Department of Mathematics
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Coxeter Quotients of Link Groups

Master Thesis of

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Contents

Introduction

One of the most classical invariants in knot theory, and also the second most important invariant in this master thesis, is the bridge index of links, introduced by Schubert [24] in 1954. A *bridge* of a link is a segment of a diagram as in Figure 1, namely an arc of the diagram that passes over at least one other (or the same) arc. The *bridge index* is the number of bridges in a diagram, minimized over all diagrams of the link. Although this definition might seem very combinatorial, it is not really. The bridge index admits a very geometrical interpretation (see Section 2.2), which makes the bridge index a fundamentally geometrical quantity.

A *meridian* of a link is a 'lollipop curve', schematically depicted in Figure 2. In this picture, the meridian is denoted by the letter m. The point x_0 is the base point of the fundamental group of the complement of the link. Usually one imagines this point sitting between the eyes of the reader. The *meridional rank* of the link is the number of meridians needed to generate the fundamental group of the link complement. This quantity is very much about the fundamental group of the link complement, or rather about the conjugacy classes corresponding to meridians.

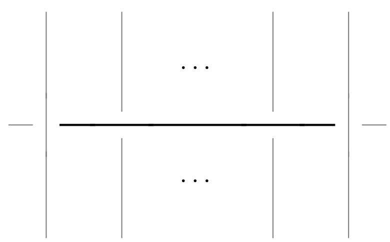


Figure 1: A bridge

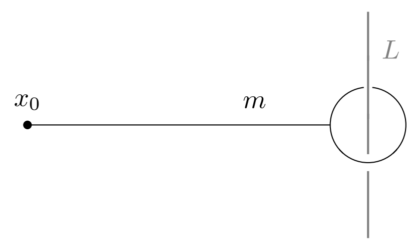


Figure 2: A meridian

Any arc in a diagram corresponds to a meridian, as can be guessed by looking at Figure 2. A variation of the classical proof of the Wirtinger Theorem (Theorem 2.5) yields that the meridional rank is bounded from above by the bridge index (Theorem 2.6). The meridional rank conjecture asks about the converse inequality, namely, whether the meridional rank of a link is equal to its bridge index. It was first asked by Cappell and Shaneson [8] in 1978. The question has also received some attention by being Problem 1.11 in Kirby's list [13]. In the meantime, it has been answered positively by Boileau and Zimmermann for knots of meridional rank two [5], using the two-fold cyclic branched cover of the link complement. It has also been answered positively by Baader, Blair and Kjuchukova for two other classes of links: twisted links, and arborescent links associated to bipartite trees with even weights [1]. A special case of both of these classes are the Pretzel links. The method used in the mentioned article is to find specific quotients of the link group which are known to not admit small generating sets from a (specific) conjugacy class, namely quotients isomorphic to Coxeter groups. A more complete list of previous results about the meridional rank conjecture is listed in Remark 3.5.

It remains an open question whether the meridional rank conjecture is true for all links. This question is interesting because it relates the algebra of the link group to the geometry of the embeddings of the link.

This master thesis aims to illustrate how the theory of quotients isomorphic to Coxeter groups can be used to prove the meridional rank conjecture for certain links. It is organized as follows. In

Section 1 we introduce and go over the basic theory of Coxeter groups. In Section 2 we do the same for knot theory. The main section of this thesis is Section 3, where the method of using Coxeter quotients to get information about the meridional rank is described. Section 4 is a case study about torus links, where it is illustrated that the method of computing the meridional rank of links using Coxeter quotients sometimes fails. The final part, Section 5, illustrates that it might be interesting to apply the theory developed in Section 3 to other settings, in this particular case to the fundamental group of the complement of ribbon disks in the 4-ball. It is a rather experimental part, but it is interesting to see some connections to the initial setting.

I am grateful to Sebastian Baader for introducing me to this fascinating subject and for his continuous effective support and helpful advice. It was great fun to write this thesis.

1 Coxeter Groups

Coxeter groups are of interest for example in geometric group theory, where they form a particularly nice family of examples. This viewpoint is explained in a book by Davis [11]. In this thesis we are going to exploit their nice properties and apply them to knot theoretic concepts such as the meridional rank, compare Section 3.3.

In this section we are going to introduce the basics of the theory of Coxeter groups, limited to just what we will use afterwards. This is by no means a proper introduction; the reader completely unfamiliar with the theory of Coxeter groups might want to read Chapter 5 from Humphreys' book [17] before reading what follows.

1.1 Coxeter Matrices and Graphs

Let S be a finite set and let $M = (m_{st})$ be a symmetric $S \times S$ matrix with coefficients in the set of natural numbers, possibly ∞, satisfying $m_{ss} = 1$ for all $s \in S$, and $m_{st} \geq 2$ if $s \neq t$. Such a matrix is called a *Coxeter matrix*. Consider the group presentation

$$W = \langle S \mid (st)^{m_{st}} = 1 \text{ for } s, t \in S \rangle,$$

where $(st)^{\infty} = 1$ means that there is no relation involving s and t. Then the group W is called the *Coxeter group* corresponding to M. The tuple (W, S) is called a *Coxeter system* and the cardinality $\#S$ is referred to as its *rank*. Since in this thesis Coxeter groups always arise with a generating set, we blur the distinction between Coxeter groups and Coxeter systems.

Another way to encode the coefficients m_{st} is to use them as weights in weighted undirected graphs, so-called *Coxeter graphs*. In this case, the nodes of the graph associated to W are the generators S, and the weight between the nodes s and t is m_{st}. The following shorthand notations are very common: If $m_{st} = 2$, then the edge between s and t is omitted. If $m_{st} = 3$, then the weight on the edge between s and t is omitted. Finally, if $m_{st} = 4$ then instead of a labeled line we draw an unlabeled double line.

Example 1.1 (Dihedral Groups)**.** The symmetry group of a regular n-gon is called the *dihedral group* of order $2n$, denoted by the symbol D_n. This group is generated by two reflections s, t whose composition is a rotation by an angle of $2\pi/n$. This is used to show that D_n is isomorphic to the Coxeter group

$$W = \langle s, t \mid (st)^n = s^2 = t^2 = 1 \rangle.$$

Thus, Coxeter groups on two generators whose product has finite order are dihedral groups.

This observation is often used to motivate the introduction of an additional dihedral group, the so-called *infinite dihedral group* with presentation

$$D_{\infty} = \langle s, t \mid s^2 = t^2 = 1 \rangle.$$

1.2 The Reflection Representation

A well-known construction leading to an easy solution of the word-problem for Coxeter groups and also yielding a geometric viewpoint for the study of Coxeter groups is a faithful representation of any

Coxeter group where its generating set acts as a set of reflections with respect to a (possibly degenerate) bilinear form.

Let (W, S) be a Coxeter system. Then we consider the bilinear form on the vector space $V = \mathbb{R}^S$ given by the matrix $B = (b_{st})$ with entries $b_{st} = -\cos \pi/m_{st}$. For a generating reflection s in S let ρ_s be the *reflection* with respect to B with root the basis vector e_s in $\mathbb{R}^S$ given by the formula

$$\rho_s(x) = x - 2B(x, e_s)e_s,$$

where x is a vector in V.

Proposition 1.2. *Let (W, S) be a Coxeter system and let $s, t \in S$. Then the reflections ρ_s and ρ_t satisfy $(\rho_s\rho_t)^{m_{st}} = 1$.*

Proof. Let n be the number of elements of S. Observe that each reflection ρ_s fixes an $(n-1)$-dimensional hyperplane, namely the orthogonal complement of the non-isotropic vector e_s. Now fix generating reflections s and t in S and consider the subspace $V_{st} = \mathbb{R}e_s \oplus \mathbb{R}e_t$, which is invariant under the maps ρ_s and ρ_t. Since the orthogonal complement of V_{st} is fixed by ρ_s and ρ_t, the order of $\rho_s\rho_t$ on V is equal to its order on V_{st}.

Let us now distinguish two cases. If m_{st} is finite, B restricted to V_{st} is positive definite. To see this, consider an arbitrary non-zero vector $v = ae_s + be_t$ in V_{st} and compute

$$\begin{aligned} B(v, v) &= a^2B(e_s, e_s) + 2abB(e_s, e_t) + b^2B(e_t, e_t) \\ &= a^2 + 2ab\text{cos}(-\pi/m_{st}) + b^2 \\ &> 0. \end{aligned}$$

Thus, V_{st} is isometric to $\mathbb{R}^2$ with the standard inner product, and the angle between the invariant subspaces of ρ_s and ρ_t is equal to $2\pi/m_{st}$. This shows that ρ_s and ρ_t are symmetries of a regular polygon and their product is a rotation by $2\pi/m_{st}$, which has order m_{st}. On the other hand, if $m_{st} = \infty$ the matrix of $\rho_s\rho_t$ is given by

$$\rho_s\rho_t = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

which has infinite order. □

This shows that the map $s \mapsto \rho_s$ induces a representation of W, called the *reflection representation* of W. We will conclude this section by showing that this representation is faithful. But first we need some preparation. To any Coxeter group W we can assign a *length function* $l : W \to \mathbb{N}$, assigning to any element w of W the minimal number of generating reflections needed to write down an expression for w. An expression for w of length $l(w)$ is called a *reduced expression* for w. To get a little bit of a feeling how the length function behaves for Coxeter groups (or more generally, for groups generated by elements of order two subject only to relations of even length, such as the reflection quotient from Section 3.1), we prove the following illustrative result.

Proposition 1.3. *Let W be a Coxeter group, $w \in W$ and $s \in S$. Then $l(ws) = l(w) \pm 1$.*

Proof. First of all, we make the observation that for any words $w, w' \in W$ we have $l(ww') \leq l(w) + l(w')$. This immediately implies that $l(ww') \geq l(w) - l(w')$ by applying the above to the words ww' and w'^{-1} instead of w and w', observing that $l(w) = l(w^{-1})$ for all w in W. We conclude that

$$l(w) - 1 \leq l(ws) \leq l(w) + 1.$$

The only thing left to show is that in fact $l(ws) \neq l(w)$. To see this, we introduce the *sign* homomorphism $\varepsilon : W \to \{\pm 1\}$ defined via $s \mapsto -1$ for all s in S. Indeed, this extends to W since all relations of W are of even length. Obviously we have $\varepsilon(w) = (-1)^{l(w)}$, and $\varepsilon(ws) \neq \varepsilon(w)$. Using this, we conclude that $l(ws) \neq l(w)$, which is precisely what we wanted to show. □

For a non-zero vector v in V we will say that v is *positive* if v is a non-negative linear combination of the basis vectors e_s. Moreover, v is *negative* if $-v$ is positive. This enables us to formulate the following important Lemma relating the length function to the reflection representation.

Lemma 1.4. *Let W be a Coxeter group, w a word in W and s a generating reflection in S. Then the vector we_s in V is positive if and only if we have that $l(ws) = l(w) + 1$.*

Proof. This is taken from [17]. Note first that it suffices to prove that whenever $l(ws) = l(w) + 1$ we have that we_s is positive, since applying this to ws instead of w proves its converse. The proof of this implication is a proof by induction on $l(w)$. Suppose $l(w) = 0$. Then w is the identity and we have that $l(ws) = l(w) + 1$ for all s.

For the inductive step, let t be any reflection in S such that $l(wt) = l(w) - 1$, taking for example t to be the last symbol in a reduced expression for w. Write $w = w'w_{st}$, where w_{st} is a word in the symbols s, t of maximal length such that the length of the expression $w'w_{st}$ is equal to $l(w)$. This exists since w_{st} equal to the empty word is in fact a (non-optimal) solution. We will now prove the Lemma by showing that the vectors $w'e_s$, $w'e_t$ and $w_{st}e_s$ are all positive, and that w_{st} is non-trivial. Taking $w_{st} = t$ shows that $l(w') < l(w)$, so we can apply the induction hypothesis to w', showing that the vectors $w'e_s$ and $w'e_t$ are both positive, provided we can show $l(w's) > l(w')$ and $l(w't) > l(w')$. Suppose toward a contradiction that $l(w's) < l(w')$. Then sw_{st} is in fact a longer word than w_{st}, contradicting the maximality of w_{st}. Similarly it is also true that $l(w't) > l(w')$.

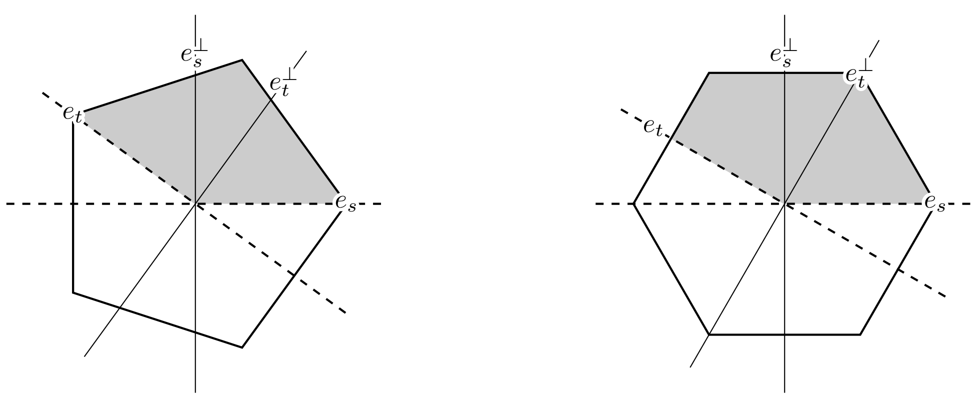


Figure 3: Illustration of the fact that e_s lands in the positive cone, indicated in grey. Note that composing reflections in the mirrors $e_s^\perp$ and $e_t^\perp$ results in a rotation by an angle of $2\pi/m_{st}$.

Finally we draw a picture to show that $v_{st}e_s$ is positive. Note first that v_{st} does not have a reduced expression ending in s, since if it did we would actually reduce its length by adding an s to the end. In particular, the length of w_{st} is strictly less than m_{st}, that is, interpreting w_{st} as a map on the plane as in the proof of Proposition 1.2, w_{st} is not rotation by an angle of π. The claim now follows from the fact that w_{st} is a rotation by at most $\pi - \pi/m_{st}$, perhaps followed by a reflection in $e_t^{\perp}$, but only if this does not result in a rotation by an angle of π. Also compare Figure 3. □

Having established how the length function is related to the action of W on the basis vectors of its geometric representation, we immediately obtain what we desired.

Theorem 1.5. *The reflection representation is faithful.*

Proof. Suppose w lies in the kernel of the reflection representation. Then obviously $we_s > 0$ for all generating reflections s in S, so by Lemma 1.4, $l(ws) = l(w) + 1$ for all s. That is, for any s we have that w does not have a reduced expression ending in s. But this implies that w is the identity. □

1.3 Roots

The main goal of this section is to show that Coxeter groups have tiny center, that is, the center of an infinite Coxeter group is trivial. To do this, our main tool will be the action of a Coxeter group on a specific subset of V called roots.

The set of *roots* of a Coxeter group W, denoted by the letter Φ, is the set of all images of the basis vectors e_s under W. The geometric observation justifying this terminology is the following. Let w be an arbitrary element of W. Then, for all generating reflections s in S, the map wsw^{-1} acts on V by a reflection whose eigenvector of eigenvalue one is the vector we_s.

Note that by Lemma 1.4, every root is either positive or negative. Writing Π for the set of positive roots, we can express this in symbols as $\Phi = \Pi \cup -\Pi$. This will be useful for a geometric characterization of the length function l on a Coxeter group. Consider the function n mapping an element $w \in W$ to the number of positive roots sent to negative roots by the action of w on V.

Proposition 1.6. *Let W be a Coxeter group. Then, for any element w of W we have that $n(w) = l(w)$.*

Proof. Let s be a generating reflection in S. We first prove that the only positive root mapped to a negative root by s is the basis vector e_s. Choose any positive root $\alpha \neq e_s$. Since α is not equal to e_s there exists a reflection t in S such that the coefficient of α corresponding to the e_t-component of α is non-zero. But the reflection s does not change the e_t-component of α, so $s\alpha$ must be positive.

We now proceed by induction on the number $l(w)$. Suppose $l(ws) > l(w)$, and that $n(w) = l(w)$. By Lemma 1.4 it follows that we_s is positive, which shows that $wse_s = -we_s$ is negative. Note that since w induces a bijection on the remaining positive roots $\Pi \backslash \{e_s\}$, we must have $n(ws) = n(w)+1 = l(ws)$. □

Recall that the *radical* $V^{\perp}$ of the bilinear form B on V is the set of vectors v' in V such that we have $B(v, v') = 0$ for all v in V. Note that $V^{\perp}$ is not affected by any reflection, so in particular $V^{\perp}$ is a W-invariant subspace. We will now prove a partial converse to this for *irreducible* Coxeter groups, that is, Coxeter groups whose associated Coxeter graph is connected.

Theorem 1.7. *Let W be an irreducible Coxeter group. Then, every proper W-invariant subspace of V is contained in $V^\perp$.*

Proof. Let V' be a W-invariant subspace of V. Suppose first that some basis vector e_s is contained in V'. Let t be adjacent to s in the Coxeter graph of W. Then te_s has a non-zero e_t-component and is contained in V', so it follows that also e_t is in V'. Proceeding like this through the connected Coxeter graph of W we obtain that V' contains a basis of V. Therefore, V' is not a proper subspace.

We have established that all proper W-invariant subspaces of V do not contain any basis vectors e_s. Let V' be such a subspace. Then V' is contained in an intersection of eigenspaces of the generating reflections s in S. But since the proper subspace V' does not contain the basis vectors e_s, we have that V' must be contained in the eigenspaces of s in S of eigenvalue 1, in other words, in the orthogonal complements of the basis vectors e_s. □

Corollary 1.8. *The center of an infinite Coxeter group is trivial.*

Proof. Let A be any endomorphism of V commuting with every element of W. We will show that A is actually multiplication with a scalar. Let $s \in S$ be any generating reflection. Since A commutes with s their eigenspaces must agree. In particular, the line $\mathbb{R}e_s$ is an eigenspace of A of eigenvalue c for some c.

Let V' be the kernel of the map $A - c$. Let $v \in V'$. Then $Av = cv$. But then we have

$$\begin{aligned}(A-c)wv &= Awv - cwv \\ &= wAv - cwv \\ &= w(A-c)v \\ &= 0,\end{aligned}$$

so V' is W-invariant. Since V' contains $\mathbb{R}e_s$ and $\mathbb{R}e_s$ is not contained in $V^\perp$, we obtain by Theorem 1.7 that $V' = V$, in other words, that A is multiplication with c.

We now consider what happens if A is actually an element of W. If $c > 0$ then all roots in Π are mapped to positive roots, so by Lemma 1.4 we have that A is the identity. If $c < 0$ we first make the following observation: Π is finite if and only if W is. Now observe that all positive roots in Π are mapped to negative roots, so the length of A as an element of W is infinite by Lemma 1.4, which is absurd. □

1.4 Conjugacy Classes

Imposing as the title of this section may sound, we are not going to be able to give a description of how conjugacy classes work in Coxeter groups. We will however be able to determine the number of conjugacy classes in the set of reflections.

Let W be a Coxeter group generated by the set S. A *reflection* is an element $w \in W$ that is conjugate to an element in S. We will write T for the set of reflections.

Proposition 1.9. *Let Γ be the Coxeter graph of a Coxeter group W, and let $s, t \in S$. Then s and t are conjugate if and only if there exists a path in Γ from s to t whose edges only consist of odd weights.*

Proof. Recall that being conjugate is an equivalence relation. By transitivity it suffices to prove, for sufficiency, that if there exists a relation $(st)^{m_{st}}$ for odd m_{st}, then s and t are conjugate. But with this relation we have for $k = (m_{st} - 1)/2$ that $t = (st)^k s(ts)^k$.

We prove the converse by contraposition. If there exists no such path between s and t, the subgraph Γ_0 of Γ consisting of the same vertices but only of the odd weighted edges is disconnected. Moreover, s and t lie in distinct path-components of Γ_0. Let $\varepsilon : W \to \{\pm 1\}$ be the homomorphism induced by sending each reflection in S that lies in the same component as s to -1, and each of the other reflections to 1. This is well-defined. To see this, note that the weights between two elements in S that are not sent to the same element under ε are always even, so the defining relators in W indeed lie in the kernel of ε considered as a map on the free group $F(S)$.

Thus, ε is a homomorphism to an abelian group mapping s and t to distinct elements. But since the image of a conjugacy class lies in one conjugacy class, this shows that s and t are not conjugate. $\square$

Let Γ_0 be as in the proof of Proposition 1.9. Then we can restate the above as follows: *There is exactly one conjugacy class in T for each component of* Γ_0. In particular, all reflections in W are conjugate if and only if Γ_0 is connected.

1.5 The Word Problem

The reflection representation gives us a way to write down elements of a Coxeter group numerically, in effect solving the word problem, whicih asks whether a given element of a group is equal to the identity. This section outlines another, more syntactic approach to the word problem. The described procedure was found by Tits [26].

Notation 1.10. Let W be a Coxeter group with generating set S and Coxeter matrix $M = (m_{st})$. Then we agree on the convention that

$$(st)^{m_{st}/2} = \begin{cases} (st)^k & \text{if } m_{st} = 2k, \\ (st)^l s & \text{if } m_{st} = 2l + 1. \end{cases}$$

E.g., if $m_{st} = 5$ we write $(st)^{5/2} = ststs$.

Theorem 1.11 (Tits)**.** *Let F be the free monoid generated by the set S. Suppose the word $w \in F$ represents the trivial element in a Coxeter group W with generating set S and Coxeter matrix $M = (m_{st})$. Then there is a sequence of moves of the following two types carrying w to the empty word.*

(*i*) *removing occurrences of s^2 from w for $s \in S$.*

(*ii*) *replacing occurrences of $(st)^{m_{st}/2}$ by $(ts)^{m_{st}/2}$ for $s, t \in S$.*

In particular, there is a sequence of the above moves carrying w to the empty word without making w longer.

1.6 Finite Coxeter Groups

Finite Coxeter groups differ from infinite ones in a few key ways. For example, with the techniques introduced in Section 1.3 we can prove the following interesting result.

Theorem 1.12. *Let W be a finite Coxeter group. Then W has a unique longest element w_0.*

Proof. Note that by Proposition 1.6 we have that the length function is bounded from above by the cardinality $|\Pi|$. Moreover, if $w \in W$ sends all e_s to negative roots, then we_s is a negative linear combination of basis vectors, whence w sends all positive roots to negative roots. In other words, if a word $w \in W$ is not of length $|\Pi|$, then there exists a basis vector e_s not sent to a negative root. But then, by Lemma 1.4 the word ws is longer than w. This shows that the length $|\Pi|$ is actually attained by some word $w_0 \in W$.

If w_1 is another word of maximal length, we have that $w_0 w_1$ maps all positive roots to positive roots, so by Proposition 1.6 we have that its length is zero, i.e., $w_0 w_1$ is the identity. It follows that the words w_0 and w_1 are equal. $\square$

It turns out that finite Coxeter groups can be completely classified. This feat was first accomplished by Coxeter [10] himself. The classification of finite irreducible Coxeter groups will be useful for us in Section 4. Since this is a well-documented exercise in combinatorial graph theory, see [17], we will omit the proof and only state the theorem.

Theorem 1.13. *Any finite irreducible Coxeter group can be found in Table 1.*

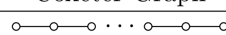
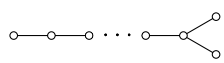
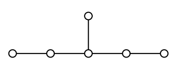
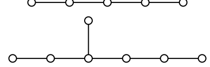
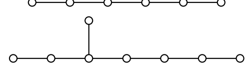
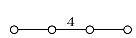
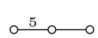
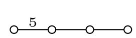
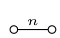
Coxeter Graph	Name	Size
	A_n	$(n+1)!$
	B_n	$2^n n!$
	D_n	$2^{n-1} n!$
	E_6	$72 \cdot 6!$
	E_7	$72 \cdot 8!$
	E_8	$192 \cdot 10!$
	F_4	1152
	H_3	120
	H_4	14400
	$I_2(n)$	$2n$

Table 1: Classification of Irreducible Finite Coxeter Groups

2 Knot Theory

We quickly recall the basic notions of knot theory. It is expected that the reader is familiar with the notion of bridge index and knows how to compute fundamental groups of link complements. All of this information can be found in Rolfsen's textbook [22].

2.1 Knots and Links

A *knot* K is the image of a smooth embedding of S^1 into S^3. A *link* L is a disjoint union of finitely many knots. Two links $L \subset S^3$ and $L' \subset S^3$ are said to be *equivalent* if there exists an orientation-preserving diffeomorphism between the pair (S^3, L) and the pair (S^3, L'), i.e., a diffeomorphism $S^3 \to S^3$ that restricts to a diffeomorphism of the links $L \to L'$.

Example 2.1 (Torus Links)**.** Let p, q be integers. Consider the universal covering $\mathbb{R}^2 \to T^2$, where the set $T^2 \subset \mathbb{R}^3$ is a standardly embedded torus. For an explicit parametrisation, consider e.g. Example 6 of Section 2-2 in Do Carmo's book [12]. Then the (p, q)-*torus link* is the image of the lines through the points $\mathbb{Z}e_1$ of slope p/q.

Example 2.2 (Pretzel Links)**.** The $(l_1, \ldots, l_k)$-*Pretzel link* is the link obtained by connecting twist regions in the way indicated in Figure 4. We will write $P(l_1, \ldots, l_k)$ for this link. An example is drawn in Figure 5.

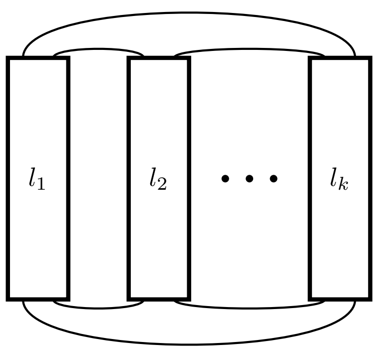


Figure 4: $P(l_1, l_2, \ldots, l_k)$

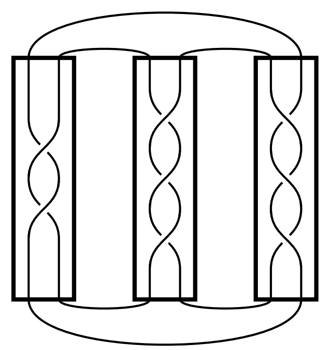


Figure 5: $P(2, 3, -3)$

2.2 The Bridge Index

Let $L \subset S^3$ be a link. The minimal number of local maxima of a smooth function $S^3 \to \mathbb{R}$ restricted to links equivalent to L is called the *bridge index* of L and referred to as $b(L)$. This can be diagrammatically characterized in the following way. Let D be a diagram of L. An *arc* in D is a segment of D beginning and ending with an undercrossing, containing no undercrossings in between. A *bridge* of D is an arc of D that contains at least one overcrossing.

Proposition 2.3. *The bridge index of a link $L \subset S^3$ is equal to the number of bridges in a diagram D of L, minimized over all diagrams.*

Proof idea. This is an exercise in [22]. One inequality is fairly straightforward: Note that a bridge of a diagram induces a maximum of the height function. Thus, the minimal number of maxima (i.e., the bridge index), is less than or equal to the number of bridges. The converse inequality is a bit trickier, involving flipping a diagram to its side and a general position argument. □

The following was first proven by Schubert [24].

Proposition 2.4. *Let K and K' be knots. Then $b(K\#K') = b(K) + b(K') - 1$, where $K\#K'$ denotes a so-called connected sum of K and K'.*

2.3 The Link Group

The *link group* $\pi(L)$ of a link $L \subset S^3$ is the fundamental group of its complement in S^3. In symbols this reads $\pi(L) = \pi_1(S^3 \setminus L)$. If L is a knot K then the link group $\pi(K)$ is referred to as its *knot group*.

The knot group essentially classifies all knots, in that if two prime knots have isomorphic knot groups then they are either equivalent or mirrors of each other [15]. Be warned that this is no longer true for links, see [22]. This suggests that studying the knot group could reveal a lot about the knot in question.

We will now describe an algorithm that computes the link group of an oriented link. Let $L \subset S^3$ be a link and let D be any diagram of L. Let $S = \{m_1, \ldots, m_k\}$ be the set of diagram meridians, oriented from right to left. A meridian m_i should be interpreted at as the meridian starting at the reader's eye, which will serve as the base point of the fundamental group, passing under the arc and going back to the reader's eye without any detour. Having assigned a generator to every crossing, we now assign a relator r to any crossing, as indicated in Figure 6.

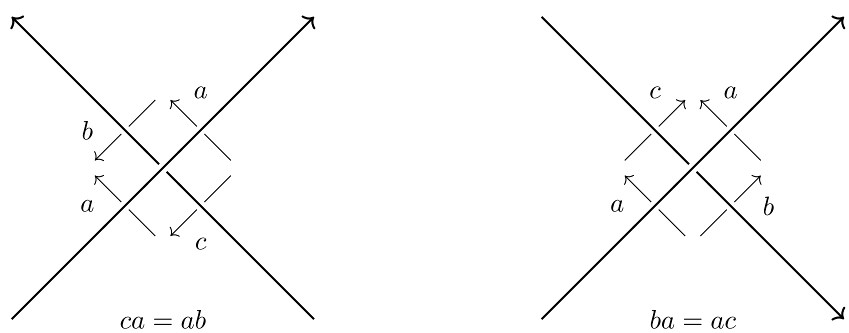


Figure 6: Relations in the Wirtinger Presentation

Since it is very well known that this procedure works, it would be reasonable to omit the proof. But we are going to copy the argument and apply it to slice disks later on (see Section 5.1), so it will be useful to go over the proof again quickly.

Theorem 2.5. *Let L be a link. Then the presentation resulting from performing the Wirtinger algorithm is a presentation of the link group $\pi(L)$.*

Proof. This is the proof given in [22]. In order to apply van Kampen's theorem [16], first we make sure that L lies entirely in a plane P, except for the crossings, where the understrand passes under the plane. Suppose all understrands reach the same maximal distance ε to P and reach it in exactly one connected component of L intersected with P^-. Here P^- is the plane parallel to P of distance ε in the down-direction, marked in grey in Figure 7. Position the base point x somewhere above P. We now subdivide S^3 as follows. Let H^+ be the closed half-space above P^- intersected with the link complement. Then H^+ contains x. Moreover, the fundamental group of H^+ is free with generating set the meridians of L. Similarly, let H^- be the closed half-space below P^-, also intersected with L. In contrast to H^+ we have that H^- is simply connected.

The fundamental group of P^- (with respect to any base point) is also free, but the generators this time are curves like γ in Figure 7. In H^-, these curves are trivial because H^- is simply connected. On the other hand, in H^+ they represent words like $b^{-1}aca^{-1}$, where a, b and c are diagram meridians (or inverses thereof) of arcs close to the crossing considered. These are exactly the Wirtinger relations. By van Kampen's Theorem, these relations yield a presentation of the link complement, which is what we wanted to show. $\square$

Theorem 2.6. *Let $L \subset S^3$ be a link. Then $\pi(L)$ is generated by $b(L)$ meridians.*

Proof. This is very similar to the proof of Theorem 2.5. This time we don't dip down for undercrossings as in Figure 7 but we stay up as long as possible after the overcrossings. Then the generators given by the van Kampen argument are precisely the bridges in the diagrammatic interpretation of bridges. $\square$

3 Coxeter Quotients

3.1 The Reflection Quotient

Let $L \subset S^3$ be a link. A *Coxeter quotient* of L is a quotient $\pi(L)/N$ of the link group which is isomorphic to a Coxeter group W such that every meridian in $\pi(L)$ corresponds to a reflection in W. In many cases, finding Coxeter quotients of links is rather straight-forward, at least with the help of a computer. This is because checking whether a specific set of Coxeter relations induces a Coxeter quotient of a link amounts to checking the validity of the defining relations of its link group.

The *reflection quotient* is the group $r(L) = \pi(L)/\langle\langle m^2 \rangle\rangle$, where m is any meridian of L and $\langle\langle m^2 \rangle\rangle$ denotes the normal subgroup of $\pi(L)$ generated by m^2. Because it is clumsy to write down a presentation in this fashion we agree to add the relations s^2 for all generators s in S when writing a presentation as $\langle S \mid R \rangle^{(2)}$. Using this notation, the reflection quotient can be defined to be

$$r(L) = \pi(L)^{(2)}.$$

Coxeter quotients of a link L are also quotients of $r(L)$. An additional reason to focus on this group instead of $\pi(L)$ is that the amount of information lost by ignoring squares of meridians is small in most cases. This can be formally justified using the following.

Theorem 3.1 (Boileau-Zimmermann [5], Theorem 1)**.** *Let K and K' be prime knots such that $r(K)$ and $r(K')$ are infinite. Then K and K' are equivalent if and only if $r(K)$ and $r(K')$ are isomorphic.*

It is easy to write down a presentation of the reflection quotient from a presentation of the link group $\pi(L) = \langle S \mid R \rangle$ of a link $L \subset S^3$. Just add the relations s^2 for all s in S, and simplify R by replacing all occurrences of s^{-1} by s and all occurrences of s^2 by the empty word. These presentations are generally shorter, easier to read and less prone to mistakes since it is no longer possible to confuse meridians with their inverses. Note in addition that the Wirtinger algorithm (see Section 2.3) can be simplified so that the relation $ca = ab$ in Figure 6 reads $acab = 1$ in the reflection quotient. We will now mainly consider the reflection quotient instead of the link group.

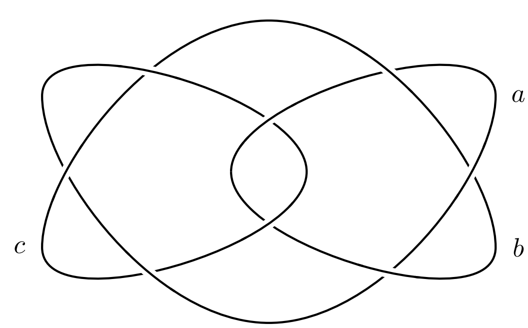


Figure 8: The knot 8_5

Example 3.2 (The Knot 8_5)**.** Consider $K = 8_5$ from Rolfsen's knot table, compare Figure 8. Then K has a Coxeter quotient with graph A_3, whose associated Coxeter group is isomorphic to the permutation group S_4. Note that by the Wirtinger algorithm, the reflection quotient (defined in Section 3.1) has a presentation

$$r(8_5) = \langle a, b, c \mid babc(ba)^2bc(ba)^4, babc(ba)^4cbabc(ab)^2a\rangle^{(2)}.$$

These relations are satisfied in the Coxeter group

$$W = \langle a, b, c \mid (ab)^2, (ac)^3, (bc)^3\rangle^{(2)}.$$

Thus, W is a quotient of $\pi_1(S^3 \setminus 8_5)$.

Note that there exist presentations of the knot group of 8_5 such that the above method does not work. For example, any presentation involving more than three meridians makes the method fail. This indicates that it is difficult to prove that a fixed link L does not admit a Coxeter quotient isomorphic to a specific Coxeter group W.

3.2 Examples of Coxeter Knots

A *Coxeter knot* is a knot K admitting a Coxeter quotient of rank $b(K)$. In Theorem 3.3 we prove that this implies that K satisfies the meridional rank conjecture. The following list of Coxeter knots in Rolfsen's table [22] was found using the following algorithm. First, compute a presentation of the reflection quotient using the Wirtinger algorithm, modified such that the presentation does not use more generators than the bridge index (which is 3 in all of the cases considered below) of the knot in question. Then, use some solution of the word problem in Coxeter groups to find all Coxeter quotients that map generating meridians to generating reflections. Note that this algorithm may not detect all Coxeter knots. A slight improvement can be obtained by considering multiple presentations, i.e., choosing different minimal generating sets of $\pi(L)$. This algorithm runs well for knots of few crossings, but takes infeasibly long for knots of crossing number 13 or higher.

Because all 2-bridge knots are Coxeter (see Section 3.4), Table 2 consists of all 3-bridge knots in Rolfsen's table with up to nine crossings that were found to be Coxeter by the above algorithm.

Knot	Coxeter Quotients
8_5	$\circ - \circ - \circ$
8_{10}	$\circ - \circ - \circ$
8_{15}	$\circ - \circ - \circ$
8_{19}	$\circ - \circ - \circ$
8_{20}	$\circ - \circ - \circ$
8_{21}	$\circ - \circ - \circ$
9_{16}	$\circ - \circ - \circ$
9_{22}	$\circ \overset{5}{-} \circ - \circ$
9_{24}	$\circ - \circ - \circ$
9_{25}	$\circ \overset{5}{-} \circ - \circ$
9_{28}	$\circ - \circ - \circ$
9_{30}	$\circ \overset{5}{-} \circ - \circ$

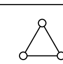
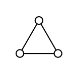
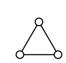
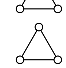
Knot	Coxeter Quotients
9_{35}	$\circ - \circ - \circ$ and 
9_{36}	$\circ \overset{5}{-} \circ - \circ$
9_{37}	$\circ - \circ - \circ$ and 
9_{40}	$\circ - \circ - \circ$ and $\circ \overset{5}{-} \circ - \circ$
9_{42}	$\circ \overset{5}{-} \circ - \circ$
9_{43}	$\circ \overset{5}{-} \circ - \circ$
9_{44}	$\circ \overset{5}{-} \circ - \circ$
9_{45}	$\circ \overset{5}{-} \circ - \circ$
9_{46}	$\circ - \circ - \circ$ and 
9_{48}	$\circ - \circ - \circ$ and 

Table 2: List of Coxeter knots with bridge index 3

The 3-bridge knots that were not found to have a rank 3 Coxeter quotient are $8_{16}, 8_{17}$, 8_{18}, 9_{29}, 9_{32}, 9_{33}, 9_{34}, $9_{38}, 9_{39}, 9_{41}, 9_{47}$ and 9_{49}. It remains open whether any of those knots admits a Coxeter quotient of rank 3. Data concerning knots of crossing number 10 can be found in Appendix A.

3.3 Meridional Rank

For a link L we define its *meridional rank* to be the least possible number of meridians of L needed to generate the group $\pi(L)$. Here, a *meridian* is any generator of $H_1(S^3 \setminus L) = \pi(L)^{\mathrm{ab}}$. We define the *Coxeter rank* of a link L, abbreviated $c(L)$, to be the maximal number n such that L has a Coxeter quotient of rank n. Abbreviating the bridge index of L by $b(L)$, and the meridional rank of L by $\mu(L)$, we have the following.

Theorem 3.3. *Let $L \subset S^3$ be any link. Then we have $c(L) \leq \mu(L) \leq b(L)$.*

Proof. The inequality $\mu(L) \leq b(L)$ is Theorem 2.6. Moreover, $c(L) \leq \mu(L)$, is a direct consequence of the fact that the number of generating reflections is determined by the Coxeter system. For a proof, see for example Lemma 2.1 in [14]. □

The following conjecture was formulated by Cappell and Shaneson [8].

Conjecture 3.4 (Meridional Rank Conjecture)**.** *The meridional rank of a link is equal to its bridge index.*

Remark 3.5 (Previous results)**.** So far, the meridional rank conjecture was proven for the following classes of links.

- Generalized Montesinos links (Boileau-Zieschang [27], 1985)
- Torus links (Rost-Zieschang [23], 1987)
- Links of meridional rank two (Boileau-Zimmermann [5], 1989)
- Generalized Montesinos links (Lustig-Moriah [20], 1991)
- A large class of iterated torus knots (Cornwell-Hemminger [9], 2014)
- Links with meridional rank three whose two-fold branched covers are graph manifolds (Boileau-Jang-Weidmann [4], 2015)
- Knots whose exteriors are graph manifolds (Boileau-Dutra-Jang-Weidmann [3], 2017)
- A family of knots with unknotting number one yet arbitrarily high bridge number (Baader-Kjuchukova [2], 2017)
- Twisted links and arborescent links associated to bipartite trees with even weights (Baader-Blair-Kjuchukova [1], 2019)

The definitions of generalized Montesinos links given in [27] and [20] do not agree.

Let us refer to a link $L \subset S^3$ that has a Coxeter quotient of rank $n = b(L)$ as a *Coxeter link*. Then we obtain the following from $c(L) \geq b(L)$ together with Theorem 3.3.

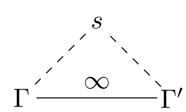
Corollary 3.6. *Coxeter links satisfy the meridional rank conjecture.*

Another application of the techniques we have developed so far lies in the following.

Theorem 3.7. *Let K and K' be knots. Then $c(K\#K') = c(K) + c(K') - 1$.*

Proof. Let W and W' be Coxeter quotients of K and K', respectively. First we choose diagrams of K and K' such that the generators of $\pi(K)$ and $\pi(K')$ that get mapped to generating reflections of W and W' are diagram meridians. Let $s \in \pi(K)$ and $s' \in \pi(K')$ each be one of those diagram meridians.

Now observe that $\pi(K\#K')$ is of the form $\pi(K) * \pi(K')/ss'$. Thus, we can explicitly write down a Coxeter quotient of $K\#K'$ as follows. Let Γ and Γ' be the Coxeter graphs associated to W and W'. Then there is a Coxeter quotient of $K\#K'$ with graph



which has rank rk $W +$ rk $W' - 1$. This proves that $c(K\#K') \geq c(K) + c(K') - 1$. The other inequality follows from the fact that any maximal Coxeter quotient of $K\#K'$ induces Coxeter quotients of K and of K' whose ranks add up to $c(K\#K') + 1$. □

Corollary 3.8. *There are knots of arbitrarily high meridional rank.*

Proof. Just take iterated connected sums of any two-bridge knot. □

Corollary 3.9. *Connected sums of Coxeter knots are Coxeter knots.*

Proof. Let $J, K \subset S^3$ be arbitrary knots. Then we have already shown that $b(J\#K) = b(J) + b(K) - 1$ and $c(J\#K) \leq b(J\#K)$. If J and K are both Coxeter knots then equality immediately follows from Theorem 3.7. □

3.4 Rank Two Quotients

In this section we have a particularly close look at dihedral quotients of link groups. As we shall see, this is closely related to a popular notion in the literature, described for example in an article by Przytycki [21].

Let L be a link. We will refer to a homomorphism $c : \pi(L) \to D_n$ mapping all meridians of L to reflections in D_n as a *Fox n-coloring*. Note that a surjective Fox n-coloring is a dihedral quotient of L. We will call a Fox n-coloring c *non-trivial* if the image of c contains more than one reflection in D_n, and we will call a link L *Fox n-colorable* if L admits a non-trivial Fox n-coloring. Note that any link admits trivial Fox n-colorings for all n.

Example 3.10 (Fox 3-colorings)**.** Let a, b, c be the reflections of D_3. Then a Fox 3-coloring of L is just a labeling of the arcs in a diagram of L such that at any crossing either only one symbol appears or all three symbols appear. The reader may convince themselves of the fact that the trefoil knot in Figure 9 is Fox 3-colorable, whereas the figure-eight knot in Figure 10 is not.

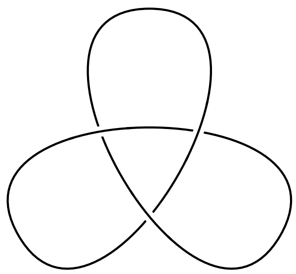


Figure 9: The Trefoil Knot

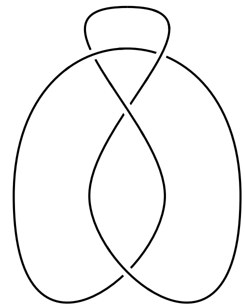


Figure 10: The Figure-Eight Knot

Let n be a prime number. We will now present an algorithm to check whether a given link L is n-colorable. First we make the following identification of the set of reflections with $\mathbb{Z}_n$. Write

$$D_n = \langle a, b \mid (ab)^n \rangle^{(2)}.$$

Then the set R of reflections is the set of elements of the form $(ab)^k a$. We will identify this reflection with the element k in $\mathbb{Z}_p$. Under this identification, the Wirtinger relation at any crossing reads

$$2k = i + j,$$

see Figure 11. Indexing the set of diagram meridians and the set of crossings in some manner we can interpret this as a system of linear equations over $\mathbb{Z}$. Let us call the matrix of this system $\widehat{C}$. Note that this system admits trivial solutions, namely the ones assigning the same element of $\mathbb{Z}$ to all diagram meridians. Ignoring this set of solutions corresponds to deleting a row and a column of $\widehat{C}$ to obtain what we will call the *coloring matrix* C.

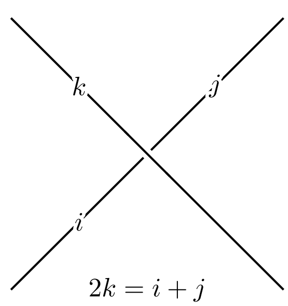


Figure 11: The Wirtinger relation at a crossing after identification of diagram meridians with $\mathbb{Z}_p$

Lemma 3.11. *Let L be any link and n be prime. Then L is Fox n-colorable if and only if n divides $\det C$, where C is the coloring matrix of L.*

Proof. The condition for the linear system described above to have non-trivial solutions is for C to be singular over $\mathbb{Z}_n$, which is obviously equivalent to n dividing $\det C$ as a matrix with integer coefficients. □

One can prove that the absolute value of the determinant of C neither depends on the diagram, nor on the choice of which row and column to delete from $\widehat{C}$. For this reason, we will refer to the determinant $|\det C|$ simply as the *determinant* of L, written $\det L$. In the literature the determinant of L is usually defined to be its Alexander polynomial evaluated at -1, but it turns out that these definitions are actually equivalent [19]. It can be shown [18] that the determinant of a 2-bridge link L is never equal to one. This implies that L has a dihedral quotient. Thus, all 2-bridge links are Coxeter.

3.5 Realization

In this section we are going to show that many Coxeter groups are Coxeter quotients of knots. In fact, *all* Coxeter groups are Coxeter quotients of links.

Proposition 3.12. *Let L be the trivial link on n components. Then the link group $\pi(L)$ is free on n generators which are meridians. In particular, any Coxeter group W is a Coxeter quotient of $\pi(L)$.*

Since this is a little underwhelming we will try to do better. The following construction is essentially due to Brunner [6].

Procedure 3.13 (Brunner's Construction)**.** Let Γ be any Coxeter graph, this time with edges labeled 2 for commuting generators and no edge for no relations (contrary to the convention above). If Γ is not planar, choose a maximal planar subgraph instead of Γ and fix an embedding of Γ into the plane.

Now consider the dual graph Γ^* of Γ whose vertices are the faces of Γ, with edges weighted m connecting vertices s, t if and only if there is an edge labeled m separating the faces of Γ corresponding to s, t. Note that Γ^* is not necessarily a simple graph, even if Γ is (e.g., if Γ is a tree).

We now interpret Γ^* as a set of instructions how to construct a surface S whose boundary will be a link $L(\Gamma^*)$ admitting a Coxeter quotient isomorphic to the Coxeter group corresponding to Γ. First, blow up the vertices of Γ^* to disks. Now replace each edge labelled m between disks by bands with m twists. This procedure is illustrated in Figure 12.

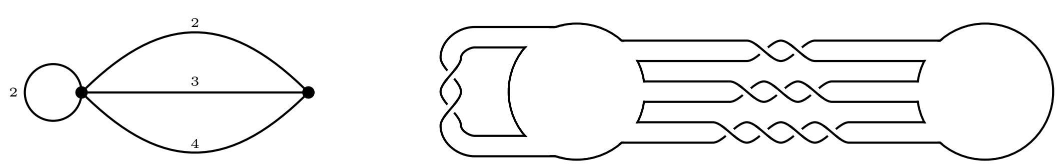


Figure 12: Brunner's construction producing from Γ^* (left) a link (right)

Beware that knots constructed in this way will not be prime.

Theorem 3.14. *The link constructed by Brunner's construction* (Procedure 3.13) *starting with a graph Γ yields a link that has a Coxeter quotient isomorphic to the Coxeter group W corresponding to a graph Δ, whose associated Coxeter group surjects onto the Coxeter group corresponding to Γ.*

A proof of this can be found in Brunner's paper [6]. Theorem 3.14 is much more interesting than the construction in Proposition 3.12. Indeed, we are almost ready to answer the following question: *What Coxeter groups are Coxeter quotients of knots?* But first, we need the following lemma.

Lemma 3.15. *Let Γ be a Coxeter graph with no cycles, i.e., a tree, whose edges are all odd. Then Brunner's construction* (Procedure 3.13) *applied to Γ yields a knot.*

Proof. Note that the dual graph of a tree is a wedge of circles, that is, a graph with only one vertex. So Brunner's construction yields a disk with bands with an odd number of twists attached. We will prove by induction on the number of bands that the boundary of this surface S has only one component.

First note that if there are no bands attached, the boundary of the disk is the unknot. This concludes the base case of the induction. Now fix an innermost band B, and start at the point p on the boundary of S that lies immediately to the left of B. Let q be the point on S that lies immediately to the right of B. Then tracing the boundary as in Figure 13 shows that there exists a path from p to q that only passes through B.

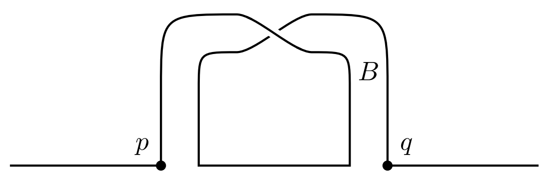


Figure 13: Tracing a band arising from Brunner's construction on a tree. Note that the situation is the same if instead of one twist there are an odd number of twists in B.

Contracting this path yields a surface with one band less. By the induction hypothesis the boundary of this is a knot. Thus, so is the boundary of S. $\square$

Theorem 3.16. *Let W be a Coxeter group. Then W is a Coxeter quotient of a knot if and only if the set T of reflections in W is a single conjugacy class.*

Proof. Let Γ be the graph of W, and let Γ_0 be the subgraph of Γ only consisting of the odd-weighted edges as in Section 1.4. Since all meridians in a knot group are conjugate it follows that if Γ_0 is disconnected, then W is not a Coxeter quotient of a knot.

Conversely, if Γ_0 is connected, apply Brunner's construction to Γ_0. This is a knot by 3.15. Moreover, it has a Coxeter quotient isomorphic to the Coxeter group $\widetilde{W}$ corresponding to the graph Γ_0 whose non-edges are replaced by edges labeled ∞. This group itself has a Coxeter quotient isomorphic to W. $\square$

4 Torus Knots

It turns out that torus knots tend to admit few Coxeter quotients. This will be discussed in Section 4.1, where it is shown that an infinite class of torus knots does not admit any non-trivial Coxeter quotients, further adding to the potentially familiar statement that *knots of determinant one admit poor representation theory*. As an upshot, in Section 4.2 we show that an infinite family of torus knots does admit high-rank Coxeter quotients.

Proposition 4.1. *Let p, q be positive integers. Then we have that the bridge index of the (p, q)-torus knot is p. In particular, the (p, q)-torus knot does not admit any Coxeter quotients of rank higher than p.*

This was first proved by Schubert himself [24]. There is also a more modern and potentially more accessible proof by Schultens [25].

4.1 Odd Weights

The qualitative features as far as Coxeter quotients go depend heavily on the weights p, q of the (p, q)-torus knots. If we require p and q to be odd, we get particularly few Coxeter quotients. In fact, if p and q are odd, the (p, q)-torus knot can only have finite Coxeter quotients, as is hinted at in the proof of Theorem 4.4. A further fact that leads us to expect few Coxeter quotients lies in the fact that if p and q are odd, then the (p, q)-torus knot has determinant one. To see this, consider the following.

Theorem 4.2. *Let p, q be odd positive integers. Then the Alexander polynomial of $K = T(p, q)$ is*

$$\Delta_K(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^q - 1)(t^p - 1)}.$$

A proof of this fact can be deduced from Example 9.15 in Burde and Zieschang's book [7]. Because the determinant of a knot is its Alexander polynomial evaluated at -1, we immediately obtain that $\det T(p, q) = 1$ whenever p and q are odd.

Corollary 4.3. *Let p, q be odd. Then the (p, q)-torus has determinant one.*

In Section 3.4 we have seen that this implies that torus knots with odd weights do not admit dihedral quotients. This brings us one step closer to the claim that torus knots with odd weights have very few Coxeter quotients. Let us first fix $p = 3$.

Theorem 4.4. *Let q be an odd positive integer such that q has no prime factor less than or equal to 5. Then the $(3, q)$-torus knot does not admit any non-trivial Coxeter quotients.*

Before proving this, we need an easy lemma.

Lemma 4.5. *The only irreducible finite Coxeter groups of rank three with cyclic abelianization have graphs A_3 or H_3. In particular, any such Coxeter group has order 24 or 120.*

Proof. This follows directly from the classification of finite irreducible Coxeter groups, see Table 1. $\square$

Proof of Theorem 4.4. By Lemma 3.11 it suffices to prove that $K = T(3, q)$ does not admit any rank three Coxeter quotients because K has determinant one. Toward a contradiction, suppose W is a Coxeter quotient of K of rank three. Since K is a knot we have that W is irreducible. We will now consider two cases.

First, if W is infinite, then W has trivial center. Let $G = \pi(K)$ and consider the quotient map $\varphi : G \to W$. Recall that this means that for any meridian m of K we have that $\varphi(m)$ is a reflection in W. Since φ is surjective we have that the center of G is mapped into the center of W, and is thus sent to the identity. Recall, e.g., from Rolfsen [22], that G has a presentation

$$G = \langle a, b \mid a^3 = b^q \rangle$$

where a and b are meridian and longitude of the torus on which K lies, respectively. Moreover, the center of G is generated by the element $a^3 = b^q$, which is a composition of an odd number of diagram meridians of K since both 3 and q are odd. But this is a contradiction: the image of $a^3 = b^q$ in W under φ is orientation-preserving because it is the identity, and orientation-reversing because it is the product of an odd number of reflections.

Finally, suppose that W is finite. Then we have that W is isomorphic to either S_4 or $\mathbb{Z}_2 \times A_5$ since K is a knot. Note that $\varphi(b)$ has order dividing $2q$, but by assumption q has no divisor that is also a divisor of the order of W, which is either $24 = 2^3 \cdot 3$ or $120 = 2^3 \cdot 3 \cdot 5$. So $\varphi(b)$ has order 2, but then $\varphi(b)$ is the unique orientation-reversing element of the center of W. Similarly, $\varphi(a)$ is the same element. But then W is trivial. □

A very similar Theorem is true for $p \geq 5$, as another close look at the classification of finite irreducible Coxeter groups in Table 1 shows.

Lemma 4.6. *No irreducible finite Coxeter group of odd rank $p \geq 5$ has an element of order $q > p$, where q is prime.*

Theorem 4.7. *Let $5 \leq p < q$ be any odd coprime integers such that q has no prime factor less than or equal to p. Then the (p, q)-torus knot does not admit any non-trivial Coxeter quotients.*

Proof. This is completely analogous to the proof of Theorem 4.4, using Lemma 4.6 instead of 4.5. □

4.2 Close Weights

This section contains an upshot showing that some torus knots in fact do have Coxeter quotients, namely the $(n, n+1)$-torus knots.

Example 4.8 ($n = 4$)**.** Consider the braid representation of the $(4, 5)$-torus knot in Figure 14.

Considering the first three strands we obtain a presentation of the reflection quotient of K, namely

$$r(K) = \langle a, b, c, d \mid adcbababcd, abadcbabcbabcd, abcbadcbabcdcbabcd \rangle^{(2)}.$$

The rules $a \mapsto (1\ 2)$, $b \mapsto (2\ 3)$, $c \mapsto (3\ 4)$, $d \mapsto (4\ 5)$ extend to a surjective homomorphism $r(K) \to S_5$, because all relators are mapped to the identity.

Theorem 4.9. *Let $n \geq 2$ be an integer. Then the $(n, n+1)$-torus knot admits a Coxeter quotient isomorphic to S_{n+1}. In particular, $(n, n+1)$-torus knots are Coxeter.*

Proof. Similarly as before we consider the braid representation of the $(n, n+1)$-torus knot in Figure 15. The first $n-1$ strands give us the following defining relations.

- $s_1 = s_n \cdots s_1 s_2 s_1 \cdots s_n$
- $s_1 s_2 s_1 = s_n \cdots s_1 s_2 s_3 s_2 s_1 \cdots s_n$
- $\ldots$
- $s_1 \cdots s_i \cdots s_1 = s_n \cdots s_1 \cdots s_{i+1} \cdots s_1 \cdots s_n$ for $1 \leq i \leq n-1$
- $\ldots$
- $s_1 \cdots s_{n-1} \cdots s_1 = s_n \cdots s_1 \cdots s_n \cdots s_1 \cdots s_n$

These relations are all satisfied under the map $s_i \mapsto (i \quad i+1)$. To see this, compute

$$s_{i+1} \cdots s_1 \cdots s_{i+1} \cdots s_1 \cdots s_{i+1} \mapsto (1 \;\; i+1)$$

which is fixed by conjugation with s_j for $j > i$. So the right hand side of the relation is equal to $(1 \;\; i+1)$. But $s_1 \cdots s_{i-1} \mapsto (1 \;\; \cdots \;\; i)$, which conjugates s_i to $(1 \;\; i+1)$, so the left hand side is also equal to $(1 \;\; i+1)$. $\square$

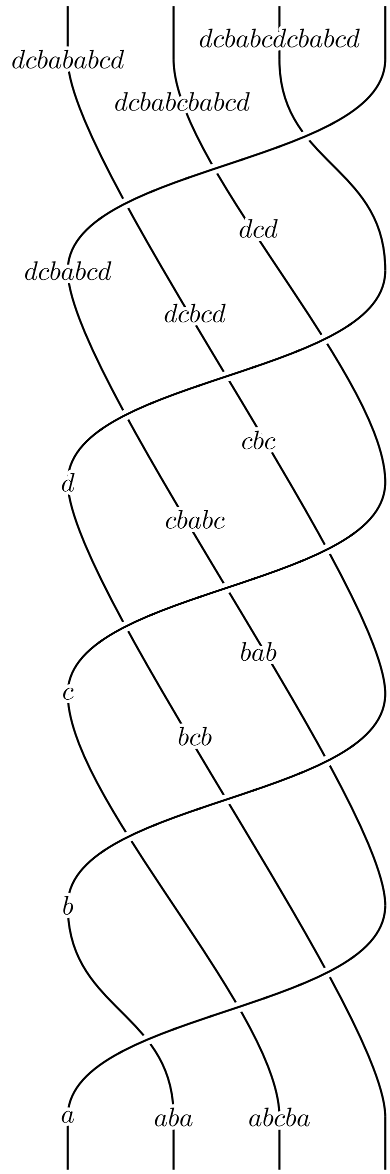


Figure 14: The case $n = 4$

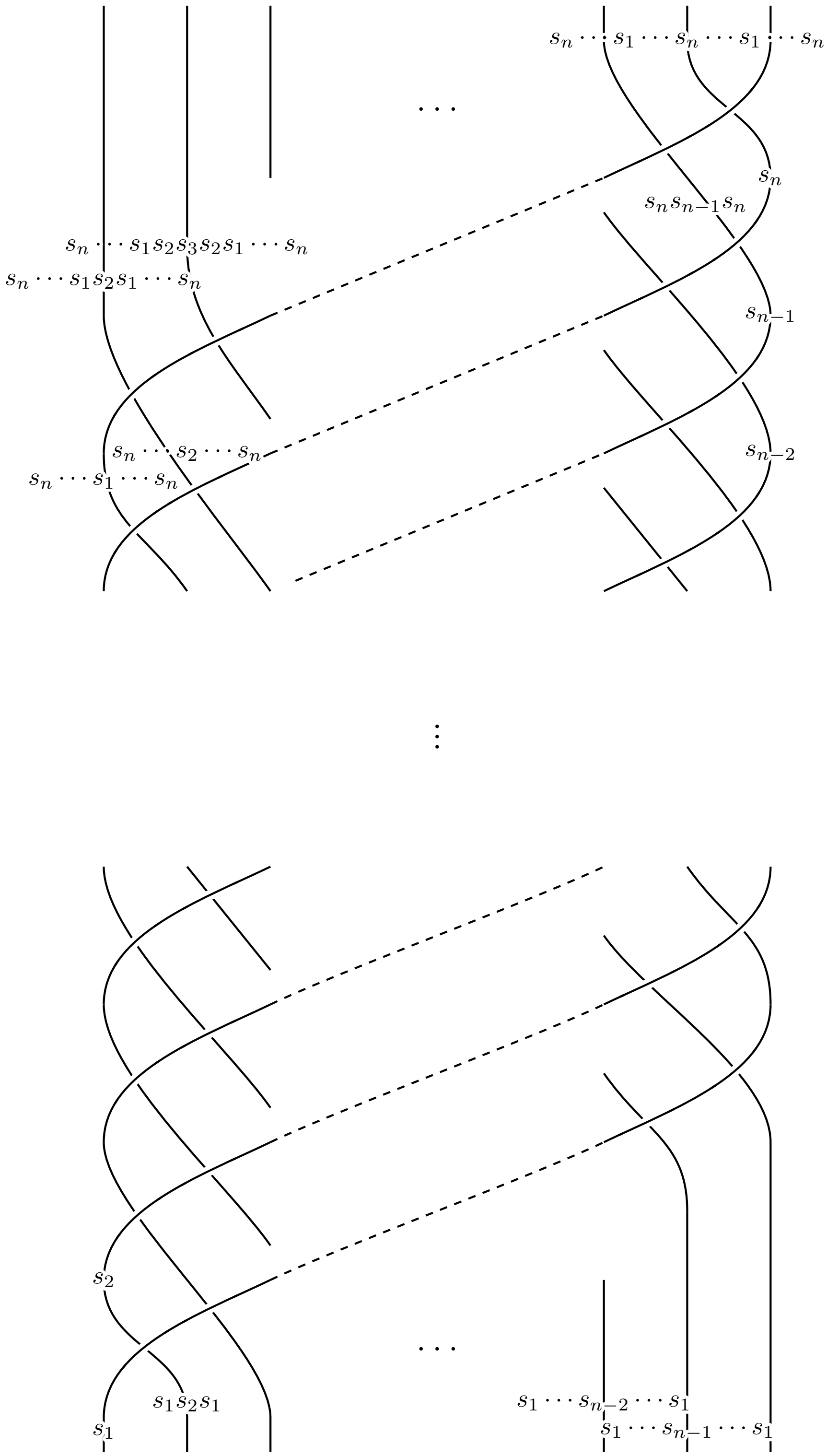


Figure 15: The braid representation of the $(n, n+1)$-torus knot

5 Ribbon Theory

To conclude this thesis, this section is devoted to applying the results and strategies of Section 2 to ribbon disks in the four-ball.

5.1 Ribbon and Slice Disks

A *ribbon disk* is a smooth immersion $D^2 \to S^3$ of a disk D^2 that only has single and double points such that it becomes injective after removing finitely many intervals from the interior of D^2. A *ribbon knot* is a knot that bounds a ribbon disk.

This can be visualized as follows. Let us call connected components of double points of a ribbon disk *ribbon singularities*. Then a ribbon singularity looks like a slit cut into the interior of the ribbon in order to allow another part of the ribbon to pass through, see Figure 16. The singularity in Figure 17 is not a ribbon singularity because the slit that we would need to cut involves a point on the boundary of the ribbon.

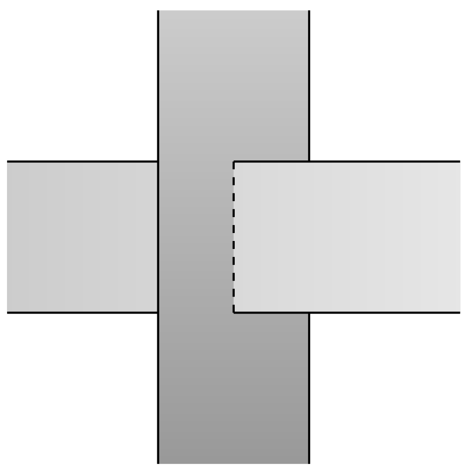


Figure 16: A ribbon singularity

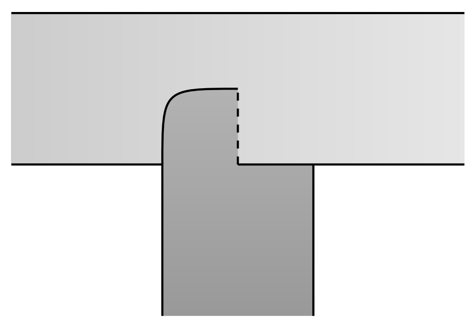


Figure 17: Not a ribbon singularity

We start by considering a particular family of ribbon knots.

Theorem 5.1. *Let K be any knot and $\bar{K}$ its mirror image. Then the connected sum $K\#\bar{K}$ is a ribbon knot.*

Proof sketch. Suppose P is the plane such that orthogonal reflection in P maps the knot $K\#\bar{K}$ onto itself. In the example in Figure 18, the plane P is represented by the dashed line. Now connect corresponding points on the knot by straight lines perpendicular to P. By a general position argument arising from the observation that projecting $K\#\bar{K}$ onto P is an immersed interval, we can adjust K such that the surface formed in this way is a ribbon disk. □

Example 5.2. Figure 19 shows the result of applying the procedure in the proof of Theorem 5.1 to the Trefoil knot.

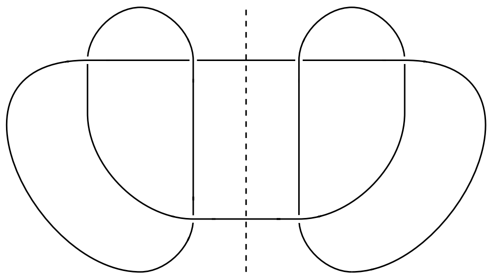


Figure 18: $3_1\#\bar{3}_1$

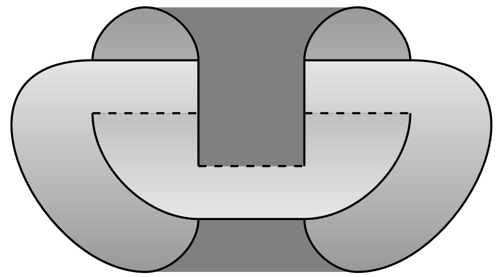


Figure 19: A ribbon disk for $3_1\#\bar{3}_1$

Let us interpret S^3 as the boundary of the unit ball D^4 in four-dimensional euclidean space. A *slice disk* is a smooth embedding of a disk D^2 into D^4 whose boundary S^1 is mapped into S^3, and whose interior B^2 is mapped into the interior B^4 of D^4. More concisely, a slice disk is a smooth embedding of the pair (D^2, S^1) into the pair (D^4, S^3). A knot that is the boundary of a slice disk is called a *slice knot*.

Theorem 5.3. *All ribbon knots are slice knots.*

Proof sketch. Let $f : D^2 \to S^3$ be the ribbon disk. We are going to add a height coordinate to f as follows. Take disjoint neighborhoods of the slits that need to be removed from D^2 in order for f to become injective. Now just add a bump function to these neighborhoods.

This procedure is schematically indicated in Figure 20. More distance from the boundary corresponds to a brighter color, and the slits we were about to cut away are dashed. The increase in brightness up until the inner large circle is to make sure that no part of the interior B^2 of D^2 is mapped to S^3. Every step we did could have been done smoothly. We have thus constructed a slice disk from a ribbon disk. $\square$

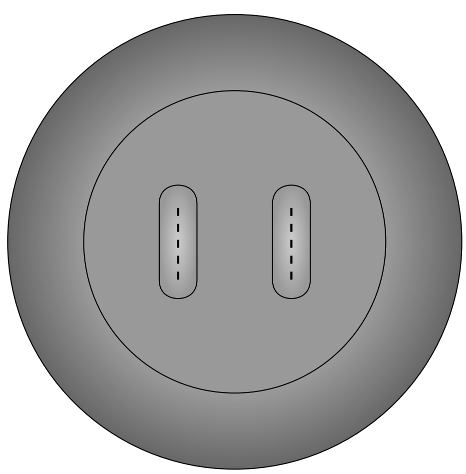


Figure 20: The disk D^2 with some slits with indications how to get an embedding into D^4

5.2 The Ribbon Disk Group

The *slice disk group* $\pi(S)$ of a slice disk S in D^4 is the fundamental group of its complement in D^4. If the slice disk is also a ribbon disk R, then $\pi(R)$ is called its *ribbon group*. We define the ribbon group of a ribbon in S^3 to be the ribbon group of its embedding into D^4 as in the proof of Theorem 5.3. In this section, we are going to give a Wirtinger-like algorithm to compute ribbon groups.

To describe our procedure, we first need to establish some terminology. Let R be a ribbon disk in S^3. A *crossing* of R is a connected component of the set of ribbon singularities, i.e., the set of double points of the immersion $D^2 \to R$. Note that (up to mirroring) each crossing has a neighborhood as in Figure 21. Denote by J the set of singularities. Then a connected component of $R \setminus J$ will be called an *arc* of R.

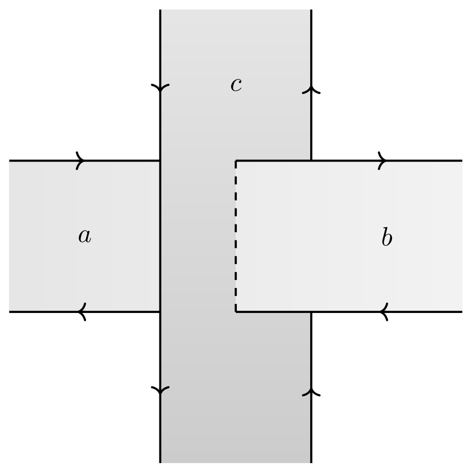


Figure 21: A neighborhood of a crossing with over-arc c and under-arcs a and b

We are now ready to describe the desired procedure, henceforth referred to as the *Ribbon Wirtinger procedure*. Let S be in bijection with the set of arcs of R. Now to any crossing as in Figure 21 assign a relation $ac = cb$. Then the presentation arising from this construction is a presentation of the ribbon group, as we will later show.

To prove that the above algorithm indeed gives rise to a presentation of the ribbon group, we will first need to address another (more complicated) procedure based on a Seifert-van Kampen argument not unlike the one we used in the proof of Theorem 2.5. First, we consider the embedding of R into D^4 as in the proof of Theorem 5.3. Let P be the three-sphere in D^4 at the height of the inner (medium-gray) disk in Figure 20. Note that P basically contains R, except for a neighborhood of the boundary of R and for neighborhoods of slits, see Figure 22 (middle right). Now P divides D^4 into two connected components. Let H^+ be the closure of the outer component and H^- the closure of the inner component. The intersection of R with a height level in H^+, P and H^-, respectively, is depicted in Figure 22.

We are now going to apply the Seifert-van Kampen Theorem to the decomposition

$$D^4 \setminus R = (H^+ \setminus R) \cup (H^- \setminus R).$$

From Figure 22 it is evident that the interior of H^+ retracts onto $S^3 \setminus \partial R$, so we get an isomorphism

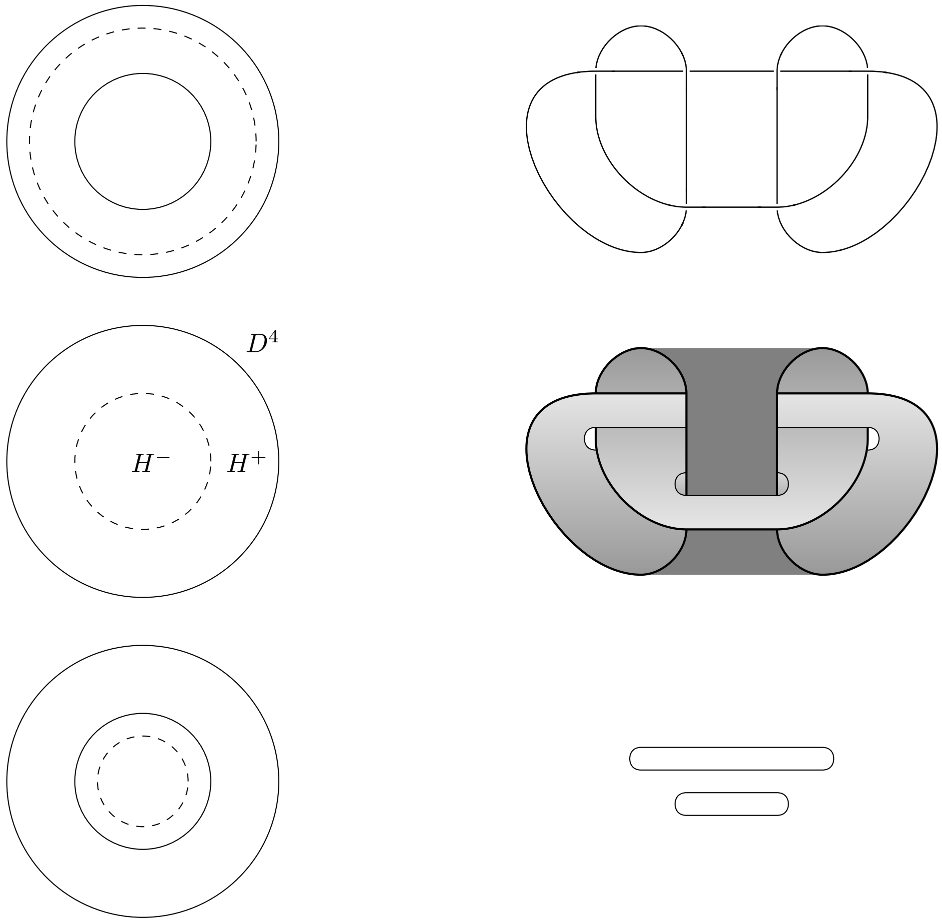


Figure 22: The embedding of R (right) into D^4 (left) at different heights

$\pi_1(H^+ \setminus R) \cong \pi(K)$, where $K = \partial R$. Similarly, $\pi_1(H^- \setminus R)$ is a free group generated by meridians of K. One immediate consequence of this is the following.

Proposition 5.4. *Let R be a ribbon disk for a knot K. Then $\pi(R)$ is a quotient of $\pi(K)$.*

Proof. Any generator of $\pi_1(H^- \setminus R)$ is represented by a meridian in $\pi_1(H^+ \setminus R)$. Thus, by the Seifert-van Kampen Theorem, $\pi(R)$ is isomorphic to a quotient of $\pi_1(H^+ \setminus R)$. $\square$

We are now going to work towards a more concrete description of $\pi(R)$ by inspecting the space $P \setminus R$, see Figure 22 (middle right). Let us first consider two meridians in $P \setminus R$ that pass through the same hole. Then, assuming they are coherently oriented, there is a homotopy between said curves in $H^- \setminus R$. This yields a set of relations, referred to as the *same-slit-relations*, identifying meridians in $\pi(K)$ passing through the same slit in $P \setminus R$. An example pair of meridians identified by the same-slit-relations is depicted in Figure 23.

The final set of relations is a little bit more subtle to see. Note that $\pi_1(P \setminus R)$ is not generated by meridians as previously discussed. In addition, we need to consider curves nullhomotopic in $H^+ \setminus P$

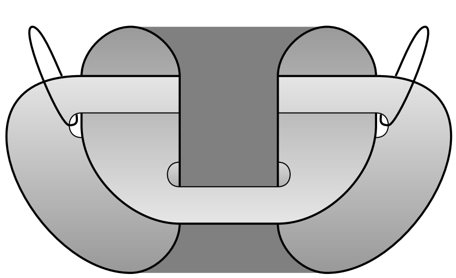


Figure 23: Same-slit-relations

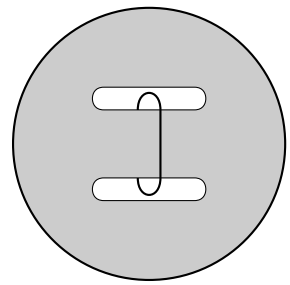


Figure 24: Through-slit-relations

that pass through slits. Including such curves into $\pi_1(H^- \setminus R)$ gives rise to the so-called *through-slit-relations*. A toy example can be found in Figure 24. We can now summarize our procedure as follows.

Lemma 5.5. *Let R be a ribbon disk for K. A presentation of $\pi(R)$ can be obtained by adding same-slit-relations and through-slit-relations to a presentation of $\pi(K)$.*

Proof. The three kinds of relations discussed correspond to a generating set of $\pi_1(P \setminus R)$. □

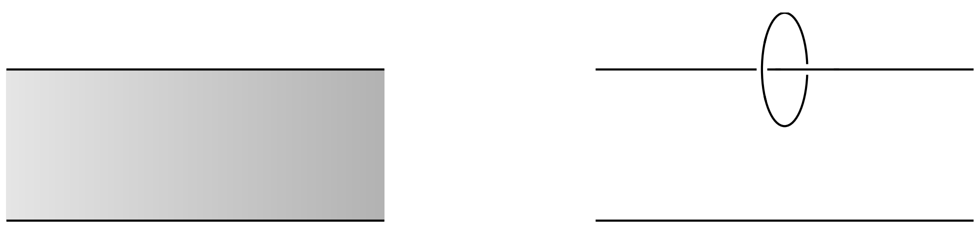


Figure 25: Interpretation of the arcs in the Ribbon Wirtinger procedure

Theorem 5.6. *The Ribbon Wirtinger procedure is correct.*

Proof sketch. The main ingredient used to check correctness is interpreting the generating set of the Ribbon Wirtinger procedure as meridians such as the meridian in Figure 25. It is then immediate that the through-slit relations, the same-slit relations and the Wirtinger relations are all satisfied. The converse is true by construction. □

Example 5.7 (Connected Sum of the Trefoil with its Mirror Image)**.** Consider the generating assignment of curves of $\pi(K)$ in Figure 26. This yields a presentation of the fundamental group of the complement of K with generators a, b and b' satisfying Wirtinger relations $aba = bab$ and $ab'a = b'ab'$.

Consider the curve γ in Figure 27, yielding the single through-slit relation $aba = bab$, which is already in our list of Wirtinger relations. Finally, the single same-slit relation is $b = b'$, as can be seen by staring at Figure 22 (middle right). This yields the presentation

$$\pi(R) = \langle a, b, b' \mid aba = bab, ab'a = b'ab', b = b' \rangle = \pi(3_1).$$

This result is actually a special case of Theorem 5.9 below.

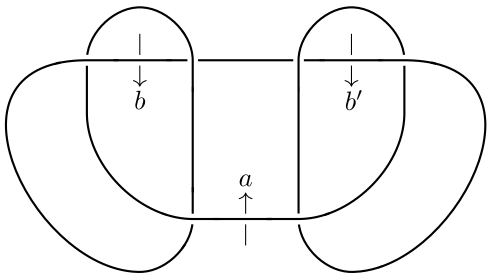


Figure 26: $3_1\#\bar{3}_1$

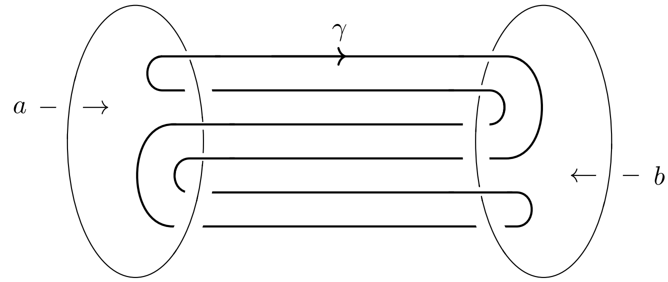


Figure 27: γ representing $b^{-1}a^{-1}b^{-1}aba$

Example 5.8 (The Knot 8_{20})**.** Consider the diagram of $K = 8_{20}$ in Figure 28. Note that K is also the $(3, -3, 2)$-pretzel knot. It is a so-called ribbon diagram, so it should not be necessary to explicitly draw the ribbon. The knot group $\pi(K)$ is generated by the meridians a, a', b, b' indicated in Figure 28.

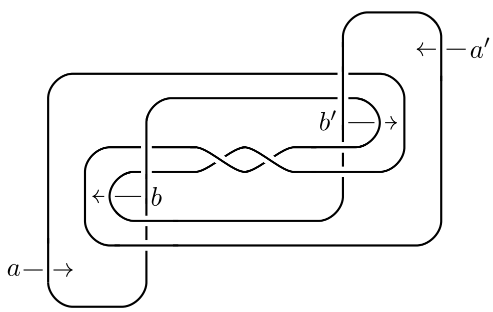


Figure 28: A ribbon diagram of 8_{20}

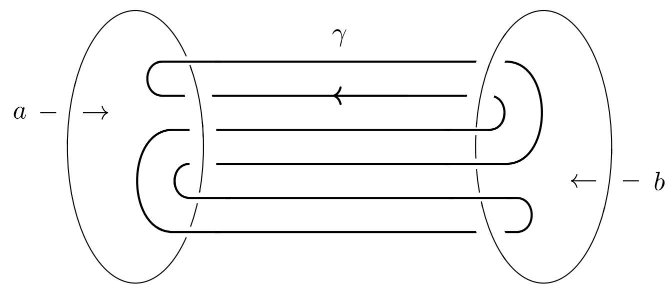


Figure 29: γ representing $abab^{-1}a^{-1}b^{-1}$

Since b can be arranged to pass through the same slit as a', we add the relation $b = a'$. Moreover, considering the curve γ in Figure 29, we see that a and b satisfy the relation $aba = bab$. Finally, after passing under the strand labeled a' we have that b' passes through the same slit as a. In symbols, $b' = b^{-1}ab$. Thus we obtain $\pi(R) = \langle a, b \mid aba = bab \rangle = \pi(3_1)$.

Theorem 5.9. *Let K be any knot and let R be the standard ribbon disk of $K\#\bar{K}$. Then $\pi(R)$ is isomorphic to the knot group $\pi(K)$.*

Proof sketch. Consider the inclusion $\iota : S^3 \setminus K\#\bar{K} \to D^4 \setminus R$. This map induces a surjective homomorphism $\iota_* : \pi(K\#\bar{K}) \twoheadrightarrow \pi(R)$. Note that sending each meridian in $\pi(K)$ to a meridian of one of the summands in $\pi(K\#\bar{K})$ defines an embedding $d : \pi(K) \hookrightarrow \pi(K\#\bar{K})$. Composition yields a map

$$\varphi : \pi(K) \hookrightarrow \pi(K\#\bar{K}) \twoheadrightarrow \pi(R).$$

The homomorphism φ is surjective because the image of d together with the kernel of ι_* generate the knot group $\pi(K\#\bar{K})$. Indeed, the same-slit relations are of the form $a^{-1}a'$, where a lies in the image of d and a' is the mirror image of a. For injectivity, we only need to prove that the through-slit relations follow from the Wirtinger relations. To see this, let us change our perspective a little bit. Consider the ribbon disk from the side. This is just a diagram of K where we remove an understrand.

For the sideways view of the ribbon disk of the connected sum of the trefoil with its mirror image in Figure 19 consider Figure 30. Let us refer to a segment of this diagram that starts and ends with overcrossings and only has undercrossings in between as an *anti-bridge*. Then anti-bridges correspond to through-slit relations. Moreover, the qualitative picture of an anti-bridge is as in Figure 31. In this picture, it is evident that the through-slit relations follow from the Wirtinger relations. □

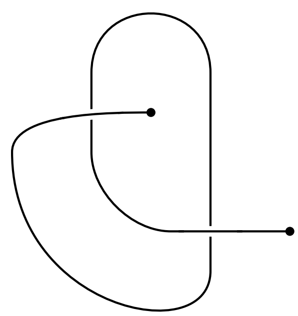


Figure 30: Sideways view of ribbon disk

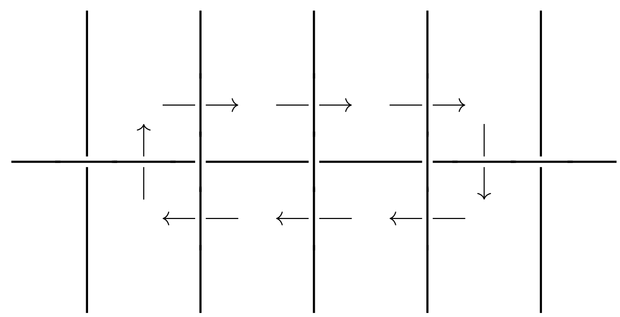


Figure 31: An anti-bridge with 3 overcrossings

5.3 Meridians and Ribbons

Similar as in the case of knots, we have that $\pi(R)$ is generated by a specific conjugacy class called meridians, consisting of curves that wrap around the ribbon disk once. For convenience, we opt for the definition that a *meridian* of a ribbon disk R is the image of a meridian of ∂R under the quotient map $\pi(\partial R) \to \pi(R)$. The least number of meridians needed to generate $\pi(R)$ is called the *meridional rank* of R, denoted $\mu(R)$. The meridional rank of the ribbon of the square knot in Example 5.7, as well as the meridional rank of the ribbon of the knot 8_{20} in Example 5.8, are two.

Continuing the analogy, think of a ribbon disk R as an immersed disk in S^3. A *bridge* is an arc of R that is involved as an over-arc in at least one crossing. We define the *bridge index* $b(R)$ to be the minimal number of bridges, minimized over ribbons equivalent to R. To make precise sense of ribbon equivalence, we should pass to the setting in D^4. The reader may carry out the details. We can now formulate the following.

Conjecture 5.10 (Meridional Rank Conjecture for Ribbons)**.** *Let R be an immersed ribbon disk in S^3. Then the meridional rank of its standard embedding into D^4 is equal to its bridge index. In symbols,*

$$\mu(R) = b(R).$$

As in the case of the meridional rank of links, one inequality can be established quite easily by diagrammatical considerations.

Proposition 5.11. *Let R be any ribbon disk. Then $\mu(R) \leq b(R)$.*

Proof. Assume that the ribbon diagram has as few bridges as possible. Let us define the notion of *bridge depth* of ribbon arcs inductively as follows. The bridge depth of a bridge is zero. For a ribbon crossing which involves a bridge as an over-arc and an under-arc for which the bridge depth is d, the bridge depth of the other arc is $d+1$ (if it was not previously defined to be $d-1$). Then any arc has an assigned bridge depth. Now consider the meridians associated to the bridges. Applying all possible Wirtinger relations d times yields an expression for an arc of depth d. Since the meridians associated to arcs generate $\pi(R)$ we get that the meridional rank $\mu(R)$ is bounded by $b(R)$. □

In fact, we can reduce a specific case of the meridional rank conjecture for ribbons to the meridional rank conjecture for knots.

Theorem 5.12. *Let K be any knot. Then the standard ribbon for $K\#\bar{K}$ satisfies the meridional rank conjecture for ribbons if K satisfies the meridional rank conjecture for knots.*

Proof sketch. The bridge index of the ribbon is at most the bridge index of K, which follows from looking at a suitable diagram of K. Namely, interpret a minimal-bridge diagram of K as the sideways view of the ribbon disk in question. Then an overcrossing of K corresponds to a singularity in a bridge in R. In other words, a bridge of K gives rise to a bridge of R. □

It is not clear whether the inequality $b(R) \leq b(K)$ is the best we can hope for, as there is no known counterexample to equality. Schematically, writing $f(R)$ for the *fusion number*, defined to be the least number of ribbon singularities of R in any projection to S^3, we can sum up the results of this chapter with the inequalities

$$\begin{array}{ccccc} c(K) \leq & \mu(K) & \leq & b(K) & \\ & \parallel & & \rotatebox{90}{$\geq$} & \\ & \mu(R) & \leq & b(R) & \leq f(R). \end{array}$$

Recall that $c(K)$ is the Coxeter rank (see Section 3.3 for the definition) rather than the crossing number. Note that this set of inequalities relies on the fact that R is the standard ribbon disk for $K\#\bar{K}$. From the inequalities it is evident that equality $b(K) = b(R)$ would imply that the meridional rank conjecture for ribbons is stronger than the meridional rank conjecture for knots, providing incentive for further research on this topic. Last but not least, we can bound the fusion number by the Coxeter rank, which is worth its own theorem.

Theorem 5.13. *Let K be a knot and let R be the standard ribbon disk for $K\#\bar{K}$. Then $c(K) \leq f(R)$.*

A Appendix

Table 3 consists of the Coxeter quotients we found for three-bridge knots of crossing number 10. It was computed using the same algorithm as in Section 3.2. No quotients were found for $10_{79}, 10_{80}, 10_{81}, 10_{83}$, 10_{86}, 10_{88}, 10_{89}, 10_{91}, 10_{92}, 10_{94}, 10_{95}, 10_{96}, 10_{97}, 10_{100}, 10_{101}, 10_{103}, 10_{104}, 10_{105}, 10_{106}, 10_{107}, 10_{109}, 10_{110}, 10_{111}, 10_{114}, 10_{115}, 10_{116}, 10_{117}, 10_{118}, 10_{119}, 10_{121}, 10_{122}, 10_{123}, 10_{148}, 10_{149}, 10_{150}, 10_{151}, 10_{152}, 10_{153}, 10_{154}, 10_{155}, 10_{156}, 10_{158}, 10_{160}, 10_{161}, 10_{162} and 10_{163}. The knots that are not mentioned are two-bridge knots.

Knot	Coxeter Quotients
10_{46}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{47}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{48}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{49}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{50}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{51}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{52}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{53}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{54}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{55}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{56}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{57}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{58}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and $\circ\overset{5}{\text{---}}\circ\overset{5}{\text{---}}\circ$
10_{59}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and $\circ\overset{5}{\text{---}}\circ\overset{5}{\text{---}}\circ$
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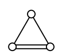
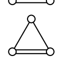
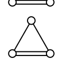
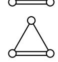
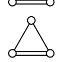
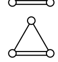
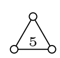
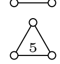
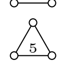
10_{61}	
10_{62}	
10_{63}	
10_{64}	
10_{65}	
10_{66}	
10_{67}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and 
10_{68}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and 
10_{69}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and 
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10_{73}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{76}	$\circ\text{---}\circ\text{---}\circ$
10_{77}	$\circ\text{---}\circ\text{---}\circ$
10_{78}	$\circ\text{---}\circ\text{---}\circ$
10_{82}	$\circ\text{---}\circ\text{---}\circ$
10_{84}	$\circ\text{---}\circ\text{---}\circ$
10_{85}	$\circ\text{---}\circ\text{---}\circ$
10_{87}	$\circ\text{---}\circ\text{---}\circ$
10_{90}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
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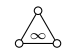
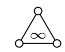
10_{98}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and 
10_{99}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and 
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10_{113}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{120}	$\circ\text{---}\circ\text{---}\circ$
10_{124}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{125}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{126}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{127}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{128}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{129}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{130}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{131}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{132}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{133}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
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10_{135}	$\circ\overset{7}{\text{---}}\circ\text{---}\circ$
10_{136}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and $\circ\overset{5}{\text{---}}\circ\overset{5}{\text{---}}\circ$
10_{137}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and $\circ\overset{5}{\text{---}}\circ\overset{5}{\text{---}}\circ$
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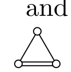
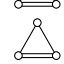
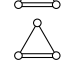
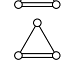
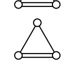
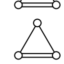
10_{139}	
10_{140}	
10_{141}	
10_{142}	
10_{143}	
10_{144}	
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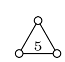
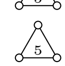
10_{146}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and 
10_{147}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$ and 
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10_{159}	$\circ\overset{5}{\text{---}}\circ\text{---}\circ$
10_{164}	$\circ\text{---}\circ\text{---}\circ$
10_{165}	$\circ\text{---}\circ\text{---}\circ$

Table 3: List of Coxeter knots with bridge index 3 and crossing number 10

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