PC3261: Classical Mechanics II

Kenneth HONG Chong Ming

Office: S16-07-06
Email: phyhcmk@nus.edu.sg

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National University of Singapore
Department of Physics
Faculty of Science

Lecture 6: d'Alembert's Principle

Constraints

- Positions of all particles of a mechanical system define the **configuration** of the system at any instant of time

- **Constraints** are limitations of a *kinematic* nature, such as *possible* positions or velocities of the particles, imposed on the mechanical system

- Given that $(\xi_1, \xi_2, \cdots, \xi_n)$ are *arbitrary* coordinates employed to specify the configuration of a mechanical system, a constraint is said to be **holonomic** if it can be expressed by a functional relation among the coordinates alone with a possible explicit time dependence

$$f(\xi_1, \xi_2, \cdots, \xi_n, t) = 0$$

- Example: a system of two particles connected by a rigid rod of length ℓ moving in the space

$$|\mathbf{r}_2 - \mathbf{r}_1|^2 - \ell^2 = 0 \quad \Rightarrow \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - \ell^2 = 0$$

Velocity-dependent constraints

- Velocity-dependent constraints are given by functional relations among the coordinates and velocities which leading to a set of differential equations

$$g(\xi_1, \cdots, \xi_n, \dot{\xi}_1, \cdots, \dot{\xi}_n, t) = 0$$

- Velocity-dependent constraints restrict the possible displacements of the mechanical system but do not impose any limits on the possible configurations

- If the velocity-dependent constraints can be reduced to the functional relations among coordinates alone with possible explicit time dependence by integration (solving differential equation), then it is said to be holonomic

- Differential equations from the velocity-dependent constraints are generally *non-integrable* which prevents them being holonomic

Example: Cylinder rolls on a straight track

- Denote the position of the center of masss of the cylinder as x and the angle of rotation about the center of mass as ϕ

- Condition for roll without slipping: R is the radius of the cylinder

$$\dot{x}(t) = R\,\dot{\phi}(t)$$

- Velocity-dependent constraint is integrable leading to a functional relation among coordinates alone

$$x(t) = R\,\phi(t) \quad \Rightarrow \quad x(t) - R\,\phi(t) = 0$$

Example: Upright disc rolls on a horizontal plane

- Denote (x, y) as the position of the disc's center of mass on the plane, θ be the angle of the plane of the disc made with the x-axis and ϕ be the angle of rotation of the disc about its symmetry axis

- Condition for roll without slipping: R is the radius of the disc

$$\begin{cases} \dot{x}(t) = R\,\dot{\phi}(t)\cos\theta(t) \\ \dot{y}(t) = R\,\dot{\phi}(t)\sin\theta(t) \end{cases}$$

- Velocity-dependent constraint is non-integrable (Frobenius integrability condition); the constraints represent restrictions on the velocities alone

- The coordinates, x, y, θ, ϕ, employed to specify the instantaneous configuration of the disc are functionally independent

Virtual displacements

- **Virtual displacements** are infinitesimal displacements that change a possible configuration to another possible configuration compatible with the constraints at a fixed instant of time

- Virtual displacements of a system of N particles:

$$f(\mathbf{r}_1(t), \cdots, \mathbf{r}_N(t), t) = 0 \quad \Rightarrow \quad f(\mathbf{r}_1(t) + \delta\mathbf{r}_1, \cdots, \mathbf{r}_N(t) + \delta\mathbf{r}_N, t) = 0$$

- Virtual displacements $\delta\mathbf{r}_i$ coincide with the real displacements $\mathrm{d}\mathbf{r}_i$ if all constraints are time independent

EXERCISE 6.1: A particle is confined to a moving surface where the equation of the surface is given by $f(\mathbf{r}, t) = 0$. Show that the virtual displacement is tangent to the surface at the same time.

$$f(\mathbf{r}(t) + \delta\mathbf{r}, t) = 0$$

$$\Rightarrow \quad f(\mathbf{r}(t), t) + \nabla f(\mathbf{r}, t) \cdot \delta\mathbf{r} = 0$$

$$\Rightarrow \quad \nabla f(\mathbf{r}, t) \cdot \delta\mathbf{r} = 0 \qquad \blacksquare$$

$$f(\mathbf{r}(t) + \mathrm{d}\mathbf{r}, t + \mathrm{d}t) = 0$$

$$\Rightarrow \quad f(\mathbf{r}(t), t) + \nabla f(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{r} + \frac{\partial f}{\partial t}\,\mathrm{d}t = 0$$

$$\Rightarrow \quad \boldsymbol{\nabla} f(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{r} + \frac{\partial f}{\partial t}\,\mathrm{d}t = 0 \qquad \blacksquare$$

Virtual work

- **Virtual work** by the force: no integration is required as the the virtual displacement is infinitesimal

$$\delta W \equiv \mathbf{F} \cdot \delta \mathbf{r}$$

- The *total* virtual work of the constraint forces is zero in most physically interesting cases

- Example: a system of two particles connected by a rigid rod of length ℓ moving in the space; $\mathbf{f}_1$ and $\mathbf{f}_2$ are constrained forces on the particles along the line connecting the particles

$$\delta W = \mathbf{f}_1 \cdot \delta \mathbf{r}_1 + \mathbf{f}_2 \cdot \delta \mathbf{r}_2 = 0$$

EXERCISE 6.2: Show that the total virtual work by the constrained forces on the two particles connected by a rigid rod moving in the space is zero.

$$f(\mathbf{r}_1, \mathbf{r}_2) = |\mathbf{r}_1 - \mathbf{r}_2|^2 - \ell^2 = 0$$

$$\Rightarrow \quad (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\delta \mathbf{r}_1 - \delta \mathbf{r}_2) = 0 \qquad \blacksquare$$

$$\mathbf{f}_1 = \lambda \, (\mathbf{r}_1 - \mathbf{r}_2) = -\mathbf{f}_2 \qquad \blacksquare$$

$$\begin{aligned} \delta W &= \mathbf{f}_1 \cdot \delta \mathbf{r}_1 + \mathbf{f}_2 \cdot \delta \mathbf{r}_2 \\ &= \mathbf{f}_1 \cdot (\delta \mathbf{r}_1 - \delta \mathbf{r}_2) \\ &= \lambda \, (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\delta \mathbf{r}_1 - \delta \mathbf{r}_2) \\ &= 0 \qquad \blacksquare \end{aligned}$$

Principle of virtual work

- Newton's second law: $\mathbf{F}_\alpha^{(\mathsf{A})}(t)$ and $\mathbf{F}_\alpha^{(\mathsf{C})}(t)$ are applied and constraint forces

$$\mathbf{F}_\alpha(t) = m_\alpha \ddot{\mathbf{r}}_\alpha(t)\,, \qquad \mathbf{F}_\alpha(t) = \mathbf{F}_\alpha^{(\mathsf{A})}(t) + \mathbf{F}_\alpha^{(\mathsf{C})}(t)$$

- Static equilibrium: $\mathbf{F}_\alpha(t) = \mathbf{0}$

$$\sum_\alpha \mathbf{F}_\alpha(t)\cdot\delta\mathbf{r}_\alpha = 0 \quad \Rightarrow \quad \sum_\alpha \mathbf{F}_\alpha^{(\mathsf{A})}(t)\cdot\delta\mathbf{r}_\alpha + \sum_\alpha \mathbf{F}_\alpha^{(\mathsf{C})}(t)\cdot\delta\mathbf{r}_\alpha = 0$$

- **Principle of virtual work**: allowing one to express the equilibrium conditions for a constrained system in terms of the applied forces alone

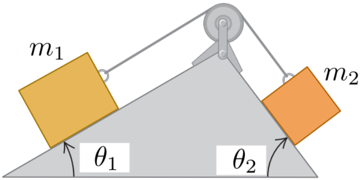
$$\sum_\alpha \mathbf{F}_\alpha^{(\mathsf{A})}(t)\cdot\delta\mathbf{r}_\alpha = 0$$

Example: Two masses on double inclined plane

- Two masses m_1 and m_2 are located each on a smooth double inclined plane with angles θ_1 and θ_2 respectively. The masses are connected by a massless and inextensible string running over a massless and frictionless pulley

- Holonomic constraint:

$$\begin{cases} f_1(\mathbf{r}_1, \mathbf{r}_2) = x_1 + x_2 - \ell = 0 \\ f_2(\mathbf{r}_1, \mathbf{r}_2) = y_1 = 0 \\ f_3(\mathbf{r}_1, \mathbf{r}_2) = y_2 = 0 \end{cases}$$



- Applied forces:

$$\mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g \sin\theta_1 \, \hat{\mathbf{e}}_{x_1} - m_1 g \cos\theta_1 \, \hat{\mathbf{e}}_{y_1} \, , \; \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g \sin\theta_2 \, \hat{\mathbf{e}}_{x_2} - m_2 g \cos\theta_2 \, \hat{\mathbf{e}}_{y_2}$$

EXERCISE 6.3: Establish the condition for equilibrium from the principle of virtual work.

$$\mathbf{F}_1^{(\mathsf{A})}(t) = m_1 g \sin\theta_1\,\hat{\mathbf{e}}_{x_1} - m_1 g \cos\theta_1\,\hat{\mathbf{e}}_{y_1}\,, \qquad \mathbf{F}_2^{(\mathsf{A})}(t) = m_2 g \sin\theta_2\,\hat{\mathbf{e}}_{x_2} - m_2 g \cos\theta_2\,\hat{\mathbf{e}}_{y_2}$$

$$\begin{cases} f_1(\mathbf{r}_1, \mathbf{r}_2) = x_1 + x_2 - \ell = 0 \\ f_2(\mathbf{r}_1, \mathbf{r}_2) = y_1 = 0 \\ f_3(\mathbf{r}_1, \mathbf{r}_2) = y_2 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \delta x_1 = -\delta x_2 \\ \delta y_1 = 0 \\ \delta y_2 = 0 \end{cases} \quad \blacksquare$$

$$\begin{cases} \delta\mathbf{r}_1 = \delta x_1\,\hat{\mathbf{e}}_{x_1} \\ \delta\mathbf{r}_2 = \delta x_2\,\hat{\mathbf{e}}_{x_2} \end{cases} \quad \blacksquare$$

$$\sum_\alpha \mathbf{F}_\alpha^{(\mathsf{A})}(t) \cdot \delta\mathbf{r}_\alpha = 0$$

$$\Rightarrow \quad m_1 g\,\delta x_1 \sin\theta_1 + m_2 g\,\delta x_2 \sin\theta_2 = 0$$

$$\Rightarrow \quad (m_1 g \sin\theta_1 - m_2 g \sin\theta_2)\,\delta x_1 = 0$$

$$\Rightarrow \quad m_1 \sin\theta_1 = m_2 \sin\theta_2 \qquad \blacksquare$$

d'Alembert's principle

- Newton's second law:

$$\mathbf{F}_\alpha(t) = m_\alpha \ddot{\mathbf{r}}_\alpha(t) \quad \Rightarrow \quad \mathbf{F}_\alpha(t) - m_\alpha \ddot{\mathbf{r}}_\alpha(t) = 0$$

- **d'Alembert's principle**: an extension of the principle of virtual work to mechanical systems in motion

$$\sum_\alpha \left[\mathbf{F}^{(\mathsf{A})}_\alpha(t) - m_\alpha \ddot{\mathbf{r}}_\alpha(t)\right] \cdot \delta\mathbf{r}_\alpha = 0$$

- d'Alembert's principle is a substantial leap forward with respect to the Newtonian approach as it excludes any reference to the constraint forces

- It is important to take note that the virtual displacements $\delta\mathbf{r}_\alpha$ are *not* independent as they have to be in harmony with the constraints

Example: Atwood's machine

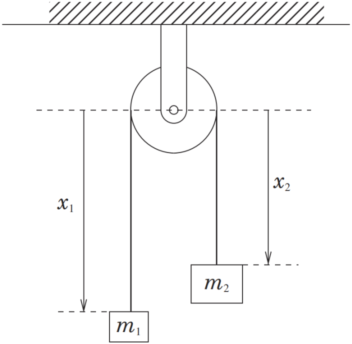
- Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless and frictionless pulley

- Holonomic constraint:

$$f(\mathbf{r}_1, \mathbf{r}_2) = x_1 + x_2 - \ell = 0$$

- Applied forces:

$$\mathbf{F}_1^{(\mathrm{A})}(t) = m_1 g\, \hat{\mathbf{e}}_1 \,, \qquad \mathbf{F}_2^{(\mathrm{A})}(t) = m_2 g\, \hat{\mathbf{e}}_2$$



EXERCISE 6.4: Use d'Alembert's principle to find the accelerations of the masses $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$.