PC3261: Classical Mechanics II

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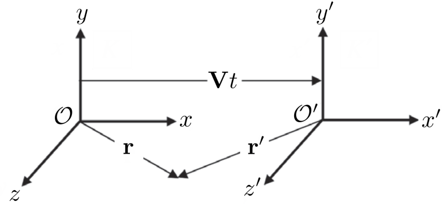
Lecture 12: Special Relativity I

Theory of relativity in Physics



- **Reference frame** is defined as an oriented system of coordinates in three-dimensional space equipped with rulers and clocks to perform measurements of position and time

- Theory of relativity establishes a *connection* between spatial and temporal measurements made in two reference frames

- Two **inertial** reference frames are arranged in a *standard configuration* where their spatial coordinate axes are aligned, their spatial origins are coincided when $t = t' = 0$ and their relative motion occurs with constant speed V along their parallel axes x and x'

Galilean relativity

- Galilean principle of relativity: laws of mechanics are the same in all inertial frames

- **Galilean boost**: constant velocity $\mathbf{V}$ is in an arbitrary direction

$$t' = t\,, \qquad \mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{V}t$$

- Equation of motion of the ith particle within a group of particles interacting via two-body central potential:

$$m_i\ddot{\mathbf{r}}(t) = -\boldsymbol{\nabla}_i \sum_j U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \quad \rightarrow \quad m_i\ddot{\mathbf{r}}'(t') = -\boldsymbol{\nabla}'_i \sum_j U'_{ij}(|\mathbf{r}'_i - \mathbf{r}'_j|)$$

- Wave equation is *not* invariant under Galilean boost

EXERCISE 12.1: Verify explicitly that the wave equation is not invariant under Galilean boost between two inertial frames with arbitrary constant relative velocity.

Postulates of special relativity

- **Principle of relativity**: The laws of Physics are the same in all inertial frames

- **Constancy of the speed of light**: The speed of light in vacuum is the same in all inertial frames regardless of the motion of its emitter or receiver

Derivation of Lorentz boost

- Linear transformation: straight lines are preserved, 20 parameters

- Coincidence of spatial and temporal origins: homogeneity of space and time, 16 parameters

- Alignment of spatial axes and choice of relative velocity along x-direction: isotropy of space, 6 parameters

$$t' = At + Bx\,, \qquad x' = Ct + Dx\,, \qquad y' = Ey\,, \qquad z' = Fz$$

- Symmetry: $(x, z) \to (-x, -z), (x', z') \to (-x', -z')$

$$y = Ey' \quad \Rightarrow \quad E^2 = 1 \quad \Rightarrow \quad E = +1$$

- Symmetry: $(x, y) \to (-x, -y), (x', y') \to (-x', -y')$

$$z = Fz' \quad \Rightarrow \quad F^2 = 1 \quad \Rightarrow \quad F = +1$$

Derivation of Lorentz boost – cont'd

- Choice of relative motion along x-direction:

$$x' = 0 \quad \Rightarrow \quad x = Vt \quad \Rightarrow \quad C = -DV$$

- Constancy of the speed of light:

$$x^2 + y^2 + z^2 = c^2 t^2 \quad \Leftrightarrow \quad x'^2 + y'^2 + z'^2 = c^2 t'^2$$

$$\Rightarrow \quad \begin{cases} -A^2c^2 + D^2V^2 = -c^2 \\ ABc^2 + D^2V = 0 \\ -B^2c^2 + D^2 = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} A = D = \dfrac{1}{\sqrt{1 - V^2/c^2}} \\ B = -\dfrac{V}{c^2} D = -\dfrac{V}{c^2\sqrt{1 - V^2/c^2}} \end{cases}$$

- Lorentz boost between frames in standard orientations: $\beta \equiv V/c$

$$ct' = \gamma\,(ct - \beta x)\;, \quad x' = \gamma\,(x - \beta ct)\;, \quad y' = y\,, \quad z' = z\,, \quad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

Composition of Lorentz boosts

- Frame 1 moves at constant velocity $\mathbf{V}_{12} = V_{12}\,\hat{\mathbf{x}}$ with respect to frame 2 and frame 2 moves at constant velocity $\mathbf{V}_{23} = V_{23}\,\hat{\mathbf{x}}$ with respect to frame 3

- Lorentz boost between frames 1 and 3:

$$ct_1 = \gamma(\beta_{13})\,(ct_3 - \beta_{13}x_3)\ , \quad x_1 = \gamma(\beta_{13})\,(x_3 - \beta_{13}ct_3)\ , \quad y_1 = y_3\,, \quad z_1 = z_3$$

- Composition rules:

$$\beta_{13} = \frac{\beta_{12} + \beta_{23}}{1 + \beta_{12}\beta_{23}}\,, \qquad \gamma(\beta_{13}) = \gamma(\beta_{12})\gamma(\beta_{23})\,(1 + \beta_{12}\beta_{23})$$

EXERCISE 12.2: Derive the composition rules for β and γ factors.

General Lorentz boost

- Axes in $\mathcal{O}$ and $\mathcal{O}'$ remain parallel but the velocity $\mathbf{V}$ of $\mathcal{O}'$ with respect to $\mathcal{O}$ is in an arbitrary direction:

$$ct' = \gamma\left(ct - \boldsymbol{\beta}\cdot\mathbf{r}\right)\ , \qquad \mathbf{r}' = \mathbf{r} + \frac{\gamma-1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{r}\right)\boldsymbol{\beta} - \gamma\boldsymbol{\beta}ct$$

- Successive boosts along the same direction of relative velocity commute and their composite is another boost

- Successive boosts along different directions of relative velocity do not commute and each of these two different composites is not another boost!

EXERCISE 12.3: Derive the Lorentz boost between two inertial frames with parallel axes and arbitrary constant relative velocity.

Spacetime interval

- **Spacetime interval**: separation between two events $(t_1, \mathbf{r}_1)$ and $(t_2, \mathbf{r}_2)$

$$\Delta s^2 \equiv (\Delta s)^2 = -c^2 (t_2 - t_1)^2 + |\mathbf{r}_2 - \mathbf{r}_1|^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

- Spacetime interval is a Lorentz invariant quantity – *frame independent* measure of the separation between two events in the spacetime

$$-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

- Classification of spacetime intervals:

$$\begin{array}{lccc} \textit{spacelike} & \Delta s^2 > 0 & \Rightarrow & \Delta x^2 + \Delta y^2 + \Delta z^2 > c^2 \Delta t^2 \\ \textit{lightlike} & \Delta s^2 = 0 & \Rightarrow & \Delta x^2 + \Delta y^2 + \Delta z^2 = c^2 \Delta t^2 \\ \textit{timelike} & \Delta s^2 < 0 & \Rightarrow & \Delta x^2 + \Delta y^2 + \Delta z^2 < c^2 \Delta t^2 \end{array}$$

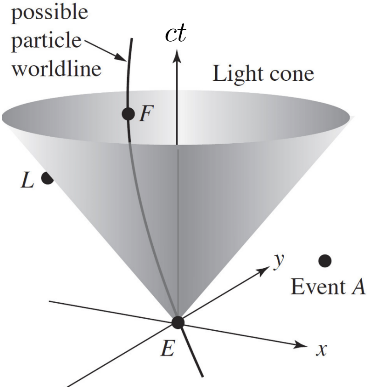
Spacetime diagram

- A **spacetime diagram** (or **Minkowski diagram**) is a convenient way to display the relationship between events in spacetime

- An event is represented by a point on the spacetime diagrams

- A particle's trajectory through spacetime, called the particle's **worldline**, is represented in a spacetime diagram by a connected sequence of events

- A **light cone** is the worldline that light, emitting from a single event and travelling in all directions, would take through spacetime

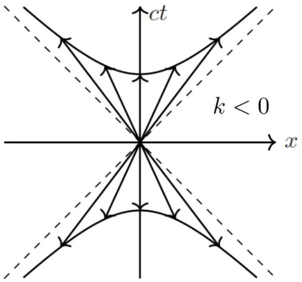


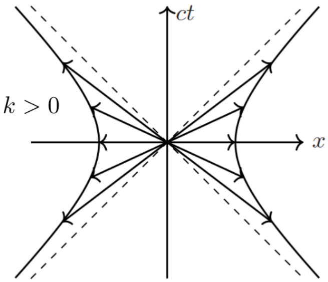
Invariant hyperbola

- All events, (t, x), that have the same spacetime interval, k, from the origin, $(0, 0)$ lie on a hyperbola in the spacetime diagram

$$\Delta s^2 = k \quad \Rightarrow \quad -c^2t^2 + x^2 = k$$

- One must not bring Euclidean geometric expectations to the Minkowski spacetime diagram!





Passive view of Lorentz boost

- ct' axis is the locus of events for which $x' = 0$: a straight line with slope $1/\beta$

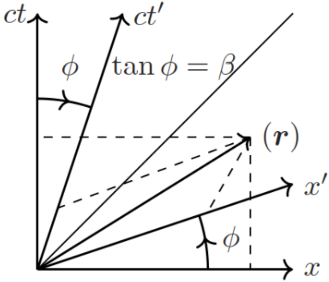
$$x' = \gamma\,(x - \beta ct) = 0 \quad \Rightarrow \quad \frac{ct}{x} = \frac{1}{\beta}$$

- x' axis is the locus of events for which $ct' = 0$: a straight line with slope β

$$ct' = \gamma\,(ct - \beta x) = 0 \quad \Rightarrow \quad \frac{ct}{x} = \beta$$

- ct' and x' axes are reflected images of each other across the light cones at the origin

- Spacetime coordinates of an event are the projections of the event along respective time and space axes: along ct and x for $\mathcal{O}$, and ct' and x' for $\mathcal{O}'$



Temporal sequence of events

- Two events which are simultaneous according to $\mathcal{O}$ might not be simultaneous according to $\mathcal{O}'$

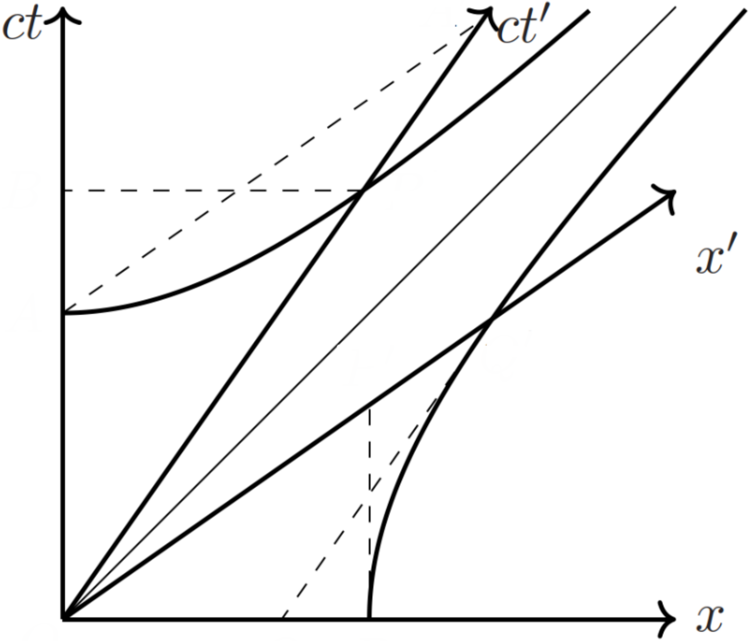
$$\Delta t = 0 \quad \Rightarrow \quad c\,\Delta t' = \gamma\,(c\,\Delta t - \beta\,\Delta x) \neq 0$$

- **Causality** is the relationship between *causes* and *effects*; **causality principle**: cause must precede its effect

- Timelike separated events: if $\Delta t > 0$, then $\Delta t' > 0$ in all *physically possible* inertial reference frames

$$\Delta s^2 < 0 \quad \Rightarrow \quad -c^2\,\Delta t^2 + \Delta x^2 < 0 \quad \Rightarrow \quad -c\,\Delta t < \Delta x < c\,\Delta t$$

$$\Delta t' \leq 0 \quad \Rightarrow \quad \gamma\,(c\,\Delta t - \beta\,\Delta x) \leq 0 \quad \Rightarrow \quad \begin{cases} \beta \leq \dfrac{c\,\Delta t}{\Delta x} \\[2ex] \beta \geq \dfrac{c\,\Delta t}{\Delta x} \end{cases} \quad \Rightarrow \quad \begin{cases} \beta < -1 \\ \beta > 1 \end{cases}$$



ct
ct'
x'
x

'Arc length' in spacetime

- Spacetime interval between two infinitesimal separated events:

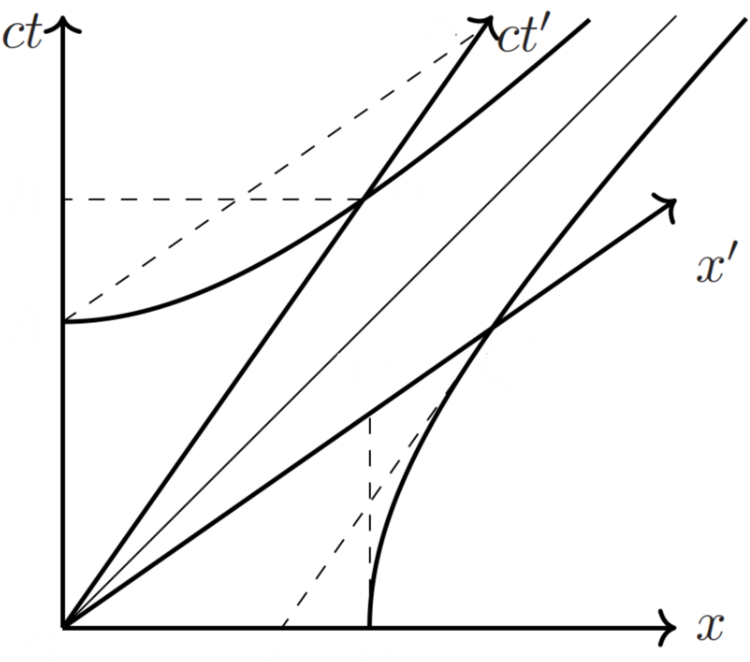
$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

- **Proper time** between two events, $(t_1, \mathbf{r}_1)$ and $(t_2, \mathbf{r}_2)$, is measured by a clock travelling along a given timelike worldline $\mathcal{C}_{12}$ connecting those events

$$\Delta\tau = \int_{\mathcal{C}_{12}} \sqrt{1 - \frac{V^2}{c^2}}\,\mathrm{d}t\,, \qquad V^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2$$

- All observers agree on the value of the proper time between the two events along the given timelike worldline

EXERCISE 12.4: Derive the expression for the proper time between two events along a given timelike worldline.



ct
ct'
x'
x

Length contraction

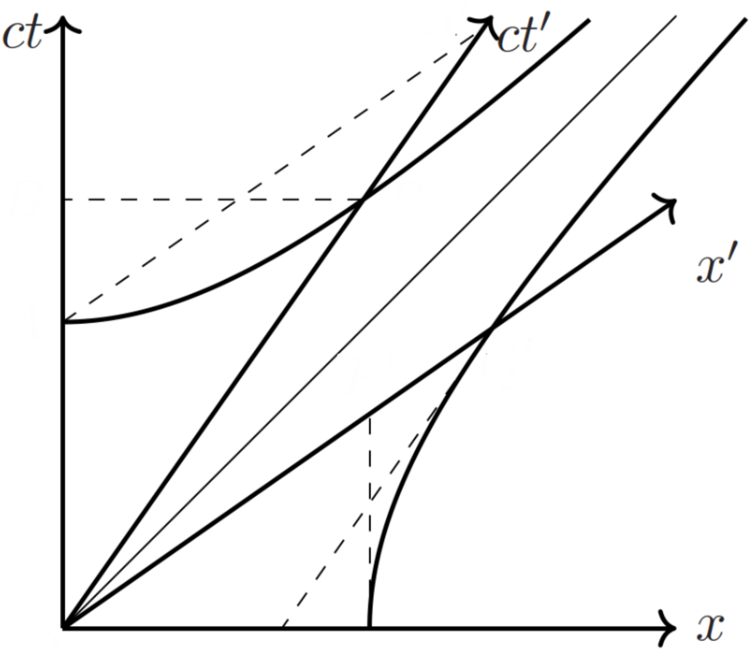
- Length of an object in any inertial frame is defined to be the spatial distance between two events located at the object's end points that are *simultaneous* in the inertial frame

- Observers in different inertial frames will disagree about the object's length as they disagree about which pairs of events are simultaneous

- Relationship between an object's length, **proper length** L_0, along a *given* direction in its own inertial frame and its length, **contracted length** L, in an inertial frame where it is observed to move with speed $v = \beta c$ in *that* direction:

$$L = L_0\sqrt{1-\beta^2}$$

EXERCISE 12.5: Derive the relationship between proper length and contracted length by using an *appropriate* Lorentz boost for coordinate differences.


ct
ct'
x'
x