

# PC3261: Classical Mechanics II

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## Lecture 6: d'Alembert's Principle

# Constraints

- Positions of all particles of a mechanical system define the **configuration** of the system at any instant of time
- **Constraints** are limitations of a *kinematic* nature, such as *possible* positions or velocities of the particles, imposed on the mechanical system
- Given that  $(\xi_1, \xi_2, \dots, \xi_n)$  are *arbitrary* coordinates employed to specify the configuration of a mechanical system, a constraint is said to be **holonomic** if it can be expressed by a functional relation among the coordinates alone with a possible explicit time dependence

$$f(\xi_1, \xi_2, \dots, \xi_n, t) = 0$$

- Example: a system of two particles connected by a rigid rod of length  $\ell$  moving in the space

$$|\mathbf{r}_2 - \mathbf{r}_1|^2 - \ell^2 = 0 \quad \Rightarrow \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - \ell^2 = 0$$

# Velocity-dependent constraints

- Velocity-dependent constraints are given by functional relations among the coordinates and velocities which leading to a set of differential equations

$$g(\xi_1, \dots, \xi_n, \dot{\xi}_1, \dots, \dot{\xi}_n, t) = 0$$

- Velocity-dependent constraints restrict the possible displacements of the mechanical system but do not impose any limits on the possible configurations
- If the velocity-dependent constraints can be reduced to the functional relations among coordinates alone with possible explicit time dependence by integration (solving differential equation), then it is said to be holonomic
- Differential equations from the velocity-dependent constraints are generally *non-integrable* which prevents them being holonomic

## Example: Cylinder rolls on a straight track

- Denote the position of the center of mass of the cylinder as  $x$  and the angle of rotation about the center of mass as  $\phi$
- Condition for roll without slipping:  $R$  is the radius of the cylinder

$$\dot{x}(t) = R \dot{\phi}(t)$$

- Velocity-dependent constraint is integrable leading to a functional relation among coordinates alone

$$x(t) = R \phi(t) \quad \Rightarrow \quad x(t) - R \phi(t) = 0$$

## Example: Upright disc rolls on a horizontal plane

- Denote  $(x, y)$  as the position of the disc's center of mass on the plane,  $\theta$  be the angle of the plane of the disc made with the  $x$ -axis and  $\phi$  be the angle of rotation of the disc about its symmetry axis
- Condition for roll without slipping:  $R$  is the radius of the disc

$$\begin{cases} \dot{x}(t) = R \dot{\phi}(t) \cos \theta(t) \\ \dot{y}(t) = R \dot{\phi}(t) \sin \theta(t) \end{cases}$$

- Velocity-dependent constraint is non-integrable (Frobenius integrability condition); the constraints represent restrictions on the velocities alone
- The coordinates,  $x, y, \theta, \phi$ , employed to specify the instantaneous configuration of the disc are functionally independent

# Virtual displacements

- **Virtual displacements** are infinitesimal displacements that change a possible configuration to another possible configuration compatible with the constraints at a fixed instant of time
- Virtual displacements of a system of  $N$  particles:

$$f(\mathbf{r}_1(t), \dots, \mathbf{r}_N(t), t) = 0 \quad \Rightarrow \quad f(\mathbf{r}_1(t) + \delta\mathbf{r}_1, \dots, \mathbf{r}_N(t) + \delta\mathbf{r}_N, t) = 0$$

- Virtual displacements  $\delta\mathbf{r}_i$  coincide with the real displacements  $d\mathbf{r}_i$  if all constraints are time independent

**EXERCISE 6.1:** A particle is confined to a moving surface where the equation of the surface is given by  $f(\mathbf{r}, t) = 0$ . Show that the virtual displacement is tangent to the surface at the same time.

# Virtual work

- **Virtual work** by the force: no integration is required as the the virtual displacement is infinitesimal

$$\delta W \equiv \mathbf{F} \cdot \delta \mathbf{r}$$

- The *total* virtual work of the constraint forces is zero in most physically interesting cases
- Example: a system of two particles connected by a rigid rod of length  $\ell$  moving in the space;  $\mathbf{f}_1$  and  $\mathbf{f}_2$  are constrained forces on the particles along the line connecting the particles

$$\delta W = \mathbf{f}_1 \cdot \delta \mathbf{r}_1 + \mathbf{f}_2 \cdot \delta \mathbf{r}_2 = 0$$

**EXERCISE 6.2:** Show that the total virtual work by the constrained forces on the two particles connected by a rigid rod moving in the space is zero.



# Principle of virtual work

- Newton's second law:  $\mathbf{F}_\alpha^{(A)}(t)$  and  $\mathbf{F}_\alpha^{(C)}(t)$  are applied and constraint forces

$$\mathbf{F}_\alpha(t) = m_\alpha \ddot{\mathbf{r}}_\alpha(t), \quad \mathbf{F}_\alpha(t) = \mathbf{F}_\alpha^{(A)}(t) + \mathbf{F}_\alpha^{(C)}(t)$$

- Static equilibrium:  $\mathbf{F}_\alpha(t) = \mathbf{0}$

$$\sum_\alpha \mathbf{F}_\alpha(t) \cdot \delta \mathbf{r}_\alpha = 0 \quad \Rightarrow \quad \sum_\alpha \mathbf{F}_\alpha^{(A)}(t) \cdot \delta \mathbf{r}_\alpha + \sum_\alpha \mathbf{F}_\alpha^{(C)}(t) \cdot \delta \mathbf{r}_\alpha = 0$$

- **Principle of virtual work:** allowing one to express the equilibrium conditions for a constrained system in terms of the applied forces alone

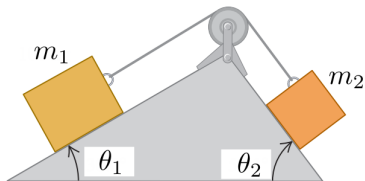
$$\sum_\alpha \mathbf{F}_\alpha^{(A)}(t) \cdot \delta \mathbf{r}_\alpha = 0$$

## Example: Two masses on double inclined plane

- Two masses  $m_1$  and  $m_2$  are located each on a smooth double inclined plane with angles  $\theta_1$  and  $\theta_2$  respectively. The masses are connected by a massless and inextensible string running over a massless and frictionless pulley

- Holonomic constraint:

$$\begin{cases} f_1(\mathbf{r}_1, \mathbf{r}_2) = x_1 + x_2 - \ell = 0 \\ f_2(\mathbf{r}_1, \mathbf{r}_2) = y_1 = 0 \\ f_3(\mathbf{r}_1, \mathbf{r}_2) = y_2 = 0 \end{cases}$$



- Applied forces:

$$\mathbf{F}_1^{(A)}(t) = m_1 g \sin \theta_1 \hat{\mathbf{e}}_{x_1} - m_1 g \cos \theta_1 \hat{\mathbf{e}}_{y_1}, \quad \mathbf{F}_2^{(A)}(t) = m_2 g \sin \theta_2 \hat{\mathbf{e}}_{x_2} - m_2 g \cos \theta_2 \hat{\mathbf{e}}_{y_2}$$

**EXERCISE 6.3:** Establish the condition for equilibrium from the principle of virtual work.

# d'Alembert's principle

- Newton's second law:

$$\mathbf{F}_\alpha(t) = m_\alpha \ddot{\mathbf{r}}_\alpha(t) \quad \Rightarrow \quad \mathbf{F}_\alpha(t) - m_\alpha \ddot{\mathbf{r}}_\alpha(t) = 0$$

- **d'Alembert's principle:** an extension of the principle of virtual work to mechanical systems in motion

$$\sum_{\alpha} \left[ \mathbf{F}_\alpha^{(A)}(t) - m_\alpha \ddot{\mathbf{r}}_\alpha(t) \right] \cdot \delta \mathbf{r}_\alpha = 0$$

- d'Alembert's principle is a substantial leap forward with respect to the Newtonian approach as it excludes any reference to the constraint forces
- It is important to take note that the virtual displacements  $\delta \mathbf{r}_\alpha$  are *not* independent as they have to be in harmony with the constraints

# Example: Atwood's machine

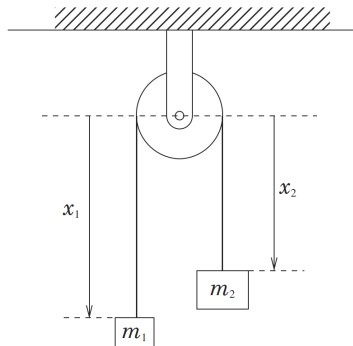
- Two masses  $m_1$  and  $m_2$  are suspended by an inextensible string which passes over a massless and frictionless pulley

- Holonomic constraint:

$$f(\mathbf{r}_1, \mathbf{r}_2) = x_1 + x_2 - \ell = 0$$

- Applied forces:

$$\mathbf{F}_1^{(A)}(t) = m_1 g \hat{\mathbf{e}}_1, \quad \mathbf{F}_2^{(A)}(t) = m_2 g \hat{\mathbf{e}}_2$$



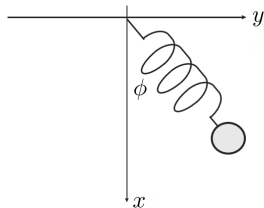
**EXERCISE 6.4:** Use d'Alembert's principle to find the accelerations of the masses  $\ddot{x}_1(t)$  and  $\ddot{x}_2(t)$ .

# Example: Pendulum with spring

- A point particle of mass  $m$  attached to a massless spring of original length  $\ell_0$  and spring constant  $k$  rotates about a frictionless pivot in a plane

- Applied forces:

$$\begin{cases} \mathbf{F}_{\text{gravity}}(t) = mg \cos \phi(t) \hat{\mathbf{e}}_\rho - mg \sin \phi(t) \hat{\mathbf{e}}_\phi \\ \mathbf{F}_{\text{spring}}(t) = -k [\rho(t) - \ell_0] \hat{\mathbf{e}}_\rho \end{cases}$$



**EXERCISE 6.5:** Use d'Alembert's principle to obtain equations of motion for  $\rho(t)$  and  $\phi(t)$ .

## Example: Spherical pendulum

- A particle of mass  $m$  is suspended by a massless wire of length  $r(t)$  to move on the surface of the spherical of radius  $r(t)$

$$r(t) = a + b \cos \omega t, \quad a > b > 0$$

- Holonomic constraint:

$$f(\mathbf{r}, t) = r(t) - a - b \cos \omega t = 0$$

- Applied force:

$$\mathbf{F}_{\text{gravity}}(t) = -mg \cos \theta(t) \hat{\mathbf{e}}_r + mg \sin \theta(t) \hat{\mathbf{e}}_\theta$$

**EXERCISE 6.6:** Use d'Alembert's principle to obtain equations of motion for  $\theta(t)$  and  $\phi(t)$ .