PC 4245 Particle physics
Tutorial 1 (solution)

1. $c = 3.00 \times 10^8\ m/s$, $\hbar = 1.055 \times 10^{-34}\ kg\, m^2/s$

$G = 6.67 \times 10^{-11}\ m^3/(kg\, s^2)$.

(a) Dimension of length $\equiv L$

Dimension of mass $\equiv M$

Dimension of time $= T$

Dimension of $c \equiv [c] = \frac{L}{T}$

$[\hbar] = [$angular momentum$] = \frac{M L^2}{T}$

For the gravitational constant G, we have

gravitational force $F = G \frac{m_1 m_2}{r^2}$

but force $F = m a$, a = acceration.

hence $G \frac{m_1 m_2}{r^2} = m_2 a$

or $G \frac{m}{r^2} = a$ i.e. $G = \frac{r^2 a}{m}$.

Thus $[G] = \frac{L^2}{M} \cdot \frac{L}{T^2} = \frac{L^3}{M T^2}$

Consider $c\ \hbar\ G$ and its dimesion

Dimension of $c^x \hbar^y G^z$

$$= \left(\frac{L}{T}\right)^x \left(\frac{ML^2}{T}\right)^y \left(\frac{L^3}{MT^2}\right)^z = L^{x+2y+3z} M^{y-z} T^{-(x+y+2z)}$$

Planck Length L_p :

$$x + 2y + 3z = 1$$
$$y - z = 0$$
$$x + y + 2z = 0$$

solution is $y = z = \frac{1}{2}, \quad x = -\frac{3}{2}$

$$\therefore L_p = c^{-3/2} \hbar^{\frac{1}{2}} G^{\frac{1}{2}} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35} \text{ m.}$$

Planck time T_p :

$$x + 2y + 3z = 0$$
$$y - z = 0$$
$$x + y + 2z = -1$$

$$\therefore \quad y = z = \frac{1}{2}, \quad x = -\frac{5}{2}$$

i.e. $$T_p = c^{-5/2} \hbar^{\frac{1}{2}} G^{\frac{1}{2}} = \sqrt{\frac{\hbar G}{c^5}}$$
$$= 5.38 \times 10^{-44} \text{ s.}$$

Planck mass M_p :

$$x + 2y + 3z = 0$$
$$y - z = 1$$
$$x + y + 2z = 0$$

solution is

$$y = \frac{1}{2}, \quad z = -\frac{1}{2}, \quad x = \frac{1}{2}$$

$$\therefore M_p = c^{\frac{1}{2}} \hbar^{\frac{1}{2}} G^{-\frac{1}{2}} = \sqrt{\frac{c\hbar}{G}}$$

$$= 2.18 \times 10^{-8} \text{ kg}$$

Planck energy $= M_p c^2 = 2.18 \times 10^{-8} \text{ kg} \cdot c^2$

$$= 1.96 \times 10^{9} \text{ Joules} = 1.23 \times 10^{19} \text{ GeV}$$

(b) Coulomb force $F_{em} = \dfrac{q_1 q_2}{4\pi\epsilon_0 r^2}$ $\qquad q_1 = q_2 = q$

$$= \frac{q^2}{4\pi\epsilon_0 r^2} = \frac{e^2}{r^2}$$

Fine structure constant $\alpha \equiv \dfrac{e^2}{\hbar c} \approx \dfrac{1}{137}$

Gravitational force $F_{Grav} = G \dfrac{m_1 m_2}{r^2}$ $\qquad m_1 = m_2 = m$

$$= G \frac{m^2}{r^2}$$

$\therefore$ Gravitational fine structure constant is

$$\alpha_G = \frac{G m^2}{\hbar c}$$

For $m = M_p$ $\qquad \alpha_G = \dfrac{G}{\hbar c} M_p^2 = \dfrac{G}{\hbar c} \cdot \dfrac{c\hbar}{G} = 1$

For $m = m_e$ (electron mass)

$$m_e c^2 \simeq \frac{1}{2}\,\text{MeV}, \qquad M_p c^2 = 1.23 \times 10^{-19}\,\text{GeV}$$

Write $m = m_e = \dfrac{m_e}{M_p} M_p$

$$= \frac{0.5\,\text{MeV}}{1.23 \times 10^{19}\,\text{GeV}} M_p = \frac{0.5 \times 10^{-3}\,\text{GeV}}{1.2 \times 10^{-19}\,\text{GeV}} M_p$$

$$= \frac{1}{2.4} \times 10^{-22} M_p = 4.2 \times 10^{-23} M_p$$

$$\therefore \alpha_G = \frac{G}{\hbar c}(m_e)^2 = \frac{G}{\hbar c}(M_p)^2 (4.2 \times 10^{-23})^2$$

$$= (4.2 \times 10^{-23})^2$$

$$= 1.76 \times 10^{-45}$$

His method doesn't require you to know the mass of an electron as it works with the planck mass derived earlier, but is longer

$$\frac{(6.67 \cdot 10^{-11})}{6.626 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}(9.109 \cdot 10^{-31})^2 \qquad = 2.78416232 \times 10^{-46}$$

PC 4245 Particle Physics
Tutorial 1 (soln.)

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1. Gell-Mann - Nishijma formula (P 130, Griffiths Eq (4.37))

$$Q = I_3 + \frac{1}{2}(A + S)$$

Q = charge,
I_3 = 3rd component of the isospin $\underset{\sim}{I}$
A = baryon number
S = strangeness.

With the discoveries of charm, bottom and top quarks, the formula can be generalized

$$Q = I_3 + \frac{1}{2}(A + S + C + B + T)$$

C = charm
B = bottomness, T = topness

Each quark has its own flavour quantum number, which is +1 if the quark has $Q = \frac{+2}{3}$ and -1 if the quark has $Q = \frac{-1}{3}$. All other quarks have a value 0 for that quantum number.

$$\therefore \quad 2Q = A + U + D + S + C + B + T$$

(the flavour quantum number has the opposite sign for antiquark)

Note that $2I_3$ is replaced by $U + D$

U = upness D = downness

Particle physics PC4245
Tutorial 1 (solutions)

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$$m_N = \frac{m_p + m_n}{2} = \frac{938.280 + 939.573}{2}$$

$$= 938.927 \text{ MeV}/c^2$$

$$m_\Sigma = \frac{m_{\Sigma^+} + m_{\Sigma^0} + m_{\Sigma^-}}{3} = \frac{1189.4 + 1192.5 + 1197.3}{3}$$

$$= 1193.1 \text{ MeV}/c^2$$

$$m_\Xi = \frac{m_{\Xi^0} + m_{\Xi^-}}{2} = \frac{1314.9 + 1321.3}{2}$$

$$= 1318.1 \text{ MeV}/c^2$$

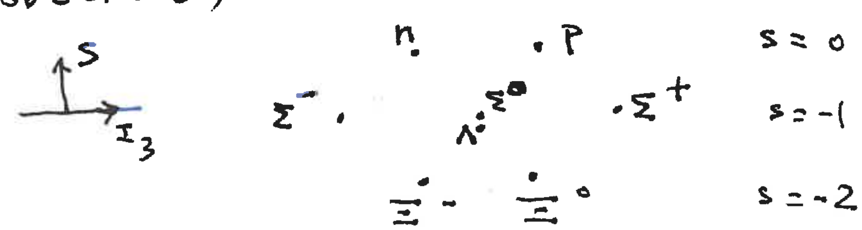
Using the mass formula, the predicted mass is

$$m_\Lambda = \frac{2(m_N + m_\Xi) - m_\Sigma}{3} = 1107.0 \text{ MeV}/c^2$$

$$m_\Lambda \text{ (observed)} = 1115.6 \text{ MeV}/c^2$$

$$\frac{m_\Lambda\text{(observed)} - m_\Lambda\text{(predicted)}}{m\text{(observed)}} = 0.8\%$$

Note: Baryon octet



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$$M_{\Delta} - M_{\Sigma^*} = 1232 - 1385 = -153\ \mathrm{MeV}/c^2$$

$$M_{\Sigma^*} - M_{\Xi^*} = 1385 - 1533 = -148\ \mathrm{MeV}/c^2$$

Average of the above two values $= -150.5$

$$\therefore\ M_{\Omega} = M_{\Xi^*} - (-150.5)$$

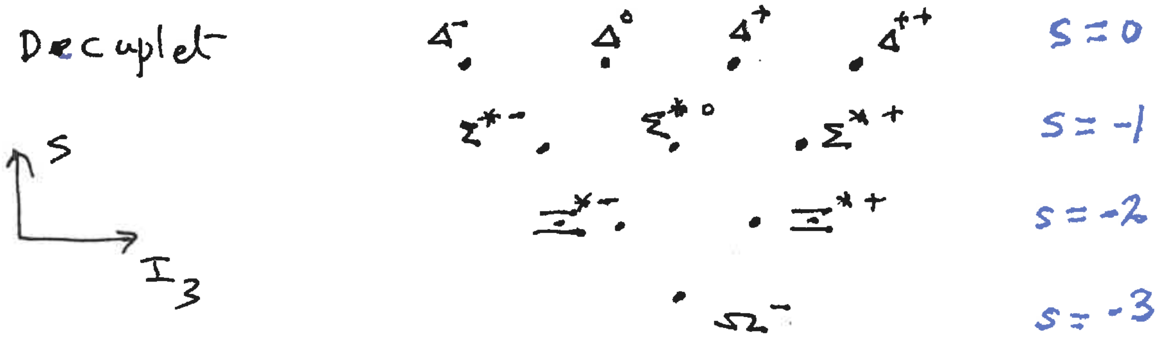
$$= 1683.5\ \mathrm{MeV}/c^2 = 1684\ \mathrm{MeV}/c^2$$

$$M_{\Omega}\ (\text{observed}) = 1672\ \mathrm{MeV}/c^2$$

$$\frac{M_{\Omega}\ (\text{predicted}) - M_{\Omega}\ (\text{observed})}{M_{\Omega}\ (\text{observed})}$$

$$= \frac{1684 - 1672}{1672} \approx 0.7\%$$

Note: Decuplet



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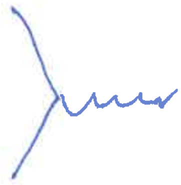
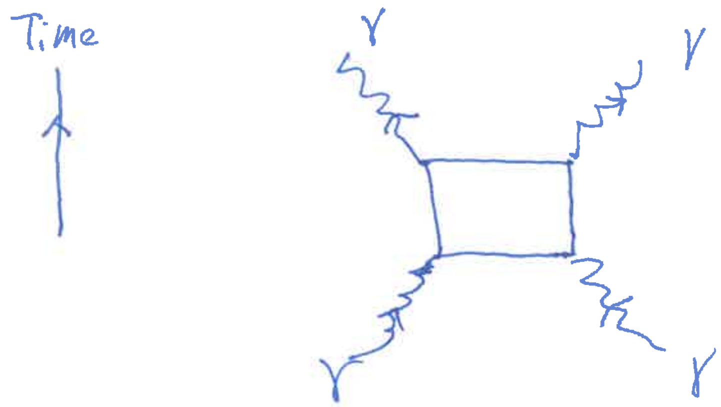
2.2
P86

Delbrück scattering $\gamma + \gamma \to \gamma + \gamma$

Using the QED vertex



the lowest order diagram is



The solid lines represent electron or positron, (virtual).

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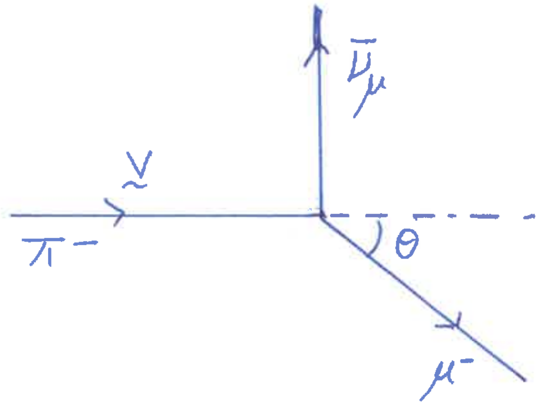
$\frac{3.15}{p.111}$



Conservation of spatial momentum:

$$|\underset{\sim}{P}_\pi| = |\underset{\sim}{P}_\mu| \cos\theta \quad \text{or} \quad P_\pi = P_\mu \cos\theta$$

$$|\underset{\sim}{P}_{\bar{\nu}_\mu}| = |\underset{\sim}{P}_\mu| \sin\theta \quad \text{or} \quad P_{\bar{\nu}_\mu} = P_\mu \sin\theta$$

Thus $\quad \tan\theta = \dfrac{P_{\bar{\nu}_\mu}}{P_\pi}$

conservation of energy $\quad E_\pi = E_{\mu^-} + E_{\bar{\nu}_\mu}$

But $\quad E_{\bar{\nu}_\mu} = |\underset{\sim}{P}_{\bar{\nu}_\mu}| c \qquad \therefore \quad \underline{P}^2_{\bar{\nu}_\mu} = \left(\dfrac{E_{\bar{\nu}_\mu}}{c}\right)^2 - |\underset{\sim}{P}_{\bar{\nu}_\mu}|^2$

$\qquad = P_{\bar{\nu}_\mu} c \qquad\qquad = 0$

$$\therefore \quad \tan\theta = \frac{E_\pi - E_{\mu^-}}{c \cdot P_\pi}$$

Conservation of 4-momentum

$$\underline{P}_\pi = \underline{P}_{\mu^-} + \underline{P}_{\bar{\nu}_\mu}$$

$$\underline{P}^2_{\bar{\nu}_\mu} = 0 = (\underline{P}_\pi - \underline{P}_{\mu^-})^2$$

$$= m_\pi^2 c^2 + m_{\mu^-}^2 c^2 - 2\, \underline{P}_\pi \cdot \underline{P}_{\mu^-}$$

$$2\underline{P}_\pi \underline{P}_{\mu^-} = 2\frac{E_\pi E_{\mu^-}}{c^2} - 2 P_\pi P_{\mu^-} \cos\theta = 2\left(\frac{E_\pi E_{\mu^-}}{c^2} - P_\pi^2\right)$$

6.

i.e.

$$\boxed{2 E_\pi E_{\mu^-} = (m_\pi^2 + m_{\mu^-}^2) c^4 + 2 P_\pi^2 c^2 .}$$

As

$$\tan\theta = \frac{E_\pi - E_{\mu^-}}{c P_\pi} = \frac{2 E_\pi^2 - 2 E_\pi E_{\mu^-}}{2 c E_\pi P_\pi}$$

$$\therefore \tan\theta = \frac{2 (E_\pi^2 - P_\pi^2 c^2) - (m_\pi^2 + m_{\mu^-}^2) c^4}{2 c E_\pi P_\pi}$$

$$= \frac{(m_\pi^2 - m_{\mu^-}^2) c^3}{2 \gamma m_\pi c^2 \gamma m_\pi V} = \frac{m_\pi^2 - m_{\mu^-}^2}{2 \gamma^2 m_\pi^2 \beta}$$

$$= \frac{1 - m_{\mu^-}^2 / m_\pi^2}{2 \beta \gamma^2}$$

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p.111

$$A + B \rightarrow C_1 + C_2 + \cdots + C_n$$

Suppose E_A is the required threshold energy of the particle A in the Lab frame (where particle B is at rest)

Total 4-momentum before collision is

$$\underline{P}_b\Big|_{Lab} = \left(\frac{E_A}{c} + m_B c,\ \underset{\sim}{p}_A\right)$$

where $\underset{\sim}{p}_A$ is the 3-momentum of A in the Lab frame.

$$\left(\underline{P}_b\Big|_{Lab}\right)^2 = \left(\frac{E_A}{c} + m_B c\right)^2 - \underset{\sim}{p}_A^2$$

$$= \left(\frac{E_A}{c} + m_B c\right)^2 - \left(\frac{E_A^2}{c^2} - m_A^2 c^2\right) = 2 E_A m_B + (m_B^2 + m_A^2) c^2 .$$

In the CM frame, if particle A energy is <u>just</u> on the threshold for production of particles $C_1, C_2, \cdots C_n$, then each of these produced particles must have zero spatial momentum.

$\therefore$ Total 4-momentum after collision in the CM frame is

$$\underline{P}_a'\Big|_{CM} = (Mc,\ 0), \qquad M = m_1 + m_2 + \cdots + m_n$$

$m_i \equiv$ rest mass of particle C_i

$$\left(\underline{P}_a'\Big|_{CM}\right)^2 = M^2 c^2 .$$

Let $\underline{P}_a\Big|_{Lab}$ be total 4-momentum after collision, then by conservation of energy-momentum,

$$\underline{P}_b\Big|_{Lab} = \underline{P}_a\Big|_{Lab}$$

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i.e. $$\left(\underline{P}_b\big|_{Lab}\right)^2 = \left(\underline{P}_a\big|_{Lab}\right)^2.$$

As $\underline{P}^2$ is an invariant i.e. it is the same in any inertial frame of reference,

$$\therefore \left(\underline{P}_a\big|_{Lab}\right)^2 = \left(\underline{P}_a'\big|_{CM}\right)^2$$

$$\therefore\ 2E_A m_B + (m_B^2 + m_A^2)c^2 = M^2 c^2$$

i.e. $$E_A = \frac{M^2 - m_B^2 - m_A^2}{2m_B} c^2$$

8. Decay $A \to B + C$, A at rest

3.19 / p112

Use conservation law: $P_A^\mu = P_B^\mu + P_C^\mu$

For energy p^0 : $m_A c = \frac{E_B}{c} + \frac{E_c}{c}$ ... (1)

m_A = rest mass of particle A

For 3-momentum p^i : $0 = \underset{\sim}{p}_B + \underset{\sim}{p}_c$ ... (2)

For a physical particle, $\left(\frac{E}{c}\right)^2 = \underset{\sim}{p}^2 + m^2 c^2$. From eq (1) and eq (2), we get

$$m_A c = \sqrt{\underset{\sim}{p}_B^2 + m_B^2 c^2} + \sqrt{\underset{\sim}{p}_c^2 + m_c^2 c^2}$$

$$\to \quad m_A^2 c^2 - 2 m_A c \sqrt{\underset{\sim}{p}_B^2 + m_B^2 c^2} + \underset{\sim}{p}_B^2 + m_B^2 c^2 = \underset{\sim}{p}_c^2 + m_c^2 c^2$$

$$\left(m_A^2 + m_B^2 - m_c^2\right) c^2 = 2 m_A c \sqrt{\underset{\sim}{p}_B^2 + m_B^2 c^2}$$

i.e.
$$4 m_A^2 \underset{\sim}{p}_B^2 = \left[(m_A^2)^2 + (m_B^2)^2 + (m_c^2)^2 - 2 m_A^2 m_B^2 - 2 m_A^2 m_c^2 - 2 m_B^2 m_c^2 \right] c^2$$

$$\to \quad |\underset{\sim}{p}_B| = \frac{\sqrt{\lambda(m_A^2, m_B^2, m_c^2)}}{2 m_A} c = |\underset{\sim}{p}_c|$$

where $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$

= triangle function

Thus part (b) of the problem is solved.

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For part (a), it is straightforward to see

$$\frac{E_B^2}{c^2} = \underset{\sim}{p}_B^2 + m_B^2 c^2 = \frac{(m_A^2 + m_B^2 - m_C^2)^2 c^2}{4 m_A^2}$$

where we have used the expression of $\underset{\sim}{p}_B^2$ from part (b).

Thus

$$E_B = \frac{(m_A^2 + m_B^2 - m_C^2)\, c^2}{2\, m_A}$$

(c) Note that $\lambda(a^2, b^2, c^2) = (a+b+c)\cdot(a+b-c)\cdot(a-b+c)\,(a-b-c)$

$$= ((a+b)^2 - c^2)\,((a-b)^2 - c^2)$$

$$= (a-b)^4 - c^2(a+b)^2 - c^2(a-b)^2 + c^4$$

$$= a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2$$

When $m_A = m_B + m_C$ i.e. $a = b + c$, $\lambda(m_A^2, m_B^2, m_C^2) = 0$

and hence $|\underset{\sim}{p}_B| = 0 = |\underset{\sim}{p}_C|$. This is because the condition $m_A = m_B + m_C$ is the threshold condition for the decay $A \to B + C$ to occur, so the particles B and C must have zero spatial momentum.

If $m_A < (m_B + m_C)$ i.e. $a < (b+c)$, $|\underset{\sim}{p}_B|$ and $|\underset{\sim}{p}_C|$ are both imaginary, so the decay $A \to B + C$ is impossible (energy is not conserved)

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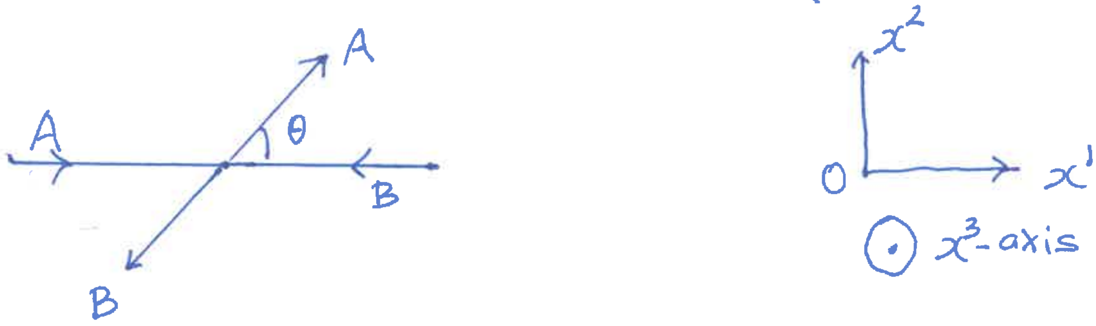
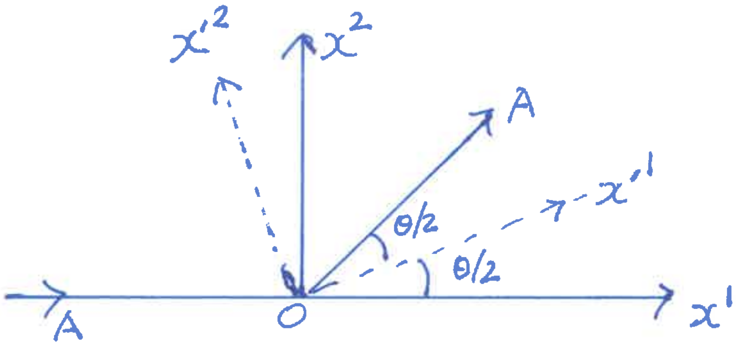
3.24 / p112

Consider elastic collision $A + B \to A + B$

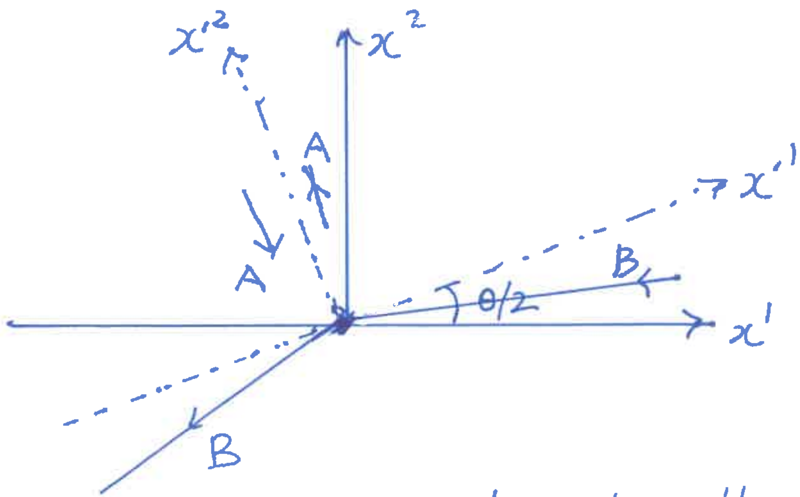
In cm frame, this collision can be depicted as



In the Breit frame, we can suppose particle A is incident along the negative x'^2-axis direction and then recoils back along the positive x'^2-axis direction:



With respect to the cm frame, the Breit frame is moving along the Ox'^1 direction so that the particle A, in the Breit frame, is incident and scattered back along the x'^2-axis. Note that x'^3-axis and x^3-axis are parallel.



The above diagram illustrates the collision as seen in the Breit frame

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Let $\underset{\sim}{P}_{A_1}$ = 4-momentum of particle A before collision. (16)

$\underset{\sim}{P}_{A_2}$ = 4-momentum of particle A after collision

since the collision is elastic, the magnitudes of the 3-momentum before and after the collision are equal and in the CM frame we have

$$|\underset{\sim}{P}_{A_1}| = |\underset{\sim}{P}_{A_2}| \equiv p$$

The speed of particle A is given by (in the CM frame)

$$\gamma_A m_A V_A = |\underset{\sim}{P}_{A_1}| = |\underset{\sim}{P}_{A_2}| = p$$

where $\gamma_A = \left[1 - \left(\frac{V_A}{c}\right)^2\right]^{-\frac{1}{2}}$

Since $E = \gamma m c^2$, we have

$$1 - \beta^2 = \left(\frac{mc^2}{E}\right)^2 \qquad \beta = \frac{v}{c}.$$

That is for particle A,

$$\left(\frac{V_A}{c}\right) = \left(1 - \frac{m_A^2 c^4}{E^2}\right)^{1/2}$$

In order for the particle A to move along the x'^2-axis in the Breit frame, we require the speed of the Breit frame along the ox'' wrt the cm frame to be equal to the component of the velocity of the particle A along the ox'' direction.

Hence, speed of the Breit frame wrt the cm frame

$$= V_A \cos\frac{\theta}{2} = c\left(1 - \frac{m_A^2 c^4}{E^2}\right)^{\frac{1}{2}} \cos\frac{\theta}{2}$$

The ox'' axis is at an angle $\frac{\theta}{2}$ wrt the ox' axis of the cm frame.

Let E' be the energy of the particle A in the Breit frame and p' be the magnitude of the momentum of the particle A in the Breit frame.

In the cm frame, we have

$$\underline{P}_{A_1} = \left(\frac{E}{c}, p, 0, 0\right)$$ = 4-momentum of particle A before collision.

$$\underline{P}_{A_2} = \left(\frac{E}{c}, p\cos\theta, p\sin\theta, 0\right)$$ = 4-momentum of particle A after collision.

In the Breit frame, the corresponding 4-momenta are

$$\underline{P}'_{A_1} = \left(\frac{E'}{c}, 0, -p', 0\right)$$

$$\underline{P}'_{A_2} = \left(\frac{E'}{c}, 0, p', 0\right)$$

3.24 p112 In the cm frame, the total 4-momentum is

$$\underline{P}_{A_1} + \underline{P}_{A_2} = \left(\frac{2E}{c},\ p(1+\cos\theta),\ p\sin\theta,\ 0 \right)$$

In the Breit frame,

$$\underline{P}'_{A_1} + \underline{P}'_{A_2} = \left(\frac{2E'}{c},\ 0,\ 0,\ 0 \right)$$

Since $(\underline{P}_{A_1} + \underline{P}_{A_2})^2$ is an invariant (value in the Breit frame same as that in the cm frame),

$$\therefore \quad \frac{4E'^2}{c^2} = \frac{4E^2}{c^2} - p^2(1+\cos\theta)^2 - p^2\sin^2\theta$$

$$= \frac{4E^2}{c^2} - 2p^2(1+\cos\theta)$$

$$E'^2 = E^2 - \frac{1}{2}p^2c^2(1+\cos\theta)$$

$$= E^2 - \frac{1}{2}c^2(1+\cos\theta)\left(\frac{E^2}{c^2} - m_A^2c^2\right)$$

$$= \frac{1}{2}\left[E^2(1-\cos\theta) + m_A^2c^4(1+\cos\theta)\right)$$

$\therefore$ Energy of the particle A in the Breit frame is

$$\frac{E}{\sqrt{2}}\left[1 - \cos\theta + \frac{m_A^2c^4}{E^2}(1+\cos\theta)\right]^{\frac{1}{2}}$$

Note : Let $\underline{P}_{B_1}$ = 4-momentum of particle B before collision

$\underline{P}_{B_2}$ = 4-momentum of particle B after collision

In the cm frame, we have

$$\underset{\sim}{p}_{A_1} + \underset{\sim}{p}_{B_1} = \underset{\sim}{p}_{A_2} + \underset{\sim}{p}_{B_2} = 0$$

But in the Breit frame,

$$\underset{\sim}{p}'_{A_1} + \underset{\sim}{p}'_{A_2} = 0$$

$$\underset{\sim}{p}'_{B_1} + \underset{\sim}{p}'_{B_2} \neq 0$$