1 The Word Problem for Coxeter Groups

To show that the fundamental groups of pretzel link complements have Coxeter quotients (Theorem 2.2) that are in some sense unique, we first consider a particular solution of the word problem for Coxeter groups (Proposition 1.2). But first, we need to agree on some

Notation 1.1. Suppose we are given a finite set $S = \{s_1, \ldots, s_n\}$ for $n \geq 2$ and a symmetric matrix $M = (m_{ij})$ of size $n \times n$, where the m_{ij} are natural numbers or ∞ such that for $i \neq j$ we have $m_{ij} \geq 2$, and for all i we have $m_{ii} = 1$. Then W is the Coxeter group presented as

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} \text{ for } i, j = 1, \ldots, n \rangle.$$

Moreover, we agree on the convention that

$$(s_i s_j)^{m_{ij}/2} = \begin{cases} (s_i s_j)^k & \text{if } m_{ij} = 2k, \\ (s_i s_j)^l s_i & \text{if } m_{ij} = 2l + 1. \end{cases}$$

E.g., if $m_{ij} = 5$ we write $(s_i s_j)^{5/2} = s_i s_j s_i s_j s_i$.

Now the solution of the word problem for W can be found in Cohen's script, Theorem 4.3.1. Here is a formulation.

Proposition 1.2 (The Word Problem for Coxeter Groups)**.** *Let M be the free monoid generated by $s_1, \ldots, s_n$. Suppose the word $w \in M$ represents the trivial element in W. Then there is a sequence of moves of the following two types carrying w to the empty word $\varepsilon \in M$.*

$$\begin{array}{ccc} (i) & (s_i)^2 & \rightsquigarrow & \varepsilon \\ (ii) & (s_i s_j)^{m_{ij}/2} & \rightsquigarrow & (s_j s_i)^{m_{ij}/2} \end{array}$$

Lemma 1.3. *If $(s_j s_i)^l = (s_k s_j)^m$, where i, j, k are pairwise distinct, then we have that m_{ij} divides l and m_{jk} divides m.*

Proof. We will use the moves (i) and (ii) from Proposition 1.2. Let

$$w = (s_i s_j)^l (s_k s_j)^m.$$

We want to show that w is trivial in W. By removing occurences of $(s_j s_i)^{m_{ij}}$ and $(s_k s_j)^{m_{jk}}$ we can assume that $l < m_{ij}$ and $m < m_{jk}$. If $l \geq m_{ij}/2$, using move (ii) on $(s_i s_j)^l$ and afterwards applying move (i) successively replaces the word $(s_i s_j)^l$ by $(s_i s_j)^{m_{ij} - l}$. So we can even assume that $l < m_{ij}/2$. Similarly, we can arrange $m < m_{jk}/2$. But then there are no more moves to apply that can decrease the number of occurences of s_i or of s_k. But this implies that $l = m = 0$ after iteratively subtracting m_{ij} and m_{jk}, respectively. □

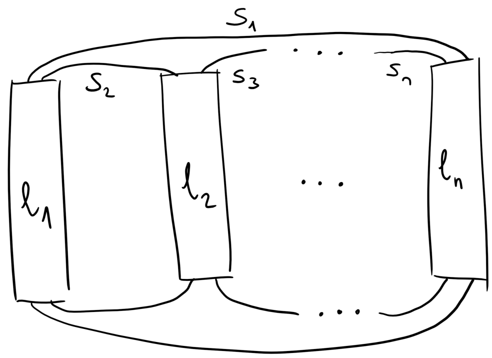

Figure 1: The generators of $\pi_1(S^3 \setminus K)$

2 Coxeter Quotients of Pretzel Link Groups

Definition 2.1. Let K be a link and $G = \pi_1(S^3 \setminus K)$. Then we say that the Coxeter group W is an *S-maximal Coxeter quotient* of G if W is a Coxeter quotient of G with generating set S, and any other Coxeter quotient of G with generating set S is isomorphic to a quotient of W.

Theorem 2.2 (Coxeter Quotients of Pretzel Link Groups)**.** *Let $K \subset S^3$ be a pretzel link. Let $S = \{s_1, \ldots, s_n\}$, where the s_i are the meridians indicated in Figure 1. Then there is a unique S-maximal Coxeter quotient of $\pi_1(S^3 \setminus K)$.*

Proof. By applying the Wirtinger algorithm and postulating $s^2 = e$ for all generators $s \in S$, one just gets the relations

$$(s_2 s_1)^{l_1} = (s_3 s_2)^{l_2} = \cdots = (s_n s_{n-1})^{l_{n-1}} = (s_1 s_n)^{l_n}.$$

By Lemma 1.3, precisely those Coxeter groups generated by S with $m_{i(i+1)}$ a divisor of l_i satisfy these relations. Now let

$$M = \begin{pmatrix} 1 & l_1 & \infty & \infty & \infty & \cdots & \infty & l_n \\ l_1 & 1 & l_2 & \infty & \infty & \cdots & \infty & \infty \\ \infty & l_2 & 1 & l_3 & \infty & \cdots & \infty & \infty \\ \infty & \infty & l_3 & 1 & \ddots & \ddots & \vdots & \vdots \\ \infty & \infty & \infty & \ddots & \ddots & \ddots & \infty & \infty \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 & l_{n-2} & \infty \\ \infty & \infty & \infty & \cdots & \infty & l_{n-2} & 1 & l_{n-1} \\ l_n & \infty & \infty & \cdots & \infty & \infty & l_{n-1} & 1 \end{pmatrix}.$$

Then the Coxeter quotient W corresponding to M is S-maximal. $\square$

3 Bipolarity of Pretzel Coxeter Groups

Definition 3.1. A Coxeter group W is called *spherical* if it is finite. Likewise, a generating set S is called *spherical* if $\langle S \rangle$ is finite. A Coxeter group is called *irreducible* if its Coxeter graph, with the convention that an edge labeled 2 is not an edge, is connected. An *odd component* is a component of the graph whose vertices are S and whose edges are the edges of the Coxeter graph with odd weights. For a subset $T \subset S$ we let $T^\perp$ be the subset of S consisting of all the $s \in S$ that commute with all $t \in T$.

Proposition 3.2. *Let K be a pretzel link with $n \geq 4$ braids, and let S be the standard generating set indicated in Figure 1, and let W be the S-maximal Coxeter quotient of the link group $\pi_1(S^3 \setminus K)$. Then W is bipolar.*

Proof. Let Γ be the graph with vertices S and an edge between s_i and s_j if m_{ij} is finite. By Theorem 1.2 in Caprace-Przytycki it suffices to show the following three things.

1. There is no spherical irreducible component of Γ.
2. There are no subsets $I \subset T$ with T irreducible and I non-empty spherical such that the subgraph induced by $I \cup T^\perp$ separates Γ.
3. If $T \subset S$ is irreducible spherical and an odd component O of S is contained in $T^\perp$, then there are adjacent $t \in O$ and $t' \in S \setminus (T \cup T^\perp)$.

First we consider 1. The graph Γ in question is A_n. This graph is connected so we just need to make sure that W is not spherical irreducible. But this follows from the presence of infinite weights, which is guaranteed by the requirement that $n \geq 4$.

For statement 2., let $T \subset S$ be irreducible. We now distinguish a few cases. If $T = \{t\}$, then $I \cup T^\perp$ consists either of just t, or of two or three adjacent vertices in Γ, in all cases their complement in Γ is connected. If we have $T = \{s, t\}$, then $T^\perp$ consists of at most one vertex adjacent in Γ to both vertices s and t, so we ultimately are in the same situations as in the previous case. Moreover, if $|T| \geq 3$, then $T^\perp$ is empty, and I has at most two elements, which cannot separate S since $n \geq 4$ and the two elements are adjacent in Γ.

Finally, for 3., if $T = \{t\}$ then $T^\perp$ is either empty, one or two non-adjacent vertices in Γ. In each case, O does not exist or is one point. In case such an O does exist, one can find such a t' because $n \geq 4$. Whenever $|T| \geq 2$ we have that $T^\perp$ is empty, so there is no such O. This exhausts all the possibilities. $\square$

4 Maximal Quotients for Fixed Generating Set

The existence of an S-maximal Coxeter quotient of Pretzel links, as shown in Theorem 2.2, is an instance of the following more general phenomenon.

Fix a finite generating set S of size n. Let $M = (m_{ij})$ and $M' = (m'_{ij})$ be Coxeter matrices. We say that M *divides* N if m_{ij} divides m'_{ij} for all i, j. This defines a partial order on the set $\mathcal{M}_S$ of Coxeter matrices with generating set S, with respect to which every subset has a join, namely

$$\bigvee_k (m_{ij}^{(k)}) = \left(\operatorname{lcm}(m_{ij}^{(1)}, m_{ij}^{(2)}, \dots)\right).$$

The following Proposition summarizes our situation.

Proposition 4.1. *The set $\mathcal{M}_n$ of Coxeter matrices of size n equipped with the partial order 'divides' is a complete join-semilattice.*

This gives us a convenient way to construct the S-maximal Coxeter quotient, as we will see in the proof of Theorem 4.3. But first, we need a Lemma.

Lemma 4.2. *Let $M^{(k)} = (m_{ij}^{(k)})$ be Coxeter matrices and let $N^{(k)}$ be the normal subgroup generated by the set of Coxeter relations corresponding to $M^{(k)}$. Then*

$$\bigcap_k N^{(k)} = N$$

where N is the normal subgroup generated by the set of Coxeter relations corresponding to $M = \bigvee_k M^{(k)}$.

Proof. Obviously $N \subset N^{(k)}$ for all k, so we only need to show the other inclusion. Suppose $r \in N^{(k)}$ for all k. First consider the case $n = 2$, i.e., the groups corresponding to $M^{(k)}$ are dihedral groups. In this case the group generated by $R^{(k)}$ is cyclic. Proceed by induction. $\square$

Theorem 4.3 (S-Maximal Quotients)**.** *If a link $L \subset S^3$ has a Coxeter quotient of size n with generating meridians $S = \{s_1, \dots, s_n\}$, then it has a unique S-maximal Coxeter quotient.*

Proof. Let $M = \bigvee \mathcal{M}_{L,S}$ where $\mathcal{M}_{L,S}$ is the set of Coxeter matrices yielding Coxeter quotients of $\pi_1(S^3 \setminus L)$ with generating meridians S. If R are the Wirtinger relations of L with respect to S, then $R \subset N^{(k)}$ for each normal subgroup $N^{(k)}$ generated by the set of relations corresponding to some matrix $M^{(k)} \in \mathcal{M}_{L,S}$. Thus, by Lemma 4.2, we also have $R \subset N$, where N is the normal subgroup generated by the relations corresponding to M. $\square$