Example: Atwood machine (yet another visit!)

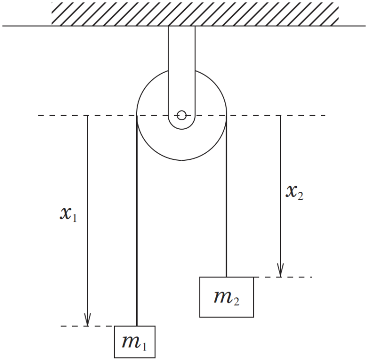
- Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless pulley with frictionless pulley

- Lagrangian:

$$\mathcal{L}(x_1, \dot{x}_1, t) = \frac{1}{2}\left(m_1 + m_2\right)\dot{x}_1^2 + \left(m_1 - m_2\right) g x_1$$

- Accelerations:

$$\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\, g = -\ddot{x}_2$$



EXERCISE 10.4: Obtain the Hamilton equations of motion for the Atwood machine and solve for the acceleration of the masses.

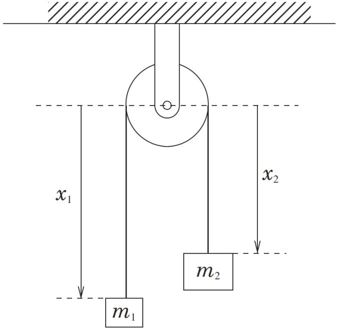
$$\mathcal{L} \equiv \mathcal{L}(x_1, \dot{x}_1, t) = \frac{1}{2}\left(m_1 + m_2\right)\dot{x}_1^2 + \left(m_1 - m_2\right) g x_1$$

$$p_{x_1} = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \left(m_1 + m_2\right)\dot{x}_1 \quad \Rightarrow \quad \dot{x}_1 = \frac{p_{x_1}}{m_1 + m_2}$$

$$\mathcal{H} \equiv \mathcal{H}(x_1, p_{x_1}, t) = \dot{x}_1\, p_{x_1} - \mathcal{L} = \frac{p_{x_1}^2}{2\left(m_1 + m_2\right)} - \left(m_1 - m_2\right) g x_1 \qquad \blacksquare$$

$$\begin{cases} \dot{x}_1 = \dfrac{\partial \mathcal{H}}{\partial p_{x_1}} = \dfrac{p_{x_1}}{m_1 + m_2} \\[2ex] \dot{p}_{x_1} = -\dfrac{\partial \mathcal{H}}{\partial x_1} = \left(m_1 - m_2\right) g \end{cases}$$

$$\Rightarrow \quad \ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\, g \qquad \blacksquare$$



Hamiltonian as a constant of motion

- Hamiltonian could be varied with time for two reasons: (1) implicit time dependence via generalized coordinates and momenta; (2) explicit time dependence

$$\mathcal{H} \equiv \mathcal{H}\left(\{q_i(t), p_i(t)\}, t\right) \quad \Rightarrow \quad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \sum_{i=1}^{M}\left(\frac{\partial \mathcal{H}}{\partial q_i}\,\dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i}\,\dot{p}_i\right) + \frac{\partial \mathcal{H}}{\partial t}$$

- Hamiltonian is a constant of motion if it has no explicit time dependence

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

- Moreover, if the kinetic energy is a homogeneous quadratic function of generalized velocities, then the Hamiltonian is the total mechanical energy

- Identification of the Hamiltonian as a constant of motion and as the total mechanical energy are two *separate* issues!

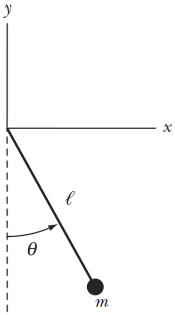
Example: Plane pendulum (revisited)

- A point particle of mass m attached to a massless rod of length ℓ rotates about a frictionless pivot in a plane

- Lagrangian:

$$\mathcal{L} \equiv \mathcal{L}(\theta, \dot{\theta}, t) = \frac{1}{2} m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

- Jacobi energy function is the total mechanical energy as the kinetic energy is a homogeneous quadratic function of generalized velocity



EXERCISE 10.5: Obtain the Hamiltonian equations of motion for the plane pendulum and identify one constant of motion.

$$\mathcal{L} \equiv \mathcal{L}(\theta, \dot{\theta}, t) = \frac{1}{2}\, m\ell^2 \dot{\theta}^2 + mg\ell \cos\theta$$

$$T(\theta, \dot{\theta}, t) = \frac{1}{2}\, m\ell^2 \dot{\theta}^2 \quad \Rightarrow \quad T(\theta, \lambda\dot{\theta}, t) = \frac{1}{2}\, m\ell^2 \left(\lambda\dot{\theta}\right)^2 = \lambda^2 T(\theta, \dot{\theta}, t) \qquad \blacksquare$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\ell^2 \dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{m\ell^2}$$

$$\mathcal{H} \equiv \mathcal{H}(\theta, p_\theta, t) = \dot{\theta}\, p_\theta - \mathcal{L} = \frac{1}{2}\, m\ell^2 \dot{\theta}^2 - mg\ell \cos\theta = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta \qquad \blacksquare$$

$$\begin{cases} \dot{\theta} = \dfrac{\partial \mathcal{H}}{\partial p_\theta} = \dfrac{p_\theta}{m\ell^2} \\[2ex] \dot{p}_\theta = -\dfrac{\partial \mathcal{H}}{\partial \theta} = -mg\ell \sin\theta \end{cases} \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{\ell}\, \sin\theta = 0 \qquad \blacksquare$$

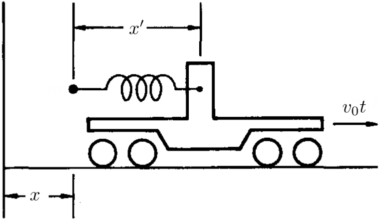
$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} = 0 \quad \Rightarrow \quad \mathcal{H} = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta = E \qquad \blacksquare$$

Choices of generalized coordinates

- Under a point transformation, the functional appearance of Lagrangian may be changed but the value of Lagrangian is not changed; nevertheless, an entirely different quantity for the Hamiltonian may be resulted!

- A point mass m is attached to a massless spring of spring constant k, the other end of which is fixed on a massless cart moving at a uniform speed v_0

- Lagrangians: $x' = x - v_0 t$



$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}\, m\dot{x}^2 - \frac{1}{2}\, k\,(x - v_0 t)^2$$

$$\Rightarrow \quad m\ddot{x} = -k\,(x - v_0 t)$$

$$\mathcal{L}(x', \dot{x}', t) = \frac{1}{2}\, m\,(\dot{x}' + v_0)^2 - \frac{1}{2}\, kx'^2 \quad \Rightarrow \quad m\ddot{x}' = -kx'$$

Choices of generalized coordinates – cont'd

- Hamiltonian: $\mathcal{H}(x, p_x, t)$ is the total mechanical energy of the system but it is not a constant of motion

$$\mathcal{H} \equiv \mathcal{H}(x, p_x, t) = \frac{p_x^2}{2m} + \frac{1}{2}\, k\, (x - v_0 t)^2$$

- Hamiltonian: $\mathcal{H}'(x', p_x', t)$ is not the total mechanical energy of the system but it is a constant of motion

$$\mathcal{H}' \equiv \mathcal{H}'(x', p_x', t) = \frac{(p_x' - m v_0)^2}{2m} + \frac{k x'^2}{2} - \frac{m v_0^2}{2}$$

- The two Hamiltonians are different in value, time dependence and functional form; however, both lead to the identical motion of the particle (convince yourself!)

PC3261: Classical Mechanics II

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Lecture 11: Hamiltonian Mechanics II

Cyclic coordinates (revisited)

- Generalized momenta associated to the cyclic coordinate is constant:

$$\frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \leftrightarrow \quad \frac{\partial \mathcal{H}}{\partial q_k} = 0 \quad \Rightarrow \quad \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k} = 0$$

- Lagrangian approach: q_2 is cyclic (p_2 is a constant of motion) but it is not necessarily true that $\dot{q}_2$ is constant; Lagrangian framework does not reduce *cleanly* to a problem with one less degrees of freedom

$$\frac{\partial \mathcal{L}}{\partial q_2} = 0 \quad \Rightarrow \quad \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2, t) \rightarrow \mathcal{L}(q_1, \dot{q}_1, \dot{q}_2, t)$$

- Hamiltonian approach: q_2 is cyclic and thus p_2 is a constant of motion; Hamiltonian framework is exactly equivalent to a problem with one less degrees of freedom!

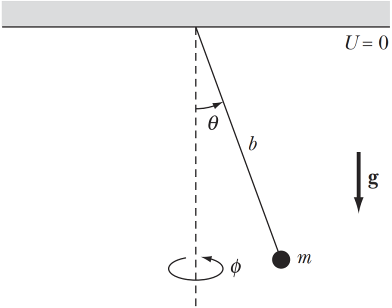
$$\frac{\partial \mathcal{H}}{\partial q_2} = 0 \quad \Rightarrow \quad \mathcal{H}(q_1, q_2, p_1, p_2, t) \rightarrow \mathcal{H}(q_1, p_1, t)$$

Example: Spherical Pendulum

- A spherical pendulum consists of a bob of mass m moving on a sphere centered on the point of support with radius $r = b$, the length of the pendulum

$$\mathcal{H} \equiv \mathcal{H}(\theta, \phi, p_\theta, p_\phi)$$

$$= \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2 \sin^2\theta} - mgb\cos\theta$$



- ϕ is an ignorable coordinate

- Two constants of motion: mechanical energy and angular momentum about the z-axis

EXERCISE 11.1: Obtain equations of motion for the spherical pendulum.

$$\mathcal{L} \equiv \mathcal{L}(\theta, \phi, \dot{\theta}, \dot{\phi}) = T - U = \frac{1}{2}\, mb^2\dot{\theta}^2 + \frac{1}{2}\, mb^2 \sin^2\theta\, \dot{\phi}^2 + mgb\cos\theta$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \dfrac{\partial \mathcal{L}}{\partial \theta} \quad \Rightarrow \quad \ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 - \dfrac{g}{b}\,\sin\theta \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \dfrac{\partial \mathcal{L}}{\partial \phi} \quad \Rightarrow \quad p_\phi \equiv \Phi = mb^2\sin^2\theta\,\dot{\phi} = \text{constant} \end{cases} \qquad \blacksquare$$

$$\Rightarrow \quad \ddot{\theta} + \frac{g}{b}\,\sin\theta - \frac{\Phi^2}{m^2b^4}\,\frac{\cos\theta}{\sin^3\theta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \quad \Rightarrow \quad h = \dot{\theta}p_\theta + \dot{\phi}p_\phi - \mathcal{L} = \frac{1}{2}\, mb^2\dot{\theta}^2 + \frac{1}{2}\, mb^2\sin^2\theta\,\dot{\phi}^2 - mgb\cos\theta = \text{constant}$$

$$\mathcal{L} \equiv \mathcal{L}(\theta, \phi, \dot{\theta}, \dot{\phi}) = T - U = \frac{1}{2}\, mb^2 \dot{\theta}^2 + \frac{1}{2}\, mb^2 \sin^2\theta\, \dot{\phi}^2 + mgb\cos\theta$$

$$\begin{cases} p_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = mb^2 \dot{\theta} \\[2ex] p_\phi = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = mb^2 \sin^2\theta\, \dot{\phi} \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{\theta} = \dfrac{p_\theta}{mb^2} \\[2ex] \dot{\phi} = \dfrac{p_\phi}{mb^2 \sin^2\theta} \end{cases}$$

$$\mathcal{H} \equiv \mathcal{H}(\theta, \phi, p_\theta, p_\phi) = T + U = \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2 \sin^2\theta} - mgb\cos\theta$$

$$\frac{\partial \mathcal{H}}{\partial \phi} = 0 \quad \Rightarrow \quad p_\phi = \Phi = \text{constant}$$

$$\mathcal{H} \equiv \mathcal{H}(\theta, p_\theta) = \frac{p_\theta^2}{2mb^2} + \frac{\Phi^2}{2mb^2 \sin^2\theta} - mgb\cos\theta$$

$$\begin{cases} \dot{\theta} = \dfrac{\partial \mathcal{H}}{\partial p_\theta} = \dfrac{p_\theta}{mb^2} \\[2ex] \dot{p_\theta} = -\dfrac{\partial \mathcal{H}}{\partial \theta} = \dfrac{\Phi^2}{mb^2}\, \dfrac{\cos\theta}{\sin^3\theta} - mgb\sin\theta \end{cases} \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{b}\, \sin\theta - \frac{\Phi^2}{m^2 b^4}\, \frac{\cos\theta}{\sin^3\theta} = 0$$

Poisson brackets

- Total time derivative of any dynamical function: $F \equiv F(\{q_i, p_i\}, t)$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_{i=1}^{M} \frac{\partial F}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial F}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} + \frac{\partial F}{\partial t} \equiv \{F, \mathcal{H}\}_{q,p} + \frac{\partial F}{\partial t}$$

- **Poisson bracket**: $F \equiv F(\{q_i, p_i\}, t)$, $G \equiv G(\{q_i, p_i\}, t)$

$$\{F, G\}_{q,p} \equiv \sum_{i=1}^{M} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

- Hamilton's canonical equation of motion in terms of Poisson brackets:

$$\begin{cases} \dot{q}_i = \dfrac{\partial \mathcal{H}}{\partial p_i} \\[2ex] \dot{p}_i = -\dfrac{\partial \mathcal{H}}{\partial q_i} \end{cases} \quad \rightarrow \quad \begin{cases} \dot{q}_i = \{q_i, \mathcal{H}\}_{q,p} \\ \dot{p}_i = \{p_i, \mathcal{H}\}_{q,p} \end{cases}$$

Poisson bracket and constant of motion

- If the Hamiltonian has no explicit time dependence, then it is a constant of motion

$$\frac{\partial \mathcal{H}}{\partial t} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \{\mathcal{H}, \mathcal{H}\}_{q,p} + \frac{\partial \mathcal{H}}{\partial t} = 0$$

- **Poisson theorem**: $F \equiv F\left(\{q_i, p_i\}, t\right)$, $G \equiv G\left(\{q_i, p_i\}, t\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{F, G\}_{q,p} = \left\{\frac{\mathrm{d}F}{\mathrm{d}t}, G\right\}_{q,p} + \left\{F, \frac{\mathrm{d}G}{\mathrm{d}t}\right\}_{q,p}$$

- If $F_1 \equiv F_1(\{q_i, p_i\}, t)$ and $F_2 \equiv F_2(\{q_i, p_i\}, t)$ are two constants of motion, then their Poisson bracket is also a constant of motion

$$\begin{cases} \dfrac{\mathrm{d}F_1}{\mathrm{d}t} = 0 \\[2ex] \dfrac{\mathrm{d}F_2}{\mathrm{d}t} = 0 \end{cases} \qquad \rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\{F_1, F_2\}_{q,p} = \left\{\frac{\mathrm{d}F_1}{\mathrm{d}t}, F_2\right\}_{q,p} + \left\{F_1, \frac{\mathrm{d}F_2}{\mathrm{d}t}\right\}_{q,p} = 0$$

Algebraic properties of Poisson bracket

- Anticommutativity:

$$\{F, G\}_{q,p} = -\{G, F\}_{q,p}$$

- Linearity:

$$\begin{cases} \{aF + bG, H\}_{q,p} = a\{F, H\}_{q,p} + b\{G, H\}_{q,p} \\ \{F, aG + bH\}_{q,p} = a\{F, G\}_{q,p} + b\{F, H\}_{q,p} \end{cases}$$

- Leibniz's rule:

$$\{FG, H\}_{q,p} = \{F, H\}_{q,p} G + F\{G, H\}_{q,p}$$

- Jacobi identity:

$$\left\{F, \{G, H\}_{q,p}\right\}_{q,p} + \left\{G, \{H, F\}_{q,p}\right\}_{q,p} + \left\{H, \{F, G\}_{q,p}\right\}_{q,p} = 0$$