PC3261: Classical Mechanics II

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Lecture 3: Linear Momentum

Linear momentum of two-particle system

- Forces are *assumed* to obey principle of superposition of forces: $\mathbf{f}_{12}$ is the force acting on m_1 due to m_2

$$\begin{cases} \mathbf{F}_1(t) = \mathbf{F}_1^{\text{ext}}(t) + \mathbf{f}_{12}(t) \\ \mathbf{F}_2(t) = \mathbf{F}_2^{\text{ext}}(t) + \mathbf{f}_{21}(t) \end{cases}$$

- Total linear momentum of the system: forces between particles are assumed to obey Newton's third law of motion

$$\mathbf{P}(t) \equiv \mathbf{p}_1(t) + \mathbf{p}_2(t) \quad \Rightarrow \quad \dot{\mathbf{P}}(t) = \mathbf{F}_1^{\text{ext}}(t) + \mathbf{F}_2^{\text{ext}}(t)$$

- Newton's second law: the time rate of change of total linear momentum of the two-particle system equals to the total *external* force acting upon it

$$\dot{\mathbf{P}}(t) = \mathbf{F}^{\text{ext}}(t)\,, \qquad \mathbf{F}^{\text{ext}}(t) \equiv \mathbf{F}_1^{\text{ext}}(t) + \mathbf{F}_2^{\text{ext}}(t)$$

Linear momentum of multi-particle system

- Total force acting on the α-particle: $\mathbf{f}_{\alpha\beta}$ is the force acting on m_α due to m_β

$$\mathbf{F}_\alpha(t) = \mathbf{F}_\alpha^{\mathsf{ext}}(t) + \sum_{\beta=1,\beta\neq\alpha}^{N} \mathbf{f}_{\alpha\beta}(t)\,, \qquad \alpha = 1, 2, 3, \cdots, N$$

- Total linear momentum of multi-particle system:

$$\mathbf{P}(t) \equiv \sum_{\alpha=1}^{N} \mathbf{p}_\alpha(t)$$

- Newton's second law: the time rate of change of total linear momentum of multi-particle system equals to the total *external* force acting upon it

$$\dot{\mathbf{P}}(t) = \mathbf{F}^{\mathsf{ext}}(t)\,, \qquad \mathsf{F}^{\mathsf{ext}}(t) \equiv \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{\mathsf{ext}}(t)$$

$$\mathbf{F}_\alpha(t) = \mathbf{F}_\alpha^{\text{ext}}(t) + \sum_{\beta=1,\beta\neq\alpha}^{N} \mathbf{f}_{\alpha\beta}(t) \quad \Rightarrow \quad \dot{\mathbf{p}}_\alpha(t) = \mathbf{F}_\alpha^{\text{ext}}(t) + \sum_{\beta=1,\beta\neq\alpha}^{N} \mathbf{f}_{\alpha\beta}(t)$$

$$\mathbf{P}(t) \equiv \sum_{\alpha=1}^{N} \mathbf{p}_\alpha(t) \quad \Rightarrow \quad \dot{\mathbf{P}}(t) \equiv \sum_{\alpha=1}^{N} \dot{\mathbf{p}}_\alpha(t)$$

$$\begin{aligned}
\dot{\mathbf{P}}(t) &= \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{\text{ext}}(t) + \sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} \mathbf{f}_{\alpha\beta}(t) \\
&= \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{\text{ext}}(t) + \sum_{\alpha=1}^{N} \sum_{\beta>\alpha} \left[\mathbf{f}_{\alpha\beta}(t) + \mathbf{f}_{\beta\alpha}(t)\right] \\
&= \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{\text{ext}}(t) \qquad \blacksquare
\end{aligned}$$

Impuse-Momentum theorem

- **Impulse-Momentum theorem** (integral form of the Newton's second law): change of total linear momentum equals to the time integral of the total external force

$$\mathbf{F}^{\text{ext}}(t) = \frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} \qquad \rightarrow \qquad \int_{t_1}^{t_2} \mathbf{F}^{\text{ext}}(t)\,\mathrm{d}t = \mathbf{P}(t_2) - \mathbf{P}(t_1)$$

- **Conservation law of linear momentum**: if the total external force on a multi-particle system is zero, then the total linear momentum of the multi-particle is a constant

$$\dot{\mathbf{P}}(t) = \mathbf{F}^{\text{ext}}(t) = \mathbf{0} \quad \Rightarrow \quad \mathbf{P}(t) = \text{constant} \quad \Rightarrow \quad \mathbf{P}(t_1) = \mathbf{P}(t_2) \quad \forall \quad t_1, t_2$$

- The validity of the conservation law of linear momentum depends crucially on the *experimental* basis of the Newton's third law!

A violation of Newton's third law???

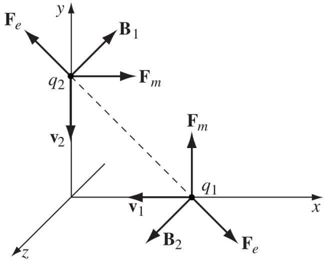
- Two point charges, q_1 and q_2, are moving at uniform velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively

- Electric fields and forces:

$$\begin{cases} \mathbf{E}_1(\mathbf{r}_2) = \dfrac{q_1}{4\pi\epsilon_0} \dfrac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ \mathbf{E}_2(\mathbf{r}_1) = \dfrac{q_2}{4\pi\epsilon_0} \dfrac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \end{cases} \Rightarrow \begin{cases} \mathbf{F}_{e,12} = q_1 \mathbf{E}_2(\mathbf{r}_1) \\ \mathbf{F}_{e,21} = q_2 \mathbf{E}_1(\mathbf{r}_2) \end{cases}$$

- Electric forces obey Newton's third law

$$\mathbf{F}_{e,12} = -\mathbf{F}_{e,21}$$



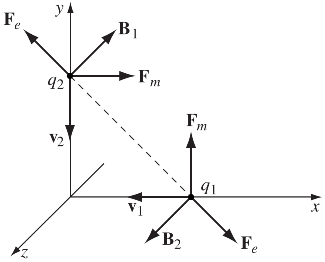
A violation of Newton's third law??? – cont'd

- Magnetic fields and forces:

$$\begin{cases} \mathbf{B}_1(\mathbf{r}_2) = \dfrac{\mu_0 q_1}{4\pi} \dfrac{\mathbf{v}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ \\ \mathbf{B}_2(\mathbf{r}_1) = \dfrac{\mu_0 q_2}{4\pi} \dfrac{\mathbf{v}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \end{cases} \Rightarrow \begin{cases} \mathbf{F}_{m,12} = q_1 \mathbf{v}_1 \times \mathbf{B}_2(\mathbf{r}_1) \\ \mathbf{F}_{m,21} = q_2 \mathbf{v}_2 \times \mathbf{B}_1(\mathbf{r}_2) \end{cases}$$

- Magnetic forces do not obey Newton's third law!

$$\mathbf{F}_{m,12} \neq -\mathbf{F}_{m,21}$$



- Electromagnetic linear momentum density: fields also possess linear momentum!

$$\mathbf{g}_{\text{EM}}(\mathbf{r}) = \epsilon_0 \, \mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})$$

System with variable mass

- Newton's second law with variable mass:

$$\mathbf{F}^{\text{ext}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,[m(t)\,\mathbf{v}(t)] \quad \xrightarrow{???} \quad \mathbf{F}^{\text{ext}}(t) = \frac{\mathrm{d}m(t)}{\mathrm{d}t}\,\mathbf{v}(t) + m\,\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t}$$

- Galilean velocity transformation: $\mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V}$

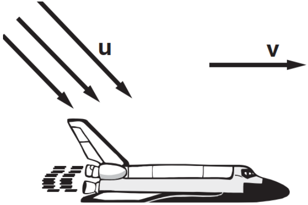
$$\frac{\mathrm{d}}{\mathrm{d}t}\,[m(t)\mathbf{v}'(t)] = \mathbf{F}^{\text{ext}}(t) \quad \not\leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\,[m(t)\mathbf{v}(t)] = \mathbf{F}^{\text{ext}}(t)$$

- There is *no* fundamental difficulty in handling any system with variable mass provided the same set of particles is included *throughout* the time interval t_1 to t_2

$$\int_{t_1}^{t_2} \mathbf{F}^{\text{ext}}(t)\,\mathrm{d}t = \mathbf{P}(t_2) - \mathbf{P}(t_1)$$

Example: Spacecraft and dust particles

- A spacecraft with mass M moves through space with constant velocity $\mathbf{v}$. The spacecraft encounters a stream of dust particles that embed themselves in the hull at rate dm/dt. The dust has velocity $\mathbf{u}$ just before it hits.



EXERCISE 3.1: Find the external force necessary to keep the spacecraft moving uniformly.

$$\begin{cases} \mathbf{P}(t) = M(t)\,\mathbf{v} + (\Delta m)\ \mathbf{u} \\ \\ \mathbf{P}(t+\Delta t) = M(t)\,\mathbf{v} + \Delta m\,\mathbf{v} \end{cases}$$

$$\Rightarrow \quad \Delta\mathbf{P} = \mathbf{P}(t+\Delta t) - \mathbf{P}(t) = (\mathbf{v} - \mathbf{u})\ \Delta m$$

$$\mathbf{F}(t) = \frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} = \lim_{\Delta\to 0} \frac{\Delta\mathbf{P}}{\Delta t} = (\mathbf{v} - \mathbf{u})\ \frac{\mathrm{d}m(t)}{\mathrm{d}t} \qquad \blacksquare$$

Newton's second law with variable mass

- A system with mass $m(t)$ moves at velovity $\mathbf{v}(t)$. Particles are added to the system at a rate $\mathrm{d}m(t)/\mathrm{d}t$. These particles have velocity $\mathbf{u}(t)$ just before entering the system.

- Newton's second law:

$$\mathbf{F}^{\text{ext}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[m(t)\,\mathbf{v}(t)\right] - \frac{\mathrm{d}m(t)}{\mathrm{d}t}\,\mathbf{u}(t)$$

$$\Rightarrow \quad m(t)\dot{\mathbf{v}}(t) = \mathbf{F}^{\text{ext}}(t) + \dot{m}(t)\left[\mathbf{u}(t) - \mathbf{v}(t)\right]$$

- Galilean invariance is preserved:

$$\mathbf{F}^{\text{ext}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[m(t)\mathbf{v}(t)\right] - \dot{m}(t)\mathbf{u}(t) \quad \leftrightarrow \quad \mathbf{F}^{\text{ext}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[m(t)\mathbf{v}'(t)\right] - \dot{m}(t)\mathbf{u}'(t)$$

$$m(t)\dot{\mathbf{v}}(t) = \mathbf{F}(t) + \dot{m}(t)\left[\mathbf{u}(t) - \mathbf{v}(t)\right]$$

$$\begin{cases} \mathbf{P}(t) = m(t)\mathbf{v}(t) + \Delta m\,\mathbf{u}(t) \\ \mathbf{P}(t+\Delta t) = [m(t) + \Delta m]\,[\mathbf{v}(t) + \Delta\mathbf{v}] \end{cases}$$

$$\Delta\mathbf{P} \equiv \mathbf{P}(t+\Delta t) - \mathbf{P}(t) = m(t)\Delta\mathbf{v} + \Delta m\,\mathbf{v}(t) + \Delta m\,\Delta\mathbf{v} - \Delta m\,\mathbf{u}(t)$$

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} &\equiv \lim_{\Delta t\to 0}\frac{\Delta\mathbf{P}}{\Delta t} \\ &= m(t)\lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t} + \left(\lim_{\Delta t\to 0}\frac{\Delta m}{\Delta t}\right)\mathbf{v}(t) + \left(\lim_{\Delta t\to 0}\Delta m\right)\left(\lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t}\right) - \left(\lim_{\Delta t\to 0}\frac{\Delta m}{\Delta t}\right)\mathbf{u}(t) \\ &= m(t)\dot{\mathbf{v}}(t) + \dot{m}(t)\mathbf{v}(t) - \dot{m}(t)\mathbf{u}(t) \qquad \blacksquare \end{aligned}$$

$$\mathbf{F}^{\text{ext}}(t) = \frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} \quad \Rightarrow \quad m(t)\dot{\mathbf{v}}(t) = \mathbf{F}(t) + \dot{m}(t)\left[\mathbf{u}(t) - \mathbf{v}(t)\right] \qquad \blacksquare$$

$$\mathbf{v}'(t) = \mathbf{v}(t) - \mathbf{V} \quad \Rightarrow \quad \dot{\mathbf{v}}'(t) = \dot{\mathbf{v}}(t)$$

$$m(t)\dot{\mathbf{v}}'(t) = \mathbf{F}(t) + \dot{m}(t)\left[\mathbf{u}'(t) - \mathbf{v}'(t)\right]$$

$$\Rightarrow \quad m(t)\dot{\mathbf{v}}(t) = \mathbf{F}(t) + \dot{m}(t)\left\{[\mathbf{u}(t) - \mathbf{V}] - [\mathbf{v}(t) - \mathbf{V}]\right\}$$

$$\Rightarrow \quad m(t)\dot{\mathbf{v}}(t) = \mathbf{F}(t) + \dot{m}(t)\left[\mathbf{u}(t) - \mathbf{v}(t)\right] \qquad \blacksquare$$

Example: Rocket in a constant gravitational field

- A rocket is taking off from rest in a uniform gravitation field $\mathbf{g} = -g\,\hat{\mathbf{e}}_z$. The fuel is ejected at a constant rate $\dot{m}(t) = -k$ at a constant exhaust speed u relative to the rocket.

- Linear momentum of the system:

$$\begin{cases} \mathbf{P}(t) = m(t)\mathbf{v}(t) \\ \mathbf{P}(t+\Delta t) = [m(t) + \Delta m]\,[\mathbf{v}(t) + \Delta\mathbf{v}] + (-\Delta m)\,[\mathbf{v}(t) + \Delta\mathbf{v} + \mathbf{u}(t+\Delta t)] \end{cases}$$

- Newton's second law:

$$m(t)\,\dot{\mathbf{v}}(t) - \mathbf{u}(t)\,\dot{m}(t) = \mathbf{F}^{\text{ext}}(t)$$

EXERCISE 3.2: Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the rocket in its subsequent motion given that the initial mass of the rocket is m_0.

$$\mathbf{g} = -g\,\hat{\mathbf{e}}_z\,, \qquad \mathbf{u} = -u\,\hat{\mathbf{e}}_z\,, \qquad \dot{m}(t) = -k \quad \Rightarrow \quad m(t) = m_0 - kt$$

$$m(t)\,\dot{\mathbf{v}}(t) - \mathbf{u}(t)\,\dot{m}(t) = \mathbf{F}^{\text{ext}}(t)$$

$$\Rightarrow \quad m(t)\,\frac{\mathrm{d}v_z(t)}{\mathrm{d}t} + \dot{m}(t)\,u = -m(t)g$$

$$\Rightarrow \quad \frac{\mathrm{d}v_z(t)}{\mathrm{d}t} = -g - u\,\frac{\dot{m}(t)}{m(t)}$$

$$\Rightarrow \quad v_z(t) = -gt - u\,\ln\frac{m(t)}{m_0}$$

$$\Rightarrow \quad v_z(t) = -gt - u\ln\left(\frac{m_0 - kt}{m_0}\right) \qquad \blacksquare$$

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = v_z(t) \quad \Rightarrow \quad z(t) = -\frac{1}{2}gt^2 - u\int_0^t \ln\left(\frac{m_0 - kt'}{m_0}\right)\,\mathrm{d}t'$$

$$\Rightarrow \quad z(t) = -\frac{1}{2}gt^2 - ut + \frac{u}{k}\,(m_0 - kt)\ln\left(\frac{m_0 - kt}{m_0}\right) \qquad \blacksquare$$

Center of mass

- Position vector of the **center of mass** of a multi-particle system:

$$\mathbf{R}_{\mathsf{CM}}(t) \equiv \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha(t)\,, \qquad M \equiv \sum_{\alpha=1}^{N} m_\alpha$$

- Velocity of the center of mass: total linear momentum of the system is equal to the linear momentum of the center of mass *as if* it were a particle of mass M with velocity $\mathbf{V}_{\mathsf{CM}}(t)$

$$\mathbf{V}_{\mathsf{CM}}(t) \equiv \dot{\mathbf{R}}_{\mathsf{CM}}(t) = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \dot{\mathbf{r}}_\alpha(t) \quad \Rightarrow \quad \mathbf{P}(t) = M\mathbf{V}_{\mathsf{CM}}(t)$$

- Acceleration of the center of mass: center of mass moves exactly *as if* it were a single particle of mass M subjected to the total external force on the system

$$\mathbf{A}_{\mathsf{CM}}(t) \equiv \ddot{\mathbf{R}}_{\mathsf{CM}}(t) = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \ddot{\mathbf{r}}_\alpha(t) \quad \Rightarrow \quad \dot{\mathbf{P}}(t) = M\,\mathbf{A}_{\mathsf{CM}}(t)$$

$$\mathbf{R}_{\mathsf{CM}}(t) = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha(t)\,, \qquad \mathbf{V}_{\mathsf{CM}}(t) = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \dot{\mathbf{r}}(t)$$

$$\mathbf{P}(t) = \sum_{\alpha=1}^{N} m_\alpha \dot{\mathbf{r}}_\alpha(t) = M\mathbf{V}_{\mathsf{CM}}(t) \qquad \blacksquare$$

Example: Projectile motion

- A rigid object consists of two masses m_1 and m_2 separated by a light rod of length L. It is thrown into the air.

- Center of mass:

$$\mathbf{R}_{\text{CM}}(t) = \frac{m_1 \mathbf{r}_1(t) + m_2 \mathbf{r}_2(t)}{m_1 + m_2}$$

- Equation of motion of the center of mass:

$$\mathbf{F}^{\text{ext}}(t) = (m_1 + m_2)\,\ddot{\mathbf{R}}_{\text{CM}}(t) \quad \Rightarrow \quad \ddot{\mathbf{R}}_{\text{CM}}(t) = \mathbf{g}$$

- The center of mass follows the parabolic trajectory of a single mass, $m_1 + m_2$, in a uniform gravitational field (motions of m_1 and m_2 about the center of mass are to be analyzed separately)

Center of mass of extended body

- Visualize mass element $\mathrm{d}m$ of volume $\mathrm{d}V$ located at position $\mathbf{r}(t)$ with mass density $\rho(\mathbf{r})$:

$$\mathbf{R}_{\mathsf{CM}}(t) = \frac{1}{M} \iiint_V \mathbf{r}(t)\,\rho(\mathbf{r})\,\mathrm{d}V$$

- Center of mass of a uniform solid (upper) hemisphere: mass M and radius R

$$\mathbf{R}_{\mathsf{CM}} = \frac{3}{8} R\,\hat{\mathbf{e}}_z$$

EXERCISE 3.3: A thin non-uniform plates lies on the xy-plane with corners $(0,0)$, $(a,0)$, $(0,b)$ and (a,b). Its surface mass density is $\sigma(x,y) = \sigma_0 xy/ab$ where σ_0 is a constant. Find its center of mass.

$$\mathbf{R}_{\mathsf{CM}}(t) = \frac{1}{M} \iiint_V \mathbf{r}(t)\,\rho(\mathbf{r})\,\mathrm{d}V\,, \qquad \mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z$$

$$\begin{aligned}
\mathbf{R}_{\mathsf{CM}} &= \frac{1}{M} \iiint_V \mathbf{r}\,\rho(\mathbf{r})\,\mathrm{d}V \\
&= \frac{1}{M} \iiint_V \left(x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z\right)\,\rho(\mathbf{r})\,\mathrm{d}V \\
&= \hat{\mathbf{e}}_x\,\frac{1}{M} \iiint_V x\,\rho(\mathbf{r})\,\mathrm{d}V + \hat{\mathbf{e}}_y\,\frac{1}{M} \iiint_V y\,\rho(\mathbf{r})\,\mathrm{d}V + \hat{\mathbf{e}}_z\,\frac{1}{M} \iiint_V z\,\rho(\mathbf{r})\,\mathrm{d}V \\
&\equiv X_{\mathsf{CM}}\,\hat{\mathbf{e}}_x + Y_{\mathsf{CM}}\,\hat{\mathbf{e}}_y + Z_{\mathsf{CM}}\,\hat{\mathbf{e}}_z \qquad \blacksquare
\end{aligned}$$

$$Z_{\mathsf{CM}} = \frac{1}{M}\int z\,\mathrm{d}m = \frac{1}{M}\int \frac{M}{V}\,z\,\mathrm{d}V = \frac{1}{V}\int z\,\pi r^2\,\mathrm{d}z$$

$$= \frac{1}{V}\int_0^R \pi z\left(R^2 - z^2\right)\,\mathrm{d}z = \frac{3}{8}\,R \qquad \blacksquare$$

$$Z_{\mathsf{CM}} = \frac{1}{M}\iiint_V z\rho\,\mathrm{d}V = \frac{1}{V}\int_{r=0}^{R}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(r\cos\theta)\;r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3}{8}\,R \qquad \blacksquare$$

$$Y_{\mathsf{CM}} = \frac{1}{M}\iiint_V y\rho\,\mathrm{d}V = \frac{1}{V}\int_{r=0}^{R}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(r\sin\theta\sin\phi)\;r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = 0 \qquad \blacksquare$$

$$\mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y\,, \qquad \sigma(\mathbf{r}) = \sigma(x,y) = \sigma_0\,\frac{xy}{ab}$$

$$M = \iint \sigma(\mathbf{r})\,\mathrm{d}A = \int_{x=0}^{a}\int_{y=0}^{b} \frac{\sigma_0}{ab}\,xy\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\,\sigma_0 ab \qquad \blacksquare$$

$$X_{\mathsf{CM}} = \frac{1}{M}\iint x\,\sigma(\mathbf{r})\,\mathrm{d}A = \frac{1}{M}\int_{x=0}^{a}\int_{y=0}^{b} x\left(\frac{\sigma_0}{ab}\,xy\right)\,\mathrm{d}x\,\mathrm{d}y = \frac{2}{3}\,a \qquad \blacksquare$$

$$Y_{\mathsf{CM}} = \frac{1}{M}\iint y\,\sigma(\mathbf{r})\,\mathrm{d}A = \frac{1}{M}\int_{x=0}^{a}\int_{y=0}^{b} y\left(\frac{\sigma_0}{ab}\,xy\right)\,\mathrm{d}x\,\mathrm{d}y = \frac{2}{3}\,b \qquad \blacksquare$$

$$\mathbf{R}_{\mathsf{CM}} = \frac{2}{3}\,a\,\hat{\mathbf{e}}_x + \frac{2}{3}\,b\,\hat{\mathbf{e}}_y \qquad \blacksquare$$