Example: Cycloidal pendulum

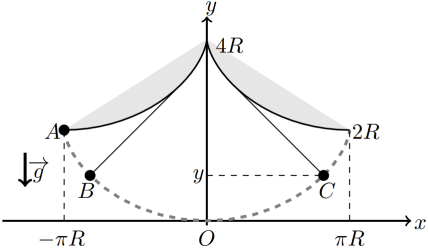
- Huygen (1673) constructed a cycloidal pendulum with a point particle of mass m and a string of length $4R$ suspended from the cusp of an inverted cycloid

- Path of point mass is a cycloid:

$$\begin{cases} x = R\,(\theta + \sin\theta) \\ y = R\,(1 - \cos\theta) \end{cases}, \qquad -\pi \le \theta \le \pi$$

- Period is independent of the amplitude!

$$T = 4\pi\sqrt{\frac{R}{g}}$$



EXERCISE 8.4: Obtain the equation of motion for cycloidal pendulum from Euler-Lagrange equation.

$$\mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y = R\left[\theta(t) + \sin\theta(t)\right]\,\hat{\mathbf{e}}_x + R\left[1 - \cos\theta(t)\right]\,\hat{\mathbf{e}}_y$$

$$T \equiv T(\theta, \dot{\theta}, t) = \frac{m}{2}\,\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t) = 2mR^2\dot{\theta}^2(t)\cos^2\frac{\theta(t)}{2}$$

$$U \equiv U(\theta) = mgy(t) = 2mgR\sin^2\frac{\theta(t)}{2}$$

$$\mathcal{L} \equiv \mathcal{L}(\theta, \dot{\theta}, t) = T - U = 2mR^2\dot{\theta}^2(t)\cos^2\frac{\theta(t)}{2} - 2mgR\sin^2\frac{\theta(t)}{2}$$

$$s(t) \equiv 4R\sin\frac{\theta(t)}{2} \quad \Rightarrow \quad \dot{s}(t) = 2R\dot{\theta}(t)\cos\frac{\theta(t)}{2}$$

$$\mathcal{L} \equiv \mathcal{L}(s, \dot{s}, t) = \frac{1}{2}\,m\dot{s}^2(t) - \frac{mg}{8R}\,s^2(t) \qquad \blacksquare$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{s}}\right) - \frac{\partial\mathcal{L}}{\partial s} = 0 \quad \Rightarrow \quad \ddot{s}(t) = -\frac{g}{4R}\,s(t) \equiv -\omega^2 s(t) \qquad \blacksquare$$

Generalized momenta

- **Generalized momenta**:

$$\mathcal{L} = \mathcal{L}(\{q_k, \dot{q}_k\}, t) \quad \Rightarrow \quad p_k \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$$

- If the Lagrange function does not depend on q_k *explicitly*, then the generalized coordinate q_k is called **cyclic coordinate** and the corresponding generalized momenta p_k is a constant of motion

$$\frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}p_k}{\mathrm{d}t} = 0$$

- Choice of generalized coordinates is adopted so that there are as many cyclic coordinates as possible and their corresponding generalized momenta are constants of motion

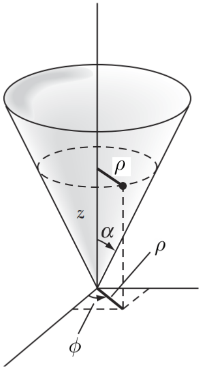
Example: Moving on a smooth cone

- A particle of mass m is constrained to move on the inside surface of a smooth cone of half-angle α. The particle is subjected to a gravitational force.

- Lagrange function: zero gravitational potential energy reference at the origin

$$\mathcal{L} = (\mathbf{r}(t), \dot{\mathbf{r}}(t), t) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

- Two degrees of freedoms

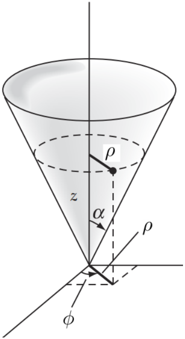


EXERCISE 8.5: Express the Lagrange function in suitable generalized coordinates and obtain the equations of motion of the particle.

$$\mathcal{L} = (\mathbf{r}(t), \dot{\mathbf{r}}(t)) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \end{cases} \quad \Rightarrow \quad \mathcal{L}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - mgz$$

$$z = \rho\cot\alpha \quad \Rightarrow \quad \mathcal{L}(\rho, \phi, \dot{\rho}, \dot{\phi}) = \frac{m}{2}\left(\dot{\rho}^2\csc^2\alpha + \rho^2\dot{\phi}^2\right) - mg\rho\cot\alpha \qquad \blacksquare$$



$$\mathcal{L} \equiv \mathcal{L}(\rho, \phi, \dot{\rho}, \dot{\phi}) = \frac{m}{2}\left(\dot{\rho}^2 \csc^2 \alpha + \rho^2 \dot{\phi}^2\right) - mg\rho \cot \alpha$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\rho^2 \dot{\phi} = \text{constant} \qquad \blacksquare$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right) = \frac{\partial \mathcal{L}}{\partial \rho} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\rho} \csc^2 \alpha\right) = m\rho\dot{\phi}^2 - mg \cot \alpha$$

$$\Rightarrow \quad \ddot{\rho} - \rho\dot{\phi}^2 \sin^2 \alpha + g \sin \alpha \cos \alpha = 0 \qquad \blacksquare$$

PC3261: Classical Mechanics II

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Lecture 9: Lagrangian Mechanics II

Homogeneous functions

- Homogeneous function of degree M: λ is *any* positive real number

$$f(\lambda x_1, \lambda x_2, \cdots, \lambda x_N) = \lambda^M f(x_1, x_2, \cdots, x_N)$$

- Examples:

$$\begin{cases} f(x,y) = \left(x^4 + 2xy^3 - 5y^4\right)\sin\dfrac{x}{y} \quad \rightarrow \quad f(\lambda x, \lambda y) = \lambda^4 f(x,y) \\[2ex] f(x,y,z) = \dfrac{C}{\sqrt{x^2+y^2+z^2}} \quad \rightarrow \quad f(\lambda x, \lambda y, \lambda z) = \lambda^{-1} f(x,y,z) \end{cases}$$

- **Euler's theorem on homogeneous function**:

$$\sum_{i=1}^{N} x_i \frac{\partial f(x_1, x_2, \cdots x_N)}{\partial x_i} = M f(x_1, x_2, \cdots x_N)$$

$$f(\lambda x_1, \lambda x_2, \cdots, \lambda x_N) = \lambda^M f(x_1, x_2, \cdots, x_N)$$

$$\Rightarrow \quad \sum_{i=1}^{N} \frac{\partial f(\lambda x_1, \lambda x_2, \cdots, \lambda x_N)}{\partial(\lambda x_i)} \frac{\partial(\lambda x_i)}{\partial \lambda} = M\lambda^{M-1} f(x_1, x_2, \cdots, x_N)$$

$$\Rightarrow \quad \sum_{i=1}^{N} \frac{\partial f(\lambda x_1, \lambda x_2, \cdots, \lambda x_N)}{\partial(\lambda x_i)} x_i = M\lambda^{M-1} f(x_1, x_2, \cdots, x_N)$$

$$\Rightarrow \quad \sum_{i=1}^{N} \frac{\partial f(\lambda x_1, \lambda x_2, \cdots, \lambda x_N)}{\partial(\lambda x_i)} x_i \Bigg|_{\lambda=1} = M\lambda^{M-1} f(x_1, x_2, \cdots, x_N)\Bigg|_{\lambda=1}$$

$$\Rightarrow \quad \sum_{i=1}^{N} \frac{\partial f(x_1, x_2, \cdots, x_N)}{\partial x_i} x_i = M f(x_1, x_2, \cdots, x_N) \qquad \blacksquare$$

Kinetic energy in terms of generalized coordinates

- Kinetic energy is a quadratic function of the generalized velocities:

$$T \equiv T(\{q_k, \dot{q}_k\}, t) = M_0(\{q_k\}, t) + \sum_{i=1}^{M} M_i(\{q_k\}, t)\, \dot{q}_i + \frac{1}{2} \sum_{i,j=1}^{M} M_{ij}(\{q_k\}, t)\, \dot{q}_i\, \dot{q}_j$$

$$\begin{cases} M_0(\{q_k\}, t) = \dfrac{1}{2} \displaystyle\sum_{\alpha=1}^{N} m_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial t} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial t} \\ M_i(\{q_k\}, t) = \displaystyle\sum_{\alpha=1}^{N} m_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial t} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \\ M_{ij}(\{q_k\}, t) = \displaystyle\sum_{\alpha=1}^{N} m_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \end{cases}$$

- Kinetic energy is a homogeneous quadratic function of the generalized velocities if $\mathbf{r}_\alpha = \mathbf{r}_\alpha(\{q_k\})$

$$T(t) = \sum_{\alpha=1}^{N} \frac{m_\alpha}{2}\, \dot{\mathbf{r}}_\alpha(t) \cdot \dot{\mathbf{r}}_\alpha(t)\,, \qquad \mathbf{r}_\alpha = \mathbf{r}_\alpha\left(\{q_k(t)\}\,, t\right)$$

$$\dot{\mathbf{r}}_\alpha(t) = \sum_{i=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\, \dot{q}_i + \frac{\partial \mathbf{r}_\alpha}{\partial t}$$

$$\begin{aligned}
T(t) &= \sum_{\alpha=1}^{N} \frac{m_\alpha}{2}\, \dot{\mathbf{r}}_\alpha(t) \cdot \dot{\mathbf{r}}_\alpha(t) \\
&= \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha \left(\sum_{i=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\, \dot{q}_i + \frac{\partial \mathbf{r}_\alpha}{\partial t} \right) \cdot \left(\sum_{j=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\, \dot{q}_j + \frac{\partial \mathbf{r}_\alpha}{\partial t} \right) \\
&= \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha\, \frac{\partial \mathbf{r}_\alpha}{\partial t} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial t} + \sum_{i=1}^{M} \left(\sum_{\alpha=1}^{N} m_\alpha\, \frac{\partial \mathbf{r}_\alpha}{\partial t} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \right) \dot{q}_i \\
&\qquad + \frac{1}{2} \sum_{i,j=1}^{M} \left(\sum_{\alpha=1}^{N} m_\alpha\, \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \right) \dot{q}_i\, \dot{q}_j \qquad \blacksquare
\end{aligned}$$

Conservation of energy

- **Jacobi energy function** is a constant of motion if the Lagrangian does not depend on time explicitly

$$h(\{q_i, \dot{q}_i\}, t) \equiv \sum_{i=1}^{M} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}(\{q_k(t), \dot{q}_k(t)\}, t)$$

- If the Lagrangian does not depend on time explicitly and the kinetic energy is a homogeneous quadratic function of generalized velocities, then the Jacobi energy function is the total mechanical energy of the system and it is a constant of motion

$$h(\{q_i, \dot{q}_i\}, t) \quad \rightarrow \quad h(\{q_i, \dot{q}_i\}) = T(\{q_i, \dot{q}_i\}) + U(\{q_i\}) = E$$

EXERCISE 9.1: Show that the Jacobi energy function, $h(\{q_i, \dot{q}_i\}, t)$ is a constant of motion if the Lagrangian does not depend on time explicitly.

$$\mathcal{L} \equiv \mathcal{L}(\{q_i(t), \dot{q}_i(t)\}, t)$$

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} &= \sum_{i=1}^{M} \left(\frac{\partial \mathcal{L}}{\partial q_i}\, \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\, \ddot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} \\
&= \sum_{i=1}^{M} \dot{q}_i \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \right] + \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{M} \dot{q}_i\, \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial \mathcal{L}}{\partial t} \\
&= 0 + \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{M} \dot{q}_i\, \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial \mathcal{L}}{\partial t}
\end{aligned}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_{i=1}^{M} \dot{q}_i\, \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} \right] = -\frac{\partial \mathcal{L}}{\partial t} \qquad \blacksquare$$

$$T \equiv T(\{q_k, \dot{q}_k\}) = \frac{1}{2} \sum_{i,j=1}^{M} M_{ij}(\{q_k\}) \; \dot{q}_i \, \dot{q}_j$$

$$T(\{q_k, \lambda\dot{q}_k\}) = \frac{1}{2} \sum_{i,j=1}^{M} M_{ij}(\{q_k\}) \; (\lambda\dot{q}_i) \; (\lambda\dot{q}_j) = \lambda^2 T(\{q_k, \dot{q}_k\})$$

$$\Rightarrow \quad \sum_{i=1}^{M} \dot{q}_i \, \frac{\partial T}{\partial \dot{q}_i} = 2T \qquad \blacksquare$$

$$\mathcal{L} \equiv \mathcal{L}\left(\{q_i, \dot{q}_i\}, t\right) = T(\{q_i, \dot{q}_i\}) - U(\{q_i\})$$

$$h(\{q_i, \dot{q}_i\}) \equiv \sum_{i=1}^{M} \dot{q}_i \, \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}(\{q_k, \dot{q}_k\}) = T(\{q_i, \dot{q}_i\}) + U(\{q_i\}) \qquad \blacksquare$$

System subjected to holonomic constraints

- System with M degrees of freedom: M independent generalized coordinates $\{q_i\}$ and M independent Euler-Lagrange equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0\,, \qquad k = 1, 2, \cdots, M$$

- System subjected to C **holonomic constraints**:

$$\psi_i(\{q_k(t)\}, t) = 0\,, \qquad i = 1, 2, \cdots, C$$

- Degree of freedom of the system is now reduced to $M - C$ and these M Euler-Lagrange equations of motion are no longer independent from each other

- One solution is to introduce $M - C$ independent generalized coordinates

Lagrange multipliers and constraints

- An alternative approach is to keep these M generalized coordinates and introduce C **Lagrange multipliers** (one for each holonomic constraint) so that there are still M independent *modified* equations of motion

- Modified Euler-Lagrange equations of motions: M second order differential equations together with C holonomic constraints to solve for M generalized coordinates and C Lagrange multipliers

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = \sum_{i=1}^{C} \lambda_i(t)\,\frac{\partial \psi_i}{\partial q_k}\,, \qquad k = 1, 2, \cdots, M$$

- **Generalized constraint forces**: an advantage of the approach with Lagrange multipliers is that the force of constraint can be determined

$$\mathcal{Q}_k^{\textsf{cons}} \equiv \sum_{i=1}^{C} \lambda_i(t)\,\frac{\partial \psi_i}{\partial q_k}\,, \qquad k = 1, 2, \cdots, M$$

Example: Atwood machine (another visit)

- Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless pulley with frictionless pulley

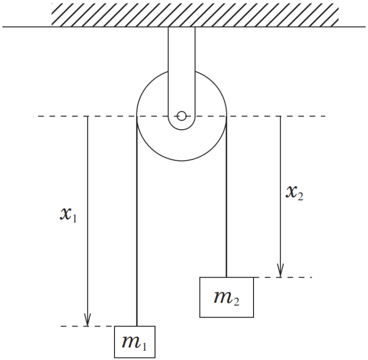
- Kinetic and potential energies:

$$T(\dot{x}_1, \dot{x}_2) = \frac{1}{2}\left(m_1\dot{x}_1^2 + m_2\dot{x}_2^2\right)$$

$$U(x_1, x_2) = -g\left(m_1x_1 + m_2x_2\right)$$

- Accelerations:

$$\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\,g = -\ddot{x}_2$$



EXERCISE 9.2: Solve for the accelerations of the masses from the Euler-Lagrange equation and determine the generalized constraint forces.

$$T(\dot{x}_1, \dot{x}_2) = \frac{1}{2}\left(m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2\right) \,, \qquad U(x_1, x_2) = -g\,(m_1 x_1 + m_2 x_2)$$

$$x_1(t) + x_2(t) = \text{constant} \quad \Rightarrow \quad \dot{x}_2(t) = -\dot{x}_1(t) \quad \Rightarrow \quad \ddot{x}_2(t) = -\ddot{x}_1(t)$$

$$\mathcal{L}(x_1, \dot{x}_1, t) = \frac{1}{2}\,(m_1 + m_2)\,\dot{x}_1^2 + (m_1 - m_2)\,g x_1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = 0 \quad \Rightarrow \quad (m_1 + m_2)\,\ddot{x}_1 - (m_1 - m_2)\,g = 0 \quad \Rightarrow \quad \ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\,g$$

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}\,m_2\dot{x}_2^2 + m_1 g x_1 + m_2 g x_2$$

$$\psi(x_1, x_2) = x_1 + x_2 - \textsf{constant} = 0$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \dfrac{\partial \mathcal{L}}{\partial x_1} = \lambda\,\dfrac{\partial \psi}{\partial x_1} \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) - \dfrac{\partial \mathcal{L}}{\partial x_2} = \lambda\,\dfrac{\partial \psi}{\partial x_2} \end{cases} \quad \Rightarrow \quad \begin{cases} m_1\ddot{x}_1 - m_1 g = \lambda \\[2ex] m_2\ddot{x}_2 - m_2 g = \lambda \end{cases}$$

$$\begin{cases} x_1 + x_2 - \textsf{constant} = 0 \\[2ex] m_1\ddot{x}_1 - m_1 g = \lambda \\[2ex] m_2\ddot{x}_2 - m_2 g = \lambda \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 = \dfrac{m_1 - m_2}{m_1 + m_2}\,g = -\ddot{x}_2 \\[2ex] \lambda = -\dfrac{2m_1 m_2}{m_1 + m_2}\,g \end{cases} \qquad \blacksquare$$

$$\begin{cases} Q_{x_1}^{\textsf{cons}} = \lambda\,\dfrac{\partial \psi}{\partial x_1} = -\dfrac{2m_1 m_2}{m_1 + m_2}\,g \\[2ex] Q_{x_2}^{\textsf{cons}} = \lambda\,\dfrac{\partial \psi}{\partial x_2} = -\dfrac{2m_1 m_2}{m_1 + m_2}\,g \end{cases} \qquad \blacksquare$$