Example: Atwood's machine

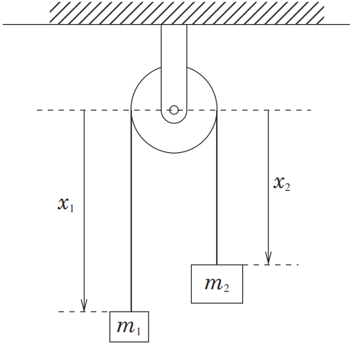
- Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless and frictionless pulley

- Holonomic constraint:

$$f(\mathbf{r}_1, \mathbf{r}_2) = x_1 + x_2 - \ell = 0$$

- Applied forces:

$$\mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g\, \hat{\mathbf{e}}_1 \,, \qquad \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g\, \hat{\mathbf{e}}_2$$



EXERCISE 6.4: Use d'Alembert's principle to find the accelerations of the masses $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$.

$$\mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g\,\hat{\mathbf{e}}_1\,, \qquad \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g\,\hat{\mathbf{e}}_2$$

$$f(\mathbf{r}_1,\mathbf{r}_2) = x_1 + x_2 - \ell = 0 \quad \Rightarrow \quad \delta x_1 = -\delta x_2 \qquad \blacksquare$$

$$f(\mathbf{r}_1,\mathbf{r}_2) = x_1 + x_2 - \ell = 0 \quad \Rightarrow \quad \ddot{x}_1(t) = -\ddot{x}_2(t) \qquad \blacksquare$$

$$\sum_\alpha \left[\mathbf{F}_\alpha^{(\mathsf{A})}(t) - m_\alpha \ddot{\mathbf{r}}_\alpha(t)\right]\cdot\delta\mathbf{r}_\alpha = 0$$

$$\Rightarrow \quad m_1\ddot{\mathbf{r}}_1(t)\cdot\delta\mathbf{r}_1 + m_2\ddot{\mathbf{r}}_2(t)\cdot\delta\mathbf{r}_2 = \mathbf{F}_1^{(\mathsf{A})}(t)\cdot\delta\mathbf{r}_1 + \mathbf{F}_2^{(\mathsf{A})}(t)\cdot\delta\mathbf{r}_2$$

$$\Rightarrow \quad m_1\ddot{x}_1(t)\,\delta x_1 + \left[-m_2\ddot{x}_1(t)\right](-\delta x_1) = m_1 g\,\delta x_1 + m_2 g\,(-\delta x_1)$$

$$\Rightarrow \quad (m_1+m_2)\,\ddot{x}_1(t)\,\delta x_1 = (m_1-m_2)\,g\,\delta x_1$$

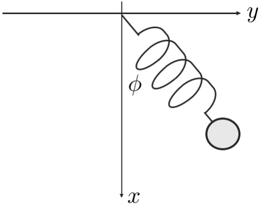
$$\Rightarrow \quad \ddot{x}_1(t) = \frac{m_1-m_2}{m_1+m_2}\,g \qquad \blacksquare$$

Example: Pendulum with spring

- A point particle of mass m attached to a massless spring of original length ℓ_0 and spring constant k rotates about a frictionless pivot in a plane

- Applied forces:

$$\begin{cases} \mathbf{F}_{\text{gravity}}(t) = mg\cos\phi(t)\,\hat{\mathbf{e}}_\rho - mg\sin\phi(t)\,\hat{\mathbf{e}}_\phi \\ \mathbf{F}_{\text{spring}}(t) = -k\,[\rho(t) - \ell_0]\,\hat{\mathbf{e}}_\rho \end{cases}$$



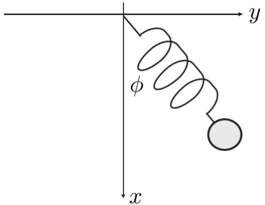
EXERCISE 6.5: Use d'Alembert's principle to obtain equations of motion for $\rho(t)$ and $\phi(t)$.

$$\mathbf{F}_{\text{gravity}}(t) = mg\cos\phi(t)\,\hat{\mathbf{e}}_\rho - mg\sin\phi(t)\,\hat{\mathbf{e}}_\phi\,, \qquad \mathbf{F}_{\text{spring}}(t) = -k\left[\rho(t) - \ell_0\right]\,\hat{\mathbf{e}}_\rho$$

$$\ddot{\mathbf{r}}(t) = \left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right]\hat{\mathbf{e}}_\rho + \left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right]\,\hat{\mathbf{e}}_\phi\,, \qquad \delta\mathbf{r} = \delta\rho\,\hat{\mathbf{e}}_\rho + \rho(t)\,\delta\phi\,\hat{\mathbf{e}}_\phi$$

$$\left[\mathbf{F}^{(\mathsf{A})}(t) - m\ddot{\mathbf{r}}(t)\right]\cdot\delta\mathbf{r} = 0$$

$$\Rightarrow \quad \begin{cases} mg\cos\phi(t) - k\left[\rho(t) - \ell_0\right] - m\left[\ddot{\rho}(t) - \rho(t)\,\dot{\phi}^2(t)\right] = 0 \\ -mg\sin\phi(t) - m\left[\rho(t)\,\ddot{\phi}(t) + 2\dot{\rho}(t)\,\dot{\phi}(t)\right] = 0 \end{cases} \qquad \blacksquare$$



Example: Spherical pendulum

- A particle of mass m is suspended by a massless wire of length $r(t)$ to move on the surface of the spherical of radius $r(t)$

$$r(t) = a + b\cos\omega t\,, \qquad a > b > 0$$

- Holonomic constraint:

$$f(\mathbf{r}, t) = r(t) - a - b\cos\omega t = 0$$

- Applied force:

$$\mathbf{F}_{\text{gravity}}(t) = -mg\cos\theta(t)\,\hat{\mathbf{e}}_r + mg\sin\theta(t)\,\hat{\mathbf{e}}_\theta$$

EXERCISE 6.6: Use d'Alembert's principle to obtain equations of motion for $\theta(t)$ and $\phi(t)$.

$$\mathbf{F}_{\mathsf{gravity}}(t) = -mg\cos\theta(t)\,\hat{\mathbf{e}}_r + mg\sin\theta(t)\,\hat{\mathbf{e}}_\theta$$

$$f(\mathbf{r}) = r(t) - a - b\cos\omega t = 0 \quad\Rightarrow\quad \dot{r}(t) = -b\omega\sin\omega t \quad\Rightarrow\quad \ddot{r}(t) = -b\omega^2\cos\omega t$$

$$\begin{aligned}
\mathbf{a}(t) = {} & \left[\ddot{r}(t) - r(t)\,\dot{\phi}^2(t)\sin^2\theta(t) - r(t)\,\dot{\theta}^2(t)\right]\hat{\mathbf{e}}_r \\
& + \left[r(t)\,\ddot{\theta}(t) + 2\dot{r}(t)\,\dot{\theta}(t) - r(t)\,\dot{\phi}^2(t)\sin\theta(t)\cos\theta(t)\right]\hat{\mathbf{e}}_\theta \\
& + \left[r(t)\,\ddot{\phi}(t)\sin\theta(t) + 2\dot{r}(t)\,\dot{\phi}(t)\sin\theta(t) + 2r(t)\,\dot{\theta}(t)\,\dot{\phi}(t)\cos\theta(t)\right]\hat{\mathbf{e}}_\phi
\end{aligned}$$

$$\delta\mathbf{r} = r(t)\,\delta\theta\,\hat{\mathbf{e}}_\theta + r(t)\sin\theta(t)\,\delta\phi\,\hat{\mathbf{e}}_\phi$$

$$\left[\mathbf{F}^{(\mathsf{A})}(t) - m\ddot{\mathbf{r}}(t)\right] \cdot \delta\mathbf{r} = 0$$

$$\Rightarrow \quad \begin{cases} mgr(t)\sin\theta(t) - mr(t)\left[r(t)\,\ddot{\theta}(t) + 2\dot{r}(t)\,\dot{\theta}(t) - r(t)\,\dot{\phi}^2(t)\sin\theta(t)\cos\theta(t)\right] = 0 \\[2ex] -mr(t)\sin\theta(t)\left[r(t)\,\ddot{\phi}(t)\sin\theta(t) + 2\dot{r}(t)\,\dot{\phi}(t)\sin\theta(t) + 2r(t)\,\dot{\theta}(t)\,\dot{\phi}(t)\cos\theta(t)\right] = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} (a + b\cos\omega t)\,\ddot{\theta}(t) - 2b\omega\dot{\theta}(t)\sin\omega t \\ \qquad\qquad - (a + b\cos\omega t)\,\dot{\phi}^2(t)\sin\theta(t)\cos\theta(t) = g\sin\theta(t) \\[2ex] (a + b\cos\omega t)\,\ddot{\phi}(t)\sin\theta(t) - 2b\omega\dot{\phi}(t)\sin\omega t\sin\theta(t) \\ \qquad\qquad + 2\,(a + b\cos\omega t)\,\dot{\theta}(t)\,\dot{\phi}(t)\cos\theta(t) = 0 \end{cases} \qquad \blacksquare$$

PC3261: Classical Mechanics II

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Lecture 7: Lagrange's Equation

Lagrange multipliers

- Holonomic constraints:

$$f_i(\mathbf{r}_1(t), \cdots, \mathbf{r}_\alpha(t), t) = 0\,, \qquad i = 1, 2, \cdots, C$$

- Constrained force: $\lambda_i(t)$ is known as the **Lagrange multipliers**

$$\mathbf{F}_\alpha^{(\mathsf{C})}(t) \equiv \sum_{i=1}^{C} \lambda_i(t)\,\frac{\partial f_i}{\partial \mathbf{r}_\alpha}$$

- d'Alembert's principle with Lagrange multipliers: all virtual displacements $\delta\mathbf{r}_\alpha$ are now be treated as independent with the introduction of Lagrange multipliers

$$\sum_\alpha \left[\mathbf{F}_\alpha^{(\mathsf{A})}(t) + \sum_{i=1}^{C} \lambda_i(t)\,\frac{\partial f_i}{\partial \mathbf{r}_\alpha} - m_\alpha \ddot{\mathbf{r}}_\alpha(t)\right] \cdot \delta\mathbf{r}_\alpha = 0$$

$$f_i(\mathbf{r}_1(t), \cdots, \mathbf{r}_\alpha(t), t) = 0\,, \qquad i = 1, 2, \cdots, C$$

$$\mathbf{F}^{(\mathsf{C})}_\alpha(t) \equiv \sum_{i=1}^{C} \lambda_i(t)\, \frac{\partial f_i}{\partial \mathbf{r}_\alpha}$$

$$\begin{aligned}
\delta W^{(\mathsf{C})} &= \sum_\alpha \mathbf{F}^{(\mathsf{C})}_\alpha(t) \cdot \delta\mathbf{r}_\alpha \\
&= \sum_\alpha \left[\sum_{i=1}^{C} \lambda_i(t)\, \frac{\partial f_i}{\partial \mathbf{r}_\alpha} \right] \cdot \delta\mathbf{r}_\alpha \\
&= \sum_{i=1}^{C} \lambda_i(t) \left[\sum_\alpha \frac{\partial f_i}{\partial \mathbf{r}_\alpha} \cdot \delta\mathbf{r}_\alpha \right] \\
&= \sum_{i=1}^{C} \lambda_i(t)\, \delta f_i \\
&= 0 \qquad \blacksquare
\end{aligned}$$

Example: Atwood machine (another visit)

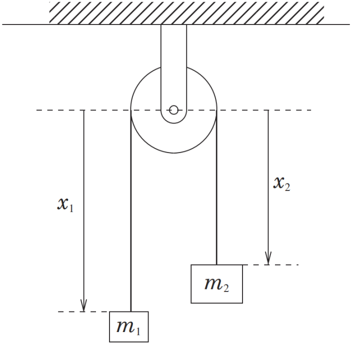
- Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless and frictionless pulley

- Holonomic constraint:

$$f(\mathbf{r}_1, \mathbf{r}_2, t) = x_1(t) + x_2(t) - \ell = 0$$

- Applied forces:

$$\mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g\,\hat{\mathbf{e}}_1\,, \qquad \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g\,\hat{\mathbf{e}}_2$$



EXERCISE 7.1: Use d'Alembert's principle with Lagrange multipliers to find the constrained forces.

$$\begin{cases} \mathbf{r}_1(t) = x_1(t)\,\hat{\mathbf{e}}_1 \\[2ex] \mathbf{r}_2(t) = x_2(t)\,\hat{\mathbf{e}}_2 \end{cases} \quad\Rightarrow\quad \begin{cases} \delta\mathbf{r}_1 = \delta x_1\,\hat{\mathbf{e}}_1 \\[2ex] \delta\mathbf{r}_2 = \delta x_2\,\hat{\mathbf{e}}_2 \end{cases} \quad\Rightarrow\quad \begin{cases} \ddot{\mathbf{r}}_1(t) = \ddot{x}_1(t)\,\hat{\mathbf{e}}_1 \\[2ex] \ddot{\mathbf{r}}_2(t) = \ddot{x}_2(t)\,\hat{\mathbf{e}}_2 \end{cases}$$

$$f(\mathbf{r}_1, \mathbf{r}_2, t) = x_1(t) + x_2(t) - \ell = 0 \quad\Rightarrow\quad \begin{cases} \dfrac{\partial f}{\partial \mathbf{r}_1} = \hat{\mathbf{e}}_1 \\[2ex] \dfrac{\partial f}{\partial \mathbf{r}_2} = \hat{\mathbf{e}}_2 \end{cases}$$

$$\mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g\,\hat{\mathbf{e}}_1\,, \qquad \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g\,\hat{\mathbf{e}}_2$$

$$\sum_\alpha \left[\mathbf{F}_\alpha^{(\mathsf{A})}(t) + \lambda(t)\,\frac{\partial f}{\partial \mathbf{r}_\alpha} - m_\alpha \ddot{\mathbf{r}}_\alpha(t)\right] \cdot \delta\mathbf{r}_\alpha = 0$$

$$\Rightarrow \quad [m_1 g + \lambda(t) - m_1 \ddot{x}_1(t)]\ \delta x_1 + [m_2 g + \lambda(t) - m_2 \ddot{x}_2(t)]\ \delta x_2 = 0$$

$$\Rightarrow \quad \begin{cases} m_1 g + \lambda(t) - m_1 \ddot{x}_1(t) = 0 \\ m_2 g + \lambda(t) - m_2 \ddot{x}_2(t) = 0 \\ x_1(t) + x_2(t) - \ell = 0 \end{cases}$$

$$x_1(t) + x_2(t) - \ell = 0 \quad \Rightarrow \quad \ddot{x}_1(t) + \ddot{x}_2(t) = 0$$

$$\begin{cases} m_1 g + \lambda(t) - m_1 \ddot{x}_1(t) = 0 \\ m_2 g + \lambda(t) - m_2 \ddot{x}_2(t) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1(t) = \dfrac{m_1 - m_2}{m_1 + m_2}\, g \\ \ddot{x}_2(t) = -\dfrac{m_1 - m_2}{m_1 + m_2}\, g \end{cases}$$

$$m_1 g + \lambda(t) - m_1 \ddot{x}_1(t) = 0 \quad \Rightarrow \quad \lambda(t) = -\frac{2 m_1 m_2}{m_1 + m_2}\, g$$

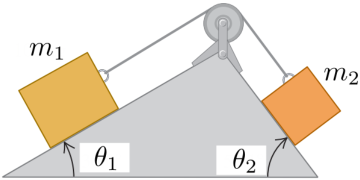
$$\begin{cases} \mathbf{F}_1^{(\mathsf{C})}(t) = \lambda(t)\, \dfrac{\partial f}{\partial \mathbf{r}_1} = -\dfrac{2 m_1 m_2}{m_1 + m_2}\, g\, \hat{\mathbf{e}}_1 \\ \mathbf{F}_2^{(\mathsf{C})}(t) = \lambda(t)\, \dfrac{\partial f}{\partial \mathbf{r}_2} = -\dfrac{2 m_1 m_2}{m_1 + m_2}\, g\, \hat{\mathbf{e}}_2 \end{cases} \qquad \blacksquare$$

Example: Double-inclined plane (another visit)

- Two masses m_1 and m_2 are located each on a smooth double inclined plane with angles θ_1 and θ_2 respectively. The masses are connected by a massless and inextensible string running over a massless and frictionless pulley

- Holonomic constraints:

$$\begin{cases} f_1(\mathbf{r}_1, \mathbf{r}_2, t) = x_1(t) + x_2(t) - \ell = 0 \\ f_2(\mathbf{r}_1, \mathbf{r}_2, t) = y_1(t) = 0 \\ f_3(\mathbf{r}_1, \mathbf{r}_2, t) = y_2(t) = 0 \end{cases}$$



- Applied forces:

$$\mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g \sin\theta_1\, \hat{\mathbf{e}}_{x_1} - m_1 g \cos\theta_1\, \hat{\mathbf{e}}_{y_1}\ ,\ \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g \sin\theta_2\, \hat{\mathbf{e}}_{x_2} - m_2 g \cos\theta_2\, \hat{\mathbf{e}}_{y_2}$$

EXERCISE 7.2: Use d'Alembert's principle with Lagrange multipliers to find the constrained forces.

$$\begin{cases} \mathbf{r}_1(t) = x_1(t)\,\hat{\mathbf{e}}_{x_1} + y_1(t)\,\hat{\mathbf{e}}_{y_1} \\ \mathbf{r}_2(t) = x_2(t)\,\hat{\mathbf{e}}_{x_2} + y_2(t)\,\hat{\mathbf{e}}_{y_2} \end{cases} \Rightarrow \begin{cases} \delta\mathbf{r}_1 = \delta x_1\,\hat{\mathbf{e}}_{x_1} + \delta y_1\,\hat{\mathbf{e}}_{y_1} \\ \delta\mathbf{r}_2 = \delta x_2\,\hat{\mathbf{e}}_{x_2} + \delta y_2\,\hat{\mathbf{e}}_{y_2} \end{cases} \Rightarrow \begin{cases} \ddot{\mathbf{r}}_1(t) = \mathbf{0} \\ \ddot{\mathbf{r}}_2(t) = \mathbf{0} \end{cases}$$

$$\begin{cases} \mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g \sin\theta_1\,\hat{\mathbf{e}}_{x_1} - m_1 g \cos\theta_1\,\hat{\mathbf{e}}_{y_1} \\ \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g \sin\theta_2\,\hat{\mathbf{e}}_{x_2} - m_2 g \cos\theta_2\,\hat{\mathbf{e}}_{y_2} \end{cases}$$

$$\begin{cases} f_1(\mathbf{r}_1, \mathbf{r}_2, t) = x_1(t) + x_2(t) - \ell = 0 \\ f_2(\mathbf{r}_1, \mathbf{r}_2, t) = y_1(t) = 0 \\ f_3(\mathbf{r}_1, \mathbf{r}_2, t) = y_2(t) = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \mathbf{F}_1^{(\mathsf{C})}(t) = \lambda_1(t)\,\dfrac{\partial f_1}{\partial \mathbf{r}_1} + \lambda_2(t)\,\dfrac{\partial f_2}{\partial \mathbf{r}_1} + \lambda_3(t)\,\dfrac{\partial f_3}{\partial \mathbf{r}_1} = \lambda_1(t)\,\hat{\mathbf{e}}_{x_1} + \lambda_2(t)\,\hat{\mathbf{e}}_{y_1} \\ \mathbf{F}_2^{(\mathsf{C})}(t) = \lambda_1(t)\,\dfrac{\partial f_1}{\partial \mathbf{r}_2} + \lambda_2(t)\,\dfrac{\partial f_2}{\partial \mathbf{r}_2} + \lambda_3(t)\,\dfrac{\partial f_3}{\partial \mathbf{r}_2} = \lambda_1(t)\,\hat{\mathbf{e}}_{x_2} + \lambda_3(t)\,\hat{\mathbf{e}}_{y_2} \end{cases}$$

$$\sum_\alpha \left[\mathbf{F}_\alpha^{(\mathsf{A})}(t) + \sum_i \lambda_i(t) \, \frac{\partial f_i}{\partial \mathbf{r}_\alpha} - m_\alpha \ddot{\mathbf{r}}_\alpha(t) \right] \cdot \delta\mathbf{r}_\alpha = 0$$

$$\Rightarrow \quad [m_1 g \sin\theta_1 + \lambda_1(t)] \; \delta x_1 + [-m_1 g \cos\theta_1 + \lambda_2(t)] \; \delta y_1$$

$$+ [m_2 g \sin\theta_2 + \lambda_1(t)] \; \delta x_2 + [-m_2 g \cos\theta_2 + \lambda_3(t)] \; \delta y_2 = 0$$

$$\Rightarrow \quad \begin{cases} m_1 g \sin\theta_1 + \lambda_1(t) = 0 \\ -m_1 g \cos\theta_1 + \lambda_2(t) = 0 \\ m_2 g \sin\theta_2 + \lambda_1(t) = 0 \\ -m_2 g \cos\theta_2 + \lambda_3(t) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} m_1 \sin\theta_1 = m_2 \sin\theta_2 \\ \lambda_1(t) = -m_1 g \sin\theta_1 = -m_2 g \sin\theta_2 \\ \lambda_2(t) = m_1 g \cos\theta_1 \\ \lambda_3(t) = m_2 g \cos\theta_2 \end{cases} \quad \blacksquare$$

$$\Rightarrow \quad \begin{cases} \mathbf{F}_1^{(\mathsf{C})}(t) = \lambda_1(t) \, \hat{\mathbf{e}}_{x_1} + \lambda_2(t) \, \hat{\mathbf{e}}_{y_1} = -m_1 g \sin\theta_1 \, \hat{\mathbf{e}}_{x_1} + m_1 g \cos\theta_1 \, \hat{\mathbf{e}}_{y_1} \\ \mathbf{F}_2^{(\mathsf{C})}(t) = \lambda_1(t) \, \hat{\mathbf{e}}_{x_2} + \lambda_3(t) \, \hat{\mathbf{e}}_{y_2} = -m_2 g \sin\theta_2 \, \hat{\mathbf{e}}_{x_2} + m_2 g \cos\theta_2 \, \hat{\mathbf{e}}_{y_2} \end{cases} \quad \blacksquare$$