Single particle in two dimensions

- Cartesian coordinates: $(q_1, q_2) \equiv (x, y)$

$$T \equiv T(x, y, \dot{x}, \dot{y}, t) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

- Generalized forces:

$$\mathcal{Q}_k(t) = \mathbf{F}(t) \cdot \frac{\partial \mathbf{r}}{\partial q_k} \quad \Rightarrow \quad \begin{cases} \mathcal{Q}_1(t) \equiv \mathcal{Q}_x(t) = \mathbf{F}(t) \cdot \dfrac{\partial \mathbf{r}}{\partial x} = \mathbf{F}(t) \cdot \hat{\mathbf{e}}_x = F_x(t) \\ \mathcal{Q}_2(t) \equiv \mathcal{Q}_y(t) = \mathbf{F}(t) \cdot \dfrac{\partial \mathbf{r}}{\partial y} = \mathbf{F}(t) \cdot \hat{\mathbf{e}}_y = F_y(t) \end{cases}$$

- Equations of motion:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{x}}\right) - \dfrac{\partial T}{\partial x} = \mathcal{Q}_x(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{y}}\right) - \dfrac{\partial T}{\partial y} = \mathcal{Q}_y(t) \end{cases} \quad \Rightarrow \quad \begin{cases} m\ddot{x}(t) = F_x(t) \\ m\ddot{y}(t) = F_y(t) \end{cases}$$

$$\mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y$$

$$\mathcal{Q}_k(t) = \mathbf{F}(t)\cdot\frac{\partial \mathbf{r}}{\partial q_k} \quad \Rightarrow \quad \begin{cases} \mathcal{Q}_1(t) = F_x(t) \\ \mathcal{Q}_2(t) = F_y(t) \end{cases}$$

$$T \equiv T(x,\dot{x},y,\dot{y},t) = \frac{m}{2}\left[\dot{x}^2(t) + \dot{y}^2(t)\right]$$

$$\begin{cases} \dfrac{\partial T}{\partial x} = 0 \\ \dfrac{\partial T}{\partial y} = 0 \end{cases}, \qquad \begin{cases} \dfrac{\partial T}{\partial \dot{x}} = m\dot{x}(t) \\ \dfrac{\partial T}{\partial \dot{y}} = m\dot{y}(t) \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{x}}\right) - \dfrac{\partial T}{\partial x} = \mathcal{Q}_x(t) \quad \Rightarrow \quad \dfrac{\mathrm{d}}{\mathrm{d}t}\left[m\dot{x}(t)\right] = F_x(t) \quad \Rightarrow \quad m\ddot{x}(t) = F_x(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{y}}\right) - \dfrac{\partial T}{\partial y} = \mathcal{Q}_y(t) \quad \Rightarrow \quad \dfrac{\mathrm{d}}{\mathrm{d}t}\left[m\dot{y}(t)\right] = F_y(t) \quad \Rightarrow \quad m\ddot{y}(t) = F_y(t) \end{cases} \qquad \blacksquare$$

Single particle in two dimensions – cont'd

- Polar coordinates: $(q_1, q_2) \equiv (\rho, \phi)$

$$T \equiv T(\rho, \phi, \dot{\rho}, \dot{\phi}, t) = \frac{m}{2}\,\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right)$$

- Generalized forces:

$$\mathcal{Q}_1(t) \equiv \mathcal{Q}_\rho(t) = F_\rho(t)\,, \qquad \mathcal{Q}_2(t) \equiv \mathcal{Q}_\phi(t) = \rho(t)\,F_\phi(t)$$

- Equations of motion:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{\rho}}\right) - \dfrac{\partial T}{\partial \rho} = \mathcal{Q}_\rho(t) \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{\phi}}\right) - \dfrac{\partial T}{\partial \phi} = \mathcal{Q}_\phi(t) \end{cases} \Rightarrow \begin{cases} m\ddot{\rho}(t) - m\rho(t)\,\dot{\phi}^2(t) = F_\rho(t) \\[2ex] m\rho^2(t)\,\ddot{\phi}(t) + 2m\rho(t)\,\dot{\rho}(t)\,\dot{\phi}(t) = \rho(t)\,F_\phi(t) \end{cases}$$

$$\mathbf{r}(t) = \rho(t)\cos\phi(t)\,\hat{\mathbf{e}}_x + \rho(t)\sin\phi(t)\,\hat{\mathbf{e}}_y$$

$$\mathcal{Q}_k(t) = \mathbf{F}(t)\cdot\frac{\partial\mathbf{r}}{\partial q_k} \quad\Rightarrow\quad \begin{cases} \mathcal{Q}_\rho(t) = \mathbf{F}(t)\cdot\dfrac{\partial\mathbf{r}}{\partial\rho} = F_\rho(t) \\[2ex] \mathcal{Q}_\phi(t) = \mathbf{F}(t)\cdot\dfrac{\partial\mathbf{r}}{\partial\phi} = \rho(t)\,F_\phi(t) \end{cases}$$

$$T \equiv T(\rho,\phi,\dot\rho,\dot\phi,t) = \frac{m}{2}\left[\dot\rho^2(t) + \rho^2(t)\,\dot\phi^2(t)\right]$$

$$\begin{cases} \dfrac{\partial T}{\partial\rho} = m\rho(t)\,\dot\phi^2(t) \\[2ex] \dfrac{\partial T}{\partial\phi} = 0 \end{cases} , \qquad \begin{cases} \dfrac{\partial T}{\partial\dot\rho} = m\dot\rho(t) \\[2ex] \dfrac{\partial T}{\partial\dot\phi} = m\rho^2(t)\,\dot\phi(t) \end{cases}$$

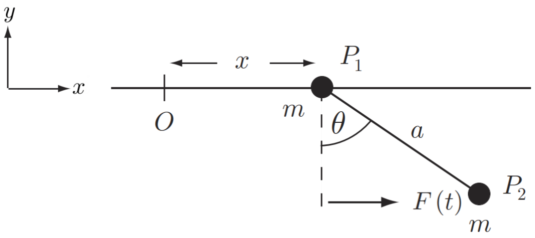
$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial\dot\rho}\right) - \dfrac{\partial T}{\partial\rho} = \mathcal{Q}_\rho(t) \quad\Rightarrow\quad m\ddot\rho(t) - m\rho(t)\,\dot\phi^2(t) = F_\rho(t) \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial\dot\phi}\right) - \dfrac{\partial T}{\partial\phi} = \mathcal{Q}_\phi(t) \quad\Rightarrow\quad m\rho^2(t)\,\ddot\phi(t) + 2m\rho(t)\,\dot\rho(t)\,\dot\phi(t) = \rho(t)\,F_\phi(t) \end{cases}$$

■

Example: A constrained two-particle system

- Two identical particles, P_1 and P_2, with mass m are connected by a light rigid rod of length a. P_1 is constrained to move along a fixed horizontal frictionless rail and the system moves in the vertical plane through the rail. An external force $F(t)\,\hat{\mathbf{e}}_x$ is acted on P_2

- Generalized coordinates: $(q_1, q_2) \equiv (x, \theta)$



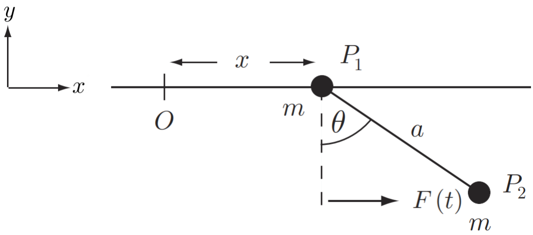
EXERCISE 7.5: Use Lagrange's equation to obtain equations of motions for $x(t)$ and $\theta(t)$.

$$\begin{cases} \mathbf{r}_1(t) = x(t)\,\hat{\mathbf{e}}_x \\ \\ \mathbf{r}_2(t) = [x(t) + a\sin\theta(t)]\;\hat{\mathbf{e}}_x - a\cos\theta(t)\,\hat{\mathbf{e}}_y \end{cases}$$

$$\begin{cases} \dot{\mathbf{r}}_1(t) = \dot{x}(t)\,\hat{\mathbf{e}}_x \\ \\ \dot{\mathbf{r}}_2(t) = \left[\dot{x}(t) + a\cos\theta(t)\,\dot{\theta}(t)\right]\;\hat{\mathbf{e}}_x + a\sin\theta(t)\,\dot{\theta}(t)\,\hat{\mathbf{e}}_y \end{cases}$$

$$T \equiv T(x, \theta, \dot{x}, \dot{\theta}, t) = \frac{m}{2}\,\dot{\mathbf{r}}_1(t)\cdot\dot{\mathbf{r}}_1(t) + \frac{m}{2}\,\dot{\mathbf{r}}_2(t)\cdot\dot{\mathbf{r}}_2(t)$$

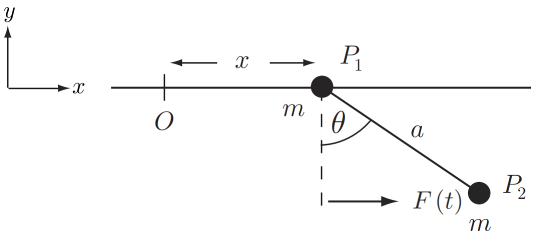
$$= m\dot{x}^2(t) + \frac{1}{2}\,ma^2\dot{\theta}^2(t) + ma\cos\theta(t)\,\dot{x}(t)\,\dot{\theta}(t)$$



$$
\begin{cases}
\mathbf{r}_1(t) = x(t)\,\hat{\mathbf{e}}_x \\
\mathbf{r}_2(t) = [x(t) + a\sin\theta(t)]\ \hat{\mathbf{e}}_x - a\cos\theta(t)\,\hat{\mathbf{e}}_y
\end{cases}
$$

$$
\begin{cases}
\mathbf{F}_1(t) = -mg\,\hat{\mathbf{e}}_y \\
\mathbf{F}_2(t) = -mg\,\hat{\mathbf{e}}_y + F(t)\,\hat{\mathbf{e}}_x
\end{cases}
$$

$$
\begin{cases}
\mathcal{Q}_x(t) = \mathbf{F}_1(t)\cdot\dfrac{\partial \mathbf{r}_1}{\partial x} + \mathbf{F}_2(t)\cdot\dfrac{\partial \mathbf{r}_2}{\partial x} = F(t) \\
\mathcal{Q}_\theta(t) = \mathbf{F}_1(t)\cdot\dfrac{\partial \mathbf{r}_1}{\partial \theta} + \mathbf{F}_2(t)\cdot\dfrac{\partial \mathbf{r}_2}{\partial \theta} = a\cos\theta(t)\,F(t) - mga\cos\theta(t)
\end{cases}
$$



$$T = m\dot{x}^2(t) + \frac{1}{2}\, ma^2\dot{\theta}^2(t) + ma\cos\theta(t)\,\dot{x}(t)\,\dot{\theta}(t)\,, \qquad \begin{cases} \mathcal{Q}_x(t) = F(t) \\ \mathcal{Q}_\theta(t) = a\cos\theta(t)\,F(t) - mga\cos\theta(t) \end{cases}$$

$$\begin{cases} \dfrac{\partial T}{\partial x} = 0 \\ \dfrac{\partial T}{\partial \dot{x}} = 2m\dot{x}(t) + ma\cos\theta(t)\,\dot{\theta}(t) \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = \mathcal{Q}_x(t)$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[2m\dot{x}(t) + ma\cos\theta(t)\,\dot{\theta}(t)\right] = F(t)$$

$$\Rightarrow \quad 2m\ddot{x}(t) + ma\cos\theta(t)\,\ddot{\theta}(t) - ma\sin\theta(t)\,\dot{\theta}^2(t) = F(t) \qquad \blacksquare$$

$$T = m\dot{x}^2(t) + \frac{1}{2}\, ma^2\dot{\theta}^2(t) + ma\cos\theta(t)\,\dot{x}(t)\,\dot{\theta}(t)\,, \quad \begin{cases} \mathcal{Q}_x(t) = F(t) \\ \mathcal{Q}_\theta(t) = a\cos\theta(t)\,F(t) - mga\cos\theta(t) \end{cases}$$

$$\begin{cases} \dfrac{\partial T}{\partial \theta} = -ma\sin\theta(t)\,\dot{x}(t)\,\dot{\theta}(t) \\ \dfrac{\partial T}{\partial \dot{\theta}} = ma^2\dot{\theta}(t) + ma\cos\theta(t)\,\dot{x}(t) \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \mathcal{Q}_\theta(t)$$

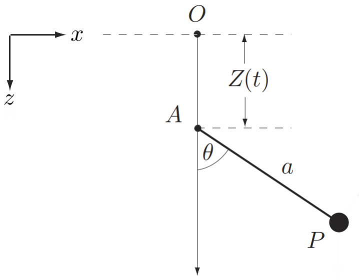
$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[ma^2\dot{\theta}(t) + ma\cos\theta(t)\,\dot{x}(t)\right] + ma\sin\theta(t)\,\dot{x}(t)\,\dot{\theta}(t) = a\cos\theta(t)\,F(t) - mga\cos\theta(t)$$

$$\Rightarrow \quad ma^2\ddot{\theta}(t) + ma\cos\theta(t)\,\ddot{x}(t) - 2ma\sin\theta(t)\,\dot{x}(t)\,\dot{\theta}(t) = a\cos\theta(t)\,F(t) - mga\sin\theta(t)$$

Example: Pendulum with an oscillating pivot

- A simple pendulum in which the pivot is made to move vertically so that its distance from the fixed origin at time t is $Z(t) = Z_0 \cos \Omega t$. The string is a light rigid rod of length a that cannot go slack

- Generalized coordinate: $q_1 \equiv \theta$



EXERCISE 7.6: Use Lagrange's equation to obtain equations of motion for $\theta(t)$.

$$\mathbf{r}(t) = a\sin\theta(t)\,\hat{\mathbf{e}}_x + \left[Z(t) + a\cos\theta(t)\right]\,\hat{\mathbf{e}}_z$$

$$\dot{\mathbf{r}}(t) = a\dot{\theta}(t)\cos\theta(t)\,\hat{\mathbf{e}}_x + \left[\dot{Z}(t) - a\dot{\theta}(t)\sin\theta(t)\right]\,\hat{\mathbf{e}}_z$$

$$T \equiv T(\theta,\dot{\theta},t) = \frac{m}{2}\,\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t) = \frac{m}{2}\left[a^2\dot{\theta}^2(t) + \dot{Z}^2(t) - 2a\dot{Z}(t)\,\dot{\theta}(t)\sin\theta(t)\right]$$

$$\frac{\partial T}{\partial\theta} = -ma\dot{Z}(t)\,\dot{\theta}(t)\cos\theta(t)\,, \qquad \frac{\partial T}{\partial\dot{\theta}} = ma^2\dot{\theta}(t) - ma\dot{Z}(t)\sin\theta(t)$$

$$\mathbf{F}(t) = mg\,\hat{\mathbf{e}}_z \quad\Rightarrow\quad \mathcal{Q}_\theta(t) = \mathbf{F}(t)\cdot\frac{\partial\mathbf{r}(t)}{\partial\theta} = -mga\cos\theta(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \mathcal{Q}_\theta(t)$$

$$\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\left[ma^2\dot{\theta}(t) - ma\dot{Z}(t)\sin\theta(t)\right] + ma\dot{Z}(t)\,\dot{\theta}(t)\cos\theta(t) = -mga\cos\theta(t)$$

$$\Rightarrow\quad \ddot{\theta}(t) - \frac{1}{a}\,\ddot{Z}(t)\sin\theta(t) = -\frac{g}{a}\,\cos\theta(t)$$

$$\Rightarrow\quad \ddot{\theta}(t) + \frac{\Omega^2 Z_0}{a}\,\cos(\Omega t)\sin\theta(t) + \frac{g}{a}\,\cos\theta(t) = 0 \qquad \blacksquare$$

PC3261: Classical Mechanics II

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Semester I, 2023/24

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National University of Singapore
Department of Physics
Faculty of Science

Lecture 8: Lagrangian Mechanics I

Conservative systems

- Applied forces are conservative:

$$U \equiv U(\{\mathbf{r}_\alpha(t)\}) \ , \qquad \mathbf{F}_\alpha^{(\mathsf{A})}(t) = -\frac{\partial U}{\partial \mathbf{r}_\alpha}$$

- Generalized forces: $U \equiv U(\{q_i\}) = U(\{\mathbf{r}_\alpha(q_i(t))\})$

$$\mathcal{Q}_k(t) = \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{(\mathsf{A})}(t) \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k} = -\sum_{\alpha=1}^{N} \frac{\partial U}{\partial \mathbf{r}_\alpha} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k} = -\frac{\partial U}{\partial q_k}$$

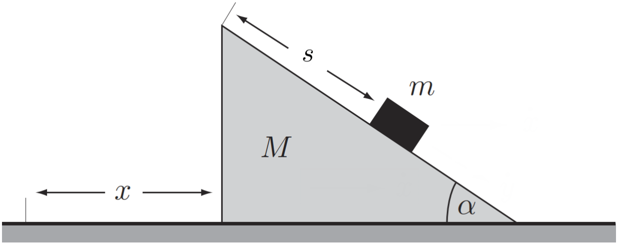
- Lagrange's equation for conservative systems:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial U}{\partial q_k} \, , \qquad k = 1, 2, \cdots, M$$

Example: A block sliding on a wedge

- A block of mass m is free to slide on the wedge of mass M which can slide on the horizontal table, both with negligible friction

- Generalized coordinates: s is the distance of the block from the top of the wedge and x is the distance of the wedge from any convenient *fixed* point on the table



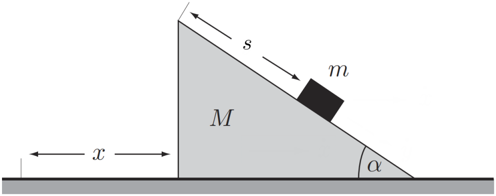
EXERCISE 8.1: Find the acceleration of the wedge, and acceleration of the block relative to the wedge from Lagrange's equation.

$$\begin{cases} \mathbf{r}_1(t) = x(t)\,\hat{\mathbf{e}}_x \\ \mathbf{r}_2(t) = [x(t) + s(t)\cos\alpha]\ \hat{\mathbf{e}}_x + [H - s(t)\sin\alpha]\ \hat{\mathbf{e}}_y \end{cases}$$

$$T \equiv T(x, s, \dot{x}, \dot{s}, t) = \frac{M}{2}\,\dot{\mathbf{r}}_1(t)\cdot\dot{\mathbf{r}}_1(t) + \frac{m}{2}\,\dot{\mathbf{r}}_2(t)\cdot\dot{\mathbf{r}}_2(t)$$

$$= \frac{M}{2}\,\dot{x}^2(t) + \frac{m}{2}\left[\dot{s}^2(t) + 2\dot{s}(t)\,\dot{x}(t)\cos\alpha + \dot{x}^2(t)\right] \qquad \blacksquare$$

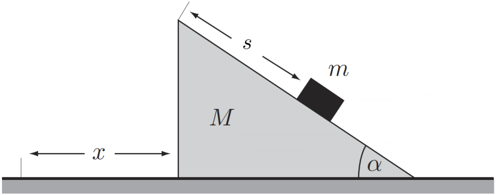
$$U \equiv U(x, s) = mgy_2(t) = mgH - mg\,s(t)\sin\alpha \qquad \blacksquare$$



$$T = \frac{M}{2}\,\dot{x}^2(t) + \frac{m}{2}\left[\dot{s}^2(t) + 2\dot{s}(t)\,\dot{x}(t)\cos\alpha + \dot{x}^2(t)\right]\;, \qquad U = mgH - mg\,s(t)\sin\alpha$$

$$\begin{cases} \dfrac{\partial T}{\partial \dot{x}} = (M+m)\dot{x}(t) + m\dot{s}(t)\cos\alpha \\[2ex] \dfrac{\partial T}{\partial x} = 0 \\[2ex] \dfrac{\partial U}{\partial x} = 0 \end{cases}$$

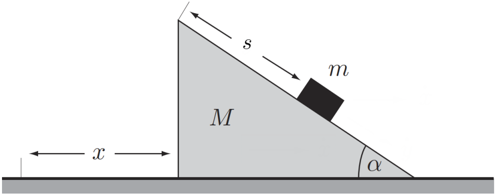
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = -\frac{\partial U}{\partial x} \quad \Rightarrow \quad (M+m)\dot{x}(t) + m\dot{s}(t)\cos\alpha = \text{constant} \qquad \blacksquare$$



$$T = \frac{M}{2}\,\dot{x}^2(t) + \frac{m}{2}\left[\dot{s}^2(t) + 2\dot{s}(t)\,\dot{x}(t)\cos\alpha + \dot{x}^2(t)\right]\,, \qquad U = mgH - mg\,s(t)\sin\alpha$$

$$\begin{cases} \dfrac{\partial T}{\partial \dot{s}} = m\dot{s}(t) + m\dot{x}(t)\cos\alpha \\[2ex] \dfrac{\partial T}{\partial s} = 0 \\[2ex] \dfrac{\partial U}{\partial s} = -mg\sin\alpha \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{s}}\right) - \frac{\partial T}{\partial s} = -\frac{\partial U}{\partial s} \quad \Rightarrow \quad m\ddot{s}(t) + m\ddot{x}(t)\cos\alpha = mg\sin\alpha \qquad \blacksquare$$



$$\begin{cases} (M+m)\dot{x}(t) + m\dot{s}(t)\cos\alpha = \text{constant} \\ m\ddot{s}(t) + m\ddot{x}(t)\cos\alpha = mg\sin\alpha \end{cases}$$

$$\Rightarrow \quad \begin{cases} (M+m)\ddot{x}(t) + m\ddot{s}(t)\cos\alpha = 0 \\ m\ddot{s}(t) + m\ddot{x}(t)\cos\alpha = mg\sin\alpha \end{cases}$$

$$\Rightarrow \quad \begin{cases} \ddot{x}(t) = -\dfrac{m}{M+m}\,\dfrac{g\sin\alpha\cos\alpha}{1-\dfrac{m\cos^2\alpha}{M+m}} \\[2ex] \ddot{s}(t) = \dfrac{g\sin\alpha}{1-\dfrac{m\cos^2\alpha}{M+m}} \end{cases} \qquad \blacksquare$$