PC3261: Classical Mechanics II

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Lecture 7: Lagrange's Equation

Lagrange multipliers

- Holonomic constraints:

$$f_i(\mathbf{r}_1(t), \cdots, \mathbf{r}_\alpha(t), t) = 0\,, \qquad i = 1, 2, \cdots, C$$

- Constrained force: $\lambda_i(t)$ is known as the **Lagrange multipliers**

$$\mathbf{F}_\alpha^{(\mathsf{C})}(t) \equiv \sum_{i=1}^{C} \lambda_i(t)\,\frac{\partial f_i}{\partial \mathbf{r}_\alpha}$$

- d'Alembert's principle with Lagrange multipliers: all virtual displacements $\delta\mathbf{r}_\alpha$ are now be treated as independent with the introduction of Lagrange multipliers

$$\sum_\alpha \left[\mathbf{F}_\alpha^{(\mathsf{A})}(t) + \sum_{i=1}^{C} \lambda_i(t)\,\frac{\partial f_i}{\partial \mathbf{r}_\alpha} - m_\alpha \ddot{\mathbf{r}}_\alpha(t)\right] \cdot \delta\mathbf{r}_\alpha = 0$$

Example: Atwood machine (another visit)

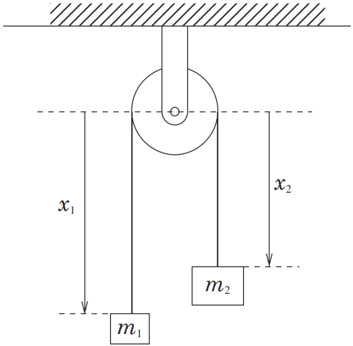
- Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless and frictionless pulley

- Holonomic constraint:

$$f(\mathbf{r}_1, \mathbf{r}_2, t) = x_1(t) + x_2(t) - \ell = 0$$

- Applied forces:

$$\mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g\,\hat{\mathbf{e}}_1\,, \qquad \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g\,\hat{\mathbf{e}}_2$$



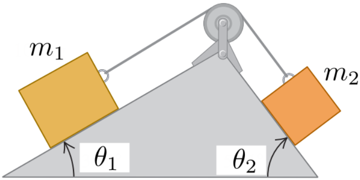
EXERCISE 7.1: Use d'Alembert's principle with Lagrange multipliers to find the constrained forces.

Example: Double-inclined plane (another visit)

- Two masses m_1 and m_2 are located each on a smooth double inclined plane with angles θ_1 and θ_2 respectively. The masses are connected by a massless and inextensible string running over a massless and frictionless pulley

- Holonomic constraints:

$$\begin{cases} f_1(\mathbf{r}_1, \mathbf{r}_2, t) = x_1(t) + x_2(t) - \ell = 0 \\ f_2(\mathbf{r}_1, \mathbf{r}_2, t) = y_1(t) = 0 \\ f_3(\mathbf{r}_1, \mathbf{r}_2, t) = y_2(t) = 0 \end{cases}$$



- Applied forces:

$$\mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g \sin\theta_1\, \hat{\mathbf{e}}_{x_1} - m_1 g \cos\theta_1\, \hat{\mathbf{e}}_{y_1}\ ,\ \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g \sin\theta_2\, \hat{\mathbf{e}}_{x_2} - m_2 g \cos\theta_2\, \hat{\mathbf{e}}_{y_2}$$

EXERCISE 7.2: Use d'Alembert's principle with Lagrange multipliers to find the constrained forces.

Degrees of freedom

- **Degrees of freedom** is the *minimum* number of *independent* coordinates that can completely specify the configuration of the mechanical system

- Holonomic constraints reduce the number of degrees of freedom of the mechanical system

- Example: a system of two particles moving in the space connected by a rigid rod of fixed length has five degrees of freedom

$$|\mathbf{r}_2 - \mathbf{r}_1|^2 - \ell^2 = 0 \quad \Rightarrow \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - \ell^2 = 0$$

- A holonomic system is a mechanical system whose constraints are all holonomic and it has as many degrees of freedom as independent coordinates necessary to specify its configuration at any instant

Generalized coordinates

- **Generalized coordinates**: a *minimal* set of independent coordinates $\{q_k\}$ to specify the configuration of the mechanical system at any instant of time

- A mechanical system consisting of N particles subject to the C holonomic constraints can be described by $M = 3N - C$ generalized coordinates $\{q_k\}$:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, q_2, \cdots, q_M, t) \ , \qquad \alpha = 1, 2, \cdots, N$$

- Generalized coordinates (θ, ϕ) for a particle restricted to the surface of a sphere with radius R undergoing uniform motion at velocity $\mathbf{u}$ relative to an inertial reference frame

$$\begin{cases} x = u_x t + R\sin\theta\cos\phi \\ y = u_y t + R\sin\theta\sin\phi \\ z = u_z t + R\cos\theta \end{cases} \quad \Rightarrow \quad (x - u_x t)^2 + (y - u_y t)^2 + (z - u_z t)^2 - R^2 = 0$$

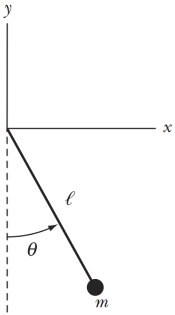
Example: Plane pendulum

- A point particle of mass m attached to a massless rod of length ℓ rotates about a frictionless pivot in a plane

- Holonomic constraints: particle is constrained to move in the xy-plane and length of the rod is fixed

$$\begin{cases} f_1(x, y, z, t) = z(t) = 0 \\ f_2(x, y, z, t) = x^2(t) + y^2(t) - \ell^2 = 0 \end{cases}$$

- One degree of freedom; two possible generalized coordinates are: (1) $q_1 = x$; and (2) $q_1 = \theta$



EXERCISE 7.3: Use d'Alembert's principle to obtain respective equations of motion for $x(t)$ and $\theta(t)$.

Generalized forces

- Generalized coordinates:

$$\mathbf{r}_\alpha \equiv \mathbf{r}_\alpha(\{q_k(t)\}, t) \;, \qquad \alpha = 1, 2, \cdots, N \,, \qquad k = 1, 2, \cdots, M$$

- Generalized forces:

$$\delta W = \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{(\mathsf{A})}(t) \cdot \delta \mathbf{r}_\alpha \equiv \sum_{k=1}^{M} \mathcal{Q}_k(t)\, \delta q_k \quad \Rightarrow \quad \mathcal{Q}_k(t) \equiv \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{(\mathsf{A})}(t) \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}$$

- Example: generalized forces associated to polar coordinates

$$\begin{cases} \mathcal{Q}_1(t) \equiv \mathcal{Q}_\rho(t) = F_x(t) \cos\phi(t) + F_y(t) \sin\phi(t) = F_\rho(t) \\ \mathcal{Q}_2(t) \equiv \mathcal{Q}_\phi(t) = -\rho(t)\, F_x(t) \sin\phi(t) + \rho(t)\, F_y(t) \cos\phi(t) = \rho(t)\, F_\phi(t) \end{cases}$$

Generalized velocities

- **Generalized velocity** associated to each generalized coordinate: $\{q_k(t), \dot{q}_k(t)\}$ are to be treated as a set of independent dynamical variables

$$\dot{q}_k(t) \equiv \frac{\mathrm{d}q_k(t)}{\mathrm{d}t}\,, \qquad k = 1, 2, \cdots, M$$

- Relationship between Cartesian velocity and generalized velocity:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(\{q_k(t)\}, t) \quad \Rightarrow \quad \dot{\mathbf{r}}_\alpha(t) = \sum_{k=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\,\dot{q}_k(t) + \frac{\partial \mathbf{r}_\alpha}{\partial t}$$

- Dot-cancellation rule: Cartesian velocity is related to the generalized velocity in the same way as the Cartesian coordinate is related to the generalized coordinate

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}_\alpha}{\partial q_k}$$

Rewriting d-Alembert's principle

- Useful result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right) = \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_k}$$

- Rewriting the inertial force term in the d'Alembert's principle:

$$\begin{aligned}
-\sum_{\alpha=1}^{N} m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha &= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right)\right]\delta q_k \\
&= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha\right)\right] - \frac{\partial}{\partial q_k}\left(\frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha\right)\right\}\delta q_k
\end{aligned}$$

EXERCISE 7.4: Obtain the expression for the inertial force term in the d'Alembert's principle.

Lagrange's equation

- Kinetic energy in terms of generalized coordinates and generalized velocities:

$$T(t) = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha(t) \cdot \dot{\mathbf{r}}_\alpha(t) \xrightarrow{\mathbf{r}_\alpha = \mathbf{r}_\alpha(\{q_k(t)\}, t)} T \equiv T(t) \equiv T\left(\{q_k, \dot{q}_k(t)\}, t\right)$$

- d'Alembrt's principle in terms of generalized coordinates:

$$\sum_{\alpha=1}^{N} \left[\mathbf{F}^{(\mathsf{A})}(t) - m_\alpha \ddot{\mathbf{r}}_\alpha(t)\right] \cdot \delta \mathbf{r}_\alpha = 0 \quad \rightarrow \quad \sum_{i=1}^{M} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - \mathcal{Q}_i\right] \delta q_i = 0$$

- Lagrange's equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T\left(\{q_k(t), \dot{q}_k(t)\}, t\right)}{\partial \dot{q}_k}\right] - \frac{\partial T\left(\{q_k(t), \dot{q}_k(t)\}, t\right)}{\partial q_k} = \mathcal{Q}_k(t), \quad k = 1, 2, \cdots, M$$

Single particle in two dimensions

- Cartesian coordinates: $(q_1, q_2) \equiv (x, y)$

$$T \equiv T(x, y, \dot{x}, \dot{y}, t) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

- Generalized forces:

$$\mathcal{Q}_k(t) = \mathbf{F}(t) \cdot \frac{\partial \mathbf{r}}{\partial q_k} \quad \Rightarrow \quad \begin{cases} \mathcal{Q}_1(t) \equiv \mathcal{Q}_x(t) = \mathbf{F}(t) \cdot \dfrac{\partial \mathbf{r}}{\partial x} = \mathbf{F}(t) \cdot \hat{\mathbf{e}}_x = F_x(t) \\ \mathcal{Q}_2(t) \equiv \mathcal{Q}_y(t) = \mathbf{F}(t) \cdot \dfrac{\partial \mathbf{r}}{\partial y} = \mathbf{F}(t) \cdot \hat{\mathbf{e}}_y = F_y(t) \end{cases}$$

- Equations of motion:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{x}}\right) - \dfrac{\partial T}{\partial x} = \mathcal{Q}_x(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{y}}\right) - \dfrac{\partial T}{\partial y} = \mathcal{Q}_y(t) \end{cases} \quad \Rightarrow \quad \begin{cases} m\ddot{x}(t) = F_x(t) \\ m\ddot{y}(t) = F_y(t) \end{cases}$$

Single particle in two dimensions – cont'd

- Polar coordinates: $(q_1, q_2) \equiv (\rho, \phi)$

$$T \equiv T(\rho, \phi, \dot{\rho}, \dot{\phi}, t) = \frac{m}{2}\,\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right)$$

- Generalized forces:

$$\mathcal{Q}_1(t) \equiv \mathcal{Q}_\rho(t) = F_\rho(t)\,, \qquad \mathcal{Q}_2(t) \equiv \mathcal{Q}_\phi(t) = \rho(t)\,F_\phi(t)$$

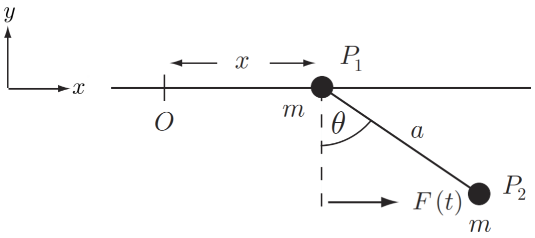
- Equations of motion:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{\rho}}\right) - \dfrac{\partial T}{\partial \rho} = \mathcal{Q}_\rho(t) \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T}{\partial \dot{\phi}}\right) - \dfrac{\partial T}{\partial \phi} = \mathcal{Q}_\phi(t) \end{cases} \Rightarrow \begin{cases} m\ddot{\rho}(t) - m\rho(t)\,\dot{\phi}^2(t) = F_\rho(t) \\[2ex] m\rho^2(t)\,\ddot{\phi}(t) + 2m\rho(t)\,\dot{\rho}(t)\,\dot{\phi}(t) = \rho(t)\,F_\phi(t) \end{cases}$$

Example: A constrained two-particle system

- Two identical particles, P_1 and P_2, with mass m are connected by a light rigid rod of length a. P_1 is constrained to move along a fixed horizontal frictionless rail and the system moves in the vertical plane through the rail. An external force $F(t)\,\hat{\mathbf{e}}_x$ is acted on P_2

- Generalized coordinates: $(q_1, q_2) \equiv (x, \theta)$

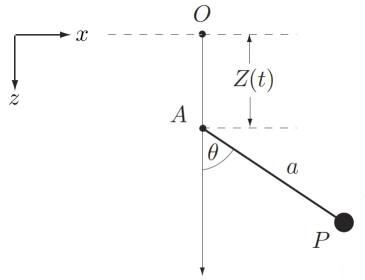


EXERCISE 7.5: Use Lagrange's equation to obtain equations of motions for $x(t)$ and $\theta(t)$.

Example: Pendulum with an oscillating pivot

- A simple pendulum in which the pivot is made to move vertically so that its distance from the fixed origin at time t is $Z(t) = Z_0 \cos \Omega t$. The string is a light rigid rod of length a that cannot go slack

- Generalized coordinate: $q_1 \equiv \theta$



EXERCISE 7.6: Use Lagrange's equation to obtain equations of motion for $\theta(t)$.