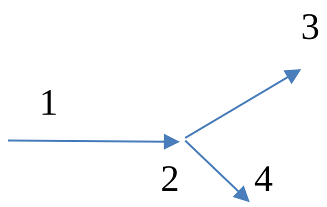
PC4245 Tutorial 3 Solution

1. This question is from the D J Griffiths, Introduction to Elementary Particles, 2nd Edition, Problem 6.8, page 223



$m_1 = m_{A,} \quad m_2 = m_B, m_3 = m_A \, , m_4 = m_B$

The differential cross section formula is

$$d\sigma = S|\mathcal{M}|^2 \frac{\hbar^2}{4} \left[(\underline{p}_1 . \underline{p}_2)^2 - (m_1 m_2 c^2)^2\right]^{-1/2} \frac{d^3 \underset{\sim}{p}_3}{(2\pi)^3 2p_3^0} \frac{d^3 \underset{\sim}{p}_4}{(2\pi)^3 2p_4^0} (2\pi)^4 \delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4)$$

In the lab frame (particle 2 at rest), see part (b) of Q2

$$\sqrt{(\underline{p}_1 \cdot \underline{p}_2)^2 - (m_1 m_2 c^2)^2} = m_2 \left|\underset{\sim}{p}_1\right| c$$

Integrating $\int d^3 \underset{\sim}{p}_4$ and writing $d^3 \underset{\sim}{p}_3 = \left|\underset{\sim}{p}_3\right|^2 \cdot d\left|\underset{\sim}{p}_3\right| \cdot d\Omega_3$,

$$\therefore \frac{d\sigma}{d\Omega} = \frac{\hbar^2}{(8\pi)^2 m_2 \left|\underset{\sim}{p}_1\right| c} \int \frac{\left|\underset{\sim}{p}_3\right|^2 \cdot d\left|\underset{\sim}{p}_3\right|}{p_3{}^0 \, p_4{}^0} |\mathcal{M}|^2 \delta(p_1^0 + p_2^0 - p_3^0 - p_4^0)$$

where $\underset{\sim}{p}_4 = \underset{\sim}{p}_1 - \underset{\sim}{p}_3 \quad \because \quad \underset{\sim}{p}_2 = 0$.

Assume recoil of particle 2 (*ie* particle B) is negligible,

$$\therefore \quad \underset{\sim}{p}_4 \approx 0 \quad \rightarrow \quad \underset{\sim}{p}_1 \approx \underset{\sim}{p}_3$$

Note: As $\underset{\sim}{p}_4 \approx 0, \quad \therefore \quad p_4^0 \approx m_4 c$. That is, $p_2^0 \approx p_4^0$, or $p_2^0 - p_4^0 \approx 0$.

Thus

$$\frac{d\sigma}{d\Omega} = \frac{\hbar^2}{(8\pi)^2 m_2 \left|\underset{\sim}{p}_1\right| c \, m_4 c} \int \frac{\left|\underset{\sim}{p}_3\right|^2 \cdot d\left|\underset{\sim}{p}_3\right|}{p_3^0} |\mathcal{M}|^2 \delta(p_1^0 - p_3^0)$$

with $\underset{\sim}{p}_4 = \underset{\sim}{p}_1 - \underset{\sim}{p}_3 \approx 0$ or $\underset{\sim}{p}_1 \approx \underset{\sim}{p}_3$.

As ${p_3^0}^2 = {\underset{\sim}{p_3}}^2 + m_3^2 c^2$, $\therefore\ dp_3^0 = \dfrac{|\underset{\sim}{p_3}| \cdot d|\underset{\sim}{p_3}|}{p_3^0}$.

$$\therefore\ \frac{d\sigma}{d\Omega} = \frac{\hbar^2}{(8\pi)^2 (m_2 c)^2 \left|\underset{\sim}{p_1}\right|} \int |\underset{\sim}{p_3}| \cdot \mathrm{d}p_3^0 \cdot |\mathcal{M}|^2 \ \delta(p_1^0 - p_3^0)\,, \qquad \because m_4 = m_2$$

$$= \left(\frac{\hbar}{8\pi\, m_2 c}\right)^2 \frac{|\underset{\sim}{p_3}|}{\left|\underset{\sim}{p_1}\right|} \cdot |\mathcal{M}|^2 \qquad \text{with } p_3^0 = p_1^0\,.$$

Now ${\underset{\sim}{p_3}}^2 = {p_3^0}^2 - m_3^2 c^2 = {p_1^0}^2 - m_3^2 c^2 = {\underset{\sim}{p_1}}^2 \quad \because m_3 = m_1$

$$\therefore\ \frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi\, m_2 c}\right)^2 \cdot |\mathcal{M}|^2 \qquad \text{with } p_3^0 = p_1^0$$

2. This question is from the D J Griffiths, Introduction to Elementary Particles, 2nd Edition, Problem 6.9, page 223

$$1 + 2 \rightarrow 3 + 4$$

Particle 2 at rest in the lab frame, particles 3 and 4 massless. We have

$$d\sigma = |\mathcal{M}|^2 \frac{S\hbar^2}{4} \left[(\underline{p}_1 \cdot \underline{p}_2)^2 - (m_1 m_2 c^2)^2\right]^{-1/2} \frac{d^3 \underset{\sim}{p_3}}{(2\pi)^3 2p_3^0} \frac{d^3 \underset{\sim}{p_4}}{(2\pi)^3 2p_4^0} (2\pi)^4 \delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4)$$

In the lab frame (particle 2 at rest),

$$\sqrt{(\underline{p}_1 \cdot \underline{p}_2)^2 - (m_1 m_2 c^2)^2} = m_2 \left|\underset{\sim}{p_1}\right| c \longrightarrow$$

Note: Given $\underset{\sim}{p}_2 = 0$,

$$\begin{aligned}
&\sqrt{(\underline{p}_1 \cdot \underline{p}_2)^2 - (m_1 m_2 c^2)^2} \\
&= \sqrt{(p_1^0 p_2^0)^2 - (\underset{\sim}{p}_1 \cdot \underset{\sim}{p}_2)^2 - (m_1 m_2 c^2)^2} \\
&= \sqrt{\left(\frac{E_1 E_2}{c^2}\right)^2 - \left(|\underset{\sim}{p}_1||\underset{\sim}{p}_2| \cos\theta\right)^2 - (m_1 m_2 c^2)^2} \\
&= \sqrt{\left(\frac{E_1 E_2}{c^2}\right)^2 - (m_1 m_2 c^2)^2} \because E_2 = m_2 c^2 \\
&= \sqrt{\left(\frac{E_1 E_2}{c^2} + m_1 E_2\right)\left(\frac{E_1 E_2}{c^2} - m_1 E_2\right)} \\
&= E_2 \sqrt{\left(\frac{E_1 + m_1 c^2}{c^2}\right)\left(\frac{E_1 - m_1 c^2}{c^2}\right)} \\
&= \frac{E_2}{c^2} \sqrt{\left({E_1}^2 - (m_1 c^2)^2\right)} \because E^2 - |\underset{\sim}{p}|^2 c^2 = m^2 c^4 \\
&= \frac{(m_2 c^2)}{c^2} |\underset{\sim}{p}_1| c \\
&= |\underset{\sim}{p}_1| m_2 c
\end{aligned}$$

hence

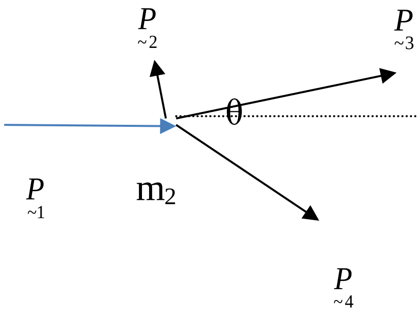
$$d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{m_2 \left|\underset{\sim}{p}_1\right| c} \cdot \frac{d^3 \underset{\sim}{p_3}}{p_3^0} \cdot \frac{d^3 \underset{\sim}{p_4}}{p_4^0} \delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4)$$

The differential cross section for observing particle 3 at the angle $(\theta\,, \phi)$ is

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{m_2 \left|\underset{\sim}{p_1}\right| c} \int \frac{|\underset{\sim}{p_3}|^2 d|\underset{\sim}{p_3}|\, d^3 \underset{\sim}{p_4}}{p_3^0 \cdot p_4^0} |\mathcal{M}|^2 \delta(p_1^0 + p_2^0 - p_3^0 - p_4^0) \cdot \delta^{(3)}(\underset{\sim}{p_1} - \underset{\sim}{p_3} - \underset{\sim}{p_4})$$

$$= \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{m_2 c \left|\underset{\sim}{p}_1\right|} \int \frac{|\underset{\sim}{p}_3|^2 d|\underset{\sim}{p}_3|}{\left|\underset{\sim}{p}_1 - \underset{\sim}{p}_3\right|} |\mathcal{M}|^2 \delta(p_1^0 + p_2^0 - \left|\underset{\sim}{p}_3\right| - \left|\underset{\sim}{p}_1 - \underset{\sim}{p}_3\right|)$$

$\because$ for massless particles, $p_3^0 = \left|\underset{\sim}{p}_3\right|, p_4^0 = \left|\underset{\sim}{p}_4\right|$, and $\underset{\sim}{p}_4 = \underset{\sim}{p}_1 - \underset{\sim}{p}_3$.



From the above figure

$$(\underset{\sim}{p}_1 - \underset{\sim}{p}_3)^2 = \underset{\sim}{p}_1{}^2 + \underset{\sim}{p}_3{}^2 - 2|\underset{\sim}{p}_1||\underset{\sim}{p}_3| \cos\theta$$

Define $p^0 = \left|\underset{\sim}{p}_3\right| + \left|\underset{\sim}{p}_1 - \underset{\sim}{p}_3\right|$

$$dp^0 = d\left|\underset{\sim}{p}_3\right| + \frac{\left(|\underset{\sim}{p}_3| - |\underset{\sim}{p}_1|\cos\theta\right) d|\underset{\sim}{p}_3|}{\sqrt{\underset{\sim}{p}_1{}^2 + \underset{\sim}{p}_3{}^2 - 2|\underset{\sim}{p}_1||\underset{\sim}{p}_3| \cos\theta}}$$

$$= d\left|\underset{\sim}{p}_3\right| \cdot \left(\frac{\left|\underset{\sim}{p}_1 - \underset{\sim}{p}_3\right| + |\underset{\sim}{p}_3| - |\underset{\sim}{p}_1|\cos\theta}{\left|\underset{\sim}{p}_1 - \underset{\sim}{p}_3\right|}\right)$$

$$\therefore \quad \frac{d\left|\underset{\sim}{p}_3\right|}{\left|\underset{\sim}{p}_1 - \underset{\sim}{p}_3\right|} = \frac{dp^0}{(p^0 - |\underset{\sim}{p}_1|\cos\theta)}$$

Thus

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{m_2 c \left|\underset{\sim}{p}_1\right|} \int \frac{|\underset{\sim}{p}_3| d|\underset{\sim}{p}_3|}{\left|\underset{\sim}{p}_1 - \underset{\sim}{p}_3\right|} |\mathcal{M}|^2 \delta(p_1^0 + p_2^0 - |\underset{\sim}{p}_3| - \left|\underset{\sim}{p}_1 - \underset{\sim}{p}_3\right|)$$

$$= \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{m_2 c \left|\underset{\sim}{p_1}\right|} \int \frac{|\underset{\sim}{p_3}|\, dp^0}{(p^0 - \ |\underset{\sim}{p_1}|\cos\theta\,)} \ \ |\mathcal{M}|^2 \delta(p_1^0 + p_2^0 - p^0)$$

$$= \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{m_2 c \left|\underset{\sim}{p_1}\right|} \ \ \frac{|\underset{\sim}{p_3}|\,|\mathcal{M}|^2}{(p_1^0 + p_2^0 - |\underset{\sim}{p_1}|\cos\theta\,)}$$

where $p_2^0 = m_2 c$, and $|\underset{\sim}{p_3}|$ is given by

$$p^0 = p_1^0 + p_2^0 = \left|\underset{\sim}{p_3}\right| - \left|\underset{\sim}{p_1} - \underset{\sim}{p_3}\right| = \left|\underset{\sim}{p_3}\right| + \sqrt{\underset{\sim}{p_1}^2 + \underset{\sim}{p_3}^2 - 2\,|\underset{\sim}{p_1}|\,|\underset{\sim}{p_3}|\,\cos\theta} \ ,$$

$$(\underset{\sim}{p_1} - \underset{\sim}{p_3})^2 = \underset{\sim}{p_1}^2 + \underset{\sim}{p_3}^2 - 2\,|\underset{\sim}{p_1}|\,|\underset{\sim}{p_3}|\,\cos\theta$$

We have

$$\underset{\sim}{p_1}^2 + \underset{\sim}{p_3}^2 - 2\,|\underset{\sim}{p_1}|\,|\underset{\sim}{p_3}|\,\cos\theta = \underset{\sim}{p_3}^2 + (p_1^0 + p_2^0)^2 \ - 2\ |\underset{\sim}{p_3}|\,(p_1^0 + p_2^0)$$

$$\therefore \ \ 2\,|\underset{\sim}{p_3}|\,\Big(\,|\underset{\sim}{p_1}|\,\cos\theta - (p_1^0 + p_2^0)\Big) = \ \underset{\sim}{p_1}^2 - (p_1^0 + p_2^0)^2$$

$$\therefore \ |\underset{\sim}{p_3}| \ = \frac{\underset{\sim}{p_1}^2 - (p_1^0 + p_2^0)^2}{2\Big(\,|\underset{\sim}{p_1}|\,\cos\theta - (p_1^0 + p_2^0)\Big)}$$

3. This question is from the D J Griffiths, Introduction to Elementary Particles, 2nd Edition, Problem 6.10, page 223

(a) $1 + 2 \rightarrow 3 + 4$

Elastic scattering. Particle 2 at rest in the lab frame, and $m_3 = m_1, m_4 = m_2$.

We have

$$d\sigma = |\mathcal{M}|^2 \frac{S\hbar^2}{4}\left[(\underline{p}_1 . \underline{p}_2)^2 - (m_1 m_2 c^2)^2\right]^{-1/2} \frac{d^3\underset{\sim}{p_3}}{(2\pi)^3 2{p_3}^0} \frac{d^3\underset{\sim}{p_4}}{(2\pi)^3 2p_4^0} (2\pi)^4 \delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4)$$

In the lab frame (particle 2 at rest),

$$\sqrt{(\underline{p}_1 \cdot \underline{p}_2)^2 - (m_1 m_2 c^2)^2} = m_2 \left|\underset{\sim}{p_1}\right| c$$

hence

$$d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{m_2 \left|\underset{\sim 1}{p}\right| c} \cdot \frac{d^3 \underset{\sim}{p_3}}{p_3^0} \cdot \frac{d^3 \underset{\sim}{p_4}}{p_4^0} \delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4)$$

The differential cross section for observing particle 3 at the angle $(\theta\ ,\phi)$ is

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{m_2 \left|\underset{\sim}{p_1}\right| c} \int \frac{|\underset{\sim}{p_3}|^2 d|\underset{\sim}{p_3}|\, d^3\underset{\sim}{p_4}}{p_3^0 \cdot p_4^0} |\mathcal{M}|^2 \delta(p_1^0 + p_2^0 - p_3^0 - p_4^0) \cdot \delta^{(3)}(\underset{\sim}{p_1} - \underset{\sim}{p_3} - \underset{\sim}{p_4})$$

$$= \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{m_2 c \left|\underset{\sim}{p_1}\right|} \int \frac{|\underset{\sim}{p_3}|^2 d|\underset{\sim}{p_3}|}{\sqrt{\underset{\sim 3}{p}^{\ 2} + m_3^2 c^2} \cdot \sqrt{(\underset{\sim}{p_1} - \underset{\sim}{p_3})^2 + m_4^2 c^2}} |\mathcal{M}|^2\, \delta(p_1^0 + p_2^0 - p_3^0 - p_4^0)$$

Define

$$p^0 = p_3^0 + p_4^0 = \sqrt{(\underset{\sim}{p_3}^2 + m_3^2 c^2)} + \sqrt{(\underset{\sim}{p_1} - \underset{\sim}{p_3})^2 + m_4^2 c^2}\ ,\quad \text{and } \varrho = |\underset{\sim 3}{p}|$$

$$dp^0 = \frac{\varrho d\varrho}{\sqrt{\varrho^2 + m_3^2 c^2}} + \frac{d\varrho\, (\varrho - |\underset{\sim}{p_1}| \cos\theta)}{\sqrt{(\underset{\sim}{p_1} - \underset{\sim}{p_3})^2 + m_4^2 c^2}}\ ,$$

$$\because (\underset{\sim}{p_1} - \underset{\sim}{p_3})^2 = \underset{\sim}{p_1}^2 + \underset{\sim}{p_3}^2 - 2|\underset{\sim}{p_1}|\,|\underset{\sim}{p_3}| \cos\theta$$

Taking out the common differential $d\varrho$, we get

$$dp^0 = d\varrho \cdot \frac{\varrho p^0 - |\underset{\sim}{p_1}| \cos\theta \cdot \sqrt{\varrho^2 + m_3^2 c^2}}{\sqrt{\varrho^2 + m_3^2 c^2} \cdot \sqrt{(\underset{\sim}{p_1} - \underset{\sim}{p_3})^2 + m_4^2 c^2}}$$

since

$$\varrho \sqrt{(\underset{\sim}{p_1} - \underset{\sim}{p_3})^2 + m_4^2 c^2} + \sqrt{\varrho^2 + m_3^2 c^2} \left(\varrho - |\underset{\sim}{p_1}| \cos\theta\right) = \varrho p^0 - |\underset{\sim}{p_1}| \cos\theta \cdot \sqrt{\varrho^2 + m_3^2 c^2}\ .$$

Hence

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S}{m_2 c\left|\underset{\sim}{p_1}\right|} \int \frac{\varrho^2\, dp^0}{\left(\varrho\, p^0 - \left|\underset{\sim 1}{p}\right| \cos\theta \sqrt{\varrho^2 + m_3^2 c^2}\right)} |\mathcal{M}|^2\, \delta(p_1^0 + p_2^0 - p^0)$$

$$= \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{m_2 c\left|\underset{\sim}{p_1}\right|} \cdot \frac{\left|\underset{\sim}{p_3}\right|^2}{\left|\underset{\sim}{p_3}\right| (p_1^0 + m_2 c) - \left|\underset{\sim}{p_1}\right| \cos\theta \cdot p_3^0}$$ ✓

where

$p_3^{0\,2} = \underset{\sim}{p_3}^2 + m_3^2 c^2$ and $\left|\underset{\sim}{p_3}\right|$ is given by

$$p_1^0 + p_2^0 = p_3^0 + p_4^0 = \sqrt{(\underset{\sim}{p_3}^2 + m_3^2 c^2)} + \sqrt{(\underset{\sim}{p_1} - \underset{\sim}{p_3})^2 + m_4^2 c^2}$$

Conservation of energy

and $p_2^0 = m_2 c$.

Squaring LHS:

$$(p_1^0 + m_2 c)^2 = \underset{\sim}{p_3}^2 + m_3^2 c^2 + \underset{\sim}{p_1}^2 + \underset{\sim}{p_3}^2 - 2\left|\underset{\sim}{p_1}\right| \left|\underset{\sim}{p_3}\right| \cos\theta + m_4^2 c^2 +$$

$$2\sqrt{(\underset{\sim}{p_3}^2 + m_3^2 c^2)} \cdot \sqrt{\underset{\sim}{p_1}^2 + \underset{\sim}{p_3}^2 - 2\left|\underset{\sim}{p_1}\right| \left|\underset{\sim}{p_3}\right| \cos\theta + m_4^2 c^2}$$

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$$(p_1^0 + m_2 c)^2 - 2\underset{\sim}{p_3}^2 - \underset{\sim}{p_1}^2 - (m_3^2 + m_4^2)c^2 + 2\left|\underset{\sim}{p_1}\right| \left|\underset{\sim}{p_3}\right| \cos\theta$$

$$= 4(\underset{\sim}{p_3}^2 + m_3^2 c^2)(\underset{\sim}{p_1}^2 + \underset{\sim}{p_3}^2 - 2\left|\underset{\sim}{p_1}\right| \left|\underset{\sim}{p_3}\right| \cos\theta + m_4^2 c^2)$$

If $m_3 = m_1, m_4 = m_2$, then

$$\left(2p_1^0 m_2 c - 2\underset{\sim}{p_3}^2 + 2\left|\underset{\sim}{p_1}\right| \cdot \left|\underset{\sim}{p_3}\right| \cos\theta\right)^2$$

$$= 4(\underset{\sim}{p_3}^2 + m_3^2 c^2)(\underset{\sim}{p_1}^2 + \underset{\sim}{p_3}^2 - 2\left|\underset{\sim}{p_1}\right| \cdot \left|\underset{\sim}{p_3}\right| \cos\theta + m_4^2 c^2)$$

or

$$p_1^{0\,2}(m_2c)^2+\underset{\sim}{p_1}^2\cdot\underset{\sim}{p_3}^2\cdot(\cos\theta)^2-2p_1^0\,m_2c\cdot\ \underset{\sim}{p_3}^2+$$

$$2p_1^0\,m_2c\left|\underset{\sim}{p_1}\right|\cdot\left|\underset{\sim}{p_3}\right|\cos\theta-2\underset{\sim}{p_3}^2\cdot\ \left|\underset{\sim}{p_1}\right|\cdot\left|\underset{\sim}{p_3}\right|\cos\theta$$

$$=\underset{\sim}{p_1}^2\cdot\underset{\sim}{p_3}^2+\underset{\sim}{p_3}^2\left(m_2^2c^2-2\ \left|\underset{\sim}{p_1}\right|\cdot\left|\underset{\sim}{p_3}\right|\cos\theta\right)+m_1^2c^2(\underset{\sim}{p_1}^2+\underset{\sim}{p_3}^2-$$
$$2\ \left|\underset{\sim}{p_1}\right|\cdot\left|\underset{\sim}{p_3}\right|\cos\theta+\ m_2^2c^2)$$

or

$$\underset{\sim}{p_1}^2c^2\,(m_2^2-m_1^2)-\ 2p_1^0\,m_2\cdot\underset{\sim}{p_3}^2+2p_1^0\,m_2\ \left|\underset{\sim}{p_1}\right|\cdot\left|\underset{\sim}{p_3}\right|\cos\theta$$

$$=\underset{\sim}{p_3}^2\underset{\sim}{p_1}^2\ \sin^2\theta+\underset{\sim}{p_3}^2\,(m_2^2+m_1^2)c^2-2m_1^2c^2\ \ \left|\underset{\sim}{p_1}\right|\cdot\left|\underset{\sim}{p_3}\right|\cos\theta$$

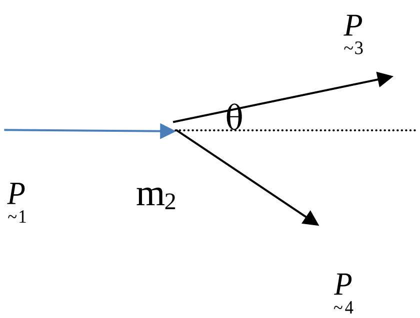
that is,

$$\underset{\sim}{p_3}^2\left(2p_1^0\,m_2+\underset{\sim}{p_1}^2\sin^2\theta+\ (m_2^2+m_1^2)c^2-\ 2\ \left|\underset{\sim}{p_1}\right|\cdot\left|\underset{\sim}{p_3}\right|\cos\theta\right)$$

$$=\underset{\sim}{p_1}^2c^2\,(m_2^2-m_1^2)\ .$$

$\left|\underset{\sim}{p_3}\right|$ can then be obtained by solving the above quadratic equation.

(b)



For massless particles

$m_1 = 0 = m_3,\ p_1^0 = |\underset{\sim}{p_1}|, p_3^0 = |\underset{\sim}{p_3}|.$

We have

$$|\underset{\sim}{p_3}|\,(p_1^0 + m_2 c) - |\underset{\sim}{p_1}|\cos\theta \cdot p_3^0 = p_3^0(p_1^0 + m_2 c) - p_1^0 \cos\theta \cdot p_3^0$$

$$= p_1^0 p_3^0 (1 - \cos\theta\,) + p_3^0 m_2 c$$

Conservation of energy $\longrightarrow$

$$p_1^0 + p_2^0 - p_3^0 = p_4^0 = \sqrt{(\underset{\sim}{p_4}{}^2 + m_4^2 c^2)} = \sqrt{(\underset{\sim}{p_1} - \underset{\sim}{p_3})^2 + m_4^2 c^2}$$

$$= \sqrt{\underset{\sim}{p_1}{}^2 + \underset{\sim}{p_3}{}^2 - 2\,|\underset{\sim}{p_1}|\,|\underset{\sim}{p_3}|\cos\theta}$$

That is,

$$(p_1^0 + m_2 c - p_3^0)^2 = {p_1^0}^2 + {p_3^0}^2 - 2p_1^0\,p_3^0 \cos\theta + m_2^2 c^2$$

or

$$2p_1^0\,p_3^0\,(-1 + \cos\theta) + 2p_1^0 m_2 c - 2p_3^0 m_2 c = 0$$

Thus

$$|\underset{\sim}{p_3}|\,(p_1^0 + m_2 c) - |\underset{\sim}{p_1}|\cos\theta \cdot p_3^0 = p_1^0\,p_3^0\,(1 - \cos\theta) + p_3^0 m_2 c = p_1^0 m_2 c\ .$$

$$\therefore\ \frac{d\sigma}{d\Omega} = S\left(\frac{\hbar\ p_3^0}{8\pi\, m_2\, c\, p_1^0}\right)^2 |M|^2$$