Degrees of freedom

- **Degrees of freedom** is the *minimum* number of *independent* coordinates that can completely specify the configuration of the mechanical system

- Holonomic constraints reduce the number of degrees of freedom of the mechanical system

- Example: a system of two particles moving in the space connected by a rigid rod of fixed length has five degrees of freedom

$$|\mathbf{r}_2 - \mathbf{r}_1|^2 - \ell^2 = 0 \quad \Rightarrow \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - \ell^2 = 0$$

- A holonomic system is a mechanical system whose constraints are all holonomic and it has as many degrees of freedom as independent coordinates necessary to specify its configuration at any instant

Generalized coordinates

- **Generalized coordinates**: a *minimal* set of independent coordinates $\{q_k\}$ to specify the configuration of the mechanical system at any instant of time

- A mechanical system consisting of N particles subject to the C holonomic constraints can be described by $M = 3N - C$ generalized coordinates $\{q_k\}$:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, q_2, \cdots, q_M, t) \ , \qquad \alpha = 1, 2, \cdots, N$$

- Generalized coordinates (θ, ϕ) for a particle restricted to the surface of a sphere with radius R undergoing uniform motion at velocity $\mathbf{u}$ relative to an inertial reference frame

$$\begin{cases} x = u_x t + R\sin\theta\cos\phi \\ y = u_y t + R\sin\theta\sin\phi \\ z = u_z t + R\cos\theta \end{cases} \quad \Rightarrow \quad (x - u_x t)^2 + (y - u_y t)^2 + (z - u_z t)^2 - R^2 = 0$$

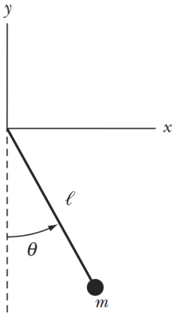
Example: Plane pendulum

- A point particle of mass m attached to a massless rod of length ℓ rotates about a frictionless pivot in a plane

- Holonomic constraints: particle is constrained to move in the xy-plane and length of the rod is fixed

$$\begin{cases} f_1(x, y, z, t) = z(t) = 0 \\ f_2(x, y, z, t) = x^2(t) + y^2(t) - \ell^2 = 0 \end{cases}$$

- One degree of freedom; two possible generalized coordinates are: (1) $q_1 = x$; or (2) $q_1 = \theta$



EXERCISE 7.3: Use d'Alembert's principle to obtain respective equations of motion for $x(t)$ and $\theta(t)$.

$$
x^2(t) + y^2(t) - \ell^2 = 0 \quad \Rightarrow \quad \begin{cases} \delta y = \dfrac{x(t)}{\sqrt{\ell^2 - x^2(t)}}\, \delta x \\[2ex] \ddot{y}(t) = \dfrac{x^2(t)\, \dot{x}^2(t)}{\left[\ell^2 - x^2(t)\right]^{3/2}} + \dfrac{\dot{x}^2(t) + x(t)\, \ddot{x}(t)}{\sqrt{\ell^2 - x^2(t)}} \end{cases}
$$

$$
\mathbf{r}(t) = x(t)\, \hat{\mathbf{e}}_x + y(t)\, \hat{\mathbf{e}}_y \quad \Rightarrow \quad \begin{cases} \delta \mathbf{r} = \delta x\, \hat{\mathbf{e}}_x + \delta y\, \hat{\mathbf{e}}_y \\[1ex] \ddot{\mathbf{r}}(t) = \ddot{x}(t)\, \hat{\mathbf{e}}_x + \ddot{y}(t)\, \hat{\mathbf{e}}_y \end{cases}
$$

$$
\mathbf{F}^{(\mathsf{A})}(t) = -mg\, \hat{\mathbf{e}}_y
$$

$$
\left[\mathbf{F}^{(\mathsf{A})}(t) - m\ddot{\mathbf{r}}(t)\right] \cdot \delta \mathbf{r} = 0 \quad \Rightarrow \quad \ddot{x}(t)\, \delta x + \ddot{y}(t)\, \delta y + g\, \delta y = 0
$$

$$
\Rightarrow \quad \ddot{x}(t) = \frac{x(t)\, \dot{x}^2(t)}{\ell^2 - x^2(t)} - \frac{g}{\ell^2}\, x(t) \sqrt{\ell^2 - x^2(t)} \qquad \blacksquare
$$

$$x^2(t) + y^2(t) - \ell^2 = 0 \quad \Rightarrow \quad \begin{cases} x(t) = \ell \sin\theta(t) \\ y(t) = -\ell \cos\theta(t) \end{cases}$$

$$\mathbf{r}(t) = x(t)\,\hat{\mathbf{e}}_x + y(t)\,\hat{\mathbf{e}}_y = \ell \sin\theta(t)\,\hat{\mathbf{e}}_x - \ell\cos\theta(t)\,\hat{\mathbf{e}}_y$$

$$\Rightarrow \quad \ddot{\mathbf{r}}(t) = \ell\left[\ddot{\theta}(t)\cos\theta(t) - \dot{\theta}^2(t)\sin\theta(t)\right]\,\hat{\mathbf{e}}_x + \ell\left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right]\,\hat{\mathbf{e}}_y$$

$$\mathbf{r}(t) = \ell\sin\theta(t)\,\hat{\mathbf{e}}_x - \ell\cos\theta(t)\,\hat{\mathbf{e}}_y \quad \Rightarrow \quad \delta\mathbf{r} = \frac{\partial\mathbf{r}}{\partial\theta}\,\delta\theta = \ell\,\delta\theta\left[\cos\theta(t)\,\hat{\mathbf{e}}_x + \sin\theta(t)\,\hat{\mathbf{e}}_y\right]$$

$$\mathbf{F}^{(\mathsf{A})}(t) = -mg\,\hat{\mathbf{e}}_y$$

$$\left[\mathbf{F}^{(\mathsf{A})}(t) - m\ddot{\mathbf{r}}(t)\right]\cdot\delta\mathbf{r} = 0$$

$$\Rightarrow \quad -mg\ell\sin\theta(t)\,\delta\theta - m\ell^2\left[\ddot{\theta}(t)\cos^2\theta(t) - \dot{\theta}^2(t)\sin\theta(t)\cos\theta(t)\right.$$
$$\left. + \ddot{\theta}(t)\sin^2\theta(t) + \dot{\theta}^2(t)\sin\theta(t)\cos\theta(t)\right]\delta\theta = 0$$

$$\Rightarrow \quad \ddot{\theta}(t) + \frac{g}{\ell}\,\sin\theta(t) = 0 \qquad \blacksquare$$

Generalized forces

- Generalized coordinates:

$$\mathbf{r}_\alpha \equiv \mathbf{r}_\alpha(\{q_k(t)\}, t) \ , \qquad \alpha = 1, 2, \cdots, N \, , \qquad k = 1, 2, \cdots, M$$

- Generalized forces:

$$\delta W = \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{(\mathsf{A})}(t) \cdot \delta \mathbf{r}_\alpha \equiv \sum_{k=1}^{M} \mathcal{Q}_k(t) \, \delta q_k \quad \Rightarrow \quad \mathcal{Q}_k(t) \equiv \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{(\mathsf{A})}(t) \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}$$

- Example: generalized forces associated to polar coordinates

$$\begin{cases} \mathcal{Q}_1(t) \equiv \mathcal{Q}_\rho(t) = F_x(t) \cos\phi(t) + F_y(t) \sin\phi(t) = F_\rho(t) \\ \mathcal{Q}_2(t) \equiv \mathcal{Q}_\phi(t) = -\rho(t) \, F_x(t) \sin\phi(t) + \rho(t) \, F_y(t) \cos\phi(t) = \rho(t) \, F_\phi(t) \end{cases}$$

$$\mathbf{r}_\alpha \equiv \mathbf{r}_\alpha\left(\{q_k(t)\}, t\right)\ , \qquad \alpha = 1, 2, \cdots, N\ , \qquad k = 1, 2, \cdots, M$$

$$\delta \mathbf{r}_\alpha = \sum_{k=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\, \delta q_k$$

$$\begin{aligned}
\delta W &= \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{(\mathsf{A})}(t) \cdot \delta \mathbf{r}_\alpha \\
&= \sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{(\mathsf{A})}(t) \cdot \left[\sum_{k=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\, \delta q_k\right] \\
&= \sum_{k=1}^{M} \left[\sum_{\alpha=1}^{N} \mathbf{F}_\alpha^{(\mathsf{A})}(t) \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right] \delta q_k \\
&\equiv \sum_{k=1}^{M} \mathcal{Q}_k(t)\, \delta q_k \qquad \blacksquare
\end{aligned}$$

$$\begin{cases} x(t) = \rho(t)\cos\phi(t) \\ y(t) = \rho(t)\sin\phi(t) \end{cases} \quad \Rightarrow \quad \mathbf{r}(t) = \rho(t)\cos\phi(t)\,\hat{\mathbf{e}}_x + \rho(t)\sin\phi(t)\,\hat{\mathbf{e}}_y$$

$$\begin{cases} \dfrac{\partial \mathbf{r}}{\partial \rho} = \cos\phi(t)\,\hat{\mathbf{e}}_x + \sin\phi(t)\,\hat{\mathbf{e}}_y \\ \dfrac{\partial \mathbf{r}}{\partial \phi} = -\rho(t)\,\sin\phi(t)\,\hat{\mathbf{e}}_x + \rho(t)\,\cos\phi(t)\,\hat{\mathbf{e}}_y \end{cases} \quad \Rightarrow \quad \begin{cases} \hat{\mathbf{e}}_\rho = \cos\phi(t)\,\hat{\mathbf{e}}_x + \sin\phi(t)\,\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_\phi = -\sin\phi(t)\,\hat{\mathbf{e}}_x + \cos\phi(t)\,\hat{\mathbf{e}}_y \end{cases}$$

$$\mathbf{F}(t) = F_x(t)\,\hat{\mathbf{e}}_x + F_y(t)\,\hat{\mathbf{e}}_y \quad \Rightarrow \quad \begin{cases} F_\rho(t) = \hat{\mathbf{e}}_\rho \cdot \mathbf{F}(t) = F_x(t)\cos\phi(t) + F_y(t)\sin\phi(t) \\ F_\phi(t) = \hat{\mathbf{e}}_\phi \cdot \mathbf{F}(t) = -F_x(t)\sin\phi(t) + F_y(t)\cos\phi(t) \end{cases}$$

$$\mathcal{Q}_k(t) = \mathbf{F}(t) \cdot \frac{\partial \mathbf{r}}{\partial q_k}$$

$$\Rightarrow \quad \begin{cases} \mathcal{Q}_\rho(t) = \mathbf{F}(t) \cdot \dfrac{\partial \mathbf{r}}{\partial \rho} = F_x(t)\cos\phi(t) + F_y(t)\sin\phi(t) = F_\rho(t) \\ \mathcal{Q}_\phi(t) = \mathbf{F}(t) \cdot \dfrac{\partial \mathbf{r}}{\partial \phi} = \rho(t)\left[-F_x(t)\sin\phi(t) + F_y(t)\cos\phi(t)\right] = \rho(t)\,F_\phi(t) \end{cases} \qquad \blacksquare$$

Generalized velocities

- **Generalized velocity** associated to each generalized coordinate: $\{q_k(t), \dot{q}_k(t)\}$ are to be treated as a set of independent dynamical variables

$$\dot{q}_k(t) \equiv \frac{\mathrm{d}q_k(t)}{\mathrm{d}t}, \qquad k = 1, 2, \cdots, M$$

- Relationship between Cartesian velocity and generalized velocity:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(\{q_k(t)\}, t) \quad \Rightarrow \quad \dot{\mathbf{r}}_\alpha(t) = \sum_{k=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_k} \dot{q}_k(t) + \frac{\partial \mathbf{r}_\alpha}{\partial t}$$

- Dot-cancellation rule: Cartesian velocity is related to the generalized velocity in the same way as the Cartesian coordinate is related to the generalized coordinate

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}_\alpha}{\partial q_k}$$

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(\{q_k(t)\}, t) \;, \qquad \alpha = 1, 2, \cdots, N \,, \quad k = 1, 2, \cdots, M$$

$$\dot{\mathbf{r}}_\alpha(t) = \sum_{k=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\, \dot{q}_k(t) + \frac{\partial \mathbf{r}_\alpha}{\partial t}$$

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k} \sum_{j=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\, \dot{q}_j(t) + \frac{\partial}{\partial \dot{q}_k} \frac{\partial \mathbf{r}_\alpha}{\partial t}$$

$$= \sum_{j=1}^{M} \left[\frac{\partial}{\partial \dot{q}_k} \left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j} \right) \dot{q}_j + \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \frac{\partial \dot{q}_j}{\partial \dot{q}_k} \right] + \frac{\partial}{\partial \dot{q}_k} \left(\frac{\partial \mathbf{r}_\alpha}{\partial t} \right)$$

$$= \sum_{j=1}^{M} \left[\frac{\partial}{\partial q_j} \left(\frac{\partial \mathbf{r}_\alpha}{\partial \dot{q}_k} \right) \dot{q}_j + \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \frac{\partial \dot{q}_j}{\partial \dot{q}_k} \right] + \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}_\alpha}{\partial \dot{q}_k} \right)$$

$$= \sum_{j=1}^{M} \left(\mathbf{0} + \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\, \delta_{jk} \right) + \mathbf{0}$$

$$= \frac{\partial \mathbf{r}_\alpha}{\partial q_k} \qquad \blacksquare$$

Rewriting d-Alembert's principle

- Useful result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right) = \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_k}$$

- Rewriting the inertial force term in the d'Alembert's principle:

$$\begin{aligned}
-\sum_{\alpha=1}^{N} m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \delta \mathbf{r}_\alpha &= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right)\right]\delta q_k \\
&= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha\right)\right] - \frac{\partial}{\partial q_k}\left(\frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha\right)\right\}\delta q_k
\end{aligned}$$

EXERCISE 7.4: Obtain the expression for the inertial force term in the d'Alembert's principle.

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(\{q_k(t)\}, t)$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right) &= \sum_{i=1}^{M} \frac{\partial}{\partial q_i}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right)\dot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right) \\
&= \sum_{i=1}^{M} \frac{\partial}{\partial q_k}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_i}\right)\dot{q}_i + \frac{\partial}{\partial q_k}\left(\frac{\partial \mathbf{r}_\alpha}{\partial t}\right) \\
&= \sum_{i=1}^{M} \frac{\partial}{\partial q_k}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_i}\dot{q}_i\right) + \frac{\partial}{\partial q_k}\left(\frac{\partial \mathbf{r}_\alpha}{\partial t}\right) \\
&= \frac{\partial}{\partial q_k}\left[\sum_{i=1}^{M}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_i}\dot{q}_i\right) + \frac{\partial \mathbf{r}_\alpha}{\partial t}\right] \\
&= \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_k} \qquad \blacksquare
\end{aligned}$$

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(\{q_k(t)\}\,, t) \quad \Rightarrow \quad \delta\mathbf{r}_\alpha = \sum_{k=1}^{M} \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\,\delta q_k$$

$$\begin{aligned}
-\sum_{\alpha=1}^{N} m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha &= -\sum_{\alpha=1}^{N}\sum_{k=1}^{M} m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\,\delta q_k \\
&= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right)\right]\,\delta q_k \\
&= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_k}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_k}\right]\,\delta q_k \\
&= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N} \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\,m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha\right)\right] - \frac{\partial}{\partial q_k}\left(\frac{1}{2}\,m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha\right)\right\}\,\delta q_k
\end{aligned}$$

Lagrange's equation

- Kinetic energy in terms of generalized coordinates and generalized velocities:

$$T(t) = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha(t) \cdot \dot{\mathbf{r}}_\alpha(t) \quad \xrightarrow{\mathbf{r}_\alpha = \mathbf{r}_\alpha(\{q_k(t)\}, t)} \quad T \equiv T(t) \equiv T\left(\{q_k, \dot{q}_k(t)\}, t\right)$$

- d'Alembrt's principle in terms of generalized coordinates:

$$\sum_{\alpha=1}^{N} \left[\mathbf{F}^{(\mathsf{A})}(t) - m_\alpha \ddot{\mathbf{r}}_\alpha(t)\right] \cdot \delta\mathbf{r}_\alpha = 0 \quad \rightarrow \quad \sum_{i=1}^{M} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - \mathcal{Q}_i\right] \delta q_i = 0$$

- Lagrange's equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T\left(\{q_k(t), \dot{q}_k(t)\}, t\right)}{\partial \dot{q}_k}\right] - \frac{\partial T\left(\{q_k(t), \dot{q}_k(t)\}, t\right)}{\partial q_k} = \mathcal{Q}_k(t)\,, \quad k = 1, 2, \cdots, M$$

$$\sum_{\alpha=1}^{N} \left[\mathbf{F}^{(\mathsf{A})}(t) - m_\alpha \ddot{\mathbf{r}}_\alpha(t)\right] \cdot \delta \mathbf{r}_\alpha = 0$$

$$\Rightarrow \quad \sum_{\alpha=1}^{N} \sum_{i=1}^{M} \left[\mathbf{F}^{(\mathsf{A})}(t) - m_\alpha \ddot{\mathbf{r}}_\alpha(t)\right] \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\, \delta q_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{M} \sum_{\alpha=1}^{N} \left\{ \mathbf{F}^{(\mathsf{A})}(t) \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial}{\partial \dot{q}_i} \left(\frac{1}{2}\, m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \right) \right] + \frac{\partial}{\partial q_i} \left(\frac{1}{2}\, m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \right) \right\} \delta q_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{M} \left[\mathcal{Q}_i - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_i} \right) + \frac{\partial T}{\partial q_i} \right] \delta q_i = 0 \qquad \blacksquare$$