1 The Ribbon Disk Group

As with a link group, the *slice disk group* $\pi(S)$ of a slice disk S in D^4 is the fundamental group of its complement in D^4. If the slice disk is in fact a ribbon R, then $\pi(R)$ is called its *ribbon group*. We define the ribbon group of a ribbon in S^3 to be the ribbon group of its embedding into D^4 as in the proof of Theorem **??**. In this section, we are going to give an Wirtinger-like algorithm to compute ribbon groups.

To describe our procedure, we first need to establish some terminology. Let R be a ribbon disk in S^3. A *crossing* is a connected component of the set of ribbon singularities, i.e., the set of double points of the immersion $D^2 \to R$. Note that each crossing has a neighborhood as in Figure 1. Denote by J the set of singularities. Then a connected component of $R \setminus J$ will be called an *arc* of R.

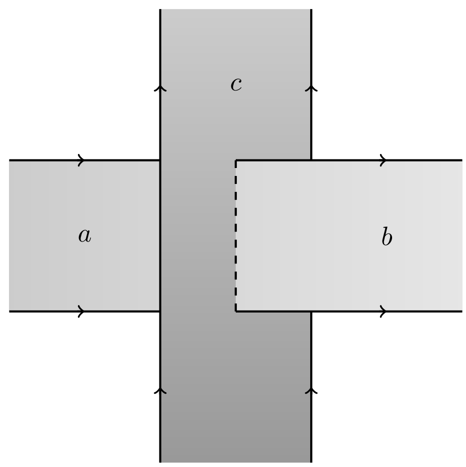

Figure 1: A neighborhood of a crossing with over-arc c and under-arcs a and b

We are now ready to describe the desired procedure, henceforth referred to as the *Ribbon Wirtinger procedure*. Let S be in bijection with the set of arcs of R. Now to any crossing as in Figure 1 assign a relation $bc = ca$. Then the presentation arising from this construction is a presentation of the ribbon group, as we will later show.

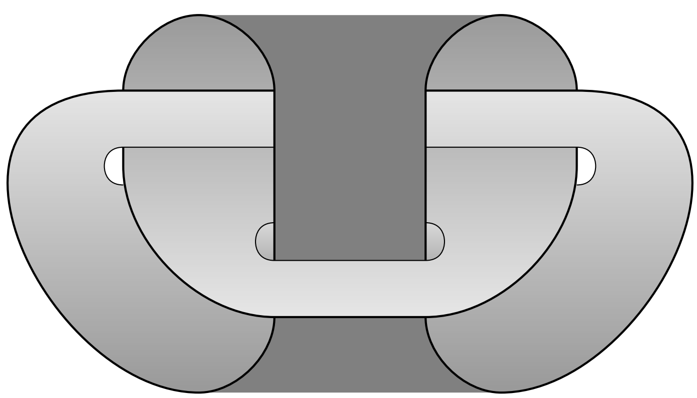

Figure 2: A ribbon disk without neighborhoods of the slits

To prove that the above algorithm indeed gives rise to a presentation of the ribbon group, we will first need to address another (more complicated) procedure based on a Seifert-van Kampen argument not unlike the one we used in the proof of Theorem **??**. First, we consider the embedding of R into D^4

as in the proof of Theorem **??**. Let P be the three-sphere in D^4 at the height of the inner (medium-gray) disk in Figure **??**. Note that P basically contains R, except for a neighborhood of the boundary of R and for neighborhoods of slits, see Figure 2. Now P divides D^4 into two connected components. Let H^+ be the closure of the outer component and H^- the closure of the inner component. The intersection of R with a height in H^+, P and H^-, respectively, is depicted in Figure 3.

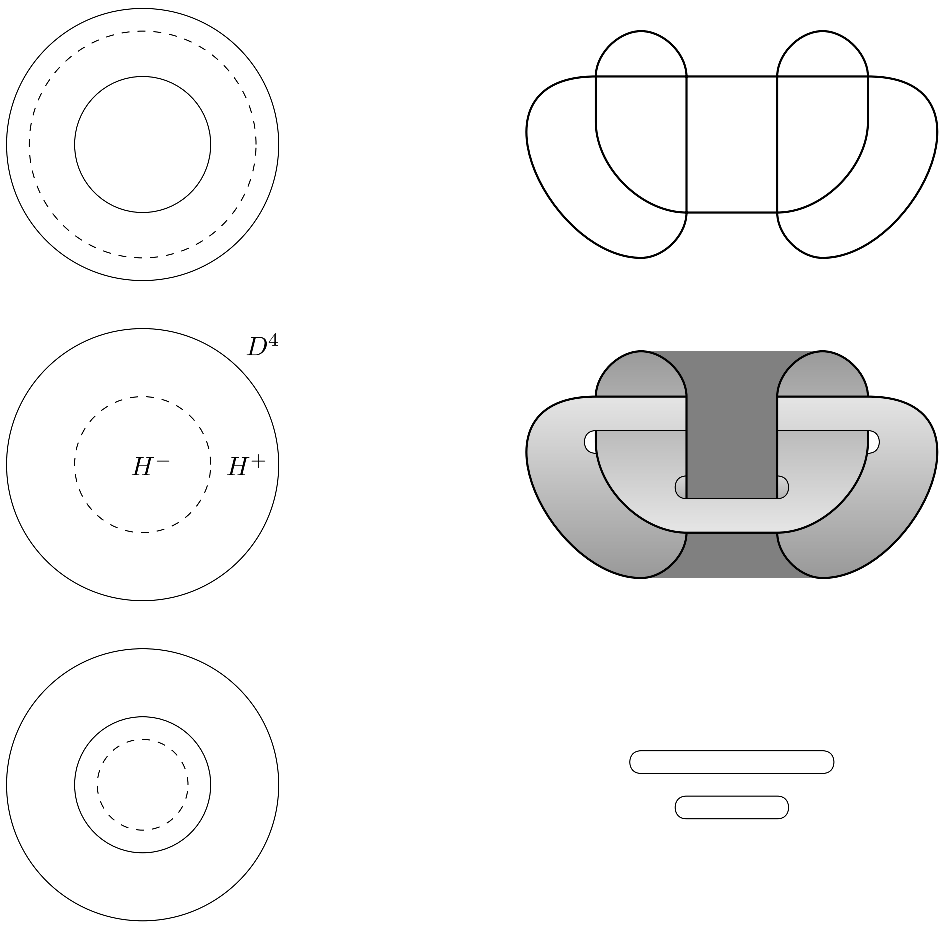

Figure 3: The embedding of R (right) into D^4 (left) at different heights

We are now going to apply the Seifert-van Kampen Theorem to the decomposition

$$D^4 \setminus R = (H^+ \setminus R) \cup (H^- \setminus R).$$

From Figure 3 it is evident that the interior of H^+ retracts onto $S^3 \setminus \partial R$, so we get an isomorphism $\pi_1(H^+ \setminus R) \cong \pi(K)$. Similarly, $\pi_1(H^- \setminus R)$ is a free group generated by meridians of K. One immediate consequence of this is the following.

Proposition 1. *Let R be a ribbon disk for a knot K. Then $\pi(R)$ is a quotient of $\pi(K)$.*

Proof. Any generator of $\pi_1(H^- \setminus R)$ is represented by a meridian in $\pi_1(H^+ \setminus R)$. Thus, by the Seifert-van Kampen Theorem, $\pi(R)$ is isomorphic to a quotient of $\pi_1(H^+ \setminus R)$. $\square$

We are now going to work towards a more concrete description of $\pi(R)$ by inspecting the space $P \setminus R$, see Figure 2. Let us first consider two meridians in $P \setminus R$ that pass through the same hole.

Then, assuming they are coherently oriented, there is a homotopy between said curves in $H^- \setminus R$. This yields a set of relations, referred to as the *same-slit-relations*, identifying meridians in $\pi(K)$ passing through the same slit in $P \setminus R$. An example pair of meridians identified by the same-slit-relations is depicted in Figure 4.

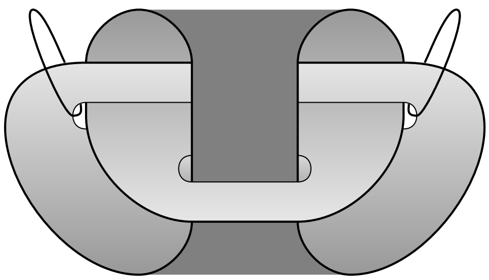

Figure 4: Same-slit-relations

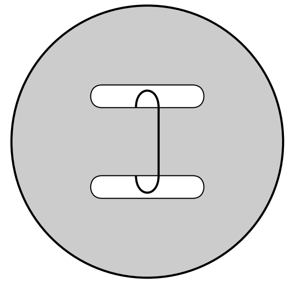

Figure 5: Through-slit-relations

The final set of relations is a little bit more subtle to see. Note that $\pi_1(P \setminus R)$ is not generated by meridians as previously discussed. In addition, we need to consider curves nullhomotopic in $H^+ \setminus P$ that pass through slits. Including such curves into $\pi_1(H^- \setminus R)$ gives rise to the so-called *through-slit-relations*. A toy example can be found in Figure 5. We can now summarize our procedure as follows.

Lemma 1. *Let R be a ribbon disk for K. A presentation of $\pi(R)$ can be obtained by adding same-slit-relations and through-slit-relations to a presentation of $\pi(K)$.*

Figure 6: Interpretation of the arcs in the Ribbon Wirtinger procedure

Theorem 1. *The Ribbon Wirtinger procedure is correct.*

Proof. The main ingredient used to check correctness is interpreting the generating set of the Ribbon Wirtinger procedure as meridians such as the meridian in Figure 6. It is then immediate that the through-slit relations, the same-slit relations and the Wirtinger relations are all satisfied. The converse is also true. □