Euler-Lagrange equation

- Rewriting Lagrange's equation for conservative systems:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_k}(T-U)\right]-\frac{\partial}{\partial q_k}(T-U)=0$$

- **Lagrange function** (or **Lagrangian**) for conservative systems:

$$\mathcal{L}\equiv\mathcal{L}\left(\{q_i(t),\dot{q}_i(t)\},t\right)\equiv T\left(\{q_i(t),\dot{q}_i(t)\},t\right)-U\left(\{q_i(t)\}\right)$$

- **Euler-Lagrange equation**: M second-order coupled ODEs, $2M$ initial conditions $\{q_i(0),\dot{q}_i(0)\}$ are required to determine *uniquely* $\{q_i(t)\}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right)-\frac{\partial\mathcal{L}}{\partial q_k}=0\,,\qquad k=1,2,\cdots,M$$

- T and U must both be expressed relative to some inertial reference frame

Single particle in three dimensions

- Lagrange function: Cartesian coordinates

$$T \equiv T(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) = \frac{1}{2}\, m \left[\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)\right] , \qquad U \equiv U(\mathbf{r}(t)) = U(x, y, z)$$

$$\mathcal{L} \equiv \mathcal{L}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \equiv T - U = \frac{1}{2}\, m \left[\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)\right] - U(x, y, z)$$

- Euler-Lagrange equation of motion: three second-order ODEs

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{x}}\right) - \dfrac{\partial L}{\partial x} = 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{y}}\right) - \dfrac{\partial L}{\partial y} = 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{z}}\right) - \dfrac{\partial L}{\partial z} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} m\ddot{x}(t) + \dfrac{\partial U}{\partial x} = 0 \\ m\ddot{y}(t) + \dfrac{\partial U}{\partial y} = 0 \\ m\ddot{z}(t) + \dfrac{\partial U}{\partial z} = 0 \end{cases} \quad \Rightarrow \quad m\ddot{\mathbf{r}}(t) = -\boldsymbol{\nabla} U$$

$$\mathcal{L} \equiv \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U = \frac{1}{2}\, m \left[\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)\right] - U(x, y, z)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0\,, \qquad i = 1, 2, 3$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}(t) \\[2ex] \dfrac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}(t) \\[2ex] \dfrac{\partial \mathcal{L}}{\partial \dot{z}(t)} = m\dot{z} \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x}(t) \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{y}}\right) = m\ddot{y}(t) \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{z}}\right) = m\ddot{z}(t) \end{cases}, \qquad \begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = -\dfrac{\partial U}{\partial x} \\[2ex] \dfrac{\partial \mathcal{L}}{\partial y} = -\dfrac{\partial U}{\partial y} \\[2ex] \dfrac{\partial \mathcal{L}}{\partial z} = -\dfrac{\partial U}{\partial z} \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad \Rightarrow \quad \begin{cases} m\ddot{x}(t) = -\dfrac{\partial U}{\partial x} \\[2ex] m\ddot{y}(t) = -\dfrac{\partial U}{\partial y} \\[2ex] m\ddot{z}(t) = -\dfrac{\partial U}{\partial z} \end{cases} \quad \Rightarrow \quad m\ddot{\mathbf{r}}(t) = -\boldsymbol{\nabla} U \qquad \blacksquare$$

Example: Plane double pendulum

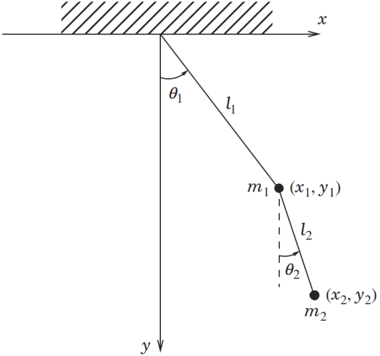
- A plane double pendulum consists of two light and inextensible rods of lengths ℓ_1 and ℓ_2 respectively. Two point masses, m_1 and m_2, are respectively attached at the end of each rod

- Holonomic constraints:

$$\begin{cases} f_1 = x_1^2 + y_1^2 - \ell_1^2 = 0 \\ f_2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - \ell_2^2 = 0 \end{cases}$$

- Generalized coordinates:

$$(q_1, q_2) \equiv (\theta_1, \theta_2)$$



EXERCISE 8.2: Obtain the equations of motion for the plane double pendulum from the Euler-Lagrange equation.

$$\begin{cases} f_1 = x_1^2 + y_1^2 - \ell_1^2 = 0 \\ \\ f_2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - \ell_2^2 = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \mathbf{r}_1(t) = \ell_1 \sin\theta_1(t)\,\hat{\mathbf{e}}_x + \ell_1 \cos\theta_1(t)\,\hat{\mathbf{e}}_y \\ \\ \mathbf{r}_2(t) = \left[\ell_1 \sin\theta_1(t) + \ell_2 \sin\theta_2(t)\right]\,\hat{\mathbf{e}}_x + \left[\ell_1 \cos\theta_1(t) + \ell_2 \cos\theta_2(t)\right]\,\hat{\mathbf{e}}_y \end{cases}$$

$$\begin{aligned} T \equiv T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t) &= \frac{m_1}{2}\,\dot{\mathbf{r}}_1(t)\cdot\dot{\mathbf{r}}_1(t) + \frac{m_2}{2}\,\dot{\mathbf{r}}_2(t)\cdot\dot{\mathbf{r}}_2(t) \\ &= \frac{m_1 + m_2}{2}\,\ell_1^2\,\dot{\theta}_1^2(t) + \frac{m_2}{2}\,\ell_2^2\,\dot{\theta}_2^2(t) + m_2\ell_1\ell_2\,\dot{\theta}_1(t)\,\dot{\theta}_2(t)\cos\left[\theta_1(t) - \theta_2(t)\right] \end{aligned}$$

$$\begin{aligned} U \equiv U(\theta_1, \theta_2) &= -m_1 g y_1(t) - m_2 g y_2(t) \\ &= -(m_1 + m_2) g \ell_1 \cos\theta_1(t) - m_2 g \ell_2 \cos\theta_2(t) \end{aligned}$$

$$
\begin{aligned}
\mathcal{L} \equiv \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t) &= T - U \\
&= \frac{m_1 + m_2}{2}\,\ell_1^2\,\dot{\theta}_1^2(t) + \frac{m_2}{2}\,\ell_2^2\,\dot{\theta}_2^2(t) + m_2\ell_1\ell_2\,\dot{\theta}_1(t)\,\dot{\theta}_2(t)\cos\left[\theta_1(t) - \theta_2(t)\right] \\
&\qquad + (m_1 + m_2)g\ell_1\cos\theta_1(t) + m_2 g\ell_2\cos\theta_2(t)
\end{aligned}
$$

$$
\begin{cases}
\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2\,\dot{\theta}_1(t) + m_2\ell_1\ell_2\,\dot{\theta}_2(t)\cos\left[\theta_1(t) - \theta_2(t)\right] \\[2ex]
\dfrac{\partial \mathcal{L}}{\partial \theta_1} = -m_2\ell_1\ell_2\,\dot{\theta}_1(t)\,\dot{\theta}_2(t)\sin\left[\theta_1(t) - \theta_2(t)\right] - (m_1 + m_2)g\ell_1\sin\theta_1(t)
\end{cases}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0
$$

$$
\begin{aligned}
\Rightarrow \quad & (m_1 + m_2)\ell_1^2\,\ddot{\theta}_1(t) + m_2\ell_1\ell_2\,\ddot{\theta}_2(t)\cos\left[\theta_1(t) - \theta_2(t)\right] \\
&\qquad + m_2\ell_1\ell_2\,\dot{\theta}_2^2(t)\sin\left[\theta_1(t) - \theta_2(t)\right] + (m_1 + m_2)g\ell_1\sin\theta_1(t) = 0 \qquad \blacksquare
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L} \equiv \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t) &= T - U \\
&= \frac{m_1 + m_2}{2}\, \ell_1^2\, \dot{\theta}_1^2(t) + \frac{m_2}{2}\, \ell_2^2\, \dot{\theta}_2^2(t) + m_2 \ell_1 \ell_2\, \dot{\theta}_1(t)\, \dot{\theta}_2(t) \cos\left[\theta_1(t) - \theta_2(t)\right] \\
&\qquad + (m_1 + m_2) g \ell_1 \cos\theta_1(t) + m_2 g \ell_2 \cos\theta_2(t)
\end{aligned}
$$

$$
\begin{cases}
\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 \ell_2^2\, \dot{\theta}_2(t) + m_2 \ell_1 \ell_2\, \dot{\theta}_1(t) \cos\left[\theta_1(t) - \theta_2(t)\right] \\[2ex]
\dfrac{\partial \mathcal{L}}{\partial \theta_2} = m_2 \ell_1 \ell_2 \dot{\theta}_1(t)\, \dot{\theta}_2(t) \sin\left[\theta_1(t) - \theta_2(t)\right] - m_2 g \ell_2 \sin\theta_2(t)
\end{cases}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0
$$

$$
\begin{aligned}
\Rightarrow \quad & m_2 \ell_2^2\, \ddot{\theta}_2(t) + m_2 \ell_1 \ell_2\, \ddot{\theta}_1(t) \cos\left[\theta_1(t) - \theta_2(t)\right] \\
&\qquad - m_2 \ell_1 \ell_2\, \dot{\theta}_1^2(t) \sin\left[\theta_1(t) - \theta_2(t)\right] + m_2 g \ell_2 \sin\theta_2(t) = 0 \qquad \blacksquare
\end{aligned}
$$

Generalized coordinates and velocities

- Configuration of a system can be geometrically represented by a *single* point in an M-dimensional space known as **configuration manifold** $\mathbb{Q}$

$$(q_1, q_2, q_3, \cdots, q_M)$$

- Euler-Lagrange equation is set of M second-order ODEs on $\mathbb{Q}$: Hessian matrix $\partial^2\mathcal{L}/\partial\dot{q}_i\,\partial\dot{q}_k$ must be non-singular

$$\sum_{i=1}^{M}\left(\frac{\partial^2\mathcal{L}}{\partial\dot{q}_i\,\partial\dot{q}_k}\,\ddot{q}_i + \frac{\partial^2\mathcal{L}}{\partial q_i\,\partial\dot{q}_k}\,\dot{q}_i\right) + \frac{\partial^2\mathcal{L}}{\partial t\,\partial\dot{q}_k} - \frac{\partial\mathcal{L}}{\partial q_k} = 0\,, \qquad k = 1, 2, \cdots, M$$

- Solution of the Euler-Lagrange equation is represented by a curve parameterized by t on $\mathbb{Q}$

$$(q_1(t), q_2(t), q_3(t), \cdots, q_M(t))$$

Generalized coordinates and velocities– cont'd

- Lagrangian is a function of both generalized coordinates and generalized velocities, $(\{q_i, \dot{q}_i\})$, living in a $2M$-dimensional space known as **tangent bundle**, $\mathbf{T}\mathbb{Q}$, of $\mathbb{Q}$; $\mathbf{T}\mathbb{Q}$ is obtained from $\mathbb{Q}$ by adjoining to each point $q \in \mathbb{Q}$ the *tangent space* $\mathbf{T}_q\mathbb{Q}$, of all possible generalized velocities at q

- Euler-Lagrange equation is a set of $2M$ first order ODEs on $\mathbf{T}\mathbb{Q}$:

$$\begin{cases} \dfrac{\mathrm{d}q_k}{\mathrm{d}t} = \dot{q}_k \\[2ex] \dfrac{\mathrm{d}\dot{q}_k}{\mathrm{d}t} = G_k(\{q_i, \dot{q}_i\}, t) \end{cases}, \qquad i, k = 1, 2, \cdots, M$$

- Solution of the Euler-Lagrange equation is represented by a curve parameterized by t on $\mathbf{T}\mathbb{Q}$:

$$(q_1(t), q_2(t), q_3(t), \cdots, q_M(t), \dot{q}_1(t), \dot{q}_2(t), \dot{q}_3(t), \cdots, \dot{q}_M(t))$$

Point transformation

- **Point transformation**: coordinate transformation between two different sets of generalized coordinates

$$q_j = q_j(\{\overline{q}_i\}, t) \quad \leftrightarrow \quad \overline{q}_i = \overline{q}_i(\{q_j\}, t), \qquad i, j = 1, 2, \cdots, M$$

- Jacobian determinant: $M \times M$ matrix

$$\frac{\partial(q_1, q_2, \cdots, q_M)}{\partial(\overline{q}_1, \overline{q}_2, \cdots, \overline{q}_M)} \equiv \begin{vmatrix} \frac{\partial q_1}{\partial \overline{q}_1} & \frac{\partial q_1}{\partial \overline{q}_2} & \cdots & \frac{\partial q_1}{\partial \overline{q}_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q_M}{\partial \overline{q}_1} & \frac{\partial q_M}{\partial \overline{q}_2} & \cdots & \frac{\partial q_M}{\partial \overline{q}_M} \end{vmatrix} \neq 0$$

- Point transformation is assumed to be invertible

Point transformation – cont'd

- Generalized velocities under point transformation:

$$\overline{q}_i = \overline{q}_i(\{q_j\}, t) \quad \Rightarrow \quad \dot{\overline{q}}_i = \sum_{j=1}^{M} \frac{\partial \overline{q}_i}{\partial q_j}\,\dot{q}_j + \frac{\partial \overline{q}_i}{\partial t} \quad \Rightarrow \quad \frac{\partial \dot{\overline{q}}_i}{\partial \dot{q}_j} = \frac{\partial \overline{q}_i}{\partial q_j}$$

- Covariance of Euler-Lagrange equation of motion under point transformation:

$$\left.\begin{array}{c} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial \mathcal{L}(\{q_k(t), \dot{q}_k(t)\}, t)}{\partial \dot{q}_i}\right] - \dfrac{\partial \mathcal{L}(\{q_k(t), \dot{q}_k(t)\}, t)}{\partial q_i} = 0 \\ \Updownarrow \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial \overline{\mathcal{L}}(\{\overline{q}_k(t), \dot{\overline{q}}_k(t)\}, t)}{\partial \dot{\overline{q}}_i}\right] - \dfrac{\partial \overline{\mathcal{L}}(\{\overline{q}_k(t), \dot{\overline{q}}_k(t)\}, t)}{\partial \overline{q}_i} = 0 \end{array}\right. , \qquad i, k = 1, 2, \cdots, M$$

EXERCISE 8.3: Show that the Euler-Lagrange equation of motion is covariant under point transformation.

$$q_j = q_j(\{\overline{q}_i\}, t)$$

$$\Rightarrow \quad \mathcal{L}(\{q_j, \dot{q}_j\}, t) = \mathcal{L}\left(\left\{q_j(\{\overline{q}_i\}, t), \dot{q}_j(\left\{\overline{q}_i, \dot{\overline{q}}_i\right\}, t)\right\}, t\right) = \overline{\mathcal{L}}(\left\{\overline{q}_i, \dot{\overline{q}}_i\right\}, t)$$

$$\frac{\partial \overline{\mathcal{L}}}{\partial \overline{q}_k} = \sum_{j=1}^{M} \left[\frac{\partial \mathcal{L}}{\partial q_j} \frac{\partial q_j}{\partial \overline{q}_k} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \overline{q}_k}\right]$$

$$\frac{\partial \overline{\mathcal{L}}}{\partial \dot{\overline{q}}_k} = \sum_{j=1}^{M} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{\overline{q}}_k} = \sum_{j=1}^{M} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \overline{q}_k}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \overline{\mathcal{L}}}{\partial \dot{\overline{q}}_k}\right) = \sum_{j=1}^{M} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) \frac{\partial q_j}{\partial \overline{q}_k} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial q_j}{\partial \overline{q}_k}\right)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\overline{\mathcal{L}}}{\partial\dot{\overline{q}}_k}\right)-\frac{\partial\overline{\mathcal{L}}}{\partial\overline{q}_k}=\sum_{j=1}^{M}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right)\frac{\partial q_j}{\partial\overline{q}_k}+\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial q_j}{\partial\overline{q}_k}\right)-\frac{\partial\mathcal{L}}{\partial q_j}\frac{\partial q_j}{\partial\overline{q}_k}-\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\frac{\partial\dot{q}_j}{\partial\overline{q}_k}\right]$$

$$=\sum_{j=1}^{M}\left\{\frac{\partial q_j}{\partial\overline{q}_k}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right)-\frac{\partial\mathcal{L}}{\partial q_j}\right]+\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial q_j}{\partial\overline{q}_k}\right)-\frac{\partial\dot{q}_j}{\partial\overline{q}_k}\right]\right\}$$

$$=0 \qquad \blacksquare$$

$$q_j=q_j(\{\overline{q}_i\},t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial q_j}{\partial\overline{q}_k}\right)=\sum_{i=1}^{M}\frac{\partial}{\partial\overline{q}_i}\left(\frac{\partial q_j}{\partial\overline{q}_k}\right)\dot{\overline{q}}_i+\frac{\partial}{\partial t}\left(\frac{\partial q_j}{\partial\overline{q}_k}\right)=\sum_{i=1}^{M}\frac{\partial}{\partial\overline{q}_k}\left(\frac{\partial q_j}{\partial\overline{q}_i}\right)\dot{\overline{q}}_i+\frac{\partial}{\partial\overline{q}_k}\left(\frac{\partial q_j}{\partial t}\right)$$

$$=\sum_{i=1}^{M}\frac{\partial}{\partial\overline{q}_k}\left(\frac{\partial q_j}{\partial\overline{q}_i}\dot{\overline{q}}_i\right)+\frac{\partial}{\partial\overline{q}_k}\left(\frac{\partial q_j}{\partial t}\right)=\frac{\partial}{\partial\overline{q}_k}\left(\sum_{i=1}^{M}\frac{\partial q_j}{\partial\overline{q}_i}\dot{\overline{q}}_i+\frac{\partial q_j}{\partial t}\right)$$

$$=\frac{\partial}{\partial\overline{q}_k}\left(\frac{\mathrm{d}q_j}{\mathrm{d}t}\right)=\frac{\partial\dot{q}_j}{\partial\overline{q}_k} \qquad \blacksquare$$