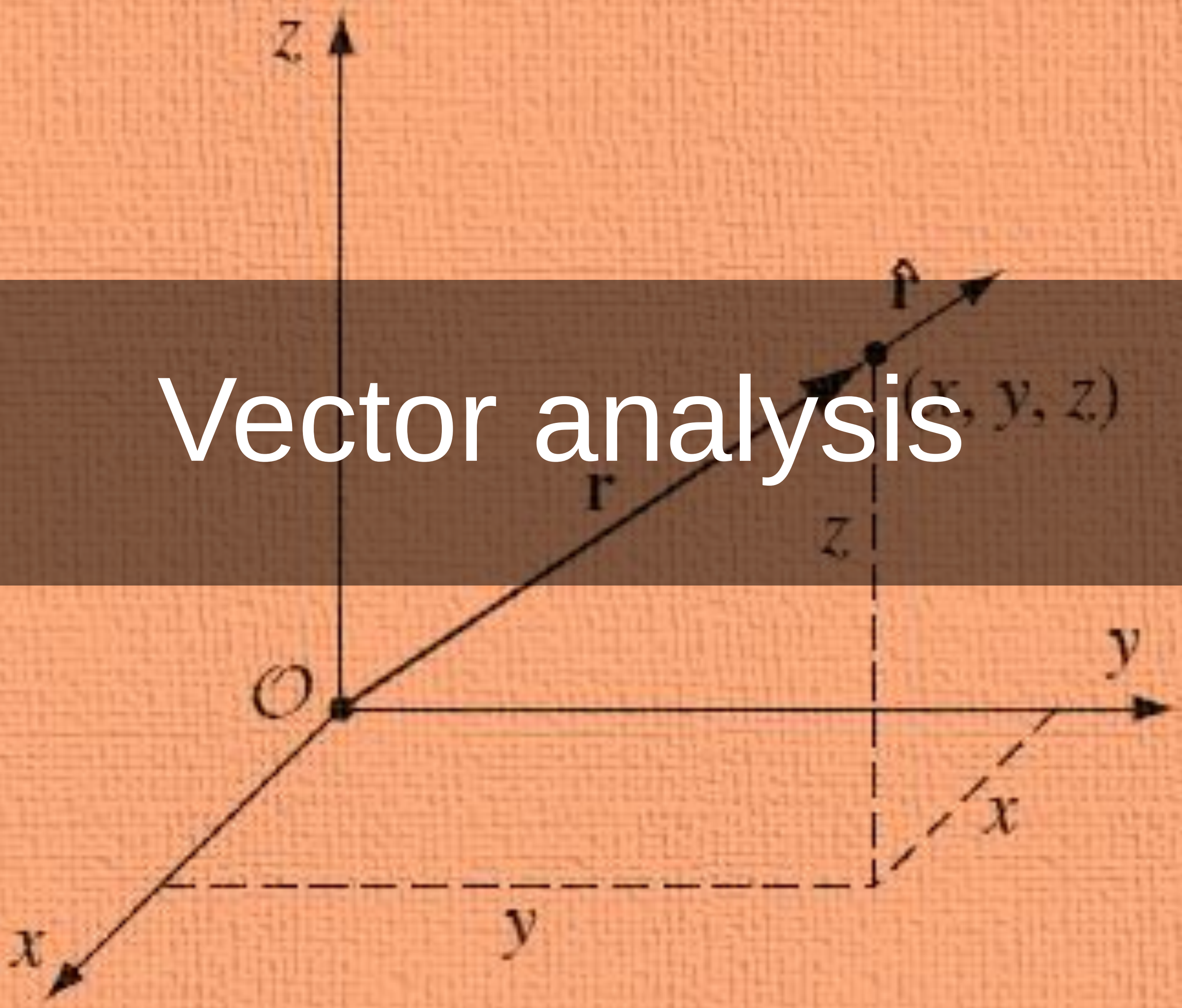
Vector analysis

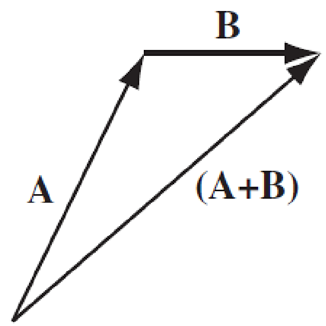


Vector algebra

Vector operations

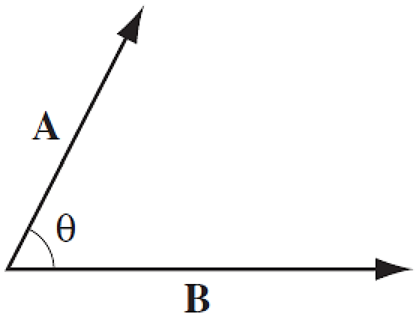
- A vector has direction and magnitude, <u>but no location</u>

- Addition of vectors
  - Commutative $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
  - Associative $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$



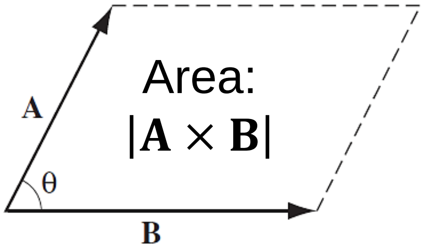
- Multiplication of vectors by a scalar
  - Distributive $a(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B}$

- Dot product of vectors $\mathbf{A}\cdot\mathbf{B}\equiv AB\cos\theta$
  - Commutative $\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$
  - Distributive $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$



Vector operations

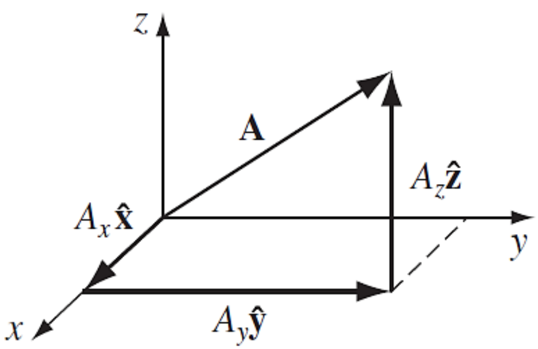
- Cross product of vectors $\mathbf{A} \times \mathbf{B} \equiv AB \sin\theta\, \hat{\mathbf{n}}$
  - Unit vector $\hat{\mathbf{n}} \perp \mathbf{A}$, $\hat{\mathbf{n}} \perp \mathbf{B}$
  - Orientation of $\hat{\mathbf{n}}$: right-hand rule
  - Distributive $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$
  - Anticommutative $(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B})$



- Component form (as opposed to abstract form) of vectors

$$\mathbf{A} = \underline{A_x}\hat{\mathbf{x}} + \underline{A_y}\hat{\mathbf{y}} + \underline{A_z}\hat{\mathbf{z}}$$

Components



  - Components: projections of the vector on three coordinate axes
  - $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are mutually perpendicular unit vectors

Vector operations

- Addition (in component form)

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= (A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}})+(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}})\\ &= (A_x+B_x)\hat{\mathbf{x}}+(A_y+B_y)\hat{\mathbf{y}}+(A_z+B_z)\hat{\mathbf{z}}.\end{aligned}$$

- Multiplication (in component form)

$$a\mathbf{A}=(aA_x)\hat{\mathbf{x}}+(aA_y)\hat{\mathbf{y}}+(aA_z)\hat{\mathbf{z}}$$

- Dot product (in component form)

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}})\cdot(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}})\\ &= A_xB_x+A_yB_y+A_zB_z.\end{aligned}$$

  - Modulus (magnitude) of a vector $\mathbf{A}\cdot\mathbf{A}=A_x^2+A_y^2+A_z^2$

$$\Rightarrow \quad A=\sqrt{A_x^2+A_y^2+A_z^2}$$

Vector operations

- Cross product (in component form)

$$\mathbf{A}\times\mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$
$$= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}$$

  - A more neat form $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

C1.cross

- Triple product
  - Scalar triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$
    - Can “rotate”

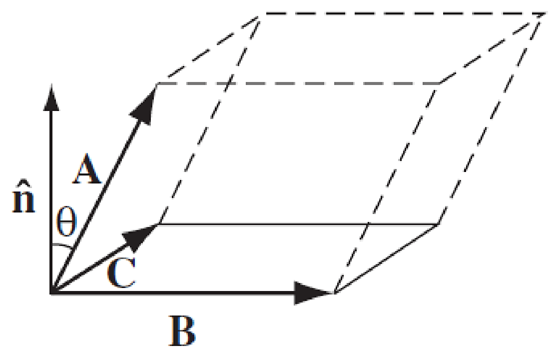
$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

  - Vector triple product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$
    - “BAC-CAB” rule

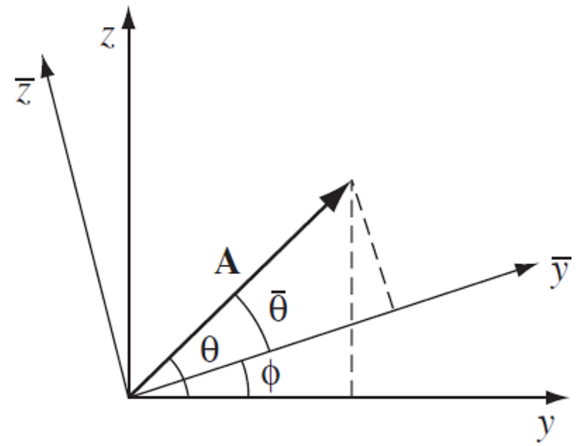
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Volume

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$



Vector operations



- Vector transformation
  - Consider axes rotation about x
    - In original coordinates

$$A_y = A\cos\theta, \qquad A_z = A\sin\theta,$$

    - In new coordinates

$$\bar{A}_y = A\cos\bar{\theta} = A\cos(\theta - \phi) = A(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$= \cos\phi A_y + \sin\phi A_z,$$

$$\bar{A}_z = A\sin\bar{\theta} = A\sin(\theta - \phi) = A(\sin\theta\cos\phi - \cos\theta\sin\phi)$$

$$= -\sin\phi A_y + \cos\phi A_z.$$

$$\Rightarrow \begin{pmatrix} \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

  - Generally, for 3D rotation

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Vector calculus

Differential calculus

- Ordinary derivative for function $f(x)$

$$\underbrace{df}_{\text{Effect}} = \underbrace{\left(\frac{df}{dx}\right)}_{\text{Slope}} \underbrace{dx}_{\text{Cause}}$$

- For a scalar function of three variables $T(x, y, z)$
  - How T varies with a movement depends on direction, and

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

- Gradient $\quad \nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$

$$dT = \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}})$$

$$= (\nabla T) \cdot (d\mathbf{l}),$$

*dT is largest when $d\mathbf{l}$ is directed along ∇T

Differential calculus

- The Del operator $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$
  - Can "act upon" functions
  - Not a vector by itself
  - But follows arithmetic rules that apply to ordinary vectors
  - Major vector operator responsible for <u>all three types of derivatives</u> in vector calculus

- The three types of derivatives
  - Gradient ∇T (Directly act on a scalar function)
  - Divergence $\nabla \cdot \mathbf{v}$ (Dot product with a vector function)
  - Curl $\nabla \times \mathbf{v}$ (Cross product with a vector function)

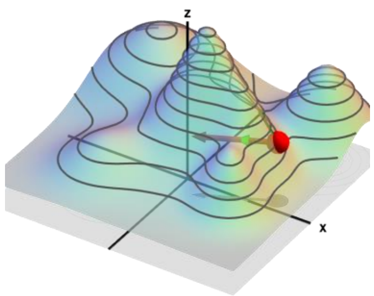
Differential calculus

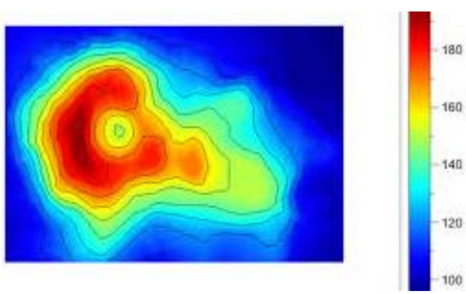
- Scalar field versus vector field (2D example)

Scalar field

$$T(x, y)$$

- A scalar number associated with each x, y location

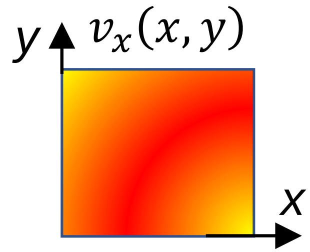


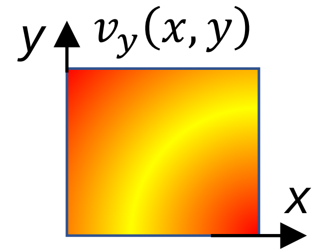


Vector field

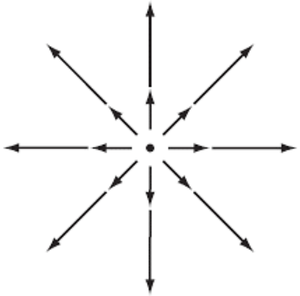
$$\mathbf{v}(x, y) = v_x(x, y)\hat{\mathbf{x}} + v_y(x, y)\hat{\mathbf{y}}$$

- A vector (with 2 components) associated with each x, y location
  - 1^{st} way to draw





  - 2^{nd} way to draw



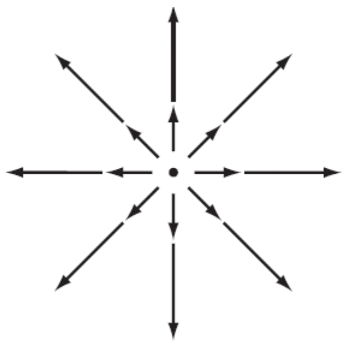
Differential calculus

- Divergence

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$$

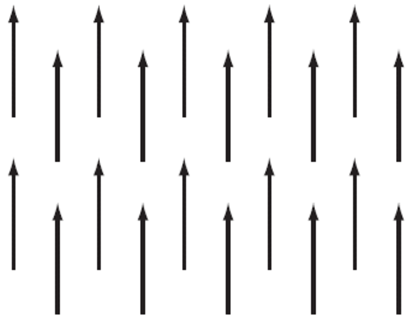
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

  - Is a measure of how much the vector spreads out (diverges) for a point in question



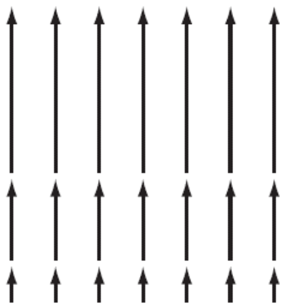
$\mathbf{v} = \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

$\nabla \cdot \mathbf{v} = 3$



$\mathbf{v} = \hat{\mathbf{z}}$

$\nabla \cdot \mathbf{v} = 0$



$\mathbf{v} = z\hat{\mathbf{z}}$

$\nabla \cdot \mathbf{v} = 1$

  - $\nabla \cdot \mathbf{v}$ can also be a scalar field

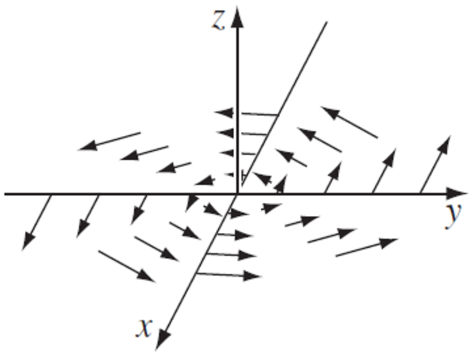
Differential calculus

- Curl

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

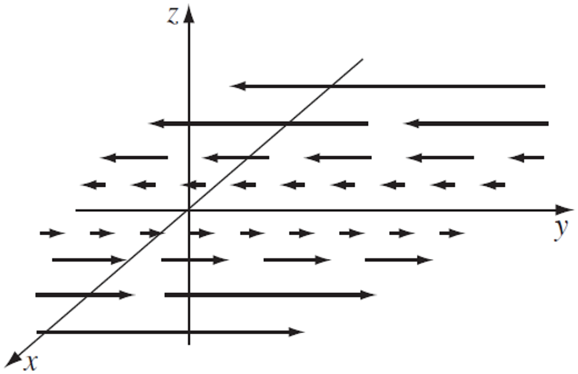
$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

  - Is a measure of how much the vector swirls around a point in question



$$\mathbf{v} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$$

$$\nabla \times \mathbf{v} = 2\hat{\mathbf{z}}$$



$$\mathbf{v} = x\hat{\mathbf{y}}$$

$$\nabla \times \mathbf{v} = \hat{\mathbf{z}}$$

  - $\nabla \times \mathbf{v}$ can also be a vector field

Differential calculus

- Ordinary derivatives versus derivatives in vector calculus
  - Trivial ones

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \Rightarrow \nabla(f+g) = \nabla f + \nabla g$$

$$\frac{d}{dx}(kf) = k\frac{df}{dx} \Rightarrow \begin{cases} \nabla(kf) = k\nabla f \\ \nabla\cdot(k\mathbf{A}) = k(\nabla\cdot\mathbf{A}) \\ \nabla\times(k\mathbf{A}) = k(\nabla\times\mathbf{A}) \end{cases}$$

  - Product rules $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

$$\Rightarrow \begin{cases} \nabla(fg) = f\nabla g + g\nabla f \\ \nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \\ \nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f) \\ \nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \\ \nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f) \\ \nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) \end{cases}$$

Differential calculus

- Ordinary derivatives versus derivatives in vector calculus
  - Quotient rules (easily derived from product rules)

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} \quad \Rightarrow \quad \begin{cases} \nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2} \\[2ex] \nabla\cdot\left(\dfrac{\mathbf{A}}{g}\right) = \dfrac{g(\nabla\cdot\mathbf{A}) - \mathbf{A}\cdot(\nabla g)}{g^2} \\[2ex] \nabla\times\left(\dfrac{\mathbf{A}}{g}\right) = \dfrac{g(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla g)}{g^2} \end{cases}$$

  - How? (Proof of a few expressions)

C1.derivatives

  - Why?

    - To make full use of the known quantities: derivatives of individual scalar & vector field

Differential calculus

- Second derivatives
  - Divergence of gradient (Laplace) $\nabla \cdot (\nabla T) = \nabla^2 T$

$$\nabla \cdot (\nabla T) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}.$$

  - Curl of gradient (always zero) $\nabla \times (\nabla T) = \mathbf{0}$
  - Gradient of divergence (nothing special) $\nabla(\nabla \cdot \mathbf{v})$
  - Divergence of curl (always zero) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$
  - Curl of curl $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$

    where $\nabla^2 \mathbf{v} \equiv (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}$

Integral calculus

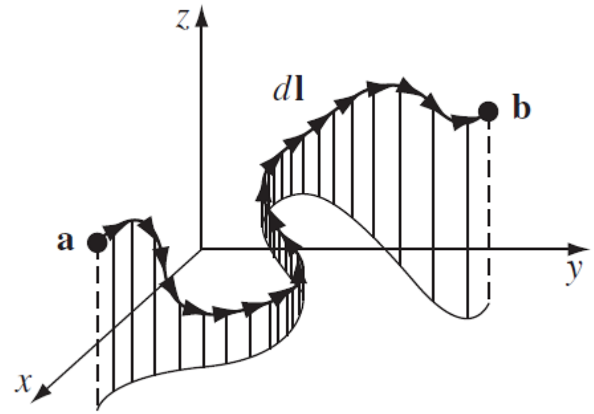
- Line integral

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l}$$

From point a to point b

Vector field

Infinitesimal displacement vector



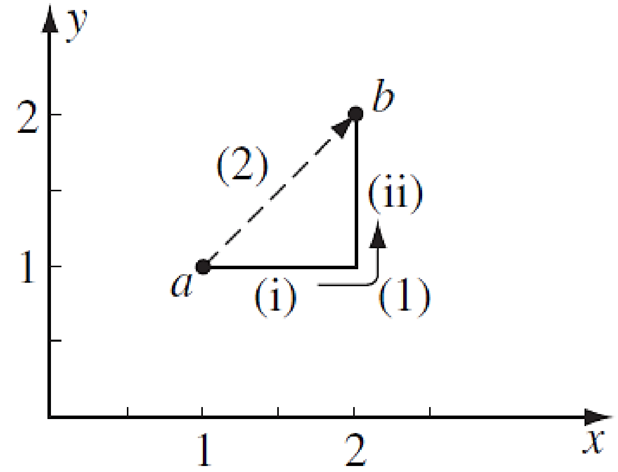
  - If path forms a closed loop, denote as $\oint \mathbf{v} \cdot d\mathbf{l}$
  - Generally line integrals depend on the path taken from a to b
  - But sometimes it can be independent from the path chosen as long as points a and b are fixed, for certain $\mathbf{v}$
  - 2D example: $\mathbf{v}(x, y) = v_x(x, y)\hat{\mathbf{x}} + v_y(x, y)\hat{\mathbf{y}}$

$$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}}$$

Integral calculus

- Line integral

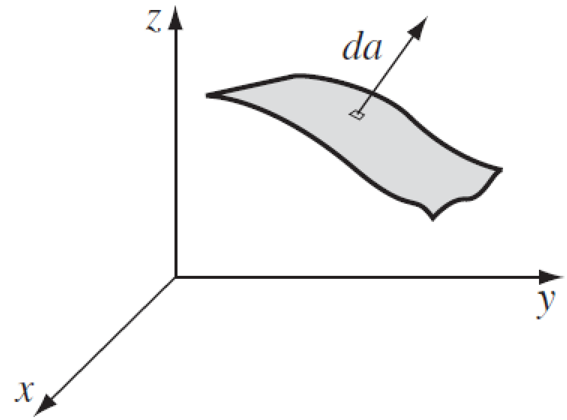
Example 1.6. Calculate the line integral of the function $\mathbf{v} = y^2\,\hat{\mathbf{x}} + 2x(y+1)\,\hat{\mathbf{y}}$ from the point $\mathbf{a} = (1, 1, 0)$ to the point $\mathbf{b} = (2, 2, 0)$, along the paths (1) and (2) in Fig. 1.21. What is $\oint \mathbf{v} \cdot d\mathbf{l}$ for the loop that goes from $\mathbf{a}$ to $\mathbf{b}$ along (1) and returns to $\mathbf{a}$ along (2)?



Integral calculus

- Surface integral

$$\underbrace{\int_S}_{\text{The specified surface}} \underbrace{\mathbf{v}}_{\text{Vector field}} \cdot \underbrace{d\mathbf{a}}_{\text{Infinitesimal patch of area}}$$

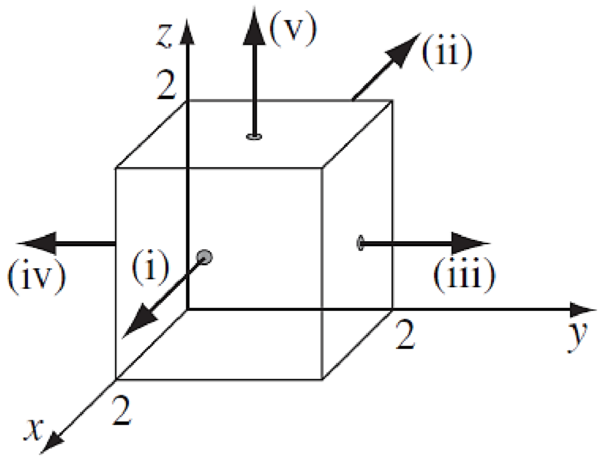


  - If surface is closed, denote as $\oint \mathbf{v} \cdot d\mathbf{a}$
  - Orientation of surface normal is intrinsically ambiguous
  - But when surface is closed, take "outward" direction as positive
  - Sometimes the integral is independent of surface chosen and is determined by the boundary line, for certain $\mathbf{v}$

Integral calculus

- Surface integral

Example 1.7. Calculate the surface integral of $\mathbf{v} = 2xz\,\hat{\mathbf{x}} + (x+2)\,\hat{\mathbf{y}} + y(z^2-3)\,\hat{\mathbf{z}}$ over five sides (excluding the bottom) of the cubical box (side 2) in Fig. 1.23. Let "upward and outward" be the positive direction, as indicated by the arrows.

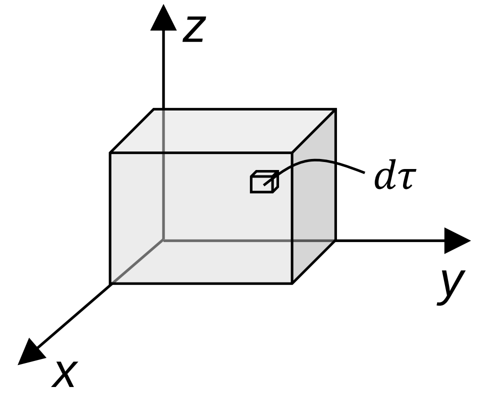



C1.exmp1.7

Integral calculus

- Volume integral

$$\underbrace{\int_{\mathcal{V}}}_{\text{The specified volume}} \underbrace{T}_{\text{Scalar field}} \, \underbrace{d\tau}_{\text{Infinitesimal volume element}}$$



  - In Cartesian coordinates $d\tau = dx\, dy\, dz$

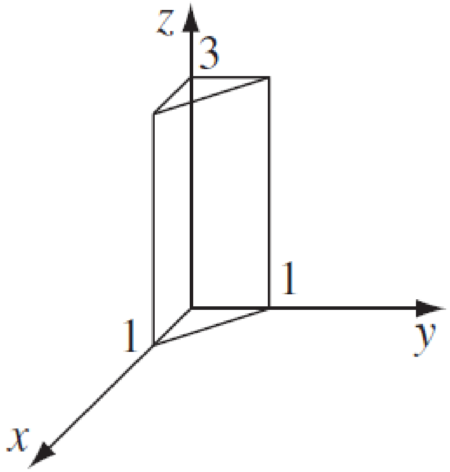
  - Integrand is usually scalar fields, but can also be vector fields (trivial extension)

$$\int \mathbf{v}\, d\tau = \int (v_x\, \hat{\mathbf{x}} + v_y\, \hat{\mathbf{y}} + v_z\, \hat{\mathbf{z}}) d\tau = \hat{\mathbf{x}} \int v_x d\tau + \hat{\mathbf{y}} \int v_y d\tau + \hat{\mathbf{z}} \int v_z d\tau$$

Integral calculus

- Volume integral

Example 1.8. Calculate the volume integral of $T = xyz^2$ over the prism in Fig. 1.24.



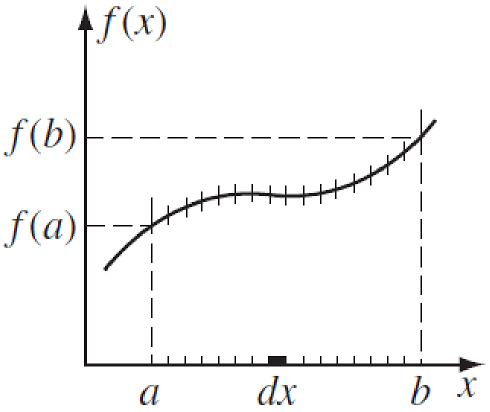
C1.exmp1.8

Integral calculus

- Fundamental theorem of calculus

$$\int_a^b \underline{F(x)}\, dx = f(b) - f(a)$$

where $df = \underline{(df/dx)}dx$



  - $F(x)dx$: infinitesimal change to function f when x changes to $x + dx$

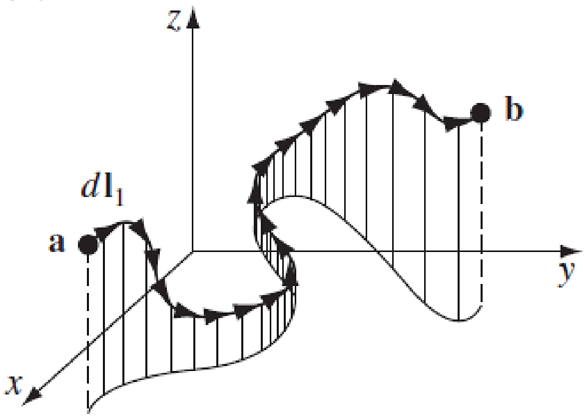
  - **Integral of a derivative over some region is given by the value of the function at the boundaries (end points)**

  - The format above appears for **all three types of derivatives** when attempting to integrate them

Integral calculus

- Fundamental theorem for gradients

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$



  - Segment by segment

$$dT = (\nabla T) \cdot d\mathbf{l}_1$$
$$dT = (\nabla T) \cdot d\mathbf{l}_2$$
$$\vdots$$

  - As long as end points are fixed, integration of a gradient is independent of the path
    - $\displaystyle \int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$
      any path
    - $\oint (\nabla T) \cdot d\mathbf{l} = 0$

Integral calculus

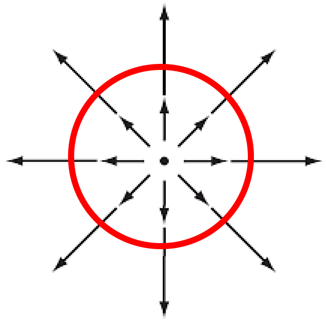
- Fundamental theorem for divergences

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$$

  - Volume integral of divergence equals a integral of the field with respect to the surface (that encloses the volume)

  - Named Gauss's theorem, Green's theorem, or divergence theorem

  - Two ways of knowing how many sources are there
    - Count the "faucets" $\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau$
    - Measure how much "water" flows out through the boundary $\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$

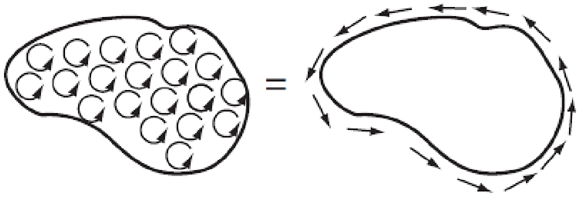


Integral calculus

- Fundamental theorem for curls

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

  - Surface integral of curl equals a integral of the field with respect to the boundary line
  - As long as the boundary line remains the same, integration of curl is independent of the choice of surface
  - Named Stokes' theorem
  - Two ways of knowing how many swirls are there
    - Count the swirls $\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$
    - Measure how much flow is following the boundary $\oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$



Integral calculus

- Integration by parts

$$\int_a^b f\left(\frac{dg}{dx}\right)dx = -\int_a^b g\left(\frac{df}{dx}\right)dx + fg\Big|_a^b$$

  - Can transfer the derivative from g to f, at the cost of a minus sign and a boundary term

- Integration by parts for gradient and curl

$$\int_{\mathcal{V}} f(\nabla\cdot\mathbf{A})\,d\tau = -\int_{\mathcal{V}} \mathbf{A}\cdot(\nabla f)\,d\tau + \oint_{\mathcal{S}} f\mathbf{A}\cdot d\mathbf{a}$$

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\int_{\mathcal{S}} f(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \int_{\mathcal{S}} [\mathbf{A}\times(\nabla f)]\cdot d\mathbf{a} + \oint_{\mathcal{P}} f\mathbf{A}\cdot d\mathbf{l}$$

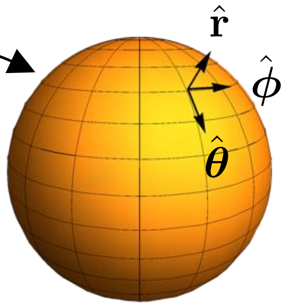
$$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$$

$$\int_{\mathcal{V}} \mathbf{B}\cdot(\nabla\times\mathbf{A})\,d\tau = \int_{\mathcal{V}} \mathbf{A}\cdot(\nabla\times\mathbf{B})\,d\tau + \oint_{\mathcal{S}} (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

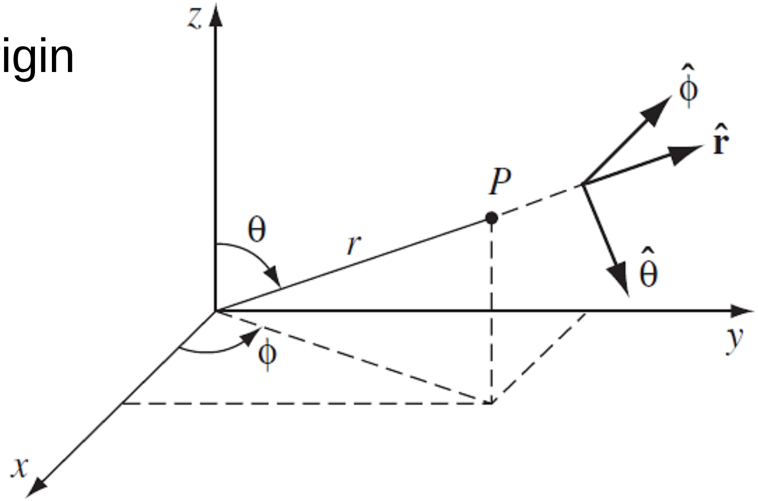
Curvilinear coordinates

- Curvilinear coordinates
  - Coordinate systems where the coordinate lines may be curved, and can be more convenient to use than Cartesian coordinates for problems adopting certain geometries



- Spherical coordinates
  - Represent a point by (r, θ, ϕ) instead of (x, y, z)
    - r: distance of the point from the origin
    - θ: angle down from the $+z$ axis
    - ϕ: azimuthal angle

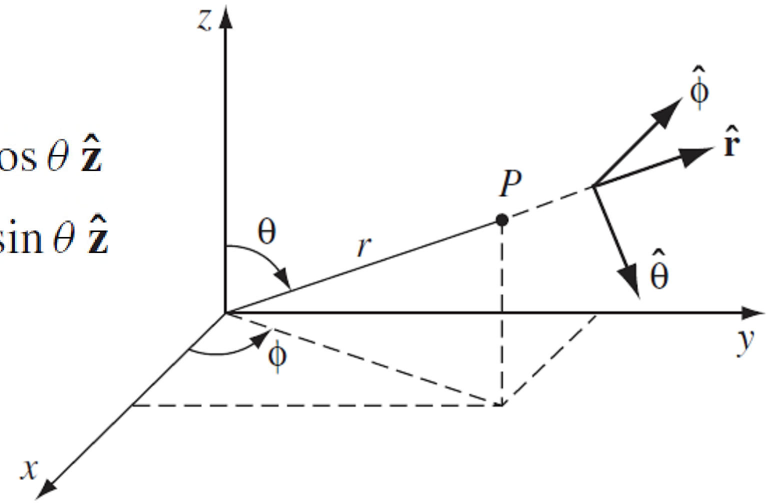
$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$



Curvilinear coordinates

- Spherical coordinates
  - Unit vectors

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$



    - Mutually orthogonal
    - a vector can be expressed as

$$\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}}$$

    - Unit vectors depend on position of the point of interest

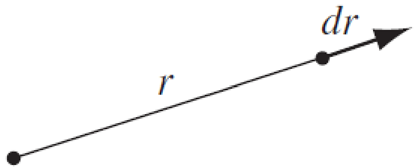
$$\hat{\boldsymbol{r}}(\theta,\phi) \qquad \hat{\boldsymbol{\theta}}(\theta,\phi) \qquad \hat{\boldsymbol{\phi}}(\theta,\phi)$$

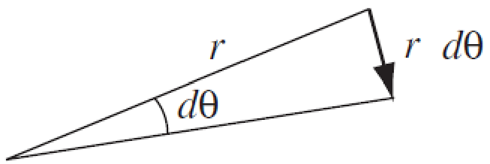
Example: charges on a sphere interacting with an electric field

Curvilinear coordinates

- Spherical coordinates
  - Infinitesimal element of length

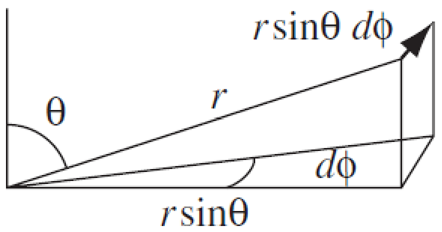
$$\begin{cases} dl_r = dr \\ dl_\theta = r\, d\theta \\ dl_\phi = r \sin\theta\, d\phi \end{cases}$$





  - Infinitesimal displacement vector

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$



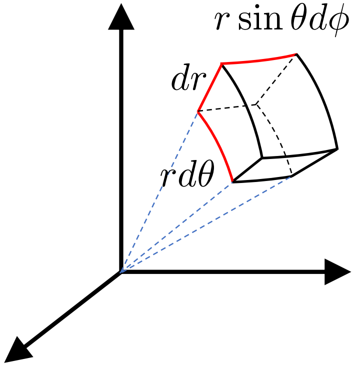
  - Infinitesimal element of volume

$$d\tau = dl_r\, dl_\theta\, dl_\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

  - Infinitesimal element of surface
    - Geometry dependent

      e.g. on a sphere

$$d\mathbf{a}_1 = dl_\theta\, dl_\phi\,\hat{\mathbf{r}} = r^2 \sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}}$$



Curvilinear coordinates

- Spherical coordinates
  - Derivatives
    - Gradient $$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$
    - Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$
    - Curl

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.$$

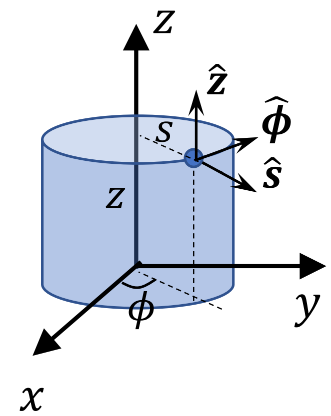
C1.spherical

    - Laplacian

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

Curvilinear coordinates

- Cylindrical coordinates
  - Coordinates
    - s: distance of the point to the z axis
    - ϕ: azimuthal angle
    - z: same as the z in Cartesian coordinates



  - Unit vectors

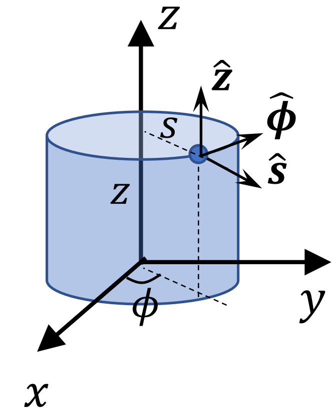
$$\begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

- Mutually orthogonal, just like all other coordinates

Curvilinear coordinates

- Cylindrical coordinates
  - Infinitesimal element of length

$$\begin{cases} dl_s = ds \\ dl_\phi = s\, d\phi \\ dl_z = dz \end{cases}$$



  - Infinitesimal displacement vector

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

  - Infinitesimal element of volume

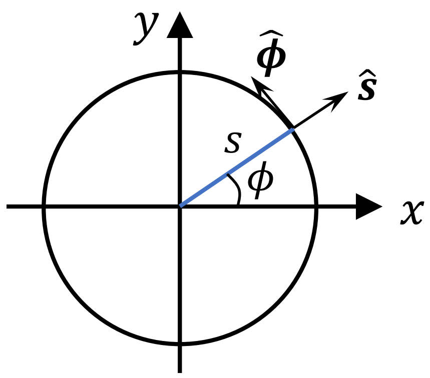
$$d\tau = s\,ds\,d\phi\,dz$$

  - Derivatives
    - Gradient $$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$
    - Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curvilinear coordinates

- Polar coordinates (in 2D)

  - Spherical coordinates without the θ, or cylindrical coordinates without the z



  - Unit vectors

$$\begin{aligned} \hat{\mathbf{s}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{aligned}$$

  - Infinitesimal element of length

$$\begin{cases} dl_s = ds \\ dl_\phi = s\,d\phi \end{cases}$$

  - Infinitesimal displacement vector $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}}$

  - Infinitesimal element of area $da = s\,ds\,d\phi$