Moving trihedral

- Tangent and normal vectors: κ is called the **curvature**

$$\hat{\mathbf{e}}_T(s) \equiv \frac{\mathrm{d}\mathbf{r}(s)}{\mathrm{d}s} \quad \Rightarrow \quad \mathbf{v}(s) = v(s)\,\hat{\mathbf{e}}_T(s)$$

$$\hat{\mathbf{e}}_N(s) \equiv \frac{1}{\kappa(s)}\,\frac{\mathrm{d}\hat{\mathbf{e}}_T(s)}{\mathrm{d}s}$$

- Binormal vector: τ is called the **torsion**

$$\hat{\mathbf{e}}_B(s) \equiv \hat{\mathbf{e}}_T(s) \times \hat{\mathbf{e}}_N(s)\,, \qquad \frac{\mathrm{d}\hat{\mathbf{e}}_B(s)}{\mathrm{d}s} \equiv -\tau(s)\,\hat{\mathbf{e}}_N(s)$$

EXERCISE 1.2: Show that the acceleration of a particle moving along a trajectory $\mathbf{r}(t)$ is give by

$$\mathbf{a}(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + \frac{v^2(t)}{\rho}\,\hat{\mathbf{e}}_N\,,$$

where $\rho \equiv 1/\kappa$ is its radius of curvature.

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}s(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathbf{r}(s)}{\mathrm{d}s} = v(t)\,\hat{\mathbf{e}}_T \qquad \blacksquare$$

$$\begin{aligned}
\mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} &= \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + v(t)\,\frac{\mathrm{d}\hat{\mathbf{e}}_T}{\mathrm{d}t} \\
&= \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + v(t)\,\frac{\mathrm{d}s(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\hat{\mathbf{e}}_T}{\mathrm{d}s} \\
&= \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + v^2(t)\,\kappa\,\hat{\mathbf{e}}_N \\
&= \frac{\mathrm{d}v(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_T + \frac{v^2(t)}{\rho}\,\hat{\mathbf{e}}_N \qquad \blacksquare
\end{aligned}$$

Example: Circular helix

- Position vector: a, b and ω are constants

$$\mathbf{r}(t) = a\cos\omega t\,\hat{\mathbf{e}}_x + a\sin\omega t\,\hat{\mathbf{e}}_y + b\omega t\,\hat{\mathbf{e}}_z$$

- Curvature and torsion: circular helix is the unique curve with non-zero constant curvature and torsion

$$\kappa(t) = \frac{a}{a^2+b^2}\,, \qquad \tau(t) = \frac{b}{a^2+b^2}$$

EXERCISE 1.3: Find the tangent, normal and binormal vectors, as well as, curvature and torsion for the circular helix.

$$\mathbf{r}(t) = a\cos\omega t\,\hat{\mathbf{e}}_x + a\sin\omega t\,\hat{\mathbf{e}}_y + b\omega t\,\hat{\mathbf{e}}_z$$

$$\dot{\mathbf{r}}(t) = -a\omega\sin\omega t\,\hat{\mathbf{e}}_x + a\omega\cos\omega t\,\hat{\mathbf{e}}_y + b\omega\,\hat{\mathbf{e}}_z$$

$$s(t) = \int_0^t |\dot{\mathbf{r}}(t)|\;\mathrm{d}t = \omega\sqrt{a^2+b^2}\,t \quad\Rightarrow\quad \frac{\mathrm{d}s(t)}{\mathrm{d}t} = \omega\sqrt{a^2+b^2}$$

$$\hat{\mathbf{e}}_T(t) = \frac{\mathrm{d}\mathbf{r}(s)}{\mathrm{d}s} = \frac{\dfrac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}}{\dfrac{\mathrm{d}s(t)}{\mathrm{d}t}} = \frac{\dot{\mathbf{r}}(t)}{\dot{s}(t)} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\omega t\,\hat{\mathbf{e}}_x + a\cos\omega t\,\hat{\mathbf{e}}_y + b\,\hat{\mathbf{e}}_z\right) \qquad \blacksquare$$

$$\hat{\mathbf{e}}_T(t) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\omega t\,\hat{\mathbf{e}}_x + a\cos\omega t\,\hat{\mathbf{e}}_y + b\,\hat{\mathbf{e}}_z\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}t} = \frac{a\omega}{\sqrt{a^2+b^2}}\left(-\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\frac{\mathrm{d}\mathbf{e}_T(t)}{\mathrm{d}s} = \frac{\dfrac{\mathrm{d}\mathbf{e}_T(t)}{\mathrm{d}t}}{\dfrac{\mathrm{d}s(t)}{\mathrm{d}t}} = \frac{a}{a^2+b^2}\left(-\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y\right) \quad \Rightarrow \quad \left|\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}s}\right| = \frac{a}{a^2+b^2}$$

$$\hat{\mathbf{e}}_N(t) = \frac{1}{\kappa(t)}\,\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}s} \quad \Rightarrow \quad \kappa(t) = \left|\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}s}\right| = \frac{a}{a^2+b^2} \qquad \blacksquare$$

$$\hat{\mathbf{e}}_N(t) = \frac{1}{\kappa(t)}\,\frac{\mathrm{d}\hat{\mathbf{e}}_T(t)}{\mathrm{d}s} = -\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y \qquad \blacksquare$$

$$\hat{\mathbf{e}}_T(t) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\omega t\,\hat{\mathbf{e}}_x + a\cos\omega t\,\hat{\mathbf{e}}_y + b\,\hat{\mathbf{e}}_z\right)\ , \quad \hat{\mathbf{e}}_N(t) = -\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y$$

$$\hat{\mathbf{e}}_B(t) = \hat{\mathbf{e}}_T(t)\times\hat{\mathbf{e}}_N(t) = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\omega t\,\hat{\mathbf{e}}_x - b\cos\omega t\,\hat{\mathbf{e}}_y + a\,\hat{\mathbf{e}}_z\right) \qquad \blacksquare$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}t} = \frac{b\omega}{\sqrt{a^2+b^2}}\left(\cos\omega t\,\hat{\mathbf{e}}_x + \sin\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}s} = \frac{\dfrac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}t}}{\dfrac{\mathrm{d}s(t)}{\mathrm{d}t}} = \frac{b}{a^2+b^2}\left(\cos\omega t\,\hat{\mathbf{e}}_x + \sin\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}s} = -\tau(t)\,\hat{\mathbf{e}}_N(t) \quad \Rightarrow \quad \tau(t) = -\hat{\mathbf{e}}_N(t)\cdot\frac{\mathrm{d}\hat{\mathbf{e}}_B(t)}{\mathrm{d}s} = \frac{b}{a^2+b^2} \qquad \blacksquare$$

$$\hat{\mathbf{e}}_N(t) = -\cos\omega t\,\hat{\mathbf{e}}_x - \sin\omega t\,\hat{\mathbf{e}}_y\,, \qquad \hat{\mathbf{e}}_B(t) = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\omega t\,\hat{\mathbf{e}}_x - b\cos\omega t\,\hat{\mathbf{e}}_y + a\,\hat{\mathbf{e}}_z\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_N(t)}{\mathrm{d}t} = \omega\left(\sin\omega t\,\hat{\mathbf{e}}_x - \cos\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\frac{\mathrm{d}\hat{\mathbf{e}}_N(t)}{\mathrm{d}s} = \frac{\dfrac{\mathrm{d}\hat{\mathbf{e}}_N(t)}{\mathrm{d}t}}{\dfrac{\mathrm{d}s(t)}{\mathrm{d}t}} = \frac{1}{\sqrt{a^2+b^2}}\left(\sin\omega t\,\hat{\mathbf{e}}_x - \cos\omega t\,\hat{\mathbf{e}}_y\right)$$

$$\hat{\mathbf{e}}_N(s)\cdot\hat{\mathbf{e}}_B(s) = 0 \quad \Rightarrow \quad \hat{\mathbf{e}}_N(s)\cdot\frac{\mathrm{d}\hat{\mathbf{e}}_B(s)}{\mathrm{d}s} + \frac{\mathrm{d}\hat{\mathbf{e}}_N(s)}{\mathrm{d}s}\cdot\hat{\mathbf{e}}_B(s) = 0$$

$$\Rightarrow \quad -\tau(s)\,\hat{\mathbf{e}}_N(s)\cdot\hat{\mathbf{e}}_N(s) + \frac{\mathrm{d}\hat{\mathbf{e}}_N(s)}{\mathrm{d}s}\cdot\hat{\mathbf{e}}_B(s) = 0 \quad \Rightarrow \quad \tau(s) = \hat{\mathbf{e}}_B(s)\cdot\frac{\mathrm{d}\hat{\mathbf{e}}_N(s)}{\mathrm{d}s}$$

$$\tau(t) = \hat{\mathbf{e}}_B(t)\cdot\frac{\mathrm{d}\hat{\mathbf{e}}_N(t)}{\mathrm{d}s} = \frac{b}{a^2+b^2} \qquad \blacksquare$$

2D polar coordinate system

- Polar coordinates: $(u_1, u_2) = (\rho, \phi)$

ρ: distance from the origin, $0 \leq \rho < \infty$

ϕ: azimuthal angle from $+x$-axis, $0 \leq \phi < 2\pi$

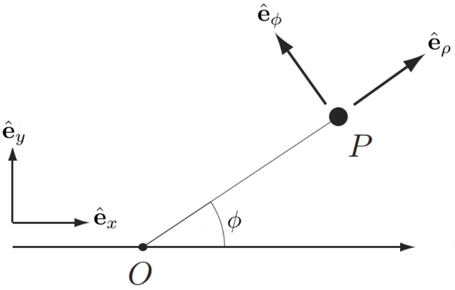
- Coordinate transformation between polar and Cartesian coordinates:

$$\begin{cases} x = \rho \cos\phi \\ y = \rho \sin\phi \end{cases} \quad \Leftrightarrow \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

- Unit basis vectors $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi)$ are *not* constant!

EXERCISE 1.4: Establish the relationship between unit basis vectors $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi)$ of the polar coordinate system and the unit basis vectors $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y)$ of the Cartesian coordinate system.

$$
\begin{cases}
\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{e}}_x + \sin\phi\,\hat{\mathbf{e}}_y \\
\hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y
\end{cases}
\qquad \blacksquare
$$



$$\begin{pmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}^{-1} \begin{pmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\phi \end{pmatrix}$$

$$\Rightarrow \quad \begin{cases} \hat{\mathbf{e}}_x = \cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_y = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \end{cases} \qquad \blacksquare$$

Kinematics in 2D polar coordinates

- Position vector:

$$\mathbf{r}(t) = \rho(t)\,\hat{\mathbf{e}}_\rho$$

- Velocity:

$$\mathbf{v}(t) = \dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi$$

- Acceleration:

$$\mathbf{a}(t) = \left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right]\hat{\mathbf{e}}_\rho + \left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right]\,\hat{\mathbf{e}}_\phi$$

EXERCISE 1.5: Express the velocity and acceleration vectors in 2D polar coordinates.

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \end{cases}, \qquad \begin{cases} \hat{\mathbf{e}}_\rho = \cos\phi(t)\,\hat{\mathbf{e}}_x + \sin\phi(t)\,\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_\phi = -\sin\phi(t)\,\hat{\mathbf{e}}_x + \cos\phi(t)\,\hat{\mathbf{e}}_y \end{cases}$$

$$\mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y = r_\rho\,\hat{\mathbf{e}}_\rho + r_\phi\,\hat{\mathbf{e}}_\phi$$

$$\begin{cases} r_\rho = \hat{\mathbf{e}}_\rho \cdot \mathbf{r}(t) = x(t)\cos\phi(t) + y(t)\sin\phi(t) = \rho(t) \\ r_\phi = \hat{\mathbf{e}}_\phi \cdot \mathbf{r}(t) = -x(t)\sin\phi(t) + y(t)\cos\phi(t) = 0 \end{cases}$$

$$\Rightarrow \quad \mathbf{r}(t) = \rho(t)\,\hat{\mathbf{e}}_\rho \qquad \blacksquare$$

$$\begin{cases} \hat{\mathbf{e}}_\rho = \cos\phi(t)\,\hat{\mathbf{e}}_x + \sin\phi(t)\,\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_\phi = -\sin\phi(t)\,\hat{\mathbf{e}}_x + \cos\phi(t)\,\hat{\mathbf{e}}_y \end{cases}$$

$$\Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}\hat{\mathbf{e}}_\rho}{\mathrm{d}t} = -\dot{\phi}(t)\sin\phi(t)\,\hat{\mathbf{e}}_x + \dot{\phi}(t)\cos\phi(t)\,\hat{\mathbf{e}}_y = \dot{\phi}(t)\,\hat{\mathbf{e}}_\phi \\ \dfrac{\mathrm{d}\hat{\mathbf{e}}_\phi}{\mathrm{d}t} = -\dot{\phi}(t)\cos\phi(t)\,\hat{\mathbf{e}}_x - \dot{\phi}(t)\sin\phi(t)\,\hat{\mathbf{e}}_y = -\dot{\phi}(t)\,\hat{\mathbf{e}}_\rho \end{cases}$$

$$\begin{aligned} \mathbf{v}(t) &= \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(t)\,\hat{\mathbf{e}}_\rho\right] \\ &= \dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi \qquad \blacksquare \end{aligned}$$

$$\mathbf{v}(t) = \dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi$$

$$\begin{cases} \dfrac{\mathrm{d}\hat{\mathbf{e}}_\rho}{\mathrm{d}t} = \dot{\phi}(t)\,\hat{\mathbf{e}}_\phi \\[2ex] \dfrac{\mathrm{d}\hat{\mathbf{e}}_\phi}{\mathrm{d}t} = \dot{\phi}(t)\,\hat{\mathbf{e}}_\rho \end{cases}$$

$$\begin{aligned} \mathbf{a}(t) &= \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi\right] \\ &= \left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right]\hat{\mathbf{e}}_\rho + \left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right]\,\hat{\mathbf{e}}_\phi \qquad \blacksquare \end{aligned}$$

Cylindrical coordinate system

- Cylindrical coordinates: $(u_1, u_2, u_3) = (\rho, \phi, z)$

ρ: polar distance from the z axis, $0 \le \rho < \infty$

ϕ: azimuthal angle from the x axis on the xy-plane, $0 \le \phi < 2\pi$

z: coordinate along the z axis, $-\infty < z < \infty$

- Coordinate transformation between cylindrical and Cartesian coordinates:

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

- Velocity and acceleration:

$$\begin{cases} \mathbf{v}(t) = \dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi + \dot{z}(t)\,\hat{\mathbf{e}}_z \\ \mathbf{a}(t) = \left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right]\hat{\mathbf{e}}_\rho + \left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right]\hat{\mathbf{e}}_\phi + \ddot{z}(t)\,\hat{\mathbf{e}}_z \end{cases}$$

Spherical coordinate system

- Spherical coordinates: $(u_1, u_2, u_3) = (r, \theta, \phi)$

r: radial distance from the origin, $0 \le r < \infty$

θ: polar angle from the z axis, $0 \le \theta \le \pi$

ϕ: azimuthal angle from the x axis on the xy-plane, $0 \le \phi < 2\pi$

- Coordinate transformation between spherical and Cartesian coordinates:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

EXERCISE 1.6: Express the spherical unit basis vectors $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi)$ in terms of Cartesian unit basis vectors $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$.

$$\mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z = r\sin\theta\cos\phi\,\hat{\mathbf{e}}_x + r\sin\theta\sin\phi\,\hat{\mathbf{e}}_y + r\cos\theta\,\hat{\mathbf{e}}_z$$

$$\begin{cases} \dfrac{\partial \mathbf{r}}{\partial r} = \sin\theta\ \cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z \\[2ex] \dfrac{\partial \mathbf{r}}{\partial \theta} = r\cos\theta\cos\phi\,\hat{\mathbf{e}}_x + r\cos\theta\sin\phi\,\hat{\mathbf{e}}_y - r\cos\theta\,\hat{\mathbf{e}}_z \\[2ex] \dfrac{\partial \mathbf{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{\mathbf{e}}_x + r\sin\theta\cos\phi\,\hat{\mathbf{e}}_y \end{cases}$$

$$\Rightarrow \quad \begin{cases} \hat{\mathbf{e}}_r \equiv \dfrac{\dfrac{\partial \mathbf{r}}{\partial r}}{\left|\dfrac{\partial \mathbf{r}}{\partial r}\right|} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z \\[4ex] \hat{\mathbf{e}}_\theta \equiv \dfrac{\dfrac{\partial \mathbf{r}}{\partial \theta}}{\left|\dfrac{\partial \mathbf{r}}{\partial \theta}\right|} = \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z \\[4ex] \hat{\mathbf{e}}_\phi \equiv \dfrac{\dfrac{\partial \mathbf{r}}{\partial \phi}}{\left|\dfrac{\partial \mathbf{r}}{\partial \phi}\right|} = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_z \end{cases} \qquad \blacksquare$$

$$\begin{cases} \hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_z \end{cases}$$

$$\begin{aligned} \hat{\mathbf{e}}_r \cdot (\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\phi) &= \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \\ &= -\sin\phi \begin{vmatrix} \sin\theta\sin\phi & \cos\theta \\ \cos\theta\sin\phi & -\sin\theta \end{vmatrix} - \cos\phi \begin{vmatrix} \sin\theta\cos\phi & \cos\theta \\ \cos\theta\cos\phi & -\sin\theta \end{vmatrix} \\ &= -\sin\phi\left(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi\right) - \cos\phi\left(-\sin^2\theta\cos\phi - \cos^2\theta\cos\phi\right) \\ &= 1 \qquad \blacksquare \end{aligned}$$

Kinematics in spherical coordinates

- Position vector:

$$\mathbf{r}(t) = r(t)\,\hat{\mathbf{e}}_r$$

- Velocity vector:

$$\mathbf{v}(t) = \dot{r}(t)\,\hat{\mathbf{e}}_r + r(t)\,\dot{\theta}(t)\,\hat{\mathbf{e}}_\theta + r(t)\,\dot{\phi}(t)\sin\theta(t)\,\hat{\mathbf{e}}_\phi$$

- Acceleration vector:

$$\begin{aligned}\mathbf{a}(t) = {} & \left[\ddot{r}(t) - r(t)\,\dot{\phi}^2(t)\sin^2\theta(t) - r(t)\,\dot{\theta}^2(t)\right]\hat{\mathbf{e}}_r \\ & + \left[r(t)\,\ddot{\theta}(t) + 2\dot{r}(t)\,\dot{\theta}(t) - r(t)\,\dot{\phi}^2(t)\sin\theta(t)\cos\theta(t)\right]\hat{\mathbf{e}}_\theta \\ & + \left[r(t)\,\ddot{\phi}(t)\sin\theta(t) + 2\dot{r}(t)\,\dot{\phi}(t)\sin\theta(t) + 2r(t)\,\dot{\theta}(t)\,\dot{\phi}(t)\cos\theta(t)\right]\hat{\mathbf{e}}_\phi\end{aligned}$$

$$\hat{\mathbf{e}}_r = \sin\theta(t)\cos\phi(t)\,\hat{\mathbf{e}}_x + \sin\theta(t)\sin\phi(t)\,\hat{\mathbf{e}}_y + \cos\theta(t)\,\hat{\mathbf{e}}_z$$

$$\begin{aligned}
\frac{\mathrm{d}\hat{\mathbf{e}}_r}{\mathrm{d}t} &= \frac{\partial\hat{\mathbf{e}}_r}{\partial\theta}\,\dot{\theta} + \frac{\partial\hat{\mathbf{e}}_r}{\partial\phi}\,\dot{\phi} \\
&= \left(\cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z\right)\dot{\theta} + \left(-\sin\theta\sin\phi\,\hat{\mathbf{e}}_x + \sin\theta\cos\phi\,\hat{\mathbf{e}}_y\right)\dot{\phi} \\
&= \dot{\theta}\,\hat{\mathbf{e}}_\theta + \sin\theta\,\dot{\phi}\,\hat{\mathbf{e}}_\phi \qquad \blacksquare
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}(t) \equiv \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\,\hat{\mathbf{e}}_r\right] \\
&= \dot{r}(t)\,\hat{\mathbf{e}}_r + r(t)\,\frac{\mathrm{d}\hat{\mathbf{e}}_r}{\mathrm{d}t} \\
&= \dot{r}(t)\,\hat{\mathbf{e}}_r + r(t)\,\dot{\theta}(t)\,\hat{\mathbf{e}}_\theta + r(t)\,\dot{\phi}(t)\,\sin\theta(t)\,\hat{\mathbf{e}}_\phi \qquad \blacksquare
\end{aligned}$$

PC3261: Classical Mechanics II

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Lecture 2: Newton's Laws of Motion

Newton's first law and inertia

- **Newton's first law**: a particle remains at rest or in uniform motion unless acted upon a force

- **Inertia** is the *resistance* of any particle to any change in its velocity and the quantitative measure of inertia is **mass**

- A mathematical description of the motion of a particle requires the choice of a **frame of reference –** a set of coordinates in space that can be used to specify the position, velocity and acceleration of the particle at any given instant of time

- A frame of reference at which Newton's first law is valid is called an **inertial frame of reference**

Newton's second law

- **Linear momentum** of a particle is defined as the product of its mass and velocity

$$\mathbf{p}(t) \equiv m\mathbf{v}(t)$$

- **Newton's second law**: a particle acted upon a force moves in such a manner that the time rate of change of linear momentum equals the force

$$\mathbf{F}(t) = \frac{\mathrm{d}\mathbf{p}(t)}{\mathrm{d}t}$$

- Both Newton's first and second laws remain exactly true in special relativity with a *suitably* redefinition of linear momentum

Newton's third law

- **Newton's third law**: if two particles exert forces on each other, these forces are equal in magnitude and opposite in direction

- **Central forces** are the forces acting along the line connecting two particles

- Velocity-dependent forces are non-central and Newton's third law *may* not apply

- Newton's third law is not valid in special relativity as the concept of absolute time is abandoned

Galilean relativity

- Two inertial frames, $\mathcal{O}$ and $\mathcal{O}'$, are oriented such that their spatial coordinate axes are parallel, their spatial origins are coincided when $t = t' = 0$ and $\mathcal{O}'$ moves at *uniform velocity* $\mathbf{V}$ with respect to $\mathcal{O}$

- **Galilean boost**:

$$\begin{cases} t' = t \\ \mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{V}t \end{cases}$$

- Galilean velocity transformation:

$$\mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V}$$

- Newton's laws are **Galilean invariance**

$$\begin{cases} \mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y + z(t)\,\hat{\mathbf{e}}_z \\ \mathbf{r}'(t') = x'(t')\,\hat{\mathbf{e}}_{x'} + y'(t')\,\hat{\mathbf{e}}_{y'} + z'(t')\,\hat{\mathbf{e}}_{z'} \end{cases}, \qquad \begin{cases} \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_{x'} \\ \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_{y'} \\ \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_{z'} \end{cases}$$

$$t' = t \qquad \Rightarrow \qquad \mathbf{r}'(t') = \mathbf{r}'(t) = x'(t)\,\hat{\mathbf{e}}_x + y'(t)\,\hat{\mathbf{e}}_y + z'(t)\,\hat{\mathbf{e}}_z$$

$$\mathbf{v}'(t') \equiv \frac{\mathrm{d}\mathbf{r}'(t')}{\mathrm{d}t'} = \frac{\mathrm{d}\mathbf{r}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}x'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_x + \frac{\mathrm{d}y'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_y + \frac{\mathrm{d}z'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_z \equiv \mathbf{v}'(t)$$

$$\mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{V}t \quad \Rightarrow \quad \frac{\mathrm{d}\mathbf{r}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{V}t) \quad \Rightarrow \quad \mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V} \qquad \blacksquare$$

$$\Rightarrow \quad \begin{cases} \mathbf{v}(t) \equiv \dfrac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_x + \dfrac{\mathrm{d}y(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_y + \dfrac{\mathrm{d}z(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_z \\ \mathbf{v}'(t) \equiv \dfrac{\mathrm{d}\mathbf{r}'(t)}{\mathrm{d}t} = \dfrac{\mathrm{d}x'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_{x'} + \dfrac{\mathrm{d}y'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_{y'} + \dfrac{\mathrm{d}z'(t)}{\mathrm{d}t}\,\hat{\mathbf{e}}_{z'} \end{cases} \qquad \blacksquare$$

$$\begin{cases} \mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y + z(t)\,\hat{\mathbf{e}}_z \\ \mathbf{r}'(t') = x'(t')\,\hat{\mathbf{e}}_{x'} + y'(t')\,\hat{\mathbf{e}}_{y'} + z'(t')\,\hat{\mathbf{e}}_{z'} \end{cases}, \qquad \begin{cases} \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_{x'} \\ \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_{y'} \\ \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_{z'} \end{cases}$$

$$t' = t \qquad \Rightarrow \qquad \mathbf{r}'(t') = \mathbf{r}'(t) = x'(t)\,\hat{\mathbf{e}}_x + y'(t)\,\hat{\mathbf{e}}_y + z'(t)\,\hat{\mathbf{e}}_z$$

$$\mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V}$$

$$\Rightarrow \quad \frac{\mathrm{d}\mathbf{v}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} \quad \Rightarrow \quad \mathbf{a}'(t) = \mathbf{a}(t) \qquad \blacksquare$$

$$\Rightarrow \quad \begin{cases} \mathbf{a}(t) \equiv \dfrac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \dfrac{\mathrm{d}x^2(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_x + \dfrac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_y + \dfrac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_z \\ \mathbf{a}'(t) \equiv \dfrac{\mathrm{d}\mathbf{v}'(t)}{\mathrm{d}t} = \dfrac{\mathrm{d}^2 x'(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_{x'} + \dfrac{\mathrm{d}^2 y'(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_{y'} + \dfrac{\mathrm{d}^2 z'(t)}{\mathrm{d}t^2}\,\hat{\mathbf{e}}_{z'} \end{cases} \qquad \blacksquare$$