Chapter 4 Symmetries (Griffiths)

Define symmetry in physics

$$\begin{array}{ccc} \text{Transformations} & \rightarrow & \text{set} \\ & & \Big\downarrow \text{ binary operation} \\ & & \text{group} \end{array}$$

A symmetry transformation in quantum mechanics leaves transition probability invariant (unchanged)

Isospin spin symmetry $SU(2)$

Find ratio of scattering cross-sections for isodoublet (nucleons, n, p) and isotriplet (pions, π^+, π^0, π^-)

Discuss discrete symmetries P, C, T

Definition of a group

We define a binary operation • on a set S

1. $\forall\alpha,\beta \in$S, $\alpha \bullet \beta \in$ S (closure property)
2. $\exists$ an identity I such that $I \bullet \alpha = \alpha = \alpha \bullet I,\ \forall\alpha \in$S
3. Associative law: $\alpha \bullet (\beta \bullet \gamma) = (\alpha \bullet \beta) \bullet \gamma,\ \forall\alpha,\beta,\gamma \in$S

A set with the above three axioms satisfied is a semigroup.

If in addition,

4. $\forall\alpha \in$S, $\exists$ an element α^{-1} such that $\alpha^{-1}\bullet\alpha = \alpha \bullet \alpha^{-1} = I$, that is, α^{-1} is the inverse of α,

then the set S is a group with respect to the binary operator •

If $\alpha \bullet \beta = \beta \bullet \alpha$, the group is commutative (Abelian). If $\alpha \bullet \beta \neq \beta \bullet \alpha$, the group is non-commutative (non-abelian), e.g. the $n \times n$ matrices form a group but is non-abelian. And the set of integers is an abelian group with respect to addition.

Consider a commutative group S(+). If the elements of S(+) form a semi group with respect to new binary operation, say multiplication (·), such that the following distributive laws hold,

$$(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$$
$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma,$$

then S(+, ·) is an integral domain.
Identity element with respect to addition = zero element
Identity element with respect to multiplication = unity element.

A ring is an integral domain without a unity element with respect to multiplication.

If S(+, ·) is a commutative group with respect to addition and also a commutative group with respect to multiplication (except the zero element has no inverse with respect to multiplication), the S(+, ·) is a field F.

Let a field F act on a commutative group V(+) by scalar multiplication $\times$ such that, $\forall \underset{\sim}{a} \in$V(+) and $\forall\alpha,\beta \in$F(+, ·), the following hold (omitting the $\times$)

1. $\alpha\underset{\sim}{a} = \underset{\sim}{a}\alpha \in$V(+),

2. $1\underset{\sim}{a} = \underset{\sim}{a} = \underset{\sim}{a}1$

3. $0\underset{\sim}{a} = \underset{\sim}{0} = \underset{\sim}{a}0$

4. $\alpha(\underset{\sim}{a} + \underset{\sim}{b}) = \alpha\underset{\sim}{a} + \alpha\underset{\sim}{b}$

5. $(\alpha + \beta)\underset{\sim}{a} = \alpha\underset{\sim}{a} + \beta\underset{\sim}{a}$
6. $\alpha(\beta\underset{\sim}{a}) = (\alpha\beta)\underset{\sim}{a} = \alpha\beta\underset{\sim}{a} \in V(+)$,

Then the set V(+) (that is closed under addition + and scalar multiplication $\times$ by elements of the field F) is called a linear vector space over the field F, and the elements of V(+) are vectors.

Define an inner product for any two elements of V(+),

$(\underset{\sim}{a}, \underset{\sim}{b}) = \underset{\sim}{a}^* \bullet \underset{\sim}{b} \in F$, $\forall\, \underset{\sim}{a}, \underset{\sim}{b} \in V(+)$, $\underset{\sim}{a}^*$=complex conjugate of $\underset{\sim}{a}$,
then V(+) is a metric linear vector space, or a linear vector space with an inner product.

A complete linear vector space with an inner product is a Hilbert space

Definition of completeness- If the limit point of any sequence in the space belongs to the space, then the space is complete.
Consider a sequence $\{u_1, u_2, u_3, \dots\}$, $\lim\limits_{n\to\infty} u_n$ is known as the limit of the sequence.
Example of incompleteness:
Consider the sequence $\left\{\frac{1}{N}, N \text{ integer}\right\}$, the limit point $\lim\limits_{N\to\infty} \frac{1}{N}=0$
is not in the sequence, hence the sequence is incomplete.

If for any two elements of a linear vector space, we can define a commutation relation, say $[\underset{\sim}{a}, \underset{\sim}{b}]$, such that

$$[\underset{\sim}{a}, \underset{\sim}{b}] = -[\underset{\sim}{b}, \underset{\sim}{a}]$$

and

$$[\underset{\sim}{a}, [\underset{\sim}{b}, \underset{\sim}{c}]] + [\underset{\sim}{b}, [\underset{\sim}{c}, \underset{\sim}{a}]] + [\underset{\sim}{c}, [\underset{\sim}{a}, \underset{\sim}{b}]] = 0 \text{ (so called Jacobi identity)}$$

are satisfied, then we have an **algebra**.

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Table 4.1 Symmetries and conservation laws.

Symmetry		Conservation law
Translation in time	$\leftrightarrow$	Energy
Translation in space	$\leftrightarrow$	Momentum
Rotation	$\leftrightarrow$	Angular momentum
Gauge transformation	$\leftrightarrow$	Charge

space time

internal

relating symmetries and conservation laws:

Noether's Theorem: Symmetries $\leftrightarrow$ Conservation laws

Every symmetry of nature yields a conservation law; conversely, every conservation law reflects an underlying symmetry. For example, the laws of physics are symmetrical with respect to *translations in time* (they work the same today as they did yesterday). Noether's theorem relates this invariance to conservation of energy. If a system is invariant under translations in space, then momentum is conserved; if it is symmetrical under rotations about a point, then angular momentum is conserved. Similarly, the invariance of electrodynamics under gauge transformations leads to conservation of charge (we call this an *internal* symmetry, in contrast to the space-time symmetries). I'm not going to *prove* Noether's theorem; the details are not terribly enlightening [1]. The important thing is the profound and beautiful *idea* that symmetries are associated with conservation laws (see Table 4.1).

I have been speaking rather casually about symmetries, and I cited some examples; but what precisely *is* a symmetry? It is an operation you can perform (at least conceptually) on a system that leaves it *invariant* – that carries it into a configuration indistinguishable from the original one. In the case of the function in Figure 4.1, changing the sign of the argument, $x \to -x$, and multiplying the whole thing by -1, $f(x) \to -f(-x)$, is a symmetry operation. For a meatier example, consider the equilateral triangle (Figure 4.2). It is carried into itself by a clockwise rotation through 120° (R_+), and by a counterclockwise rotation through 120° (R_-), by flipping it about the vertical axis a (R_a), or around the axis through b (R_b), or c (R_c). Is that all? Well, doing *nothing* (I) obviously leaves it invariant, so this too is a symmetry operation, albeit a pretty trivial one. And then we could combine operations – for example, rotate clockwise through 240°. But that's the same as rotating *counter* clockwise by 120° (i.e. $R_+^2 = R_-$). As it turns out, we have already identified all the distinct symmetry operations on the equilateral triangle (see Problem 4.1).

The set of all symmetry operations (on a particular system) has the following properties:

1. *Closure*: If R_i and R_j are in the set, then the *product*, R_iR_j – meaning: first perform R_j, then perform R_i* – is also in the set; that is, there exists some R_k such that $R_iR_j = R_k$.

* Note the 'backwards' ordering. Think of the symmetry operations as acting on a system to their right: $R_iR_j(\Delta) = R_i[R_j(\Delta)]$; R_j acts first, and then R_i acts on the result.

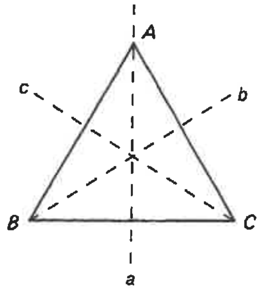


Fig. 4.2 Symmetries of the equilateral triangle.

2. *Identity*: There is an element I such that $IR_i = R_i I = R_i$ for all elements R_i.
3. *Inverse*: For every element R_i there is an *inverse*, R_i^{-1}, such that $R_i R_i^{-1} = R_i^{-1} R_i = I$.
4. *Associativity*: $R_i(R_j R_k) = (R_i R_j) R_k$.

These are the defining properties of a mathematical *group*. Indeed, group theory may be regarded as the systematic study of symmetries. Note that group elements need not *commute*: $R_i R_j \neq R_j R_i$, in general. If all the elements *do* commute, the group is called *Abelian*. Translations in space and time form Abelian groups; rotations (in three dimensions) do *not* [2]. Groups can be *finite* (like the triangle group, which has just six elements) or *infinite* (for example, the set of integers, with addition playing the role of group 'multiplication'). We shall encounter *continuous* groups (such as the group of all rotations in a plane), in which the elements depend on one or more continuous parameters* (the angle of rotation, in this case), and *discrete* groups, in which the elements may be labeled by an index that takes on only integer values (all finite groups are, of course, discrete).

As it turns out, most of the groups of interest in physics can be formulated as groups of *matrices*. The Lorentz group, for instance, consists of the set of 4×4 Λ matrices introduced in Chapter 3. In elementary particle physics, the most common groups are of the type mathematicians call $U(n)$: the collection of all unitary $n \times n$ matrices (see Table 4.2). (A *unitary matrix* is one whose inverse is equal to its transpose conjugate: $U^{-1} = \tilde{U}^*$.) If we restrict ourselves further to unitary matrices with determinant 1, the group is called $SU(n)$. (The S stands for 'special', which just means 'determinant 1'.) If we limit ourselves to *real* unitary matrices, the group is $O(n)$. (O stands for 'orthogonal'; an orthogonal matrix is one whose inverse is equal to its transpose: $O^{-1} = \tilde{O}$.) Finally, the group of real, orthogonal, $n \times n$ matrices of determinant 1 is $SO(n)$; $SO(n)$ may be thought of as the group of all *rotations* in a space of n dimensions. Thus, $SO(3)$ describes the

* If this dependence takes the form of an *analytic* function, it is called a *Lie* group. All of the continuous groups one encounters in physics are Lie groups [3].

(6)

Table 4.2 Important symmetry groups.

Group name	Dimension	Matrices in group
$U(n)$	$n \times n$	unitary ($\tilde{U}^* U = 1$)
$SU(n)$	$n \times n$	unitary, determinant 1
$O(n)$	$n \times n$	orthogonal ($\tilde{O}O = 1$)
$SO(n)$	$n \times n$	orthogonal, determinant 1

rotational symmetry of our world, a symmetry that is related by Noether's theorem to the conservation of angular momentum. Indeed, the entire quantum theory of angular momentum is really closet group theory. It so happens that $SO(3)$ is almost identical in mathematical structure to $SU(2)$, which is the most important *internal* symmetry in elementary particle physics. So the theory of angular momentum, to which we turn next, will actually serve us twice.

SO(3) for usual 3-dim space

SU(2) for Hilbert space

One final thing. Every group G can be *represented* by a group of *matrices*: for every group element a there is a corresponding matrix M_a, and the correspondence respects group multiplication, in the sense that if $ab = c$, then $M_a M_b = M_c$. A representation need not be 'faithful': there may be many distinct group elements represented by the same matrix. (Mathematically, the group of matrices is *homomorphic*, but not necessarily *isomorphic*, to G.) Indeed, there is a trivial case, in which we represent every element by the 1×1 *unit* matrix (which is to say, the number 1). If G is a group of matrices, such as $SU(6)$ or $O(18)$, then it is (obviously) a representation of *itself* – we call it the *fundamental* representation. But there will, in general, be many other representations, by matrices of various dimensions. For example, $SU(2)$ has representations of dimension 1 (the trivial one), 2 (the matrices themselves), 3, 4, 5, and in fact *every* positive integer. A major problem in group theory is the characterization of all the representations of a given group.

Adjoint representation

Of course, you can always construct a new representation by combining two old ones, thus

$$M_a = \begin{pmatrix} \boxed{M_a^{(1)}} & \text{(zeros)} \\ \text{(zeros)} & \boxed{M_a^{(2)}} \end{pmatrix}$$

But we don't count this separately; when we list the representations of a group, we are talking about the so-called *irreducible* representations, which *cannot* be decomposed into block-diagonal form. Actually, you have already encountered several examples of group representations, probably without realizing it: an ordinary scalar belongs to the one-dimensional representation of the rotation group, $SO(3)$, and a vector belongs to the three-dimensional representation; four-vectors belong to the four-dimensional representation of the Lorentz group; and the curious geometrical arrangements of Gell-Mann's Eightfold Way correspond to irreducible representations of the group $SU(3)$.

Representations.

A representation of a group G is a homomorphism of G onto a group of linear operators acting on a linear vector space,

$$D(g_i)\, D(g_j) = D(g_i g_j)$$

If a represenation is isomorphic to the group it is a faithful representation

A ray representation: $D(g_i)$ and $e^{i\alpha_i} D(g_i)$, α_i = real, are allowed

$$D(g_i)\, D(g_j) = e^{i\alpha_{ij}}\, D(g_i g_j)$$

α_{ij} = arbitary real number which can depend on the group elements g_i and g_j

If α_{ij} is restricted to take only a finite number of values, the representation is multiple-valued

E.g. Double-valued representation: $\alpha_{ij} = 0$ or $\alpha_{ij} = \pi$, i.e

$$D(g_i)\, D(g_j) = \pm\, D(g_i g_j)$$

Two representations are equivalent if one can be transformed into the other by a similarity transformation

A representation of a finite or compact Lie group can be transformed into a unitary representation by a similarity transformation

Reducible
$$D(g) = \begin{pmatrix} D_1(g) & X(g) \\ 0 & D_2(g) \end{pmatrix}$$

Fully reducible
$$D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}$$

A representation of a finite or compact Lie group is fully reducible

$$D = D^{(1)} \oplus D^{(2)} \oplus \cdots$$

conjugate representation

$\bar{D}$ = conjugate representation of D if we take the complex conjugate of the matrices of D

$$\bar{D}(g) = (D(g))^{*}$$ ← complex conjugate

Clebsch-Gordan Coefficents

Addition of angular momenta

$$\underset{\sim}{J}_1 \qquad \underset{\sim}{J}_2$$

$$|\alpha_1 j_1 m_1\rangle = basis\ \ for\ \ \underset{\sim}{J}_1^2\ \ and\ \ J_{1z}$$

$$|\alpha_2 j_2 m_2\rangle = basis\ \ for\ \ \underset{\sim}{J}_2^2\ \ and\ \ J_{2z}$$

$J_{13} = J_{1\,3}$ ← 3rd component; 1 ← particle 1

The base vectors

$$|\alpha\, j_1\, j_2\, m_1\, m_2\rangle \equiv |\alpha_1\, j_1\, m_1\rangle |\alpha_2\, j_2\, m_2\rangle$$

$\alpha,\ j_1, j_2$ fixed,

m_1, m_2 vary $\qquad -j_1 \le m_1 \le j_1$

$\qquad\qquad\qquad -j_2 \le m_2 \le j_2$

span the subspace $\xi(\alpha, j_1, j_2)$.

$$\underset{\sim}{J}^2 = (\underset{\sim 1}{J} + \underset{\sim 2}{J})^2 \quad \textit{and } J_z \ \textit{ act on } \ \xi(\alpha, j_1, j_2)$$

Since $\underset{\sim 1}{J}^2$ and $\underset{\sim 2}{J}^2$ commute with $\underset{\sim}{J}^2$ *and* J_z, can also use the base

$$|\alpha\, j_1\, j_2\, j\, m\rangle, \quad \alpha, j_1, j_2 \ \textit{fixed}$$

$$j, m \ \text{vary}$$

$$|j_1 - j_2| \le j \le (j_1 + j_2)$$

$$-j \le m \le j$$

to generate the same subspace $\xi(\alpha, j_1, j_2)$.

The two bases are related:

$$|\alpha\, j_1\, j_2\, j\, m\rangle = \sum_{m_2=-j_2}^{j_2} \sum_{m_1=-j_1}^{j_1} |\alpha\, j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2 | j m\rangle$$

$$|\alpha\, j_1\, j_2\, m_1\, m_2\rangle = \sum_{m=-j}^{j} \sum_{j=|j_1-j_1|}^{(j_1+j_2)} |\alpha\, j_1\, j_2\, j\, m\rangle\langle j\, m | j_1\, j_2\, m_1\, m_2\rangle$$

$\langle j_1\, j_2\, m_1\, m_2 | j\, m\rangle = \langle j\, m | j_1\, j_2\, m_1\, m_2\rangle^* \equiv$ Clebsch-Gordan coefficents

Meaning of C.G. coeffs

(i) relating two basis vectors (just like Fourier transform)

(ii) $\langle j_1\, j_2\, m_1\, m_2 | j\, m\rangle$= probability amplitude of finding the state $|j_1\, j_2\, m_1\, m_2\rangle$ when the system is in state $|jm\rangle$

Properties of C.G. coeffs

(1) Selection rule:

$$\langle \alpha\, j_1 j_2 m_1 m_2 | jm\rangle = 0 \;\; unless$$

$$m_1 + m_2 = m \;\; and \;\; |j_1 - j_2| \le j \le (j_1 + j_2)$$

(2) Phase convention: require

$$\langle j_1\, j_2\, j_1\, m_2 | j\, j\rangle \text{ real and } \ge 0$$

$$\mathrm{m}_2 = j - j_1$$

$$j = |j_1 - j_2|,\ |j_1 - j_2| + 1 \ldots \qquad (j_1 + j_2)$$

Note: When $m_1 = j_1$ *and* $m = j$, it does <u>not</u> necessarily imply $m_2 = j_2$ since $j \ne (j_1 + j_2)$ in general

(3) Reality: All C.G. coeffs can be obtained from

$$\langle j_1\, j_2\, j_1\, m_2 | j\, j \rangle$$

$\therefore$ all C.G. coeffs are real

(4) Orthogonality

$$\sum_{m_1\, m_2} \langle j_1\, j_2\, m_1\, m_2 | j\, m \rangle \langle j_1\, j_2\, m_1\, m_2 | j'\, m' \rangle = \delta_{jj'}\, \delta_{mm'}$$

$$\sum_{j\ m} \langle j_1\, j_2\, m_1\, m_2 | j\, m \rangle \langle j_1\, j_2\, m_1'\, m_2' | j\, m \rangle = \delta_{m_1 m_1'}\, \delta_{m_2 m_2'}$$

Wigner-Eckart theorem

In a standard representation $\{\underset{\sim}{J}^2, J_z\}$ whose basis vectors are denoted by $|\tau\, j m\rangle$,

The matrix element $\langle \tau\, j\, m | T_g^{(k)} | \tau' j' m' \rangle$

of the qth standard component of a given kth order irreducible tensor operator, T$^{(k)}$, is equal to the product of the Clebsch-Gordan coefficient.

$$\langle j' k\, m' q | jm \rangle$$

by a quantity independent of *m*, *m*′ and *q* $(q = -k, -k+1, \ldots. + k)$

$$\langle \tau\, j\, m | T_q^{(k)} | \tau' j' m' \rangle = \frac{1}{\sqrt{2j+1}} \langle \tau j \,||\, T^{(k)} \,||\, \tau' j' \rangle \Box \langle j' k\, m' q | jm \rangle$$

$\langle \tau j \,||\, T^{(k)} \,||\, \tau' j' \rangle$= reduced matrix element

$\langle j' k\, m' q | jm \rangle$=Clebsch-Gordan coefficient

$\neq 0$ only if $m = m' + q$ and $|j - j'| \leq k \leq j + j'$

For a scalar operator *S* $\langle \tau\, jm | s | \tau' j' m' \rangle = \delta_{jj'} \delta_{mm'} S_{\tau\tau'}^{(j)}$

$S_{\tau\tau'}^{(j)}$ independent of m and m'

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. .			

Steps for use (J <=> I):
1) Match I_1 × I_2 of outgoing state to the top-left numbers to find the relevant table
2) Start reading the row to find matching m_1 and m_2
3) Move one box to the right and you get the relevant coefficient (see pg18 for e.g.)
4) | I_1 I_2 I M>
To fill in values for I and M, move up the box you are currently at and fill in using the given values

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

1/2×1/2

m_1 m_2	1, +1
+1/2 +1/2	1

m_1 m_2	1, 0	0, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	1, −1
−1/2 −1/2	1

1×1/2

m_1 m_2	3/2, +3/2
+1 +1/2	1

m_1 m_2	3/2, +1/2	1/2, +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	3/2, −1/2	1/2, −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	3/2, −3/2
−1 −1/2	1

2×1/2

m_1 m_2	5/2, +5/2
+2 +1/2	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2, +1/2	3/2, +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2, −1/2	3/2, −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2, −3/2	3/2, −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2, −5/2
−2 −1/2	1

2×1

m_1 m_2	3, +3
+2 +1	1

m_1 m_2	3, +2	2, +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	3, +1	2, +1	1, +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	3, 0	2, 0	1, 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	3, −1	2, −1	1, −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	3, −2	2, −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	3, −3
−2 −1	1

1×1

m_1 m_2	2, +2
+1 +1	1

m_1 m_2	2, +1	1, +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	2, 0	1, 0	0, 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	2, −1	1, −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	2, −2
−1 −1	1

3/2×1/2

m_1 m_2	2, +2
+3/2 +1/2	1

m_1 m_2	2, +1	1, +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	2, 0	1, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	2, −1	1, −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	2, −2
−3/2 −1/2	1

3/2×1

m_1 m_2	5/2, +5/2
+3/2 +1	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	5/2, −3/2	3/2, −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	5/2, −5/2
−3/2 −1	1

3/2×3/2

m_1 m_2	3, +3
+3/2 +3/2	1

m_1 m_2	3, +2	2, +2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	3, +1	2, +1	1, +1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	3, 0	2, 0	1, 0	0, 0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	3, −1	2, −1	1, −1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	3, −2	2, −2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	3, −3
−3/2 −3/2	1

2×3/2

m_1 m_2	7/2, +7/2
+2 +3/2	1

m_1 m_2	7/2, +5/2	5/2, +5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	7/2, −5/2	5/2, −5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	7/2, −7/2
−2 −3/2	1

2×2

m_1 m_2	4, +4
+2 +2	1

m_1 m_2	4, +3	3, +3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	4, +2	3, +2	2, +2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	4, +1	3, +1	2, +1	1, +1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	4, −1	3, −1	2, −1	1, −1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	4, −2	3, −2	2, −2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	4, −3	3, −3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	4, −4
−2 −2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^2_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

proton and neutron can be regarded as two different states of a nucleon. This is isospin symmetry and the symmetry group is SU(2).

Isospin symmetry is a good symmetry for strongly interacting particles.

It can classify hadrons into (iso) multiplets.

E.g. singlet Λ, doublet (n, p), triplet (π^-, π^0, π^+)

isospin symmetry can also be used to relate scattering cross-sections of one isomultiplet to another isomultiplet, among members of the isomultiplets. We show this by an example.

E.g. Consider scattering of pions (isotriplet) with nucleons (isodoublet), we restrict to 2 incident particles to 2 outgoing particles.

There are 6 elastic processes

charge conserved and KE conserved

$$\pi^{\pm 0} p \rightarrow \pi^{\pm 0} p \; ; \quad \pi^{\pm 0} n \rightarrow \pi^{\pm 0} n \, ,$$

4 charge exchange scattering

$\pi^+ n \rightarrow \pi^0 p$, $\qquad \pi^0 p \longrightarrow \pi^+ n$

$\pi^0 n \longrightarrow \pi^- p$, $\qquad \pi^- p \rightarrow \pi^0 n$

the rxn is possible as long as charge is conserved

We show that all these 10 cross sections are related, due to underlying SU(2) isospin symmetry.

For example, consider

$$\pi^+ p \to \pi^+ p \quad \text{(i)}$$ KE & charge conserved

$$\pi^- p \to \pi^- p \quad \text{(ii)}$$ KE & charge conserved

$$\pi^- p \to \pi^0 n \quad \text{(iii)}$$ charge conserved

like how the H-atom states can be characterized by nlm quantum no., we will use isospin to charactertize the incoming (in-state) and outgoing (out-state) states in scattering cross sectn

To compute scattering cross section, we need scatt. amp.,

$$\text{scatt amplitude} = {}_{out}\langle \quad | \quad \rangle_{in} \qquad | \;\rangle_{in} = \text{in-state}$$

these are states

For our example, specify the in-state and out-state in terms of isospins, or better still, total isospins.

Consider process (i), the individual isospin of the particle involved is known

$$p = |\tfrac{1}{2}, \tfrac{1}{2}\rangle, \qquad \pi^+ = |1, +1\rangle$$

$$\uparrow \; I^2 \quad \uparrow \; I_3$$

{like ang. mom $|j\,m\rangle$, we have $|I\,m\rangle$}

So the in-state in process (i) is given by

$$\pi^+ p \to |1\;+1\rangle|\tfrac{1}{2}\;\tfrac{1}{2}\rangle$$

$$= |\underset{I_1}{1}\;\underset{I_2}{\tfrac{1}{2}}\;\underset{m_{I_1}}{1}\;\underset{m_{I_2}}{\tfrac{1}{2}}\rangle$$

the m_1 and m_2 here is the m of I (isospin), not the same quantity as m used in angular momentum

Express in terms of total isospin quantum numbers.

Recall $\underset{(j_1\, m_1)}{\underset{\sim}{J_1}}, \; \underset{(j_2\, m_2)}{\underset{\sim}{J_2}} \longrightarrow \underset{(j\; m)}{\underset{\sim}{J}} = \underset{\sim}{J_1} + \underset{\sim}{J_2}$

Addition of two angular momenta, $\underset{\sim}{J_1}$, $\underset{\sim}{J_2}$

$$j = (j_1+j_2),\quad j_1+j_2-1,\ \cdots\ |j_1-j_2|$$

$$m = m_1 + m_2$$

Clebsch-Gordan expansion,

$$|j_1 j_2\ m_1 m_2\rangle = \sum_{j,m} |j_1 j_2 j m\rangle \langle j m | j_1 j_2 m_1 m_2\rangle$$

$$|j_1 j_2\ j m\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle \langle m_1 m_2 | j_1 j_2 j m\rangle$$

So express $\pi^+ p = |\underset{I_1}{\overset{\uparrow}{1}}\ \underset{I_2}{\overset{\uparrow}{\tfrac12}}\ \underset{m_{I_1}}{\overset{\uparrow}{1}}\ \underset{m_{I_2}}{\overset{\uparrow}{\tfrac12}}\rangle$

in terms of $|I_1\ I_2\ I\ m\rangle$

since $I_1 = 1$, $I_2 = \frac12$ $\therefore$ $I = \frac32, \frac12$,

Using Clebsch-Gordan table

$$\pi^+ p = |\underset{I_1}{\overset{\uparrow}{1}}\ \underset{I_2}{\overset{\uparrow}{\tfrac12}}\ \underset{m_{I_1}}{\overset{\uparrow}{1}}\ \underset{m_{I_2}}{\overset{\uparrow}{\tfrac12}}\rangle$$

$$= |\underset{I_1}{\overset{\uparrow}{1}}\ \underset{I_2}{\overset{\uparrow}{\tfrac12}}\ \underset{I}{\overset{\uparrow}{\tfrac32}}\ \underset{m}{\overset{\uparrow}{\tfrac32}}\rangle_{in}$$

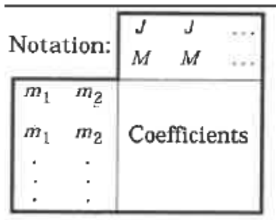
Similarly, for process (i), the out-state is

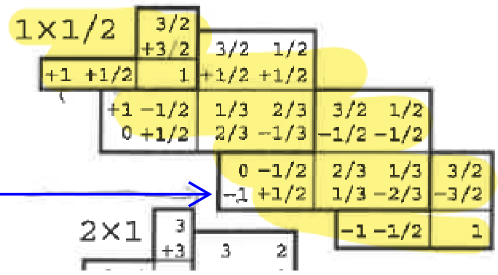
$$\pi^+ p = |1\ \tfrac12\ 1\ \tfrac12\rangle = |1\ \tfrac12\ \tfrac32\ \tfrac32\rangle_{out}$$

So scatt. amp for (i) is

$$\mathcal{M}_{(i)} = {}_{out}\langle 1\ \tfrac12\ \tfrac32\ \tfrac32 | 1\ \tfrac12\ \tfrac32\ \tfrac32\rangle_{in}$$

Important: TOTAL isospin is conserved in strong interaction but individual isospin values need not be the same





Calc ratio of scattering amplitudes / cross sections here

(QFT needed to find individual SA/CS but symmetry lets us find the ratio of the SA/CS without complicated calcs

For (ii) $\pi^- p \to \pi^- p$

$$\pi^- p = \underset{I_1\ \ m_{I_1}}{|1\ -1\rangle}\ \underset{I_2\ \ m_{I_2}}{|\tfrac{1}{2}\ +\tfrac{1}{2}\rangle}$$

$$\equiv |\underset{I_1}{1}\ \underset{I_2}{\tfrac{1}{2}}\ \underset{m_{I_1}}{-1}\ \underset{m_{I_2}}{+\tfrac{1}{2}}\rangle$$

e.g.

$$\underset{\text{Table}}{=} \sqrt{\tfrac{1}{3}}\ |1\ \tfrac{1}{2}\ \tfrac{3}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\ |1\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

scatt. amp

$$\mathcal{M}_{(ii)} = \tfrac{1}{3}\ {}_{\text{out}}\langle 1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\ |\ 1\ \tfrac{1}{2}\ \tfrac{3}{2}\ -\tfrac{1}{2}\rangle_{\text{in}}$$

$$+ \tfrac{2}{3}\ {}_{\text{out}}\langle 1\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\ |\ 1\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{\text{in}}$$

HW

For (iii) $\pi^- p \to \pi^0 n$

$$\pi^0 n = |1\ 0\rangle\ |\tfrac{1}{2}\ \tfrac{-1}{2}\rangle = |1\ \tfrac{1}{2}\ 0\ \tfrac{-1}{2}\rangle$$

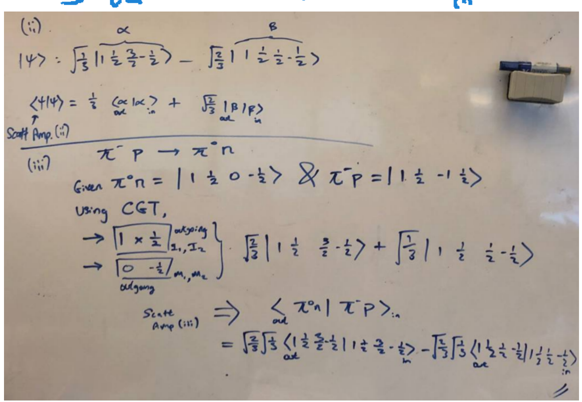
$$= \sqrt{\tfrac{2}{3}}\ |1\ \tfrac{1}{2}\ \tfrac{3}{2}\ -\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\ |1\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

(To check, HW)

scatt. amp $= {}_{\text{out}}\langle \pi^0 n\ |\ \pi^- p\rangle_{\text{in}} = \mathcal{M}_{(iii)}$

$$= \tfrac{\sqrt{2}}{3}\ {}_{\text{out}}\langle 1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\ |\ 1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{-1}{2}\rangle_{\text{in}} - \tfrac{\sqrt{2}}{3}\ {}_{\text{out}}\langle 1\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\ |\ 1\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle_{\text{in}}$$

The 3 scatt. amps $\mathcal{M}_{(i)}$, $\mathcal{M}_{(ii)}$ related as will be shown below.



$$\mathcal{M}_{(i)} : \mathcal{M}_{(ii)} : \mathcal{M}_{(iii)} = {}_{out}\langle \overset{I}{\tfrac{3}{2}}\ \overset{m}{\tfrac{3}{2}} | \overset{I}{\tfrac{3}{2}}\ \overset{m}{\tfrac{3}{2}} \rangle_{in} :$$

$$\left(\tfrac{1}{3}\ {}_{out}\langle \tfrac{3}{2}\ \tfrac{-1}{2} | \tfrac{3}{2}\ \tfrac{-1}{2} \rangle_{in} + \tfrac{2}{3}\ {}_{out}\langle \tfrac{1}{2}\ \tfrac{-1}{2} | \tfrac{1}{2}\ \tfrac{-1}{2} \rangle_{in} \right) :$$

$$\left(\tfrac{\sqrt{2}}{3}\ {}_{out}\langle \tfrac{3}{2}\ \tfrac{-1}{2} | \tfrac{3}{2}\ \tfrac{-1}{2} \rangle_{in} - \tfrac{\sqrt{2}}{3}\ \langle \tfrac{1}{2}\ \tfrac{-1}{2} | \tfrac{1}{2}\ \tfrac{-1}{2} \rangle_{in} \right)$$

Put

$${}_{out}\langle \tfrac{3}{2}\ \tfrac{3}{2} | \tfrac{3}{2}\ \tfrac{3}{2} \rangle_{in} = \mathcal{M}_{\frac{3}{2}}$$

different m

$$\overset{?}{=} {}_{out}\langle \tfrac{3}{2}\ \tfrac{-1}{2} | \tfrac{3}{2}\ \tfrac{-1}{2} \rangle_{in}$$

Why? = Wigner-Eckart theorem

i.e. $\mathcal{M}_{3/2}$ depends on $I = 3/2$ but not on m

$$\langle \tfrac{1}{2}\ -\tfrac{1}{2} | \tfrac{1}{2}\ -\tfrac{1}{2} \rangle = \mathcal{M}_{\frac{1}{2}}$$

WET used to equate the top <out|in> with the middle <out|in> as both being M_3/2 because they have the same j (doesn't matter that their m is diff) See WET in pg 13 dirac delta notation)

$$\mathcal{M}_{(i)} : \mathcal{M}_{(ii)} : \mathcal{M}_{(iii)} = \mathcal{M}_{\frac{3}{2}} : \left(\tfrac{1}{3} \mathcal{M}_{\frac{3}{2}} + \tfrac{2}{3} \mathcal{M}_{\frac{1}{2}} \right) :$$

$$\left(\tfrac{\sqrt{2}}{3} \mathcal{M}_{\frac{3}{2}} - \tfrac{\sqrt{2}}{3} \mathcal{M}_{\frac{1}{2}} \right)$$

scattering amplitude depends on energy of incident particles,

At CM energy 1232 MeV^2, $\mathcal{M}_{\frac{3}{2}} \gg \mathcal{M}_{\frac{1}{2}}$,

Then

$$\mathcal{M}_{(i)} : \mathcal{M}_{(ii)} : \mathcal{M}_{(iii)} = \left(1 : \tfrac{1}{3} : \tfrac{\sqrt{2}}{3} \right)$$

$$= (3 : 1 : \sqrt{2})$$

Cross section $= |\mathcal{M}|^2$

$$\sigma_{(i)} : \sigma_{(ii)} : \sigma_{(iii)} = |\mathcal{M}_{3/2}|^2 : \left|\frac{1}{3}\mathcal{M}_{3/2} + \frac{2}{3}\mathcal{M}_{\frac{1}{2}}\right|^2$$

$$: \left|\frac{\sqrt{2}}{3}\mathcal{M}_{3/2} - \frac{\sqrt{2}}{3}\mathcal{M}_{\frac{1}{2}}\right|^2$$

$$\simeq |\mathcal{M}_{\frac{3}{2}}|^2 : \left|\frac{1}{3}\mathcal{M}_{3/2}\right|^2 : \left|\frac{\sqrt{2}}{3}\mathcal{M}_{\frac{3}{2}}\right|^2$$

M_3/2 can be ignored now, just compare coeff

$$= 1 : \frac{1}{9} : \frac{2}{9} = 9 : 1 : 2$$

If we are interested in the cross section ratio of $\sigma_{\pi^+ p}$ and $\sigma_{\pi^- p}$ for the 3 processes

σ(i) / [σ(ii) + σ(iii)]

$$\frac{\sigma_{\pi^+ p}}{\sigma_{\pi^- p}} = \frac{9}{1+2} = 3$$

where $\sigma_{\pi^- p} = \sigma_{(ii)} + \sigma_{(iii)}$

The calculated ratio agrees with the experimental result. see Fig in page 21

We now extend isospin SU(2) to higher flavour symmetries, SU(3), SU(4), ... SU(6)

2

$$\sigma_a : \sigma_c : \sigma_j = 9|\mathcal{M}_3|^2 : |\mathcal{M}_3 + 2\mathcal{M}_1|^2 : 2|\mathcal{M}_3 - \mathcal{M}_1|^2 \tag{4.49}$$

0.6×10^{-23} s

At a *CM* energy of 1232 MeV there occurs a famous and dramatic bump in pion-nucleon scattering, first discovered by Fermi in 1951;[7] here the pion and nucleon join to form a short-lived "resonance" state—the Δ. We know the Δ carries $I = \frac{3}{2}$, so we expect that at this energy $\mathcal{M}_3 \gg \mathcal{M}_1$, and hence

$$\sigma_a : \sigma_c : \sigma_j = 9 : 1 : 2 \tag{4.50}$$

Experimentally, it is easier to measure the total cross sections, so (c) and (j) are combined:

$$\frac{\sigma_{tot}(\pi^+ + p)}{\sigma_{tot}(\pi^- + p)} = 3 \tag{4.51}$$

As you can see in Figure 4.6, this prediction is well satisfied by the data.

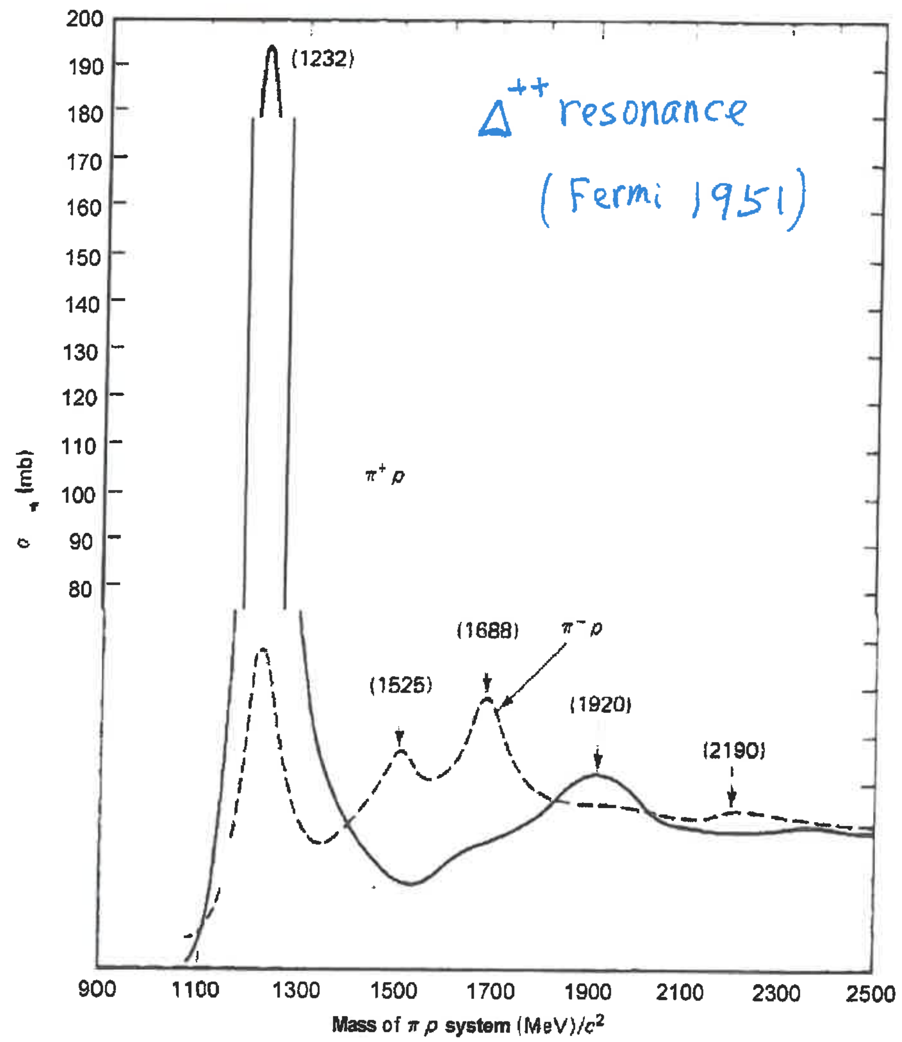


Figure 4.6 Total cross sections for π^+p (solid line) and π^-p (dashed line) scattering. (Source: S. Gasiorowicz, Elementary Particle Physics (New York: Wiley, copyright © 1966, page 294. Reprinted by permission of John Wiley and Sons, Inc.)

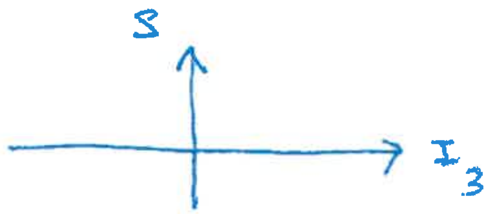
Quark flavour

J^2 = Casimir operator

Originally (Heisenberg, 1932) isospin was introduced to classify elementary particles into doublet (p, n), or triplet (π^+, π^0, π^-) etc. The isospin group is $SU(2)$

$|jm\rangle$, $|I\, m_I\rangle$

In early 1960, many more elementary particles were found, $SU(2)$ isospin as a classification scheme is not adequate. A new quantum number, strangeness S, was introduced.



Many particles can then be accommodated into representations of a bigger symmetry group $SU(3)$

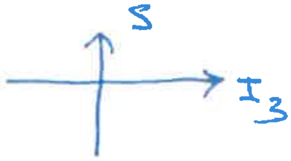
Mesons form singlet or octet representations of $SU(3)$

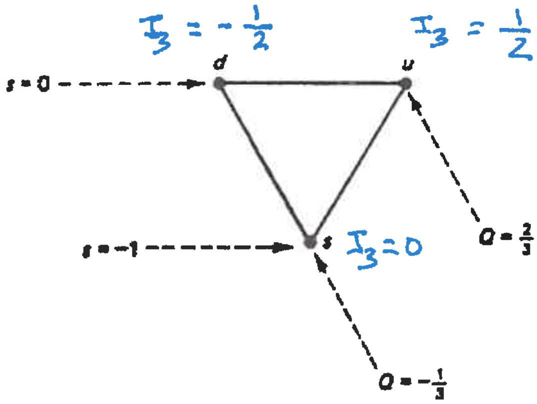
Baryons form singlet, octet (eight fold way) decuplet representations of $SU(3)$

Questions were then raised why only these three types of representation of $SU(3)$ are realized by elementary particles at that time?

The quark model (3 quarks) explains this. Mesons are made of quark and antiquark. Baryon are made of 3 quarks

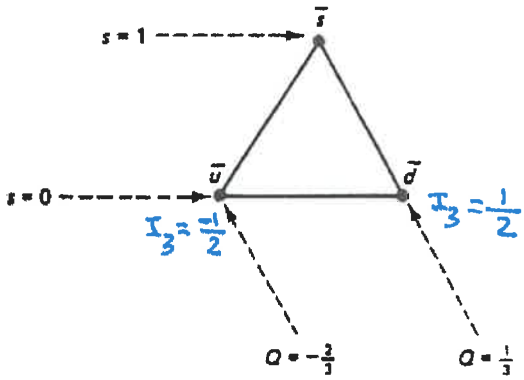
Assume the 3 quarks form the fundamental respresentation of SU(3), triplet 3





The Quarks

Antiquarks form the conjugate representation, denoted by $\bar{3}$



The Antiquarks

From the fundamental representation, one can construct higher dimensional representation:

$$3 \otimes \bar{3} = 1 \oplus 8$$

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$

1 = singlet

8 = Octet

10 = Decuplet

wrt SU(3) transformations

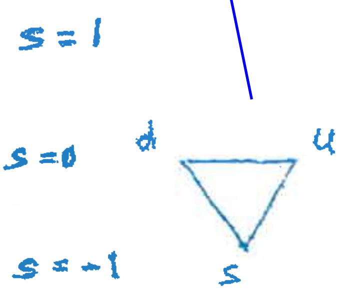
Outer multiplication of two matrices

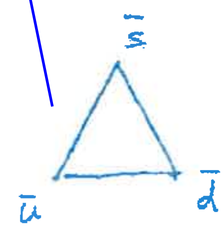
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

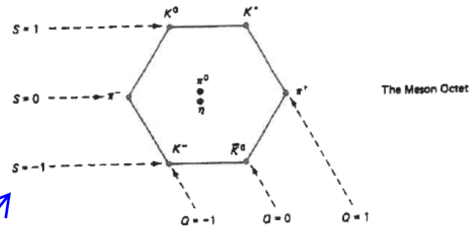
$$= \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

Mesons are made of quark and antiquark

$$3 \otimes \bar{3} = 8 \oplus 1$$







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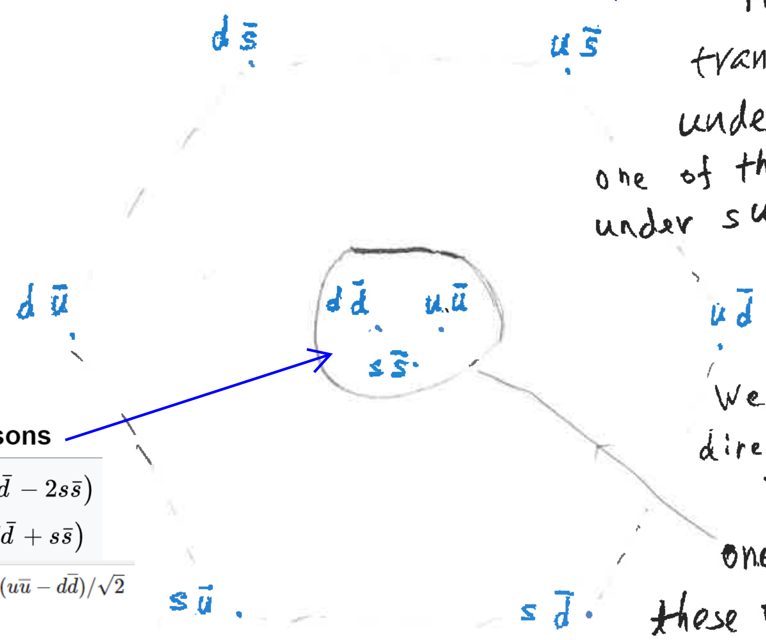


Among the nine states, nonet, 8 of them transform into each other under SU(3) transformation

one of them is invariant under SU(3) transformations.

So the nonet can be decomposed into an octet plus a singlet.

We say the nonet is a direct sum $\oplus$ of $\underline{8}$ and $\underline{1}$

$$3 \otimes \bar{3} = 1 \oplus 8$$

one of combinations of these three is unchanged under SU(3) transformations, that is singlet wrt SU(3)

Eta and eta prime mesons

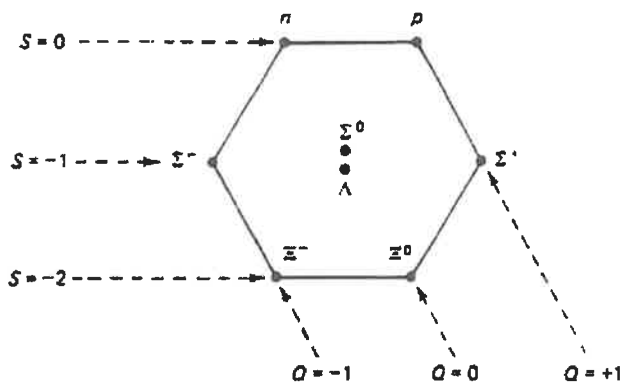
Composition	$\eta : \approx \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right)$
	$\eta' : \approx \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right)$
π^0	$(u\bar{u} - d\bar{d})/\sqrt{2}$

For baryons:

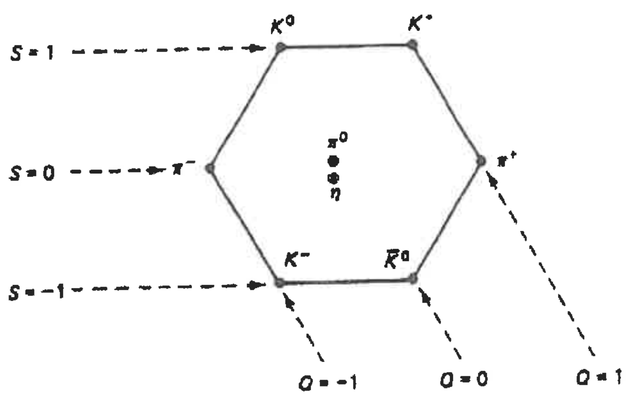
$$3 \otimes 3 \otimes 3 = \underbrace{\underline{10}}_{\text{Decuplet}} \oplus 8 \oplus 8 \oplus 1$$

Note: The above nonet is a direct sum of octet and a singlet, Octet, singlet wrt SU(3)

The octet consists of 2 isodoublets, 1 isotriplet and 1 isosinglet wrt SU(2)



The Baryon Octet



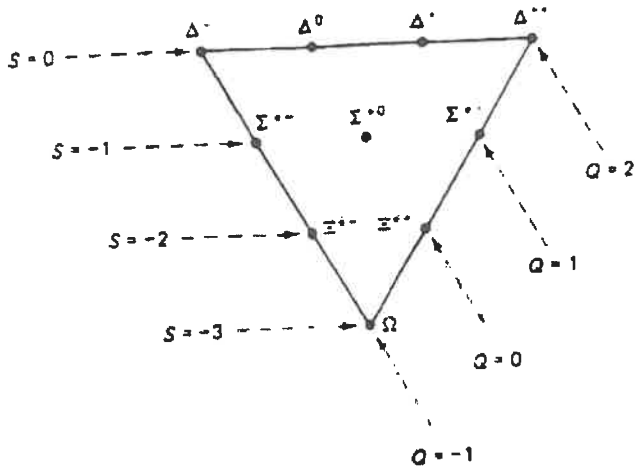
The Meson Octet

For baryons:

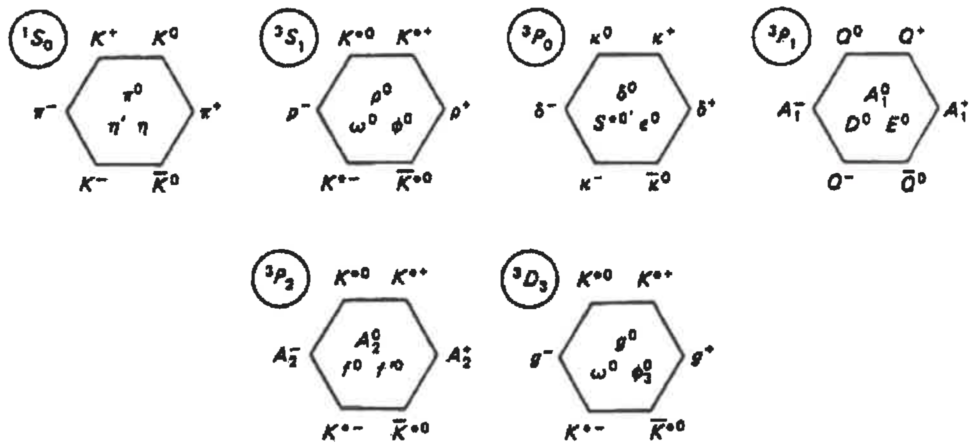
$$3 \otimes 3 \otimes 3 = \underbrace{10}_{\text{fr}} \oplus 8 \oplus 8 \oplus 1$$

Decuplet

(26)



The Baryon Decuplet



$^{2S+1}L_j$

Established meson nonets. Obviously, we are running out of letters. It is customary to distinguish different particles represented by the same letter by indicating the mass parenthetically (in MeV/c^2), thus $K^*(892)$, $K^*(1430)$, $K^*(1650)$, and so on. In this figure the supermultiplets are labeled in spectroscopic notation At present, there are no complete baryon supermultiplets beyond the octet and decuplet, although there are many partially filled diagrams.

So SU(3) scheme with 3 quarks (u, d, s) as the fundamental constituents of matter was a very good scheme for classifying mesons and baryons into SU(3) singlets, octets, decuplets

Charm quark was discovered in Nov. 1974

Then SU(3) was extended to SU(4) scheme with four constituent quarks: u, d, c, s.

And when bottom quark b and top quark t were discovered, SU(6) is used to include b, t quarks

Unfortunately all these higher groups are badly broken, due to the large mass differences among the 6 quarks. Members of the multiplet have very different masses.

In the SU(3) scheme, proton and neutron almost same mass, so are the pions (π^+, π^0, π^-, triplet)

Depending on the circumstances, one assigns
effective (constituent) mass or
current (bare) mass to quarks

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Since it is not physically possible even at solar-interior temperatures to "strip naked" any quark of its covering, it is a matter of legitimate doubt whether current quarks are actual or real, or merely a convenient but unrealistic and abstract notion. High energy particle accelerators provide a demonstration that the idea of a "naked quark" is in some sense real: If the current quark imbedded in one constituent quark is hit inside its covering with large momentum, the current quark accelerates through its evanescent covering and leaves it behind, at least temporarily producing a "naked" or undressed quark,[citation needed] showing that to some extent the idea is realistic (*see* glueball *for speculations about what happens to the dressing of virtual particles that gets left behind*).

The current quark mass means the mass of the constituent quark with the mass of the respective constituent quark covering subtracted away. (isolated quarks)

current

Table 4.4 Quark masses (MeV/c^2)

	Quark flavor	Bare mass	Effective mass
SU(1)	u	2	336
SU(2)	d	5	340
SU(3)	s	95	486
SU(4)	c	1300	1550
SU(5)	b	4200	4730
SU(6)	t	174 000	177 000

constituent

Constituent quarks are valence quarks for which the correlations for the description of hadrons by means of gluons and sea-quarks are put into effective quark masses of these valence quarks.

Diff in masses increase for heavier quarks

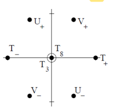
Warning: These numbers are somewhat speculative and model dependent [12].

Is invariant mass the same as effective mass?
No, not the same. Particle physics mainly concerned with bare and effective but Invariant mass is another concept

A symmetry is a transformation; in SU(2), after applying isospin transformation, the up and down quarks are symmetric (good symmetry hence protons and neutrons are isospin-symmetric).
But in SU(3), after applying SU(3) transformation, {u,d,s} are symmetric but the symmetry is 'broken' because the bare vs effective mass of s quark has a difference of greater than 3%

The three root vectors E_α and their conjugates $E_{-\alpha}$ span the six dimensions orthogonal to $\mathcal{H}$. These make transitions between the weights. The root which takes $d \to u$, which we used to call T^+, is now $E_{(1,0)} = \frac{1}{\sqrt{2}}(T_1 + iT_2)$. We also have $E_{(1/2,\sqrt{3}/2)} = \frac{1}{\sqrt{2}}(T_4 + iT_5)$, sometimes called V_+, which takes $s \to u$, and $E_{(-1/2,\sqrt{3}/2)} = \frac{1}{\sqrt{2}}(T_6 + iT_7)$, also called U_+, which takes $s \to d$.

So the root space of the generators looks like the figure, with the roots forming a regular hexagon, with angles between them of $60° \times n$. From the diagram we see T_- generates doublets on V_+ and U_- as required by $\frac{\vec{\alpha}_{T_+} \cdot \vec{\alpha}_{V_+}}{\vec{\alpha}^2_{T_+}} = \frac{n}{2} = \frac{1}{2}$. Similarly U_- generates doublets ("U spin doublets") starting with V_+ or T_-.



Taken from Rutgers University, SU(3) (see pdf in PC4245)

However, there is an important *caveat* in this neat hierarchy: isospin, $SU(2)$, is a very 'good' symmetry; the members of an isospin multiplet differ in mass by at most 2 or 3%, which is about the level at which electromagnetic corrections would be expected.* But the Eightfold Way, $SU(3)$, is a badly 'broken' symmetry; mass splittings within the baryon octet are around 40%. The symmetry breaking is even worse when we include charm; the $\Lambda_c^+(udc)$ weighs more than twice the $\Lambda(uds)$, although they are in the same $SU(4)$ supermultiplet. It is worse still with bottom, and absolutely terrible with top, which doesn't form bound states at all.

Why is isospin such a good symmetry, the Eightfold Way fair, and flavor $SU(6)$ so poor? The Standard Model blames it all on the quark masses. Now, the theory of quark masses is a slippery business, given the fact that they are not accessible to direct experimental measurement. Various arguments [9] suggest that the u and d quarks are *intrinsically* very light, about 10 times the mass of the electron. However, within the confines of a hadron, their *effective* mass is much greater. The precise value, in fact, depends on the context; it tends to be a little higher in baryons than in mesons (more on this in Chapter 5). In somewhat the same way, the effective inertia of a spoon is greater when you're stirring honey than when you're stirring tea, and in either case it exceeds the true mass of the spoon. Generally speaking, the effective mass of a quark in a hadron is about 350 MeV/c^2 greater than its bare mass [10] (see Table 4.4). Compared to this, the quite different *bare* masses of up and down quarks are practically irrelevant; they *function* as though they had identical masses. But the s quark is distinctly heavier, and the c, b, and t quarks are widely separated. Apart from the differences in quark masses, the strong interactions treat all flavors equally. Thus isospin is a good symmetry because the effective u and d masses are so nearly equal (which is to say, on a more fundamental level, because their *bare* masses are so small); the Eightfold Way is a *fair* symmetry because the effective mass of the strange quark is not *too* far from that of the u and d. But

strong interaction

* Indeed, it used to be thought that isospin was an *exact* symmetry of the strong interactions, and *all* of the symmetry breaking was attributable to electromagnetic contamination. The fact that the n–p mass splitting is in the wrong direction to be purely electromagnetic was troubling, however, and we now believe that $SU(2)$ is only an *approximate* symmetry of the strong interactions.