# Conservative forces and potential energies

- A force $\mathbf{F}$ acting on a particle is **conservative** if and only if it satisfies two conditions:
  1. $\mathbf{F}$ depends only on particle's position $\mathbf{r}$, that is $\mathbf{F} = \mathbf{F}(\mathbf{r}(t))$
  2. For any two points $1$ and $2$, the work $W(1 \to 2)$ by force $\mathbf{F}$ is the same for all paths between $1$ and $2$

- **Potential energy** associated to a given conservative force is defined to be the negative of the work done by the conservative force if the particle moves from the *reference* point $\mathbf{r}_0$ to the point of interest $\mathbf{r}$

$$U(\mathbf{r}) \equiv -W\left(\mathbf{r}_0 \to \mathbf{r}\right) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}'$$

- **Total mechanical energy**, $E(t) \equiv U(\mathbf{r}(t)) + T(t)$, is conserved if all forces acting on the particle are conservative:

$$T(t_2) - T(t_1) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \quad \Rightarrow \quad U(\mathbf{r}(t_1)) + T(t_1) = U(\mathbf{r}(t_2)) + T(t_2)$$

# Potential energy for uniform gravitational force

- Work by a uniform force only depends on the net displacement, $\mathbf{r}_2 - \mathbf{r}_1$, not on the particular path taken from $\mathbf{r}_1$ to $\mathbf{r}_2$

- Uniform gravitational force:

$$\mathbf{F}(\mathbf{r}) = -mg\,\hat{\mathbf{e}}_z$$

- Potential energy associated with uniform gravitational field:

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = mg\,(z - z_0)$$

- Choosing the zero reference of gravitational potential energy at ground level $z_0 = 0$, then the uniform gravitational potential energy depends only on the height above the ground

# Conservative force and gradient of potential energy

- Infinitesimal work by a conservative force:

$$W\left(\mathbf{r}\to\mathbf{r}+\mathrm{d}\mathbf{r}\right)=\begin{cases}\mathbf{F}(\mathbf{r})\cdot\mathrm{d}\mathbf{r}=F_x(\mathbf{r})\,\mathrm{d}x+F_y(\mathbf{r})\,\mathrm{d}y+F_z(\mathbf{r})\,\mathrm{d}z\\[2ex] -\left[U(\mathbf{r}+\mathrm{d}\mathbf{r})-U(\mathbf{r})\right]=-\dfrac{\partial U(\mathbf{r})}{\partial x}\,\mathrm{d}x-\dfrac{\partial U(\mathbf{r})}{\partial y}\,\mathrm{d}y-\dfrac{\partial U(\mathbf{r})}{\partial z}\,\mathrm{d}z\end{cases}$$

- Conservative force in terms of gradient of potential energy:

$$\mathbf{F}(\mathbf{r})=-\frac{\partial U(\mathbf{r})}{\partial x}\,\hat{\mathbf{e}}_x-\frac{\partial U(\mathbf{r})}{\partial y}\,\hat{\mathbf{e}}_y-\frac{\partial U(\mathbf{r})}{\partial z}\,\hat{\mathbf{e}}_z=-\boldsymbol{\nabla}U(\mathbf{r})$$

- Total mechanical energy is a constant of motion:

$$E(t)\equiv U(\mathbf{r}(t))+T(t)\quad\Rightarrow\quad\frac{\mathrm{d}E(t)}{\mathrm{d}t}=0$$

**EXERCISE 5.4:** Show that the total mechanical energy with time-independent potential energy is a constant of motion.

$$E(t) \equiv U(\mathbf{r}(t)) + T(t)\,, \qquad \mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla} U(\mathbf{r})\,, \qquad \mathbf{F}(\mathbf{r}(t)) = m\ddot{\mathbf{r}}(t)$$

$$\begin{aligned}
\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[U(\mathbf{r}(t)) + \frac{m}{2}\,\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)\right] \\
&= \boldsymbol{\nabla} U(\mathbf{r}(t))\cdot\dot{\mathbf{r}}(t) + m\ddot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t) \\
&= 0 \qquad \blacksquare
\end{aligned}$$

# Elastic potential energy

- One dimensional force, $\mathbf{F}(\mathbf{r}) = F(x)\,\hat{\mathbf{e}}_x$, is always conservative (why??)

- Elastic force in one dimension: $k$ is the spring constant and $x_0$ is the equilibrium position

$$\mathbf{F}(\mathbf{r}) = -k\,(x - x_0)\;\hat{\mathbf{e}}_x$$

- Elastic potential energy: zero reference of elastic potential energy is chosen at equilibrium position

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}')\cdot \mathrm{d}\mathbf{r}' = \frac{1}{2}\,k\,(x - x_0)^2$$

- Elastic force from elastic potential energy:

$$\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla} U(\mathbf{r}) = -k\,(x - x_0)\;\hat{\mathbf{e}}_x$$

# Several conservative forces

- Total conservative forces on the particle: principle of superposition of forces

$$\mathbf{F}_{\mathsf{c}}(\mathbf{r}) = \sum_i \mathbf{F}_{\mathsf{c},i}(\mathbf{r})$$

- Work-energy theorem: all forces on the particle are conservative

$$T(t_2) - T(t_1) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \sum_i \mathbf{F}_{\mathsf{c},i}(\mathbf{r}(t)) \cdot \mathrm{d}\mathbf{r}$$

$$= \sum_i \left[ U_i(\mathbf{r}(t_1)) - U_i(\mathbf{r}(t_2)) \right]$$

- Total mechanical energy is a constant of motion:

$$E(t) \equiv \sum_i U_i(\mathbf{r}(t)) + T(t) \quad \Rightarrow \quad \frac{\mathrm{d}E(t)}{\mathrm{d}t} = 0$$

# Non-conservative forces

- Work on the particle by non-conservative forces:

$$W_{\mathsf{nc}}\left(\mathbf{r}_1 \to \mathbf{r}_2\right) = \int_{\mathcal{C}_{1\to 2}} \sum_j \mathbf{F}_{\mathsf{nc,j}}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} = \sum_j W_{\mathsf{nc},j}\left(\mathbf{r}_1 \to \mathbf{r}_2\right)$$

- Work-energy theorem: total mechanical energy is not conserved and the change in total mechanical energy is the work by non-conservative forces

$$\begin{aligned}
T(t_2) - T(t_1) &= \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} \\
&= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \sum_i \mathbf{F}_{\mathsf{c},i}(\mathbf{r}(t)) \cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_{1\to 2}} \sum_j \mathbf{F}_{\mathsf{nc},j}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} \\
&= \sum_i \left[U_i(\mathbf{r}(t_1)) - U_i(\mathbf{r}(t_2))\right] + W_{\mathsf{nc}}\left(\mathbf{r}_1 \to \mathbf{r}_2\right)
\end{aligned}$$

$$\Rightarrow \quad \sum_i U_i(\mathbf{r}(t_1)) + T(t_1) + W_{\mathsf{nc}}\left(\mathbf{r}_1 \to \mathbf{r}_2\right) = \sum_i U_i(\mathbf{r}(t_2)) + T(t_2)$$

# Condition for conservative forces

- **Stoke's theorem**: integral of the curl of a vector field over an open surface $\mathcal{S}$ is equal to the circulation of the vector field around the curve $\partial\mathcal{S}$ bounding the surface $\mathcal{S}$

$$\iint_{\mathcal{S}} [\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})] \cdot \mathrm{d}\mathbf{a} = \oint_{\partial\mathcal{S}} \mathbf{A}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

- Work by conservative force is the same for all paths between $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$W(\mathbf{r}_1 \to \mathbf{r}_2) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}'_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

$$\int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} - \int_{\mathcal{C}'_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0 \quad \Rightarrow \quad \oint \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0$$

- Conservative force is **irrotational**:

$$\boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}) = \mathbf{0}$$

# Spherically symmetric central force is conservative

- Spherically symmetric central force is irrotational:

$$\mathbf{F}(\mathbf{r}) = f(r)\,\hat{\mathbf{e}}_r \quad \Rightarrow \quad \boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}) = \mathbf{0}$$

- Potential energy function associated with spherically symmetric central force:

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = -\int_{r_0}^{r} f(r')\,\mathrm{d}r' \equiv U(r)$$

- Obtaining spherically symmetric central force from potential energy:

$$\mathbf{F}(r) = -\boldsymbol{\nabla} U(\mathbf{r}) = f(r)\,\hat{\mathbf{e}}_r$$

**EXERCISE 5.5:** The electrostatic force on a point charge $q$ located at $\mathbf{r}$ due to a fixed point charge $Q$ at the origin is given by $\mathbf{F}(\mathbf{r}) = Qq/\left(4\pi\epsilon_0 r^2\right)\,\hat{\mathbf{e}}_r$. Show that it is conservative and find the corresponding potential energy.

$$\mathbf{F}(\mathbf{r}) = f(r)\,\hat{\mathbf{e}}_r\,, \qquad U(\mathbf{r}) = -\int_{r_0}^{r} f(r')\,\mathrm{d}r'$$

$$\mathrm{d}\mathbf{r} = \mathrm{d}r\,\hat{\mathbf{e}}_r + r\,\mathrm{d}\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,\mathrm{d}\phi\,\hat{\mathbf{e}}_\phi \equiv h_1\,\mathrm{d}r\,\hat{\mathbf{e}}_r + h_2\,\mathrm{d}\theta\,\hat{\mathbf{e}}_\theta + h_3\,\mathrm{d}\phi\,\hat{\mathbf{e}}_\phi$$

$$\boldsymbol{\nabla}\times\mathbf{F}(\mathbf{r}) = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\,\hat{\mathbf{e}}_r & h_2\,\hat{\mathbf{e}}_\theta & h_3\,\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ h_1 F_r & h_2 F_\theta & h_3 F_\phi \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\,\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f(r) & 0 & 0 \end{vmatrix} = \mathbf{0} \qquad \blacksquare$$

$$\begin{aligned}\boldsymbol{\nabla}U(\mathbf{r}) &= \frac{1}{h_1}\,\frac{\partial U(\mathbf{r})}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{h_2}\,\frac{\partial U(\mathbf{r})}{\partial \theta}\,\hat{\mathbf{e}}_\theta + \frac{1}{h_3}\,\frac{\partial U(\mathbf{r})}{\partial \phi}\,\hat{\mathbf{e}}_\phi \\ &= \frac{\partial U(r)}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\,\frac{\partial U(r)}{\partial \theta}\,\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\,\frac{\partial U(r)}{\partial \phi}\,\hat{\mathbf{e}}_\phi \\ &= \frac{\mathrm{d}}{\mathrm{d}r}\left[-\int_{r_0}^{r} f(r')\,\mathrm{d}r'\right]\hat{\mathbf{e}}_r = -f(r)\,\hat{\mathbf{e}}_r \qquad \blacksquare\end{aligned}$$

$$\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla}U(\mathbf{r}) = f(r)\,\hat{\mathbf{e}}_r \qquad \blacksquare$$

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{Qq}{r^2}\,\hat{\mathbf{e}}_r \equiv f(r)\,\hat{\mathbf{e}}_r$$

$$\boldsymbol{\nabla}\times\mathbf{F}(\mathbf{r}) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\,\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f(r) & 0 & 0 \end{vmatrix} = \mathbf{0} \qquad \blacksquare$$

$$U(r) = -\int_{r_0}^{r} f(r')\,\mathrm{d}r' = -\frac{Qq}{4\pi\epsilon_0}\int_{r_0}^{r}\frac{1}{r'^2}\mathrm{d}r' = \frac{Qq}{4\pi\epsilon_0}\left(\frac{1}{r}-\frac{1}{r_0}\right) \qquad \blacksquare$$

# Time-dependent potential energy

- Irrotational time-dependent force:

$$\boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}, t) = \mathbf{0}$$

- Time-dependent potential energy:

$$U(\mathbf{r}, t) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}', t) \cdot \mathrm{d}\mathbf{r}'$$

- Total mechanical energy is *not* a constant of motion!

$$E(t) \equiv T(t) + U(\mathbf{r}(t), t) \quad \Rightarrow \quad \frac{\mathrm{d}E(t)}{\mathrm{d}t} \neq 0$$

**EXERCISE 5.6:** Show that the total mechanical energy with time-dependent potential energy is not a constant of motion.

$$T(t) = \frac{1}{2}\, m\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)\,, \qquad U(\mathbf{r},t) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}',t)\cdot \mathrm{d}\mathbf{r}'$$

$$E(t) = T(t) + U(\mathbf{r}(t),t) \quad \Rightarrow \quad \mathrm{d}E = \mathrm{d}T + \mathrm{d}U$$

$$\mathrm{d}T = m\ddot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)\,\mathrm{d}t = \mathbf{F}(\mathbf{r}(t),t)\cdot\dot{\mathbf{r}}(t)\,\mathrm{d}t = -\boldsymbol{\nabla}U(\mathbf{r}(t),t)\cdot\mathrm{d}\mathbf{r}$$

$$\mathrm{d}U = \boldsymbol{\nabla}U(\mathbf{r}(t),t)\cdot\mathrm{d}\mathbf{r} + \frac{\partial U(\mathbf{r}(t),t)}{\partial t}\,\mathrm{d}t$$

$$\mathrm{d}E = \mathrm{d}T + \mathrm{d}U = \frac{\partial U(\mathbf{r}(t),t)}{\partial t}\,\mathrm{d}t \quad \Rightarrow \quad \frac{\mathrm{d}E(t)}{\mathrm{d}t} = \frac{\partial U(\mathbf{r}(t),t)}{\partial t} \neq 0 \qquad \blacksquare$$

# Work-energy theorem for multi-particle system

- Total force acting on the $\alpha$-particle: $\mathbf{f}_{\alpha\beta}$ is the force acting on $m_\alpha$ due to $m_\beta$

$$\mathbf{F}_\alpha(t) = \mathbf{F}_\alpha^{\text{ext}}(t) + \sum_{\beta=1,\beta\neq\alpha}^{N} \mathbf{f}_{\alpha\beta}(t)\,, \qquad \alpha = 1, 2, 3, \cdots, N$$

- Total kinetic energy of multi-particle system:

$$T(t) \equiv \sum_{\alpha=1}^{N} \frac{1}{2}\, m_\alpha \mathbf{v}_\alpha(t) \cdot \mathbf{v}_\alpha(t)$$

- Work-energy theorem: total work by all external and internal forces during a given time interval is equal to the change in the kinetic energy of the multi-particle system during this time interval

$$T(t_2) - T(t_1) = \sum_{\alpha=1}^{N} \int_{t_1}^{t_2} \mathbf{F}_\alpha^{\text{ext}}(t) \cdot \dot{\mathbf{r}}_\alpha(t)\, \mathrm{d}t + \sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} \int_{t_1}^{t_2} \mathbf{f}_{\alpha\beta}(t) \cdot \dot{\mathbf{r}}_\alpha(t)\, \mathrm{d}t$$

# External conservative forces

- External conservative forces acting on the $\alpha$-particle:

$$\mathbf{F}^{\mathsf{ext}}_{\mathsf{c},\alpha}(t) = \sum_i \mathbf{F}_{\mathsf{c},i}(\mathbf{r}_\alpha(t)) = -\sum_i \boldsymbol{\nabla}_\alpha U_i(\mathbf{r}_\alpha(t))$$

- Total work by external conservative forces acting on the $\alpha$-particle:

$$W_{\mathsf{c},\alpha}\left(\mathbf{r}_{\alpha,1} \to \mathbf{r}_{\alpha,2}\right) = -\sum_i \int_{\mathbf{r}_{\alpha,1}}^{\mathbf{r}_{\alpha,2}} \boldsymbol{\nabla}_\alpha U_i(\mathbf{r}_\alpha(t)) \cdot \mathrm{d}\mathbf{r}_\alpha$$

$$= \sum_i \left[U_i\left(\mathbf{r}_\alpha(t_1)\right) - U_i\left(\mathbf{r}_\alpha(t_2)\right)\right]$$

- Total external potential energy of multi-particle system:

$$U^{\mathsf{ext}}(\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t)) \equiv \sum_{\alpha=1}^{N} \sum_i U_i\left(\mathbf{r}_\alpha(t)\right)$$

# External non-conservative forces

- External non-conservative forces acting on the $\alpha$-particle:

$$\mathbf{F}^{\text{ext}}_{\text{nc},\alpha}(t) = \sum_j \mathbf{F}_{\text{nc},j}(\mathbf{r}_\alpha(t), \dot{\mathbf{r}}_\alpha(t), t)$$

- Total work by external non-conservative forces acting on the $\alpha$-particle:

$$W_{\text{nc},\alpha}\left(\mathbf{r}_{\alpha,1} \to \mathbf{r}_{\alpha,2}\right) = \sum_j \int_{\mathcal{C}_{\mathbf{r}_{\alpha,1}\to\mathbf{r}_{\alpha,2}}} \mathbf{F}_{\text{nc},j}(\mathbf{r}_\alpha(t), \dot{\mathbf{r}}_\alpha(t), t) \cdot \mathrm{d}\mathbf{r}_\alpha$$

- Total work by external non-conservative forces acting on multi-particle system:

$$W_{\text{nc}}\left(\mathbf{r}_{1,1} \to \mathbf{r}_{1,2}, \cdots, \mathbf{r}_{N,1} \to \mathbf{r}_{N,2}\right) \equiv \sum_{\alpha=1}^{N} W_{\text{nc},\alpha}\left(\mathbf{r}_{\alpha,1} \to \mathbf{r}_{\alpha,2}\right)$$

# Internal forces

- Internal force acting on $\alpha$-particle due to $\beta$-particle is conservative:

$$\mathbf{f}_{\alpha\beta}(t) = -\boldsymbol{\nabla}_\alpha U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t)|\right) , \qquad \mathbf{r}_{\alpha\beta}(t) \equiv \mathbf{r}_\alpha(t) - \mathbf{r}_\beta(t)$$

- Total work by pair of internal forces:

$$\int_{t_1}^{t_2} \mathbf{f}_{\alpha\beta}(t)\cdot\dot{\mathbf{r}}_\alpha(t)\,\mathrm{d}t + \int_{t_1}^{t_2} \mathbf{f}_{\beta\alpha}(t)\cdot\dot{\mathbf{r}}_\beta(t)\,\mathrm{d}t = U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t_1)|\right) - U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t_2)|\right)$$

- Total work by internal forces:

$$\sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\int_{t_1}^{t_2} \mathbf{f}_{\alpha\beta}(t)\cdot\dot{\mathbf{r}}_\alpha(t)\,\mathrm{d}t = U^{\mathsf{int}}(\mathbf{r}_1(t_1),\cdots,\mathbf{r}_N(t_1)) - U^{\mathsf{int}}(\mathbf{r}_1(t_2),\cdots,\mathbf{r}_N(t_2))$$

$$U^{\mathsf{int}}(\mathbf{r}_1(t),\cdots,\mathbf{r}_N(t)) \equiv \sum_{\alpha=1}^{N}\sum_{\beta>\alpha}^{N} U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t)|\right)$$

$$U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t)|\right) = U_{\beta\alpha}\left(|\mathbf{r}_{\beta\alpha}(t)|\right) \;, \qquad \mathbf{r}_{\alpha\beta}(t) \equiv \mathbf{r}_\alpha(t) - \mathbf{r}_\beta(t)$$

$$\begin{aligned}
\mathbf{f}_{\alpha\beta}(t) &= -\boldsymbol{\nabla}_\alpha U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t)|\right) \\
&= +\boldsymbol{\nabla}_\beta U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t)|\right) \\
&= +\boldsymbol{\nabla}_\beta U_{\beta\alpha}\left(|\mathbf{r}_{\beta\alpha}(t)|\right) \\
&= -\mathbf{f}_{\beta\alpha}(t)
\end{aligned}$$

$$
\begin{aligned}
&\int_{t_1}^{t_2} \mathbf{f}_{\alpha\beta}(t) \cdot \dot{\mathbf{r}}_\alpha(t)\,\mathrm{d}t + \int_{t_1}^{t_2} \mathbf{f}_{\beta\alpha}(t) \cdot \dot{\mathbf{r}}_\beta(t)\,\mathrm{d}t \\
&= -\int_{t_1}^{t_2} \boldsymbol{\nabla}_\alpha U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t)|\right) \cdot \mathrm{d}\mathbf{r}_\alpha - \int_{t_1}^{t_2} \boldsymbol{\nabla}_\beta U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t)|\right) \cdot \mathrm{d}\mathbf{r}_\beta \\
&= -\int_{t_1}^{t_2} \mathrm{d}U_{\alpha\beta} \\
&= U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t_1)|\right) - U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t_2)|\right)
\end{aligned}
$$

# Work-energy theorem for multi-particle system – cont'd

- Total potential energy of multi-particle system:

$$U(\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t)) \equiv U^{\text{ext}}(\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t)) + U^{\text{int}}(\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t))$$

- Work-energy theorem:

$$\begin{aligned} &U(\mathbf{r}_1(t_1), \cdots, \mathbf{r}_N(t_1)) + T(t_1) + W_{\text{nc}}\left(\mathbf{r}_{1,1} \to \mathbf{r}_{1,2}, \cdots, \mathbf{r}_{N,1} \to \mathbf{r}_{N,2}\right) \\ &= U(\mathbf{r}_1(t_2), \cdots, \mathbf{r}_N(t_2)) + T(t_2) \end{aligned}$$

- Total mechanical energy is not conserved due to the time-dependent potential energies and/or work by non-conservative forces

# Example: A star with two planets

- Gravitational force acting on point mass $m_1$ due to another point mass $m_2$:

$$\mathbf{F}_{12}\left(\mathbf{r}_1(t)\right) = -\frac{Gm_1m_2}{\left|\mathbf{r}_1(t)-\mathbf{r}_2(t)\right|^3}\left[\mathbf{r}_1(t)-\mathbf{r}_2(t)\right]$$

- A star of very large mass $M$ is orbited by two planets of masses $m_1$ and $m_2$

$$U^{\text{ext}}(\mathbf{r}_1(t),\mathbf{r}_2(t)) = -\frac{GMm_1}{r_1(t)} - \frac{GMm_2}{r_2(t)}\,, \quad U^{\text{int}}(\mathbf{r}_1(t),\mathbf{r}_2(t)) = -\frac{Gm_1m_2}{r_{12}(t)}$$

- Total mechanical energy:

$$E(t) = \frac{1}{2}\,m_1\dot{\mathbf{r}}_1(t)\cdot\dot{\mathbf{r}}_1(t) + \frac{1}{2}\,m_2\dot{\mathbf{r}}_2(t)\cdot\dot{\mathbf{r}}_2(t) - GM\left[\frac{m_1}{r_1(t)}+\frac{m_2}{r_2(t)}\right] - \frac{Gm_1m_2}{r_{12}(t)}$$

- If $E(0)<0$, is it possible for a planet to escape to infinity?

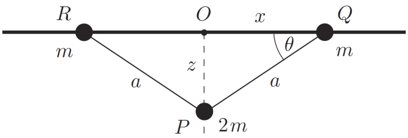
# Example: A constrained three-particle system

- A ball $P$ of mass $2m$ suspended by two light inextensible strings of length $a$ from two sliders $Q$ and $R$, each of mass $m$, which can move on a smooth horizontal rail. The system moves symmetrically so that $O$, the midpoint of $Q$ and $R$, remains fixed and $P$ moves on the downward vertical through $O$. Initially, the system is released from rest with the three particles in a straight line and with the strings taut. Ignore gravitational forces between masses.

- Tension forces exerted by the inextensible strings do zero work in total (WHY?)

- Total mechanical energy:

$$E(t) = ma^2\dot{\theta}^2(t) - 2mga\sin\theta(t)$$




**EXERCISE 5.7:** Derive the first order differential equation governing the dynamics of the system.

$$[\mathbf{r}_\alpha(t) - \mathbf{r}_\beta(t)] \cdot [\mathbf{r}_\alpha(t) - \mathbf{r}_\beta(t)] = \text{constant} \quad \Rightarrow \quad [\mathbf{r}_\alpha(t) - \mathbf{r}_\beta(t)] \cdot [\dot{\mathbf{r}}_\alpha(t) - \dot{\mathbf{r}}_\beta(t)] = 0$$

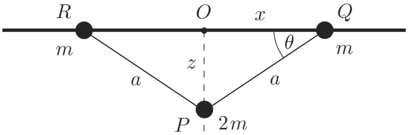
$$\begin{aligned}
&\int_{t_1}^{t_2} \mathbf{f}_{\alpha\beta}(t) \cdot \dot{\mathbf{r}}_\alpha(t)\,\mathrm{d}t + \int_{t_1}^{t_2} \mathbf{f}_{\beta\alpha}(t) \cdot \dot{\mathbf{r}}_\beta(t)\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \mathbf{f}_{\alpha\beta}(t) \cdot [\dot{\mathbf{r}}_\alpha(t) - \dot{\mathbf{r}}_\beta(t)]\,\mathrm{d}t \\
&= 0 \qquad \blacksquare
\end{aligned}$$

$$
\begin{cases}
\mathbf{r}_1(t) = \mathbf{r}_R(t) = -a\cos\theta(t)\,\hat{\mathbf{e}}_y \\
\mathbf{r}_2(t) = \mathbf{r}_Q(t) = +a\cos\theta(t)\,\hat{\mathbf{e}}_y \\
\mathbf{r}_3(t) = \mathbf{r}_P(t) = -a\sin\theta(t)\,\hat{\mathbf{e}}_z
\end{cases}
\quad\Rightarrow\quad
\begin{cases}
\dot{\mathbf{r}}_1(t) = +a\dot{\theta}(t)\sin\theta(t)\,\hat{\mathbf{e}}_y \\
\dot{\mathbf{r}}_2(t) = -a\dot{\theta}(t)\sin\theta(t)\,\hat{\mathbf{e}}_y \\
\dot{\mathbf{r}}_3(t) = -a\dot{\theta}(t)\cos\theta(t)\,\hat{\mathbf{e}}_z
\end{cases}
$$

$$
T(t) = \sum_{\alpha=1}^{3} \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha(t)\cdot\dot{\mathbf{r}}_\alpha(t) = ma^2\dot{\theta}^2(t)
$$

$$
\mathbf{F}_{\mathsf{c}}^{\mathsf{ext}}(t) = \sum_{\alpha=1}^{3} \mathbf{F}_{\mathsf{c},\alpha}^{\mathsf{ext}}(\mathbf{r}_\alpha(t)) = -mg\,\hat{\mathbf{e}}_z - mg\,\hat{\mathbf{e}}_z - 2mg\,\hat{\mathbf{e}}_z
$$

$$
U^{\mathsf{ext}}(\mathbf{r}_1(t),\mathbf{r}_2(t),\mathbf{r}_3(t)) = \sum_{\alpha=1}^{3} U(\mathbf{r}_\alpha(t)) = 0 + 0 - 2mga\sin\theta(t)
$$



$$\mathbf{F}^{\mathsf{ext}}_{\mathsf{nc}}(t) = \sum_{\alpha=1}^{3} \mathbf{F}^{\mathsf{ext}}_{\mathsf{nc},\alpha}(t) = N_1(t)\,\hat{\mathbf{e}}_z + N_2(t)\,\hat{\mathbf{e}}_z$$

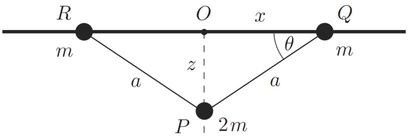
$$W_{\mathsf{nc}}\left(\mathbf{r}_{1,1} \to \mathbf{r}_{1,2}, \mathbf{r}_{2,1} \to \mathbf{r}_{2,2}, \mathbf{r}_{3,1} \to \mathbf{r}_{3,2}\right)$$

$$= \int_{\mathcal{C}_{\mathbf{r}_{1,1}\to\mathbf{r}_{1,2}}} N_1(t)\,\hat{\mathbf{e}}_z \cdot \dot{\mathbf{r}}_1(t)\,\mathrm{d}t + \int_{\mathcal{C}_{\mathbf{r}_{2,1}\to\mathbf{r}_{2,2}}} N_2(t)\,\hat{\mathbf{e}}_z \cdot \dot{\mathbf{r}}_2(t)\,\mathrm{d}t = 0$$

$$\int_{t_1}^{t_2} \mathbf{T}_{13}(t) \cdot \dot{\mathbf{r}}_1(t)\,\mathrm{d}t + \int_{t_1}^{t_2} \mathbf{T}_{31}(t) \cdot \dot{\mathbf{r}}_3(t)\,\mathrm{d}t = 0$$

$$\int_{t_1}^{t_2} \mathbf{T}_{23}(t) \cdot \dot{\mathbf{r}}_2(t)\,\mathrm{d}t + \int_{t_1}^{t_2} \mathbf{T}_{32}(t) \cdot \dot{\mathbf{r}}_3(t)\,\mathrm{d}t = 0$$

$$U^{\mathsf{int}}(\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t)) = 0$$



$$
\begin{aligned}
U(\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t)) &= U^{\mathsf{ext}}(\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t)) + U^{\mathsf{int}}(\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t)) \\
&= -2mga\sin\theta(t)
\end{aligned}
$$

$$
E(t) = U(\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t)) + T(t) = -2mga\sin\theta(t) + ma^2\dot{\theta}^2(t)
$$

$$
E \equiv E(0) = 0
$$

$$
E(t) = E \quad \Rightarrow \quad -2mga\sin\theta(t) + ma^2\dot{\theta}^2(t) = 0 \quad \Rightarrow \quad \dot{\theta}^2(t) - \frac{2g}{a}\sin\theta(t) = 0 \qquad \blacksquare
$$

# Kinetic energy of multi-particle system

- Total kinetic energy of multi-particle system:

$$T(t) \equiv \sum_{\alpha=1}^{N} T_\alpha(t) = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha(t) \cdot \dot{\mathbf{r}}_\alpha(t)$$

- Total kinetic energy of multi-particle system in the center-of-mass frame:

$$T'(t) \equiv \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\mathbf{r}}'_\alpha(t) \cdot \dot{\mathbf{r}}'_\alpha(t)$$

- Total kinetic energy of multi-particle system equals to the sum of kinetic energy of the center-of-mass and kinetic energy relative to the center-of-mass frame:

$$T(t) = T_{\mathsf{CM}}(t) + T'(t) = \frac{1}{2} M \dot{\mathbf{R}}_{\mathsf{CM}}(t) \cdot \dot{\mathbf{R}}_{\mathsf{CM}}(t) + \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\mathbf{r}}'_\alpha(t) \cdot \dot{\mathbf{r}}'_\alpha(t)$$

$$T(t) = \sum_{\alpha=1}^{N} \frac{1}{2}\, m_\alpha \dot{\mathbf{r}}_\alpha(t) \cdot \dot{\mathbf{r}}_\alpha(t)\,, \qquad \mathbf{r}_\alpha(t) = \mathbf{R}_{\mathsf{CM}}(t) + \mathbf{r}'_\alpha(t)$$

$$\begin{aligned}
T(t) &= \sum_{\alpha=1}^{N} \frac{1}{2}\, m_\alpha \dot{\mathbf{r}}_\alpha(t) \cdot \dot{\mathbf{r}}_\alpha(t) \\
&= \sum_{\alpha=1}^{N} \frac{1}{2}\, m_\alpha \left[\dot{\mathbf{R}}_{\mathsf{CM}}(t) + \dot{\mathbf{r}}'_\alpha(t)\right] \cdot \left[\dot{\mathbf{R}}_{\mathsf{CM}}(t) + \dot{\mathbf{r}}'_\alpha(t)\right] \\
&= \frac{1}{2}\left(\sum_{\alpha=1}^{N} m_\alpha\right) \dot{\mathbf{R}}_{\mathsf{CM}}(t) \cdot \dot{\mathbf{R}}_{\mathsf{CM}}(t) + \dot{\mathbf{R}}_{\mathsf{CM}}(t) \cdot \left(\sum_{\alpha=1}^{N} m_\alpha \dot{\mathbf{r}}'_\alpha(t)\right) + \sum_{\alpha=1}^{N} \frac{1}{2}\, m_\alpha \dot{\mathbf{r}}'_\alpha(t) \cdot \dot{\mathbf{r}}'_\alpha(t) \\
&= \frac{1}{2}\, M \dot{\mathbf{R}}_{\mathsf{CM}}(t) \cdot \dot{\mathbf{R}}_{\mathsf{CM}}(t) + \sum_{\alpha=1}^{N} \frac{1}{2}\, m_\alpha \dot{\mathbf{r}}'_\alpha(t) \cdot \dot{\mathbf{r}}'_\alpha(t) \qquad \blacksquare
\end{aligned}$$