Chapter 7 QED

Part I

QED study interaction of a charged particle with a photon

This is a general approach for both macro and micro systems

1. Equation of motion for the charged particle (as a free particle)

wavefunction of a free particle is a plane wave ---->

2. Equation of motion for the photon (as a free particle)

3. Interaction between the charged particle and the photon

classically, interaction of electric current $\underset{\sim}{j}$ and $\underset{\sim}{E}$, $\underset{\sim}{B}$

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Chapter 7 Griffiths QED Part I

Basically we study the dynamics of charged particles with electromagnetic field; Interaction of a charged particle e.g. electron with a photon in QED

Classically, equation of motion is needed. for particles

Particles obey Newton's law $\underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt}$ (1687)

The electromagnetic field $(\underset{\sim}{E}, \underset{\sim}{B})$ obeys the Maxwell eqn (1865)

Gauss' Law $\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon_0}$ ρ = charge density

Faraday's Law $\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$

Ampere's Law $c^2 (\nabla \times \underset{\sim}{B}) = \frac{\underset{\sim}{j}}{\epsilon_0} + \frac{\partial \underset{\sim}{E}}{\partial t}$ $\underset{\sim}{j}$ = current density

(collectively, Ampere-Maxwell Law)

this term introduced by maxwell; no source of Bfield

$\nabla \cdot \underset{\sim}{B} = 0$

Magnetic monopoles

his notation for ×

Lorentz force equation $\underset{\sim}{F} = q(\underset{\sim}{E} + \underset{\sim}{v} \wedge \underset{\sim}{B})$

The above 3 sets of equations answer all problems of charged particles interacting with $\underset{\sim}{E}, \underset{\sim}{B}$.

Quantum mechanically, the Newton eqn is replaced by Schrödinger equation or by the Dirac equation if including relativistic effect. The Maxwell equation can be taken over quantum mechanically by using the gauge field $A_\mu(x)$

To study the interaction of a photon with an electron, first find free photon solution (plane wave) and also free electron solution (plane wave). After free particle solutions are obtained, we solve the interaction (Hamiltonian) by using Feynman diagramatic technique, basically a perturbation method.

Now first put Maxwell's equations in relativistically covariant form:

$$\underset{\sim}{E} = -\nabla V - \frac{\partial \underset{\sim}{A}}{\partial t}$$

V = electric potential

$\underset{\sim}{A}$ = magnetic vector potential

By convention $(\nabla V)^i = \partial_i V = \frac{\partial V}{\partial x^i}$, $\therefore E^i = -\frac{\partial V}{\partial x^i} - \frac{\partial A^i}{\partial t}$

$$\underset{\sim}{B} = \nabla \wedge \underset{\sim}{A}$$

$$B^i = (\nabla \wedge \underset{\sim}{A})^i = \varepsilon^{ijk} \partial_j A^k = -\varepsilon^{ijk} \partial_j A_k$$

$(\nabla \wedge A)^i = \varepsilon^{ijk} \nabla_j A^k = \varepsilon^{ijk} \partial_j A^k$

put V and $\underset{\sim}{A}$ together as a 4-vector, A_μ

$$\boxed{A = \left(\frac{V}{c}, \underset{\sim}{A}\right) = (A^0, \underset{\sim}{A})}$$

Introduce electromagnetic field tensor $F_{\mu\nu}$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mu, \nu = 0, 1, 2, 3$$

$$= \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

this is the covariant form

Check

$$E^i = c F^{i0}$$

$$B^i = -\frac{1}{2} \varepsilon^{ijk} F_{jk}$$

(HW)

$g^{00} = +1$, $g^{11} = -1 = g^{22} = g^{33}$

The 4 Maxwell equations become (3)

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad (j^0 = \rho c) \quad \underset{\sim}{j} = (j^0, \underset{\sim}{j})$$

$$\ldots \qquad (1)$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \text{--- sourceless}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2!} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

where $\tilde{F}$ = dual of F

Look for free photon solution from eq (1), i.e. want to find $A_\mu(\underset{\sim}{x})$ for a free photon, $A_\mu(\underset{\sim}{x})$ = gauge field

First note equation (1) has a gauge degree of freedom because a new gauge field $A'_\mu(\underset{\sim}{x})$ defined by

$$A'_\mu(\underset{\sim}{x}) = A_\mu(\underset{\sim}{x}) + \partial_\mu \lambda(\underset{\sim}{x})$$

$\lambda(\underset{\sim}{x})$ = smooth function

can lead to the same $F_{\mu\nu}$:

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$$

$$= F_{\mu\nu} + \partial_\mu \partial_\nu \lambda(\underset{\sim}{x}) - \partial_\nu \partial_\mu \lambda(\underset{\sim}{x})$$

$$= F_{\mu\nu}$$

Can introduce conditions to make $A_\mu(\underset{\sim}{x})$ unique. First impose Lorentz condition

$\underset{\sim}{P} = \frac{\hbar}{i}\nabla$ $\quad \partial_\mu A^\mu = 0 \quad \longrightarrow \quad \partial_\mu \partial^\mu \lambda(\underset{\sim}{x}) = 0$

$P_\mu = i\hbar\partial_\mu \quad \longrightarrow \quad P \cdot A = 0$

still not sufficient to specify $A^\mu(x)$ uniquely.

Next use Coulomb gauge condition to make

$$A^0 = 0$$

$\underset{\sim}{P} \cdot \underset{\sim}{A} = 0 \quad \nabla \cdot \underset{\sim}{A} = 0$

With the Lorentz condition and Coulomb gauge, $\underset{\sim}{A}$ has 2 independent components

The free photon equation

$$\partial_\mu F^{\mu\nu} = 0, \qquad \partial_\mu F^{\mu\nu} = j^\nu = 0$$

Look at $\partial_\mu F^{\mu\nu} = 0$

$\partial_\mu = \frac{\partial}{\partial x^\mu}$

$\partial^\mu = \frac{\partial}{\partial x_\mu}$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$$

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

HW

$$\partial^\nu \partial_\mu A^\mu = 0$$

$\because \partial_\mu A^\mu = 0$ (Lorentz condition)

$$\rightarrow \quad \partial_\mu \partial^\mu A^\nu = 0$$

$$\rightarrow \quad \Box^2 A^\nu = 0, \quad \cdots (2)$$

$\Box^2 =$ D'Alembertian

$\equiv \partial_\mu \partial^\mu$

$= \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2$

solution is Ansatz

$$A_\mu(x) = \text{Const} \cdot e^{-iP\cdot x/\hbar}\, \varepsilon_\mu(P) \quad \cdots (3)$$

$$p = \hbar k \qquad k = (k^0, \underset{\sim}{k}) \qquad k^0 = \frac{\omega}{c}$$

plane wave in S.E. $\psi(\underset{\sim}{x}) = e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}$

Expression (3) is a solution of eq (2) iff

$$|\mathrm{p}|^2 = \left(\frac{E}{c}\right)^2 - \underset{\sim}{p}^2 = 0$$

Hence, $\underset{\sim}{p}^2 = m_0 c^2$

$p^2 = 0$ i.e. $p^0 = |\underset{\sim}{p}|$ (HW)

and also (3) must satisfy the Lorentz condition

$$\partial_\mu A^\mu = 0$$

$\underbrace{\partial_\mu \partial^\mu A^\nu = 0}_{(2)}$ || ansatz $\underbrace{A_\mu(x) = B e^{-i p\cdot x/\hbar} \cdot \varepsilon_\mu(p)}_{(3)}$ where $B \in \mathbb{R}$

Putting (3) into (2) : $\left(\frac{-ip}{\hbar}\right)^2 B e^{-ip\cdot x/\hbar} \varepsilon_\mu(p) + B e^{-ip\cdot x/\hbar} \underbrace{\Box^2 \varepsilon_\mu(p)}_{0}$

$= -\frac{|p|^2}{\hbar} B e^{-ip\cdot x/\hbar} \cdot \varepsilon_\mu(p) + 0$

For (3) to be a soln of (2), $|p|^2 = 0$,

consequently $|p|^2 = p^0 - \underset{\sim}{p} = 0$

$\underset{\sim}{p} = p^0$ // shown

as expected of massless photons

The Lorentz condition also requires that $\underbrace{\partial_\mu A^\mu}_{(4)} = 0$,

Putting (3) into (4) : $\frac{-i|p|}{\hbar} B e^{-ip\cdot x/\hbar} \varepsilon_\mu(p) + B e^{-ip\cdot x/\hbar} \underbrace{\left(\partial_\mu \varepsilon_\mu(p)\right)}_{0} = 0$

$\frac{-i}{\hbar} |p| \varepsilon_\mu(p) B e^{-ip\cdot x/\hbar} = 0$

For (3) to be a soln of (4), $|p| \varepsilon_\mu(p) = 0$ // shown

$$p \cdot \varepsilon = p_\mu \varepsilon^\mu = 0$$

For coulomb gauge

$$\varepsilon^0 = 0 \quad \because \quad A^0 = 0$$

$\longrightarrow$ $\underset{\sim}{p} \cdot \underset{\sim}{\varepsilon} = 0 \xrightarrow{\text{hence}} (\nabla \cdot \underset{\sim}{A} = 0)$

so the free photon is

$$\longrightarrow \quad \underset{\sim}{A}(\underset{\sim}{x}) = \text{const}\; e^{-ip\cdot x/\hbar}\, \underset{\sim}{\varepsilon}$$

and $p^2 = 0$, $\varepsilon^0 = 0$, $\underset{\sim}{p} \cdot \underset{\sim}{\varepsilon} = 0$

If photon propagates along x^3- direction,

$$\underset{\sim}{p} = (0, 0, p)$$

Then solutions for $\underset{\sim}{p} \cdot \underset{\sim}{\varepsilon} = 0$ are given by

$$\underset{\sim}{\varepsilon}_{(1)} = (1, 0, 0), \qquad \varepsilon_{(2)} = (0, 1, 0)$$

The polarization $\underset{\sim}{\varepsilon}$ is perpendicular to the photon propagation direction

The two solutions $\underset{\sim}{\varepsilon}_{(1)} = (1, 0, 0)$, $\underset{\sim}{\varepsilon}_{(2)} = (0, 1, 0)$ describe linearly polarized em field (Transverse polarization)

For circular polarization, the polarization vector $\underset{\sim}{\varepsilon}(\underset{\sim}{p})$ can be written as

$$\underset{\sim}{\varepsilon}_{(\pm)} = \mp \frac{\varepsilon_{(1)} \pm i\,\varepsilon_{(2)}}{\sqrt{2}}$$

$\underset{\sim}{\varepsilon}_{+}$ = RH circularly polarized $= \frac{-1}{\sqrt{2}}(1, i, 0)$

$\underset{\sim}{\varepsilon}_{-}$ = LH circularly polarized $= \frac{1}{\sqrt{2}}(1, -i, 0)$

Thus we have obtained free photon solution

$$A_\mu(\underline{x}) = (\text{Constant}) \cdot e^{-i\underline{P}\cdot\underline{x}/\hbar}\ \varepsilon_\mu(\underline{P}), \quad \underline{P}^2 = 0$$

In the Coulomb gauge, $\varepsilon_0(\underline{P}) = 0 = A_0(\underline{x})$, $\nabla\cdot\underset{\sim}{A}(\underline{x}) = 0$, the $\underset{\sim}{\varepsilon}(\underline{p})$ is as given above.

A C. T. Wu and Chen Ning Yang (6a)

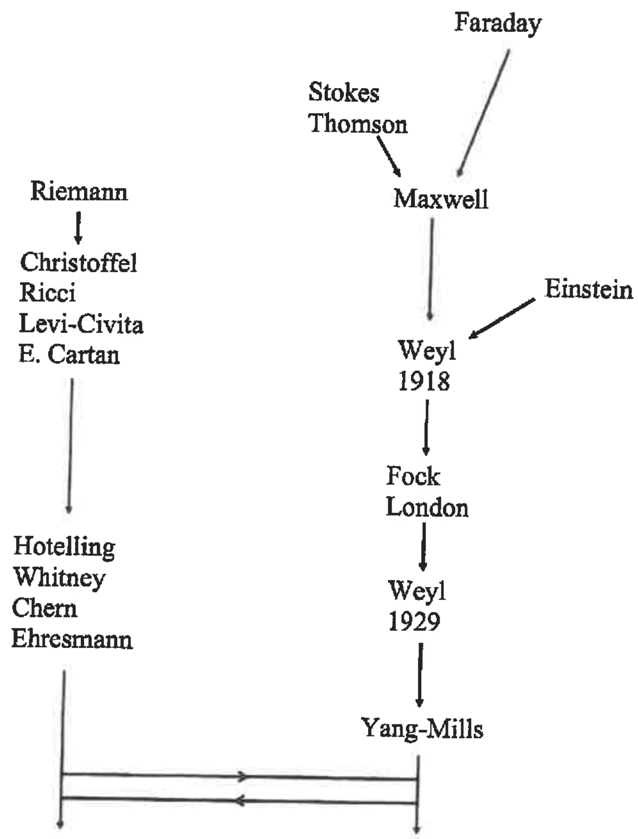

Fig. 1. Flow of ideas in the evolution of the concept of the vector potential.

describable beautifully and precisely by field theories, and that all these theories have mathematical structures required by the concept of symmetry. Hence the principle: *symmetry dictates interaction*. The conceptual history of this remarkable development is the subject of the present paper.

Playing an important part in this history is the vector potential **A**, which first made its appearance in the 19th century. There was certain freedom, now called gauge freedom in its definition, which was early recognized as a simple but somewhat annoying mathematical property. It is this freedom which has now metamorphosed into the key symmetry principle that dictates the exact equations describing the fundamental forces of nature.

Very remarkably, the mathematics of this symmetry principle was in the meantime developed by geometers in the theory of *fiber bundles*, entirely independently of the developments in physics. When this became known, a renewed cross-fertilization of basic ideas between the disciplines of physics and mathematics happily resulted.

Throughout this paper our emphasis is on the early motivation and evolution of the key ideas. There is a vast literature about various aspects of the history we

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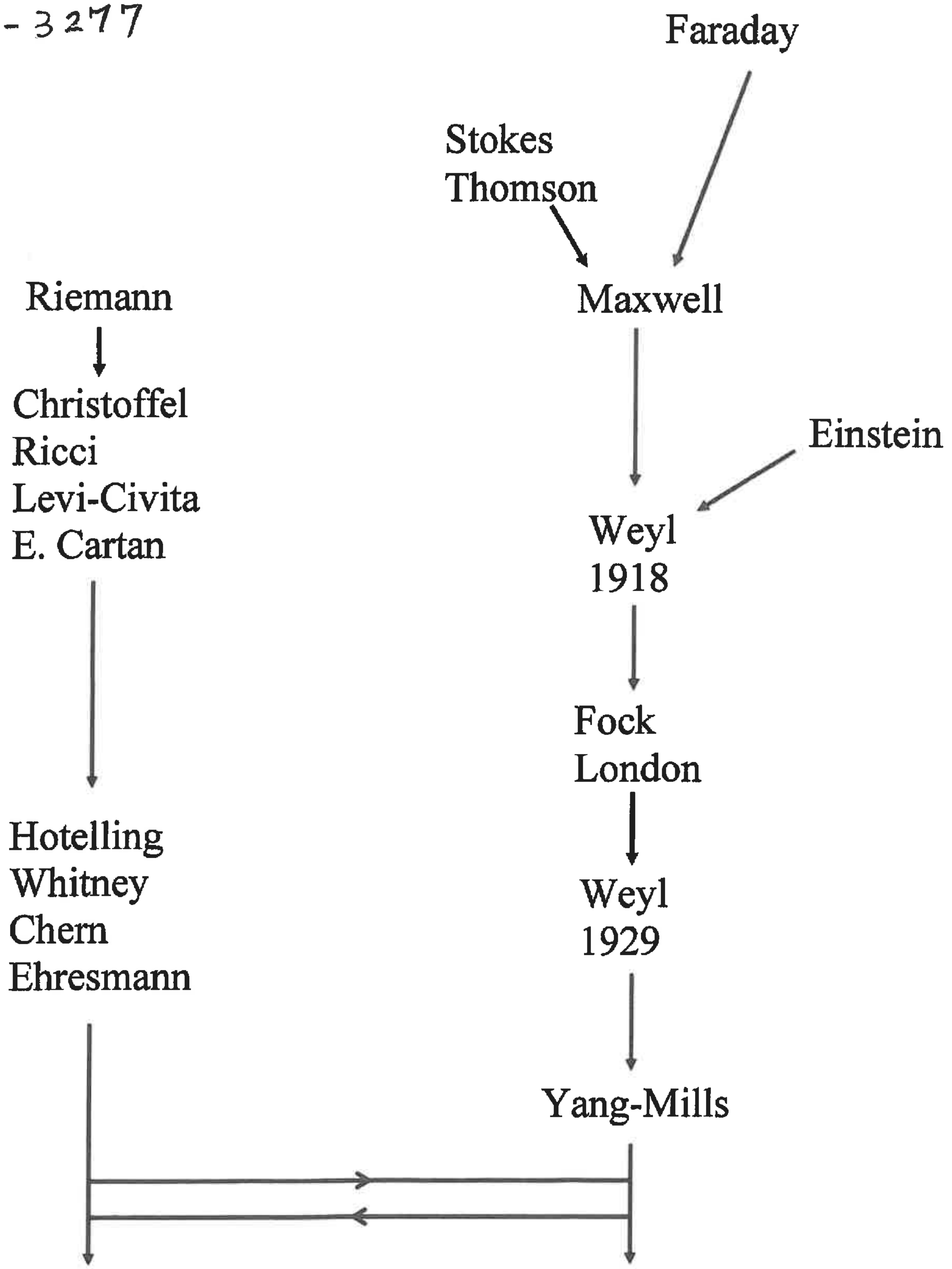

Figure 1

Relativistic equation for electron

In trying to unify Special Relativity with SE, we must first address that SE treats time and space differently

The familiar Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(\underset{\sim}{x}) = H \psi(\underset{\sim}{x})$$

$$E = \frac{\underset{\sim}{p}^2}{2m}$$

$$H = \frac{\underset{\sim}{p}^2}{2m} \quad \text{(free electron kinetic energy)}$$

$$\underset{\sim}{p} = \frac{\hbar}{i} \nabla$$

is relativistically not correct because time is first order derivative whilst space second order derivative, so time and space not treated equally

Two ways to 'derive' relativistically 'correct' equations:

(i) $\frac{\partial^2}{\partial t^2}$, $\frac{\partial^2}{\partial x^{i\,2}}$ both second order

(ii) $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x^i}$ both first order

First way, change $\frac{\partial}{\partial t}$ in Schrödinger equation to $\frac{\partial^2}{\partial t^2}$.

In Special Relativity, for a free particle,

$$P^2 = m^2 c^2 \quad \text{ie} \quad p^{0\,2} - \underset{\sim}{p}^2 = m^2 c^2$$

The 'correct' equation should be

$$\boxed{P^2 \, \psi(\underset{\sim}{x}) = m^2 c^2 \, \psi(\underset{\sim}{x})} \qquad (4)$$

$$P_\mu = i\hbar \partial_\mu = i\hbar \frac{\partial}{\partial x^\mu} \rightarrow P_0 = i\hbar \frac{\partial}{\partial x^0} = i\hbar \frac{1}{c} \frac{\partial}{\partial t}, \quad P_i = i\hbar \frac{\partial}{\partial x^i}$$

$$p^i = -P_i = \frac{\hbar}{i} \frac{\partial}{\partial x^i}$$

Then the 'correct' equation eq (4) becomes

$$\left(\Box^2 + \frac{m^2 c^2}{\hbar^2} \right) \psi(\underset{\sim}{x}) = 0 \qquad \cdots \qquad (5) \qquad \text{HW}$$

check work

known as Klein-Gordon equation

plane wave solution

$$\psi(x) = \text{const}\ e^{-i\,\underline{p}\cdot\underline{x}/\hbar}$$

$$\underline{p}^2 = m^2c^2, \quad p^0 = \pm\sqrt{\underset{\sim}{p}^2 + m^2c^2}$$

This allows $-$ve energy $p^0 = -\sqrt{\underset{\sim}{p}^2 + m^2c^2}$

in the plane wave solution. This is the

#1 first difficulty about the K. G. eqn

Next is $\psi(\underline{x})$ a wave function (probability amplitude) ?

In S.E., prob. density $= |\psi|^2 = \psi^*\psi = \rho$

probability current density

$$\underset{\sim}{j} = \frac{1}{2m}\left(\psi^*\underset{\sim}{p}\psi + (\underset{\sim}{p}\psi)^*\psi\right)$$

$$\underset{\sim}{p} = \frac{\hbar}{i}\nabla$$

should we do the same for the K. G.- $\psi(\underline{x})$?

If do the same, then $\partial_\mu j^\mu \neq 0$

i.e. prob. is not conserved.

In order to ensure $\partial_\mu j^\mu = 0$ for the K.G.. case,
(conservation of probability)

one puts

$$j^{\mu} = \frac{1}{2m}\left(\phi^* p^{\mu}\phi + (p^{\mu}\phi)^*\phi\right) \quad \cdots (6)$$

change K.G $\psi(\underline{x})$ to $\phi(\underline{x})$

This definition does lead to

use chain rule and symmetry of second derivatives to get D'Alembertian?

$$\partial_{\mu} j^{\mu} = 0 \qquad (HW)$$

But problem remains because

$$\rho = \frac{j^0}{c} = \frac{1}{2mc}\left(\phi^* p^0\phi + (p^0\phi)^*\phi\right)$$

$p^0 = i\hbar \frac{\partial}{c\partial t}$

as obtained from j^0 (defined by eq (6))

can be -ve. That means probability density ρ can be -ve, not allowed!

So K.G. equation is wrong if $\phi(\underline{x})$ is a prob. amp. However nowadays we regard K.G equation is relativistically correct for spin 0 particle such as pion but then here $\phi(x)$ is interpreted as a field operator.

Historically, this is also known as the second quantization

Next comes the Dirac equation. (10)

change S. E. ($\frac{\partial}{\partial t}$, $\frac{\partial^2}{\partial x^2}$) to
schrödinger Equation

($\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$) 1st order derivatives.

S. E. $$i\hbar\partial_t \psi(\underline{x}) = \frac{\underset{\sim}{p}^2}{2m} \psi(\underline{x})$$

K. G $$\underline{p}^2 \phi(\underline{x}) = m^2 c^2 \phi(\underline{x})$$

How to change 2nd order derivative in space $\frac{\partial^2}{\partial x^2}$ to 1st order $\frac{\partial}{\partial x}$?

Take square root of the operator $\underline{p}^2$

$$\underline{p}^2 = -\hbar^2 \Box^2 = -\hbar^2 \partial_\mu \partial^\mu$$

Let $\psi(\underline{x})$ be multicomponent $\psi_i(\underline{x})$, $i = 1, 2, \cdots N$,

$$\psi(\underline{x}) \rightarrow \psi(\underline{x}) = \begin{pmatrix} \psi_1(\underline{x}) \\ \vdots \\ \psi_N(\underline{x}) \end{pmatrix}$$

i.e. $\sqrt{\underline{p}^2}$ must be a matrix

Dirac introduced $\not{p}$

we treat this as an operator $$\not{p} = p_\mu \gamma^\mu, \quad \mu = 0, 1, 2, 3$$

and obtained the Dirac equation

$$\not{p}\, \psi(\underline{x}) = mc\, \psi(\underline{x}), \quad \psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix}$$

Then look for plane wave solution and (1)

also construct prob. current density $j_\mu(x)$

s. t. $\partial_\mu j^\mu = 0$

As $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}$ so γ^μ $(\mu = 0, 1, 2, 3)$

must be $N \times N$ matrices, $\mu = 0, 1, 2, 3$

It turns out γ^μ is not a 4-vector

so $\not{p} = p_\mu \gamma^\mu$ is not a scalar although

p_μ is a 4-vector.

$\not{p} = p_\mu \gamma^\mu = \gamma^\mu p_\mu$ is not a scalar wrt

Lorentz transformations

Dirac equation

$$\not{p}\, \psi(\underline{x}) = mc\, \psi(\underline{x})$$

literally taking the square root of the

K. G. eqn $\quad \underline{p}^2\, \phi(\underline{x}) = m^2 c^2 \phi(\underline{x})$

$$\not{p} = p_\mu \gamma^\mu \qquad \mu = 0, 1, 2, 3$$

$$\psi(\underline{x}) = \begin{pmatrix} \psi_1(\underline{x}) \\ \psi_2(\underline{x}) \\ \vdots \\ \psi_N(\underline{x}) \end{pmatrix}$$

$$\gamma^\mu = N \times N \text{ matrix}$$

Today study properties of γ^μ and find plane wave solution of the Dirac eqn.

Properties of γ^μ:

1st the Dirac equation must yield

$$\underline{p}^2 = m^2 c^2$$

in order to be consistent with sp. Relativity for a free particle.

For this, we 'square' the Dirac eqn

Apply $\not{p}$ to $\not{p}\psi = mc\psi$

$$\not{p}^2 \psi = \not{p}\, mc\psi = mc\, \not{p}\psi = m^2 c^2 \psi$$

mc is a constant

we say that there is no x term in the N x N matrix γ so that P_μ and γ commute and can be rearranged

(13)

$$\not{p}^2 = \not{p}\not{p} = P_\mu \gamma^\mu \cdot P_\nu \gamma^\nu$$

$$P_\mu = i\hbar\frac{\partial}{\partial x^\mu}$$

$$= P_\mu P_\nu \gamma^\mu \gamma^\nu$$

$$= P_\nu P_\mu \gamma^\nu \gamma^\mu$$

$$g_{\mu\nu} = g_{\nu\mu}$$
$$\gamma^\mu\gamma^\nu = \gamma^\nu\gamma^\mu$$

(Recall: want $\not{p}^2 \to \underline{p}^2$

$$= \frac{1}{2}\left(P_\mu P_\nu \gamma^\mu \gamma^\nu + P_\nu P_\mu \gamma^\nu \gamma^\mu\right)$$

$$P_\mu P_\nu = P_\nu P_\mu$$

second order derivatives commute, pull out

$$= P_\mu P_\nu \frac{1}{2}[\gamma^\mu, \gamma^\nu]_+$$

← anticommutator

$$[A, B]_+ = AB + BA$$

cf $\underline{p}^2 = P_\mu P^\mu = g_{\mu\nu} p^\mu p^\nu$

In order for $\not{p}^2 = \underline{p}^2$ then demand

$$\frac{1}{2}[\gamma^\mu, \gamma^\nu]_+ \overset{!}{=} g^{\mu\nu}$$

$$\text{i.e.} \quad [\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$$

which defines the Dirac matrix γ^μ

What are the properties of γ^μ, $\mu = 0, 1, 2, 3$?

(1) $\gamma^{0\,2} = 1$, $\gamma^{i\,2} = -1$, $i = 1, 2, 3$

Proof: $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

put $\mu = 0 = \nu \to 2\gamma^{0\,2} = 2g^{00} = 2 \quad \because g^{00} = +1$

i.e. $\gamma^{0\,2} = 1$

The correct equation of motion for a relativistic particle with spin $\frac{1}{2}$ is the Dirac equation

$$\not{p}\,\psi(x) = mc\,\psi(x)$$

$$\not{p} = p_\mu \gamma^\mu = \gamma^\mu p_\mu, \qquad p_\mu = i\hbar\partial_\mu = i\hbar\frac{\partial}{\partial x^\mu}$$

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$$

The anticommutator for the Dirac matrix γ^μ defines γ^μ. One can show

(i)-(iii) can be derived

(i) $\gamma^{0\,2} = 1$, $\gamma^{i\,2} = -1$, $i = 1, 2, 3$

(ii) $\mathrm{Tr}\,\gamma^\mu = 0$, $\mu = 0, 1, 2, 3$

(iii) γ^μ is $N \times N$ matrix, N = even integer

Dirac put $N = 4$ because N=6 and greater didn't have any useful meaning

(iv) Hermiticity of γ^μ,

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

can define $\beta = \gamma^0$, $\quad \underset{\sim}{\alpha} = \gamma^0 \underset{\sim}{\gamma}$, or $\alpha^i = \gamma^0 \gamma^i$

$i = 1, 2, 3$

Then $\quad \beta^+ = \beta$, $\quad \alpha^{i+} = \alpha^i$

the Hamiltonian of a free Dirac particle is

$$H = c \, \underset{\sim}{\alpha} \cdot \underset{\sim}{p} + \beta m c^2$$

(v) Representations of γ^μ, $\quad \mu = 0, 1, 2, 3$

The Dirac representation

2 x 2 identity matrix

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli matrix

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note added:

Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underset{\sim}{\gamma} = \begin{pmatrix} 0 & \underset{\sim}{\sigma} \\ -\underset{\sim}{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Italian physicist

Majorana representation

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}. \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note in the Majorana representation, all elements of the gamma (γ) matrices are imaginary number

Derive properties of the Dirac matrix γ^μ

from the defining equation

$$[\gamma^\mu, \gamma^\nu]_+ = 2 g^{\mu\nu} , \qquad \mu, \nu = 0, 1, 2, 3$$

(i) $\mu = 0 = \nu$

$$[\gamma^0, \gamma^0]_+ = 2 g^{00} = 2 \qquad \because g^{00} = +1$$

$$\therefore \quad \gamma^{0\,2} = \gamma^0 \gamma^0 = 1 \quad \text{(identity matrix)}$$

Similarly, $\quad \gamma^{1\,2} = \gamma^1 \gamma^1 = -1 \qquad \because$ (HW) $g^{11} = -1$

$$\rightarrow \quad \gamma^{i\,2} = -1 , \qquad i = 1, 2, 3$$

(HW)

(ii) $\mu = 0, \quad \nu = i , \qquad i = 1, 2, 3$

$$[\gamma^0, \gamma^i]_+ = 2 g^{0i} = 0$$

$$\gamma^0 \gamma^i = - \gamma^i \gamma^0$$

Multiply γ^0 on both sides

$$\gamma^0 \gamma^0 \gamma^i = - \gamma^0 \gamma^i \gamma^0$$

$$\therefore \quad \gamma^i = - \gamma^0 \gamma^i \gamma^0 \qquad \because \gamma^{0\,2} = 1$$

Taking trace of both sides,

$$\mathrm{Tr}\,\gamma^i = \mathrm{Tr}\,(-\gamma^0\gamma^i\gamma^0) = -\mathrm{Tr}(\gamma^0\gamma^i\gamma^0)$$

$$= -\mathrm{Tr}(\gamma^0\gamma^0\gamma^i) \quad \because \mathrm{Tr}\,AB = \mathrm{Tr}\,BA$$

$$= -\mathrm{Tr}\,\gamma^i \quad \because {\gamma^0}^2 = 1$$

$$\therefore \quad \mathrm{Tr}\,\gamma^i = 0 \qquad i = 1, 2, 3$$

similarly $\mathrm{Tr}\,\gamma^0 = 0$ (HW) same as above, apply gamma^nu instead and rearrange

Thus $\mathrm{Tr}\,\gamma^\mu = 0, \quad \mu = 0, 1, 2, 3.$

(iii) Let γ^μ be a $N \times N$ matrix

show $N =$ even integer

Use $\gamma^i = -\gamma^0\gamma^i\gamma^0$

Taking determinant both sides

$$\det\gamma^i = \det(-\gamma^0\gamma^i\gamma^0) = (-1)^N \det(\gamma^0\gamma^i\gamma^0)$$

As $\det AB = \det BA$,

$$\det(\gamma^0\gamma^i\gamma^0) = \det(\gamma^i\gamma^0\gamma^0) = \det\gamma^i$$

$$\because {\gamma^0}^2 = 1$$

Thus $\det\gamma^i \overset{!}{=} (-1)^N \det\gamma^i$

For an N by N matrix, the determinant involves taking the sum of several products each made up of N terms; if N is even then (-1) has no effect but if N is odd then the overall determinant changes by a factor of (-1)

$\therefore$ N = even integer,

$N = 2, 4, 6, 8 \ldots$

Dirac chose $N = 4$

From now onwards, put $N=4$, i.e.

$\gamma^\mu = 4 \times 4$ matrix

and the wavefunction $\psi(\underset{\sim}{x})$

$$\psi(x) = \begin{pmatrix} \psi_1(\underset{\sim}{x}) \\ \psi_2(\underset{\sim}{x}) \\ \psi_3(\underset{\sim}{x}) \\ \psi_4(\underset{\sim}{x}) \end{pmatrix}$$

(iv) Is γ^μ Hermitian?

To answer this, find the Dirac Hamiltonian first from the Dira equation

Dirac equation is

$\gamma^\mu p_\mu = p_\mu \gamma^\mu$

$$\not{p}\,\psi(\underset{\sim}{x}) = mc\,\psi(\underset{\sim}{x}) \qquad \not{p} = p_\mu \gamma^\mu$$

$$\therefore \underline{(\gamma^0 p^0 - \underset{\sim}{\gamma} \cdot \underset{\sim}{p})}\,\psi(\underset{\sim}{x}) = mc\,\psi(\underset{\sim}{x})$$

$$\gamma^0 p^0 \psi(x) = (\underset{\sim}{\gamma} \cdot \underset{\sim}{p} + mc)\psi(x)$$

Multiply both sides by γ^0,

mc is a constant

$\gamma^{0\,2} = 1$

$$p^0 \psi(x) = (\gamma^0 \underset{\sim}{\gamma} \cdot \underset{\sim}{p} + mc\gamma^0)\psi(x)$$

cf Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(x) = H\psi(x)$$

As $p_\mu = i\hbar \frac{\partial}{\partial x^\mu}$

$\therefore$

$$i\hbar \frac{\partial}{\partial x^0}\psi(x) = (\gamma^0 \underset{\sim}{\gamma} \cdot \underset{\sim}{p} + mc\gamma^0)\psi(x)$$

$$i\hbar \frac{\partial}{\partial t}\psi(x) = (c\gamma^0 \underset{\sim}{\gamma} \cdot \underset{\sim}{p} + mc^2\gamma^0)\psi(x)$$

$x^0 = ct$

Dirac Hamiltonian

$\therefore$

$$H = c\gamma^0 \underset{\sim}{\gamma} \cdot \underset{\sim}{p} + mc^2\gamma^0$$

$$= c\,\underset{\sim}{\alpha} \cdot \underset{\sim}{p} + \beta mc^2, \qquad \underset{\sim}{\alpha} \equiv \gamma^0 \underset{\sim}{\gamma}$$

$$\beta \equiv \gamma^0$$

Now $\underline{H^\dagger = H \text{ and using } \underset{\sim}{p}^\dagger = \underset{\sim}{p}}$

$\therefore \quad \underset{\sim}{\alpha}^\dagger = \underset{\sim}{\alpha}, \qquad \beta^\dagger = \beta$

can show $\gamma^\mu = \gamma^0 \gamma^{\mu\dagger} \gamma^0$ (HW)

$$\gamma^{0\dagger} = \gamma^0, \qquad \gamma^i = \gamma^0 \gamma^i \gamma^0$$

$$H^\dagger = H$$

$$H = c\,\underset{\sim}{\alpha}\cdot\underset{\sim}{p} + \beta m c^2, \qquad \underset{\sim}{p}^\dagger = \underset{\sim}{p}$$

$$\underset{\sim}{\alpha}^\dagger = \underset{\sim}{\alpha}, \qquad \beta^\dagger = \beta$$

$$\beta = \gamma^0 \qquad \therefore\ \gamma^{0\dagger} = \gamma^0$$

Now

$$\underset{\sim}{\alpha} = \gamma^0\,\underset{\sim}{\gamma}, \qquad \underset{\sim}{\alpha}^\dagger = \underset{\sim}{\gamma}^\dagger\,\gamma^0$$

$$\underset{\sim}{\alpha} = \underset{\sim}{\alpha}^\dagger \quad\rightarrow\quad \gamma^0\,\underset{\sim}{\gamma} = \underset{\sim}{\gamma}^\dagger\,\gamma^0$$

$$\gamma^0\,\gamma^\dagger\,\gamma^0 = \gamma^{02}\,\underset{\sim}{\gamma} = \underset{\sim}{\gamma}$$

can write (HW)

$$\gamma^\mu = \gamma^0\,\gamma^{\mu\dagger}\,\gamma^0$$

$$\gamma^{\mu\dagger} = \gamma^0\,\gamma^\mu\,\gamma^0$$