PC3261: Classical Mechanics II

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Lecture 10: Hamiltonian Mechanics I

Legendre transformation

- Conjugate pair of variables: (u, x) and (v, y) are conjugate pairs

$$\mathrm{d}f = u\,\mathrm{d}x + v\,\mathrm{d}y$$

- **Legendre transformation** converts a function with dependence on variable(s) to another function with dependence on conjugate variable(s)

$$f(x,y) \quad \Rightarrow \quad \mathrm{d}f = \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}x + \frac{\partial f(x,y)}{\partial y}\,\mathrm{d}y \equiv u(x,y)\,\mathrm{d}x + v(x,y)\,\mathrm{d}y$$

$$f(x,y) \to g(u,y) \equiv f(x(u,y),y) - x(u,y)u$$

- Examples: thermodynamic internal energy $E(S,V)$ to Helmholtz free energy $F(T,V)$, internal energy $E(S,V)$ to Gibbs free energy $G(T,P)$, internal energy to enthalpy $H(S,P)$, etc

EXERCISE 10.1: Starting from $g = g(u,y)$, perform a Legendre transformation to another function $h = h(x,v)$.

$$\mathrm{d}f = u(x,y)\,\mathrm{d}x + v(x,y)\,\mathrm{d}y$$

$$u = u(x,y) = \left(\frac{\partial f}{\partial x}\right)_y \quad \Rightarrow \quad x = x(u,y)$$

$$g(u,y) \equiv f(x(u,y),y) - x(u,y)u$$

$$\mathrm{d}g = \mathrm{d}f - x\,\mathrm{d}u - u\,\mathrm{d}x$$

$$\Rightarrow \quad \mathrm{d}g = (u\,\mathrm{d}x + v\,\mathrm{d}y) - x\,\mathrm{d}u - u\,\mathrm{d}x$$

$$\Rightarrow \quad \mathrm{d}g = -x\,\mathrm{d}u + v\,\mathrm{d}y$$

$$\Rightarrow \quad g(u,y) \qquad \blacksquare$$

$$\mathrm{d}f = u\,\mathrm{d}x + v\,\mathrm{d}y \equiv \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$$

$$g = g(u,y) \equiv f(x(u,y)) - x(u,y)u \quad \Rightarrow \quad \mathrm{d}g = -x\,\mathrm{d}u + v\,\mathrm{d}y$$

$$\begin{cases} u = u(x,y) = \left(\dfrac{\partial f}{\partial x}\right)_y \\[2ex] v = v(x,y) = \left(\dfrac{\partial f}{\partial y}\right)_x \end{cases} \quad \Rightarrow \quad \begin{cases} u = u(x,v) \\ y = y(x,v) \end{cases}$$

$$h = h(x,v) \equiv g(u(x,v), y(x,v)) + u(x,v)x - y(x,v)v$$

$$\Rightarrow \quad \mathrm{d}h = (-x\,\mathrm{d}u + v\,\mathrm{d}y) + (u\,\mathrm{d}x + x\,\mathrm{d}u) - (y\,\mathrm{d}v + v\,\mathrm{d}y) = u\,\mathrm{d}x - y\,\mathrm{d}v \qquad \blacksquare$$

$$h = h(x,v) \equiv -u(x,v)x + y(x,v)v - g(u(x,v), y(x,v))$$

$$\Rightarrow \quad \mathrm{d}h = -(u\,\mathrm{d}x + x\,\mathrm{d}u) + (y\,\mathrm{d}v + v\,\mathrm{d}y) - (-x\,\mathrm{d}u + v\,\mathrm{d}y) = -u\,\mathrm{d}x + y\,\mathrm{d}v \qquad \blacksquare$$

Hamiltonian function

- Lagrangian function:

$$\mathcal{L} \equiv \mathcal{L}\left(\{q_i(t), \dot{q}_i(t)\}, t\right) \quad \Rightarrow \quad \mathrm{d}\mathcal{L} = \sum_{i=1}^{M} \left(\frac{\partial \mathcal{L}}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,\mathrm{d}\dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t$$

- **Hamiltonian function** (or **Hamiltonian**) has explicit dependences on generalized coordinates q_i, generalized momenta p_i and time t

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}\left(\{q_i(t), p_i(t)\}, t\right) \\ &\equiv \sum_{i=1}^{M} \dot{q}_i\left(\{q_k(t), p_k(t)\}, t\right)\, p_i(t) - \mathcal{L}\left(\{q_i(t), \dot{q}_i\left(\{q_k(t), p_k(t)\}, t\right)\}, t\right) \end{aligned}$$

- Be cautious on the flip of signs in the Legendre transformation from Lagrangian to Hamiltonian!

Hamiltonian function – cont'd

- Generalized momenta:

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \Rightarrow \quad \dot{q}_i = \dot{q}_i\left(\{q_k(t), p_k(t)\}, t\right)$$

- A couple of extra steps (not necessarily trivial) in the construction of Hamiltonian from the Lagrangian are to write down the generalized momenta p_i and solve for the generalized velocities in terms of generalized coordinates, generalized momenta and time, $\dot{q}_i(\{q_k, p_k\}, t)$

- The set of generalized coordinates and generalized momenta, $\{q_k(t), p_k(t)\}$, used in Hamiltonian mechanics is generally known as **canonical coordinates**

EXERCISE 10.2: Construct the Hamiltonian for a particle of mass m subjected to a conservative central force field with potential energy $U(r)$ using usual polar coordinates r and ϕ as generalized coordinates.

$$\mathcal{L} \equiv \mathcal{L}(r, \phi, \dot{r}, \dot{\phi}, t) = \frac{1}{2}\, m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) - U(r)$$

$$\begin{cases} p_r = \dfrac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \\[2ex] p_\phi = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2 \dot{\phi} \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{r} = \dfrac{p_r}{m} \\[2ex] \dot{\phi} = \dfrac{p_\phi}{mr^2} \end{cases} \qquad \blacksquare$$

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}(r, \phi, p_r, p_\phi, t) \\ &\equiv \sum_i p_i\, \dot{q}_i(r, \phi, p_r, p_\phi, t) - \mathcal{L}(r, \phi, \dot{r}(r, \phi, p_r, p_\phi, t), \dot{\phi}(r, \phi, p_r, p_\phi, t), t) \\ &= p_r \dot{r} + p_\phi \dot{\phi} - \left[\frac{1}{2}\, m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) - U(r)\right] \\ &= \frac{1}{2m} \left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + U(r) \qquad \blacksquare \end{aligned}$$

Hamilton equations of motion

- **Hamilton equations of motion** (or **canonical equations of motion**):

$$\begin{cases} \dot{q}_i = \dfrac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial \mathcal{H}}{\partial q_i} \end{cases}, \qquad i = 1, 2, \cdots, M\,, \qquad \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t}$$

- Hamiltonian approach gives $(2M+1)$ first-order differential equations instead of the M second-order differential equations in the Lagrangian approach

- Solution of Hamiltonian equations of motion is represented by a curve parameterized by t in the $2M$-dimensional space known as **cotangent bundle** $\mathbf{T}^*\mathbb{Q}$

$$(q_1(t), q_2(t), \cdots, q_M(t), p_1(t), p_2(t), \cdots, p_M(t))$$

EXERCISE 10.3: Derive Hamilton equations of motion.

$$\mathcal{H} \equiv \mathcal{H}\left(\{q_i(t), p_i(t)\}, t\right) \quad \Rightarrow \quad \mathrm{d}\mathcal{H} = \sum_{i=1}^{M} \left(\frac{\partial \mathcal{H}}{\partial q_i}\, \mathrm{d}q_i + \frac{\partial \mathcal{H}}{\partial p_i}\, \mathrm{d}p_i \right) + \frac{\partial \mathcal{H}}{\partial t}\, \mathrm{d}t$$

$$\mathcal{H} \equiv \sum_{i=1}^{M} \dot{q}_i\, p_i - \mathcal{L}\left(\{q_i(t), \dot{q}_i(t)\}, t\right)$$

$$\Rightarrow \quad \mathrm{d}\mathcal{H} = \sum_{i=1}^{M} \left(\dot{q}_i\, \mathrm{d}p_i + p_i\, \mathrm{d}\dot{q}_i\right) - \left[\sum_{i=1}^{M} \left(\frac{\partial \mathcal{L}}{\partial q_i}\, \mathrm{d}q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\, \mathrm{d}\dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}\, \mathrm{d}t \right]$$

$$= \sum_{i=1}^{M} \left(\dot{q}_i\, \mathrm{d}p_i - \dot{p}_i\, \mathrm{d}q_i\right) - \frac{\partial \mathcal{L}}{\partial t}\, \mathrm{d}t$$

$$\sum_{i=1}^{M} \left(\frac{\partial \mathcal{H}}{\partial q_i}\, \mathrm{d}q_i + \frac{\partial \mathcal{H}}{\partial p_i}\, \mathrm{d}p_i \right) + \frac{\partial \mathcal{H}}{\partial t}\, \mathrm{d}t = \sum_{i=1}^{M} \left(\dot{q}_i\, \mathrm{d}p_i - \dot{p}_i\, \mathrm{d}q_i\right) - \frac{\partial \mathcal{L}}{\partial t}\, \mathrm{d}t$$

$$\Rightarrow \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}\,, \qquad \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t} \qquad \blacksquare$$