PC4245

chapter 4 part II

Discrete symmetries

P = parity, space inversion (mirror reflection)

C = charge conjugation (positive charge $\rightleftharpoons$ negative charge)

T = time reversal (motion reversal, $\underset{\sim}{p} \to -\underset{\sim}{p}$)

First discuss space inversion, P

Review symmetry transformation in quantum mechanics

Introduce space inversion in 3-dimensional physical space, then as an operator in quantum mechanics

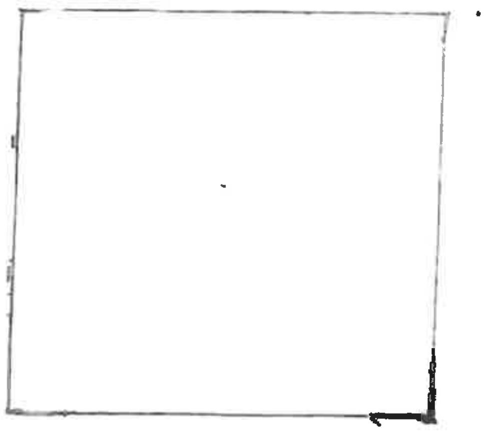
Parity operator π, π unitary, Hermitian (observable)

Downfall of parity conservation in weak interaction

C. S. Wu experiment 1956

parity broken in weak decay.

A square mirror puzzle



L - R symmetrical

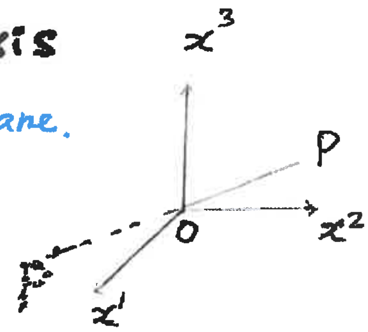
U - d symmetrical

However,
image is LR reversed
but not ud reversed

Space inversion :

$$\underset{\sim}{x} \longrightarrow \underset{\sim}{x}' = -\underset{\sim}{x}$$

Same as mirror reflection plus a rotation of 180° about an axis



Consider a point P in front of the x^3-x^1 plane.

Its coordinates $= (0, x^2, x^3) = \underset{\sim}{x}_P$

Mirror reflection on the x^3-x^1 plane,

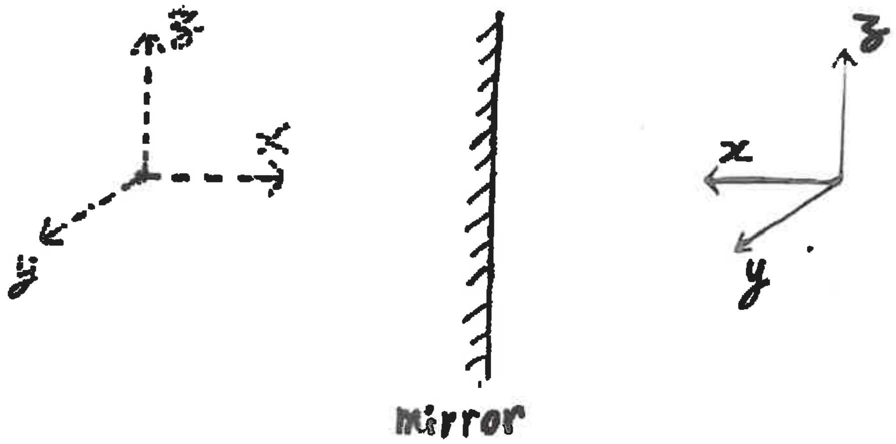
$$\underset{\sim}{x}_P \rightarrow \underset{\sim}{x}_P^M = (0, -x^2, x^3).$$

Rotation about the x^2-axis by 180°,

$$\underset{\sim}{x}_P^M \rightarrow \underset{\sim}{x}_P^{MR} = (0, -x^2, -x^3). = -\underset{\sim}{x}_P$$

Mirror reflection : Why L-R inverted and not Up-down inverted ?

It is the axis normal to the mirror that is inverted

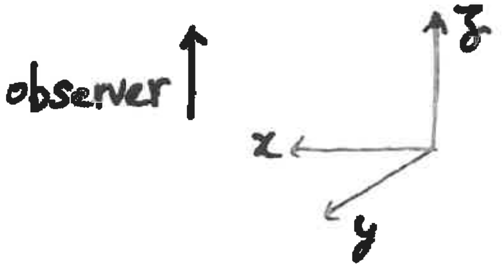


Observer facing ← x axis sees the ↙ y-axis on his RHS

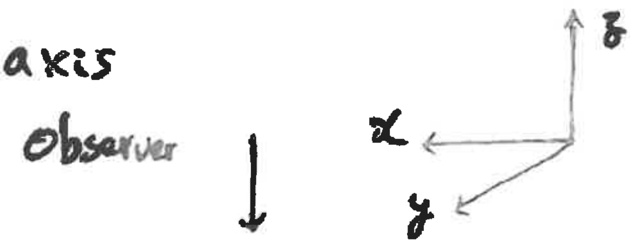
But when the observer facing the mirror image (facing the ⇠ x axis) the ⇠ y-axis is on his LHS

In the above the observer turns 180° about the z-axis

Consider the observer facing ← x-axis again
He sees the z-axis above his head.



Now the observer turn 180° about the y-axis



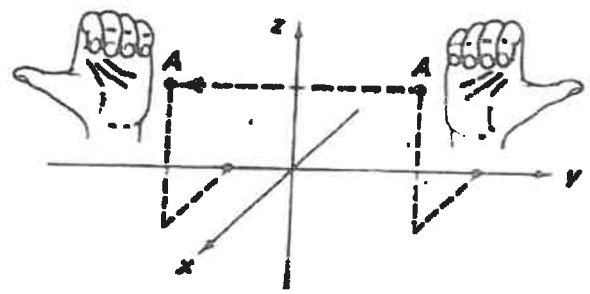
to face the mirror
He faces ----> x axis

He then sees the z axis
below his feet
The image is up side down !

Because we move on 2-dimensional plane, so
usually we turn about z-axis and
mirror image appears to be L-R inverted

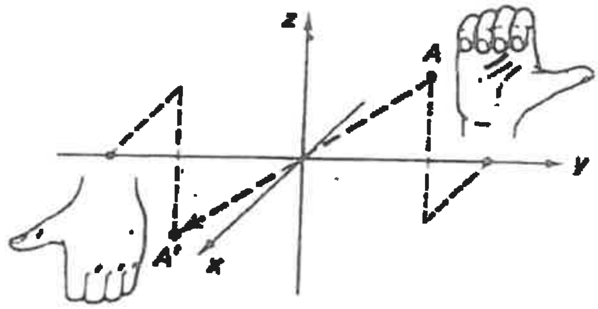
If we are free to move in 3-dimensional
space, then we can easily turn about the
y-axis and the mirror image would
appear up-down inverted.

Recap: Mirror reflection is just one of
coordinate axes inverted.
whether Left-Right or Up-down
depends on how the observer viewing the
image



(a) Reflection (in the x-z plane)
$(x, y, z) \rightarrow (x, -y, z)$

This requires where to put the mirror



(b) Inversion $(x, y, z) \rightarrow (-x, -y, -z)$

No mirror needed

Figure 4.11 Reflections and inversions.

Symmetry Transformation in Quantum Mechanics

In QM, state $|\psi\rangle$ and operators are the key elements in analyzing a physical problem. Clearly, a symmetry transformation is associated with an operator in Hilbert space.

Define symmetry transformation in QM:
A symmetry transformation operator U is a 1-1 mapping that maps a dynamically possible state, say $|\psi\rangle$, to another dynamically possible state $|\psi'\rangle$, namely $U:|\psi\rangle \to |\psi'\rangle = U|\psi\rangle$, such that the transition probability is preserved, i.e. no change in the transition probability.

$$\text{Transition probability from state } |\psi\rangle \text{ to state } |\varphi\rangle = |\langle\varphi|\psi\rangle|^2$$
$$\text{Transition probability from state } |\psi'\rangle \text{ to state } |\varphi'\rangle = |\langle\varphi'|\psi'\rangle|^2$$
$$\text{Transition probability is preserved means: } |\langle\varphi|\psi\rangle|^2 = |\langle\varphi'|\psi'\rangle|^2$$

In other words, the transition probability before applying the symmetry transformation U is the same as the transition probability after the applying the same symmetry transformation.

From the definition, we can show that
(i) U is unitary.
(ii) U is linear or anti-linear.
(iii) If U does not depend on time explicitly, then $[U, H] = 0$

Proof:

(i) An operator A is unitary if $A^+ = A^{-1}$.
We want to show that symmetry transformation U is unitary.
Given that $|\langle\varphi|\psi\rangle|^2 = |\langle\varphi'|\psi'\rangle|^2$, $|\psi'\rangle = U|\psi\rangle$, $|\varphi'\rangle = U|\varphi\rangle \Rightarrow \langle\varphi'| = \langle\varphi|U^+$, then
$$|\langle\varphi'|\psi'\rangle|^2 = |\langle\varphi|U^+U|\psi\rangle|^2 = |\langle\varphi|\psi\rangle|^2$$

This is true for any arbitrary state $|\varphi\rangle$ and $|\psi\rangle$, hence $U^+U = 1$.

By associativity rule $a\cdot(b\cdot c) = (a\cdot b)\cdot c$, we can show that $UU^+ = 1$.
From the definition of an inverse operator, $U^{-1}U = 1 = UU^{-1}$, so we have $U^{-1} = U^+$, i.e. U is unitary.

(ii) To show U is linear or anti-linear, we consider a state $|\psi\rangle$ and a state $\alpha|\psi\rangle$, where α is a complex number.

U is linear if $U(\alpha|\psi\rangle) = \alpha(U|\psi\rangle)$;
U is anti-linear if $U(\alpha|\psi\rangle) = \alpha^*(U|\psi\rangle)$, where $\alpha^* =$ complex conjugate of α.

Given $|\langle\varphi|\psi\rangle|^2 = |\langle\varphi'|\psi'\rangle|^2$, one can have either $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$ or $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^*$. Then U is linear if $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$; U is anti-linear if $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^*$.

Consider first the case $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$,
LHS: $\langle\varphi'|\psi'\rangle = \langle\varphi'|U|\psi\rangle$, let $|\psi\rangle = \lambda|\Omega\rangle$, where λ = constant, then

$$\langle\varphi'|\psi'\rangle = \langle\varphi'|U\lambda|\Omega\rangle$$

RHS: $\langle\varphi|\psi\rangle = \langle\varphi|\lambda|\Omega\rangle = \lambda\langle\varphi|\Omega\rangle = \lambda\langle\varphi'|\Omega'\rangle = \lambda\langle\varphi'|U|\Omega\rangle = \langle\varphi'|\lambda U|\Omega\rangle$

Since $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$, then $\langle\varphi'|U\lambda|\Omega\rangle = \langle\varphi'|\lambda U|\Omega\rangle$.

As $\langle\varphi'|$ and $|\Omega\rangle$ are arbitrary, so $U\lambda = \lambda U$, i.e. U is linear.

If we start from $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^*$ instead of $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle$, then we can show $U\lambda = \lambda^* U$, i.e. U is anti-linear.
LHS=$\langle\varphi'|\psi'\rangle$=$\langle\varphi'|U|\psi\rangle = \langle\varphi'|U\lambda|\omega\rangle$
RHS= $\langle\varphi|\psi\rangle^* = (\langle\varphi|\lambda|\omega\rangle)^* = (\lambda\langle\varphi|\omega\rangle)^* = \lambda^*\langle\varphi|\omega\rangle^* = \lambda^*\langle\varphi'|\omega'\rangle = \lambda^*\langle\varphi'|U|\omega\rangle = \langle\varphi'|\lambda^* U|\omega\rangle$

LHS=RHS gives $\langle\varphi'|U\lambda|\omega\rangle$=$\langle\varphi'|\lambda^* U|\omega\rangle$, that is, $U\lambda|\omega\rangle$=$\lambda^* U|\omega\rangle$since $\langle\varphi'|$ is arbitrary. Or, $U\lambda = \lambda^* U$.
Note: Put $|\omega\rangle = a|\omega_1\rangle + b|\omega_2\rangle$, then $U\lambda|\omega\rangle = U\lambda(a|\omega_1\rangle + b|\omega_2\rangle) = U(\lambda a|\omega_1\rangle + \lambda b|\omega_2\rangle) = U\lambda a|\omega_1\rangle + U\lambda b|\omega_2\rangle$

$$= \lambda^* a^* U|\omega_1\rangle + \lambda^* b^* U|\omega_2\rangle$$

That is equivalent to $U(a|\varphi\rangle + b|\psi\rangle) = a^* U|\varphi\rangle + b^* U|\psi\rangle$.

So we have shown that U is either linear or anti-linear.

A linear unitary operator is usually called a unitary operator. An anti-linear unitary operator is called anti-unitary operator. In nature, most of the symmetry transformations are associated with unitary operators. Time reversal and charge conjugation are associated with anti-unitary operators.

(iii) To show $[U,H] = 0$ if $\frac{\partial U}{\partial t} = 0$, we consider 2 dynamically possible states $|\psi\rangle$ and $|\psi'\rangle = U|\psi\rangle$.

By definition, a dynamically possible state is a state that satisfies the TDSE:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \quad \text{-- (1)}$$

$$i\hbar\frac{\partial}{\partial t}|\psi'\rangle = H|\psi'\rangle \quad \text{-- (2)}$$

From the 2nd equation above, we have LHS:

$$i\hbar\frac{\partial}{\partial t}U|\psi\rangle = \underbrace{i\hbar\left(\frac{\partial U}{\partial t}\right)|\psi\rangle + Ui\hbar\frac{\partial}{\partial t}|\psi\rangle}_{\text{chain rule}} = \underbrace{i\hbar\left(\frac{\partial U}{\partial t}\right)|\psi\rangle + UH|\psi\rangle}_{\text{sub using (1)}}$$

For RHS: $\quad H|\psi'\rangle = HU|\psi\rangle$

Since LHS = RHS, we have

$$i\hbar\cancelto{0}{\left(\frac{\partial U}{\partial t}\right)}|\psi\rangle + UH|\psi\rangle = HU|\psi\rangle \quad \longleftarrow \quad \text{Take U as linear}$$

If U does not depend on time explicitly, i.e. $\frac{\partial U}{\partial t} = 0$, then we have

$$(UH - HU)|\psi\rangle = 0$$

As $|\psi\rangle$ is any dynamically possible state, so we have $[U, H] = 0$.

What if U is anti-linear?

$$i\hbar\frac{\partial}{\partial t}|\psi'\rangle = H|\psi'\rangle = HU|\psi\rangle$$

If U is linear: $\quad UH|\psi\rangle = HU|\psi\rangle, \quad [U, H] = 0$

If U is anti-linear: $\quad HU|\psi\rangle = HU|\psi\rangle$

$$-UH|\psi\rangle = HU|\psi\rangle, \quad \{U, H\} = 0$$

changing the position requires taking the complex conjugate space inversion can be represented by a 3×3 matrix (8)

$$\mathcal{R}_- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\underset{\sim}{x} \longrightarrow \underset{\sim}{x}' = \mathcal{R}_-[\underset{\sim}{x}] = -\underset{\sim}{x}$$

In 3-dim physical space, rotations form a group SO(3). If reflections included, then group is O(3) := orthogonal group in 3 dimensions.

Mirror Reflection $\rightarrow$ chirality = handedness.

Apply mirror reflection to a physical system, the state of the physical system may or may not change

In QM state is described by $|\Psi\rangle$ and mirror reflection is by an operator

A tran $\mathcal{R}_-$ in 3-dim. space induces an operator π in Hilbert space: π is defined:

$$\underset{\sim}{x} \rightarrow \underset{\sim}{x}' = \pi \underset{\sim}{x} \pi^{-1} = -\underset{\sim}{x} \rightarrow \pi \underset{\sim}{x} = -\underset{\sim}{x} \pi$$

$$\underset{\sim}{p} \rightarrow \underset{\sim}{p}' = \pi \underset{\sim}{p} \pi^{-1} = -\underset{\sim}{p}$$

$$\underset{\sim}{J} \rightarrow \underset{\sim}{J}' = \pi \underset{\sim}{J} \pi^{-1} = \underset{\sim}{J}$$

(Ballentine: QM, A modern approach WSPC)

In QM, π = parity operator

From definition of π, we have

(i) π is unitary, linear

can show $\pi^2 = 1$ ← identity

$\rightarrow \quad \pi^{-1} = \pi$

But $\pi^{-1} = \pi^{+}$ (unitary)

i.e. $\pi = \pi^{+}$ i.e. π is Hermitian

But π^2 = identity, hence eigenvalues

$= +1$ or -1 (H w)

Given Eigenvalue problem,

$\psi(x) = c^2 \psi(x)$

Apply π^2 on LHS: $\pi \pi (\psi(x))$

$= \pi \, \psi(-x)$

$= -(\psi(-x))$

$= \psi(x)$

Since $\pi^2(\psi(x)) = \psi(x)$

then $c^2 = 1$

$c = \pm 1$

eigenvalue of the parity operator π is known as parity.

Note: We start with π as a transformation operator, but now it becomes an observable

(ii) π acts on state $|\psi\rangle$:

$$\pi : |\psi\rangle \rightarrow |\psi'\rangle = \pi |\psi\rangle$$

specifically $|\underset{\sim}{x}\rangle \rightarrow |\underset{\sim}{x}'\rangle = \pi |\underset{\sim}{x}\rangle \overset{\text{can}}{\underset{\text{show}}{=}} |-\underset{\sim}{x}\rangle$

proof: Given $\hat{\underset{\sim}{x}} |\underset{\sim}{x}\rangle = \underset{\sim}{x} |\underset{\sim}{x}\rangle$

$\hat{x} \, \pi |x\rangle = -\pi \hat{x} |x\rangle$ (by defn of π, $\pi x = -x \pi$ anticommutes)

$= -\pi x |x\rangle = -x \, \pi |x\rangle$

the operator

i.e. $\pi|x\rangle$ is an eigenstate of $\hat{x}$ with eigenvalue $-x$

But $\pi|-x\rangle = -x|-x\rangle$

$\Rightarrow \quad \pi|x\rangle = |-x\rangle$

$\rightarrow$ If π_c = coord. representation of π: $\langle x|\pi|\psi\rangle = \pi_c\langle x|\psi\rangle = \pi_c\psi(x)$

Notation for operators

$$\pi_c\,\psi(x) = \psi(-x)$$

Proof: As $\pi|x\rangle = |-x\rangle$, $\therefore \langle x|\pi^{\dagger} = \langle -x|$ i.e. $\langle x|\pi = \langle -x|$ $\because \pi^{\dagger}=\pi$

Thus $\langle x|\pi|\psi\rangle = \langle -x|\psi\rangle = \psi(-x)$. $\langle x|\pi|\psi\rangle = \pi_c\langle x|\psi\rangle = \pi_c\psi(x)$

(iii)(a) If a physical system is in the eigenstate of π, then this physical system cannot have a dipole moment $\underset{\sim}{d} = q\,\underset{\sim}{x}$, q = charge

Electric

(HW)

(b) If a physical system has a space inversion symmetry, and if the state of the physical system is a nondegenerate eigenstate of its Hamiltonian H, then the dipole moment $\underset{\sim}{d}$ of this system $= 0$

(iii) Proof:

The physical system is invariant under space inversion.
$\rightarrow [H, \pi] = 0.$

suppose $|\psi\rangle$ is a non-degenerate eigenstate of the Hamiltonian, then physical system in this state has zero electric dipole moment, $\langle\psi|\underset{\sim}{d}|\psi\rangle = 0$

Proof: Given $H|\psi\rangle = E|\psi\rangle$, E = nondegenerate

and $[\pi, H] = 0$

under parity operator π, $|\psi\rangle \rightarrow |\psi'\rangle = \pi|\psi\rangle.$

$$H|\psi'\rangle = H\pi|\psi\rangle = \pi H|\psi\rangle \quad \because [\pi, H] = 0$$
$$= \pi E|\psi\rangle = E\pi|\psi\rangle = E|\psi'\rangle$$

so $|\psi'\rangle$ and $|\psi\rangle$ have same energy value E.

But E is nondegenerate, $\therefore |\psi'\rangle$ and $|\psi\rangle$ must be the same state, $\therefore |\psi'\rangle = k|\psi\rangle$, k = constant $\neq 0$

under π, $\underset{\sim}{d} \rightarrow \underset{\sim}{d}' = \pi \underset{\sim}{d} \pi^{-1} = -\underset{\sim}{d} \quad \because \underset{\sim}{d} = q\underset{\sim}{x}$

As the physical system has space inversion symmetry, the expectation value is unchanged under π.

so $\langle\psi'|\underset{\sim}{d}'|\psi'\rangle = \langle\psi|\underset{\sim}{d}|\psi\rangle$

$$LHS = \langle\psi'|\underset{\sim}{d}'|\psi'\rangle = k k^* \langle\psi|-\underset{\sim}{d}|\psi\rangle$$
$$= -|k|^2 \langle\psi|\underset{\sim}{d}|\psi\rangle$$

$$RHS = \langle\psi|\underset{\sim}{d}|\psi\rangle$$

$\therefore (1 + |k|^2)\langle\psi|\underset{\sim}{d}|\psi\rangle = 0$

i.e. $\langle\psi|\underset{\sim}{d}|\psi\rangle = 0$

(iv) Intrinsic Parity for a particle

In strong and electromagnetic interactions, parity is conserved i.e. parity is a good quantum number so it is useful to assign parity quantum numbers for particles participating in strong and electromagnetic interactions, so hadrons are assigned parity but not lepton.

Hydrogen atom parity example

Familiar example from quantum mechanics course, is hydrogen atom. State of H atom $\psi(\underset{\sim}{x}) = \psi(r, \theta, \phi)$

$= \text{constant} \cdot \text{Laguerre polynomial} \cdot \text{spherical harmonics}$

$= f(r)\, Y_l^m(\theta, \phi)$.

Under space inversion $\underset{\sim}{x} \to -\underset{\sim}{x}$, $(r, \theta, \phi) \to (r, \pi-\theta, \phi+\pi)$

$\psi(r, \theta, \phi) = \text{const}\, f(r) Y_l^m(\theta, \phi) \to Y_l^m(\pi-\theta, \phi+\pi) = (-)^l\, Y_l^m(\theta, \phi)$

i.e. the state of the H atom is an eigenstate of the parity operator π with eigenvalue $(-1)^l$ = parity, $l = 0, 1, 2, \cdots$

Along the same line, hadrons can have intrinsic parity. By convention, nucleon (p, n) has parity $+1$

$\to$ quark has parity $+1$, antiquark parity -1.

Mesons made out of quark, antiquark,

parity of meson $= (+1)(-1)\cdot(-1)^l = (-)^{l+1}$, $l = 0, 1, 2 \cdots$

l = relative orbital motion of quark and antiquark

Parity of photon $= -1$ $\because$ momentum $\underset{\sim}{p}$ and $\underset{\sim}{p} - q\underset{\sim}{A}$

$\underset{\sim}{A} \to$ photon field

under space inversion $\underset{\sim}{p} \to -\underset{\sim}{p}$

Baryons made out of 3 quarks,

parity of a baryon

$= \pi(q)\,\pi(q)\,\pi(q)\,\pi(\text{relative motion})$

$= \pi(\text{relative motion}), \qquad \pi(q) = +1$

For 2 particle relative motion, the parity is $(-1)^{\ell}$, ℓ = orbital quantum number of the relative motion of the 2 particles

For 3 particle in relative motion, the parity of the relative motion is not so simple.

particles 1, 2 $\rightarrow \ell$
particles 2, 3 $\rightarrow \ell'$
particles 1, 3 $\rightarrow \ell''$

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For integer j, the rank j tensor that transforms as $(-1)^{j+1}$ under space inversion ~~is~~ a pseudo tensor

Mirror reflection

pseudo tensor $(-)^{j+1}$, $j = 0, 1, 2, \cdots$

0 - scalar
1 - vector
2 - tensor

$\underset{\sim}{J}^2$ = Casimir operator for $SO(3)$

Orthogonal group $O(3)$ → 2 casimir $\underset{\sim}{J}^2$, π

Table 4.5 Scalars and vectors under parity

Physical properties are one of the following six ranked tensors (0,1,2)

Scalar	:	$P(s) = s$
Pseudoscalar	:	$P(p) = -p$
Vector (or polar vector)	:	$P(v) = -v$
Pseudovector (or axial vector)	:	$P(a) = a$
Tensor		
Pseudotensor		

just as they are classified by spin, charge, isospin, strangeness, and so on. According to quantum field theory, the parity of a fermion (half-integer spin) must be *opposite* to that of the corresponding antiparticle, while the parity of a boson (integer spin) is the *same* as its antiparticle. We take the *quarks* to have *positive* intrinsic parity, so the antiquarks are negative.* The parity of a composite system in its ground state is the *product* of the parities of its constituents (we say that parity is a 'multiplicative' quantum number, in contrast to charge, strangeness, and so on, which are 'additive').† Thus the baryon octet and decuplet have positive parity, $(+1)^3$, whereas the pseudoscalar and vector meson nonets have negative parity, $(-1)(+1)$. (The prefix 'pseudo' tells you the parity of the particles.) For an excited state (of two particles) there is an extra factor of $(-1)^l$, where l is the orbital angular momentum [18]. Thus, in general, the mesons carry a parity of $(-1)^{l+1}$ (see Table 4.6). Meanwhile, the *photon* is a *vector* particle (it is represented by the vector potential A^μ); its spin is 1 and its intrinsic parity is -1.

$\underset{\sim}{P} - q\underset{\sim}{A}$

The mirror symmetry of strong and electromagnetic interactions means that parity is conserved in all such processes. Originally, everyone took it for granted that the same goes for the weak interactions as well. But a disturbing paradox arose in the early fifties, known as the 'tau–theta puzzle'. Two strange mesons, called at the time τ and θ, appeared to be identical in every respect – same mass, same spin (zero), same charge, and so on – except that one of them decayed into two pions and the other into three pions, states of opposite parity:

θ, $\tau \to K^0, \bar{K}^0$

$$K^+ \quad \theta^+ \to \pi^+ + \pi^0 \qquad (P = (-1)^2 = +1)$$

$$K^+ \quad \tau^+ \to \left\{\begin{matrix} \pi^+ + \pi^0 + \pi^0 \\ \pi^+ + \pi^+ + \pi^- \end{matrix}\right\} \qquad (P = (-1)^3 = -1) \qquad (4.53)$$

* This choice is completely arbitrary; we could just as well do it the other way around. Indeed, in principle we could assign positive parity to some quark flavors and negative to others. This would lead to a different set of hadronic parities, but the *conservation* of parity would still hold. The rule stated here is obviously the *simplest*, and it leads to the conventional assignments.

† There is less to this distinction than meets the eye; in a sense, it results from a notational anomaly. Scrupulous consistency would require that we write the parity operator in exponential form, $P = e^{i\pi K}$, with the operator K playing a role analogous to, say, spin (Equation 4.28). The eigenvalues of K would be 0 and 1, corresponding to +1 and −1 for P, and multiplication of parities would correspond to addition of K.

✓

The problem of θ, τ particles was resolved by proposing parity is not conserved in weak decay