Chapter 6 Griffiths

Decays and cross section (scattering)

Experimentally, the spectroscopic investigation (spectral lines) provides information about bound states of the particles (e. g. hydrogen atom H as bound state of e^- and p). Another approach is scattering of the particles, observe decay of these particles.

scattering can reveal the nature of particle interaction

→ Formulate decay process and scattering mathematically.

A typical decay process : $1 \to 2 + 3 + 4 \cdots + N$

Define decay rate = probability a particle decay per unit time $= \Gamma$

The probability a particle will decay in time $\delta t = \Gamma \cdot \delta t$

If there are N particles at time t, then the number of particles will decay in time $\delta t = N \cdot \Gamma \cdot \delta t$

$$\to \quad \delta N = - N \Gamma \delta t$$ a dN loss equal to NΓδt

$$\frac{dN}{dt} = - \Gamma N$$

solving

$\rightarrow$ $$N = N_0 e^{-\Gamma t}$$

N_0 = # of particles at time $t=0$.

Mean life time of a particle $= \tau$

$$= \frac{\text{sum of the lifetimes of all the decayed particles}}{\text{sum of all the decayed particles}}$$

$$= \frac{1}{\Gamma} \quad \text{(as shown below)}$$

Suppose at time t, we have N particles and at time $t + \delta t$, δN particles decay away, that means life time of all δN particles $= t \cdot \delta N$ (each of the δN particles has a lifetime t).

$$\tau = \frac{\int_{N_0}^{0} t\, dN}{\int_{N_0}^{0} dN}$$

(HW) $$= \frac{1}{\Gamma}$$

$$\frac{dN}{dt} = -\Gamma N \Rightarrow N = N_0 e^{-\Gamma t}$$

$$\tau = \frac{\text{Sum of lifetimes of all decayed}}{\text{Sum of all decayed}}$$

1. Consider at time t, N particles exist
2. Consider at time $t + \delta t$, $N - \delta N$ particles exist
3. Hence, δN particles had lifetime t and so

Sum of lifetimes of all decayed $= t\,\delta N$

$$\therefore \tau = \frac{\int_{N_0}^{0} t\, dN}{\int_{N_0}^{0} dN}$$

$$= \frac{\int_0^\infty t(-\Gamma N\, dt)}{\int_0^\infty -\Gamma N\, dt}$$

$$= -\frac{\int_0^\infty t \Gamma N_0 e^{-\Gamma t}\, dt}{\int_0^\infty \cancel{\Gamma N_0}\, e^{-\Gamma t}\, dt}$$

$$= -\frac{1}{\left[\frac{1}{-\Gamma} e^{-\Gamma t}\right]_0^\infty} \left(t e^{-\Gamma t} \frac{1}{-\Gamma} \Big|_0^\infty - \int_0^\infty e^{-\Gamma t} \frac{1}{-\Gamma}\, dt \right)$$

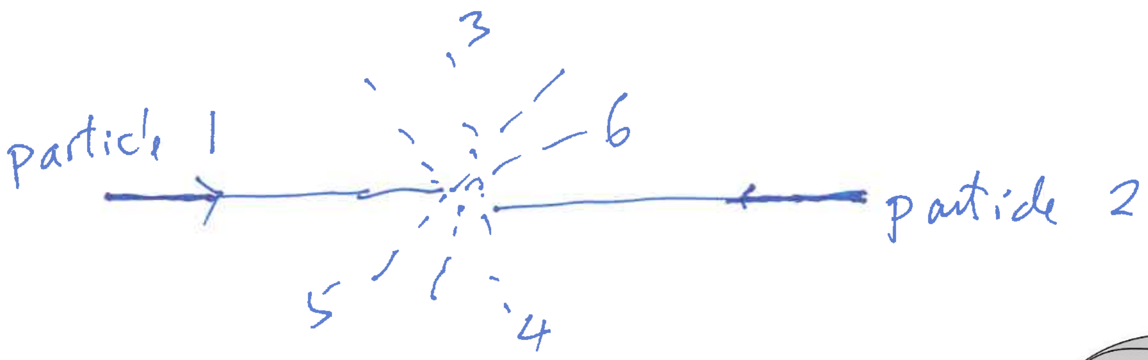
$$= -\frac{1}{\frac{1}{\Gamma}((-0) - (-1))} \left([-0-0] + \left[\frac{1}{\Gamma^2} e^{-\Gamma t}\right]_0^\infty \right)$$

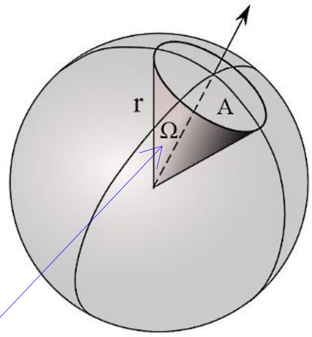
$$= \Gamma \left(\frac{1}{\Gamma^2}(0 - (-1)) \right)$$

$$= \frac{1}{\Gamma} \text{ // shown}$$

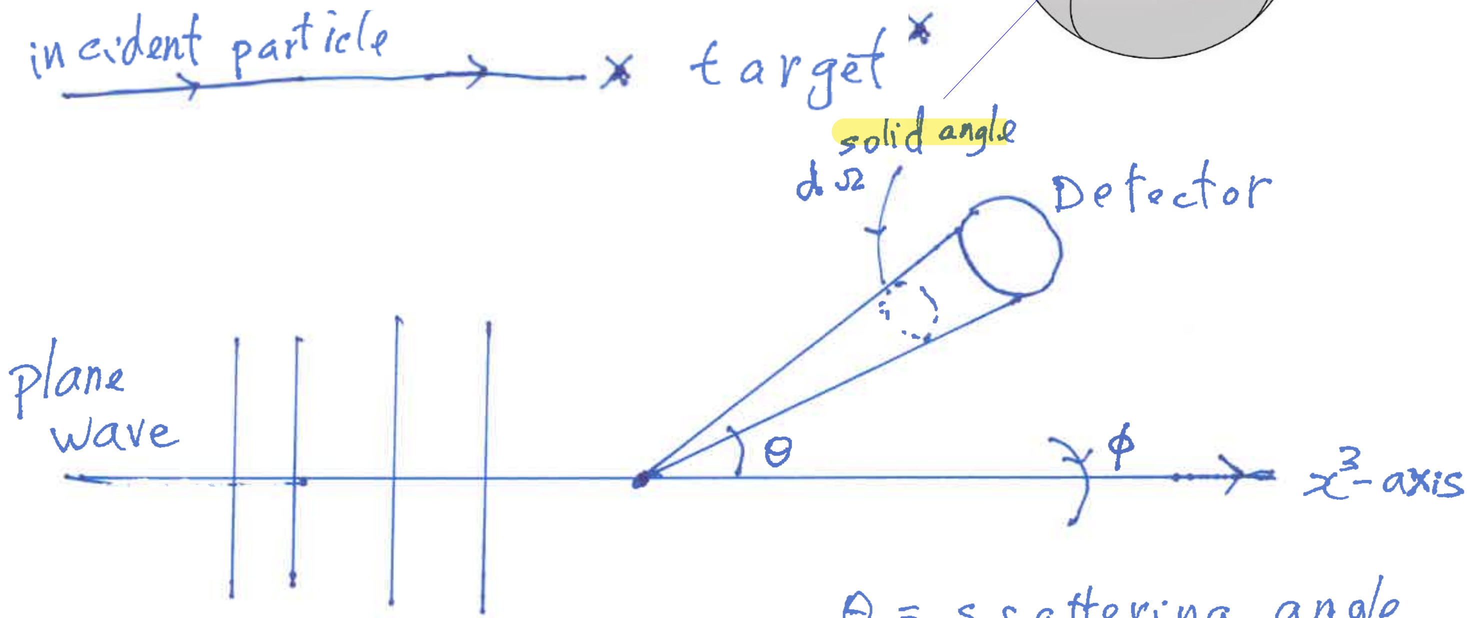
Find the half-life of ---------------->

scattering $\quad 1 + 2 \rightarrow 3 + 4 + 5 + \ldots$





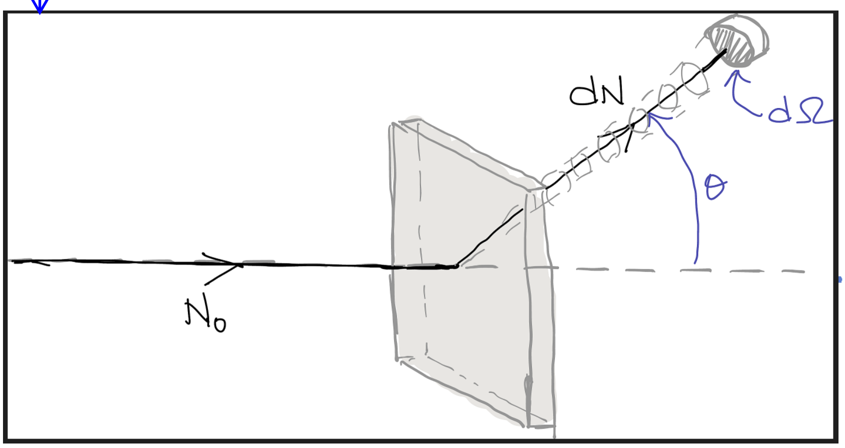
Lab frame



θ = scattering angle

scattering amplitude

$$\text{Differential cross section} = \frac{\text{scattered flux}}{\text{Incident flux}} = \frac{d\sigma}{d\Omega}$$



σ is the total interaction cross-section and can be thought of as the "strength" of the interaction between incident particles and target

Incident flux = # of particles incident per unit area per unit time (number)

= I_0 (probability current density $\vec{j}$)

scattered flux = # of particles scattered into the solid angle direction (θ, ϕ) per solid angle per unit time [the detector is at the angular position (θ, ϕ)]

= $I(\theta, \phi)$

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{I(\theta, \phi)}{I_0},$$

Total cross section $\sigma = \int_{\text{all angles}} \frac{d\sigma}{d\Omega} d\Omega$

Today discuss how to compute Γ and $\frac{d\sigma}{d\Omega}$

Γ = decay rate

To do that we need a formula from quantum mechanics, the Fermi golden rule. We quote:

→ Transition probability per unit time = $\boxed{\frac{2\pi}{\hbar} \cdot |M|^2 \cdot \text{phase space factor}}$

Compute $|M|^2$ from dynamics, M = scattering amplitude

phase space factor from kinematics

$\frac{2\pi}{\hbar} |V_{f_0 i}|^2 \rho(E_{f_0}) t$, $E_{f_0} \simeq E_i$

$$\frac{2\pi}{\hbar} |\underbrace{V_{f_0 i}}_{\langle f_0 | V | i \rangle}|^2 \rho(E_{f_0}) \ , \quad E_{f_0} \simeq E_i$$

final state with $E_{f_0} = E_i$ — initial state

$\mathcal{M}$ = scattering amplitude (matrix element)

can be obtained by solving equation of motion or using Feynman diagrams with Feynman rules.

phase space factor denotes the states available for the finally produced particles to occupy

The larger the phase space factor, the more likely the process will be.

Using this transition probability formula, one can derive, the differential decay rate $d\Gamma$

For decay of a single particle $\underline{P}_1 = (m_1 c, 0)$ (stationary)

$$d\Gamma = \frac{S}{2\hbar m_1} |\mathcal{M}|^2 \cdot (2\pi)^4 \delta^{(4)}(\underline{P}_1 - \underline{P}_2 - \underline{P}_3 \cdots \underline{P}_N) \cdot$$

$\delta^{(4)}$ = product of 4 Dirac delta functions for overall conservation of 4-momentum

$$\prod_{j=2}^{N} \left(\frac{d^4 \underline{P}_j}{(2\pi)^4} \theta(p_j^0)(2\pi)\, \delta(\underline{P}_j^2 - m_j^2 c^2) \right)$$

$\delta(\underline{P}_j^2 - m_j^2 c^2)$: mass shell condition for each particle

S = statistical factor

$= \frac{1}{j!}$ if there are j identical particles produced

same thing

Decay of a single particle (at rest, $\underset{\sim}{p}_1 = (m_1 c, \underset{\sim}{0})$)

dynamics of the transition

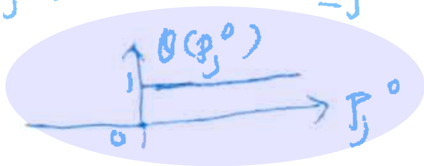
kinematics of the transition

$$d\Gamma = \frac{S}{2\hbar m_1} |\mathcal{M}|^2 \cdot (2\pi)^4 \delta^{(4)}(\underset{\sim}{p}_1 - \underset{\sim}{p}_2 - \underset{\sim}{p}_3 \cdots - \underset{\sim}{p}_n) \cdot$$

$$\prod_{j=2}^{n} \frac{d^4 \underset{\sim}{p}_j}{(2\pi)^4} \theta(p_j^0)(2\pi)\delta(\underset{\sim}{p}_j^2 - m_j^2 c^2)$$

$$d^4 \underset{\sim}{p}_j = dp_j^0\, dp_j^1\, dp_j^2\, dp_j^3 = dp_j^0\, d^3 \underset{\sim}{p}_j$$

step function; =1 if Energy > 0 else = 0



S = statistical factor

$= \frac{1}{j!}$ if there are j identical particles produced

e.g. if there are $3\,\pi^0$, $4\,\pi^-$, $5\,\pi^+$ in the final produced particles, then $S = \frac{1}{3!\,4!\,5!}$

$\theta(x)$ = step function, $\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

$\underset{\sim}{p}_j^2 = m_j^2 c^2$ means particle j is in its mass shell. $(p_j^{0\,2} - \underset{\sim}{p}_j^2 = m_j^2 c^2)$

$$\left(\frac{E}{c}\right)^2 - \mathbf{p}^2 = m^2c^2$$

$$\int dp_j^0\, \theta(p_j^0)\, \delta(\underset{\sim}{p}_j^2 - m_j^2 c^2)$$

$$= \frac{1}{2 p_j^0}$$

using $\delta(x^2 - a^2) = \frac{1}{2|a|}(\delta(x-a) + \delta(x+a))$

(shown later)

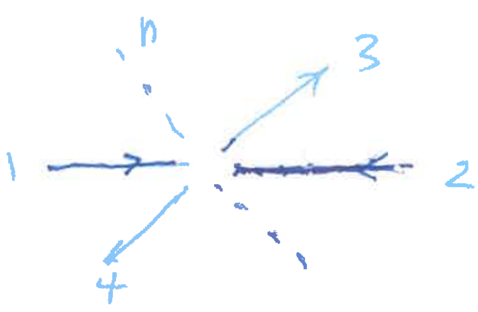
After integrating away $\int dp_j^0$, the differential decay rate is

$$\Gamma = \frac{S}{2\hbar m_1} \int |M|^2 (2\pi)^4 \, \delta^{(4)}(\underline{p}_1 - \underline{p}_2 - \underline{p}_3 \cdots \underline{p}_n) \cdot$$

$$\cdot \prod_{j=2}^{n} \left(\frac{1}{2 p_j^0} \frac{d^3 \underset{\sim}{p}_j}{(2\pi)^3} \right),$$

$$p_j^0 = \sqrt{\underset{\sim}{p}_j^2 + m_j^2 c^2}$$

scattering



$$1 + 2 \longrightarrow 3 + 4 + \cdots + n$$

The formula is

$$d\sigma = \frac{S\,\hbar^2}{4\cdot\sqrt{(\underline{p}_1\cdot\underline{p}_2)^2 - (m_1 m_2 c^2)^2}}\cdot |\mathcal{M}|^2 \cdot$$

$$(2\pi)^4\,\delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4 \cdots - \underline{p}_n)\cdot$$

$$\prod_{j=3}^{n} \underline{\frac{d^4 \underline{p}_j}{(2\pi)^4}\,\theta(p_j^0)\cdot 2\pi\,\delta(\underline{p}_j^2 - m_j^2 c^2)}$$

Integrating $\int dp_j^0$

$\rightarrow$

$$d\sigma = \frac{S\,\hbar^2}{4\sqrt{(\underline{p}_1\cdot\underline{p}_2)^2 - (m_1 m_2 c^2)^2}}\cdot |\mathcal{M}|^2 \cdot$$

$$(2\pi)^4\,\delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4 \cdots - \underline{p}_n)\cdot$$

$$\prod_{j=3}^{n} \frac{d^3 p_j}{(2\pi)^3\, 2p_j^0}$$

Basically, learn how to reduce 4-dimensional integral to 3-dimensional, then to 1-dimensional integral

To show

$$\delta(x^2 - a^2) = \frac{1}{2|a|}\left(\delta(x-a) + \delta(x+a)\right)$$

Proof

By definition, for a smooth function $f(x)$,

$$\int_{-b}^{b} f(x)\,\delta(x-a)\,dx = \begin{cases} f(a) & \text{if } a \in [-b, b] \\ 0 & \text{if } a \notin [-b, b] \end{cases}$$

LHS

$$= \int_{-\infty}^{\infty} f(x)\,\delta(x^2 - a^2)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,\delta(x^2 - a^2)\,dx + \int_{0}^{\infty} f(x)\,\delta(x^2 - a^2)\,dx$$

$$= \int_{0}^{\infty} f(-x)\,\delta(x^2 - a^2)\,dx + \int_{0}^{\infty} f(x)\,\delta(x^2 - a^2)\,dx$$

$$= \int_{0}^{\infty} f(-\sqrt{y})\,\delta(y - a^2)\,\frac{dy}{2\sqrt{y}} + \int_{0}^{\infty} f(\sqrt{y})\,\delta(y - a^2)\,\frac{dy}{2\sqrt{y}}$$

$$= f(-a)\frac{1}{2a} + f(a)\frac{1}{2a} \qquad \text{assuming } a \geq 0$$

$$= \frac{1}{2|a|}\left(f(-a) + f(a)\right)$$

$$\text{RHS} = \int_{-\infty}^{\infty} f(x) \cdot \frac{1}{2|a|}\left(\delta(x-a) + \delta(x+a)\right) dx$$

$$RHS = \int_{-\infty}^{0} f(x) \frac{1}{2|a|} \left(\delta(x-a) + \delta(x+a) \right) dx$$

$$+ \int_{0}^{\infty} f(x) \frac{1}{2|a|} \left(\delta(x-a) + \delta(x+a) \right) dx$$

$$= \frac{1}{2|a|} \int_{-\infty}^{0} f(x)\, \delta(x+a)\, dx + \frac{1}{2|a|} \int_{0}^{\infty} f(x)\, \delta(x-a)\, dx$$

assume $a \geq 0$

$$= \frac{1}{2a} f(-a) + \frac{1}{2a} f(a)$$

$$= \frac{1}{2|a|} \left(f(-a) + f(a) \right)$$

q.e.d.

$$\int_{-\infty}^{\infty} dp^0\, \theta(p^0)\, \delta(\underline{p}^2 - m^2c^2)\, f(p^0)$$

$\underline{p}^2 = p^{0\,2} - \underline{p}^2$

$$= \int_{-\infty}^{\infty} dp^0\, \theta(p^0) \frac{1}{2|a|} \left[\delta(p^0 - a) + \delta(p^0 + a) \right] f(p^0)$$

$p^0 = a = \left(\underline{p}^2 + m^2c^2 \right)^{\frac{1}{2}}$

$$= \frac{1}{2|a|} \int_{-\infty}^{\infty} dp^0\, \delta(p^0 - a)\, f(p^0)$$

$$= \frac{1}{2|a|} f(a), \qquad p^0 = a = \sqrt{\underline{p}^2 + m^2c^2}$$

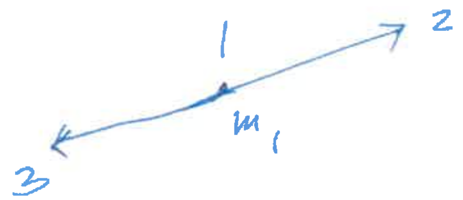
$a \geq 0$

$$\therefore \int_{-\infty}^{\infty} dp^0\, \theta(p^0)\, \delta(\underline{p}^2 - m^2c^2) = \frac{1}{2p^0}, \qquad p^0 = a$$

Consider 2-particle decays

$$1 \to 2 + 3$$

Assume particle 1 at rest and decays



The decay rate is given by (page 6a)

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \, \delta^{(4)}(\underline{p}_1 - \underline{p}_2 - \underline{p}_3) \cdot$$

$$\cdot \prod_{j=2}^{3} \left(\frac{1}{2 p_j^0} \frac{d^3 \underset{\sim}{p}_j}{(2\pi)^3} \right)$$

scattering amplitude $\mathcal{M}$
$= \mathcal{M}(\underline{p}_1, \underline{p}_2, \underline{p}_3 \ldots)$

$$= \frac{S}{8\pi^2 \hbar m_1} \int |\mathcal{M}|^2 \, \delta^{(4)}(\underline{p}_1 - \underline{p}_2 - \underline{p}_3) \frac{d^3 \underset{\sim}{p}_2}{2 p_2^0} \cdot \frac{d^3 \underset{\sim}{p}_3}{2 p_3^0}$$

$$= \frac{S}{8\pi^2 \hbar m_1} \int |\mathcal{M}|^2 \, \delta(p_1^0 - p_2^0 - p_3^0) \, \delta^{(3)}(\underset{\sim}{p}_1 - \underset{\sim}{p}_2 - \underset{\sim}{p}_3) \cdot$$

$$\cdot \frac{d^3 \underset{\sim}{p}_2}{2 p_2^0} \frac{d^3 \underset{\sim}{p}_3}{2 p_3^0}$$

Taking the lab frame (frame of ref where p1 at rest),

As the decaying particle 1 is at rest,

$\underset{\sim}{P}_1 = 0.$

$\therefore$

$$\Gamma = \frac{S}{8\pi^2 \hbar m_1} \int |M|^2 \, \delta(p_1^0 - p_2^0 - p_3^0) \, \delta^{(3)}(-\underset{\sim}{p}_2 - \underset{\sim}{p}_3) \cdot \frac{d^3 \underset{\sim}{p}_2}{2 p_2^0} \, \frac{d^3 \underset{\sim}{p}_3}{2 p_3^0}$$

$\delta^{(3)}(\underset{\sim}{p}_2 + \underset{\sim}{p}_3)$ $\because \delta(-x) = \delta(x)$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

$$= \frac{S}{8\pi^2 \hbar m_1} \int |M|^2 \, \delta(p_1^0 - p_2^0 - p_3^0) \, \frac{d^3 \underset{\sim}{p}_2}{2 p_2^0 \cdot 2 p_3^0}$$

where $\underset{\sim}{p}_3 = -\underset{\sim}{p}_2$

$p_3^0 = \sqrt{\underset{\sim}{p}_3^2 + m_3^2 c^2}$

The volume differential $d^3 \underset{\sim}{p}_2$ can be written as

sinθdθdΦ in spherical coords

$$d^3 \underset{\sim}{p}_2 = |\underset{\sim}{p}_2|^2 \cdot d|\underset{\sim}{p}_2| \cdot \underline{d\Omega_{\underset{\sim}{p}_2}} \qquad \left(d^3 \underset{\sim}{x} = r^2 \, dr \, d\Omega \right)$$

Integrating $d\Omega_{\underset{\sim}{p}_2}$ and assuming $|M|^2$ does not depend on $\Omega_{\underset{\sim}{p}_2}$, we have

$$\Gamma = \frac{S}{8\pi\hbar m_1} \int |M|^2 \, \delta(p_1^0 - p_2^0 - p_3^0) \cdot \frac{|\underset{\sim}{p}_2|^2 \cdot d|\underset{\sim}{p}_2|}{p_2^0 \cdot p_3^0}$$

where $\underset{\sim}{p}_3 = -\underset{\sim}{p}_2$

Changing the integration variable by defining

$$p^0 = p_2^0 + p_3^0$$

Under diff situations, this change of variable needs to be adjusted to perform the integral as desired

$$\therefore \; dp^0 = dp_2^0 + dp_3^0$$

$$p_i^{0\,2} = \underset{\sim}{p}_i^2 + m_i^2 c^2$$

$i = 2, 3$

Next using chain rule,

$$\frac{d}{dp_i^0}(p_i^0) = \frac{d}{dp_i^0}\sqrt{|\underset{\sim}{p}_2|^2 + m_i c}$$

$$= \frac{1}{2\sqrt{|\underset{\sim}{p}_2|^2 + m_i c}} \, 2|\underset{\sim}{p}_i| d\underset{\sim}{p}_i$$

$$= \frac{|\underset{\sim}{p}_i| d\underset{\sim}{p}_i}{p_i^0}, \qquad \text{for } i = 2, 3$$

Hence, $dp_2^0 = \dfrac{|\underset{\sim}{p}_2| \cdot d|\underset{\sim}{p}_2|}{p_2^0}$ and $dp_3^0 = \dfrac{|\underset{\sim}{p}_3| \cdot d|\underset{\sim}{p}_3|}{p_3^0}$

$$= \frac{|\underset{\sim}{p}_2| \cdot d|\underset{\sim}{p}_2|}{p_2^0} + \frac{|\underset{\sim}{p}_3| \cdot d|\underset{\sim}{p}_3|}{p_3^0}$$

$$= \frac{p_2^0 + p_3^0}{p_2^0 \cdot p_3^0} \cdot |\underset{\sim}{p}_2| \cdot d|\underset{\sim}{p}_2|$$

$\because \underset{\sim}{p}_3 = -\underset{\sim}{p}_2$

$\therefore |\underset{\sim}{p}_3| = |\underset{\sim}{p}_2|$

$$\therefore \; \frac{dp^0}{p^0} = \frac{|\underset{\sim}{p}_2| \cdot d|\underset{\sim}{p}_2|}{p_2^0 \cdot p_3^0}$$

Note: we have changed integration variable $|\underset{\sim}{p}|$ to p^0

Thus

$$P = \frac{S}{8\pi\hbar \cdot m_1} \int |\mathcal{M}|^2 \, \delta(p_1^0 - p^0) \cdot \frac{|\underset{\sim}{p}_2| \cdot dp^0}{p^0}$$

$$= \frac{S}{8\pi\hbar \cdot m_1} |\mathcal{M}|^2 \frac{|\underset{\sim}{p}_2|}{p_1^0} \quad \text{where } \underset{\sim}{p}_3 = -\underset{\sim}{p}_2$$

$$p^0 \equiv p_2^0 + p_3^0 = p_1^0$$

As particle 1 is at rest, $p_1^0 = m_1 c$

$\therefore$

$$\Gamma = \frac{S}{8\pi\hbar \cdot m_1^2 c} |\mathcal{M}|^2 \cdot |\underset{\sim}{p}_2|$$

$$\text{where } \underset{\sim}{p}_3 = -\underset{\sim}{p}_2$$

$$p_2^0 + p_3^0 = p_1^0 = m_1 c$$

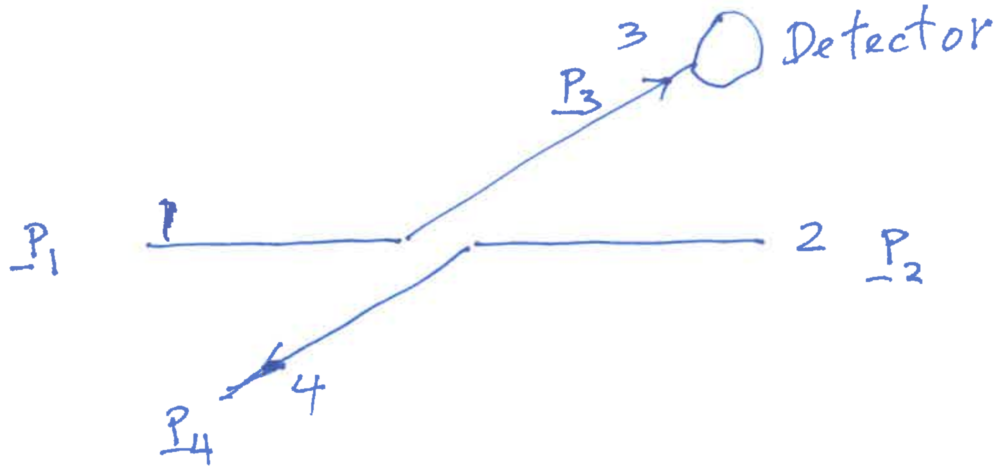
To find $|\underset{\sim}{p}_2|$:

As $p_1^0 = p_2^0 + p_3^0$, $\therefore$

$$m_1 c = \sqrt{\underset{\sim}{p}_2^2 + m_2^2 c^2} + \sqrt{\underset{\sim}{p}_3^2 + m_3^2 c^2} \quad \because \underset{\sim}{p}_3 = -\underset{\sim}{p}_2$$

$$\therefore \underset{\sim}{p}_2^2 = \frac{c^2}{4m_1^2} \cdot \left[(m_1^4 + m_2^4 + m_3^4) - 2(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2) \right] \quad \text{(HW)}$$

Consider 2 particles to 2 particles scattering



Using the Fermi golden rule, the differential cross section can be written as (page 7)

$\mathcal{M}$ = scattering amplitude

$$d\sigma = \frac{S\,\hbar^2}{4\cdot\sqrt{(\underline{P}_1\cdot\underline{P}_2)^2-(m_1 m_2 c^2)^2}}\cdot|\mathcal{M}|^2\,.$$

$$\cdot(2\pi)^4\,\delta^{(4)}(\underline{P}_1+\underline{P}_2-\underline{P}_3-\underline{P}_4)\,.$$

$$\prod_{j=3}^{4}\underbrace{\frac{d^4 P_j}{(2\pi)^4}(2\pi)\,\delta(\underline{P}_j^2-m_j^2c^2)\cdot\Theta(P_j^0)}$$

Integrating away the energy P_j^0

We get

$$d\sigma = \frac{S\hbar^2}{4\sqrt{(\underset{\sim}{p}_1\cdot\underset{\sim}{p}_2)^2-(m_1 m_2 c^2)^2}}\cdot|\mathcal{M}|^2\cdot$$

$$(2\pi)^4\,\delta^{(4)}(\underset{\sim}{p}_1+\underset{\sim}{p}_2-\underset{\sim}{p}_3-\underset{\sim}{p}_4)\cdot\prod_{j=3}^{4}\underline{\frac{d^3\underset{\sim}{p}_j}{(2\pi)^3}\;\frac{1}{2p_j^0}}$$

$$= \frac{s\hbar^2}{4\cdot\sqrt{(\underset{\sim}{p}_1\cdot\underset{\sim}{p}_2)^2-(m_1 m_2 c^2)^2}}\,|\mathcal{M}|^2\cdot(2\pi)^4\,\delta^{(4)}(\underset{\sim}{p}_1+\underset{\sim}{p}_2-\underset{\sim}{p}_3-\underset{\sim}{p}_4)$$

$$\frac{d^3\underset{\sim}{p}_3}{(2\pi)^3}\;\frac{d^3\underset{\sim}{p}_4}{(2\pi)^3}\cdot\frac{1}{2p_3^0}\cdot\frac{1}{2p_4^0}$$

Integrating away $\int d^3\underset{\sim}{p}_4$ by using the Dirac delta function $\delta^{(3)}(\underset{\sim}{p}_1+\underset{\sim}{p}_2-\underset{\sim}{p}_3-\underset{\sim}{p}_4)$,

$$d\sigma = \frac{s\hbar^2}{4\cdot\sqrt{(\underset{\sim}{p}_1\cdot\underset{\sim}{p}_2)^2-(m_1 m_2 c^2)^2}}\cdot|\mathcal{M}|^2\cdot$$

$$\delta(p_1^0+p_2^0-p_3^0-p_4^0)\cdot\frac{d^3\underset{\sim}{p}_3}{(2\pi)^2}\cdot\frac{1}{4\,p_3^0\cdot p_4^0}$$

where $\underset{\sim}{p}_4 = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 - \underset{\sim}{p}_3$

We assume the detector is detecting particle 3

so

$$d^3\underset{\sim}{p}_3 = |\underset{\sim}{p}_3|^2 \cdot d|\underset{\sim}{p}_3| \cdot d\Omega_{\underset{\sim}{p}_3}$$

dx = r^2 dr sinθdΦdθ
= r^2 dr dΩ

and compute $\dfrac{d\sigma}{d\Omega_{\underset{\sim}{p}_3}}$

Note how we don't integrate out dΩ this time because we are interested in the differential cross-section so we bring it over to LHS. If we integrate it out we will get the total cross-section instead.

We write

$$\frac{d\sigma}{d\Omega_{\underset{\sim}{p}_3}} = \frac{s\hbar^2}{4\sqrt{(\underset{\sim}{p}_1\cdot\underset{\sim}{p}_2)^2 - (m_1 m_2 c^2)^2}} \cdot$$

$$\cdot \int \frac{|\underset{\sim}{p}_3|^2 \cdot d|\underset{\sim}{p}_3|}{(4\pi)^2\, p_3^0\, p_4^0} \cdot |M|^2 \cdot \delta(p_1^0 + p_2^0 - p_3^0 - p_4^0)$$

In CM frame,

$\underset{\sim}{p}_1 + \underset{\sim}{p}_2 = 0 \therefore \underset{\sim}{p}_4 = -\underset{\sim}{p}_3$

$$\underset{\sim}{p}_4 = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 - \underset{\sim}{p}_3$$
$$= -\underset{\sim}{p}_3 \quad (\text{CM frame})$$

Changing the integrating variable $|\underset{\sim}{p}_3|$ by defining

$$p^0 = p_3^0 + p_4^0,$$

$$dp^0 = dp_3^0 + dp_4^0$$

$$= \frac{|\mathbf{p}_3| \cdot d|\mathbf{p}_3|}{p_3^0} + \frac{|\mathbf{p}_3| \cdot d|\mathbf{p}_3|}{p_4^0}$$

$$\because \quad p_3^0 = \sqrt{\mathbf{p}_3^2 + m_3^2 c^2}$$

$$\mathbf{p}_4 = -\mathbf{p}_3$$

$$= \frac{p^0}{p_3^0 \cdot p_4^0} \cdot |\mathbf{p}_3| \cdot d|\mathbf{p}_3|$$

$$\therefore \quad \frac{dp^0}{p^0} = \frac{|\mathbf{p}_3| \cdot d|\mathbf{p}_3|}{p_3^0 \cdot p_4^0}$$

We get

$$\frac{d\sigma}{d\Omega} = \frac{s \hbar^2}{4 \cdot \sqrt{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - (m_1 m_2 c^2)^2}} \cdot \frac{1}{(4\pi)^2} \cdot$$

$$\int |\mathbf{p}_3| \cdot \frac{dp^0}{p^0} \cdot |\mathcal{M}|^2 \cdot \delta(p_1^0 + p_2^0 - p^0)$$

Compare from decays: $\Gamma = \frac{S}{8\pi\hbar \cdot m_1} \int |\mathcal{M}|^2 \, \delta(p_1^0 - p^0) \cdot \frac{|\mathbf{p}_2| \cdot dp^0}{p^0}$

Integrating $\int dp^0$,

This expression can be condensed if one of the initial particles have zero momentum (see Tut 3)

$$\frac{d\sigma}{d\Omega} = \frac{s\hbar^2}{(8\pi)^2 \sqrt{(\underline{P}_1 \cdot \underline{P}_2)^2 - (m_1 m_2 c^2)^2}} \cdot \frac{|\mathcal{M}|^2 \cdot |\underset{\sim}{p}_3|}{(P_1^0 + P_2^0)}$$

$$\underset{\sim}{p}_3 = -\underset{\sim}{p}_4 \quad (\text{CM frame})$$

$$p^0 = P_3^0 + P_4^0 = P_1^0 + P_2^0$$

Only unknown is $|\underset{\sim}{p}_3|$

We can find $|\underset{\sim}{p}_3|$ by using

$$P_1^0 + P_2^0 = P_3^0 + P_4^0$$

$$= \sqrt{\underset{\sim}{p}_3^2 + m_3^2 c^2} + \sqrt{\underset{\sim}{p}_3^2 + m_4^2 c^2}$$

As $(P_1^0 + P_2^0)$ is fixed and known, so can get $|\underset{\sim}{p}_3|$ from the above relation

$$\underset{\sim}{p}_3^2 = \frac{\left(K^2 + (m_4^2 - m_3^2) c^2\right)^2}{4 K^2} - m_4^2 c^2$$

$$K \equiv P_1^0 + P_2^0$$