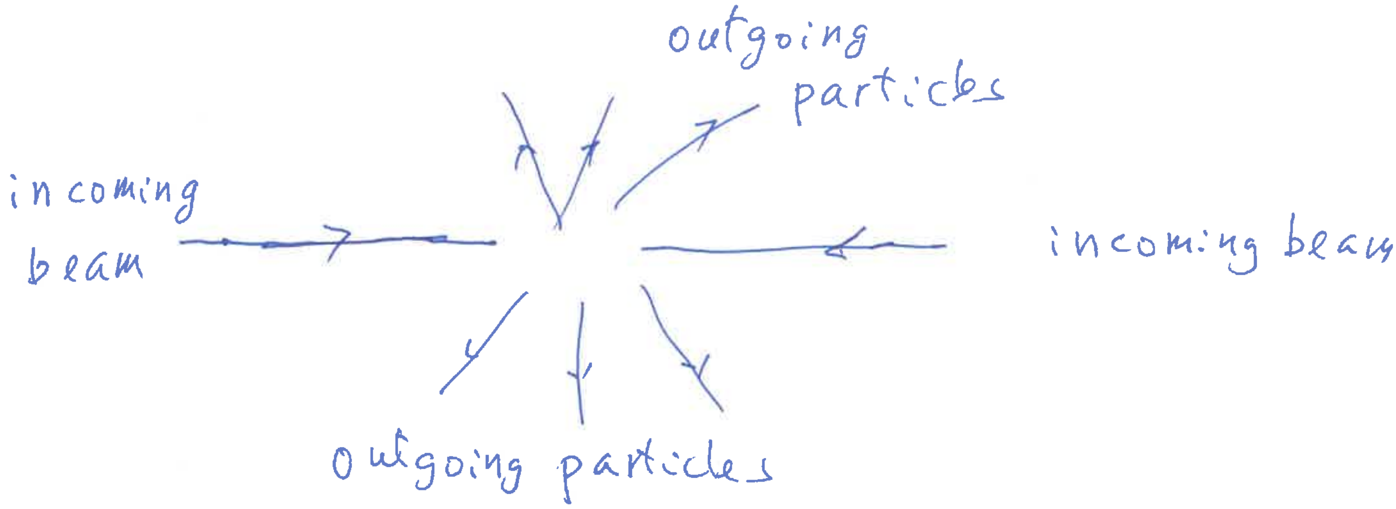
Relativistic Kinematics.

In particle physics, particle reactions involve high energy, e.g in the collider, violent



collisions. Thus the reactions are relativistic

We review special relativity in 4-vector notations and study simple examples in high energy collisions.

Special Relativity:

Frames of reference

Postulates of special Relativity

Galilean and Lorentz transformations

Matrix representation

Definition of general Lorentz transformatio

Metric tensor $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$.

Frames of reference

Fundamental to the study of physics is

frame of reference

Non inertial frames are frames in the presence of external forces, e.g. rotating frames (merry-go-round) or frames under linear acceleration (lifts)

Inertial frames in which external forces are absent, e.g.

A spaceship freely falling in gravitational field experiences no external force is an ideal inertial frame.

Postulates

1. Principle of relativity: All inertial frames of reference are equivalent.

Newtonian relativity: equivalent under Galilean transformations

Newton, principia 1967

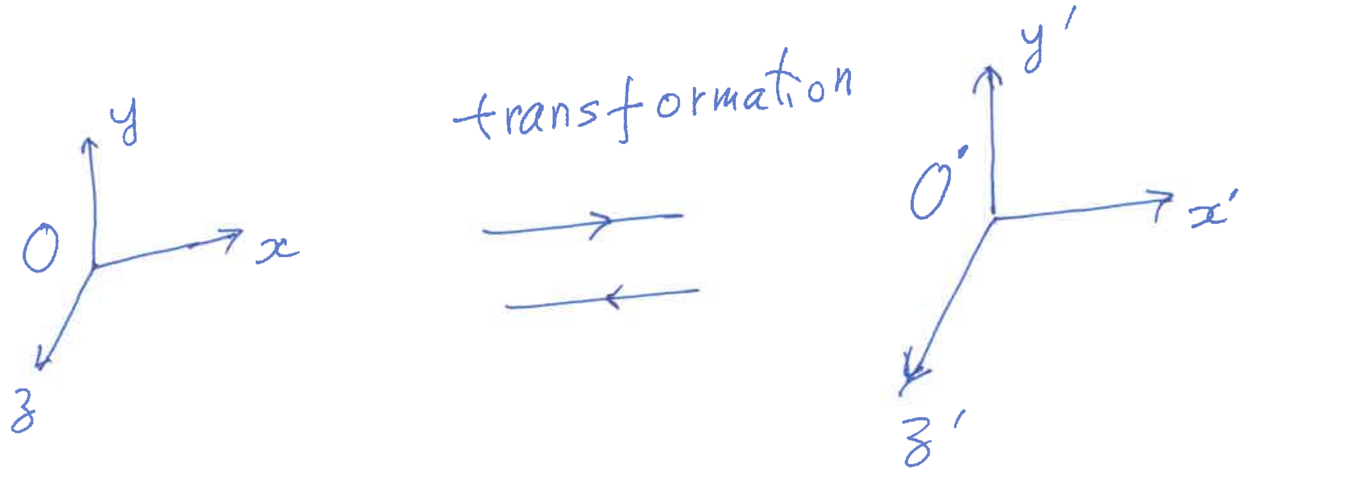
Einsteinian relativity: equivalent under Lorentz transformations

Einstein Special Relativity 1905

2. speed of light c is the same in any inertial frame of reference

Michelson - Morley experiment 1887

Trasformations between two inertial frames O and O'



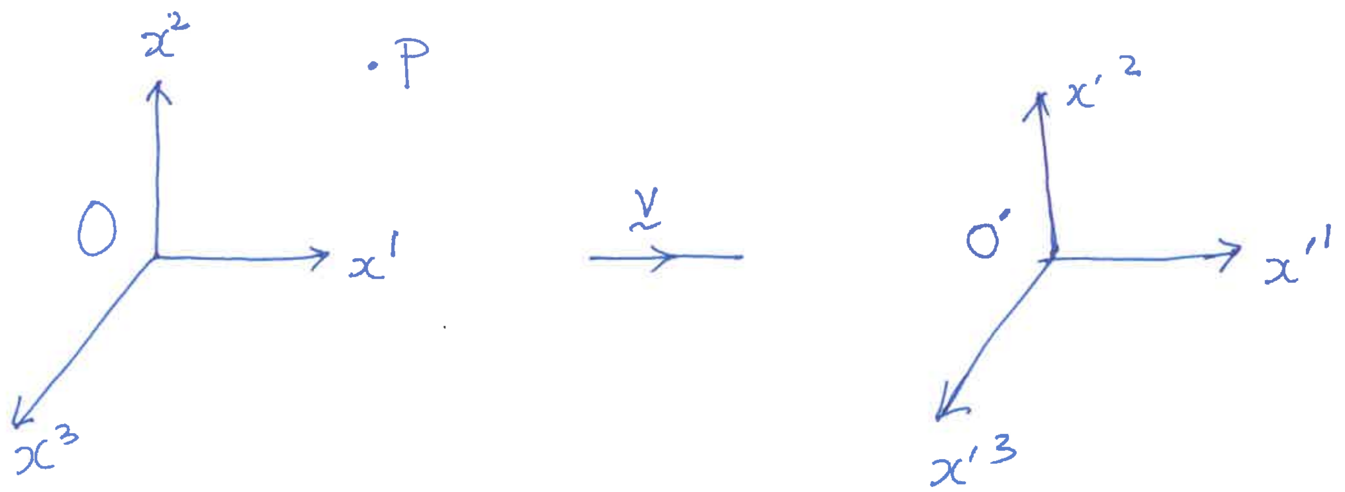
inertial frame O inertial frame O'

For convenience, change x, y, z to x^1, x^2, x^3

and time t to $x^0 \equiv ct$

c = speed of light

Assume at time $t = 0 = t'$, O frame and O' frame coincide with respective axes parallel to each other, also O' frame moves along the x^1-axis of O frame



Consider an event (a particle) at point P of space-time

Coordinates of P in O frame

$(t, \underset{\sim}{x})$, $\quad \underset{\sim}{x} = (x^1, x^2, x^3)$

Coordinates of P in O' frame

$(t', \underset{\sim}{x}')$, $\quad \underset{\sim}{x}' = (x'^1, x'^2, x'^3)$

Galilean transformation

$$\underset{\sim}{x}' = \underset{\sim}{x} - \underset{\sim}{v}\, t$$

$$t' = t$$

$\underset{\sim}{v}$ = velocity of O′ frame with respect to O frame

intuitively obvious.

Time is absolute, $t' = t$ (no change)

Space is relative, $\Delta \underset{\sim}{x}' \neq \Delta \underset{\sim}{x}$

Under Galilean transformations, speed of light can be different for different inertial frame observers, but the Michelson - Morley experiment indicates speed of light is constant for all inertial frame observers.

Hence the Galilean transformation is not the right transformation between two inertial frames.

Note that the Newton second law of the motion, the equation of motion $\underset{\sim}{F} = m \ddot{\underset{\sim}{x}}$, is covariant with respect to Galilean transformation, but not the Maxwell equations.

The principle of relativity (all inertial frames of reference are equivalent) together with the requirement that speed of light is constant in inertial frames lead to the Lorentz transformation, which is the right transformation between any two inertial frames.

Assume at time $t = 0 = t'$, O' frame and O frame coincides with respective axes parallel to each other, also O' frame moves along the x^1-axis of O frame.

The Lorentz transformation is

$$x'^1 = \gamma (x^1 - \beta x^0) \qquad \beta = \frac{v}{c}$$

$$x'^2 = x^2$$

$$x'^3 = x^3 \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$x'^0 = \gamma (x^0 - \beta x^1) \qquad x^0 = ct$$

$$x'^0 = ct'$$

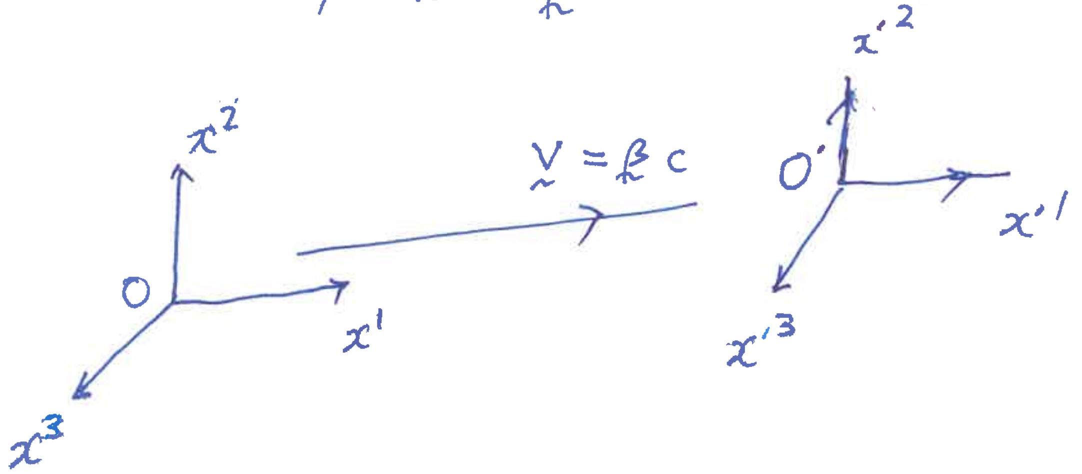
spatial coordinates and time coordinates mix, x'^1 contains x^1 and x^0, x'^0 contains x^0 and x^1.

space and time both relative. →

c (speed of light) is a constant.

Write down Lorentz transformation along any ~~coordinate axis~~ direction, that is $\underset{\sim}{\beta} = \frac{\underset{\sim}{v}}{c}$, not just along x^1-axis

Lorentz transformation along any spatial direction with velocity $\underset{\sim}{V} \equiv \underset{\sim}{\beta} c$



Along x^1- axis

$$x'^1 = \gamma(x^1 - \beta x^0), \quad x'^2 = x^2, \quad x'^3 = x^3$$

$$x'^0 = \gamma(x^0 - \beta x^1). \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Note : spatial components perpendicular to $\underset{\sim}{V}$ unchanged (in this case, x^2, x^3)

Resolve $\underset{\sim}{x} = (x^1, x^2, x^3) = \underset{\sim}{x}_\perp + \underset{\sim}{x}_{/\!/}$

$$\underset{\sim}{x}_{/\!/} = \frac{\underset{\sim}{x} \cdot \underset{\sim}{\beta}}{|\underset{\sim}{\beta}|^2} \underset{\sim}{\beta}, \qquad x_\perp \cdot \underset{\sim}{\beta} = 0$$

Thus

$$\boxed{\underset{\sim}{x}'_\perp = \underset{\sim}{x}_\perp}$$

$$\underset{\sim}{x}'_{/\!/} = \gamma(\underset{\sim}{x}_{/\!/} - \underset{\sim}{\beta} x^0)$$

$$x'^0 = \gamma(x^0 - \underset{\sim}{\beta} \cdot \underset{\sim}{x})$$

$$\boxed{\underset{\sim}{x}' = \underset{\sim}{x}'_{\perp} + \underset{\sim}{x}'_{\parallel}}$$

$$= \underset{\sim}{x}_{\perp} + \gamma\left(\underset{\sim}{x}_{\parallel} - \underset{\sim}{\beta}\, x^0\right)$$

$$= \underset{\sim}{x} + (\gamma - 1)\underset{\sim}{x}_{\parallel} - \gamma \underset{\sim}{\beta}\, x^0$$

$$= \underset{\sim}{x} + (\gamma - 1)\frac{\underset{\sim}{x} \cdot \underset{\sim}{\beta}}{|\underset{\sim}{\beta}|^2}\underset{\sim}{\beta} - \gamma \underset{\sim}{\beta}\, x^0$$

$$\boxed{x'^0 = \gamma\left(x^0 - \underset{\sim}{\beta} \cdot \underset{\sim}{x}\right).}$$

$$\underset{\sim}{\beta} = \frac{\underset{\sim}{v}}{c}$$

This is the general form of x'^0

$$\gamma = \frac{1}{\sqrt{1 - \underset{\sim}{\beta}^2}}$$

Before proceeding further, first note that Galilean transformation and Lorentz transformation can be written as matrix

put $\underset{\sim}{x} = (x^0, \underset{\sim}{x}),$ $\quad \underset{\sim}{x} = 4$ component

$\underset{\sim}{x} = (x^1, x^2, x^3)$ 3-component

For Galilean transformation along x^1-axis

$$x'^1 = x^1 - vt, \quad x'^2 = x^2, \quad x'^3 = x^3,$$

$$t' = t$$

$$\begin{pmatrix} t' \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}$$

Different values v will give different Galilean transformations

Verify all Galilean transformations form a group i.e. satisfy 4 axioms of a group (H W)

known as the Galilean group

Home work

Defn of a group

A set S of elements $\{a, b, c \cdots d\}$

with a binary operation $\cdot$

such that (s.t)

(1) closure: If $a \in S$, $b \in S$,

then $a \cdot b \in S$

(2) $\exists$ (there exists) an identity I

$$I \cdot a = a = a \cdot I \quad \text{for any } a \in S$$

(3) Associativity:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(4) $\exists$ an iverse a^{-1} for any a

$$a^{-1} \cdot a = I \text{ (identity)}$$

$$= a \cdot a^{-1}$$

Group, usually denoted by G, is commonly used in physics; many transformations in physics form a Group. E.g., rotations form a rotation group denoted by $SO(3)$. Lorentz transformations form a group denoted by $SO(3,1)$.

similarly the Lorentz transformation along the x^1-axis can be written in a matrix form

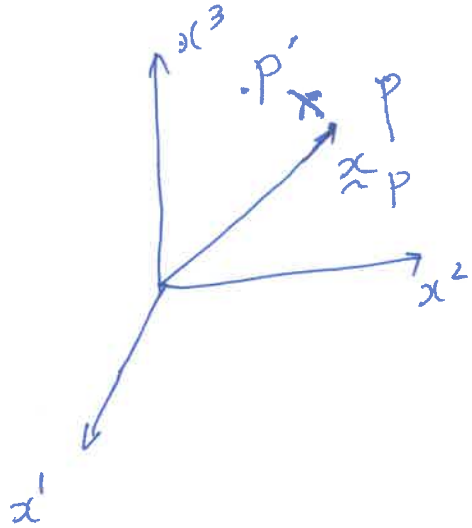
$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

All Lorentz transformations form a group, the Lorentz group (HW)

We now proceed to find the most general Lorentz transformation

we take cue from rotation transformation in 3 dimensional space

Position vector in 3 dimensional space is denoted by $\underset{\sim}{x}_p = (x_p^1, x_p^2, x_p^3)$



rotation $\mathcal{R}$, P moves to P'

$$\underset{\sim}{x}_p \xrightarrow{\mathcal{R}} \underset{\sim}{x}_{p'} = \mathcal{R}\,\underset{\sim}{x}_p$$

Distance of the point P before rotation

$$= {x_p^1}^2 + {x_p^2}^2 + {x_p^3}^2 \quad \cdots \quad (1)$$

After rotation $\mathcal{R}$, P moves to P', the distance of the point P' from the origin

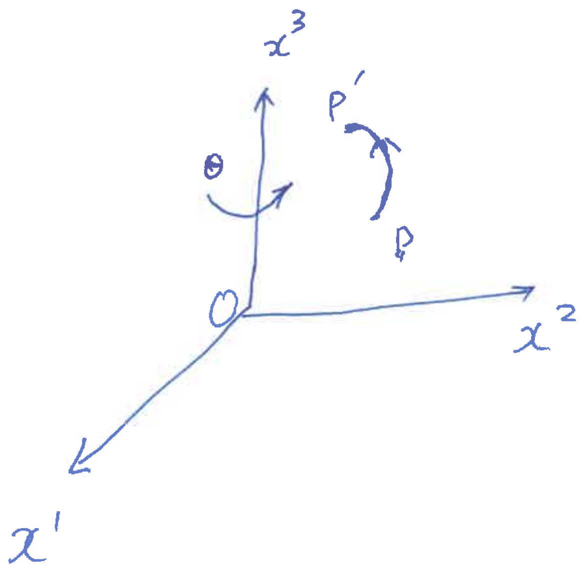
$$= {x_{p'}^1}^2 + {x_{p'}^2}^2 + {x_{p'}^3}^2. \quad \cdots \quad (2)$$

It is found : distance before rotation, eq (1)

= distance after rotation, eq (2).

We say spatial distance in 3 dimensional space is invariant under spatial rotation.

For a rotation about the x^3-axis (z-axis) by an angle θ, the rotation matrix is given by



$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It can be easily verified that for the Lorentz transformation

$$x'^0 = \gamma(x^0 - \beta x^1), \quad x'^1 = \gamma(x^1 - \beta x^0),$$
$$x'^2 = x^2, \qquad x'^3 = x^3$$

the quantity $\left(x^{0^2} - x^{1^2} - x^{2^2} - x^{3^2}\right)$ is the same before and after the Lorentz transformation stated above.

In fact, one finds the interval (Δs) as defined by

$$\Delta s^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2,$$

$$\Delta \underset{\sim}{x} = \underset{\sim}{x}_P - \underset{\sim}{x}_Q, \qquad P, Q \text{ two points}$$

$$\Delta x^0 = x^0_P - x^0_Q, \qquad \text{(events) in spacetime}$$

is unchanged (invariant) under the above Lorentz transformation (HW)

We can now introduce a general Lorentz transformation as a linear transformation that preserves the interval Δs^2.

A transformation Λ is linear iff

$$\Lambda(a \underline{x}_P + b \underline{x}_Q) = a \Lambda \underline{x}_P + b \Lambda \underline{x}_Q, \qquad a, b = \text{constants}$$

A Lorentz tran is a linear transformation that preserves the interval

$$\Delta s^2 = \Delta \underline{x} \cdot \Delta \underline{x} = \Delta x^{0\,2} - (\Delta \underline{x})^2$$

One denotes the Lorentz tran as $(\Lambda, \underline{a})$

$$\underline{x} \to \underline{x}' = \Lambda \underline{x}$$ (Homogeneous Lorentz tran)

or $$\underline{x}' = \Lambda \underline{x} + \underline{a}$$ (inhomogeneous Lorentz transformation = Poincaré tran.)

$\underline{a}$ = constant 4-vector

so $(\Lambda, \underline{a})$ transformation preserves the interval

$$\Delta \underline{x}' \cdot \Delta \underline{x}' = \Delta \underline{x} \cdot \Delta \underline{x}$$

For simplicity, discuss homogeneous Lorentz tran

Λ :

$$\underline{x} \to \underline{x}' = \Lambda \underline{x}$$

$\to$ $$s^2 = \underline{x} \cdot \underline{x} = \text{interval}$$

Λ preserves $\underline{x} \cdot \underline{x} = \underline{x}^2 = \left(x^{0\,2} - x^{1\,2} - x^{2\,2} - x^{3\,2}\right)$

i.e. $\underline{x}'^2 = \underline{x}^2$

First linear : $\Lambda(a\underline{x}_1 + b\underline{x}_2) = a\Lambda\underline{x}_1 + b\Lambda\underline{x}_2$

a, b are constant

The transformation $\underline{x}' = \Lambda \underline{x}$ can be written in component form

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \qquad \mu = 0, 1, 2, 3 \quad \nu = 0, 1, 2, 3$$

summation convention:

repeated indices means summation

ν runs from 0, 1, 2, 3

Thus $$x'^{\mu} = \Lambda^{\mu}{}_{0}\, x^{0} + \Lambda^{\mu}{}_{1}\, x^{1} + \Lambda^{\mu}{}_{2}\, x^{2} + \Lambda^{\mu}{}_{3}\, x^{3}$$ (HW)

From $\underline{x}'^2 = \underline{x}^2$, we can derive a relation for Λ

$$\underline{x}'^{2} = (\Lambda \underline{x})\cdot(\Lambda \underline{x}) = \underline{x}^{2}$$

$$(\Lambda x)^{\mu}\cdot(\Lambda x)_{\mu}$$

$$\left[\rightarrow \left(\Lambda^{\mu}{}_{\alpha}\, x^{\alpha}\right)\left(\Lambda_{\mu}{}^{\beta}\, x_{\beta}\right) = \underline{x}^{2}\right]$$

To proceed further, need to introduce metric tensor g

$$\underline{x}^{2} = x^{0\,2} - x^{1\,2} - x^{2\,2} - x^{3\,2}$$

$$\overset{HW}{=} g_{\mu\nu}\, x^{\mu} x^{\nu} \quad \text{if } g^{00} = +1,\ g^{11} = g^{22} = g^{33} = -1$$

$$g^{\mu\nu} = 0 \ \forall\ \mu \neq \nu$$

$g_{\mu\nu}$ tells us how to measure 'distance'

In ordinary 3-dim space

$$\underset{\sim}{x}^2 = x^{1\,2} + x^{2\,2} + x^{3\,2}$$

$$= g_{ij}\, x^i x^j, \qquad i, j = 1, 2, 3$$

$g_{ij} = 0$ except $i = j$

then $g_{11} = g_{22} = g_{33}$
$= +1$

g_{ij} = metric tensor, which defines Euclidean geometry in 3-dim space, if $g_{ij} = \delta_{ij}$

In 4-dim spacetime, the metric tensor is $g_{\mu\nu}$, where $g_{\mu\nu} = 0 \;\; \forall\, \mu \neq \nu$

and $g_{00} = +1, \quad g_{11} = -1 = g_{22} = g_{33}$

which defines Minkowski geometry or the Minkowski space time

In general $g_{\mu\nu} \rightarrow$ Riemannian geometry

Now go back to $\Lambda^\mu{}_\nu$

$$\underset{\sim}{x}'^2 = g_{\mu\nu}\, x'^\mu x'^\nu \qquad O' \text{ frame}$$

$$\underset{\sim}{x}^2 = g_{\mu\nu}\, x^\mu x^\nu \qquad O \text{ frame}$$

Note: $g_{\mu\nu}$ same for both O' frame and O frame
$\therefore$ same spacetime manifold, same geometry

$$\underline{x}'^2 = g_{\mu\nu} x'^\mu x'^\nu \qquad (x'^\mu = \Lambda^\mu{}_\nu x^\nu)$$

$$= g_{\mu\nu} \Lambda^\mu{}_\alpha x^\alpha \Lambda^\nu{}_\beta x^\beta$$

$$= g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta$$

$$\underline{x}^2 = g_{\alpha\beta} x^\alpha x^\beta$$

$$\because \quad \underline{x}'^2 = \underline{x}^2$$

$$\therefore \quad \underline{g_{\mu\nu} \cdot \Lambda^\mu{}_\alpha \cdot \Lambda^\nu{}_\beta = g_{\alpha\beta}}$$

this is the relation Λ must satisfy in order for Λ to be a Lorentz transformation.

HW: what are the $\Lambda^\mu{}_\nu$ for the Lorentz transformation along x'-axis

$$x'^0 = \gamma(x^0 - \beta x^1)$$

$$x'^1 = \gamma(x^1 - \beta x^0), \qquad x'^2 = x^2, \quad x'^3 = x^3$$

compare with $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $\therefore$

$$\Lambda^0{}_0 = \gamma, \qquad \Lambda^0{}_1 = -\gamma\beta$$

Write down the rest

$$\Lambda^\mu{}_\nu = ? \qquad \text{(HW)}$$

Some properties of Lorentz tran Λ

From definition $\underline{x} \to \underline{x}' = \Lambda \underline{x}$

In cpt form $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$

cf = compare

(Cf: 3-dimensional rotation

$\underset{\sim}{x} \to \underset{\sim}{x}' = \mathcal{R}\, \underset{\sim}{x}$

$\to x'_i = \mathcal{R}_{ij}\, x_j$

$\mathcal{R}_{ij}$ = 3 × 3 matrix)

So represent $\Lambda^{\mu}{}_{\nu}$ by a 4 × 4 matrix

Define a 4×4 matrix $(\Lambda)_{\mu\nu} \equiv \Lambda^{\mu}{}_{\nu}$

Thus in matrix form, for a Lorentz tran along x^1-axis

$$(\Lambda) = \begin{array}{c|cccc} & x^0, & x^1, & x^2, & x^3 \\ \hline x^0 & & & & \\ x^1 & & & & \\ x^2 & & & & \\ x^3 & & & & \end{array} \qquad \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

HW

$$(\Lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathcal{R} & \\ 0 & & & \end{pmatrix}$$

spatial rotation

$\mathcal{R}$ 3×3 matrix

$$(\Lambda_s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ space inversion

$$(\Lambda_t) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Time inversion

$$(\Lambda_{st}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ spacetime inversion

Any general Lorentz transformation Λ must satisfy

$$g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

which can be written in matrix form.

Define matrix

$$((g))_{\mu\nu} = g_{\mu\nu}$$

$$((\Lambda))_{\mu\nu} = \Lambda^{\mu}{}_{\nu}$$

Then we have

$$(g)_{\mu\nu} \; (\Lambda)_{\mu\alpha} \; (\Lambda)_{\nu\beta} = (g)_{\alpha\beta} \, .$$

$$(\Lambda^t)_{\alpha\mu} \; (g)_{\mu\nu} \; (\Lambda)_{\nu\beta} = (g)_{\alpha\beta} \, ,$$

Λ^t = transpose of Λ

$$\longrightarrow \quad \Lambda^t \, g \, \Lambda = g$$

Taking determinant,

$$\det (\Lambda^t g \Lambda) = \det (g)$$

$$\longrightarrow \quad \det \Lambda = \pm 1 \qquad (\text{HW})$$

cf: R = rotation in 3-dim space, $\det R = +1$

Next can show

$$\Lambda^0{}_0 > +1 \quad \text{or} \quad \Lambda^0{}_0 < -1$$

(HW)