

# Deep Learning for Individual Heterogeneity \*

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## Abstract

This paper integrates deep neural networks (DNNs) into structural economic models to increase flexibility and capture rich heterogeneity while preserving interpretability. Economic structure and machine learning are complements in empirical modeling, not substitutes: DNNs provide the capacity to learn complex, non-linear heterogeneity patterns, while the structural model ensures the estimates remain interpretable and suitable for decision making and policy analysis. We start with a standard parametric structural model and then enrich its parameters into fully flexible functions of observables, which are estimated using a particular DNN architecture whose structure reflects the economic model. We illustrate our framework by studying demand estimation in consumer choice. We show that by enriching a standard demand model we can capture rich heterogeneity, and further, exploit this heterogeneity to create a personalized pricing strategy. This type of optimization is not possible without economic structure, but cannot be heterogeneous without machine learning. Finally, we provide theoretical justification of each step in our proposed methodology. We first establish non-asymptotic bounds and convergence rates of our structural deep learning approach. Next, a novel and quite general influence function calculation allows for feasible inference via double machine learning in a wide variety of contexts. These results may be of interest in many other contexts, as they generalize prior work.

**Keywords:** Deep Learning, Structural Modeling, Heterogeneity, Machine Learning, Neural Networks, Influence Functions, Neyman Orthogonality, Semiparametric Inference, Double Machine Learning.

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# 1 Introduction

Structural economic models are designed to obey theoretical and domain-specific restrictions emanating from the discipline. As a consequence, when taken to data, estimates and inferences from structural models are (economically) interpretable and directly useful for decisions, counterfactuals, and policy making. Crucially, in many economic decision problems, the structure of the model and the discipline it imposes afford meaningful interpolation and extrapolation which, in turn, facilitates the construction of counterfactuals and ultimately optimization.

Structural models are, however, potentially incomplete. Theory often does not specify every aspect of the framework we need to conduct empirical analysis. One critical element, and our focus in this paper, is the specification is heterogeneity. Even in cases where the presence of heterogeneity is implied by economic reasoning, the form and type is often unknown. As such, researchers must choose, or search for, a specification for heterogeneity that respects the structural assumptions of the model, as well as the practical considerations imposed by the data.

Often, the choice of the manner in which heterogeneity is modeled reflects the underlying objectives of the researcher. There are scenarios where one wishes to merely “control” for heterogeneity, since the key constructs of interest are not directly tied to it. In these cases, heterogeneity is treated as a nuisance parameter (or function). More recently, however, there has been a push to “exploit” heterogeneity by constructing individualized or personalized policies. Here, heterogeneity itself becomes a key quantity of interest. In these cases, we also will need to predict heterogeneity (for a new observation), and, as such, usual control-type methods (e.g. fixed effects) become less relevant. Targeting and personalization must be done on the basis of observable characteristics.

We argue that both the discipline of economic structure and the flexibility of machine learning (ML) tools are essential for heterogeneity-dependent constructs such as personalization policies, targeting, and counterfactuals. Both have their strengths and weaknesses, but, by combining them appropriately, one can use the strengths of each to remedy the shortcomings of the other. On one hand, theory and structure make optimization feasible and reliable, but do not dictate heterogeneity patterns. On the other hand, ML cannot learn economic structure with finite data (as argued and demonstrated below). Therefore, ML cannot take the place of economic theory and constraints, but flexibly learning patterns and heterogeneity is precisely the strength of modern ML. Thus, while ML cannot replace structural modeling, it can be useful to augment and enrich structure. We argue that economic structure and machine learning are *complements* in empirical modeling, not *substitutes*.

We show how to embed machine learning, in the form of modern deep neural networks (DNNs), into economic modeling. Our approach begins with a structural model imposed by the researcher. The model relates the outcomes  $\mathbf{y}_i$  to the covariates of interest  $\mathbf{t}_i$  (treatments or policy-relevant variables) and depends on parameters of interest  $\boldsymbol{\theta}$ , which are to be estimated from the data. We suppose this structural model is captured by a loss function  $\ell(\mathbf{y}_i, \mathbf{t}_i, \boldsymbol{\theta})$ . The structural model encodes restrictions and constraints on the data generating process and the parameters  $\boldsymbol{\theta}$  have direct economic interpretations. From an econometric point of view, this is a standard parametric estimation and inference problem.

We then enrich the model by changing the parameters  $\boldsymbol{\theta}$  into *parameter functions*  $\boldsymbol{\theta}(\mathbf{x})$ , which are fully flexible functions of a vector of observed characteristics  $\mathbf{x}_i$ . The new model is then  $\ell(\mathbf{y}_i, \mathbf{t}_i, \boldsymbol{\theta}(\mathbf{x}_i))$  (see Figure 1 for a visual depiction of our framework). This change adds flexibility while maintaining all the structure and meaning of the original model. The incompleteness of theory is reflected in the flexibility of treating  $\boldsymbol{\theta}(\mathbf{x}_i)$  as a fully unknown function. This is exactly where the strength of machine learning is exploited, and how ML complements economic structure. Either one alone is insufficient: structure without machine learning would lose the richness and fail to capture important patterns in the data, while naively applying machine learning without structure would lose the interpretability and the utility in policy/decision making. As an aside, we note that this framework does nest some prior approaches to controlling for heterogeneity, but that is not our focus here.

The structural model allows for optimization, but the ML-enriched structure allows for individual-level optimization (i.e. for each unique  $\mathbf{x}$ ). This is useful for decision making and policy analysis at the individual level, answering “who gets what” questions regarding targeting and “what who gets” questions of personalization. The former is useful, for example, in deciding to which type of individuals we allocate a scarce resource or treatment, while the latter focuses on the assignment of different treatments to each individual.

We introduce a novel yet simple structural deep learning approach to estimate the parameter functions. The key to this approach is the architecture is depicted in Figure 1. Neural networks allow the structure of the model to be directly encoded in the estimation through the network architecture. The expressive power of deep neural networks comes from the hidden layers. In standard approaches this flexibility is used to learn regression functions or form predictions, which determine both the target and the loss functions. In our approach, the hidden layers are directed through a *parameter layer* so that the power of ML is used entirely for learning the parameter functions. These then enter the *model layer* according to the structural model in order to optimize the loss. The required change to the architecture is intuitive and simple to implement. (See Figure B.1 for a direct comparison to regression.) The neural network optimizes the parameter functions to minimize the

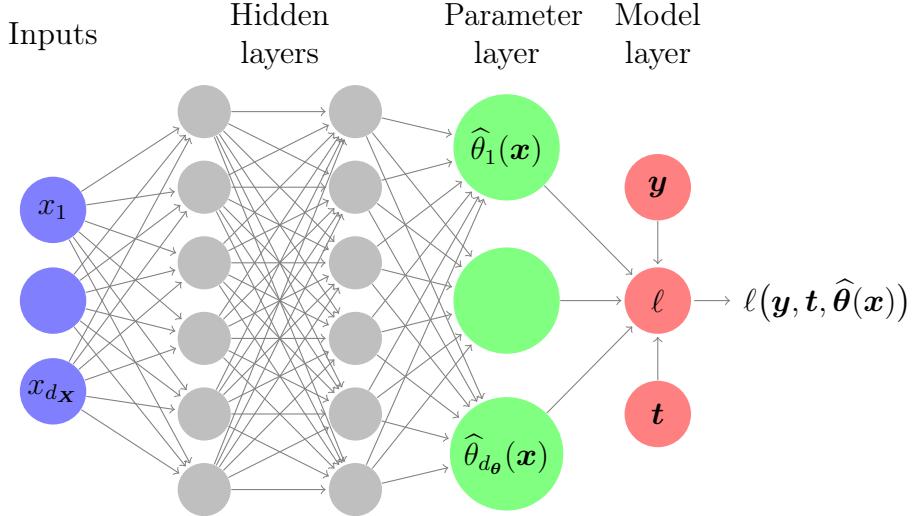


Figure 1: **Structural deep learning.** A schematic depiction of the deep neural network architecture for estimating the parameter functions  $\boldsymbol{\theta}(\mathbf{x})$  in the structural economic model  $\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}(\mathbf{X}))$  defined in Eqn. (3.5).

same structural loss, not prediction loss. This formalizes exactly how machine learning and structure are complementary. The ML enriches the economic model, filling in gaps left by theory. Simultaneously the economics aids the implementation of ML to estimate structural objects with economic meaning and interpretation. Our method allows for learning any economically interesting parameter, such as coefficients, variances, elasticities, et cetera, all as rich, heterogeneous functions.

To further build intuition, consider optimization of the network in Figure 1. Neural network estimation relies on using gradients to find optima, proceeding by back propagation through the network. Our idea is simply to structure the final layers of this network, as shown in Figure 1, optimizing the loss through the parameter functions, instead of optimizing prediction directly, but the computation is the same. This intuition also goes the other direction: parametric structural estimation of  $\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta})$  would be optimized in exactly the same way, but the gradients would stop at the parameter layer. Our idea simply extends this through additional layers to add heterogeneity.

We theoretically justify this part of the methodology by proving nonasymptotic bounds, and implied convergence rates, for the structured deep neural networks (Theorem 1). This result generalizes Farrell, Liang, and Misra (2021). These rates depend only on the dimension of the heterogeneity  $\mathbf{x}_i$ , because the relationship of  $t_i$  to  $y_i$  is not learned from the data. Our bounds apply to many settings of interest, requiring only mild conditions on the loss function  $\ell(\cdot)$  and standard smoothness requirements, and thus may be of interest outside structural modeling.

We obtain valid inference by applying the double machine learning methodology of Chernozhukov et al. (2018a), where the requisite orthogonal score is obtained from a novel and general influence function calculation (Theorem 2). Our insight here exploits the fact that we explicitly enrich a parametric model to obtain a two-step semiparametric setting, and in this case we show that *ordinary* derivatives can continue to characterize the influence function, just as in the parametric case. The influence function is thus known for a very wide class of models and parameters, and need not be newly derived in each case. Further, the only thing required is the value of these derivatives at the data points, not as functions in general. These may be known in advance or, importantly for practice, obtained using automatic differentiation engine built into neural network estimation without any human derivations needed. Collectively, this means we can deliver the influence function, and thus obtain valid inference via double machine learning, in more contexts than previously possible. This hopefully lessens the barriers to obtaining post-ML inference in practice.

We illustrate our methods and results by revisiting and extending the analysis of Bertrand et al. (2010). The data is from a large scale field experiment run on behalf of a financial institution in South Africa. Consumers were sent marketing material for short terms loans where a number of features of the advertising content and the interest rate offered were randomized. As in the original paper, we employ a binary choice model, one of the workhorse models in applied economics, but we enrich the model to capture heterogeneity. We then model demand as a function of prices and marketing efforts, and once the demand function is estimated, we compute optimal prices as a function of characteristics (third degree price discrimination) and conduct inference on quantities such as counterfactual profits. This involves applying double machine learning ideas to objects not available in closed form (the solution to a fixed point problem), which illustrates the generality of our inference method.

The rest of the paper is organized as follows. We next review some related literature. Section 2 then shows a simple, motivating example demonstrating why neither structural models nor machine learning individually suffice. Our framework for structured deep learning and subsequent inference is described in Section 3. We then apply these ideas in Section 4. Section 5 summarizes the theoretical results and Section 6 concludes. The Appendix gives all proofs, further discussion of related contexts, and thorough discussion of the important and intuitive special case of generalized linear models in order to illustrate the ideas and connect to other results.

## 1.1 Related Literature

Our work touches on several areas of economics, econometrics, and statistics, to which we cannot hope to do justice in a few paragraphs. We will give an overview here, with some further discussion is given throughout the paper.

First, at a broad level, our work is related to the recent interest in integrating ML into economic research. A large part, if not all, of this work has used ML to learn regression functions or make predictions, even if these feature in other economic contexts. ML is often a substitute for nonparametric regression, and succeeds where classical methods fail due to complexities or limitations of the data. Outside of pure prediction, causal inference is often the end goal, but our work, being motivated by structure, extends to a breath of economic contexts. For recent reviews in this space, with differing focuses, see [Varian \(2014\)](#), [Bajari et al. \(2015\)](#), [Mullainathan and Spiess \(2017\)](#), and [Athey and Imbens \(2019\)](#).

Although it is not our focus per se, our work is related to literature on heterogeneous treatment effects ([Athey and Imbens, 2016](#); [Chernozhukov et al., 2018b](#)) and through it to the topics of personalization and targeting ([Dubé and Misra, 2023](#); [Agrawal et al., 2022](#); [Hitsch, Misra, and Zhang, 2024](#)). Our embedding of deep neural networks into structural models affords the estimation of heterogeneous parameter functions across a broad array of applications and the potential to generate personalized policies. We demonstrate its use in the context of personalized interest rates in our application section.

On the theoretical front, the structured deep learning approach we propose, and the accompanying nonasymptotic bounds, connect our work to the recent literature studying the statistical properties of ML methods. Our theoretical results build directly upon and generalize [Farrell, Liang, and Misra \(2021\)](#). Conceptually, [Athey, Tibshirani, and Wager \(2019\)](#) and [Foster and Syrgkanis \(2023\)](#) may be the most closely related, as both focus on quantities other than prediction functions, use models beyond regression, and utilize orthogonal scores as key ingredients. The contexts and results are different, as we focus on deep learning and semiparametric inference. [Athey, Tibshirani, and Wager \(2019\)](#) study random forests and nonparametric inference, while [Foster and Syrgkanis \(2023\)](#) are concerned with risk bounds under weak conditions.

The use of influence functions explicitly with the goal of obtaining valid inference under weaker conditions is not new, whether for classical nonparametrics ([Cattaneo, 2010](#)) or machine learning ([Farrell, 2015](#)). Our work contributes most directly to the recent literature applying the double machine learning (DML) method of [Chernozhukov et al. \(2018a\)](#). DML combines sample splitting with an orthogonal score, which an influence function provides. Many applications of DML stick to contexts with well-known influence functions or hand-derive a new score. Our contribution here is the derivation of a generic influence function

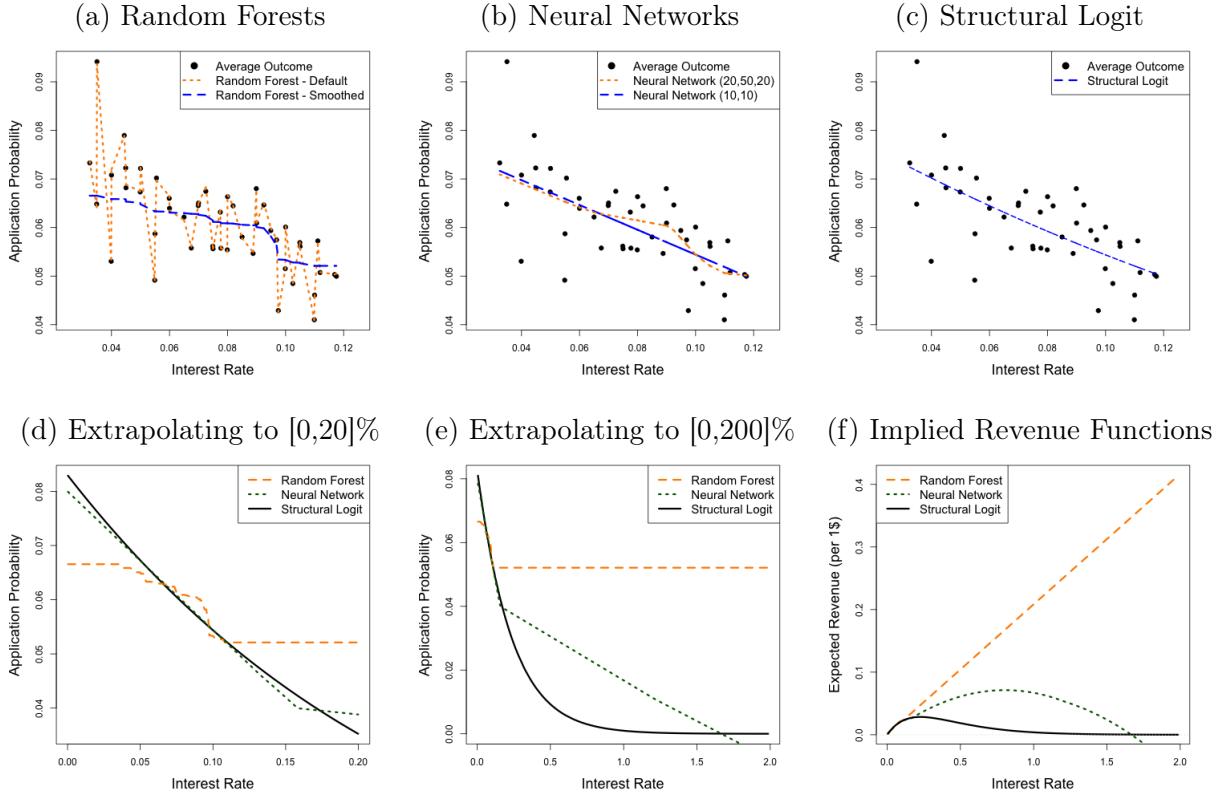
that covers a broad class of models and can be computed automatically. We share this goal with the “auto-DML” method (see Remark 3.4 below); our approach is different and has relative strengths and weaknesses.

## 2 The Importance of Structure

Most modern applications of personalized policy design require the ability to recover heterogeneous responses to some treatment(s) from data, and the facility to use those in an optimization framework to obtain improvements in the counterfactual. While the former can be done via use of the ML toolkit, the latter requires that the estimated response functions obey certain economic curvature and shape restrictions. As such, our choices are to either learn these constraints from data or impose them via structure. We demonstrate, with a simple example, that learning economic structure from finite date can be impossible but that structure is invaluable for decision making problems. To make the point clear we ignore heterogeneity and rely on the reader to extrapolate to the fact that the lessons here become even more important when recovering heterogeneity or personalization, because this creates greater demands on data. Our key point is that, structure imposes appropriate economic restrictions and constraints, and consequently data is used more efficiently to learn the heterogeneity.

The value of structure in economic modeling is well understood. Indeed, the half-century old Lucas Critique is a (major) example of the value of structural thinking. With the power of ML, there is a renewed temptation to learn everything from data alone. This view has its origins in the machine learning literature, which is focused on prediction, but has seeped into other contexts. [Zadrozny \(2004\)](#) writes, in the context of classification, that the ML community is “interested in the predictive performance of the model and not in making conclusions about the underlying mechanisms that generate the data.” So successful is machine learning, and deep learning in particular, at prediction tasks, that it can be tempting to assume that any problem can be tackled with accurate enough predictions. In the modern age of large language models and generative AI, this approach is even more appealing. But in contrast with those contexts, where prediction is sufficient, economic decision making cannot proceed without the “underlying mechanisms”, that is, without structure and economics. The danger in naively applying ML in decision making is that *economic assumptions and structure* are replaced with the *statistical and computational assumptions and structure* of estimators.

To illustrate, we will use the data of [Bertrand et al. \(2010\)](#), which described and analyzed more fully in Section 4. This context is ideal because, by focusing on a subset of the data, we find that this is simultaneously a straightforward demand estimation problem and



**Figure 2: Structural modeling and machine learning for demand functions.** Panels (a), (b), and (c) show estimated demand functions using random forests, neural networks, and a structural binary choice model, respectively, using the data of [Bertrand et al. \(2010\)](#). Panels (d) and (e) show the extrapolated demand functions and (f) the implied revenue functions.

straightforward prediction task, making both structural modeling and machine learning well-motivated. The context is demand for a short-term loan given its interest rate, and our goal is to estimate demand as a function of price and then derive the optimal interest rate offer. For 40,507 individuals we observe the binary outcome  $y_i \in \{0, 1\}$  indicating a loan application decision and the policy relevant variable  $t_i = r_i$  giving the interest rate offered. The good being “purchased” is the loan and its price is captured by the interest rate. The interest rate  $t_i = r_i$  is randomized, so there are no endogeneity concerns, and is furthermore continuous, taking on 46 unique values. With 40,507 observations on two variables, this is a tailor-made classification task, and the naive ML view would hold that if conversion for any interest rate offered can be predicted accurately, then demand function can be recovered, and finally the interest rate can be optimized. However, as we will see, neglecting the “underlying mechanisms that generate the data” will lead to a failure. If the unstructured approach fails in this context, adding heterogeneity is only more difficult.

Figure 2 shows three different approaches to estimating the demand function in this data. In each case, the dots show the empirical application rate (average purchasing decision) at each offered interest rate. First we consider two ML approaches which treat this as purely a predication/classification task, and are thus based on the model

$$\mathbb{P}[Y=1 \mid R=r] = \eta(r) \quad (2.1)$$

for an unknown function  $\eta(r)$  to be nonparametrically estimated. Panel (a) shows two random forest estimates using the `ranger` package in R. The dotted orange line uses the default settings. This appears to be undersmoothed and thus the blue dashed line forces a smoother fit using an ad-hoc restriction on the tree depth. It is easy to (by eye) reject the default random forest as an unreasonable estimate of the demand function, although it should be noted that even this judgment requires economic theory, not statistical/ML criteria: nothing in machine learning says the curve should be smooth or downward-sloping. Panel (b) shows two neural network fits. The dotted orange line uses three hidden layers with 20, 50, and 20 nodes, respectively, for a flexible fit. Note that despite the richness, the fit does not appear to be overly complex. The blue dashed line uses two layers with 10 nodes each for a smoother fit (matching Section 4). Finally, panel (c) replaces (2.1) with the basic workhorse structural model in industrial organization, namely a linear randomly utility model with a logistic errors, so that

$$\mathbb{P}[Y=1 \mid R=r] = G(\theta_1^* + \theta_2^* r) := \frac{1}{1 + \exp(-[\theta_1^* + \theta_2^* r])}, \quad (2.2)$$

where  $G(u)$  is the logit function and  $\theta_1^*$  and  $\theta_2^*$  are the intercept and slope, respectively.

The smoothed-out forest and both neural networks produce curves that could reasonably be demand functions. However, as we continue with the pricing problem, it becomes clear how different are these fits. Panels (d) and (e) show the extrapolated estimated demand functions of the smoothed forest, the (10,10) neural network, and the binary choice model to interest rates of zero to 20% (panel (d)) and zero to 200% (panel (e)). Panel (f) then shows the implied revenue for each demand function estimate. While this does not include costs, it is clear what each estimate implies for an optimal price (see Section 4 for more realistic profit optimization).

The forest fit is perhaps the most striking. Demand is completely flat for any higher interest rate, and therefore revenue grows without bound, and so the optimal price is infinite. This is both mechanical and a general phenomenon: a forest is an average of trees, which are piecewise constant, and therefore all extrapolation is based on a flat line, no matter the context or data. The neural network closely resembles the binary choice model on the support of the data, but when we extrapolate we see that the invisible complexities of the network

yield very different demand and revenue curves. In contrast to both of these, the structural model gives well-behaved demand and revenue curves and yields a reasonable optimal price.

To conclude this example, several remarks are in order. (i) Students of all fields, including econometrics, statistics, and machine learning, are taught never to extrapolate a statistical fit outside the support of the data, and here we see why in dramatic fashion. Note that the same logic applies to interpolation. However, our goal of price optimization *requires* inter-and extra-polation, which by definition is based on assumptions and not upon the data. (ii) One can dismiss this simple example with the argument that no reasonable decision maker sets an infinite price or that the demand/revenue functions look obviously “wrong”, but again, these conclusions come from using economic structure to inform the ML and data, not the other way around. (iii) The lessons in this example are unrelated to statistical uncertainty. Different data would yield different fits, but the issues would persist, or perhaps manifest differently (except the forest, which always yields infinite prices, and thus has zero statistical uncertainty). (iv) Finally, it is worth noting that structural reasoning is commonly used in machine learning to improve prediction tasks. The structure of image recognition tasks is directly encoded into convolutional neural networks. The transformer architecture ([Vaswani et al., 2017](#)) is behind the recent success of large language models. In sum, structure at its core means constraints and restrictions, and this example shows that statistical structure and assumptions may not be appropriate or sufficient for economic decision making.

## 3 Embedding Deep Learning in Structural Models

### 3.1 Enriched Structural Model

We now turn to our approach to enrich structural models with machine learning. We describe our structured deep neural network architecture, which bakes the model into the ML, and how second-step inference can be done using double machine learning. The prior section shows that ML is not suitable for learning structure, which is by definition the opposite of flexibility. Our goal here is to use ML for what it is good at, recovering complex heterogeneity, while retaining all the advantages of the structural model.

The starting point is a standard parametric structural model, described by

$$\boldsymbol{\theta}^* = \arg \min_{\boldsymbol{\theta} \in \Theta} \mathbb{E} [\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta})], \quad (3.1)$$

where the loss function  $\ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta})$  encodes the researcher’s economic restrictions on how the outcomes  $\mathbf{Y} \in \mathbb{R}^{d_Y}$  relate to the variables of interest  $\mathbf{T} \in \mathbb{R}^{d_T}$ , depending on parameters

$\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^{d_\theta}$ .<sup>1</sup> The “treatment”  $\mathbf{T}$  variables can be randomized or not and can be continuous, discrete, or mixed; the model can be causal or not. In the first step the structural parameters  $\boldsymbol{\theta}$  are estimated from the data by solving the empirical analogue of (3.1) over the appropriate parameter space  $\Theta$ .

In a second step, inference is conducted on an object of the form

$$\mathbb{E}[\mathbf{H}(\mathbf{X}, \boldsymbol{\theta}, \tilde{\mathbf{t}})], \quad (3.2)$$

for a known function  $\mathbf{H}$ , that depends on the parameters, covariates, and possibly some value of interest  $\tilde{\mathbf{t}}$  for the policy relevant variables. The function  $\mathbf{H}$  can be the parameters themselves, quantities such as marginal effects, elasticities, measures of surplus, and can encompass optimization and other such operations. For example,  $\tilde{\mathbf{t}}$  can be an optimal price, as in our empirical application in Section 4. Technically  $\tilde{\mathbf{t}}$  can be subsumed into the definition of  $\mathbf{H}$ , but it is expositionally useful to make explicit.

Other than restricting to per-observation losses and smooth functions (so that derivatives exist), the combination of Equations (3.1) and (3.2) encompass a wide variety of parametric models, including many M- and Z- estimation problems, such regression models, quasi/pseudo-likelihoods, or generalized estimating equations and accompanying second step objects like marginal effects, average elasticities, and other economic quantities (Newey and McFadden, 1994; Ackerberg et al., 2007).

The positives of structural modeling are widely appreciated, and demonstrated above. The model incorporates theory-based restrictions and constraints. This means that the estimated parameters, or transformations thereof, are interpretable and directly useful in policy analysis, decision making, and the formation of counterfactuals. The two-step nature of the problem reflects exactly these types of analyses, where second step counterfactuals are obtained after estimating primitives of the model in the first stage. It also means that the data is used more efficiently, since structure implies restrictions, which will be crucial when embedding machine learning or nonparametrics.

The major drawback of Eqn. (3.1) is rigidity: it does not allow for any heterogeneity in the relationship between  $\mathbf{Y}$  and  $\mathbf{T}$ . We want to allow for heterogeneity in a way that is flexible but maintains the economic structure. This could be for robustness, as results will be biased and subsequent decisions or analyses will be erroneous if heterogeneity exists but is neglected. This type of concern has been the focus of much recent research, particularly in program evaluation (Chernozhukov et al., 2018b; Wager and Athey, 2018; De Chaisemartin and d’Haultfoeuille,

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<sup>1</sup>Notation. Vectors and matrices will be written in boldface. Capital letters are used for population random variables; lower case for realizations. The expectation operator with respect to the true data generating process is denoted  $\mathbb{E}[\cdot]$ . True values use a superscript  $\star$ . The  $L_2$  norm for a function  $g(\mathbf{x})$  is  $\|g\|_2 = \mathbb{E}[g(\mathbf{X})^2]^{1/2}$ .

2023). Further, as mentioned earlier, capturing and exploiting heterogeneity is key in modern targeting and personalization contexts, which is also an active area of research.

Our approach recasts the parameters  $\boldsymbol{\theta}$  as *parameter functions*  $\boldsymbol{\theta}(\mathbf{X})$  that are fully flexible in observed covariates  $\mathbf{X} \in \mathbb{R}^{d_x}$ . That is, we assume the true first stage structural model is

$$\boldsymbol{\theta}^*(\cdot) = \arg \min_{\boldsymbol{\theta} \in \mathcal{F}} \mathbb{E}[\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}(\mathbf{X}))], \quad (3.3)$$

for a function class  $\mathcal{F}$ , which obeys standard restrictions (Assumption 2 below). Equation (3.3) is a specific, though quite general, formulation of nonparametric M-estimation (Gallant and Nychka, 1987). Crucially all the economic structure is maintained: whatever the interpretation of the parameter  $\boldsymbol{\theta}$ , the same holds for  $\boldsymbol{\theta}^*(\mathbf{x})$  for individuals of “type”  $\mathbf{X} = \mathbf{x}$ . These are not “nuisance” functions. The view, particularly common in the realm of inference after ML, is that first step functions are literally a nuisance, i.e. something annoying that must be dealt with, but are not interesting. We object to this view: in many applications the learned heterogeneity is actually the most interesting part, and because the  $\boldsymbol{\theta}^*(\mathbf{x})$  are interpretable functions in our framework, using them is straightforward.

The second step parameter of interest is correspondingly enriched, to be

$$\boldsymbol{\mu}^* = \mathbb{E}[\mathbf{H}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}), \hat{\mathbf{t}})]. \quad (3.4)$$

To keep exposition simple we focus on the case of averages, i.e. where the parameter of interest is available in closed form. Replacing this step with parametric GMM is straightforward but notationally more cumbersome (see Remark 3.1). Our results may be useful for nonparametric inference as well (Remark 3.2). Equations (3.3) and (3.4) together define a broad class of two-step semiparametric settings, matching the generality of (3.1) and (3.2). Appendix C shows some examples, both familiar and new. Finally, note that  $\boldsymbol{\mu}^*$  is defined using  $\boldsymbol{\theta}^*(\mathbf{X})$ , but in some decision making cases this may not be appropriate, as discussed in Remark 4.1.

## 3.2 Structural Deep Learning

To estimate the parameter function  $\boldsymbol{\theta}^*(\mathbf{x})$  we solve the empirical analogue of (3.3) where minimization is over a class of *structural* deep neural networks, i.e. those with architecture shown in Figure 1 as discussed above. We define

$$\widehat{\boldsymbol{\theta}}(\cdot) = \arg \min_{\boldsymbol{\theta} \in \mathcal{F}_{\text{DNN}}} \frac{1}{n} \sum_{i=1}^n \ell(\mathbf{y}_i, \mathbf{t}_i, \boldsymbol{\theta}(\mathbf{x}_i)), \quad (3.5)$$

where  $\mathcal{F}_{\text{DNN}}$  is the class of deep neural networks that encodes not only the overall architecture but also tuning parameter choices of the width and depth of the network and the shape parameter given by the activation function. We focus on the standard fully connected feedforward neural network with ReLU activation for the hidden layers, but the same idea applies to other cases. To save space, we will not review deep learning basics here. Recent textbook treatments include: [Goodfellow, Bengio, and Courville \(2016\)](#) and [James et al. \(2021\)](#) for introductions and examples, [Roberts, Yaida, and Hanin \(2022\)](#) for theory, and [Keydana \(2023\)](#) for implementation. Theorem 1 in Section 5 gives the convergence of these  $\hat{\theta}(\cdot)$ .

Our structural deep learning approach can be applied to any model/loss  $\ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x}))$  and is easy to implement. Only a few lines of code need be changed relative standard neural network implementations to enforce the structural model and force the power of the network to learn the parameter functions rather than optimize the loss directly. This “structural compatibility” is one argument in favor of using deep neural networks for this modeling. There are several further reasons why neural networks are well suited to this task. (1) Perhaps most obviously, deep learning is a state of the art ML method and brings with it all the advantages: expressive power, the capacity for high dimensionality and flexible interactions, and moreover, the ability to use novel data such as images or text, which could be included in the definition of  $\mathbf{X}$ . (2) From a practical point of view, neural networks can handle discrete covariates seamlessly without affecting the convergence rate nor the implementation. While in theory including discrete covariates does not impact the convergence rate for many nonparametric estimators, obtaining these estimates in practice typically requires special care and custom methods ([Racine and Li, 2004](#); [Ma, Racine, and Yang, 2015](#)). (3) Deep learning is built on automatic differentiation, which allows us to obtain the necessary influence function computationally for free for any  $\boldsymbol{\mu}^*$  from (3.4) based on any first step (3.3) (see Section 3.3) making double machine learning conceptually straightforward to apply in any problem.

Of course, DNNs are not the only method that could be used to recover the parameter functions, nor do we claim any formal optimality property. The economic model holds globally, and we wish to match this in estimation, as it is important to learn counterfactual quantities in the second step, but other methods have the same property, including global series methods such as splines or polynomials and modern basis-function style methods including ridge regression and lasso.<sup>2</sup> All these estimators impose the structural model globally and in a computationally simple way. This may be one reason why series methods were often used

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<sup>2</sup>The term “global” is slightly ambiguous, because methods like splines or partitioning are global smoothers and are structurally compatible, but use only data local to the evaluation point, more like a traditional kernel ([Cattaneo and Farrell, 2013](#); [Cattaneo, Farrell, and Feng, 2020](#)).

in classic nonparametric M estimation (see [Gallant and Nychka \(1987\)](#) for pioneering early work and [Chen \(2007\)](#) for further theory). Also [Chen \(2007\)](#) views neural networks as sieves for generic nonparametric M estimation, as do we. But these methods lack advantages (1), (2), and (3) from the prior paragraph. Classical methods often do not work for modern applications with more than several covariates. Selection methods require pre-specification of bases, including interaction effects.

Although it is not always as straightforward, it is possible to use “local” methods to learn the value of the parameter functions at a point, i.e.  $\boldsymbol{\theta}^*(\mathbf{x})$  for some  $\mathbf{x}$ , and much recent work has been done in this area. [Fan and Zhang \(2008\)](#) discuss kernels and local polynomials while [Zeileis, Hothorn, and Hornik \(2008\)](#), [Athey, Tibshirani, and Wager \(2019\)](#), [Nekipelov, Novosad, and Ryan \(2019\)](#), and [Chatla and Shmueli \(2020\)](#) use trees and random forests; all share our goal of learning non-prediction function. [Athey, Tibshirani, and Wager \(2019\)](#) and [Foster and Syrgkanis \(2023\)](#) are perhaps the most recent closest antecedents to our work, as both move ML away from prediction and both use orthogonal scores as a key ingredient. [Athey, Tibshirani, and Wager \(2019\)](#) study random forests in a similar class of problems to (3.3). Random forests also handle higher dimensional problems and flexible interactions, sharing advantage (1) above, and by their nature share advantage (2). As such, the forests of [Athey, Tibshirani, and Wager \(2019\)](#) are probably the closest substitute for neural networks in our setting. The goal in that paper is inference on the nonparametric object  $\boldsymbol{\theta}^*(\mathbf{x})$  at a point  $\mathbf{X} = \mathbf{x}$ , which is different than our two-step problem. Influence functions based on ordinary derivatives are used there, though in the estimation step to aid with tree splitting, rather than for inference per se, as we do below. [Foster and Syrgkanis \(2023\)](#) are focused on risk minimization, as opposed to inference, but consider a similar class of models and use orthogonal scores to obtain improved properties. Their estimation target more closely resembles our two step problem, though a key innovation in their case is performing this in one step, rather than estimating the primitives and then studying different counterfactual questions. Their first and second step parameters can also be more general objects than ours in some ways.

Finally, other recent work has considered the combination of deep learning with structural models. Examples in different contexts include [Norets \(2012\)](#), [Igami \(2020\)](#), [Chen, Didisheim, and Scheidegger \(2021\)](#), [Kaji, Manresa, and Pouliot \(2023\)](#), and [Wei and Jiang \(2025\)](#). Often, the goal is estimation of a parametric structural model and deep learning methods are applied to learn the mapping of data to parameters, which is quite different from our use of neural networks to enrich parameters for heterogeneity and personalization.

### 3.3 Inference

For second step inference on  $\boldsymbol{\mu}^* = \mathbb{E}[\mathbf{H}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}), \tilde{\mathbf{t}})]$ , we apply and contribute to the recent semiparametric inference literature and in particular the now-common double machine learning (DML) method of Chernozhukov et al. (2018a). DML is a generic method for obtaining semiparametric inference that combines two ingredients: sample splitting and a Neyman orthogonal score. Asymptotic Normality then follows under weak conditions on first step estimators, which is particularly important for ML-based first steps, because relatively little is known of their mathematical and statistical properties. Typically,  $L_2$  convergence rates are sufficient (along with mild regularity conditions). To save space, we defer to Chernozhukov et al. (2018a), Newey and Robins (2018), and Chernozhukov et al. (2022a) for complete discussion of the method and further references. As usual with DML, our results apply to any first-step estimator that has fast enough rates. However, an automatic differentiation engine may be convenient for computing the influence function when it is not known from prior work.

While sample splitting is straightforward both conceptually and in applications, orthogonal scores are not, and this is where our contribution lies. The key characterization of a Neyman orthogonal score is that it has a zero derivative with respect to the first stage parameters (see Section 5), which translates to less sensitivity to first step error. Though more general conceptually, Neyman orthogonal scores often come from influence functions where this zero derivative property is ensured. Knowing the influence function, or orthogonal score, is required for using DML, and this is often a hurdle in applications. For this reason many papers stick to known examples (e.g., average treatment effects or partially linear models), list references where scores are derived, or hand-derive new scores (typical examples are Belloni, Chernozhukov, and Hansen (2014), Farrell (2015), and Chernozhukov et al. (2022a)). We contribute to this inference method by showing that an influence function for  $\boldsymbol{\mu}^*$  is automatically available for any combination of  $\ell(\cdot)$  and  $\mathbf{H}(\cdot)$  in Equations (3.3) and (3.4) provided their ordinary derivatives exist. We give the form of this influence function and discuss how it can be automatically computed in applications, even when it cannot be derived precisely or written down. This makes it easy to deploy in practice, because  $\ell(\cdot)$  and  $\mathbf{H}(\cdot)$  are defined by the researcher, and the rest can proceed automatically, requiring only nonparametric regression. A key insight into applicability is that the orthogonal score need not be known as a function per se, we only require that its value be computable at each observation. Our goal and results here are closest to the series of work on “auto-DML” (Remark 3.4).

To state the influence function result, and thus estimation and inference for  $\boldsymbol{\mu}^*$ , we define relevant derivatives, which may be known or found with automatic differentiation. Let

$\mathbf{H}_\theta(\mathbf{x}, \cdot; \tilde{\mathbf{t}})$  be the  $d_\mu \times d_\theta$  Jacobian of  $\mathbf{H}$  with respect to  $\theta$ . Let  $\ell_\theta(\mathbf{y}, \mathbf{t}, \cdot)$  be the gradient of  $\ell$  with respect to  $\theta$  and  $\ell_{\theta\theta}(\mathbf{y}, \mathbf{t}, \cdot)$  be the matrix of second derivatives. These are *ordinary* derivatives, not functional derivatives, and are thus computable automatically, even if they are evaluated at the value of the function  $\theta^*(\mathbf{x})$ .<sup>3</sup> Then an influence function that applies to any combination of enriched model (3.3) and parameter of interest (3.4) is  $\psi(\mathbf{y}, \mathbf{t}, \mathbf{x}, \theta, \Lambda) - \mu^*$ , with

$$\psi(\mathbf{y}, \mathbf{t}, \theta, \Lambda) = \mathbf{H}(\mathbf{x}, \theta(\mathbf{x}); \tilde{\mathbf{t}}) - \mathbf{H}_\theta(\mathbf{x}, \theta(\mathbf{x}); \tilde{\mathbf{t}})\Lambda(\mathbf{x})^{-1}\ell_\theta(\mathbf{y}, \mathbf{t}, \theta(\mathbf{x})), \quad (3.6)$$

where  $\Lambda(\mathbf{x}) = \mathbb{E}[\ell_{\theta\theta}(\mathbf{y}, \mathbf{t}, \theta(\mathbf{x})) \mid \mathbf{X} = \mathbf{x}]$  the population conditional Hessian of  $\ell$ , all evaluated at  $\theta = \theta(\mathbf{x})$  and is the “other” nonparametric object that generally arises in the correction term. The (inverse) propensity score is perhaps the most familiar example, and our requirements on  $\Lambda(\mathbf{x})$  mirror that case exactly. Theorem 2 below justifies this result and discusses it further. Influence functions are being used in a wide variety of contexts, and our derivation may be of independent interest (Remark 5.3).

To build intuition, it is useful to note that our result has precisely the same form as its parametric counterpart, but appropriately generalized. For the parametric two step inference problem of (3.1) and (3.2), the influence function is known to be  $\mathbf{H}(\mathbf{x}, \theta; \tilde{\mathbf{t}}) - \mathbf{H}_\theta(\mathbf{x}, \theta; \tilde{\mathbf{t}})\Lambda^{-1}\ell_\theta(\mathbf{y}, \mathbf{t}, \theta)$ , where  $\Lambda = \mathbb{E}[\ell_{\theta\theta}(\mathbf{y}, \mathbf{t}, \theta)]$  (Newey and McFadden, 1994). Equation (3.6) is the same, and equally general, but enriched and hence conditional on  $\mathbf{X} = \mathbf{x}$ . This tight connection helps with implementation, because if the original structural model is understood by the researcher, so is the enriched version. For example, identification in the parametric case typically requires  $\Lambda^{-1}$  to exist, and in the enriched version this must hold for all “types”  $\mathbf{x}$ . This is often a matter of assuming a positive conditional variance, rather than marginal variance. Appendix C gives some examples and special cases.

Obtaining the estimate  $\hat{\mu}$  and standard errors follows standard ideas of DML (Chernozhukov et al., 2018a). The only wrinkle is that  $\Lambda(\mathbf{x})$  generally depends on  $\theta(\mathbf{x})$ , and so

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<sup>3</sup>To be precise,  $\ell_\theta(\mathbf{y}, \mathbf{t}, \theta(\mathbf{x}))$  is the  $d_\theta$ -vector of first derivatives with respect to the parameter, evaluated at the number  $\theta(\mathbf{x})$ , as in

$$\ell_\theta(\mathbf{y}, \mathbf{t}, \theta(\mathbf{x})) = \left. \frac{\partial \ell(\mathbf{y}, \mathbf{t}, \mathbf{b})}{\partial \mathbf{b}} \right|_{\mathbf{b}=\theta(\mathbf{x})}.$$

The Jacobian  $\mathbf{H}_\theta(\mathbf{x}, \cdot; \tilde{\mathbf{t}})$  is similar. Then  $\ell_{\theta\theta}(\mathbf{y}, \mathbf{t}, \theta(\mathbf{x}))$  is the  $d_\theta \times d_\theta$  matrix of second derivatives, with  $\{k_1, k_2\}$  element given by

$$\left[ \ell_{\theta\theta}(\mathbf{y}, \mathbf{t}, \theta(\mathbf{x})) \right]_{k_1, k_2} = \left. \frac{\partial^2 \ell(\mathbf{y}, \mathbf{t}, \mathbf{b})}{\partial b_{k_1} \partial b_{k_2}} \right|_{\mathbf{b}=\theta(\mathbf{x})},$$

where  $b_{k_1}$  and  $b_{k_2}$  are the respective elements of the place-holder  $\mathbf{b}$ . The use of standard differentiation in these contexts has been used in some prior work, though to our knowledge not paired with automatic differentiation to obtain feasible inference in such a broad set of models. One can obtain the influence function using functional derivatives and then evaluate these at the true function  $\theta^*(\cdot)$ , but our point is that one only needs the ordinary derivative evaluated at the number  $\theta^*(\mathbf{x})$  (or the data points more specifically).

three-way splitting is technically required. Note that throughout, particularly when paired with automatic differentiation, note that we do not need to know the *functions*  $\mathbf{H}_\theta(\mathbf{x}, \theta(\mathbf{x}); \tilde{\mathbf{t}})$ ,  $\ell_\theta(\mathbf{y}, \mathbf{t}, \theta(\mathbf{x}))$ , and  $\ell_{\theta\theta}(\mathbf{y}, \mathbf{t}, \hat{\theta}(\mathbf{x}))$ ; it suffices to know the *values*  $\mathbf{H}_\theta(\mathbf{x}_i, \hat{\theta}(\mathbf{x}_i); \tilde{\mathbf{t}})$ ,  $\ell_\theta(\mathbf{y}_i, \mathbf{t}_i, \hat{\theta}(\mathbf{x}_i))$ , and  $\ell_{\theta\theta}(\mathbf{y}_i, \mathbf{t}_i, \hat{\theta}(\mathbf{x}_i))$ , which are readily available.

DML is now quite standard, so we keep the description brief. The data are divided into  $K$  disjoint subsets, denoted  $\mathcal{I}_k$ , of equal size. Then

$$\hat{\mu} = \frac{1}{K} \sum_{k=1}^K \hat{\mu}_k, \quad \hat{\mu}_k = \frac{1}{|\mathcal{I}_k|} \sum_{i \in \mathcal{I}_k} \psi(\mathbf{y}_i, \mathbf{t}_i, \hat{\theta}_k(\mathbf{x}_i), \hat{\Lambda}_k(\mathbf{x}_i)), \quad (3.7)$$

where  $|\mathcal{I}_k|$  and  $\hat{\theta}_k(\mathbf{x}_i)$  and  $\hat{\Lambda}_k(\mathbf{x}_i)$  are obtained using observations in  $\mathcal{I}_k^c$ . If  $\Lambda(\mathbf{x})$  depends on  $\theta(\mathbf{x})$ , then  $\mathcal{I}_k^c$  is divided in two and  $\hat{\theta}_k(\mathbf{x}_i)$  and  $\hat{\Lambda}_k(\mathbf{x}_i)$  are obtained on separate samples. This further splitting is required in theory in general (see Remark 3.3 and Appendix B for common exceptions) but in practice may make no difference. Also, short stacking may yield improved performance (Ahrens et al., 2025). From the optimization of (3.5), we can obtain the *values*  $\ell_{\theta\theta}(\mathbf{y}_i, \mathbf{t}_i, \hat{\theta}(\mathbf{x}_i))$  automatically. In other words,  $\ell_{\theta\theta}(\mathbf{y}_i, \mathbf{t}_i, \hat{\theta}(\mathbf{x}_i))$  is simply a column of data which we nonparametrically regress on  $\mathbf{x}_i$  to obtain  $\hat{\Lambda}(\mathbf{x}_i)$ . This is identical to the more standard practice of obtaining the functional derivatives, characterizing the nonparametric object that is  $\Lambda(\mathbf{x})$ , and then estimating it: we are not doing a numerical approximation. Numerical differentiation can be even simpler and faster, however. The same ideas apply to all other (potentially) unknown functions. Finally, the asymptotic variance  $\Psi = \mathbb{V}[\psi(\mathbf{Y}, \mathbf{T}, \mathbf{X}, \theta^\star, \Lambda)]$  can be consistently estimated by

$$\hat{\Psi} = \frac{1}{K} \sum_{k=1}^K \frac{1}{|\mathcal{I}_k|} \sum_{i \in \mathcal{I}_k} \left( \psi(\mathbf{y}_i, \mathbf{t}_i, \hat{\theta}_k(\mathbf{x}_i), \hat{\Lambda}_k(\mathbf{x}_i)) - \hat{\mu}_k \right)^2. \quad (3.8)$$

Theorem 2 validates this procedure. In theory, the size of each subset is proportional to  $n$  and thus sample splitting does not impact convergence rates or precision, but in practice this can be a poor approximation. For classical kernel estimators, Velez (2024) proves that larger  $K$  yields better results in an asymptotic framework with  $K \rightarrow \infty$ . For machine learning methods, this can be computationally costly, but we find in applications that large  $K$  is more stable. With small  $K$ , the different  $\hat{\theta}_k(\mathbf{x})$  can be quite different. Finally, sample splitting is not always needed, as shown by Farrell, Liang, and Misra (2021) for post-DNN average treatment effects (under essentially the same assumptions as here). It would be useful to extend that argument to more general models; see Chen, Syrgkanis, and Austern (2022) for relevant theory.

**Remark 3.1** (Two Step GMM). Our first-step correction can also be used in GMM settings. The second step may be a set of moment conditions  $\mathbb{E}[\tilde{\mathbf{H}}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}), \boldsymbol{\mu}^*, \tilde{\mathbf{t}})] = 0$  for some  $\tilde{\mathbf{H}}$ . The correction factor then takes the form  $\phi(\mathbf{y}, \mathbf{t}, \mathbf{x}, \boldsymbol{\Lambda}, \boldsymbol{\theta}) = \tilde{\mathbf{H}}_{\boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}), \boldsymbol{\mu}^*, \tilde{\mathbf{t}}) \boldsymbol{\Lambda}(\mathbf{x})^{-1} \boldsymbol{\ell}_{\boldsymbol{\theta}}(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x}))$ . Following Chernozhukov et al. (2022a), adding this correction to the original moments yields the orthogonal moment conditions  $\mathbb{E}[\tilde{\mathbf{H}}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}), \boldsymbol{\mu}^*, \tilde{\mathbf{t}}) - \phi(\mathbf{y}, \mathbf{t}, \mathbf{x}, \boldsymbol{\Lambda}, \boldsymbol{\theta})] = 0$ . Chernozhukov et al. (2022a), extending Chernozhukov et al. (2018a), show that the advantages of DML carry over to GMM based on these moments. Our methodology applies here, including the use of automatic differentiation if needed. Asymptotic normality will follow by applying Chernozhukov et al. (2022a) instead of Chernozhukov et al. (2018a). 

**Remark 3.2.** Semenova and Chernozhukov (2021) and Colangelo and Lee (2023) use orthogonal scores for two-step nonparametric inference (following ML) and we conjecture that the same could be done in the enriched structural models considered here. This would be valuable for future research. See also Remark B.3. 

**Remark 3.3** (Notes on  $\boldsymbol{\Lambda}(\mathbf{x})$ ). The function  $\boldsymbol{\Lambda}(\mathbf{x})$  is a nuisance in the truest sense: it is required only because we use influence functions as a tool to obtain valid inference. In our case,  $\boldsymbol{\Lambda}(\mathbf{x})$  is always low-dimensional and consists only of regressions, not conditional density functions. In some cases obtaining  $\boldsymbol{\Lambda}(\mathbf{x})$  is simplified. If  $\mathbf{T}$  is randomly assigned, or more generally independent of  $\mathbf{X}$ , then  $\boldsymbol{\Lambda}(\mathbf{x})$  can often be computed or estimated more simply, though it may remain a function of  $\mathbf{x}$ . Even if not randomly assigned, if  $\mathbf{T}$  is known to be assigned based on a subvector of  $\mathbf{X}$ , such as in targeting problems, this can be imposed on the estimation. Inverse functions are standard in semiparametric inference, and in practice this piece is often the most difficult. The challenge is generally model- and data- specific, unrelated to the choice of nonparametric/ML method. Often some form of regularization is used. The most widely known is trimming the propensity score, which has received rigorous study (Ma and Wang, 2020). Extending this to our setting would be useful. 

**Remark 3.4** (Auto-DML). The recent work on “auto-DML” shares our goal of applying the DML method without having to derive a new orthogonal score each time. This recent work (Chernozhukov, Newey, and Singh, 2022a,b; Chernozhukov et al., 2022a,b, 2023, 2024a,b) shows that the correction term of the influence function satisfies certain moment conditions which can then be taken to data to obtain feasible inference. In the context of (3.6), this amounts to estimating the quantity  $\mathbf{H}_{\boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}); \tilde{\mathbf{t}}) \boldsymbol{\Lambda}(\mathbf{x})^{-1}$ . There are different versions of this method, but broadly speaking it is as general and widely applicable as ours. Our approach

can be applied to many second stage parameters, since the first step correction need only be estimated once. We do require that  $\boldsymbol{\theta}^*(\mathbf{x})$  enter the second stage only through its evaluation at a data point. Both methods require a second nonparametric estimation of a nuisance function. The auto-DML method estimates the inverse directly, which might yield more stable performance than estimating  $\boldsymbol{\Lambda}(\mathbf{x})$  and then inverting. On the other hand, examining  $\widehat{\boldsymbol{\Lambda}}(\mathbf{x})$  is often a key step in the analysis and helps diagnose identification, such as examining estimated propensity scores to evaluate the overlap assumption. ♣

## 4 Application: Advertising and Personalized Interest Rates

### 4.1 Empirical Context

In this section we use our framework to replicate and extend [Bertrand et al. \(2010\)](#). The data is from a large scale field experiment run on behalf of a financial institution in South Africa. Consumers were sent marketing material for short terms loans where a number of features of the advertising content and the interest rate offered were randomized (full details are left to that paper). For the purposes of our analysis we will focus on the interest rate offered as the treatment variable, and denote the scalar  $T = R$ . We will treat the characteristics of the advertising assigned to customers as covariates ( $\mathbf{X}_a$ ) rather than treatments. These assigned ad characteristics will be used along with a set of customer demographics ( $\mathbf{X}_d$ ) in our analysis. We collectively refer to these as  $\mathbf{X} = \{\mathbf{X}_d, \mathbf{X}_a\}$  and use them to calibrate our measures of heterogeneity in the analysis below. The key outcome variable ( $Y$ ) is the indicator for whether or not the consumer applied for the loan. We use a binary choice model, one of the workhorse models in applied economics. Other relevant variables available in the data include an indicator of loan default ( $D$ ) and the loan amount ( $L$ ). While the full data set has  $N = 53,194$  individual observations, for the purposes of our illustration we will limit ourselves to high-risk customers that form the bulk of the data. As such,  $N = 40,507$ . Of these, only 2,371 have  $Y = 1$ , which makes estimation more difficult.

In what follows, we conduct a series of analysis. First, we estimate a binary choice model of loan application allowing for heterogeneity in the parameter vector via our structured DNNs. We use these to compute the marginal effect of interest rate. We then use the results of the model (with some additional assumptions) to construct optimal personalized interest rate offers and compute the expected profits from implementing the personalization scheme.

## 4.2 Model and Implementation

Our setup adapts the framework outlined in Bertrand et al. (2010) and assumes that consumers have a utility

$$u = \theta_1^*(\mathbf{x}_d, \mathbf{x}_a) + \theta_2^*(\mathbf{x}_d)r + \varepsilon,$$

where  $\boldsymbol{\theta}^*(\mathbf{x}) = (\theta_1^*(\mathbf{x}_d, \mathbf{x}_a), \theta_2^*(\mathbf{x}_d))'$  are the vector of parameter functions and, (in our earlier notation)  $\mathbf{T} = R$ . We further assume that  $\varepsilon$  is Logistic distributed, which gives the standard Logit probabilities of response. Let  $\mathbf{r}_1 = (1, r)'$ . Then the response is assumed to be

$$\mathbb{P}[Y=1 \mid \mathbf{X}=\mathbf{x}, R=r] = G(\boldsymbol{\theta}^*(\mathbf{x})' \mathbf{r}_1) = \frac{1}{1 + \exp(-[\theta_1^*(\mathbf{x}_d, \mathbf{x}_a) + \theta_2^*(\mathbf{x}_d)r])}, \quad (4.1)$$

where  $G(u)$  is the logit function. This is exactly the enriched version of structural model (2.2) used in Section 2.

Using these probabilities we can construct the log-likelihood as

$$y \log(\mathbb{P}[Y=1 \mid \mathbf{X}=\mathbf{x}, R=r]) + (1-y) \log(\mathbb{P}[Y=0 \mid \mathbf{X}=\mathbf{x}, R=r]),$$

which is a heterogeneity enriched version of the standard workhorse binary choice model. The negative of this log-likelihood serves as the loss (3.3) for our problem. One can easily verify the high-level assumptions in this setting, particularly given that the binary choice model is widely studied and well understood. For example, it is straightforward to show that  $\Lambda(\mathbf{x}) = \mathbb{E}[G(\boldsymbol{\theta}^*(\mathbf{x})' \mathbf{R}_1)(1 - G(\boldsymbol{\theta}^*(\mathbf{x})' \mathbf{R}_1)) \mathbf{R}_1 \mathbf{R}_1' \mid \mathbf{X}=\mathbf{x}]$ , where  $\mathbf{R}_1 = (1, R)'$ , and will be invertible under standard and commonly used economic assumptions. More discussion is given in Appendix B.

We implement Figure 1 to maximize the likelihood with a simple network with two hidden layers of 10 nodes each. The simplicity of the network architecture is driven by the fact many dimensions of  $\mathbf{X} = \{\mathbf{X}_d, \mathbf{X}_a\}$  are binary, so there is less functional approximation required, and that we have a smallish dataset ( $N = 40,507$ ). We use the Torch interface in R (Keydana (2023)) to construct the computational graph and optimize the likelihood using the ADAM optimizer. For inference purposes we use 50-fold cross fitting, using all but one fold to estimate  $\hat{\boldsymbol{\theta}}(\mathbf{x})$  and the remaining fold to obtain  $\hat{\boldsymbol{\mu}}$ .<sup>4</sup>

## 4.3 Parameters of Interest and Associated Results

We will use our estimated  $\hat{\boldsymbol{\theta}}(\mathbf{x})$  and novel influence function to explore two derived quantities, i.e. two different  $\boldsymbol{\mu}^*$  of (3.4). First, we examine the marginal effect of the the focal treatment

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<sup>4</sup>Replication files are available at <https://github.com/maxhfarrell/FLM2>.

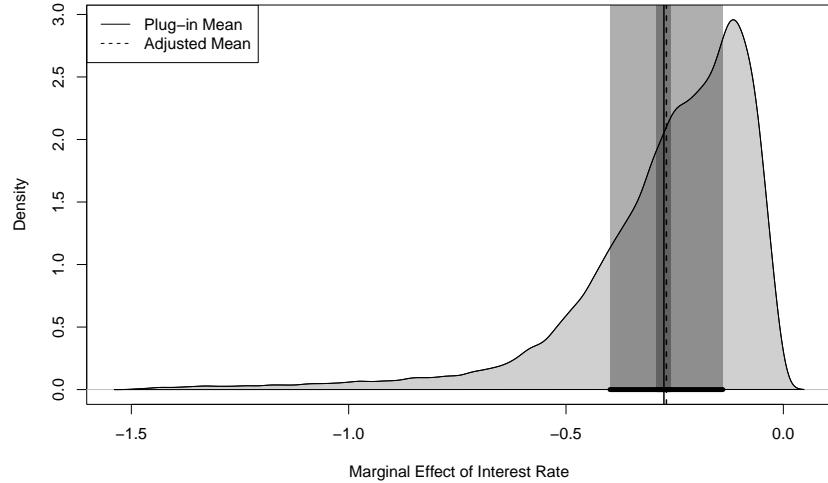


Figure 3: Marginal Effect of Advertising Content

(interest rate offer). We note that our Logit specifications is slightly different from [Bertrand et al. \(2010\)](#), who use a Probit specification. Second, we turn to a more ambitious goal of personalization and profit maximization, making more full use of the power of the framework. Additional examples include (i) the price elasticity at a price (here, interest rate)  $\tilde{r}$ , which sets  $H = (1 - G(\theta_1 + \theta_2\tilde{r}))\theta_2\tilde{r}$ ; (ii) a measure of willingness to pay obtained by taking  $H = \theta_2/\theta_1$ ; and (iii) expected consumer welfare,  $H = -\log(1 + \exp(\theta_1 + \theta_2\tilde{r}))/\theta_2$ . These are standard second-step objects of interest in choice models, and can be immediately used in our framework by evaluating each using the parameter functions  $\boldsymbol{\theta}(\mathbf{x})$ . Importantly, without our explicit use of a structural model, characterizing these quantities and obtaining inference would be difficult.

#### 4.3.1 Marginal Effects

With our structure, the average marginal effect (AME) of any treatment can be written in closed form. Let  $\tilde{r}$  be a given value of the interest rate (the treatment) and  $\tilde{\mathbf{r}}_1 = (1, \tilde{r})'$ . Then the parameter of interest is

$$\text{AME}(\tilde{r}) = \mathbb{E} \left[ \frac{\partial G(\boldsymbol{\theta}' \mathbf{r}_1)}{\partial r} \Big|_{\boldsymbol{\theta}^*(\mathbf{X})' \tilde{\mathbf{r}}_1} \right] = \mathbb{E} [G(\boldsymbol{\theta}^*(\mathbf{X})' \tilde{\mathbf{r}}_1) (1 - G(\boldsymbol{\theta}^*(\mathbf{X})' \tilde{\mathbf{r}}_1)) \theta_2^*(\mathbf{X})].$$

We set  $\tilde{r}$  to the sample average (0.084) for simplicity. From the data we obtain the point estimate  $\widehat{\text{AME}}(0.084) = -0.269$  using (3.7), with corresponding 95% confidence interval  $(-0.399, -0.139)$  obtained from (3.8). The original analysis in [Bertrand et al. \(2010\)](#) found a

marginal effect of  $-0.29$  (see Table III therein). Further, since we use a subset of the data, we refit a simple probit model on the subset and obtain a marginal effect of  $-0.2505$ . Both these estimates fall within our confidence interval, which we interpret as the original findings being robust to heterogeneity for this parameter.

We plot the distribution of the conditional average marginal effects along with their means and confidence intervals in Figure 3. The lightest grey shows a kernel-smoothed density estimate. The shaded region is our 95% confidence interval, while the darkest shading shows the confidence interval one would obtain ignoring first step estimation (not using the influence function adjustment). The plot shows considerable heterogeneity uncovered by the DNNs. This shows that although the estimate of the average is robust to heterogeneity, there is considerable potential for personalization.

#### 4.3.2 Optimal Personalized Offers

We now demonstrate how the estimated heterogeneity can be translated into personalized offers and the simplicity with which one can conduct inference on quantities of interest. We examine the mean of personalized interest rate offers and the expected profits from personalization. Note that it poses no issue that the corresponding  $H$  functions are not available in closed form.

To construct profits we have to make some assumptions about the decision process of the firm and construct some auxiliary measures. First we will create a simple, parametric model for loan default probability. Given the rarity of defaults, the sample size is too small to uncover meaningful heterogeneity, and therefore we assume that the probability of default  $D = 1$  given an interest rate  $R = r$  is also logistic:

$$\mathbb{P}[D=1 \mid R=r] = \frac{1}{1 + \exp(-[\delta_1^* + \delta_2^*r])}.$$

We estimate the parameters  $(\delta_1^*, \delta_2^*)$  from data and, for convenience in this illustration, take these parameters as given.<sup>5</sup>

To write the firm's expected profit for a given consumer, let  $L$  be the loan amount,  $M$  be the loan term ( $= 4$ ), and assume the outside option of the money being loaned is to obtain a rate of return  $r_0$ , we set to 0.01 in the analysis. Since we focus on optimizing the interest rate for fixed values of the parameters, let  $\mathbb{P}[Y=1 \mid \mathbf{X}=\mathbf{x}, R=r] = P(r)$  and

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<sup>5</sup>With richer data, one could apply the estimation and inference framework using a bivariate outcome of application and default,  $\mathbf{Y} = (Y, D)'$  and enrich  $(\delta_1^*, \delta_2^*)$  to include heterogeneity. Accounting for the estimation of  $\delta_1^*(\mathbf{x})$  and  $\delta_2^*(\mathbf{x})$  would be then automatic. With only 280 defaults observed, 0.7% of observations, this is not possible.

$\mathbb{P}[D=1 \mid R=r] = D(r)$ . Then profits are

$$\Pi(r) = L \left( \underbrace{P(r)(M(1 - D(r))r - D(r))}_{\text{Expected profit given loan}} + \underbrace{(1 - P(r))Mr_0}_{\text{Outside option if no loan}} \right). \quad (4.2)$$

Some discussion of this profit function is warranted. The profit function has two components - first the part where given a loan being initiated we have expected revenues. This expectation is computed by taking into account the revenue stream under non-default as well as the possibility of loss based on default (assuming no recovery of funds). The second component reflects the opportunity cost of the loan. We do not intend for this profit to be a perfect representation of reality but simply to illustrate the manner in which our methods can be applied realistic structural settings.

To find the optimal interest rate, we obtain the first order condition as per the usual optimization machinery. This yields

$$0 = \frac{d\Pi}{dr} = L \left[ \dot{P}(r)(M(1 - D(r))r - D(r) - Mr_0) + P(r)(M(1 - D(r)) - Mr\dot{D}(r) - \dot{D}(r)) \right],$$

where  $\dot{P}$  and  $\dot{D}$  represent derivatives with respect to their scalar arguments. This profit function is smooth in  $r$  but there exists the possibility that it is not uni-modal. Based on simulations we verified that, for parameters where  $\dot{P} < 0$  and  $\dot{D} > 0$  (as would be expected in this context) and for  $r \in [0, r_{\max} = 0.25]$  the profit function is uni-modal and a unique  $r^*$  obtains. To explore this we define a representation of the fixed point problem as

$$r = r + \frac{d\Pi(r)}{dr}. \quad (4.3)$$

This is an implicit function which will show a unique fixed point  $r_{\text{opt}}$  if right hand side is decreasing in  $r$ . Figure 4 presents a visual representation of Equation (4.3). Each light grey curve corresponds to a distinct consumer profile  $\mathbf{x}_i$  and its intersection with the  $y = x$  line represents the fixed point  $r_{\text{opt}}$  (the scale of the axes is different so  $y = x$  is not at  $45^\circ$ ). The density then represents the kernel density of the optimal personalized offers  $r_{\text{opt}}(\mathbf{x}_i)$  across consumers. We note that while the fixed points are only shown for a subset of customers (to avoid clutter) the density is computed across the entire sample.

The reader should note that even though  $r_{\text{opt}}$  is not available in closed form it remains a smooth function of the parameters  $\boldsymbol{\theta}$ , which is all that is required for our method to apply. We can therefore provide inference for any statistic of the form (3.4). As a simple example, Figure 4 shows estimation and inference for  $\mu^* = \mathbb{E}[r_{\text{opt}}(\boldsymbol{\theta}^*(\mathbf{X}))]$ , the average of optimal offers. We obtain a point estimate of 14.12%, with a 95% confidence interval [12.0%, 16.23%].

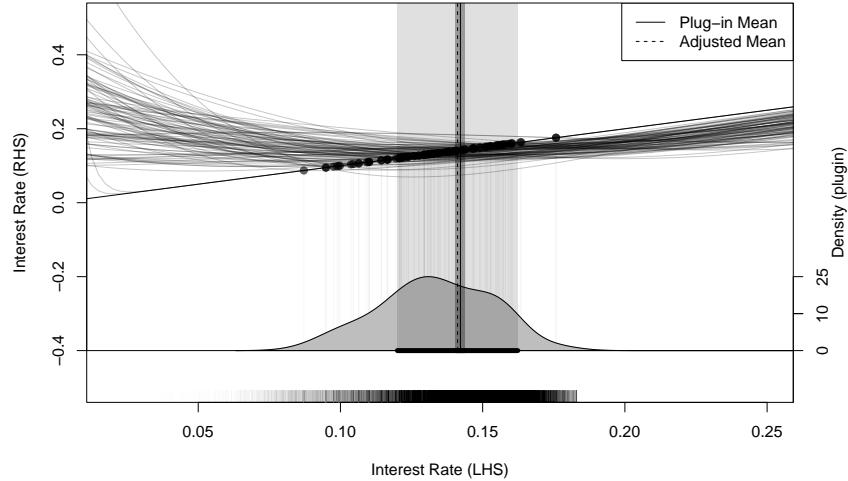


Figure 4: Optimal Personalized Interest Rate Offers

This is shown in the figure, along with the plug-in interval as before.

Next we study expected profits from setting the optimal personalized interest rate, i.e.,  $\mathbb{E}[\Pi(r_{\text{opt}}(\boldsymbol{\theta}^*(\mathbf{X})))]$ . From (4.2), this is expressed as

$$\mu^* = \mathbb{E}[L[G(r_{\text{opt}}(\boldsymbol{\theta}^*(\mathbf{X}))) (M(1 - D(r_{\text{opt}}(\boldsymbol{\theta}^*(\mathbf{X}))))r - D(r_{\text{opt}}(\boldsymbol{\theta}^*(\mathbf{X})))) + (1 - G(r_{\text{opt}}(\boldsymbol{\theta}^*(\mathbf{X}))))Mr_0]]$$

We can apply our framework directly to this estimand, despite the complications (again, this is a smooth function  $H$  but not expressible in closed form), using automatic or numerical differentiation. We can also appeal to the envelope theorem, which ensures that  $\partial\Pi/\partial r|_{r=r_{\text{opt}}} = 0$ . As such, the influence function for expected profits can be constructed in closed form (conditional on  $r_{\text{opt}}$ , not  $\boldsymbol{\theta}$ ). All three approaches yield nearly identical results.

We standardize the results (for plotting) and interpret the expected profit construct as the net expected income from offering a \$1 loan at a personalized interest rate to each potential customer. We find that  $\hat{\mu} = \$0.0497$  with a 95% confidence interval of  $[\$0.0459, \$0.0535]$ .

Figure 5 depicts the density of profits across customers along with the estimate and confidence interval for the mean. Several features are notable here. First, we note that the estimate  $\hat{\mu}$ , which includes the influence function adjustment, is outside the naive confidence interval based on the plug-in estimator. Statistically, this indicates that the bias correction from the influence function adjustment is large relative to the variance, that is, first stage noise from  $\hat{\boldsymbol{\theta}}(\mathbf{x})$  shifts the asymptotic distribution substantially, demonstrating the importance of double machine learning. The same phenomenon was shown in a partially linear model in Chernozhukov et al. (2018a) and also appears in nonparametric bias correction contexts

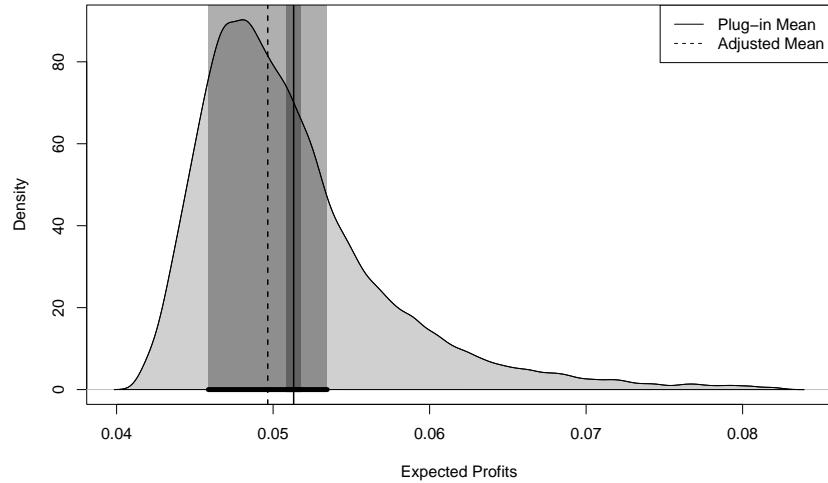


Figure 5: Expected Profits from Personalized Interest Rate Offers

(Cattaneo et al., 2024a). See Remark 4.1 for related discussion.

It is worth noting that the discrepancy between the estimates arise primarily on account of the curvature of the profit function. The gradient of the profit function is steep around the estimated parameter and perturbations therein result in large changes in the influence component. The shape of the profit density is driven by a complex interaction of various components - the distribution of marginal effects of interest rate (Figure 3), coupled with default propensities, the optimal prices, and the formulation of profits. Even with such complexity, the profits (and prices) are well behaved and economically meaningful. We caution the reader again that our application is an illustration and ignores a number of other factors that might have bearing on the firm's and consumers' decision problems.

In sum, this application showcases the simplicity with which (parametric) structural economic models can be enriched to incorporate nonparametric heterogeneity via deep neural networks, and how the results can be used directly for economic decision making and policy analysis based on personalization. While a full-fledged application would incorporate additional features and economic nuances, this proof of concept nonetheless showcases quite a difficult estimation and inference problem.

**Remark 4.1** (Implementation Uncertainty). Throughout the application we use DML, building on our novel influence function, to provide valid inference for second step parameters. However, in some real-world decision making contexts, this may be inappropriate. Consider expected profits. Above, we studied  $\mu^* = \mathbb{E}[\Pi(r_{\text{opt}}(\boldsymbol{\theta}^*(\mathbf{X})))]$ . For the firm, this corresponds to the profits they can expect from implementing *true* optimal personalization, based on

$\boldsymbol{\theta}^*(\mathbf{x})$ . We therefore require our novel influence function to account for the estimation error in the first stage, i.e., the fact that we use  $\widehat{\boldsymbol{\theta}}(\mathbf{x})$  instead of  $\boldsymbol{\theta}^*(\mathbf{x})$ . However, from the firm's point of view, if they choose to implement the strategy  $r_{\text{opt}}(\widehat{\boldsymbol{\theta}}(\mathbf{x}))$ , it is more natural to consider  $\widehat{\boldsymbol{\theta}}(\mathbf{x})$  as fixed and set the parameter of interest accordingly as  $\tilde{\mu} = \mathbb{E}[\Pi(r_{\text{opt}}(\widehat{\boldsymbol{\theta}}(\mathbf{X})))]$ , because this corresponds to the profits they can expect from what they would actually implement. In this case, DML and the influence function correction are not necessary, and the plug-in can be used directly. ♣

## 5 Theoretical Results

### 5.1 Bounds for Structural Deep Learning

We first provide theoretical guarantees for the structural deep neural networks of Section 3.2. Our theory generalizes [Farrell, Liang, and Misra \(2021\)](#) to the structural setting. We impose two assumptions. For the loss function, we require Lipschitz continuity in general and, near the truth, sufficient curvature. Neither are restrictive and both are common in the nonparametric M estimation literature (cf [Chen \(2007\)](#) and others, where further references and use of other norms are discussed). These conditions are for estimation of  $\boldsymbol{\theta}^*(\mathbf{x})$ ; further assumptions will be required for inference.

**Assumption 1.** Suppose that  $\boldsymbol{\theta}^*(\mathbf{x})$  are nonparametrically identified in (3.3), uniformly bounded, and that there are constants  $c_1$ ,  $c_2$ , and  $C_\ell$  that are bounded and bounded away from zero, such that for arbitrary  $\boldsymbol{\theta}(\mathbf{x})$  and  $\tilde{\boldsymbol{\theta}}(\mathbf{x})$ , the loss obeys  $|\ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x})) - \ell(\mathbf{y}, \mathbf{t}, \tilde{\boldsymbol{\theta}}(\mathbf{x}))| \leq C_\ell \|\boldsymbol{\theta}(\mathbf{x}) - \tilde{\boldsymbol{\theta}}(\mathbf{x})\|_2$  and

$$c_1 \mathbb{E} [\|\boldsymbol{\theta}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2] \leq \mathbb{E}[\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}(\mathbf{X}))] - \mathbb{E}[\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}^*(\mathbf{X}))] \leq c_2 \mathbb{E} [\|\boldsymbol{\theta}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2].$$

These requirements will often be implied by restrictions on the gradient and Hessian of the loss, or on the matrix  $\boldsymbol{\Lambda}(\mathbf{x})$ . Such restrictions are natural in our setting, since they are commonly applied to parametric structural models; the same conditions readily transfer to the enriched setting. Differentiability is not required here and thus our results can be used in nonsmooth cases (for example [Tambwekar et al. \(2022\)](#) and [Padilla, Tansey, and Chen \(2022\)](#) apply the theory of [Farrell, Liang, and Misra \(2021\)](#) to quantile regression and the same extension could be done here), however, differentiability will be required for inference later and may help in verification of these conditions.

The data generating process is assumed to obey the following conditions. Let  $\mathbf{W} = (\mathbf{Y}', \mathbf{T}', \mathbf{X}')'$  be the population random variables. Denote by  $\mathbf{X}_c$  the continuously distributed

elements of  $\mathbf{X}$ , with  $d_c = \dim(\mathbf{X}_c)$ , and take the rest to be binary random variables, without loss of generality.

**Assumption 2.** (i) The elements of  $\mathbf{W}$  are bounded random variables. (ii)  $\mathbf{X}_c$  has compact, connected support, taken to be  $[-1, 1]^{d_c}$ . (iii) As functions of  $\mathbf{x}_c$ ,  $\theta_k^*(\mathbf{x}) \in \mathcal{W}^{p,\infty}([-1, 1]^{d_c})$ , for  $k = 1, \dots, d_\theta$ , where for positive integers  $p$  and  $q$ , define the Hölder ball  $\mathcal{W}^{p,\infty}([-1, 1]^q)$  of functions  $h : \mathbb{R}^q \rightarrow \mathbb{R}$  with smoothness  $p \in \mathbb{N}_+$  as

$$\mathcal{W}^{p,\infty}([-1, 1]^q) := \left\{ h : \max_{\mathbf{r}, |\mathbf{r}| \leq p} \text{ess sup}_{\mathbf{v} \in [-1, 1]^q} |D^\mathbf{r} h(\mathbf{v})| \leq 1 \right\},$$

where  $\mathbf{r} = (r_1, \dots, r_q)$ ,  $|\mathbf{r}| = r_1 + \dots + r_q$  and  $D^\mathbf{r} h$  is the weak derivative. (iv)  $\max_{k \leq d_\theta} \sup_{\mathbf{x}} |\theta_k^*(\mathbf{x})| < M$ , for some positive  $M$ .

These are typical assumptions for nonparametric estimation and are similar to [Farrell, Liang, and Misra \(2021\)](#). Part (iii) of this assumption restricts to smooth functions, which are known to be approximable by deep neural networks ([Yarotsky, 2017, 2018; Hanin, 2017](#)).

We now state the main result for structural deep learning. We specialize this theorem to the standard implementation using deep and wide multi-layer perceptrons (fully connected, feedforward neural networks) and set the width and depth specifically for the fastest rate. Equation (A.4) in Appendix A shows a more general bound that is agnostic about the type of approximation, and hence the type of network. This can be used to obtain faster rates or cover fixed-width, very deep networks (see Section 2.3 of [Farrell, Liang, and Misra \(2021\)](#) for discussion).

**Theorem 1.** Let  $\mathbf{w}_i$ ,  $i = 1, \dots, n$ , be a random sample that obeys Assumptions 1 and 2. Define  $\widehat{\boldsymbol{\theta}}$  as the estimator found by solving (3.5), where the class  $\mathcal{F}_{\text{DNN}}$  is a feedforward, fully connected network with ReLU activation structured according to Figure 1, with parameter functions bounded by  $2M$ , width  $J \asymp n^{(d_c)/2(p+d_c)} \log^2 n$ , and depth  $L \asymp \log n$ . Then

$$\|\widehat{\theta}_k - \theta_k^*\|_{L_2(\mathbf{X})}^2 \leq C \cdot \left\{ n^{-\frac{p}{p+d_c}} \log^8 n + \frac{\log \log n}{n} \right\}$$

and

$$\mathbb{E}_n \left[ \left( \widehat{\theta}_k - \theta_k^* \right)^2 \right] \leq C \cdot \left\{ n^{-\frac{p}{p+d_c}} \log^8 n + \frac{\log \log n}{n} \right\},$$

for  $n$  large enough, with probability  $1 - \exp\{-n^{\frac{d_c}{p+d_c}} \log^8 n\}$ , for  $k = 1, \dots, d_\theta$ , where the constant  $C$  may depend on the  $\dim(\mathbf{W})$ ,  $d_\theta$ , and other fixed quantities in Assumptions 1 and 2.

The result of Theorem 1 speaks directly to the nonparametric M estimation literature. This theorem takes deep learning away from prediction and toward learning economically meaningful parameters. It shows that deep nets enjoy the same properties of other nonparametric/ML methods, but with the advantages of structure. A theoretical drawback is that for a given smoothness level, this rate is not optimal. Obtaining the optimal rate, or studying different norms, would be a useful extension. The bounds are sharp enough for inference and reflects the excellent empirical performance.

## 5.2 Influence Function and DML

We now turn to our influence function result. Influence functions have a long history in econometrics. Newey (1994) remains the seminal treatment. We defer to that work and Ichimura and Newey (2022) for background and the theory of influence functions, including regularity conditions for their existence. Our goal is not to contribute to the theory of influence functions per se, but rather to use the tools of this theory to obtain the methodological result of a broadly applicable influence function, to enable two-step semiparametric inference under weak conditions. Our main result is a calculation made possible by applying the ideas in these works, chiefly Newey (1994). That is, we view the influence function as a tool for obtaining feasible inference, rather than an object of interest in its own right (such as for studying efficiency). This viewpoint is implicit in recent work on inference after ML (e.g., Belloni, Chernozhukov, and Hansen, 2014; Farrell, 2015; Chernozhukov et al., 2018a) but it is worthwhile to make it explicit to better understand how this mode of thinking allows us to cover such a wide range of applications and apply automatic differentiation.

The assumption we impose next is mostly standard and ensures sufficient regularity for our influence function to be calculated and for asymptotic Normality of the resulting estimator. One conceptual point is that further assumptions will be needed for a causal interpretation, such as unconfoundedness or conditional exogeneity.

**Assumption 3.** *The following conditions hold on the distribution of  $\mathbf{W}$ , uniformly in the given conditioning elements. (i) Equation (3.3) holds and identifies  $\boldsymbol{\theta}^*(\mathbf{x})$ , where  $\ell(\mathbf{w}, \boldsymbol{\theta})$  is thrice continuously differentiable with respect to  $\boldsymbol{\theta}$ . (ii)  $\mathbb{E}[\ell_{\boldsymbol{\theta}}(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}(\mathbf{x})) | \mathbf{X} = \mathbf{x}, \mathbf{T} = \mathbf{t}] = 0$ . (iii)  $\boldsymbol{\Lambda}(\mathbf{x})$  is invertible with bounded inverse. (iv) The parameter  $\boldsymbol{\mu}^*$  of Equation (3.4) is identified and pathwise differentiable and  $\mathbf{H}$  is thrice continuously differentiable in  $\boldsymbol{\theta}$ . (v)  $\mathbf{H}(\mathbf{X}, \boldsymbol{\theta}(\mathbf{X}); \tilde{\mathbf{t}})$  and  $\ell_{\boldsymbol{\theta}}(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}(\mathbf{X}))$  possess  $q > 4$  finite absolute moments and positive variances.*

The most important assumptions here are that the first order condition of (3.3) holds,  $\boldsymbol{\theta}^*(\mathbf{x})$  is identified, and that  $\boldsymbol{\mu}^*$  is pathwise differentiable. The latter keeps focus on regular

semiparametric contexts. The former follows our idea to take a well-defined parametric model, for which such identification would hold, and enrich the model with machine learning. Conditional mean restrictions are a particularly popular case (see Appendix B). Condition (iii) will often be implied by other conditions on the model, such as in the case of logistic regression if  $\mathbb{P}[Y=1 \mid \mathbf{X}=\mathbf{x}, \mathbf{T}=\mathbf{t}]$  is bounded away from zero and one (which in turn may be implied by conditions on  $\mathbf{X}$ ,  $\mathbf{T}$ , and the functions  $\boldsymbol{\theta}^*$ , such as boundedness). Or, in the context of treatment effects we need the standard overlap condition. Some version of the condition of positive variance, or invertibility of  $\Lambda(\mathbf{x})$ , is quite standard in semiparametric problems.

The following result justifies our inference method. Other than the novel influence function, it is an application of existing DML theory (Chernozhukov et al., 2018a). Let  $\mathbf{0}_d$  be the  $d$ -long zero vector and  $\mathbf{I}_d$  be the  $d$ -square identity matrix.

**Theorem 2.** *Let  $\mathbf{w}_i$ ,  $i = 1, \dots, n$ , be a random sample that obeys Assumption 3. Assume  $\|\widehat{\boldsymbol{\theta}}_{k_1} - \boldsymbol{\theta}_{k_1}^*\|_{L_2(\mathbf{X})} = o_P(n^{-1/4})$  and  $\|[\widehat{\Lambda}]_{k_1, k_2} - [\Lambda]_{k_1, k_2}\|_{L_2(\mathbf{X})} = o_P(n^{-1/4})$  for all  $k_1, k_2 \in \{1, \dots, d_\theta\}$ , and that  $\widehat{\Lambda}(\mathbf{x}_i)$  is uniformly invertible. Then (i) (3.6) gives a Neyman orthogonal score and (ii) the DML-based  $\widehat{\boldsymbol{\mu}}$  and  $\widehat{\boldsymbol{\Psi}}$  of (3.7) and (3.8) obey*

$$\sqrt{|\mathcal{C}|} \widehat{\boldsymbol{\Psi}}^{-1/2} (\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}^*) = \frac{1}{\sqrt{|\mathcal{C}|}} \sum_{i \in \mathcal{C}} \boldsymbol{\Psi}^{-1/2} \boldsymbol{\psi}(\mathbf{w}_i, \boldsymbol{\theta}^*(\mathbf{x}_i), \Lambda(\mathbf{x}_i)) / \sqrt{n} + o_p(1) \xrightarrow{d} \mathcal{N}(\mathbf{0}_{d_\mu}, \mathbf{I}_{d_\mu}).$$

Here we impose the standard rate conditions on the first step estimators, allowing for valid inference after using any sufficiently accurate nonparametric/ML method. Deep learning remains a preferred choice in structural modeling, as discussed above. The assumptions used here intended to be simple and familiar, but are not minimal. The rate condition on  $\Lambda(\mathbf{x})$  can be weakened as shown by Chernozhukov et al. (2022a). Cattaneo, Jansson, and Ma (2019) show that in some problems computationally intensive procedures can be used to weaken first step assumptions. Further, our distributional approximation is first-order invariant to the first step estimator, as is typical, and this can be a poor finite sample approximation. More refined approximations, which account for the first step explicitly, have been obtained in simple cases (Cattaneo, Crump, and Jansson, 2014; Cattaneo, Jansson, and Newey, 2018; Cattaneo et al., 2024c) but it is not clear if the same can be done here. Lastly, we note that similar debiasing correction terms appear for inference in high-dimensional models (Belloni, Chernozhukov, and Hansen, 2014; Javanmard and Montanari, 2014; Zhang and Zhang, 2014) and for loss functions directly (Foster and Syrgkanis, 2023).

**Remark 5.1** (Efficiency). In many cases, our influence function matches the efficient one. When the original model is based on a likelihood or exponential family, and (3.3) and (3.4)

contain all information, we conjecture that efficiency is always obtained following Remark 4.1 of [Mammen and van de Geer \(1997\)](#). In general, however, our estimator result is not guaranteed to be efficient. For example, in partially linear models we only obtain efficiency under homoskedasticity assumptions ([Appendix C](#)). ♣

**Remark 5.2** (High Dimensional Parametric Approach). An alternative approach in two-step inference would be to consider  $\widehat{\boldsymbol{\theta}}(\mathbf{x}_i)$  as a parametric model (where the weights and biases of the deep net are the parameters) and apply parametric two-step estimation. This is shown to be a valid approximation to the semiparametric case in some contexts by [Ackerberg, Chen, and Hahn \(2012\)](#). Applying this idea to deep learning may be valid, but is practically infeasible as the number of parameters is too large and the estimator too complex. For example, the equivalent of  $\Lambda(\mathbf{x})$  would be a square matrix of dimension equal to the number of parameters in the deep net, which can be extremely large. Computing and inverting such a matrix may be impossible. ♣

**Remark 5.3** (Other Uses of Influence Functions). Influence functions have appeared in many different contexts in statistics recently and our results can potentially be used to extend these methods to new contexts. Here we list a few examples. (i) [Athey and Wager \(2021\)](#) study policy optimization and show that using an orthogonal score yields faster reminder rates in terms of welfare just as for inference. Our score could be used to bring their insight into new areas. (ii) [Koh and Liang \(2017\)](#) use influence functions to try to understand “black-box” ML methods. Extending this to economic contexts would be valuable in applied research and policy evaluation. (iii) [Robins et al. \(2008\)](#) use higher-order influence functions to obtain refined semiparametric inference. We conjecture that using automatic differentiation could be used to obtain higher order inference just as with our first order results. (iv) [Firpo, Fortin, and Lemieux \(2009\)](#) rely on influence functions for distributional statistics, and could potentially be generalized to other models. ♣

## 6 Conclusion

Structural modeling is a workhorse of empirical economic research. We have shown how to enrich these models with deep learning to capture rich heterogeneity, filling in the gaps left by economic theory. This method combines the strength of structural modeling and the strength of machine learning. We established convergence rates for structured deep learning and valid inference using a novel influence function calculation. Our method represents a

step toward easier and more rigorous use of machine learning in economic research, but is far from complete. Shape restrictions are on major form of structure arising from economic theory (see Chetverikov, Santos, and Shaikh (2018) for a recent review). Extending our methods to include impose shape constraints is an important step for future research. From an implementation point of view, there is also a lot of ground to cover for deep neural networks, including penalization and regularization, tuning parameter choices, and robust computation.

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## Appendix A Proofs

### A.1 Bounds Structural Deep Learning – Theorem 1

Here we prove Theorem 1, using the proof method of Farrell, Liang, and Misra (2021). Some details will be deferred to that paper.

Define  $\boldsymbol{\theta}_n \in \mathcal{F}_{\text{DNN}}$  as the best approximation to  $\boldsymbol{\theta}^*$  in the class of DNNs as defined in the theorem statement and let  $\epsilon_n$  denote the error of the approximation:

$$\boldsymbol{\theta}_n = \arg \min_{\boldsymbol{\theta} \in \mathcal{F}_{\text{DNN}}} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_\infty, \quad \epsilon_n = \|\boldsymbol{\theta}_n - \boldsymbol{\theta}^*\|_\infty.$$

For the MLP case, under Assumption 2 this error is controlled by the width and depth, and we specify to this case at the end of the proof. For now we leave the error generic to allow for other approximation assumptions (such as other smoothness classes) and other architectures.

By Assumption 1 and that  $\widehat{\boldsymbol{\theta}}$  optimizes  $\ell$  over  $\mathcal{F}_{\text{DNN}}$  in the data,

$$\begin{aligned} & c_1 \mathbb{E} \left[ \|\widehat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2 \right] \\ & \leq \mathbb{E}[\ell(\mathbf{Y}, \mathbf{T}, \widehat{\boldsymbol{\theta}}(\mathbf{X}))] - \mathbb{E}[\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}^*(\mathbf{X}))] \\ & \leq \mathbb{E}[\ell(\mathbf{Y}, \mathbf{T}, \widehat{\boldsymbol{\theta}}(\mathbf{X}))] - \mathbb{E}[\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}^*(\mathbf{X}))] - \mathbb{E}_n[\ell(\mathbf{Y}, \mathbf{T}, \widehat{\boldsymbol{\theta}}(\mathbf{X}))] + \mathbb{E}_n[\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}_n(\mathbf{X}))] \\ & = (\mathbb{E} - \mathbb{E}_n) \left[ \ell(\mathbf{Y}, \mathbf{T}, \widehat{\boldsymbol{\theta}}(\mathbf{X})) - \ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}^*(\mathbf{X})) \right] + \mathbb{E}_n [\ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}_n(\mathbf{X})) - \ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}^*(\mathbf{X}))]. \end{aligned}$$

Applying Farrell, Liang, and Misra (2021, Equation (A.2)) to the second term of the last line above, we find that with probability  $1 - e^{-\gamma}$

$$\begin{aligned} & c_1 \mathbb{E} \left[ \|\widehat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2 \right] \\ & \leq (\mathbb{E} - \mathbb{E}_n) \left[ \ell(\mathbf{Y}, \mathbf{T}, \widehat{\boldsymbol{\theta}}(\mathbf{X})) - \ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}^*(\mathbf{X})) \right] + c_2 \epsilon_n^2 + \epsilon_n \sqrt{\frac{2C_\ell^2 \gamma}{n}} + \frac{7C_\ell M \gamma}{n}. \quad (\text{A.1}) \end{aligned}$$

We now apply the localization-based analysis of Farrell, Liang, and Misra (2021) to the first

term above and then collect the results. Suppose that for some  $r_0$ ,  $\mathbb{E}[\|\widehat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2]^{1/2} \leq r_0$ , which can always be attained given the boundedness. Let  $\mathcal{F}_{\text{DNN}}^0$  be the subset of  $\mathcal{F}_{\text{DNN}}$  such that  $\boldsymbol{\theta} \in \mathcal{F}_{\text{DNN}}^0$  if  $\mathbb{E}[\|\boldsymbol{\theta}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2]^{1/2} \leq r_0$ . Then by Theorem 2.1 in [Bartlett, Bousquet, and Mendelson \(2005\)](#), for  $\mathcal{G} = \{g = \ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x})) - \ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}^*(\mathbf{x})) : \boldsymbol{\theta} \in \mathcal{F}_{\text{DNN}}^0\}$ , we find that, with probability at least  $1 - 2e^{-\gamma}$ , the empirical process term of (A.1) is bounded as

$$(\mathbb{E} - \mathbb{E}_n) [\ell(\mathbf{Y}, \mathbf{T}, \widehat{\boldsymbol{\theta}}(\mathbf{X})) - \ell(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}^*(\mathbf{X}))] \leq 6\mathbb{E}_\eta R_n \mathcal{G} + \sqrt{\frac{2C_\ell^2 r_0^2 \gamma}{n}} + \frac{23 \cdot 3MC_\ell \gamma}{3} \frac{\gamma}{n}, \quad (\text{A.2})$$

where

$$R_n \mathcal{G} = \sup_{g \in \mathcal{G}} \frac{1}{n} \sum_{i=1}^n \eta_i g(\mathbf{w}_i) = \sup_{\boldsymbol{\theta} \in \mathcal{F}_{\text{DNN}}^0} \frac{1}{n} \sum_{i=1}^n \eta_i (\ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x})) - \ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}^*(\mathbf{x}))) .$$

is the empirical Rademacher complexity and  $\mathbb{E}_\eta R_n \mathcal{G}$  is its expectation holding fixed the data, i.e. over the i.i.d. Rademacher variables  $\eta_i$ . The argument given in Section A.2.2 of [Farrell, Liang, and Misra \(2021\)](#) does not apply directly to  $\mathbb{E}_\eta R_n \mathcal{G}$  because  $\boldsymbol{\theta}$  is vector valued. Instead, we replace Lemma 2 therein with [Maurer \(2016, Corollary 1\)](#), which in our context yields (below  $\eta_{ik}$ 's denote i.i.d. Rademacher random variables)

$$\begin{aligned} \mathbb{E}_\eta \sup_{\boldsymbol{\theta} \in \mathcal{F}_{\text{DNN}}^0} \frac{1}{n} \sum_{i=1}^n \eta_i (\ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x})) - \ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}^*(\mathbf{x}))) &\leq \sqrt{2}C_\ell \mathbb{E}_\eta \sup_{\boldsymbol{\theta} \in \mathcal{F}_{\text{DNN}}^0} \sum_{k=1}^{d_\boldsymbol{\theta}} \frac{1}{n} \sum_{i=1}^n \eta_{ik} (\theta_k(\mathbf{x}_i) - \theta_k^*(\mathbf{x}_i)) \\ &\leq \sqrt{2}C_\ell \sum_{k=1}^{d_\boldsymbol{\theta}} \mathbb{E}_\eta \sup_{\theta_k \in \mathcal{F}_{\text{DNN}, k}^0} \frac{1}{n} \sum_{i=1}^n \eta_{ik} (\theta_k(\mathbf{x}_i) - \theta_k^*(\mathbf{x}_i)), \end{aligned}$$

with the second inequality following because the class of DNNs  $\mathcal{F}_{\text{DNN}}$  we use is decomposable with respect to each coordinate, and therefore we can bound one coordinate at a time.

We then apply Section A.2.1 and Lemmas 3 and 4 of [Farrell, Liang, and Misra \(2021\)](#) to the term for each component function  $\theta_k$ ,  $k = 1, \dots, d_\boldsymbol{\theta}$ , yielding

$$\mathbb{E}_\eta \sup_{\theta_k \in \mathcal{F}_{\text{DNN}, k}^0} \frac{1}{n} \sum_{i=1}^n \eta_{ik} (\theta_k(\mathbf{x}_i) - \theta_k^*(\mathbf{x}_i)) \leq 32r_0 \sqrt{\frac{\text{Pdim}(\mathcal{F}_{\text{DNN}, k})}{n} \left( \log \frac{2eM}{r_0} + \frac{3}{2} \log n \right)},$$

with probability  $1 - \exp^{-\gamma}$ , where  $\text{Pdim}(\mathcal{F})$  is the pseudo-dimension of the class  $\mathcal{F}$ . Therefore, whenever  $r_0 \geq 1/n$  and  $n \geq (2eM)^2$ ,

$$\mathbb{E}_\eta \sup_{\boldsymbol{\theta} \in \mathcal{F}_{\text{DNN}}^0} \frac{1}{n} \sum_{i=1}^n \eta_i (\ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x})) - \ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}^*(\mathbf{x}))) \leq K r_0 \sqrt{\frac{\text{Pdim}(\mathcal{F}_{\text{DNN}})}{n} \log n},$$

for a constant  $K$  that depends on  $C_\ell$  and  $d_{\boldsymbol{\theta}}$ .

This last bound is then combined with (A.2) and put into (A.1) and we find that

$$\begin{aligned} & c_1 \mathbb{E} [\|\hat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2] \\ & \leq 6K r_0 \sqrt{\frac{\text{Pdim}(\mathcal{F}_{\text{DNN}})}{n} \log n} + \sqrt{\frac{2C_\ell^2 r_0^2 \gamma}{n}} + \frac{23 \cdot 3MC_\ell}{3} \frac{\gamma}{n} + c_2 \epsilon_n^2 + \epsilon_n \sqrt{\frac{2C_\ell^2 \gamma}{n}} + \frac{7C_\ell M \gamma}{n} \\ & \leq r_0 \left( 6K \sqrt{\frac{\text{Pdim}(\mathcal{F}_{\text{DNN}})}{n} \log n} + \sqrt{\frac{2C_\ell^2 \gamma}{n}} \right) + c_2 \epsilon_n^2 + \epsilon_n \sqrt{\frac{2C_\ell^2 \gamma}{n}} + K_2 \frac{\gamma}{n} \\ & \leq r_0 \left( K_1 \sqrt{\frac{QL \log(Q)}{n} \log n} + \sqrt{\frac{2C_\ell^2 \gamma}{n}} \right) + c_2 \epsilon_n^2 + \epsilon_n \sqrt{\frac{2C_\ell^2 \gamma}{n}} + K_2 \frac{\gamma}{n}, \end{aligned} \quad (\text{A.3})$$

for constants  $K_1$  and  $K_2$ , where the final inequality applies Theorem 6 in [Bartlett et al. \(2017\)](#) to bound the pseudo-dimension of ReLU networks in terms of their depth  $L$  and total parameters  $Q$ .

The bound of Equation (A.3), reached under the assumption that  $\mathbb{E}[\|\hat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2]^{1/2} \leq r_0$ , provides the key input into Sections A.2.3 and A.2.4 of [Farrell, Liang, and Misra \(2021\)](#), which now go through with only change to the constants to capture the dependence on  $d_{\boldsymbol{\theta}}$ . Following those steps exactly we find that with probability  $1 - e^{-\gamma_1}$ ,

$$\begin{aligned} \mathbb{E} [\|\hat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2] & \leq C \left( \frac{QL \log(Q)}{n} \log n + \frac{\log \log n + \gamma_1}{n} + \epsilon_n^2 \right) \\ \mathbb{E}_n [\|\hat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X})\|_2^2] & \leq C' \left( \frac{QL \log(Q)}{n} \log n + \frac{\log \log n + \gamma_1}{n} + \epsilon_n^2 \right), \end{aligned} \quad (\text{A.4})$$

for positive constants  $C$  and  $C'$  which do not depend on  $n$  but depend on the constants given in Assumption 1 and as well as the dimensionalities, including  $d_{\boldsymbol{\theta}}$ .

To specialize this result to the MLP case, for which the total number of parameters obeys

$Q \leq CJ^2L$ , we use the approximation result from Theorem 1 of [Yarotsky \(2017\)](#), or its restatement in Lemma 7 of [Farrell, Liang, and Misra \(2021\)](#). This result tell us that for each  $\theta_k^*$ , the following holds for width  $J$ , depth  $L$ , and approximation error  $\epsilon_n$ :

$$J = J(\epsilon_n) \leq Q(\epsilon_n)L(\epsilon_n) \leq C^2\epsilon_n^{-\frac{d_c}{p}}(\log(1/\epsilon_n) + 1)^2,$$

$$L = L(\epsilon_n) \leq C \cdot (\log(1/\epsilon_n) + 1).$$

Therefore, a network that is  $d_\theta$  times wider can yield the same approximation for  $\boldsymbol{\theta}^*$ .

Importantly, only  $d_c$  matters here. To see why, suppose  $X_1$  is binary. Then for two smooth,  $(d_{\mathbf{X}} - 1)$ -dimensional functions  $\theta_{k,1}^*$  and  $\theta_{k,0}^*$ , it holds that  $\theta_k^*(\mathbf{x}) = x_1\theta_{k,1}^*(x_2, \dots, x_{d_{\mathbf{X}}}) + (1 - x_1)\theta_{k,0}^*(x_2, \dots, x_{d_{\mathbf{X}}})$ . Adding a single node to each hidden layer allows the network to pass forward the input  $x_1$  and multiply it with two separate learned functions just prior to the parameter, giving exactly  $\widehat{\theta}_k(\mathbf{x}) = x_1\widehat{\theta}_{k,1}(x_2, \dots, x_{d_{\mathbf{X}}}) + (1 - x_1)\widehat{\theta}_{k,0}(x_2, \dots, x_{d_{\mathbf{X}}})$ . (Intuitively, this is similar to the combination of  $\widehat{\alpha}(\mathbf{x})$  and  $\widehat{\beta}(\mathbf{x})$  in Figure B.1, with  $\mathbf{t}$  there playing the role of the binary  $x_1$  here.) The same argument can be applied to every category of the discrete data and to each function to be learned. Since  $d_{\mathbf{X}}$  is fixed, this results in only a constant increase in the width of the network. Put together, we take  $\epsilon_n = n^{-\frac{p}{2(p+d_c)}}$ , i.e.  $J \asymp n^{\frac{d_c}{2(p+d_c)}} \log^2 n$ ,  $L \asymp \log n$ , and we obtain the final result.  $\square$

## A.2 Influence Function and Asymptotic Normality

To prove Theorem 2 we first derive the influence function, then we apply standard DML results to obtain the limiting distribution. Our derivation of the influence function applies [Newey \(1994\)](#). For deeper treatments of influence functions, including efficiency bounds and conditions for existence, see [Newey \(1990\)](#), [Newey \(1994\)](#), [van der Vaart \(1998\)](#), Chapter 25, and [Ichimura and Newey \(2022\)](#).

The starting point is a parametric submodel, indexed by a parameter  $\eta$ . Because our first stage (3.3) is explicitly based on enriching the parametric structural model (3.1), our

submodels are as well-behaved as the original model and derivatives in the submodel are well-understood ordinary derivatives of the original structural model. For the calculation, distributions and other nonparametric objects are indexed by  $\eta$ , and thus we define  $\boldsymbol{\theta}(\mathbf{x}; \eta)$  and  $\boldsymbol{\mu}^*(\eta)$  as

$$\boldsymbol{\theta}^*(\cdot; \eta) = \arg \min_{\mathbf{b}} \int \ell(\mathbf{w}, \mathbf{b}(\mathbf{x})) f_{\mathbf{w}}(\mathbf{w}; \eta) d\mathbf{w} \quad (\text{A.5})$$

and

$$\boldsymbol{\mu}^*(\eta) = \int \mathbf{H}(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}; \eta); \tilde{\mathbf{t}}) f_{\mathbf{x}}(\mathbf{x}; \eta) d\mathbf{x}, \quad (\text{A.6})$$

where  $f_{\mathbf{w}}$  and  $f_{\mathbf{x}}$  are the distributions of  $\mathbf{w} = (\mathbf{y}', \mathbf{t}', \mathbf{x}')'$  and  $\mathbf{x}$  respectively. For notational simplicity, we will assume throughout the derivation that such densities exist. The true data generating process is obtained at  $\eta = 0$ . When evaluating at  $\eta = 0$  we will often omit the dependence on  $\eta$ , such as  $f_{\mathbf{x}}(\mathbf{x}; \eta) = f_{\mathbf{x}}(\mathbf{x})$ ,  $\boldsymbol{\theta}(\mathbf{x}; 0) = \boldsymbol{\theta}^*(\mathbf{x})$ , or  $\mathbb{E}[\cdot]$  for expectations with respect to the true distribution.

The pathwise derivative approach proceeds, as in Newey (1994) and others, by finding a function  $\boldsymbol{\psi}(\mathbf{w})$  such that

$$\frac{\partial \boldsymbol{\mu}(\eta)}{\partial \eta} \Big|_{\eta=0} = \mathbb{E}[\boldsymbol{\psi}(\mathbf{W}) S(\mathbf{W})], \quad (\text{A.7})$$

for the (true) score  $S(\mathbf{w}) = S(\mathbf{w}; \eta)|_{\eta=0}$ .

The first step is differentiating (A.6) with respect to the parameter  $\eta$ , and evaluating this at  $\eta = 0$ . The product rule and the chain rule yield

$$\begin{aligned} \frac{\partial \boldsymbol{\mu}(\eta)}{\partial \eta} \Big|_{\eta=0} &= \frac{\partial}{\partial \eta} \left\{ \int \mathbf{H}(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}; \eta); \tilde{\mathbf{t}}) f_{\mathbf{x}}(\mathbf{x}; \eta) d\mathbf{x} \right\} \Big|_{\eta=0} \\ &= \int \mathbf{H}(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}; 0); \tilde{\mathbf{t}}) \frac{\partial f_{\mathbf{x}}(\mathbf{x}; \eta)}{\partial \eta} \Big|_{\eta=0} d\mathbf{x} + \int \frac{\partial \mathbf{H}(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}; \eta); \tilde{\mathbf{t}})}{\partial \eta} \Big|_{\eta=0} f_{\mathbf{x}}(\mathbf{x}; 0) d\mathbf{x}, \\ &= \int \mathbf{H}(\mathbf{x}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}}) \frac{\partial f_{\mathbf{x}}(\mathbf{x}; \eta)}{\partial \eta} \Big|_{\eta=0} d\mathbf{x} + \int \mathbf{H}_{\boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}}) \boldsymbol{\theta}_{\eta}(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}, \end{aligned} \quad (\text{A.8})$$

where  $\boldsymbol{\theta}_{\eta}(\mathbf{x}) = \boldsymbol{\theta}_{\eta}(\mathbf{x}; 0)$  is the  $d_{\boldsymbol{\theta}}$ -vector gradient of  $\boldsymbol{\theta}(\mathbf{x}; \eta)$  with respect to  $\eta$ , evaluated at

$\eta = 0$ , given by

$$\boldsymbol{\theta}_\eta(\mathbf{x}; 0) = \frac{\partial \boldsymbol{\theta}(\mathbf{x}; \eta)}{\partial \eta} \Big|_{\eta=0},$$

and  $\mathbf{H}_{\boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}})$  is the  $d_{\boldsymbol{\mu}} \times d_{\boldsymbol{\theta}}$  Jacobian of  $\mathbf{H}$  with respect to  $\boldsymbol{\theta}$ , evaluated at  $\eta = 0$ , that is, the matrix with  $\{h, k\}$  element, for  $h = 1, \dots, d_{\boldsymbol{\mu}}, k = 1, \dots, d_{\boldsymbol{\theta}}$ , given by

$$[\mathbf{H}_{\boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}})]_{h,k} = \frac{\partial H_h(\mathbf{x}, \mathbf{b}; \tilde{\mathbf{t}})}{\partial b_k} \Big|_{\mathbf{b}=\boldsymbol{\theta}(\mathbf{x}; 0)},$$

with  $H_h$  the  $h^{\text{th}}$  element of  $\mathbf{H}$  and  $b_k$  the  $k$  element of  $\mathbf{b}$ . For intuition, note that element  $h = 1, \dots, d_{\boldsymbol{\mu}}$  of the  $d_{\boldsymbol{\mu}}$ -vector  $\mathbf{H}_{\boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}})\boldsymbol{\theta}_\eta(\mathbf{x})$  is

$$\frac{\partial \mathbf{H}_h}{\partial \eta} \Big|_{\eta=0} = \sum_{k=1}^{d_{\boldsymbol{\theta}}} \frac{\partial \mathbf{H}_h(\mathbf{x}, \mathbf{b}; \tilde{\mathbf{t}})}{\partial b_k} \Big|_{\mathbf{b}=\boldsymbol{\theta}(\mathbf{x}; 0)} \frac{\partial \boldsymbol{\theta}_k(\mathbf{x}; \eta)}{\partial \eta} \Big|_{\eta=0}.$$

We will show that both terms of Equation (A.8) above can be written as expectations of products with the full score  $S(\mathbf{w})$ , as required by (A.7). We will often use the standard facts that scores are mean zero and that

$$S(\mathbf{y}, \mathbf{x}, \mathbf{t}) = S(\mathbf{y}, \mathbf{t} \mid \mathbf{x}) + S(\mathbf{x}). \quad (\text{A.9})$$

The first term of Equation (A.8) is

$$\begin{aligned} \int \mathbf{H}(\mathbf{x}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}}) \frac{\partial f_{\mathbf{x}}(\mathbf{x}; \eta)}{\partial \eta} \Big|_{\eta=0} d\mathbf{x} &= \mathbb{E} [\mathbf{H}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}}) S(\mathbf{X})] \\ &= \mathbb{E} [\mathbf{H}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}}) S(\mathbf{Y}, \mathbf{X}, \mathbf{T})], \end{aligned} \quad (\text{A.10})$$

where the first equality holds because the marginal score obeys  $S(\mathbf{x})f_{\mathbf{x}}(\mathbf{x}) = \partial f_{\mathbf{x}}(\mathbf{x}; \eta)/\partial \eta|_{\eta=0}$  and the second equality follows from the usual mean zero property of scores and (A.9):

$$\mathbb{E} [\mathbf{H}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{x}), \tilde{\mathbf{t}}) S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X})] = \mathbb{E} [\mathbf{H}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{x}), \tilde{\mathbf{t}}) \mathbb{E}[S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X}) \mid \mathbf{X}]] = 0.$$

This first term is then the standard ‘‘plug-in’’ portion of the influence function, that is, the term that would appear if  $\boldsymbol{\theta}^*(\mathbf{x})$  were known (or if  $\widehat{\boldsymbol{\theta}}(\mathbf{x})$  were fixed). The second term of Equation (A.8) will give rise to the correction factor that accounts for the nonparametric estimation.

To find this correction factor, we must find  $\boldsymbol{\theta}_\eta(\mathbf{x}) = \partial\boldsymbol{\theta}(\mathbf{x}; \eta)/\partial\eta|_{\eta=0}$ . This is a key step in the derivation, and crucially leverages the structure of the model  $\ell$  and the fact that  $\ell$  depends on  $\boldsymbol{\theta}(\cdot)$  only through evaluation at a single point and only through  $\mathbf{X}$ . We will use these facts to derive an expression for  $\partial\boldsymbol{\theta}(\mathbf{x}; \eta)/\partial\eta$ , which involves the appropriate scores and then may be substituted into (A.8) to yield the required form.

We begin with the fact that the first order condition holds as an identity in  $\eta$  and conditional on  $\mathbf{X}$ . That is, as an identity in  $\eta$ ,

$$\mathbb{E}_\eta [\ell_\boldsymbol{\theta}(\mathbf{W}, \boldsymbol{\theta}(\mathbf{x}; \eta)) | \mathbf{X} = \mathbf{x}] \equiv 0, \quad (\text{A.11})$$

where  $\ell_\boldsymbol{\theta}$  is the  $d_\boldsymbol{\theta}$ -vector gradient of  $\ell$  with respect to  $\boldsymbol{\theta}$ , given by

$$\ell_\boldsymbol{\theta}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x}; \eta)) = \left. \frac{\partial \ell(\mathbf{w}, \mathbf{b})}{\partial \mathbf{b}} \right|_{\mathbf{b}=\boldsymbol{\theta}(\mathbf{x}; \eta)}.$$

The expectation is also indexed by  $\eta$  in the submodel, as the density depends on  $\eta$ . To be explicit, as an identity in  $\eta$  we have

$$\int \left. \frac{\partial \ell(\mathbf{w}, \mathbf{b})}{\partial \mathbf{b}} \right|_{\mathbf{b}=\boldsymbol{\theta}(\mathbf{x}; \eta)} f_{\mathbf{y}, \mathbf{t} | \mathbf{x}}(\mathbf{y}, \mathbf{t}; \eta | \mathbf{x}) d\mathbf{y} d\mathbf{t} \equiv 0.$$

Define  $\ell_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x}; \eta))$  as the  $d_\boldsymbol{\theta} \times d_\boldsymbol{\theta}$  matrix of second derivatives of  $\ell(\mathbf{w}, \mathbf{b})$  with respect to  $\mathbf{b}$ , evaluated at  $\mathbf{b} = \boldsymbol{\theta}(\mathbf{x}; \eta)$ . That is,  $\ell_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x}; \eta))$  has  $\{k_1, k_2\}$  element given by

$$[\ell_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x}; \eta))]_{k_1, k_2} = \left. \frac{\partial^2 \ell(\mathbf{w}, \mathbf{b})}{\partial b_{k_1} \partial b_{k_2}} \right|_{\mathbf{b}=\boldsymbol{\theta}(\mathbf{x}; \eta)},$$

where  $b_{k_1}$  and  $b_{k_2}$  are the respective elements of  $\mathbf{b}$ . With this notation, differentiating the above identity with respect to  $\eta$  and applying the chain rule we find

$$\begin{aligned} \int \frac{\partial \ell(\mathbf{w}, \mathbf{b}(\mathbf{x}))}{\partial \mathbf{b}} \Big|_{\mathbf{b}=\boldsymbol{\theta}(\mathbf{x}; \eta)} & \frac{\partial f_{\mathbf{y}, \mathbf{t}|\mathbf{x}}(\mathbf{y}, \mathbf{t}; \eta \mid \mathbf{x})}{\partial \eta} d\mathbf{y} dt \\ & + \int \ell_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x}; \eta)) \boldsymbol{\theta}_\eta(\mathbf{x}; \eta) f_{\mathbf{y}, \mathbf{t}|\mathbf{x}}(\mathbf{y}, \mathbf{t}; \eta \mid \mathbf{x}) d\mathbf{y} dt = 0, \end{aligned}$$

where the second term captures the derivatives of  $\ell_{\boldsymbol{\theta}}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x}; \eta))$  with respect to  $\eta$ , and recall,  $\boldsymbol{\theta}_\eta(\mathbf{x}; \eta)$  is the  $d_{\boldsymbol{\theta}}$ -vector gradient of  $\boldsymbol{\theta}$  with respect to  $\eta$ , and is the key ingredient.

Evaluating this result at  $\eta = 0$ , we obtain

$$\mathbb{E} [\ell_{\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X}) \mid \mathbf{X}] + \mathbb{E} [\ell_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) \boldsymbol{\theta}_\eta(\mathbf{x}) \mid \mathbf{X}] = 0, \quad (\text{A.12})$$

where  $S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X})$  is the conditional score and is obtained because  $S(\mathbf{y}, \mathbf{t} \mid \mathbf{x}) f_{\mathbf{y}, \mathbf{t}|\mathbf{x}}(\mathbf{y}, \mathbf{t} \mid \mathbf{x}) = \partial f_{\mathbf{y}, \mathbf{t}|\mathbf{x}}(\mathbf{y}, \mathbf{t}; \eta \mid \mathbf{x}) / \partial \eta \Big|_{\eta=0}$ . Rearranging (A.12), and using that  $\boldsymbol{\theta}$  is only a function of  $\mathbf{X}$ , gives

$$\mathbb{E} [\ell_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) \mid \mathbf{X}] \boldsymbol{\theta}_\eta(\mathbf{x}) = -\mathbb{E} [\ell_{\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X}) \mid \mathbf{X}].$$

Then, because  $\Lambda(\mathbf{x}) := \mathbb{E} [\ell_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) \mid \mathbf{X} = \mathbf{x}]$  is invertible, we have

$$\begin{aligned} \boldsymbol{\theta}_\eta(\mathbf{x}) &= -\mathbb{E} [\ell_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) \mid \mathbf{X}]^{-1} \mathbb{E} [\ell_{\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X}) \mid \mathbf{X}] \\ &= -\mathbb{E} [\Lambda(\mathbf{x})^{-1} \ell_{\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X}) \mid \mathbf{X}]. \end{aligned}$$

Substituting this into the second term of Equation (A.8) and applying iterated expectations, we have

$$\begin{aligned} \int \mathbf{H}_{\boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}}) \boldsymbol{\theta}_\eta(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} &= -\mathbb{E} [\mathbf{H}_{\boldsymbol{\theta}}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}); \tilde{\mathbf{t}}) \mathbb{E} [\Lambda(\mathbf{X})^{-1} \ell_{\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X}) \mid \mathbf{X}]] \\ &= -\mathbb{E} [\mathbb{E} [\mathbf{H}_{\boldsymbol{\theta}}(\mathbf{X}, \boldsymbol{\theta}(\mathbf{X}); \tilde{\mathbf{t}}) \Lambda(\mathbf{X})^{-1} \ell_{\boldsymbol{\theta}}(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X}) \mid \mathbf{X}]] \end{aligned}$$

$$= -\mathbb{E} \left[ \mathbf{H}_\theta(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}); \tilde{\mathbf{t}}) \Lambda(\mathbf{X})^{-1} \ell_\theta(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{Y}, \mathbf{T} \mid \mathbf{X}) \right].$$

Next, because the first order condition holds conditionally,

$$\begin{aligned} \mathbb{E} \left[ \mathbf{H}_\theta(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}); \tilde{\mathbf{t}}) \Lambda(\mathbf{X})^{-1} \ell_\theta(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{X}) \right] \\ = \mathbb{E} \left[ \mathbf{H}_\theta(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}); \tilde{\mathbf{t}}) \Lambda(\mathbf{X})^{-1} \mathbb{E} [\ell_\theta(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) \mid \mathbf{X}] S(\mathbf{X}) \right]. \end{aligned}$$

Therefore, continuing from the previous display and applying (A.9), the second term of Equation (A.8) is of the required form:

$$-\mathbb{E} \left[ \mathbf{H}_\theta(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}); \tilde{\mathbf{t}}) \Lambda(\mathbf{X})^{-1} \ell_\theta(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{Y}, \mathbf{T}, \mathbf{X}) \right] \quad (\text{A.13})$$

Combining Equations (A.10) and (A.13) with (A.8), we find that

$$\begin{aligned} \frac{\partial \boldsymbol{\mu}(\eta)}{\partial \eta} \Big|_{\eta=0} &= \mathbb{E} [\mathbf{H}(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}}) S(\mathbf{Y}, \mathbf{X}, \mathbf{T})] \\ &\quad - \mathbb{E} \left[ \mathbf{H}_\theta(\mathbf{X}, \boldsymbol{\theta}^*(\mathbf{X}); \tilde{\mathbf{t}}) \Lambda(\mathbf{X})^{-1} \ell_\theta(\mathbf{W}, \boldsymbol{\theta}^*(\mathbf{x})) S(\mathbf{Y}, \mathbf{T}, \mathbf{X}) \right]. \quad (\text{A.14}) \end{aligned}$$

Thus we have verified Equation (A.7) with

$$\psi(\mathbf{w}) = \mathbf{H}(\mathbf{x}, \boldsymbol{\theta}^*(\mathbf{x}); \tilde{\mathbf{t}}) - \mathbf{H}_\theta(\mathbf{x}, \boldsymbol{\theta}^*(\mathbf{X}); \tilde{\mathbf{t}}) \Lambda(\mathbf{x})^{-1} \ell_\theta(\mathbf{w}, \boldsymbol{\theta}^*(\mathbf{x})). \quad (\text{A.15})$$

This is not an influence function as it lacks the appropriate centering, but of course  $\mathbb{E}[\boldsymbol{\mu}^* S(\mathbf{W})] = \boldsymbol{\mu}^* \mathbb{E}[S(\mathbf{W})] = 0$ , and thus we can freely center this  $\psi(\mathbf{t})$  and still obey (A.7).  $\square$

### A.2.1 Asymptotic Normality

The asymptotic Normality of Theorem 2 follows from Theorems 3.1 and 3.2 of Chernozhukov et al. (2018a) upon verifying Assumptions 3.1 and 3.2 therein. Assumption 3.1(a) holds for

$\psi(\mathbf{y}, \mathbf{t}, \mathbf{x}, \boldsymbol{\theta}, \Lambda) - \boldsymbol{\mu}^*$  given in Equation (3.6): the first term of  $\psi$  has mean  $\boldsymbol{\mu}^*$  by definition in (3.4) while the second is (conditionally) mean zero as assumed in Assumption 3, with  $\Lambda(\mathbf{x})^{-1}$  uniformly bounded. Assumption 3.1(b), linearity, holds by definition of (3.4) and the form of the score in Equation 3.6. Assumption 3.1(c) holds by Assumption 3, in particular the assumed smoothness and the nonsingularity of  $\Lambda(\mathbf{x})$ . Assumption 3.1(d), Neyman orthogonality, is verified directly by the calculation of the influence function. Assumption 3.1(e) holds trivially as the matrix  $J_0$  therein is the identity. Assumption 3.2, parts (b) and (d) follow directly from the moment conditions imposed. Conditions (a) and (c) follow from Equations (3.7) and (3.8) of Chernozhukov et al. (2018a) and the assumed convergence of the first stage estimates.  $\square$

## Appendix B Generalized Linear Models

To help illustrate our results, and because it is a leading use case, this section specializes to modeling the conditional mean with a linear index. This model will also help us link to prior work, both in nonparametrics and semiparametrics.

The parametric structural model in this case is determined by the conditional mean restriction on a scalar outcome  $Y$  given a vector of explanatory variables  $\mathbf{T}$ . Assume that

$$\mathbb{E}[Y \mid \mathbf{T} = \mathbf{t}] = G(\alpha^* + \boldsymbol{\beta}^{*\prime} \mathbf{t}), \quad (\text{B.1})$$

for an inverse link function  $G(u) : \mathbb{R} \rightarrow \mathbb{R}$ . The full model may come with assumptions about variances and other parts of the data generating process. Standard examples include linear and logistic regression. Slope coefficients  $\boldsymbol{\beta}^*$  in such models, along with marginal effects in nonlinear models, are among the most commonly studied objects in empirical research.

To enrich this model with ML, we change the intercept and slope to the parameter

functions  $\boldsymbol{\theta}^*(\mathbf{x}) = (\alpha^*(\mathbf{x}), \beta^*(\mathbf{x})')'$  and assume that

$$\mathbb{E}[Y | \mathbf{T} = \mathbf{t}, \mathbf{X} = \mathbf{x}] = G(\alpha^*(\mathbf{x}) + \beta^*(\mathbf{x})'\mathbf{t}). \quad (\text{B.2})$$

This formulation maintains the structural relationship between  $\mathbf{T}$  and  $Y$  but allows for rich heterogeneity in  $\mathbf{X}$ . In contrast to our structured approach, the naive ML approach would treat  $\mathbf{X}$  and  $\mathbf{T}$  as equally informative about  $Y$ , and assume that for an unknown  $\eta(\mathbf{t}, \mathbf{x})$  to be estimated

$$\mathbb{E}[Y | \mathbf{T} = \mathbf{t}, \mathbf{X} = \mathbf{x}] = G(\eta(\mathbf{x}, \mathbf{t})), \quad (\text{B.3})$$

or often simply  $E[Y | \mathbf{T} = \mathbf{t}, \mathbf{X} = \mathbf{x}] = \eta(\mathbf{x}, \mathbf{t})$ , without even the structure of the inverse link.

**Remark B.1** (Origins of Model (B.2)). Models of the form (B.2) are common in the literature, referred to as a “varying coefficient” model (Cleveland, Grosse, and Shyu, 1991; Hastie and Tibshirani, 1993), “functional coefficient” model (Chen and Tsay, 1993), or “smooth coefficient” model (Li et al., 2002), and falls into the class of “extended linear models” as in Stone et al. (1997). O’Hagan (1978) may be the earliest treatment. Our results apply to all these cases as well as other similar models including generalized additive models or further restrictions such as partially linear models. See also Remarks B.2 and B.3 and other discussion in Appendix C. ♣

Deep learning architectures to estimate these two models are shown in Figure B.1. The loss function is the same in both cases. Panel (a) is the standard ML approach as available in standard software. All information in  $(\mathbf{X}', \mathbf{T}')$  is fed into the hidden layers. Panel (b) specializes Figure 1 to force the expressivity to learn the slope and intercept parameter functions in order to reduce the loss. The figure illustrates how easy it is to implement the structural approach, with only an extra line or two of code. As an aside, Figure B.1 also illustrates how our method (and theory) applies to generalized additive models, where the different components of  $\boldsymbol{\theta}^*$  are known to rely on different subsets of  $\mathbf{X}$ : we simply sever the

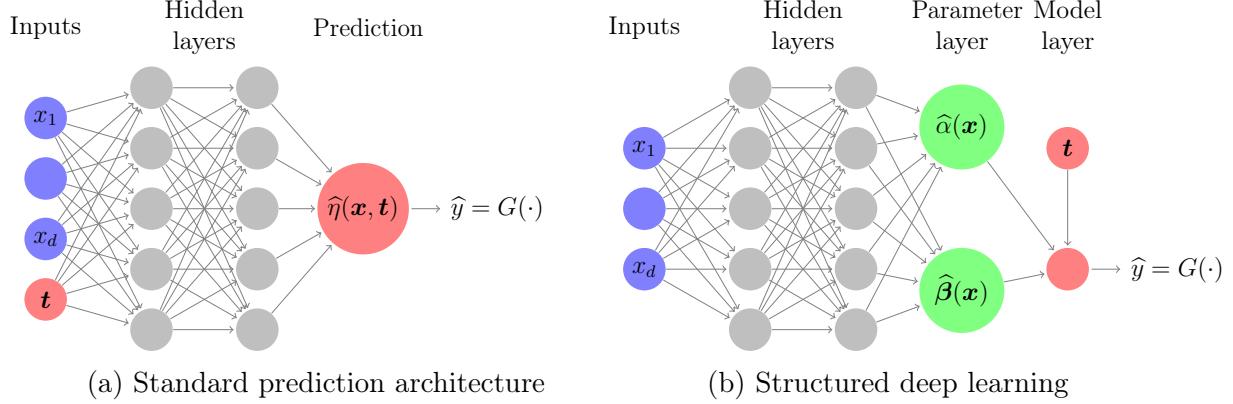


Figure B.1: Comparing the standard prediction-focused architecture to learning parameter functions using the structured approach, matching the models of (B.3) and (B.2) respectively.

appropriate links, so that separate networks feed into the parameter layer nodes according to the model.

Interpreting  $\beta^*(\mathbf{x})$  in (B.2) is the same as interpreting the original (homogeneous) slope  $\beta^*$  in (B.1). In contrast, (B.3) is unstructured and uninterpretable. Here  $\eta(\mathbf{x}, \mathbf{t})$  is truly a nuisance function, where “nuisance” is taken literally to mean annoying and uninteresting. Economically, as demonstrated in Section 2, the lack of structure makes this output not useful. It is difficult, if not impossible, to recover many second stage objects without structure, beyond (weighted) average derivatives. Further, from a statistical point of view,  $\eta(\mathbf{x}, \mathbf{t})$  can only be learned at a much slower rate, governed by  $d_{\mathbf{X}} + d_{\mathbf{T}}$ , compared to learning  $\beta^*(\mathbf{x})$  which depends only upon  $d_{\mathbf{X}}$ . Though generally assumed away in theory since all dimensions are finite, this difference matters in applications. All in all, it is better to view (B.2) as an enriched version of (B.1) rather than a restricted version of (B.3).

## B.1 Deep Neural Networks for Generalized Linear Models

The theoretical results for deep neural networks (this subsection) and inference (next subsection) for the special case of the model (B.2) are useful to illustrate the required assumptions and compare to prior work. Further, these special cases are directly useful in many applications, given the popularity of the model.

For first step estimation, we can verify the high level conditions using the following familiar, primitive assumptions.

**Assumption 4.** (i) *The conditional expectation  $G(\alpha^*(\mathbf{x}) + \beta^*(\mathbf{x})'\mathbf{t})$  enters the loss through a known, real-valued transformation  $g(\cdot)$ , where (i)  $g$  and  $G$  are continuously invertible and  $g/\|g\|_\infty$  and  $G/\|G\|_\infty$  belong to  $\mathcal{W}^{p,\infty}([-1, 1])$ , for  $p \geq 3$ . (ii) Assumption 1 holds with  $\ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x}))$  replaced by  $\ell(\mathbf{y}, g)$ , and the conditions therein apply to the scalar argument  $g$ . (iii) The eigenvalues of  $\mathbb{E}[\mathbf{T}\mathbf{T}' \mid \mathbf{X} = \mathbf{x}]$  are bounded and bounded away from zero uniformly in  $\mathbf{x}$ .*

Condition (i) ensures that the loss function is sufficiently smooth while (ii) and (iii) ensures the curvature through the standard positive variance condition. These conditions are familiar from the parametric case, where, for example, assuming that  $\mathbb{E}[\mathbf{T}\mathbf{T}']$  is positive definite would be standard.

Specializing Theorem 1 to this case, we have the following result.

**Corollary B.1.** *Let the conditions of Theorem 1 and Assumption 4 hold. Then for a DNN structured according to Figure B.1,  $\|\widehat{\alpha} - \alpha^*\|_{L_2(\mathbf{X})}^2 = O(n^{-\frac{p}{p+d_c}} \log^8(n))$  and  $\|\widehat{\beta}_k - \beta^*\|_{L_2(\mathbf{X})}^2 = O(n^{-\frac{p}{p+d_c}} \log^8(n))$  for  $k = 1, \dots, d_T$ . Further  $\|G(\widehat{\alpha}(\mathbf{x}) + \widehat{\beta}(\mathbf{x})'\mathbf{t}) - G(\alpha^*(\mathbf{x}) + \beta^*(\mathbf{x})'\mathbf{t})\|_{L_2(\mathbf{X})}^2 = O(n^{-\frac{p}{p+d_c}} \log^8(n))$ .*

Here we give two simple results (cf the more full statement in Theorem 1). First, we show that we can estimate the heterogeneous intercept and slope parameters at the appropriate rate, depending on the (continuous) dimension of the heterogeneity. This is direct from Theorem 1. This is required as economic constructs depend on these parameters, rather than on the conditional expectation function as a whole. But we also give a result for the mean function in the enriched model (B.2). This result may be of independent interest, as it establishes that structural deep learning has good statistical properties in varying coefficient models, additive models, and other such cases. It is also useful for comparing to the more typical use of inference after ML, where the (prediction) function  $\mathbb{E}[Y|\mathbf{x}, \mathbf{t}]$  would be unstructured

and would be learnable at a rate dependent on  $d_{\mathbf{X}} + d_{\mathbf{T}}$ . For example, it is much easier (statistically) to estimate  $\mathbb{E}[\beta^*(\mathbf{X})]$  than the average derivatives  $\mathbb{E}[\partial G(\eta(\mathbf{X}, \mathbf{T}))/\partial \mathbf{t}]$ , even though both are linear summaries of the dependence of  $Y$  on  $\mathbf{T}$ .

**Remark B.2** (Adaptivity). Specific forms of deep neural networks are known to be adaptive to structures like (B.2) in the sense that if the structure holds, then even if estimation is done under the generic (B.3), the estimator will still obtain the same rate as though (B.2) was imposed (Bach, 2017; Bauer and Kohler, 2019; Schmidt-Hieber, 2020). This is perhaps not surprising, since neural networks are based on compositions and (B.2) is based on a composition of simpler functions. This is still not useful for our purpose because  $\theta^*(\mathbf{x}) = (\alpha^*(\mathbf{x}), \beta^*(\mathbf{x}))'$  cannot be recovered from an adaptive procedure and used in the second stage. The estimator *statistically* adapts to the structure, but not *economically*. Economic structure is enforced on the data, not discovered from the data, as demonstrated in Section 2. Further, experience shows that, despite the (asymptotic) theory of adaptivity, finite sample performance is improved by structure. ♣

**Remark B.3** (CATE Estimation). A large recent strand of machine learning literature studies the conditional average treatment effect (CATE) function, which corresponds to  $\beta^*(\mathbf{x})$  in (B.2) with identity  $G(u)$  and a scalar, binary  $\mathbf{T}$  (so that (B.2) is without loss of generality, i.e. equivalent to (B.3)). Farrell, Liang, and Misra (2021) study this case. Under certain conditions  $\beta^*(\mathbf{x})$ , being the difference between two prediction functions, can be recovered faster than the typical nonparametric bound of Stone (1982), which is itself faster than Corollary B.1. Generally, this is possible when  $\beta(\mathbf{x})$  is somehow “simpler” than its components, such as being smoother or lower dimensional. See, for example, Shalit, Johansson, and Sontag (2017) and Nie and Wager (2021) for early important work, Kennedy (2023) for fast rates and much other discussion, Foster and Syrgkanis (2023) for global bounds including sharp risk bounds, and finally Kennedy et al. (2024) for quantifying the minimax bound precisely. Some of these methods rely on orthogonal scores, and so it would

be interesting to see if our score and neural networks could be adapted to provide similar rate improvements in other economic contexts. Athey, Tibshirani, and Wager (2019) is also related here, though adaptivity is not discussed therein. ♣

## B.2 Influence Function for Generalized Linear Models

Specializing our influence function to case of (B.2) helps connect with past work. To save notation, define  $\mathbf{T}_1 = (1, \mathbf{T}')'$ ,  $G_0(\mathbf{x}, \mathbf{t}) = G(\alpha^*(\mathbf{x}) + \beta^*(\mathbf{x})'\mathbf{t})$ , and  $\dot{G}_0(\mathbf{x}, \mathbf{t}) = dG(u)/du$  at  $u = \alpha^*(\mathbf{x}) + \beta^*(\mathbf{x})'\mathbf{t}$ . If we take the standard approach where  $\ell_{\boldsymbol{\theta}}(y, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x})) = \mathbf{T}_1(G_0(\mathbf{x}, \mathbf{t}) - y)$ , the influence function is

$$\psi(y, \mathbf{t}, \mathbf{x}, \boldsymbol{\theta}, \Lambda) = \mathbf{H}(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}); \tilde{\mathbf{t}}) + \mathbf{H}_{\boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}); \tilde{\mathbf{t}})^{-1} \mathbf{T}_1(y - G_0(\mathbf{x}, \mathbf{t})), \quad (\text{B.4})$$

with  $\Lambda(\mathbf{x}) = \mathbb{E}[\dot{G}_0(\mathbf{x}, \mathbf{T}) \mathbf{T}_1 \mathbf{T}_1' \mid \mathbf{X} = \mathbf{x}]$ . Further, if  $d_{\mathbf{T}} = d_{\boldsymbol{\mu}} = 1$ , so that  $\beta^* = (\alpha^*, \beta^*)'$ , this can be written

$$\begin{aligned} \psi(y, t, \mathbf{x}, \boldsymbol{\theta}, \Lambda) &= H(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}); \tilde{t}) \\ &+ \frac{H_{\alpha}(\mathbf{x})(\lambda_2(\mathbf{x}) - \lambda_1(\mathbf{x})t) + H_{\beta}(\mathbf{x})(\lambda_0(\mathbf{x})t - \lambda_1(\mathbf{x}))}{\lambda_2(\mathbf{x})\lambda_0(\mathbf{x}) - \lambda_1(\mathbf{x})^2} (y - G_0(\mathbf{x}, t)), \end{aligned} \quad (\text{B.5})$$

where  $\lambda_k(\mathbf{x}) = \mathbb{E}[\dot{G}_0(\mathbf{x}, T) T^k \mid \mathbf{X} = \mathbf{x}], k = \{0, 1, 2\}$ ,  $H_{\alpha}(\mathbf{x}) = \partial H(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}); \tilde{t})/\partial \alpha$ , and  $H_{\beta}(\mathbf{x}) = \partial H(\mathbf{x}, \boldsymbol{\theta}(\mathbf{x}); \tilde{t})/\partial \beta$ . In nonlinear models the need for three-way sample splitting is clear: the regressand of  $\Lambda(\mathbf{x}) = \mathbb{E}[\dot{G}_0(\mathbf{x}, T) \mathbf{T}_1 \mathbf{T}_1' \mid \mathbf{X} = \mathbf{x}]$  depends on the unknown  $\boldsymbol{\theta}^*(\mathbf{x})$  through  $\dot{G}_0(\mathbf{x}, T)$ , which must be estimated in a first step. For linear models, three splits are not necessary.

This result recovers several known influence functions. For linear models, it matches Hahn (1998) for binary treatments and Graham and de Xavier Pinto (2022) for continuous. For mean square projections in general, see Newey (1994). For semiparametric regression, where  $\beta^*$  is assumed constant, our result gives valid inference but does not match the efficient

influence function (Mammen and van de Geer, 1997) because we do not impose the constancy.

We can also effortlessly obtain new results. For example, in Section 4 we conduct inference on the fully flexible average marginal effects in a logistic regression, where  $\mu^* = \mathbb{E}[\dot{G}(\alpha^*(\mathbf{X}) + \beta^*(\mathbf{X})'\mathbf{T})\beta^*(\mathbf{X})]$ . In this case the function  $\mathbf{H}$  and its derivatives are available in closed form, and the derivatives of the loss are well known, but again, this is not necessary. We also use (B.4) for a function  $\mathbf{H}$  that is not available in closed form (Spall, 1986; Jorgensen, 1993).

### B.3 Proof of Corollary B.1

For this derivation, define  $\mathbf{T}_1 = (1, \mathbf{T}')'$  and recall that  $\boldsymbol{\theta}^* = (\alpha^*, \beta^{*\prime})'$  (and similarly for realizations, estimators, etc). First, consider identification. Since  $G$  is invertible and conditional expectations are always identified, the quantity  $G^{-1}(\mathbb{E}[Y | \mathbf{X} = \mathbf{x}, \mathbf{T} = \mathbf{t}])$  is identified. Suppose that  $\boldsymbol{\theta}^*(\mathbf{x})$  is not identified. Then there exists  $\boldsymbol{\theta}_1(\mathbf{x})$  and  $\boldsymbol{\theta}_2(\mathbf{x})$  such that  $G^{-1}(\mathbb{E}[Y | \mathbf{X} = \mathbf{x}, \mathbf{T} = \mathbf{t}]) = \boldsymbol{\theta}_1(\mathbf{x})'\mathbf{T}_1 = \boldsymbol{\theta}_2(\mathbf{x})'\mathbf{T}_1$  a.e. or equivalently that for  $\boldsymbol{\theta}_*(\mathbf{x}) = \boldsymbol{\theta}_1(\mathbf{x}) - \boldsymbol{\theta}_2(\mathbf{x})$ ,  $\boldsymbol{\theta}_*(\mathbf{x})'\mathbf{T}_1 = 0$ . But  $\boldsymbol{\theta}_*(\mathbf{x})'\mathbf{T}_1 = 0$  a.e. implies that

$$0 = \mathbb{E}[(\boldsymbol{\theta}_*(\mathbf{x})'\mathbf{T}_1)^2 | \mathbf{X} = \mathbf{x}] = \boldsymbol{\theta}_*(\mathbf{x})'\mathbb{E}[\mathbf{T}_1\mathbf{T}_1' | \mathbf{X}]\boldsymbol{\theta}_*(\mathbf{x}),$$

but because the middle matrix is positive definite, this means that  $\boldsymbol{\theta}_*(\mathbf{x})$  is zero. For linear  $G$ , this argument is given in Huang and Shen (2004) and elsewhere.

The estimation bounds follow immediately from Theorem 1, given the conditions of Assumption 4. The fact that  $\mathbb{E}[\mathbf{T}_1\mathbf{T}_1' | \mathbf{X}]$  is (uniformly) positive yields

$$\begin{aligned} &= \mathbb{E}_{\mathbf{X}} \left[ \left( \widehat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X}) \right)' \mathbb{E}[\mathbf{T}_1\mathbf{T}_1' | \mathbf{X}] \left( \widehat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X}) \right) \right] \\ &\geq C \mathbb{E}_{\mathbf{X}} \left[ \left( \widehat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X}) \right)' \left( \widehat{\boldsymbol{\theta}}(\mathbf{X}) - \boldsymbol{\theta}^*(\mathbf{X}) \right) \right]. \end{aligned}$$

This verifies the curvature condition on the loss function. The continuity condition holds

because the loss is smooth in  $g$  and the linear index can be recovered from  $g(G(\boldsymbol{\theta}(\mathbf{x})'\mathbf{T}_1))$ . The structure of the network ensures that the network and the smoothness of the loss imply that the approximation and bounds immediately apply to the function  $g(G(\boldsymbol{\theta}(\mathbf{x})'\mathbf{T}_1))$ , and the smoothness of these functions mean that the linear index  $\boldsymbol{\theta}(\mathbf{x})'\mathbf{T}_1$  can be recovered.  $\square$

## Appendix C Examples and Discussion

In this section we discuss a few special cases of our methodology to build intuition and connect to prior work, particularly other work on semiparametric inference. These examples help place our work in the literature, illustrate how familiar models can be enriched with deep learning, and show how the identification assumptions required on parametric models carry over.

### C.1 Average Effect of a Binary Treatment

Perhaps the most well known and commonly used influence function is for the average effect of a binary treatment in observational data. The history of this influence function is illustrative: it was characterized precisely first for the purposes of efficiency considerations (Hahn, 1998), later used to show that certain plug-in estimators could be efficient (Hirano, Imbens, and Ridder, 2003; Imbens, Newey, and Ridder, 2007) under strong assumptions, then recently used to obtain valid (and in this case efficient) inference under weaker first-stage conditions (Cattaneo, 2010; Farrell, 2015). This influence function matches the doubly robust estimator (Robins, Rotnitzky, and Zhao, 1994, 1995) and has been used in other contexts to obtain sharper results, see Remarks B.3 and 5.3.

Here we have a scalar outcome and  $\mathbf{T} = T = \{0, 1\}$  is the scalar binary treatment indicator. Let  $Y(t)$  be the potential outcome under treatment  $T = t$  and assume that the observed outcome  $Y = TY(1) + (1 - T)Y(0)$  along with unconfoundedness. The parameter of interest is  $\mu^* = \mathbb{E}[Y(1) - Y(0)]$ . In a randomized experiment without additional covariates, one estimates

$\mu^*$  with a difference in means, which is equivalent to running a regression of the observed outcome  $Y$  on the dummy variable  $T$ . The enriched structural version is then (B.2), with  $G(u) = u$ , so that  $\mathbb{E}[Y | \mathbf{x}, \mathbf{t}] = \alpha^*(\mathbf{x}) + \beta^*(\mathbf{x})\mathbf{t}$ . In this notation,  $\beta^*(\mathbf{x}) = \mathbb{E}[Y(1) - Y(0) | \mathbf{x}]$  is the conditional average treatment effect (CATE) function.

The average treatment effect  $\mu^* = \mathbb{E}[\beta^*(\mathbf{X})]$  is defined via  $\mathbf{H}(\mathbf{x}, \boldsymbol{\theta}^*; \tilde{t}) = \beta^*$ . Equation (B.5) matches Hahn (1998), because in this case  $H_\alpha = 0$ ,  $H_\beta = 1$ ,  $\lambda_0 = 1$ , and  $\lambda_1(\mathbf{x}) = \lambda_2(\mathbf{x}) = \mathbb{P}[T = 1 | \mathbf{X} = \mathbf{x}] := p(\mathbf{x})$ , the propensity score, and so by adding and subtracting  $p(\mathbf{x})$  and using the fact that  $(1 - t)t = 0$ , we have

$$\begin{aligned}\psi(\mathbf{w}, \boldsymbol{\theta}, \boldsymbol{\Lambda}) &= \beta(\mathbf{x}) + \frac{\dot{H}_1(\mathbf{x})(\lambda_2(\mathbf{x}) - \lambda_1(\mathbf{x})t) + \dot{H}_2(\mathbf{x})(\lambda_0(\mathbf{x})t - \lambda_1(\mathbf{x}))}{\lambda_2(\mathbf{x})\lambda_0(\mathbf{x}) - \lambda_1(\mathbf{x})^2}(y - G(\boldsymbol{\theta}(\mathbf{x})'\mathbf{t})) \\ &= \beta(\mathbf{x}) + \frac{(t - p(\mathbf{x}))(y - \alpha(\mathbf{x}) - \beta(\mathbf{x})t)}{p(\mathbf{x}) - p(\mathbf{x})^2} \\ &= \beta(\mathbf{x}) + \frac{[(1 - p(\mathbf{x}))t - p(\mathbf{x})(1 - t)](y - \alpha(\mathbf{x}) - \beta(\mathbf{x})t)}{p(\mathbf{x})(1 - p(\mathbf{x}))} \\ &= \beta(\mathbf{x}) + \frac{(1 - p(\mathbf{x}))t(y - \alpha(\mathbf{x}) - \beta(\mathbf{x})t)}{p(\mathbf{x})(1 - p(\mathbf{x}))} - \frac{p(\mathbf{x})(1 - t)(y - \alpha(\mathbf{x}) - \beta(\mathbf{x})t)}{p(\mathbf{x})(1 - p(\mathbf{x}))} \\ &= \beta(\mathbf{x}) + \frac{t(y - \alpha(\mathbf{x}) - \beta(\mathbf{x})t)}{p(\mathbf{x})} - \frac{(1 - t)(y - \alpha(\mathbf{x}))}{(1 - p(\mathbf{x}))}.\end{aligned}$$

In this example, the standard overlap assumption, that the propensity score is bounded away from zero and one, ensures that  $\boldsymbol{\Lambda}(\mathbf{x})^{-1}$  is well behaved: the determinant of  $\boldsymbol{\Lambda}(\mathbf{x}) = p(\mathbf{x})(1 - p(\mathbf{x}))$ , the initial denominator above.

It is straightforward to extend this example in a number of directions. Additional mean parameters could be added to cover average treatment effects for specific treatment groups or multi-valued treatments (see Cattaneo (2010) and Cattaneo and Farrell (2011) for inference using classical nonparametrics (series) and Farrell (2015) for machine learning (group lasso) results).

Beyond means, the framework makes it easy to consider the variance of  $Y(1)$  versus that of  $Y(0)$ , to assess riskiness of treatments over the population, by taking a quasi-likelihood approach, i.e., letting  $\ell(\mathbf{y}, \mathbf{t}, \boldsymbol{\theta}(\mathbf{x}))$  include the variance instead of simply fitting least squares.

The conditions for convexity of this loss are well-known from likelihood theory and can be directly used here.

## C.2 Partially Linear Models

Partially linear models are a common case for semiparametric inference, dating to the seminal work of [Robinson \(1988\)](#). Here (B.2) holds, but with  $\beta^*(\mathbf{x}) = \beta^*$  constant. Restricting the slope to be constant rules out all treatment effect heterogeneity and is a strong assumption that should be used with caution.

For simplicity, consider the case with a scalar treatment variable. If  $\beta^*$  is the parameter of interest, (B.5) shows that  $\psi(\mathbf{w}, \boldsymbol{\theta}^*, \boldsymbol{\Lambda}) - \beta^*$  is

$$\left[ \lambda_2(\mathbf{x}) - \frac{\lambda_1(\mathbf{x})^2}{\lambda_0(\mathbf{x})} \right]^{-1} \left( t - \frac{\lambda_1(\mathbf{x})}{\lambda_0(\mathbf{x})} \right) \left( y - G(\alpha^*(\mathbf{x}) + \beta^*t) \right).$$

We must assume that  $\lambda_2(\mathbf{x})\lambda_0(\mathbf{x}) \neq \lambda_1(\mathbf{x})^2$ , which for identity  $G$  requires positive conditional variance of  $T$ . In nonlinear models the conditional moments will be weighted by  $\dot{G}$  if we have used the appropriate loss. In some cases the nonsingularity will follow from other regularity conditions, such as for the logistic link, where  $\dot{G} = G(1 - G)$  and the Hessian is invertible under bounded covariates and we use the log-likelihood.

Partially linear models have featured prominently in the recent literature on high dimensional and ML settings, particularly with identity link. The pioneering work of [Belloni, Chernozhukov, and Hansen \(2014\)](#) established valid inference after lasso selection in this context. [Chernozhukov et al. \(2018a\)](#) use it as the leading example of their generic results, and present several different Neyman orthogonal scores that could be used, none of which agree with ours due to the fact that we do not impose constant slope. [Cattaneo, Jansson, and Newey \(2018\)](#) study high dimensional linear models, including many-terms series settings, establish a more refined distributional approximation, and study standard error (in)validity. For the case of nonlinear link function, [Carroll et al. \(1997\)](#) and [Mammen and van de Geer \(1997\)](#)

are closest to our work, while [Belloni, Chernozhukov, and Wei \(2016\)](#) study high-dimensional sparse models. Not imposing the constant slope means we do not attain the efficiency bound ([Mammen and van de Geer, 1997](#)) in general, but only under restrictions on the variance, such as  $Y$  being homoskedastic in  $\mathbf{T}$  in the linear case.

However, it is trivial to impose a constant slope by changing the architecture in Figure B.1 so that only  $\alpha(\mathbf{x})$  is flexible. Establishing that such a slope estimate is root- $n$  consistent is beyond the scope here, but appears natural as the functional approximation holds without essential change. Deriving the statistical properties of this estimate would be interesting future work.

Finally, our results could be used to study other components of the partially linear setting. For example, in both empirical finance [Cattaneo et al. \(2020\)](#) and applied microeconomics [Cattaneo et al. \(2024b\)](#) the function  $\alpha(\mathbf{x})$  is of direct interest and we could conduct inference on the average.

### C.3 Continuous Treatment Variables

[Wooldridge \(2004\)](#) and [Graham and de Xavier Pinto \(2022\)](#) consider the linear model case and discuss conditions for a causal interpretation of  $\mathbb{E}[\boldsymbol{\beta}^*(\mathbf{X})]$ . Our result in (B.4) matches the locally efficient influence function of [Graham and de Xavier Pinto \(2022\)](#). [Hirshberg and Wager \(2019\)](#) use a different approach to recover the average effect, but briefly discuss double robustness. [Chernozhukov et al. \(2019\)](#) use the model with the goal of policy targeting.

Our method could be used to enrich other structural models in this context. As a first example, consider the so-called Berry logit ([Berry, 1994](#)) model for demand. Here the outcome is the market share distribution across firms. The researcher observes  $\{Y_{jm}\}$  which represent a collection of  $j = 0 \dots J$  market shares across  $m = 1 \dots M$  markets. The objective is then to model these as a function of firm (marketing) decisions  $\mathbf{T}_{jm}$  (see e.g. [Nevo \(2001\)](#)). We can introduce heterogeneity across markets by allowing for the marketing effects to be moderated by consumer characteristics  $\mathbf{x}_m$ , so that we can write a collection of  $(J - 1)$  equations as

follows

$$\mathbb{E} \left[ \log \left( \frac{Y_{jm}}{Y_{0m}} \right) \middle| \mathbf{X} = \mathbf{x}_m, \mathbf{T}_{jm} = \mathbf{t}_{jm} \right] = \alpha_j^*(\mathbf{x}_m) + \boldsymbol{\beta}^*(\mathbf{x}_m)'(\mathbf{t}_{jm} - \mathbf{t}_{0m}).$$

Stacking these equations and the corresponding data allows us to construct an estimator for  $\alpha_{0j}(\mathbf{x}_m)$  and  $\boldsymbol{\beta}^*(\mathbf{x}_m)$ . We could extend this to include instruments (Okui et al., 2012).

Second, consider the Cobb-Douglas production function with heterogeneous parameters, which is given by  $Y = CK^{\theta_1^*(\mathbf{x})}L^{\theta_2^*(\mathbf{x})}$ . The standard approach is to take logs and assume

$$\mathbb{E} [\log Y \mid \mathbf{X} = \mathbf{x}, K = k, L = l] = \log C + \theta_1^*(\mathbf{x}) \log k + \theta_2^*(\mathbf{x}) \log l.$$

We can then estimate the structural parameters using our deep learning approach and conduct inference to decide if on average the technology exhibits increasing, constant, or decreasing returns to scale by computing  $\boldsymbol{\mu}^* = \mathbb{E}[\theta_1^*(\mathbf{x}) + \theta_2^*(\mathbf{x})]$ . The Cobb-Douglas specification has also been used in demand settings and marketing mix models and the framework described above would be readily applicable there as well.

## C.4 Fractional Outcomes

The case of nonlinear (B.2) (nonidentity  $G(u)$ ) is less well studied. The empirical application in Sections 2 and 4 uses a logistic link. To give another example, consider a fractional outcome model where  $Y$  is continuous but restricted to lie in  $[0, 1]$ . Following the seminal treatment of Papke and Wooldridge (1996), we take a quasi-likelihood approach, using logistic distribution with mean given by Equation (B.2):  $\mathbb{E}[Y \mid \mathbf{T}=\mathbf{t}, \mathbf{X}=\mathbf{x}] = G(\alpha^*(\mathbf{x}) + \boldsymbol{\beta}^*(\mathbf{x})'t)$ . Papke and Wooldridge (1996) explicitly advocate the use of structure to ensure that the outcomes remain on the unit interval and argue that this specification is valid even at the endpoints and is more practically relevant than transformations of the dependent variable.

In the application of Papke and Wooldridge (1996), the data is at the firm level, the outcome is 401(k) participation, and the policy variable is the firm's rate of contribution

matching. The quantities of interest are the marginal effect of the match rate on participation and the degree to which this marginal effect exhibits diminishing patterns. Our framework makes it trivial to enrich this model with heterogeneity across firms and then conduct inference on average marginal effects or the average change in the marginal effect, given by

$$\text{AME}(\tilde{t}) = \mathbb{E} \left[ \frac{\partial \mathbb{E}[Y | \mathbf{X}, t]}{\partial t} \Big|_{t=\tilde{t}} \right] \quad \text{and} \quad \text{ACME}(\tilde{t}) = \mathbb{E} \left[ \frac{\partial^2 \mathbb{E}[Y | \mathbf{X}, t]}{\partial t^2} \Big|_{t=\tilde{t}} \right].$$

Because of the structure of the model, these are easily recovered in the form of  $\boldsymbol{\mu}^*$ , by taking  $H_{\text{AME}}(\mathbf{x}, \boldsymbol{\theta}; \tilde{t}) = \beta^* G_0(\mathbf{x}, \mathbf{t})(1 - G_0(\mathbf{x}, \mathbf{t}))$  and  $H_{\text{ACME}}(\mathbf{x}, \boldsymbol{\theta}; \tilde{t}) = \beta^{*2} G_0(\mathbf{x}, \mathbf{t})(1 - G_0(\mathbf{x}, \mathbf{t}))(1 - 2G_0(\mathbf{x}, \mathbf{t}))$ , respectively.

It is useful to contrast our model with the unstructured, naive ML approach, as in (B.3). We impose structure on the nature of the effect of  $\mathbf{T}$  on  $\mathbf{Y}$ . For unrestricted effects of continuous treatments using influence functions, see [Kennedy et al. \(2017\)](#) and [Colangelo and Lee \(2023\)](#). The unrestricted model may increase the generality of the results but can make inference and interpretation more difficult. A common parameter of interest in such cases is the average derivative ([Powell, Stock, and Stoker, 1989](#); [Newey and Stoker, 1993](#)). This represents the average of a linear approximation of an unstructured relationship of  $\mathbf{T}$  to  $Y$ . Our approach is perhaps more direct and transparent: if a linear summary is of interest in the end, we directly enrich the linear approximation, rather than recover it from a more complex object. This contrast can also be seen in the fractional outcome models, where recovering the second derivative of a complex  $G(\hat{\eta}(\mathbf{x}, \mathbf{t}))$  could be challenging but our approach is transparent and simple.

#### C.4.1 Application – American Community Survey

To illustrate this idea, we revisit the American Community Survey data studied by [Cattaneo et al. \(2024a\)](#).<sup>6</sup> This is a simple, and narrow, empirical exercise, intended only as an example. We will apply our method to the setting of Figure 4(a) in [Cattaneo et al. \(2024a\)](#). The

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<sup>6</sup>Replication files are available at <https://github.com/maxhfarrell/FLM2>.

data is at the zip code tabulation area level and has  $n = 27,985$  samples. The outcome  $Y$  is the percentage of individuals without health insurance and the treatment variable  $T$ , which is of course not randomized, is a binary indicator for low and high population density states, defined as those with population densities below or above 100 people per square mile, respectively. Density is defined as the average population per square mile, and the data is available from the Census Bureau. The heterogeneity variable  $X$  is per capita income.

[Cattaneo et al. \(2024a\)](#) are concerned with nonparametric properties and inference of the functions  $G(\alpha^*(x))$  and  $G(\alpha^*(x) + \beta^*(x))$ , in the notation of (B.2). Here we will conduct semiparametric inference on the average effect of density:

$$\mu^* = \mathbb{E}[G(\alpha^*(X) + \beta^*(X)) - G(\alpha^*(X))].$$

With  $X$  being scalar and continuous and nearly 30,000 observations, this is a fairly simple nonparametric problem. We estimate  $\alpha^*(x)$  and  $\beta^*(x)$  using structured neural networks, as in Figure B.1 (or Figure 1 more generally) using two hidden layers with 20 nodes each. For  $\Lambda(x)$  we use same network, but unstructured as this is regression. We additionally implement short stacking of [Ahrens et al. \(2025\)](#) using neural nets, random forests, and linear regression, as an example. We use 50-fold cross fitting with three-way splitting and two-way splitting (where  $\hat{\alpha}(x)$ ,  $\hat{\beta}(x)$ , and  $\hat{\Lambda}(x)$  are fit on the same data), but also no splitting at all.

The results are shown in Figure C.1 and Table C.1. Figure C.1 shows the first stage: the estimated  $G(\hat{\alpha}(x))$  and  $G(\hat{\alpha}(x) + \hat{\beta}(x))$  using neural networks (solid lines) and binscatter regressions (dots). This estimation is done using the full sample. The shaded regions are robust bias corrected confidence bands for the binscatter regressions. The binscatter results are obtained using `binsreg` R package ([Cattaneo et al., 2025](#)) and are identical to Figure 4(a) of [Cattaneo et al. \(2024a\)](#). We see that the neural networks and binscatter regressions are similar estimates of the unknown functions.

Table C.1 shows the semiparametric inference results. The point estimates are quite

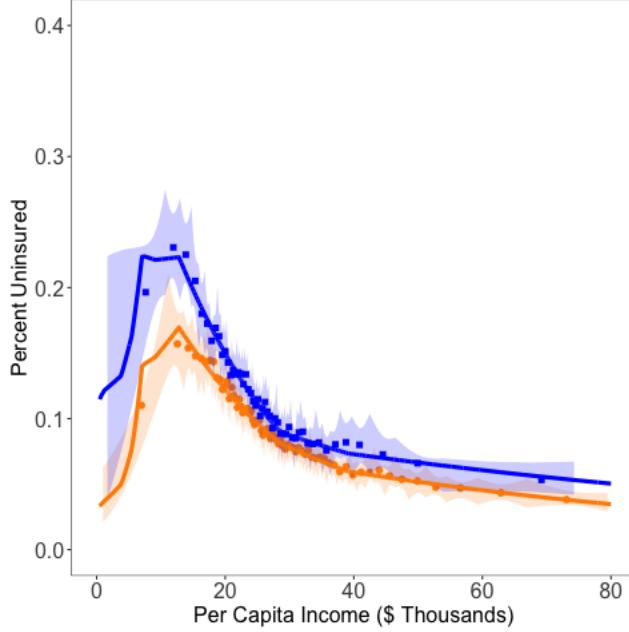


Figure C.1: **Fractional Outcome Model with Binary Treatment – First Stage.** This figure uses the ACS data of [Cattaneo et al. \(2024a\)](#) to compare areas in low density states (blue) and high density states (orange). Low density states are defined as those with average population per square mile below 100. The dots and shaded region are the point estimates and robust bias corrected 95% confidence bands from the original paper. The solid lines show structured neural network estimates.

consistent across the different approaches. The standard errors vary, and we see much smaller standard errors without cross fitting. This may be because cross fitting introduces extra variation (across the samples) which might be reduced with repeated splitting and median aggregation. Note that the short-stacking standard errors are also valid for the row above, i.e., without short stacking. Further, under strong enough assumptions that the plug-in estimate is asymptotically normal, the row below will yield valid standard errors, though this may be unrealistic in this setting. The bottom two rows, using three-way splitting, are the only results that are fully theoretically justified, however, as discussed above, the other may be as well.

Table C.1: **Fractional Outcome Model with Binary Treatment – Semiparametric Inference.** This table shows point estimates and standard errors for semiparametric inference on  $\mu^* = \mathbb{E}[G(\alpha^*(X) + \beta^*(X)) - G(\alpha^*(X))]$ , comparing uninsuredness rates between low and high density states, with heterogeneity in per capita income. Low density states are defined as those with average population per square mile below 100.

Method	Point Estimate	Standard Error
Plug-in	0.019	—
No cross fitting	0.020	0.0005
2-way cross fitting	0.014	0.0044
... w/ short stacking	0.017	0.0042
3-way cross fitting	0.026	0.0043
... w/ short stacking	0.021	0.0036

## C.5 Tobit

The type I Tobit model is well-studied in the parametric case, and so it serves as a useful example to illustrate how existing knowledge can be used to interpret the assumptions required in the enriched case.

The observed outcome is  $Y = \max(0, Y^*)$ , where in the parametric, homogeneous case  $Y^*$  is Gaussian given  $\mathbf{T} = \mathbf{t}$  with mean  $\alpha^* + \boldsymbol{\beta}^{*\prime} \mathbf{t}$  and variance  $\sigma^2$ . The enriched version has mean  $\alpha^*(\mathbf{x}) + \boldsymbol{\beta}^*(\mathbf{x})' \mathbf{t}$  and variance  $\sigma^2(\mathbf{x})$  (in practice it can help to fit  $\sigma^2(\mathbf{x}) = \exp\{\tilde{\sigma}(\mathbf{x})\}$ ). As usual, we work with the transformed parameters  $\boldsymbol{\theta}^*(\mathbf{x}) = (\boldsymbol{\theta}_1^*(\mathbf{x})', \theta_2^*(\mathbf{x}))'$ , with  $\boldsymbol{\theta}_1^*(\mathbf{x}) = (\alpha^*(\mathbf{x}), \boldsymbol{\beta}^*(\mathbf{x})')/\sigma(\mathbf{x})$  and  $\theta_2^*(\mathbf{x}) = \sigma^{-1}(\mathbf{x})$ . See [Amemiya \(1985\)](#) and [Wooldridge \(2010\)](#) for textbook treatments.

The gradient and Hessian are cumbersome but known. These can be used both for understanding the assumptions required but also, if desired, in the computation. Let  $\mathbb{1}_0 = \mathbb{1}\{Y^* \leq 0\}$  and  $\mathbb{1}_1 = \mathbb{1}\{Y^* > 0\}$ . Let  $\phi$  and  $\Phi$  denote the Gaussian density and distribution functions. With  $\mathbf{T}_1 = (1, \mathbf{t}')'$ , the gradient (score) terms are

$$\ell_{\boldsymbol{\theta}_1}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x})) = \mathbb{1}_0 \frac{\phi(\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1) \mathbf{T}_1}{1 - \Phi(\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1)} - \mathbb{1}_1 (\theta_2(\mathbf{x}) y - \boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1) \mathbf{T}_1'$$

and

$$\ell_{\theta_2}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x})) = -\mathbb{1}_1 \theta_2(\mathbf{x})^{-1} + \mathbb{1}_1 (\theta_2(\mathbf{x}) y - \boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1) y.$$

The second derivatives are

$$\ell_{\theta_1 \theta_1}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x})) = -\mathbb{1}_0 \frac{\phi(\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1)(\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1) \mathbf{T}_1 \mathbf{T}_1'}{1 - \Phi(\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1)} + \mathbb{1}_0 \frac{\phi(\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1)^2 \mathbf{T}_1 \mathbf{T}_1 \mathbf{T}_1'}{[1 - \Phi(\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1)]^2} + \mathbb{1}_1 \mathbf{T}_1 \mathbf{T}_1',$$

$$\ell_{\theta_2 \theta_2}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x})) = -\mathbb{1}_1 \theta_2(\mathbf{x})^{-2} + \mathbb{1}_1 y^2, \quad \text{and} \quad \ell_{\theta_1 \theta_2}(\mathbf{w}, \boldsymbol{\theta}(\mathbf{x})) = \mathbb{1}_1 y \mathbf{T}_1.$$

That the gradients are conditionally mean zero can be directly verified. The matrix  $\boldsymbol{\Lambda}(\mathbf{x})^{-1}$  exists because  $\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1 - \phi(\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1)/[1 - \Phi(\boldsymbol{\theta}_1(\mathbf{x})' \mathbf{T}_1)] > 0$ , using exactly the logic from parametric models (Donald, 1990; Olsen, 1978; Amemiya, 1985). The remaining assumptions required would be standard for nonparametrics/ML, such as smoothness and boundedness.

## C.6 Instrumental Variables

For multiple first stage objects, the influence function correction terms are generally additive (Newey, 1994). A good example is when  $\mathbf{T}$  is endogenous and instrumental variables are available. Suppose there is single endogenous variable  $T$  and a single instrument  $Z$  and the researcher is applying two stage least squares. We enrich this to allow for fully flexible observed heterogeneity in the effects of the instrument and the endogenous variable, arriving at the two-equation model

$$Y = \theta_1^*(\mathbf{X}) + \theta_2^*(\mathbf{X})T + V, \tag{C.1}$$

$$T = \zeta_1^*(\mathbf{X}) + \zeta_2^*(\mathbf{X})Z + U, \tag{C.2}$$

where  $\mathbb{E}[V \mid \mathbf{X}, Z] = \mathbb{E}[U \mid \mathbf{X}, Z] = 0$ . For estimation, and moreover, derivation of an orthogonal score, we simply plug (C.2) into (C.1) to obtain the reduced form equation

$$Y = \alpha^*(\mathbf{X}) + \beta^*(\mathbf{X})Z + \tilde{V}, \tag{C.3}$$

$$\alpha^*(\mathbf{x}) = \theta_1^*(\mathbf{x}) + \theta_2^*(\mathbf{x})\zeta_1^*(\mathbf{x}), \quad \beta^*(\mathbf{x}) = \theta_2^*(\mathbf{x})\zeta_2^*(\mathbf{x}), \quad \tilde{V} = \theta_2^*(\mathbf{X})U + V.$$

This approach directly generalizes two-stage least squares to handle high-dimensional, complex heterogeneity, but notice that the linearity structure is maintained. To estimate the first stage parameter functions we simply apply (3.3) where the loss is the sum of two squared losses and the architecture in Figure B.1 is used twice. The leading case for inference would be the average partial effect  $\mu^* = \mathbb{E}[\theta_2^*(\mathbf{X})] = \mathbb{E}[\beta^*(\mathbf{X})/\zeta_2^*(\mathbf{X})]$ . We again see the familiarity of the assumptions required: we need strong instruments, which here means we need  $\zeta_2^*(\mathbf{X})$  to be nowhere zero. This is a strong assumption, and may not be tenable in applications. A constant slope can be assumed to make this assumption more plausible, at the cost of restricting heterogeneity. Any other function  $H(\mathbf{X}, \alpha^*, \beta^*, \zeta_1^*, \zeta_2^*; \tilde{\mathbf{t}})$  could be used.

The influence function will be of the standard form in this case, with essentially two copies of the linear case above. Define  $\boldsymbol{\theta}^* = (\alpha^*, \beta^*, \zeta_1^*, \zeta_2^*)'$ ,  $\mathbf{w} = (y, t, z)$ ,  $\mathbf{t}_1 = (1, t)'$ ,  $\mathbf{z}_1 = (1, z)'$  and  $\mathbf{I}_2$  the  $2 \times 2$  identity matrix. Then

$$\ell_{\boldsymbol{\theta}}(\mathbf{w}, \boldsymbol{\theta}^*(\mathbf{x})) = - \begin{pmatrix} y - \alpha^*(\mathbf{x}) - \beta^*(\mathbf{x})z \\ t - \zeta_1^*(\mathbf{x}) - \zeta_2^*(\mathbf{x})z \end{pmatrix} \otimes \mathbf{z} \quad \text{and} \quad \ell_{\boldsymbol{\theta}\boldsymbol{\theta}}(\mathbf{w}, \boldsymbol{\theta}^*(\mathbf{x})) = \mathbf{I}_2 \otimes \mathbf{z}_1 \mathbf{z}_1'.$$

Therefore  $\boldsymbol{\Lambda}(\mathbf{x}) = \mathbf{I}_2 \otimes \boldsymbol{\Lambda}_Z(\mathbf{x})$ , where  $\boldsymbol{\Lambda}_Z(\mathbf{x}) = \mathbb{E}[\mathbf{z}_1 \mathbf{z}_1' \mid \mathbf{X} = \mathbf{x}]$ .

As before, our score is not the only possibility for estimation and inference IV models. Our approach aims for ease of use and transparency, both by sticking to the two stage least squares and by the structural compatibility of deep learning. Restricting to homogeneous effects, Chernozhukov et al. (2018a) study partially linear IV models and study three different possibly orthogonal scores, each requiring different functions in the first step and in the correction term. The same could be done in our unrestricted model. As in partially linear models, if any parameters are constant this can be enforced in the architecture. It is not obvious which approach is best.