We have already obtained a correct relativistic equation for a spin $\frac{1}{2}$ particle, the Dirac equation

$$\not{p}\,\psi(x) = mc\,\psi(x) \qquad \not{p} = p_\mu \gamma^\mu$$

We now want to construct a free particle solution of the Dirac equation.

Recall:

In non-relativistic quantum mechanics, the equation of motion is the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(\underset{\sim}{x}) = H\psi(\underset{\sim}{x}), \qquad H = \frac{\underset{\sim}{p}^2}{2m} + V(\underset{\sim}{x})$$

For a free particle $H = \frac{\underset{\sim}{p}^2}{2m}$, no potential force field, $V(\underset{\sim}{x}) = 0$, $\rightarrow$ $i\hbar\frac{\partial}{\partial t}\psi(\underset{\sim}{x}) = -\frac{\hbar^2}{2m}\nabla^2\psi(\underset{\sim}{x})$

The free particle is a plane wave

$$\psi(\underset{\sim}{x}, t) = \text{const.}\; e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)} \quad \text{or } e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega t)}$$

$$\underset{\sim}{p} = \hbar\underset{\sim}{k}, \qquad E = \hbar\omega, \qquad E = \frac{\underset{\sim}{p}^2}{2m}$$

Note: $e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$, $e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}+\omega t)}$ not allowed photon is described by the Maxwell equation

$$\partial_\mu \partial^\mu \underline{A}(\underline{x}) = 0 \qquad \text{or} \qquad \Box^2 \underline{A}(\underline{x}) = 0$$

$\Box^2$ = D'lembertian

$$\partial_\mu A^\mu(\underline{x}) = 0 \qquad \text{Lorentz condition}$$

Free photon is a plane wave

$$A_\mu(x) = \text{const}\ e^{-i\underline{p}\cdot x/\hbar}\ \varepsilon_\mu(\underline{p}),\quad \underline{p}^2 = 0$$

or $\underline{A}(x) = \text{const}\ e^{-i\underline{p}\cdot\underline{x}/\hbar}\ \underline{\varepsilon}(\underline{p})$

and $\partial_\mu A^\mu(\underline{x}) = 0 \longrightarrow \underline{p}\cdot\underline{\varepsilon}(\underline{p}) = 0$

$\underline{\varepsilon}(\underline{p})$ = polarization

The relativistic spin-0 particle is described by the Klein-Gordon equation

$$\underline{p}^2\ \phi(\underline{x}) = m^2c^2\ \phi(\underline{x}),\quad \underline{p}^2 = -\hbar^2\Box^2$$

The free particle is a plane-wave

$$\phi(x) = \text{const}\ e^{-i(\underline{p}\cdot x)/\hbar},\quad \underline{p}^2 = m^2c^2$$

spin 0 particle or scalar particle or pseudo-scalar particle, e.g. π^0, π^+, π^- mesons

Construct the free particle solution of the Dirac equation.

$$\not{p}\,\psi(\underline{x}) = mc\,\psi(\underline{x})$$

The plane wave solution can be written as

$$\psi(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar}\,u(\underline{p})$$

or

$$\psi_\alpha(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar}\,u_\alpha(\underline{p}),$$

$$\alpha = 1, 2, 3, 4$$

$$u(\underline{p}) = \begin{pmatrix} u_1(\underline{p}) \\ u_2(\underline{p}) \\ u_3(\underline{p}) \\ u_4(\underline{p}) \end{pmatrix}$$

The unknowns are $\underline{p}$ and $u(\underline{p})$

4-momentum of the particle

bispinor

Substituting

$$\psi(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar}\, u(\underline{p})$$

into the Dirac equation

$$\gamma^\mu P_\mu \psi(\underline{x}) = mc\, \psi(\underline{x}),$$

we get

$$\gamma^\mu P_\mu\, u(\underline{p}) = mc\, u(\underline{p}),$$

P_μ are four numbers, not a differential operator.

Using the Dirac representation for the matrix γ^μ,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

$$\gamma^\mu i\hbar\partial_\mu \psi(\underline{x}) = mc\,\psi(\underline{x}) \rightarrow \gamma^\mu P_\mu\, u(\underline{p}) = mc\, u(\underline{p})$$

we have

$$(\gamma^0 P_0 + \gamma^i P_i)\, u(\underline{p}) = mc\, u(\underline{p})$$

$$\rightarrow (\gamma^0 p^0 - \gamma^i p^i)\, u(\underline{p}) = mc\, u(\underline{p})$$

By convention $\gamma^i p^i = \underset{\sim}{\gamma}\cdot\underset{\sim}{p}$

$$\begin{pmatrix} p^0 & -\underset{\sim}{\sigma}\cdot\underset{\sim}{p} \\ \underset{\sim}{\sigma}\cdot\underset{\sim}{p} & -p^0 \end{pmatrix} u(\underset{\sim}{p}) = mc\, u(\underset{\sim}{p})$$

$$\because \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\underset{\sim}{\gamma}\cdot\underset{\sim}{p} = \begin{pmatrix} 0 & \underset{\sim}{\sigma}\cdot\underset{\sim}{p} \\ -\underset{\sim}{\sigma}\cdot\underset{\sim}{p} & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} p^0 - mc & -\underset{\sim}{\sigma}\cdot\underset{\sim}{p} \\ \underset{\sim}{\sigma}\cdot\underset{\sim}{p} & -p^0 - mc \end{pmatrix} u(\underset{\sim}{p}) = 0$$

Nontrivial solution for $u(\underset{\sim}{p})$ iff,

$$\begin{vmatrix} p^0 - mc & -\underset{\sim}{\sigma}\cdot\underset{\sim}{p} \\ \underset{\sim}{\sigma}\cdot\underset{\sim}{p} & -p^0 - mc \end{vmatrix} = 0$$

or

$$(p^0 - m^2c^2) - (\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^2 = 0$$

But $(\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^2 = \underset{\sim}{p}^2$

so

$$p^0 = \pm\sqrt{\underset{\sim}{p}^2 + m^2c^2}$$

$$p^0 = \frac{E}{c}$$

Having obtained $\underline{p} = (p^0, \underset{\sim}{p})$, we now find the bispinor $u(\underline{p})$

Two cases (i) $p^0 > 0$, (ii) $p^0 < 0$

(i) $p^0 = + \sqrt{\underset{\sim}{p}^2 + m^2 c^2}$

We want to solve

$$\begin{pmatrix} p^0 & -\underset{\sim}{\sigma} \cdot \underset{\sim}{p} \\ \underset{\sim}{\sigma} \cdot \underset{\sim}{p} & -p^0 \end{pmatrix} u(\underline{p}) = mc\, u(\underline{p})$$

convenient to write $u(\underline{p})$ as

$$u(\underline{p}) = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}, \qquad w^1 = \begin{pmatrix} u_1(\underline{p}) \\ u_2(\underline{p}) \end{pmatrix}, \qquad w^2 = \begin{pmatrix} u_3(\underline{p}) \\ u_4(\underline{p}) \end{pmatrix}$$

hence

$$p^0 w^1 - \underset{\sim}{\sigma} \cdot \underset{\sim}{p}\, w^2 = mc\, w^1$$
$$\underset{\sim}{\sigma} \cdot \underset{\sim}{p}\, w^1 - p^0 w^2 = mc\, w^2$$

One can solve for w^1 in terms of w^2 or vice versa.

For case (i) $p^0 > 0$, more convenient to express w^2 in terms of w^1, so use

$$\underset{\sim}{\sigma} \cdot \underset{\sim}{p}\, w^1 - p^0 w^2 = mc\, w^2$$

$$w^2 = \frac{\underset{\sim}{\sigma} \cdot \underset{\sim}{p}}{p^0 + mc} w^1$$

Thus

$$u(\underset{\sim}{p}) = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} w^1 \\ \frac{\underset{\sim}{\sigma} \cdot \underset{\sim}{p}}{p^0 + mc} w^1 \end{pmatrix}$$

$$w^1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1, u_2 \text{ arbitrary}$$

Two linearly independent solutions for w^1, e.g.

$$w^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad w^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is convenient to write the two linearly independent positive energy solutions as

$$u_+^{(s)}(\underline{p}) = \begin{pmatrix} w^s \\ \dfrac{\underline{\sigma}\cdot\underline{p}}{p^0+mc} w^s \end{pmatrix}, \qquad s = 1, 2$$

+ve energy
$p^0 > 0$

For convenience or normalization, can require

$$w^{s\dagger} w^s = 1$$

Thus we have obtained the positive energy free particle solution of the Dirac equation,

$$\psi(\underline{x}) = e^{-i\,\underline{p}\cdot\underline{x}/\hbar}\, u_+^{(s)}(\underline{p}), \qquad s = 1, 2$$

$$u_+^{(s)}(\underline{p}) = \underset{k}{\text{constant}} \begin{pmatrix} w^s \\ \dfrac{\underline{\sigma}\cdot\underline{p}}{p^0+mc} w^s \end{pmatrix}$$

Normalization convention: $u_+^{(s)\dagger} u_+^{(s)} = 2p^0$
$p^0 > 0$

$$|k|^2 \left(w^{s\dagger},\ \left(\frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s\right)^\dagger \right) \begin{pmatrix} w^s \\ \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s \end{pmatrix} = 2p^0 \quad (9)$$

$$\rightarrow \quad |k| = ? \quad \text{Ans: } \sqrt{p^0+mc}$$

$$|k|^2 \left(w^{s\dagger}\cdot w^s + \left(\frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s\right)^\dagger \cdot \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s \right.$$

$$= 2p^0$$

$$w^{s\dagger} w^s = 1$$

$$|k|^2 \left(1 + w^{s\dagger} \left(\frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc}\right)^\dagger \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s \right)$$

$$= 2p^0$$

$$|k|^2 \left(1 + w^{s\dagger} \frac{(\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^\dagger(\underset{\sim}{\sigma}\cdot\underset{\sim}{p})}{(p^0+mc)^2} w^s \right) = 2p^0$$

$$|k|^2 \left(1 + w^{s\dagger} \frac{\underset{\sim}{p}^2}{(p^0+mc)^2} w^s \right) = 2p^0$$

$$\because (\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^\dagger = \underset{\sim}{\sigma}\cdot\underset{\sim}{p} \quad \text{HW}$$

$$\text{and } (\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^2 = \underset{\sim}{p}^2 \quad \text{HW}$$

$$|k|^2 \left(1 + \frac{\underset{\sim}{p}^2}{(p^0+mc)^2} \right) = 2p^0 \quad \because w^{s\dagger} w^s = 1$$

$$|k|^2 \left(1 + \frac{p^{0\,2} - m^2c^2}{(p^0+mc)^2} \right) = 2p^0$$

$$|k|^2 \left(1 + \frac{p^0 - mc}{p^0 + mc} \right) = 2p^0$$

$$|k|^2 \quad \frac{2p^0}{p^0+mc} = 2p^0$$

$$|k| = \sqrt{p^0 + mc}$$

i.e.

$$u_+^{(s)} = \sqrt{p^0+mc} \begin{pmatrix} w^s \\ \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s \end{pmatrix} \qquad s = 1, 2.$$

So +ve energy $p^0 = \sqrt{\underset{\sim}{p}^2 + m^2c^2}$ has been constructed explicitly.

Now negative energy soln

$$p^0 = -\sqrt{\underset{\sim}{p}^2 + m^2c^2}$$

$$\psi(x) = e^{-i(p\cdot x)/\hbar} \; u_-^{(s)}(p) \quad (1)$$

$$u_-^{(s)}(p) = \sqrt{mc - p^0} \begin{pmatrix} \dfrac{\sigma \cdot p}{p^0 - mc} w^s \\ w^s \end{pmatrix} \qquad s = 1, 2$$

(HW)

Reinterpret $u_-^{(s)}(p)$ by putting

$$p \to -p$$

$$u_-^{(s)}(-p) = \sqrt{mc + p^0} \begin{pmatrix} \dfrac{\sigma \cdot p}{p^0 + mc} w^s \\ w^s \end{pmatrix} \cdots (+)'$$

$$\psi(x) = e^{i p\cdot x/\hbar} \; u_-^{(s)}(-p)$$

which is regarded as a solution for an anti particle e^+ (positron) of positive energy.

The free Dirac particle can be written also as

$$\psi(x) = e^{+ip\cdot x/\hbar} \; V(p)$$

Can solve $p^0 = \pm\sqrt{p^2 + m^2c^2}$

For $p^0 = +\sqrt{\underset{\sim}{p}^2 + m^2c^2}$

$$V_+^{(s)}(\underset{\sim}{p}) = \sqrt{p^0 + mc} \begin{pmatrix} \dfrac{\underset{\sim}{\sigma} \cdot \underset{\sim}{p}}{p^0 + mc} w^s \\ w^s \end{pmatrix} \quad \text{(HW)}$$

$$V_+^{(s)\dagger}(\underset{\sim}{p})\, V_+^{(s)}(\underset{\sim}{p}) = 2p^0 \;\text{--}\; (x)$$

Compare (+) and (x), identify

$$V_+^{(1)}(\underset{\sim}{p}) = u_-^{(2)}(-\underset{\sim}{p})$$

$$V_+^{(2)}(\underset{\sim}{p}) = -u_-^{(1)}(-\underset{\sim}{p})$$

A general solution is $\psi(\underset{\sim}{x}) = a e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}/\hbar} u(\underset{\sim}{p}) + b e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}/\hbar} V(\underset{\sim}{p})$

a, b constant

Why 2 l. i. solutions for $u_+^{(s)}$ or $u_-^{(s)}$ (or why energy p^0 is doubly degenerate?)

i.e. for the same energy p^0, can have 2 l. i. solutions i.e. p^0 (energy) is doubly degenerate $\rightarrow$ $\exists$ other observable that commutes with the Dirac Hamiltonian

This observable is the helicity operator $h(\underset{\sim}{p})$

$$h(\underset{\sim}{p}) = \underset{\sim}{\Sigma} \cdot \frac{\underset{\sim}{p}}{|\underset{\sim}{p}|}$$ here $\underset{\sim}{p}$ is not an operator

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\sigma} & 0 \\ 0 & \underset{\sim}{\sigma} \end{pmatrix}$$

Dirac spin operator $\underset{\sim}{S} = \frac{\hbar}{2} \Sigma$

(compare with Schrödinger $\underset{\sim}{S} = \frac{\hbar}{2} \underset{\sim}{\sigma}$)

Can show $[h(\underset{\sim}{p}), H] = 0$

$$H = c\,\underset{\sim}{\alpha} \cdot \underset{\sim}{p} + \beta m c^2$$

$\rightarrow$ $h^2(\underset{\sim}{p}) = 1$ (identity operator) (HW)

i.e. eigenvalues of $h(p) = \pm 1$

which can be used to differentiate the two l. i. soln of the same energy.

What are scalar, vectors and tensors in the Dirac formulation?

· Scalar, Vector and tensor constructed from $\psi(\underline{x})$

$\bar{\psi}(\underline{x})\,\psi(\underline{x})$ scalar

$\bar{\psi}(\underline{x})\,\gamma^5\,\psi(\underline{x})$ pseudoscalar

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{HW}$$

(Dirac representation)

$\bar{\psi}(\underline{x})\,\gamma^\mu\,\psi(\underline{x})$ vector

$\bar{\psi}(\underline{x})\,\gamma^5\gamma^\mu\,\psi(\underline{x})$ pseudo vector

$\bar{\psi}\,\sigma^{\mu\nu}\,\psi$ tensor

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

The prob. current density $j^\mu = c\,\bar{\psi}\,\gamma^\mu\,\psi$ is a 4-vector

$\bar{\psi}(\underline{x}) = \psi^\dagger(\underline{x})\,\gamma^0$ = Dirac adjoint of $\psi(\underline{x})$

$\psi^\dagger(\underline{x})$ = Hermitian conjugate (or adjoint) of $\psi(\underline{x})$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$$

$$\bar{\psi} = \psi^\dagger \gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

Here 1 = 2×2 identity matrix

As $j^\mu = c\,\bar{\psi}\gamma^\mu \psi$,

$$\therefore \quad j^0 = c\,\bar{\psi}\gamma^0\psi = c\,(\psi_1^*\psi_1, \psi_2^*\psi_2, \psi_3^*\psi_3, \psi_4^*\psi_4)$$

Thus in the Dirac case, the probability density j^0 (25) is positive, unlike the Klein. Gordon case.

One can check the 4- probability current density j^μ (25) does satisfy the continuity equation $\partial_\mu j^\mu$ (25) $= 0$, thus probability is conserved

To check probability conservation: $\partial_\mu j^\mu = 0$

prob. current density for the Dirac eqn is defined

$$j^\mu = c\,\bar{\psi}\gamma^\mu\psi$$

want to show $\partial_\mu j^\mu = 0$ conservation of prob.

from the equation of motion.

$$\not{p}\,\psi = mc\,\psi, \qquad \not{p} = p_\mu\gamma^\mu = \gamma^\mu p_\mu$$

$$p_\mu = i\hbar\partial_\mu$$

$$\partial_\mu j^\mu = c(\partial_\mu\bar{\psi})\gamma^\mu\psi + c\,\bar{\psi}\gamma^\mu(\partial_\mu\psi)$$

$$= c\,\underline{\partial_\mu\bar{\psi}}\gamma^\mu\psi + c\,\bar{\psi}\,mc\psi/(i\hbar) \quad \text{-- (1)}$$

Eq of motion for $\bar{\psi} \equiv \psi^\dagger\gamma^0$ (Dirac adjoint)

Taking the adjoint of the Dirac equation

$$\gamma^\mu \cdot i\hbar\partial_\mu\psi = mc\,\psi$$

$$-i\hbar\partial_\mu\psi^\dagger\cdot\gamma^{\mu\dagger} = mc\,\psi^\dagger$$

Recall $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

$$\therefore\ -i\hbar\,\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 = mc\,\psi^\dagger$$

Multiply γ^0 from the right and as $\gamma^{0\,2} = 1$,

we have

$$-i\hbar\partial_\mu \psi^\dagger \gamma^0 \gamma^\mu = mc\, \psi^\dagger \gamma^0$$

Now $\bar{\psi} \equiv \psi^\dagger \gamma^0$ the Dirac adjoint,

$$-i\hbar\partial_\mu \bar{\psi}\, \gamma^\mu = mc\, \bar{\psi} \qquad (2)$$

i.e.

$$(p_\mu \bar{\psi})\, \gamma^\mu = -\, mc\, \bar{\psi}$$

cf $\quad \not{p}\, \psi = mc\, \psi$

substituting eq(2) into eq(1), we finally arrive

at

$$\partial_\mu j^\mu = \frac{c\, mc\, \bar{\psi}}{-i\hbar} \cdot \psi + mc^2 \frac{\bar{\psi}\, \psi}{i\hbar} = 0$$

the continuity equation for the 4-current density

$j^\mu(x)$

Charge conjugation in the Dirac formulation

Consider a charged particle (electron),

$$\not{p}\,\psi(\underline{x}) = mc\,\psi(\underline{x})$$

In the presence of an em field $A_\mu(\underline{x})$

$$\underline{P} \to \underline{P} - q\underline{A}$$

the Dirac eqn becomes

$$(\not{p} - q\not{A})\,\psi = mc\,\psi$$

$$\not{A} = A_\mu \gamma^\mu$$

$$\gamma^\mu (i\hbar\partial_\mu - qA_\mu)\psi = mc\,\psi$$

Taking adjoint

$$(-i\hbar\partial_\mu - qA_\mu)\psi^+ \gamma^{\mu+} = mc\,\psi^+ \qquad \gamma^{\mu+} = \gamma^0\gamma^\mu\gamma^0$$

$$(i\hbar\partial_\mu + qA_\mu)\bar{\psi}\cdot\gamma^\mu = -mc\,\bar{\psi}$$

Taking transpose

$$\gamma^{\mu t}(i\hbar\partial_\mu + qA_\mu)\bar{\psi}^t = -mc\,\bar{\psi}^t$$

If $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$, then $[\gamma^{\mu t}, \gamma^{\nu t}]_+ = 2g^{\mu\nu}$

$$\to \quad \gamma^{\mu t} = -C^{-1}\gamma^\mu C$$

$$\rightarrow \quad C^{-1}\gamma^{\mu} C\,(i\hbar\partial_{\mu} + qA_{\mu})\,\overline{\psi}^{t} = mc\,\overline{\psi}^{t}$$

$$\rightarrow \quad \gamma^{\mu}(i\hbar\partial_{\mu} + qA_{\mu})\,C\,\overline{\psi}^{t} = mc\,C\,\overline{\psi}^{t}$$

Define the charge conjugate Dirac ψ

$$\psi_c \equiv C\,\overline{\psi}^{t} = C\,(\psi^{\dagger}\gamma^{0})^{t}$$

$$= C\,\gamma^{0t}\,\psi^{*} \qquad \gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= C\,\gamma^{0}\,\psi^{*}$$

The charge conjugate equation of

$$(\not{p} - q\not{A})\,\psi = mc\,\psi \qquad \dagger \text{ HW}$$

is

$$(\not{p} + q\not{A})\,\psi_c = mc\,\psi_c$$

Explicit expression for the charge conjugation operator

$$C = i\gamma^{2}\gamma^{0}$$

Summary

e^- $\qquad\qquad$ e^+

Wave functions

$$\psi(\underset{\sim}{x}) = e^{-i\underset{\sim}{p}\cdot\frac{\underset{\sim}{x}}{\hbar}}\, u^{(s)}(\underset{\sim}{p}) \qquad\qquad \psi(\underset{\sim}{x}) = e^{i\underset{\sim}{p}\cdot\frac{\underset{\sim}{x}}{\hbar}}\, v^{(s)}(\underset{\sim}{p})$$

s=1 spin up
s= 2 spin down

s=1 spin down
s=2 spin up

and

$$(\not{p} - mc)u = 0 \qquad\qquad (\not{p} + mc)v = 0$$

$$\bar{u}(\not{p} - mc) = 0 \qquad\qquad \bar{v}(\not{p} + mc) = 0$$

Orthonormality

$$\bar{u}^{(s_1)}\, u^{(s_2)} = 2mc\, \delta_{s_1 s_2} \qquad\qquad \bar{v}^{(s_1)}\, v^{(s_2)} = -2mc\, \delta_{s_1 s_2}$$

$s_1, s_2 = 1, 2$

Completeness

$$\sum_{s=1}^{2} u^{(s)}\, \bar{u}^{(s)} = (\not{p} + mc) \qquad\qquad \sum_{s=1}^{2} v^{(s)}\, \bar{v}^{(s)} = (\not{p} - mc)$$

Photon

Plane Wave

$$A^{\mu}(\underset{\sim}{x}) = e^{-i\underset{\sim}{p}\cdot\frac{\underset{\sim}{x}}{\hbar}}\ \varepsilon^{\mu}_{(s)}$$

s=1, 2 for the two polarization states

Polarization vector $\varepsilon^{\mu}_{(s)}$ statistics $\quad p_{\mu}\, \varepsilon^{\mu}_{(s)} = 0$.

Orthonormality

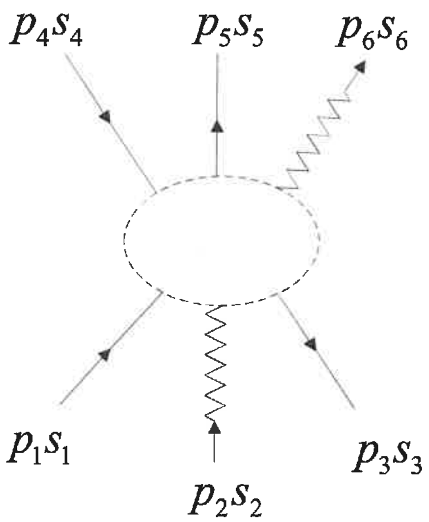
$$\varepsilon^{\mu^*}_{(s_1)} \quad \varepsilon_{\mu(s_2)} = \delta_{s_1 s_2}$$

Coulomb gauge $\quad \varepsilon^\circ = 0, \quad \underset{\sim}{\varepsilon} \cdot \underset{\sim}{p} = 0$

Completeness

$$\sum_{s=1}^{2} (\varepsilon_{(s)})_i \, (\varepsilon^*_{(s)})_j = \delta_{ij} - \hat{p}_i \hat{p}_j \qquad \hat{p}_i = p_i / |\underset{\sim}{p}|$$

Feynman rules QED



Notations

Label external lines by momentum p_i and spin s_i,

Label internal lines by momenta q_i

Arrows on external fermion lines indicate

e^- (forward in time) $\qquad$ e^+ (backward in time)

Arrows on internal fermion lines are assigned so that direction of the flow of 4-momenta through the diagram is kept.

Arrows on external photon lines point forward; for internal photon lines, the choice is arbitrary.

(i) External lines

e^- incoming $: u$

outgoing $: \bar{u}$

e^+ incoming $: \bar{v}$

outgoing $: v$

γ incoming $: \varepsilon^{\mu}$

outgoing $: \varepsilon^{\mu^*}$

(ii)Vertex

Each vertex contributes a factor $ig\gamma^{\mu}$

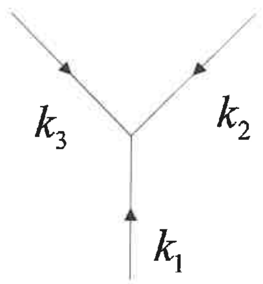
g = dimensionless coupling constant = $\sqrt{4\pi\alpha}$

(iii) Propagators (internal lines)

$$e^- \ or \ e^+ : \frac{i}{\not{q} - mc} = \frac{i(\not{q} + mc)}{q^2 - m^2c^2}$$

γ : $\quad \dfrac{-ig_{\mu\nu}}{q^2}$

(iv) Conservation of 4 - momentum P_μ:



For each vertex, write $\quad (2\pi)^4\delta^{(4)}(\underset{\_}{k}_1 + \underset{\_}{k}_2 + \underset{\_}{k}_3)$

(v) Integrate over internal momenta

$$\int \frac{d^4q}{(2\pi)^4}$$

(vi) Cancel the overall delta function

$$(2\pi)^4\delta^{(4)}(\underset{\_}{p}_1 + \underset{\_}{p}_2 \ldots \underset{\_}{p}_n)$$

what remains is the $-i\mathcal{M}$, $\mathcal{M}$= scattering amplitude

(vii) Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) $e^{-\prime}s(ore^{+\prime}s)$

or of an incoming e^- with an outgoing e^+ (or vice versa)

(viii) Charge is conserved at each vertex.

Lepton number etc must also be conserved.

(ix) For a closed fermion loop, include a factor -1 and take the trace.