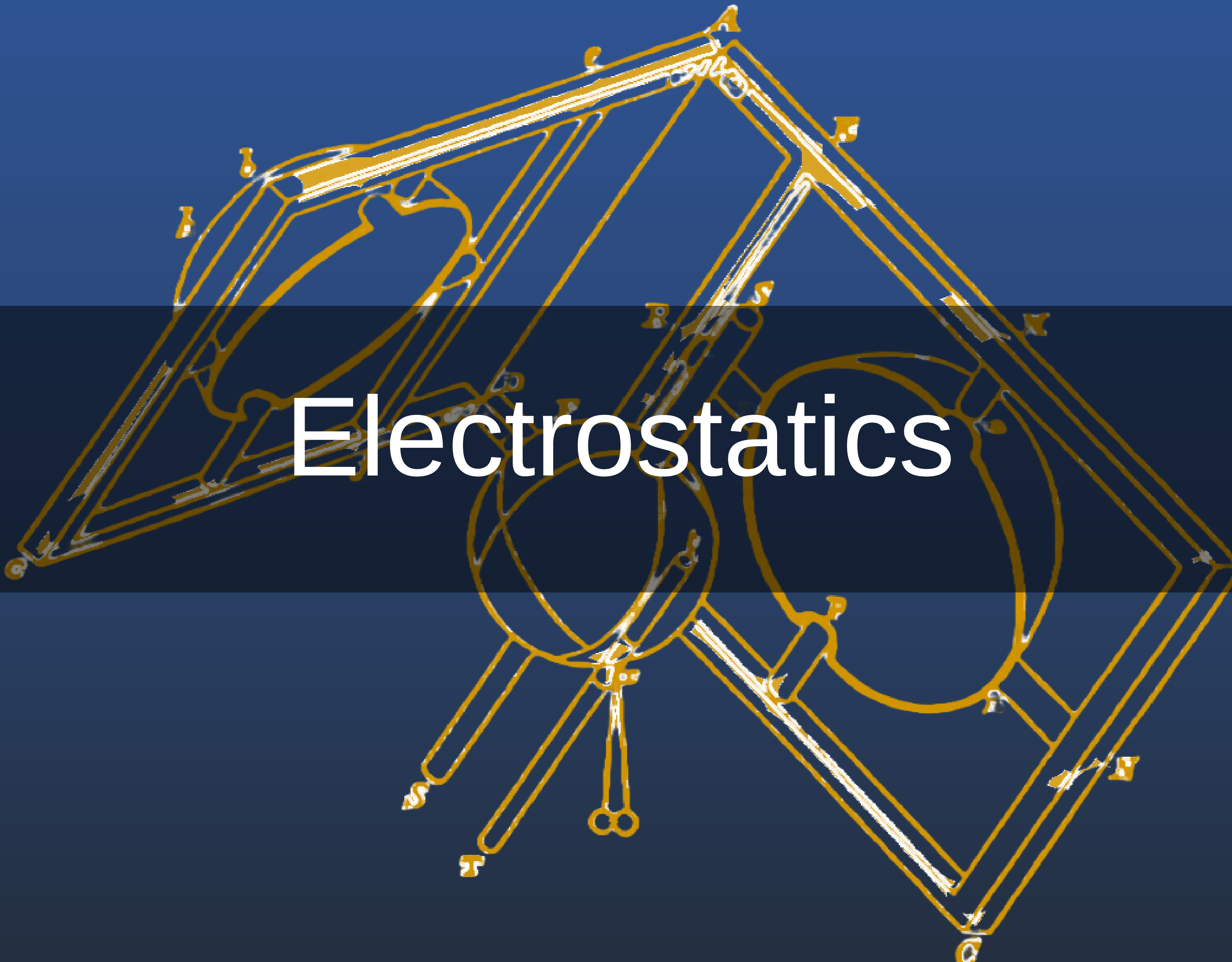
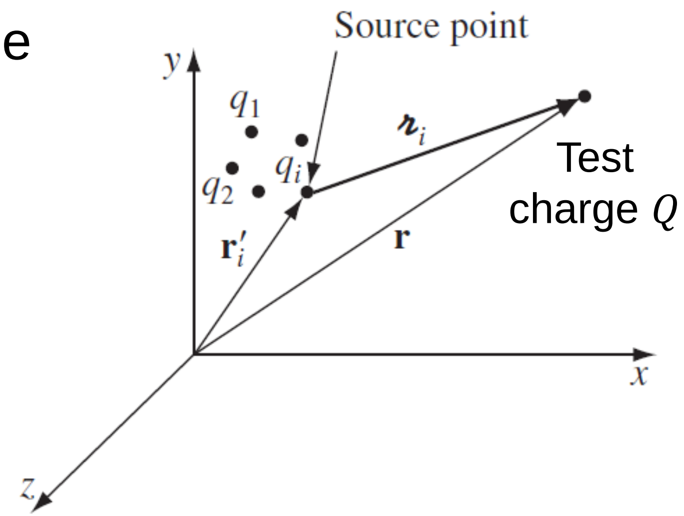
Electrostatics



Charge, electric field, and potential

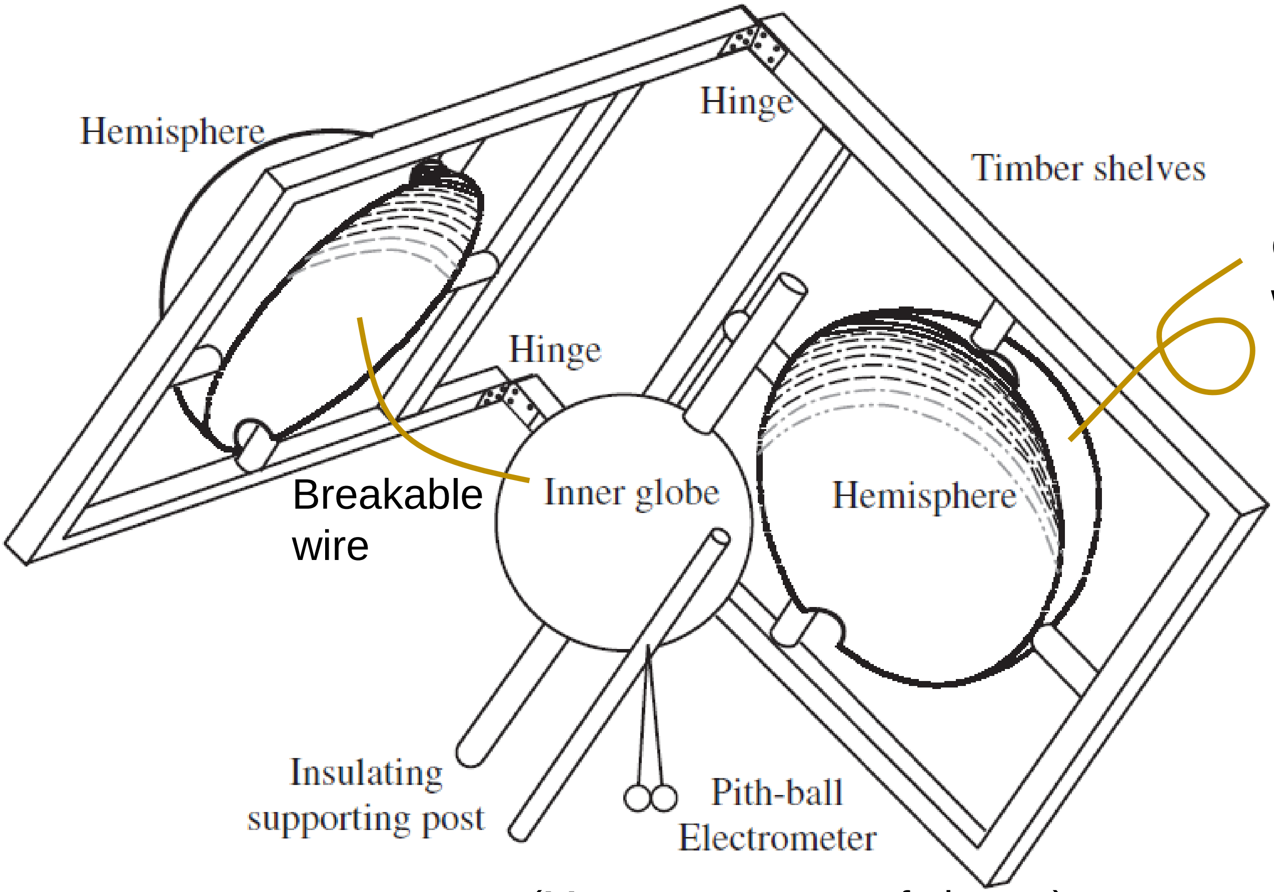
Coulomb's law

- Force of n source charges on a test charge
  - Force from source charge q_i acting on test charge Q
    - Coulomb's law $\boxed{F_i = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_i Q}{\boldsymbol{r}_i^2}\widehat{\boldsymbol{r}_i}}$
    - Permittivity $\varepsilon_0 = 8.85 \times 10^{12}$ C²/(N m²)
    - Separation vector $\boldsymbol{r}_i = \boldsymbol{r} - \boldsymbol{r}_i'$
    - Location of Q: $\boldsymbol{r}$, location of q_i: $\boldsymbol{r}_i'$
  - Principle of superposition
    - Total force acting on test charge $F = \sum_{i=1}^{n} F_i$
    - Not a necessity, but an experimental fact



* $\boldsymbol{\mathscr{r}}$ in textbook is typed as $\boldsymbol{r}$ in our slides (Cursive "r")

Coulomb’s law



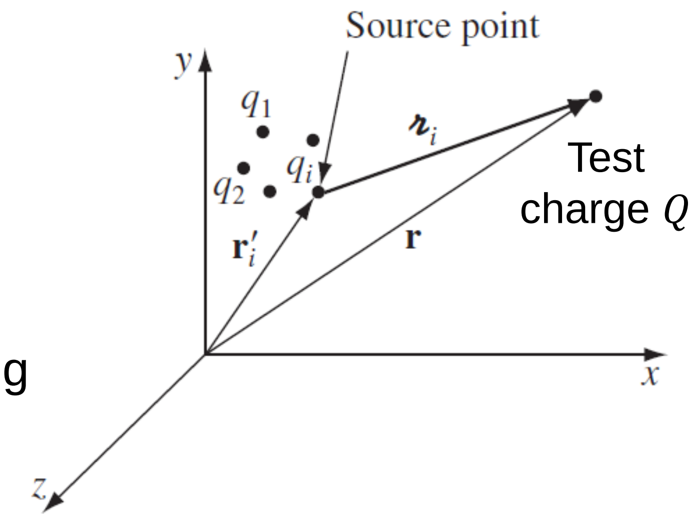
Cavendish’s apparatus for determining $F \propto r^{-2}$ in Coulomb’s law

Electric field induced by charge

- Relation of force and electric field

$$\boxed{\mathbf{F} = Q\mathbf{E}}$$

  - Electric field: force per unit charge
  - Real physical entity, as a vector field filling the space around charges
  - Negated theory of "ether"



- Electric field induced by discrete charges

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i$$

  - Separation vector $\boldsymbol{\mathscr{r}}_i = \boldsymbol{r} - \boldsymbol{r}'_i$, contains $\boldsymbol{r}$
  - Principle of superposition also holds

Electric field induced by charge

- Electric field induced by continuous charge distribution

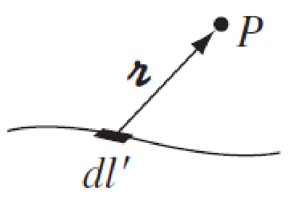
$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dq}$$

  - Add up contributions from infinitesimal charge elements dq
  - Three ways dq can be distributed

Line charge

$dq \to \lambda\, dl'$

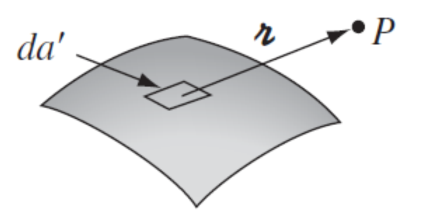
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dl'$$



Surface charge

$dq \to \sigma\, da'$

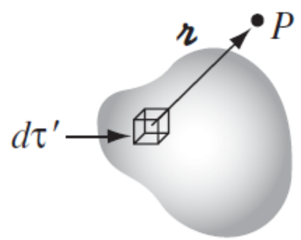
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,da'$$



Volume charge

$dq \to \rho\, d\tau'$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

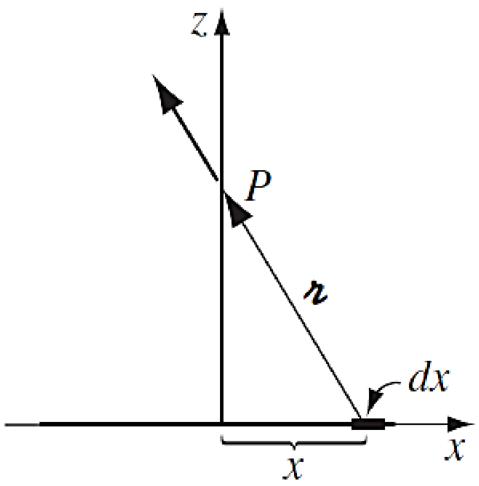


*λ, σ, ρ: charge per unit length, area, volume

Electric field induced by charge

- Electric field induced by continuous charge distribution

Example 2.2. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ (Fig. 2.6).



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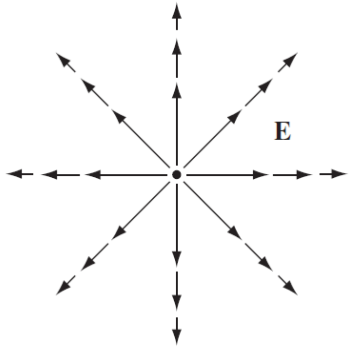
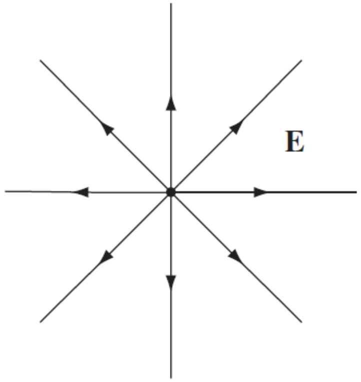
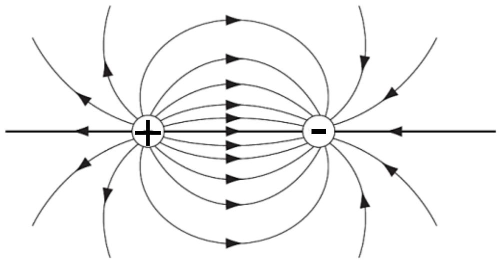
  - Integration sometimes can get formidable, need to device new tools to simplify problems.

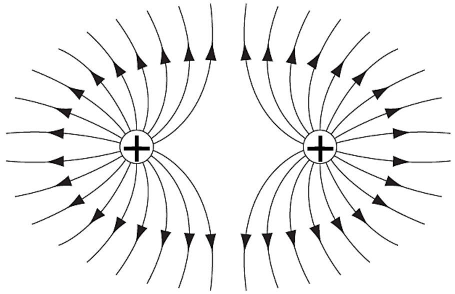
Gauss's law

- Electric field lines
  - Source charge q at the origin $\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}}$
  - Draw vector field – field falls off like $1/r^2$
  - Connect up the arrows – electric field lines
    - Direction of line indicates field direction
    - Density of line indicates field magnitude
  - Field lines begin from positive charges and end on negative ones

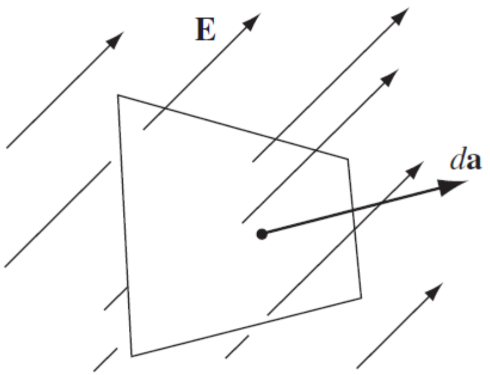








Gauss's law



- Electric field flux $\quad \Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$
  - A measure of the number of field lines passing through an area
- Gauss's law
  - The flux through any closed surface is a measure of the total charge inside

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot \underbrace{(r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}})}_{d\boldsymbol{a}} = \frac{1}{\epsilon_0} q$$

Spherical surface of radius r

  - The surface integral can be any shape, not necessarily spherical
  - Multiple charges $\quad \oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n}\left(\oint \mathbf{E}_i \cdot d\mathbf{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\epsilon_0} q_i\right)$

$$\Rightarrow \boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}}$$

(Q_{enc}: total charge enclosed in the integrated surface)

Gauss's law

- Gauss's law
  - Gauss's law in the differential form

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$ (integral form)

Divergence theorem

Consider volume distribution

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau \qquad Q_{\text{enc}} = \int_{\mathcal{V}} \rho\, d\tau$$

$$\Rightarrow \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau$$

$$\Rightarrow \boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho}$$ (differential form)

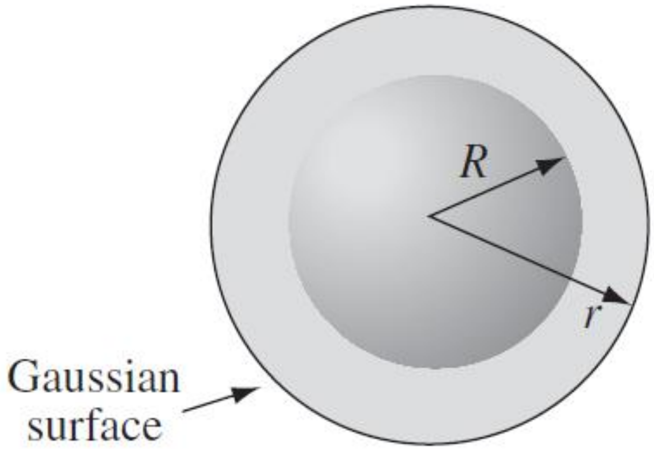
    - Differential form more compact, but integral form easier to use

  - Use of Gauss's law to calculate electric field
    - Need (1) Gauss's law in integral form and (2) symmetry arguments

Gauss's law

- Application of Gauss's law

Example 2.3. Find the field outside a uniformly charged solid sphere of radius R and total charge q.

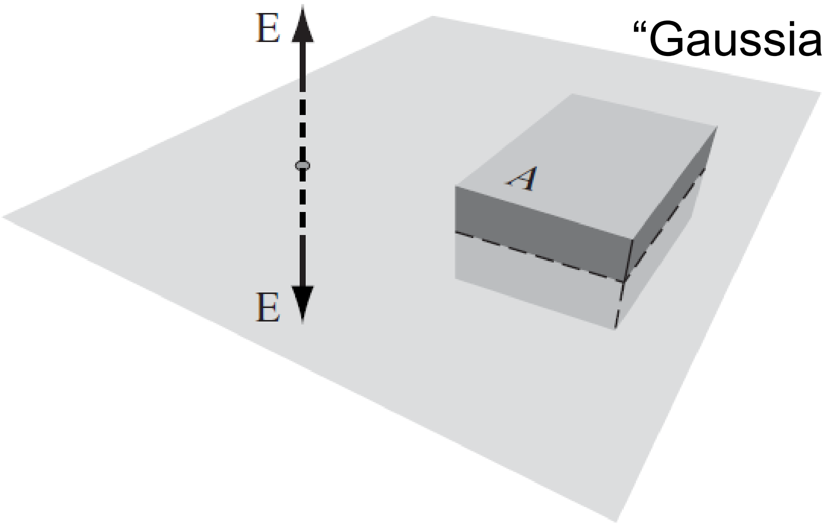


  - The field outside the sphere is exactly the same as it would have been if all the charge had been concentrated at the center

Gauss's law

- Application of Gauss's law

Example 2.5. An infinite plane carries a uniform surface charge σ. Find its electric field.



Divergence of electric field

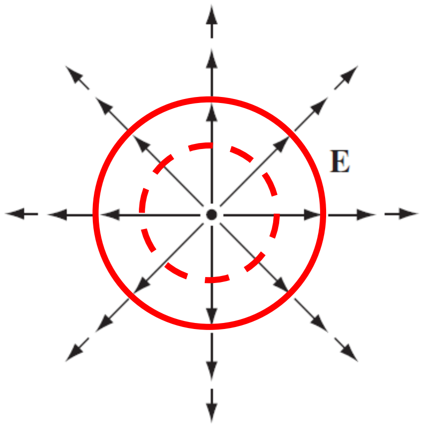
- Directly calculate divergence
  - According to Coulomb's law

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')\,d\tau'$$

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \boxed{\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)}\rho(\mathbf{r}')\,d\tau'$$

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \nabla\cdot\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0$$

> ? This seems to contradict the Gauss's law, what went wrong



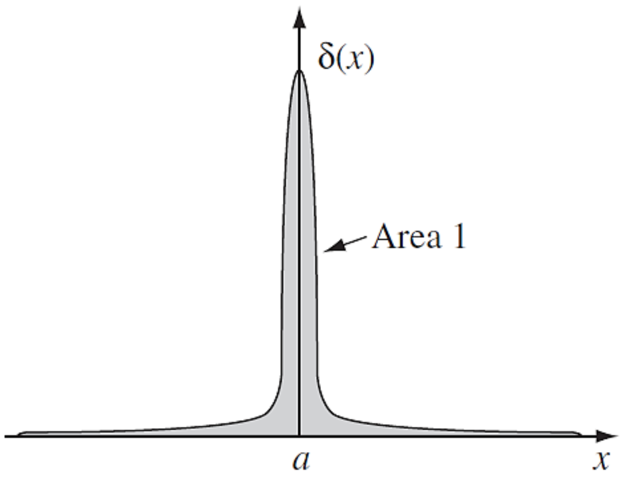
  - The derivation above is correct anywhere but the origin ($r = 0$), where the divergence should go to infinity
    - Consider special case of point charge and Gauss's law with varying volume to integrate

Divergence of electric field

- Delta function
  - Infinitely high, infinitesimally narrow
  - 1D Delta function

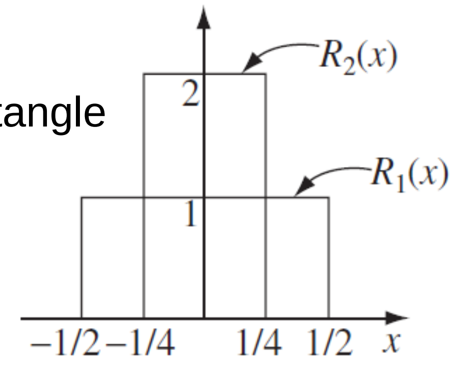
$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

with $$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

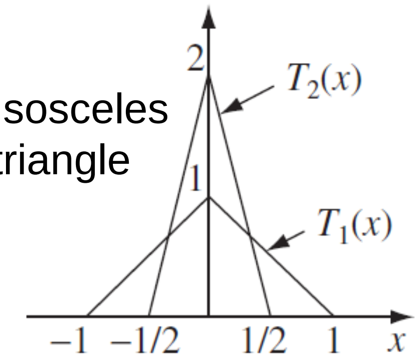


- Can be understood as the limit of a sequence of functions

Rectangle



Isosceles triangle



Divergence of electric field

- Delta function
  - 1D Delta function
    - When in an integral, “picks out” the value of a function

      Since $\delta(x)$ anywhere 0 but at $x = 0$

      [$f(x)$ being an ordinary function not going to infinity]

      $$f(x)\delta(x) = f(0)\delta(x)$$

      $$\Rightarrow \int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0)$$

      And, one can shift $\delta(x)$ to $\delta(x-a)$ to pick out another one

      $$f(x)\delta(x-a) = f(a)\delta(x-a)$$

      $$\Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)}$$

    - A frequently used expression $\delta(kx) = \dfrac{1}{|k|}\delta(x)$

Divergence of electric field

- Delta function
  - 3D Delta function $\boxed{\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z)}$

    with $\displaystyle\int_{\text{all space}} \delta^3(\mathbf{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$

    - Picks out a function value $\displaystyle\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\,d\tau = f(\mathbf{a})$

- Back to calculating divergence of electric field

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\rho(\mathbf{r}')\,d\tau'$$

$$\Big\downarrow\ \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

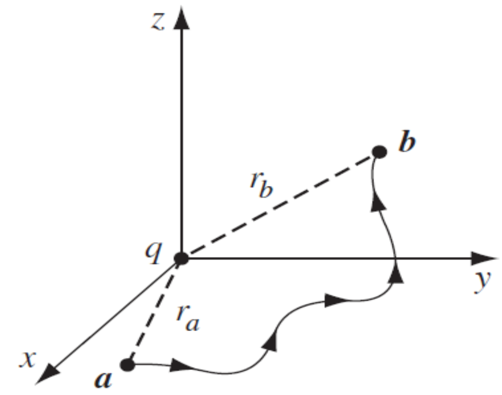
C2.delta

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int 4\pi\delta^3(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\,d\tau' = \frac{1}{\epsilon_0}\rho(\mathbf{r})$$

Gauss's law recovered

Curl of electric field

- Calculate curl for point charge at origin



$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2}dr = \left.\frac{-1}{4\pi\epsilon_0}\frac{q}{r}\right|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

- For any closed loop ($r_a = r_b$) $\oint \mathbf{E}\cdot d\mathbf{l} = 0$

$\Rightarrow$ $\boxed{\nabla \times \mathbf{E} = \mathbf{0}}$ due to Stoke's theorem

Stoke's theorem

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}$$

- Any static charge distribution

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla \times \mathbf{E}_1) + (\nabla \times \mathbf{E}_2) + \ldots = \mathbf{0}$$

Electric potential

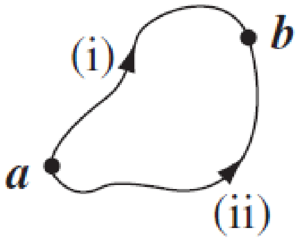
- Vector field **E** cannot take arbitrary form
  - Crucial constraint: $\nabla \times \mathbf{E} = \mathbf{0}$

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \qquad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} \qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

? Any chance the vector field can be described more easily?

- Electric potential: $\boxed{V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}}$
  - Unit: joules per coulomb
  - $\mathcal{O}$: a reference point (usually taken as infinity)
  - Integral does not depend on path
    - $\nabla \times \mathbf{E} = \mathbf{0}$
    - $\oint \mathbf{E} \cdot d\mathbf{l} = 0$
    - $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$ is path independent

Equivalent statements



Electric potential

- Electric potential: $V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$
  - Potential difference between two points is more meaningful

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$
$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

On the other hand, the theorem for gradient gives

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} \quad \Rightarrow \quad \boxed{\mathbf{E} = -\nabla V}$$

  - Scalar field V gives full information of vector field **E**
  - Can be off by a constant if choosing a different reference point

$$V'(\mathbf{r}) = -\int_{\mathcal{O}'}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{O}'}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = K + V(\mathbf{r})$$

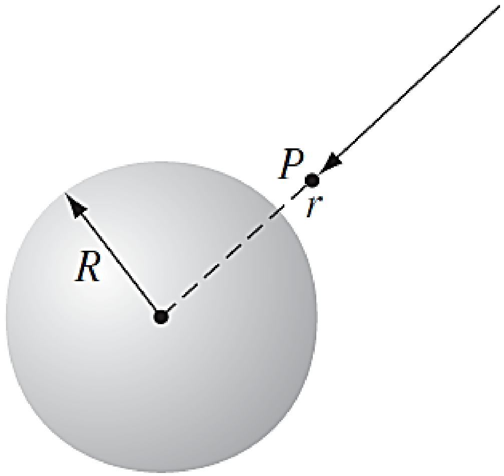
new

Electric potential

- Application of electric potential

Example. Find the potential of a point charge q at origin

Example 2.7. Find the potential inside and outside a spherical shell of radius R (Fig. 2.31) that carries a uniform surface charge. Set the reference point at infinity.



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Electric potential

- Poisson's equation of potential
  - Poisson's equation $\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$

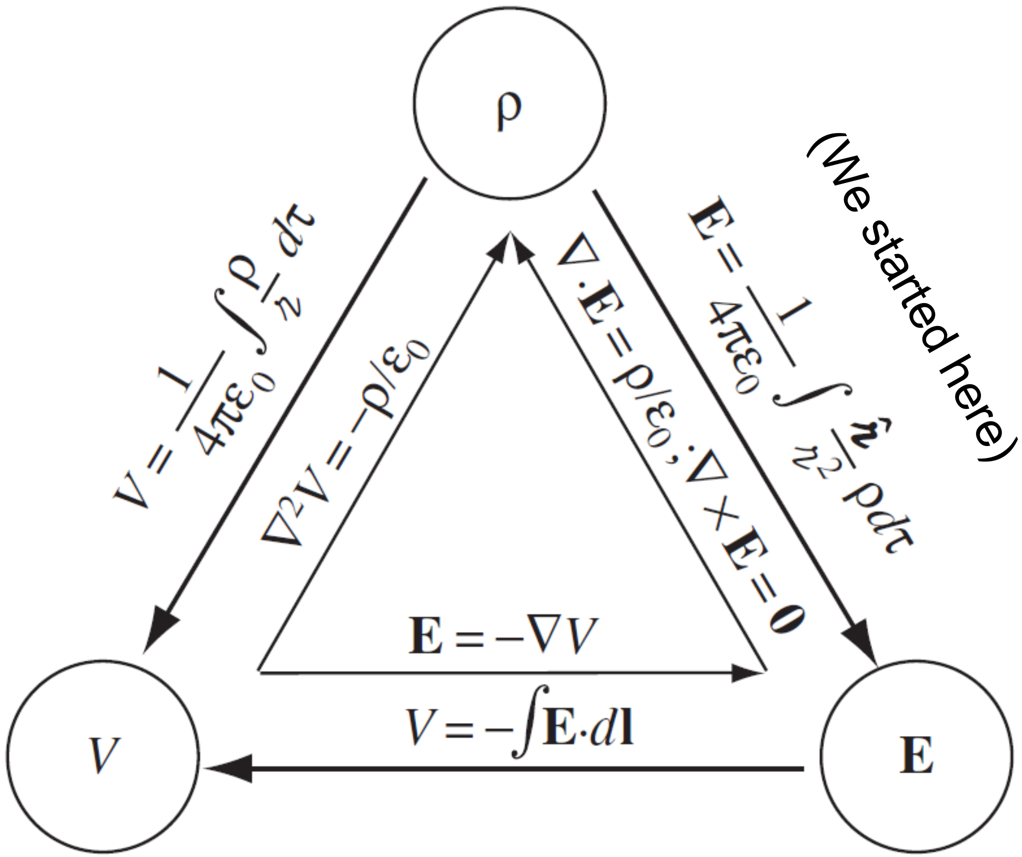
    $$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$
  - In regions with no charge, Laplace's equation $\boxed{\nabla^2 V = 0}$
  - Curl of a gradient always zero $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = \mathbf{0}$

- Potential of a localized charge distribution
  - Pick infinity as the reference point $\mathcal{O} = \infty$
  - Principle of superposition holds $V = V_1 + V_2 + \ldots$
  - Discrete charges $\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i}}$
  - Continuous charge $\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'}$

    $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} \, d\tau'$$

    Can check

Charge, electric field, and potential

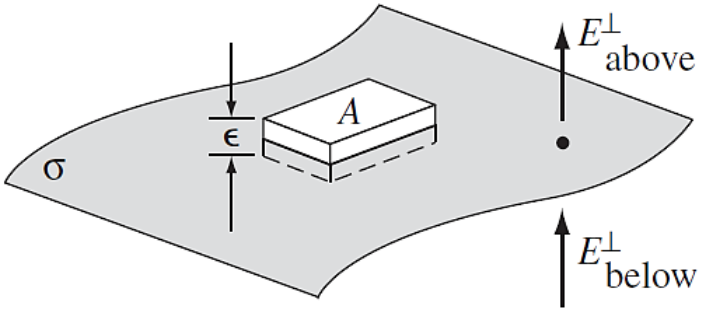


Differential equations need boundary conditions to solve

Boundary conditions

- Boundary conditions of **E** across a 2D charged surface
  - Normal component of **E**

    "Gaussian pillbox" with $\varepsilon \to 0$

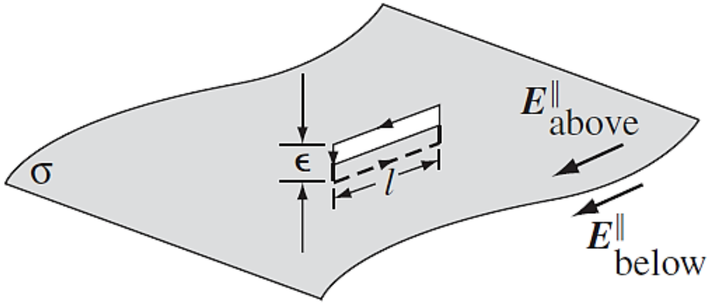


$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \sigma A$$

$$\Rightarrow \boxed{E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma}$$

  - Tangential component of **E**

    Thin loop with $\varepsilon \to 0$



$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\Rightarrow \boxed{\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}}}$$

  - Summarizing above $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \dfrac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$

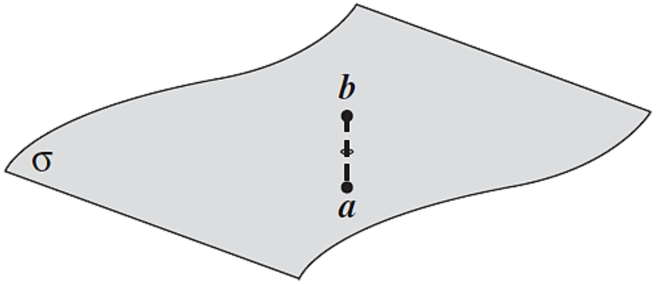
Boundary conditions

- Boundary conditions of V across a 2D charged surface
  - Potential is continuous (across any boundary)

$$V_{\text{above}} - V_{\text{below}} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

Path length → 0

$$\boxed{V_{\text{above}} = V_{\text{below}}}$$



  - Gradient of potential is discontinuous

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

$$\mathbf{E} = -\nabla V$$

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0}\sigma\hat{\mathbf{n}}$$

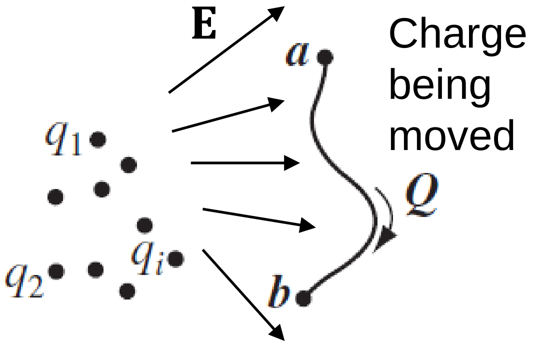
$$\Rightarrow \boxed{\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0}\sigma}$$

where we define normal derivative of V

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

Energy in electrostatics

- Work done to move a charge
  - Integrate force over distance



$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

$$= Q[V(\mathbf{b}) - V(\mathbf{a})]$$

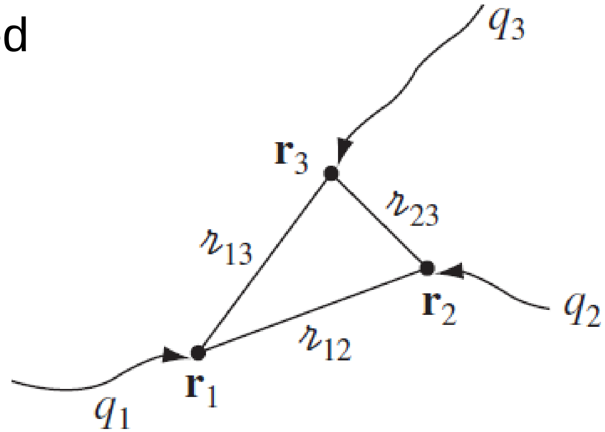
  - Electrostatic force is conservative (path independent)
  - Can confirm the unit of electric potential
  - Work for bringing from infinitely far to $\boldsymbol{r}$

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

$$W = QV(\mathbf{r})$$ with the potential reference point set to infinity

Energy in electrostatics

- Energy of a point charge configuration
  - Equals to the work required to bring charges together from infinity
    - First charge q_1 to $\boldsymbol{r_1}$, no work required
    - q_2 to $\mathbf{r_2}$ $\quad W_2 = \dfrac{1}{4\pi\epsilon_0} q_2 \left(\dfrac{q_1}{\mathscr{r}_{12}}\right)$
    - q_3 to $\mathbf{r_3}$ $\quad W_3 = \dfrac{1}{4\pi\epsilon_0} q_3 \left(\dfrac{q_1}{\mathscr{r}_{13}} + \dfrac{q_2}{\mathscr{r}_{23}}\right)$
    - $W = W_1 + W_2 + W_3$



  - Total work (energy) for n charges

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{\mathscr{r}_{ij}} \right)}_{V(\mathbf{r}_i)}$$

Count once for each pair

$$\boxed{W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i)}$$

Potential q_i feels due to all **other** charges

Energy in electrostatics

- Energy of a continuous charge distribution
  - Generalize point charge equation to

$$W = \frac{1}{2}\int \rho V \, d\tau$$

with V: actual potential, without excluding the charge of interest

$$\Big\downarrow \; \rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

$$W = \frac{\epsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V \, d\tau$$

$\Big\downarrow$ Integrate by parts $\int_{\mathcal{V}} f(\nabla \cdot \mathbf{A})\, d\tau = -\int_{\mathcal{V}} \mathbf{A} \cdot (\nabla f)\, d\tau + \oint_{\mathcal{S}} f\mathbf{A} \cdot d\mathbf{a}$

$$W = \frac{\epsilon_0}{2}\left[-\int \mathbf{E} \cdot (\nabla V)\, d\tau + \oint V\mathbf{E} \cdot d\mathbf{a}\right] = \frac{\epsilon_0}{2}\left(\int_{\mathcal{V}} E^2 \, d\tau + \underbrace{\oint_{\mathcal{S}} V\mathbf{E} \cdot d\mathbf{a}}_{\text{Vanishes when } \mathcal{V} \to \infty}\right)$$

$$\Rightarrow \boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau}$$

  - <u>Cannot</u> be directly compared to equation of point charge, see textbook

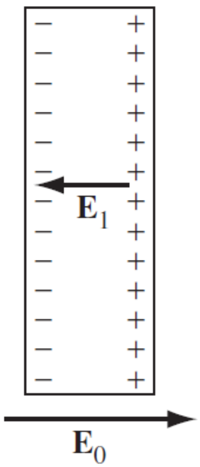
Conductor electrostatics

- Conductors
  - Free electrons – solid-state metals and doped semiconductors
  - Free ions – Electrolyte, salt water, lithium ion battery
  - Unlimited supply of free charges, which are free to move

- Electrostatics of perfect conductors
  - $\mathbf{E} = 0$ inside a conductor
    - If not, charge will flow to induce a new surface charge distribution that exactly cancels the internal field
  - $\rho = 0$ (net charge volume density) inside a conductor
    - Because $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$
  - A conductor is an equipotential
    - For any two points, $V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$

Conductor electrostatics

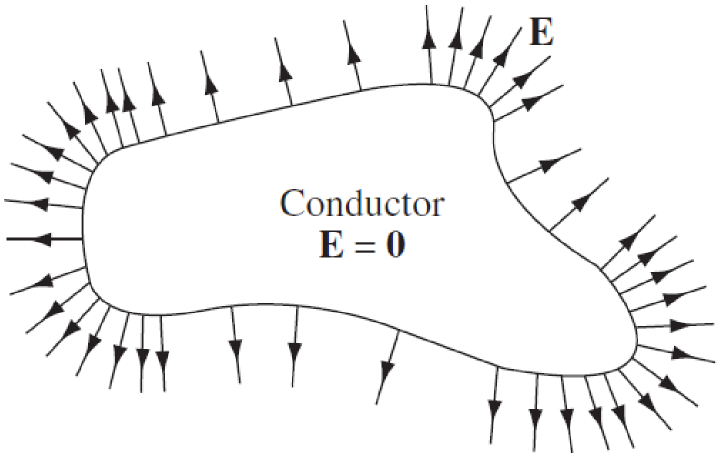
- Electrostatics of perfect conductors
  - Any net charge only resides on the surface (minimizes energy)
  - Surface net charges serves to cancel the internal field



  - $\mathbf{E}$ is always perpendicular to the surface, just outside the conductor
    - Recall boundary conditions

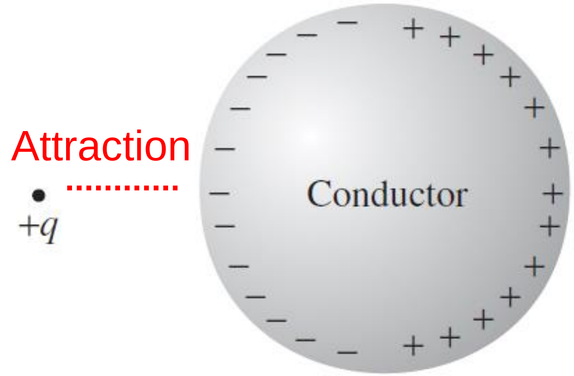
$$E^{\perp}_{\text{above}} - \cancelto{0}{E^{\perp}_{\text{below}}} = \frac{1}{\epsilon_0}\sigma$$

$$\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} = 0$$

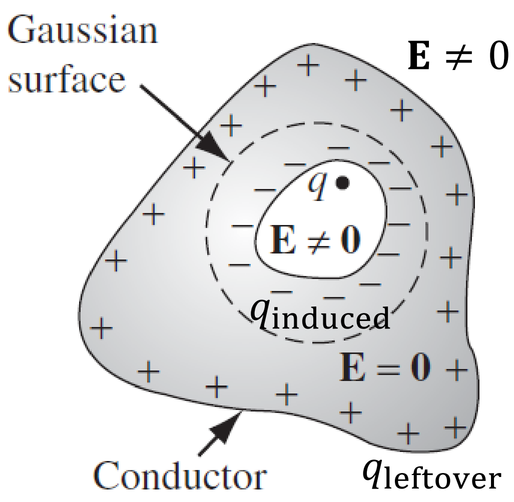


Conductor electrostatics

- Induced charges
  - Charge placed outside a metal
    - Induced charge serves to cancel field inside conductor
    - Net force of attraction



  - Charge in the cavity of a hollow metal
    - Inside the cavity: $\mathbf{E} \neq 0$
    - Induced charge $q_{\text{induced}} = -q$ at inner wall
    - Inside the conductor: $\mathbf{E} = 0$
    - Leftover charge $q_{\text{leftover}} = q$ at outer wall
    - Outside the conductor: $\mathbf{E} \neq 0$



Conductor electrostatics

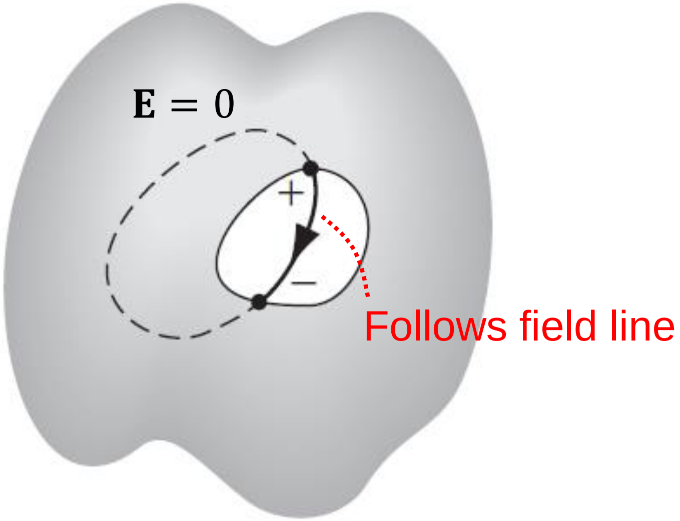
- Induced charges
  - Faraday cage
    - If no charge is placed in the cavity of a hollow conductor, $\mathbf{E} = 0$ in the cavity regardless of the outside conditions

      If not, can construct a loop of integration, whose trajectory in the cavity follows the field line

      ⇒ $\oint \mathbf{E} \cdot d\mathrm{l} \neq 0$

      ⇒ Contradicts $\nabla \times \mathbf{E} = 0$

      ⇒ $\mathbf{E} = 0$

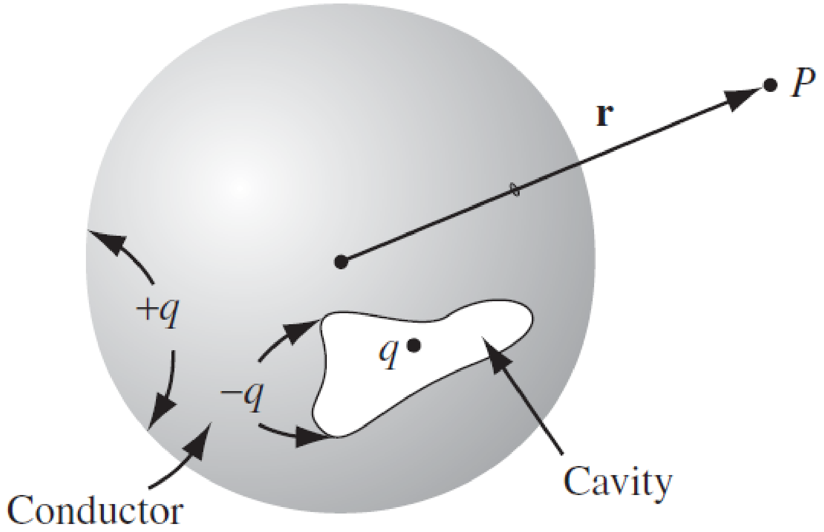


    - Protects sensitive apparatus inside the cavity by shielding out external electric fields

Conductor electrostatics

- Induced charges

Example 2.10. An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it (Fig. 2.46). Somewhere within the cavity is a charge q. *Question:* What is the field outside the sphere?



Conductor electrostatics

- Surface charge and force on a conductor
  - Boundary conditions

$$\begin{cases} \mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \dfrac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \\[2ex] \dfrac{\partial V_{\text{above}}}{\partial n} - \dfrac{\partial V_{\text{below}}}{\partial n} = -\dfrac{1}{\epsilon_0}\sigma \end{cases} \quad \xrightarrow{\text{On the surface of a perfect conductor}} \quad \begin{cases} \boxed{\begin{array}{l} \mathbf{E} = \dfrac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \\[2ex] \sigma = -\epsilon_0 \dfrac{\partial V}{\partial n} \end{array}} \end{cases}$$

  - Force (per unit area) exerted on the conductor
    - Can prove (textbook p.104) : for any surface across which is discontinuous, force needs to be calculated by

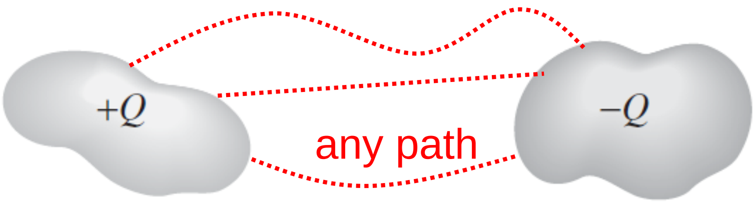
$$\mathbf{f} = \sigma \mathbf{E}_{\text{average}} = \frac{1}{2}\sigma(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}})$$

    - For conductors $\mathbf{f} = \dfrac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}}$

Conductor electrostatics

- Capacitors
  - We can define a potential difference between two conductors, without specifying locations of the integral

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}$$



  - Although $\mathbf{E}$ is geometry dependent, we know $\mathbf{E} \propto Q$, and $V \propto Q$
  - Can define ratio as capacitance $C \equiv \dfrac{Q}{V}$
    - A purely geometrical quantity, determined by shapes, sizes, and separation of the two conductors
    - Unit: farads (F), or Coulomb per volt
    - Always positive

Potentials

Laplace equation

- Why Laplace equation is of interest
  - Three ways to solve electrostatic problems
    - $\mathbf{E}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, d\tau'$ Involves lengthy integrations
    - $V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(\mathbf{r}')}{\mathscr{r}}\, d\tau'$
    - $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$ + boundary conditions

    Similar
  - Many problems are only concerned with charge-free regions
    - Laplace equation $\nabla^2 V = 0$ plus boundary conditions
    - Charges can exist elsewhere
    - Laplace equation has ubiquitous usage: electrostatics, theory of gravity, magnetism, theory of heat…

Laplace equation

- Laplace equation in 1D

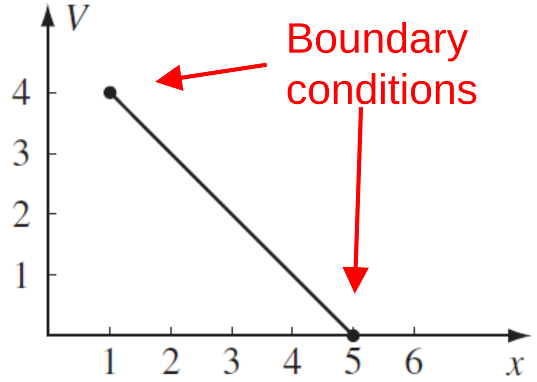
$$\frac{d^2V}{dx^2} = 0$$

  - Solution: $V(x) = mx + b$ where m and b are constants

  - Trivial solution, but two notable features (generalizable to higher-dimension equations)

    - $V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$ <u>for any a</u>

    - Solution has no local maximum or minimum, extrema must exist at boundaries



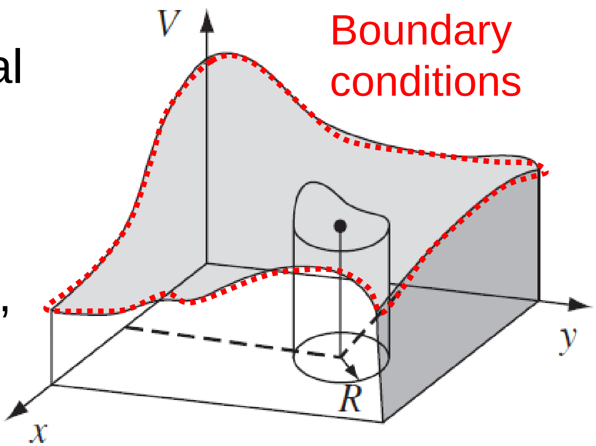
Laplace equation

- Laplace equation in 2D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

  - Partial differential equation, where general solutions of the closed form is not possible

  - $V(x, y)$ is the average of those around the point
    - $V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V\, dl$ where path is a circle centered at (x, y)
    - Method of relaxation to reach a numerical solution by iteratively using the equation above



  - Solution has no local maximum or minimum, extrema must exist at boundaries

Laplace equation

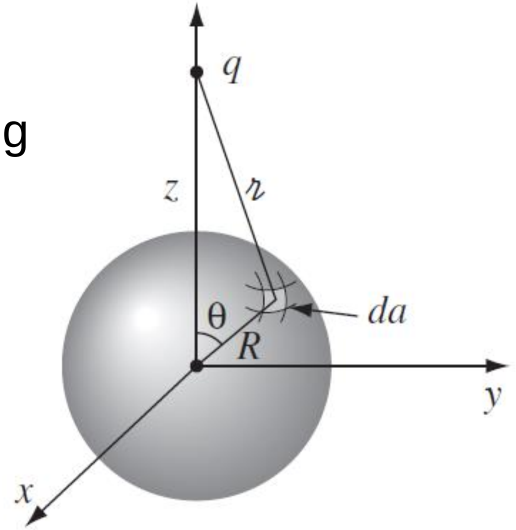
- Laplace equation in 3D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

  - $V(\mathbf{r})$ is the average of V over a spherical surface centered at $\mathbf{r}$
    - $V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V\, da$ where area is a sphere centered at (x, y, z)

Example. Verify the above property by calculating the potential induced by a single charge q

C2.3DLaplace



  - Solution has no local maximum or minimum

Uniqueness theorems

- The first uniqueness theorem

> The solution to Laplace's equation in some volume is uniquely determined if V is specified on the boundary surface.

  - Relevant to apparatus whose parts are connected to battery or ground
  - Proof by contradiction

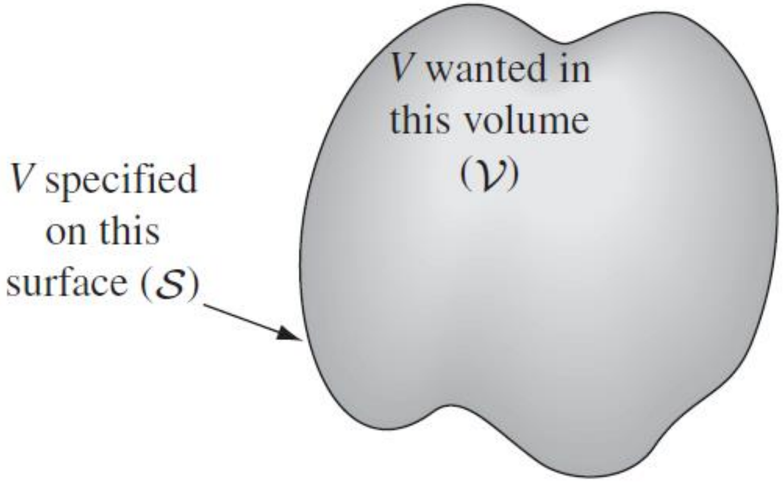
Suppose there are two solutions

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0$$

Define $V_3 \equiv V_1 - V_2$

Then $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$

with $V_3 = 0$ on all boundaries



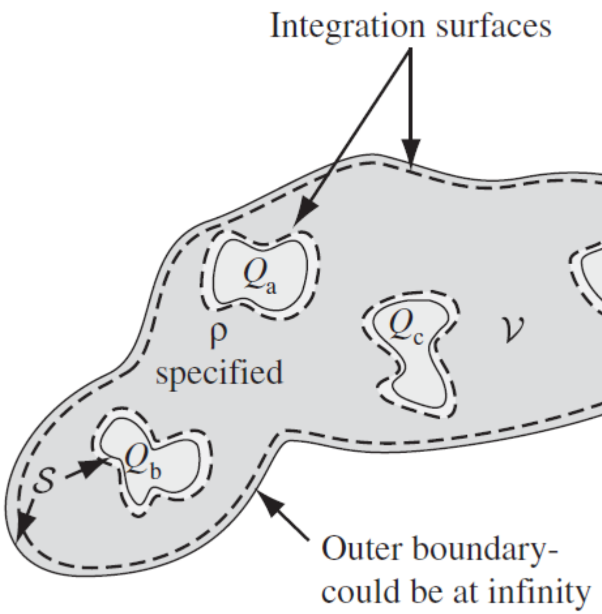
⇨ $V_3 = 0$ everywhere as there is no extrema expect on boundaries

⇨ $V_1 = V_2$

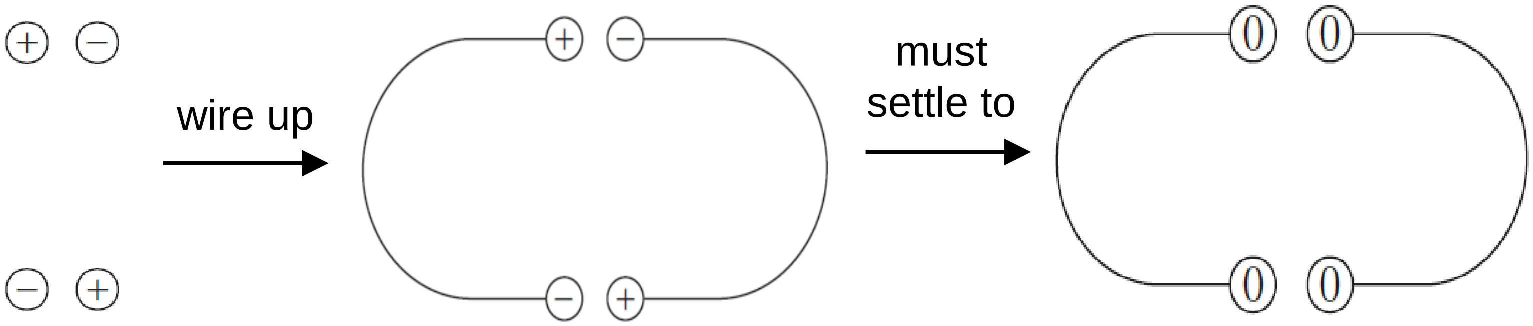
Uniqueness theorems

- The second uniqueness theorem

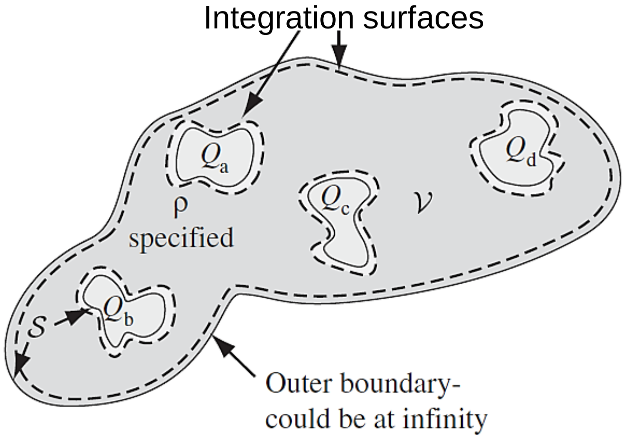
  In a volume surrounded by conductors with a specified charge density, the electric field is uniquely determined if the total charge (not the charge distribution) on each conductor is given.



  - Useful for conductor electrostatics
  - Purcell's example



Uniqueness theorems



- The second uniqueness theorem
  - Proof by contradiction

    Suppose there are two solutions

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0}\rho, \qquad \nabla \cdot \mathbf{E}_2 = \frac{1}{\epsilon_0}\rho$$

$$\Rightarrow \oint_{i\text{th conducting surface}} \mathbf{E}_{1,2} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_i \qquad \oint_{\text{outer boundary}} \mathbf{E}_{1,2} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}$$

Define $\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2$ then $\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0$

Consider the following expression (with product rule and intuition)

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3(\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2$$

$$\Rightarrow \int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{E}_3)\, d\tau = \oint_{\mathcal{S}} V_3 \mathbf{E}_3 \cdot d\mathbf{a} = -\int_{\mathcal{V}} (E_3)^2\, d\tau$$

also

$$= V_3 \oint \mathbf{E}_3 \cdot d\mathbf{a} = 0$$

$\Rightarrow$ $E_3 = 0$ everywhere

Three ways to solve the Laplace equation in the charge-free region

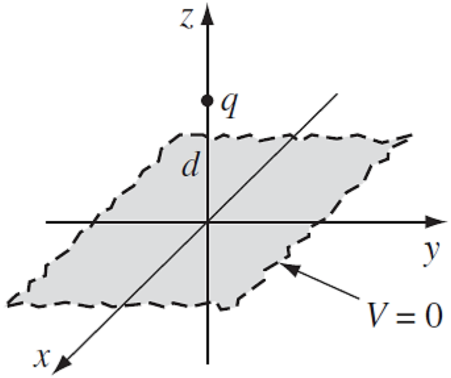
1. The method of images

2. Separation of variables

3. Multipole expansion

The method of images

- Usually works for problems involving charge(s) and conductor(s)
- The classical image problem
  - Point charge q held a distance d from a grounded ($V = 0$) conducting plate (infinitely large)
  - Ask: the potential in the region above the plane
  - The 1st uniqueness theorem applicable - V at all boundaries known
    - $V = 0$ at $z = 0$
    - $V = 0$ at $\sqrt{x^2 + y^2 + z^2} \to \infty$
    - Can guess a $V(x, y, z)$ that is consistent with the Poisson's equation (in the region of interest) and these requirements

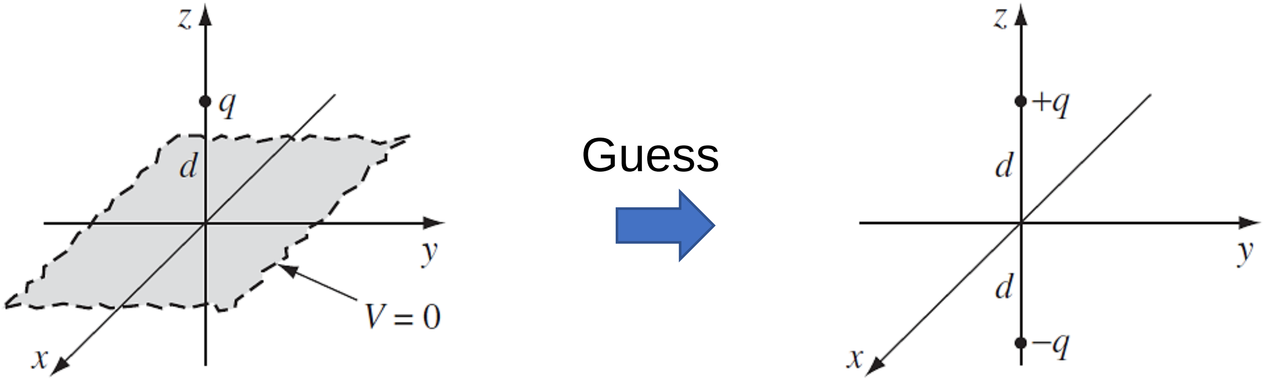


The method of images

- The classical image problem
  - Guess: potential due to two point charges at $(0,0,\pm d)$ in free space

$$V(x,y,z)=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

    - Charge configuration same as original problem at $z>0$ ✓
    - $V=0$ at $z=0$ ✓
    - $V=0$ at $\sqrt{x^2+y^2+z^2}\rightarrow\infty$ ✓

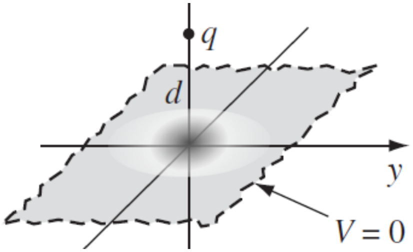


The method of images

- The classical image problem

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

  - Induced surface charge on the conducting plate



$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0}$$

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}$$

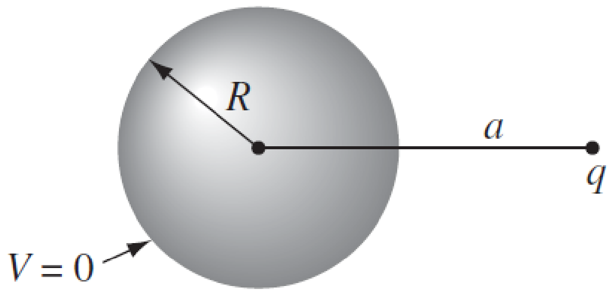
$$\sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

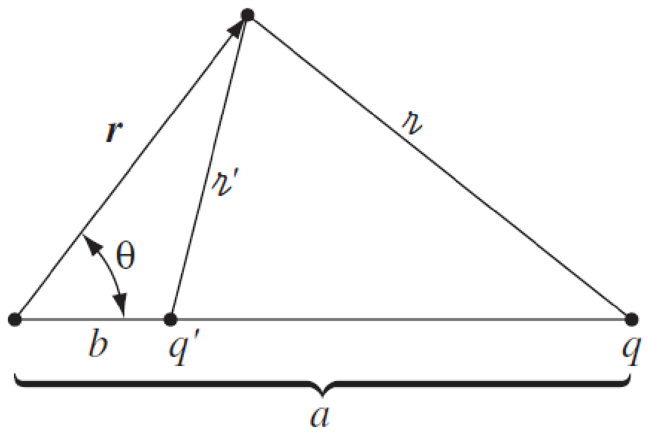
    - Total surface charge (calculate with polar coordinate)

$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r\,dr\,d\phi = \frac{qd}{\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q$$

The method of images

Example 3.2. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R (Fig. 3.12). Find the potential outside the sphere.



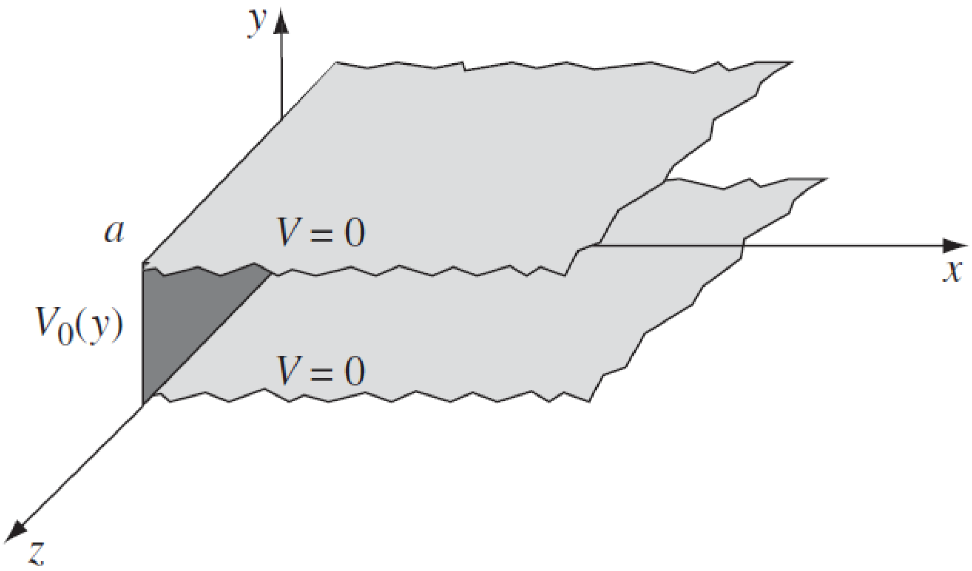


C2.exmp3.2

Separation of variables

- Assume general solutions are products of single-variable functions
  - Applicability strongly depends on problem
- Separation of variables in Cartesian coordinates

Example 3.3. Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$ (Fig. 3.17). The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."



Separation of variables

- Example 3.3

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$

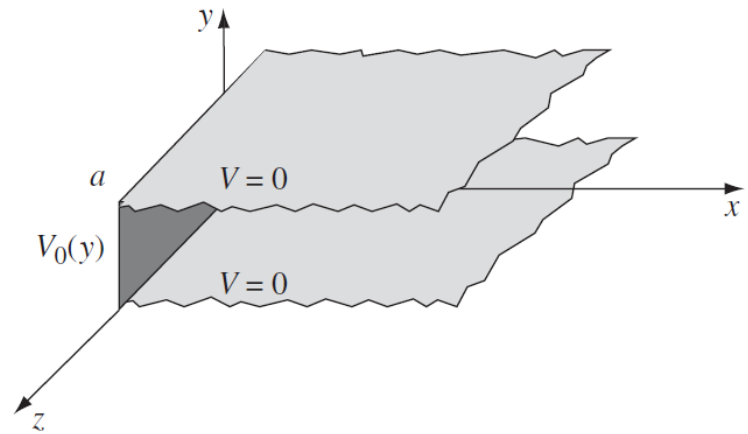
  - Boundary conditions

    (1) $V(x,y) = 0$ when $y = 0$

    (2) $V(x,y) = 0$ when $y = a$

    (3) $V(x,y) = V_0(y)$ when $x = 0$

    (4) $V(x,y) \to 0$ when $x \to \infty$



  - Trial solution $V(x,y) = X(x)Y(y)$

$$\Rightarrow \quad \underline{\frac{1}{X}\frac{d^2X}{dx^2}} + \underline{\frac{1}{Y}\frac{d^2Y}{dy^2}} = 0$$

Two terms only depend on x or y

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \text{with} \quad C_1 + C_2 = 0$$

Separation of variables

- Example 3.3

  - Trial solution $V(x,y) = X(x)Y(y)$

> **?** What would happen if you assume $C_1 < 0$, $C_2 > 0$

    - Suppose $C_1 = k^2 > 0$, $C_2 = -k^2 < 0$ (k being a constant)

$$\Rightarrow \begin{cases} \dfrac{d^2X}{dx^2} = k^2 X \\ \dfrac{d^2Y}{dy^2} = -k^2 Y \end{cases} \Rightarrow \begin{cases} X(x) = Ae^{kx} + Be^{-kx} \\ Y(y) = C\sin ky + D\cos ky \end{cases}$$

$$\Rightarrow V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

  - Determine A, B, C, D by matching boundary conditions

(4) $V(x,y) \to 0$ when $x \to \infty$ $\Rightarrow$ $A = 0$

(1) $V(x,y) = 0$ when $y = 0$ $\Rightarrow$ $D = 0$

(2) $V(x,y) = 0$ when $y = a$ $\Rightarrow$ $k = \dfrac{n\pi}{a}$ $(n = 1,2,3 \ldots)$

Separation of variables

- Example 3.3
  - Determine A, B, C, D by matching boundary conditions

$$V(x, y) = Ce^{-kx} \sin ky \qquad (k = \frac{n\pi}{a})$$

(3) $V(x,y) = V_0(y)$ when $x = 0$, how to match this condition?

- Create a linear combination of solutions

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

(C_n: coefficients to be determined)

$$\Rightarrow \underbrace{V(0, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a)}_{\text{Fourier series expansion of } V_0(y)} = V_0(y)$$

$$\Rightarrow C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy$$

C2.Fourier

Separation of variables

- Example 3.3

  - Examine the solution

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

with $$C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy$$

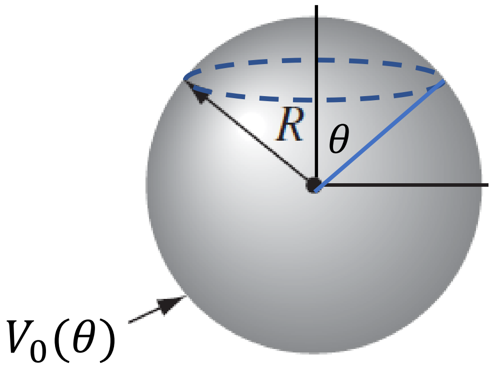
    - The assumption of $V(x, y) = X(x)Y(y)$ was bodacious
    - Validity of this method highly depend on the system's geometry
    - The final solution written as a linear combination of trial solutions is not separated in variables

Separation of variables

- Separation of variables in spherical coordinates
  - Applicable to round objects

Example 3.6. The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside the sphere.

Example 3.7. The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R, but this time we are asked to find the potential *outside*, assuming there is no charge there.



Separation of variables

- Examples 3.6 and 3.7

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

$\downarrow$ Problem & solution must be ϕ-independent

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

- Boundary conditions

  (1) $V(R,\theta)=V_0(\theta)$ at the surface of the sphere

  (2) $V(0,\theta)$ finite

  (3) $V(r,\theta)\to 0$ when $r\to\infty$

- Trial solution $V(r,\theta)=R(r)\Theta(\theta)$

Two terms only depend on r or θ

$$\Rightarrow \underline{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}+\underline{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}=0$$

Separation of variables

- Examples 3.6 and 3.7
    - Trial solution $V(r,\theta) = R(r)\Theta(\theta)$

$$\Rightarrow \begin{cases} \dfrac{1}{R}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right) = l(l+1) \\ \dfrac{1}{\Theta\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d\Theta}{d\theta}\right) = -l(l+1) \end{cases} \Rightarrow \begin{cases} R(r) = Ar^l + \dfrac{B}{r^{l+1}} \\ \Theta(\theta) = P_l(\cos\theta) \end{cases}$$

$$\Rightarrow V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta)$$

$\Rightarrow$ General solution $\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)}$

- Legendre polynomial $P_l(x) \equiv \dfrac{1}{2^l l!}\left(\dfrac{d}{dx}\right)^l (x^2-1)^l$

$$\begin{aligned} P_0(x) &= 1 & P_3(x) &= (5x^3-3x)/2 \\ P_1(x) &= x & P_4(x) &= (35x^4-30x^2+3)/8 \\ P_2(x) &= (3x^2-1)/2 & P_5(x) &= (63x^5-70x^3+15x)/8 \end{aligned}$$

Separation of variables

- Examples 3.6 and 3.7
  - Determine A_l, B_l by matching boundary conditions

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

- For $V(r,\theta)$ inside the sphere

(2) $V(0,\theta)$ finite $\Rightarrow$ $B_l = 0$

(1) $V(R,\theta) = V_0(\theta)$ $\Rightarrow$ $V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta)$

Orthogonality of Legendre polynomial

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = \frac{2\delta_{ll'}}{2l+1}$$

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

Separation of variables

- Examples 3.6 and 3.7
  - Determine A_l, B_l by matching boundary conditions

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

    - For $V(r,\theta)$ outside the sphere

(3) $V(\infty,\theta) \to 0 \quad \Rightarrow \quad A_l = 0$

(1) $V(R,\theta) = V_0(\theta) \quad \Rightarrow \quad V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta)$

$\downarrow$ Orthogonality of Legendre polynomial

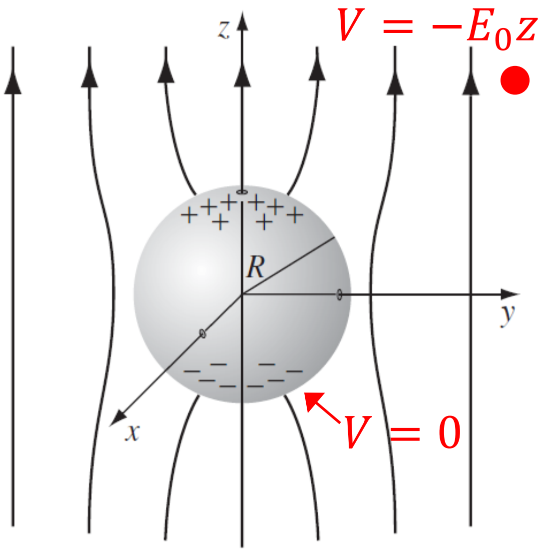
$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

Separation of variables

Example 3.8. An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The field will push positive charge to the "northern" surface of the sphere, and—symmetrically—negative charge to the "southern" surface (Fig. 3.24). This induced charge, in turn, distorts the field in the neighborhood of the sphere. Find the potential in the region outside the sphere.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Far away:
$V = -E_0 z + C$



$V = 0$

Boundary conditions

(1) $V(R,\theta) = 0$ at the conductor

$$\Rightarrow \quad A_l R^l + \frac{B_l}{R^{l+1}} = 0$$

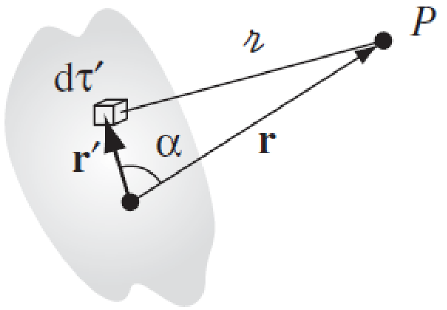
$$\Rightarrow \quad V(r,\theta) = \sum_{l=0}^{\infty} A_l\left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right)P_l(\cos\theta)$$

(2) $V(\infty,\theta) \rightarrow -E_0 r\cos\theta + C$

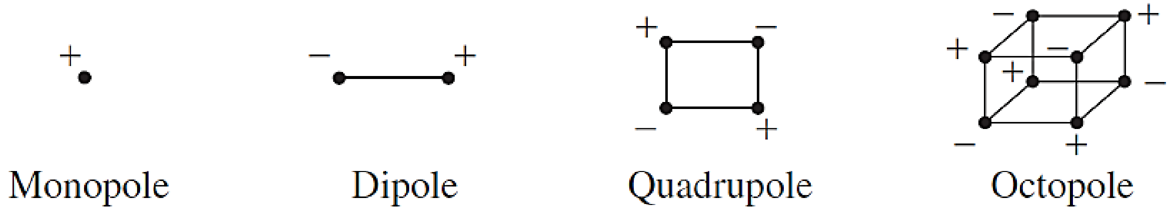
$\Rightarrow \quad A_1 = -E_0$ All other terms zero

Multipole expansion

- Why not exact solution but use an expansion?
  - Exact solution $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$
    - For many problems hard to integrate, because of $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$
    - For many problem not necessary, if one only cares about solution for large $\boldsymbol{r}$
  - Multipole expansion
    - An expansion that examines from low to high order multipole contributions progressively



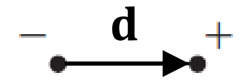
  - Types of multipoles
    - Monopole, dipole, quadrupole, octupole



Multipole expansion

- Types of multipoles
  - Monopole = total charge, $Q = \sum_i q_i$, or $Q = \int \rho(\boldsymbol{r}')d\tau'$
  - Dipole between 2 charges

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i \quad \text{or} \quad \mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau'$$



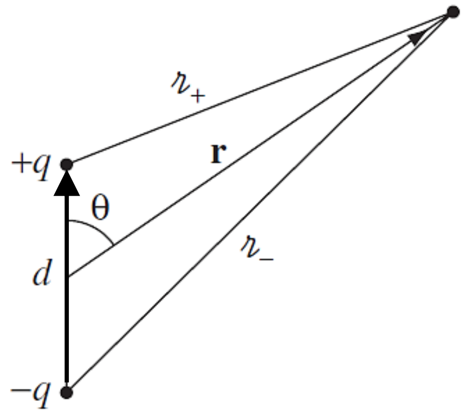
$$\boxed{\mathbf{p} = q\mathbf{d}}$$

(points from $-q$ to $+q$)

Example 3.10. A (physical) **electric dipole** consists of two equal and opposite charges ($\pm q$) separated by a distance d. Find the approximate potential at points far from the dipole.

- Dipole field at large $\boldsymbol{r}$: $\boxed{V_{\text{dip}}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}}$

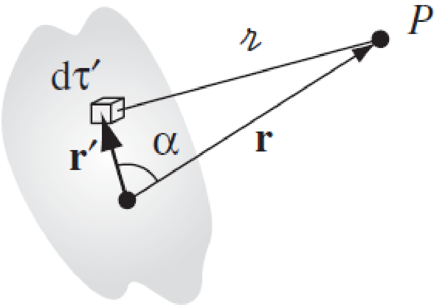
C2.Dipole



Multipole expansion

- Formal multipole expansion

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$



  - Focus on expanding $1/\mathscr{r}$

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$\mathscr{r} = r\sqrt{1+\epsilon}$ where $\epsilon \equiv \left(\dfrac{r'}{r}\right)\left(\dfrac{r'}{r} - 2\cos\alpha\right)$

Taylor expansion knowing $\epsilon \ll 1$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right)$$

Multipole expansion

- Formal multipole expansion
  - Focus on expanding $1/\mathscr{r}$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 \right.$$
$$\left. - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \dots\right]$$

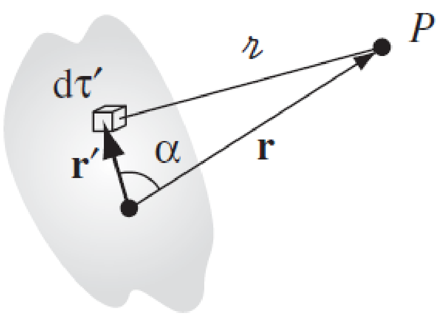
$\Big\downarrow$ Sort by different powers of r'/r

$$= \frac{1}{r}\left[\underline{1} + \left(\frac{r'}{r}\right)\underline{(\cos\alpha)} + \left(\frac{r'}{r}\right)^2 \underline{\left(\frac{3\cos^2\alpha - 1}{2}\right)} \right.$$
$$\left. + \left(\frac{r'}{r}\right)^3 \underline{\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right)} + \dots\right] \qquad = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

In the form of Legendre polynomials

Multipole expansion

- Formal multipole expansion



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau'$$

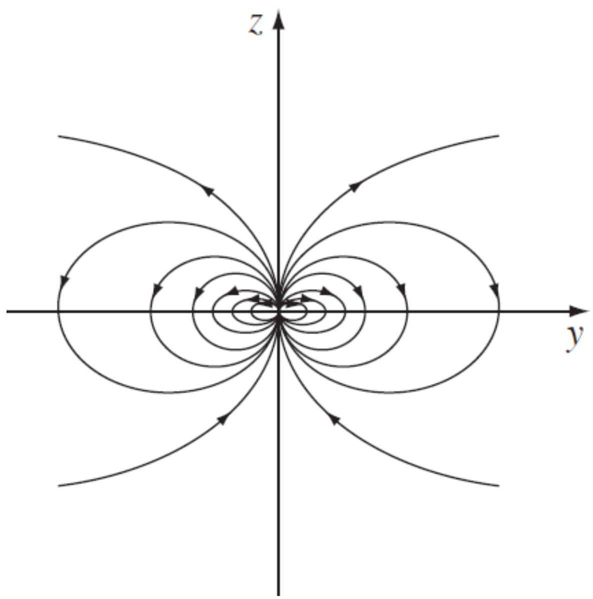
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\underbrace{\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau'}_{\text{Monopole}} + \underbrace{\frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau'}_{\text{Dipole}}\right.$$
$$\left. + \underbrace{\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau'}_{\text{Quadrupole}} + \ldots\right]$$

- Merits of multipole expansion
  - r moved out of the integral ($\alpha = \theta$ polar angle if we set $\boldsymbol{r}$ along $\hat{z}$)
  - Can truncate to finite terms
- Caveat: terms can depend on choice of origin (such as dipole term)

Multipole expansion

- Pure dipole vs physical dipole

Pure dipole

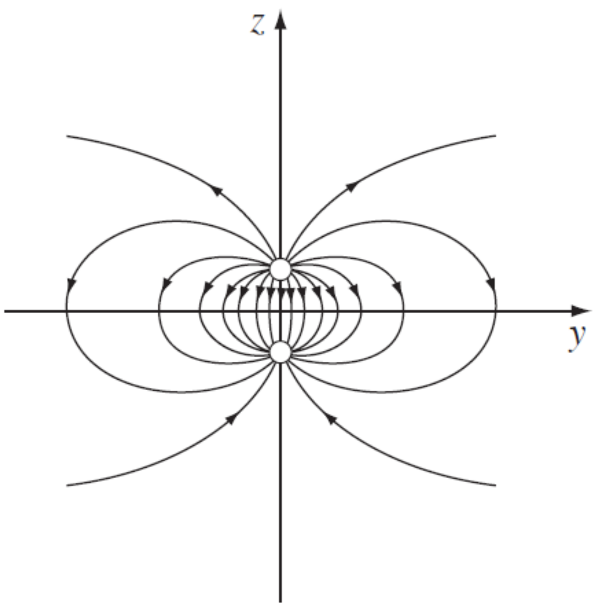


$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

Assumes $r \gg d$ always holds

$\mathbf{p} = q\mathbf{d}$ but take $q \to \infty$, $d \to 0$

Physical dipole



Deviations appear when closing up onto the dipole

$$\mathbf{p} = q\mathbf{d}$$

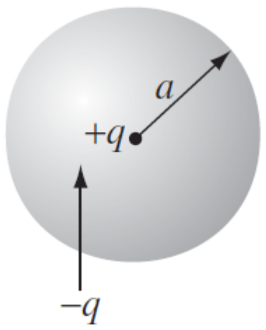
Electric fields in matter

Dielectrics

- Electrostatic responses of materials
  - Conductors: unlimited supply of free charges, which are free to move
  - Dielectrics: insulators in which all charges are attached to specific atoms

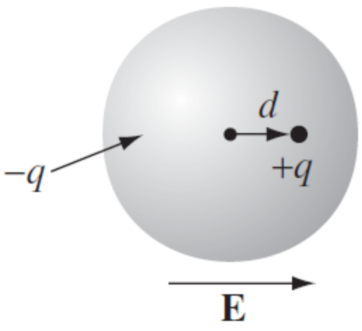
- Two ways of inducing macroscopic polarizations in dielectrics
  - Induced dipole in neutral atoms /molecules

Without $\mathbf{E}$



Electron cloud around central nucleus

With $\mathbf{E}$

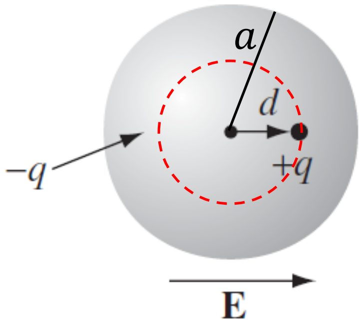


Electron cloud displaced from central nucleus

Dielectrics

- Two ways of inducing macroscopic polarizations in dielectrics
  - Induced dipole in neutral atoms/molecules
    - Field from e⁻ cloud at the $+q$ location

      Apply Gauss's law for sphere with radius d



$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

$$\Rightarrow \quad 4\pi d^2 E_e = \frac{1}{\epsilon_0} q \frac{4/3 \cdot \pi d^3}{4/3 \cdot \pi a^3} \quad \Rightarrow \quad E_e = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}$$

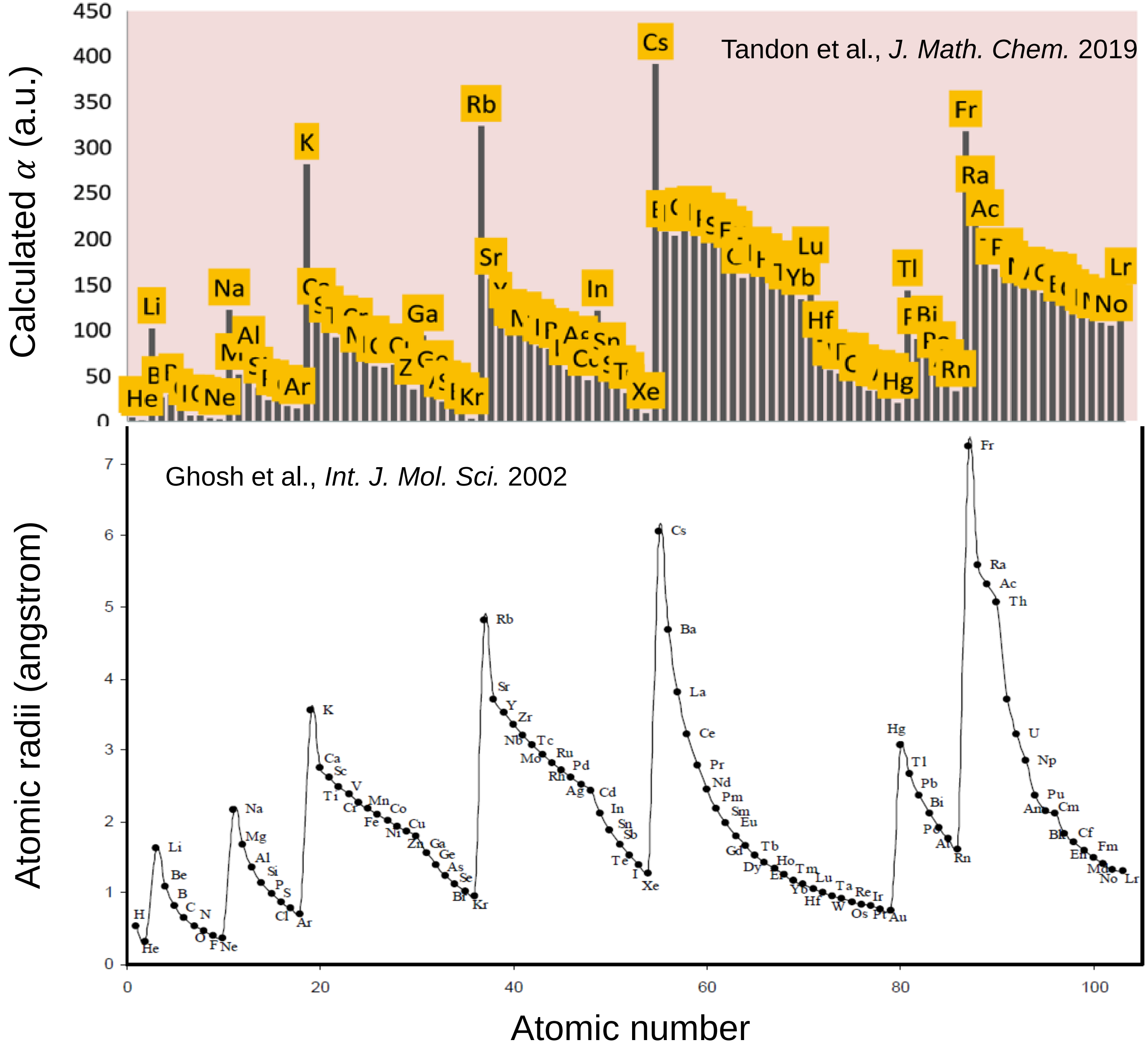
- In equilibrium, external field $E = E_e$

$$\Rightarrow \quad E = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}$$

coefficient bridging applied field with induced dipole moment

$$\Rightarrow \quad qd = p = \underline{(4\pi\epsilon_0 a^3)} E$$

Polarizability α (in au)



Dielectrics

- Two ways of inducing macroscopic polarizations in dielectrics
  - Induced dipole in neutral atoms/molecules
    - Isotropic atoms/molecules

$$\boxed{\mathbf{p} = \alpha \mathbf{E}}$$

Induced dipole moment

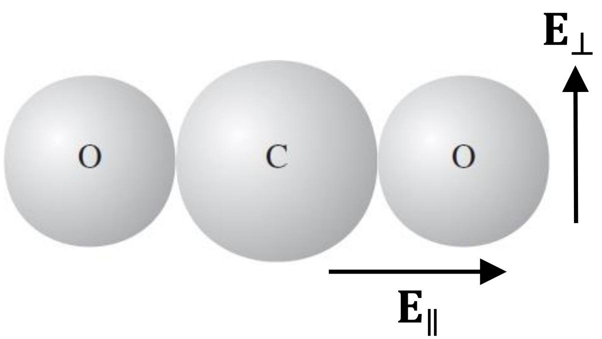
Polarizability

Applied electric field

    - Anisotropic molecules

$$\mathbf{p} = \overleftrightarrow{\alpha} \cdot \mathbf{E}$$

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

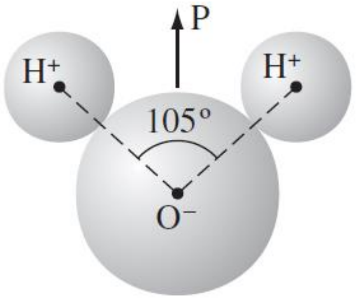


$\overleftrightarrow{\alpha}$ tensor structure can be simplified based on molecular symmetry

Dielectrics

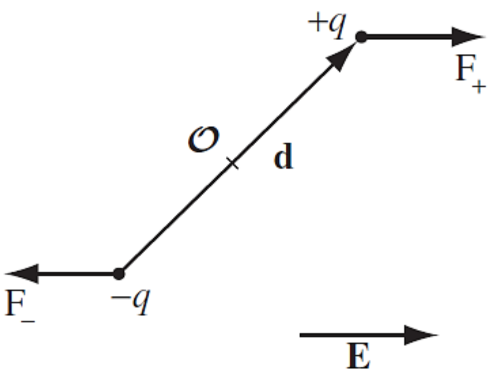
- Two ways of inducing macroscopic polarizations in dielectrics
  - Alignment of polar molecules
    - Polar molecule: molecules with built-in permanent dipole moments

      Example : H_2O (examine centers of + and – charges)



    - Torque in a uniform field

$$
\begin{aligned}
\mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\
&= \left[(\mathbf{d}/2) \times (q\mathbf{E})\right] + \left[(-\mathbf{d}/2) \times (-q\mathbf{E})\right] \\
&= q\mathbf{d} \times \mathbf{E} \\
&= \mathbf{p} \times \mathbf{E}
\end{aligned}
$$

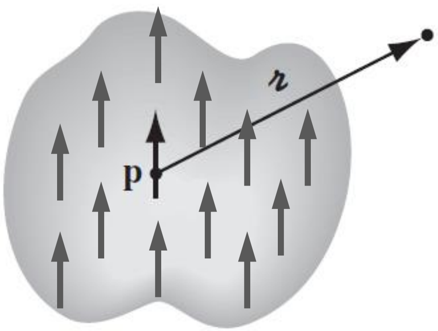


Torque tends to align $\mathbf{p}$ parallel to $\mathbf{E}$

Polarization

- Dielectrics as an ensemble of atoms/molecules responding to $\mathbf{E}$
  - Polarization: dipole moment per unit volume

$$\mathbf{P} = \frac{1}{V}\sum_i \mathbf{p}_i$$



* $\mathbf{P}$ can be nonuniform: denoted by $\mathbf{P}(\mathbf{r}')$

  - Potential of a polarized dielectric
    - Neglect the cause of the polarization (will be considered later)
    - Potential around a single (pure) dipole $V_{\text{dip}}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$
    - Potential induced by many dipoles

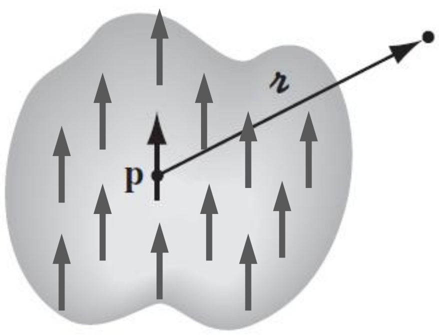
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

* $d\mathbf{p} = \mathbf{P}(\mathbf{r}')\,d\tau'$

Dipole moment in the small element of volume

Polarization

- Dielectrics as an ensemble of atoms/molecules responding to $\mathbf{E}$
  - Potential of a polarized dielectric



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

$$\Big\downarrow \quad \nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

C2.Delprime

$$= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

$\Big\downarrow$ Integrate by parts

$$= \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{1}{\mathscr{r}}\mathbf{P}\cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}}\,da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}}\,d\tau'$$

Surface normal

* with $\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}$

$\rho_b \equiv -\nabla\cdot\mathbf{P}$

Polarization

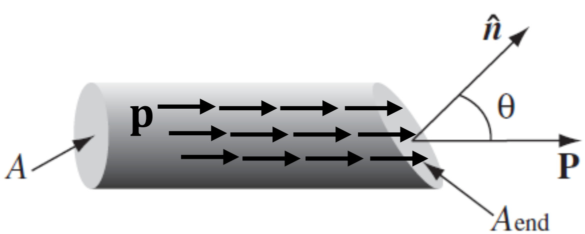
- Dielectrics as an ensemble of atoms/molecules responding to $\mathbf{E}$
  - Potential of a polarized dielectric

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}}\, da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}}\, d\tau'$$

    - Density of bound charges $\begin{cases} \text{surface: } & \boxed{\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}} \\ \text{volume: } & \boxed{\rho_b \equiv -\nabla\cdot\mathbf{P}} \end{cases}$

  - Physical picture of bound charges

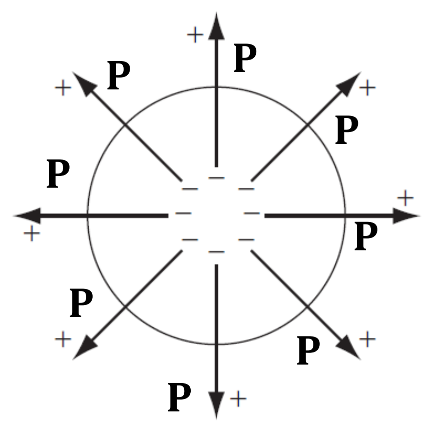
Surface bound charge (suppose uniform P)



Dangling charges on the surface

C2.surfacecharge

Volume bound charge



Examine

$$\oint_{\mathcal{S}} \mathbf{P}\cdot d\mathbf{a}'$$

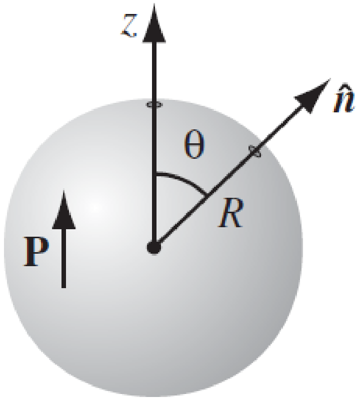
$$\int_{\mathcal{V}} (\nabla'\cdot\mathbf{P})\, d\tau'$$

Polarization

- Dielectrics as an ensemble of atoms/molecules responding to $\mathbf{E}$
    - Electric field produced by a polarized dielectric

Example 4.2. Find the electric field produced by a uniformly polarized sphere of radius R.

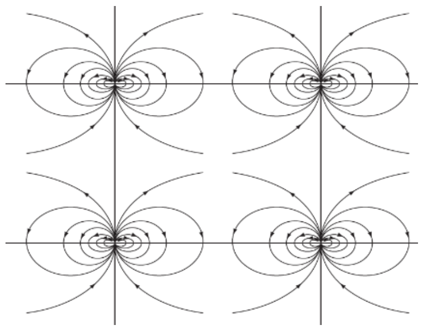
C2.exmp4.2



Example 3.9. A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

Polarization

- Two implicit approximations
  - Averaged out nonuniform microscopic field



$$\mathbf{P} = \frac{1}{V}\sum_i \mathbf{p}_i$$

$\mathbf{P}$

Overall object≫
V ≫ atom size

  - Physical dipole approximated by pure dipole
    - $V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\mathcal{V}} \dfrac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$ applicable only when $\mathscr{r} \gg d$

Electric displacement

- Add the cause and the effect of polarization
  - Total electric field

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

    - $\mathbf{E}$: total electric field, not just the portion generated by polarization
    - ρ: total charge density
    - ρ_b: bound charge density, due to polarization
    - ρ_f: free charge density that we control, not a result of polarization

$$\Rightarrow \quad \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

  - The electric displacement $\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}}$

$$\Rightarrow \quad \boxed{\nabla \cdot \mathbf{D} = \rho_f}$$

    - Gauss's law for displacement field $\boxed{\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}}$

Enclosed total free charge

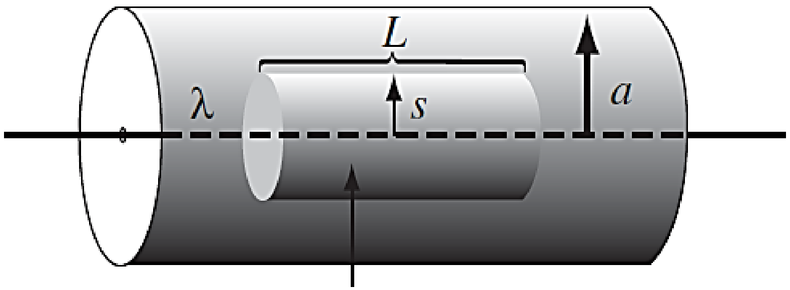
Electric displacement

- Divergence and curl of electric displacement
  - The electric displacement $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$
    - $\nabla \cdot \mathbf{D} = \rho_f$
    - Gauss's law for displacement field $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$
    - $\nabla \times \mathbf{D} = \epsilon_0 (\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P} \neq \mathbf{0}$
    - No Coulomb's law for displacement field $\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi} \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho_f(\mathbf{r}')\, d\tau'$
    - No potential for displacement field

Electric displacement

- Application of electric displacement

Example 4.4. A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a (Fig. 4.17). Find the electric displacement.



Gaussian surface

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\Rightarrow \quad D(2\pi s L) = \lambda L$$

$$\Rightarrow \quad \mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}$$

For both inside and outside the rubber

Outside the rubber, $\mathbf{P} = 0$

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$$

Inside the rubber, $\mathbf{P} \neq 0$

$$\mathbf{E} = \frac{1}{\varepsilon_0}\mathbf{D} - \mathbf{P}$$

? What is missing?

Linear dielectrics

- An infinite regress
  - Applied $\mathbf{E}$ induces $\mathbf{P}$, $\mathbf{P}$ produces field, and adds to $\mathbf{E}$, which in turn modifies $\mathbf{P}$…
  - Breaking out from this regress requires

    (1) an equation linking total $\mathbf{P}$ with total $\mathbf{E}$ (applied plus $\mathbf{P}$-induced)

    (2) starting by analyzing $\mathbf{D}$, which is related to free charges

- Linear dielectrics
  - $\boxed{\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}}$ (χ_e: electric susceptibility)

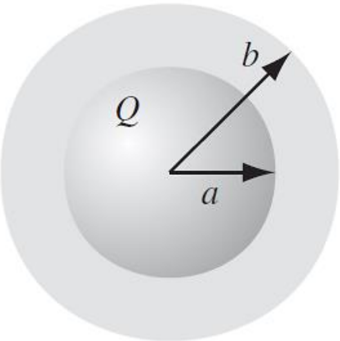
    $\Rightarrow$ $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}$
  - $\boxed{\mathbf{D} = \epsilon \mathbf{E} \quad (\epsilon = \epsilon_0 (1 + \chi_e)\text{: permittivity})}$
  - Relative permittivity $\boxed{\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}}$
    - Also called the dielectric constant

Material	Dielectric Constant
Benzene	2.28
Diamond	5.7-5.9
Salt	5.9
Silicon	11.7
Methanol	33.0
Water	80.1
Ice (-30° C)	104
$KTaNbO_3$ (0° C)	34,000

Linear dielectrics

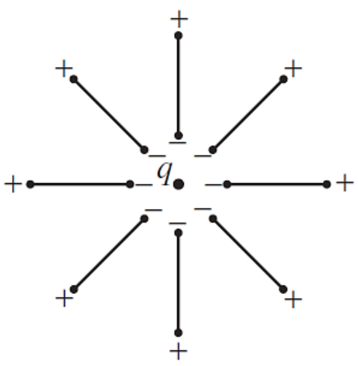
Example 4.5. A metal sphere of radius a carries a charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

C2.exmp4.5



Example: calculate the field produced by a free charge q embedded in a large dielectric (with permittivity of ϵ)

$$\mathbf{E} = \frac{1}{4\pi\underline{\epsilon}}\frac{q}{r^2}\hat{\mathbf{r}}$$



Screening effect by bound charges

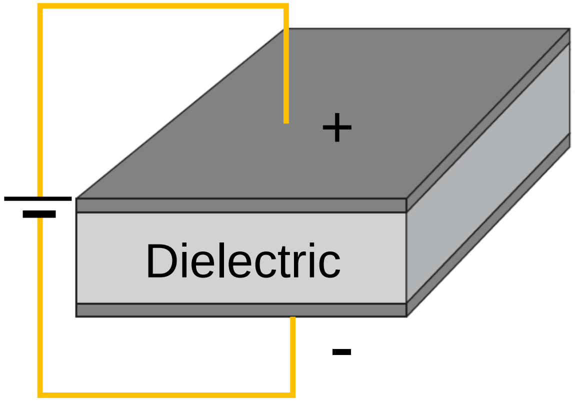
For metal, view the $\varepsilon \to \infty$ limit

Use of dielectrics: capacitors

- Capacitor with dielectrics
  - $\mathbf{D}$ derived from Gauss law is $\mathbf{E}$ as if the dielectric does not exist

$$\nabla \cdot \mathbf{D} = \rho_f \xrightarrow{\text{Gauss's law}} \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\Rightarrow \mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}}$$



  - $\mathbf{E}$ in dielectric is reduced from $\mathbf{E}$ in vacuum

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}}$$

  - Potential V reduced from that in vacuum

$$V = V_{\text{vac}}/\epsilon_r \quad \text{since we have} \quad V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}$$

  - Capacitance increased $C = \epsilon_r C_{\text{vac}}$

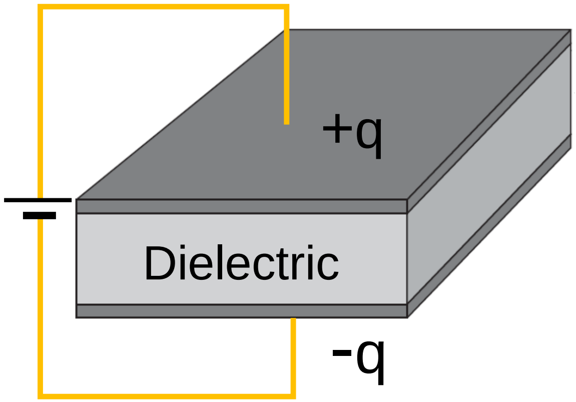
Use of dielectrics: capacitors

- Energy stored in capacitors
  - Consider charging from q to $q + dq$

$$dW = V(q)\,dq = \left(\frac{q}{C}\right) dq$$

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C}$$

$$\boxed{W = \frac{1}{2} C V^2}$$



  - With dielectric as filling, energy stored in a capacitor is boosted

- Electrostatic energy in dielectrics $\boxed{W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\, d\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau}$
  - Compare with equation in vacuum $W = \frac{\epsilon_0}{2}\int E^2\, d\tau$

Boundary conditions

- Boundary conditions reexamined in the context of dielectrics
  - Earlier findings still hold, but σ needs to include free and bound charges

$$\begin{cases} E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \dfrac{1}{\epsilon_0}\sigma \\ \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}} \end{cases}$$

  - Easier to use the boundary conditions of $\mathbf{D}$

Surface free charge density

$$\begin{cases} \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\ \nabla \times \mathbf{D} = \nabla \times \mathbf{P} \end{cases} \Rightarrow \boxed{\begin{cases} D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f \\ \mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}} \end{cases}}$$

And, for linear dielectrics

$$\begin{cases} \epsilon_{\text{above}} E^{\perp}_{\text{above}} - \epsilon_{\text{below}} E^{\perp}_{\text{below}} = \sigma_f \\ \epsilon_{\text{above}} \dfrac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \dfrac{\partial V_{\text{below}}}{\partial n} = -\sigma_f \\ V_{\text{above}} = V_{\text{below}} \end{cases}$$