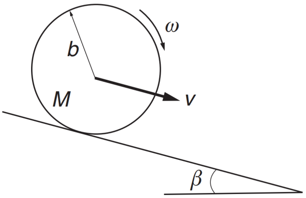
Example: Drum rolling down a plane

- A uniform drum of radius b and mass M rolls without slipping down a plane inclined at angle β.



- Translation of the center of mass:

$$\begin{cases} M g \sin\beta - f = M\ddot{X}_{\mathsf{CM}}(t) \\ N - M g \cos\beta = M\ddot{Y}_{\mathsf{CM}}(t) \end{cases}$$

- Motion with no slipping: the contact is very rough $f \leq \mu_s N$

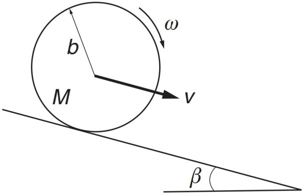
$$\dot{X}_{\mathsf{CM}}(t) = b\dot{\phi}(t) = b\omega(t) \quad \Rightarrow \quad \ddot{X}_{\mathsf{CM}}(t) = b\ddot{\phi}(t) = b\dot{\omega}(t)$$

EXERCISE 4.5: Find the drum's acceleration along the plane.

$$\mathbf{R}_{\mathsf{CM}}(t) = X_{\mathsf{CM}}(t)\,\hat{\mathbf{e}}_x + b\,\hat{\mathbf{e}}_y$$

$$\begin{cases} \mathbf{W}(t) = Mg\sin\beta\,\hat{\mathbf{e}}_x - Mg\cos\beta\,\hat{\mathbf{e}}_y \\ \mathbf{f}(t) = -f(t)\,\hat{\mathbf{e}}_x \\ \mathbf{N}(t) = N(t)\,\hat{\mathbf{e}}_y \end{cases}$$

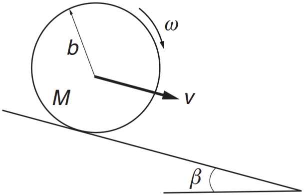
$$\begin{cases} \mathbf{r}_W(t) = X_{\mathsf{CM}}(t)\,\hat{\mathbf{e}}_x + b\,\hat{\mathbf{e}}_y \\ \mathbf{r}_f(t) = X_{\mathsf{CM}}(t)\,\hat{\mathbf{e}}_x \\ \mathbf{r}_N(t) = X_{\mathsf{CM}}(t)\,\hat{\mathbf{e}}_x \end{cases}$$



$$\mathbf{F}(t) = M\ddot{\mathbf{R}}_{\mathsf{CM}}(t)$$

$$\Rightarrow \quad \begin{cases} Mg\sin\beta - f(t) = M\ddot{X}_{\mathsf{CM}}(t) \\ N(t) - Mg\cos\beta = M\ddot{Y}_{\mathsf{CM}}(t) \end{cases}$$

$$\Rightarrow \quad \begin{cases} Mg\sin\beta - f(t) = M\ddot{X}_{\mathsf{CM}}(t) \\ N - Mg\cos\beta = 0 \end{cases}$$

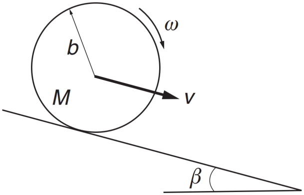


$$\boldsymbol{\tau}'^{\text{ext}}(t) = \dot{\mathbf{L}}'(t) + M\,[\mathbf{R}_{\text{CM}}(t) - \mathbf{R}(t)] \times \ddot{\mathbf{R}}(t)\,, \quad \mathbf{L}(t) = \mathbf{R}_{\text{CM}}(t) \times \mathbf{P}(t) + \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t)$$

$$\boldsymbol{\mathcal{T}}_{\text{CM}}(t) = \sum_i [\mathbf{r}_i(t) - \mathbf{R}_{\text{CM}}(t)] \times \mathbf{F}_i(t) = -bf(t)\,\hat{\mathbf{e}}_z$$

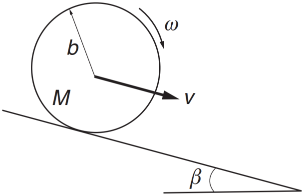
$$\mathbf{L}_{\text{CM}}(t) = \mathbf{L}^{\text{spin}}(t) = -\frac{1}{2}\,Mb^2\omega(t)\,\hat{\mathbf{e}}_z$$

$$\boldsymbol{\mathcal{T}}_{\text{CM}}(t) = \dot{\mathbf{L}}_{\text{CM}}(t) \quad \Rightarrow \quad bf(t) = \frac{1}{2}\,Mb^2\dot{\omega}(t) \qquad \blacksquare$$



$$\begin{cases} Mg\sin\beta - f(t) = M\ddot{X}_{\mathsf{CM}}(t) \\ N(t) - Mg\cos\beta = 0 \\ bf(t) = \dfrac{1}{2}\,Mb^2\,\dot{\omega}(t) \\ \ddot{X}_{\mathsf{CM}}(t) = b\dot{\omega}(t) \end{cases}$$

$$\Rightarrow \quad \begin{cases} \ddot{X}_{\mathsf{CM}}(t) = \dfrac{2}{3}\,g\sin\beta \\ f(t) = \dfrac{1}{3}\,Mg\sin\beta \end{cases} \qquad \blacksquare$$



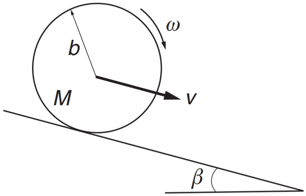
$$\boldsymbol{\tau}'^{\text{ext}}(t) = \dot{\mathbf{L}}'(t) + M\left[\mathbf{R}_{\text{CM}}(t) - \mathbf{R}(t)\right]\times\ddot{\mathbf{R}}(t)\,, \quad \mathbf{L}(t) = \mathbf{R}_{\text{CM}}(t)\times\mathbf{P}(t) + \sum_{\alpha=1}^{N}\mathbf{r}'_\alpha(t)\times m_\alpha\dot{\mathbf{r}}'_\alpha(t)$$

$$\mathcal{T}_{\text{origin}}(t) = \sum_i \mathbf{r}_i(t)\times\mathbf{F}_i(t) = -\left[Mg\cos\beta\, X_{\text{CM}}(t) + Mgb\sin\beta - N(t)\,X_{\text{CM}}(t)\right]\hat{\mathbf{e}}_z$$

$$\mathbf{L}_{\text{origin}}(t) = \mathbf{L}^{\text{orbital}}(t) + \mathbf{L}^{\text{spin}}(t) = -\left[Mb\dot{X}_{\text{CM}}(t) + \frac{1}{2}Mb^2\omega(t)\right]\hat{\mathbf{e}}_z$$

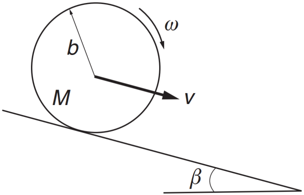
$$\mathcal{T}_{\text{origin}}(t) = \dot{\mathbf{L}}_{\text{origin}}(t)$$

$$\Rightarrow \quad Mg\cos\beta\, X_{\text{CM}}(t) + Mgb\sin\beta - N(t)\,X_{\text{CM}}(t) = Mb\ddot{X}_{\text{CM}}(t) + \frac{1}{2}Mb^2\dot{\omega}(t) \qquad \blacksquare$$



$$
\begin{cases}
Mg\sin\beta - f(t) = M\ddot{X}_{\mathsf{CM}}(t) \\[2ex]
N(t) - Mg\cos\beta = 0 \\[2ex]
Mg\cos\beta\, X_{\mathsf{CM}}(t) + Mgb\sin\beta - N(t)\, X_{\mathsf{CM}}(t) = Mb\ddot{X}_{\mathsf{CM}}(t) + \dfrac{1}{2}\, Mb^2\dot{\omega}(t) \\[2ex]
\ddot{X}_{\mathsf{CM}}(t) = b\dot{\omega}(t)
\end{cases}
$$

$$
\Rightarrow \quad \begin{cases}
\ddot{X}_{\mathsf{CM}}(t) = \dfrac{2}{3}\, g\sin\beta \\[2ex]
f(t) = \dfrac{1}{3}\, Mg\sin\beta
\end{cases} \qquad \blacksquare
$$



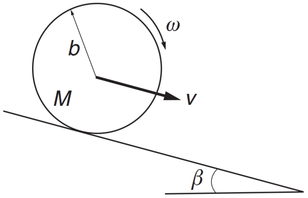
$$\boldsymbol{\tau}'^{\text{ext}}(t) = \dot{\mathbf{L}}'(t) + M\left[\mathbf{R}_{\text{CM}}(t) - \mathbf{R}(t)\right] \times \ddot{\mathbf{R}}(t)\,, \quad \mathbf{L}(t) = \mathbf{R}_{\text{CM}}(t) \times \mathbf{P}(t) + \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t)$$

$$\boldsymbol{\mathcal{T}}_{\text{contact}}(t) = \sum_i \left[\mathbf{r}_i(t) - \mathbf{r}_{\text{contact}}(t)\right] \times \mathbf{F}_i(t) = -Mgb\sin\beta\,\hat{\mathbf{e}}_z$$

$$\mathbf{L}_{\text{contact}}(t) = \mathbf{L}^{\text{orbital}}(t) + \mathbf{L}^{\text{spin}}(t) = -\left[Mb\dot{X}_{\text{CM}}(t) + \frac{1}{2}Mb^2\omega(t)\right]\hat{\mathbf{e}}_z$$

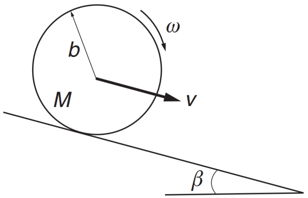
$$\boldsymbol{\mathcal{T}}_{\text{contact}}(t) = \dot{\mathbf{L}}_{\text{contact}}(t)$$

$$\Rightarrow \quad Mgb\sin\beta = Mb\ddot{X}_{\text{CM}}(t) + \frac{1}{2}Mb^2\dot{\omega}(t) \qquad \blacksquare$$



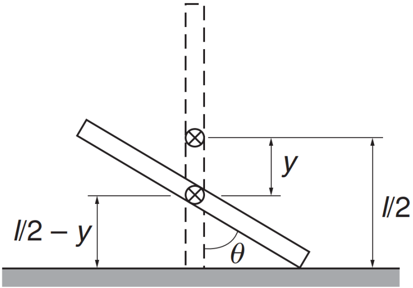
$$
\begin{cases}
Mg\sin\beta - f(t) = M\ddot{X}_{\mathsf{CM}}(t) \\[2ex]
N(t) - Mg\cos\beta = 0 \\[2ex]
Mgb\sin\beta = Mb\ddot{X}_{\mathsf{CM}}(t) + \dfrac{1}{2}\,Mb^2\dot{\omega}(t) \\[2ex]
\ddot{X}_{\mathsf{CM}}(t) = b\dot{\omega}(t)
\end{cases}
$$

$$
\Rightarrow \quad \begin{cases}
\ddot{X}_{\mathsf{CM}}(t) = \dfrac{2}{3}\,g\sin\beta \\[2ex]
f(t) = \dfrac{1}{3}\,Mg\sin\beta
\end{cases} \qquad \blacksquare
$$



Example: The falling stick

- A uniform stick of length ℓ and mass M, initially upright on a frictionless table, starts falling



EXERCISE 4.6: Find the normal force from table as a function of θ from the vertical.

$$\mathbf{F}(t) = M\ddot{\mathbf{R}}_{\mathsf{CM}}(t) \quad \Rightarrow \quad \begin{cases} 0 = M\ddot{X}_{\mathsf{CM}}(t) \\ \\ N(t) - Mg = M\ddot{Y}_{\mathsf{CM}}(t) \end{cases}$$

$$\ddot{X}_{\mathsf{CM}}(t) = 0 \quad \Rightarrow \quad \dot{X}_{\mathsf{CM}}(t) = \text{constant} = 0 \quad \Rightarrow \quad X_{\mathsf{CM}}(t) = \text{constant} = 0 \qquad \blacksquare$$

$$Y_{\mathsf{CM}}(t) = \frac{\ell}{2}\,\cos\theta(t) \quad \Rightarrow \quad \dot{Y}_{\mathsf{CM}}(t) = -\frac{\ell}{2}\,\dot{\theta}(t)\sin\theta(t)$$

$$\Rightarrow \quad \ddot{Y}_{\mathsf{CM}}(t) = -\frac{\ell}{2}\,\ddot{\theta}(t)\sin\theta(t) - \frac{\ell}{2}\,\dot{\theta}^2(t)\cos\theta(t)$$

$$N(t) = Mg + M\ddot{Y}_{\mathsf{CM}}(t) = Mg - \frac{1}{2}\,M\ell\left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right]$$

$$\mathcal{T}_{\mathsf{CM}}(t) = I_{\mathsf{CM}}\ddot{\theta}(t) \quad \Rightarrow \quad \frac{\ell}{2}\,N(t)\,\sin\theta(t) = \frac{1}{12}\,M\ell^2\,\ddot{\theta}(t) \quad \Rightarrow \quad N(t) = \frac{1}{6}\,M\ell\,\frac{\ddot{\theta}(t)}{\sin\theta(t)}$$

$$\begin{cases} N(t) = Mg - \dfrac{1}{2}\, M\ell \left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right] \\[2ex] N(t) = \dfrac{1}{6}\, M\ell\, \dfrac{\ddot{\theta}(t)}{\sin\theta(t)} \end{cases}$$

$$\Rightarrow \quad \ddot{\theta}(t) = \frac{6g}{\ell}\, \sin\theta(t) - 3\sin\theta(t)\left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{\theta}^2(t)\right] = 2\dot{\theta}(t)\,\ddot{\theta}(t) = 2\dot{\theta}(t)\left\{\frac{6g}{\ell}\, \sin\theta(t) - 3\sin\theta(t)\left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right]\right\}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{\theta}^2(t)\right] = -\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{12g}{\ell}\, \cos\theta(t) + 3\left[\dot{\theta}(t)\, \sin\theta(t)\right]^2\right\}$$

$$\Rightarrow \quad \dot{\theta}^2(t) + \frac{12g}{\ell}\, \cos\theta(t) + 3\left[\dot{\theta}(t)\, \sin\theta(t)\right]^2 = C$$

$$\begin{cases} \theta(0) = 0 \\[2ex] \dot{\theta}(0) = 0 \end{cases} \quad \Rightarrow \quad \dot{\theta}^2(0) + \frac{12g}{\ell}\, \cos\theta(0) + 3\left[\dot{\theta}(0)\, \sin\theta(0)\right]^2 = C \quad \Rightarrow \quad C = \frac{12g}{\ell}$$

$$\dot{\theta}^2(t) + \frac{12g}{\ell}\, \cos\theta(t) + 3\left[\dot{\theta}(t)\, \sin\theta(t)\right]^2 = \frac{12g}{\ell} \quad \Rightarrow \quad \dot{\theta}^2(t) = \frac{12g}{\ell}\, \frac{1-\cos\theta(t)}{1+3\sin^2\theta(t)} \qquad \blacksquare$$

$$
\begin{cases}
N(t) = Mg - \dfrac{1}{2}\, M\ell \left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right] \\[2ex]
N(t) = \dfrac{1}{6}\, M\ell\, \dfrac{\ddot{\theta}(t)}{\sin\theta(t)}
\end{cases}
$$

$$
\dot{\theta}^2(t) = \frac{12g}{\ell}\,\frac{1-\cos\theta(t)}{1+3\sin^2\theta(t)} \quad \Rightarrow \quad \ddot{\theta}(t) = \frac{6g}{\ell}\,\sin\theta(t)\,\frac{4-6\cos\theta(t)+3\cos^2\theta(t)}{\left[1+3\sin^2\theta(t)\right]^2}
$$

$$
N(t) = Mg - \frac{1}{2}\, M\ell \left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right] = \frac{4-6\cos\theta(t)+3\cos^2\theta(t)}{\left[1+3\sin\theta(t)\right]^2}\, Mg \qquad \blacksquare
$$

PC3261: Classical Mechanics II

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Semester I, 2023/24

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NUS National University of Singapore
Department of Physics
Faculty of Science

Lecture 5: Work and Energy

Kinetic energy and work

- **Kinetic energy**:

$$T(t) \equiv \frac{1}{2} m\mathbf{v}(t) \cdot \mathbf{v}(t)$$

- **Work** by the force on the particle during a time interval:

$$W_{1\to 2} \equiv \int_{t=t_1}^{t_2} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \dot{\mathbf{r}}(t)\, \mathrm{d}t$$

- **Work-energy theorem**: total work by the forces during a given time interval is equal to the change in the kinetic energy of the particle during this time interval

$$T(t_2) - T(t_1) = \int_{t=t_1}^{t_2} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \dot{\mathbf{r}}(t)\, \mathrm{d}t$$

$$\mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) = m\,\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t}$$

$$\Rightarrow \quad \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathbf{v}(t) = m\,\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} \cdot \mathbf{v}(t)$$

$$\Rightarrow \quad \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\, m\mathbf{v}(t) \cdot \mathbf{v}(t)\right] \qquad \blacksquare$$

Work as a line integral

- Work $W_{1\to 2}$ on the particle by the force $\mathbf{F}$ is given by the line integral of $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ along its trajectory $\mathcal{C}_{1\to 2}$ from point $\mathbf{r}_1$ to point $\mathbf{r}_2$:

$$W(\mathbf{r}_1 \to \mathbf{r}_2) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r}$$

- Work-energy theorem: change in the kinetic energy of a particle as it moves from points 1 to 2 is the work by the *net* force on the particle

$$T(t_2) - T(t_1) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r}$$

- Work by the net force is the sum of works done by respective forces:

$$W\left(\mathbf{r}_1 \to \mathbf{r}_2\right) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_{1\to 2}} \sum_i \mathbf{F}_i(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r}$$

$$= \sum_i \int_{\mathcal{C}_{1\to 2}} \mathbf{F}_i(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} = \sum_i W_i\left(\mathbf{r}_1 \to \mathbf{r}_2\right)$$

Example: Work by a uniform force

- Uniform force: $\mathbf{F}(\mathbf{r}) = F_0\,\hat{\mathbf{e}}_n$, F_0 is a constant and $\hat{\mathbf{e}}_n$ is a constant unit vector

- Work by the uniform force on the particle moving from $\mathbf{r}_1$ to $\mathbf{r}_2$ along an *arbitrary* path: θ is the angle between $\hat{\mathbf{e}}_n$ and $\mathbf{r}_2 - \mathbf{r}_1$

$$W\left(\mathbf{r}_1 \to \mathbf{r}_2\right) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = F_0\,\hat{\mathbf{e}}_n \cdot (\mathbf{r}_2 - \mathbf{r}_1) = F_0\,|\mathbf{r}_2 - \mathbf{r}_1|\cos\theta$$

- Work by a uniform force only depends on the net displacement, $\mathbf{r}_2 - \mathbf{r}_1$, not on the particular path taken from $\mathbf{r}_1$ to $\mathbf{r}_2$!

- Work by a uniform force around a closed path is zero: $\mathcal{C}_{1\to 2} \neq -\mathcal{C}_{2\to 1}$

$$W\left(\mathbf{r}_2 \to \mathbf{r}_1\right) = \int_{\mathcal{C}_{2\to 1}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = F_0\,\hat{\mathbf{e}}_n \cdot (\mathbf{r}_1 - \mathbf{r}_2) = -W\left(\mathbf{r}_1 \to \mathbf{r}_2\right)$$

$$
\begin{aligned}
W\left(\mathbf{r}_1 \rightarrow \mathbf{r}_2\right) &= \int_{\mathcal{C}_{1\rightarrow 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_{1\rightarrow 2}} F_0\, \hat{\mathbf{e}}_n \cdot \mathrm{d}\mathbf{r} \\
&= F_0\, \hat{\mathbf{e}}_n \cdot \int_{\mathcal{C}_{1\rightarrow 2}} \mathrm{d}\mathbf{r} \\
&= F_0\, \hat{\mathbf{e}}_n \cdot \left[\hat{\mathbf{e}}_x \int_{\mathcal{C}_{1\rightarrow 2}} \mathrm{d}x + \hat{\mathbf{e}}_y \int_{\mathcal{C}_{1\rightarrow 2}} \mathrm{d}y + \hat{\mathbf{e}}_z \int_{\mathcal{C}_{1\rightarrow 2}} \mathrm{d}z\right] \\
&= F_0\, \hat{\mathbf{e}}_n \cdot \left[\left(x_2 - x_1\right)\, \hat{\mathbf{e}}_x + \left(y_2 - y_1\right)\, \hat{\mathbf{e}}_y + \left(z_2 - z_1\right)\, \hat{\mathbf{e}}_z\right] \\
&= F_0\, \hat{\mathbf{e}}_n \cdot \left(\mathbf{r}_2 - \mathbf{r}_1\right) \qquad \blacksquare
\end{aligned}
$$

Example: Inverted pendulum

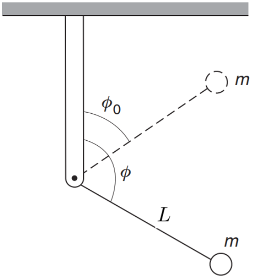
- A pendulum consists of a light rigid rod of length L pivoted at one end with mass m attached at the other end. The pendulum is released from rest at angle ϕ_0

- Equation of motion:

$$\frac{\mathrm{d}^2\phi(t)}{\mathrm{d}t^2} = \frac{g}{L}\sin\phi(t)$$

- Maximum speed is achieved by letting the pendulum fall from $\phi_0 = 0$ to the bottom $\phi = \pi$:

$$v_{\max} = 2\sqrt{gL}$$



EXERCISE 5.1: Obtain the speed of the mass m when the rod is at an angle ϕ from work-energy theorem.

$$\mathbf{r}(t) = L\sin\phi(t)\,\hat{\mathbf{e}}_y + L\cos\phi(t)\,\hat{\mathbf{e}}_z \quad\Rightarrow\quad \mathrm{d}\mathbf{r} = L\cos\phi(t)\,\mathrm{d}\phi\,\hat{\mathbf{e}}_y - L\sin\phi(t)\,\mathrm{d}\phi\,\hat{\mathbf{e}}_z$$

$$\mathbf{F}(t) = \mathbf{W}(t) + \mathbf{N}(t) = -mg\,\hat{\mathbf{e}}_z - \left[N(t)\sin\phi(t)\,\hat{\mathbf{e}}_y + N(t)\cos\phi(t)\,\hat{\mathbf{e}}_z\right]$$

$$\begin{aligned} W\left(\mathbf{r}(0)\to\mathbf{r}(t)\right) &= \int_{\mathbf{r}(0)}^{\mathbf{r}(t)} \mathbf{F}(t)\cdot\mathrm{d}\mathbf{r} = \int_{\mathbf{r}(0)}^{\mathbf{r}(t)} \mathbf{W}(t)\cdot\mathrm{d}\mathbf{r} + \int_{\mathbf{r}(0)}^{\mathbf{r}(t)} \mathbf{N}(t)\cdot\mathrm{d}\mathbf{r} \\ &= \int_{\phi(0)}^{\phi(t)} mgL\sin\phi(t)\,\mathrm{d}\phi = mgL\left[\cos\phi(0) - \cos\phi(t)\right] \qquad \blacksquare \end{aligned}$$

$$T(t) - T(0) = W\left(\mathbf{r}(0)\to\mathbf{r}(t)\right) \;\Rightarrow\; \frac{m}{2}\,v^2(t) - \frac{m}{2}\,v^2(0) = mgL\left[\cos\phi(0) - \cos\phi(t)\right]$$

$$\Rightarrow\quad v(t) = \sqrt{2gL\left[\cos\phi(0) - \cos\phi(t)\right] + v^2(0)} \qquad \blacksquare$$

$$\mathbf{r}(t) = L\sin\phi(t)\,\hat{\mathbf{e}}_y + L\cos\phi(t)\,\hat{\mathbf{e}}_z$$

$$\begin{cases} \hat{\mathbf{e}}_\rho = \sin\phi(t)\,\hat{\mathbf{e}}_y + \cos\phi(t)\,\hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\phi = \cos\phi(t)\,\hat{\mathbf{e}}_y - \sin\phi(t)\,\hat{\mathbf{e}}_z \end{cases}, \qquad \begin{cases} \mathbf{r}(t) = L\,\hat{\mathbf{e}}_\rho \\ \dot{\mathbf{r}}(t) = L\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi \\ \ddot{\mathbf{r}}(t) = -L\dot{\phi}^2(t)\,\hat{\mathbf{e}}_\rho + L\ddot{\phi}(t)\,\hat{\mathbf{e}}_\phi \end{cases}$$

$$\mathbf{F}(t) = \mathbf{W}(t) + \mathbf{N}(t) = [-mg\cos\phi(t)\,\hat{\mathbf{e}}_\rho + mg\sin\phi(t)\,\hat{\mathbf{e}}_\phi] - N(t)\,\hat{\mathbf{e}}_\rho$$

$$\mathbf{F}(t) = m\ddot{\mathbf{r}}(t) \quad \Rightarrow \quad \begin{cases} N(t) + mg\cos\phi(t) = mL\dot{\phi}^2(t) \\ mg\sin\phi(t) = mL\ddot{\phi}(t) \end{cases} \qquad \blacksquare$$

Example: Escape speed

- Gravitational force acting on a mass m at a distance r from the center of Earth of mass M:

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^2}\,\hat{\mathbf{e}}_r$$

- Mass m is projected from the surface of the Earth $r = R_e$ with an initial speed v_0 at an angle α from the vertical

- Escape speed for the mass m to escape Earth's gravitational field is independent of the launching direction:

$$v_{\text{escape}} = \sqrt{2gR_e}$$

EXERCISE 5.2: Obtain the expression for the escape speed from work-energy theorem. Assume gravitational force is the only force and ignore the rotation of the Earth.

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^2}\,\hat{\mathbf{e}}_r\,, \qquad \mathrm{d}\mathbf{r} = \mathrm{d}r\,\hat{\mathbf{e}}_r + r\,\mathrm{d}\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,\mathrm{d}\phi\,\hat{\mathbf{e}}_\phi$$

$$W\left(\mathbf{r}_1 \to \mathbf{r}_2\right) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r})\cdot \mathrm{d}\mathbf{r} = -\int_{r_1}^{r_2} \frac{GMm}{r^2}\,\mathrm{d}r$$

$$T(t) - T(0) = W\left(\mathbf{r}_0 \to \mathbf{r}\right) \quad \Rightarrow \quad \frac{m}{2}\,v^2(t) - \frac{m}{2}\,v^2(0) = GMm\left[\frac{1}{r(t)} - \frac{1}{r(0)}\right]$$

$$\Rightarrow \quad v^2(0) = v^2(t) - 2GM\left[\frac{1}{r(t)} - \frac{1}{r(0)}\right] \qquad \blacksquare$$

$$\begin{cases} r(t) \to \infty \\ v(t) = 0 \end{cases} \quad \Rightarrow \quad v^2(0) = \frac{2GM}{R_e} \qquad \blacksquare$$

Example: Pendulum motion

- A point mass of mass m is attached at the end of the massless string of length L. It is released from $\theta = \theta_0$ with $\dot{\theta} = 0$ at $t = 0$

- Work-energy theorem: θ_0 is the maximum angular displacement of the point mass

$$\frac{1}{2} L\dot{\theta}^2(t) = g\cos\theta(t) - g\cos\theta_0$$

- Small angle approximation: $\theta_0 \ll 1$

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t\right)$$

EXERCISE 5.3: Obtain the first-order differential equation for $\theta(t)$ governing the dynamics of the point mass. Assuming small angles, $\theta_0 \ll 1$, solve for $\theta(t)$.

$$\mathbf{r}(t) = L\sin\theta(t)\,\hat{\mathbf{e}}_y + L\cos\theta(t)\,\hat{\mathbf{e}}_z\,, \qquad \mathbf{W}(t) = mg\,\hat{\mathbf{e}}_z$$

$$\dot{\mathbf{r}}(t) = L\dot{\theta}(t)\cos\theta(t)\,\hat{\mathbf{e}}_y - L\dot{\theta}(t)\sin\theta(t)\,\hat{\mathbf{e}}_z \quad \Rightarrow \quad T(t) = \frac{m}{2}\,\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t) = \frac{1}{2}\,mL^2\dot{\theta}^2(t)$$

$$\mathbf{F}(t) = \mathbf{W}(t) + \mathbf{N}(t) = mg\,\hat{\mathbf{e}}_z + N(t)\left[-\sin\theta(t)\,\hat{\mathbf{e}}_y - \cos\theta(t)\,\hat{\mathbf{e}}_z\right]$$

$$W\left(\mathbf{r}(0)\to\mathbf{r}(t)\right) = \int_{\mathbf{r}(0)}^{\mathbf{r}(t)} \mathbf{F}(t)\cdot\mathrm{d}\mathbf{r} = -\int_{\theta(0)}^{\theta(t)} mgL\sin\theta(t)\,\mathrm{d}\theta$$

$$T(t) - T(0) = W\left(\mathbf{r}(0)\to\mathbf{r}(t)\right)$$

$$\Rightarrow \quad \frac{1}{2}\,mL^2\dot{\theta}^2(t) - \frac{1}{2}\,mL^2\dot{\theta}^2(0) = mgL\left[\cos\theta(t) - \cos\theta(0)\right]$$

$$\Rightarrow \quad \frac{1}{2}\,L\dot{\theta}^2(t) = g\cos\theta(t) - g\cos\theta_0 \qquad \blacksquare$$

$$\frac{1}{2}\,L\dot{\theta}^2(t) = g\cos\theta(t) - g\cos\theta_0$$

$$\Rightarrow \quad \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = -\sqrt{\frac{2g}{L}\left[\cos\theta(t) - \cos\theta_0\right]}$$

$$\Rightarrow \quad \sqrt{\frac{2g}{L}}\int_0^t \mathrm{d}t = -\int_{\theta_0}^{\theta(t)} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\Rightarrow \quad \sqrt{\frac{2g}{L}}\int_0^t \mathrm{d}t = -\sqrt{2}\int_{\theta_0}^{\theta(t)} \frac{\mathrm{d}\theta}{\sqrt{\theta_0^2 - \theta^2}}$$

$$\Rightarrow \quad \sqrt{\frac{g}{L}}\int_0^t \mathrm{d}t = -\int_{\theta_0}^{\theta(t)} \frac{1}{\theta_0}\,\frac{\mathrm{d}\theta}{\sqrt{1 - \theta^2/\theta_0^2}}$$

$$\Rightarrow \quad \sqrt{\frac{g}{L}}\,t = -\sin^{-1}\left(\frac{\theta(t)}{\theta_0}\right) + \frac{\pi}{2}$$

$$\Rightarrow \quad \theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{L}}\,t\right) \qquad \blacksquare$$

Example: Pendulum motion – cont'd

- Incomplete elliptical integral of the first kind:

$$F(\varphi; k) \equiv \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}}\,, \qquad 0 \le k^2 \le 1\,, \qquad 0 \le \varphi \le \frac{\pi}{2}$$

- Amplitude-dependent period of the pendulum motion:

$$T = 4\sqrt{\frac{L}{g}}\, F\left(\frac{\pi}{2}; \sin\frac{\theta_0}{2}\right)$$

- Series expansion:

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2} + \mathcal{O}\left(\sin^6\frac{\theta_0}{2}\right)\right]$$

$$\frac{1}{2}\, L\dot{\theta}^2(t) = g\cos\theta(t) - g\cos\theta_0$$

$$\Rightarrow \quad \sqrt{\frac{2g}{L}}\int_0^{T/4} \mathrm{d}t = -\int_{\theta_0}^{0} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\Rightarrow \quad T = 4\sqrt{\frac{L}{2g}}\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} \qquad \blacksquare$$

$$\cos\theta - \cos\theta_0 = \left(1 - 2\sin^2\frac{\theta}{2}\right) - \left(1 - 2\sin^2\frac{\theta_0}{2}\right) = 2\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

$$\sin\alpha \equiv \frac{\sin\dfrac{\theta}{2}}{\sin\dfrac{\theta_0}{2}} \quad \Rightarrow \quad \sin\frac{\theta_0}{2}\cos\alpha\,\mathrm{d}\alpha = \frac{1}{2}\,\cos\frac{\theta}{2}\,\mathrm{d}\theta$$

$$T = 4\sqrt{\frac{L}{2g}}\int_0^{\pi/2} \frac{1}{\sqrt{2}\sin\dfrac{\theta_0}{2}\sqrt{1-\sin^2\alpha}}\,\frac{2\sin\dfrac{\theta_0}{2}\cos\alpha}{\cos\dfrac{\theta}{2}}\,\mathrm{d}\alpha$$

$$= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{\mathrm{d}\alpha}{\sqrt{1-\sin^2\dfrac{\theta_0}{2}\sin^2\alpha}} = 4\sqrt{\frac{L}{g}}\,F\left(\frac{\pi}{2};\sin\frac{\theta_0}{2}\right) \qquad \blacksquare$$

$$F\left(\frac{\pi}{2};k\right) = \int_0^{\pi/2} \frac{\mathrm{d}\alpha}{\sqrt{1-k^2\sin^2\alpha}}\,, \qquad k \equiv \sin\frac{\theta_0}{2}$$

$$\begin{aligned}
\left(1-k^2\sin^2\alpha\right)^{-1/2} &= 1 + \left(\frac{1}{2}\right)k^2\sin^2\alpha + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)k^4\sin^4\alpha + \mathcal{O}\left(k^6\right) \\
&= 1 + \frac{1}{2}\,k^2\sin^2\alpha + \frac{3}{8}\,k^4\sin^4\alpha + \mathcal{O}\left(k^6\right)
\end{aligned}$$

$$\int_0^{\pi/2}\sin^2\alpha\,\mathrm{d}\alpha = \frac{\pi}{4}\,, \qquad \int_0^{\pi/2}\sin^4\alpha\,\mathrm{d}\alpha = \frac{3\pi}{16}$$

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{\mathrm{d}\alpha}{\sqrt{1-\sin^2\dfrac{\theta_0}{2}\sin^2\alpha}}$$

$$= 4\sqrt{\frac{L}{g}}\left\{\int_0^{\pi/2}\int \mathrm{d}\alpha + \frac{1}{2}\sin^2\frac{\theta_0}{2}\int_0^{\pi/2}\sin^2\alpha\,\mathrm{d}\alpha + \frac{3}{8}\sin^4\frac{\theta_0}{2}\int_0^{\pi/2}\sin^4\alpha\,\mathrm{d}\alpha + \mathcal{O}\left(\sin^6\frac{\theta_0}{2}\right)\right\}$$

$$= 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2} + \mathcal{O}\left(\sin^6\frac{\theta_0}{2}\right)\right] \qquad \blacksquare$$

$$\left(1-k^2\sin^2\alpha\right)^{-1/2}=\sum_{n=0}^{\infty}\frac{(2n)!}{\left(2^n\right)^2\left(n!\right)^2}\,k^{2n}\sin^{2n}\alpha$$

$$\int_0^{\pi/2}\sin^{2n}\alpha\,\mathrm{d}\alpha=\frac{(2n)!}{\left(2^n\right)^2\left(n!\right)^2}\,\frac{\pi}{2}$$

$$\begin{aligned}
T&=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{\mathrm{d}\alpha}{\sqrt{1-\sin^2\dfrac{\theta_0}{2}\sin^2\alpha}}\\
&=4\sqrt{\frac{L}{g}}\sum_{n=0}^{\infty}\frac{(2n)!}{\left(2^n\right)^2\left(n!\right)^2}\,\sin^{2n}\frac{\theta_0}{2}\int_0^{\pi/2}\sin^{2n}\alpha\,\mathrm{d}\alpha\\
&=4\sqrt{\frac{L}{g}}\sum_{n=0}^{\infty}\frac{(2n)!}{\left(2^n\right)^2\left(n!\right)^2}\,\sin^{2n}\frac{\theta_0}{2}\,\frac{(2n)!}{\left(2^n\right)^2\left(n!\right)^2}\,\frac{\pi}{2}\\
&=2\pi\sqrt{\frac{L}{g}}\sum_{n=0}^{\infty}\frac{\left[(2n)!\right]^2}{2^{4n}\left(n!\right)^4}\,\sin^{2n}\frac{\theta_0}{2}\qquad\blacksquare
\end{aligned}$$