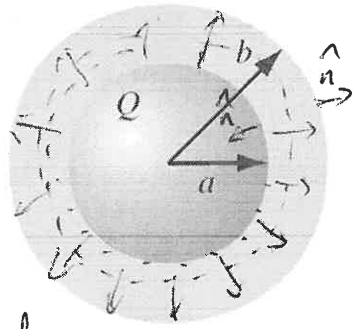
Example 4.5. A metal sphere of radius a carries a charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).



for $a < r \leq b$

$\oint \vec{D} \cdot d\vec{a} = Q_{enc}$

$\Rightarrow D \cdot 4\pi r^2 = Q$

$\Rightarrow \vec{D} = \frac{Q}{4\pi r^2} \hat{r}$

$\Rightarrow \vec{E}_{in} = \frac{1}{\epsilon} \vec{D} = \frac{Q}{4\pi \epsilon r^2} \hat{r}$

$= \frac{Q}{4\pi \epsilon_0 \epsilon_r r^2} \hat{r}$

for $r > b$ $\quad \vec{D} = \frac{Q}{4\pi r^2} \hat{r}$

$\Rightarrow \vec{E}_{out} = \frac{1}{\epsilon_0} \vec{D} = \frac{Q}{4\pi \epsilon_0 r^2} \hat{r}$

$$V = -\int_\infty^b \vec{E}_{out} \cdot d\vec{r} - \int_b^a \vec{E}_{in} \cdot d\vec{r}$$

$$= -\int_\infty^b \frac{Q}{4\pi\epsilon_0 r^2} dr - \int_b^a \frac{Q}{4\pi\epsilon r^2} dr$$

$$= \frac{Q}{4\pi} \left[\frac{1}{\epsilon_0} \left(\frac{1}{b} - 0 \right) + \frac{1}{\epsilon} \left(\frac{1}{a} - \frac{1}{b} \right) \right]$$

$$= \frac{Q}{4\pi} \left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right)$$

$\vec{P}$ in dielectric $\quad \vec{P} = \epsilon_0 \chi_e \vec{E}_{in} = \frac{\epsilon_0 \chi_e Q}{4\pi \epsilon r^2} \hat{r}$

$$\sigma_b = \vec{P} \cdot \hat{n} = \begin{cases} \frac{\epsilon_0 \chi_e Q}{4\pi \epsilon b^2} & \text{at outer surface} \\ -\frac{\epsilon_0 \chi_e Q}{4\pi \epsilon a^2} & \text{at inner surface} \end{cases}$$