PC3261: Classical Mechanics II

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Lecture 8: Lagrangian Mechanics I

Conservative systems

- Applied forces are conservative:

$$U \equiv U(\{\mathbf{r}_\alpha(t)\}) \ , \qquad \mathbf{F}^{(\mathsf{A})}_\alpha(t) = -\frac{\partial U}{\partial \mathbf{r}_\alpha}$$

- Generalized forces: $U \equiv U(\{q_i\}) = U(\{\mathbf{r}_\alpha(q_i(t))\})$

$$\mathcal{Q}_k(t) = \sum_{\alpha=1}^{N} \mathbf{F}^{(\mathsf{A})}_\alpha(t) \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k} = -\sum_{\alpha=1}^{N} \frac{\partial U}{\partial \mathbf{r}_\alpha} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k} = -\frac{\partial U}{\partial q_k}$$

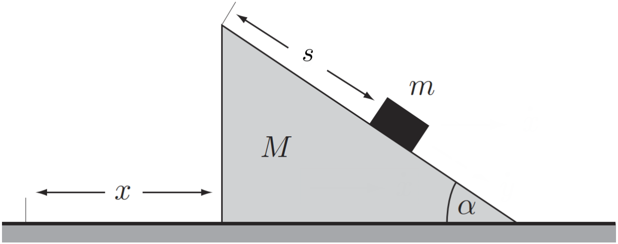
- Lagrange's equation for conservative systems:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial U}{\partial q_k} \, , \qquad k = 1, 2, \cdots, M$$

Example: A block sliding on a wedge

- A block of mass m is free to slide on the wedge of mass M which can slide on the horizontal table, both with negligible friction

- Generalized coordinates: s is the distance of the block from the top of the wedge and x is the distance of the wedge from any convenient *fixed* point on the table



EXERCISE 8.1: Find the acceleration of the wedge, and acceleration of the block relative to the wedge from Lagrange's equation.

Euler-Lagrange equation

- Rewriting Lagrange's equation for conservative systems:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_k}(T-U)\right]-\frac{\partial}{\partial q_k}(T-U)=0$$

- **Lagrange function** (or **Lagrangian**) for conservative systems:

$$\mathcal{L}\equiv\mathcal{L}\left(\left\{q_i(t),\dot{q}_i(t)\right\},t\right)\equiv T\left(\left\{q_i(t),\dot{q}_i(t)\right\},t\right)-U\left(\left\{q_i(t)\right\}\right)$$

- **Euler-Lagrange equation**: M second-order coupled ODEs, $2M$ initial conditions $\{q_i(0),\dot{q}_i(0)\}$ are required to determine *uniquely* $\{q_i(t)\}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right)-\frac{\partial \mathcal{L}}{\partial q_k}=0\,,\qquad k=1,2,\cdots,M$$

- T and U must both be expressed relative to some inertial reference frame

Single particle in three dimensions

- Lagrange function: Cartesian coordinates

$$T \equiv T(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) = \frac{1}{2}\, m \left[\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)\right] , \qquad U \equiv U(\mathbf{r}(t)) = U(x, y, z)$$

$$\mathcal{L} \equiv \mathcal{L}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \equiv T - U = \frac{1}{2}\, m \left[\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)\right] - U(x, y, z)$$

- Euler-Lagrange equation of motion: three second-order ODEs

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{x}}\right) - \dfrac{\partial L}{\partial x} = 0 \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{y}}\right) - \dfrac{\partial L}{\partial y} = 0 \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{z}}\right) - \dfrac{\partial L}{\partial z} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} m\ddot{x}(t) + \dfrac{\partial U}{\partial x} = 0 \\[2ex] m\ddot{y}(t) + \dfrac{\partial U}{\partial y} = 0 \\[2ex] m\ddot{z}(t) + \dfrac{\partial U}{\partial z} = 0 \end{cases} \quad \Rightarrow \quad m\ddot{\mathbf{r}}(t) = -\boldsymbol{\nabla} U$$

Example: Plane double pendulum

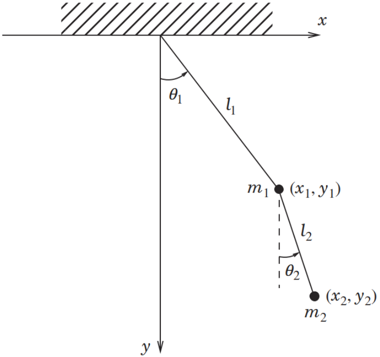
- A plane double pendulum consists of two light and inextensible rods of lengths ℓ_1 and ℓ_2 respectively. Two point masses, m_1 and m_2, are respectively attached at the end of each rod

- Holonomic constraints:

$$\begin{cases} f_1 = x_1^2 + y_1^2 - \ell_1^2 = 0 \\ f_2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - \ell_2^2 = 0 \end{cases}$$

- Generalized coordinates:

$$(q_1, q_2) \equiv (\theta_1, \theta_2)$$



EXERCISE 8.2: Obtain the equations of motion for the plane double pendulum from the Euler-Lagrange equation.

Generalized coordinates and velocities

- Configuration of a system can be geometrically represented by a *single* point in an M-dimensional space known as **configuration manifold** $\mathbb{Q}$

$$(q_1, q_2, q_3, \cdots, q_M)$$

- Euler-Lagrange equation is set of M second-order ODEs on $\mathbb{Q}$: Hessian matrix $\partial^2 \mathcal{L} / \partial \dot{q}_i \, \partial \dot{q}_k$ must be non-singular

$$\sum_{i=1}^{M} \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \, \partial \dot{q}_k} \, \ddot{q}_i + \frac{\partial^2 \mathcal{L}}{\partial q_i \, \partial \dot{q}_k} \, \dot{q}_i \right) + \frac{\partial^2 \mathcal{L}}{\partial t \, \partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = 0 \,, \qquad k = 1, 2, \cdots, M$$

- Solution of the Euler-Lagrange equation is represented by a curve parameterized by t on $\mathbb{Q}$

$$(q_1(t), q_2(t), q_3(t), \cdots, q_M(t))$$

Generalized coordinates and velocities– cont'd

- Lagrangian is a function of both generalized coordinates and generalized velocities, $(\{q_i, \dot{q}_i\})$, living in a $2M$-dimensional space known as **tangent bundle**, $\mathbf{T}\mathbb{Q}$, of $\mathbb{Q}$; $\mathbf{T}\mathbb{Q}$ is obtained from $\mathbb{Q}$ by adjoining to each point $q \in \mathbb{Q}$ the *tangent space* $\mathbf{T}_q\mathbb{Q}$, of all possible generalized velocities at q

- Euler-Lagrange equation is a set of $2M$ first order ODEs on $\mathbf{T}\mathbb{Q}$:

$$\begin{cases} \dfrac{\mathrm{d}q_k}{\mathrm{d}t} = \dot{q}_k \\[2ex] \dfrac{\mathrm{d}\dot{q}_k}{\mathrm{d}t} = G_k(\{q_i, \dot{q}_i\}, t) \end{cases}, \qquad i, k = 1, 2, \cdots, M$$

- Solution of the Euler-Lagrange equation is represented by a curve parameterized by t on $\mathbf{T}\mathbb{Q}$:

$$(q_1(t), q_2(t), q_3(t), \cdots, q_M(t), \dot{q}_1(t), \dot{q}_2(t), \dot{q}_3(t), \cdots, \dot{q}_M(t))$$

Point transformation

- **Point transformation**: coordinate transformation between two different sets of generalized coordinates

$$q_j = q_j(\{\bar{q}_i\}, t) \quad \leftrightarrow \quad \bar{q}_i = \bar{q}_i(\{q_j\}, t), \qquad i, j = 1, 2, \cdots, M$$

- Jacobian determinant: $M \times M$ matrix

$$\frac{\partial(q_1, q_2, \cdots, q_M)}{\partial(\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_M)} \equiv \begin{vmatrix} \dfrac{\partial q_1}{\partial \bar{q}_1} & \dfrac{\partial q_1}{\partial \bar{q}_2} & \cdots & \dfrac{\partial q_1}{\partial \bar{q}_M} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial q_M}{\partial \bar{q}_1} & \dfrac{\partial q_M}{\partial \bar{q}_2} & \cdots & \dfrac{\partial q_M}{\partial \bar{q}_M} \end{vmatrix} \neq 0$$

- Point transformation is assumed to be invertible

Point transformation – cont'd

- Generalized velocities under point transformation:

$$\overline{q}_i = \overline{q}_i(\{q_j\}, t) \quad \Rightarrow \quad \dot{\overline{q}}_i = \sum_{j=1}^{M} \frac{\partial \overline{q}_i}{\partial q_j} \dot{q}_j + \frac{\partial \overline{q}_i}{\partial t} \quad \Rightarrow \quad \frac{\partial \dot{\overline{q}}_i}{\partial \dot{q}_j} = \frac{\partial \overline{q}_i}{\partial q_j}$$

- Covariance of Euler-Lagrange equation of motion under point transformation:

$$\left.\begin{array}{c} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial \mathcal{L}(\{q_k(t), \dot{q}_k(t)\}, t)}{\partial \dot{q}_i}\right] - \dfrac{\partial \mathcal{L}(\{q_k(t), \dot{q}_k(t)\}, t)}{\partial q_i} = 0 \\ \Updownarrow \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial \overline{\mathcal{L}}(\{\overline{q}_k(t), \dot{\overline{q}}_k(t)\}, t)}{\partial \dot{\overline{q}}_i}\right] - \dfrac{\partial \overline{\mathcal{L}}(\{\overline{q}_k(t), \dot{\overline{q}}_k(t)\}, t)}{\partial \overline{q}_i} = 0 \end{array}\right. , \qquad i, k = 1, 2, \cdots, M$$

EXERCISE 8.3: Show that the Euler-Lagrange equation of motion is covariant under point transformation.

Example: Cycloidal pendulum

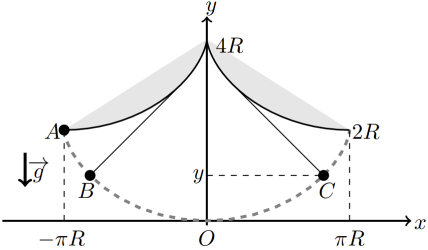
- Huygen (1673) constructed a cycloidal pendulum with a point particle of mass m and a string of length $4R$ suspended from the cusp of an inverted cycloid

- Path of point mass is a cycloid:

$$\begin{cases} x = R\,(\theta + \sin\theta) \\ y = R\,(1 - \cos\theta) \end{cases}, \qquad -\pi \leq \theta \leq \pi$$

- Period is independent of the amplitude!

$$T = 4\pi\sqrt{\frac{R}{g}}$$



EXERCISE 8.4: Obtain the equation of motion for cycloidal pendulum from Euler-Lagrange equation.

Generalized momenta

- **Generalized momenta**:

$$\mathcal{L} = \mathcal{L}(\{q_k, \dot{q}_k\}, t) \quad \Rightarrow \quad p_k \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$$

- If the Lagrange function does not depend on q_k *explicitly*, then the generalized coordinate q_k is called **cyclic coordinate** and the corresponding generalized momenta p_k is a constant of motion

$$\frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}p_k}{\mathrm{d}t} = 0$$

- Choice of generalized coordinates is adopted so that there are as many cyclic coordinates as possible and their corresponding generalized momenta are constants of motion

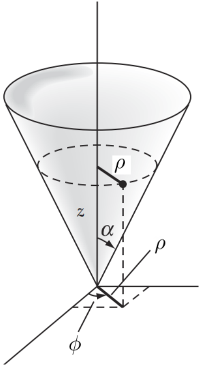
Example: Moving on a smooth cone

- A particle of mass m is constrained to move on the inside surface of a smooth cone of half-angle α. The particle is subjected to a gravitational force.

- Lagrange function: zero gravitational potential energy reference at the origin

$$\mathcal{L} = (\mathbf{r}(t), \dot{\mathbf{r}}(t), t) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

- Two degrees of freedoms



EXERCISE 8.5: Express the Lagrange function in suitable generalized coordinates and obtain the equations of motion of the particle.