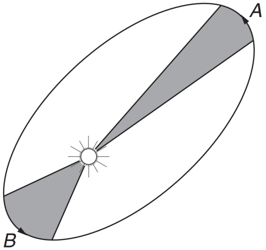
Example: Kepler's second law

- Johannes Kepler announced his first two laws of planetary motion in 1609. The first law is that the orbits of the planets are ellipses

- The second is the law of equal areas: the area swept out by the radius vector from the Sun to a planet in a given time is the same for any location of the planet in its orbit

- The orbit of a planet is confined to a plane as the torque due to gravitational force on the planet about the Sun is zero

$$\mathbf{F}(\mathbf{t}) = f(r)\,\hat{\mathbf{e}}_r \quad \Rightarrow \quad \boldsymbol{\mathcal{T}}(t) = \mathbf{r}(t) \times \mathbf{F}(t) = \mathbf{0}$$



EXERCISE 4.2: Show that the rate at which area is swept out is constant.

$$\mathbf{F}(t) = f(r)\,\hat{\mathbf{e}}_r\,, \qquad \mathbf{r}(t) = r(t)\,\hat{\mathbf{e}}_r$$

$$\boldsymbol{\mathcal{T}}(t) = \mathbf{r}(t) \times \mathbf{F}(t) = \mathbf{0} \qquad \blacksquare$$

$$\mathbf{r}(t) = \rho(t)\,\hat{\mathbf{e}}_\rho \quad \Rightarrow \quad \mathbf{v}(t) = \dot{\rho}(t)\,\hat{\mathbf{e}}_\rho + \rho(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_\phi$$

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t) = m\rho^2(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_z$$

$$\dot{\mathbf{L}}(t) = \boldsymbol{\mathcal{T}}(t) \quad \Rightarrow \quad \mathbf{L}(t) = m\,\rho^2(t)\,\dot{\phi}(t)\,\hat{\mathbf{e}}_z = \text{constant}$$

$$\Rightarrow \quad L_z(t) \equiv m\rho^2(t)\,\dot{\phi}(t) = \text{constant} \qquad \blacksquare$$

$$L_z = m\rho^2(t)\,\dot{\phi}(t) = \textsf{constant}$$

$$\Delta A \approx \frac{1}{2}\,[\rho(t) + \Delta\rho]\,[\rho(t)\,\Delta\phi] = \frac{1}{2}\,\rho^2(t)\,\Delta\phi + \frac{1}{2}\,\rho(t)\,\Delta\rho\,\Delta\phi$$

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} \equiv \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2}\,\rho^2(t)\,\dot{\phi}(t) = \frac{L_z}{2m} = \textsf{constant} \qquad \blacksquare$$

Orbital and spin angular momentum

- Total angular momentum of multi-particle system about the origin: sum of the angular momentum of the center of mass relative to the origin and the angular momentum of the motion relative to the center-of-mass frame

$$\mathbf{L}(t) = \mathbf{R}_{\mathsf{CM}}(t) \times \mathbf{P}(t) + \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t)$$

- Orbital and spin angular momentum:

$$\mathbf{L}^{\mathsf{orbital}}(t) \equiv \mathbf{R}_{\mathsf{CM}}(t) \times \mathbf{P}(t)\,, \qquad \mathbf{L}^{\mathsf{spin}}(t) \equiv \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t)$$

- Time rate of change of total angular momentum about the center-of-mass frame equals to the total external torque about the center-of-mass frame!

$$\dot{\mathbf{L}}'(t) = \boldsymbol{\mathcal{T}}'^{\mathsf{ext}}(t)$$

$$\mathbf{L}(t) = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{p}_\alpha(t)\,, \qquad \mathbf{r}'_\alpha(t) = \mathbf{r}_\alpha(t) - \mathbf{R}_{\mathsf{CM}}(t)$$

$$\mathbf{L}(t) = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{p}_\alpha(t) = \sum_{\alpha=1}^{N} \left[\mathbf{R}_{\mathsf{CM}}(t) + \mathbf{r}'_\alpha(t)\right] \times m_\alpha \left[\dot{\mathbf{R}}_{\mathsf{CM}}(t) + \dot{\mathbf{r}}'_\alpha(t)\right]$$

$$= \sum_{\alpha=1}^{N} \mathbf{R}_{\mathsf{CM}}(t) \times m_\alpha \dot{\mathbf{R}}_{\mathsf{CM}}(t) + \sum_{\alpha=1}^{N} \mathbf{R}_{\mathsf{CM}}(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t) + \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{R}}_{\mathsf{CM}}(t)$$

$$+ \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t)$$

$$= \mathbf{R}_{\mathsf{CM}}(t) \times M\dot{\mathbf{R}}_{\mathsf{CM}}(t) + \mathbf{R}_{\mathsf{CM}}(t) \times \left[\sum_{\alpha=1}^{N} m_\alpha \dot{\mathbf{r}}'_\alpha(t)\right] + \left[\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}'_\alpha(t)\right] \times \dot{\mathbf{R}}_{\mathsf{CM}}(t)$$

$$+ \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t)$$

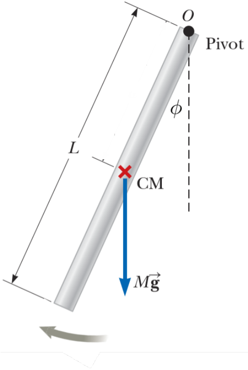
$$= \mathbf{R}_{\mathsf{CM}}(t) \times M\dot{\mathbf{R}}_{\mathsf{CM}}(t) + \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t) \qquad \blacksquare$$

$$\mathbf{L}'(t) = \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t)\,, \qquad \mathbf{r}'_\alpha(t) = \mathbf{r}_\alpha(t) - \mathbf{R}_{\mathsf{CM}}(t)$$

$$\begin{aligned}
\dot{\mathbf{L}}'(t) &= \sum_{\alpha=1}^{N} \dot{\mathbf{r}}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t) + \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha \times m_\alpha \ddot{\mathbf{r}}'_\alpha(t) \\
&= \mathbf{0} + \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \ddot{\mathbf{r}}_\alpha(t) - \left[\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}'_\alpha(t)\right] \times \ddot{\mathbf{R}}_{\mathsf{CM}}(t) \\
&= \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times \mathbf{F}_\alpha(t) \\
&= \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times \mathbf{F}^{\mathsf{ext}}_\alpha(t) + \sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} \mathbf{r}'_\alpha(t) \times \mathbf{f}_{\alpha\beta}(t) \\
&= \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times \mathbf{F}^{\mathsf{ext}}_\alpha(t) \\
&= \boldsymbol{\mathcal{T}}'^{\mathsf{ext}}(t) \qquad \blacksquare
\end{aligned}$$

Example: Swinging rod

- A long uniform rod of length L and mass M hangs at its end from a pivot point about which it is free to swing in a vertical plane like a physical pendulum



EXERCISE 4.3: Calculate orbital and spin angular momentum of the rod. Also, calculate directly the total angular momentum of the rod about the pivot point.

$$\mathbf{R}_{\mathsf{CM}}(t) = \frac{L}{2}\sin\phi(t)\,\hat{\mathbf{e}}_y + \frac{L}{2}\cos\phi(t)\,\hat{\mathbf{e}}_z$$

$$\Rightarrow \quad \mathbf{V}_{\mathsf{CM}}(t) = \frac{L}{2}\,\dot{\phi}(t)\cos\phi(t)\,\hat{\mathbf{e}}_y - \frac{L}{2}\,\dot{\phi}(t)\sin\phi(t)\,\hat{\mathbf{e}}_z$$

$$\begin{aligned}
\mathbf{L}^{\mathsf{orbital}}(t) &= \mathbf{R}_{\mathsf{CM}}(t) \times \mathbf{P}(t) = M\mathbf{R}_{\mathsf{CM}}(t) \times \mathbf{V}_{\mathsf{CM}}(t) \\
&= -\frac{ML^2}{4}\,\dot{\phi}(t)\sin^2\phi(t)\,\hat{\mathbf{e}}_x - \frac{ML^2}{4}\,\dot{\phi}(t)\cos^2\phi(t)\,\hat{\mathbf{e}}_x \\
&= -\frac{1}{4}\,ML^2\dot{\phi}(t)\,\hat{\mathbf{e}}_x \qquad \blacksquare
\end{aligned}$$

$$\mathbf{r}'_\alpha(t) = r'_\alpha \sin\phi(t)\,\hat{\mathbf{e}}'_y + r'_\alpha \cos\phi(t)\,\hat{\mathbf{e}}'_z$$

$$\Rightarrow \quad \dot{\mathbf{r}}'_\alpha(t) = r'_\alpha \dot{\phi}(t) \cos\phi(t)\,\hat{\mathbf{e}}'_y - r'_\alpha \dot{\phi}(t) \sin\phi(t)\,\hat{\mathbf{e}}'_z \qquad \blacksquare$$

$$\mathbf{L}'_\alpha(t) = m_\alpha \mathbf{r}'_\alpha(t) \times \dot{\mathbf{r}}'_\alpha(t) = -m_\alpha r'^2_\alpha \dot{\phi}(t)\,\hat{\mathbf{e}}'_x$$

$$\mathbf{L}^{\mathsf{spin}}(t) = -2\int_{r'=0}^{L/2} r'^2 \dot{\phi}(t)\,\mathrm{d}m\,\hat{\mathbf{e}}'_x = -\frac{1}{12}\,ML^2\dot{\phi}(t)\,\hat{\mathbf{e}}'_x \qquad \blacksquare$$

$$\mathbf{r}_\alpha(t) = r_\alpha \sin\phi(t)\,\hat{\mathbf{e}}_y + r_\alpha \cos\phi(t)\,\hat{\mathbf{e}}_z$$

$$\Rightarrow \quad \dot{\mathbf{r}}_\alpha(t) = r_\alpha \dot{\phi}(t) \cos\phi(t)\,\hat{\mathbf{e}}_y - r_\alpha \dot{\phi}(t) \sin\phi(t)\,\hat{\mathbf{e}}_z \qquad \blacksquare$$

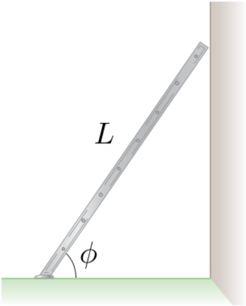
$$\mathbf{L}_\alpha(t) = m_\alpha \mathbf{r}_\alpha(t) \times \dot{\mathbf{r}}_\alpha(t) = -m_\alpha r_\alpha^2 \dot{\phi}(t)\,\hat{\mathbf{e}}_x$$

$$\mathbf{L}(t) = -\int_{r=0}^{L} r^2 \dot{\phi}(t)\,\mathrm{d}m\,\hat{\mathbf{e}}_x = -\frac{1}{3}\,ML^2\dot{\phi}(t)\,\hat{\mathbf{e}}_x \qquad \blacksquare$$

$$\begin{cases} \mathbf{L}^{\text{orbital}}(t) = -\dfrac{1}{4} ML^2 \dot{\phi}(t)\,\hat{\mathbf{e}}_x \\[2ex] \mathbf{L}^{\text{spin}}(t) = -\dfrac{1}{12} ML^2 \dot{\phi}(t)\,\hat{\mathbf{e}}'_x \end{cases} \quad \Rightarrow \quad \mathbf{L}^{\text{orbital}}(t) + \mathbf{L}^{\text{spin}}(t) = \mathbf{L}(t) \quad \checkmark$$

Example: Sliding ladder

- A uniform ladder of length L is supported by a smooth horizontal floor and leaning against a smooth vertical wall. The ladder is released from rest with $\phi = 30^\circ$. The ladder is assumed to remain in contact with the wall in its subsequent motion.



EXERCISE 4.4: Obtain a second order differential equation for $\phi(t)$ governing the motion of the ladder.

$$\mathbf{R}_{\mathsf{CM}}(t) = \frac{L}{2}\cos\phi(t)\,\hat{\mathbf{e}}_y + \frac{L}{2}\sin\phi(t)\,\hat{\mathbf{e}}_z$$

$$\Rightarrow \quad \mathbf{V}_{\mathsf{CM}}(t) = -\frac{L}{2}\,\dot{\phi}(t)\sin\phi(t)\,\hat{\mathbf{e}}_y + \frac{L}{2}\,\dot{\phi}(t)\cos\phi(t)\,\hat{\mathbf{e}}_z$$

$$\Rightarrow \; \mathbf{A}_{\mathsf{CM}}(t) = -\frac{L}{2}\left[\dot{\phi}^2(t)\cos\phi(t) + \ddot{\phi}(t)\sin\phi(t)\right]\hat{\mathbf{e}}_y + \frac{L}{2}\left[-\dot{\phi}^2(t)\sin\phi(t) + \ddot{\phi}(t)\cos\phi(t)\right]\hat{\mathbf{e}}_z$$

$$\mathbf{F}(t) = N_{\mathsf{wall}}(t)\,\hat{\mathbf{e}}_y + \left[N_{\mathsf{floor}}(t) - Mg\right]\,\hat{\mathbf{e}}_z$$

$$\mathbf{F}(t) = M\mathbf{A}_{\mathsf{CM}}(t) \quad \Rightarrow \quad \begin{cases} N_{\mathsf{wall}}(t) = -\dfrac{ML}{2}\left[\dot{\phi}^2(t)\cos\phi(t) + \ddot{\phi}(t)\sin\phi(t)\right] \\[2ex] N_{\mathsf{floor}}(t) = Mg + \dfrac{ML}{2}\left[-\dot{\phi}^2(t)\sin\phi(t) + \ddot{\phi}(t)\cos\phi(t)\right] \end{cases} \qquad \blacksquare$$

$$\mathbf{L}'(t) = \sum_{\alpha=1}^{N} \mathbf{r}'_\alpha(t) \times m_\alpha \dot{\mathbf{r}}'_\alpha(t)$$

$$\mathbf{r}'_\alpha(t) = -r'_\alpha \cos\phi(t)\,\hat{\mathbf{e}}'_y + r'_\alpha \sin\phi(t)\,\hat{\mathbf{e}}'_z \qquad \blacksquare$$

$$\dot{\mathbf{r}}'_\alpha(t) = r'_\alpha \dot{\phi}(t) \sin\phi(t)\,\hat{\mathbf{e}}'_y + r'_\alpha \dot{\phi}(t) \cos\phi(t)\,\hat{\mathbf{e}}'_z \qquad \blacksquare$$

$$\mathbf{L}'_\alpha(t) = m_\alpha \mathbf{r}'_\alpha(t) \times \dot{\mathbf{r}}'_\alpha(t) = -m_\alpha r'^2_\alpha \dot{\phi}(t)\,\hat{\mathbf{e}}'_x$$

$$\mathbf{L}'(t) = -2\int_{r'=0}^{L/2} r'^2 \dot{\phi}(t)\,\mathrm{d}m\,\hat{\mathbf{e}}'_x = -\frac{1}{12}\,ML^2\dot{\phi}(t)\,\hat{\mathbf{e}}'_x \qquad \blacksquare$$

$$\begin{cases} N_{\text{wall}}(t) = -\dfrac{ML}{2}\left[\dot\phi^2(t)\cos\phi(t) + \ddot\phi(t)\sin\phi(t)\right] \\[2ex] N_{\text{floor}}(t) = Mg + \dfrac{ML}{2}\left[-\dot\phi^2(t)\sin\phi(t) + \ddot\phi(t)\cos\phi(t)\right] \end{cases}, \qquad \mathbf{L}'(t) = -\frac{1}{12}\,ML^2\dot\phi(t)\,\hat{\mathbf{e}}'_x$$

$$\begin{aligned} \boldsymbol{\tau}'^{\text{ext}}(t) &= \left[-\frac{L}{2}\cos\phi(t)\,\hat{\mathbf{e}}'_y + \frac{L}{2}\sin\phi(t)\,\hat{\mathbf{e}}'_z\right] \times N_{\text{wall}}(t)\,\hat{\mathbf{e}}'_y \\ &\qquad + \left[\frac{L}{2}\cos\phi(t)\,\hat{\mathbf{e}}'_y - \frac{L}{2}\sin\phi(t)\,\hat{\mathbf{e}}'_z\right] \times N_{\text{floor}}(t)\,\hat{\mathbf{e}}'_z \\ &= \frac{L}{2}\left[-N_{\text{wall}}(t)\sin\phi(t) + N_{\text{floor}}(t)\cos\phi(t)\right]\,\hat{\mathbf{e}}'_x \\ &= \frac{ML^2}{4}\left[\ddot\phi(t) + \frac{2g}{L}\cos\phi(t)\right]\,\hat{\mathbf{e}}'_x \qquad \blacksquare \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{L}}'(t) = \boldsymbol{\tau}'^{\text{ext}}(t) \quad &\Rightarrow \quad -\frac{1}{12}ML^2\ddot\phi(t) = \frac{ML^2}{4}\left[\ddot\phi(t) + \frac{2g}{L}\cos\phi(t)\right] \\ &\Rightarrow \quad \ddot\phi(t) + \frac{3}{2}\,\frac{g}{L}\cos\phi(t) = 0 \qquad \blacksquare \end{aligned}$$

Torque about an *arbitrary* point

- Torque about an arbitrary point $\mathcal{O}'$ with position vector $\mathbf{R}(t)$ about the origin $\mathcal{O}$ of an inertial frame: assignment 2

$$\boldsymbol{\tau}'^{\text{ext}}(t) = \dot{\mathbf{L}}'(t) + M\left[\mathbf{R}_{\text{CM}}(t) - \mathbf{R}(t)\right] \times \ddot{\mathbf{R}}(t)$$

- Coordinate axes relative to $\mathcal{O}'$ are still assumed to be parallel to the coordinate axes of the inertial frame

- The extra term vanishes if any of the following three conditions is satisfied:
  - $\mathbf{R}(t) = \mathbf{R}_{\text{CM}}(t)$
  - $\ddot{\mathbf{R}}(t) = \mathbf{0}$
  - $\mathbf{R}_{\text{CM}}(t) - \mathbf{R}(t)$ is parallel to $\ddot{\mathbf{R}}(t)$

Torque and angular momentum about an axis

- Component of the torque about the orign in the direction of a unit vector $\hat{\mathbf{e}}_n$ is called the torque about the axis through the origin

$$\hat{\mathbf{e}}_n \cdot \boldsymbol{\tau}(t) = \hat{\mathbf{e}}_n \cdot [\mathbf{r}(t) \times \mathbf{F}(t)] = [\hat{\mathbf{e}}_n \times \mathbf{r}(t)] \cdot \mathbf{F}(t) \equiv \rho(t)\,[\hat{\mathbf{e}}_\phi \cdot \mathbf{F}(t)]$$

- ρ is the perpendicular distance of the point to the axis and $\hat{\mathbf{e}}_\phi$ is the azimuthal diretion around the axis (fixed by the right-hand rule)

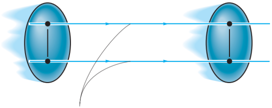
- Component of the angular momentum about the orign in the direction of a unit vector $\hat{\mathbf{e}}_n$:

$$\hat{\mathbf{e}}_n \cdot \boldsymbol{\ell}(t) = \hat{\mathbf{e}}_n \cdot [\mathbf{r}(t) \times \mathbf{p}(t)] = [\hat{\mathbf{e}}_n \times \mathbf{r}(t)] \cdot \mathbf{p}(t) = \rho(t)\,[\hat{\mathbf{e}}_\phi \cdot \mathbf{p}(t)]$$

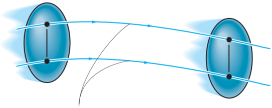
Planar motion of rigid body

- A **rigid body** is a system of particles for which the distances between the particles remained unchanged

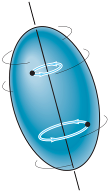
- A rigid body executes **planar motion** when all parts of the body move in parallel planes and the plane of motion is typically chosen to be the plane consisting of the center of mass



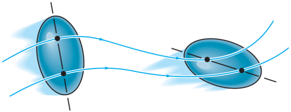
Path of rectilinear translation



Path of curvilinear translation



Rotation about a fixed axis



General plane motion

Kinematics of rotation about a fixed axis

- **Chasles theorem** asserts that any displacement of a *rigid body* can be decomposed into two *independent* motions: a translation of the center of mass and a rotation around the center of mass

- When a rigid body rotates about a fixed axis, all points (other than those on the axis) move in a concentric circles about the fixed axis

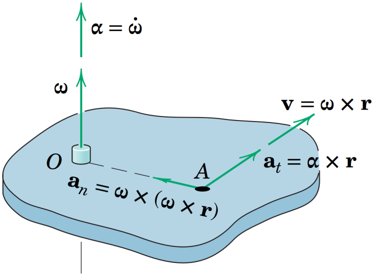
- Velocity and acceleration:

$$\boldsymbol{\omega}(t) = \omega(t)\,\hat{\mathbf{e}}_n$$

$$\mathbf{v}(t) = \boldsymbol{\omega}(t) \times \mathbf{r}(t)$$

$$\mathbf{a}(t) = \boldsymbol{\alpha}(t) \times \mathbf{r}(t) + \boldsymbol{\omega}(t) \times [\boldsymbol{\omega}(t) \times \mathbf{r}(t)]$$

$$\omega(t) = \dot{\phi}(t)\,, \quad \alpha(t) = \dot{\omega}(t) = \ddot{\phi}(t)$$



Angular momentum of rigid body about a fixed axis

- Angular momentum about a fixed axis:

$$\hat{\mathbf{e}}_n \cdot \mathbf{L}(t) = \hat{\mathbf{e}}_n \cdot \left[\sum_i m_i \mathbf{r}_i(t) \times \mathbf{v}_i(t)\right] = \left(\sum_i m_i \rho_i^2\right) \omega(t)$$

- Conservation of angular momentum about a fixed axis: If the external forces acting on a system have zero torque about a fixed axis, then the angular momentum of the system about that axis is conserved

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{\mathbf{e}}_n \cdot \mathbf{L}(t)\right] = \hat{\mathbf{e}}_n \cdot \frac{\mathrm{d}\mathbf{L}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\hat{\mathbf{e}}_n}{\mathrm{d}t} \cdot \mathbf{L}(t) = \hat{\mathbf{e}}_n \cdot \boldsymbol{\mathcal{T}}^{\text{ext}}(t)$$

- If the external forces acting on a system have zero torque about a moving axis with a constant direction through the center of mass, then the angular momentum of the system about that axis is conserved

Moment of inertia

- **Moment of inertia** of rigid body about a fixed axis:

$$I \equiv \sum_i m_i \rho_i^2 \qquad \to \qquad I = \int \rho^2 \, \mathrm{d}m$$

- Angular momentum about a fixed axis:

$$L_n(t) = I\omega(t) \quad \not\to \quad \mathbf{L}(t) = I\boldsymbol{\omega}(t)$$

- Equations of motion for planar motion of rigid body: choose the plane of motion to be xy-plane and axis of rotation to be z-axis through the center of mass

$$\begin{cases} M \dfrac{\mathrm{d}V_x(t)}{\mathrm{d}t} = F_x(t) \\ M \dfrac{\mathrm{d}V_y(t)}{\mathrm{d}t} = F_y(t) \end{cases}, \qquad I_{\mathsf{CM}} \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \mathcal{T}_{\mathsf{CM}}^{\mathsf{ext}}(t)$$

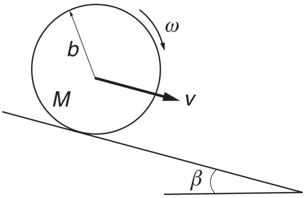
Two theorems for moment of inertia

- **Perpendicular-axis theorem**: the moment of inertia of any planar object about an axis normal to the plane of the object is equal to the sum of the moments of inertia about any two mutually perpendicular axes passing through the given axis and lying in the plane of the object

- **Parallel-axis theorem**: the moment of inertia of a rigid body about any axis is equal to the moment of inertia about a parallel axis passing through the center of mass plus the product of the mass of the body and the square of the distance between the two axes

Example: Drum rolling down a plane

- A uniform drum of radius b and mass M rolls without slipping down a plane inclined at angle β.



- Translation of the center of mass:

$$\begin{cases} Mg\sin\beta - f = M\ddot{X}_{\text{CM}}(t) \\ N - Mg\cos\beta = M\ddot{Y}_{\text{CM}}(t) \end{cases}$$

- Motion with no slipping: the contact is very rough $f \le \mu_s N$

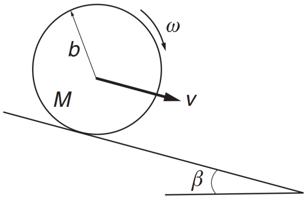
$$\dot{X}_{\text{CM}}(t) = b\dot{\phi}(t) = b\omega(t) \quad \Rightarrow \quad \ddot{X}_{\text{CM}}(t) = b\ddot{\phi}(t) = b\dot{\omega}(t)$$

EXERCISE 4.5: Find the drum's acceleration along the plane.

$$\mathbf{R}_{\mathsf{CM}}(t) = X_{\mathsf{CM}}(t)\,\hat{\mathbf{e}}_x + b\,\hat{\mathbf{e}}_y$$

$$\begin{cases} \mathbf{W}(t) = Mg\sin\beta\,\hat{\mathbf{e}}_x - Mg\cos\beta\,\hat{\mathbf{e}}_y \\ \mathbf{f}(t) = -f(t)\,\hat{\mathbf{e}}_x \\ \mathbf{N}(t) = N(t)\,\hat{\mathbf{e}}_y \end{cases}$$

$$\begin{cases} \mathbf{r}_W(t) = X_{\mathsf{CM}}(t)\,\hat{\mathbf{e}}_x + b\,\hat{\mathbf{e}}_y \\ \mathbf{r}_f(t) = X_{\mathsf{CM}}(t)\,\hat{\mathbf{e}}_x \\ \mathbf{r}_N(t) = X_{\mathsf{CM}}(t)\,\hat{\mathbf{e}}_x \end{cases}$$



$$\mathbf{F}(t) = M\ddot{\mathbf{R}}_{\mathsf{CM}}(t)$$

$$\Rightarrow \quad \begin{cases} Mg\sin\beta - f(t) = M\ddot{X}_{\mathsf{CM}}(t) \\ N(t) - Mg\cos\beta = M\ddot{Y}_{\mathsf{CM}}(t) \end{cases}$$

$$\Rightarrow \quad \begin{cases} Mg\sin\beta - f(t) = M\ddot{X}_{\mathsf{CM}}(t) \\ N - Mg\cos\beta = 0 \end{cases}$$