Composition of Lorentz boosts

- Frame 1 moves at constant velocity $\mathbf{V}_{12} = V_{12}\,\hat{\mathbf{x}}$ with respect to frame 2 and frame 2 moves at constant velocity $\mathbf{V}_{23} = V_{23}\,\hat{\mathbf{x}}$ with respect to frame 3

- Lorentz boost between frames 1 and 3:

$$ct_1 = \gamma(\beta_{13})\,(ct_3 - \beta_{13} x_3)\ , \quad x_1 = \gamma(\beta_{13})\,(x_3 - \beta_{13} ct_3)\ , \quad y_1 = y_3\,, \quad z_1 = z_3$$

- Composition rules:

$$\beta_{13} = \frac{\beta_{12} + \beta_{23}}{1 + \beta_{12}\beta_{23}}\,, \qquad \gamma(\beta_{13}) = \gamma(\beta_{12})\gamma(\beta_{23})\,(1 + \beta_{12}\beta_{23})$$

EXERCISE 12.2: Derive the composition rules for β and γ factors.

$$\begin{cases} \begin{pmatrix} ct_1 \\ x_1 \end{pmatrix} = \gamma(\beta_{12}) \begin{pmatrix} 1 & -\beta_{12} \\ -\beta_{12} & 1 \end{pmatrix} \begin{pmatrix} ct_2 \\ x_2 \end{pmatrix} \\ \\ \begin{pmatrix} ct_2 \\ x_2 \end{pmatrix} = \gamma(\beta_{23}) \begin{pmatrix} 1 & -\beta_{23} \\ -\beta_{23} & 1 \end{pmatrix} \begin{pmatrix} ct_3 \\ x_3 \end{pmatrix} \end{cases}$$

$$\Rightarrow \quad \begin{pmatrix} ct_1 \\ x_1 \end{pmatrix} = \gamma(\beta_{12})\gamma(\beta_{23}) \begin{pmatrix} 1 & -\beta_{12} \\ -\beta_{12} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta_{23} \\ -\beta_{23} & 1 \end{pmatrix} \begin{pmatrix} ct_3 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \gamma(\beta_{12})\gamma(\beta_{23}) \left[(1 + \beta_{12}\beta_{23})\, x_3 - (\beta_{12} + \beta_{23})\, ct_3 \right] \\ &= \gamma(\beta_{12})\gamma(\beta_{23})\, (1 + \beta_{12}\beta_{23}) \left(x_3 - \frac{\beta_{12} + \beta_{23}}{1 + \beta_{12}\beta_{23}} ct_3 \right) \equiv \gamma(\beta_{13})\, (x_3 - \beta_{13} ct_3) \qquad \blacksquare \end{aligned}$$

$$\begin{aligned} 1 - \beta_{13}^2 &= 1 - \left(\frac{\beta_{12} + \beta_{23}}{1 + \beta_{12}\beta_{23}} \right)^2 = \frac{\left(1 - \beta_{12}^2\right)\left(1 - \beta_{23}^2\right)}{\left(1 + \beta_{12}\beta_{23}\right)^2} \\ &= \left[\frac{1}{\gamma(\beta_{12})\gamma(\beta_{23})\, (1 + \beta_{12}\beta_{23})} \right]^2 = \frac{1}{\gamma^2(\beta_{13})} \qquad \blacksquare \end{aligned}$$

General Lorentz boost

- Axes in $\mathcal{O}$ and $\mathcal{O}'$ remain parallel but the velocity $\mathbf{V}$ of $\mathcal{O}'$ with respect to $\mathcal{O}$ is in an arbitrary direction:

$$ct' = \gamma\left(ct - \boldsymbol{\beta}\cdot\mathbf{r}\right)\ , \qquad \mathbf{r}' = \mathbf{r} + \frac{\gamma - 1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{r}\right)\boldsymbol{\beta} - \gamma\boldsymbol{\beta}ct$$

- Successive boosts along the same direction of relative velocity commute and their composite is another boost

- Successive boosts along different directions of relative velocity do not commute and each of these two different composites is not another boost!

EXERCISE 12.3: Derive the Lorentz boost between two inertial frames with parallel axes and arbitrary constant relative velocity.

$$\mathbf{r} = \mathbf{r}_\parallel + \mathbf{r}_\perp\,, \qquad \mathbf{r}_\parallel = \left(\hat{\boldsymbol{\beta}}\cdot\mathbf{r}\right)\hat{\boldsymbol{\beta}} = \frac{(\boldsymbol{\beta}\cdot\mathbf{r})\,\boldsymbol{\beta}}{\beta^2}$$

$$ct' = \gamma\,(ct - \boldsymbol{\beta}\cdot\mathbf{r}) \qquad \blacksquare$$

$$\mathbf{r}'_\parallel = \gamma\left(\mathbf{r}_\parallel - \boldsymbol{\beta}ct\right)\,, \qquad \mathbf{r}'_\perp = \mathbf{r}_\perp \qquad \blacksquare$$

$$\begin{aligned}
\mathbf{r}' &= \mathbf{r}'_\parallel + \mathbf{r}'_\perp \\
&= \gamma\left(\mathbf{r}_\parallel - \boldsymbol{\beta}ct\right) + \left(\mathbf{r} - \mathbf{r}_\parallel\right) \\
&= \mathbf{r} + (\gamma - 1)\,\mathbf{r}_\parallel - \gamma\boldsymbol{\beta}ct \\
&= \mathbf{r} + \frac{\gamma - 1}{\beta^2}\,(\boldsymbol{\beta}\cdot\mathbf{r})\,\boldsymbol{\beta} - \gamma\boldsymbol{\beta}ct \qquad \blacksquare
\end{aligned}$$

$$\boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{r}) = \boldsymbol{\beta}\,(\boldsymbol{\beta} \cdot \mathbf{r}) - \mathbf{r}\,(\boldsymbol{\beta} \cdot \boldsymbol{\beta})$$

$$\mathbf{r}' = \mathbf{r} + \frac{\gamma - 1}{\beta^2}\,(\boldsymbol{\beta} \cdot \mathbf{r})\,\boldsymbol{\beta} - \gamma\boldsymbol{\beta}ct = \gamma\,(\mathbf{r} - \boldsymbol{\beta}ct) + \frac{\gamma - 1}{\beta^2}\,\boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{r}) \qquad \blacksquare$$

Spacetime interval

- **Spacetime interval**: separation between two events $(t_1, \mathbf{r}_1)$ and $(t_2, \mathbf{r}_2)$

$$\Delta s^2 \equiv (\Delta s)^2 = -c^2 (t_2 - t_1)^2 + |\mathbf{r}_2 - \mathbf{r}_1|^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

- Spacetime interval is a Lorentz invariant quantity – *frame independent* measure of the separation between two events in the spacetime

$$-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

- Classification of spacetime intervals:

$$\begin{array}{llcl} \textit{spacelike} & \Delta s^2 > 0 & \Rightarrow & \Delta x^2 + \Delta y^2 + \Delta z^2 > c^2 \Delta t^2 \\ \textit{lightlike} & \Delta s^2 = 0 & \Rightarrow & \Delta x^2 + \Delta y^2 + \Delta z^2 = c^2 \Delta t^2 \\ \textit{timelike} & \Delta s^2 < 0 & \Rightarrow & \Delta x^2 + \Delta y^2 + \Delta z^2 < c^2 \Delta t^2 \end{array}$$

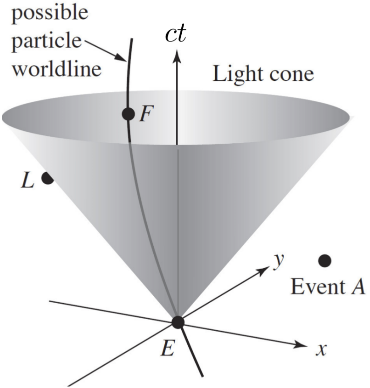
Spacetime diagram

- A **spacetime diagram** (or **Minkowski diagram**) is a convenient way to display the relationship between events in spacetime

- An event is represented by a point on the spacetime diagrams

- A particle's trajectory through spacetime, called the particle's **worldline**, is represented in a spacetime diagram by a connected sequence of events

- A **light cone** is the worldline that light, emitting from a single event and travelling in all directions, would take through spacetime

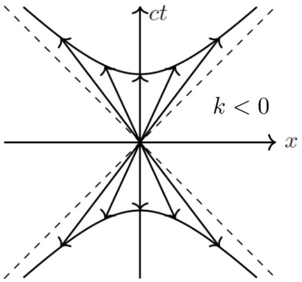


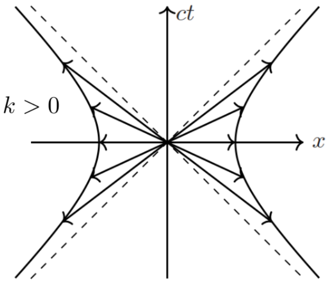
Invariant hyperbola

- All events, (t, x), that have the same spacetime interval, k, from the origin, $(0, 0)$ lie on a hyperbola in the spacetime diagram

$$\Delta s^2 = k \quad \Rightarrow \quad -c^2 t^2 + x^2 = k$$

- One must not bring Euclidean geometric expectations to the Minkowski spacetime diagram!





Passive view of Lorentz boost

- ct' axis is the locus of events for which $x' = 0$: a straight line with slope $1/\beta$

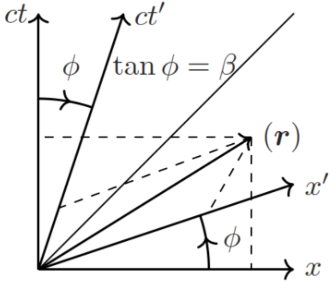
$$x' = \gamma\,(x - \beta ct) = 0 \quad \Rightarrow \quad \frac{ct}{x} = \frac{1}{\beta}$$

- x' axis is the locus of events for which $ct' = 0$: a straight line with slope β

$$ct' = \gamma\,(ct - \beta x) = 0 \quad \Rightarrow \quad \frac{ct}{x} = \beta$$

- ct' and x' axes are reflected images of each other across the light cones at the origin

- Spacetime coordinates of an event are the projections of the event along respective time and space axes: along ct and x for $\mathcal{O}$, and ct' and x' for $\mathcal{O}'$



Temporal sequence of events

- Two events which are simultaneous according to $\mathcal{O}$ might not be simultaneous according to $\mathcal{O}'$

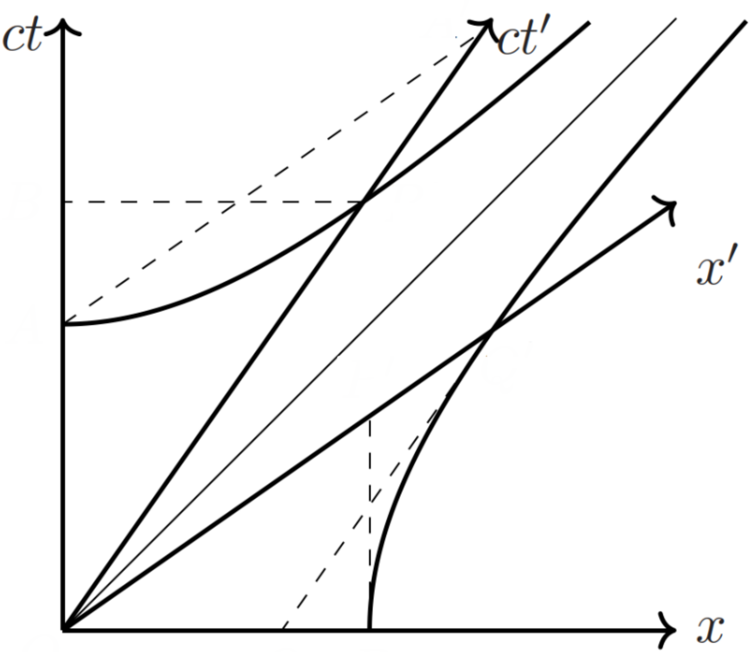
$$\Delta t = 0 \quad \Rightarrow \quad c\,\Delta t' = \gamma\,(c\,\Delta t - \beta\,\Delta x) \neq 0$$

- **Causality** is the relationship between *causes* and *effects*; **causality principle**: cause must precede its effect

- Timelike separated events: if $\Delta t > 0$, then $\Delta t' > 0$ in all *physically possible* inertial reference frames

$$\Delta s^2 < 0 \quad \Rightarrow \quad -c^2\,\Delta t^2 + \Delta x^2 < 0 \quad \Rightarrow \quad -c\,\Delta t < \Delta x < c\,\Delta t$$

$$\Delta t' \leq 0 \quad \Rightarrow \quad \gamma\,(c\,\Delta t - \beta\,\Delta x) \leq 0 \quad \Rightarrow \quad \begin{cases} \beta \leq \dfrac{c\,\Delta t}{\Delta x} \\[2ex] \beta \geq \dfrac{c\,\Delta t}{\Delta x} \end{cases} \quad \Rightarrow \quad \begin{cases} \beta < -1 \\ \beta > 1 \end{cases}$$


ct
ct′
x′
x

'Arc length' in spacetime

- Spacetime interval between two infinitesimal separated events:

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

- **Proper time** between two events, $(t_1, \mathbf{r}_1)$ and $(t_2, \mathbf{r}_2)$, is measured by a clock travelling along a given timelike worldline $\mathcal{C}_{12}$ connecting those events

$$\Delta\tau = \int_{\mathcal{C}_{12}} \sqrt{1 - \frac{V^2}{c^2}}\,\mathrm{d}t\,, \qquad V^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2$$

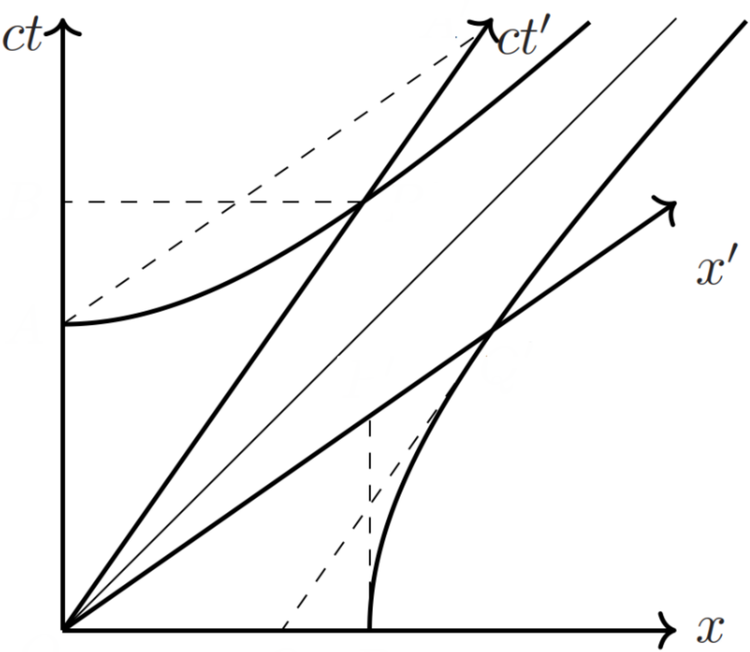
- All observers agree on the value of the proper time between the two events along the given timelike worldline

EXERCISE 12.4: Derive the expression for the proper time between two events along a given timelike worldline.

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

$$\mathrm{d}s'^2 = -c^2\,\mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2 = -c^2\,\mathrm{d}t'^2$$

$$\begin{aligned}
\Delta\tau &= \int_{\mathcal{C}_{12}} \mathrm{d}t' = \frac{1}{c}\int_{\mathcal{C}_{12}} \sqrt{-\mathrm{d}s'^2} = \frac{1}{c}\int_{\mathcal{C}_{12}} \sqrt{-\mathrm{d}s^2} \\
&= \int_{\mathcal{C}_{12}} \sqrt{\mathrm{d}t^2 - \frac{1}{c^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)} \\
&= \int_{\mathcal{C}_{12}} \sqrt{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]}\,\mathrm{d}t \\
&= \int_{\mathcal{C}_{12}} \sqrt{1 - \frac{V^2}{c^2}}\,\mathrm{d}t \qquad \blacksquare
\end{aligned}$$


ct
ct'
x'
x

Length contraction

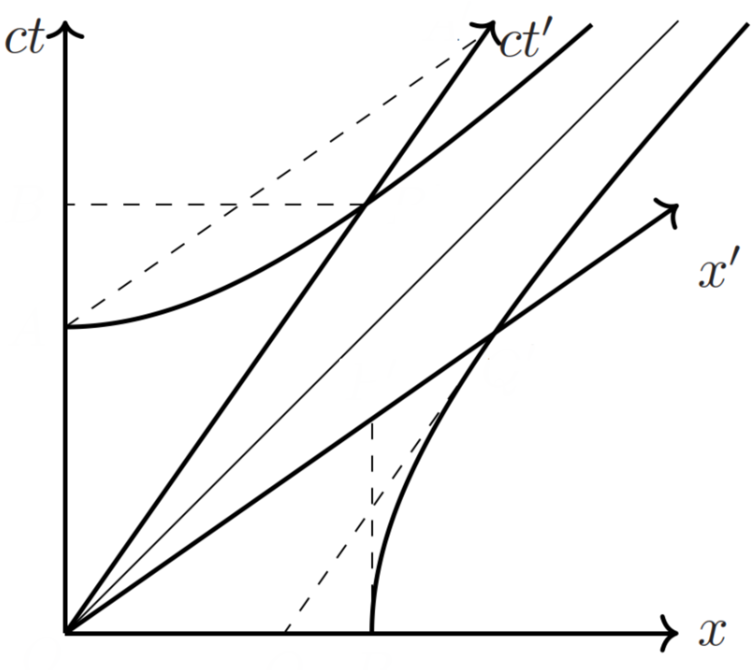
- Length of an object in any inertial frame is defined to be the spatial distance between two events located at the object's end points that are *simultaneous* in the inertial frame

- Observers in different inertial frames will disagree about the object's length as they disagree about which pairs of events are simultaneous

- Relationship between an object's length, **proper length** L_0, along a *given* direction in its own inertial frame and its length, **contracted length** L, in an inertial frame where it is observed to move with speed $v = \beta c$ in *that* direction:

$$L = L_0\sqrt{1-\beta^2}$$

EXERCISE 12.5: Derive the relationship between proper length and contracted length by using an *appropriate* Lorentz boost for coordinate differences.


ct
ct′
x′
x

left end : $A(t' = 0, x' = 0, y' = 0, z' = 0)\,,$ right end : $B(t' = 0, x' = L_0, y' = 0, z' = 0)$

left end : $A(t = 0, x = 0, y = 0, z = 0)\,,$ right end : $C(t = 0, x = L, y = 0, z = 0)$

$$x'_C - x'_A = \gamma\,(x_C - x_A) - \gamma\beta\,(t_C - t_A)$$

$$\Rightarrow \quad L_0 = \gamma L$$

$$\Rightarrow \quad L = L_0\sqrt{1 - \beta^2} \qquad \blacksquare$$