Fundamental Poisson brackets

- Fundamental Poisson brackets:

$$\{q_i, q_j\}_{q,p} = 0\,, \qquad \{p_i, p_j\}_{q,p} = 0\,, \qquad \{q_i, p_j\}_{q,p} = \delta_{ij}$$

- Canonical quantization: $\{\ ,\ \}_{q,p} \to [\ ,\ ]/\mathrm{i}\hbar$, $\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

$$[\hat{q}_i, \hat{q}_j] = 0\,, \quad [\hat{p}_i, \hat{p}_j] = 0\,, \quad [\hat{q}_i, \hat{p}_j] = \mathrm{i}\hbar\,\delta_{ij}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\,\hat{A}_{\mathsf{H}}(t) = \frac{1}{\mathrm{i}\hbar}\left[\hat{A}_{\mathsf{H}}, \hat{H}_{\mathsf{H}}\right] + \left(\frac{\partial \hat{A}_{\mathsf{S}}}{\partial t}\right)_{\mathsf{H}}$$

- Poisson brackets for the components of the angular momentum:

$$L_k = \sum_{i,j=1}^{3} \epsilon_{ijk} x_i p_j \qquad \to \qquad \{L_i, L_j\}_{q,p} = \sum_{k=1}^{3} \epsilon_{ijk} L_k$$

EXERCISE 11.2: Evaluate $\{\mathbf{r}, \mathbf{n}\cdot\mathbf{L}\}_{q,p}$ where $\mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z$ and $\mathbf{n} = n_x\,\hat{\mathbf{e}}_x + n_y\,\hat{\mathbf{e}}_y + n_z\,\hat{\mathbf{e}}_z$ is a constant vector.

$$
\begin{aligned}
\{x_i, \mathbf{n}\cdot\mathbf{L}\}_{q,p} &= \sum_{j=1}^{3} \{x_i, n_j L_j\}_{q,p} \\
&= \sum_{j=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} \epsilon_{jrs} n_j \{x_i, x_r p_s\}_{q,p} \\
&= \sum_{j=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} \epsilon_{jrs} n_j x_r \{x_i, p_s\}_{q,p} + \sum_{j=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} \epsilon_{jrs} n_j \{x_i, x_r\}_{q,p} p_s \\
&= \sum_{j=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} \epsilon_{jrs} n_j x_r \delta_{is} + 0 \\
&= \sum_{j=1}^{3}\sum_{r=1}^{3} \epsilon_{jri} n_j x_r \\
&= (\mathbf{n}\times\mathbf{r})_i \qquad \blacksquare
\end{aligned}
$$

Example: Projectile motion

- A projectile with mass m is moving on the vertical xy-plane in a uniform gravitational field

$$\mathcal{H} \equiv \mathcal{H}(x, y, p_x, p_y, t) = \frac{p_x^2 + p_y^2}{2m} + mgy$$

- Two constants of motion:

$$\begin{cases} F_1 \equiv y - \dfrac{p_y t}{m} - \dfrac{1}{2}\, g t^2 \\ F_2 \equiv x - \dfrac{p_x t}{m} \end{cases}$$

EXERCISE 11.3: Show that F_1 and F_2 are constants of motion. Find the other three constants of motion.

$$\mathcal{H} \equiv \mathcal{H}(x, y, p_x, p_y, t) = \frac{p_x^2 + p_y^2}{2m} + mgy$$

$$\begin{cases} \dot{x} = \{x, \mathcal{H}\}_{q,p} = \dfrac{\partial \mathcal{H}}{\partial p_x} = \dfrac{p_x}{m} \\ \\ \dot{p}_x = \{p_x, \mathcal{H}\}_{q,p} = -\dfrac{\partial \mathcal{H}}{\partial x} = 0 \end{cases}, \qquad \begin{cases} \dot{y} = \{y, \mathcal{H}\}_{q,p} = \dfrac{\partial \mathcal{H}}{\partial p_y} = \dfrac{p_y}{m} \\ \\ \dot{p}_y = \{p_y, \mathcal{H}\}_{q,p} = -\dfrac{\partial \mathcal{H}}{\partial y} = -mg \end{cases}$$

$$F_1 \equiv y - \frac{p_y t}{m} - \frac{1}{2}\, g t^2$$

$$\frac{\mathrm{d}F_1}{\mathrm{d}t} = \{F_1, \mathcal{H}\}_{q,p} + \frac{\partial F_1}{\partial t} = \{y, \mathcal{H}\}_{q,p} - \frac{t}{m}\{p_y, \mathcal{H}\}_{q,p} + \left(-\frac{p_y}{m} - gt\right) = 0 \qquad \blacksquare$$

$$F_2 \equiv x - \frac{p_x t}{m}$$

$$\frac{\mathrm{d}F_2}{\mathrm{d}t} = \{F_2, \mathcal{H}\}_{q,p} + \frac{\partial F_2}{\partial t} = \{x, \mathcal{H}\}_{q,p} - \frac{t}{m}\{p_x, \mathcal{H}\}_{q,p} + \left(-\frac{p_x}{m}\right) = 0 \qquad \blacksquare$$

$$\mathcal{H} \equiv \mathcal{H}(x, y, p_x, p_y, t) = \frac{p_x^2 + p_y^2}{2m} + mgy$$

$$\begin{cases} \dot{x} = \{x, \mathcal{H}\}_{q,p} = \dfrac{\partial \mathcal{H}}{\partial p_x} = \dfrac{p_x}{m} \\[2ex] \dot{p}_x = \{p_x, \mathcal{H}\}_{q,p} = -\dfrac{\partial \mathcal{H}}{\partial x} = 0 \end{cases}, \qquad \begin{cases} \dot{y} = \{y, \mathcal{H}\}_{q,p} = \dfrac{\partial \mathcal{H}}{\partial p_y} = \dfrac{p_y}{m} \\[2ex] \dot{p}_y = \{p_y, \mathcal{H}\}_{q,p} = -\dfrac{\partial \mathcal{H}}{\partial y} = -mg \end{cases}$$

$$F_3 \equiv \mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + mgy \quad \Rightarrow \quad \frac{\mathrm{d}F_3}{\mathrm{d}t} = \{\mathcal{H}, \mathcal{H}\}_{q,p} + \frac{\partial \mathcal{H}}{\partial t} = 0 \qquad \blacksquare$$

$$F_4 \equiv \{F_1, H\}_{q,p} = \left\{y - \frac{p_y t}{m} - \frac{1}{2}\, g t^2, \mathcal{H}\right\}_{q,p} = \{y, \mathcal{H}\}_{q,p} - \frac{t}{m}\{p_y, \mathcal{H}\}_{q,p} = \frac{p_y}{m} + gt \qquad \blacksquare$$

$$F_5 \equiv \{F_2, H\}_{q,p} = \left\{x - \frac{p_x t}{m}, \mathcal{H}\right\}_{q,p} = \{x, \mathcal{H}\}_{q,p} - \frac{t}{m}\{p_x, \mathcal{H}\}_{q,p} = \frac{p_x}{m} \qquad \blacksquare$$

Integrable systems

- The notion of **integrability** of a mechanical system refers to the possibility of *explicitly* solving its equations of motion

- The s dynamical variables $F_1(\{q_k, p_k\}), \cdots, F_s(\{q_k, p_k\})$ are said to be in **involution** if the Poison bracket of any two of them is zero

$$\{F_i, F_j\}_{q,p} = 0\,, \qquad i, j = 1, 2, \cdots, s$$

- A Hamiltonian system with m degrees of freedom is said to be integrable if there exist m independent constants of the motion in involution

$$\begin{cases} \dfrac{\mathrm{d}F_i}{\mathrm{d}t} = 0\,, & i = 1, 2, \cdots, m \\ \{F_i, F_j\}_{q,p} = 0\,, & i, j = 1, 2, \cdots, m \end{cases}$$

Lagrangian versus Hamiltonian mechanics

- Euler-Lagrange equations of motion are covariant under a point transformation:

$$q_i = q_i\left(\{Q_j\}, t\right) \quad \rightarrow \quad \mathcal{L}\left(\{q_i, \dot{q}_i\}, t\right) = \mathcal{L}'\left(\{Q_i, \dot{Q}_i\}, t\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}'}{\partial \dot{Q}_i}\right) - \frac{\partial \mathcal{L}'}{\partial Q_i} = 0$$

- Hamilton equations of motion is also covariant under a point transformation *provided* that the *new* Hamiltonian (known as **Kamiltonian**) is constructed using the *new* conjugate momentum via a Legendre transformation

$$P_i = \frac{\partial \mathcal{L}'}{\partial \dot{Q}_i} \quad \rightarrow \quad \mathcal{K} \equiv \sum_{i=1}^{M} \dot{Q}_i\, P_i - \mathcal{L}' \neq \mathcal{H} \quad \rightarrow \quad \begin{cases} \dot{Q}_i = \dfrac{\partial \mathcal{K}}{\partial P_i} \\[2ex] \dot{P}_i = -\dfrac{\partial \mathcal{K}}{\partial Q_i} \end{cases}$$

Canonical transformation

- Hamilton equations of motion is, generally, covariant under a **canonical transformation** which is the change of canonical coordinates (generalized coordinates and generalized momenta are being treated under equal footing)

$$\begin{cases} Q_i \equiv Q_i(\{q_j, p_j\}, t) \\ P_i \equiv P_i(\{q_j, p_j\}, t) \end{cases} \quad \rightarrow \quad \begin{cases} \dot{Q}_i = \dfrac{\partial \mathcal{K}}{\partial P_i} \\ \dot{P}_i = -\dfrac{\partial \mathcal{K}}{\partial Q_i} \end{cases}$$

- **Phase space Lagrangian**: $2M$ independent generalized coordinates $\{q_k, p_k\}$

$$\tilde{\mathcal{L}} \equiv \tilde{\mathcal{L}}(\{q_i, p_i, \dot{q}_i, \dot{p}_i\}, t) \equiv \sum_{k=1}^{M} p_k \dot{q}_k - \mathcal{H}(\{q_k, p_k\}, t)$$

Canonical transformation – cont'd

- $2M$ Euler-Lagrange equations associated to the phase space Lagrangian give $2M$ Hamilton's canonical equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}_k}\right) - \dfrac{\partial \tilde{\mathcal{L}}}{\partial q_k} = 0 \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \tilde{\mathcal{L}}}{\partial \dot{p}_k}\right) - \dfrac{\partial \tilde{\mathcal{L}}}{\partial p_k} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}p_k}{\mathrm{d}t} - \left(-\dfrac{\partial \mathcal{H}}{\partial q_k}\right) = 0 \\[2ex] 0 - \left(\dot{q}_k - \dfrac{\partial \mathcal{H}}{\partial p_k}\right) = 0 \end{cases}$$

- Euler-Lagrange equations associated to the phase space Lagrangian are invariant under a gauge transformation and hence ensuring the same Hamilton's equations are obtained

$$\tilde{\mathcal{L}}'(\{q_i, p_i, \dot{q}_i, \dot{p}_i\}, t) = \tilde{\mathcal{L}}(\{q_i, p_i, \dot{q}_i, \dot{p}_i\}, t) + \frac{\mathrm{d}\Lambda(\{q_i, p_i\}, t)}{\mathrm{d}t}$$

Canonical transformation and generating function

- $4M + 1$ variables $(\{q_i, p_i, Q_i, P_i\}, t)$ subjected to $2M$ transformation equations, $Q_i(\{q_j, p_j\}, t)$ and $P_i(\{q_j, p_j\}, t)$, leads to flexible choices of $2M + 1$ independent variables

- The gauge function, $\Lambda(\{q_i, p_i\}, t)$, is known as the **generating function** which generates the canonical transformation

- Four basic classes of generating functions: (Question: What is the relationship between different classes of generating functions?)

$$\begin{cases} \text{Type 1: } \Lambda_1 \equiv \Lambda_1(\{q_i, Q_i\}, t) \\ \text{Type 2: } \Lambda_2 \equiv \Lambda_2(\{q_i, P_i\}, t) \\ \text{Type 3: } \Lambda_3 \equiv \Lambda_3(\{p_i, Q_i\}, t) \\ \text{Type 4: } \Lambda_4 \equiv \Lambda_4(\{p_i, P_i\}, t) \end{cases}$$

Example: Harmonic oscillator

- Hamilton equations of motion:

$$\mathcal{H}(q,p) = \frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m} \Rightarrow \begin{cases} \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q \\[2ex] \dot{q} = \dfrac{\partial \mathcal{H}}{\partial p} = \dfrac{p}{m} \end{cases} \Rightarrow \begin{cases} \ddot{q} = -\omega^2 q \\ \ddot{p} = -\omega^2 p \end{cases}$$

- Type 1 generating function: this canonical transformation effectively exchanges the role of the coordinate and momentum!

$$\Lambda_1 \equiv \Lambda_1(q,Q,t) = qQ \qquad \Rightarrow \qquad \begin{cases} Q \equiv Q(q,p,t) = -p \\ P \equiv P(q,p,t) = q \end{cases}$$

EXERCISE 11.4: Obtain the canonical transformation generated by $\Lambda_1(q,Q,t) = qQ$ and the Kamiltonian equations of motion.

$$\mathcal{H}(q,p) = \frac{1}{2}\, m\omega^2 q^2 + \frac{p^2}{2m}$$

$$\Lambda_1 \equiv \Lambda_1(q, p(q,Q,t), t) = \Lambda_1(q,Q,t) = qQ \quad \Rightarrow \quad \frac{\mathrm{d}\Lambda_1}{\mathrm{d}t} = \frac{\partial \Lambda_1}{\partial q}\,\dot{q} + \frac{\partial \Lambda_1}{\partial Q}\,\dot{Q} + \frac{\partial \Lambda_1}{\mathrm{d}t}$$

$$\tilde{\mathcal{L}}'(q,p,\dot{q},\dot{p},t) = \tilde{\mathcal{L}}(q,p,\dot{q},\dot{p},t) + \frac{\mathrm{d}\Lambda_1(q,p,t)}{\mathrm{d}t}$$

$$\Rightarrow \quad P\dot{Q} - \mathcal{K}(Q,P,t) = p\dot{q} - \mathcal{H}(q,p,t) + \frac{\partial \Lambda_1}{\partial q}\,\dot{q} + \frac{\partial \Lambda_1}{\partial Q}\,\dot{Q} + \frac{\partial \Lambda_1}{\mathrm{d}t}$$

$$\Rightarrow \quad \begin{cases} p \equiv p(q,Q,t) = -\dfrac{\partial \Lambda_1}{\partial q} = -Q \\[2ex] P \equiv P(q,Q,t) = \dfrac{\partial \Lambda_1}{\partial Q} = q \\[2ex] \mathcal{K}(Q,P,t) = \mathcal{H}(q,p,t) - \dfrac{\partial \Lambda_1}{\partial t} \end{cases} \quad \Rightarrow \quad \begin{cases} Q \equiv Q(q,p,t) = -p \\[2ex] P \equiv P(q,p,t) = q \\[2ex] \mathcal{K} \equiv \mathcal{K}(Q,P,t) = \dfrac{1}{2}\, m\omega^2 P^2 + \dfrac{Q^2}{2m} \end{cases}$$

$$\begin{cases} \dot{Q} = \dfrac{\partial \mathcal{K}}{\partial P} = m\omega^2 P \\[2ex] \dot{P} = -\dfrac{\partial \mathcal{K}}{\partial Q} = -\dfrac{Q}{m} \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{Q} = -\omega^2 Q \\[1ex] \ddot{P} = -\omega^2 P \end{cases} \qquad \blacksquare$$