1 The π-Orbifold Group

Definition 1.1. Let $L \subset S^3$ be a link, and let $m \in \pi_1(S^3 \setminus L)$ be a meridian. Then the *π-orbifold group* of L is

$$O(L) = \pi_1(S^3 \setminus L)/\langle\langle m^2 \rangle\rangle,$$

where $\langle\langle m^2 \rangle\rangle$ is the normal subgroup of $\pi_1(S^3 \setminus L)$ generated by m^2. The link L is said to be *sufficiently complicated* if $O(L)$ is infinite.

Example 1.2. Note that if K is a 2-bridge knot, then $O(K)$ is a dihedral group of finite order and hence finite. There are also 3-bridge knots K for which $O(K)$ is finite: see Table 1.

K	$\#O(K)$
8_5	336
8_{10}	528
8_{19}	48
8_{20}	144
8_{21}	240

Table 1: Some finite π-orbifold groups of 3-bridge knots with 8 crossings.

It is not clear whether Table 1 contains all 3-bridge knots with 8 crossings with finite π-orbifold group.

Definition 1.3. Let $L \subset S^3$ be a link. The *meridional rank* of L is the minimal number of meridians required to generate $\pi_1(S^3 \setminus L)$. The *involutuary rank* of L is the minimal number of involutions required to generate $O(L)$.

Lemma 1.4. *The involutary rank of a link $L \subset S^3$ is bounded from above by the meridional rank of L.*

Proof. Let $\langle m_1, \ldots, m_r | R \rangle$ be a Wirtinger presentation for L such that the meridional rank of L is r. Then the equivalence classes of $m_1, \ldots, m_r$ in the π-orbifold group $O(L)$ are involutions that generate $O(L)$. □

Conjecture 1.5. The meridional rank and the involutary rank are equal.

Definition 1.6. A link $L \subset S^3$ is *Coxeter* if $O(L)$ admits a Coxeter quotient of rank s, where s is the involutary rank of L. Here, a *Coxeter quotient* of $O(L)$ is a quotient of $O(L)$ that is a Coxeter group whose generating set consists of equivalence classes of meridians of $\pi_1(S^3 \setminus L)$. The *Coxeter rank* of L is the maximal rank of a Coxeter quotient of $O(L)$.

Lemma 1.7. *The Coxeter rank of a link $L \subset S^3$ is bounded from above by the involutary rank of L.*

Proof. Let q be the Coxeter rank of L, and let $W = \langle m_1, \dots, m_q | R \rangle$ be a Coxeter presentation of a maximal Coxeter quotient of $O(L)$. Then W is not generated by less than q involutions (consider the fixed subspaces of the generators in the reflection representation of W), so neither is $O(L)$. Thus, the Coxeter rank is bounded from above by the involutary rank. 1.4. $\square$

Example 1.8. It is an interesting question whether the Coxeter rank of a link $L \subset S^3$ is always equal to its bridge index. Sadly, this is not always the case: the knot $L = 8_{18}$ has bridge index 3 but Coxeter rank 2. This is because $O(8_{18})$ does not admit irreducible representations into $SO(3)$, see Raphael Zentner: Question 10.2 in *A CLASS OF KNOTS WITH SIMPLE SU(2)-REPRESENTATIONS.*