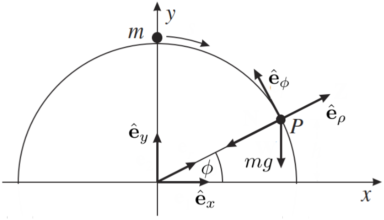
Example: Particle on a hemisphere

- A particle of mass m is located at the "North pole" of a smooth hemisphere of radius R fixed on the ground. The particle slides down the hemisphere after a small kick.

- Particle is constrained to move on the hemisphere before breaking off:

$$\rho(t) = R \quad \Rightarrow \quad \begin{cases} \dot{\rho}(t) = 0 \\ \ddot{\rho}(t) = 0 \end{cases}$$



EXERCISE 2.3: Find the angle and the speed at which the particle breaks off from the hemisphere.

$$\mathbf{F}(t) = \left[N(t) - mg\sin\phi(t)\right]\,\hat{\mathbf{e}}_\rho(t) - mg\cos\phi(t)\,\hat{\mathbf{e}}_\phi(t)$$

$$\begin{cases} N(t) - mg\sin\phi(t) = -mR\,\dot{\phi}^2(t) \\ -mg\cos\phi(t) = mR\,\ddot{\phi}(t) \end{cases}$$

$$-\,mg\sin\phi(t_0) = -mR\,\dot{\phi}^2(t_0) \quad\Rightarrow\quad \dot{\phi}^2(t_0) = \frac{g}{R}\sin\phi(t_0) \qquad \blacksquare$$

$$\ddot{\phi}(t) = -\frac{g}{R}\cos\phi(t) \quad\Rightarrow\quad \frac{\mathrm{d}\dot{\phi}(\phi)}{\mathrm{d}\phi}\,\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = -\frac{g}{R}\cos\phi(t)$$

$$\Rightarrow\quad \int_{\dot{\phi}'=0}^{\dot{\phi}(t_0)} \dot{\phi}'\,\mathrm{d}\dot{\phi}' = -\frac{g}{R}\int_{\phi'=\pi/2}^{\phi(t_0)} \cos\phi'\,\mathrm{d}\phi'$$

$$\Rightarrow\quad \frac{1}{2}\,\dot{\phi}^2(t_0) = -\frac{g}{R}\,\left[\sin\phi(t_0) - 1\right] \qquad \blacksquare$$

$$\begin{cases} \dot{\phi}^2(t_0) = \dfrac{g}{R}\sin\phi(t_0) \\ \\ \dfrac{1}{2}\,\dot{\phi}^2(t_0) = -\dfrac{g}{R}\,\left[\sin\phi(t_0) - 1\right] \end{cases}$$

$$\Rightarrow \quad \begin{cases} \sin\phi(t_0) = \dfrac{2}{3} \\ \\ \dot{\phi}^2(t_0) = \dfrac{2g}{3R} \end{cases} \qquad \blacksquare$$

$$\phi(t_0) = \sin^{-1}\frac{2}{3} \approx 42^\circ \qquad \blacksquare$$

$$\mathbf{v}(t_0) = \dot{\rho}(t_0)\,\hat{\mathbf{e}}_\rho + \rho(t_0)\,\dot{\phi}(t_0)\,\hat{\mathbf{e}}_\phi = -\sqrt{\frac{2Rg}{3}}\,\hat{\mathbf{e}}_\phi \quad \Rightarrow \quad v(t_0) = \sqrt{\frac{2Rg}{3}} \qquad \blacksquare$$

Projectile with resistance

- Linear resistance: $\mathbf{F} = -mk\mathbf{v}$, $k \geq 0$

- Equation of motion:

$$\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = -g\,\hat{\mathbf{e}}_z - k\mathbf{v}(t)$$

- Initial conditions:

$$\mathbf{r}(0) = (x_0, y_0, z_0)\,, \qquad \mathbf{v}(0) = (0, v_0\cos\theta_0, v_0\sin\theta_0)$$

- Equation of motion in Cartesian coordinates:

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -kv_x(t)\,, \qquad \frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} = -kv_y(t)\,, \qquad \frac{\mathrm{d}^2z(t)}{\mathrm{d}t^2} = -g - kv_z(t)$$

Projectile with resistance: x-direction

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -k v_x(t)\,, \qquad x(0) = x_0\,, \qquad v_x(0) = 0$$

- Solving for $v_x(t)$:

$$\frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = -k v_x(t) \qquad \Rightarrow \qquad v_x(t) = 0$$

- Solving for $x(t)$:

$$v_x(t) = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}x(t)}{\mathrm{d}t} = 0 \qquad \Rightarrow \qquad x(t) = x_0$$

- Motion along the x-direction is essentially stationary

Projectile with resistance: y-direction

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = -k v_y(t)\,, \qquad y(0) = y_0\,, \qquad v_y(0) = v_0 \cos\theta_0$$

- Solving:

$$v_y(t) = v_0 \cos\theta_0\, \mathrm{e}^{-kt}\,, \qquad y(t) = y_0 + \frac{v_0 \cos\theta_0}{k}\left(1 - \mathrm{e}^{-kt}\right)$$

- Zero-friction limit: $k \to 0$

$$v_y(t) \to v_0 \cos\theta_0\,, \qquad y(t) \to y_0 + v_0\,(\cos\theta_0)\,t$$

EXERCISE 2.4: Obtain short-time and long-time behaviours for $v_y(t)$ and $y(t)$.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = -k v_y(t)\,, \qquad y(0) = y_0\,, \qquad v_y(0) = v_0 \cos\theta_0$$

$$\frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = -k v_y(t) \quad \Rightarrow \quad \int_{v_y' = v_0\cos\theta_0}^{v_y} \frac{\mathrm{d}v_y'}{v_y'} = -k \int_{t'=0}^{t} \mathrm{d}t$$

$$\Rightarrow \quad v_y(t) = v_0 \cos\theta_0\, \mathrm{e}^{-kt} \qquad \blacksquare$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = v_0 \cos\theta_0\, \mathrm{e}^{-kt} \quad \Rightarrow \quad \int_{y'=y_0}^{y} \mathrm{d}y' = \int_{t'=0}^{t} v_0 \cos\theta_0\, \mathrm{e}^{-kt'}\, \mathrm{d}t'$$

$$\Rightarrow \quad y(t) = y_0 + \frac{v_0 \cos\theta_0}{k} \left(1 - \mathrm{e}^{-kt}\right) \qquad \blacksquare$$

$$v_y(t) = v_0 \cos\theta_0 \, \mathrm{e}^{-kt} \, , \qquad y(t) = y_0 + \frac{v_0 \cos\theta_0}{k} \left(1 - \mathrm{e}^{-kt}\right)$$

$$t \ll \frac{1}{k} \qquad \rightarrow \qquad \begin{cases} v_y(t) \rightarrow v_0 \cos\theta_0 \, (1 - kt) \\ y(t) \rightarrow y_0 + v_0 \, (\cos\theta_0) \, t - \dfrac{1}{2} k v_0 \, (\cos\theta_0) \, t^2 \end{cases} \qquad \blacksquare$$

$$t \gg \frac{1}{k} \qquad \rightarrow \qquad \begin{cases} v_y(t) \rightarrow 0 \\ y(t) \rightarrow y_0 + \dfrac{v_0 \cos\theta_0}{k} \end{cases} \qquad \blacksquare$$

$$\frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = -k v_y(t) \quad \Rightarrow \quad \frac{\mathrm{d}v_y(y)}{\mathrm{d}y} \, \frac{\mathrm{d}y(t)}{\mathrm{d}t} = -k v_y(t)$$

$$\Rightarrow \quad \int_{v_y' = v_0 \cos\theta_0}^{v_y} \mathrm{d}v_y' = - \int_{y' = y_0}^{y} k \, \mathrm{d}y' \quad \Rightarrow \quad v_y(y) = v_0 \cos\theta_0 - k \, (y - y_0) \qquad \blacksquare$$

Projectile with resistance: z-direction

$$\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = -g - k v_z(t)\,, \qquad z(0) = z_0\,, \qquad v_z(0) = v_0 \sin\theta_0$$

- Solving:

$$v_z(t) = \left(v_0 \sin\theta_0 + \frac{g}{k}\right) \mathrm{e}^{-kt} - \frac{g}{k}\,, \qquad z(t) = z_0 + \frac{1}{k}\left(v_0 \sin\theta_0 + \frac{g}{k}\right)\left(1 - \mathrm{e}^{-kt}\right) - \frac{gt}{k}$$

- Short-time behaviour:

$$v_z(t) \to v_0 \sin\theta_0 - (g + k v_0 \sin\theta_0)\, t\,, \quad z(t) \to z_0 + v_0 (\sin\theta_0)\, t - \frac{1}{2}(g + k v_0 \sin\theta_0)\, t^2$$

- Long-time behaviour:

$$v_z(t) \to -\frac{g}{k}\,, \qquad z(t) \to z_0 + \frac{1}{k}\left(v_0 \sin\theta_0 + \frac{g}{k}\right) - \frac{gt}{k}$$

$$\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = -g - k v_z(t)\,, \qquad z(0) = z_0\,, \qquad v_z(0) = v_0 \sin\theta_0$$

$$\frac{\mathrm{d}v_z(t)}{\mathrm{d}t} = -g - k v_z(t) \quad \Rightarrow \quad \int_{v_z' = v_0 \sin\theta_0}^{v_z} \frac{\mathrm{d}v_z'}{g + k v_z'} = -\int_{t'=0}^{t} \mathrm{d}t'$$

$$\Rightarrow \quad \frac{1}{k} \ln \frac{g + k v_z(t)}{g + k v_0 \sin\theta_0} = -t \quad \Rightarrow \quad v_z(t) = \left(v_0 \sin\theta_0 + \frac{g}{k}\right) \mathrm{e}^{-kt} - \frac{g}{k} \qquad \blacksquare$$

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \left(v_0 \sin\theta_0 + \frac{g}{k}\right) \mathrm{e}^{-kt} - \frac{g}{k}$$

$$\Rightarrow \quad \int_{z'=z_0}^{z} \mathrm{d}z' = \int_{t'=0}^{t} \left[\left(v_0 \sin\theta_0 + \frac{g}{k}\right) \mathrm{e}^{-kt'} - \frac{g}{k}\right] \mathrm{d}t'$$

$$\Rightarrow \quad z(t) = z_0 + \frac{1}{k}\left(v_0 \sin\theta_0 + \frac{g}{k}\right)\left(1 - \mathrm{e}^{-kt}\right) - \frac{gt}{k} \qquad \blacksquare$$

Projectile with resistance: horizontal range

- Time of the flight: $z_0 = 0$

$$z(T) = 0 \quad \Rightarrow \quad (kv_0 \sin\theta_0 + g)\left(1 - \mathrm{e}^{-kT}\right) - kgT = 0$$

- Dimensionless resistance parameter:

$$\epsilon \equiv \frac{kv_0}{g} \quad \Rightarrow \quad (\epsilon \sin\theta_0 + 1)\left(1 - \mathrm{e}^{-kT}\right) - kT = 0$$

- Perturbation calculation for *weak* friction: $\epsilon \ll 1$

$$T = \frac{2v_0 \sin\theta_0}{g}\left[1 + c_1\epsilon + c_2\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)\right]$$

- Values for c_1 and c_2 are to be determined

Projectile with resistance: horizontal range – cont'd

- Substitution, series expansion and solving:

$$T = \frac{2v_0 \sin\theta_0}{g} \left[1 - \frac{1}{3}\,\epsilon \sin\theta_0 + \frac{2}{9}\,\epsilon^2 \sin^2\theta_0 + \mathcal{O}\left(\epsilon^3\right)\right]$$

- Horizontal range: $y_0 = 0$

$$R \equiv y(T) = \frac{v_0 \cos\theta_0}{k}\left(1 - \mathrm{e}^{-kT}\right)$$

- Substitutions and series expansion:

$$R = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} \left[1 - \frac{4}{3}\,\epsilon \sin\theta_0 + \frac{14}{9}\,\epsilon^2 \sin^2\theta_0 + \mathcal{O}\left(\epsilon^3\right)\right]$$

EXERCISE 2.5: Complete the perturbation calculations to obtain the expression for R up to ϵ^2.

$$(\epsilon \sin\theta_0 + 1)\left(1 - \mathrm{e}^{-kT}\right) - kT = 0\,, \qquad \epsilon \equiv \frac{kv_0}{g}\,, \qquad T = \frac{2v_0 \sin\theta_0}{g}\left(1 + c_1\epsilon + c_2\epsilon^2\right)$$

$$\begin{aligned}
&(\epsilon \sin\theta_0 + 1)\left(1 - \mathrm{e}^{-kT}\right) - kT = 0 \\
\Rightarrow \quad &(\epsilon \sin\theta_0 + 1)\left(1 - \mathrm{e}^{-\epsilon g T/v_0}\right) - \epsilon\,\frac{gT}{v_0} = 0 \\
\Rightarrow \quad &(\epsilon \sin\theta_0 + 1)\left\{1 - \exp\left[-\frac{\epsilon g}{v_0}\,\frac{2v_0 \sin\theta_0}{g}\left(1 + c_1\epsilon + c_2\epsilon^2\right)\right]\right\} \\
&\qquad - \frac{\epsilon g}{v_0}\,\frac{2v_0 \sin\theta_0}{g}\left(1 + c_1\epsilon + c_2\epsilon^2\right) = 0 \\
\Rightarrow \quad &(\epsilon \sin\theta_0 + 1)\left\{1 - \exp\left[-2\epsilon \sin\theta_0\left(1 + c_1\epsilon + c_2\epsilon^2\right)\right]\right\} - 2\epsilon \sin\theta_0\left(1 + c_1\epsilon + c_2\epsilon^2\right) = 0
\end{aligned}$$

$$f(\epsilon) \equiv (\epsilon \sin\theta_0 + 1)\left\{1 - \exp\left[-2\epsilon\sin\theta_0\left(1 + c_1\epsilon + c_2\epsilon^2\right)\right]\right\} - 2\epsilon\sin\theta_0\left(1 + c_1\epsilon + c_2\epsilon^2\right)$$

$$\Rightarrow \quad \begin{cases} f^{(0)}(0) = 0 \\ f^{(1)}(0) = 0 \\ f^{(2)}(0) = 0 \\ f^{(3)}(0) = -4\sin^2\theta_0\,(3c_1 + \sin\theta_0) \\ f^{(4)}(0) = 16\sin^2\theta_0\left(\sin^2\theta_0 - 3c_1^2 - 3c_2\right) \end{cases}$$

$$\Rightarrow \quad f(\epsilon) = \frac{1}{3!}\left(-4\sin^2\theta_0\right)(3c_1 + \sin\theta_0)\,\epsilon^3 + \frac{1}{4!}\left(16\sin^2\theta_0\right)\left(\sin^2\theta_0 - 3c_1^2 - 3c_2\right)\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)$$

$$f(\epsilon) = 0 \quad \Rightarrow \quad \begin{cases} 3c_1 + \sin\theta_0 = 0 \\ \sin^2\theta_0 - 3c_1^2 - 3c_2 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = -\dfrac{1}{3}\,\sin\theta_0 \\ c_2 = \dfrac{2}{9}\,\sin^2\theta_0 \end{cases} \quad \blacksquare$$

$$R = \frac{v_0 \cos\theta_0}{k}\left(1 - \mathrm{e}^{-kT}\right), \qquad \epsilon \equiv \frac{kv_0}{g}, \qquad T = \frac{2v_0 \sin\theta_0}{g}\left(1 + c_1\epsilon + c_2\epsilon^2\right)$$

$$\begin{aligned} R &= \frac{v_0 \cos\theta_0}{k}\left(1 - \mathrm{e}^{-kT}\right) = \frac{v_0^2 \cos\theta_0}{\epsilon g}\left(1 - \mathrm{e}^{-\epsilon g T/v_0}\right) \\ &= \frac{v_0^2 \cos\theta_0}{\epsilon g}\left\{1 - \exp\left[-\frac{\epsilon g}{v_0}\,\frac{2v_0 \sin\theta_0}{g}\left(1 + c_1\epsilon + c_2\epsilon^2\right)\right]\right\} \\ &= \frac{v_0^2 \cos\theta_0}{\epsilon g}\left\{1 - \exp\left[-2\epsilon \sin\theta_0\left(1 + c_1\epsilon + c_2\epsilon^2\right)\right]\right\} \qquad \blacksquare \end{aligned}$$

$$g(\epsilon) \equiv \frac{v_0^2 \cos\theta_0}{\epsilon g} \left\{1 - \exp\left[-2\epsilon \sin\theta_0 \left(1 + c_1\epsilon + c_2\epsilon^2\right)\right]\right\}$$

$$\Rightarrow \quad \begin{cases} g^{(0)}(0) = \dfrac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} \\[2ex] g^{(1)}(0) = \dfrac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} \left(c_1 - \sin\theta_0\right) \\[2ex] g^{(2)}(0) = \dfrac{4v_0^2 \sin\theta_0 \cos\theta_0}{3g} \left(2 + 3c_2 - 6c_1 \sin\theta_0 - 2\cos^2\theta_0\right) \end{cases}$$

$$\begin{cases} c_1 = -\dfrac{1}{3} \sin\theta_0 \\[2ex] c_2 = \dfrac{2}{9} \sin^2\theta_0 \end{cases} \quad \Rightarrow \quad \begin{cases} g^{(0)}(0) = \dfrac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} \\[2ex] g^{(1)}(0) = -\dfrac{8v_0^2 \sin^2\theta_0 \cos\theta_0}{3g} \\[2ex] g^{(2)}(0) = \dfrac{56v_0^2 \sin^3\theta_0 \cos\theta_0}{9g} \end{cases} \quad \blacksquare$$

$$\Rightarrow \quad R = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} \left[1 - \frac{4}{3}\epsilon \sin\theta_0 + \frac{14}{9}\epsilon^2 \sin^2\theta_0 + \mathcal{O}\left(\epsilon^3\right)\right] \quad \blacksquare$$

Linear homogeneous ODEs

- n-order homogeneous equation with constant coefficients: $a_n \neq 0$

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1\, y^{(1)}(x) + a_0 y^{(0)}(x) = 0$$

- Characteristics equation: n-degree polynomial of λ

$$y(x) = \mathrm{e}^{\lambda x} \quad \Rightarrow \quad a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

- Characteristic roots give linearly independent solutions:
  - λ is a real root with no degeneracy: $\mathrm{e}^{\lambda x}$ is the solution
  - λ is a real root with doubly degeneracy: $\mathrm{e}^{\lambda x}$ and $x\mathrm{e}^{\lambda x}$ are solutions
  - $\lambda = \alpha \pm \mathrm{i}\beta$ are complex root with no degeneracy: $\mathrm{e}^{\alpha x} \sin \beta x$ and $\mathrm{e}^{\alpha x} \cos \beta x$ are solutions
  - $\lambda = \alpha \pm \mathrm{i}\beta$ are complex root with doubly degeneracy: $\mathrm{e}^{\alpha x} \sin \beta x$, $x\mathrm{e}^{\alpha x} \sin \beta x$, $\mathrm{e}^{\alpha x} \cos \beta x$ and $x\mathrm{e}^{\alpha x} \cos \beta x$ are solutions

Linear homogeneous ODEs – cont'd

- **Wronskian** of a set of n functions $\{f_1(x), \cdots, f_n(x)\}$:

$$W[f_1, f_2, \cdots, f_n](x) \equiv \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

- General solution: $\{y_n(x)\}$ is a set of linearly independent solutions

$$W[y_1, y_2, \cdots, y_n](x) \neq 0$$

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_{n-1} y_{n-1}(x) + C_n y_n(x)$$

- Constants C_i are to be determined from initial/boundary conditions

Charge in magnetic field

- A point charge of mass m and charge q is moving in a region of uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{e}}_y$

- Equation of motion in Cartesian coordinates: $\omega \equiv qB_0/m$

$$m\,\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = q\,\mathbf{v}(t)\times\mathbf{B} \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\omega\,\dfrac{\mathrm{d}z(t)}{\mathrm{d}t} \\[2ex] \dfrac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} = 0 \\[2ex] \dfrac{\mathrm{d}^2z(t)}{\mathrm{d}t^2} = \omega\,\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \end{cases}$$

- Initial conditions:

$$\mathbf{r}(0) = (x_0, y_0, z_0)\ , \qquad \mathbf{v}(0) = (0, v_{y0}, v_{z0})$$

Charge in magnetic field: y-direction

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = 0\,, \qquad y(0) = y_0\,, \qquad v_y(0) = v_{y0}$$

- Solving for $v_y(t)$:

$$\frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = 0 \qquad \Rightarrow \qquad v_y(t) = v_{y0}$$

- Solving for $y(t)$:

$$v_y(t) = v_{y0} \qquad \Rightarrow \qquad \frac{\mathrm{d}y(t)}{\mathrm{d}t} = v_{y0} \qquad \Rightarrow \qquad v_y(t) = v_{y0}t + y_0$$

- Motion along the y-direction is essentially uniform

Charge in magnetic field: x and z-directions

- Coupled differential equations:

$$\begin{cases} \dfrac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\omega \, \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} \\[2ex] \dfrac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = \omega \, \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \end{cases} , \qquad \begin{cases} x(0) = x_0 \,, \quad v_x(0) = 0 \\ z(0) = z_0 \,, \quad v_z(0) = v_{z0} \end{cases}$$

- Decoupling and solving:

$$\begin{cases} x(t) = C_1 \cos \omega t + C_2 \sin \omega t + C_0 \\ z(t) = D_1 \cos \omega t + D_2 \sin \omega t + D_0 \end{cases}$$

- Question: Are C_1, C_2, C_0, D_1, D_2 and D_0 all independent from each other?

EXERCISE 2.6: Obtain the general solutions for the coupled differential equations for $x(t)$ and $z(t)$.

$$\begin{cases} \dfrac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\omega \, \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = \omega \, \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \end{cases} \Rightarrow \begin{cases} \dfrac{\mathrm{d}^3 x(t)}{\mathrm{d}t^3} = -\omega^2 \, \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}^3 z(t)}{\mathrm{d}t^3} = -\omega^2 \, \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} \end{cases}$$

$$x(t) = \mathrm{e}^{\lambda t} \quad \Rightarrow \quad \frac{\mathrm{d}^3 x(t)}{\mathrm{d}t^3} = -\omega^2 \, \frac{\mathrm{d}x(t)}{\mathrm{d}t} \quad \Rightarrow \quad \lambda \left(\lambda^2 + \omega^2\right) = 0 \quad \Rightarrow \quad \lambda = 0 \, , \pm \mathrm{i}\omega \qquad \blacksquare$$

$$x(t) = A\mathrm{e}^{+\mathrm{i}\omega t} + B + C\mathrm{e}^{-\mathrm{i}\omega t} = (A + C) \cos \omega t + \mathrm{i} \, (A - C) \sin \omega t + B \qquad \blacksquare$$

Charge in magnetic field: x and z-directions – cont'd

- Eliminating dependencies:

$$\begin{cases} x(t) = C_1 \cos \omega t + C_2 \sin \omega t + C_0 \\ z(t) = -C_2 \cos \omega t + C_1 \sin \omega t + D_0 \end{cases}$$

- Imposing initial conditions for $x(t)$ and $z(t)$:

$$\begin{cases} x(0) = x_0 \quad \Rightarrow \quad C_0 = x_0 - C_1 \\ z(0) = z_0 \quad \Rightarrow \quad D_0 = z_0 \end{cases}$$

- Imposing initial conditions for $\dot{x}(t)$ and $\dot{z}(t)$:

$$\begin{cases} \dot{x}(0) = 0 \quad \Rightarrow \quad C_2 = 0 \\ \dot{z}(0) = v_{z0} \quad \Rightarrow \quad C_1 = \dfrac{v_{z0}}{\omega} \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\omega \, \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} \\[2ex] \dfrac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = \omega \, \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \end{cases} , \qquad \begin{cases} x(t) = C_1 \cos\omega t + C_2 \sin\omega t + C_0 \\[2ex] z(t) = D_1 \cos\omega t + D_2 \sin\omega t + D_0 \end{cases}$$

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\omega \, \frac{\mathrm{d}z(t)}{\mathrm{d}t}$$

$$\Rightarrow \quad -\omega^2 C_1 \cos\omega t - \omega^2 C_2 \sin\omega t = -\omega \left(-\omega D_1 \sin\omega t + \omega D_2 \cos\omega t\right)$$

$$\Rightarrow \quad \begin{cases} -\omega^2 C_1 = -\omega^2 D_2 \\[2ex] -\omega^2 C_2 = \omega^2 D_1 \end{cases} \quad \Rightarrow \quad \begin{cases} D_2 = C_1 \\[2ex] D_1 = -C_2 \end{cases} \qquad \blacksquare$$

$$\begin{vmatrix} \sin\omega t & \cos\omega t \\[1ex] \omega\cos\omega t & -\omega\sin\omega t \end{vmatrix} = -\omega \neq 0 \qquad \blacksquare$$

Charge in magnetic field: trajectory

- Position and velocity:

$$\begin{cases} x(t) = \dfrac{v_{z0}}{\omega}\cos\omega t + x_0 - \dfrac{v_{z0}}{\omega} \\ y(t) = v_{y0}t + y_0 \\ z(t) = \dfrac{v_{z0}}{\omega}\sin\omega t + z_0 \end{cases}, \qquad \begin{cases} v_x(t) = -v_{z0}\sin\omega t \\ v_y(t) = v_{y0} \\ v_z(t) = v_{z0}\cos\omega t \end{cases}$$

- Trajectory of the point charge is a circular helix of radius mv_{z0}/qB_0 centered at $(x, z) = (x_0 - mv_{z0}/qB_0, z_0)$

$$\left[x(t) - \left(x_0 - \frac{mv_{z0}}{qB_0}\right)\right]^2 + [z(t) - z_0]^2 = \left(\frac{mv_{z0}}{qB_0}\right)^2$$