Example: Harmonic oscillator

- Hamilton equations of motion:

$$\mathcal{H}(q,p) = \frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m} \;\Rightarrow\; \begin{cases} \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q \\[2ex] \dot{q} = \dfrac{\partial \mathcal{H}}{\partial p} = \dfrac{p}{m} \end{cases} \;\Rightarrow\; \begin{cases} \ddot{q} = -\omega^2 q \\ \ddot{p} = -\omega^2 p \end{cases}$$

- Type 1 generating function: this canonical transformation effectively exchanges the role of the coordinate and momentum!

$$\Lambda_1 \equiv \Lambda_1(q,Q,t) = qQ \qquad \Rightarrow \qquad \begin{cases} Q \equiv Q(q,p,t) = -p \\ P \equiv P(q,p,t) = q \end{cases}$$

EXERCISE 11.4: Obtain the canonical transformation generated by $\Lambda_1(q,Q,t) = qQ$ and the Kamiltonian equations of motion.

$$\mathcal{H}(q,p) = \frac{1}{2}\,m\omega^2 q^2 + \frac{p^2}{2m}$$

$$\Lambda_1 \equiv \Lambda_1(q, p(q,Q,t), t) = \Lambda_1(q,Q,t) = qQ \quad \Rightarrow \quad \frac{\mathrm{d}\Lambda_1}{\mathrm{d}t} = \frac{\partial \Lambda_1}{\partial q}\,\dot{q} + \frac{\partial \Lambda_1}{\partial Q}\,\dot{Q} + \frac{\partial \Lambda_1}{\mathrm{d}t}$$

$$\tilde{\mathcal{L}}'(q,p,\dot{q},\dot{p},t) = \tilde{\mathcal{L}}(q,p,\dot{q},\dot{p},t) + \frac{\mathrm{d}\Lambda_1(q,p,t)}{\mathrm{d}t}$$

$$\Rightarrow \quad P\dot{Q} - \mathcal{K}(Q,P,t) = p\dot{q} - \mathcal{H}(q,p,t) + \frac{\partial \Lambda_1}{\partial q}\,\dot{q} + \frac{\partial \Lambda_1}{\partial Q}\,\dot{Q} + \frac{\partial \Lambda_1}{\mathrm{d}t}$$

$$\Rightarrow \quad \begin{cases} p \equiv p(q,Q,t) = -\dfrac{\partial \Lambda_1}{\partial q} = -Q \\[2ex] P \equiv P(q,Q,t) = \dfrac{\partial \Lambda_1}{\partial Q} = q \\[2ex] \mathcal{K}(Q,P,t) = \mathcal{H}(q,p,t) - \dfrac{\partial \Lambda_1}{\partial t} \end{cases} \quad \Rightarrow \quad \begin{cases} Q \equiv Q(q,p,t) = -p \\[2ex] P \equiv P(q,p,t) = q \\[2ex] \mathcal{K} \equiv \mathcal{K}(Q,P,t) = \dfrac{1}{2}\,m\omega^2 P^2 + \dfrac{Q^2}{2m} \end{cases}$$

$$\begin{cases} \dot{Q} = \dfrac{\partial \mathcal{K}}{\partial P} = m\omega^2 P \\[2ex] \dot{P} = -\dfrac{\partial \mathcal{K}}{\partial Q} = -\dfrac{Q}{m} \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{Q} = -\omega^2 Q \\[1ex] \ddot{P} = -\omega^2 P \end{cases} \qquad \blacksquare$$

Canonicality

- A transformation is canonical if and only the fundamental Poisson brackets are invariant:

$$\{Q_i, Q_j\}_{q,p} = 0\,, \qquad \{P_i, P_j\}_{q,p} = 0\,, \qquad \{Q_i, P_j\}_{q,p} = \delta_{ij}$$

- Solving harmonic oscillator by guessing at a strategic canonical transformation:

$$\begin{cases} q \equiv q(Q,P,t) = \sqrt{\dfrac{2P}{m\omega}}\,\sin Q \\[2ex] p \equiv p(Q,P,t) = \sqrt{2m\omega P}\,\cos Q \end{cases} \quad \Rightarrow \quad \mathcal{K}(Q,P,t) = \omega P$$

- A practical convenient strategy for tackling a dynamical system is to find/guess a canonical transformation to simplify the Hamiltonian and then verify the canonicality using the Poisson bracket!

EXERCISE 11.5: Solve for $q(t)$ and $p(t)$ via $Q(t)$ and $P(t)$.

$$
\begin{cases}
q \equiv q(Q, P, t) = \sqrt{\dfrac{2P}{m\omega}}\, \sin Q \\[2ex]
p \equiv p(Q, P, t) = \sqrt{2m\omega P}\, \cos Q
\end{cases}
$$

$$
\begin{aligned}
\{q, p\}_{Q,P} &= \frac{\partial q}{\partial Q}\,\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\,\frac{\partial p}{\partial Q} \\
&= \left(\sqrt{\frac{2P}{m\omega}}\, \cos Q\right) \left(\sqrt{\frac{m\omega}{2P}}\, \cos Q\right) - \left(\sqrt{\frac{1}{2m\omega P}}\, \sin Q\right) \left(-\sqrt{2m\omega P}\, \sin Q\right) \\
&= 1 \qquad \blacksquare
\end{aligned}
$$

$$
\{q, q\}_{Q,P} = \frac{\partial q}{\partial Q}\,\frac{\partial q}{\partial P} - \frac{\partial q}{\partial P}\,\frac{\partial q}{\partial Q} = 0 \qquad \blacksquare
$$

$$
\{p, p\}_{Q,P} = \frac{\partial p}{\partial Q}\,\frac{\partial p}{\partial P} - \frac{\partial p}{\partial P}\,\frac{\partial p}{\partial Q} = 0 \qquad \blacksquare
$$

$$\mathcal{K}(Q, P, t) = \omega P$$

$$\begin{cases} \dot{Q} = \dfrac{\partial \mathcal{K}}{\partial P} = \omega \\[2ex] \dot{P} = -\dfrac{\partial \mathcal{K}}{\partial Q} = 0 \end{cases} \Rightarrow \quad \begin{cases} Q(t) = \omega t + Q(0) \\ P(t) = P(0) \end{cases} \qquad \blacksquare$$

$$\begin{cases} q = \sqrt{\dfrac{2P}{m\omega}}\, \sin Q \\[2ex] p = \sqrt{2m\omega P}\, \cos Q \end{cases} \Rightarrow \quad \begin{cases} q(t) = \sqrt{\dfrac{2Q(0)}{m\omega}}\, \sin\left[\omega t + Q(0)\right] \\[2ex] p(t) = \sqrt{2m\omega\, P(0)}\, \cos\left[\omega t + Q(0)\right) \end{cases} \qquad \blacksquare$$

Liouville's theorem

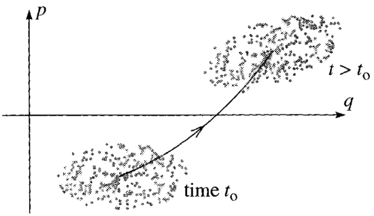
- Continuity equation: ρ is the volume charge density and $\mathbf{J}$ is the volume current density in E&M

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

- Hamiltonian mechanics: $\rho \equiv \rho\left(\{q_i, p_i\}, t\right)$ is the density of points in the phase space and the corresponding "current density" is defined by $\sum_{i=1}^{M} \rho\left(\dot{q}_i + \dot{p}_i\right)$

- **Liouville's theorem**: density of points in the phase space corresponding to the time evolution of the systems remains constant during the time evolution

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$$



$$\rho \equiv \rho\left(\{q_i, p_i\}, t\right)$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \hat{\mathbf{f}}_i \cdot \hat{\mathbf{f}}_j = \delta_{ij}\,, \qquad \hat{\mathbf{e}}_i \cdot \hat{\mathbf{f}}_j = 0\,, \qquad i, j = 1, 2, \cdots, M$$

$$\mathbf{J} = \sum_{i=1}^{M} \rho\,\dot{q}_i\,\hat{\mathbf{e}}_i + \rho\,\dot{p}_i\,\hat{\mathbf{f}}_i\,, \qquad \boldsymbol{\nabla} \equiv \sum_{i=1}^{M} \hat{\mathbf{e}}_i\,\frac{\partial}{\partial q_i} + \hat{\mathbf{f}}_j\,\frac{\partial}{\partial p_i}$$

$$\boldsymbol{\nabla} \cdot \mathbf{J} = \sum_{i=1}^{M} \frac{\partial}{\partial q_i}\left(\rho\,\dot{q}_i\right) + \frac{\partial}{\partial p_i}\left(\rho\,\dot{p}_i\right) = \sum_{i=1}^{M} \frac{\partial \rho}{\partial q_i}\,\dot{q}_i + \frac{\partial \rho}{\partial p_i}\,\dot{p}_i + \rho\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right)$$

$$= \sum_{i=1}^{M} \frac{\partial \rho}{\partial q_i}\,\dot{q}_i + \frac{\partial \rho}{\partial p_i}\,\dot{p}_i + \rho\left(\frac{\partial^2 \mathcal{H}}{\partial q_i\,\partial p_i} - \frac{\partial^2 \mathcal{H}}{\partial p_i\,\partial q_i}\right) = \sum_{i=1}^{M} \frac{\partial \rho}{\partial q_i}\,\dot{q}_i + \frac{\partial \rho}{\partial p_i}\,\dot{p}_i$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \sum_{i=1}^{M} \frac{\partial \rho}{\partial q_i}\,\dot{q}_i + \frac{\partial \rho}{\partial p_i}\,\dot{p}_i = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \qquad \blacksquare$$

PC3261: Classical Mechanics II

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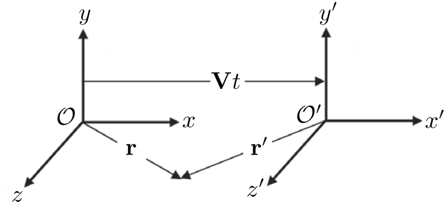
Lecture 12: Special Relativity I

Theory of relativity in Physics



- **Reference frame** is defined as an oriented system of coordinates in three-dimensional space equipped with rulers and clocks to perform measurements of position and time

- Theory of relativity establishes a *connection* between spatial and temporal measurements made in two reference frames

- Two **inertial** reference frames are arranged in a *standard configuration* where their spatial coordinate axes are aligned, their spatial origins are coincided when $t = t' = 0$ and their relative motion occurs with constant speed V along their parallel axes x and x'

Galilean relativity

- Galilean principle of relativity: laws of mechanics are the same in all inertial frames

- **Galilean boost**: constant velocity $\mathbf{V}$ is in an arbitrary direction

$$t' = t\,, \qquad\qquad \mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{V}t$$

- Equation of motion of the ith particle within a group of particles interacting via two-body central potential:

$$m_i\ddot{\mathbf{r}}(t) = -\boldsymbol{\nabla}_i \sum_j U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \quad \rightarrow \quad m_i\ddot{\mathbf{r}}'(t') = -\boldsymbol{\nabla}'_i \sum_j U'_{ij}(|\mathbf{r}'_i - \mathbf{r}'_j|)$$

- Wave equation is *not* invariant under Galilean boost

EXERCISE 12.1: Verify explicitly that the wave equation is not invariant under Galilean boost between two inertial frames with arbitrary constant relative velocity.

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right]\psi(\mathbf{r},t)=0$$

$$\begin{cases} t'=t \\ \mathbf{r}'=\mathbf{r}-\mathbf{V}t \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial}{\partial t}=\dfrac{\partial t'}{\partial t}\dfrac{\partial}{\partial t'}+\dfrac{\partial \mathbf{r}'}{\partial t}\cdot\dfrac{\partial}{\partial \mathbf{r}'}=\dfrac{\partial}{\partial t'}-\mathbf{V}\cdot\dfrac{\partial}{\partial \mathbf{r}'} \\ \dfrac{\partial}{\partial \mathbf{r}}=\dfrac{\partial \mathbf{r}'}{\partial \mathbf{r}}\cdot\dfrac{\partial}{\partial \mathbf{r}'}+\dfrac{\partial t'}{\partial \mathbf{r}}\dfrac{\partial}{\partial t'}=\dfrac{\partial}{\partial \mathbf{r}'} \end{cases} \quad \blacksquare$$

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right]\psi(\mathbf{r},t)=0$$

$$\Rightarrow \quad \left[-\frac{1}{c^2}\left(\frac{\partial}{\partial t'}-\mathbf{V}\cdot\frac{\partial}{\partial \mathbf{r}'}\right)\left(\frac{\partial}{\partial t'}-\mathbf{V}\cdot\frac{\partial}{\partial \mathbf{r}'}\right)+\frac{\partial}{\partial \mathbf{r}'}\cdot\frac{\partial}{\partial \mathbf{r}'}\right]\psi'(\mathbf{r}',t')=0$$

$$\Rightarrow \quad \left[-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}+\nabla'^2+\frac{2}{c^2}\left(\mathbf{V}\cdot\boldsymbol{\nabla}'\right)\frac{\partial}{\partial t'}-\frac{1}{c^2}\left(\mathbf{V}\cdot\boldsymbol{\nabla}'\right)^2\right]\psi'(\mathbf{r}',t')=0 \quad \blacksquare$$

Postulates of special relativity

- **Principle of relativity**: The laws of Physics are the same in all inertial frames

- **Constancy of the speed of light**: The speed of light in vacuum is the same in all inertial frames regardless of the motion of its emitter or receiver

Derivation of Lorentz boost

- Linear transformation: straight lines are preserved, 20 parameters

- Coincidence of spatial and temporal origins: homogeneity of space and time, 16 parameters

- Alignment of spatial axes and choice of relative velocity along x-direction: isotropy of space, 6 parameters

$$t' = At + Bx\,, \qquad x' = Ct + Dx\,, \qquad y' = Ey\,, \qquad z' = Fz$$

- Symmetry: $(x, z) \to (-x, -z), (x', z') \to (-x', -z')$

$$y = Ey' \quad \Rightarrow \quad E^2 = 1 \quad \Rightarrow \quad E = +1$$

- Symmetry: $(x, y) \to (-x, -y), (x', y') \to (-x', -y')$

$$z = Fz' \quad \Rightarrow \quad F^2 = 1 \quad \Rightarrow \quad F = +1$$

Derivation of Lorentz boost – cont'd

- Choice of relative motion along x-direction:

$$x' = 0 \quad \Rightarrow \quad x = Vt \quad \Rightarrow \quad C = -DV$$

- Constancy of the speed of light:

$$x^2 + y^2 + z^2 = c^2 t^2 \quad \Leftrightarrow \quad x'^2 + y'^2 + z'^2 = c^2 t'^2$$

$$\Rightarrow \quad \begin{cases} -A^2c^2 + D^2V^2 = -c^2 \\ ABc^2 + D^2V = 0 \\ -B^2c^2 + D^2 = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} A = D = \dfrac{1}{\sqrt{1 - V^2/c^2}} \\ B = -\dfrac{V}{c^2}D = -\dfrac{V}{c^2\sqrt{1 - V^2/c^2}} \end{cases}$$

- Lorentz boost between frames in standard orientations: $\beta \equiv V/c$

$$ct' = \gamma\,(ct - \beta x)\ , \quad x' = \gamma\,(x - \beta ct)\ , \quad y' = y\,, \quad z' = z\,, \quad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$