gamma$

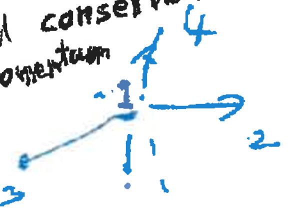
For decay of a single particle $\underline{P}_1 = (m_1 c, 0)$

(stationary)

$$d\Gamma = \frac{S}{2\hbar m_1} |\mathcal{M}|^2 \cdot$$

$$(2\pi)^4 \delta^{(4)}(\underline{P}_1 - \underline{P}_2 - \underline{P}_3 \cdots \underline{P}_N) \cdot$$

= product of 4 Dirac delta functions for overall conservation of 4 - momentum



$$\prod_{j=2}^{N} \left(\frac{d^4 \underline{P}_j}{(2\pi)^4} \theta(P_j^0)(2\pi)\, \delta(\underline{P}_j^2 - m_j^2 c^2) \right)$$

mass shell condition for each particle

S = statistical factor

$= \frac{1}{j!}$ if there are j identical particles produced

· Decay of a single particle (at rest, $\underline{p}_1 = (m_1 c, 0)$)

$$d\Gamma = \frac{S}{2\hbar m_1} |M|^2 \cdot (2\pi)^4 \delta^{(4)}(\underline{p}_1 - \underline{p}_2 - \underline{p}_3 \cdots - \underline{p}_n) \cdot$$

$$dp_j^0\, dp_j^1\, dp_j^2\, dp_j^3 = dp_j^0\, d^3\underline{p}_j$$

$$\cdot \prod_{j=2}^{n} \frac{d^4 \underline{p}_j}{(2\pi)^4} \theta(p_j^0)(2\pi)\, \delta(\underline{p}_j^2 - m_j^2 c^2)$$

S = statistical factor

$= \frac{1}{j!}$ if there are j identical particles produced

e.g. if there are $3\,\pi^0$, $4\,\pi^-$, $5\,\pi^+$ in the final produced particles, then $S = \frac{1}{3!\,4!\,5!}$

$\theta(x)$ = step function, $\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

$\underline{p}_j^2 = m_j^2 c^2$ means particle j is in its mass shell. $(p_j^{0\,2} - \underline{p}_j^2 = m_j^2 c^2)$

$$\int dp_j^0\, \theta(p_j^0)\, \delta(\underline{p}_j^2 - m_j^2 c^2) \,,$$

$$= \frac{1}{2 p_j^0}$$

using $\delta(x^2 - a^2) = \frac{1}{2|a|}(\delta(x-a) + \delta(x+a))$
(shown later)

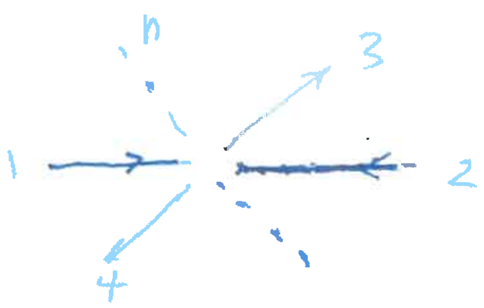
After integrating away $\int dp_j^0$, the differential decay rate is

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \, \delta^{(4)}(\underline{p}_1 - \underline{p}_2 - \underline{p}_3 \cdots \underline{p}_n) \cdot$$

$$\cdot \prod_{j=2}^{n} \left(\frac{1}{2 p_j^0} \frac{d^3 \underset{\sim}{p}_j}{(2\pi)^3} \right),$$

$$p_j^0 = \sqrt{\underset{\sim}{p}_j^2 + m_j^2 c^2}$$

scattering



$$1 + 2 \longrightarrow 3 + 4 + \cdots + n$$

The formula is

$$d\sigma = \frac{s\,\hbar^2}{4 \cdot \sqrt{(\underline{P}_1 \cdot \underline{P}_2)^2 - (m_1 m_2 c^2)^2}} \cdot |M|^2 \cdot$$

$$(2\pi)^4 \delta^{(4)}(\underline{P}_1 + \underline{P}_2 - \underline{P}_3 - \underline{P}_4 \cdots - \underline{P}_n) \cdot$$

$$\prod_{j=3}^{n} \frac{d^4 \underline{P}_j}{(2\pi)^4} \theta(p_j^0) \cdot 2\pi \; \delta(\underline{P}_j^2 - m_j^2 c^2)$$

Integrating $\xrightarrow{}$ $\int dp_j^0$

$$d\sigma = \frac{s\,\hbar^2}{4\sqrt{(\underline{P}_1 \cdot \underline{P}_2)^2 - (m_1 m_2 c^2)^2}} \cdot |M|^2 \cdot$$

$$(2\pi)^4 \delta^{(4)}(\underline{P}_1 + \underline{P}_2 - \underline{P}_3 - \underline{P}_4 \cdots - \underline{P}_n) \cdot$$

$$\prod_{j=3}^{n} \frac{d^3 p_j}{(2\pi)^3 \; 2 p_j^0}$$

Basically, learn how to reduce 4-dimensional integral to 3-dimensional, then to 1-dimensional integral

To show

$$\delta(x^2 - a^2) = \frac{1}{2|a|}(\delta(x-a) + \delta(x+a))$$

Proof

By definition, for a smooth function $f(x)$,

$$\int_{-b}^{b} f(x)\,\delta(x-a)\,dx = \begin{cases} f(a) & \text{if } a \in [-b, b] \\ 0 & \text{if } a \notin [-b, b] \end{cases}$$

LHS

$$= \int_{-\infty}^{\infty} f(x)\,\delta(x^2 - a^2)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,\delta(x^2 - a^2)\,dx + \int_{0}^{\infty} f(x)\,\delta(x^2 - a^2)\,dx$$

$$= \int_{0}^{\infty} f(-x)\,\delta(x^2 - a^2)\,dx + \int_{0}^{\infty} f(x)\,\delta(x^2 - a^2)\,dx$$

$$= \int_{0}^{\infty} f(-\sqrt{y})\,\delta(y - a^2)\frac{dy}{2\sqrt{y}} + \int_{0}^{\infty} f(\sqrt{y})\,\delta(y - a^2)\frac{dy}{2\sqrt{y}}$$

$$= f(-a)\frac{1}{2a} + f(a)\frac{1}{2a} \qquad \text{assuming } a \geqslant 0$$

$$= \frac{1}{2|a|}(f(-a) + f(a))$$

$$\text{RHS} = \int_{-\infty}^{\infty} f(x) \cdot \frac{1}{2|a|}(\delta(x-a) + \delta(x+a))\,dx$$

$$RHS = \int_{-\infty}^{0} f(x) \frac{1}{2|a|}\left(\delta(x-a) + \delta(x+a)\right) dx$$

$$+ \int_{0}^{\infty} f(x) \frac{1}{2|a|}\left(\delta(x-a) + \delta(x+a)\right) dx$$

$$= \frac{1}{2|a|} \int_{-\infty}^{0} f(x)\, \delta(x+a)\, dx + \frac{1}{2|a|} \int_{0}^{\infty} f(x)\, \delta(x-a)\, dx$$

assume $a \geqslant 0$

$$= \frac{1}{2a} f(-a) + \frac{1}{2a} f(a)$$

$$= \frac{1}{2|a|} \left(f(-a) + f(a)\right)$$

q.e.d.

$$\int_{-\infty}^{\infty} dp^0\, \Theta(p^0)\, \delta(\underline{p}^2 - m^2c^2)\, f(p^0)$$

$$\underline{p}^2 = p^{0\,2} - \underset{\sim}{p}^2$$

$$= \int_{-\infty}^{\infty} dp^0\, \theta(p^0) \frac{1}{2|a|} \left[\delta(p^0 - a) + \delta(p^0 + a)\right] f(p^0)$$

$$p^0 = a = \left(\underset{\sim}{p}^2 + m^2c^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{2|a|} \int_{-\infty}^{\infty} dp^0\, \delta(p^0 - a)\, f(p^0)$$

$$= \frac{1}{2|a|} f(a), \qquad p^0 = a = \sqrt{\underset{\sim}{p}^2 + m^2c^2}$$

$a \geqslant 0$

$$\therefore \int_{-\infty}^{\infty} dp^0\, \theta(p^0)\, \delta(\underline{p}^2 - m^2c^2) = \frac{1}{2p^0}, \qquad p^0 = a$$

Consider 2-particle decays

$$1 \rightarrow 2 + 3$$

Assume particle 1 at rest and decays

The decay rate is given by (page ⑥ⓐ)

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \, \delta^{(4)}(\underline{p}_1 - \underline{p}_2 - \underline{p}_3) \cdot$$

$$\cdot \prod_{j=2}^{3} \left(\frac{1}{2 p_j^0} \frac{d^3 \underset{\sim}{p}_j}{(2\pi)^3} \right)$$

scattering amplitude $\mathcal{M}$
$= \mathcal{M}(\underline{p}_1, \underline{p}_2, \underline{p}_3 \ldots)$

$$= \frac{S}{8\pi^2 \hbar m_1} \int |\mathcal{M}|^2 \, \delta^{(4)}(\underline{p}_1 - \underline{p}_2 - \underline{p}_3) \frac{d^3 \underset{\sim}{p}_2}{2 p_2^0} \cdot \frac{d^3 \underset{\sim}{p}_3}{2 p_3^0}$$

$$= \frac{S}{8\pi^2 \hbar m_1} \int |\mathcal{M}|^2 \, \delta(p_1^0 - p_2^0 - p_3^0) \, \delta^{(3)}(\underset{\sim}{p}_1 - \underset{\sim}{p}_2 - \underset{\sim}{p}_3) \cdot$$

$$\cdot \frac{d^3 \underset{\sim}{p}_2}{2 p_2^0} \; \frac{d^3 \underset{\sim}{p}_3}{2 p_3^0}$$