PC3261: Classical Mechanics II

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Lecture 10: Hamiltonian Mechanics I

Legendre transformation

- Conjugate pair of variables: (u, x) and (v, y) are conjugate pairs

$$\mathrm{d}f = u\,\mathrm{d}x + v\,\mathrm{d}y$$

- **Legendre transformation** converts a function with dependence on variable(s) to another function with dependence on conjugate variable(s)

$$f(x,y) \quad \Rightarrow \quad \mathrm{d}f = \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}x + \frac{\partial f(x,y)}{\partial y}\,\mathrm{d}y \equiv u(x,y)\,\mathrm{d}x + v(x,y)\,\mathrm{d}y$$

$$f(x,y) \to g(u,y) \equiv f(x(u,y),y) - x(u,y)u$$

- Examples: thermodynamic internal energy $E(S,V)$ to Helmholtz free energy $F(T,V)$, internal energy $E(S,V)$ to Gibbs free energy $G(T,P)$, internal energy to enthalpy $H(S,P)$, etc

EXERCISE 10.1: Starting from $g = g(u,y)$, perform a Legendre transformation to another function $h = h(x,v)$.

Hamiltonian function

- Lagrangian function:

$$\mathcal{L} \equiv \mathcal{L}\left(\{q_i(t), \dot{q}_i(t)\}, t\right) \quad \Rightarrow \quad \mathrm{d}\mathcal{L} = \sum_{i=1}^{M} \left(\frac{\partial \mathcal{L}}{\partial q_i} \, \mathrm{d}q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \, \mathrm{d}\dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} \, \mathrm{d}t$$

- **Hamiltonian function** (or **Hamiltonian**) has explicit dependences on generalized coordinates q_i, generalized momenta p_i and time t

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}\left(\{q_i(t), p_i(t)\}, t\right) \\ &\equiv \sum_{i=1}^{M} \dot{q}_i\left(\{q_k(t), p_k(t)\}, t\right) \, p_i(t) - \mathcal{L}\left(\{q_i(t), \dot{q}_i\left(\{q_k(t), p_k(t)\}, t\right)\}, t\right) \end{aligned}$$

- Be cautious on the flip of signs in the Legendre transformation from Lagrangian to Hamiltonian!

Hamiltonian function – cont'd

- Generalized momenta:

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \Rightarrow \quad \dot{q}_i = \dot{q}_i \left(\{q_k(t), p_k(t)\}, t \right)$$

- A couple of extra steps (not necessarily trivial) in the construction of Hamiltonian from the Lagrangian are to write down the generalized momenta p_i and solve for the generalized velocities in terms of generalized coordinates, generalized momenta and time, $\dot{q}_i(\{q_k, p_k\}, t)$

- The set of generalized coordinates and generalized momenta, $\{q_k(t), p_k(t)\}$, used in Hamiltonian mechanics is generally known as **canonical coordinates**

EXERCISE 10.2: Construct the Hamiltonian for a particle of mass m subjected to a conservative central force field with potential energy $U(r)$ using usual polar coordinates r and ϕ as generalized coordinates.

Hamilton equations of motion

- **Hamilton equations of motion** (or **canonical equations of motion**):

$$\begin{cases} \dot{q}_i = \dfrac{\partial \mathcal{H}}{\partial p_i} \\[2ex] \dot{p}_i = -\dfrac{\partial \mathcal{H}}{\partial q_i} \end{cases}, \qquad i = 1, 2, \cdots, M\,, \qquad\qquad \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t}$$

- Hamiltonian approach gives $(2M+1)$ first-order differential equations instead of the M second-order differential equations in the Lagrangian approach

- Solution of Hamiltonian equations of motion is represented by a curve parameterized by t in the $2M$-dimensional space known as **cotangent bundle** $\mathbf{T}^*\mathbb{Q}$

$$(q_1(t), q_2(t), \cdots, q_M(t), p_1(t), p_2(t), \cdots, p_M(t))$$

EXERCISE 10.3: Derive Hamilton equations of motion.

Example: Atwood machine (yet another visit!)

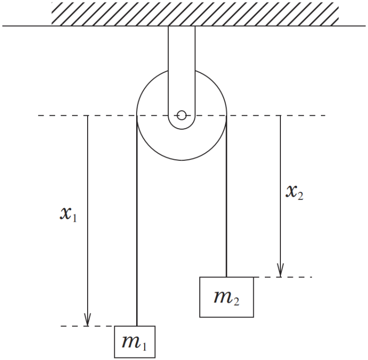
- Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless pulley with frictionless pulley

- Lagrangian:

$$\mathcal{L}(x_1, \dot{x}_1, t) = \frac{1}{2}(m_1 + m_2)\,\dot{x}_1^2 + (m_1 - m_2)\,g x_1$$

- Accelerations:

$$\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\,g = -\ddot{x}_2$$



EXERCISE 10.4: Obtain the Hamilton equations of motion for the Atwood machine and solve for the acceleration of the masses.

Hamiltonian as a constant of motion

- Hamiltonian could be varied with time for two reasons: (1) implicit time dependence via generalized coordinates and momenta; (2) explicit time dependence

$$\mathcal{H} \equiv \mathcal{H}\left(\{q_i(t), p_i(t)\}, t\right) \quad \Rightarrow \quad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \sum_{i=1}^{M} \left(\frac{\partial \mathcal{H}}{\partial q_i}\, \dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i}\, \dot{p}_i \right) + \frac{\partial \mathcal{H}}{\partial t}$$

- Hamiltonian is a constant of motion if it has no explicit time dependence

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

- Moreover, if the kinetic energy is a homogeneous quadratic function of generalized velocities, then the Hamiltonian is the total mechanical energy

- Identification of the Hamiltonian as a constant of motion and as the total mechanical energy are two *separate* issues!

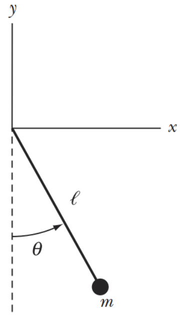
Example: Plane pendulum (revisited)

- A point particle of mass m attached to a massless rod of length ℓ rotates about a frictionless pivot in a plane

- Lagrangian:

$$\mathcal{L} \equiv \mathcal{L}(\theta, \dot{\theta}, t) = \frac{1}{2} m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

- Jacobi energy function is the total mechanical energy as the kinetic energy is a homogeneous quadratic function of generalized velocity



EXERCISE 10.5: Obtain the Hamiltonian equations of motion for the plane pendulum and identify one constant of motion.

Choices of generalized coordinates

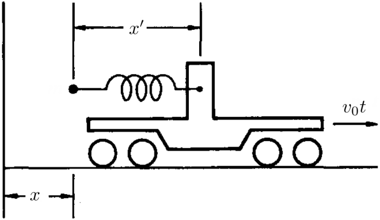
- Under a point transformation, the functional appearance of Lagrangian may be changed but the value of Lagrangian is not changed; nevertheless, an entirely different quantity for the Hamiltonian may be resulted!

- A point mass m is attached to a massless spring of spring constant k, the other end of which is fixed on a massless cart moving at a uniform speed v_0

- Lagrangians: $x' = x - v_0 t$

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k \left(x - v_0 t\right)^2$$

$$\Rightarrow \quad m\ddot{x} = -k\left(x - v_0 t\right)$$



$$\mathcal{L}(x', \dot{x}', t) = \frac{1}{2} m \left(\dot{x}' + v_0\right)^2 - \frac{1}{2} kx'^2 \quad \Rightarrow \quad m\ddot{x}' = -kx'$$

Choices of generalized coordinates – cont'd

- Hamiltonian: $\mathcal{H}(x, p_x, t)$ is the total mechanical energy of the system but it is not a constant of motion

$$\mathcal{H} \equiv \mathcal{H}(x, p_x, t) = \frac{p_x^2}{2m} + \frac{1}{2}\, k \left(x - v_0 t\right)^2$$

- Hamiltonian: $\mathcal{H}'(x', p_x', t)$ is not the total mechanical energy of the system but it is a constant of motion

$$\mathcal{H}' \equiv \mathcal{H}'(x', p_x', t) = \frac{\left(p_x' - m v_0\right)^2}{2m} + \frac{k x'^2}{2} - \frac{m v_0^2}{2}$$

- The two Hamiltonians are different in value, time dependence and functional form; however, both lead to the identical motion of the particle (convince yourself!)