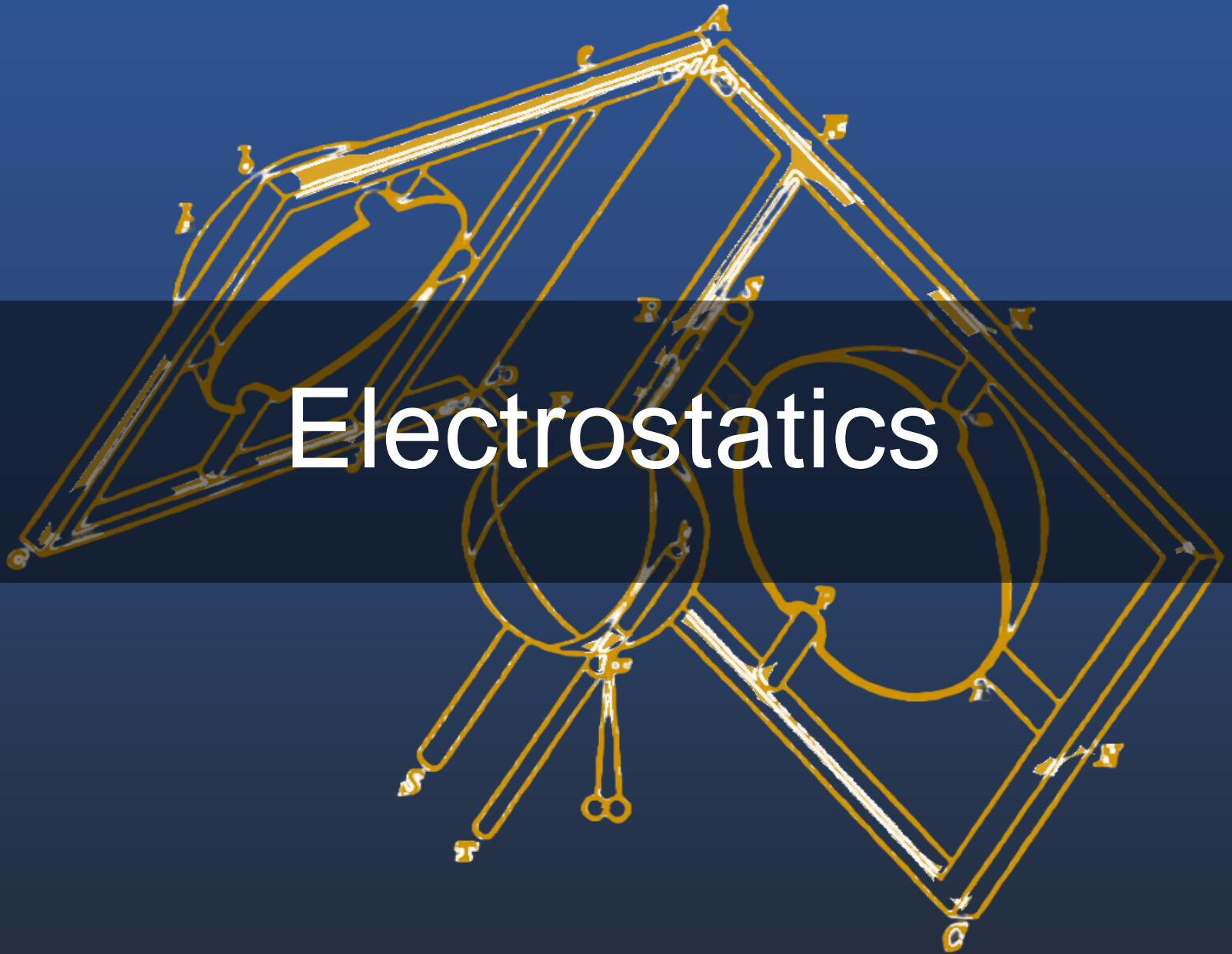


Electrostatics



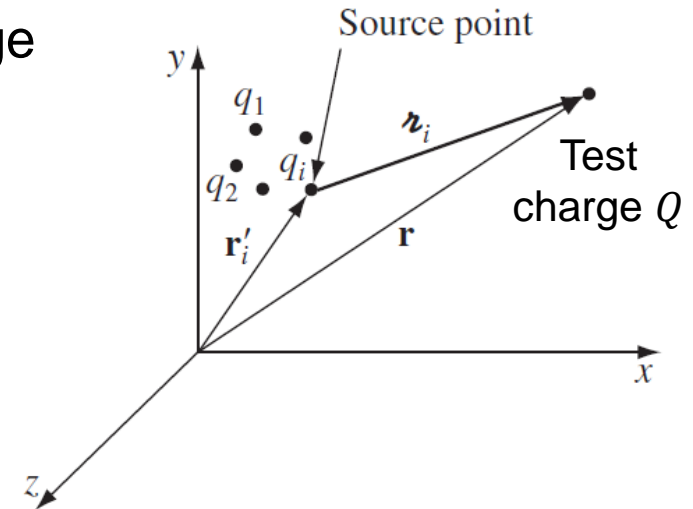
Charge, electric field, and potential

Coulomb's law

- Force of n source charges on a test charge

- Force from source charge q_i acting on test charge Q

- Coulomb's law
$$F_i = \frac{1}{4\pi\epsilon_0} \frac{q_i Q}{r_i^2} \hat{\mathbf{r}}_i$$
- Permittivity $\epsilon_0 = 8.85 \times 10^{12} \text{ C}^2/(\text{N m}^2)$
- Separation vector $\mathbf{r}_i = \mathbf{r} - \mathbf{r}'_i$
- Location of Q : \mathbf{r} , location of q_i : \mathbf{r}'_i

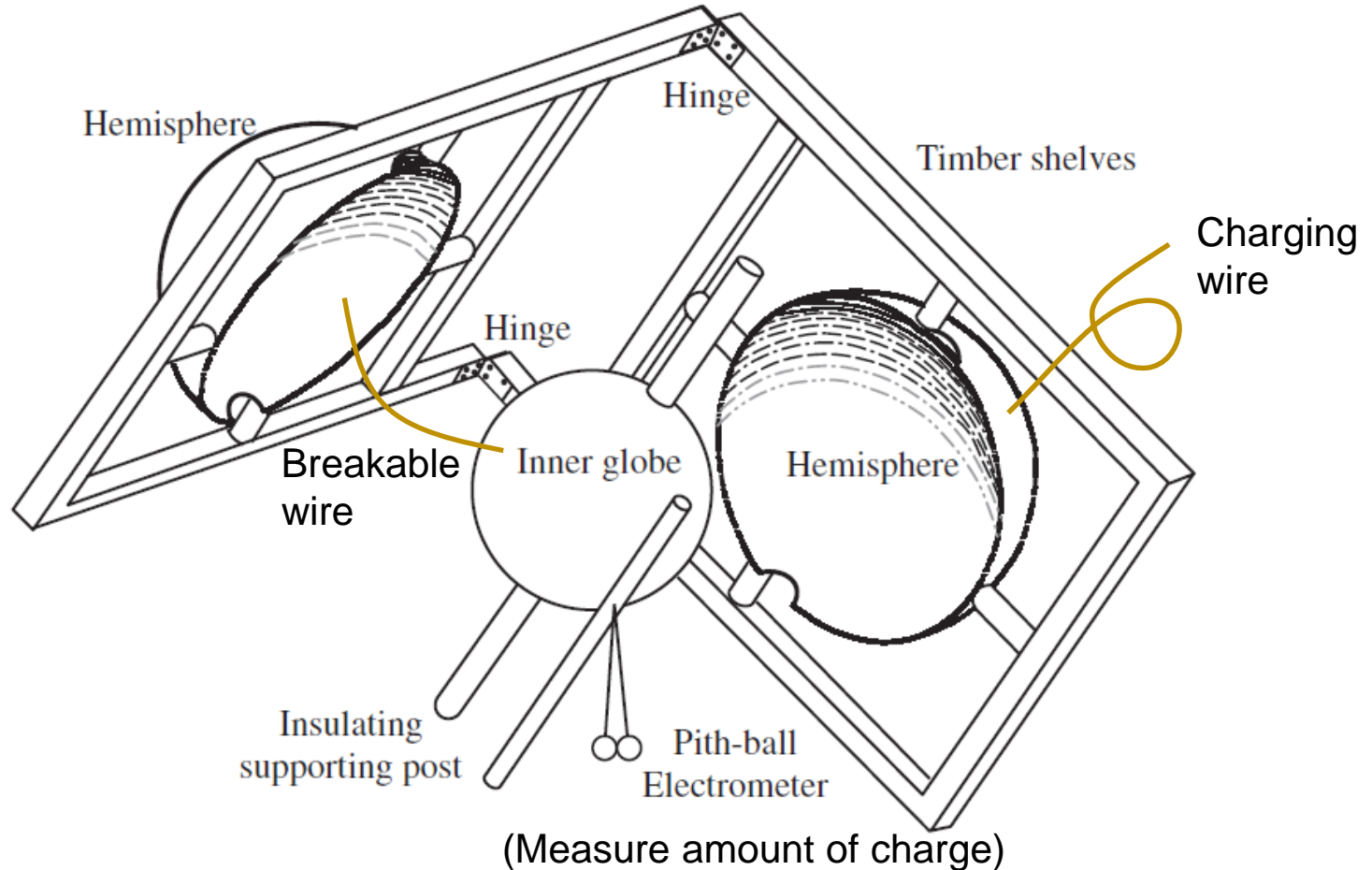


- Principle of superposition

- Total force acting on test charge
$$F = \sum_{i=1}^n F_i$$
- Not a necessity, but an experimental fact

* \mathbf{r} in textbook is typed as \mathbf{r} in our slides (Cursive “r”)

Coulomb's law



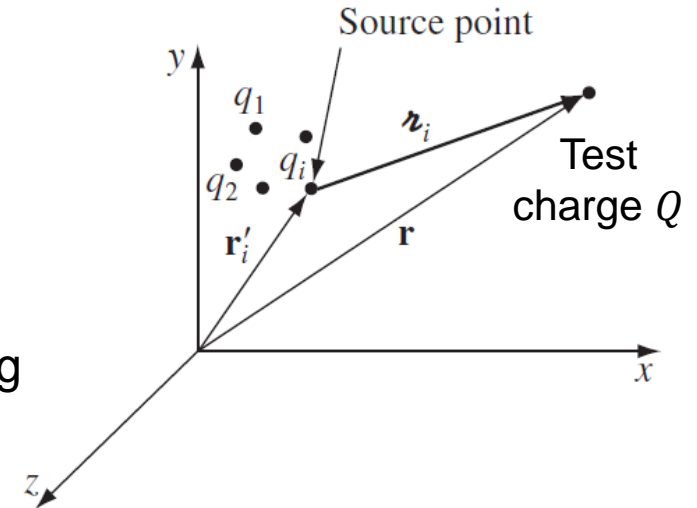
Cavendish's apparatus for determining $F \propto r^{-2}$ in Coulomb's law

Electric field induced by charge

- Relation of force and electric field

$$\mathbf{F} = Q\mathbf{E}$$

- Electric field: force per unit charge
- Real physical entity, as a vector field filling the space around charges
- Negated theory of “ether”



- Electric field induced by discrete charges

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i$$

- Separation vector $\mathbf{r}_i = \mathbf{r} - \mathbf{r}'_i$, contains \mathbf{r}
- Principle of superposition also holds

Electric field induced by charge

- Electric field induced by continuous charge distribution

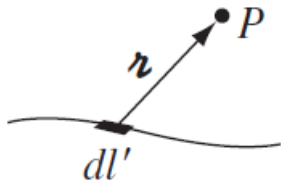
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r^2} \hat{\mathbf{r}} dq$$

- Add up contributions from infinitesimal charge elements dq
- Three ways dq can be distributed

Line charge

$$dq \rightarrow \lambda dl'$$

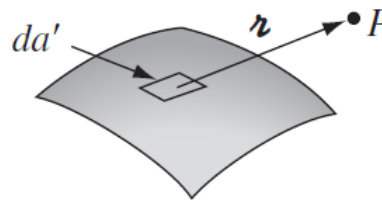
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\mathbf{r}')}{r^2} \hat{\mathbf{r}} dl'$$



Surface charge

$$dq \rightarrow \sigma da'$$

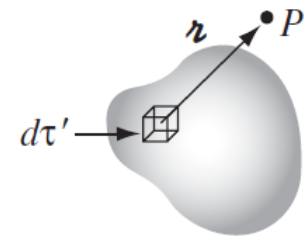
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{r^2} \hat{\mathbf{r}} da'$$



Volume charge

$$dq \rightarrow \rho d\tau'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} d\tau'$$

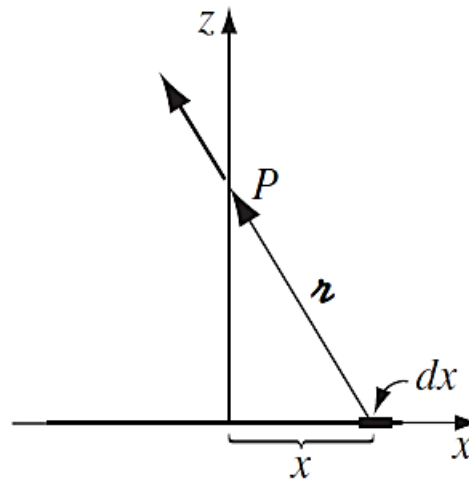


* λ , σ , ρ : charge per unit length, area, volume

Electric field induced by charge

- Electric field induced by continuous charge distribution

Example 2.2. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ (Fig. 2.6).



- Integration sometimes can get formidable, need to device new tools to simplify problems.

Gauss's law

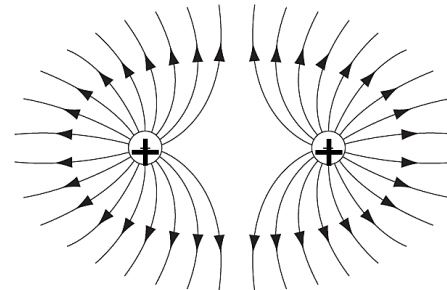
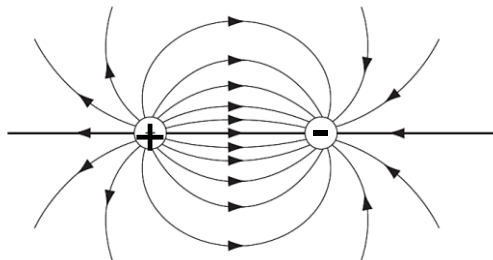
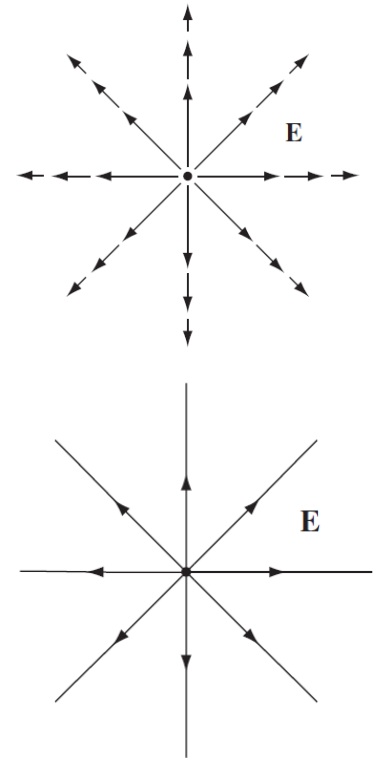
- Electric field lines

- Source charge q at the origin

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

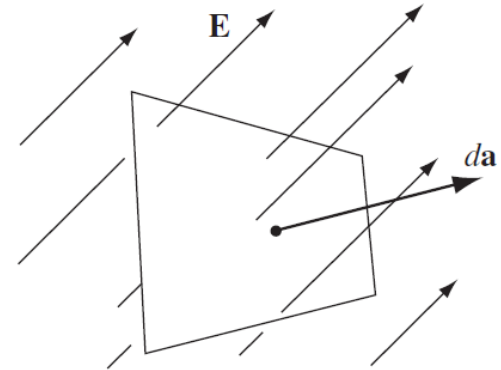
- Draw vector field – field falls off like $1/r^2$
- Connect up the arrows – electric field lines
 - Direction of line indicates field direction
 - Density of line indicates field magnitude

- Field lines begin from positive charges and end on negative ones



Gauss's law

- Electric field flux $\Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$
 - A measure of the number of field lines passing through an area



- Gauss's law
 - The flux through any closed surface is a measure of the total charge inside

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r^2} \hat{\mathbf{r}} \right) \cdot \underbrace{(r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}})}_{d\mathbf{a}} = \frac{1}{\epsilon_0} q$$

↑
↑
 Spherical surface of radius r

- The surface integral can be any shape, not necessarily spherical

- Multiple charges $\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^n \left(\oint \mathbf{E}_i \cdot d\mathbf{a} \right) = \sum_{i=1}^n \left(\frac{1}{\epsilon_0} q_i \right)$

➔

 $\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$
 (Q_{enc} : total charge enclosed in the integrated surface)

Gauss's law

- Gauss's law
 - Gauss's law in the differential form

$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$ (integral form)

Divergence theorem \swarrow \searrow Consider volume distribution

$\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{E}) d\tau$ $Q_{\text{enc}} = \int_V \rho d\tau$

$\Rightarrow \int_V (\nabla \cdot \mathbf{E}) d\tau = \int_V \left(\frac{\rho}{\epsilon_0} \right) d\tau$

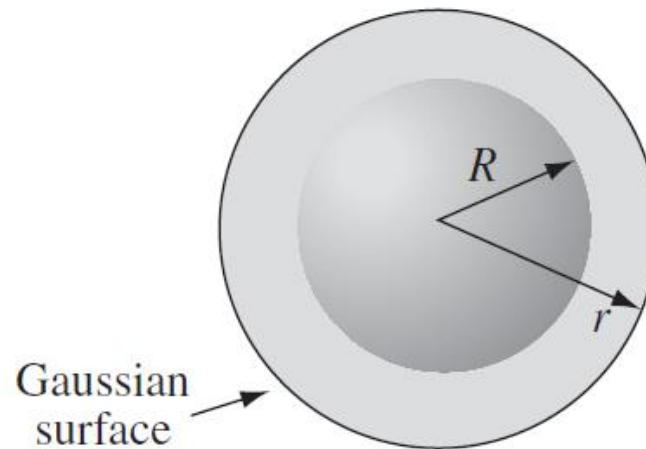
$\Rightarrow \boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho}$ (differential form)

- Differential form more compact, but integral form easier to use
- Use of Gauss's law to calculate electric field
 - Need (1) Gauss's law in integral form and (2) symmetry arguments

Gauss's law

- Application of Gauss's law

Example 2.3. Find the field outside a uniformly charged solid sphere of radius R and total charge q .



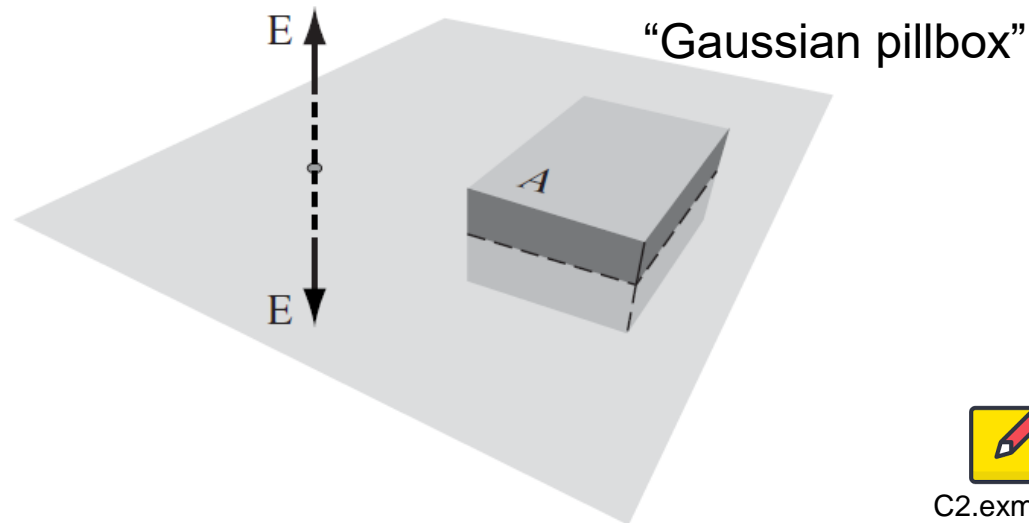
C2.exmp2.3

- The field outside the sphere is exactly the same as it would have been if all the charge had been concentrated at the center

Gauss's law

- Application of Gauss's law

Example 2.5. An infinite plane carries a uniform surface charge σ . Find its electric field.



C2.exmp2.5

Divergence of electric field

- Directly calculate divergence
 - According to Coulomb's law

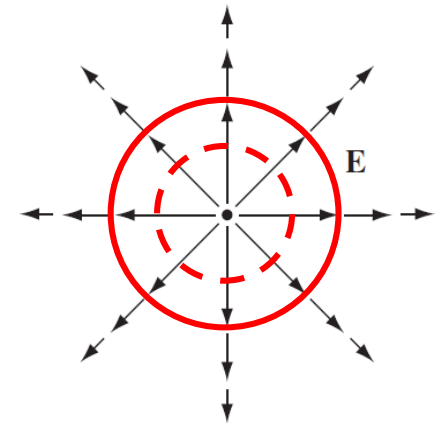
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\mathbf{r}}}{r^2} \rho(\mathbf{r}') d\tau'$$

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) \rho(\mathbf{r}') d\tau'$$

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{1}{r^2} \right) = 0$$

- The derivation above is correct anywhere but the origin ($r = 0$), where the divergence should go to infinity
 - Consider special case of point charge and Gauss's law with varying volume to integrate

? This seems to contradict the Gauss's law, what went wrong

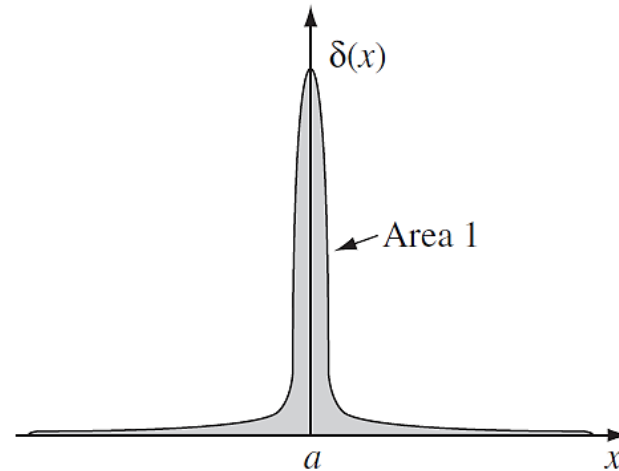


Divergence of electric field

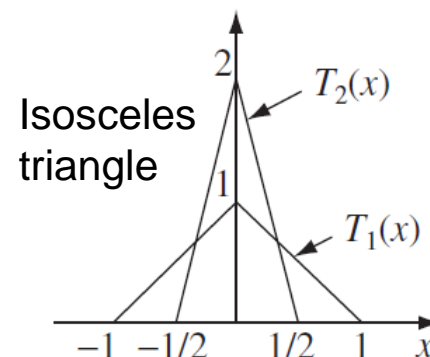
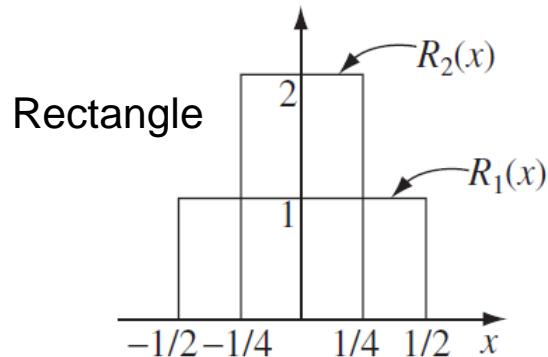
- Delta function
 - Infinitely high, infinitesimally narrow
 - 1D Delta function

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

$$\text{with } \int_{-\infty}^{\infty} \delta(x) dx = 1$$



- Can be understood as the limit of a sequence of functions



Divergence of electric field

- Delta function

- 1D Delta function

- When in an integral, “picks out” the value of a function

Since $\delta(x)$ anywhere 0 but at $x = 0$

$$f(x)\delta(x) = f(0)\delta(x)$$

[$f(x)$ being an ordinary function not going to infinity]

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\delta(x) dx = f(0) \int_{-\infty}^{\infty} \delta(x) dx = f(0)$$

And, one can shift $\delta(x)$ to $\delta(x - a)$ to pick out another one

$$f(x)\delta(x - a) = f(a)\delta(x - a)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\delta(x - a) dx = f(a)$$

- A frequently used expression $\delta(kx) = \frac{1}{|k|}\delta(x)$

Divergence of electric field

- Delta function

- 3D Delta function $\delta^3(\mathbf{r}) = \delta(x) \delta(y) \delta(z)$

with $\int_{\text{all space}} \delta^3(\mathbf{r}) d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x) \delta(y) \delta(z) dx dy dz = 1$

- Picks out a function value $\int_{\text{all space}} f(\mathbf{r}) \delta^3(\mathbf{r} - \mathbf{a}) d\tau = f(\mathbf{a})$

- Back to calculating divergence of electric field

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) \rho(\mathbf{r}') d\tau'$$

$$\downarrow \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r})$$



$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi \delta^3(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\tau' = \frac{1}{\epsilon_0} \rho(\mathbf{r})$$

Gauss's law
recovered

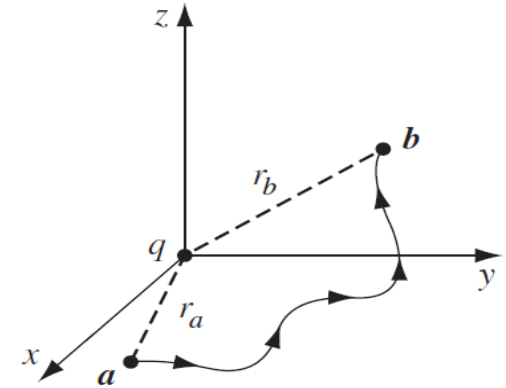
Curl of electric field

- Calculate curl for point charge at origin

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin\theta d\phi \hat{\boldsymbol{\phi}}$$

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{q}{r^2} dr = \left. \frac{-1}{4\pi\epsilon_0} \frac{q}{r} \right|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right)$$



- For any closed loop ($r_a = r_b$) $\oint \mathbf{E} \cdot d\mathbf{l} = 0$

➡ $\nabla \times \mathbf{E} = \mathbf{0}$ due to Stoke's theorem

Stoke's theorem

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

- Any static charge distribution

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \dots) = (\nabla \times \mathbf{E}_1) + (\nabla \times \mathbf{E}_2) + \dots = \mathbf{0}$$

Electric potential

- Vector field \mathbf{E} cannot take arbitrary form

- Crucial constraint: $\nabla \times \mathbf{E} = \mathbf{0}$

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \quad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$



Any chance the vector field can be described more easily?

- Electric potential: $V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$

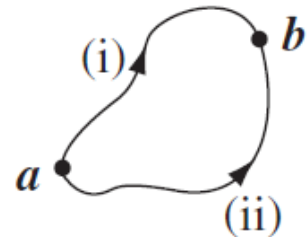
- Unit: joules per coulomb
- \mathcal{O} : a reference point (usually taken as infinity)
- Integral does not depend on path

- $\nabla \times \mathbf{E} = \mathbf{0}$

- $\oint \mathbf{E} \cdot d\mathbf{l} = 0$

- $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ is path independent

} Equivalent statements



Electric potential

- Electric potential: $V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$
 - Potential difference between two points is more meaningful

$$\begin{aligned} V(\mathbf{b}) - V(\mathbf{a}) &= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l} \\ &= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} \end{aligned}$$

On the other hand, the theorem for gradient gives

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} \quad \Rightarrow \quad \boxed{\mathbf{E} = -\nabla V}$$

- Scalar field V gives full information of vector field \mathbf{E}
- Can be off by a constant if choosing a different reference point

$$V'(\mathbf{r}) = -\int_{\mathcal{O}'}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{O}'}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = K + V(\mathbf{r})$$

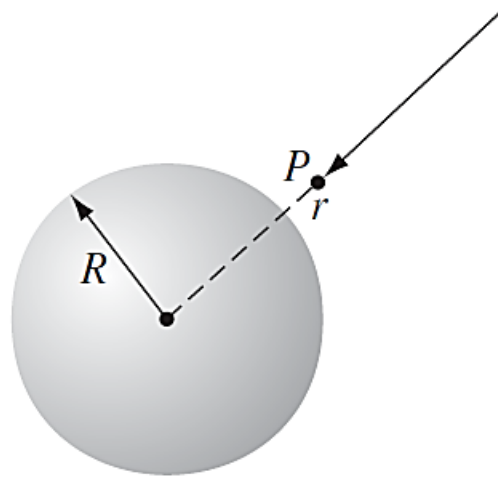
new \rightarrow

Electric potential

- Application of electric potential

Example. Find the potential of a point charge q at origin

Example 2.7. Find the potential inside and outside a spherical shell of radius R (Fig. 2.31) that carries a uniform surface charge. Set the reference point at infinity.



C2.exmp2.7

Electric potential

- Poisson's equation of potential

- Poisson's equation $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

- In regions with no charge, Laplace's equation $\nabla^2 V = 0$

- Curl of a gradient always zero $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = \mathbf{0}$

- Potential of a localized charge distribution

- Pick infinity as the reference point $\mathcal{O} = \infty$

- Principle of superposition holds $V = V_1 + V_2 + \dots$

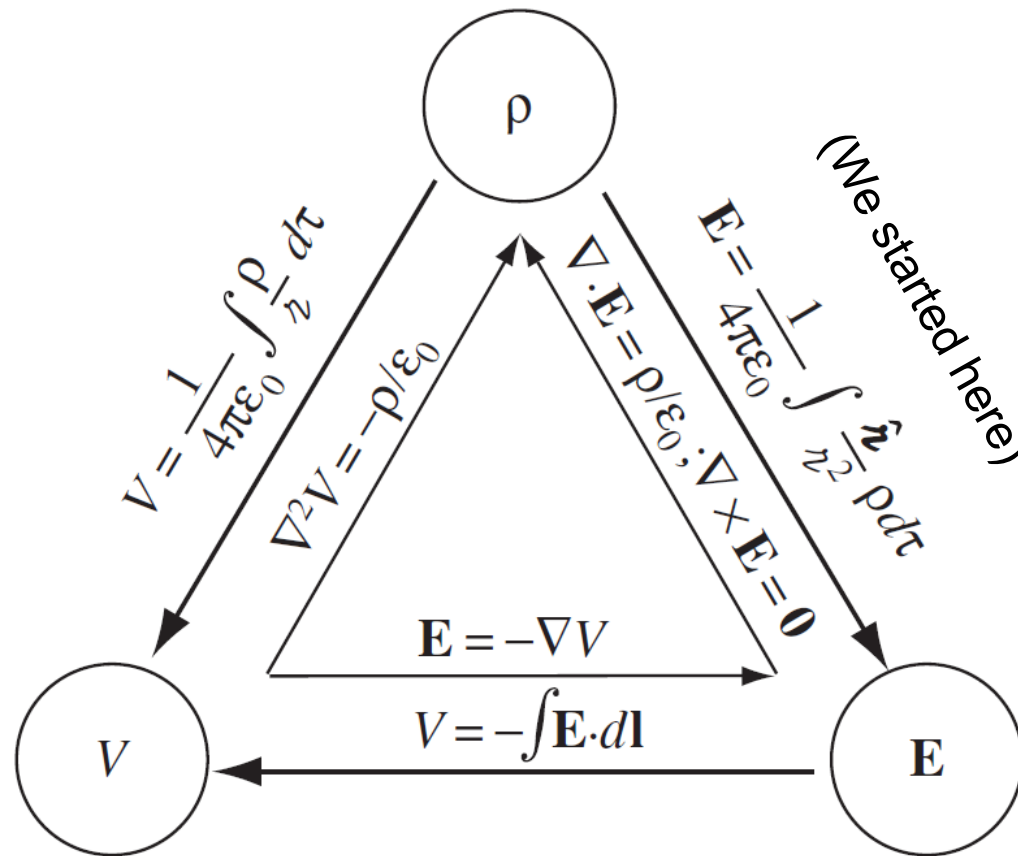
- Discrete charges $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i}$

- Continuous charge $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} d\tau'$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} d\tau'$$

Can check

Charge, electric field, and potential



Differential equations need boundary conditions to solve

Boundary conditions

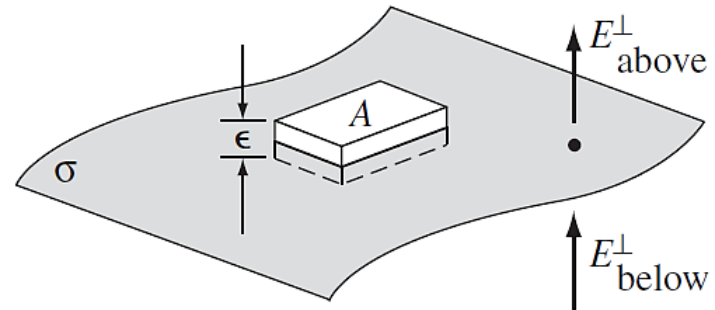
- Boundary conditions of \mathbf{E} across a 2D charged surface

- Normal component of \mathbf{E}

“Gaussian pillbox” with $\varepsilon \rightarrow 0$

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \sigma A$$

$$\Rightarrow E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = \frac{1}{\epsilon_0} \sigma$$

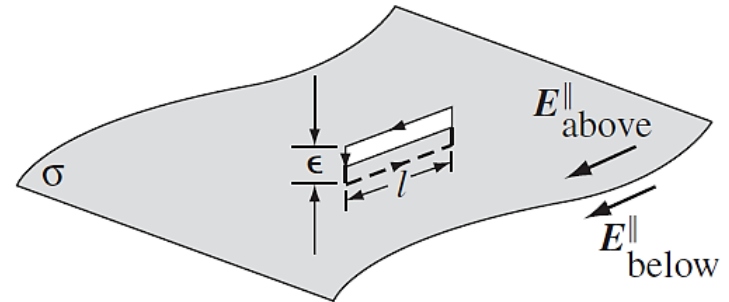


- Tangential component of \mathbf{E}

Thin loop with $\varepsilon \rightarrow 0$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\Rightarrow \mathbf{E}_{\text{above}}^{\parallel} = \mathbf{E}_{\text{below}}^{\parallel}$$



- Summarizing above $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$

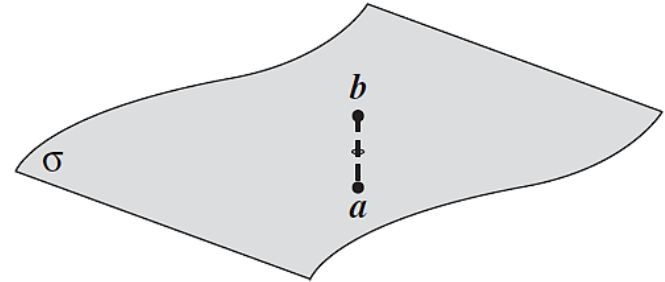
Boundary conditions

- Boundary conditions of V across a 2D charged surface
 - Potential is continuous (across any boundary)

$$V_{\text{above}} - V_{\text{below}} = - \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

↓
Path length $\rightarrow 0$

$V_{\text{above}} = V_{\text{below}}$



- Gradient of potential is discontinuous

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$$

↓
 $\mathbf{E} = -\nabla V$

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0} \sigma \hat{\mathbf{n}}$$

➡

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0} \sigma$$

where we define normal derivative of V

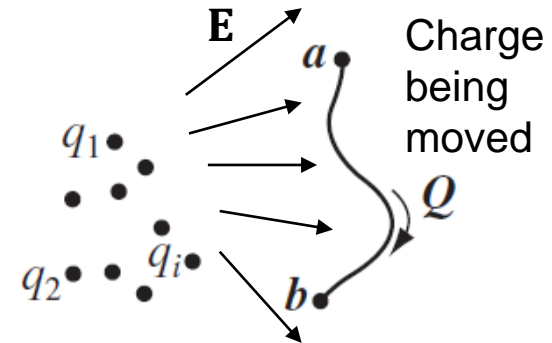
$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

Energy in electrostatics

- Work done to move a charge

- Integrate force over distance

$$\begin{aligned} W &= \int_a^b \mathbf{F} \cdot d\mathbf{l} = -Q \int_a^b \mathbf{E} \cdot d\mathbf{l} \\ &= Q[V(\mathbf{b}) - V(\mathbf{a})] \end{aligned}$$



- Electrostatic force is conservative (path independent)
- Can confirm the unit of electric potential
- Work for bringing from infinitely far to \mathbf{r}

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

$$W = QV(\mathbf{r}) \quad \text{with the potential reference point set to infinity}$$

Energy in electrostatics

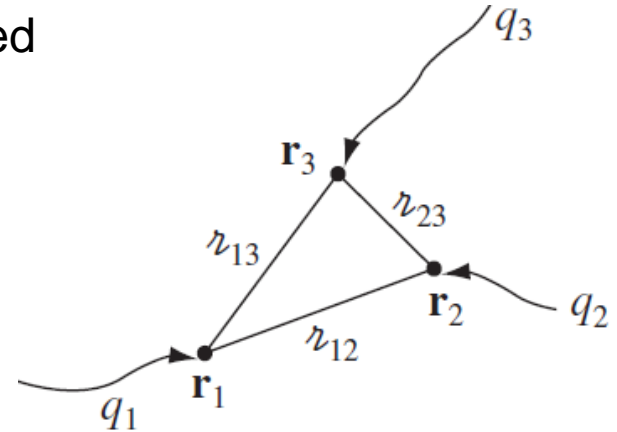
- Energy of a point charge configuration
 - Equals to the work required to bring charges together from infinity

- First charge q_1 to \mathbf{r}_1 , no work required

- q_2 to \mathbf{r}_2 $W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{r_{12}} \right)$

- q_3 to \mathbf{r}_3 $W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right)$

- $W = W_1 + W_2 + W_3$



- Total work (energy) for n charges

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \sum_{j>i}^n \frac{q_i q_j}{r_{ij}} = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^n \sum_{j \neq i}^n \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^n q_i \left(\underbrace{\sum_{j \neq i}^n \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}}}_{V(\mathbf{r}_i)} \right)$$

Count once for each pair

➡ $W = \frac{1}{2} \sum_{i=1}^n q_i V(\mathbf{r}_i)$

Potential q_i feels due to all **other** charges

Energy in electrostatics

- Energy of a continuous charge distribution
 - Generalize point charge equation to

$$W = \frac{1}{2} \int \rho V d\tau$$

with V : actual potential, without
excluding the charge of interest

$$\downarrow \quad \rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

$$W = \frac{\epsilon_0}{2} \int (\nabla \cdot \mathbf{E}) V d\tau$$

$$\downarrow \quad \text{Integrate by parts} \quad \int_V f(\nabla \cdot \mathbf{A}) d\tau = - \int_V \mathbf{A} \cdot (\nabla f) d\tau + \oint_S f \mathbf{A} \cdot d\mathbf{a}$$

$$W = \frac{\epsilon_0}{2} \left[- \int \mathbf{E} \cdot (\nabla V) d\tau + \oint V \mathbf{E} \cdot d\mathbf{a} \right] = \frac{\epsilon_0}{2} \left(\int_V E^2 d\tau + \underbrace{\oint_S V \mathbf{E} \cdot d\mathbf{a}}_{\text{Vanishes when } \mathcal{V} \rightarrow \infty} \right)$$

$$\Rightarrow \boxed{W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau}$$

- Cannot be directly compared to equation of point charge, see textbook

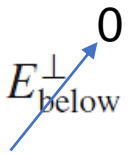
Conductor electrostatics

- Conductors
 - Free electrons – solid-state metals and doped semiconductors
 - Free ions – Electrolyte, salt water, lithium ion battery
 - Unlimited supply of free charges, which are free to move
- Electrostatics of perfect conductors
 - $\mathbf{E} = 0$ inside a conductor
 - If not, charge will flow to induce a new surface charge distribution that exactly cancels the internal field
 - $\rho = 0$ (net charge volume density) inside a conductor
 - Because $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$
 - A conductor is an equipotential
 - For any two points, $V(\mathbf{b}) - V(\mathbf{a}) = - \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$

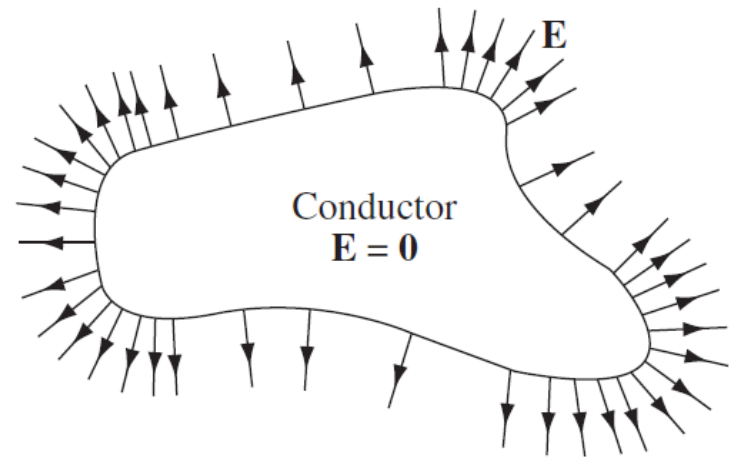
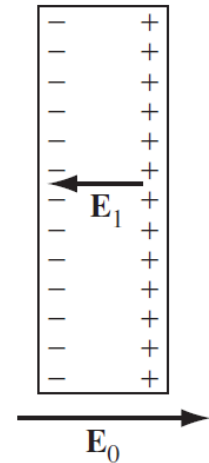
Conductor electrostatics

- Electrostatics of perfect conductors
 - Any net charge only resides on the surface (minimizes energy)
 - Surface net charges serves to cancel the internal field
 - \mathbf{E} is always perpendicular to the surface, just outside the conductor
 - Recall boundary conditions

$$E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = \frac{1}{\epsilon_0} \sigma$$



$$\mathbf{E}_{\text{above}}^{\parallel} = \mathbf{E}_{\text{below}}^{\parallel} = 0$$

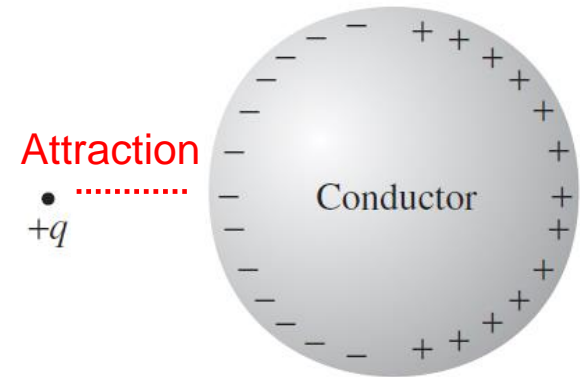


Conductor electrostatics

- Induced charges

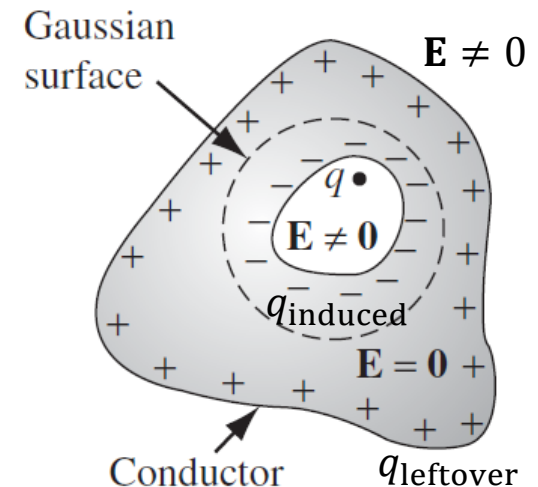
- Charge placed outside a metal

- Induced charge serves to cancel field inside conductor
- Net force of attraction



- Charge in the cavity of a hollow metal

- Inside the cavity: $\mathbf{E} \neq 0$
- Induced charge $q_{\text{induced}} = -q$ at inner wall
- Inside the conductor: $\mathbf{E} = 0$
- Leftover charge $q_{\text{leftover}} = q$ at outer wall
- Outside the conductor: $\mathbf{E} \neq 0$



Conductor electrostatics

- Induced charges

- Faraday cage

- If no charge is placed in the cavity of a hollow conductor, $\mathbf{E} = 0$ in the cavity regardless of the outside conditions

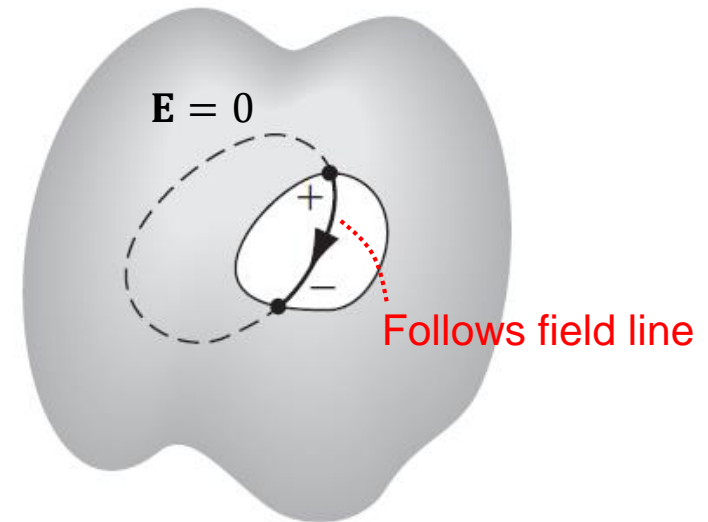
If not, can construct a loop of integration, whose trajectory in the cavity follows the field line

➡ $\oint \mathbf{E} \cdot d\mathbf{l} \neq 0$

➡ Contradicts $\nabla \times \mathbf{E} = 0$

➡ $\mathbf{E} = 0$

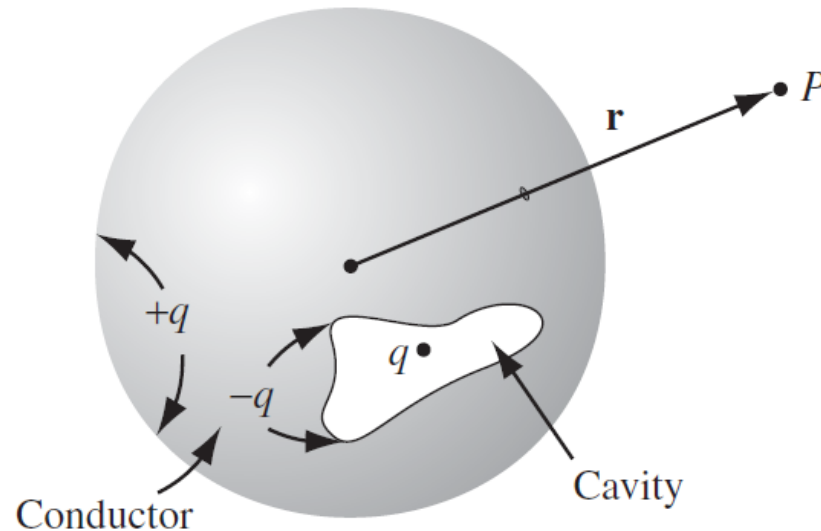
- Protects sensitive apparatus inside the cavity by shielding out external electric fields



Conductor electrostatics

- Induced charges

Example 2.10. An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it (Fig. 2.46). Somewhere within the cavity is a charge q . *Question:* What is the field outside the sphere?



Conductor electrostatics

- Surface charge and force on a conductor
 - Boundary conditions

$$\left\{ \begin{array}{l} \mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \\ \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0} \sigma \end{array} \right. \xrightarrow{\text{On the surface of a perfect conductor}} \left\{ \begin{array}{l} \mathbf{E} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \\ \sigma = -\epsilon_0 \frac{\partial V}{\partial n} \end{array} \right.$$

- Force (per unit area) exerted on the conductor
 - Can prove (textbook p.104) : for any surface across which is discontinuous, force needs to be calculated by

$$\mathbf{f} = \sigma \mathbf{E}_{\text{average}} = \frac{1}{2} \sigma (\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}})$$

- For conductors $\mathbf{f} = \frac{1}{2\epsilon_0} \sigma^2 \hat{\mathbf{n}}$

Conductor electrostatics

- Capacitors

- We can define a potential difference between two conductors, without specifying locations of the integral

$$V = V_+ - V_- = - \int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}$$



- Although \mathbf{E} is geometry dependent, we know $\mathbf{E} \propto Q$, and $V \propto Q$
- Can define ratio as capacitance $C \equiv \frac{Q}{V}$
 - A purely geometrical quantity, determined by shapes, sizes, and separation of the two conductors
 - Unit: farads (F), or Coulomb per volt
 - Always positive

Potentials

Laplace equation

- Why Laplace equation is of interest

- Three ways to solve electrostatic problems

- $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} d\tau'$ Involves lengthy integrations

- $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} d\tau'$

- $\nabla^2 V = -\frac{\rho}{\epsilon_0} + \text{boundary conditions}$

} Similar

- Many problems are only concerned with charge-free regions

- Laplace equation $\nabla^2 V = 0$ plus boundary conditions

- Charges can exist elsewhere

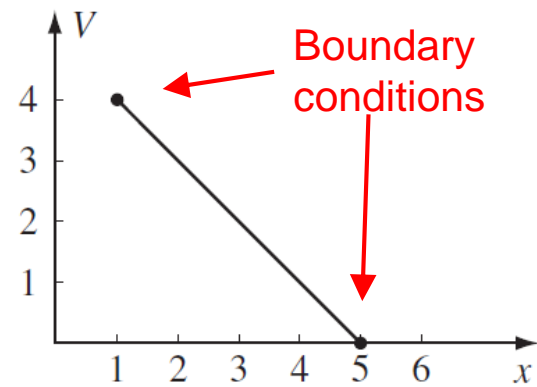
- Laplace equation has ubiquitous usage: electrostatics, theory of gravity, magnetism, theory of heat...

Laplace equation

- Laplace equation in 1D

$$\frac{d^2 V}{dx^2} = 0$$

- Solution: $V(x) = mx + b$ where m and b are constants
- Trivial solution, but two notable features (generalizable to higher-dimension equations)
 - $V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$ for any a
 - Solution has no local maximum or minimum, extrema must exist at boundaries



Laplace equation

- Laplace equation in 2D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

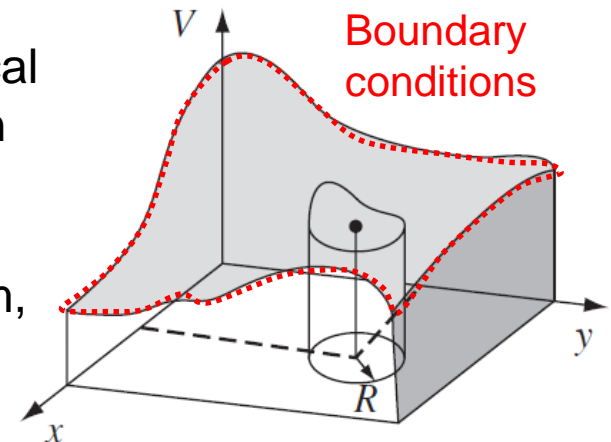
- Partial differential equation, where general solutions of the closed form is not possible

- $V(x, y)$ is the average of those around the point

- $V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V dl$ where path is a circle centered at (x, y)

- Method of relaxation to reach a numerical solution by iteratively using the equation above

- Solution has no local maximum or minimum, extrema must exist at boundaries



Laplace equation

- Laplace equation in 3D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- $V(\mathbf{r})$ is the average of V over a spherical surface centered at \mathbf{r}

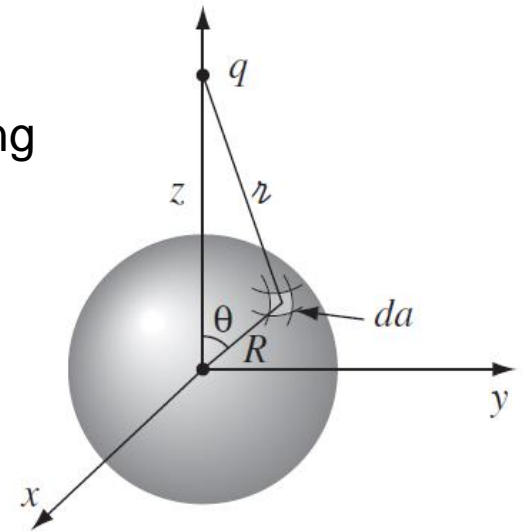
- $V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V da$ where area is a sphere centered at (x, y, z)

Example. Verify the above property by calculating the potential induced by a single charge q



C2.3DLaplace

- Solution has no local maximum or minimum



Uniqueness theorems

- The first uniqueness theorem

The solution to Laplace's equation in some volume is uniquely determined if V is specified on the boundary surface.

- Relevant to apparatus whose parts are connected to battery or ground
- Proof by contradiction

Suppose there are two solutions

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0$$

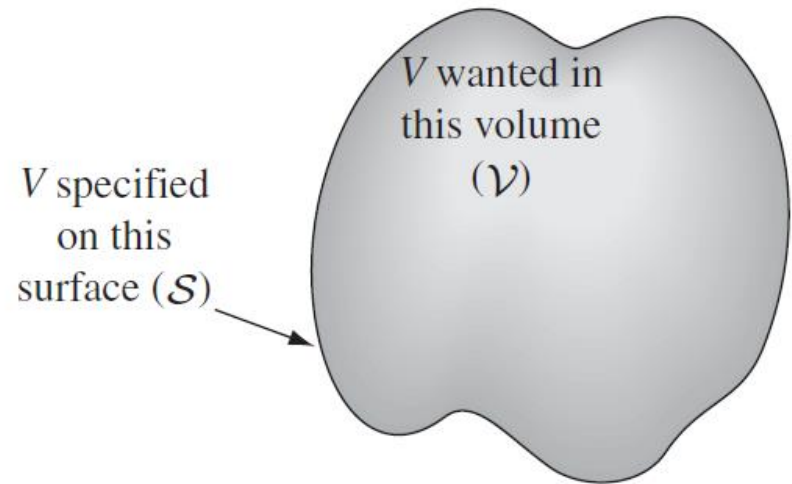
Define $V_3 \equiv V_1 - V_2$

$$\text{Then } \nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

with $V_3 = 0$ on all boundaries

➡ $V_3 = 0$ everywhere as there is no extrema except on boundaries

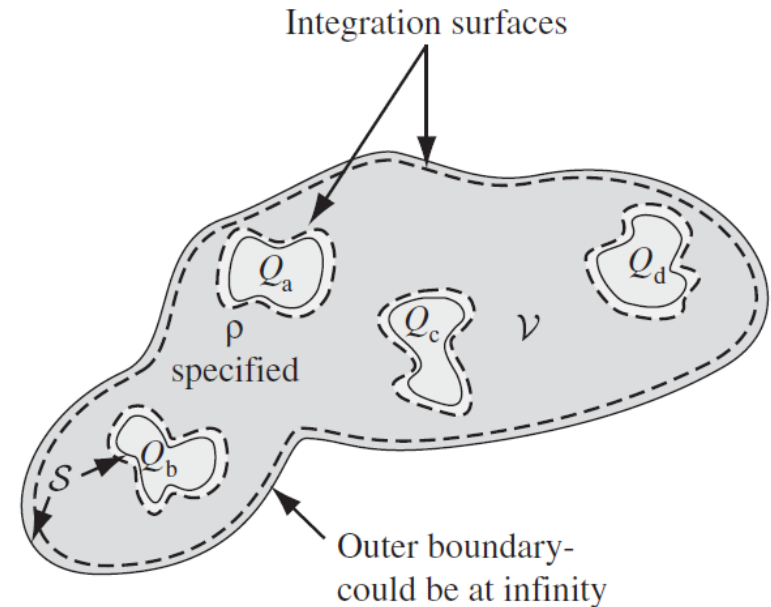
➡ $V_1 = V_2$



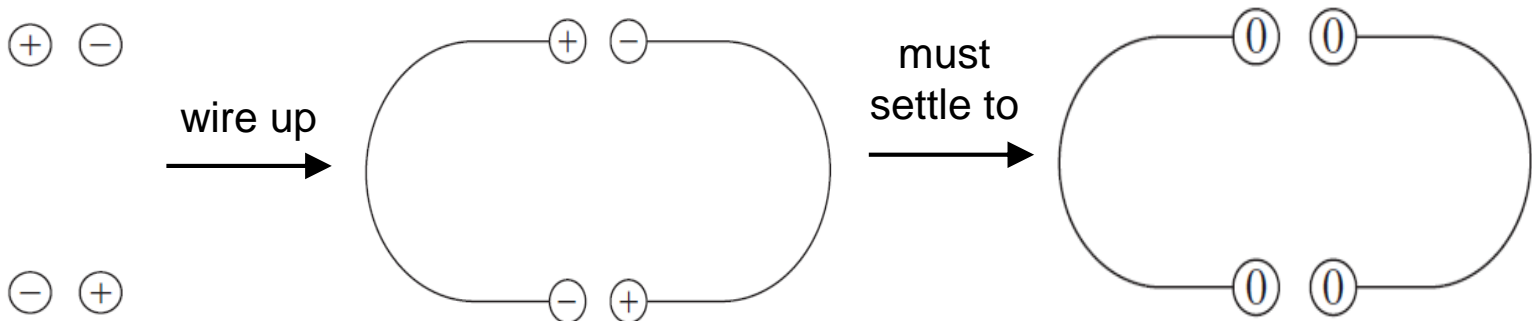
Uniqueness theorems

- The second uniqueness theorem

In a volume surrounded by conductors with a specified charge density, the electric field is uniquely determined if the total charge (not the charge distribution) on each conductor is given.



- Useful for conductor electrostatics
- Purcell's example



Uniqueness theorems

- The second uniqueness theorem

- Proof by contradiction

Suppose there are two solutions

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0} \rho, \quad \nabla \cdot \mathbf{E}_2 = \frac{1}{\epsilon_0} \rho$$

$$\Rightarrow \oint_{i \text{ th conducting surface}} \mathbf{E}_{1,2} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_i \quad \oint_{\text{outer boundary}} \mathbf{E}_{1,2} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}$$

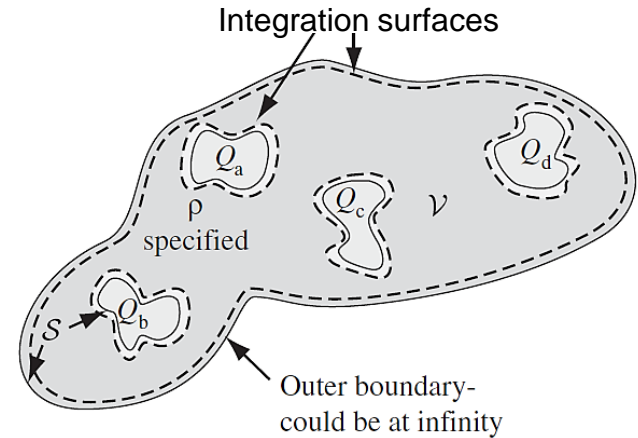
Define $\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2$ then $\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0$

Consider the following expression (with product rule and intuition)

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3 (\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2$$

$$\Rightarrow \int_V \nabla \cdot (V_3 \mathbf{E}_3) d\tau = \oint_S V_3 \mathbf{E}_3 \cdot d\mathbf{a} = - \int_V (E_3)^2 d\tau$$

$$\left. \begin{array}{l} \text{also} \downarrow \\ = V_3 \oint \mathbf{E}_3 \cdot d\mathbf{a} = 0 \end{array} \right\} \Rightarrow E_3 = 0 \text{ everywhere}$$

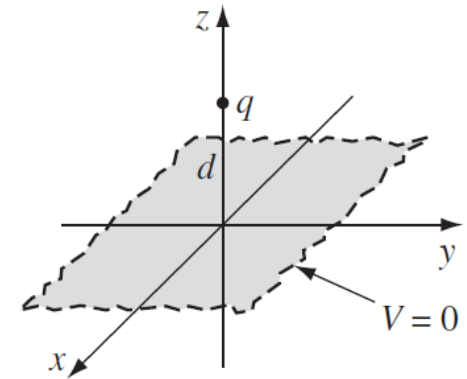


Three ways to solve the Laplace equation in the charge-free region

1. The method of images
2. Separation of variables
3. Multipole expansion

The method of images

- Usually works for problems involving charge(s) and conductor(s)
- The classical image problem
 - Point charge q held a distance d from a grounded ($V = 0$) conducting plate (infinitely large)
 - Ask: the potential in the region above the plane
 - The 1st uniqueness theorem applicable - V at all boundaries known
 - $V = 0$ at $z = 0$
 - $V = 0$ at $\sqrt{x^2 + y^2 + z^2} \rightarrow \infty$
 - Can guess a $V(x, y, z)$ that is consistent with the Poisson's equation (in the region of interest) and these requirements

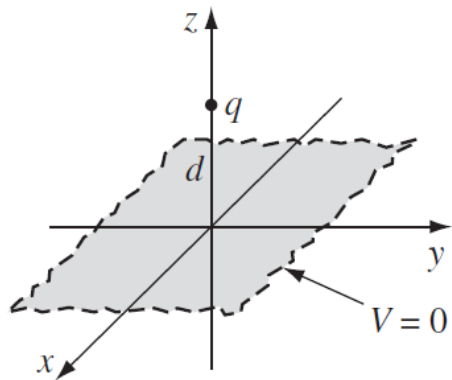


The method of images

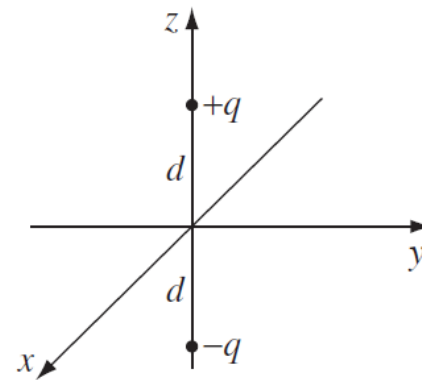
- The classical image problem
 - Guess: potential due to two point charges at $(0,0,\pm d)$ in free space

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z + d)^2}} \right]$$

- Charge configuration same as original problem at $z > 0$ ✓
- $V = 0$ at $z = 0$ ✓
- $V = 0$ at $\sqrt{x^2 + y^2 + z^2} \rightarrow \infty$ ✓



Guess



The method of images

- The classical image problem

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z + d)^2}} \right]$$

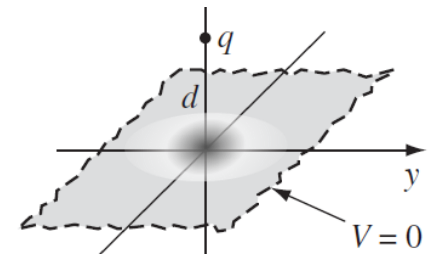
- Induced surface charge on the conducting plate

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0}$$

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{-q(z - d)}{[x^2 + y^2 + (z - d)^2]^{3/2}} + \frac{q(z + d)}{[x^2 + y^2 + (z + d)^2]^{3/2}} \right\}$$

↓

$$\sigma(x, y) = \frac{-qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}$$

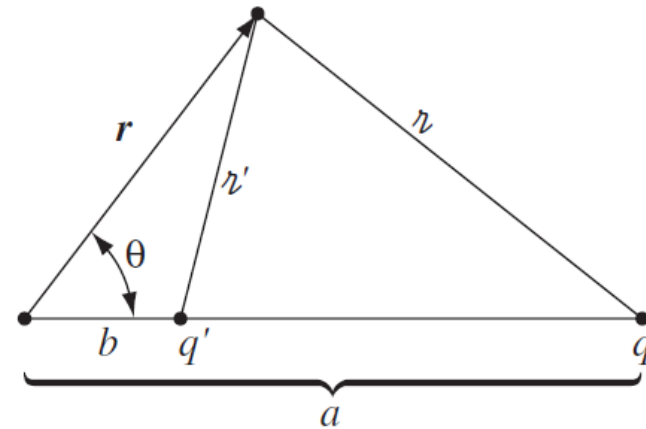
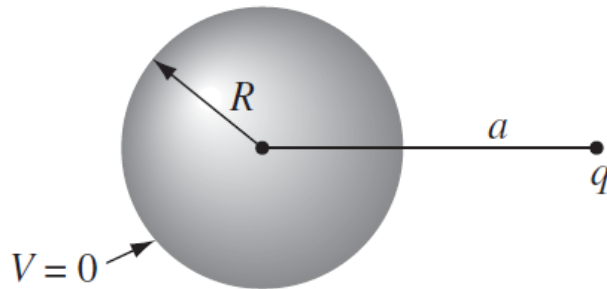


- Total surface charge (calculate with polar coordinate)

$$Q = \int_0^{2\pi} \int_0^\infty \frac{-qd}{2\pi(r^2 + d^2)^{3/2}} r dr d\phi = \left. \frac{qd}{\sqrt{r^2 + d^2}} \right|_0^\infty = -q$$

The method of images

Example 3.2. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R (Fig. 3.12). Find the potential outside the sphere.

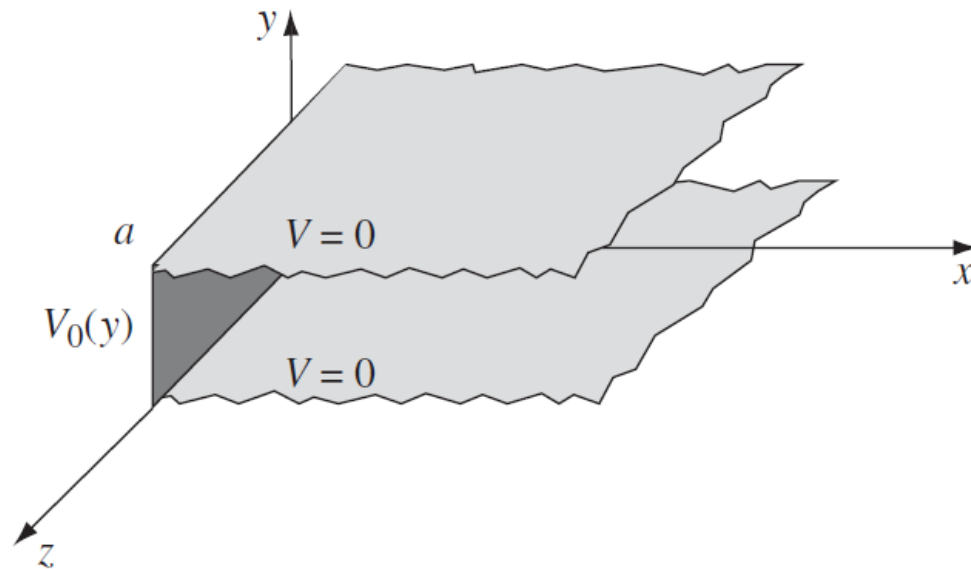


C2.exmp3.2

Separation of variables

- Assume general solutions are products of single-variable functions
 - Applicability strongly depends on problem
- Separation of variables in Cartesian coordinates

Example 3.3. Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$ (Fig. 3.17). The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this “slot.”



Separation of variables

- Example 3.3

$$\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = 0$$

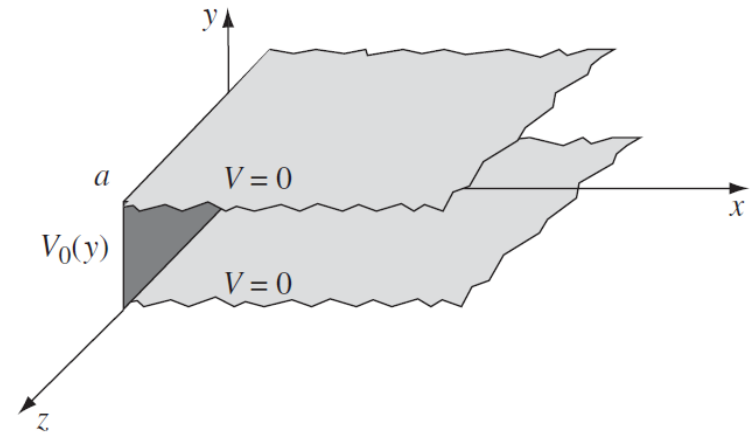
- Boundary conditions

(1) $V(x, y) = 0$ when $y = 0$

(2) $V(x, y) = 0$ when $y = a$

(3) $V(x, y) = V_0(y)$ when $x = 0$

(4) $V(x, y) \rightarrow 0$ when $x \rightarrow \infty$



- Trial solution $V(x, y) = X(x)Y(y)$

➡ $\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$ Two terms only depend on x or y

➡ $\frac{1}{X} \frac{d^2 X}{dx^2} = C_1$ and $\frac{1}{Y} \frac{d^2 Y}{dy^2} = C_2$, with $C_1 + C_2 = 0$

Separation of variables

- Example 3.3

- Trial solution $V(x, y) = X(x)Y(y)$

? What would happen if you assume $C_1 < 0, C_2 > 0$

- Suppose $C_1 = k^2 > 0, C_2 = -k^2 < 0$ (k being a constant)

$$\begin{aligned} \Rightarrow \left\{ \begin{array}{l} \frac{d^2 X}{dx^2} = k^2 X \\ \frac{d^2 Y}{dy^2} = -k^2 Y \end{array} \right. & \Rightarrow \left\{ \begin{array}{l} X(x) = Ae^{kx} + Be^{-kx} \\ Y(y) = C \sin ky + D \cos ky \end{array} \right. \end{aligned}$$

$$\Rightarrow V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)$$

- Determine A, B, C, D by matching boundary conditions

$$(4) V(x, y) \rightarrow 0 \text{ when } x \rightarrow \infty \Rightarrow A = 0$$

$$(1) V(x, y) = 0 \text{ when } y = 0 \Rightarrow D = 0$$

$$(2) V(x, y) = 0 \text{ when } y = a \Rightarrow k = \frac{n\pi}{a} \quad (n = 1, 2, 3 \dots)$$

Separation of variables

- Example 3.3
 - Determine A, B, C, D by matching boundary conditions

$$V(x, y) = C e^{-kx} \sin ky \quad (k = \frac{n\pi}{a})$$

(3) $V(x, y) = V_0(y)$ when $x = 0$, how to match this condition?

- Create a linear combination of solutions

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a) \quad (C_n: \text{coefficients to be determined})$$

$$\Rightarrow V(0, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$

Fourier series expansion of $V_0(y)$

$$\Rightarrow C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a) dy$$



C2.Fourier

Separation of variables

- Example 3.3
 - Examine the solution

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$\text{with } C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a) dy$$

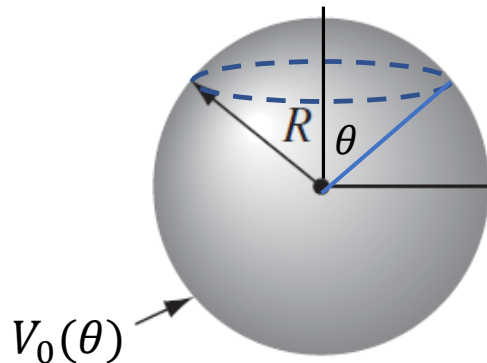
- The assumption of $V(x, y) = X(x)Y(y)$ was bodacious
- Validity of this method highly depend on the system's geometry
- The final solution written as a linear combination of trial solutions is not separated in variables

Separation of variables

- Separation of variables in spherical coordinates
 - Applicable to round objects

Example 3.6. The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R . Find the potential inside the sphere.

Example 3.7. The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R , but this time we are asked to find the potential *outside*, assuming there is no charge there.



Separation of variables

- Examples 3.6 and 3.7

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$



Problem & solution must be ϕ -independent

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

- Boundary conditions

(1) $V(R, \theta) = V_0(\theta)$ at the surface of the sphere

(2) $V(0, \theta)$ finite

(3) $V(R, \theta) \rightarrow 0$ when $R \rightarrow \infty$

- Trial solution $V(r, \theta) = R(r)\Theta(\theta)$

Two terms only depend on r or θ

$$\Rightarrow \frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

Separation of variables

- Examples 3.6 and 3.7

- Trial solution $V(r, \theta) = R(r)\Theta(\theta)$

$$\Rightarrow \begin{cases} \frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = l(l+1) \\ \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = -l(l+1) \end{cases} \Rightarrow \begin{cases} R(r) = Ar^l + \frac{B}{r^{l+1}} \\ \Theta(\theta) = P_l(\cos \theta) \end{cases}$$

$$\Rightarrow V(r, \theta) = \left(Ar^l + \frac{B}{r^{l+1}} \right) P_l(\cos \theta)$$

$$\Rightarrow \text{General solution } V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

- Legendre polynomial $P_l(x) \equiv \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = (3x^2 - 1)/2$$

$$P_3(x) = (5x^3 - 3x)/2$$

$$P_4(x) = (35x^4 - 30x^2 + 3)/8$$

$$P_5(x) = (63x^5 - 70x^3 + 15x)/8$$

Separation of variables

- Examples 3.6 and 3.7
 - Determine A_l, B_l by matching boundary conditions

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

- For $V(r, \theta)$ inside the sphere

(2) $V(0, \theta)$ finite $\Rightarrow B_l = 0$

(1) $V(R, \theta) = V_0(\theta) \Rightarrow V(R, \theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta) = V_0(\theta)$

Orthogonality of Legendre polynomial

$$\int_{-1}^1 P_l(x) P_{l'}(x) dx = \int_0^\pi P_l(\cos \theta) P_{l'}(\cos \theta) \sin \theta d\theta = \frac{2\delta_{ll'}}{2l+1}$$
$$A_l = \frac{2l+1}{2R^l} \int_0^\pi V_0(\theta) P_l(\cos \theta) \sin \theta d\theta$$

Separation of variables

- Examples 3.6 and 3.7
 - Determine A_l, B_l by matching boundary conditions

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

- For $V(r, \theta)$ outside the sphere

$$(3) V(\infty, \theta) \rightarrow 0 \quad \Rightarrow \quad A_l = 0$$

$$(1) V(R, \theta) = V_0(\theta) \quad \Rightarrow \quad V(R, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta) = V_0(\theta)$$

Orthogonality
of Legendre
polynomial

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^\pi V_0(\theta) P_l(\cos \theta) \sin \theta d\theta$$

Multipole expansion

- Why not exact solution but use an expansion?

- Exact solution
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r} \rho(\mathbf{r}') d\tau'$$

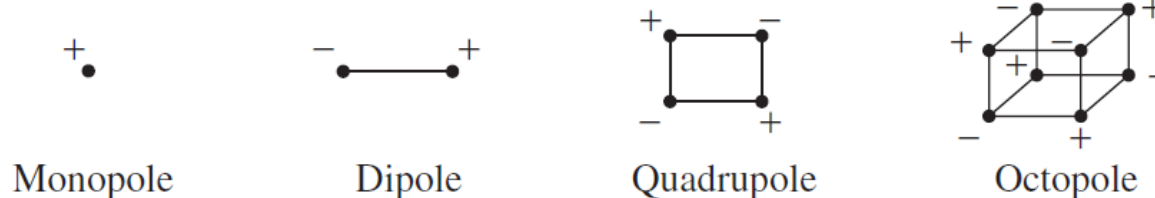
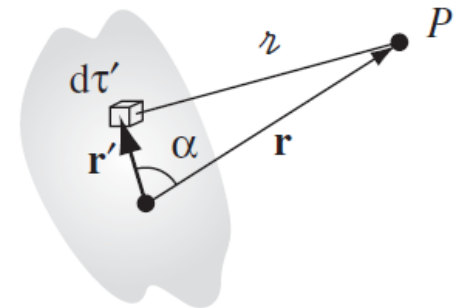
- For many problems hard to integrate, because of $r = |\mathbf{r} - \mathbf{r}'|$
 - For many problem not necessary, if one only cares about solution for large r

- Multipole expansion

- An expansion that examines from low to high order multipole contributions progressively

- Types of multipoles

- Monopole, dipole, quadrupole, octupole

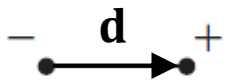


Multipole expansion

- Types of multipoles

- Monopole = total charge, $Q = \sum_i q_i$, or $Q = \int \rho(\mathbf{r}') d\tau'$
- Dipole between 2 charges

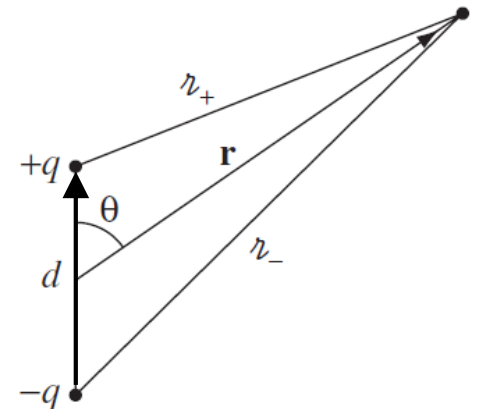
$$\mathbf{p} = \sum_{i=1}^n q_i \mathbf{r}'_i \quad \text{or} \quad \mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}') d\tau'$$



$\mathbf{p} = q\mathbf{d}$
(points from $-q$ to $+q$)

Example 3.10. A (physical) **electric dipole** consists of two equal and opposite charges ($\pm q$) separated by a distance d . Find the approximate potential at points far from the dipole.

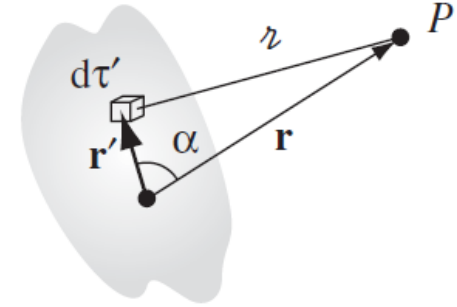
- Dipole field at large \mathbf{r} : $V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$



Multipole expansion

- Formal multipole expansion

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r} \rho(\mathbf{r}') d\tau'$$



- Focus on expanding $1/r$

$$r^2 = r^2 + (r')^2 - 2rr' \cos \alpha = r^2 \left[1 + \left(\frac{r'}{r} \right)^2 - 2 \left(\frac{r'}{r} \right) \cos \alpha \right]$$

$$\downarrow \quad r = r \sqrt{1 + \epsilon} \quad \text{where} \quad \epsilon \equiv \left(\frac{r'}{r} \right) \left(\frac{r'}{r} - 2 \cos \alpha \right)$$

Taylor expansion knowing $\epsilon \ll 1$

$$\frac{1}{r} = \frac{1}{r} (1 + \epsilon)^{-1/2} = \frac{1}{r} \left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots \right)$$

Multipole expansion

- Formal multipole expansion
 - Focus on expanding $1/r$

$$\frac{1}{r} = \frac{1}{r} \left[1 - \frac{1}{2} \left(\frac{r'}{r} \right) \left(\frac{r'}{r} - 2 \cos \alpha \right) + \frac{3}{8} \left(\frac{r'}{r} \right)^2 \left(\frac{r'}{r} - 2 \cos \alpha \right)^2 - \frac{5}{16} \left(\frac{r'}{r} \right)^3 \left(\frac{r'}{r} - 2 \cos \alpha \right)^3 + \dots \right]$$



Sort by different powers of r'/r

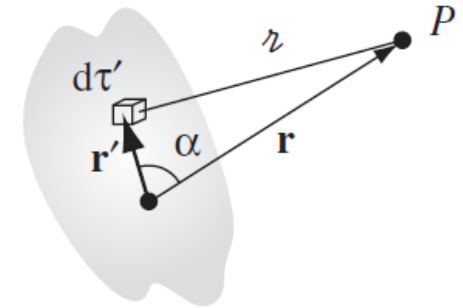
$$= \frac{1}{r} \left[1 + \left(\frac{r'}{r} \right) \underline{(\cos \alpha)} + \left(\frac{r'}{r} \right)^2 \underline{\left(\frac{3 \cos^2 \alpha - 1}{2} \right)} + \left(\frac{r'}{r} \right)^3 \underline{\left(\frac{5 \cos^3 \alpha - 3 \cos \alpha}{2} \right)} + \dots \right] = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n(\cos \alpha)$$

In the form of Legendre polynomials

Multipole expansion

- Formal multipole expansion

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos \alpha) \rho(\mathbf{r}') d\tau'$$



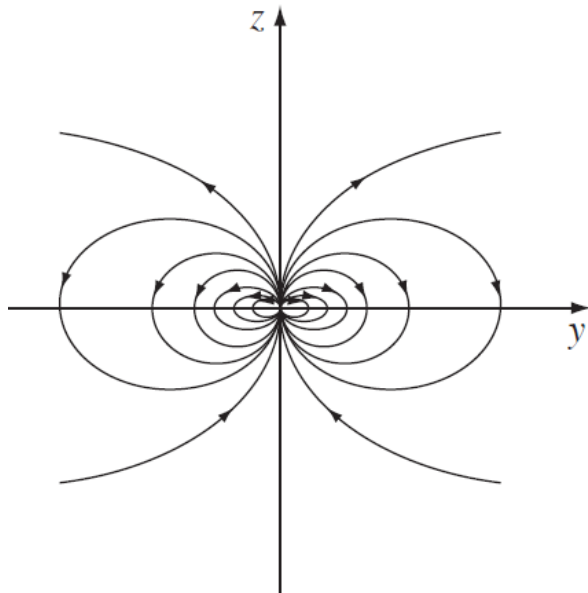
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[\underbrace{\frac{1}{r} \int \rho(\mathbf{r}') d\tau'}_{\text{Monopole}} + \underbrace{\frac{1}{r^2} \int r' \cos \alpha \rho(\mathbf{r}') d\tau'}_{\text{Dipole}} + \underbrace{\frac{1}{r^3} \int (r')^2 \left(\frac{3}{2} \cos^2 \alpha - \frac{1}{2} \right) \rho(\mathbf{r}') d\tau'}_{\text{Quadrupole}} + \dots \right]$$

- Merits of multipole expansion
 - r moved out of the integral ($\alpha = \theta$ polar angle if we set \mathbf{r} along \hat{z})
 - Can truncate to finite terms
- Caveat: terms can depend on choice of origin (such as dipole term)

Multipole expansion

- Pure dipole vs physical dipole

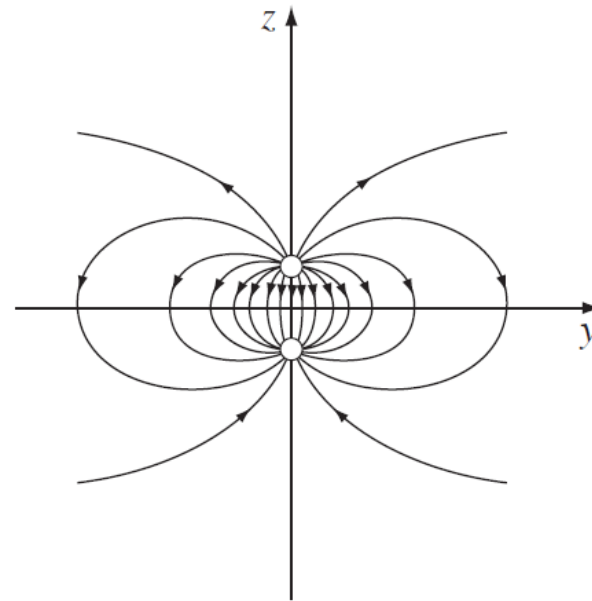
Pure dipole



$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

Assumes $r \gg d$ always holds
 $\mathbf{p} = q\mathbf{d}$ but take $q \rightarrow \infty$, $d \rightarrow 0$

Physical dipole



Deviations appear when
closing up onto the dipole

$$\mathbf{p} = q\mathbf{d}$$