Particle physics PC4245

Tutorial 2 solutions

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Angular momentum of the electron

$= I\dot{\theta} = \frac{2}{5} m v r$ (Moment of inertia of a sphere $= I = \frac{2}{5} m r^2$)

$= \frac{1}{2}\hbar$

$r = 10^{-18}$ m, $\hbar = 6.582 \times 10^{-22}$ MeV s
$= 1.055 \times 10^{-34}$ J

mass $m = 0.511$ MeV/c^2 $= 9.110 \times 10^{-31}$ kg

$\therefore$ speed $= \frac{1}{2}\, 10^{18} \cdot \frac{1}{9.110} \cdot 10^{31} \cdot 1.055 \times 10^{-34} \cdot \frac{5}{2}$

$= \frac{5}{2} \frac{0.5275}{9.110} \times 10^{15}$ m/s $= 1.45 \times 10^{14}$ m/s

$> 3 \times 10^{8}$ m/s $= c$ (speed of light)

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using the eigenstates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ of S_z as the basis, we write

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$$

(a) The eigenstates of S_x with eigenvalues $\pm\frac{\hbar}{2}$ are respectively

$$\chi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm 1\end{pmatrix}$$

Any spinor $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ can be written in terms of $\chi_\pm$

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = a\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + b\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

with $a = \frac{\alpha+\beta}{\sqrt{2}}$, $b = \frac{\alpha-\beta}{\sqrt{2}}$

Thus for the electron state $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$, we find

$$a = \frac{3}{\sqrt{10}}, \qquad b = \frac{-1}{\sqrt{10}}$$

or
$$\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix} = \frac{3}{\sqrt{20}}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{\sqrt{20}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{3}{\sqrt{10}}\chi_+ - \frac{1}{\sqrt{10}}\chi_-$$

probability of getting eigenvalue $+\frac{\hbar}{2}$ for S_x is $\frac{9}{10}$

$-\frac{\hbar}{2}$ for S_x is $\frac{1}{10}$

(b) The eigenstates of $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$

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are

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}$$

with eigenvalues $\pm\frac{\hbar}{2}$ respectively.

For any spinor $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, one can write

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = a\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} + b\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

For $\alpha = \frac{1}{\sqrt{5}}$, $\beta = \frac{2}{\sqrt{5}}$, it is easy to see

$$a = \frac{1}{\sqrt{10}}(1-2i) \quad , \quad b = \frac{1}{\sqrt{10}}(1+2i)$$

The probability of getting $\frac{\hbar}{2}$ when measuring S_y for the electron in the state $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\ 2\end{pmatrix}$ is $|a|^2 = \frac{1}{2}$

Similarly the probability of getting $-\frac{\hbar}{2}$ is $|b|^2 = \frac{1}{2}$

(c) Eigenstates of S_z with eigenvalues $\pm\frac{\hbar}{2}$ are respectively $\begin{pmatrix}1\\ 0\end{pmatrix}$ and $\begin{pmatrix}0\\ 1\end{pmatrix}$

So $$\frac{1}{\sqrt{5}}\begin{pmatrix}1\\ 2\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\ 0\end{pmatrix} + \frac{2}{\sqrt{5}}\begin{pmatrix}0\\ 1\end{pmatrix}$$

probability of getting $+\frac{\hbar}{2}$ for S_z is $\frac{1}{5}$

" " " $-\frac{\hbar}{2}$ " " is $\frac{4}{5}$.

3. The commutation relations for the spin $\frac{1}{2}$ angular momentum are

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$$[s_i, s_j] = i\hbar\, \varepsilon_{ijk}\, s_k$$

$$[s_i, \underset{\sim}{s}^2] = 0, \quad \underset{\sim}{s}^2 = s_i s_i = s_x^2 + s_y^2 + s_z^2$$

Common eigenstates of s_z and s^2 are

$$|+\rangle \equiv |\tfrac{1}{2}\ +\tfrac{1}{2}\rangle, \qquad |-\rangle \equiv |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

With respect to this basis, $s^2 = \frac{3}{4}\hbar^2$, $s_z^2 = \frac{1}{4}\hbar^2$

and $s_\pm^2 = 0$,

where $s_\pm \equiv s_x \pm i s_y$

As $s_\pm^2 = (s_x^2 - s_y^2) \pm i[s_x, s_y]_+ = 0$

$\therefore\ s_x^2 = s_y^2 = \frac{1}{4}\hbar^2$ and $[s_x, s_y]_+ = 0$

Since $[s_z, s_\pm]_+ |i\rangle = 0$ for $i = +, -$

$\therefore\ [s_z, s_\pm]_+ = 0$

This leads to $[s_z, s_x]_+ = [s_z, s_y]_+ = 0$

Defining $s_i = \frac{\hbar}{2}\sigma_i$, we get (a) and (b):

(a) As $s_i^2 = \frac{\hbar^2}{4}$ $\quad\therefore\ \sigma_i^2 = 1$

(b) As $[s_i, s_j] = i\hbar\, \varepsilon_{ijk}\, s_k$ and $[s_i, s_j]_+ = 0$,

We have $s_i s_j = \frac{1}{2} i\hbar\, \varepsilon_{ijk}\, s_k \longrightarrow \sigma_i \sigma_j = i\, \varepsilon_{ijk}\, \sigma_k$

Combing (a) and (b) from the above,

$$\sigma_i \sigma_j = \delta_{ij} + i\, \varepsilon_{ijk}\, \sigma_k$$

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Note that instead of the above method, one can use the explicit matrix representation of σ_i :

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

to verify (a) and (b).

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(a) $$e^{i\pi\sigma_z/2} = 1 + \frac{i\pi\sigma_z}{2} + \frac{1}{2!}\left(\frac{i\pi\sigma_z}{2}\right)^2 + \frac{1}{3!}\left(\frac{i\pi\sigma_z}{2}\right)^3 + \frac{1}{4!}\left(\frac{i\pi\sigma_z}{2}\right)^4 + \cdots$$

$$= 1 + \frac{i\pi\sigma_z}{2} - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{i}{3!}\left(\frac{\pi}{2}\right)^3\sigma_z + \frac{1}{4!}\left(\frac{\pi}{2}\right)^4 + \cdots$$

$\because \sigma_z^2 = 1$

$$= \cos\frac{\pi}{2} + i\sigma_z \sin\frac{\pi}{2} = i\sigma_z$$

(b) The matrix U is $U = \exp\left(-i\frac{\pi}{2}\sigma_y\right)$

$$= -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Spin up spinor is $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, spin down spinor is $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Clearly $U\chi_+ = \chi_-$

(c) $U = \exp\left(-\frac{i\theta}{2}\underset{\sim}{\sigma}\cdot\underset{\sim}{n}\right)$, $\underset{\sim}{n}$ = unit vector specifying the direction of rotation

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4.2! p155 Expanding by power series,

$$U = 1 - i\frac{\theta}{2}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{n} + \frac{1}{2!}\left(\frac{-i\theta}{2}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{n}\right)^2 + \frac{1}{3!}\left(\frac{-i\theta}{2}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{n}\right)^3$$

$$+ \frac{1}{4}\left(\frac{-i\theta}{2}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{n}\right)^4 + \cdots$$

$$= 1 - i\frac{\theta}{2}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{n} + \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{i}{3!}\left(\frac{\theta}{2}\right)^3\underset{\sim}{\sigma}\cdot\underset{\sim}{n} + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + \cdots,$$

$$= \cos\frac{\theta}{2} - i(\underset{\sim}{\sigma}\cdot\underset{\sim}{n})\sin\frac{\theta}{2}$$

since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Note: $(\underset{\sim}{\sigma}\cdot\underset{\sim}{n})^2 = 1$, $(\underset{\sim}{\sigma}\cdot\underset{\sim}{n})^3 = \underset{\sim}{\sigma}\cdot\underset{\sim}{n}$.

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From p.36 of the text, the isospin state of Σ^{*0} is $|1\,0\rangle$

From p.36 of the text, the isospin states of $\Sigma^+, \Sigma^0, \Sigma^-$ are respectively $|1\,1\rangle$, $|1\,0\rangle$, $|1\,-1\rangle$.

Writing $|I_1\, I_2\, m_{I_1}\, m_{I_2}\rangle \equiv |I_1\, m_{I_1}\rangle\, |I_2\, m_{I_2}\rangle$

$$|I_1\, I_2\, I\, m\rangle \equiv |I\, m\rangle$$

and using the table of Clebsch-Gordan coefficients, we have

$$\Sigma^+\pi^- : |1\,1\rangle\,|1\,-1\rangle = \frac{1}{\sqrt{6}}|2\,0\rangle + \frac{1}{\sqrt{2}}|1\,0\rangle + \frac{1}{\sqrt{3}}|0\,0\rangle$$

$$\Sigma^0\pi^0 : |1\,0\rangle\,|1\,0\rangle = \sqrt{\tfrac{2}{3}}|2\,0\rangle - \frac{1}{\sqrt{3}}|0\,0\rangle$$

$$\Sigma^-\pi^+ : |1\,-1\rangle\,|1\,1\rangle = \frac{1}{\sqrt{6}}|2\,0\rangle - \frac{1}{\sqrt{2}}|1\,0\rangle + \frac{1}{\sqrt{3}}|0\,0\rangle$$

(a) $\Sigma^{*0} \rightarrow \Sigma^+\pi^-$:

scattering amplitude $\mathcal{M}_a = \frac{1}{\sqrt{2}}\mathcal{M}_1$

$\mathcal{M}_1 = {}_{out}\langle 1\,0\,|\,1\,0\rangle_{in}$ = scattering amplitude for isospin multiplet with total isospin $I = 1$.

(b) $\Sigma^{*0} \rightarrow \Sigma^0\pi^0$: $\quad \mathcal{M}_b = 0$

(c) $\Sigma^{*} \rightarrow \Sigma^-\pi^+$: $\quad \mathcal{M}_c = \frac{-1}{\sqrt{2}}\mathcal{M}_1$

$\therefore\ \sigma_a : \sigma_b : \sigma_c = \frac{1}{2} : 0 : \frac{1}{2}$ (cross sections ratio)

$\therefore$ 50 disintegrations are $\Sigma^+\pi^-$

50 disintegrations are $\Sigma^-\pi^+$

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(i) πN scattering amplitudes

(b) $\pi^0 + p \to \pi^0 + p$: $\mathcal{M}_b = \frac{2}{3}\mathcal{M}_3 + \frac{1}{3}\mathcal{M}_1$

(d) $\pi^+ + n \to \pi^+ + n$: $\mathcal{M}_d = \frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1 = \mathcal{M}_c$

(e) $\pi^0 + n \to \pi^0 + n$: $\mathcal{M}_e = \frac{2}{3}\mathcal{M}_3 + \frac{1}{3}\mathcal{M}_1 = \mathcal{M}_b$

(g) $\pi^+ + n \to \pi^0 + p$: $\mathcal{M}_g = \frac{\sqrt{2}}{3}\mathcal{M}_3 - \frac{\sqrt{2}}{3}\mathcal{M}_1 = \mathcal{M}_j$

(h) $\pi^0 + p \to \pi^+ + n$: $\mathcal{M}_h = \frac{\sqrt{2}}{3}\mathcal{M}_3 - \frac{\sqrt{2}}{3}\mathcal{M}_1 = \mathcal{M}_j$

(i) $\pi^0 + n \to \pi^- + p$: $\mathcal{M}_i = \frac{\sqrt{2}}{3}\mathcal{M}_3 - \frac{\sqrt{2}}{3}\mathcal{M}_1 = \mathcal{M}_j$

(ii) We need only to consider $\sigma_a, \sigma_b, \sigma_c, \sigma_j$ as many processes have same scattering cross sections.

From the above,

$$\sigma_a : \sigma_b : \sigma_c : \sigma_j = 9|\mathcal{M}_3|^2 : |2\mathcal{M}_3 + \mathcal{M}_1|^2 : |\mathcal{M}_3 + 2\mathcal{M}_1|^2 : 2|\mathcal{M}_3 - \mathcal{M}_1|^2$$

(iii) If $\mathcal{M}_3 \gg \mathcal{M}_1$

$$\sigma_a : \sigma_b : \sigma_c : \sigma_j = 9 : 4 : 1 : 2$$

7 Baryon cannot be converted to antibaryon

4.38 as baryon number is conserved.

p157 Mesons have zero baryon number, so

interconvertion is possible for neutral mesons.

For meson made out of $q\bar{q}$ (quark - antiquark) pair, then antimeson ($\bar{q}q$) is same as meson, e.g. the ϕ meson ($s\bar{s}$), $\phi = \bar{\phi}$.

We consider neutral mesons made out of $q_1 \bar{q}_2$ pairs such that sum of the electric charge of quark q_1 and the electric charge of antiquark $\bar{q}_2$ = 0.

The possible neutral mesons are therefore

$d\bar{s} \leftrightarrow s\bar{d}$; $d\bar{b} \leftrightarrow b\bar{d}$; $s\bar{b} \leftrightarrow b\bar{s}$

$u\bar{c} \leftrightarrow c\bar{u}$ (Note: top quark does not form bound state)

which may interconvert.

Neutron and antineutron have different baryon number (neutron, +1; antineutron, -1), so they cannot be linearly superimposed to form another baryon.

The neutral strange vector mesons K^{0*} and $\overline{K^{0*}}$ decay readily by strong interaction, they have no time to interconvert.