Lagrange multipliers and constraints – cont'd

- Redefine Lagrangian to include holonomic constraints:

$$\mathcal{L}'(\{q_i(t), \dot{q}_i(t), \lambda_j(t)\}, t) \equiv \mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t) - \sum_{j=1}^{C} \lambda_j(t)\, \psi_j(\{q_i(t)\}, t)$$

- Euler-Lagrange equations of motion:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}'}{\partial \dot{q}_k}\right) - \dfrac{\partial \mathcal{L}'}{\partial q_k} = 0\,, & k = 1, 2, \cdots, M \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}'}{\partial \dot{\lambda}_j}\right) - \dfrac{\partial \mathcal{L}'}{\partial \lambda_j} = 0\,, & j = 1, 2, \cdots, C \end{cases}$$

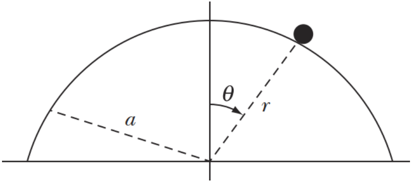
- The definition of Lagrangian for a system is not unique but the bottom line is that it must give the correct equations of motion of the system!

Example: Particle on a hemisphere (revisited)

- A particle of mass m starts at rest on top of a smooth fixed hemisphere of radius a

- Lagrangian:

$$\mathcal{L}(r,\theta,\dot{r},\dot{\theta}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$



- Holonomic constraint:

$$\psi(r,\theta) = r(t) - a = 0$$

EXERCISE 9.3: Determine the angle at which the particle leaves the hemisphere from the Euler-Lagrange equation.

$$\mathcal{L}(r,\theta,\dot r,\dot\theta) = \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2\right) - mgr\cos\theta\,, \qquad \psi(r,\theta) = r - a = 0$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot r}\right) - \dfrac{\partial \mathcal{L}}{\partial r} = \lambda\,\dfrac{\partial \psi}{\partial r} \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot\theta}\right) - \dfrac{\partial \mathcal{L}}{\partial \theta} = \lambda\,\dfrac{\partial \psi}{\partial \theta} \end{cases} \Rightarrow \quad \begin{cases} m\ddot r - mr\dot\theta^2 + mg\cos\theta = \lambda \\[2ex] mr^2\ddot\theta + 2mr\dot r\dot\theta - mgr\sin\theta = 0 \end{cases}$$

$$\begin{cases} r - a = 0 \\[1ex] m\ddot r - mr\dot\theta^2 + mg\cos\theta = \lambda \\[1ex] mr^2\ddot\theta + 2mr\dot r\dot\theta - mgr\sin\theta = 0 \end{cases} \quad\Rightarrow\quad \begin{cases} \lambda = mg\cos\theta - ma\dot\theta^2 \\[2ex] \ddot\theta = \dfrac{g}{a}\,\sin\theta \end{cases}$$

$$\ddot\theta = \frac{g}{a}\,\sin\theta \quad\Rightarrow\quad \dot\theta\,\frac{\mathrm{d}\dot\theta}{\mathrm{d}\theta} = \frac{g}{a}\,\sin\theta \quad\Rightarrow\quad \frac{\dot\theta^2}{2} = -\frac{g}{a}\,\cos\theta + \frac{g}{a} \qquad \blacksquare$$

$$\mathcal{Q}_r^{\mathsf{cons}} = \lambda\,\frac{\partial\psi}{\partial r} = \lambda = mg\,(3\cos\theta - 2) \quad\Rightarrow\quad \mathcal{Q}_r^{\mathsf{cons}} = 0 \quad\Rightarrow\quad \theta_0 = \cos^{-1}\frac{2}{3} \qquad \blacksquare$$

$$\mathcal{L}'(r,\theta,\dot{r},\dot{\theta},\lambda) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta - \lambda\left(r-a\right)$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}'}{\partial \dot{r}}\right) - \dfrac{\partial \mathcal{L}'}{\partial r} = 0 \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}'}{\partial \dot{\theta}}\right) - \dfrac{\partial \mathcal{L}'}{\partial \theta} = 0 \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}'}{\partial \dot{\lambda}}\right) - \dfrac{\partial \mathcal{L}'}{\partial \lambda} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta - \lambda = 0 \\[2ex] mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0 \\[2ex] r - a = 0 \end{cases}$$

Generalized non-conservative forces

- Generalized non-conservative forces:

$$\mathcal{Q}_k^{\text{nc}} = \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{\text{nc}} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}$$

- Euler-Lagrange equations of motion with both constraint forces and non-conservative forces:

$$\mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t) \equiv T(\{q_i(t), \dot{q}_i(t)\}, t) - U(\{q_i(t)\}, t)$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = \mathcal{Q}_k^{\text{cons}} + \mathcal{Q}_k^{\text{nc}}, \qquad k = 1, 2, \cdots, M$$

EXERCISE 9.4: A simple pendulum of mass m and length ℓ is subjected to linear resistance force $\mathbf{F} = -\gamma \mathbf{v}$ with $\gamma > 0$. Obtain the equations of motion of this pendulum with suitable generalized coordinate(s).

$$\begin{cases} x = \ell \sin\theta \\ \\ y = -\ell\cos\theta \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x} = \ell\dot{\theta}\cos\theta \\ \\ \dot{y} = \ell\dot{\theta}\sin\theta \end{cases}$$

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}\, m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

$$\mathbf{F} = -\gamma\mathbf{v} = -\gamma\ell\dot{\theta}\cos\theta\,\hat{\mathbf{e}}_x - \gamma\ell\dot{\theta}\sin\theta\,\hat{\mathbf{e}}_y$$

$$\mathcal{Q}_\theta = \mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial\theta} = -\gamma\ell\dot{\theta}\cos\theta\,(\ell\cos\theta) - \gamma\ell\dot{\theta}\sin\theta\,(\ell\sin\theta) = -\gamma\ell^2\dot{\theta} \qquad \blacksquare$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \mathcal{Q}_\theta \quad \Rightarrow \quad \ddot{\theta} + \frac{\gamma}{m}\,\dot{\theta} + \frac{g}{\ell}\,\sin\theta = 0 \qquad \blacksquare$$

Generalized potential function

- Generalized forces that can be derived from a **generalized potential function** $\mathcal{U}(\{q_i(t), \dot{q}_i(t)\}, t)$:

$$\mathcal{Q}_k = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{U}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{U}}{\partial q_k}$$

- Lagrangian:

$$\mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t) = T(\{q_i(t), \dot{q}_i(t)\}, t) - \mathcal{U}(\{q_i(t), \dot{q}_i(t)\}, t)$$

- Euler-Lagrange equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0\,, \qquad k = 1, 2, \cdots, M$$

Charge in external electromagnetic field

- Potential formulation in classical electrodynamics:

$$\mathbf{E}(\mathbf{r},t) = -\boldsymbol{\nabla}\phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}\,, \qquad \mathbf{B}(\mathbf{r},t) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r},t)$$

- Lorentz force:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) \quad \Rightarrow \quad F_i = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{x}_i}\left(q\phi - q\mathbf{A}\cdot\mathbf{v}\right)\right] - \frac{\partial}{\partial x_i}\left(q\phi - q\mathbf{A}\cdot\mathbf{v}\right)$$

- Lagrangian for charge in external electromagnetic field:

$$\mathcal{L}(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) = \frac{m}{2}\,\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t) - q\,\phi(\mathbf{r},t) + q\,\dot{\mathbf{r}}(t)\cdot\mathbf{A}(\mathbf{r},t)$$

- Generalized momentum is the mechanical momentum $m\dot{\mathbf{r}}$ plus a magnetic term $q\mathbf{A}$ which paves its way in the quantum theory of a charged particle in a magnetic field!

Gauge symmetry

- Gauge transformation: $\Lambda(\{q_i(t)\},t)$ is known as a **gauge function**

$$\mathcal{L}(\{q_i(t),\dot{q}_i(t)\},t) \to \overline{\mathcal{L}}(\{q_i(t),\dot{q}_i(t)\},t) = \mathcal{L}(\{q_i(t),\dot{q}_i(t)\},t) + \frac{\mathrm{d}\Lambda(\{q_i(t)\},t)}{\mathrm{d}t}$$

- Invariance of Euler-Lagrange equation under gauge transformation:

$$\mathcal{L} \equiv \mathcal{L}(\{q_i(t),\dot{q}_i(t)\},t)\ , \qquad \overline{\mathcal{L}} \equiv \overline{\mathcal{L}}(\{q_i(t),\dot{q}_i(t)\},t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \frac{\partial\mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\overline{\mathcal{L}}}{\partial\dot{\overline{q}}_i}\right) - \frac{\partial\overline{\mathcal{L}}}{\partial\overline{q}_i}$$

- Two Lagrangians, which are differed by a total time derivative of an arbitrary function of generalized coordinates and time, give identical equations of motion

EXERCISE 9.5: Show that Galilean transformation is a gauge transformation for the Lagrangian of a system of N particles interacting via central potentials. Identify the gauge function.

$$\mathcal{L}\left(\{\mathbf{r}_\alpha,\dot{\mathbf{r}}_\alpha\}\right)=\sum_{\alpha=1}^{N}\frac{1}{2}\,m_\alpha\dot{\mathbf{r}}_\alpha\cdot\dot{\mathbf{r}}_\alpha-\frac{1}{2}\sum_{\alpha=1}^{N}\sum_{\beta\neq\alpha}U_{\alpha\beta}\left(|\mathbf{r}_\alpha-\mathbf{r}_\beta|\right)$$

$$\mathbf{r}_\alpha(t)\to\mathbf{r}'_\alpha(t)=\mathbf{r}_\alpha(t)+\mathbf{V}t$$

$$\Rightarrow\quad\dot{\mathbf{r}}'_\alpha(t)=\dot{\mathbf{r}}_\alpha(t)+\mathbf{V}$$

$$\mathcal{L}'\left(\left\{\mathbf{r}'_\alpha,\dot{\mathbf{r}}'_\alpha\right\}\right)=\sum_{\alpha=1}^{N}\frac{1}{2}\,m_\alpha\left(\dot{\mathbf{r}}'_\alpha-\mathbf{V}\right)\cdot\left(\dot{\mathbf{r}}'_\alpha-\mathbf{V}\right)-\frac{1}{2}\sum_{\alpha=1}^{N}\sum_{\beta\neq\alpha}U_{\alpha\beta}\left(\left|\mathbf{r}'_\alpha-\mathbf{r}'_\beta\right|\right)$$

$$=\mathcal{L}\left(\left\{\mathbf{r}'_\alpha,\dot{\mathbf{r}}'_\alpha\right\}\right)-\sum_{\alpha=1}^{N}m_\alpha\dot{\mathbf{r}}'_\alpha\cdot\mathbf{V}+\sum_{\alpha=1}^{N}\frac{1}{2}\,m_\alpha\mathbf{V}\cdot\mathbf{V}$$

$$=\mathcal{L}\left(\left\{\mathbf{r}'_\alpha,\dot{\mathbf{r}}'_\alpha\right\}\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(-\sum_{\alpha=1}^{N}m_\alpha\mathbf{r}'_\alpha\cdot\mathbf{V}+\sum_{\alpha=1}^{N}\frac{1}{2}\,m_\alpha\mathbf{V}\cdot\mathbf{V}t\right)\qquad\blacksquare$$