PC3261: Classical Mechanics II

Kenneth HONG Chong Ming

Office: S16-07-06
Email: phyhcmk@nus.edu.sg

Semester I, 2023/24

Latest update: October 26, 2023 8:56pm

NUS National University of Singapore
Department of Physics
Faculty of Science

Lecture 11: Hamiltonian Mechanics II

Cyclic coordinates (revisited)

- Generalized momenta associated to the cyclic coordinate is constant:

$$\frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad \leftrightarrow \quad \frac{\partial \mathcal{H}}{\partial q_k} = 0 \quad \Rightarrow \quad \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k} = 0$$

- Lagrangian approach: q_2 is cyclic (p_2 is a constant of motion) but it is not necessarily true that $\dot{q}_2$ is constant; Lagrangian framework does not reduce *cleanly* to a problem with one less degrees of freedom

$$\frac{\partial \mathcal{L}}{\partial q_2} = 0 \quad \Rightarrow \quad \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2, t) \rightarrow \mathcal{L}(q_1, \dot{q}_1, \dot{q}_2, t)$$

- Hamiltonian approach: q_2 is cyclic and thus p_2 is a constant of motion; Hamiltonian framework is exactly equivalent to a problem with one less degrees of freedom!

$$\frac{\partial \mathcal{H}}{\partial q_2} = 0 \quad \Rightarrow \quad \mathcal{H}(q_1, q_2, p_1, p_2, t) \rightarrow \mathcal{H}(q_1, p_1, t)$$

Example: Spherical Pendulum

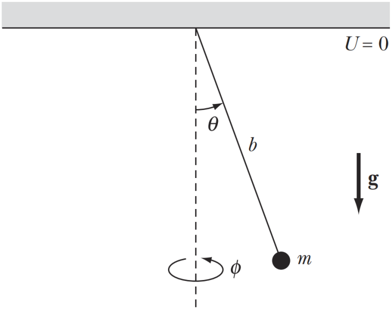
- A spherical pendulum consists of a bob of mass m moving on a sphere centered on the point of support with radius $r = b$, the length of the pendulum

$$\mathcal{H} \equiv \mathcal{H}(\theta, \phi, p_\theta, p_\phi)$$

$$= \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2 \sin^2\theta} - mgb\cos\theta$$

- ϕ is an ignorable coordinate

- Two constants of motion: mechanical energy and angular momentum about the z-axis



EXERCISE 11.1: Obtain equations of motion for the spherical pendulum.

Poisson brackets

- Total time derivative of any dynamical function: $F \equiv F(\{q_i, p_i\}, t)$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_{i=1}^{M} \frac{\partial F}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial F}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} + \frac{\partial F}{\partial t} \equiv \{F, \mathcal{H}\}_{q,p} + \frac{\partial F}{\partial t}$$

- **Poisson bracket**: $F \equiv F(\{q_i, p_i\}, t)$, $G \equiv G(\{q_i, p_i\}, t)$

$$\{F, G\}_{q,p} \equiv \sum_{i=1}^{M} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

- Hamilton's canonical equation of motion in terms of Poisson brackets:

$$\begin{cases} \dot{q}_i = \dfrac{\partial \mathcal{H}}{\partial p_i} \\[2ex] \dot{p}_i = -\dfrac{\partial \mathcal{H}}{\partial q_i} \end{cases} \quad \rightarrow \quad \begin{cases} \dot{q}_i = \{q_i, \mathcal{H}\}_{q,p} \\ \dot{p}_i = \{p_i, \mathcal{H}\}_{q,p} \end{cases}$$

Poisson bracket and constant of motion

- If the Hamiltonian has no explicit time dependence, then it is a constant of motion

$$\frac{\partial \mathcal{H}}{\partial t} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \{\mathcal{H}, \mathcal{H}\}_{q,p} + \frac{\partial \mathcal{H}}{\partial t} = 0$$

- **Poisson theorem**: $F \equiv F\left(\{q_i, p_i\}, t\right)$, $G \equiv G\left(\{q_i, p_i\}, t\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{F, G\}_{q,p} = \left\{\frac{\mathrm{d}F}{\mathrm{d}t}, G\right\}_{q,p} + \left\{F, \frac{\mathrm{d}G}{\mathrm{d}t}\right\}_{q,p}$$

- If $F_1 \equiv F_1(\{q_i, p_i\}, t)$ and $F_2 \equiv F_2(\{q_i, p_i\}, t)$ are two constants of motion, then their Poisson bracket is also a constant of motion

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}F_1}{\mathrm{d}t} = 0 \\[2ex] \dfrac{\mathrm{d}F_2}{\mathrm{d}t} = 0 \end{array}\right. \qquad \rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\{F_1, F_2\}_{q,p} = \left\{\frac{\mathrm{d}F_1}{\mathrm{d}t}, F_2\right\}_{q,p} + \left\{F_1, \frac{\mathrm{d}F_2}{\mathrm{d}t}\right\}_{q,p} = 0$$

Algebraic properties of Poisson bracket

- Anticommutativity:

$$\{F, G\}_{q,p} = -\{G, F\}_{q,p}$$

- Linearity:

$$\begin{cases} \{aF + bG, H\}_{q,p} = a\{F, H\}_{q,p} + b\{G, H\}_{q,p} \\ \{F, aG + bH\}_{q,p} = a\{F, G\}_{q,p} + b\{F, H\}_{q,p} \end{cases}$$

- Leibniz's rule:

$$\{FG, H\}_{q,p} = \{F, H\}_{q,p} G + F\{G, H\}_{q,p}$$

- Jacobi identity:

$$\left\{F, \{G, H\}_{q,p}\right\}_{q,p} + \left\{G, \{H, F\}_{q,p}\right\}_{q,p} + \left\{H, \{F, G\}_{q,p}\right\}_{q,p} = 0$$

Fundamental Poisson brackets

- Fundamental Poisson brackets:

$$\{q_i, q_j\}_{q,p} = 0\,, \qquad \{p_i, p_j\}_{q,p} = 0\,, \qquad \{q_i, p_j\}_{q,p} = \delta_{ij}$$

- Canonical quantization: $\{\ ,\ \}_{q,p} \to [\ ,\ ]/\mathrm{i}\hbar$, $\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

$$[\hat{q}_i, \hat{q}_j] = 0\,, \quad [\hat{p}_i, \hat{p}_j] = 0\,, \quad [\hat{q}_i, \hat{p}_j] = \mathrm{i}\hbar\,\delta_{ij}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\,\hat{A}_{\mathsf{H}}(t) = \frac{1}{\mathrm{i}\hbar}\left[\hat{A}_{\mathsf{H}}, \hat{H}_{\mathsf{H}}\right] + \left(\frac{\partial \hat{A}_{\mathsf{S}}}{\partial t}\right)_{\mathsf{H}}$$

- Poisson brackets for the components of the angular momentum:

$$L_k = \sum_{i,j=1}^{3} \epsilon_{ijk} x_j p_k \qquad \to \qquad \{L_i, L_j\}_{q,p} = \sum_{k=1}^{3} \epsilon_{ijk} L_k$$

EXERCISE 11.2: Evaluate $\{\mathbf{r}, \mathbf{n}\cdot\mathbf{L}\}_{q,p}$ where $\mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z$ and $\mathbf{n} = n_x\,\hat{\mathbf{e}}_x + n_y\,\hat{\mathbf{e}}_y + n_z\,\hat{\mathbf{e}}_z$ is a constant vector.

Example: Projectile motion

- A projectile with mass m is moving on the vertical xy-plane in a uniform gravitational field

$$\mathcal{H} \equiv \mathcal{H}(x, y, p_x, p_y, t) = \frac{p_x^2 + p_y^2}{2m} + mgy$$

- Two constants of motion:

$$\begin{cases} F_1 \equiv y - \dfrac{p_y t}{m} - \dfrac{1}{2}\, g t^2 \\ F_2 \equiv x - \dfrac{p_x t}{m} \end{cases}$$

EXERCISE 11.3: Show that F_1 and F_2 are constants of motion. Find the other three constants of motion.

Integrable systems

- The notion of **integrability** of a mechanical system refers to the possibility of *explicitly* solving its equations of motion

- The s dynamical variables $F_1(\{q_k, p_k\}), \cdots, F_s(\{q_k, p_k\})$ are said to be in **involution** if the Poison bracket of any two of them is zero

$$\{F_i, F_j\}_{q,p} = 0\,, \qquad i,j = 1,2,\cdots,s$$

- A Hamiltonian system with m degrees of freedom is said to be integrable if there exist m independent constants of the motion in involution

$$\begin{cases} \dfrac{\mathrm{d}F_i}{\mathrm{d}t} = 0\,, & i = 1,2,\cdots,m \\ \{F_i, F_j\}_{q,p} = 0\,, & i,j = 1,2,\cdots,m \end{cases}$$

Lagrangian versus Hamiltonian mechanics

- Euler-Lagrange equations of motion are covariant under a point transformation:

$$q_i = q_i\left(\{Q_j\}, t\right) \quad \rightarrow \quad \mathcal{L}\left(\{q_i, \dot{q}_i\}, t\right) = \mathcal{L}'\left(\{Q_i, \dot{Q}_i\}, t\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}'}{\partial \dot{Q}_i}\right) - \frac{\partial \mathcal{L}'}{\partial Q_i} = 0$$

- Hamilton equations of motion is also covariant under a point transformation *provided* that the *new* Hamiltonian (known as **Kamiltonian**) is constructed using the *new* conjugate momentum via a Legendre transformation

$$P_i = \frac{\partial \mathcal{L}'}{\partial \dot{Q}_i} \quad \rightarrow \quad \mathcal{K} \equiv \sum_{i=1}^{M} \dot{Q}_i\, P_i - \mathcal{L}' \neq \mathcal{H} \quad \rightarrow \quad \begin{cases} \dot{Q}_i = \dfrac{\partial \mathcal{K}}{\partial P_i} \\[2ex] \dot{P}_i = -\dfrac{\partial \mathcal{K}}{\partial Q_i} \end{cases}$$

Canonical transformation

- Hamilton equations of motion is, generally, covariant under a **canonical transformation** which is the change of canonical coordinates (generalized coordinates and generalized momenta are being treated under equal footing)

$$\begin{cases} Q_i \equiv Q_i(\{q_j, p_j\}, t) \\ P_i \equiv P_i(\{q_j, p_j\}, t) \end{cases} \quad \rightarrow \quad \begin{cases} \dot{Q}_i = \dfrac{\partial \mathcal{K}}{\partial P_i} \\ \dot{P}_i = -\dfrac{\partial \mathcal{K}}{\partial Q_i} \end{cases}$$

- **Phase space Lagrangian**: $2M$ independent generalized coordinates $\{q_k, p_k\}$

$$\tilde{\mathcal{L}} \equiv \tilde{\mathcal{L}}(\{q_i, p_i, \dot{q}_i, \dot{p}_i\}, t) \equiv \sum_{k=1}^{M} p_k \dot{q}_k - \mathcal{H}(\{q_k, p_k\}, t)$$

Canonical transformation – cont'd

- $2M$ Euler-Lagrange equations associated to the phase space Lagrangian give $2M$ Hamilton's canonical equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}_k}\right) - \dfrac{\partial \tilde{\mathcal{L}}}{\partial q_k} = 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \tilde{\mathcal{L}}}{\partial \dot{p}_k}\right) - \dfrac{\partial \tilde{\mathcal{L}}}{\partial p_k} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{\mathrm{d}p_k}{\mathrm{d}t} - \left(-\dfrac{\partial \mathcal{H}}{\partial q_k}\right) = 0 \\ 0 - \left(\dot{q}_k - \dfrac{\partial \mathcal{H}}{\partial p_k}\right) = 0 \end{cases}$$

- Euler-Lagrange equations associated to the phase space Lagrangian are invariant under a gauge transformation and hence ensuring the same Hamilton's equations are obtained

$$\tilde{\mathcal{L}}'(\{q_i, p_i, \dot{q}_i, \dot{p}_i\}, t) = \tilde{\mathcal{L}}(\{q_i, p_i, \dot{q}_i, \dot{p}_i\}, t) + \frac{\mathrm{d}\Lambda(\{q_i, p_i\}, t)}{\mathrm{d}t}$$

Canonical transformation and generating function

- $4M + 1$ variables $(\{q_i, p_i, Q_i, P_i\}, t)$ subjected to $2M$ transformation equations, $Q_i(\{q_j, p_j\}, t)$ and $P_i(\{q_j, p_j\}, t)$, leads to flexible choices of $2M + 1$ independent variables

- The gauge function, $\Lambda(\{q_i, p_i\}, t)$, is known as the **generating function** which generates the canonical transformation

- Four basic classes of generating functions: (Question: What is the relationship between different classes of generating functions?)

$$\begin{cases} \text{Type 1:} \;\; \Lambda_1 \equiv \Lambda_1(\{q_i, Q_i\}, t) \\ \text{Type 2:} \;\; \Lambda_2 \equiv \Lambda_2(\{q_i, P_i\}, t) \\ \text{Type 3:} \;\; \Lambda_3 \equiv \Lambda_3(\{p_i, Q_i\}, t) \\ \text{Type 4:} \;\; \Lambda_4 \equiv \Lambda_4(\{p_i, P_i\}, t) \end{cases}$$

Example: Harmonic oscillator

- Hamilton equations of motion:

$$\mathcal{H}(q,p) = \frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m} \;\Rightarrow\; \begin{cases} \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q \\[2ex] \dot{q} = \dfrac{\partial \mathcal{H}}{\partial p} = \dfrac{p}{m} \end{cases} \;\Rightarrow\; \begin{cases} \ddot{q} = -\omega^2 q \\[1ex] \ddot{p} = -\omega^2 p \end{cases}$$

- Type 1 generating function: this canonical transformation effectively exchanges the role of the coordinate and momentum!

$$\Lambda_1 \equiv \Lambda_1(q,Q,t) = qQ \qquad \Rightarrow \qquad \begin{cases} Q \equiv Q(q,p,t) = -p \\[1ex] P \equiv P(q,p,t) = q \end{cases}$$

EXERCISE 11.4: Obtain the canonical transformation generated by $\Lambda_1(q,Q,t) = qQ$ and the Kamiltonian equations of motion.

Canonicality

- A transformation is canonical if and only the fundamental Poisson brackets are invariant:

$$\{Q_i, Q_j\}_{q,p} = 0\,, \qquad \{P_i, P_j\}_{q,p} = 0\,, \qquad \{Q_i, P_j\}_{q,p} = \delta_{ij}$$

- Solving harmonic oscillator by guessing at a strategic canonical transformation:

$$\begin{cases} q \equiv q(Q,P,t) = \sqrt{\dfrac{2P}{m\omega}}\,\sin Q \\[2ex] p \equiv p(Q,P,t) = \sqrt{2m\omega P}\,\cos Q \end{cases} \quad \Rightarrow \quad \mathcal{K}(Q,P,t) = \omega P$$

- A practical convenient strategy for tackling a dynamical system is to find/guess a canonical transformation to simplify the Hamiltonian and then verify the canonicality using the Poisson bracket!

EXERCISE 11.5: Solve for $q(t)$ and $p(t)$ via $Q(t)$ and $P(t)$.

Liouville's theorem

- Continuity equation: ρ is the volume charge density and $\mathbf{J}$ is the volume current density in E&M

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

- Hamiltonian mechanics: $\rho \equiv \rho\left(\{q_i, p_i\}, t\right)$ is the density of points in the phase space and the corresponding "current density" is defined by $\sum_{i=1}^{M} \rho\left(\dot{q}_i + \dot{p}_i\right)$

- **Liouville's theorem**: density of points in the phase space corresponding to the time evolution of the systems remains constant during the time evolution

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$$