# K-Resilient Nash Equilibria in Multi-Player Concurrent Reachability Games

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# Outline

# Introduction

- In today's world, we are surrounded by complex systems.
- We use automated devices for many tasks in our daily life.
- In many cases, failure to ensure the correct functioning of these devices can lead to heavy losses.
- An ATM machine working incorrectly can cause heavy financial loss.
- Incorrect functioning of the flight control systems installed in an aeroplane can put a risk on the lives of people aboard.
- Therefore, it is very important to ensure that a system functions correctly and has the desired properties.
- While system testing prior to usage can help find and resolve errors, it cannot ensure that the system is error free.

# Introduction

- To ensure that the system is free of design errors, formal specification and verification techniques come in play.
- Formal specification and verification techniques work by mathematically modelling the system and its specification and then mathematically verifying that the system model has the desired properties.
- There are many approaches to formal specification and verification.
- One such approach is to use game theoretic concepts.
- Systems that involve multiple agents interacting with each other can be modelled as multi-player games.
- Game theoretic solution concepts can then be used to study the properties of such systems.

# Games, Strategies and Equilibria

- A game consists of a finite number of players.
- Every player has a set of strategies to choose from. Players play from among the strategies available to them.
- A strategy profile is a combination of strategies, one for each player.
- A strategy profile decides the complete outcome of the game.
- For every outcome of the game, each player has an associated payoff.
- Players are rational and play to maximize their respective payoffs.
- Players may also be allowed to play mixed (randomized) strategies. In such cases, expected values of payoffs are used to characterize the game.

# Games, Strategies and Equilibria

- A game can be represented using a matrix which lists the payoff vectors for every possible combination of pure strategies of players.
- Such a representation is called normal form representation.

**Table :** *Rock-Paper-Scissors game in normal form.*

| | Rock | Paper | Scissors |
|---|---|---|---|
| Rock | 0 , 0 | -1 , 1 | 1 , -1 |
| Paper | 1 , -1 | 0 , 0 | -1 , 1 |
| Scissors | -1 , 1 | 1 , -1 | 0 , 0 |

- Table 1 shows the normal form representation of a two player rock-paper-scissors game.

# Games, Strategies and Equilibria

- Normal form representation of games is pretty exhaustive and is often inconvenient.
- If the number of players and available strategies is large, normal form representation is practically not possible.
- Various other succinct representations have been proposed.
- Games played on graphs is one such representation.
- In this representation, a system is modelled as a state transition diagram.
- The states represent various configurations of the system.
- These games may be turn-based games or concurrent games.

# Games, Strategies and Equilibria

- In turn-based games, in each state, only one player decides the next state.
- In concurrent games, in each state, every player chooses an action from a set of allowed actions. The combination of actions (one for each player) then determines the next state.
- The payoffs of the players are defined with respect to the runs (paths) in this graph.
- Players may have qualitative objectives (0-1 objectives) like reachability objectives, safety objectives, Büchi objectives etc.
- Players may also have quantitative objectives where a real valued payoff is associated to each run.
- This may be done by defining for each player, a preference ordering on the runs of the graph.

# Games, Strategies and Equilibria

- A strategy for a player is a mapping that associates every finite path in the graph to an action available to the player in the last state of that path.
- A strategy profile is a tuple of strategies, one for each player.
- A strategy profile determines the complete outcome of the game.
- A Nash equilibrium is a strategy profile such that no player has a benefit on unilaterally deviating from its strategy.
- If mixed strategies are allowed, every game has at least one Nash equilibrium [16].
- In the rock-paper-scissors game shown in Table 1, there is a mixed strategy Nash equilibrium when both the players randomize uniformly between the three pure strategies available to them.

# Games, Strategies and Equilibria

- A *k-resilient* Nash equilibrium is a strategy profile such that, if at most $k$ players deviate from their respective strategies, none of the deviating player can benefit.
- Such a strategy profile is *resilient* to deviations by upto at most $k$ players [1, 2].
- Thus, a Nash equilibrium is actually a *1-resilient* Nash equilibrium.
- A *k-resilient* Nash equilibrium may not exist in general [1, 2].

# Our Contribution

- We study non-deterministic multi-player concurrent reachability games as defined in [6, 7].
- We prove some properties that characterize *k-resilient* Nash equilibria in these games when only pure strategies are allowed.
- We then use these properties to develop algorithms for checking existence of *k-resilient* Nash equilibrium in finite non-deterministic multi-player concurrent reachability games in untimed and timed settings when only pure strategies are allowed.
- Our algorithms are in 2-EXPTIME.
- The properties that we prove also contain all the necessary information to compute a *k-resilient* Nash equilibrium if it exists.

# Motivation

- A Nash equilibrium is susceptible to deviations by more than one player.
- Players can form coalitions and deviate in order to increase their payoffs.
- This may disturb the stability of the system.
- Therefore, it makes sense to study solution concepts that are resistant to deviations by coalitions of players also.
- One might assume that a coalition will deviate only if all the members of the coalition have some incentive and some equilibrium concepts have been proposed considering only such deviations [4, 5, 15].
- A *k-resilient* Nash equilibrium is an even stronger notion as it also considers deviations where even only one player of the coalition can benefit.

# Motivation

- Abraham et al. [1] discuss several reasons to consider such deviations.
- There may be situations where only one player effectively controls the coalition.
- This can happen in a network if a player can hijack nodes in the network.
- A player may threaten other players or persuade them to deviate by promising side payments.
- This notion can also take care of situations where players make arbitrary moves.
- Abraham et al. [1, 2] consider that there would be some kind of communication among the deviating coalition.

# Motivation

- We claim that deviations by multiple players are possible even without any prior communication.
- In real world situations, players might exhibit risk taking behaviour.
- In a Nash equilibrium, if a player is not happy with his payoff, he might want to take a risk by deviating in the hope that some other player(s) would also deviate (reasoning similarly), and hence he may be benefited.
- While taking such risks, he might take into account his loss if any, in case no one else deviates and his deviation is unilateral.
- He may be ready to bear this loss considering his dissatisfaction with his current payoff and the benefits he may gain if the risk turns out in his favour.

# Motivation

- In case of a weak Nash equilibrium, a player may take such risks without the fear of any loss in case no one else deviates and his deviation is unilateral.
- In games with qualitative objectives such as reachability games, a player can have the payoff of either 0 or 1.
- In such games, if a player has a payoff of 0 in the Nash equilibrium selected, he may like to deviate and take a risk because he has nothing to lose.
- Thus, the risk taking ability can be a threat to the stability of the system.

# Motivation

- Now consider a strategy profile which is a *k-resilient* Nash equilibrium.
- In this case, if a player takes such a risk, it would be required that more than $k$ players deviate for the risk to turn out in his favour.
- Higher the value of $k$, lower are the chances that the risk turns out in his favour.
- Therefore higher the resilience, lower is the risk taking ability of the players and hence more stable is the system.

# Related Work

- Games played on graphs have been extensively used in formal specification and verification techniques.
- Henzinger [14] reviews some important work in this area.
- There has also been a lot of work on Nash equilibria and other related solution concepts (like subgame-perfect equilibria and secure equilibria) in turn-based multi-player games.
- Grädel and Ummels [11] present a very nice survey of the work in this area.
- An important result is that every deterministic turn-based game with Borel winning conditions has a pure strategy Nash equilibrium [10].

# Related Work

- Recently, there has been a focus on multi-player concurrent games.
- The existence problem of Nash equilibrium in finite non-deterministic multi-player concurrent reachability games is NP-complete [6, 7, 9].
- The existence of Nash equilibrium is decidable in polynomial time for concurrent games with Büchi objectives [8, 9].
- For timed games with reachability objectives, existence problem of Nash equilibrium is EXPTIME-complete [6, 7, 9].
- Similar results exist for games with safety objectives, co-Büchi objectives, Rabin objectives and parity objectives [9].

# Related Work

- Various solution concepts have been proposed to take care of deviations by more than one player.
- Halpern [12, 13] gives a brief review of these solution concepts.
- In [4], a concept of strong equilibrium is proposed which is a strategy profile such that no coalition of players can deviate in such a way that all its members benefit.
- In [5], coalition-proof Nash equilibrium is proposed which is a strategy profile such that no coalition has a self enforcing deviation.
- A self enforcing deviation is a deviation by a coalition such that all its members benefit from the deviation and no subset of this coalition has a further self enforcing deviation possible.

# Related Work

- According to the definitions, every strong equilibrium is also a coalition-proof Nash equilibrium.
- In [15], coalition-proof correlated equilibrium is studied which is similar to coalition-proof Nash equilibrium but considers correlated strategies.
- All these solution concepts consider only those deviations where all the members of the deviating coalition can benefit from the deviation.
- However, recently Abraham et al. [1, 2] proposed the solution concept of *k-resilient* Nash equilibrium which also considers the deviations in which even one member of the coalition can benefit.

# Related Work

- The reasons to consider such deviations as given in [1] have been discussed earlier.
- Abraham et al. [1, 2] prove that *k-resilient* Nash equilibria exist in rational secret sharing and multi-party computation games with mediators, with some bounds on the values of $k$ and $n$ (total number of players).
- They also give various bounds and conditions on when can the mediator be simulated using cheap talk.
- In all these works, some kind of prior communication among the coalition is assumed before the deviation.
- However, we feel that deviation by multiple players is also possible without any prior communication due to the risk taking behaviour of players as explained earlier.

# Preliminaries

**Definition (Transition System)**

A transition system $S$ is defined as a 2-tuple, $S = (States, Edg)$ where:

- $States$ is a possibly uncountable set of states.
- $Edg \subseteq States \times States$ is the set of transitions (edges).

- A finite transition system is a transition system in which the set of states ($States$) is finite.

# Preliminaries

- A path $\pi$ in $S$, is a non-empty sequence $(s_i)_{0 \leq i < n}$ (where $n \in \mathbb{N} \cup \{+\infty\}$) of states of $S$ such that $(s_i, s_{i+1}) \in Edg$ for every $i < n - 1$.
- The length of a path $\pi = (s_i)_{0 \leq i < n}$ is denoted by $|\pi|$. $|\pi| = n - 1$.
- The set of finite paths (or *histories*) of $S$ is denoted by $Hists_S$.
- The set of infinite paths (or *plays*) of $S$ is denoted by $Plays_S$.
- The set of all the paths of $S$ is denoted by $Paths_S$. $Paths_S = Hists_S \cup Plays_S$.
- Given a path $\pi = (s_i)_{0 \leq i < n}$ and an integer $j < n$, the $j^{th}$ prefix of $\pi$ (denoted by $\pi_{\leq j}$) is the finite path $(s_i)_{0 \leq i < j+1}$.
- If a path $\pi$ is a history (a finite path), the last state of $\pi$ is denoted by $last(\pi)$. $last(\pi) = s_{|\pi|}$.

# Multi-Player Concurrent Reachability Games

**Definition (Multi-Player Concurrent Reachability Game)**

A non-deterministic multi-player concurrent reachability game $G$ is defined as a 7-tuple, $G = (States, Edg, Agt, Act, Mov, Tab, \Omega)$.

- $(States, Edg)$ is a transition system.
- $Agt$ is a finite set of players (agents).
- $Act$ is a possibly uncountable set of actions.
- $Mov : States \times Agt \rightarrow 2^{Act} \setminus \{\emptyset\}$ is a mapping that indicates the actions available to a given player in a given state.
- $Tab : States \times Act^{Agt} \rightarrow 2^{Edg} \setminus \{\emptyset\}$ is a mapping that associates a given tuple of actions of players in a given state to the resulting set of transitions. It is required that if $(s', s'') \in Tab(s, (m_A)_{A \in Agt})$, then $s' = s$.

# Multi-Player Concurrent Reachability Games

- $\Omega : Agt \rightarrow 2^{States}$ is a mapping that assigns to each agent, a set of states, which is the reachability objective of that agent (i.e., the agent wants to reach at least one of these states).
- In a game $G$, from some state $s$, each player $A$ selects one action $m_A$ from its set $Mov(s, A)$ of allowed actions.
- The tuple of actions $(m_A)_{A \in Agt}$ thus formed is called a move and is written as $m_{Agt}$.
- This results in a set of transitions $Tab(s, (m_A)_{A \in Agt})$ (or simply $Tab(s, m_{Agt})$).
- One of these transitions is applied (non-deterministically) which gives the next state of the game.
- In this way, the game continues to form a path $\pi$ in its underlying transition system.

# Multi-Player Concurrent Reachability Games

- We associate a payoff vector $v_{Agt} = (v_A)_{A \in Agt}$ to every path $\pi$ in $G$.
- If a path $\pi$ visits $\Omega(A)$ then we let $v_A(\pi) = 1$, otherwise $v_A(\pi) = 0$ where $v_A(\pi)$ is the payoff received by player $A$, when the path $\pi$ is taken in $G$.
- As in [6, 7], we use the notations $Hist_G$, $Play_G$ and $Path_G$ for the set of *histories*, *plays* and *paths* respectively, in the underlying transition system of $G$.
- We also write $Hist_G(s)$, $Play_G(s)$ and $Path_G(s)$ for respective subsets of *histories*, *plays* and *paths* starting in state $s$.

# Multi-Player Concurrent Reachability Games

## Definition (Strategy)

In a game $G$, a strategy for a player $A \in Agt$ is a mapping $\sigma_A : Hist_G \rightarrow Act$ such that for every $\pi \in Hist_G$, $\sigma_A(\pi) \in Mov(last(\pi), A)$.

- Given a subset of agents (also called a *coalition*) $P \subseteq Agt$, a strategy $\sigma_P$ for the coalition $P$ is a tuple of strategies, one for each player in $P$ ($\sigma_P = (\sigma_A)_{A \in P}$).

## Definition (Strategy Profile)

In a game $G$, a strategy profile $\sigma_{Agt}$ is a tuple of strategies, one for each player in $Agt$ (i.e., a strategy profile is a strategy for the coalition $Agt$). $\sigma_{Agt} = (\sigma_A)_{A \in Agt}$.

# Multi-Player Concurrent Reachability Games

- With respect to the sets of strategies, following notations are used in this thesis:
  - $Strat_G^A$: Set of all strategies for a player $A \in Agt$ in a game $G$.
  - $Strat_G^P$: Set of all strategies for a coalition $P \subseteq Agt$ in a game $G$.
  - $Strat_G^{Agt}$: Set of all strategy profiles in a game $G$.
- We only consider pure strategies in this thesis.
- In a game $G$, for a coalition $P$ and a strategy $\sigma_P$ (for $P$), a path $\pi = (s_i)_{0 \leq i \leq |\pi|}$ is said to be compatible with the strategy $\sigma_P$, if, for every $j \leq |\pi| - 1$, there is a move $m_{Agt} = (m_A)_{A \in Agt}$ such that:
  - $m_A \in Mov(s_j, A)$ for every $A \in Agt$.
  - $m_A = \sigma_A(\pi_{\leq j})$ for every $A \in P$.
  - $(s_j, s_{j+1}) \in Tab(s_j, m_{Agt})$

# Multi-Player Concurrent Reachability Games

- In a game $G$, the paths that are compatible with a strategy $\sigma_P$ (of a coalition $P$) are also called *outcomes* of the strategy $\sigma_P$.
- We use the notation $Out_G(\sigma_P)$ to denote the set of outcomes of strategy $\sigma_P$.
- The set of finite outcomes of strategy $\sigma_P$ is denoted by $Out^f_G(\sigma_P)$.
- The set of infinite outcomes of strategy $\sigma_P$ is denoted by $Out^\infty_G(\sigma_P)$.
- $Out_G(\sigma_P) = Out^f_G(\sigma_P) \cup Out^\infty_G(\sigma_P)$.
- We write $Out_G(s, \sigma_P)$, $Out^f_G(s, \sigma_P)$ and $Out^\infty_G(s, \sigma_P)$ for the respective sets of outcomes, finite outcomes and infinite outcomes of strategy $\sigma_P$ starting in state $s$.

# Multi-Player Concurrent Reachability Games

- In a game $G$, given a move $m_{Agt}$ and an action $m'_B$ for some player $B$, we write $m_{Agt}[B \to m'_B]$ for the move $n_{Agt}$ with $n_A = m_A$ when $A \neq B$ and $n_B = m'_B$.
- In a game $G$, given a move $m_{Agt}$ and an action tuple $m'_P$ for a coalition $P$ ($m'_P = (m'_A)_{A \in P}$), we write $m_{Agt}[P \to m'_P]$ for the move $n_{Agt}$ with $n_A = m_A$ for every $A \notin P$ and $n_A = m'_A$ for every $A \in P$.
- In a game $G$, given a strategy profile $\alpha_{Agt} \in Strat_G^{Agt}$ and a strategy $\alpha'_B$ for some player $B$ ($\alpha'_B \in Strat_G^B$), we write $\alpha_{Agt}[B \to \alpha'_B]$ for the strategy profile $\beta_{Agt}$ with $\beta_A = \alpha_A$ when $A \neq B$ and $\beta_B = \alpha'_B$.

# Multi-Player Concurrent Reachability Games

- In a game $G$, given a strategy profile $\alpha_{Agt} \in Strat_G^{Agt}$ and a strategy $\alpha'_P$ for a coalition $P$ ($\alpha'_P \in Strat_G^P$), we write $\alpha_{Agt}[P \to \alpha'_P]$ for the strategy profile $\beta_{Agt}$ with $\beta_A = \alpha_A$ for every $A \notin P$ and $\beta_A = \alpha'_A$ for every $A \in P$.
- In a game $G$, let $k$ be a non-negative integer such that $k \leq |Agt|$ ($k$ is a non-negative integer less than or equal to the number of players in $G$). We use the notation $2_k^{Agt}$ to denote the set of all subsets of $Agt$ of size at most $k$.

$$2_k^{Agt} = \{P \mid P \in 2^{Agt} \; and \; |P| \leq k\}$$

# Multi-Player Concurrent Reachability Games

## Definition (K-Resilient Pseudo-Nash Equilibrium)

Given a non-deterministic concurrent reachability game $G$ and a state $s$ in $G$, a *k-resilient* pseudo-Nash equilibrium (for some $k \leq |Agt|$) in $G$ from $s$ is a pair $(\sigma_{Agt}, \pi)$ where $\sigma_{Agt} \in Strat_G^{Agt}$ and $\pi \in Out_G(s, \sigma_{Agt})$, such that, for every $P \in 2_k^{Agt}$ and every $\sigma'_P \in Strat_G^P$, it holds:

$$\forall \pi' \in Out_G(s, \sigma_{Agt}[P \rightarrow \sigma'_P]). \ \forall B \in P. \ v_B(\pi') \leq v_B(\pi)$$

Such an outcome $\pi$ is called a *k-optimal* play for the strategy profile $\sigma_{Agt}$.

# Multi-Player Concurrent Reachability Games

- The payoff vector of a *k-resilient* pseudo-Nash equilibrium $(\sigma_{Agt}, \pi)$ is a tuple $(v_A(\pi))_{A \in Agt}$ where $v_A(\pi) = 1$ if $\pi$ visits $\Omega(A)$ and $v_A(\pi) = 0$ otherwise.
- In case of deterministic concurrent reachability games, there is only one outcome of a given strategy profile ($\pi$ is uniquely determined by $\sigma_{Agt}$) and hence *k-resilient* pseudo-Nash equilibria coincide with real *k-resilient* Nash equilibria as defined in [1, 2]: they are strategy profiles such that, if at most $k$ players deviate from their respective strategies, none of the deviating player can benefit.

# Multi-Player Concurrent Reachability Games

- In case of non-deterministic concurrent reachability games, a strategy profile $\sigma_{Agt}$ for a *k-resilient* pseudo-Nash equilibrium may give rise to several outcomes.
- The choice of playing the *k-optimal* play $\pi$ is then made cooperatively by all the players.
- Once a strategy profile is fixed, non-determinism is resolved by all the players by choosing one of the possible outcomes in such a way that if at most $k$ players deviate from their respective strategies or choice of outcome, none of the deviating player can benefit.

# Characterizing K-Resilient Nash Equilibria

## Definition (K-Suspect Coalitions)

In a game $G$, for some $k \leq |Agt|$, the set of *k-suspect coalitions* for an edge $e = (s, s')$, given a move $m_{Agt}$ (denoted as $k - Susp_G(e, m_{Agt})$) is defined as the set:

$$k - Susp_G(e, m_{Agt}) = \{P \in 2^{Agt}_k \mid \forall B \in P.\ \exists m'_B \in Mov(s, B) \\ s.t.\ e \in Tab(s, m_{Agt}[P \to m'_P]) \\ \text{where } m'_P = (m'_B)_{B \in P} \text{ is an action tuple for } P\}$$

- For an edge $e = (s, s')$, given a move $m_{Agt}$, if $e \in Tab(s, m_{Agt})$ then $k - Susp_G(e, m_{Agt}) = 2^{Agt}_k$.

# Characterizing K-Resilient Nash Equilibria

## Definition (K-Suspect Coalitions)

In a game $G$, for some $k \leq |Agt|$, the set of *k-suspect coalitions* for a finite path $\pi = (s_i)_{i \leq |\pi|}$, given a strategy profile $\sigma_{Agt}$ (denoted as $k - Susp_G(\pi, \sigma_{Agt})$) is defined as the set:

$$k - Susp_G(\pi, \sigma_{Agt}) = \bigcap_{i < |\pi|} k - Susp_G((s_i, s_{i+1}), (\sigma_A(\pi_{\leq i}))_{A \in Agt})$$

# Characterizing K-Resilient Nash Equilibria

**Lemma (1)**

*In a game $G$, for some $k \leq |Agt|$, given $\sigma_{Agt} \in Strat_G^{Agt}$, $\pi \in Hist_G$ and a coalition $P$ of size at most $k$ ($P \in 2_k^{Agt}$), the following three propositions are equivalent:*
*(a)* $P \in k - Susp_G(\pi, \sigma_{Agt})$
*(b)* $\exists \sigma'_P \in Strat_G^P.\ \pi \in Out_G^f(\sigma_{Agt}[P \rightarrow \sigma'_P])$
*(c)* $\pi \in Out_G^f((\sigma_A)_{A \in Agt \setminus P})$

- For a finite path $\pi$, given a strategy profile $\sigma_{Agt}$, if $\pi \in Out_G^f(\sigma_{Agt})$ then $k - Susp_G(\pi, \sigma_{Agt}) = 2_k^{Agt}$.

# Characterizing K-Resilient Nash Equilibria

- In a game $G$, for some $k \leq |Agt|$, consider the set of ordered pairs $(P, X)$ where $P \subseteq Agt$ is a coalition and $X \subseteq 2_k^{Agt}$.
- Let us define a partial ordering on the set of ordered pairs $(P, X)$ as follows:

$$(P', X') \leq (P, X) \textit{ iff } P' \subseteq P \textit{ and } X' \subseteq X$$

- The relation '$\leq$' is trivially reflexive, transitive and anti-symmetric.

# Characterizing K-Resilient Nash Equilibria

## Definition (K-Repellor Set)

In a game $G$, for some $k \leq |Agt|$, given $P \subseteq Agt$ and $X \subseteq 2_k^{Agt}$, the *k-repellor set* of the ordered pair $(P, X)$ (denoted as $k - Rep_G(P, X)$) is defined inductively as follows: as the base case, we let $k - Rep_G(\emptyset, X) = States$ for every $X \subseteq 2_k^{Agt}$. Then assuming that $k - Rep_G(P', X')$ has been defined for every $(P', X') \precneq (P, X)$, we let $k - Rep_G(P, X)$ be the largest set satisfying the following two conditions:

(a) $\forall B \in P.\ k - Rep_G(P, X) \cap \Omega(B) = \emptyset$

(b) $\forall s \in k - Rep_G(P, X).\ \exists m_{Agt} \in Act^{Agt}.\ \forall s' \in States.$
$s' \in k - Rep_G(P', X')$

where $X' = k - Susp_G((s, s'), m_{Agt}) \cap X$ and $P' = P \bigcap \left( \bigcup_{W \in X'} W \right)$

# Characterizing K-Resilient Nash Equilibria

- Intuitively, in a game $G$, for some $k \leq |Agt|$, given $P \subseteq Agt$ and $X \subseteq 2_k^{Agt}$, $k - Rep_G(P, X)$ is the set of states from where players can cooperate to stay in $k - Rep_G(P, X)$, thus never satisfying the objectives of players in $P$, such that if any coalition $Q \in X$ deviates from its strategy and breaks the cooperation, it won't help fulfilling the objectives of players in $P \cap Q$
- Intuitively, in $k - Rep_G(P, X)$, $P$ is the set of players whose objectives must not be fulfilled by deviations from coalitions in $X$.

# Characterizing K-Resilient Nash Equilibria

**Lemma (2)**

*In a game $G$, for some $k \leq |Agt|$, given $P, Q \subseteq Agt$ and $X, Y \subseteq 2_k^{Agt}$. If $Q \subseteq P$ and $Y \subseteq X$, then $k - Rep_G(P, X) \subseteq k - Rep_G(Q, Y)$.*

*Proof (Sketch).* This lemma can be easily proved by induction on the ordered pair $(P, X)$.

# Characterizing K-Resilient Nash Equilibria

## Definition (K-Secure Move)

In a game $G$, for some $k \leq |Agt|$, given $P \subseteq Agt$, $X \subseteq 2^{Agt}_k$ and a state $s$, the set of *k-secure moves* for the tuple $(s, P, X)$ (denoted as $k - Secure_G(s, P, X)$) is defined as:

$$k - Secure_G(s, P, X) = \left\{ m_{Agt} \in Act^{Agt} \;\middle|\; \forall s' \in States. \right.$$
$$s' \in k - Rep_G(P', X') \text{ where } X' = k - Susp_G((s, s'), m_{Agt}) \cap X$$
$$\left. \text{and } P' = P \bigcap \left( \bigcup_{W \in X'} W \right) \right\}$$

- Intuitively, $k - Secure_G(s, P, X)$ is the set of moves available from state $s \in k - Rep_G(P, X)$ for staying in $k - Rep_G(P, X)$ (hence the word *k-secure*).

# Characterizing K-Resilient Nash Equilibria

***A special case:***

- In a game $G$, for some $k \leq |Agt|$, given $P \subseteq Agt$, $X \subseteq 2_k^{Agt}$ and a state $s$. Consider the case $P \bigcap \left( \bigcup_{W \in X} W \right) \neq P$ (i.e., there is at least one player in $P$ which is not a member of any coalition in $X$).
- In this case playing a *k-secure move* from $k - Rep_G(P, X)$ may take us outside $k - Rep_G(P, X)$.
- This may happen as follows: if a *k-secure move* $m_{Agt} \in k - Secure_G(s, P, X)$ is played and a transition $(s, s')$ is applied as a result, then according to condition (b) of definition of k-repellor sets: $s' \in k - Rep_G(P', X')$

# Characterizing K-Resilient Nash Equilibria

- where $X' = k - Susp_G((s, s'), m_{Agt}) \cap X = 2_k^{Agt} \cap X = X$.
- and $P' = P \bigcap \left( \bigcup_{W \in X'} W \right) = P \bigcap \left( \bigcup_{W \in X} W \right) \neq P \ (P' \subsetneq P)$.
- Therefore, $s' \in k - Rep_G(P', X)$ but $s'$ is not necessarily in $k - Rep_G(P, X)$. From Lemma 2, we have $k - Rep_G(P, X) \subseteq k - Rep_G(P', X)$.
- Thus, taking a *k-secure move* from $k - Rep_G(P, X)$ in such cases does not necessarily mean staying in $k - Rep_G(P, X)$.
- Although, this seems to violate our intuitive meaning of *k-secure moves*, this is not something undesirable. Its just a special case.

# Characterizing K-Resilient Nash Equilibria

- This special case does not affect our results because, in this thesis, we will only be considering the cases where $P \bigcap \left( \bigcup_{W \in X} W \right) = P$ (i.e., every player in $P$ is a member of at least one coalition in $X$).
- In such cases taking a *k-secure move* from $k - Rep_G(P, X)$ results in staying in $k - Rep_G(P, X)$ as expected according to the intuitive meaning of *k-secure moves*.

# Characterizing K-Resilient Nash Equilibria

**Definition (Transition System $K - S_G(P, X)$)**

In a game $G$, for some $k \leq |Agt|$, given $P \subseteq Agt$ and $X \subseteq 2^{Agt}_k$, we define the transition system $k - S_G(P, X) = (States, Edg')$ as follows: $(s, s') \in Edg'$ iff there exists some move $m_{Agt} \in k - Secure_G(s, P, X)$ such that $(s, s') \in Tab(s, m_{Agt})$. Note in particular that every $s \in k - Rep_G(P, X)$ has an outgoing transition in $k - S_G(P, X)$.

# Characterizing K-Resilient Nash Equilibria

**Lemma (3)**

*In a game $G$, for some $k \leq |Agt|$, given $s \in States$, $P \subseteq Agt$ and $X \subseteq 2_k^{Agt}$ such that $P \bigcap \left( \bigcup_{W \in X} W \right) = P$. Then $s \in k - Rep_G(P, X)$ if and only if there exists an infinite path $\pi$ in $k - S_G(P, X)$ starting from $s$.*

*Proof (Sketch).*
($\Rightarrow$) Starting from $s \in k - Rep_G(P, X)$, we can play *k-secure moves* to always stay in $k - Rep_G(P, X)$ thus forming an infinite path in $k - S_G(P, X)$.
($\Leftarrow$) If $\pi$ is an infinite path in $k - S_G(P, X)$ then every state of $\pi$ is in $k - Rep_G(P, X)$. In particular $s \in k - Rep_G(P, X)$.

# Characterizing K-Resilient Nash Equilibria

**Lemma (4)**

*In a game $G$, for some $k \leq |Agt|$, given $s \in States$, $P \subseteq Agt$ and $X \subseteq 2_k^{Agt}$ such that $P \bigcap \left( \bigcup_{W \in X} W \right) = P$. Let $\pi \in Play_G(s)$ be an infinite play with initial state $s$. Then $\pi$ is a path in $k - S_G(P, X)$ if and only if there exists $\sigma_{Agt} \in Strat_G^{Agt}$ such that $\pi \in Out_G(s, \sigma_{Agt})$ and for every $Q \in X$ and every $\sigma'_Q \in Strat_G^Q$, it holds that:*
*$\forall \pi' \in Out_G(s, \sigma_{Agt}[Q \to \sigma'_Q]).\ \pi'$ does not visit $\Omega(B)$ for every $B \in P \cap Q$.*

*Proof (Sketch).* $(\Rightarrow)$

- Assume that $\pi = (s_i)_{i \geq 0}$ is an infinite play in $k - S_G(P, X)$.

# Characterizing K-Resilient Nash Equilibria

- Let us define the following partial functions:
  - $c : States \times 2^{Agt} \times 2^{2^{Agt}_k} \rightarrow Act^{Agt}$: for every tuple $(s, R, Y)$ such that $s \in k - Rep_G(R, Y)$, we let $c(s, R, Y) = m_{Agt}$ for some move $m_{Agt} \in k - Secure_G(s, R, Y)$.
  - $d : \mathbb{N} \rightarrow Act^{Agt}$: for every $i \in \mathbb{N}$, we let $d(i) = m_{Agt}$ for some move $m_{Agt} \in k - Secure_G(s_i, P, X)$ such that $(s_i, s_{i+1}) \in Tab(s_i, m_{Agt})$. This is well defined because $\pi$ is a play in $k - S_G(P, X)$.
- The strategy profile $\sigma_{Agt}$ is defined as follows:
  - On prefixes of $\pi$, we let $(\sigma_A(\pi_{\leq i}))_{A \in Agt} = d(i)$ for every $i \in \mathbb{N}$.
  - For any $\pi' = (s'_i)_{i \leq |\pi'|}$ that is not a prefix of $\pi$, we let $(\sigma_A(\pi'))_{A \in Agt} = c(s'_{|\pi'|}, P', X')$ where

$$X' = k - Susp_G(\pi', \sigma_{Agt}) \cap X \text{ and } P' = P \bigcap \left( \bigcup_{W \in X'} W \right).$$

# Characterizing K-Resilient Nash Equilibria

- By construction, we have $\pi \in Out_G(s, \sigma_{Agt})$.
- Let $Q \in X$ and let $\sigma'_Q \in Strat_G^Q$.
- Let $\pi' \in Out_G(s, \sigma_{Agt}[Q \to \sigma'_Q])$.
- Assuming $\pi' = (s'_i)_{i \geq 0}$, it can be shown by induction on $i$ that $s'_i \in k - Rep_G(P \cap Q, \{Q\})$ for every $i \in \mathbb{N}$.
- Hence $\pi'$ does not visit $\Omega(B)$ for every $B \in P \cap Q$.

($\Leftarrow$) This direction can be easily proved by induction on the ordered pair $(P, X)$.

# Characterizing K-Resilient Nash Equilibria

**Theorem (5)**

*Let $G$ be a non-deterministic multi-player concurrent reachability game and let $s \in States$. There is a k-resilient pseudo-Nash equilibrium in $G$ from $s$ with a payoff vector $v_{Agt} = (v_A)_{A \in Agt}$ if and only if, letting $P = \{A \in Agt \mid v_A = 0\}$, there is an infinite path $\pi$ in $k - S_G(P, 2_k^{Agt})$ which starts in $s$ and visits $\Omega(A)$ for every $A \notin P$. Furthermore, $\pi$ is the k-optimal play for this k-resilient pseudo-Nash equilibrium.*

*Proof (Sketch).* Both the directions of this theorem can be easily proved using Lemma 4 and the definition of *k-resilient* pseudo-Nash equilibrium.

# Characterizing K-Resilient Nash Equilibria

**Theorem (6)**

*In a game $G$, if $\pi$ is an infinite path in $k - S_G(P, 2_k^{Agt})$ from a state $s$ visiting $\Omega(A)$ for every $A \notin P$, then there is a k-resilient pseudo-Nash equilibrium $(\sigma_{Agt}, \pi)$ where the strategy profile $\sigma_{Agt}$ consists in playing k-secure moves in the transition system $k - S_G(P \cap P', 2_k^{Agt} \cap X')$ for some $P' \subseteq Agt$ and $X' \subseteq 2_k^{Agt}$ satisfying the condition* $(P \cap P') \bigcap \left( \bigcup_{W \in 2_k^{Agt} \cap X'} W \right) = (P \cap P')$.

# Characterizing K-Resilient Nash Equilibria

*Proof (Sketch).*

- The strategy profile $\sigma_{Agt}$ should contain $\pi$ as one of its outcomes. This can be done by selecting relevant *k-secure moves* from $k - S_G(P, 2_k^{Agt})$.
- For a history that is out of $\pi$ but still in $k - Rep_G(P, 2_k^{Agt})$, $\sigma_{Agt}$ is defined to select *k-secure moves* to stay in $k - Rep_G(P, 2_k^{Agt})$.
- For a history $\pi'$ out of $k - Rep_G(P, 2_k^{Agt})$, we compute $X' = X \cap k - Susp_G(\pi', \sigma_{Agt})$ and $P' = P \bigcap \left( \bigcup_{W \in X'} W \right)$ and the strategy profile $\sigma_{Agt}$ is defined to select *k-secure moves* in $k - S_G(P \cap P', 2_k^{Agt} \cap X')$.

# Application to Finite Games

- A finite non-deterministic multi-player concurrent reachability game is a game in which the underlying transition system $((States, Edg))$ is finite and the set of actions $(Act)$ is also finite.
- Given a finite game $G$, a state $s$ in $G$ and a positive integer $k \leq |Agt|$, we can apply Theorem 5 to develop an algorithm for checking existence of a *k-resilient* pseudo-Nash equilibrium in the game $G$ from state $s$.

# Application to Finite Games

**Algorithm 1** Existence of a *k-resilient* pseudo-Nash equilibrium

**Input:** A finite non-deterministic multi-player concurrent reachability game $G$, a state $s$ in $G$ and a positive integer $k \leq |Agt|$.

**Output:** A boolean value: **true** if there exists a *k-resilient* pseudo-Nash equilibrium in $G$ from $s$; **false** otherwise.

```
1: for every possible payoff vector v_Agt = (v_A)_{A∈Agt} do
2:     P ← {A ∈ Agt | v_A = 0}
3:     Compute k − Rep_G(P, 2_k^Agt)
4:     Construct k − S_G(P, 2_k^Agt)
5:     for every play π in k − S_G(P, 2_k^Agt) starting from s do
6:         if π visits Ω(A) for every A ∉ P then
7:             return true
8:         end if
9:     end for
10: end for
11: return false
```

# Application to Finite Games

- The correctness of Algorithm 1 follows from Theorem 5.
- Algorithm 1 runs in time exponential with respect to the number of states and size of transition table but doubly exponential time with respect to the number of agents.
- Overall, Algorithm 1 runs in time doubly exponential with respect to the size of input.
- Algorithm 1 is in 2-EXPTIME.
- In case a *k-resilient* pseudo-Nash equilibrium exists in a game, it can be computed using the generic method given in Theorem 6.

# Timed Games

- For formal specification and verification of real time systems that involve timing constraints (such as flight control systems installed in an aeroplane), we need to include the timing constraints while modelling the system specification.
- If a system desires some properties that include timing constraints, we should be able to include the concept of 'time' in the system specification.
- In order to verify that the system has such desired properties (that include timing constraints), the system model must be able to capture such properties completely in its specification.
- For this purpose, Alur and Dill [3] define a concept of timed automata.

# Timed Games

- Timed automata are a natural extension of finite automata in the sense that transitions in timed automata not just read a symbol (over an alphabet) but also take into account the 'time' at which a symbol is read.
- If a symbol represents an event, then transitions in timed automata depend not only on the occurrence of an event, but also on the time of occurrence of an event.
- Thus, timed automata are able to capture properties that involve timing constraints and hence are useful in formal specification and verification of real time systems.

# Timed Games

- Timed games are defined in a way similar to timed automata, but include necessary game theoretic concepts.
- Real time systems that involve multiple agents interacting with each other can be modelled as multi-player timed games.
- Game theoretic solution concepts can then be used to study the properties of such systems.
- In this thesis, we are interested in multi-player timed concurrent reachability games as defined in [6, 7].

# Timed Games

## Definition (Clock Valuation)

Consider a finite set of clocks $\chi$. A valuation $v$ over the finite set of clocks $\chi$ is an application $v : \chi \rightarrow \mathbb{R}_+$ that assigns to each clock $x \in \chi$, a positive real number that signifies the units of time since the clock $x$ was last reset.

- If $v$ is a valuation over the set of clocks $\chi$ and $t \in \mathbb{R}_+$, then $v + t$ is the valuation that assigns to each $x \in \chi$, the value $v(x) + t$.
- If $v$ is a valuation over the set of clocks $\chi$ and $\varphi \subseteq \chi$, then $[\varphi \leftarrow 0]v$ is the valuation that assigns the value 0 to each $y \in \varphi$ and the value $v(x)$ to each $x \in \chi \setminus \varphi$.

# Timed Games

## Definition (Clock Constraint)

A clock constraint over the set of clocks $\chi$ is a formula built on the grammar $\zeta(\chi) \ni g ::= x \backsim c \mid g \wedge g$, where $x$ ranges over $\chi$, $\backsim \in \{<, \leq, =, \geq, >\}$ and $c$ is an integer.

- A valuation $v$ over a set of clocks $\chi$ satisfies a clock constraint $g$ over $\chi$ if on assigning the values $v(x)$ to each $x \in \chi$, the clock constraint $g$ evaluates to true.
- If a valuation $v$ satisfies a clock constraint $g$, we write it as $v \models g$.

# Timed Games

**Definition (Timed Concurrent Reachability Game)**

A multi-player timed concurrent reachability game $G$ is defined as a 7-tuple, $G = (Loc, \chi, Inv, Trans, Agt, Owner, \Omega)$.

- $Loc$ is a finite set of locations.
- $\chi$ is a finite set of clocks.
- $Inv : Loc \rightarrow \zeta(\chi)$ assigns an invariant to each location. If $l$ is a location and $Inv(l) = g$, it means that while the game is at location $l$, the valuation of clocks in $\chi$ must satisfy the clock constraint $g$.

# Timed Games

- $Trans \subseteq Loc \times \zeta(\chi) \times 2^{\chi} \times Loc$ is the set of transitions. If $\delta = (l, g, z, l')$ is a transition, it means that the transition $\delta$ is firable only if the clock constraint $g$ evaluates to true. Further, on firing the transition $\delta$, the game goes from location $l$ to location $l'$ and the clocks in the set $z$ are reset to 0.
- $Agt$ is a finite set of agents (or players).
- $Owner : Trans \rightarrow Agt$ assigns an agent to each transition. If $Owner(\delta) = A$, it means that only player $A$ can fire the transition $\delta$.
- $\Omega : Agt \rightarrow 2^{Loc}$ assigns to each agent, a set of locations which is the reachability objective of that agent (i.e., the agent wants to reach at least one of these locations).

# Timed Games

- A state of the game is an ordered pair $(l, v)$ where $l \in Loc$ and $v$ is a valuation over the set $\chi$ of clocks, provided that $v \models Inv(l)$.
- The game starts from an initial state $s_0 = (l, \mathbf{0})$, where $\mathbf{0}$ is a valuation that assigns the value 0 to every clock $x \in \chi$ and is assumed to satisfy $Inv(l)$.
- From each state $(l, v)$, every player $A$ chooses a non-negative real number $d$ and a transition $\delta = (l, g, z, l')$ with the intended meaning that the player $A$ wants to delay for $d$ time units and then fire the transition $\delta = (l, g, z, l')$.

# Timed Games

- There are various natural restrictions on these choices:
  - Spending $d$ time units in $l$ must be allowed, i.e., $v + d' \models Inv(l)$ for every $0 \leq d' \leq d$. As the invariants are convex, this is equivalent to having only $v + d \models Inv(l)$.
  - Player $A$ must be the owner of the transition $\delta = (l, g, z, l')$, i.e., $Owner(\delta) = A$.
  - The transition $\delta = (l, g, z, l')$ is firable after $d$ time units, i.e., $v + d \models g$.
  - The invariant of $l'$ must be satisfied when entering $l'$, i.e., $[z \leftarrow 0](v + d) \models Inv(l')$.
- If (and only if) there is no such possible choice for some player $A$, then $A$ chooses a null action (denoted by $\perp$).

# Timed Games

- Given a tuple of choices $m_{Agt}$ of all the players, with $m_A \in (\mathbb{R}_+ \times Trans) \cup \{\perp\}$, a player $B$ such that $d_B = min\{d_A \mid A \in Agt$ and $m_A = (d_A, \delta_A)\}$ is selected non-deterministically, and the corresponding transition $\delta_B = (l, g_B, z_B, l')$ is applied leading to a new state $(l', [z_B \leftarrow 0](v + d_B))$.

# Semantics of Timed Games

- With a multi-player timed concurrent reachability game $G = (Loc, \chi, Inv, Trans, Agt, Owner, \Omega)$, we can associate the infinite non-deterministic multi-player concurrent reachability game $G' = (States, Edg, Agt, Act, Mov, Tab, \Omega')$.
- $States = \{(l, v) \mid l \in Loc,\ v : \chi \to \mathbb{R}_+ \text{ such that } v \models Inv(l)\}$.
- $s_0 = (l_0, \mathbf{0})$ is the initial state.
- The set of transitions *Trans* in $G$ give rise to the set of edges *Edg* in $G'$ as follows: for every $d \in \mathbb{R}_+$, every $\delta = (l, g, z, l')$ in *Trans* and every $(l, v) \in States$ such that $v + d \models Inv(l) \wedge g$ and $[z \leftarrow 0](v + d) \models Inv(l')$, there is an edge $((l, v), (l', [z \leftarrow 0](v + d)))$ in *Edg*.
- The set of actions is $Act = \{(d, \delta) \mid d \in \mathbb{R}_+,\ \delta \in Trans\} \cup \{\perp\}$.

# Semantics of Timed Games

- An action $(d, \delta)$ (where $\delta = (l, g, z, l'))$ is allowed to player $A$ in state $(l, v)$ iff the following conditions hold:
    - $(l, v + d) \in States$ (this is the case when $v + d \models Inv(l)$).
    - $\delta = (l, g, z, l')$ is such that $Owner(\delta) = A$.
    - $v + d \models g$.
    - $[z \leftarrow 0](v + d) \models Inv(l')$

  Then $Mov((l, v), A)$ is the set of actions allowed to player $A$ in state $(l, v)$ when this set is non empty, and $Mov((l, v), A) = \{\perp\}$ otherwise.

# Semantics of Timed Games

- Given a state $(l, v) \in States$ and a tuple of actions $(m_A)_{A \in Agt}$ (a move $m_{Agt}$) allowed from this state, $Tab((l, v), m_{Agt})$ is defined as the set:

$$\Big\{((l,v),(l',v')) \;\Big|\; \exists B.\ d_B = min\{d_A \mid A \in Agt \text{ and } m_A = (d_A, \delta_A)\} \\ \text{and } \delta_B = (l, g_B, z_B, l') \\ \text{and } v' = [z_B \leftarrow 0](v + d_B)\Big\}$$

- For every $A \in Agt$,
$\Omega'(A) = \{(l, v) \mid (l, v) \in States \text{ and } l \in \Omega(A)\}$.

# Semantics of Timed Games

- Multi-player timed concurrent reachability games inherit the notions of path, history, play, strategy, strategy profile, outcome and *k-resilient* pseudo-Nash equilibrium via the semantics described above.
- In this thesis, we consider only non-blocking multi-player timed concurrent reachability games. Non-blocking games are games in which for every state $(l, v)$, at least one player has an allowed action:

$$\prod_{A \in Agt} Mov((l, v), A) \neq \{(\perp)_{A \in Agt}\}$$

# Region Games

## Definition (Region Game)

Let $G = (Loc, \chi, Inv, Trans, Agt, Owner, \Omega)$ be a multi-player timed concurrent reachability game. We define a region game $G_R$ associated to $G$ as
$G_R = (States_R, Edg_R, Agt, Act_R, Mov_R, Tab_R, \Omega_R)$.

- $States_R = \{(l, r) \in Loc \times \Re \mid r \models Inv(l)\}$ where $\Re$ is the set of clock regions.
- $Edg_R$ is the set of transitions of the region automaton underlying $G$.
- $Act_R = \{(r, p, \delta) \mid r \in \Re,\ p \in \{1, 2, 3\}$ and $\delta \in Trans\} \cup \{\perp\}$.

# Region Games

- $Mov_R : States_R \times Agt \to 2^{Act_R} \setminus \{\emptyset\}$ is such that:

$$Mov_R((l,r),A) = \Big\{(r',p,\delta) \;\Big|\; r' \in Succ(r),\ r' \models Inv(l), \\ p \in \{1,2,3\} \text{ if } r' \text{ is time-elapsing, else } p = 1, \\ \delta = (l,g,z,l') \in Trans \text{ is such that } r' \models g \\ \text{and } [z \leftarrow 0]r' \models Inv(l') \text{ and } Owner(\delta) = A\Big\}$$

if it is non-empty and $Mov_R((l,r),A) = \{\perp\}$ otherwise. Roughly, the index $p$ allows the players to say if they want to play first, second or later if their region is selected.

# Region Games

- $Tab_R : States_R \times Act_R^{Agt} \rightarrow 2^{Edg_R} \setminus \{\emptyset\}$ is such that for every $(l, r) \in States_R$ and every $m_{Agt} \in \prod_{A \in Agt} Mov_R((l, r), A)$, if we write $r'$ for $min\{r_A \mid m_A = (r_A, p_A, \delta_A)\}$ and $p'$ for $min\{p_A \mid m_A = (r', p_A, \delta_A)\}$, then

$$Tab_R((l, r), m_{Agt}) = \Big\{((l, r), (l_B, [z_B \leftarrow 0]r_B)) \,\Big|\, m_B = (r_B, p_B, \delta_B) \text{ with } r_B = r',\ p_B = p' \text{ and } \delta_B = (l, g_B, z_B, l_B)\Big\}$$

- For every $A \in Agt$, $\Omega_R(A) = \{(l, r) \mid (l, r) \in States_R \text{ and } l \in \Omega(A)\}$

# K-Resilient Nash Equilibria in Timed Games

**Lemma (7)**

*Consider two games $G$ and $G'$ involving the same set of agents ($Agt$) with reachability objectives given as $\Omega$ and $\Omega'$ respectively. For some $k \leq |Agt|$, assume that there exists a binary relation $\backsim_k$ between states of $G$ and states of $G'$ such that if $s \backsim_k s'$, then:*

- *For every $A \in Agt$, if $s' \in \Omega'(A)$ then $s \in \Omega(A)$.*
- *For every move $m_{\mathbf{Agt}}$ in $G$, there exists a move $m'_{\mathbf{Agt}}$ in $G'$ such that:*
  - *For every $t'$ in $G'$, there is $t \backsim_k t'$ in $G$ s.t. $k - Susp_{\mathbf{G'}}((s', t'), m'_{\mathbf{Agt}}) \subseteq k - Susp_{\mathbf{G}}((s, t), m_{\mathbf{Agt}})$.*
  - *For every $(s, t) \in Tab(s, m_{\mathbf{Agt}})$, there is a $(s', t') \in Tab(s', m'_{\mathbf{Agt}})$ s.t. $t \backsim_k t'$.*

*Then for every $P \subseteq Agt$ and every $X \subseteq 2^{\mathbf{Agt}}_{\mathbf{k}}$ such that $P \bigcap \left( \bigcup_{\mathbf{W} \in \mathbf{X}} W \right) = P$ and for every $s$ and $s'$ such that $s \backsim_k s'$, it holds:*

1. *If $s \in k - Rep_{\mathbf{G}}(P, X)$, then $s' \in k - Rep_{\mathbf{G'}}(P, X)$.*
2. *For every $(s, t) \in Edg_{\mathbf{k-Rep}}$, there exists $(s', t') \in Edg'_{\mathbf{k-Rep}}$ s.t. $t \backsim_k t'$, where $Edg_{\mathbf{k-Rep}}$ and $Edg'_{\mathbf{k-Rep}}$ are the set of edges in transition systems $k - S_{\mathbf{G}}(P, X)$ and $k - S_{\mathbf{G'}}(P, X)$ respectively.*

*Proof (Sketch).* This lemma can be easily proved by induction on the ordered pair $(P, X)$.

# K-Resilient Nash Equilibria in Timed Games

- It can be proved that a timed game $G$ and its associated region game $G_R$ simulate each other in the sense of Lemma 7 via relation $\backsim_k$ for every $k \leq |Agt|$.
- It can be proved that for every $k \leq |Agt|$, there is a binary relation $\backsim_k$ (as defined in Lemma 7), such that the the relation $\backsim_k$ exists between states of $G$ and states of $G_R$ and the relation $\backsim_k$ also exists between states of $G_R$ and states of $G$.
- In particular, for every state $(l, v)$ in $G$ and every state $(l, r)$ in $G_R$, if $r$ is the region containing $v$ then $(l, v) \backsim_k (l, r)$ and $(l, r) \backsim_k (l, v)$ for every $k \leq |Agt|$.
- This proof is possible by using the translations $\lambda$ (which maps moves in $G$ to equivalent moves in $G_R$) and $\mu$ (which maps moves in $G_R$ to equivalent moves in $G$) as defined in [6, 7].

# K-Resilient Nash Equilibria in Timed Games

- From Lemma 7, it can be inferred that for every $P \subseteq Agt$ and every $X \subseteq 2^{Agt}_k$ such that $P \bigcap \left( \bigcup_{W \in X} W \right) = P$ and for every state $(l, v)$ in $G$ and corresponding state $(l, r)$ in $G_R$ such that $r$ is the region containing $v$, the following two results hold:
    1. $(l, v) \in k - Rep_G(P, X) \Leftrightarrow (l, r) \in k - Rep_{G_R}(P, X)$.
    2. $((l, v), (l', v')) \in Edg^G_{k-Rep} \Leftrightarrow ((l, r), (l', r')) \in Edg^{G_R}_{k-Rep}$ where $r'$ is the region containing $v'$ and $Edg^G_{k-Rep}$ and $Edg^{G_R}_{k-Rep}$ are the set of edges in the transition systems $k - S_G(P, X)$ and $k - S_{G_R}(P, X)$ respectively.

# K-Resilient Nash Equilibria in Timed Games

**Theorem (8)**

*Let $G$ be a multi-player timed concurrent reachability game and let $G_R$ be its associated region game. Then for every $k \leq |Agt|$, there is a k-resilient pseudo-Nash equilibrium in $G$ from $(s, \boldsymbol{0})$ if and only if there is a k-resilient pseudo-Nash equilibrium in $G_R$ from $(s, [\boldsymbol{0}])$ where $[\boldsymbol{0}]$ is the region associated to valuation $\boldsymbol{0}$. Furthermore, this equivalence is constructive.*

*Proof (Sketch).*

- We have established that a timed game $G$ and its associated region game $G_R$ simulate each other in the sense of Lemma 7 via relation $\backsim_k$ for every $k \leq |Agt|$.

# K-Resilient Nash Equilibria in Timed Games

- It follows that for every $P \subseteq Agt$, there is a path $\rho$ in $k - S_G(P, 2_k^{Agt})$ if and only if there is a corresponding path $\rho'$ in $k - S_{G_R}(P, 2_k^{Agt})$ which visits exactly the same regions visited by $\rho$.
- Therefore, there is an infinite path $\rho$ in $k - S_G(P, 2_k^{Agt})$ from $(s, \mathbf{0})$ which visits $\Omega(A)$ for every $A \notin P$, if and only if, there is an infinite path $\rho'$ in $k - S_{G_R}(P, 2_k^{Agt})$ from $(s, [\mathbf{0}])$ which visits $\Omega_R(A)$ for every $A \notin P$.
- From Theorem 5, we conclude that there is a *k-resilient* pseudo-Nash equilibrium in $G$ from $(s, \mathbf{0})$ if and only if there is a *k-resilient* pseudo-Nash equilibrium in $G_R$ from $(s, [\mathbf{0}])$.

# K-Resilient Nash Equilibria in Timed Games

- By Theorem 8, we can infer that the translation of a multi-player timed concurrent reachability game to its associated region game preserves a *k-resilient* pseudo-Nash equilibrium (if one exists).
- Hence, in order to check the existence of *k-resilient* pseudo-Nash equilibrium in a multi-player timed concurrent reachability game, we can translate the timed game to its associated region game and then check the existence of *k-resilient* pseudo-Nash equilibrium in the region game.
- As the region games in this case are finite non-deterministic multi-player concurrent reachability games, we can apply Algorithm 1 to check the existence of *k-resilient* pseudo-Nash equilibrium in the region games.

# K-Resilient Nash Equilibria in Timed Games

- The translation of a multi-player timed concurrent reachability game to its associated region game results in an exponential blow-up in the number of states and an exponential blow-up in the size of transition table but the number of agents is unchanged.
- Recall that Algorithm 1 runs in time exponential with respect to the number of states and the size of transition table but doubly exponential with respect to the number of agents.
- Therefore, the algorithm for timed games runs in time doubly exponential with respect to the number of states, size of transition table and the number of agents. Hence the algorithm is doubly exponential with respect to the size of input and is still in 2-EXPTIME.

# Conclusion

- We studied non-deterministic multi-player concurrent reachability games as defined in [6, 7].
- We proved some properties that characterize *k-resilient* Nash equilibria in these games when only pure strategies are allowed.
- We then used these properties to develop algorithms for checking existence of *k-resilient* Nash equilibrium in finite non-deterministic multi-player concurrent reachability games in untimed and timed settings when only pure strategies are allowed.
- Our algorithms are in 2-EXPTIME.
- The properties that we proved also contain all the necessary information to compute a *k-resilient* Nash equilibrium if it exists.

# Future Work

- We have not been able to prove the optimality of our algorithms. It may be interesting to check if the algorithms can be made better using non-determinism or if 2-EXPTIME (complexity class of proposed algorithms) is actually the lower bound for the problems.
- We have considered games where only pure strategies are allowed. The work may be extended to games where players can have mixed strategies.
- We have considered games with reachability objectives only. The work may be extended to study *k-resilient* Nash equilibria in games with other qualitative objectives like safety objectives, Büchi objectives, co-Büchi objectives, Rabin objectives and parity objectives.

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# THANK YOU!!