1 Reflection Representation

Consider a real vector space V of dimension n. A **linear reflection** is a linear map $A \in \mathrm{GL}(V)$ fixing an $(n-1)$-dimensional hyperplane called its **mirror** and one eigenvector of eigenvalue -1, called its **root**.

Let (W, S) be a Coxeter system of type $M = (m_{ij})$. Then a **reflection representation** is a linear representation $\rho : W \to \mathrm{GL}(V)$ obtained as follows. Writing $S = \{s_1, \dots, s_n\}$, consider a real vector space V of dimension n and a basis $\{e_1, \dots, e_n\}$. Let B be a symmetric bilinear form satisfying

$$B(e_i, e_j) = -\cos(\pi/m_{ij})$$

if $m_{ij} < \infty$. Otherwise, if $m_{ij} = \infty$ we just assume $B(e_i, e_j) \leq 1$. The flexibility there is useful to guarantee irreducibility (See Davis for why this is allowed). Then define the representation $\rho : W \to \mathrm{GL}(V)$ by $s_i \mapsto \rho_i$, where

$$\rho_i(x) = x - 2B(x, e_i)e_i.$$

Proposition 1.1. *Any reflection representation obtained by the preceeding construction defines a faithful linear representation.*

Proposition 1.2. *If B is non-degenerate then no non-zero point of V is fixed by all reflections.*

2 Pretzel Links

Let $L = P(l_1, \dots, l_n)$ be a Pretzel link. We assume $|l_i| \geq 2$. We associate to L the Coxeter group

$$W = \langle s_1, \dots, s_n \mid (s_2 s_1)^{l_1}, (s_3 s_2)^{l_2}, \dots, (s_n s_{n-1})^{l_{n-1}}, (s_1 s_n)^{l_n} \rangle.$$

We will call groups of this form **Pretzel Coxeter groups**. Their Coxeter matrix is

$$M = \begin{pmatrix} 1 & l_1 & \infty & \infty & \infty & \cdots & \infty & l_n \\ l_1 & 1 & l_2 & \infty & \infty & \cdots & \infty & \infty \\ \infty & l_2 & 1 & l_3 & \infty & \cdots & \infty & \infty \\ \infty & \infty & l_3 & 1 & \ddots & \ddots & \vdots & \vdots \\ \infty & \infty & \infty & \ddots & \ddots & \ddots & \infty & \infty \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 & l_{n-2} & \infty \\ \infty & \infty & \infty & \cdots & \infty & l_{n-2} & 1 & l_{n-1} \\ l_n & \infty & \infty & \cdots & \infty & \infty & l_{n-1} & 1 \end{pmatrix}.$$

Proposition 2.1. *The Pretzel Coxeter group associated to a Pretzel link L is a quotient of the link group $\pi_1(S^3 \setminus L)$.*

Lemma 2.2. *Let X, Y be real square matrices of the same size, and suppose X is invertible. Then, for all but finitely many $\lambda \in \mathbb{R}$ the matrix $\lambda X + Y$ is invertible.*

Proof. The function $\mu \mapsto \det(X + \mu Y)$ is a non-zero polynomial, so $X + \mu Y$ is invertible for almost all μ in $\mathbb{R}$. Dividing by μ and letting $\lambda = \pm\mu^{-1}$ we see that whenever $X + \mu Y$ is invertible, so is $\lambda X + Y$. $\square$

Proposition 2.3. *Let $L = P(l_1, \ldots, l_n)$ be a Pretzel link. Then the Pretzel Coxeter group associated to L admits a linear reflection representation of degree n such that the only point fixed by all reflections is zero.*

Proof. This includes multiple applications of Lemma 2.2. If $n = 2$, then B is positive definite and thus non-degenerate. If $n = 3$ then B might note be positive definite, but one can easily check with Gauss elimination that B is non-degenerate.

Next, we consider the case $n = 4$. It is not strictly necesarry to consider this case separately but it serves to illustrate the strategy to come. Let

$$X_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 1 & b_1 & 0 & b_4 \\ b_1 & 1 & b_2 & 0 \\ 0 & b_2 & 1 & b_3 \\ b_4 & 0 & b_3 & 1 \end{pmatrix},$$

where $b_i = -cos(\pi/l_i)$. By Lemma 2.2 there exists $\lambda \leq -1$ such that the bilinear form B with matrix $\lambda X + Y$ is invertible, in which case the radical of B is trivial. Thus, the reflection representation determined by B is such that no point except zero is fixed. This concludes the proof for $n = 4$.

For $n = 5$ note similarly that the matrix

$$X_5 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

is invertible. Defining Y_5 similarly as in the case $n = 4$ yields the desired result.

Now let n be arbitrary. Define Z_n to be the matrix of the standard bilinear form (introduced e.g. in Humphreys) corresponding to the Coxeter matrix M. More concretely we have

$$Z_n = \begin{pmatrix} 1 & b_1 & -1 & -1 & -1 & \cdots & -1 & b_n \\ b_1 & 1 & b_2 & -1 & -1 & \cdots & -1 & -1 \\ -1 & b_2 & 1 & b_3 & -1 & \cdots & -1 & -1 \\ -1 & -1 & b_3 & 1 & \ddots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \ddots & \ddots & \ddots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 & b_{n-2} & x \\ -1 & -1 & -1 & \cdots & -1 & b_{n-2} & 1 & b_{n-1} \\ b_n & -1 & -1 & \cdots & -1 & -1 & b_{n-1} & 1 \end{pmatrix}$$

where $b_i = -\cos(\pi/l_i)$. Let X be the block matrix

$$X_{2k} = \begin{pmatrix} 0 & I_{n/2} \\ I_{n/2} & 0 \end{pmatrix}, \quad X_{2k+5} = \begin{pmatrix} 0 & 0 & I_{k/2} \\ 0 & X_5 & 0 \\ I_{k/2} & 0 & 0 \end{pmatrix}$$

where I_m is the identity matrix of size m and X_5 is as in the case $n = 5$. Then let $Y = Z + X$ and as above apply Lemma 2.2. $\square$

Corollary 2.4. *The meridional rank of a Pretzel link $L = P(l_1, \dots, l_n)$, where we assume $l_i \geq 2$, is equal to n.*

Proof. Since $\pi_1(S^3 \setminus L)$ is generated by the obvious meridians we have that the meridional rank is less than or equal to n. Moreover, the meridional rank is bounded from below by n because by irreducibility of the reflection representation the Pretzel Coxeter quotient associated to L is not generated by less than n reflections. Thus the meridional rank is equal to n. $\square$

Conjecture 2.5. *Any other Coxeter quotient of $\pi_1(S^3 \setminus L)$ is a quotient of the Pretzel Coxeter group.*

Sadly this is not true, because the $(3,3,3)$-Pretzel knot has an S_3-quotient.

Question 2.6. *Can this be applied to produce a rigid cover of $\pi_1(S^3 \setminus L)$ or of S^3 branched along L?*

This is probably not so interesting.

3 Realisation of Coxeter Groups

Question 3.1. *Let W be a Coxeter group. Does there exist a link $L \subset S^3$ such that W is a quotient of $\pi_1(S^3 \setminus L)$?*

Yes. The trivial link.

Proposition 3.2. *If $K \subset S^3$ is a knot and W is a quotient of $\pi_1(S^3 \setminus K)$, then at most one edge in the Coxeter diagram of W is labeled 2.*

Proof. If not then $\pi_1(S^3 \setminus K)$ has a quotient of the form $(\mathbb{Z}/2)^k$ for some $k \geq 2$. But the abelianization of the π-orbifold quotient is isomorphic to $\mathbb{Z}/2$, which does not have a quotient isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$. □