Center-of-mass frame

- **Center-of-mass frame** is a reference frame at which the center of mass remains at the origin:

$$\mathbf{r}'_\alpha(t) \equiv \mathbf{r}_\alpha(t) - \mathbf{R}_{\mathsf{CM}}(t) \quad \Rightarrow \quad \mathbf{R}'_{\mathsf{CM}}(t) = \mathbf{0}$$

- Velocity of the center of mass in the center-of-mass frame: center of mass is stationary in the center-of-mass frame

$$\mathbf{V}'_{\mathsf{CM}}(t) = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \dot{\mathbf{r}}'_\alpha(t) = \mathbf{0}$$

- Acceleration of the center of mass in the center-of-mass frame:

$$\mathbf{A}'_{\mathsf{CM}}(t) = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \ddot{\mathbf{r}}'_\alpha(t) = \mathbf{0}$$

$$\mathbf{R}_{\mathsf{CM}}(t) = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha(t)\,, \qquad \mathbf{r}'_\alpha(t) = \mathbf{r}_\alpha(t) - \mathbf{R}_{\mathsf{CM}}(t)$$

$$\begin{aligned}
\mathbf{R}'_{\mathsf{CM}}(t) &= \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}'_\alpha(t) \\
&= \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha(t) - \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{R}_{\mathsf{CM}}(t) \\
&= \mathbf{0} \qquad \blacksquare
\end{aligned}$$

$$\mathbf{R}_{\mathsf{CM}}(t) = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}'_\alpha(t) = \mathbf{0} \quad \Rightarrow \quad \sum_{\alpha=1}^{N} m_\alpha \dot{\mathbf{r}}'_\alpha(t) = \mathbf{0}$$

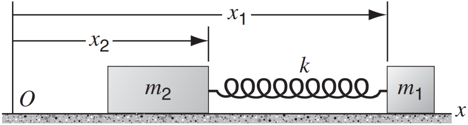
$$\Rightarrow \quad \mathbf{V}'_{\mathsf{CM}}(t) = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \dot{\mathbf{r}}'_\alpha(t) = \mathbf{0} \qquad \blacksquare$$

Example: Two-body oscillations

- Two idential blocks 1 and 2 each of mass m slide without friction on a straight track. They are connected by a massless spring with unstretched length L_0 and spring constant k. Initially, the system is at rest. At $t = 0$, block 1 is hit sharply giving it an instanteneous velocity v_0 to the right.

- Equations of motion in the center-of-mass frame:

$$\begin{cases} m\ddot{x}_1'(t) = -k\left[x_1'(t) - x_2'(t) - L_0\right] \\ m\ddot{x}_2'(t) = +k\left[x_1'(t) - x_2'(t) - L_0\right] \end{cases}$$



EXERCISE 3.4: Find the velocities of each block at later times with respect to the track.

$$\mathbf{F}^{\mathsf{ext}}(t) = M\ddot{\mathbf{R}}_{\mathsf{CM}}(t) = \mathbf{0} \quad \Rightarrow \quad \ddot{\mathbf{R}}_{\mathsf{CM}}(t) = \mathbf{0}$$

$$X_{\mathsf{CM}}(t) = \frac{m_1 x_1(t) + m_2 x_2(t)}{m_1 + m_2} = \frac{1}{2}\left[x_1(t) + x_2(t)\right]$$

$$\mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{R}_{\mathsf{CM}}(t) \quad \Rightarrow \quad \begin{cases} x_1'(t) = x_1(t) - X_{\mathsf{CM}}(t) = \frac{1}{2}\left[x_1(t) - x_2(t)\right] \\ x_2'(t) = x_2(t) - X_{\mathsf{CM}}(t) = -\frac{1}{2}\left[x_1(t) - x_2(t)\right] \end{cases}$$

$$x_1(t) - x_2(t) - L_0 = x_1'(t) - x_2'(t) - L_0$$

$$\begin{cases} \mathbf{F}_1(t) = m_1\ddot{\mathbf{r}}_1'(t) \\ \mathbf{F}_2(t) = m_2\ddot{\mathbf{r}}_2'(t) \end{cases} \quad \Rightarrow \quad \begin{cases} m\ddot{x}_1'(t) = -k\left[x_1'(t) - x_2'(t) - L_0\right] \\ m\ddot{x}_2'(t) = +k\left[x_1'(t) - x_2'(t) - L_0\right] \end{cases} \quad \blacksquare$$

$$\begin{cases} m\ddot{x}_1'(t) = -k\left[x_1'(t) - x_2'(t) - L_0\right] \\ m\ddot{x}_2'(t) = +k\left[x_1'(t) - x_2'(t) - L_0\right] \end{cases}$$

$$\xrightarrow{u \equiv x_1' - x_2' - L_0} \quad m\ddot{u}(t) + 2ku(t) = 0$$

$$\Rightarrow \quad u(t) = A\cos\omega t + B\sin\omega t \qquad \omega \equiv \sqrt{\frac{2k}{m}}$$

$$\begin{cases} u(0) = x_1(0) - x_2(0) - L_0 = 0 \\ \dot{u}(0) = \dot{x}_1(0) - \dot{x}_2(0) = v_0 \end{cases} \quad \Rightarrow \quad u(t) = \frac{v_0}{\omega}\sin\omega t \qquad \blacksquare$$

$$u(t) = \frac{v_0}{\omega} \sin \omega t \,, \qquad u(t) \equiv x'_1(t) - x'_2(t) - L_0$$

$$\begin{cases} x'_1(t) = \dfrac{1}{2} \left[x_1(t) - x_2(t) \right] \\[2ex] x'_2(t) = -\dfrac{1}{2} \left[x_1(t) - x_2(t) \right] \end{cases} \quad \Rightarrow \quad \dot{x}'_1(t) = -\dot{x}'_2(t)$$

$$\dot{u}(t) = \dot{x}'_1(t) - \dot{x}'_2(t) = v_0 \,\cos \omega t \quad \Rightarrow \quad \dot{x}'_1(t) = -\dot{x}'_2(t) = \frac{v_0}{2}\,\cos \omega t \qquad \blacksquare$$

$$\ddot{X}_{\mathsf{CM}}(t) = 0 \quad \Rightarrow \quad \dot{X}_{\mathsf{CM}}(t) = \dot{X}_{\mathsf{CM}}(0) = \frac{1}{2} \left[\dot{x}_1(0) + \dot{x}_2(0) \right] = \frac{v_0}{2} \qquad \blacksquare$$

$$\begin{cases} \dot{x}'_1(t) = \dot{x}_1(t) - \dot{X}_{\mathsf{CM}}(t) \\[2ex] \dot{x}'_2(t) = \dot{x}_2(t) - \dot{X}_{\mathsf{CM}}(t) \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x}_1(t) = \dot{x}'_1(t) + \dot{X}_{\mathsf{CM}}(t) = \dfrac{v_0}{2}\,(1 + \cos \omega t) \\[2ex] \dot{x}_2(t) = \dot{x}'_2(t) + \dot{X}_{\mathsf{CM}}(t) = \dfrac{v_0}{2}\,(1 - \cos \omega t) \end{cases} \qquad \blacksquare$$

PC3261: Classical Mechanics II

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Lecture 4: Angular Momentum

Angular momentum and torque

- **Angular momentum** of a particle *about* the origin:

$$\boldsymbol{\ell}(t) \equiv \mathbf{r}(t) \times \mathbf{p}(t)$$

- **Torque** (or **moment of force**) due to force acting on particle about the origin:

$$\boldsymbol{\tau}(t) \equiv \mathbf{r}(t) \times \mathbf{F}(t)$$

- **Rotational Newton's second law**: time rate of change of angular momentum of a particle about the origin is equal to the total torque about the origin

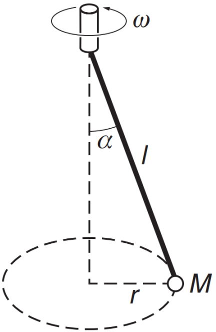
$$\dot{\boldsymbol{\ell}}(t) = \dot{\mathbf{r}}(t) \times \mathbf{p}(t) + \mathbf{r}(t) \times \dot{\mathbf{p}}(t) \quad \Rightarrow \quad \dot{\boldsymbol{\ell}}(t) = \boldsymbol{\tau}(t)$$

Example: Conical pendulum

- Mass M is fixed to the end of a light rod of length L that is pivoted to swing from the end of a hub that rotates at constant angular frequency ω. The mass moves with steady speed in a circular path of constant radius.

- Net external force on the mass:

$$\mathbf{F}(t) = -Mg\tan\alpha\,\hat{\mathbf{e}}_\rho$$



EXERCISE 4.1: Verify that the relation $\boldsymbol{\tau}(t) = \dot{\boldsymbol{\ell}}(t)$ is satisfied for the following two origins: (1) center of the circular plane of motion; and (2) pivot point on the axis.

$$\mathbf{r}(t) = L\sin\alpha\,\hat{\mathbf{e}}_\rho\,, \qquad \mathbf{F}(t) = -Mg\tan\alpha\,\hat{\mathbf{e}}_\rho\,, \qquad \mathbf{v}(t) = L\omega\sin\alpha\,\hat{\mathbf{e}}_\phi$$

$$\boldsymbol{\tau}(t) = \mathbf{r}(t)\times\mathbf{F}(t) = (L\sin\alpha\,\hat{\mathbf{e}}_\rho)\times(-Mg\tan\alpha\,\hat{\mathbf{e}}_\rho) = \mathbf{0} \qquad \blacksquare$$

$$\boldsymbol{\ell}(t) = \mathbf{r}(t)\times\mathbf{p}(t) = (L\sin\alpha\,\hat{\mathbf{e}}_\rho)\times(ML\omega\sin\alpha\,\hat{\mathbf{e}}_\phi) = ML^2\omega\sin^2\alpha\,\hat{\mathbf{e}}_z \qquad \blacksquare$$

$$\dot{\boldsymbol{\ell}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(Mr^2\omega\,\hat{\mathbf{e}}_z\right) = \mathbf{0} \qquad \checkmark$$

$$\mathbf{r}(t) = L\sin\alpha\,\hat{\mathbf{e}}_\rho\,, \quad \mathbf{F}(t) = -Mg\tan\alpha\,\hat{\mathbf{e}}_\rho\,, \quad \mathbf{v}(t) = L\omega\sin\alpha\,\hat{\mathbf{e}}_\phi\,, \quad \mathbf{r}_{\mathsf{pivot}}(t) = L\cos\alpha\,\hat{\mathbf{e}}_z$$

$$\begin{aligned}\boldsymbol{\tau}_{\mathsf{pivot}}(t) &= [\mathbf{r}(t) - \mathbf{r}_{\mathsf{pivot}}(t)] \times \mathbf{F}(t) \\ &= [L\sin\alpha\,\hat{\mathbf{e}}_\rho - L\cos\alpha\,\hat{\mathbf{e}}_z] \times (-Mg\tan\alpha\,\hat{\mathbf{e}}_\rho) \\ &= MgL\sin\alpha\,\hat{\mathbf{e}}_\phi \qquad \blacksquare\end{aligned}$$

$$\begin{aligned}\boldsymbol{\ell}_{\mathsf{pivot}}(t) &= [\mathbf{r}(t) - \mathbf{r}_{\mathsf{pivot}}(t)] \times \mathbf{p}(t) \\ &= [L\sin\alpha\,\hat{\mathbf{e}}_\rho - L\cos\alpha\,\hat{\mathbf{e}}_z] \times (ML\omega\sin\alpha\,\hat{\mathbf{e}}_\phi) \\ &= ML^2\omega\sin^2\alpha\,\hat{\mathbf{e}}_z + ML^2\omega\sin\alpha\cos\alpha\,\hat{\mathbf{e}}_\rho \qquad \blacksquare\end{aligned}$$

$$\dot{\boldsymbol{\ell}}_{\mathsf{pivot}}(t) = ML^2\omega^2\sin\alpha\cos\alpha\,\hat{\mathbf{e}}_\phi = MgL\sin\alpha\,\hat{\mathbf{e}}_\phi \qquad \checkmark$$

Angular momentum of multi-particle system

- Total angular momentum of multi-particle system about the origin:

$$\mathbf{L}(t) \equiv \sum_{\alpha=1}^{N} \boldsymbol{\ell}_\alpha(t) = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{p}_\alpha(t)$$

- Time rate of change of total angular momentum of multi-particle system about the origin equals to the total torque on the system about the origin:

$$\dot{\mathbf{L}}(t) = \sum_{\alpha=1}^{N} \dot{\boldsymbol{\ell}}_\alpha(t) = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{F}_\alpha(t) = \boldsymbol{\mathcal{T}}(t)\,, \qquad \boldsymbol{\mathcal{T}}(t) \equiv \sum_{\alpha=1}^{N} \boldsymbol{\tau}_\alpha(t)$$

- Rotational Newton's second law: internal forces are central

$$\dot{\mathbf{L}}(t) = \boldsymbol{\mathcal{T}}^{\text{ext}}(t)\,, \qquad \boldsymbol{\mathcal{T}}^{\text{ext}}(t) \equiv \sum_{\alpha=1}^{N} \boldsymbol{\tau}_\alpha^{\text{ext}}(t) = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{F}_\alpha^{\text{ext}}(t)$$

$$\mathbf{L}(t) = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{p}_\alpha(t)\,, \qquad \dot{\mathbf{p}}_\alpha(t) = \mathbf{F}_\alpha^{\mathsf{ext}}(t) + \sum_{\beta=1,\beta\neq\alpha}^{N} \mathbf{f}_{\alpha\beta}(t)$$

$$\begin{aligned}
\dot{\mathbf{L}}(t) &= \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{F}_\alpha^{\mathsf{ext}}(t) + \sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} \mathbf{r}_\alpha(t) \times \mathbf{f}_{\alpha\beta}(t) \\
&= \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{F}_\alpha^{\mathsf{ext}}(t) + \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} \left[\mathbf{r}_\alpha(t) \times \mathbf{f}_{\alpha\beta}(t) + \mathbf{r}_\beta(t) \times \mathbf{f}_{\beta\alpha}(t)\right] \\
&= \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{F}_\alpha^{\mathsf{ext}}(t) + \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} \left[\mathbf{r}_\alpha(t) \times \mathbf{f}_{\alpha\beta}(t) - \mathbf{r}_\beta(t) \times \mathbf{f}_{\alpha\beta}(t)\right] \\
&= \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{F}_\alpha^{\mathsf{ext}}(t) + \sum_{\alpha=1}^{N} \sum_{\beta>\alpha}^{N} \left[\mathbf{r}_\alpha(t) - \mathbf{r}_\beta(t)\right] \times \mathbf{f}_{\alpha\beta}(t) \\
&= \sum_{\alpha=1}^{N} \mathbf{r}_\alpha(t) \times \mathbf{F}_\alpha^{\mathsf{ext}}(t) \qquad \blacksquare
\end{aligned}$$

Example: Gravitational torque

- Total gravitational torque on an object about the origin:

$$\boldsymbol{\mathcal{T}}(t) = \int \rho(\mathbf{r})\, \mathbf{r}(t) \times \mathbf{g}(\mathbf{r})\, \mathrm{d}V$$

- Uniform gravitational field:

$$\boldsymbol{\mathcal{T}}(t) = \left[\int \rho(\mathbf{r})\, \mathbf{r}(t)\, \mathrm{d}V\right] \times \mathbf{g} = \mathbf{R}_{\mathsf{CM}}(t) \times M\mathbf{g}$$

- **Center of gravity**: a point at which the total weight of the body is supposed to be concentrated

$$\mathbf{R}_{\mathsf{CG}}(t) \times \int \rho(\mathbf{r})\, \mathbf{g}(\mathbf{r})\, \mathrm{d}V \equiv \int \rho(\mathbf{r})\, \mathbf{r}(t) \times \mathbf{g}(\mathbf{r})\, \mathrm{d}V$$