

PC3261: Classical Mechanics II

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Lecture 1: Kinematics

Kronecker delta symbol

Notes

- **Kronecker delta symbol:** completely symmetric

$$\delta_{ij} = \delta_{ji}, \quad \delta_{ij} \equiv \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}, \quad i, j = 1, 2, 3$$

- Useful identities:

$$A_i = \sum_{j=1}^3 \delta_{ij} A_j, \quad \sum_{k=1}^3 \delta_{ik} \delta_{kj} = \delta_{ij}, \quad \sum_{i=1}^3 \sum_{j=1}^3 \delta_{ij} = 3$$

Levi-Civita symbol

Notes

- **Levi-Civita symbol:** completely anti-symmetric

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj}, \quad \epsilon_{123} \equiv +1, \quad i, j, k = 1, 2, 3$$

- Product of Levi-Civita symbols:

$$\epsilon_{ijk} \epsilon_{mnr} = \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{ir} \\ \delta_{jm} & \delta_{jn} & \delta_{jr} \\ \delta_{km} & \delta_{kn} & \delta_{kr} \end{vmatrix}$$

- Useful identities:

$$\sum_{k=1}^3 \epsilon_{ijk} \epsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}, \quad \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{mjk} \epsilon_{njk} = 2\delta_{mn}, \quad \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk} \epsilon_{ijk} = 6$$

Cartesian coordinate system

Notes

- Cartesian coordinates: $(x_1, x_2, x_3) \equiv (x, y, z)$

$$-\infty < x < \infty, \quad -\infty < y < \infty, \quad -\infty < z < \infty$$

- Cartesian unit basis vectors: $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \equiv (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \quad \rightarrow \quad \begin{cases} \hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_y \cdot \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z = 1 \\ \hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_y \cdot \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_x = 0 \end{cases}$$

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \sum_{k=1}^3 \epsilon_{ijk} \hat{\mathbf{e}}_k \quad \rightarrow \quad \begin{cases} \hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_y \times \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x = \hat{\mathbf{e}}_y \end{cases}$$

- Cartesian unit basis vectors are constant

Position vector

Notes

- **Position** of a particle in the space is specified by a vector relative to the *spatial origin* of a given *reference frame* known as **position vector**
- Position vector in the Cartesian coordinate system: (x, y, z) are the Cartesian coordinates of the particle

$$\mathbf{r} = x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y + z \hat{\mathbf{e}}_z = \sum_{i=1}^3 x_i \hat{\mathbf{e}}_i$$

- Motion of the particle traces a **trajectory** in the space and can be described mathematically by an one-dimensional **curve**
- Trajectory of the motion of particle can be specified by the position vector *parameterized* by **time** relative to the *temporal origin* of the reference frame

$$\mathbf{r}(t) = x(t) \hat{\mathbf{e}}_x + y(t) \hat{\mathbf{e}}_y + z(t) \hat{\mathbf{e}}_z = \sum_{i=1}^3 x_i(t) \hat{\mathbf{e}}_i$$

Velocity vector

Notes

- **Velocity vector:** rate of change of the position vector with respect to time

$$\mathbf{v}(t) \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \equiv \frac{d\mathbf{r}(t)}{dt} \equiv \dot{\mathbf{r}}(t)$$

- Velocity vector is *tangent* to the trajectory of the particle at any given instant of time

- **Speed:** magnitude of the velocity vector

$$v(t) \equiv |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)}$$

- Cartesian coordinate system:

$$\dot{\mathbf{r}}(t) = \dot{x}(t) \hat{\mathbf{e}}_x + \dot{y}(t) \hat{\mathbf{e}}_y + \dot{z}(t) \hat{\mathbf{e}}_z \quad \Rightarrow \quad \dot{r}(t) \equiv |\dot{\mathbf{r}}(t)| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}$$

Acceleration vector

Notes

- **Acceleration vector:** rate of change of the velocity vector with respect to time

$$\mathbf{a}(t) \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{v}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} \equiv \frac{d\mathbf{v}(t)}{dt} \equiv \dot{\mathbf{v}}(t) = \frac{d^2\mathbf{r}(t)}{dt^2} \equiv \ddot{\mathbf{r}}(t)$$

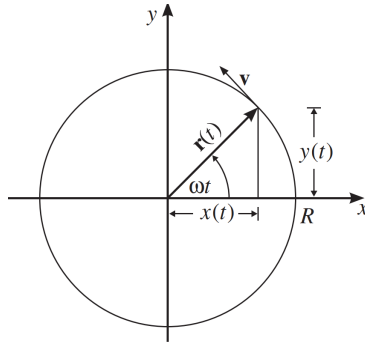
- Cartesian coordinate system:

$$\ddot{\mathbf{r}}(t) = \ddot{x}(t) \hat{\mathbf{e}}_x + \ddot{y}(t) \hat{\mathbf{e}}_y + \ddot{z}(t) \hat{\mathbf{e}}_z \quad \Rightarrow \quad \ddot{r}(t) \equiv |\ddot{\mathbf{r}}(t)| = \sqrt{\ddot{x}^2(t) + \ddot{y}^2(t) + \ddot{z}^2(t)}$$

Example: Uniform circular motion

Notes

- A particle moves in a circle lying in the xy plane (centered at the origin and radius R) with constant angular speed ω counter-clockwise as viewed from $+z$ axis. The particle is on the $+x$ axis at $t = 0$



EXERCISE 1.1: Find the particle's velocity and acceleration vectors. What are the magnitude and direction of the particle's acceleration?

Another mathematical description of trajectory

Notes

- Trajectory of the motion of particle can also be represented mathematically by the position vector parameterized by **arc length** along the trajectory

- Arc length:

$$s(t) = \int_0^t ds = \int_0^t |\mathbf{dr}| = \int_0^t \sqrt{\left[\frac{dx(t)}{dt}\right]^2 + \left[\frac{dy(t)}{dt}\right]^2 + \left[\frac{dz(t)}{dt}\right]^2} dt$$

- Speed:

$$v(t) = |\mathbf{v}(t)| = \left| \frac{d\mathbf{r}(t)}{dt} \right| = \frac{ds(t)}{dt}$$

- A set of three orthogonal unit vectors, parameterized by arc length, can be constructed at each point of the trajectory

Moving trihedral

Notes

- Tangent and normal vectors: κ is called the **curvature**

$$\hat{\mathbf{e}}_T(s) \equiv \frac{d\mathbf{r}(s)}{ds} \Rightarrow \mathbf{v}(s) = v(s) \hat{\mathbf{e}}_T(s)$$

$$\hat{\mathbf{e}}_N(s) \equiv \frac{1}{\kappa(s)} \frac{d\hat{\mathbf{e}}_T(s)}{ds}$$

- Binormal vector: τ is called the **torsion**

$$\hat{\mathbf{e}}_B(s) \equiv \hat{\mathbf{e}}_T(s) \times \hat{\mathbf{e}}_N(s), \quad \frac{d\hat{\mathbf{e}}_B(s)}{ds} \equiv -\tau(s) \hat{\mathbf{e}}_N(s)$$

EXERCISE 1.2: Show that the acceleration of a particle moving along a trajectory $\mathbf{r}(t)$ is give by

$$\mathbf{a}(t) = \frac{dv(t)}{dt} \hat{\mathbf{e}}_T + \frac{v^2(t)}{\rho} \hat{\mathbf{e}}_N,$$

where $\rho \equiv 1/\kappa$ is its radius of curvature.

Example: Circular helix

Notes

- Position vector: a , b and ω are constants

$$\mathbf{r}(t) = a \cos \omega t \hat{\mathbf{e}}_x + a \sin \omega t \hat{\mathbf{e}}_y + b \omega t \hat{\mathbf{e}}_z$$

- Curvature and torsion: circular helix is the unique curve with non-zero constant curvature and torsion

$$\kappa(t) = \frac{a}{a^2 + b^2}, \quad \tau(t) = \frac{b}{a^2 + b^2}$$

EXERCISE 1.3: Find the tangent, normal and binormal vectors, as well as, curvature and torsion for the circular helix.

2D polar coordinate system

Notes

- Polar coordinates: $(u_1, u_2) = (\rho, \phi)$

ρ : distance from the origin, $0 \leq \rho < \infty$

ϕ : azimuthal angle from $+x$ -axis, $0 \leq \phi < 2\pi$

- Coordinate transformation between polar and Cartesian coordinates:

$$\begin{cases} x = \rho \cos \phi \\ y = \rho \sin \phi \end{cases} \Leftrightarrow \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1} \left(\frac{y}{x} \right) \end{cases}$$

- Unit basis vectors $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi)$ are *not* constant!

EXERCISE 1.4: Establish the relationship between unit basis vectors $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi)$ of the polar coordinate system and the unit basis vectors $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y)$ of the Cartesian coordinate system.

Kinematics in 2D polar coordinates

Notes

- Position vector:

$$\mathbf{r}(t) = \rho(t) \hat{\mathbf{e}}_\rho$$

- Velocity:

$$\mathbf{v}(t) = \dot{\rho}(t) \hat{\mathbf{e}}_\rho + \rho(t) \dot{\phi}(t) \hat{\mathbf{e}}_\phi$$

- Acceleration:

$$\mathbf{a}(t) = [\ddot{\rho}(t) - \rho(t) \dot{\phi}^2(t)] \hat{\mathbf{e}}_\rho + [\rho(t) \ddot{\phi}(t) + 2\dot{\rho}(t) \dot{\phi}(t)] \hat{\mathbf{e}}_\phi$$

EXERCISE 1.5: Express the velocity and acceleration vectors in 2D polar coordinates.

Cylindrical coordinate system

Notes

- Cylindrical coordinates: $(u_1, u_2, u_3) = (\rho, \phi, z)$

ρ : polar distance from the z axis, $0 \leq \rho < \infty$

ϕ : azimuthal angle from the x axis on the xy -plane, $0 \leq \phi < 2\pi$

z : coordinate along the z axis, $-\infty < z < \infty$

- Coordinate transformation between cylindrical and Cartesian coordinates:

$$\begin{cases} x = \rho \cos \phi \\ y = \rho \sin \phi \\ z = z \end{cases} \Leftrightarrow \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

- Velocity and acceleration:

$$\begin{cases} \mathbf{v}(t) = \dot{\rho}(t) \hat{\mathbf{e}}_\rho + \rho(t) \dot{\phi}(t) \hat{\mathbf{e}}_\phi + \dot{z}(t) \hat{\mathbf{e}}_z \\ \mathbf{a}(t) = [\ddot{\rho}(t) - \rho(t) \dot{\phi}^2(t)] \hat{\mathbf{e}}_\rho + [\rho(t) \ddot{\phi}(t) + 2\dot{\rho}(t) \dot{\phi}(t)] \hat{\mathbf{e}}_\phi + \ddot{z}(t) \hat{\mathbf{e}}_z \end{cases}$$

Spherical coordinate system

Notes

- Spherical coordinates: $(u_1, u_2, u_3) = (r, \theta, \phi)$

r : radial distance from the origin, $0 \leq r < \infty$

θ : polar angle from the z axis, $0 \leq \theta \leq \pi$

ϕ : azimuthal angle from the x axis on the xy -plane, $0 \leq \phi < 2\pi$

- Coordinate transformation between spherical and Cartesian coordinates:

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

EXERCISE 1.6: Express the spherical unit basis vectors $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi)$ in terms of Cartesian unit basis vectors $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$.

Kinematics in spherical coordinates

Notes

- Position vector:

$$\mathbf{r}(t) = r(t) \hat{\mathbf{e}}_r$$

- Velocity vector:

$$\mathbf{v}(t) = \dot{r}(t) \hat{\mathbf{e}}_r + r(t) \dot{\theta}(t) \hat{\mathbf{e}}_\theta + r(t) \dot{\phi}(t) \sin \theta(t) \hat{\mathbf{e}}_\phi$$

- Acceleration vector:

$$\begin{aligned} \mathbf{a}(t) = & \left[\ddot{r}(t) - r(t) \dot{\phi}^2(t) \sin^2 \theta(t) - r(t) \dot{\theta}^2(t) \right] \hat{\mathbf{e}}_r \\ & + \left[r(t) \ddot{\theta}(t) + 2\dot{r}(t) \dot{\theta}(t) - r(t) \dot{\phi}^2(t) \sin \theta(t) \cos \theta(t) \right] \hat{\mathbf{e}}_\theta \\ & + \left[r(t) \ddot{\phi}(t) \sin \theta(t) + 2\dot{r}(t) \dot{\phi}(t) \sin \theta(t) + 2r(t) \dot{\theta}(t) \dot{\phi}(t) \cos \theta(t) \right] \hat{\mathbf{e}}_\phi \end{aligned}$$

Notes