Examples

(i) $e^- \mu^- \to e^- \mu^-$ electron muon scattering

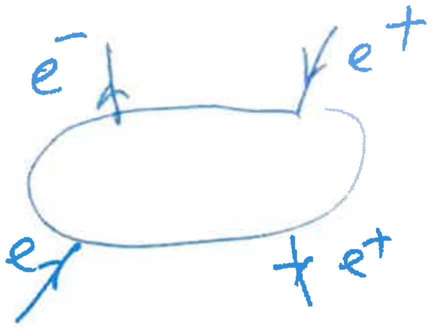
(ii) $e^- e^- \to e^- e^-$ Møller scattering

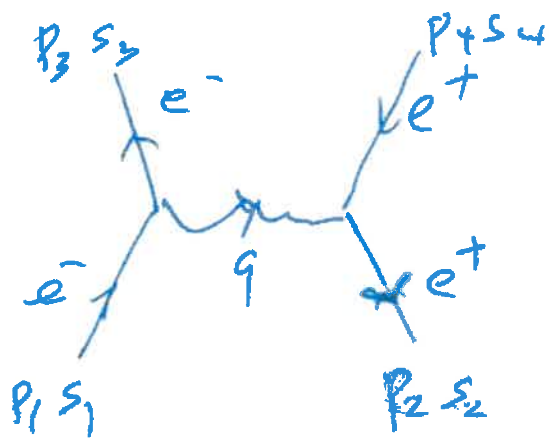
(iii) $e^- e^+ \to e^- e^+$ Bhabha scattering

(iv) $e^- \gamma \to e^- \gamma$ Compton scattering

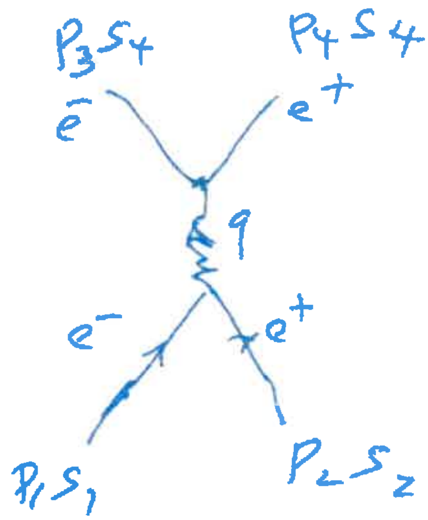
(v) $\underset{\underline{u}}{\underset{\uparrow}{e^-}} \underset{\underline{V}}{\underset{\uparrow}{e^+}} \to \underset{\underline{\varepsilon}}{\underset{\uparrow}{\gamma}} \gamma$ annihilation

(iii) $e^- e^+ \to e^- e^+$ Bhabha scatt.





(i)



(ii)

Compute $\mathcal{M}_{(i)}$

$$\int \frac{d^4 q}{(2\pi)^4} \, \bar v(2)\, i g \gamma^\nu v(4) \, \frac{-i g_{\mu\nu}}{q^2} \, (2\pi)^4 \delta^{(4)}(p_1 - q - p_3)\, \bar u(3)\, i g \gamma^\mu u(1) \, (2\pi)^4 \delta^{(4)}(q + p_2 - p_4) = ?$$

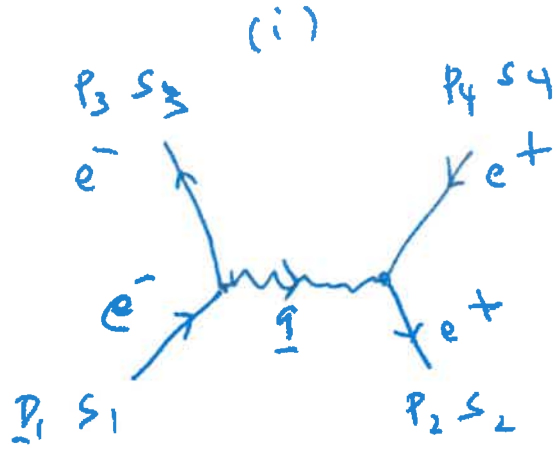
HW

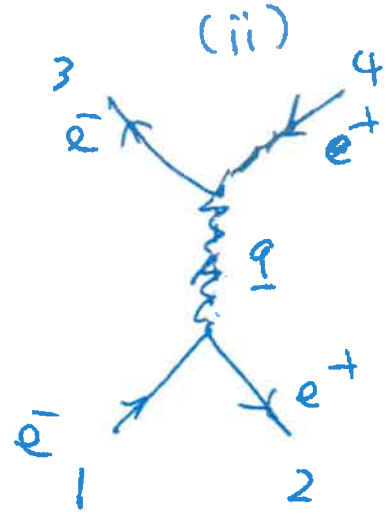
$$\mathcal{M}_{(i)} = \frac{-g^2}{(p_1 - p_3)^2} \left[\bar u(3) \gamma^\mu u(1)\right] \left[\bar v(2) \gamma_\mu v(4)\right]$$

HW

Time

Feynman diagram





Find scatt. amp $\mathcal{M}(i)$

number

$$\int \frac{d^4 q}{(2\pi)^4}\left(\bar{v}(2)\, i g\gamma^\nu v(4)\right)\frac{-i g_{\mu\nu}}{q^2}\left(\bar{u}(3)\, i g\gamma^\mu u(1)\right)$$

$$(2\pi)^4\delta^{(4)}(p_1-p_3-q)\cdot(2\pi)^4\delta^{(4)}(q+p_2-p_4)$$

$$= i g^2 (2\pi)^4 \delta^{(4)}(p_1-p_3-p_4+p_2)\cdot$$

$$\bar{v}(2)\gamma^\nu v(4)\frac{g_{\mu\nu}}{(p_1-p_3)^2}\bar{u}(3)\gamma^\mu u(1)$$

$$\mathcal{M}_{(i)} = -g^2\, \bar{v}(2)\gamma_\mu v(4)\cdot\frac{1}{(p_1-p_3)^2}\bar{u}(3)\gamma^\mu u(1)$$

$$\gamma_\mu \equiv g_{\mu\nu}\gamma^\nu$$

$\gamma_0 = \gamma^0,\quad \gamma_i = -\gamma^i,$

$i = 1, 2, 3$

$$\int \frac{d^4q}{(2\pi)^4}\, \bar{u}(3)\, ig\gamma^\nu\, v(4) \cdot \frac{-ig_{\mu\nu}}{q^2}\, \bar{v}(2)\, ig\gamma^\mu\, u(1)$$

$$(2\pi)^4\, \delta^{(4)}(p_1 - q + p_2)\,(2\pi)^4\, \delta^{(4)}(q - p_3 - p_4)$$

$$= (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\; i g^2\, \bar{u}(3)\gamma^\nu v(4) \cdot \frac{g_{\mu\nu}}{\underset{p_1+p_2}{q^2}}$$

$$\bar{v}(2)\, \gamma^\mu\, u(1)$$

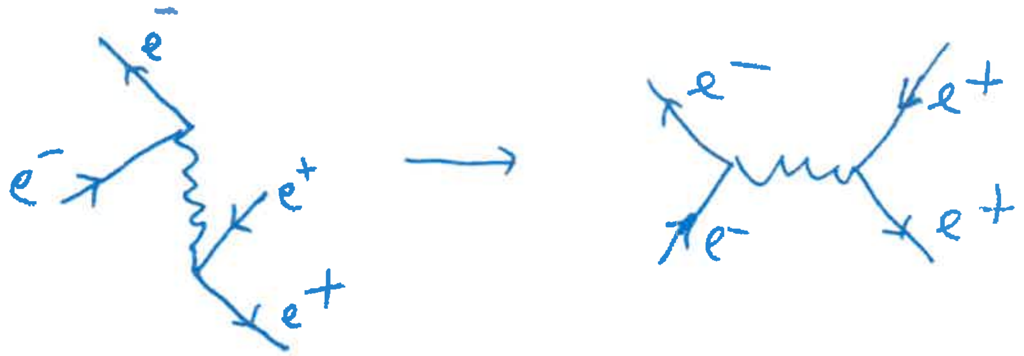
$$\rightarrow \quad \mathcal{M}_{(ii)} = -g^2\, \bar{u}(3)\gamma_\mu v(4) \cdot \frac{1}{(p_1 + p_2)^2} \cdot$$

$$\bar{v}(2)\gamma^\mu u(1)$$

Should we add $\mathcal{M}_{(ii)}$ to $\mathcal{M}_{(i)}$ or should we subtract?

This depends on whether the two diagrams can be obtained from each other by (i) interchanging the two incoming identical particles, or (ii) interchanging the two outgoing identical particles, or (iii) interchanging an incoming e^- with an outgoing e^+ (anti-particle) or vice versa

In diagram (ii), interchange outgoing e^+ with incoming e^-



That means the first diagram can be obtained from the 2nd diagram by using crossing symmetry.

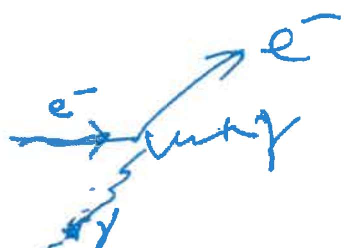
Can show 2nd diagram can be obtained from 1st diagram by crossing symmetry (HW)

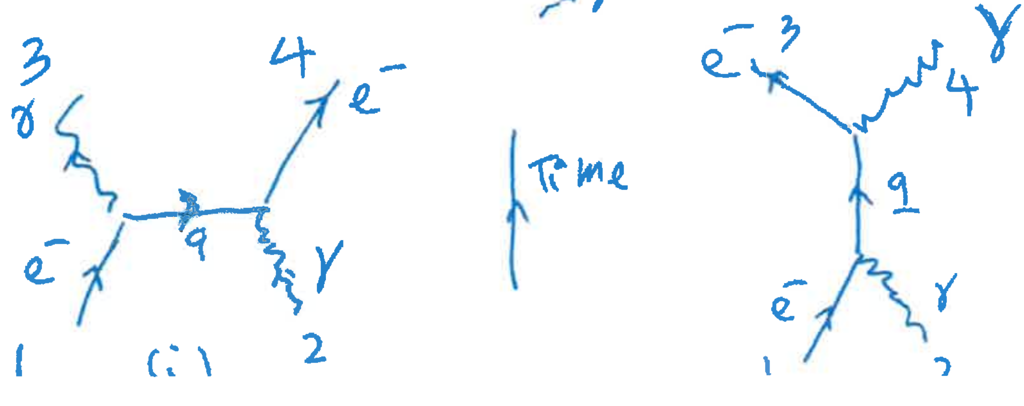
So the total scatt. amp is

$$\mathcal{M} = \mathcal{M}_{(i)} - \mathcal{M}_{(ii)}$$

Now do

(iv) $e^- \gamma \to e^- \gamma$





(i) (ii)

$$\int \frac{d^4 q}{(2\pi)^4} \, \bar{u}(4) \, i g \gamma^\nu \, \varepsilon_\nu(2) \, \frac{i}{\not{q} - mc} \, \varepsilon^*_\mu(3) \, i g \gamma^\mu u(1)$$

$$(2\pi)^4 \delta^{(4)}(\underline{p}_1 - \underline{k}_3 - \underline{q}) \cdot (2\pi)^4 \delta^{(4)}(\underline{q} + \underline{k}_2 - \underline{p}_4)$$

$$\mathcal{M}_{(i)} = g^2 \, \bar{u}(4) \cdot \gamma^\nu \varepsilon_\nu(2) \, \frac{1}{(\not{p}_1 - \not{k}_3 - mc)} \cdot \varepsilon^*_\mu(3) \gamma^\mu u(1)$$

$$\equiv g^2 \, \bar{u}(4) \, \not{\varepsilon}(2) \, \frac{1}{\not{p}_1 - \not{k}_3 - mc} \, \not{\varepsilon}^*(3) \, u(1)$$

For 2nd diagram

$$\int \frac{d^4 q}{(2\pi)^4} \, \bar{u}(3) \, i g \gamma^\nu \, \varepsilon^*_\nu(4) \, \frac{i}{\not{q} - mc} \, \varepsilon_\mu(2) \, i g \gamma^\mu u(1)$$

$$(2\pi)^4 \delta^{(4)}(\underline{p}_1 + \underline{k}_2 - \underline{q}) \, (2\pi)^4 \delta^{(4)}(\underline{q} - \underline{p}_3 - \underline{k}_4)$$

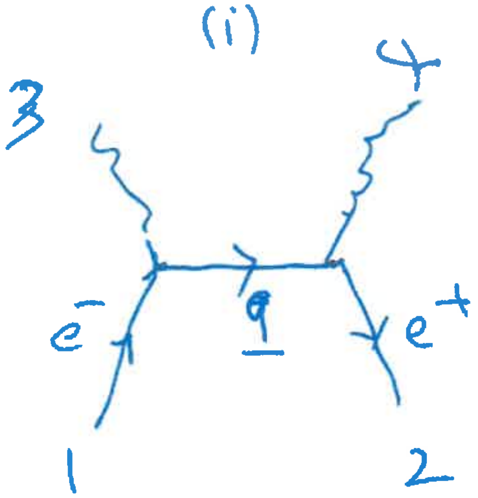
$$\mathcal{M}_{(ii)} = g^2 \, \bar{u}(3) \, \not{\varepsilon}(4) \, \frac{1}{(\not{p}_1 + \not{k}_2 - mc)} \, \not{\varepsilon}(2) \, u(1)$$

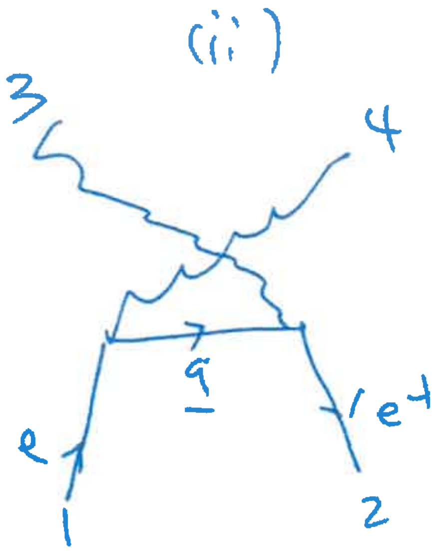
$$\mathcal{M} = \mathcal{M}_{(i)} + \mathcal{M}_{(ii)}$$

Last process

(v) $e^+ e^- \to \gamma\gamma$

Feynman diagrams





For diagram (i)

$$\int \frac{d^4 \underline{q}}{(2\pi)^4} \bar{v}(2)\, ig\gamma^{\nu} \varepsilon^*_{\nu}(4) \frac{i}{\not{q} - mc} \varepsilon^*_{\mu}(3)\, ig\gamma^{\mu} u(1)$$

$$(2\pi)^4 \delta^{(4)}(\underline{p}_1 - \underline{q} - \underline{k}_3)(2\pi)^4 \delta^{(4)}(\underline{q} + \underline{p}_2 - \underline{k}_4)$$

$$\mathcal{M}_{(i)} = g^2\, \bar{v}(2) \not{\varepsilon}^*(4) \frac{1}{\not{p}_1 - \not{k}_3 - mc} \not{\varepsilon}^*(3)\, u(1)$$

For diagram (ii)

$$\mathcal{M}_{(ii)} = g^2\, \bar{v}(2) \not{\varepsilon}^*(3) \frac{1}{\not{p}_1 - \not{k}_4 - mc} \not{\varepsilon}^*(4)\, u(1)$$

(HW)

$$\mathcal{M} = \mathcal{M}_{(i)} + \mathcal{M}_{(ii)}$$

$$\mathcal{M}_{(i)} = g^2\, \bar{v}(2)\, \not{\epsilon}^*(4)\, \frac{1}{\not{p}_1 - \not{k}_3 - mc}\, \not{\epsilon}^*(3)\, u(1)$$

Note: $\not{k}_3 \equiv k_{3\mu}\gamma^\mu$

$$= g^2\, \bar{v}(2)\, \not{\epsilon}^*(4)\, \frac{(\not{p}_1 - \not{k}_3 + mc)}{(\underline{p}_1 - \underline{k}_3)^2 - m^2c^2}\, \not{\epsilon}^*(3)\, u(1)$$

$$= \frac{g^2}{(\underline{p}_1 - \underline{k}_3)^2 - m^2c^2}\, \bar{v}(2)\, \not{\epsilon}^*(4)\, (\not{p}_1 - \not{k}_3 + mc)\, \not{\epsilon}^*(3)\, u(1)$$

Diagram (ii) HW

$$\mathcal{M}_{(ii)} = g^2\, \bar{v}(2)\, \not{\epsilon}^*(3)\, \frac{1}{\not{p}_1 - \not{k}_4 - mc}\, \not{\epsilon}^*(4)\, u(1)$$

Total scattering amplitude

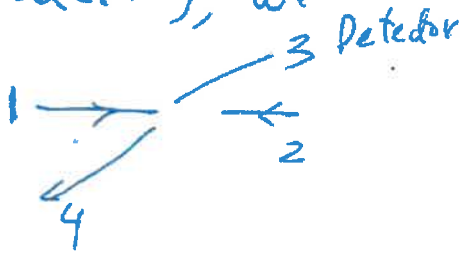
$$\mathcal{M} = \mathcal{M}_{(i)} + \mathcal{M}_{(ii)}$$

=

Next discuss differential crosssection. $\frac{d\sigma}{d\Omega}$

Recall the scattering cross section can be written as a product of dynamic part (scattering amplitude) and the kinematic part (phase space factor)

For a 2 particle to 2 particle scattering, we have shown



$$\frac{d\sigma}{d\Omega_3} =$$

$$\frac{d\sigma}{d\Omega_3} = \frac{S\,\hbar^2}{64\pi^2\sqrt{(\underset{\sim}{p}_1\cdot\underset{\sim}{p}_2)^2-(m_1 m_2 c^2)^2}}\,\frac{|\mathcal{M}|^2\,|\underset{\sim}{p}_3|}{(p_1^0+p_2^0)}\Bigg|_{\underset{\sim}{p}_4=-\underset{\sim}{p}_3}$$

$$\underset{\sim}{p}_3^{\,2} = \frac{\left(\alpha^2+(m_4^2-m_3^2)c^2\right)^2}{4\,\alpha^2} - m_4^2 c^2$$

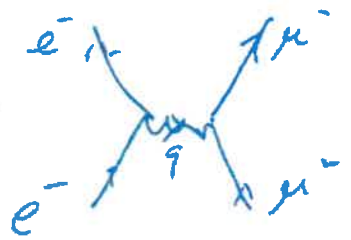
$$\alpha \equiv p_1^0 + p_2^0$$

In this expression, the only unknown is $|\mathcal{M}|^2$.

In many experiments, the detector just count the number of particles and the spins (polarizations) are not measured. If that is the case, then we must compute $\mathcal{M}$ for every possible spin (for the $2\to 2$ process, that means we have to compute $\mathcal{M}$ for spin s_1, s_2, s_3, s_4 and then sum up) i.e. we compute the average over spins of incident particles and summation over final spins

$$\langle |\mathcal{M}|^2 \rangle \equiv \frac{1}{4}\sum_{s_1,s_2,s_3,s_4} |\mathcal{M}|^2$$

Consider $e^-\mu^- \to e^-\mu^-$



$$\mathcal{M} = \frac{-g^2}{(\underset{\sim}{p}_1-\underset{\sim}{p}_3)^2}\,\bar{u}(3)\gamma^\nu u(1)\cdot g_{\mu\nu}\,\bar{u}(4)\gamma^\mu u(2)$$

Instead of doing the summation for 16 $|\mathcal{M}|^2$ we can use Casimir's trick to avoid computing each of the 16 $|\mathcal{M}|^2$ and then summing.

$$|\mathcal{M}|^2 = \mathcal{M}\,\mathcal{M}^*$$

$$\mathcal{M} = \alpha \quad \bar{u}(3)\,\gamma_\mu\, u(1)\;\bar{u}(4)\,\gamma^\mu u(2)$$

$$\alpha \equiv \frac{-g^2}{(\underline{p}_1 - \underline{p}_3)^2}$$

$$\mathcal{M}\,\mathcal{M}^* = \alpha^2 \quad \bar{u}(3)\,\gamma_\mu\, u(1)\cdot\,\bar{u}(4)\,\gamma^\mu\, u(2)\,\cdot$$

$$\Big(\underline{\bar{u}(3)\,\gamma_\nu\, u(1)}\;\; \underline{\bar{u}(4)\,\gamma^\nu\, u(2)}\Big)^*$$

number

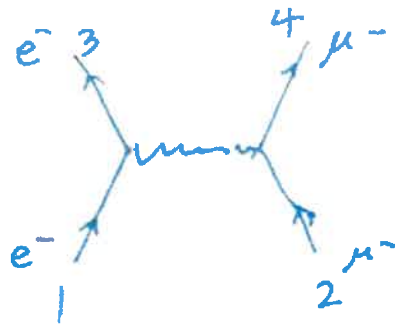
$$= \alpha^2 \;\bar{u}(3)\,\gamma_\mu\, u(1)\;\bar{u}(4)\,\gamma^\mu\; u(2)$$

$$u(2)^\dagger\;\gamma^{\nu\dagger}\;\bar{u}(4)^\dagger\,\cdot\;\; u(1)^\dagger\;\gamma_\nu^\dagger\;\;\bar{u}(3)^\dagger$$

Note: $\bar{u}^\dagger = (u^\dagger\gamma^0)^\dagger = \gamma^{0\dagger} u = \gamma^0 u$

$\gamma_\nu^\dagger \;= \gamma^0\,\gamma_\nu\,\gamma^0$

$$e^- + \mu^- \to e^- + \mu^-$$



$$\mathcal{M} = \frac{-g^2}{(\underline{P}_1 - \underline{P}_3)^2} \left(\bar{u}(3)\gamma^\mu u(1)\right)\cdot\left(\bar{u}(4)\gamma_\mu u(2)\right)$$

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{s_1 s_2 s_3 s_4} |\mathcal{M}|^2$$

$$= \frac{g^4}{4(\underline{P}_1 - \underline{P}_3)^4} \sum_{\substack{s_1 s_2 \\ s_3 s_4}} \left(\bar{u}(3)\gamma^\mu \underline{u(1)}\right)\left(\bar{u}(4)\gamma_\mu u(2)\right)\cdot \left(\underline{\bar{u}(1)}\gamma^\nu u(3)\right)\cdot\left(\bar{u}(2)\gamma_\nu u(4)\right)$$

completeness of bispinor

$$\sum_s u^{(s)}(\underline{p})\cdot \bar{u}^{(s)}(\underline{p}) = (\not{p} + mc)$$

$$\therefore \langle |\mathcal{M}|^2 \rangle = \frac{1}{4}\cdot\frac{g^4}{(\underline{P}_1 - \underline{P}_3)^4}\cdot \sum_{s_3 s_4} \left(\bar{u}(3)\gamma^\mu(\not{p}_1 + m_1 c)\gamma^\nu u(3)\right)\cdot \left(\bar{u}(4)\gamma_\mu(\not{p}_2 + m_2 c)\gamma_\nu u(4)\right)$$

$$= \frac{1}{4}\,\frac{g^4}{(\underline{P}_1 - \underline{P}_3)^2}\cdot \mathrm{Tr}\left[\gamma^\mu(\not{p}_1 + m_1 c)\gamma^\nu(\not{p}_3 + m_3 c)\right]\cdot \mathrm{Tr}\left[\gamma_\mu(\not{p}_2 + m_2 c)\gamma_\nu(\not{p}_4 + m_4 c)\right]$$

$$\mathrm{Tr}\,\gamma^\mu = 0, \qquad \mu = 0, 1, 2, 3$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha) = 0, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\gamma^{5\,2} = 1$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta)$$

$$= 4\left(g^{\mu\nu} g^{\alpha\beta} + g^{\beta\mu} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\beta}\right)$$

$$\mathrm{Tr}\left[\gamma^\mu (\not{p}_1 + m_1 c)\gamma^\nu (\not{p}_3 + m_3 c)\right]$$

$$= \mathrm{Tr}\left[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^\mu \not{p}_1 \gamma^\nu m_3 c + m_1 c \gamma^\mu \gamma^\nu \not{p}_3 + m_1 m_3 c^2 \gamma^\mu \gamma^\nu\right]$$

$$= \mathrm{Tr}\left[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 + m_1 \cdot m_3 c^2 \gamma^\mu \gamma^\nu\right]$$

$$\because \mathrm{Tr}\,\gamma^\mu \gamma^\alpha \gamma^\nu = 0$$

Using Casimir's trick, have computed

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4}\, \alpha \sum_{s_1 s_2 s_3 s_4} \left(\qquad \right)$$

for the $e^- \mu^- \to e^- \mu^-$ process

Using formula from the previous lecture

$$\mathrm{Tr}\left[\gamma_\mu (\not{p}_4 + m_4 c) \gamma_\nu (\not{p}_2 + m_2 c) \right]$$

$$\overset{HW}{=} 4 \left[p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu} (\underline{p}_2 \underline{p}_4 - m_2 m_4 c^2) \right]$$

$\rightarrow$

$$\langle |\mathcal{M}|^2 \rangle = \frac{g^4}{(\underline{p}_1 - \underline{p}_3)^4} \frac{1}{4} \mathrm{Tr}\left[\gamma^\mu (\not{p}_1 + m_1 c) \gamma^\nu (\not{p}_3 + m_3 c) \right] \cdot \mathrm{Tr}\left[\gamma_\mu (\not{p}_2 + m_2 c) \cdot \gamma_\nu (\not{p}_4 + m_4 c) \right]$$

$$\overset{HW}{=} \frac{4 g^4}{(\underline{p}_1 - \underline{p}_3)^4} \left[p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu} (\underline{p}_2 \underline{p}_4 - m_2 m_4 c^2) \right] \cdot$$

$$\cdot \left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} (\underline{p}_1 \underline{p}_3 - m_1 m_3 c^2) \right]$$

$$\overset{HW}{=} \frac{8 g^4}{(\underline{p}_1 - \underline{p}_3)^4} \Big[(\underline{p}_1 \cdot \underline{p}_2)(\underline{p}_3 \cdot \underline{p}_4) + (\underline{p}_2 \underline{p}_3)(\underline{p}_1 \underline{p}_4)$$

$$- (\underline{p}_2 \cdot \underline{p}_4) m_1 m_3 c^2 - (\underline{p}_1 \underline{p}_3) m_2 m_4 c^2$$

$$+ 2 (m_1 m_2 m_3 m_4) c^4 \Big]$$

$$m_1 = m_3 = m_e$$

$$m_2 = m_4 = m_\mu$$

When computing scattering amplitude $\mathcal{M}$ for Feynman's diagrams of one loop or higher number of loops, one always encounters integrals that are divergent.

These integrals are divergent because of

(i) integrand not well-behaved

(ii) lower limit of the integral

(iii) upper limit of the integral.

One can introduce different techniques to render these divergent integrals to become finite. This is called regularization, and usually parameters must be introduced to make the divergent integrals finite.

The parameters are arbitrary and must be gotten rid of.

(b)

These arbitrary parameters are usually gotten rid of by absorbing them into the physical quantities like charge, mass and coupling constant.

The procedure to get rid of the arbitrary parameters consistently (not just 1-loop level but also all higher loops) is known as renormalization program