Validity in distributive ℓ-monoids

As explained by Simon Santschi

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Details can be found in [1]. For a formula

$$\bigwedge_i s_i \leq \bigvee_j t_j$$

consider the set $\mathrm{is}(s_1, \ldots, s_n, t_1, \ldots, t_m)$ of initial subterms of the s_i and t_j (including the empty term corresponding to the identity). Let

$$\Sigma = \{u \leq v \Rightarrow ux \leq vx \mid ux \neq vx \in \mathrm{is}(s_1, \ldots, s_n, t_1, \ldots, t_m)\}\,.$$

Then we have $|\Sigma| \leq N(N-1)$ where $N = |\,\mathrm{is}(s_1, \ldots, s_n, t_1, \ldots, t_m)|$. Moreover, define

$$\mathsf{fail} = \{s_i > t_j \mid i = 1, \ldots, n, j = 1, \ldots, m\}\,.$$

Then ask Z3 whether $\Sigma \cup \mathsf{fail}$ is satisfiable in $\langle \mathbb{N}, \leq, \rangle$ since this is equivalent to

$$\mathsf{DLM} \not\models \bigwedge_i s_i \leq \bigvee_j t_j.$$

As a possible optimization, consider $\Sigma \cup \mathsf{fail} \cup \mathsf{bd}$ instead, where

$$\mathsf{bd} = \{u \leq |\,\mathrm{is}(s_1, \ldots, s_n, t_1, \ldots, t_m)| \mid u \in \mathrm{is}(s_1, \ldots, s_n, t_1, \ldots, t_m)\}\,.$$

This might actually make it slower, though.

References

[1] Almudena Colacito, Nikolaos Galatos, George Metcalfe, and Simon Santschi, *From distributive l-monoids to l-groups, and back again*, 2021.