PC3261: Classical Mechanics II

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Lecture 5: Work and Energy

Kinetic energy and work

- **Kinetic energy**:

$$T(t) \equiv \frac{1}{2} m\mathbf{v}(t) \cdot \mathbf{v}(t)$$

- **Work** by the force on the particle during a time interval:

$$W_{1\to 2} \equiv \int_{t=t_1}^{t_2} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \dot{\mathbf{r}}(t)\, \mathrm{d}t$$

- **Work-energy theorem**: total work by the forces during a given time interval is equal to the change in the kinetic energy of the particle during this time interval

$$T(t_2) - T(t_1) = \int_{t=t_1}^{t_2} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \dot{\mathbf{r}}(t)\, \mathrm{d}t$$

Work as a line integral

- Work $W_{1\to 2}$ on the particle by the force $\mathbf{F}$ is given by the line integral of $\mathbf{F}\cdot \mathrm{d}\mathbf{r}$ along its trajectory $\mathcal{C}_{1\to 2}$ from point $\mathbf{r}_1$ to point $\mathbf{r}_2$:

$$W(\mathbf{r}_1 \to \mathbf{r}_2) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r}$$

- Work-energy theorem: change in the kinetic energy of a particle as it moves from points 1 to 2 is the work by the *net* force on the particle

$$T(t_2) - T(t_1) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r}$$

- Work by the net force is the sum of works done by respective forces:

$$W\left(\mathbf{r}_1 \to \mathbf{r}_2\right) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_{1\to 2}} \sum_i \mathbf{F}_i(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r}$$

$$= \sum_i \int_{\mathcal{C}_{1\to 2}} \mathbf{F}_i(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} = \sum_i W_i\left(\mathbf{r}_1 \to \mathbf{r}_2\right)$$

Example: Work by a uniform force

- Uniform force: $\mathbf{F}(\mathbf{r}) = F_0\,\hat{\mathbf{e}}_n$, F_0 is a constant and $\hat{\mathbf{e}}_n$ is a constant unit vector

- Work by the uniform force on the particle moving from $\mathbf{r}_1$ to $\mathbf{r}_2$ along an *arbitrary* path: θ is the angle between $\hat{\mathbf{e}}_n$ and $\mathbf{r}_2 - \mathbf{r}_1$

$$W\left(\mathbf{r}_1 \to \mathbf{r}_2\right) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = F_0\,\hat{\mathbf{e}}_n \cdot (\mathbf{r}_2 - \mathbf{r}_1) = F_0\,|\mathbf{r}_2 - \mathbf{r}_1|\cos\theta$$

- Work by a uniform force only depends on the net displacement, $\mathbf{r}_2 - \mathbf{r}_1$, not on the particular path taken from $\mathbf{r}_1$ to $\mathbf{r}_2$!

- Work by a uniform force around a closed path is zero: $\mathcal{C}_{1\to 2} \neq -\mathcal{C}_{2\to 1}$

$$W\left(\mathbf{r}_2 \to \mathbf{r}_1\right) = \int_{\mathcal{C}_{2\to 1}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = F_0\,\hat{\mathbf{e}}_n \cdot (\mathbf{r}_1 - \mathbf{r}_2) = -W\left(\mathbf{r}_1 \to \mathbf{r}_2\right)$$

Example: Inverted pendulum

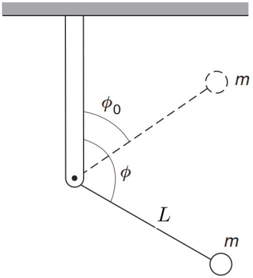
- A pendulum consists of a light rigid rod of length L pivoted at one end with mass m attached at the other end. The pendulum is released from rest at angle ϕ_0

- Equation of motion:

$$\frac{\mathrm{d}^2\phi(t)}{\mathrm{d}t^2} = \frac{g}{L}\sin\phi(t)$$

- Maximum speed is achieved by letting the pendulum fall from $\phi_0 = 0$ to the bottom $\phi = \pi$:

$$v_{\max} = 2\sqrt{gL}$$



EXERCISE 5.1: Obtain the speed of the mass m when the rod is at an angle ϕ from work-energy theorem.

Example: Escape speed

- Gravitational force acting on a mass m at a distance r from the center of Earth of mass M:

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^2}\,\hat{\mathbf{e}}_r$$

- Mass m is projected from the surface of the Earth $r = R_e$ with an initial speed v_0 at an angle α from the vertical

- Escape speed for the mass m to escape Earth's gravitational field is independent of the launching direction:

$$v_{\text{escape}} = \sqrt{2gR_e}$$

EXERCISE 5.2: Obtain the expression for the escape speed from work-energy theorem. Assume gravitational force is the only force and ignore the rotation of the Earth.

Example: Pendulum motion

- A point mass of mass m is attached at the end of the massless string of length L. It is released from $\theta = \theta_0$ with $\dot{\theta} = 0$ at $t = 0$

- Work-energy theorem: θ_0 is the maximum angular displacement of the point mass

$$\frac{1}{2} L\dot{\theta}^2(t) = g\cos\theta(t) - g\cos\theta_0$$

- Small angle approximation: $\theta_0 \ll 1$

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t\right)$$

EXERCISE 5.3: Obtain the first-order differential equation for $\theta(t)$ governing the dynamics of the point mass. Assuming small angles, $\theta_0 \ll 1$, solve for $\theta(t)$.

Example: Pendulum motion – cont'd

- Incomplete elliptical integral of the first kind:

$$F(\varphi;k) \equiv \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}\,, \qquad 0 \le k^2 \le 1\,, \qquad 0 \le \varphi \le \frac{\pi}{2}$$

- Amplitude-dependent period of the pendulum motion:

$$T = 4\sqrt{\frac{L}{g}}\,F\left(\frac{\pi}{2};\sin\frac{\theta_0}{2}\right)$$

- Series expansion:

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}\sin^2\frac{\theta_0}{2}+\frac{9}{64}\sin^4\frac{\theta_0}{2}+\mathcal{O}\left(\sin^6\frac{\theta_0}{2}\right)\right]$$

Conservative forces and potential energies

- A force $\mathbf{F}$ acting on a particle is **conservative** if and only if it satisfies two conditions:
  1. $\mathbf{F}$ depends only on particle's position $\mathbf{r}$, that is $\mathbf{F} = \mathbf{F}(\mathbf{r}(t))$
  2. For any two points 1 and 2, the work $W(1 \to 2)$ by force $\mathbf{F}$ is the same for all paths between 1 and 2

- **Potential energy** associated to a given conservative force is defined to be the negative of the work done by the conservative force if the particle moves from the *reference* point $\mathbf{r}_0$ to the point of interest $\mathbf{r}$

$$U(\mathbf{r}) \equiv -W\left(\mathbf{r}_0 \to \mathbf{r}\right) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}'$$

- **Total mechanical energy**, $E(t) \equiv U(\mathbf{r}(t)) + T(t)$, is conserved if all forces acting on the particle are conservative:

$$T(t_2) - T(t_1) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \quad \Rightarrow \quad U(\mathbf{r}(t_1)) + T(t_1) = U(\mathbf{r}(t_2)) + T(t_2)$$

Potential energy for uniform gravitational force

- Work by a uniform force only depends on the net displacement, $\mathbf{r}_2 - \mathbf{r}_1$, not on the particular path taken from $\mathbf{r}_1$ to $\mathbf{r}_2$

- Uniform gravitational force:

$$\mathbf{F}(\mathbf{r}) = -mg\,\hat{\mathbf{e}}_z$$

- Potential energy associated with uniform gravitational field:

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = mg\,(z - z_0)$$

- Choosing the zero reference of gravitational potential energy at ground level $z_0 = 0$, then the uniform gravitational potential energy depends only on the height above the ground

Conservative force and gradient of potential energy

- Infinitesimal work by a conservative force:

$$W\left(\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r}\right) = \begin{cases} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = F_x(\mathbf{r})\,\mathrm{d}x + F_y(\mathbf{r})\,\mathrm{d}y + F_z(\mathbf{r})\,\mathrm{d}z \\ -\left[U(\mathbf{r} + \mathrm{d}\mathbf{r}) - U(\mathbf{r})\right] = -\dfrac{\partial U(\mathbf{r})}{\partial x}\,\mathrm{d}x - \dfrac{\partial U(\mathbf{r})}{\partial y}\,\mathrm{d}y - \dfrac{\partial U(\mathbf{r})}{\partial z}\,\mathrm{d}z \end{cases}$$

- Conservative force in terms of gradient of potential energy:

$$\mathbf{F}(\mathbf{r}) = -\frac{\partial U(\mathbf{r})}{\partial x}\,\hat{\mathbf{e}}_x - \frac{\partial U(\mathbf{r})}{\partial y}\,\hat{\mathbf{e}}_y - \frac{\partial U(\mathbf{r})}{\partial z}\,\hat{\mathbf{e}}_z = -\boldsymbol{\nabla} U(\mathbf{r})$$

- Total mechanical energy is a constant of motion:

$$E(t) \equiv U(\mathbf{r}(t)) + T(t) \quad \Rightarrow \quad \frac{\mathrm{d}E(t)}{\mathrm{d}t} = 0$$

EXERCISE 5.4: Show that the total mechanical energy with time-independent potential energy is a constant of motion.

Elastic potential energy

- One dimensional force, $\mathbf{F}(\mathbf{r}) = F(x)\,\hat{\mathbf{e}}_x$, is always conservative (why??)

- Elastic force in one dimension: k is the spring constant and x_0 is the equilibrium position

$$\mathbf{F}(\mathbf{r}) = -k\,(x - x_0)\;\hat{\mathbf{e}}_x$$

- Elastic potential energy: zero reference of elastic potential energy is chosen at equilibrium position

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = \frac{1}{2}\,k\,(x - x_0)^2$$

- Elastic force from elastic potential energy:

$$\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla} U(\mathbf{r}) = -k\,(x - x_0)\;\hat{\mathbf{e}}_x$$

Several conservative forces

- Total conservative forces on the particle: principle of superposition of forces

$$\mathbf{F}_{\mathsf{c}}(\mathbf{r}) = \sum_i \mathbf{F}_{\mathsf{c},i}(\mathbf{r})$$

- Work-energy theorem: all forces on the particle are conservative

$$T(t_2) - T(t_1) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \sum_i \mathbf{F}_{\mathsf{c},i}(\mathbf{r}(t)) \cdot \mathrm{d}\mathbf{r}$$

$$= \sum_i \left[U_i(\mathbf{r}(t_1)) - U_i(\mathbf{r}(t_2)) \right]$$

- Total mechanical energy is a constant of motion:

$$E(t) \equiv \sum_i U_i(\mathbf{r}(t)) + T(t) \quad \Rightarrow \quad \frac{\mathrm{d}E(t)}{\mathrm{d}t} = 0$$

Non-conservative forces

- Work on the particle by non-conservative forces:

$$W_{\mathsf{nc}}\left(\mathbf{r}_1 \to \mathbf{r}_2\right) = \int_{\mathcal{C}_{1\to 2}} \sum_j \mathbf{F}_{\mathsf{nc,j}}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} = \sum_j W_{\mathsf{nc},j}\left(\mathbf{r}_1 \to \mathbf{r}_2\right)$$

- Work-energy theorem: total mechanical energy is not conserved and the change in total mechanical energy is the work by non-conservative forces

$$\begin{aligned}
T(t_2) - T(t_1) &= \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} \\
&= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \sum_i \mathbf{F}_{\mathsf{c},i}(\mathbf{r}(t)) \cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_{1\to 2}} \sum_j \mathbf{F}_{\mathsf{nc},j}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \cdot \mathrm{d}\mathbf{r} \\
&= \sum_i \left[U_i(\mathbf{r}(t_1)) - U_i(\mathbf{r}(t_2))\right] + W_{\mathsf{nc}}\left(\mathbf{r}_1 \to \mathbf{r}_2\right)
\end{aligned}$$

$$\Rightarrow \quad \sum_i U_i(\mathbf{r}(t_1)) + T(t_1) + W_{\mathsf{nc}}\left(\mathbf{r}_1 \to \mathbf{r}_2\right) = \sum_i U_i(\mathbf{r}(t_2)) + T(t_2)$$

Condition for conservative forces

- **Stoke's theorem**: integral of the curl of a vector field over an open surface $\mathcal{S}$ is equal to the circulation of the vector field around the curve $\partial\mathcal{S}$ bounding the surface $\mathcal{S}$

$$\iint_{\mathcal{S}} [\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})] \cdot \mathrm{d}\mathbf{a} = \oint_{\partial\mathcal{S}} \mathbf{A}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

- Work by conservative force is the same for all paths between $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$W(\mathbf{r}_1 \to \mathbf{r}_2) = \int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}'_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

$$\int_{\mathcal{C}_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} - \int_{\mathcal{C}'_{1\to 2}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0 \quad \Rightarrow \quad \oint \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0$$

- Conservative force is **irrotational**:

$$\boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}) = \mathbf{0}$$

Spherically symmetric central force is conservative

- Spherically symmetric central force is irrotational:

$$\mathbf{F}(\mathbf{r}) = f(r)\,\hat{\mathbf{e}}_r \quad \Rightarrow \quad \boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}) = \mathbf{0}$$

- Potential energy function associated with spherically symmetric central force:

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = -\int_{r_0}^{r} f(r')\,\mathrm{d}r' \equiv U(r)$$

- Obtaining spherically symmetric central force from potential energy:

$$\mathbf{F}(r) = -\boldsymbol{\nabla} U(\mathbf{r}) = f(r)\,\hat{\mathbf{e}}_r$$

EXERCISE 5.5: The electrostatic force on a point charge q located at $\mathbf{r}$ due to a fixed point charge Q at the origin is given by $\mathbf{F}(\mathbf{r}) = Qq/\left(4\pi\epsilon_0 r^2\right)\,\hat{\mathbf{e}}_r$. Show that it is conservative and find the corresponding potential energy.

Time-dependent potential energy

- Irrotational time-dependent force:

$$\boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}, t) = \mathbf{0}$$

- Time-dependent potential energy:

$$U(\mathbf{r}, t) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}', t) \cdot \mathrm{d}\mathbf{r}'$$

- Total mechanical energy is *not* a constant of motion!

$$E(t) \equiv T(t) + U(\mathbf{r}(t), t) \quad \Rightarrow \quad \frac{\mathrm{d}E(t)}{\mathrm{d}t} \neq 0$$

EXERCISE 5.6: Show that the total mechanical energy with time-dependent potential energy is not a constant of motion.

Work-energy theorem for multi-particle system

- Total force acting on the α-particle: $\mathbf{f}_{\alpha\beta}$ is the force acting on m_α due to m_β

$$\mathbf{F}_\alpha(t) = \mathbf{F}_\alpha^{\mathsf{ext}}(t) + \sum_{\beta=1,\beta\neq\alpha}^{N} \mathbf{f}_{\alpha\beta}(t)\,, \qquad \alpha = 1, 2, 3, \cdots, N$$

- Total kinetic energy of multi-particle system:

$$T(t) \equiv \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \mathbf{v}_\alpha(t) \cdot \mathbf{v}_\alpha(t)$$

- Work-energy theorem: total work by all external and internal forces during a given time interval is equal to the change in the kinetic energy of the multi-particle system during this time interval

$$T(t_2) - T(t_1) = \sum_{\alpha=1}^{N} \int_{t_1}^{t_2} \mathbf{F}_\alpha^{\mathsf{ext}}(t) \cdot \dot{\mathbf{r}}_\alpha(t)\,\mathrm{d}t + \sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} \int_{t_1}^{t_2} \mathbf{f}_{\alpha\beta}(t) \cdot \dot{\mathbf{r}}_\alpha(t)\,\mathrm{d}t$$

External conservative forces

- External conservative forces acting on the α-particle:

$$\mathbf{F}^{\mathsf{ext}}_{\mathsf{c},\alpha}(t) = \sum_i \mathbf{F}_{\mathsf{c},i}(\mathbf{r}_\alpha(t)) = -\sum_i \boldsymbol{\nabla}_\alpha U_i(\mathbf{r}_\alpha(t))$$

- Total work by external conservative forces acting on the α-particle:

$$\begin{aligned} W_{\mathsf{c},\alpha}\left(\mathbf{r}_{\alpha,1} \to \mathbf{r}_{\alpha,2}\right) &= -\sum_i \int_{\mathbf{r}_{\alpha,1}}^{\mathbf{r}_{\alpha,2}} \boldsymbol{\nabla}_\alpha U_i(\mathbf{r}_\alpha(t)) \cdot \mathrm{d}\mathbf{r}_\alpha \\ &= \sum_i \left[U_i\left(\mathbf{r}_\alpha(t_1)\right) - U_i\left(\mathbf{r}_\alpha(t_2)\right)\right] \end{aligned}$$

- Total external potential energy of multi-particle system:

$$U^{\mathsf{ext}}(\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t)) \equiv \sum_{\alpha=1}^{N} \sum_i U_i\left(\mathbf{r}_\alpha(t)\right)$$

External non-conservative forces

- External non-conservative forces acting on the α-particle:

$$\mathbf{F}^{\text{ext}}_{\text{nc},\alpha}(t) = \sum_j \mathbf{F}_{\text{nc},j}(\mathbf{r}_\alpha(t), \dot{\mathbf{r}}_\alpha(t), t)$$

- Total work by external non-conservative forces acting on the α-particle:

$$W_{\text{nc},\alpha}\left(\mathbf{r}_{\alpha,1} \to \mathbf{r}_{\alpha,2}\right) = \sum_j \int_{\mathcal{C}_{\mathbf{r}_{\alpha,1} \to \mathbf{r}_{\alpha,2}}} \mathbf{F}_{\text{nc},j}(\mathbf{r}_\alpha(t), \dot{\mathbf{r}}_\alpha(t), t) \cdot \mathrm{d}\mathbf{r}_\alpha$$

- Total work by external non-conservative forces acting on multi-particle system:

$$W_{\text{nc}}\left(\mathbf{r}_{1,1} \to \mathbf{r}_{1,2}, \cdots, \mathbf{r}_{N,1} \to \mathbf{r}_{N,2}\right) \equiv \sum_{\alpha=1}^{N} W_{\text{nc},\alpha}\left(\mathbf{r}_{\alpha,1} \to \mathbf{r}_{\alpha,2}\right)$$

Internal forces

- Internal force acting on α-particle due to β-particle is conservative:

$$\mathbf{f}_{\alpha\beta}(t) = -\boldsymbol{\nabla}_\alpha U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t)|\right) , \qquad \mathbf{r}_{\alpha\beta}(t) \equiv \mathbf{r}_\alpha(t) - \mathbf{r}_\beta(t)$$

- Total work by pair of internal forces:

$$\int_{t_1}^{t_2} \mathbf{f}_{\alpha\beta}(t)\cdot\dot{\mathbf{r}}_\alpha(t)\,\mathrm{d}t + \int_{t_1}^{t_2} \mathbf{f}_{\beta\alpha}(t)\cdot\dot{\mathbf{r}}_\beta(t)\,\mathrm{d}t = U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t_1)|\right) - U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t_2)|\right)$$

- Total work by internal forces:

$$\sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\int_{t_1}^{t_2} \mathbf{f}_{\alpha\beta}(t)\cdot\dot{\mathbf{r}}_\alpha(t)\,\mathrm{d}t = U^{\text{int}}(\mathbf{r}_1(t_1),\cdots,\mathbf{r}_N(t_1)) - U^{\text{int}}(\mathbf{r}_1(t_2),\cdots,\mathbf{r}_N(t_2))$$

$$U^{\text{int}}(\mathbf{r}_1(t),\cdots,\mathbf{r}_N(t)) \equiv \sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N} U_{\alpha\beta}\left(|\mathbf{r}_{\alpha\beta}(t)|\right)$$

Work-energy theorem for multi-particle system – cont'd

- Total potential energy of multi-particle system:

$$U(\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t)) \equiv U^{\text{ext}}(\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t)) + U^{\text{int}}(\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t))$$

- Work-energy theorem:

$$\begin{aligned} &U(\mathbf{r}_1(t_1), \cdots, \mathbf{r}_N(t_1)) + T(t_1) + W_{\text{nc}}\left(\mathbf{r}_{1,1} \to \mathbf{r}_{1,2}, \cdots, \mathbf{r}_{N,1} \to \mathbf{r}_{N,2}\right) \\ &= U(\mathbf{r}_1(t_2), \cdots, \mathbf{r}_N(t_2)) + T(t_2) \end{aligned}$$

- Total mechanical energy is not conserved due to the time-dependent potential energies and/or work by non-conservative forces

Example: A star with two planets

- Gravitational force acting on point mass m_1 due to another point mass m_2:

$$\mathbf{F}_{12}\left(\mathbf{r}_1(t)\right) = -\frac{Gm_1m_2}{\left|\mathbf{r}_1(t) - \mathbf{r}_2(t)\right|^3}\left[\mathbf{r}_1(t) - \mathbf{r}_2(t)\right]$$

- A star of very large mass M is orbited by two planets of masses m_1 and m_2

$$U^{\mathsf{ext}}(\mathbf{r}_1(t), \mathbf{r}_2(t)) = -\frac{GMm_1}{r_1(t)} - \frac{GMm_2}{r_2(t)}\,, \quad U^{\mathsf{int}}(\mathbf{r}_1(t), \mathbf{r}_2(t)) = -\frac{Gm_1m_2}{r_{12}(t)}$$

- Total mechanical energy:

$$E(t) = \frac{1}{2}\,m_1\dot{\mathbf{r}}_1(t)\cdot\dot{\mathbf{r}}_1(t) + \frac{1}{2}\,m_2\dot{\mathbf{r}}_2(t)\cdot\dot{\mathbf{r}}_2(t) - GM\left[\frac{m_1}{r_1(t)} + \frac{m_2}{r_2(t)}\right] - \frac{Gm_1m_2}{r_{12}(t)}$$

- If $E(0) < 0$, is it possible for a planet to escape to infinity?

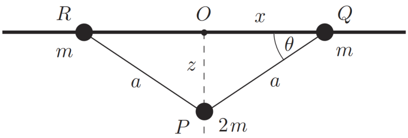
Example: A constrained three-particle system

- A ball P of mass $2m$ suspended by two light inextensible strings of length a from two sliders Q and R, each of mass m, which can move on a smooth horizontal rail. The system moves symmetrically so that O, the midpoint of Q and R, remains fixed and P moves on the downward vertical through O. Initially, the system is released from rest with the three particles in a straight line and with the strings taut. Ignore gravitational forces between masses.

- Tension forces exerted by the inextensible strings do zero work in total (WHY?)

- Total mechanical energy:

$$E(t) = ma^2\dot{\theta}^2(t) - 2mga\sin\theta(t)$$



EXERCISE 5.7: Derive the first order differential equation governing the dynamics of the system.

Kinetic energy of multi-particle system

- Total kinetic energy of multi-particle system:

$$T(t) \equiv \sum_{\alpha=1}^{N} T_\alpha(t) = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha(t) \cdot \dot{\mathbf{r}}_\alpha(t)$$

- Total kinetic energy of multi-particle system in the center-of-mass frame:

$$T'(t) \equiv \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\mathbf{r}}'_\alpha(t) \cdot \dot{\mathbf{r}}'_\alpha(t)$$

- Total kinetic energy of multi-particle system equals to the sum of kinetic energy of the center-of-mass and kinetic energy relative to the center-of-mass frame:

$$T(t) = T_{\mathsf{CM}}(t) + T'(t) = \frac{1}{2} M \dot{\mathbf{R}}_{\mathsf{CM}}(t) \cdot \dot{\mathbf{R}}_{\mathsf{CM}}(t) + \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\mathbf{r}}'_\alpha(t) \cdot \dot{\mathbf{r}}'_\alpha(t)$$