Equation of motion

- Second order ordinary differential equation: $\mathbf{r}(0) = \mathbf{r}_0$, $\dot{\mathbf{r}}(0) = \mathbf{v}_0$

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \qquad \rightarrow \qquad \begin{cases} \mathbf{r}(t) = ? \\ \dot{\mathbf{r}}(t) = ?? \end{cases}$$

- Cartesian coordinates:

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \qquad \Rightarrow \qquad \begin{cases} m\ddot{x}(t) = F_x(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\ddot{y}(t) = F_y(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\ddot{z}(t) = F_z(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \end{cases}$$

- Polar coordinates:

$$m\ddot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \quad \Rightarrow \quad \begin{cases} m\left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right] = F_\rho(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ m\left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right] = F_\phi(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \end{cases}$$

First order separable ordinary differential equation

- General form:

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = f(x)\,g(y)$$

- Implicit **general solution**: existence of an *arbitrary* constant in the solution

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

First order linear ordinary differential equation

- Standard form: $a_1(x) \neq 0$

$$a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + a_0(x)\,y(x) = f(x)$$

- **Integrating factor** $\mu(x)$: integration constant is irrelevant

$$\mu(x)\,a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + \mu(x)\,a_0(x)\,y(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\,a_1(x)\,y(x)\right]$$

$$\Rightarrow \quad \mu(x) = \frac{1}{a_1(x)} \exp\left[\int^x \frac{a_0(\xi)}{a_1(\xi)}\,\mathrm{d}\xi\right]$$

- General solution: c is an arbitrary integration constant

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\,a_1(x)\,y(x)\right] = \mu(x)\,f(x) \quad \Rightarrow \quad y(x) = \frac{1}{\mu(x)\,a_1(x)}\left[\int^x \mu(\xi)\,f(\xi)\,\mathrm{d}\xi + c\right]$$

$$a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + a_0(x)\,y(x) = f(x)$$

$$\mu(x)\,a_1(x)\,\frac{\mathrm{d}y(x)}{\mathrm{d}x} + \mu(x)\,a_0(x)\,y(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\,a_1(x)\,y(x)\right]$$

$$\Rightarrow \quad \mu(x)\,a_0(x) = \mu(x)\,\frac{\mathrm{d}a_1(x)}{\mathrm{d}x} + \frac{\mathrm{d}\mu(x)}{\mathrm{d}x}\,a_1(x)$$

$$\Rightarrow \quad \frac{\mathrm{d}\mu}{\mu(x)} = \frac{a_0(x)}{a_1(x)}\,\mathrm{d}x - \frac{\mathrm{d}a_1}{a_1(x)}$$

$$\Rightarrow \quad \ln\mu(x) = \int^x \frac{a_0(\xi)}{a_1(\xi)}\,\mathrm{d}\xi - \ln a_1(x)$$

$$\Rightarrow \quad \mu(x) = \frac{1}{a_1(x)}\exp\left[\int^x \frac{a_0(\xi)}{a_1(\xi)}\,\mathrm{d}\xi\right] \qquad \blacksquare$$

Special case: $F_x = F_x(t)$

- Solving for $v_x(t)$: $v_x(0) = v_{x0}$

$$m\ddot{x}(t) = F_x(t) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(t) \quad \Rightarrow \quad m\int_{v_x'=v_{x0}}^{v_x} \mathrm{d}v_x' = \int_{t'=0}^{t} F_x(t')\,\mathrm{d}t'$$

$$\Rightarrow \quad v_x(t) = v_{x0} + \frac{1}{m}\int_{t'=0}^{t} F_x(t')\,\mathrm{d}t'$$

- Solving for $x(t)$: $x(0) = x_0$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v_x(t) \quad \Rightarrow \quad \int_{x'=x_0}^{x} \mathrm{d}x' = \int_{t'=0}^{t} v_x(t')\,\mathrm{d}t'$$

$$\Rightarrow \quad x(t) = x_0 + v_{x0}t + \frac{1}{m}\int_{t'=0}^{t}\left[\int_{t''=0}^{t'} F_x(t'')\,\mathrm{d}t''\right]\mathrm{d}t'$$

Special case: $F_x = F_x(x)$

- Solving for $v_x(x)$: $x = x(t) \leftrightarrow t = t(x)$

$$m\ddot{x}(t) = F_x(x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F_x(x)$$

$$\Rightarrow \quad mv_x(x)\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x} = F_x(x) \quad \Rightarrow \quad m\int_{v_x'=v_{x0}}^{v_x} v_x'\,\mathrm{d}v_x' = \int_{x'=x_0}^{x} F_x(x')\,\mathrm{d}x'$$

$$\Rightarrow \quad v_x^2(x) = v_{x0}^2 + \frac{2}{m}\int_{x'=x_0}^{x} F_x(x')\,\mathrm{d}x'$$

- Solving for $x(t)$: $x = x(t) \leftrightarrow t = t(x)$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v_x(x) \quad \Rightarrow \quad \int_{x'=x_0}^{x} \frac{\mathrm{d}x'}{v_x(x')} = \int_{t'=0}^{t} \mathrm{d}t'$$

$$\Rightarrow \quad t = \int_{x'=x_0}^{x} \frac{\mathrm{d}x'}{v_x(x')} \quad \Rightarrow \quad x(t)$$

Special case: $F_x = F_x(v_x)$

- Solving for $v_x(t)$:

$$m\ddot{x}(t) = F_x(v_x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(v_x)$$

$$\Rightarrow \quad m\int_{v'_x=v_{x0}}^{v_x} \frac{\mathrm{d}v'_x}{F_x(v'_x)} = \int_{t'=0}^{t} \mathrm{d}t' \quad \Rightarrow \quad v_x(t) \quad \Rightarrow \quad x(t)$$

- Solving for $v_x(x)$:

$$m\ddot{x}(t) = F_x(v_x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = F_x(v_x) \quad \Rightarrow \quad m\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F_x(v_x)$$

$$\Rightarrow \quad mv_x(x)\,\frac{\mathrm{d}v_x(x)}{\mathrm{d}x} = F_x(v_x) \quad \Rightarrow \quad m\int_{v'_x=v_{x0}}^{v_x} \frac{v'_x}{F_x(v'_x)}\,\mathrm{d}v'_x = \int_{x'=x_0}^{x} \mathrm{d}x'$$

$$\Rightarrow \quad v_x(x) \quad \Rightarrow \quad x(t)$$

Example: Double Atwood machine

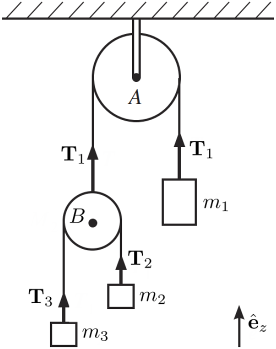
- A mass m_1 hangs at one end of a string that is led over a pulley A. The other end carries another pulley B which in tern carries a string with the masses m_2 and m_3 fixed to its ends. All pulleys and strings are assumed to be massless. Also, all strings are inextensible.

- Inextensible strings:

$$\mathbf{a}_{1A} = -\mathbf{a}_{BA}\,, \qquad \mathbf{a}_{2B} = -\mathbf{a}_{3B}$$

- Massless strings and pulleys:

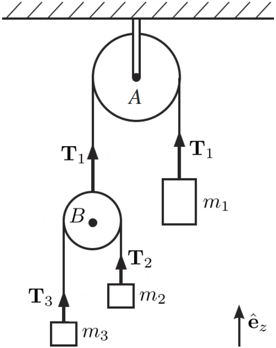
$$T_2 = T_3 = T\,, \qquad T_1 = 2T_2 = 2T_3 = 2T$$



EXERCISE 2.1: Find the acceleration of all masses.

$$
\begin{cases}
\mathbf{a}_{1A} = -\mathbf{a}_{BA} \\
\mathbf{a}_{2B} = -\mathbf{a}_{3B}
\end{cases}
$$

$$
\mathbf{a}_{2B} = -\mathbf{a}_{3B} \quad \Rightarrow \quad a_2 + a_1 = -\left(a_3 + a_1\right) \quad \Rightarrow \quad a_1 = -\frac{1}{2}\left(a_2 + a_3\right) \qquad \blacksquare
$$



$$\begin{cases} T_1 - m_1 g = m_1 a_1 \\ T_2 - m_2 g = m_2 a_2 \\ T_3 - m_3 g = m_3 a_3 \end{cases} \Rightarrow \begin{cases} 2T - m_1 g = -\dfrac{m_1}{2}\left(a_2 + a_3\right) \\ T - m_2 g = m_2 a_2 \\ T - m_3 g = m_3 a_3 \end{cases}$$

$$\Rightarrow \begin{cases} a_2 = -\dfrac{4m_2m_3 + m_1\left(m_2 - 3m_3\right)}{m_1\left(m_2 + m_3\right) + 4m_2m_3}\, g \\ a_3 = -\dfrac{4m_2m_3 + m_1\left(m_3 - 3m_2\right)}{m_1\left(m_2 + m_3\right) + 4m_2m_3}\, g \\ T = \dfrac{4m_1m_2m_3}{m_1\left(m_2 + m_3\right) + 4m_2m_3}\, g \end{cases} \qquad \blacksquare$$

$$a_1 = -\frac{1}{2}\left(a_2 + a_3\right) = \frac{4m_2m_3 - m_1\left(m_2 + m_3\right)}{m_1\left(m_2 + m_3\right) + 4m_2m_3}\, g \qquad \blacksquare$$

Example: Two masses on inclined plane

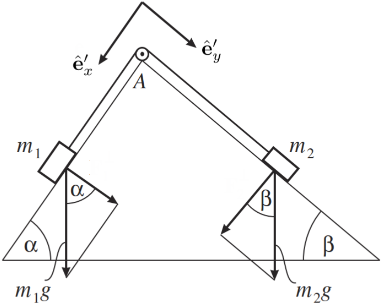
- Two masses m_1 and m_2 are lying each on one of two joined inclined planes with angles α and β with the horizontal. Both inclined planes and the horizontal make a right-angle triangle. The two masses are connected by a massless and inextensible string running over a massless and fixed pulley. The coefficients of kinetic friction of both planes are μ_k.

- Inextensible string:

$$\mathbf{a}_1 = a\,\hat{\mathbf{e}}'_x\,, \qquad \mathbf{a}_2 = -a\,\hat{\mathbf{e}}'_y$$

- Massless string and pulley:

$$\mathbf{T}_1 = -T\,\hat{\mathbf{e}}'_x\,, \qquad \mathbf{T}_2 = -T\,\hat{\mathbf{e}}'_y$$

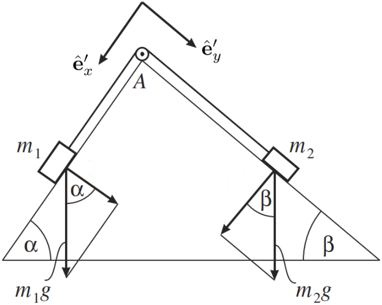


EXERCISE 2.2: Find the acceleration of the masses.

$$\mathbf{F}_1 = (m_1 g \sin\alpha - T - \mu_k N_1)\ \hat{\mathbf{e}}'_x + (m_1 g \cos\alpha - N_1)\ \hat{\mathbf{e}}'_y$$

$$\mathbf{F}_1 = m_1 \mathbf{a}_1 \quad \Rightarrow \quad \begin{cases} m_1 g \sin\alpha - T - \mu_k N_1 = m_1 a \\ m_1 g \cos\alpha - N_1 = 0 \end{cases}$$

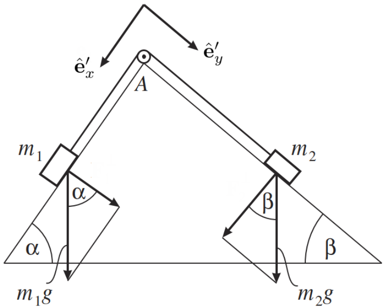
$$\Rightarrow \quad m_1 g \sin\alpha - T - \mu_k m_1 g \cos\alpha = m_1 a \qquad \blacksquare$$



$$\mathbf{F}_2 = (m_2 g\cos\beta - N_2)\ \hat{\mathbf{e}}'_x + (m_2 g\sin\beta - T + \mu_k N_2)\ \hat{\mathbf{e}}'_y$$

$$\mathbf{F}_2 = m_2\mathbf{a}_2 \quad \Rightarrow \quad \begin{cases} m_2 g\cos\beta - N_2 = 0 \\ m_2 g\sin\beta - T + \mu_k N_2 = -m_2 a \end{cases}$$

$$\Rightarrow \quad m_2 g\sin\beta - T + \mu_k m_2 g\cos\beta = -m_2 a \qquad \blacksquare$$



$$\begin{cases} m_1 g \sin\alpha - T - \mu_k m_1 g \cos\alpha = m_1 a \\ m_2 g \sin\beta - T + \mu_k m_2 g \cos\beta = -m_2 a \end{cases}$$

$$\Rightarrow \quad \begin{cases} a = \dfrac{(m_1 \sin\alpha - m_2 \sin\beta) - \mu_k \left(m_1 \cos\alpha + m_2 \cos\beta\right)}{m_1 + m_2}\, g \\ T = \dfrac{m_1 m_2 g}{m_1 + m_2} \left[(\sin\alpha + \sin\beta) - \mu_k \left(\cos\alpha - \cos\beta\right)\right] \end{cases} \qquad \blacksquare$$

$$\mu_k \to 0 \qquad \Rightarrow \qquad a \to \frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2}\, g \qquad \blacksquare$$

$$\alpha = \beta = \frac{\pi}{2} \qquad \Rightarrow \qquad a \to \frac{m_1 - m_2}{m_1 + m_2}\, g \qquad \blacksquare$$