We have already obtained a correct relativistic equation for a spin $\frac{1}{2}$ particle, the Dirac equation

$$\not{p}\,\psi(x) = mc\,\psi(x) \qquad \not{p} = p_\mu \gamma^\mu$$

We now want to construct a free particle solution of the Dirac equation.

Recall:

In non-relativistic quantum mechanics, the equation of motion is the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(\underset{\sim}{x}) = H\psi(\underset{\sim}{x}), \qquad H = \frac{\underset{\sim}{p}^2}{2m} + V(\underset{\sim}{x})$$

For a free particle $H = \frac{\underset{\sim}{p}^2}{2m}$, no potential force field, $V(\underset{\sim}{x}) = 0$. $\rightarrow$ $i\hbar \frac{\partial}{\partial t}\psi(\underset{\sim}{x}) = -\frac{\hbar^2}{2m}\nabla^2\psi(\underset{\sim}{x})$

The free particle is a plane wave

$$\psi(\underset{\sim}{x}, t) = \text{const.}\; e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)} \text{ or } e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{x} + \omega t)}$$

$$\underset{\sim}{p} = \hbar\underset{\sim}{k}, \quad E = \hbar\omega, \quad E = \frac{\underset{\sim}{p}^2}{2m}$$

Note: $e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$, $e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} + \omega t)}$ not allowed photon is described by the Maxwell equation

$$\partial_\mu \partial^\mu \underline{A}(\underline{x}) = 0 \qquad \text{or} \quad \Box^2 \underline{A}(\underline{x}) = 0$$

$\Box^2$ = D'lembertian

$$\partial_\mu A^\mu(\underline{x}) = 0 \qquad \text{Lorentz condition}$$

Free photon is a plane wave

$$A_\mu(\underline{x}) = \text{const}\; e^{-i\underline{p}\cdot\underline{x}/\hbar}\; \varepsilon_\mu(\underline{p}), \quad \underline{p}^2 = 0$$

or $\underline{A}(\underline{x}) = \text{const}\; e^{-i\underline{p}\cdot\underline{x}/\hbar}\; \underline{\varepsilon}(\underline{p})$

and $\partial_\mu A^\mu(\underline{x}) = 0 \longrightarrow \underline{p}\cdot\underline{\varepsilon}(\underline{p}) = 0$

$\underline{\varepsilon}(\underline{p})$ = polarization

The relativistic spin-0 particle is described by the Klein-Gordon equation

$$\underline{p}^2\,\phi(\underline{x}) = m^2c^2\,\phi(\underline{x}), \quad \underline{p}^2 = -\hbar^2\Box^2$$

The free particle is a plane-wave

$$\phi(\underline{x}) = \text{const}\; e^{-i(\underline{p}\cdot\underline{x})/\hbar}, \quad \underline{p}^2 = m^2c^2$$

spin 0 particle or scalar particle or pseudo-scalar particle, e.g. π^0, π^+, π^- mesons

Construct the free particle solution of the Dirac equation.

$$\not{p}\,\psi(\underline{x}) = mc\,\psi(\underline{x})$$

The plane wave solution can be written as

$$\psi(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar}\,u(\underline{p})$$

or

$$\psi_\alpha(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar}\,u_\alpha(\underline{p}),$$

$$\alpha = 1, 2, 3, 4$$

$$u(\underline{p}) = \begin{pmatrix} u_1(\underline{p}) \\ u_2(\underline{p}) \\ u_3(\underline{p}) \\ u_4(\underline{p}) \end{pmatrix}$$

We want to find the solutions to this to understand the bispinor Dirac wavefunction
Each of the four components represent wavefunctions that live in Hilbert space

The unknowns are $\underline{p}$ and $u(\underline{p})$

4-momentum of the particle

bispinor

The dirac wavefunction is a bispinor in Hilbert space
Note: Bispinors are not vectors or tensors, they are their own thing

Substituting

$$\psi(\underline{x}) = e^{-i\underline{p}\cdot\underline{x}/\hbar}\, u(\underline{p})$$

into the Dirac equation

$$\gamma^\mu p_\mu \psi(\underline{x}) = mc\, \psi(\underline{x}),$$

sub in, cancel the e term

we get

$$\gamma^\mu p_\mu u(\underline{p}) = mc\, u(\underline{p}),$$

p_μ are four numbers, not a differential operator.

Using the Dirac representation for the matrix γ^μ,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

We have 2 solns

$$(\gamma^0 p_0 + \gamma^i p_i)\, u(\underline{p}) = mc\, u(\underline{p})$$

$$(\gamma^0 p^0 - \gamma^i p^i)\, u(\underline{p}) = mc\, u(\underline{p})$$

$$\underset{\sim}{\gamma}\cdot\underset{\sim}{p} = \gamma^i p^i = -\gamma^i p_i$$

If LHS is a matrix, RHS must also be a matrix which is rep by the mc(identity matrix)

$$\begin{pmatrix} p^0 & -\underset{\sim}{\sigma}\cdot\underset{\sim}{p} \\ \underset{\sim}{\sigma}\cdot\underset{\sim}{p} & -p^0 \end{pmatrix} u(\underset{\sim}{p}) = \underline{mc}\; u(\underset{\sim}{p})\, \underline{\mathbb{1}}$$

moving the diagonal mc over to LHS,

or

$$\begin{pmatrix} p^0 - mc & -\underset{\sim}{\sigma}\cdot\underset{\sim}{p} \\ \underset{\sim}{\sigma}\cdot\underset{\sim}{p} & -p^0 - mc \end{pmatrix} \begin{pmatrix} u_1(\underset{\sim}{p}) \\ u_2(\underset{\sim}{p}) \\ u_3(\underset{\sim}{p}) \\ u_4(\underset{\sim}{p}) \end{pmatrix} = 0$$

Nontrivial solution for $u(\underset{\sim}{p})$ iff,

$$\begin{vmatrix} p^0 - mc & -\underset{\sim}{\sigma}\cdot\underset{\sim}{p} \\ \underset{\sim}{\sigma}\cdot\underset{\sim}{p} & -p^0 - mc \end{vmatrix} = 0$$

or

$$(p^0 - m^2c^2) - (\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^2 = 0$$

But $(\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^2 = \underset{\sim}{p}^2$

$$\mathbf{p}\cdot\boldsymbol{\sigma} = p_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} p_z & (p_x - ip_y) \\ (p_x + ip_y) & -p_z \end{pmatrix}$$
$$(\mathbf{p}\cdot\boldsymbol{\sigma})^2 = \begin{pmatrix} p_z^2 + (p_x - ip_y)(p_x + ip_y) & p_z(p_x - ip_y) - p_z(p_x - ip_y) \\ p_z(p_x + ip_y) - p_z(p_x + ip_y) & (p_x + ip_y)(p_x - ip_y) + p_z^2 \end{pmatrix} = \mathbf{p}^2 \quad (7.39)$$

so

$$p^0 = \pm\sqrt{\underset{\sim}{p}^2 + m^2c^2}$$

$$p^0 = \frac{E}{c}$$

- Having obtained $p = (p^0, \underset{\sim}{p})$, we now find the bispinor $u(p)$

Two cases (i) $p^0 > 0$, (ii) $p^0 < 0$

(i) $p^0 = + \sqrt{\underset{\sim}{p}^2 + m^2 c^2}$

we want to solve

$$\begin{pmatrix} p^0 & -\underset{\sim}{\sigma} \cdot \underset{\sim}{p} \\ \underset{\sim}{\sigma} \cdot \underset{\sim}{p} & -p^0 \end{pmatrix} u(p) = mc\, u(p)$$

convenient to write $u(p)$ as

$$u(p) = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}, \qquad w^1 = \begin{pmatrix} u_1(p) \\ u_2(p) \end{pmatrix}, \qquad w^2 = \begin{pmatrix} u_3(p) \\ u_4(p) \end{pmatrix}$$

hence

$$\boxed{\begin{aligned} p^0 w^1 - \underset{\sim}{\sigma} \cdot \underset{\sim}{p}\, w^2 &= mc\, w^1 \\ \underset{\sim}{\sigma} \cdot \underset{\sim}{p}\, w^1 - p^0 w^2 &= mc\, w^2 \end{aligned}}$$

One can solve for w^1 in terms of w^2 or vice versa.

For case (i) $p^0 > 0$, more convenient to express w^2 in terms of w^1, so use

$$\underset{\sim}{\sigma} \cdot \underset{\sim}{p}\, w^1 - p^0 w^2 = mc\, w^2$$

$$\boxed{w^2 = \frac{\underset{\sim}{\sigma} \cdot \underset{\sim}{p}}{p^0 + mc}\, w^1}$$

Thus

$$u(\underset{\sim}{p}) = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} w^1 \\ \dfrac{\underset{\sim}{\sigma} \cdot \underset{\sim}{p}}{p^0 + mc}\, w^1 \end{pmatrix}$$

$$w^1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1, u_2 \text{ arbitrary}$$

Two linearly independent solutions for w^1, e.g.

$$w^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad w^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is convenient to write the two linearly independent positive energy solutions as

$$u_{+}^{(s)}(\underset{\sim}{p}) = \begin{pmatrix} w^s \\ \dfrac{\underset{\sim}{\sigma} \cdot \underset{\sim}{p}}{p^0 + mc} w^s \end{pmatrix}, \quad s = 1, 2$$

+ve energy
$p^0 > 0$

For convenience or normalization, can require

$$w^{s\dagger} w^s = 1$$

Thus we have obtained the positive energy free particle solution of the Dirac equation,

$$\psi(\underset{\sim}{x}) = e^{-i \underset{\sim}{p} \cdot \underset{\sim}{x} / \hbar} \, u_{+}^{(s)}(\underset{\sim}{p}), \quad s = 1, 2$$

$$u_{+}^{(s)}(\underset{\sim}{p}) = \underset{k}{\text{constant}} \begin{pmatrix} w^s \\ \dfrac{\underset{\sim}{\sigma} \cdot \underset{\sim}{p}}{p^0 + mc} w^s \end{pmatrix}$$

Normalization convention: $u_{+}^{(s)\dagger} u_{+}^{(s)} = 2p^0$
$p^0 > 0$

$k \to k^*$
$w^s \to w^{s\dagger}$
$\left(\dfrac{\underset{\sim}{\sigma} \cdot p}{p^0 + mc}\right)^\dagger = \left(\dfrac{\underset{\sim}{\sigma} \cdot p}{p^0 + mc}\right)$

$$|k|^2 \left(w^{s\dagger}, \left(\frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s\right)^\dagger \right) \begin{pmatrix} w^s \\ \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s \end{pmatrix} = 2p^0 \quad (9)$$

$\rightarrow$ $|k| = ?$ Ans: $\sqrt{p^0+mc}$ (HW)

Answer below

$$|k|^2 \left(w^{s\dagger}\cdot w^s + \left(\frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s\right)^\dagger \cdot \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s \right) = 2p^0$$

$$w^{s\dagger} w^s = 1$$

$$|k|^2 \left(1 + w^{s\dagger} \left(\frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc}\right)^\dagger \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc} w^s \right) = 2p^0$$

$$|k|^2 \left(1 + w^{s\dagger} \frac{(\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^\dagger(\underset{\sim}{\sigma}\cdot\underset{\sim}{p})}{(p^0+mc)^2} w^s \right) = 2p^0$$

$$|k|^2 \left(1 + w^{s\dagger} \frac{\underset{\sim}{p}^2}{(p^0+mc)^2} w^s \right) = 2p^0$$

$$\mathbf{p}\cdot\boldsymbol{\sigma} = p_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + p_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + p_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
$$= \begin{pmatrix} p_z & (p_x - ip_y) \\ (p_x + ip_y) & -p_z \end{pmatrix}$$
$$(\mathbf{p}\cdot\boldsymbol{\sigma})^2 = \begin{pmatrix} p_z^2 + (p_x - ip_y)(p_x + ip_y) & p_z(p_x - ip_y) - p_z(p_x - ip_y) \\ p_z(p_x + ip_y) - p_z(p_x + ip_y) & (p_x + ip_y)(p_x - ip_y) + p_z^2 \end{pmatrix} = \mathbf{p}^2 \quad (7.39)$$

$\because$ $(\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^\dagger = \underset{\sim}{\sigma}\cdot\underset{\sim}{p}$ HW

and $(\underset{\sim}{\sigma}\cdot\underset{\sim}{p})^2 = \underset{\sim}{p}^2$ HW

$$|k|^2 \left(1 + \frac{\underset{\sim}{p}^2}{(p^0+mc)^2} \right) = 2p^0$$

$\because \ w^{s\dagger} w^s = 1$

$$|k|^2 \left(1 + \frac{p^{0\,2} - m^2c^2}{(p^0+mc)^2} \right) = 2p^0$$

$$|k|^2 \left(1 + \frac{p^0 - mc}{p^0+mc} \right) = 2p^0$$

$$|k|^2 \ \frac{2p^0}{p^0+mc} = 2p^0$$

$$\underline{|k| = \sqrt{p^0+mc}}$$

i.e.

$$u_+^{(s)} = \sqrt{p^0+mc} \begin{pmatrix} w^s \\ \dfrac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p^0+mc}\, w^s \end{pmatrix}$$

$s = 1, 2.$

So +ve energy $p^0 = \sqrt{\underset{\sim}{p}^2 + m^2c^2}$ has been constructed explicitly.

Now negative energy soln

$$p^0 = -\sqrt{\underset{\sim}{p}^2 + m^2c^2}$$

$$\psi(x) = e^{-i(p\cdot x)/\hbar}\, u_-^{(s)}(p) \qquad (11)$$

$$u_-^{(s)}(p) = \sqrt{mc - p^0}\begin{pmatrix} \dfrac{\underset{\sim}{\sigma}\cdot \underset{\sim}{p}}{p^0 - mc}\, w^s \\ w^s \end{pmatrix} \qquad s=1,2 \qquad (HW)$$

Reinterpret $u_-^{(s)}(p)$ by putting

$$\underset{\sim}{p} \to -\underset{\sim}{p}$$

$$\underset{\sim}{\sigma}\cdot(-\underset{\sim}{p}) = -\sigma\cdot \underset{\sim}{p}$$
$$-p^0 - mc = -(p^0 + mc)$$

$$u_-^{(s)}(-p) = \sqrt{mc + p^0}\begin{pmatrix} \dfrac{\underset{\sim}{\sigma}\cdot \underset{\sim}{p}}{p^0 + mc}\, w^s \\ w^s \end{pmatrix} \cdots (+)'$$

$$\psi(x) = e^{i\underset{\sim}{p}\cdot \underset{\sim}{x}/\hbar}\, u_-^{(s)}(-\underset{\sim}{p})$$

which is regarded as a solution for an anti particle e^+ (positron) of positive energy.

the free Dirac particle can be written also as

$$\psi(x) = e^{+i\underset{\sim}{p}\cdot \underset{\sim}{x}/\hbar}\, V(\underset{\sim}{p})$$

Can solve $p^0 = \pm\sqrt{\underset{\sim}{p}^2 + m^2c^2}$

For $p^0 = +\sqrt{\underline{p}^2 + m^2c^2}$

$$V_+^{(s)}(\underline{p}) = \sqrt{p^0 + mc}\begin{pmatrix} \dfrac{\underline{\sigma}\cdot\underline{p}}{p^0+mc}\, w^s \\ w^s \end{pmatrix} \quad \text{(HW)} \qquad \text{-- (x)}$$

Compare (+) and (x), identify

$$V_+^{(1)}(\underline{p}) = u_-^{(2)}(-\underline{p})$$

$$V_+^{(2)}(\underline{p}) = -\,u_-^{(1)}(-\underline{p})$$

A general solution is $\psi(x) = a e^{-ip\cdot x/\hbar} u(\underline{p}) + b e^{ip\cdot x/\hbar} V(\underline{p})$

a, b constant

Why 2 l. i. solutions for $u_+^{(s)}$ or $u_-^{(s)}$ (or why energy p^0 is doubly degenerate?)

i.e. for the same energy p^0, can have 2 l. i. solutions i.e. p^0 (energy) is doubly degenerate $\rightarrow$ $\exists$ other observable that commutes with the Dirac Hamiltonian

This observable is the helicity operator $h(\underline{p})$

QED II stopped here in Week 12

Helicity operator

$$h(p) = \underset{\sim}{\Sigma} \cdot \frac{\underset{\sim}{p}}{|\underset{\sim}{p}|}$$

here $\underset{\sim}{p}$ is not an operator

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\sigma} & 0 \\ 0 & \underset{\sim}{\sigma} \end{pmatrix}$$

Dirac spin operator $\underset{\sim}{S} = \frac{\hbar}{2} \Sigma_i$

(compare with schrödinger $\underset{\sim}{S} = \frac{\hbar}{2} \underset{\sim}{\sigma}$)

Can show $[h(\underset{\sim}{p}), H] = 0$

$$H = c\, \underset{\sim}{\alpha} \cdot \underset{\sim}{p} + \beta m c^2$$

$\rightarrow$ $h^2(\underset{\sim}{p}) = 1$ (identity operator) (HW)

i.e. eigenvalues of $h(\underset{\sim}{p}) = \pm 1$

which can be used to differentiate the two l. i. soln of the same energy.

What are scalar, vectors and tensors in the Dirac formulation?

· scalar, vector and tensor constructed from $\psi(\underline{x})$

Tested but won't be asked to derive

$\bar{\psi}(\underline{x})\,\psi(\underline{x})$ scalar

$\bar{\psi}(\underline{x})\,\gamma^5\,\psi(\underline{x})$ pseudoscalar

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

Chirality and helicity are the same ONLY when describing massless particles (eigenvalues same)
In all other cases, chirality and helicity are DIFFERENT

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

HW

(Dirac representation)

$\bar{\psi}(\underline{x})\,\gamma^\mu\,\psi(\underline{x})$ vector

$\bar{\psi}(\underline{x})\,\gamma^5\gamma^\mu\,\psi(\underline{x})$ pseudo vector

$\bar{\psi}\,\sigma^{\mu\nu}\,\psi$ tensor

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$$

The prob. current density $j^\mu = c\,\bar{\psi}\,\gamma^\mu\,\psi$ is a 4-vector

$\bar{\psi}(\underline{x}) = \psi^\dagger(\underline{x})\,\gamma^0$ = Dirac adjoint of $\psi(\underline{x})$

$\psi^\dagger(\underline{x})$ = Hermitian conjugate (or adjoint) of $\psi(\underline{x})$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \psi^+ = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$$

$$\bar{\psi} = \psi^+ \gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

Here 1 = 2x2 identity matrix

As $\quad j^\mu = c\,\bar{\psi}\gamma^\mu\psi$,

$$\therefore \quad j^0 = c\,\bar{\psi}\gamma^0\psi = c\,(\psi_1^*\psi_1, \psi_2^*\psi_2, \psi_3^*\psi_3, \psi_4^*\psi_4)$$

Thus in the Dirac case, the probability density $j^0(x)$ is positive, unlike the Klein-Gordon case.

One can check the 4-probability current density $j^\mu(x)$ does satisfy the continuity equation $\partial_\mu j^\mu(x) = 0$, thus probability is conserved

To check probability conservation: $\partial_\mu j^\mu = 0$

prob. current density for the Dirac eqn is defined

$$j^\mu = c\,\bar{\psi}\gamma^\mu\psi$$

want to show $\partial_\mu j^\mu = 0$ conservation of prob.

from the equation of motion.

$$\not{p}\psi = mc\psi, \qquad \not{p} = p_\mu\gamma^\mu = \gamma^\mu p_\mu$$
$$p_\mu = i\hbar\partial_\mu$$

$$\partial_\mu j^\mu = c(\partial_\mu\bar{\psi})\gamma^\mu\psi + c\bar{\psi}\gamma^\mu(\partial_\mu\psi)$$

$$= c\,\underline{\partial_\mu\bar{\psi}}\gamma^\mu\psi + c\bar{\psi}\,mc\psi/(i\hbar) \quad \text{-- (1)}$$

Eq of motion for $\bar{\psi} \equiv \psi^\dagger\gamma^0$ (Dirac adjoint)

Taking the adjoint of the Dirac equation

$$\gamma^\mu \cdot i\hbar\partial_\mu\psi = mc\,\psi$$

$$-i\hbar\partial_\mu\psi^\dagger \cdot \gamma^{\mu\dagger} = mc\,\psi^\dagger$$

Recall $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

$$\therefore\ -i\hbar\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 = mc\,\psi^\dagger$$

Multiply γ^0 from the right and as $\gamma^{0\,2} = 1$,

we have

$$-i\hbar \partial_\mu \psi^\dagger \gamma^0 \gamma^\mu = mc\, \psi^\dagger \gamma^0$$

Now $\bar{\psi} \equiv \psi^\dagger \gamma^0$ the Dirac adjoint,

$$-i\hbar \partial_\mu \bar{\psi}\, \gamma^\mu = mc\, \bar{\psi} \qquad (2)$$

i.e.

$$\underline{(p_\mu \bar{\psi})\, \gamma^\mu = -\, mc\, \bar{\psi}}$$

compare from Dirac Eqn
$\not{p}\, \psi(x) = mc\, \psi(x)$

substituting eq (2) into eq (1), we finally arrive

at

$$\partial_\mu j^\mu = \frac{c\, mc\, \bar{\psi}}{-i\hbar} \cdot \psi + mc^2 \frac{\bar{\psi}\, \psi}{i\hbar} = 0$$

the continuity equation for the 4-current density

$j^\mu(x)$

Charge conjugation in the Dirac formulation

Consider a charged particle (electron),

$$\not{p}\,\psi(x) = mc\,\psi(x)$$

In the presence of an em field $A_\mu(x)$

$$\underline{p} \rightarrow \underline{p} - q\underline{A}$$

the Dirac eqn becomes

$$(\not{p} - q\not{A})\,\psi = mc\,\psi$$

$$\not{A} = A_\mu \gamma^\mu$$

$$\gamma^\mu (i\hbar\partial_\mu - q A_\mu)\psi = mc\,\psi$$

Taking adjoint

$$(-i\hbar\partial_\mu - qA_\mu)\psi^\dagger \gamma^{\mu\dagger} = mc\,\psi^\dagger \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

$$(i\hbar\partial_\mu + qA_\mu)\,\bar{\psi}\,\gamma^\mu = -mc\,\bar{\psi}$$

Taking transpose

$$\gamma^{\mu t}(i\hbar\partial_\mu + qA_\mu)\,\bar{\psi}^t = -mc\,\bar{\psi}^t$$

If $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$, then $[\gamma^{\mu t}, \gamma^{\nu t}]_+ = 2g^{\mu\nu}$

$$\rightarrow \quad \gamma^{\mu t} = -C^{-1}\gamma^\mu C$$

$$\rightarrow \quad C^{-1}\gamma^{\mu} C\,(i\hbar\partial_\mu + qA_\mu)\,\bar{\psi}^t = mc\,\bar{\psi}^t$$

$$\longrightarrow \quad \gamma^{\mu}(i\hbar\partial_\mu + qA_\mu)\,C\,\bar{\psi}^t = mc\;C\bar{\psi}^t$$

Define the charge conjugate Dirac ψ

$$\psi_c \equiv C\,\bar{\psi}^t = C\,(\psi^{+}\gamma^0)^t$$

$$= C\,\gamma^{0t}\,\psi^{*} \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= C\,\gamma^0\,\psi^{*}$$

The charge conjugate equation of

$$(\not{p} - q\not{A})\,\psi = mc\,\psi \qquad \dagger\ HW$$

is $$(\not{p} + q\not{A})\,\psi_c = mc\,\psi_c$$

Explicit expression for the charge conjugation operator

$$\boxed{C = i\gamma^2\gamma^0}$$