PC3261: Classical Mechanics II

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Lecture 9: Lagrangian Mechanics II

Homogeneous functions

- Homogeneous function of degree M: λ is *any* positive real number

$$f(\lambda x_1, \lambda x_2, \cdots, \lambda x_N) = \lambda^M f(x_1, x_2, \cdots, x_N)$$

- Examples:

$$\begin{cases} f(x,y) = \left(x^4 + 2xy^3 - 5y^4\right)\sin\dfrac{x}{y} \quad \rightarrow \quad f(\lambda x, \lambda y) = \lambda^4 f(x,y) \\[2ex] f(x,y,z) = \dfrac{C}{\sqrt{x^2+y^2+z^2}} \quad \rightarrow \quad f(\lambda x, \lambda y, \lambda z) = \lambda^{-1} f(x,y,z) \end{cases}$$

- **Euler's theorem on homogeneous function**:

$$\sum_{i=1}^{N} x_i \frac{\partial f(x_1, x_2, \cdots x_N)}{\partial x_i} = M f(x_1, x_2, \cdots x_N)$$

Kinetic energy in terms of generalized coordinates

- Kinetic energy is a quadratic function of the generalized velocities:

$$T \equiv T(\{q_k, \dot{q}_k\}, t) = M_0(\{q_k\}, t) + \sum_{i=1}^{M} M_i(\{q_k\}, t)\ \dot{q}_i + \frac{1}{2} \sum_{i,j=1}^{M} M_{ij}(\{q_k\}, t)\ \dot{q}_i\, \dot{q}_j$$

$$\begin{cases} M_0(\{q_k\}, t) = \dfrac{1}{2} \displaystyle\sum_{\alpha=1}^{N} m_\alpha \, \dfrac{\partial \mathbf{r}_\alpha}{\partial t} \cdot \dfrac{\partial \mathbf{r}_\alpha}{\partial t} \\[2ex] M_i(\{q_k\}, t) = \displaystyle\sum_{\alpha=1}^{N} m_\alpha \, \dfrac{\partial \mathbf{r}_\alpha}{\partial t} \cdot \dfrac{\partial \mathbf{r}_\alpha}{\partial q_i} \\[2ex] M_{ij}(\{q_k\}, t) = \displaystyle\sum_{\alpha=1}^{N} m_\alpha \, \dfrac{\partial \mathbf{r}_\alpha}{\partial q_i} \cdot \dfrac{\partial \mathbf{r}_\alpha}{\partial q_j} \end{cases}$$

- Kinetic energy is a homogeneous quadratic function of the generalized velocities if $\mathbf{r}_\alpha = \mathbf{r}_\alpha(\{q_k\})$

Conservation of energy

- **Jacobi energy function** is a constant of motion if the Lagrangian does not depend on time explicitly

$$h(\{q_i, \dot{q}_i\}, t) \equiv \sum_{i=1}^{M} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}(\{q_k(t), \dot{q}_k(t)\}, t)$$

- If the Lagrangian does not depend on time explicitly and the kinetic energy is a homogeneous quadratic function of generalized velocities, then the Jacobi energy function is the total mechanical energy of the system and it is a constant of motion

$$h(\{q_i, \dot{q}_i\}, t) \quad \rightarrow \quad h(\{q_i, \dot{q}_i\}) = T(\{q_i, \dot{q}_i\}) + U(\{q_i\}) = E$$

EXERCISE 9.1: Show that the Jacobi energy function, $h(\{q_i, \dot{q}_i\}, t)$ is a constant of motion if the Lagrangian does not depend on time explicitly.

System subjected to holonomic constraints

- System with M degrees of freedom: M independent generalized coordinates $\{q_i\}$ and M independent Euler-Lagrange equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0\,, \qquad k = 1, 2, \cdots, M$$

- System subjected to C **holonomic constraints**:

$$\psi_i(\{q_k(t)\}, t) = 0\,, \qquad i = 1, 2, \cdots, C$$

- Degree of freedom of the system is now reduced to $M - C$ and these M Euler-Lagrange equations of motion are no longer independent from each other

- One solution is to introduce $M - C$ independent generalized coordinates

Lagrange multipliers and constraints

- An alternative approach is to keep these M generalized coordinates and introduce C **Lagrange multipliers** (one for each holonomic constraint) so that there are still M independent *modified* equations of motion

- Modified Euler-Lagrange equations of motions: M second order differential equations together with C holonomic constraints to solve for M generalized coordinates and C Lagrange multipliers

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = \sum_{i=1}^{C} \lambda_i(t)\, \frac{\partial \psi_i}{\partial q_k}\,, \qquad k = 1, 2, \cdots, M$$

- **Generalized constraint forces**: an advantage of the approach with Lagrange multipliers is that the force of constraint can be determined

$$\mathcal{Q}_k^{\mathsf{cons}} \equiv \sum_{i=1}^{C} \lambda_i(t)\, \frac{\partial \psi_i}{\partial q_k}\,, \qquad k = 1, 2, \cdots, M$$

Example: Atwood machine (another visit)

- Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless pulley with frictionless pulley

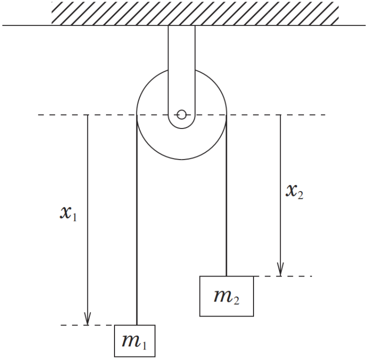
- Kinetic and potential energies:

$$T(\dot{x}_1, \dot{x}_2) = \frac{1}{2}\left(m_1\dot{x}_1^2 + m_2\dot{x}_2^2\right)$$

$$U(x_1, x_2) = -g\left(m_1 x_1 + m_2 x_2\right)$$

- Accelerations:

$$\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\, g = -\ddot{x}_2$$



EXERCISE 9.2: Solve for the accelerations of the masses from the Euler-Lagrange equation and determine the generalized constraint forces.

Lagrange multipliers and constraints – cont'd

- Redefine Lagrangian to include holonomic constraints:

$$\mathcal{L}'(\{q_i(t), \dot{q}_i(t), \lambda_j(t)\}, t) \equiv \mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t) - \sum_{j=1}^{C} \lambda_j(t)\, \psi_j(\{q_i(t)\}, t)$$

- Euler-Lagrange equations of motion:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}'}{\partial \dot{q}_k}\right) - \dfrac{\partial \mathcal{L}'}{\partial q_k} = 0\,, & \qquad k = 1, 2, \cdots, M \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}'}{\partial \dot{\lambda}_j}\right) - \dfrac{\partial \mathcal{L}'}{\partial \lambda_j} = 0\,, & \qquad j = 1, 2, \cdots, C \end{cases}$$

- The definition of Lagrangian for a system is not unique but the bottom line is that it must give the correct equations of motion of the system!

Example: Particle on a hemisphere (revisited)

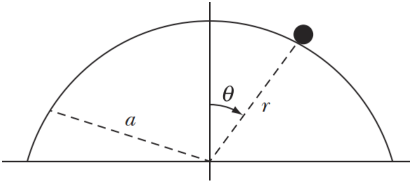
- A particle of mass m starts at rest on top of a smooth fixed hemisphere of radius a

- Lagrangian:

$$\mathcal{L}(r,\theta,\dot{r},\dot{\theta}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

- Holonomic constraint:

$$\psi(r,\theta) = r(t) - a = 0$$



EXERCISE 9.3: Determine the angle at which the particle leaves the hemisphere from the Euler-Lagrange equation.

Generalized non-conservative forces

- Generalized non-conservative forces:

$$\mathcal{Q}_k^{\mathsf{nc}} = \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{\mathsf{nc}} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}$$

- Euler-Lagrange equations of motion with both constraint forces and non-conservative forces:

$$\mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t) \equiv T(\{q_i(t), \dot{q}_i(t)\}, t) - U(\{q_i(t)\}, t)$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = \mathcal{Q}_k^{\mathsf{cons}} + \mathcal{Q}_k^{\mathsf{nc}}, \qquad k = 1, 2, \cdots, M$$

EXERCISE 9.4: A simple pendulum of mass m and length ℓ is subjected to linear resistance force $\mathbf{F} = -\gamma \mathbf{v}$ with $\gamma > 0$. Obtain the equations of motion of this pendulum with suitable generalized coordinate(s).

Generalized potential function

- Generalized forces that can be derived from a **generalized potential function** $\mathcal{U}(\{q_i(t), \dot{q}_i(t)\}, t)$:

$$\mathcal{Q}_k = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{U}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{U}}{\partial q_k}$$

- Lagrangian:

$$\mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t) = T(\{q_i(t), \dot{q}_i(t)\}, t) - \mathcal{U}(\{q_i(t), \dot{q}_i(t)\}, t)$$

- Euler-Lagrange equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0\,, \qquad k = 1, 2, \cdots, M$$

Charge in external electromagnetic field

- Potential formulation in classical electrodynamics:

$$\mathbf{E}(\mathbf{r},t) = -\boldsymbol{\nabla}\phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}\,, \qquad \mathbf{B}(\mathbf{r},t) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r},t)$$

- Lorentz force:

$$\mathbf{F} = q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) \quad \Rightarrow \quad F_i = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{x}_i}\left(q\phi - q\mathbf{A}\cdot\mathbf{v}\right)\right] - \frac{\partial}{\partial x_i}\left(q\phi - q\mathbf{A}\cdot\mathbf{v}\right)$$

- Lagrangian for charge in external electromagnetic field:

$$\mathcal{L}(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) = \frac{m}{2}\,\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t) - q\,\phi(\mathbf{r},t) + q\,\dot{\mathbf{r}}(t)\cdot\mathbf{A}(\mathbf{r},t)$$

- Generalized momentum is the mechanical momentum $m\dot{\mathbf{r}}$ plus a magnetic term $q\mathbf{A}$ which paves its way in the quantum theory of a charged particle in a magnetic field!

Gauge symmetry

- Gauge transformation: $\Lambda(\{q_i(t)\}, t)$ is known as a **gauge function**

$$\mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t) \to \overline{\mathcal{L}}(\{q_i(t), \dot{q}_i(t)\}, t) = \mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t) + \frac{\mathrm{d}\Lambda(\{q_i(t)\}, t)}{\mathrm{d}t}$$

- Invariance of Euler-Lagrange equation under gauge transformation:

$$\mathcal{L} \equiv \mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t) \ , \qquad \overline{\mathcal{L}} \equiv \overline{\mathcal{L}}(\{q_i(t), \dot{q}_i(t)\}, t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \overline{\mathcal{L}}}{\partial \dot{\overline{q}}_i}\right) - \frac{\partial \overline{\mathcal{L}}}{\partial \overline{q}_i}$$

- Two Lagrangians, which are differed by a total time derivative of an arbitrary function of generalized coordinates and time, give identical equations of motion

EXERCISE 9.5: Show that Galilean transformation is a gauge transformation for the Lagrangian of a system of N particles interacting via central potentials. Identify the gauge function.