1 Covering Spaces

Recall 1.1. Let X, Y be topological spaces and let $p : Y \to X$ be a normal covering map. Then, the group $\mathrm{Aut}(p)$ of deck transformations of p is isomorphic to the quotient $\pi_1(X)/p_*(\pi_1(Y))$.

Conversely, suppose we are given a normal subgroup H of the fundamental group $\pi_1(X)$. We want to construct a normal covering space Y with a covering map $p : Y \to X$ such that $\mathrm{Aut}(p)$ is isomorphic to the quotient $\pi_1(X)/H$. To do this, let $\widetilde{X}$ be the universal cover of X. Consider the universal covering map $\widetilde{p} : \widetilde{X} \to X$ of X. Then the group $\mathrm{Aut}(\widetilde{p})$ of deck transformations of $\widetilde{p}$ is isomorphic to $\pi_1(X)$. Thus, we can consider H as a subgroup of $\mathrm{Aut}(\widetilde{p})$, which is itself a subgroup of the group $\mathrm{Homeo}(\widetilde{X})$. Consider now the orbit space $Y = H\backslash\widetilde{X}$. Because the universal covering map $\widetilde{p}$ is invariant under H we obtain a covering map $p : Y \to X$. Since $\widetilde{X}$ is simply connected, $\pi_1(Y)$ is isomorphic to H. This shows that $\mathrm{Aut}(p)$ is isomorphic to $\pi_1(X)/H$.

The following theorem summarizes what we have shown.

Theorem 1.2. *Let X be a topological space. Then there is a bijection between the set of normal coverings $p : Y \to X$ of X and normal subgroups H of the fundamental group $\pi_1(X)$ of X.*

Remark 1.3. The construction above would also work if instead of choosing the universal cover $\widetilde{X}$ we chose a space Z for which we have that $\pi_1(Z)$ is a normal subgroup of H under the correct identifications.

2 Coxeter Covers