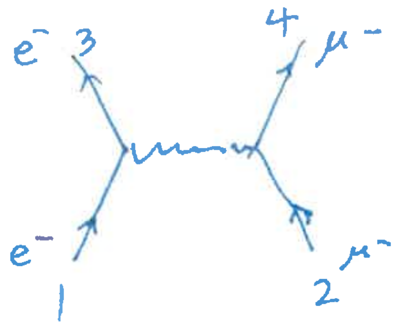
PC 4245

Part IV

0.1

(4) Electron-muon scattering

$$e^- + \mu^- \rightarrow e^- + \mu^-$$



$$\mathcal{M} = \frac{-g^2}{(\underline{P}_1 - \underline{P}_3)^2} \left(\bar{u}(3)\gamma^\mu u(1)\right)\cdot\left(\bar{u}(4)\gamma_\mu u(2)\right)$$

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{s_1 s_2 s_3 s_4} |\mathcal{M}|^2$$

$$= \frac{g^4}{4(\underline{P}_1 - \underline{P}_3)^4} \sum_{\substack{s_1 s_2 \\ s_3 s_4}} \left(\bar{u}(3)\gamma^\mu \underline{u(1)}\right)\left(\bar{u}(4)\gamma_\mu u(2)\right)\cdot \left(\underline{\bar{u}(1)}\gamma^\nu u(3)\right)\cdot\left(\bar{u}(2)\gamma_\nu u(4)\right)$$

completeness of bispinor

$$\sum_s u^{(s)}(\underline{P})\cdot \bar{u}^{(s)}(\underline{P}) = (\not{p} + mc)$$

$$\therefore \langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \frac{g^4}{(\underline{P}_1 - \underline{P}_3)^4} \cdot \sum_{s_3 s_4} \left(\bar{u}(3)\gamma^\mu(\not{p}_1 + m_1 c)\gamma^\nu u(3)\right)\cdot \left(\bar{u}(4)\gamma_\mu(\not{p}_2 + m_2 c)\gamma_\nu u(4)\right)$$

$$= \frac{1}{4}\frac{g^4}{(\underline{P}_1 - \underline{P}_3)^2} \cdot \mathrm{Tr}\left[\gamma^\mu(\not{p}_1 + m_1 c)\gamma^\nu(\not{p}_3 + m_3 c)\right]\cdot \mathrm{Tr}\left[\gamma_\mu(\not{p}_2 + m_2 c)\gamma_\nu(\not{p}_4 + m_4 c)\right]$$

$$\mathrm{Tr}\,\gamma^\mu = 0, \qquad \mu = 0, 1, 2, 3$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha) = 0,$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\gamma^{5\,2} = 1$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta)$$

$$= 4(g^{\mu\nu} g^{\alpha\beta} + g^{\beta\mu} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\beta})$$

$$\mathrm{Tr}[\gamma^\mu (\not{p}_1 + m_1 c) \gamma^\nu (\not{p}_3 + m_3 c)]$$

$$= \mathrm{Tr}[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^\mu \not{p}_1 \gamma^\nu m_3 c + m_1 c \gamma^\mu \gamma^\nu \not{p}_3$$

$$+ m_1 m_3 c^2 \gamma^\mu \gamma^\nu]$$

$$= \mathrm{Tr}[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 + m_1 m_3 c^2 \gamma^\mu \gamma^\nu]$$

$$\because \mathrm{Tr}\,\gamma^\mu \gamma^\alpha \gamma^\nu = 0$$

From Griffiths,
"The factor of 1/4 is included because we want the average over the initial spins; since there are two particles, each with two allowed spin orientations, the average is a quarter of the sum."

Casimir trick takes advantage of the completeness of dirac spinors to compute |M|^2 without having to sum all 16 individual amplitudes

Using Casimir's trick, have computed

$$\langle |M|^2 \rangle = \frac{1}{4} \alpha \sum_{s_1 s_2 s_3 s_4} (\qquad)$$

for the $e^- \mu^- \to e^- \mu^-$ process

Using formula from the previous lecture

$$Tr[\gamma_\mu (\not{p}_4 + m_4 c) \gamma_\nu (\not{p}_2 + m_2 c)]$$

$$\stackrel{HW}{=} 4 [p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu} (p_2 p_4 - m_2 m_4 c^2)]$$

$$\rightarrow$$

$$\langle |M|^2 \rangle = \frac{g^4}{(p_1 - p_3)^4} \frac{1}{4} Tr[\gamma^\mu (\not{p}_1 + m_1 c) \gamma^\nu (\not{p}_3 + m_3 c)] Tr[\gamma_\mu (\not{p}_2 + m_2 c) \cdot \gamma_\nu (\not{p}_4 + m_4 c)]$$

$$\stackrel{HW}{=} \frac{4 g^4}{(p_1 - p_3)^4} [p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu} (p_2 p_4 - m_2 m_4 c^2)] \cdot$$

$$\cdot [p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} (p_1 p_3 - m_1 m_3 c^2)]$$

$$\stackrel{HW}{=} \frac{8 g^4}{(p_1 - p_3)^4} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_2 \cdot p_3)(p_1 p_4) - (p_2 \cdot p_4) m_1 m_3 c^2 - (p_1 \cdot p_3) m_2 m_4 c^2 + 2 (m_1 m_2 m_3 m_4) c^4]$$

$$m_1 = m_3 = m_e$$

$$m_2 = m_4 = m_\mu$$

Today we discuss a 1-loop Feynman diagram for the scattering process

$$e^- + \mu^- \to e^- + \mu^-$$

The 1-loop Feynman integral is a divergent integral.

It can be made finite by a regularization, using a cut-off parameter M.

The cut-off parameter, M, can then be absorbed by the coupling constant g

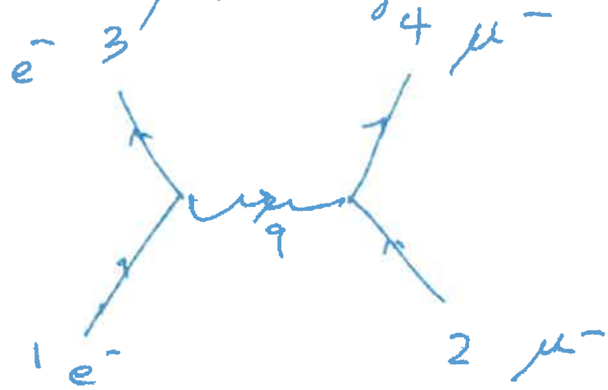
$$g \to g_R \quad \text{renormalized coupling constant}$$

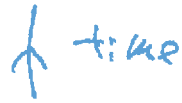
We will also discuss hadron production in e^+e^- collisions, the first section of chapter 8, Griffiths.

Today discuss 1-loop diagram

Consider a simple process $e^- \mu^- \to e^- \mu^-$

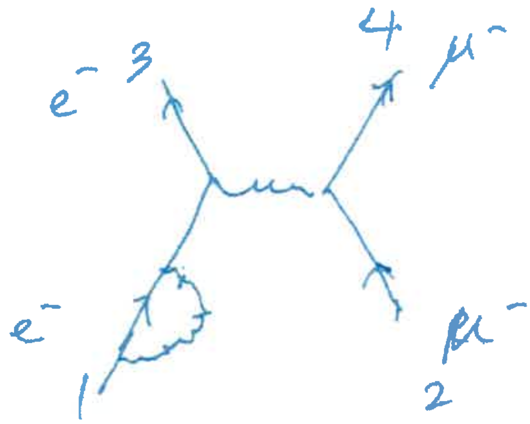
At tree-level, only one diagram:

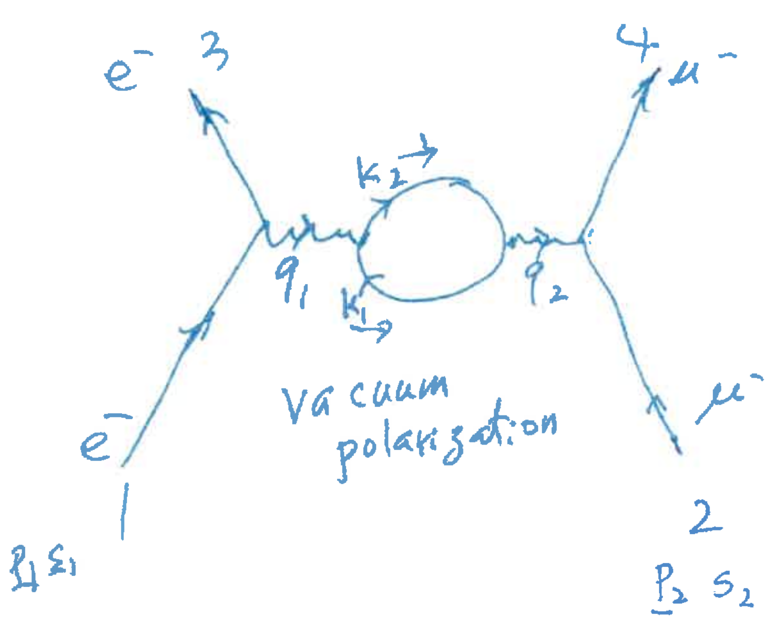




Construct one-loop diagrams by using the QED vertex  . Many possibilities

e. g.





Follow the usual procedure, write down the mathematical expression for the 1-loop diagram (vacuum polarization diagram)

$$(-)\mathrm{Tr}$$

$$\frac{-ig_{\alpha\beta}}{q_2^2}\left(ig\gamma^\alpha \frac{i}{\not{k}_2-mc}\, ig\gamma^\nu \frac{i}{\not{k}_1-mc}\right)\frac{-ig_{\mu\nu}}{q_1^2}\,\bar{u}(3)\, ig\gamma^\mu u(1)$$

$$(2\pi)^4\delta^{(4)}(q_1-k_2-k_1)\,(2\pi)^4\delta^{(4)}(p_1-p_3-q_1)\,\bar{u}(4)\, ig\gamma^\beta u(2)$$

$$(2\pi)^4\delta^{(4)}(q_2+p_2-p_4)\,(2\pi)^4\delta^{(4)}(k_1+k_2-q_2)$$

$$\int\frac{d^4q_2}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4q_1}{(2\pi)^4}$$

$$= -g^4\int d^4q_1\, d^4q_2\, d^4k_2\, d^4k_1 \cdot \bar{u}(4)\gamma^\beta u(2)\frac{1}{q_2^2}\cdot$$

$$\mathrm{Tr}\left(\gamma_\beta \frac{1}{\not{k}_2-mc}\gamma^\nu\frac{1}{\not{k}_1-mc}\right)\frac{1}{q_1^2}\,\bar{u}(3)\gamma_\nu u(1)\cdot$$

$$\delta^{(4)}(q_1-k_2-k_1)\,\underline{\delta^{(4)}(p_1-p_3-q_1)}\,\delta^{(4)}(q_2+p_2-p_4)\,\delta^{(4)}(k_1+k_2-q_2)$$

Integrating out $\int d^4 q_1$, we get

$$= -g^4 \int d^{(4)}q_2\, d^{(4)}k_1\, d^{(4)}k_2\;\; \bar{u}(4)\gamma^\beta u(2)\, \frac{1}{q_2^2}$$

$$\mathrm{Tr}\left[\gamma_\beta \frac{1}{\not{k}_2 - mc}\gamma_\nu \frac{1}{\not{k}_1 - mc}\right] \cdot \frac{1}{(\underline{p}_1 - \underline{p}_3)^2}\, \bar{u}(3)\gamma^\nu u(1)$$

$$\delta^{(4)}(\underline{p}_1 - \underline{p}_3 - \underline{k}_2 - \underline{k}_1) \cdot \underline{\delta^{(4)}(\underline{q}_2 + \underline{p}_2 - \underline{p}_4)}\; \delta^{(4)}(\underline{k}_1 + \underline{k}_2 - \underline{q}_2)$$

Integrate out $\int d^4 q_2$

$$= -g^4 \int d^{(4)}\underline{k}_1\, d^{(4)}\underline{k}_2 \cdot \bar{u}(4)\gamma^\beta u(2) \cdot \frac{1}{(p_4 - p_2)^2}\,.$$

$$\mathrm{Tr}\left[\gamma_\beta \frac{1}{\not{k}_2 - mc}\gamma_\nu \frac{1}{\not{k}_1 - mc}\right] \frac{1}{(\underline{p}_1 - \underline{p}_3)^2}\, \bar{u}(3)\gamma^\nu u(1)$$

$$\underline{\delta^{(4)}(\underline{p}_1 - \underline{p}_3 - \underline{k}_2 - \underline{k}_1)}\; \delta^{(4)}(\underline{k}_1 + \underline{k}_2 - \underline{p}_4 + \underline{p}_2)$$

Integrate out $\int d^4 \underline{k}_2$

$$= -g^4 \int d^{(4)}\underline{k}_1\;\; \bar{u}(4)\gamma^\beta u(2) \cdot \frac{1}{(\underline{p}_4 - \underline{p}_2)^2}\,.$$

$$\mathrm{Tr}\left[\gamma_\beta \frac{1}{\not{p}_1 - \not{p}_3 - \not{k}_1 - mc}\gamma_\nu \frac{1}{\not{k}_1 - mc}\right] \frac{\bar{u}(3)\gamma^\nu u(1)}{(\underline{p}_1 - \underline{p}_3)^2}$$

$$\delta^{(4)}(\underline{k}_1 + \underline{p}_1 - \underline{p}_3 - \underline{k}_1 - \underline{p}_4 + \underline{p}_2)$$

change

$\underline{d^{(4)}\underline{k}_1} \longrightarrow d^{(4)}\underline{k}$ why?

(5)

$$= -g^4 \int d^4\underline{k} \; \frac{\bar{u}(4)\gamma^\beta u(2)}{(\underline{P}_4 - \underline{P}_2)^2} \; \mathrm{Tr}\left[\gamma_\beta \frac{1}{\not{P}_1 - \not{P}_3 - \not{k} - m_c} \gamma_\nu \frac{1}{\not{k} - m_c}\right] \cdot$$

$$\frac{\bar{u}(3)\gamma^\nu u(1)}{(\underline{P}_1 - \underline{P}_3)^2} \quad \delta^{(4)}(\underline{P}_1 + \underline{P}_2 - \underline{P}_3 - \underline{P}_4)$$

Following the rule, the scattering amp. at 1-loop level (change k_1 to k)

$$\mathcal{M}_{1\text{-loop}} = -ig^4 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(4)\gamma^\beta u(2)}{(\underline{P}_4 - \underline{P}_2)^2} \cdot$$

$$\mathrm{Tr}\left[\gamma_\beta \frac{1}{\not{P}_1 - \not{P}_3 - \not{k} - m_c} \gamma_\nu \cdot \frac{1}{\not{k} - m_c}\right] \cdot$$

$$\longrightarrow \not{k} = k_\mu \gamma^\mu$$

$$\frac{\bar{u}(3)\gamma^\nu u(1)}{(\underline{P}_1 - \underline{P}_3)^2} \qquad (\underline{P}_1 + \underline{P}_2 = \underline{P}_3 + \underline{P}_4)$$

Define $(\underline{P}_1 - \underline{P}_3)^2$ = momentum transfer square

$$= t = (\underline{P}_2 - \underline{P}_4)^2$$

$$I_{\beta\nu} = \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\beta \frac{1}{\not{P}_1 - \not{P}_3 - \not{k} - m_c} \gamma_\nu \frac{1}{\not{k} - m_c}\right]$$

$$\to \mathcal{M}_{1\text{-loop}} = -ig^4 \frac{1}{t^2} \bar{u}(4)\gamma^\beta u(2) \, I_{\beta\nu} \, \bar{u}(3)\gamma^\nu u(1)$$

$$\mathcal{M}_{tree} = -g^2\, \bar{u}(4)\, \gamma_\mu\, u(2)\, \frac{1}{t}\, \bar{u}(3)\, \gamma^\mu\, u(1) \quad (6)$$

The total scatt amplitude

$$\mathcal{M} = \mathcal{M}_{tree} + \mathcal{M}_{1\text{-loop}}$$

Look at

$$I_{\beta\nu} = \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left[\gamma_\beta \frac{1}{\not p_1 - \not p_3 - \not k - mc} \gamma_\nu \frac{1}{\not k - mc}\right].$$

$$\int d^4\underline{k} = \int_0^\infty dk^0\, dk^1\, dk^2\, dk^4 \qquad d^3\underline{x} = r^2\, dr \cdot d\Omega$$

Note: $\frac{1}{\not k - mc} = \frac{\not k + mc}{(\not k - mc)(\not k + mc)} \overset{HW}{=} \frac{\not k + mc}{k^2 - m^2c^2}$

Integrand $\sim \frac{1}{k^2}$

As $k \to 0$, lower limit no problem

As $k \to \infty$, $\int_0^\infty \frac{d^4k}{k^2} \to \int_0^\infty k\, dk \sim \left[\frac{k^2}{2}\right]_0^\infty$ ($d^4k \to k^3 dk \cdot d\Omega_k$)

quadratic divergent

the integral behaves as $k^2 \to \infty$

i.e the integral is divergent
ultra violet

If $k \to 0$, integral divergent, we say infrared divergent

If $k \to \infty$, integral is divergent, we say (7)
ultraviolet divergent.

so for the expression $I_{\beta\nu}$ we encounter
ultraviolet divergence

The standard way of resolving the divergence problem is to render the integral finite

this is known as regularization.

There are several regularization procedures, e.g. dimensional regularization.

For the ultra violet divergence in this case, we introduce a cut off parameter at the upper limit of the integral.

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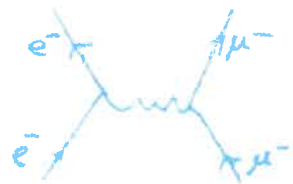
$e^- + \mu^- \to e^- + \mu^-$

Tree diagram



$$\mathcal{M}_{tree} = -g^2\, \bar{u}(4)\gamma^\mu u(2) \cdot \frac{g_{\mu\nu}}{t} \cdot \bar{u}(3)\gamma^\nu u(1)$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = q^2$$

1-loop diagram (vacuum polarization)

$$\mathcal{M}_{1\text{-}loop} = -ig^4\, \bar{u}(4)\gamma^\mu u(2)\, \frac{I_{\mu\nu}}{t^2}\, \bar{u}(3)\gamma^\nu u(1)$$

The regularized $I_{\mu\nu}$

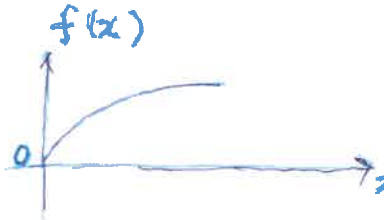
$$I^{\mu\nu} = \int_0^M \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left(\gamma^\mu \frac{1}{\not p_1 - \not p_3 - \not k - mc}\gamma^\nu \frac{1}{\not k - mc}\right)$$

regularized by the upper cut-off parameter M

$$= g^{\mu\nu} \frac{it}{12\pi^2}\left[\ln\frac{M^2}{m^2} - f\left(\frac{-t}{m^2c^2}\right)\right],$$

$f\left(\frac{-t}{m^2c^2}\right)$ = a finite function, $f(0) = 0$

not easy to sh



$$\mathcal{M}_{1\text{-loop}} = g^4 \, \underline{\bar{u}(4)\gamma^\mu u(2)} \, \frac{g_{\mu\nu}}{t} \cdot \frac{1}{12\pi^2} \cdot \left[\ln\frac{M^2}{m^2} - f\left(\frac{-t}{m^2c^2}\right)\right]$$

number

$$\underline{\bar{u}(3)\gamma^\nu u(1)}$$

$\therefore$ $\mathcal{M}_{tree} + \mathcal{M}_{1\text{-loop}}$ = total amplitude up to 1-loop

$$= -\bar{u}(4)\gamma^\mu u(2) \frac{g_{\mu\nu}}{t} \bar{u}(3)\gamma^\nu u(1) \cdot \left[\left(g^2 - g^4\frac{1}{12\pi^2}\ln\frac{M^2}{m^2}\right)\right.$$

$$\left. + g^4 \frac{1}{12\pi^2} f\left(\frac{-t}{m^2c^2}\right)\right]$$

Define renormalized coupling constant g_R by

$$g_R^2 \equiv g^2 - g^4 \frac{1}{12\pi^2} \ln\frac{M^2}{m^2}$$

As $f(0) = 0$, we can define a t-dependent $g_R(t)$

$$g_R^2(t) = g^2 - g^4\frac{1}{12\pi^2}\ln\frac{M^2}{m^2} + g^4\frac{1}{12\pi^2} f\left(\frac{-t}{m^2c^2}\right)$$

$$= g_R^2(0) + g^4\frac{1}{12\pi^2} f\left(\frac{-t}{m^2c^2}\right), \quad g_R^2(0) = g_R^2, \quad g^4 \approx g_R^4 = g_R^4(0)$$

As $g^2 = 4\pi\alpha = 4\pi\frac{e^2}{\hbar c}$, $\therefore \alpha(t) = \alpha(0)\left[1 + \frac{\alpha(0)}{3\pi}\cdot f\left(\frac{-t}{m^2c^2}\right)\right]$

$$g_R^2 \equiv g^2 - g^4 \frac{1}{12\pi^2} \ln \frac{M^2}{m^2}$$

As $f(0) = 0$, we can define a

t- dependent $g_R(t)$:

$$g_R^2(t) = g^2 - g^4 \frac{1}{12\pi^2} \ln \frac{M^2}{m^2} + g^4 \frac{1}{12\pi^2} f\left(\frac{-t}{m^2c^2}\right)$$

$$= g_R^2 + g^4 \frac{1}{12\pi^2} f\left(\frac{-t}{m^2c^2}\right)$$

$$= g_R^2(0) + g^4 \frac{1}{12\pi^2} \cdot f\left(\frac{-t}{m^2c^2}\right)$$

$$g_R^2(0) = g_R^2$$

As $g^2 = 4\pi\alpha = 4\pi \frac{e^2}{\hbar c}$

$$\therefore \alpha(t) = \alpha(0)\left[1 + \frac{\alpha(0)}{3\pi} f\left(\frac{-t}{m^2c^2}\right)\right]$$

Thus

$$\mathcal{M}_{tree} + \mathcal{M}_{1\text{-}loop}$$

$$= -\bar{u}(4)\gamma^{\mu}u(2)\ \frac{g_{\mu\nu}}{t}\ \bar{u}(3)\gamma^{\nu}u(1)\left[g_R^2 + g^4\frac{1}{12\pi^2}f\left(\frac{-t}{m^2c^2}\right)\right]$$

$$= -\bar{u}(4)\gamma^{\mu}u(2)\ \frac{g_{\mu\nu}}{t}\ \bar{u}(3)\gamma^{\nu}u(1)\quad g_R^2(t)$$

$$g_R^2(0) \equiv g_R^2$$

$$= -g_R^2(t)\,\bar{u}(4)\gamma^{\mu}u(2)\ \frac{g_{\mu\nu}}{t}\ \bar{u}(3)\gamma^{\nu}u(1)$$

same as $\mathcal{M}_{tree}$ with g^2 being replaced by $g_R^2(t)$

$g_R(t)$ is the renormalized coupling constant and dependent on the momentum transfer square t

i.e. $g_R(t)$ is the running coupling constant, t dependent

The renormalization procedure can be performed to two-loop and higher order loop diagrams consistently for QED and the results agree with experiment measurements to astonishing accuracy. QED most precious theory.

When computing scattering amplitude $\mathcal{M}$ for Feynman's diagrams of one loop or higher number of loops, one always encounters integrals that are divergent.

These integrals are divergent because of

(i) integrand not well-behaved

(ii) lower limit of the integral

(iii) upper limit of the integral.

One can introduce different techniques to render these divergent integrals to become finite. This is called regularization, and usually parameters must be introduced to make the divergent integrals finite.

The parameters are arbitrary and must be gotten rid of.

These arbitrary parameters are usually gotten rid of by absorbing them into the physical quantities like charge, mass and coupling constant.

The procedure to get rid of the arbitrary parameters consistently (not just 1-loop level but also all higher loops) is known as renormalization program