As we have already learned how to describe a free photon $A_\mu(x)$ and a free electron $\psi(x)$ as plane wave solutions of the Maxwell equations and the Dirac equation respectively, we can proceed to study their interaction.

The interaction is dictated by the gauge symmetry or the principle of gauge invariance.

Instead of using Hamiltonian

$$H = H_{photon} + H_{electron} + H_I$$

Lagrangian density is used

$$\mathcal{L} = \mathcal{L}_{photon} + \mathcal{L}_{electron} + \mathcal{L}_I$$

$$\mathcal{L}_I = q\, j_\mu A^\mu = q\, j^\mu A_\mu$$

$$j^\mu = c\, \bar{\psi}(x) \cdot \gamma^\mu \psi(x)$$

We shall proceed the study, using Feynman rules and Feynman diagrams.

Instead of using quantum field theoretic method to derive the transition amplitude (scattering amplitude) and hence the differential cross section, we use a diagramatic method, the Feynman diagram

For any physical process, we first sketch the Feynman diagram for the process (we learned in chapter 2). Then using a dictionary (Feynman rules), each piece of the diagram can be translated to mathematical expression (symbol)

These mathematical expressions are joined up together to give the scattering amplitude.

We now list out the Feynman rules

Using examples, we illustrate how scattering amplitudes can be derived from a Feynman diagram using Feynman rules

Summary

e^-

e^+

Wave functions

u is an unknown bispinor for charged particle which we need to find to know Ψ(x_ubar), similar to how we found ε(p_ubar) for the photon case

$$\psi(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar} u^{(s)}(\underline{p})$$

s=1 spin up
s= 2 spin down

$$(\not{p} - mc)u = 0$$

$$\bar{u}(\not{p} - mc) = 0 \quad \bar{u} = u^\dagger \gamma^0$$

$$\psi(\underline{x}) = e^{i\underline{p}\cdot\underline{x}/\hbar} v^{(s)}(\underline{p})$$

s=1 spin down
s=2 spin up

$$(\not{p} + mc)v = 0$$

$$\bar{v}(\not{p} + mc) = 0 \quad \bar{v} = v^\dagger \gamma^0$$

Orthonormality

untested $\bar{u}^{(s_1)} u^{(s_2)} = 2mc\delta_{s_1 s_2}$

$u^\dagger(p)\, u(p) = 2|p^0|$

$$\bar{v}^{(s_1)} v^{(s_2)} = -2mc\delta_{s_1 s_2}$$

$s_1, s_2 = 1, 2$

Completeness $\sum_i |i\rangle\langle i| = 1$ (completeness of a basis)

first encounter with completeness likely in QM1, completeness of a basis

$$\sum_{s=1}^{2} u^{(s)} \bar{u}^{(s)} = (\not{p} + mc)$$

$$\sum_{s=1}^{2} v^{(s)} \bar{v}^{(s)} = (\not{p} - mc)$$

Photon

Plane Wave

$$A^\mu(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar} \varepsilon^\mu_{(s)},$$ s=1, 2 for the two polarization states

Polarization vector ε^{μ} satisfies, $p_{\mu}\varepsilon^{\mu} = 0$ Lorentz condition

Orthonormality

$$\varepsilon^{\mu^*}_{s_1}\varepsilon_{\mu(s_2)} = \delta_{s_1 s_2}$$

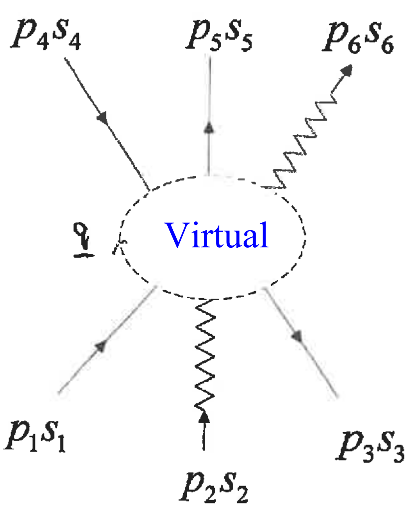
Coulomb gauge $\varepsilon^{\circ} = 0, \quad \underset{\sim}{\varepsilon} \cdot \underset{\sim}{p} = 0$

Completeness

$$\sum_{s=1}^{2} (\varepsilon_{(s)})_i (\varepsilon^*_{(s)})_j = \delta_{ij} - \hat{p}_i \hat{p}_j \qquad \hat{p}_i = p_i / |\underset{\sim}{p}|$$

Feynman rules QED

Real



Real

Time

These are states of the incmg and outgg particles. Only relevant quantum numbers are the momentum and spin

Notations

Label external lines by momentum p_i and spin s_i,

Label internal lines by momenta q_i in this chapter, q represents 4-momentum

Arrows on external fermion lines indicate

e^- (forward in time) $\qquad$ e^+ (backward in time)

particle $\qquad$ anti particle

Arrows on internal fermion lines are assigned so that direction of the flow of 4-momenta through the diagram is kept.

Arrows on external photon lines point forward; for internal photon lines, the choice is arbitrary.

must remember!

(i) External lines

Comparison with harmonic oscillator

e^-	incoming	: u	annihilation
	outgoing	: $\bar{u}$	creation
e^+	incoming	: $\bar{v}$	
	outgoing	: v	
γ	incoming	: ε^{μ}	
	outgoing	: ε^{μ^*}	

(ii)Vertex

Each vertex contributes a factor $ig\gamma^{\mu}$

g= dimensionless coupling constant $=\sqrt{4\pi\alpha}$

$$\alpha = \frac{e^2}{\hbar c} = \frac{q_e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137}$$

(iii) Propagators (internal lines)

HW

$$e^- \text{ or } e^+ : \quad \frac{i}{\not q - mc} = \frac{i(\not q + mc)}{q^2 - m^2c^2}$$

$\not q = q_\mu \gamma^\mu$

$\frac{i}{\not q - mc}$, where $\not q = q_\mu \gamma^\mu$

$= \frac{i(\not q + mc)}{(\not q - mc)(\not q + mc)}$

Note: $[\not q, mc] = 0$ as mc is a constant and

$q_\mu\gamma^\mu \cdot q_\nu\gamma^\nu = q_\mu q^\nu \gamma^\mu \gamma_\nu = q_\nu q^\mu \gamma^\nu \gamma_\mu = \frac{1}{2}(q_\mu q^\nu \gamma^\mu\gamma_\nu + q_\nu q^\mu \gamma^\nu \gamma_\mu) = \not q^2$

$= \frac{i(\not q + mc)}{\not q^2 - m^2c^2}$

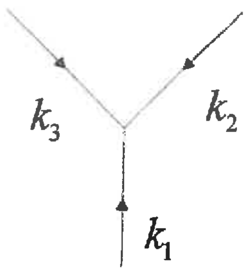
$$\text{photon } \gamma : \quad \frac{-ig_{\mu\nu}}{q^2}$$

(iv) Conservation of 4 - momentum P_μ:

Time

p_2 $\quad \mu, k$

$p_1 \quad (2\pi)^4 \delta^{(4)}(\underline{p}_1 - \underline{p}_2 - \underline{k})$



For each vertex, write $\quad (2\pi)^4 \delta^{(4)}(\underline{k}_1 + \underline{k}_2 + \underline{k}_3)$

(v) Integrate over internal momenta

$$\boxed{\int \frac{d^4 q}{(2\pi)^4}}$$

(vi) Cancel the overall delta function

$$(2\pi)^4 \delta^{(4)}(\underline{p}_1 + \underline{p}_2 \dots \underline{p}_n)$$

what remains is the $-iM$. Multiply by i, and obtain
M= scattering amplitude

(vii) Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) e^-'s (or e^+'s)
or of an incoming e^- with an outgoing e^+ (or vice versa)

crossing symmetry

(viii) Charge is conserved at each vertex.
Lepton number etc must also be conserved.

(ix) For a closed fermion loop, include a factor -1 and take the trace.

Examples

(i) $e^- \mu^- \to e^- \mu^-$ electron muon scattering

(ii) $e^- e^- \to e^- e^-$ Møller scattering

(iii) $e^- e^+ \to e^- e^+$ Bhabha scattering

(iv) $e^- \gamma \to e^- \gamma$ Compton scattering

(v) $e^- e^+ \to \gamma \gamma$ annihilation

$\uparrow \quad \uparrow \qquad \uparrow$

$\underline{u} \quad \underline{v} \qquad \underline{\varepsilon}$

Calculating scattering amplitudes at the tree level for the following processes:

(1) $e^- \mu^- \to e^- \mu^-$ Electron-muon scattering

(2) $e^- e^- \to e^- e^-$ Møller scatt

(3) $e^+ e^- \to e^+ e^-$ Bhabha scatt.

(4) $e^- \gamma \to e^- \gamma$ Compton scatt.

(5) $e^+ e^- \to \gamma \gamma$ photo production

The scattering amplitude is a function of the initial and final states, meaning that

$$\mathcal{M} = \mathcal{M}(p_1, s_1, p_2 s_2 ; p_3, s_3, p_4, s_4)$$

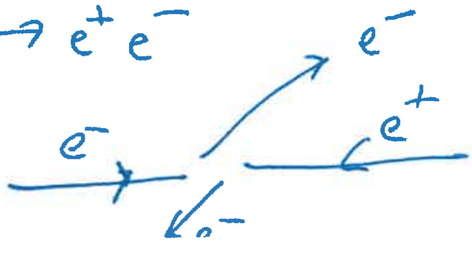
differential cross section $\frac{d\sigma}{d\Omega}$ = kinematic part $\cdot\, |\mathcal{M}|^2$

dynamic part

Casimir trick to sum $|\mathcal{M}|^2$

Proceed to compute $\mathcal{M}$

(3) $e^+ e^- \to e^+ e^-$



We show, by using examples, how the scattering amplitude $\mathcal{M}$ can be computed from the Feynman diagram using the Feynman rules.

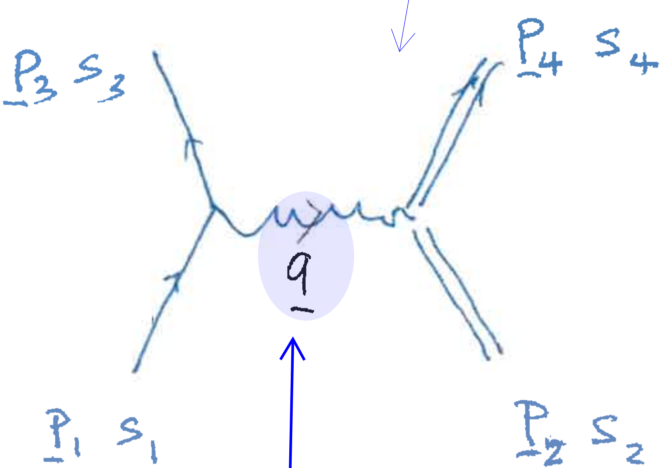
To begin with, we consider a simple process

(1) electron - muon scattering

$e^- \mu^- \rightarrow e^- \mu^-$

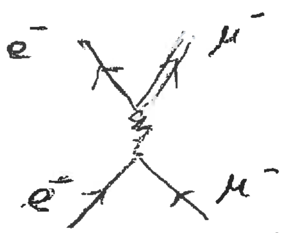
Interaction governed by the vertex

At the tree-level, only one possibility of joining up the lines with the only allowed vertex

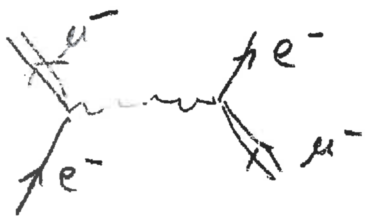


direction of virtual particle doesn't matter in drawing of diagrams and in calculation

every vertex must have 1 in, 1 out and 1 int. line



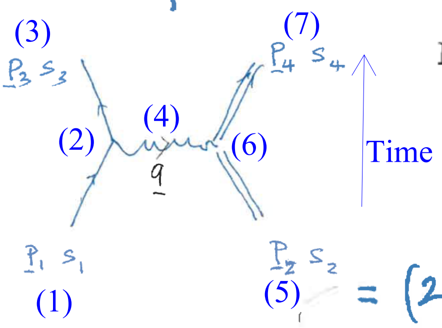
not allowed
(charge not conserved)



not allowed
(lepton number conservation)

Use Feynman's rules to translate the diagram into mathmatical expression. AND FOLLOW THE TRACK (see moller scattering order of writing the outgoing particles)

Read the diagram from left to Right and below to Top. Write the expressions from Right to left.



Each vertex contributes a factor $ig\gamma^\mu$ $g=\sqrt{4\pi\alpha},\ \alpha=\frac{1}{137}$

Photons: $\frac{-ig_{\mu\nu}}{q^2}$

Any arbitrary index can be used

$$= \underbrace{(2\pi)^4\delta^{(4)}(P_1 - P_3 - q)}_{(8)} \cdot \underbrace{\frac{-ig_{\mu\nu}}{q^2}}_{(4)} \underbrace{\bar{u}(P_3, s_3)}_{(3)} \cdot \underbrace{ig\gamma^\mu}_{(2)} \cdot \underbrace{u(P_1, s_1)}_{(1)}$$

$$\underbrace{\int \frac{d^4q}{\cancel{(2\pi)^4}}}_{(10)} \underbrace{\cancel{(2\pi)^4}\delta^{(4)}(q + P_2 - P_4)}_{(9)} \underbrace{\bar{u}(P_4, s_4)}_{(7)} \underbrace{ig\gamma^\nu}_{(6)} \underbrace{u(P_2, s_2)}_{(5)}$$

$$= ig^2 (2\pi)^4 \underbrace{\int d^4q}_{(10)} \underbrace{\bar{u}(P_4, s_4)\gamma^\nu u(P_2, s_2)}_{(765)} \cdot \underbrace{\delta^{(4)}(P_1 - P_3 - q)}_{(8)} \cdot$$

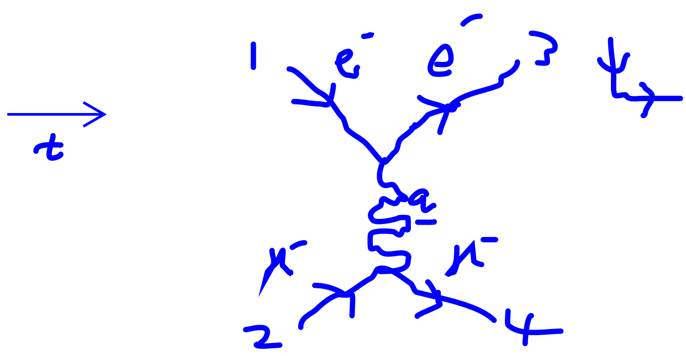
complex number

$$\cdot \underbrace{\frac{g_{\mu\nu}}{q^2}}_{(4)} \cdot \underbrace{\bar{u}(P_3, s_3)\gamma^\mu u(P_1, s_1)}_{(321)} \underbrace{\delta^{(4)}(q + P_2 - P_4)}_{(9)}$$

int away (8), re-express (4), (9) and (10)

$$= ig^2 (2\pi)^4 \underbrace{\bar{u}(P_4, s_4)\gamma^\nu u(P_2, s_2)}_{(765)} \underbrace{\frac{g_{\mu\nu}}{(P_1 - P_3)^2}}_{(4,10)\to(4)} \cdot$$

$$\underbrace{\bar{u}(P_3, s_3)\gamma^\mu u(P_1, s_1)}_{(321)} \cdot \underbrace{\delta^{(4)}(P_1 - P_3 + P_2 - P_4)}_{(9)}$$



Throw away the overall $\overset{(9)}{\underline{\delta\text{elta function}}}$ for 4-momentum conservation, we get

$$-i\mathcal{M} = i g^2 \overset{(765)}{\underline{\bar{u}(\underline{P}_4\ s_4)\,\gamma^\nu\, u(\underline{P}_2\ s_2)}}\ \overset{(4)}{\frac{g_{\mu\nu}}{(\underline{P}_1 - \underline{P}_3)^2}}\cdot$$

$$\overset{(321)}{\underline{\bar{u}(\underline{P}_3,\ s_3)\,\gamma^\mu\, u(\underline{P}_1\ s_1)}}$$

Multiplying by i to get $\mathcal{M}$, the scattering amplitude

$$\mathcal{M} = -g^2\, \overset{(765)}{\underline{\bar{u}(\underline{P}_4,\ s_4)\,\gamma^\nu\, u(\underline{P}_2,\ s_2)}}\cdot \overset{(4)}{\frac{g_{\mu\nu}}{(\underline{P}_1 - \underline{P}_3)^2}}$$

$$\overset{(321)}{\underline{\bar{u}(\underline{P}_3,\ s_3)\,\gamma^\mu\, u(\underline{P}_1,\ s_1)}}$$

If u and $\bar{u}$ are known explicitly, then

1x4 row 4x1 column

$(\bar{u}\ \gamma^\nu\ u)$ is just a complex number

4x1 sq. matrix

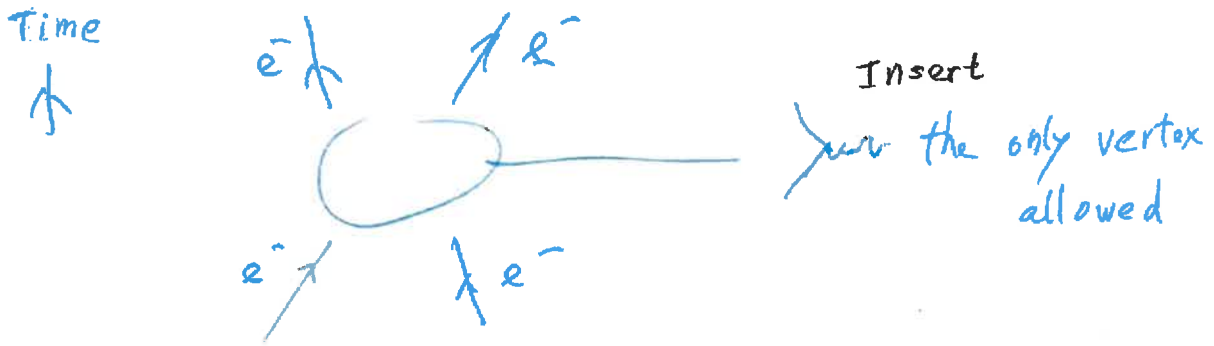
i.e. if all the u, $\bar{u}$ are known explicitly, then the scattering amplitude $\mathcal{M}$ is just a complex number.

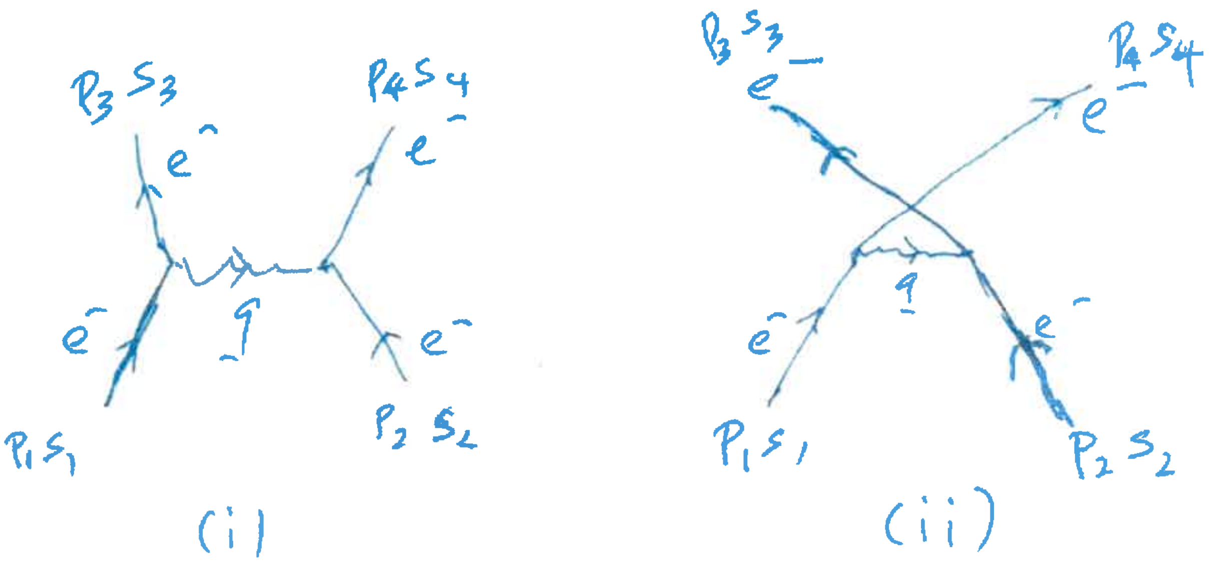
simplify the notations : $(\underline{P}_1, s_1) \to (1)$, $(\underline{P}_i, s_i) \to (i)$

$$\boxed{\mathcal{M} = -g^2\, \underbrace{\bar{u}(4)\,\gamma^\nu\, u(2)}_{(765)}\ \overset{(4)}{\frac{g_{\mu\nu}}{(\underline{P}_1 - \underline{P}_3)^2}}\cdot \underbrace{\bar{u}(3)\,\gamma^\mu\, u(1)}_{(321)}}$$

continue to get scattering amplitude of a physical process by using Feynman diagrams.

(ii) $e^- e^- \to e^- e^-$ (2) Möller scattering





2 diagrams $\longrightarrow$ 2 amplitudes $\mathcal{M}_{(i)}, \mathcal{M}_{(ii)}$

For diagram (i), it is like the $e^-\mu^- \to e^-\mu^-$, so we copy the result from previous example

6

$$\mathcal{M}_{(i)} = -g^2\, \bar{u}(4)\gamma^\nu u(2) \frac{g_{\mu\nu}}{(p_1 - p_3)^2} \bar{u}(3)\gamma^\mu u(1)$$

$$= \frac{-g^2}{q^2} \bar{u}(4)\gamma^\mu u(2) \cdot \bar{u}(3)\gamma_\mu u(1).$$

$(p_1 - p_3)^2$ = momentum transfer square $\equiv q^2 = t$

$\gamma_\mu \equiv g_{\mu\nu}\gamma^\nu$

$$\mathcal{M}_{(ii)} = \frac{-g^2}{q^2} \bar{u}(3)\gamma^\mu u(2) \cdot \bar{u}(4)\gamma_\mu u(1)$$

(H.W)

FOLLOW THE TRACKS OF THE DIAGRAM, not the type of particle

$q^2 = (p_1 - p_4)^2$

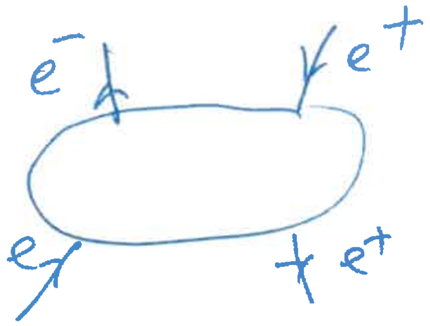
$$\mathcal{M} = \mathcal{M}_{(i)} - \mathcal{M}_{(ii)}$$

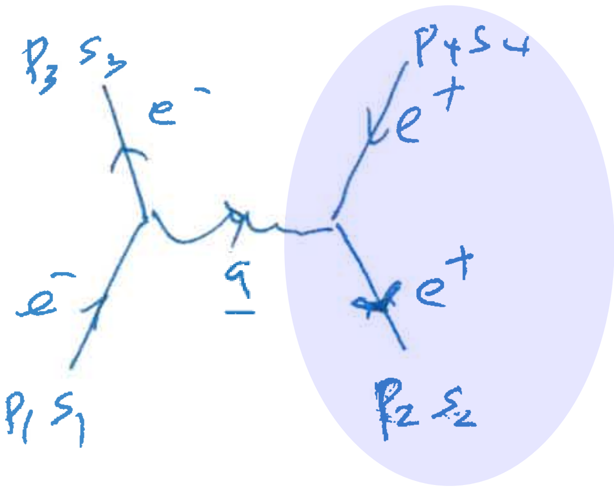
$g_{\mu\nu}\gamma^\nu = \gamma_\mu$

$$= -g^2 \Big[\bar{u}(4)\gamma^\mu u(2) \frac{1}{q^2} \bar{u}(3)\gamma_\mu u(1)$$

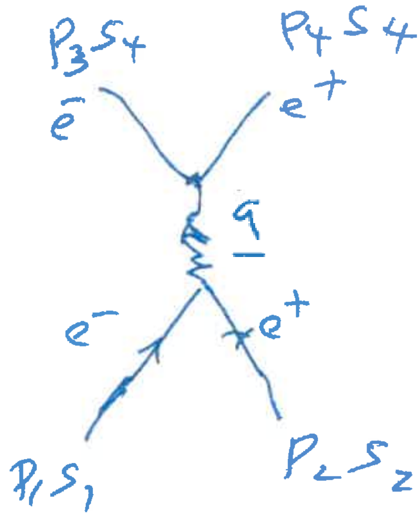
$$- \bar{u}(3)\gamma^\mu u(2) \cdot \frac{1}{q'^2} \bar{u}(4)\gamma_\mu u(1) \Big]$$

(iii) $e^- e^+ \to e^- e^+$ (3) Bhabha scatt. (7)





(i)



(ii)

Compute $\mathcal{M}_{(i)}$

!!! Write (4) before (2) because going forward in time for vertices containing antiparticles begin from (4) and ends at (2)

$$\bar{v}(2)\, i g \gamma^{\nu}\, v(4)\, \frac{-i g_{\mu\nu}}{q^2}\, (2\pi)^4 \delta^{(4)}(p_1 - q - p_3)\, \bar{u}(3)\, i g \gamma^{\mu}\, u(1)$$

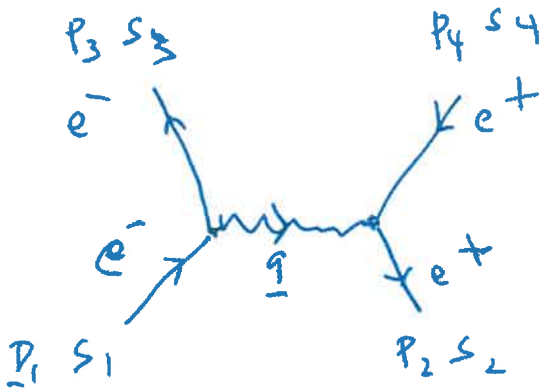
$$\int \frac{d^4 q}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q + p_2 - p_4) = ?$$ HW

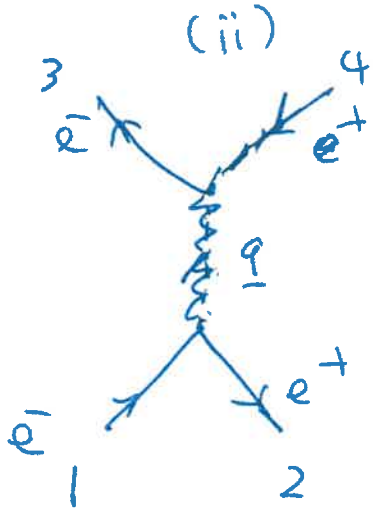
$$\mathcal{M}_{(i)} = \frac{-g^2}{(p_1 - p_3)^2} \left[\bar{u}(3) \gamma^{\mu} u(1)\right] \left[\bar{v}(2) \gamma_{\mu} v(4)\right]$$

HW

Time

Feynman diagram





Find scatt. amp $\mathcal{M}(i)$

Complex numbers

$$\int \frac{d^4q}{(2\pi)^4}\left(\bar{v}(2)\, ig\gamma^\nu\, v(4)\right)\frac{-ig_{\mu\nu}}{q^2}\left(\bar{u}(3)\, ig\gamma^\mu\, u(1)\right)$$

$$(2\pi)^4\,\delta^{(4)}(p_1-p_3-q)\cdot(2\pi)^4\,\delta^{(4)}(q+p_2-p_4)$$

$$= i g^2 (2\pi)^4\,\delta^{(4)}(p_1-p_3-p_4+p_2)\cdot$$

$$\bar{v}(2)\gamma^\nu v(4)\,\frac{g_{\mu\nu}}{(p_1-p_3)^2}\,\bar{u}(3)\,\gamma^\mu\, u(1)$$

$$\mathcal{M}_{(i)} = -g^2\ \bar{v}(2)\gamma_\mu V(4)\cdot\frac{1}{(p_1-p_3)^2}\,\bar{u}(3)\gamma^\mu u(1)$$

$$\gamma_\mu \equiv g_{\mu\nu}\gamma^\nu$$

$$\gamma_0=\gamma^0,\quad \gamma_i=-\gamma^i,$$

$$i=1,2,3$$

$$\int \frac{d^4 q}{(2\pi)^4} \, \bar{u}(3)\, ig\gamma^\nu \, v(4) \cdot \frac{-ig_{\mu\nu}}{q^2} \, \bar{v}(2)\, ig\gamma^\mu \, u(1)$$

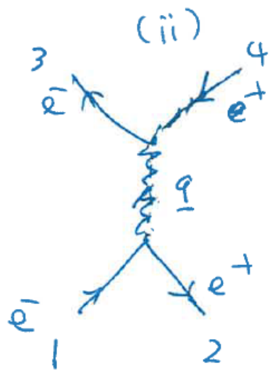
$$(2\pi)^4 \, \delta^{(4)}(p_1 - q + p_2)(2\pi)^4 \, \delta^{(4)}(q - p_3 - p_4)$$

$$= (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \; i g^2 \, \bar{u}(3)\gamma^\nu v(4) \cdot \frac{g_{\mu\nu}}{\underbrace{q}_{p_1+p_2}{}^2}$$

$$\bar{v}(2)\, \gamma^\mu \, u(1)$$

$$\rightarrow \quad \mathcal{M}_{(ii)} = -g^2 \, \bar{u}(3)\, \gamma_\mu \, v(4) \cdot \frac{1}{(p_1 + p_2)^2} \cdot$$

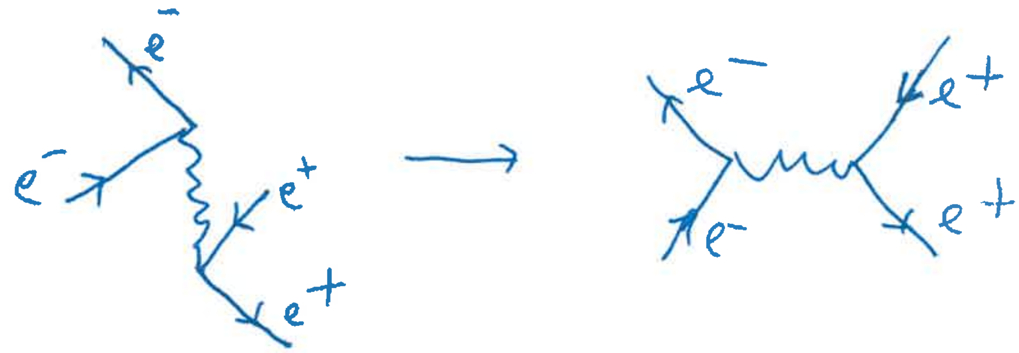
$$\bar{v}(2)\gamma^\mu u(1)$$



should we add $\mathcal{M}_{(ii)}$ to $\mathcal{M}_{(i)}$ or should we subtract? substract, can exchange e+ with e-

This depends on whether the two diagrams can be obtained from each other by (i) interchanging the two incoming identical particles, or (ii) interchanging the two outgoing identical particles, or (iii) interchanging an incoming e^- with an outgoing e^+ (anti-particle) or vice versa

In diagram (ii), interchange outgoing e^+ with incoming e^-



That means the first diagram can be obtained from the 2nd diagram by using crossing symmetry.

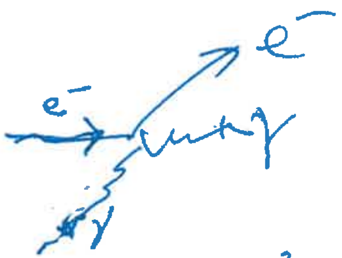
Can show 2nd diagram can be obtained from 1st diagram by crossing symmetry (HW)

So the total scatt. amp. is

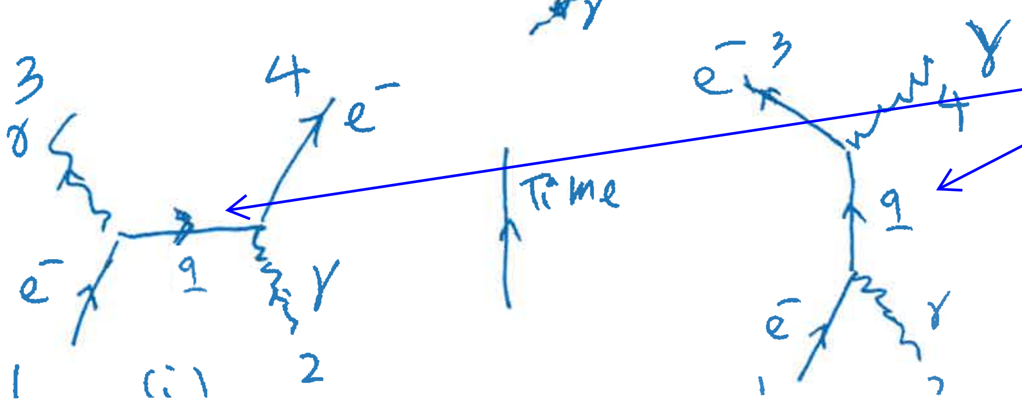
$$\mathcal{M} = \mathcal{M}_{(i)} - \mathcal{M}_{(ii)}$$

Now do

(iv) $e^-\gamma \to e^-\gamma$



(4) Compton Scattering



$$\boxed{e^- \text{ or } e^+ : \frac{i}{\not{q} - mc}}$$

$$\int \frac{d^4q}{(2\pi)^4} \, \bar{u}(4) \, ig\gamma^\nu \, \varepsilon_\nu(2) \, \frac{i}{\not{q} - mc} \, \varepsilon^*_\mu(3) \, ig\gamma^\mu u(1)$$

$$(2\pi)^4 \delta^{(4)}(\underline{p}_1 - \underline{k}_3 - \underline{q}) \cdot (2\pi)^4 \delta^{(4)}(\underline{q} + \underline{k}_2 - \underline{p}_4)$$

$$\mathcal{M}_{(i)} = g^2 \, \bar{u}(4) \cdot \gamma^\nu \varepsilon_\nu(2) \frac{1}{(\not{p}_1 - \not{k}_3 - mc)} \cdot \varepsilon^*_\mu(3) \gamma^\mu u(1)$$

$$\equiv g^2 \, \bar{u}(4) \, \not{\varepsilon}(2) \frac{1}{\not{p}_1 - \not{k}_3 - mc} \not{\varepsilon}^*(3) \, u(1)$$

explained in pg 245

$$[\bar{u}(a)\Gamma_2 u(b)]^* = [u(a)^\dagger\gamma^0\Gamma_2 u(b)]^\dagger = u(b)^\dagger\Gamma_2^\dagger\gamma^{0\dagger}u(a) \quad (7.115)$$

Now, $\gamma^{0\dagger} = \gamma^0$, and $(\gamma^0)^2 = 1$, so

$$[\bar{u}(a)\Gamma_2 u(b)]^* = u(b)^\dagger\gamma^0\gamma^0\Gamma_2^\dagger\gamma^0 u(a) = \bar{u}(b)\bar{\Gamma}_2 u(a) \quad (7.116)$$

where* $\bar{\Gamma}_2 \equiv \gamma^0\Gamma_2^\dagger\gamma^0$ (7.117)

Thus $G = [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(b)\bar{\Gamma}_2 u(a)]$ (7.118)

For 2nd diagram

just replace $\varepsilon(4)$ with $\varepsilon(3)$

$$\int \frac{d^4q}{(2\pi)^4} \, \bar{u}(3) \, ig\gamma^\nu \, \varepsilon^*_\nu(4) \, \frac{i}{\not{q} - mc} \, \varepsilon_\mu(2) \, ig\gamma^\mu u(1)$$

$$(2\pi)^4 \delta^{(4)}(\underline{p}_1 + \underline{k}_2 - \underline{q}) \, (2\pi)^4 \delta^{(4)}(\underline{q} - \underline{p}_3 - \underline{k}_4)$$

$$\mathcal{M}_{(ii)} = g^2 \, \bar{u}(3) \, \not{\varepsilon}^*(4) \frac{1}{(\not{p}_1 + \not{k}_2 - mc)} \not{\varepsilon}(2) \, u(1)$$

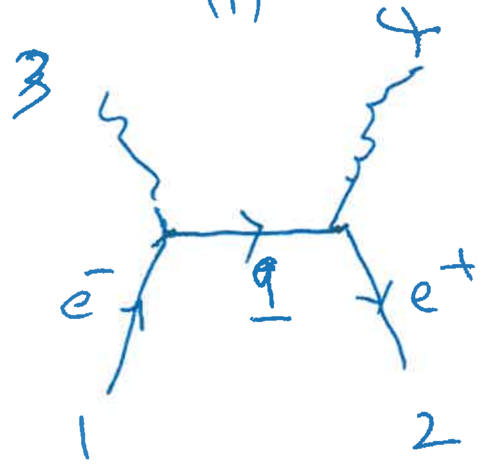
$$\mathcal{M} = \mathcal{M}_{(i)} + \mathcal{M}_{(ii)}$$

Last process

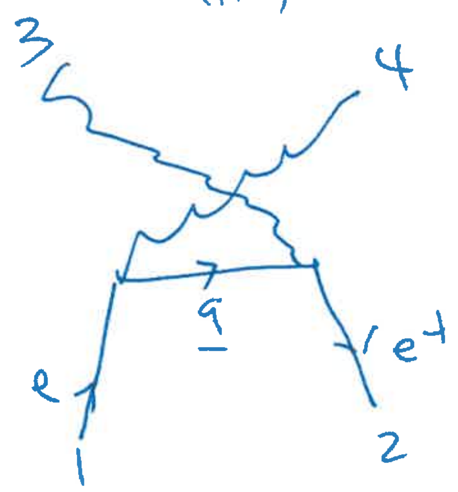
(V) $e^+e^- \to \gamma\gamma$

Feynman diagrams

(i)



(ii)



For diagram (i)

$$\int \frac{d^4 \underline{q}}{(2\pi)^4}\; \bar{v}(2)\, ig\gamma^\nu \epsilon^*_\nu(4)\, \frac{i}{\not{q} - mc}\, \epsilon^*_\mu(3)\, ig\gamma^\mu u(1)$$

$$(2\pi)^4 \delta^{(4)}(\underline{p}_1 - \underline{q} - \underline{k}_3)(2\pi)^4 \delta^{(4)}(\underline{q} + \underline{p}_2 - \underline{k}_4)$$

$$\mathcal{M}_{(i)} = g^2\, \bar{v}(2)\, \not{\epsilon}^*(4)\, \frac{1}{\not{p}_1 - \not{k}_3 - mc}\, \not{\epsilon}^*(3)\, u(1)$$

For diagram (ii)

$$\mathcal{M}_{(ii)} = g^2\, \bar{v}(2)\, \not{\epsilon}^*(3)\, \frac{1}{\not{p}_1 - \not{k}_4 - mc}\, \not{\epsilon}^*(4)\, u(1)$$

(HW)

$$\mathcal{M} = \mathcal{M}_{(i)} + \mathcal{M}_{(ii)}$$

$$\mathcal{M}_{(i)} = g^2\, \bar{v}(2)\, \not{\epsilon}^*(4) \frac{1}{\not{p}_1 - \not{k}_3 - mc} \not{\epsilon}^*(3)\, u(1)$$

Note: $\not{k}_3 \equiv k_{3\mu}\gamma^\mu$

$$= g^2\, \bar{v}(2)\, \not{\epsilon}^*(4) \frac{(\not{p}_1 - \not{k}_3 + mc)}{(p_1 - k_3)^2 - m^2c^2} \not{\epsilon}^*(3)\, u(1)$$

$$= \frac{g^2}{(p_1 - k_3)^2 - m^2c^2}\, \bar{v}(2)\, \not{\epsilon}^*(4)(\not{p}_1 - \not{k}_3 + mc)\not{\epsilon}^*(3)\, u(1)$$

Diagram (ii) HW

$$\mathcal{M}_{(ii)} = g^2\, \bar{v}(2)\, \not{\epsilon}^*(3) \frac{1}{\not{p}_1 - \not{k}_4 - mc} \not{\epsilon}^*(4)\, u(1)$$

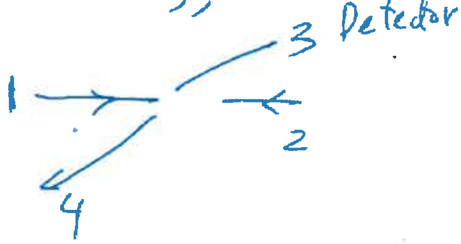
Total scattering amplitude

$$\mathcal{M} = \mathcal{M}_{(i)} + \mathcal{M}_{(ii)}$$

Next discuss differential crosssection. $\frac{d\sigma}{d\Omega}$

Recall the scattering cross section can be written as a product of dynamic part (scattering amplitude) and the kinematic part (phase space factor)

For a 2 particle to 2 particle scattering, we have shown



$$\frac{d\sigma}{d\Omega_3} =$$

$$\frac{d\sigma}{d\Omega_3} = \frac{s\,\hbar^2}{64\pi^2\sqrt{(\underset{\sim}{p}_1\cdot \underset{\sim}{p}_2)^2-(m_1 m_2 c^2)^2}}\;\frac{|\mathcal{M}|^2\,|\underset{\sim}{p}_3|}{(p_1^{\,0}+p_2^{\,0})}\Bigg|_{\underset{\sim}{p}_4=-\underset{\sim}{p}_3}$$

$$\underset{\sim}{p}_3^{\,2} = \frac{\left(\alpha^2+(m_4^2-m_3^2)c^2\right)^2}{4\alpha^2} - m_4^2c^2$$

$$\alpha \equiv p_1^{\,0} + p_2^{\,0}$$

In this expression, the only unknown is $|\mathcal{M}|^2$.

In many experiments, the detector just count the number of particles and the spins (polarizations) are not measured. If that is the case, then we must compute $\mathcal{M}$ for every possible spin (for the $2\to 2$ process, that means we have to compute $\mathcal{M}$ for spin s_1, s_2, s_3, s_4 and then sum up) i.e.

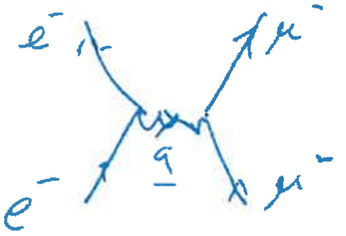
We compute the average over spins of incident particles and summation over final spins

$$\boxed{\langle |\mathcal{M}|^2\rangle \equiv \frac{1}{4}\sum_{s_1,s_2,s_3,s_4} |\mathcal{M}|^2}$$

From Griffiths,
"The factor of 1/4 is included because we want the average over the initial spins; since there are two particles, each with two allowed spin orientations, the average is a quarter of the sum."

NOTE: if we know the spin of the incoming states, no need for factor of 1/4

Consider $e^-\mu^- \to e^-\mu^-$



$$\mathcal{M} = \frac{-g^2}{(\underset{\sim}{p}_1-\underset{\sim}{p}_3)^2}\,\bar{u}(3)\gamma^\nu u(1)\cdot g_{\mu\nu}\,\bar{u}(4)\gamma^\mu u(2)$$

Instead of doing the summation for 16 $|\mathcal{M}|^2$ we can use Casimir's trick to avoid computing each of the 16 $|\mathcal{M}|^2$ and then summing.

$$|\mathcal{M}|^2 = \mathcal{M}\,\mathcal{M}^*$$

$$\mathcal{M} = \alpha\ \bar{u}(3)\,\gamma_\mu\,u(1)\ \bar{u}(4)\,\gamma^\mu u(2)$$

$$\alpha \equiv \frac{-g^2}{(\underline{p}_1 - \underline{p}_3)^2}$$

$$\mathcal{M}\,\mathcal{M}^* = \alpha^2\ \bar{u}(3)\,\gamma_\mu\,u(1)\cdot\bar{u}(4)\,\gamma^\mu\,u(2)\cdot$$

$$\underbrace{\left(\bar{u}(3)\,\gamma_\nu\,u(1)\ \bar{u}(4)\,\gamma^\nu\,u(2)\right)^*}_{\text{number}}$$

Since the transpose of a column matrix is a row matrix, and the complex conjugate of each element remains the same, Ψ^* = Ψ^†

$$= \alpha^2\ \bar{u}(3)\,\gamma_\mu\,u(1)\ \bar{u}(4)\,\gamma^\mu\,u(2)$$

$$\underline{u(2)^\dagger\ \gamma^{\nu\dagger}\ \bar{u}(4)^\dagger\cdot\ u(1)^\dagger\ \gamma_\nu^\dagger\ \bar{u}(3)^\dagger}$$

Note: $\bar{u}^\dagger = (u^\dagger\gamma^0)^\dagger = \gamma^{0\dagger}u = \gamma^0 u$

$\gamma_\nu^\dagger = \gamma^0\gamma_\nu\gamma^0$