PC3261: Classical Mechanics II

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Lecture 13: Special Relativity II

General Lorentz boost (revisited)

- Lorentz boost of the spacetime coordinates in terms of matrices:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \Lambda^t{}_t & \Lambda^t{}_x & \Lambda^t{}_y & \Lambda^t{}_z \\ \Lambda^x{}_t & \Lambda^x{}_x & \Lambda^x{}_y & \Lambda^x{}_z \\ \Lambda^y{}_t & \Lambda^y{}_x & \Lambda^y{}_y & \Lambda^y{}_z \\ \Lambda^z{}_t & \Lambda^z{}_x & \Lambda^z{}_y & \Lambda^z{}_z \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

- Non-zero components of the Lorentz boost matrix between two inertial frames in standard orientation:

$$\Lambda^t{}_t = \Lambda^x{}_x = \gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \quad \Lambda^t{}_x = \Lambda^x{}_t = -\gamma\beta = -\frac{\beta}{\sqrt{1-\beta^2}}\,, \quad \Lambda^y{}_y = \Lambda^z{}_z = 1$$

EXERCISE 13.1: Find the matrix components of general Lorentz boost between two inertial frames with parallel axes and arbitrary relative velocity.

Abstract indices

- **Contravariant** components of **four-position** vector:

$$x^\alpha = (ct, \mathbf{r}) = (ct, x, y, z)$$

- Greek-letter (Latin-letter) indices for spacetime (space) components:

$$x^\alpha = (x^0, x^i) \qquad \rightarrow \qquad \begin{cases} x^0 \equiv ct \\ x^1 \equiv x \\ x^2 \equiv y \\ x^3 \equiv z \end{cases}$$

- Lorentz boost of the spacetime coordinates using abstract indices:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \Lambda^t{}_t & \Lambda^t{}_x & \Lambda^t{}_y & \Lambda^t{}_z \\ \Lambda^x{}_t & \Lambda^x{}_x & \Lambda^x{}_y & \Lambda^x{}_z \\ \Lambda^y{}_t & \Lambda^y{}_x & \Lambda^y{}_y & \Lambda^y{}_z \\ \Lambda^z{}_t & \Lambda^z{}_x & \Lambda^z{}_y & \Lambda^z{}_z \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad \leftrightarrow \qquad x'^\alpha = \sum_{\beta=0}^{3} \Lambda^\alpha{}_\beta x^\beta$$

Einstein summation convention

- If the same Greek-letter or Latin-letter index appears *exactly once as a superscript and exactly once as a subscript* in any single term of an expression, the term is to be summed over all possible values of that index

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \rightarrow \qquad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$x^{\mu} x_{\mu} = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3$$

$$x^i x_i = x^1 x_1 + x^2 x_2 + x^3 x_3$$

- **Free index** is free to assign any value; **dummy index** is to take all values

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \rightarrow \qquad \begin{cases} \mu \text{ is a free index} \\ \nu \text{ is a dummy index} \end{cases}$$

$$\mu = 1 : \qquad x'^{1} = \Lambda^{1}{}_{\nu} x^{\nu} = \Lambda^{1}{}_{0} x^{0} + \Lambda^{1}{}_{1} x^{1} + \Lambda^{1}{}_{2} x^{2} + \Lambda^{1}{}_{3} x^{3}$$

Rules for indices

- Number and letter of free indices: Every term in an equation must have the same number of free indices and must use the same letter for these free indices across all terms

$$A^\mu = B^\nu \quad \text{(WRONG)} \qquad\qquad A^2 = \eta_{\alpha\beta} A^\mu A^\nu \quad \text{(WRONG)}$$

$$A^\mu = \Lambda^\mu{}_\nu B^\nu + \Xi^\mu{}_\alpha \Theta^\nu{}_\beta B_\nu C^{\alpha\beta} \quad \text{(CORRECT)}$$

- Renaming indices: (1) rename every occurrence of the letter of free (dummy) index in the equation (single term of the equation); (2) avoid using a letter already used in the equation (single term of the equation) when renaming free (dummy) indices

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \rightarrow \quad x'^\alpha = \Lambda^\mu{}_\nu x^\nu \quad \text{or} \quad x'^\nu = \Lambda^\nu{}_\nu x^\nu \quad \text{(WRONG)}$$

$$A^\mu = \Lambda^\mu{}_\nu B^\nu + \Xi^\mu{}_\alpha \Theta^\nu{}_\beta B_\nu C^{\alpha\beta} \;\rightarrow\; A^\nu = \Lambda^\nu{}_\mu B^\mu + \Xi^\nu{}_\mu \Theta^\alpha{}_\beta B_\alpha C^{\mu\beta} \quad \text{(CORRECT)}$$

- Write it out explicitly whenever in doubt!

Kronecker delta

- **Kronecker delta**: 16-component object

$$\delta^0{}_0 = \delta^1{}_1 = \delta^2{}_2 = \delta^3{}_3 = 1\,, \qquad \delta^\mu{}_\nu = 0 \text{ if } \mu \neq \nu$$

- Expressing the identity matrix in abstract-index notation:

$$\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda} = \mathbf{I} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-1} \qquad \Leftrightarrow \qquad (\Lambda^{-1})^\mu{}_\alpha \Lambda^\alpha{}_\nu = \delta^\mu{}_\nu = \Lambda^\mu{}_\alpha \left(\Lambda^{-1}\right)^\alpha{}_\nu$$

- Summing over either index of the Kronecker delta is equivalent to replacing the value of the summed index with the value of the Kronecker delta's other index:

$$\delta^\mu{}_\nu A^\nu = A^\mu\,, \qquad \delta^\mu{}_\alpha \eta_{\mu\nu} = \eta_{\alpha\nu}\,, \qquad \ldots$$

Minkowski metric tensor

- **Minkowski metric tensor**:

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_{tt} & \eta_{tx} & \eta_{ty} & \eta_{tz} \\ \eta_{xt} & \eta_{xx} & \eta_{xy} & \eta_{xz} \\ \eta_{yt} & \eta_{yx} & \eta_{yy} & \eta_{yz} \\ \eta_{zt} & \eta_{zx} & \eta_{zy} & \eta_{zz} \end{pmatrix} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \leftrightarrow \quad \begin{cases} \eta_{00} = -1 \\ \eta_{11} = \eta_{22} = \eta_{33} = 1 \\ \eta_{\mu\nu} = 0 \text{ if } \mu \neq \nu \end{cases}$$

- **Covariant** components of four-position vector: index lowering

$$x_\alpha \equiv \eta_{\alpha\beta} x^\beta = (x_0, x_i) = (-ct, \mathbf{r}) = (-ct, x, y, z)$$

- Spacetime interval in terms of Minkowski metric tensor:

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}x^\mu\,\mathrm{d}x_\mu = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

Inverse Minkowski metric tensor

- Raising index by **inverse Minkowski metric tensor** $\eta^{\alpha\beta}$:

$$x^\alpha \equiv \eta^{\alpha\beta} x_\beta \,, \qquad \eta^{\alpha\beta} \equiv \left(\eta^{-1}\right)^{\alpha\beta}$$

- Transformation of the covariant components of spacetime coordinates:

$$x'_\alpha = \Lambda_\alpha{}^\beta x_\beta \,, \qquad \Lambda_\alpha{}^\beta \equiv \eta_{\alpha\mu} \Lambda^\mu{}_\nu \eta^{\nu\beta}$$

- Relationship between $\Lambda^\alpha{}_\beta$ and $\Lambda_\alpha{}^\beta$:

$$\Lambda_\alpha{}^\beta = \left(\Lambda^{-1}\right)^\beta{}_\alpha$$

EXERCISE 13.2: Find the components of $\boldsymbol{\Lambda}^{-1}$ in terms of the components of $\boldsymbol{\Lambda}$.

Lorentz transformation

- Invariance of spacetime interval:

$$\eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \eta_{\alpha\beta}\,\mathrm{d}x'^\alpha\,\mathrm{d}x'^\beta \qquad \Rightarrow \qquad \eta_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu}$$

- All spacetime transformations $\Lambda^\alpha{}_\beta$ satisfying $\eta_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu}$ form a group known as **Lorentz group** $\mathrm{O}(1,3)$

- Spatial rotation about z-axis belongs to Lorentz group:

$$\boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Four-tensors

- Transformation of components of a **four-tensor** (or **Lorentz tensor**) of rank N: $N = n_1 + n_2$

$$T'^{\alpha_1 \cdots \alpha_{n_1}}{}_{\beta_1 \cdots \beta_{n_2}} = \Lambda^{\alpha_1}{}_{\mu_1} \cdots \Lambda^{\alpha_{n_1}}{}_{\mu_{n_1}} \Lambda_{\beta_1}{}^{\nu_1} \cdots \Lambda_{\beta_{n_2}}{}^{\nu_{n_2}} T^{\mu_1 \cdots \mu_{n_1}}{}_{\nu_1 \cdots \nu_{n_2}}$$

- Scalar is a Lorentz tensor of rank 0 (or **Lorentz scalar**)

- Four-vector is a Lorentz tensor of rank 1 (or **Lorentz vector**):

$$A'^{\alpha} = \Lambda^{\alpha}{}_{\mu} A^{\mu} \,, \qquad A'_{\alpha} = \Lambda_{\alpha}{}^{\mu} A_{\mu}$$

- Lorentz tensor of rank 2:

$$T'^{\alpha\beta} = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} T^{\mu\nu} \,, \qquad T'_{\alpha\beta} = \Lambda_{\alpha}{}^{\mu} \Lambda_{\beta}{}^{\nu} T_{\mu\nu} \,, \qquad T'^{\alpha}{}_{\beta} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\beta}{}^{\nu} T^{\mu}{}_{\nu}$$

EXERCISE 13.3: Given that A^{μ} and B_{μ} are Lorentz vectors, show that $A^{\alpha} B_{\alpha}$ is a Lorentz scalar and $A^{\alpha} B_{\beta}$ is a Lorentz tensor of rank 2.

Four-tensors – cont'd

- Minkowski metric tensor is a Lorentz tensor of rank 2:

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \eta'_{\alpha\beta}\,\mathrm{d}x'^\alpha\,\mathrm{d}x'^\beta \quad \Rightarrow \quad \eta'_{\alpha\beta} = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu \eta_{\mu\nu}$$

- Kronecker delta is an **invariant** Lorentz tensor of rank 2:

$$\delta'^\alpha{}_\beta = \Lambda^\alpha{}_\mu \Lambda_\beta{}^\nu \delta^\mu{}_\nu = \delta^\alpha{}_\beta$$

- Lowering and raising indices:

$$T^{\mu_1}{}_\beta{}^{\mu_2}{}_{\nu_1\nu_2\nu_3} = \eta_{\alpha\beta} T^{\mu_1\alpha\mu_2}{}_{\nu_1\nu_2\nu_3}\,, \qquad T_{\mu_1}{}^\beta{}_{\mu_2}{}^{\nu_1\nu_2\nu_3} = \eta^{\alpha\beta} T_{\mu_1\alpha\mu_2}{}^{\nu_1\nu_2\nu_3}$$

EXERCISE 13.4: Show that the Minkowski metric tensor is an invariant Lorentz tensor of rank 2.

Four-vectors

- Contravariant components of **four-vector**:

$$A^\alpha \equiv (A^0, \mathbf{A}) = (A^0, A^i)$$

- Transformation of contravariant components of four-vector under general Lorentz boost:

$$A'^\alpha = \Lambda^\alpha{}_\beta A^\beta \qquad \rightarrow \qquad \begin{cases} A'^0 = \gamma\left(A^0 - \boldsymbol{\beta}\cdot\mathbf{A}\right) \\ \mathbf{A}'_\parallel = \gamma\left(\mathbf{A}_\parallel - \boldsymbol{\beta}A^0\right) \\ \mathbf{A}'_\perp = \mathbf{A}_\perp \end{cases}$$

- Covariant components of four-vector:

$$A_\alpha \equiv \eta_{\alpha\beta}A^\beta\,, \qquad A_\alpha = (A_0, A_i) = (-A^0, A^i) = (-A^0, \mathbf{A})$$

Four-vectors – cont'd

- Transformation of covariant components of four-vector under Lorentz transformation:

$$A'_\alpha = \Lambda_\alpha{}^\mu A_\mu$$

- $A^\alpha A_\alpha$ is a Lorentz scalar:

$$\begin{aligned} A'^\alpha A'_\alpha &= -\left(A'^0\right)^2 + \left(A'^1\right)^2 + \left(A'^2\right)^2 + \left(A'^3\right)^2 \\ &= -\left(A^0\right)^2 + \left(A^1\right)^2 + \left(A^2\right)^2 + \left(A^3\right)^2 \\ &= A^\mu A_\mu \end{aligned}$$

- Classification of four-vectors:

$$A^\alpha A_\alpha \begin{cases} < 0\,, & \text{timelike} \\ = 0\,, & \text{lightlike} \\ > 0\,, & \text{spacelike} \end{cases}$$

Four-gradient

- Covariant components of **four-gradient**:

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

- **d'Alembert operator** is a Lorentz scalar differential operator:

$$\Box \equiv \partial^\alpha \partial_\alpha = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Wave equations:

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right]\psi(\mathbf{r}, t) = 0 \qquad \rightarrow \qquad \Box\,\psi(x) = 0$$

EXERCISE 13.5: Derive the transformation of derivatives with respect to space and time under general Lorentz boost.

Four-velocity

- **Four-velocity**: $\mathbf{u}$ is the velocity of the particle

$$U^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = (U^0, \mathbf{U}) = \gamma(u)\,(c, \mathbf{u}) = \gamma(u)\,\big(c, u^i\big)\ , \qquad \gamma(u) \equiv \frac{1}{\sqrt{1 - u^2/c^2}}$$

- Four-velocity is a time-like four-vector:

$$U^\alpha U_\alpha = \eta_{\alpha\beta} U^\alpha U^\beta = -c^2 < 0$$

- Transformation of the contravariant components of four-velocity under general Lorentz boost:

$$U'^\alpha = \Lambda^\alpha{}_\beta U^\beta \qquad \to \qquad \begin{cases} \gamma(u') = \gamma\gamma(u)\,(1 - \boldsymbol{\beta}\cdot\mathbf{u}/c) \\ \gamma(u')\,\mathbf{u}'_\parallel = \gamma\gamma(u)\,\big(\mathbf{u}_\parallel - \boldsymbol{\beta} c\big) \\ \gamma(u')\,\mathbf{u}'_\perp = \gamma(u)\,\mathbf{u}_\perp \end{cases}$$

Velocity transformation

- Velocity in terms of four-velocity:

$$u^i = c\,\frac{U^i}{U^0}$$

- Lorentz velocity transformation:

$$\mathbf{u}'_{\parallel} = \frac{\mathbf{u}_{\parallel} - \boldsymbol{\beta} c}{1 - \boldsymbol{\beta}\cdot\mathbf{u}/c}\,, \qquad \mathbf{u}'_{\perp} = \frac{\mathbf{u}_{\perp}}{\gamma\,(1 - \boldsymbol{\beta}\cdot\mathbf{u}/c)}$$

- Invariance of the speed of light:

$$u_{\parallel}^2 + u_{\perp}^2 = c^2 \quad \Leftrightarrow \quad u_{\parallel}'^2 + u_{\perp}'^2 = c^2$$

EXERCISE 13.6: Show that $u_{\parallel}^2 + u_{\perp}^2 < c^2$ implies $u_{\parallel}'^2 + u_{\perp}'^2 < c^2$.

Four-momentum

- **Four-momentum**: m and $\mathbf{u}$ are the rest mass and velocity of the particle

$$P^\alpha \equiv mU^\alpha = (P^0, \mathbf{P}) = \gamma(u)\, m(c, \mathbf{u}) = \gamma(u)\, m(c, u^i)$$

- Relativistic energy and relativistic momentum:

$$E \equiv P^0 c = \gamma(u)\, mc^2\,, \qquad\qquad \mathbf{P} = \gamma(u)\, m\mathbf{u}$$

- Transformation of relativistic energy and relativistic momentum under general Lorentz boost:

$$P'^\alpha = \Lambda^\alpha{}_\beta P^\beta \qquad \rightarrow \qquad \begin{cases} E' = \gamma\,(E - c\,\boldsymbol{\beta}\cdot\mathbf{P}) \\ \mathbf{P}'_\parallel = \gamma\left(\mathbf{P}_\parallel - \dfrac{1}{c}\,\boldsymbol{\beta} E\right) \\ \mathbf{P}'_\perp = \mathbf{P}_\perp \end{cases}$$

Four-momentum – cont'd

- Relationship between relativistic energy and momentum: **on-shell condition**

$$P^\alpha P_\alpha = -m^2c^2 \quad \Rightarrow \quad E^2 - c^2\mathbf{P}\cdot\mathbf{P} = m^2c^4$$

- Relationship between relativistic momentum and velocity:

$$\mathbf{u} = \frac{c^2}{E}\,\mathbf{P}$$

- Four-momentum of photon: particle with zero rest mass

$$P^\alpha = \left(\frac{E}{c}, \frac{E}{c^2}\,\mathbf{u}\right), \qquad\qquad \mathbf{u}\cdot\mathbf{u} = c^2$$

- **Conservation of four-momentum** is equivalent to both conservation of relativistic energy and relativistic momentum

Four-acceleration

- **Four-acceleration**: $\mathbf{a}$ is the acceleration of the particle

$$\mathcal{A}^\alpha \equiv \frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = \left(\mathcal{A}^0, \boldsymbol{\mathcal{A}}\right) = \gamma^2(u)\,(0, \mathbf{a}) + \gamma^4(u)\,\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\,(c, \mathbf{u})$$

- Four-acceleration is a spacelike four-vector

$$\mathcal{A}^\alpha \mathcal{A}_\alpha = \frac{1}{\left(1 - u^2/c^2\right)^2}\left[\mathbf{a}\cdot\mathbf{a} + \frac{1}{c^2\left(1 - u^2/c^2\right)}\,(\mathbf{a}\cdot\mathbf{u})^2\right] > 0$$

- Any motion with $\mathcal{A}^\alpha \mathcal{A}_\alpha =$ constant gives uniformly accelerated motion in special relativity

- Lorentz acceleration transformation:

$$\mathbf{a}'_\parallel = \frac{\mathbf{a}_\parallel}{\gamma^3\left(1 - \boldsymbol{\beta}\cdot\mathbf{u}/c\right)^3}\,, \qquad \mathbf{a}'_\perp = \frac{\mathbf{a}_\perp}{\gamma^2\left(1 - \boldsymbol{\beta}\cdot\mathbf{u}/c\right)^2} + \frac{(\boldsymbol{\beta}\cdot\mathbf{a})\,\mathbf{u}_\perp}{c\gamma^2\left(1 - \boldsymbol{\beta}\cdot\mathbf{u}/c\right)^3}$$