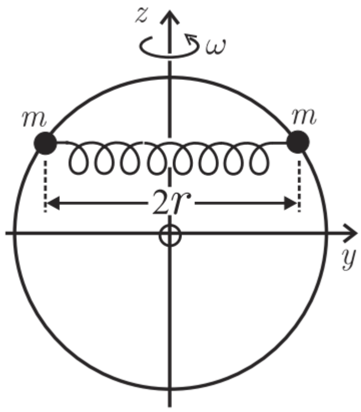
PC3261: Classical Mechanics II
Assignment 6

1. [40 pts] Consider a system consisting of two beads, a massless spring and a circular light hoop. The beads are connected by the spring and they slide without friction on the hoop. The hoop lies in the yz-plane with its center at the origin of the coordinate system. The yz-plane is horizontal and the spring is parallel to the y-axis. Each bead has mass m, the force constant of the spring is k and the radius of the hoop is R. The equilibrium length $2r_0$ of the spring is less that the diameter of the hoop, i.e. $r_0 < R$. Suppose the hoop rotates about the z-axis with a constant angular speed ω.



(a) Express the Lagrangian in terms of cylindrical coordinates and show that it can be written in the one-dimensional form:

$$\mathcal{L} = \frac{1}{2}\,\mu\dot{z}^2 - V_{\text{eff}}(z)\,,$$

where μ is the position-dependent effective mass and V_{eff} is the one-dimensional effective potential. Determine the expressions for μ and $V_{\text{eff}}(z)$.

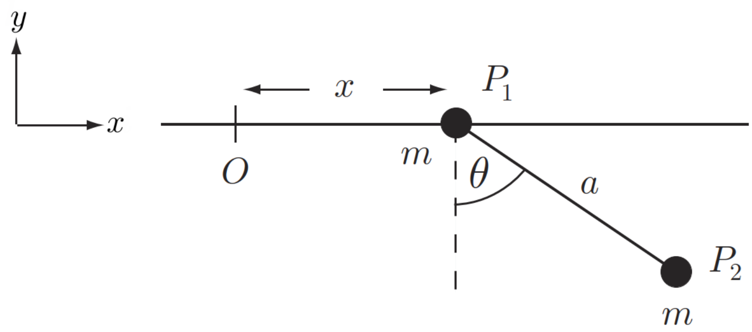
(b) Determine the equilibrium points z_ω of the beads. And, determine the stability of these equilibrium points. Show that there exists a critical angular speed ω_{crit} at which the stability of these equilibrium points should depend on whether $\omega > \omega_{\text{crit}}$, $\omega = \omega_{\text{crit}}$ or $\omega < \omega_{\text{crit}}$ respectively.

(c) Now, suppose the axis of rotation of the hoop is turned through an angle α about the y-axis. Determine the effect of a uniform gravitational field $\mathbf{g} = -g\,\hat{\mathrm{e}}_x$ on the above results.

(d) Determine the angular frequencies of small oscillations about the equilibrium points when $\alpha = 0$. Express the results in terms of ω, ω_{crit} and $\omega_0 \equiv \sqrt{2k/m}$.

Remark: This system is a mechanical equivalent of a thermodynamic system where the state is characterized by three physical variables: z_ω, ω and g. This simple system provides insights into topics like spontaneous symmetry breaking, phase transitions, order parameters and critical exponents in the thermodynamic system. The angular speed ω is analogous to the temperature in a thermodynamic system. The equilibrium positions z_ω play the role of an order parameter which 'spontaneously' acquires a non-zero value growing as $\sqrt{\omega_{crit} - \omega}$ just below ω_{crit}. The critical exponent for this order parameter is $1/2$ which is very familiar in the Landau theory for various systems with second-order phase transitions. The role of the gravitational force is to cause an explicit symmetry breaking illustrating the difference between explicit and spontaneous symmetry breaking.

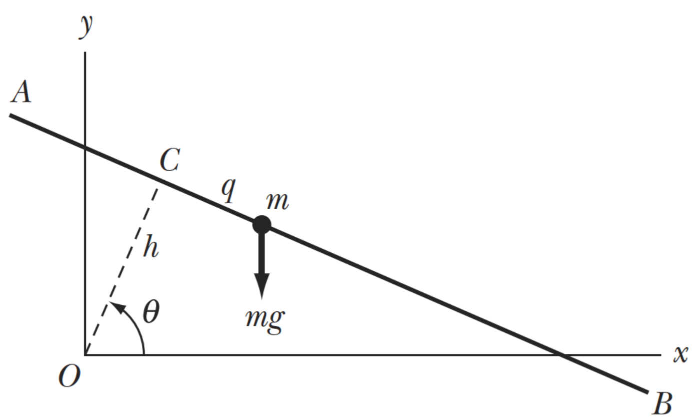
2. [30 pts] A system consists of two identical particles P_1 and P_2 of mass m connected by a light inextensible string of length a. The particle P_1 is constrained to move along a fixed smooth horizontal rail and the whole system moves under uniform gravity in the vertical plane through the rail.



(a) Using x and θ as generalized coordinates, find the Hamiltonian of the system.

(b) Hence, obtain differential equations for x and θ governing the dynamics of the system.

3. [30 pts] A particle of mass m can slide freely along a light wire AB whose perpendicular distance to the origin O is h. The line OC rotates about the origin at a constant angular speed Ω. The position of the particle can be described in terms of the angle θ and the distance q to the point C. The initial conditions are $\theta(0) = 0$, $q(0) = 0$ and $\dot{q}(0) = 0$.



(a) Using q as the generalized coordinate, find the Hamiltonian of the system.

(b) Is the total mechanical energy given by the Hamiltonian? Is the total mechanical energy conserved? Explain.

(c) Obtain equation of motion using Hamilton equation of motion. Solve for $q(t)$.

Last updated: October 28, 2023 10:25 am